THE SPATIAL THEORY OF LINEAR ELASTIC MEMBERS BY DIRECT KINEMATIC METHOD

THE SPATIAL THEORY OF LINEAR ELASTIC MEMBERS
BY DIRECT KINEMATIC METHOD

By
MITHILESH KUMAR SINHA, B.Sc.(Eng.)

A Thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
Master of Engineering

McMaster University
February 1970

MASTER OF ENGINEERING (1970)
(Civil Engineering)

McMASTER UNIVERSITY
Hamilton, Ontario

TITLE: The Spatial Theory of Linear Elastic Members by Direct Kinematic Method

AUTHOR: Mithilesh Kumar Sinha, B.Sc.(Eng.), Patna University

SUPERVISOR: Professor G. Æ. Oravas

NUMBER OF PAGES: ix, 103

SCOPE AND CONTENTS:

In this work the direct, kinematic, small-displacement theory has been developed for the analysis of thin, elastic members which are curved and twisted in their natural configurations. Principles of continuum mechanics have been used to derive the equations of equilibrium. Throughout this investigation the three-dimensional aspect of the problem is preserved. Local kinematic compatibility of the displacement field has been investigated by the formal Saint-Venant's method. This development serves to substantiate the validity of the kinematic tridimensional approach. By the judicious neglection of small terms of higher order throughout this analysis, the basic system of equations arrived at by the author admit favourable comparison with the existing equations by other authors.

ACKNOWLEDGEMENTS

The author takes this opportunity to express his sincere and profound gratitude to his research supervisor, Dr. G. Æ. Oravas, not only for his guidance but also for his inspiration and advice inside and outside of this research work. The financial support for this research in the form of N.R.C. grant is gratefully acknowledged. The author wishes to extend his sincere thanks to Dr. Leslie C. McLean, a former graduate student in the Department of Civil Engineering and Engineering Mechanics, whose preliminary investigations of this problem form the basis of this thesis. In addition, the author thanks Mrs. H. Kennelly, who typed the entire manuscript.

TABLE OF CONTENTS

NOTATION

A	(normal) cross-sectional area of slender members
$A(\mathcal{R}_n), A(\mathcal{R}_b)$	first area moments about normal $\bar{E}_n$ and binomial $\bar{E}_b$ axes
$d\bar{A}$	directed differential surface element
$\mathcal{D}_t, \mathcal{D}_n, \mathcal{D}_b$	derivative operators referred to mobile Euler base
$\bar{e}_x, \bar{e}_y, \bar{e}_z$	base vectors of inertial frame of reference
$\bar{e}_t, \bar{e}_n, \bar{e}_b = \{\bar{e}_i\}$	unit vectors of mobile Euler base in deformed configuration
$\bar{E}_t, \bar{E}_n, \bar{E}_b = \{\bar{E}_i\}$	unit vectors of mobile Euler base in undeformed configuration
E	Euler (Young) modulus of elasticity
$\bar{f}$	body force intensity
$d\bar{F}$	differential force vector
$F_{tt}(\sigma), F_{tn}(\sigma), F_{tb}(\sigma)$	components of stress resultant vector
$\bar{F}_t(\sigma)$	stress resultant vector
G, μ	shear modulus of elasticity
I_{tt}, I_{nn}, I_{bb}	second area moments about $\bar{E}_t, \bar{E}_n, \bar{E}_b$ base vectors respectively
$\bar{\bar{I}} = \bar{E}_r \bar{I}_r = I_{rs} \bar{E}_r \bar{E}_s$	second area moment tensor
m_1, m_2	arbitrary functions of S

$M_{tt}(\sigma), M_{tn}(\sigma)$	components of stress couple tensor
$M_{tb}(\sigma)$	$\bar{\bar{M}}(\sigma) = \bar{E}_r \bar{M}_r(\sigma) = M_{rs}(\sigma)\bar{E}_r\bar{E}_s$
$d\bar{M}$	differential moment vector
$\bar{M}_t(\sigma) = \bar{E}_t \cdot \bar{\bar{M}}(\sigma)$	stress couple vector
$\bar{\bar{M}}(\sigma) = \bar{E}_r \bar{M}_r(\sigma) = M_{rs}(\sigma)\bar{E}_r\bar{E}_s$	stress couple tensor
n	subscript for component referred to $\bar{E}_n$ vector
$\bar{n}$	vector normal to external surface of body (slender member)
p, P	arbitrary point on elastica in deformed and undeformed configurations respectively
$\bar{\bar{P}}$	auxiliary tensor
$\mathcal{R}_n, \mathcal{R}_b$	components of vector $\bar{\mathcal{R}}$ referred to $\bar{E}_n$ and binormal $\bar{E}_b$ axes of the Euler mobile directed base
$\bar{r}^\circ, \bar{R}^\circ$	position vectors of centroid of cross section referred to fixed directed base in deformed and undeformed configurations respectively
$\bar{r}, \bar{R}$	position vectors of arbitrary point in member-space referred to fixed directed base in deformed and undeformed configurations respectively
$\bar{\mathcal{r}}, \bar{\mathcal{R}}$	position vectors of arbitrary point in member space referred to directed mobile Euler base in deformed and undeformed states respectively

s, S	parametric arc-lengths in deformed and un-deformed configurations respectively
$\bar{\bar{S}}$	Saint-Venant compatibility tensor
t	subscript for component referred to $\bar{E}_t$ vector; time parameter
T	torsion of space curve in un-deformed state
$\bar{T}$	torsion vector in un-deformed state
$\bar{U}^{\circ}$	displacement vector for centroid of cross section, and elastica
$\bar{U}$	displacement vector for arbitrary point in member space
dv	differential volume of slender member
v	total volume of slender member

δ	variational operator
$\bar{\bar{\varepsilon}}$	strain tensor
$\varepsilon_{ij}=\bar{E}_i\bar{E}_j:\bar{\bar{\varepsilon}}$	strain tensor components
θ	rotation
$\bar{\theta} = \theta\bar{E}_t$	rotational vector
κ, K	curvatures of elastica in deformed and undeformed configurations respectively
$\lambda=\frac{2\nu}{1-2\nu}\cdot\mu$	Lamé's Second Elastic Coefficient
$\mu=G$	Lamé's First Elastic Coefficient (=shear modulus of elasticity)
ν	Poisson's ratio

ρ	mass density
Σ	total external surface area of slender members
$\bar{\sigma}_t = \bar{E}_t \cdot \bar{\bar{\sigma}}$	stress vector
$\bar{\bar{\sigma}} = \bar{E}_r \bar{\sigma}_r = \sigma_{rs} \bar{E}_r \bar{E}_s$	stress tensor
$\sigma_{ij} = \bar{E}_i \bar{E}_j : \bar{\bar{\sigma}}$	stress tensor components
τ	torsion of elastica in deformed configuration
$\bar{E}_t \cdot \frac{\partial \bar{U}}{\partial \bar{R}} = \frac{\partial \bar{U}}{\partial S} = \bar{\phi}$	directional displacement derivative along undeformed elastica
ϕ_t, ϕ_n, ϕ_b	components of directional displacement derivative
ψ	warping function
$d\Omega$	infinitesimal arc length along the perimeter of arbitrary cross-section
Ω	perimeter of arbitrary cross-section

CHAPTER 1

INTRODUCTION

§(1.1) General

The subject of this study is the deformation of elastic members whose longitudinal axes represent general curves in space. The direct method of tensor analysis has been used throughout the analysis as the most appropriate mathematical tool. In this investigation Euler's kinematical concept is used to establish the state of strain in the slender members. In order to prove the compatibility of the assumed displacement field Saint Venant's method has been pursued in its direct form. The geometric space occupied by the slender member has been treated as a finite portion of an elastic continuum to which the Cauchy Axioms of Motion are applicable. Finally, the generalised Bernoulli-Euler Equation for the slender elastic members is obtained through suitable simplifications. The entire analysis is carried out within the limitations of the small-displacement theory. The present investigation pursues a systematic mathematical procedure which yields readily to a conceptual interpretation of deformation-phenomena of slender elastic members. This method of analysis is considerably different from conventional procedures and facilitates a more descriptive appreciation of the deformation process as a tridimensional phenomenon. An attempt has been made for consistency in neglecting small terms of higher order in all expressions. The class of problems of slender members discussed in this thesis is restricted to those cases in which shear deformations and the centre-line stretching are considered to be negligibly small.

§(1.2) A Historical Introduction:

The first rational enquiry of the phenomena of resistance of slender member was made by Galileo Galilei (1564-1642). In the course of his investigation he treated solids as inelastic material subject to brittle rupture, not being in possession of any constitutive law connecting displacement with the force that produces them. Galilei attempted to determine the limiting strength of a beam in rupture without deformation whose one end was fixed into a wall and subjected to its own weight or applied weight. The celebrated constitutive elastic law was enunciated by Robert Hooke (1635-1703) in 1678. Later on the tridimensional constitutive equations for molecular isotropic elastic solids were given by Claude-Louis-Marie-Henri Navier (1785-1836) in 1821.

In 1685 Gottfried Wilhelm Leibniz gave the first mathematical analysis of the tension in the interior fibres of a loaded beam based on the assumption that this tension varies linearly across the cross-section. He concluded, in a special case, that the bending moment is proportional to the second area moment of the cross-section of the beam. The second area moment appears for the first time in Leibniz's work, which contains the first application of infinitesimal calculus to mechanics of solids.

Jacob Bernoulli (1655-1705) in 1691 to 1705 investigated problems concerning the resistance of bent rods. He was inspired by Leibniz's paper of 1685 which initiated the modern mathematical theory of elasticity by connecting the constitutive Hooke's law to the stretching of the fibres in the cross-section of the beam in the Galilei's problem of the cantilever beam. Bernoulli assumed, like Hooke and Huygens before him, that the resistance of a bent rod was due to the extension and contraction of its longitudinal filaments. In the course of these researches Bernoulli saw that a relation giving the ratio (force)/(area) or *mean stress*, as a function of strain characterizes a material rather than a particular specimen of material. This was in 1704 and marked the earliest occurrence of a true *stress-strain* relation and a material property of a deformable medium. It is interesting to note that only Bernoulli's reluctance to put any faith in the Hooke's law, which contradicted his experiments on guts, kept him from introducing the so-called Elastic modulus E.

The investigations of Jacob Bernoulli were taken up by Antoine Parent (1666-1716), an unusual scientist by being a capable theoretician and a talented experimenter. Parent tackled the original Leibniz's problem in 1713 and demonstrated, for the first time, the existence of the interior *shear stresses*. He balanced the internal stress resultants apart from balancing the applied moments and internal stress couples in the analysis of the loaded beam and correctly located the neutral line by this equilibrium criterion. According to Truesdell[(1)*], Parent's excellent work, which really foreshadows the stress principle, was published obscurely, drew no attention, and consequently had no influence on the development of the theory of beams.

Leonhard Euler (1707-1783) in his treatment of the problems of *elastica*, from 1735 to 1774, evolved the concept of the undeformable longitudinal fiber which resists bending (the most common method of present day analysis). In 1728 both Euler and Daniel Bernoulli (1700-1782), nephew of Jacob Bernoulli, independently established a unified theory of elastica:

$$M = \frac{B}{r}$$

where B denotes the modulus of bending or flexural rigidity and r is the radius of curvature of the bent beam. Daniel Bernoulli was the first to linearize and integrate this equation in 1735. In 1727, in an earlier unpublished work on bells considered as a cluster of curved beam segments, Euler was the first to deduce the Jacob Bernoulli equation of bending from the Hooke's law for extension of the beam fibres,

---

*Numbers in parantheses refer to the References on page 86.

assuming like Jacob Bernoulli the inextensible neutral fibre or elastica to be on the concave side of the curved beam. However, this error which was present in all of Euler's researches did not have any effect on his results concerning elastica and stability, because Euler never specialized sufficiently to have any need of using it. In this work Euler became the first scientist to define *modulus of extension*, E, as the material property of beam. (Young, who studied Euler's work, missed Euler's point and erroneously gave his modulus, which was not a material property, but depended upon the size of the specimen). Again it was Euler in 1774 who gave his famous study of skew elastica in space and established the *kimematic flexural formula*

$$\bar{M} = B\left(\frac{d^2\bar{R}}{dS^2} \times \frac{d\bar{R}}{dS}\right) = \frac{B}{r}\bar{B}$$

involving the osculating plane and the concept of binormal $\bar{B}$; yet another paper on a more extensive study of the spatial threads presented in 1782 (published posthumously in 1786) contained many general results of differential geometry of spatial curves including the mobile Euler directed base $\{\bar{t},\bar{n},\bar{b}\}$.

Gaspard Monge (1764-1818) was the first to differentiate between the two types of curvatures, i.e. the curvature and torsion.

Jean-Joseph Fourier (1768-1830) also differentiated between the two curvatures for the curve, and communicated it to Michel-Ange Lancret (1774-1807). Lancret published a memoir in 1805 in which he establishes differential expressions for the two curvatures which were called *angle de torsion* and *angle de courbure* by Louis-Légér Vallée (1784-1864) in 1825. Their finite forms of expression were established by Augustin-Louis Cauchy (1789-1857) in 1826 where the *première courbure* is $\frac{1}{r}$ and the *seconde courbure* is $\frac{1}{R}$.

The study of helical curves in terms of spatial coordinates was initiated as early as in 1724 by Henri Pitot (1695-1771), who introduced the term *courbe a double courbure* for these curves.

Alexis-Claude Clairaut (1713-1765) studies special space curves as intersections of surfaces, but not as independent entities, in his famous monograph in 1731. His work inspired the investigations of Euler and Monge.

Johann Christian Martin Bartels (1769-1836), of the University of Dorpat in Estonia,resumed Euler's investigations of the spatial curves in 1824. Bartels employed the mobile Euler directed base and even went further than Euler by employing a modified form of the convected local directed base.

Bartels was successful in establishing the complete elementary theory of space curves expressed in extrinsic coordinates in the *Bartels Fundamental Formulas for Space Curves*

$$\begin{cases} \bar{n}\cdot d\bar{t} = -\bar{t}\cdot d\bar{n} = |d\bar{t}| = \kappa \; ds \\ \bar{b}\cdot d\bar{n} = -\bar{n}\cdot d\bar{b} = |d\bar{b}| = \tau \; ds \\ \bar{t}\cdot d\bar{b} = -\bar{b}\cdot d\bar{t} \end{cases}$$

A fully equivalent set of formulas to Bartels' formulas was independently discovered by Jean-Frédéric Frenet (1816-1888) in his doctoral dissertation of 1847. Frenet formulas expressed the arc rate of changes of the vectors in the mobile Euler directed base.

Since Bartels' pioneering work remained unknown in Western Europe, these fundamental equations completing the elementary theory of space curves are known as the *Frenet-Formulas*. Even Frenet's work remained unknown in France until Joseph Alfred Serret (1819-1885) found them independently in 1851. Therefore, sometimes the Bartels Formulas are also called the Frenet-Serret Formulas. The researches of Frenet and Serret on space curves were stimulated by an investigation of Adhemar-Jean-Claude Barré de Saint-Venant (1797-1886) on spatial curves in connection with his studies of spatial elastic bars in 1845.

Jean-Gaston Darboux (1842-1917) introduced explicitly the kinematical method of space curves in 1866 and extensively employed it in his lectures of 1887-1896.

Ernesto Cesàro (1859-1906) studied the intrinsic geometry of spatial curves by direct methods in his monograph of 1896.

In 1776 Charles-Augustin de Coulomb (1736-1806) contributed significantly for the theory of stress in the flexure of beams of constant cross sections. He wrote down all conditions of equilibrium for the forces acting upon the cross-section of a loaded beam and proved that shear stress was not only possible but *necessary*. Coulomb neglected deformation and restricted his attention to statics of beams. He was the first to carry out extensive experiments for torsional behaviour of thin wires in 1777-1784 and found that the torque was proportional to small twist. However, he made no attempt at a mathematical theory of torsion.

Cauchy was the first to give a general theory of the stress tensor principle independent of the material constitution of the continuum in his memoirs of 1823 and 1827. It is interesting to note that the basic results of the stress tensor principle were already present in an unpublished memoir on the double refraction of light in elastic translucent media by Augustin-Jean Fresnel (1788-1827) written in 1822, which was known to Cauchy. Already in 1766 Euler, in a treatise on the mechanics of perfect fluids, introduced and interpreted the components of the rate of deformation tensor for the fluid continuum.

While all the basic ideas necessary for the general theory of stress had been proposed by 1773, there was no sign of that other necessary ingredient of a full theory of elasticity, namely, the theory of strain . Barré de Saint-Venant was the first to derive the local *kinematic compatibility condition* for elastic isotropic material as a function of the state of strain in 1860*. In 1892 Eugenio Beltrami (1835-1900) obtained the compatibility condition in the absence of body forces $\bar{f}=0$, for *linearly* elastic isotropic materials as a function of the state of stress expressed by the components of the stress tensor $\bar{\bar{\sigma}} = \sigma_{rs} \bar{E}_r \bar{E}_s$. The compatibility condition in terms of stresses and the body forces was obtained by Luigi Donati (1846-1932)

*"Élasticité des Solides" *Bulletin, Societé Philomatique de Paris*, [Extrait de precesverbos seances pendent l'année (1860)], pp. 77-80

in 1894 as an extremal of the Potential Energy Functional of the linearly elastic solid expressed in terms of the Beltrami stress functions. He also obtained strain tensor components as an extremal of the Potential Energy Functional.

Augustus Edward Hough Love[2] (1863-1940) in his celebrated book studied the problem of slender members by the use of classical methods. He made the approximation that the elastica remains unstretched in its displaced configuration. The concept of principal flexo-torsional axes was introduced by Love for defining the orientation of cross-sections relative to the space curves.

DiPrima and Handelman[3] employed vector methods to derive the equations of vibration of twisted beams under the assumption that strains due to shear stresses could be neglected. They obtained the natural frequency of a cantilever twisted beam and pointed out some analogy in the analysis of twisted and untwisted beams. Massoud[4] in a short note on the equation of motion for any twisted and curved beams also used vector methods and gave simple methods for the derivation of the equation of motion of incomplete elastic rings.

The problem of curved and twisted rods was also analysed by Tso[5]. His equations of motion are based on the Newton Second Axiom of Motion in their application to linear continuum in a configuration of a space curve. The couplings between the three types of motion i.e., extensional, flexural and torsional were shown explicitly by him in the spatial case of space curve possessing the form of a circular helix.

CHAPTER 2

DIFFERENTIAL GEOMETRY OF THE ELASTICA

§(2.1) Definition

Elastica is defined to be a longitudinal fibre embedded in the slender member which passes through the centroid of the cross-section and remains *invariant* in length in the process of deformation.

§(2.2) Kinematics of Elastica

The undeformed configuration of the slender members is assigned to possess both curvature and torsion, i.e., the elastica (or, the centre-line) being a space curve of general nature.

With reference to figure (2.1) let P be any generic point on the elastica with radius vector $\bar{R}^{\circ}$ referred to the inertial frame of reference $\{\bar{e}_x, \bar{e}_y, \bar{e}_z\}$. S denotes the arc-length parameter of the space curve in undeformed configuration.

On applying the fundamental concepts of differential geometry to the *undeformed* configuration of elastica the following equations for the unit base vectors are obtained:

the unit tangent vector -

$$\bar{E}_t = \frac{d\bar{R}^{\circ}}{dS}$$

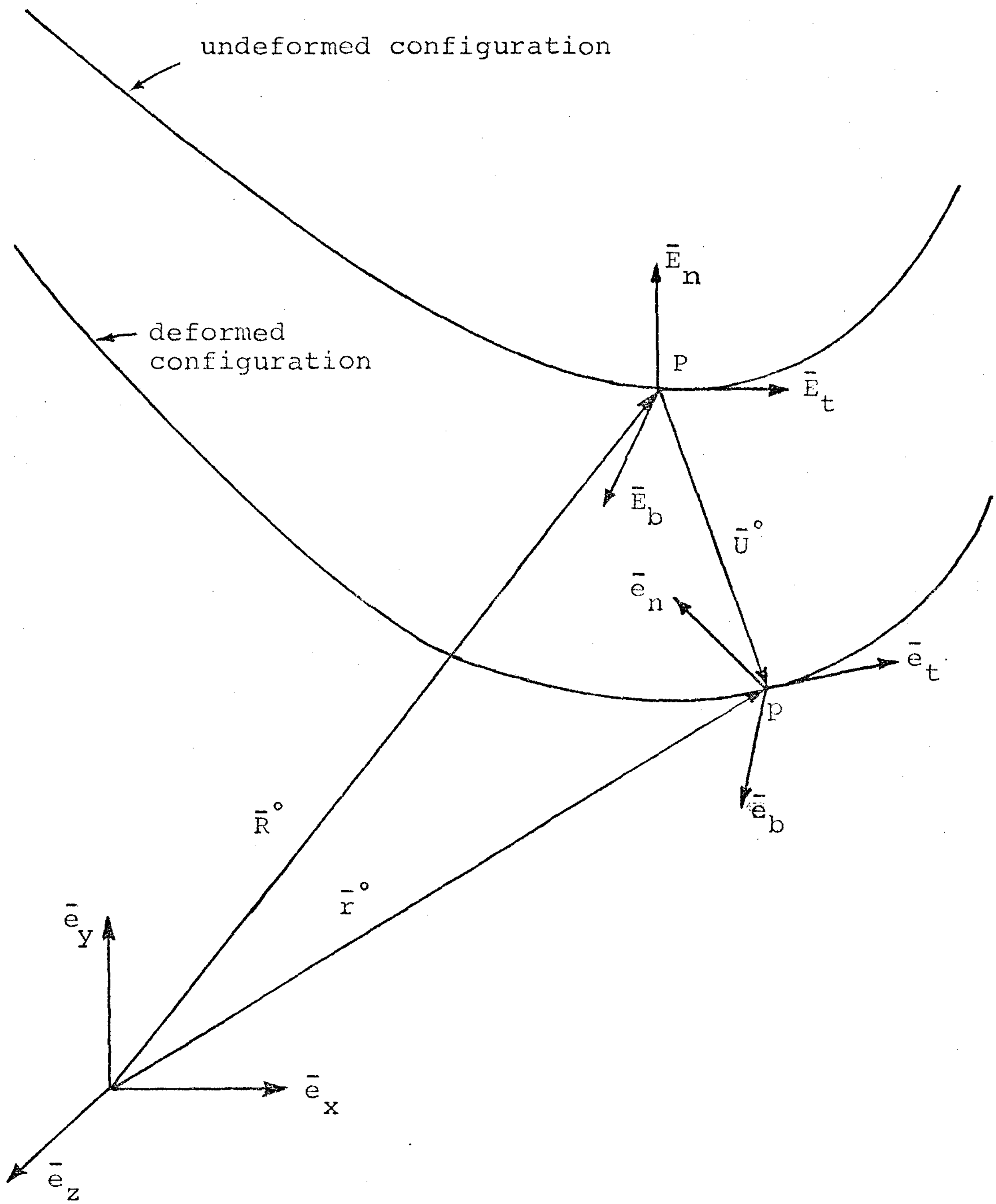

undeformed configuration
deformed configuration
$\bar{E}_n$
P
$\bar{E}_t$
$\bar{E}_b$
$\bar{U}^\circ$
$\bar{e}_n$
$\bar{e}_t$
p
$\bar{e}_b$
$\bar{R}^\circ$
$\bar{r}^\circ$
$\bar{e}_y$
$\bar{e}_x$
$\bar{e}_z$

Figure (2.1)

the unit normal vector

$$\bar{E}_n = \frac{\dfrac{d^2\bar{R}^\circ}{ds^2}}{\left|\dfrac{d^2\bar{R}^\circ}{ds^2}\right|}$$

and the unit binormal vector (by Euler's definition)

$$\bar{E}_b = \bar{E}_t \times \bar{E}_n$$

The set of unit base vectors $\{\bar{E}_t, \bar{E}_n, \bar{E}_b\}$ constitute a dextral set of local vector triad known as the Mobile Euler Directed Base.

The Darboux-vector is defined by

$$\bar{D} = T\bar{E}_t + K\bar{E}_b = \bar{T} + \bar{K}$$

where

$T \equiv$ Torsion of space curve at the generic point P

$K \equiv$ Curvature of space curve at the generic point P

$\bar{T} \equiv T\bar{E}_t$ – Torsion vector

$\bar{K} \equiv K\bar{E}_b$ – Curvature vector

Let the generic point P, measured by the position vector $\bar{R}^\circ$ after the deformation be the point p , measured by position vector $\bar{r}^\circ$. In general, the unit vectors of the directed Euler base embedded in the slender member do not remain orthogonal to each other. However, in a deformation process a system of orthonormal vectors $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$ can be constructed for the deformed configuration by a similar method to the undeformed configuration. Let $\bar{U}^\circ = \bar{r}^\circ - \bar{R}^\circ$ be the displacement vector of the generic point P of elastica and s, the parametric arc-length of the space curve in its deformed configuration. Hence, for the *deformed* configuration:

the tangent vector

$$\bar{e}_t = \frac{d\bar{r}^\circ}{ds}$$

the normal vector

$$\bar{e}_n = \frac{d^2\bar{r}^\circ}{ds^2} \Big/ \left|\frac{d^2\bar{r}^\circ}{ds^2}\right|$$

and the binormal vector (by Euler's definition)

$$\bar{e}_b = \bar{e}_t \times \bar{e}_n$$

The unit vectors $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$ constitute a dextral set of a local vector triad known as the Mobile Euler Directed Base.

The Darboux-vector, in this case, is

$$\bar{d} = \tau\bar{e}_t + \kappa\bar{e}_b = \bar{\tau} + \bar{\kappa}$$

where

τ - torsion of space curve at the generic point p

κ - Curvature of space curve at the generic point p

$\bar{\tau}, \bar{\kappa}$ - torsion and curvature vectors respectively.

§(2.3) Kinematics of Deformation

In order to specify the directed base $\{\bar{e}_t, \bar{e}_n, \bar{e}_b\}$ in terms of the parameters of the undeformed configuration and to prove within the limitations of the definition of elastica that the vectors $\bar{e}_i$ are orthonormal, a number of fundamental relations can be established.

The arc-length measured along the undeformed elastica $\bar{R}^{\circ}$ is prescribed by the fundamental form

$$dS^2 = d\bar{R}^{\circ} \cdot d\bar{R}^{\circ}$$

Similarly, the arc-length along the deformed elastica $\bar{r}^{\circ}$ is prescribed by the fundamental form

$$d\delta^2 = d\bar{r}^{\circ} \cdot d\bar{r}^{\circ}$$

Again, with reference to figure (2.1)

$$\bar{r}^{\circ} = \bar{R}^{\circ} + \bar{U}^{\circ}$$

Thus,

$$\begin{aligned} d\delta^2 = d\bar{r}^{\circ} \cdot d\bar{r}^{\circ} &= d(\bar{R}^{\circ} + \bar{U}^{\circ}) \cdot d(\bar{R}^{\circ} + \bar{U}^{\circ}) \\ &= d\bar{R}^{\circ} \cdot d\bar{R}^{\circ} + 2d\bar{R}^{\circ} \cdot d\bar{U}^{\circ} + d\bar{U}^{\circ} \cdot d\bar{U}^{\circ} \\ &= dS^2 + 2d\bar{R}^{\circ} \cdot d\bar{U}^{\circ} + d\bar{U}^{\circ} \cdot d\bar{U}^{\circ} \end{aligned}$$

or

$$\left(\frac{d\delta}{dS}\right)^2 = 1 + 2\,\frac{d\bar{R}^{\circ}}{dS} \cdot \frac{d\bar{U}^{\circ}}{dS} + \frac{d\bar{U}^{\circ}}{dS} \cdot \frac{d\bar{U}^{\circ}}{dS}$$

If it is assumed that the displacements are small, confirming the linear theory, then the non-linear terms in $\bar{U}^{\circ}$ are neglected. Hence, the above equation assumes the form

$$\frac{d\bar{r}^{\circ} \cdot d\bar{r}^{\circ}}{d\bar{R}^{\circ} \cdot d\bar{R}^{\circ}} = \left(\frac{d\delta}{dS}\right)^2 \doteq 1 + 2\,\frac{d\bar{R}^{\circ}}{dS} \cdot \frac{d\bar{U}^{\circ}}{dS} = 1 + 2\,\bar{E}_t \cdot \frac{d\bar{U}^{\circ}}{dS} = 1 + 2\varepsilon_{tt}^{\circ}$$

where, $\overset{\circ}{\varepsilon}_{tt}$ is the strain of the elastica in the direction of the tangent vector $\bar{E}_t$. According to the basic definition of the invariant length of elastica [see § (2.1)], $\overset{\circ}{\varepsilon}_{tt}$ is zero for mathematical analysis. This is justified in the sense that $\overset{\circ}{\varepsilon}_{tt}$, i.e. the longitudinal strain, is a very small quantity in the small-displacement theory in comparison with unity, and, therefore, it can be neglected.

Hence, the Hypothesis of the Invariant Length of Elastica implies

$$d\mathcal{s}^2 \doteq dS^2$$

or

$$d\bar{r}^{\circ} \cdot d\bar{r}^{\circ} \doteq d\bar{R}^{\circ} \cdot d\bar{R}^{\circ}$$

It should be noted that the above result has led to the definition of the elastica.

Therefore,

$$\bar{e}_t = \frac{d\bar{r}^{\circ}}{d\mathcal{s}} \doteq \frac{d\bar{r}^{\circ}}{dS} = \frac{d}{dS}(\bar{R}^{\circ} + \bar{U}^{\circ}) = \frac{d\bar{R}^{\circ}}{dS} + \frac{d\bar{U}^{\circ}}{dS}$$

or

$$\bar{e}_t = \bar{E}_t + \frac{d\bar{U}^{\circ}}{dS} \qquad \ldots\ldots (2.1)$$

It can now be shown that vector $\bar{e}_t$ is a unit vector to the same degree of approximation.

$$\bar{e}_t \cdot \bar{e}_t = (\bar{E}_t + \frac{d\bar{U}^{\circ}}{dS}) \cdot (\bar{E}_t + \frac{d\bar{U}^{\circ}}{dS})$$

$$= \bar{E}_t \cdot \bar{E}_t + 2\bar{E}_t \cdot \frac{d\bar{U}^{\circ}}{dS} + \frac{d\bar{U}^{\circ}}{dS} \cdot \frac{d\bar{U}^{\circ}}{dS}$$

$$\doteq 1 + 2\,\overset{\circ}{\varepsilon}_{tt}$$

if the non-linear term $(\frac{d\bar{U}^{\circ}}{dS} \cdot \frac{d\bar{U}^{\circ}}{dS})$ is neglected. Since, by definition $\overset{\circ}{\varepsilon}_{tt} = 0$, then

$$\bar{e}_t \cdot \bar{e}_t = 1$$

and the vector $\bar{e}_t$ is a unit tangent vector.

The curvature vector of the deformed elastica

$$\bar{k} = k\,\bar{e}_n = \frac{d^2\bar{r}^{\circ}}{ds^2} = \frac{d}{ds}\left(\frac{d\bar{r}^{\circ}}{ds}\right) = \frac{d}{ds}\,\bar{e}_t \doteq \frac{d}{dS}\,\bar{e}_t$$

Substituting for $\bar{e}_t$ from equation (2.1)

$$\bar{k} = \frac{d}{dS}\left(\bar{E}_t + \frac{d\bar{U}^{\circ}}{dS}\right) = \bar{D} \times \left(\bar{E}_t + \frac{d\bar{U}^{\circ}}{dS}\right)$$

Since $\frac{d\bar{E}_t}{dS} = \bar{D} \times \bar{E}_t = K\bar{E}_n$ (see Appendix A)

$$\bar{k} = K\bar{E}_n + \frac{d^2\bar{U}^{\circ}}{dS^2} = k\,\bar{e}_n = \kappa\bar{e}_n \quad .... \quad (2.2)$$

Hence

$$\bar{k} \cdot \bar{k} = \kappa^2 = \left(K\bar{E}_n + \frac{d^2\bar{U}^{\circ}}{dS^2}\right) \cdot \left(K\bar{E}_n + \frac{d^2\bar{U}^{\circ}}{dS^2}\right)$$

$$= K^2(\bar{E}_n \cdot \bar{E}_n) + 2K\,\bar{E}_n \cdot \frac{d^2\bar{U}^{\circ}}{dS^2} + \frac{d^2\bar{U}^{\circ}}{dS^2} \cdot \frac{d^2\bar{U}^{\circ}}{dS^2}$$

$$= K^2 + 2\,K\,\bar{E}_n \cdot \frac{d^2\bar{U}^{\circ}}{dS^2}\,, \quad \left[\text{neglecting } \left(\frac{d^2\bar{U}^{\circ}}{dS^2} \cdot \frac{d^2\bar{U}^{\circ}}{dS^2}\right)\right]$$

or

$$\kappa^2 = K^2\left[1 + \frac{2}{K}\left(\bar{E}_n \cdot \frac{d^2\bar{U}^{\circ}}{dS^2}\right)\right]$$

or

$$\kappa = K\left[1 + \frac{2}{K}\left(\bar{E}_n \cdot \frac{d^2\bar{U}^{\circ}}{dS^2}\right)\right]^{\frac{1}{2}}$$

Using the Binomial Theorem for the expansion of the terms in the bracket, and neglecting higher order terms the following result is obtained

$$\kappa = K + \bar{E}_n \cdot \frac{d^2 \bar{\overset{\circ}{U}}}{dS^2} \qquad \ldots . (2.3)$$

The displacement vector $\bar{\overset{\circ}{U}}$ can be referred to the Euler directed mobile base $\{\bar{E}_t, \bar{E}_n, \bar{E}_b\}$

$$\bar{\overset{\circ}{U}} = \overset{\circ}{U}_t \, \bar{E}_t + \overset{\circ}{U}_n \, \bar{E}_n + \overset{\circ}{U}_b \, \bar{E}_b \qquad \ldots . (2.4)$$

Differentiating equation (2.4) with respect to S and using Frenet-Serret formulas for derivatives of mobile base vectors $\{\bar{E}_i\}$ (see Appendix A)

$$\frac{d\bar{\overset{\circ}{U}}}{dS} = \left(\frac{d\overset{\circ}{U}_t}{dS} - K\overset{\circ}{U}_n\right)\bar{E}_t + \left(\frac{d\overset{\circ}{U}_n}{dS} + K\overset{\circ}{U}_t - T\overset{\circ}{U}_b\right) \bar{E}_n + \left(\frac{d\overset{\circ}{U}_b}{dS} + T\,\overset{\circ}{U}_n\right)\bar{E}_b \qquad \ldots . (2.5)$$

The following new quantities are defined:

$$\left.\begin{aligned} \phi_t &= \frac{d\bar{\overset{\circ}{U}}}{dS} \cdot \bar{E}_t = \frac{d\overset{\circ}{U}_t}{dS} - K\,\overset{\circ}{U}_n \\ \phi_n &= \frac{d\bar{\overset{\circ}{U}}}{dS} \cdot \bar{E}_n = \frac{d\overset{\circ}{U}_n}{dS} + K\,\overset{\circ}{U}_t - T\,\overset{\circ}{U}_b \\ \phi_b &= \frac{d\bar{\overset{\circ}{U}}}{dS} \cdot \bar{E}_b = \frac{d\overset{\circ}{U}_b}{dS} + T\,\overset{\circ}{U}_n \end{aligned}\right\} \qquad \ldots . \quad (2.6)$$

and

Kinematically, the quantities ϕ_t, ϕ_n, ϕ_b admit physical interpretation as the projections of the directional displacement derivative $\frac{d\bar{U}^{\circ}}{dS}$ on the Euler mobile base vectors $\bar{E}_t, \bar{E}_n, \bar{E}_b$ respectively.

By definition, the elastica remains unextended, i.e. $\phi_t = \overset{\circ}{\varepsilon}_{tt} = 0$. Thus, the condition of unextended centre-line is expressed as

$$\phi_t = \frac{dU_t^{\circ}}{dS} - K\, U_n^{\circ} = 0 \qquad \text{....(2.7)}$$

Using this condition, equation (2.5) can be written as

$$\frac{d\bar{U}^{\circ}}{dS} = \phi_n\, \bar{E}_n + \phi_b\, \bar{E}_b$$

Substituting the above in equation (2.1) yields

$$\bar{e}_t = \bar{E}_t + \phi_n\, \bar{E}_n + \phi_b\, \bar{E}_b \qquad \text{....(2.8)}$$

From equation (2.3)

$$\kappa = K + \bar{E}_n \cdot \frac{d^2\bar{U}^{\circ}}{dS^2} = K + \bar{E}_n \cdot \frac{d}{dS}\left(\frac{d\bar{U}^{\circ}}{dS}\right)$$

$$= K + \bar{E}_n \cdot \frac{d}{dS}\left(\phi_n\, \bar{E}_n + \phi_b\, \bar{E}_b\right)$$

$$= K + \bar{E}_n \cdot \left[\frac{d\phi_n}{dS}\, \bar{E}_n + \phi_n(-K\bar{E}_t + T\bar{E}_b) + \frac{d\phi_b}{dS}\, \bar{E}_b + (-T\bar{E}_n)\, \phi_b\right]$$

or

$$\kappa = K + \frac{d\phi_n}{dS} - T\phi_b \qquad \text{....(2.9)}$$

where the kinematic Frenet-Serret Formulas have been used (see Appendix A). Again, from equation (2.2)

$$\bar{k} = \kappa\bar{e}_n = K\bar{E}_n + \frac{d^2\bar{U}^\circ}{dS^2}$$

or

$$\bar{e}_n = \frac{1}{\kappa}[K\bar{E}_n + \frac{d\phi_n}{dS}\bar{E}_n + \phi_n(-K\bar{E}_t + T\bar{E}_b) + \frac{d\phi_b}{dS}\bar{E}_b + (-T\bar{E}_n)\phi_b]$$

$$= \frac{1}{\kappa}[(-K\phi_n)\bar{E}_t + (K + \frac{d\phi_n}{dS} - T\phi_b)\bar{E}_n + (\frac{d\phi_b}{dS} + T\phi_n)\bar{E}_b]$$

Since

$$\kappa = K + \frac{d\phi_n}{dS} - T\phi_b \quad \text{(see equation 2.9)}$$

therefore,

$$\bar{e}_n = -\frac{K}{\kappa}\phi_n\bar{E}_t + \bar{E}_n + \frac{1}{\kappa}(\frac{d\phi_b}{dS} + T\phi_n)\bar{E}_b$$

It can be written in a more simplified form as

$$\bar{e}_n = \bar{E}_n + m_1\bar{E}_t + m_2\bar{E}_b \qquad ....(2.10)$$

where

$$m_1 = -\frac{K}{\kappa}\phi_n$$

$$m_2 = \frac{1}{\kappa}[\frac{d\phi_b}{dS} + T\phi_n]$$

It will now be shown that the vectors $\bar{e}_t, \bar{e}_n$ are orthogonal. From equations (2.1) and (2.2)

$$\bar{e}_t \cdot \bar{e}_n = (\bar{E}_t + \frac{d\bar{U}^\circ}{dS}) \cdot \frac{1}{\kappa}(K\bar{E}_n + \frac{d^2\bar{U}^\circ}{dS^2})$$

Taking the dot products and neglecting the small term

$$(\frac{d\bar{U}^\circ}{dS} \cdot \frac{d^2\bar{U}^\circ}{dS^2})\frac{1}{\kappa}$$

yields

$$\bar{e}_t \cdot \bar{e}_n = \frac{1}{\kappa}[\bar{E}_t \cdot \frac{d^2\bar{U}^\circ}{dS^2} + \frac{d\bar{U}^\circ}{dS} \cdot K\bar{E}_n]$$

$$= \frac{1}{\kappa} [\frac{d}{dS} (\bar{E}_t \cdot \frac{d\bar{U}^{\circ}}{dS})]$$

$$= \frac{1}{\kappa} [\frac{d}{dS} (\overset{\circ}{\varepsilon}_{tt})] = 0$$

Hence, vectors $\bar{e}_t$ and $\bar{e}_n$ are orthogonal to each other to the order of approximation employed. The third vector $\bar{e}_b$ of the Euler Triad is defined by the dextral rule as

$$\bar{e}_b = \bar{e}_t \times \bar{e}_n$$

Substituting for vectors $\bar{e}_t$ and $\bar{e}_n$ from equations (2.8) and (2.10) respectively gives

$$\bar{e}_b = (\bar{E}_t + \phi_n\bar{E}_n + \phi_b\bar{E}_n) \times (m_1\bar{E}_t + \bar{E}_n + m_2\bar{E}_b)$$

Using the distributive law of cross product and neglecting products of two small quantities like $m_1\Phi_n$, $m_1\Phi_b$, $m_2\Phi_n$, $m_2\Phi_b$, the approximate expression for $\bar{e}_b$ is as follows:

$$\bar{e}_b = -\phi_b\bar{E}_t - m_2\bar{E}_n + \bar{E}_b \qquad \text{....(2.11)}$$

Finally, the torsional component of space curve in undeformed and deformed configurations can be related. From the fundamental concepts of differential geometry in deformed configuration

$$\frac{d\bar{e}_b}{ds} = - \tau\bar{e}_n \quad \text{(See Appendix A)}$$

Post dot-multiplying it with $\bar{e}_n$ yields

$$\tau = -\frac{d\bar{e}_b}{ds}\cdot\bar{e}_n \doteq -\frac{d\bar{e}_b}{dS}\cdot\bar{e}_n$$

Substituting expressions for vectors $\bar{e}_n$ and $\bar{e}_b$ from equations (2.10) and (2.11) respectively gives

$$\tau = -\frac{d}{dS}\,[-\phi_b\bar{E}_t - m_2\bar{E}_n + \bar{E}_b]\cdot[m_1\bar{E}_t + \bar{E}_n + m_2\bar{E}_b]$$

or

$$\tau = (\frac{d\phi_b}{dS} - m_2K)m_1 + (\phi_bK + \frac{dm_2}{dS} + T) + (m_2)^2T$$

Neglecting terms like m_1m_2K, $(m_2)^2T$, $m_1\frac{d\phi_b}{dS}$, this torsional relation appears approximately as

$$\tau \doteq T + \frac{dm_2}{dS} + K\phi_b \qquad \text{....(2.12)}$$

From the results of this chapter it is seen that kinematical quantities in the deformed configuration can be represented in terms of the corresponding quantities in the undeformed configuration augmented by their first variation. In symbolic form

$$\bar{e}_i = \bar{E}_i + \delta\bar{E}_i \quad (\text{for } i = t,n,b)$$

$$\kappa = K + \delta K$$

and $$\tau = T + \delta T$$

The variational quantities from equations (2.8), (2.10), (2.11), (2.9) and (2.12) are given as follows:

$$\left.\begin{aligned}
\delta\bar{E}_t &= \phi_n\bar{E}_n + \phi_b\bar{E}_b \\
\delta\bar{E}_n &= m_1\bar{E}_t + m_2\bar{E}_b \\
\delta E_b &= -\phi_b\bar{E}_t - m_2\bar{E}_n \\
\delta K &= \frac{d\phi_n}{dS} - T\phi_b \\
\delta T &= \frac{dm_2}{dS} + K\phi_b
\end{aligned}\right\} \qquad \ldots.(2.13)$$

CHAPTER 3

STRAIN TENSOR

§(3.1) Introduction

In the last chapter geometric properties of the elastica have been investigated in detail. It was concluded therein that each of the kinematical quantities in the deformed configuration can be represented as the same kinematical quantity in the undeformed configuration plus its first variation. By considering a characteristic fibre in the geometric space occupied by the slender member, the state of strain for the continuum shall be established.

§(3.2) Definition

A characteristic parallel fibre is defined as a fibre which is always a constant distance from the elastica. This distance of the characteristic fibre shall be denoted by position vector $\bar{\mathcal{R}}$ in the normal plane $(\bar{E}_n, \bar{E}_b)$ of the Euler Triad. (see figure 3.1). Thus the parallel fibre is located by the position vector $\bar{R} = \overset{\circ}{\bar{R}} + \bar{\mathcal{R}}$ in the undeformed member.

§(3.3) Assumption

It is assumed that the normal cross-section of the slender members retains its shape after deformation, i.e. there

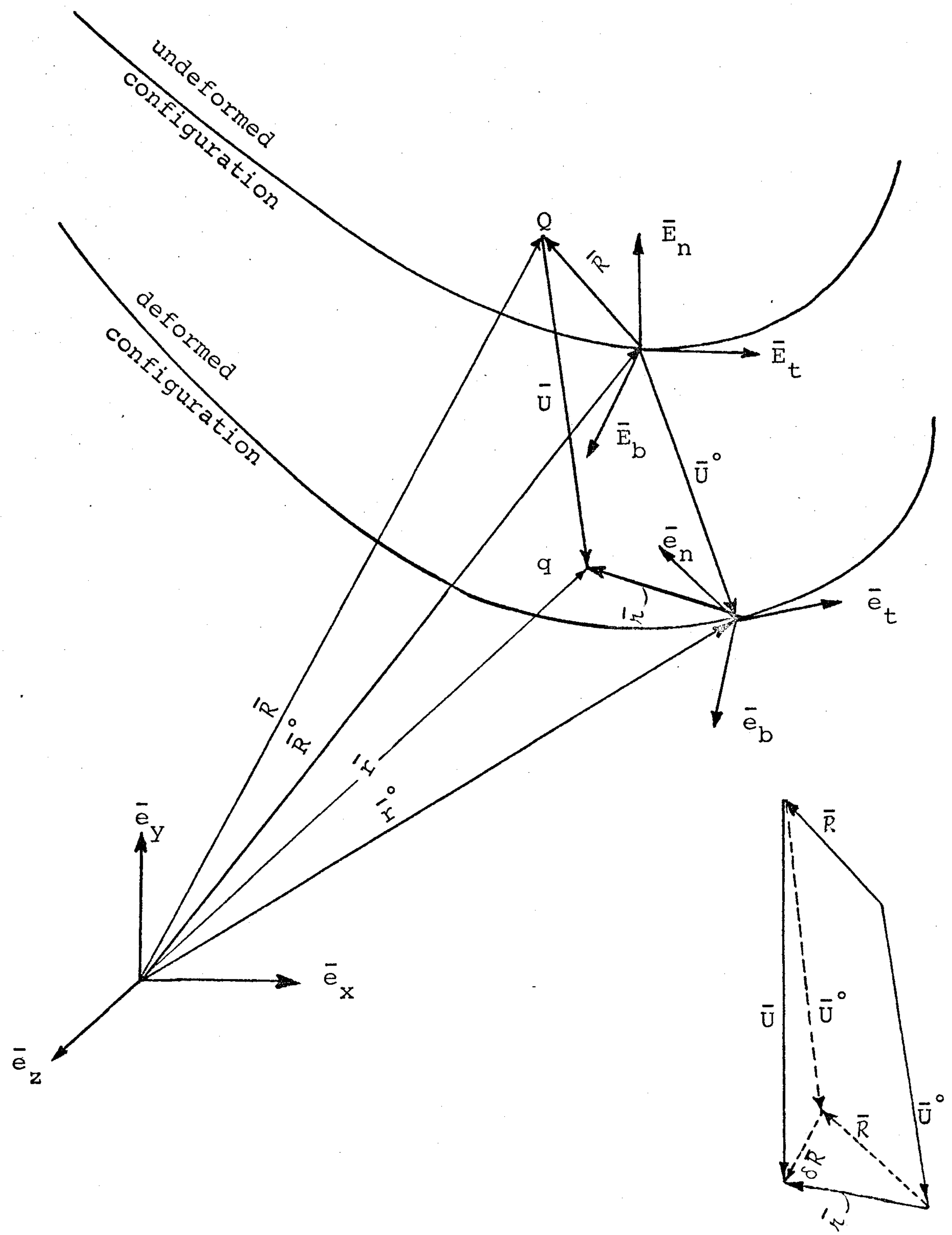


Figure (3.1)

Figure (3.2)

is no stretching in the plane of the cross-section.

§(3.4) Deformation of the Characteristic Parallel Fibre

With reference to figure (3.1) $\{\bar{E}_t, \bar{E}_n, \bar{E}_b\}$ represent the undeformed Euler Triad and $\bar{R}$ denotes the position vector of the characteristic parallel fibre at point Q. It should be noted that the vector $\bar{R} = R_n\bar{E}_n + R_b\bar{E}_b$ is referred to the undeformed normal plane containing $\bar{E}_n, \bar{E}_b$ as axes. Position vector to the generic point in the normal cross-section referred to the inertial base $\{\bar{e}_x, \bar{e}_y, \bar{e}_z\}$ is $\bar{R}$. After the deformation, the vector $\bar{R}$ is transformed into vector $\bar{r}$ and the corresponding position vector referred to the inertial base $\{\bar{e}_x, \bar{e}_y, \bar{e}_z\}$ of the point q is $\bar{r} = \bar{r}^{\circ} + \bar{r}$.

Therefore, in the undeformed state

$$\bar{R} = R_n\bar{E}_n + R_b\bar{E}_b \qquad \ldots.(3.1)$$

Let $\bar{\theta} = \theta\bar{E}_t$, be the rotation of $\bar{R}$ to the generic point Q of the normal cross-section when it passes from undeformed to deformed equilibrium state. It is to be noted here that $\bar{\theta}$ is the independent rotation vector of the normal cross-section. It does not represent the natural rotation of the Euler Triad because of the presence of natural torsion in the slender members.

From figure (3.2), vector equation

$$\bar{R} + \bar{U} = \bar{U}^{\circ} + \bar{r}$$

or

$$\bar{U} = \bar{U}^{\circ} + \bar{R} - \bar{r}$$
$$= \bar{U}^{\circ} + \delta\bar{R}$$
$$= \bar{U}^{\circ} + \delta(R_n\bar{E}_n + R_b\bar{E}_b)$$

Now, imposing the condition of nondeforming cross-section in its own plane and adding the rotation vector in the above equation, displacement vector $\bar{U}$ for a generic point (like Q) is given by

$$\bar{U} = \bar{U}^{\circ} + R_n\ \delta\bar{E}_n + R_b\delta\bar{E}_b + \bar{\theta}\times\bar{R} + (\frac{\partial\theta}{\partial S} + \delta T)\psi\bar{E}_t \quad \text{....(3.2)}$$

where $(\frac{\partial\theta}{\partial S} + \delta T)\psi\bar{E}_t$ denotes the terminal displacement of $\bar{R}$ in the tangential direction, $\psi = \psi(R_n, R_b)$ is an unprescribed function of the cross-section called the warping functions.

Substituting for $\bar{R}$ from equation (3.1), equation (3.2) can be written as

$$\bar{U} = \bar{U}^{\circ} + R_n(\delta\bar{E}_n + \theta\bar{E}_b) + R_b(\delta\bar{E}_b - \theta\bar{E}_n) + (\frac{\partial\theta}{\partial S} + \delta T)\psi\bar{E}_t \quad \text{...(3.3)}$$

§(3.5) Directed Derivative $\frac{\partial}{\partial\bar{R}}$ for the Slender Member in Geometric Space

The total differential of the displacement vector $\bar{U}$ can be written

$$d\bar{U} = \frac{\partial\bar{U}}{\partial S}\,dS + \frac{\partial\bar{U}}{\partial R_n}\,dR_n + \frac{\partial\bar{U}}{\partial R_b}\,dR_b \quad \text{....(3.4)}$$

where $R_n = \bar{R}\cdot\bar{E}_n$, $R_b = \bar{R}\cdot\bar{E}_b$

Again, with reference to figure (3.1)

$$\bar{R} = \bar{R}^{\circ} + \bar{R} = \bar{R}^{\circ} + R_n\ \bar{E}_n + R_b\bar{E}_b$$

And the differential of vector $\bar{R}$

$$d\bar{R} = d\bar{R}^{\circ} + d(R_n\bar{E}_n + R_b\bar{E}_b)$$
$$= d\bar{R}^{\circ} + dR_n\bar{E}_n + R_n d\bar{E}_n + dR_b\bar{E}_b + R_b d\bar{E}_b$$
$$= dS\bar{E}_t + dR_n\bar{E}_n + R_n(-K\bar{E}_t + T\bar{E}_b)dS + dR_b\bar{E}_b$$
$$+ R_b(-T\bar{E}_n)dS$$

or

$$d\bar{R} = (1-KR_n)dS\bar{E}_t + (dR_n - TR_b dS)\bar{E}_n + (dR_b + TR_n dS)\bar{E}_b \quad \ldots.(3.5)$$

Since the total differential of $\bar{U}$ may also be written as

$$d\bar{U} = d\bar{R}\cdot\frac{\partial\bar{U}}{\partial\bar{R}} \quad \ldots.(3.6)$$

Then the semi-direct form of the directed derivative $\frac{\partial}{\partial\bar{R}}$ can be evaluated by equating the two differentials.

The directed derivative $\frac{\partial}{\partial\bar{R}}$ is assumed in the following general form as an unprescribed vectorial operator

$$\frac{\partial}{\partial\bar{R}} \equiv \bar{E}_t\mathcal{D}_t(\ ) + \bar{E}_n\mathcal{D}_n(\ ) + \bar{E}_b\mathcal{D}_b(\ ) \quad \ldots.(3.7)$$

where $\mathcal{D}_t, \mathcal{D}_n, \mathcal{D}_b$ represent undefined scalar operators. Therefore, from equations (3.5), (3.6) and (3.7)

$$d\bar{U} = [(1-KR_n)dS\bar{E}_t + (dR_n - TR_b dS)\bar{E}_n + (dR_b + TR_n dS)\bar{E}_b]$$
$$\cdot[\bar{E}_t\mathcal{D}_t(\ ) + \bar{E}_n\mathcal{D}_n(\ ) + \bar{E}_b\mathcal{D}_b(\ )](\bar{U}) \quad \ldots.(3.8)$$

Equating equations (3.4) and (3.8)

$$\frac{\partial \bar{U}}{\partial S} dS + \frac{\partial \bar{U}}{\partial R_n} dR_n + \frac{\partial \bar{U}}{\partial R_b} dR_b = [(1 - KR_n)dS\, \bar{E}_t + (dR_n - TR_b dS)\bar{E}_n + (dR_b + TR_n dS)\bar{E}_b] \cdot [\bar{E}_t \mathcal{D}_t(\bar{U}) + \bar{E}_n \mathcal{D}_n(\bar{U}) + \bar{E}_b \mathcal{D}_b(\bar{U})]$$

A simplification of the right hand side by carrying out the dot product results in the following equation

$$\frac{\partial \bar{U}}{\partial S} dS + \frac{\partial \bar{U}}{\partial R_n} dR_n + \frac{\partial \bar{U}}{\partial R_b} dR_b = [(1-KR_n)\mathcal{D}_t(\bar{U}) - TR_b\mathcal{D}_n(\bar{U}) + TR_n\mathcal{D}_b(\bar{U})]dS + \mathcal{D}_n(\bar{U})\, dR_n + \mathcal{D}_b(\bar{U})\, dR_b$$

On transposition

$$[\frac{\partial(\bar{U})}{\partial S} - (1-KR_n)\mathcal{D}_t(\bar{U}) + TR_b\mathcal{D}_n(\bar{U}) - TR_n\mathcal{D}_b(\bar{U})]dS + [\frac{\partial \bar{U}}{\partial R_n} - \mathcal{D}_n(\bar{U})]dR_n + [\frac{\partial \bar{U}}{\partial R_b} - \mathcal{D}_b(\bar{U})]dR_b = 0$$

Since the scalar variables dS, dR_n, dR_b are independent, then this solution must be valid if only dS, dR_n and dR_b in succession are assumed to be nonvanishing. This condition implies vanishing of all coefficients

$$\frac{\partial \bar{U}}{\partial S} - (1-KR_n)\mathcal{D}_t(\bar{U}) + TR_b\mathcal{D}_n(\bar{U}) - TR_n\mathcal{D}_b(\bar{U}) = 0$$

$$\frac{\partial \bar{U}}{\partial R_n} - \mathcal{D}_n(\bar{U}) = 0$$

$$\frac{\partial \bar{U}}{\partial R_b} - \mathcal{D}_b(\bar{U}) = 0$$

From the above equations the unknown derivative operators can be specified in terms of the parameters of the member-space

$$\mathcal{D}_t(\ ) \equiv \frac{1}{(1-KR_n)}\left[\frac{\partial}{\partial S}(\ ) + TR_b \frac{\partial}{\partial R_n}(\ ) - TR_n \frac{\partial}{\partial R_b}\right]$$

$$\mathcal{D}_n(\ ) \equiv \frac{\partial}{\partial R_n}(\ )$$

$$\mathcal{D}_b(\ ) \equiv \frac{\partial}{\partial R_b}(\ )$$

Hence the directed derivative operator for the undeformed member space emerges in the form

$$\frac{\partial}{\partial \bar{R}} \equiv \frac{\bar{E}_t}{(1-KR_n)}\left[\frac{\partial}{\partial S}(\ ) + TR_b \frac{\partial}{\partial R_n}(\ ) - TR_n \frac{\partial}{\partial R_b}(\ )\right] + \bar{E}_n \frac{\partial}{\partial R_n}(\ ) + \bar{E}_b \frac{\partial}{\partial R_b}(\ ) \qquad \ldots.(3.9)$$

§(3.6) <u>Strain Tensor</u>

The linear strain tensor $\bar{\bar{\varepsilon}}$ for a continuum is given by

$$\bar{\bar{\varepsilon}} = \frac{1}{2}\left(\frac{\partial \bar{U}}{\partial \bar{R}} + \frac{\bar{U}\partial}{\partial \bar{R}}\right) \qquad \text{(see Appendix B)}$$

In order to evaluate $\bar{\bar{\varepsilon}}$, the displacement gradient $\frac{\partial \bar{U}}{\partial \bar{R}}$ is required. Allowing the directed derivative $\frac{\partial}{\partial \bar{R}}$ from equation (3.9) to operate on displacement function $\bar{U}$

$$\frac{\partial \bar{U}}{\partial \bar{R}} = \frac{\bar{E}_t}{(1-K\mathcal{R}_n)} \left[\frac{\partial \bar{U}}{\partial S} + T\mathcal{R}_b \frac{\partial \bar{U}}{\partial \mathcal{R}_n} - T\mathcal{R}_n \frac{\partial \bar{U}}{\partial \mathcal{R}_b}\right] + \bar{E}_n \frac{\partial \bar{U}}{\partial \mathcal{R}_n}$$

$$+ \bar{E}_b \frac{\partial \bar{U}}{\partial \mathcal{R}_b} \qquad \ldots.(3.10)$$

where $\bar{U}$ is given by equation (3.3).

Evaluating the first term

$$\frac{\partial \bar{U}}{\partial S} = \frac{\partial}{\partial S}\left[\bar{U}^{\circ} + \mathcal{R}_n(\delta\bar{E}_n + \theta\bar{E}_b) + \mathcal{R}_b(\delta\bar{E}_b - \theta\bar{E}_n) + \left(\frac{\partial \theta}{\partial S} + \delta T\right)\psi\bar{E}_t\right]$$

Substituting the values of variational quantities $\delta\bar{E}_n$ and $\delta\bar{E}_b$ from equation (2.13)

$$\frac{\partial \bar{U}}{\partial S} = \frac{\partial \bar{U}^{\circ}}{\partial S} + \mathcal{R}_n\left[\frac{\partial}{\partial S}(m_1\bar{E}_t + m_2\bar{E}_b) + \frac{\partial}{\partial S}(\theta\bar{E}_t)\right]$$

$$+ \mathcal{R}_b\left[\frac{\partial}{\partial S}(-\phi_b\bar{E}_t - m_2\bar{E}_n) - \frac{\partial}{\partial S}(\theta\bar{E}_n)\right]$$

$$+ \frac{\partial}{\partial S}\left[\left(\frac{\partial \theta}{\partial S} + \delta T\right)\psi\bar{E}_t\right]$$

Using the kinematical Frenet-Serret formulas for derivatives of the base vectors of the Euler Triad $\frac{\partial \bar{E}_i}{\partial S} = \bar{D}\times\bar{E}_i$ and collecting terms of independent base vectors $\bar{E}_i$

$$\frac{\partial \bar{U}}{\partial S} = \left[\mathcal{R}_n\left(\frac{\partial m_1}{\partial S}\right) + \mathcal{R}_b\left(m_2K + K\theta - \frac{\partial \phi_b}{\partial S}\right) + \psi\frac{\partial}{\partial S}\left(\frac{\partial \theta}{\partial S} + \delta T\right)\right]\bar{E}_t$$

$$+ \left[\phi_n + \mathcal{R}_n\,(m_1K - Tm_2 - T\theta) - \mathcal{R}_b\left(K\phi_b + \frac{\partial m_2}{\partial S} + \frac{\partial \theta}{\partial S}\right)\right.$$

$$\left. + \left(\frac{\partial \theta}{\partial S} + \delta T\right)\psi K\right]\bar{E}_n + \left[\phi_b + \mathcal{R}_n\left(\frac{\partial m_2}{\partial S} + \frac{\partial \theta}{\partial S}\right)\right.$$

$$\left. - \mathcal{R}_b(m_2T + \theta T)\right]\bar{E}_b$$

The second term

$$\frac{\partial \bar{U}}{\partial R_n} = \frac{\partial}{\partial R_n} [\bar{U}^{\circ} + R_n(\delta \bar{E}_n + \theta \bar{E}_b) + R_b(\delta \bar{E}_b - \theta \bar{E}_n) + (\frac{\partial \theta}{\partial S} + \delta T)\psi \bar{E}_t]$$

Substituting the values of variational quantities $\delta \bar{E}_n$, $\delta \bar{E}_b$ from equation (2.13) in the equation above and carrying out partial derivatives

$$\frac{\partial \bar{U}}{\partial R_n} = (m_1 \bar{E}_t + m_2 \bar{E}_b + \theta \bar{E}_b) + (\frac{\partial \theta}{\partial S} + \delta T) \frac{\partial \psi}{\partial R_n} \bar{E}_t$$

or

$$\frac{\partial \bar{U}}{\partial R_n} = [m_1 + (\frac{\partial \theta}{\partial S} + \delta T) \frac{\partial \psi}{\partial R_n}] \bar{E}_t + (m_2 + \theta) \bar{E}_b \ .$$

The third term $\frac{\partial \bar{U}}{\partial R_b}$ is obtained in a similar manner.

$$\frac{\partial \bar{U}}{\partial R_b} = [- \phi_b + (\frac{\partial \theta}{\partial S} + \delta T) \frac{\partial \psi}{\partial R_b}] \bar{E}_t - (m_2 + \theta) \bar{E}_n$$

Substituting these expressions for $\frac{\partial \bar{U}}{\partial S}$, $\frac{\partial \bar{U}}{\partial R_n}$, $\frac{\partial \bar{U}}{\partial R_b}$ in equation (3.10) and collecting coefficients of independent dyads $\bar{E}_i \bar{E}_j$ the following expression for the displacement gradient $\frac{\partial \bar{U}}{\partial \bar{R}}$ is obtained

$$\frac{\partial \bar{U}}{\partial \bar{R}} = \frac{\bar{E}_t}{(1-KR_n)} \Big[[R_n \{\frac{\partial m_1}{\partial S} + T\phi_b - T(\frac{\partial \theta}{\partial S} + \delta T) \frac{\partial \psi}{\partial R_b}\}$$

$$+ R_b \{m_2 K + \theta K - \frac{\partial \phi_b}{\partial S} + T m_1 + T (\frac{\partial \theta}{\partial S} + \delta T) \frac{\partial \psi}{\partial R_n} \}$$

$$+ \{\frac{\partial^2 \theta}{\partial S^2} + \frac{\partial}{\partial S} (\delta T)\}\psi] \bar{E}_t + [\phi_n + R_n\{m_1 K\}$$

$$- R_b (K\phi_b + \frac{\partial m_2}{\partial S} + \frac{\partial \theta}{\partial S}) + (\frac{\partial \theta}{\partial S} + \delta T)\psi K] \bar{E}_n$$

$$+ [\phi_b + R_n (\frac{\partial m_2}{\partial S} + \frac{\partial \theta}{\partial S})] \bar{E}_b \Big]$$

$$+ \bar{E}_n \Big[[m_1 + (\frac{\partial \theta}{\partial S} + \delta T) \frac{\partial \psi}{\partial R_n}] \bar{E}_t + [(m_2 + \theta)] \bar{E}_b \Big]$$

$$+ \bar{E}_b \Big[[-\phi_b + \frac{\partial \theta}{\partial S} + \delta T) \frac{\partial \psi}{\partial R_b}] \bar{E}_t - [(m_2 + \theta)] \bar{E}_n \Big]$$

....(3.11)

Symbolically, the displacement gradient $\frac{\partial \bar{U}}{\partial \bar{R}}$ can be written

$$\bar{\bar{U}} = \frac{\partial \bar{U}}{\partial \bar{R}} = \begin{bmatrix} U_{tt} \bar{E}_t \bar{E}_t + U_{tn} \bar{E}_t \bar{E}_n + U_{tb} \bar{E}_t \bar{E}_b \\ + U_{nt} \bar{E}_n \bar{E}_t + U_{nn} \bar{E}_n \bar{E}_n + U_{nb} \bar{E}_n \bar{E}_b \\ + U_{bt} \bar{E}_b \bar{E}_t + U_{bn} \bar{E}_b \bar{E}_n + U_{bb} \bar{E}_b \bar{E}_b \end{bmatrix}$$

The conjugate displacement gradient tensor

$$\frac{\bar{U}\partial}{\partial \bar{R}} = (\frac{\partial \bar{U}}{\partial \bar{R}})_c$$

hence

$$\bar{\bar{U}}_c = \frac{\bar{U}\partial}{\partial \bar{R}} = \begin{bmatrix} U_{tt} \bar{E}_t \bar{E}_t + U_{tn} \bar{E}_n \bar{E}_t + U_{tb} \bar{E}_b \bar{E}_t \\ + U_{nt} \bar{E}_t \bar{E}_n + U_{nn} \bar{E}_n \bar{E}_n + U_{nb} \bar{E}_b \bar{E}_n \\ + U_{bt} \bar{E}_t \bar{E}_b + U_{bn} \bar{E}_n \bar{E}_b + U_{bb} \bar{E}_b \bar{E}_b \end{bmatrix}$$

and the linear strain tensor

$$\bar{\bar{\varepsilon}}=\frac{1}{2}[\frac{\partial\bar{U}}{\partial\bar{R}}+\frac{\bar{U}\partial}{\partial\bar{R}}] = \begin{bmatrix} U_{tt}\ \bar{E}_t\bar{E}_t+\frac{1}{2}(U_{tn}+U_{nt})\bar{E}_t\bar{E}_n+\frac{1}{2}(U_{tb}+U_{bt})\bar{E}_t\bar{E}_b \\ +\frac{1}{2}(U_{nt}+U_{tn})\bar{E}_n\bar{E}_t+\ U_{nn}\bar{E}_n\bar{E}_n\ +\frac{1}{2}(U_{nb}+U_{bn})\bar{E}_n\bar{E}_b \\ +\frac{1}{2}(U_{bt}+U_{tb})\bar{E}_b\bar{E}_t+\frac{1}{2}(U_{bn}+U_{nb})\bar{E}_b\bar{E}_n+\ U_{bb}\ \bar{E}_b\bar{E}_b \end{bmatrix}$$

or, symbolically

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_{tt}\ \bar{E}_t\bar{E}_t + \varepsilon_{tn}\ \bar{E}_t\bar{E}_n + \varepsilon_{tb}\ \bar{E}_t\bar{E}_b \\ +\varepsilon_{nt}\ \bar{E}_n\bar{E}_t + \varepsilon_{nn}\ \bar{E}_n\bar{E}_n + \varepsilon_{nb}\ \bar{E}_n\bar{E}_n \\ +\varepsilon_{bt}\ \bar{E}_b\bar{E}_t + \varepsilon_{bn}\ \bar{E}_b\bar{E}_b + \varepsilon_{bb}\ \bar{E}_b\bar{E}_b \end{bmatrix}$$

where the components of the strain tensor ε_{ij} are specified by equation (3.11). The strain components appear as follows:

$$\left.\begin{aligned}
\varepsilon_{tt} &= \frac{1}{(1-K\mathcal{R}_n)}\ [\mathcal{R}_n\ \{\frac{\partial m_1}{\partial S} + T\phi_b - T(\frac{\partial\theta}{\partial S} + \delta T)\ \frac{\partial\psi}{\partial\mathcal{R}_b}\ \} \\
&\quad + \mathcal{R}_b\ \{m_2K + \theta K - \frac{\partial\phi_b}{\partial S} + Tm_1+T(\frac{\partial\theta}{\partial S}+\delta T)\ \frac{\partial\psi}{\partial\mathcal{R}_n}\} \\
&\quad + \psi\ \{\frac{\partial}{\partial S}\ (\frac{\partial\theta}{\partial S} + \delta T)\}] \\
\varepsilon_{tn} &= \frac{1}{2}\ [\frac{1}{(1-K\mathcal{R}_n)}\{\phi_n + \mathcal{R}_n(m_1K)-\mathcal{R}_b(K\phi_b + \frac{\partial m_2}{\partial S} + \frac{\partial\theta}{\partial S}) \\
&\quad + (\frac{\partial\theta}{\partial S} + \delta T)\psi K\} + \{m_1+ (\frac{\partial\theta}{\partial S} + \delta T)\ \frac{\partial\psi}{\partial\mathcal{R}_n}\}] \\
\varepsilon_{tb} &= \frac{1}{2}[\frac{1}{(1-K\mathcal{R}_n)}\ \{\phi_b + \mathcal{R}_n(\frac{\partial m_2}{\partial S} + \frac{\partial\theta}{\partial S})\} \\
&\quad + \{-\ \phi_b + (\frac{\partial\theta}{\partial S} + \delta T)\ \frac{\partial\psi}{\partial\mathcal{R}_b}\}] \\
\varepsilon_{nt} &= \varepsilon_{tn}
\end{aligned}\right\} \quad \ldots.(3.12)$$

$$\left.\begin{aligned}\varepsilon_{nn} &= 0\\ \varepsilon_{nb} &= \tfrac{1}{2}[\{(m_2+\theta)\}+\{-(m_2+\theta)\}] = 0 = \varepsilon_{bn}\\ \varepsilon_{bt} &= \varepsilon_{tb}\end{aligned}\right\}$$

....(3.12)

The final strain tensor emerges as follows

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_{tt}\ \bar{E}_t\bar{E}_t & + \varepsilon_{tn}\ \bar{E}_t\bar{E}_n & + \varepsilon_{tb}\ \bar{E}_t\bar{E}_b \\ +\varepsilon_{nt}\ \bar{E}_n\bar{E}_t\ + & 0 & + \quad 0 \\ +\varepsilon_{bt}\ \bar{E}_b\bar{E}_t\ + & 0 & + \quad 0 \end{bmatrix}$$

The absence of the strain tensor components ε_{nn}, ε_{nb}, ε_{bn} and ε_{bb} is justified because of the assumption involved in §(3.3), which implied that the (normal) cross-section does not deform in its own plane. Therefore, the strain components corresponding to such strains are absent in the strain tensor.

§(3.7) Approximate Measures of Strains

When 'warping' is neglected, then the function $\psi=0$, and the corresponding strain components are denoted 'simplified' strains. They are as follows:

$$\left\{\begin{aligned}\varepsilon_{tt} &= \frac{1}{(1-KR_n)}\,[R_n(\frac{\partial m_1}{\partial S} + T\phi_b) + R_b(m_2K + \theta K - \frac{\partial\phi_b}{\partial S} + Tm_1)]\\ \varepsilon_{tn} &= \varepsilon_{nt} = \frac{1}{2}\,[m_1 + \frac{1}{(1-KR_n)}\,\{\phi_n + R_n(m_1K) - R_b(K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})\}]\\ \varepsilon_{tb} &= \varepsilon_{bt} = \frac{1}{2}\,[-\phi_b + \frac{1}{(1-KR_n)}\,\{\phi_b + R_n\,(\frac{dm_2}{dS} + \frac{d\theta}{dS})\}]\end{aligned}\right.$$

....(3.13)

Above equations can be further simplified algebraically. Re-writing,

$$\varepsilon_{tn} = \frac{1}{2} [m_1 + \frac{1}{(1-KR_n)} \{\phi_n + R_n(m_1K) - R_b(K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})\}]$$

$$= \frac{1}{2(1-KR_n)} [m_1 - m_1KR_n + \phi_n + R_n(m_1K) - R_b(K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})]$$

Substituting for m_1 from page (19)

$$= \frac{1}{2(1-KR_n)} [-\frac{K}{\kappa}\phi_n + \phi_n - R_b(K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})]$$

$$= \frac{1}{2(1-KR_n)} [\frac{\delta K}{\kappa}\phi_n - R_b(K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})]$$

Neglecting $\frac{\delta K}{\kappa}\phi_n$ as a small quantity compared with the terms retained yields

$$\varepsilon_{tn} = -\frac{R_b}{2(1-KR_n)} (K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})$$

Similarly

$$\varepsilon_{tb} = \frac{1}{2} [-\phi_b + \frac{1}{(1-KR_n)} \{\phi_b + R_n(\frac{dm_2}{dS} + \frac{d\theta}{dS})\}]$$

$$= \frac{1}{2(1-KR_n)} [-\phi_b + R_nK\phi_b + \phi_b + R_n(\frac{dm_2}{dS} + \frac{d\theta}{dS})]$$

or

$$\varepsilon_{tb} = \frac{R_n}{2(1-KR_n)} (K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})$$

Since $KR_n << 1$, the approximation $1-KR_n \doteq 1$ is appropriate in the equations given above, which reduce them to the following form:

$$\left.\begin{aligned}
\varepsilon_{tt} &= R_n \left(\frac{dm_1}{dS} + T\phi_b\right) + R_b\left(m_2K + \theta K - \frac{d\phi_b}{dS} + Tm_1\right) \\
\varepsilon_{tn} &= \varepsilon_{nt} = -\frac{R_b}{2}\left(K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS}\right) \\
\varepsilon_{tb} &= \varepsilon_{bt} = \frac{R_n}{2}\left(K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS}\right)
\end{aligned}\right\} \qquad \ldots.(3.14)$$

CHAPTER 4

THE SAINT VENANT KINEMATIC COMPATIBILITY EQUATIONS

§(4.1) Introduction

A unique strain tensor $\bar{\bar{\varepsilon}}(\bar{U}) = \bar{\bar{\varepsilon}}(\bar{R})$ for $\bar{U} = \bar{U}(\bar{R})$ is defined for a prescribed, single valued and sufficiently smooth displacement field $\bar{U}(\bar{R})$ by the relationship

$$\bar{\bar{\varepsilon}} = \frac{1}{2}\left(\frac{\partial\bar{U}}{\partial\bar{R}} + \frac{\bar{U}\partial}{\partial\bar{R}}\right) \text{ (see Appendix B)} \qquad \ldots.(4.1)$$

As long as the prescribed $\bar{U}(\bar{R})$ is a continuous, single valued vector point function (apart from arbitrary rigid body displacement) and possesses the single valued, continuous derivatives indicated, the strain tensor $\bar{\bar{\varepsilon}}(\bar{U})$ is a unique, continuous tensor field function.

In the reverse case, however, when the strain tensor $\bar{\bar{\varepsilon}}(\bar{R})$ is prescribed and a single valued displacement function $\bar{U}(\bar{R})$ satisfying the definition above is sought as the unknown, then, the strain tensor $\bar{\bar{\varepsilon}}$ must satisfy a certain condition in order that it might have been produced from a single valued, continuous displacement field $\bar{U}(\bar{R})$ according to the definition above.

A prescribed, single valued $\bar{U}(\bar{R})$ thus defines a unique $\bar{\bar{\varepsilon}}$, but the converse is not necessarily true. Obviously, the condition which must be satisfied by the prescribed $\bar{\bar{\varepsilon}}$ in order that it is a unique tensor function of $\bar{U}$ according to the definition above must result from equation (4.1), yet such relation should be independent of the unprescribed displacement function $\bar{U}$. Such a criterion on $\bar{\bar{\varepsilon}}$ can be easily obtained by a formal method of eliminating $\bar{U}$. This consists of taking the double curl on both sides of equation (4.1). Therefore,

$$\frac{\partial}{\partial \bar{R}} \times \bar{\bar{\varepsilon}} \times \frac{\partial}{\partial \bar{R}} = \frac{\partial}{\partial \bar{R}} \times \frac{1}{2} \left(\frac{\partial \bar{U}}{\partial \bar{R}} + \frac{\bar{U}\partial}{\partial \bar{R}}\right) \times \frac{\partial}{\partial \bar{R}} = \bar{\bar{0}}$$

for all $\bar{U}$, as

$$\frac{\bar{U}\partial}{\partial \bar{R}} \times \frac{\partial}{\partial \bar{R}} = \frac{\partial}{\partial \bar{R}} \times \frac{\partial \bar{U}}{\partial \bar{R}} = \bar{\bar{0}}$$

Thus, the equation

$$\frac{\partial}{\partial \bar{R}} \times \bar{\bar{\varepsilon}} \times \frac{\partial}{\partial \bar{R}} = \bar{\bar{0}} \qquad \text{....(4.2)}$$

prescribes the relation which must be satisfied by $\bar{\bar{\varepsilon}}$ in order that $\bar{\bar{\varepsilon}}$ is described by a continuous, single valued, displacement field $\bar{U}(\bar{R})$ in accordance with the definition of the strain tensor as a function of $\bar{U}$.

The tensor

$$\bar{\bar{S}} = \frac{\partial}{\partial \bar{R}} \times \bar{\bar{\varepsilon}} \times \frac{\partial}{\partial \bar{R}}$$

is called the local kinematic Saint-Venant compatibility tensor.

§(4.2) Compatibility Equations

The directed derivative in the geometric space of slender member has already been obtained in chapter 3, equation (3.9). If 'warping' is neglected, the 'simplified' strain tensor components $\varepsilon_{ij} = \bar{\bar{\varepsilon}}: \bar{E}_i\bar{E}_j$ are given by equations (3.13)

Taking the curl of $\bar{\bar{\varepsilon}}$ as a *prefactor* and evaluating the derivatives of the Euler Triad by the kinematic Frenet-Serret formulas (see Appendix A), an auxiliary tensor $\bar{\bar{P}}$ will result

$$\frac{\partial \times \bar{\bar{\varepsilon}}}{\partial \bar{R}} = P_{ij}\ \bar{E}_i\bar{E}_j = \bar{\bar{P}}$$

Taking a curl of $\bar{\bar{P}}$ as a *postfactor* yields the local Saint-Venant kinematic compatibility condition

$$\bar{\bar{S}} = \frac{\bar{\bar{P}}\times\partial}{\partial \bar{R}} = \left(\frac{\partial\times\bar{\bar{\varepsilon}}}{\partial \bar{R}}\right) \times \frac{\partial}{\partial \bar{R}} = \bar{\bar{0}}$$

A typical calculation with the first component of the strain tensor $\bar{\bar{\varepsilon}}$ is shown in detail:

$$\frac{\partial}{\partial \bar{R}} \times \varepsilon_{tt}\ \bar{E}_t\bar{E}_t = \frac{\bar{E}_t}{(1-K\mathcal{R}_n)}\left[\frac{\partial}{\partial S} \times (\varepsilon_{tt}\ \bar{E}_t\bar{E}_t) + T\mathcal{R}_b \frac{\partial}{\partial \mathcal{R}_n} \times (\varepsilon_{tt}\ \bar{E}_t\bar{E}_t)\right.$$

$$\left. - T\mathcal{R}_n \frac{\partial}{\partial \mathcal{R}_b} \times (\varepsilon_{tt}\ \bar{E}_t\bar{E}_t)\right] + \bar{E}_n \frac{\partial}{\partial \mathcal{R}_n} \times (\varepsilon_{tt}\ \bar{E}_t\bar{E}_t)$$

$$+ \bar{E}_b \frac{\partial}{\partial \mathcal{R}_b} \times (\varepsilon_{tt}\ \bar{E}_t\bar{E}_t)$$

It is noted that

$$\bar{E}_i \neq \bar{E}_i(R_j) \text{ where } i = t,n,b$$
$$j = t,n$$

Therefore,

$$\frac{\partial}{\partial \bar{R}} \times \varepsilon_{tt} \bar{E}_t\bar{E}_t = \frac{\bar{E}_t}{(1-KR_n)} \times [\frac{\partial \varepsilon_{tt}}{\partial S} \bar{E}_t\bar{E}_t + \varepsilon_{tt} \frac{\partial \bar{E}_t}{\partial S} \bar{E}_t + \varepsilon_{tt} \bar{E}_t \frac{\partial \bar{E}_t}{\partial S}$$

$$+ TR_b \frac{\partial \varepsilon_{tt}}{\partial R_n} \bar{E}_t\bar{E}_t - TR_n \frac{\partial \varepsilon_{tt}}{\partial R_b} \bar{E}_t\bar{E}_t]$$

$$+ \bar{E}_n \times \frac{\partial \varepsilon_{tt}}{\partial R_n} \bar{E}_t\bar{E}_t + \bar{E}_b \times \frac{\partial \varepsilon_{tt}}{\partial R_b} \bar{E}_t\bar{E}_t$$

Using the kinematical Serret-Frenet formulas (Appendix A) and then carrying out the cross products yields the following result:

$$\frac{\partial}{\partial \bar{R}} \times \varepsilon_{tt} \bar{E}_t\bar{E}_t = \frac{K\varepsilon_{tt}}{(1-KR_n)} \bar{E}_b\bar{E}_t - \frac{\partial \varepsilon_{tt}}{\partial R_n} \bar{E}_b\bar{E}_t + \frac{\partial \varepsilon_{tt}}{\partial R_b} \bar{E}_n\bar{E}_t$$

Performing similar operations with the remaining components of the tensor $\bar{\bar{\varepsilon}}$ and accumulating the coefficients of the tensor dyads $\bar{E}_i\bar{E}_j$ reveals the tensor $\bar{\bar{P}}$ as follows:

$$\bar{\bar{P}} = \begin{bmatrix} P_{tt} \bar{E}_t\bar{E}_t + P_{tn} \bar{E}_t\bar{E}_n + P_{tb} \bar{E}_t\bar{E}_b \\ + P_{nt} \bar{E}_n\bar{E}_t + P_{nn} \bar{E}_n\bar{E}_n + P_{nb} \bar{E}_n\bar{E}_b \\ + P_{bt} \bar{E}_b\bar{E}_t + P_{bn} \bar{E}_b\bar{E}_n + P_{bb} \bar{E}_b\bar{E}_b \end{bmatrix}$$

where

$$\begin{cases}
P_{tt} = \dfrac{\partial \varepsilon_{bt}}{\partial R_n} - \dfrac{\partial \varepsilon_{nt}}{\partial R_b} \\
P_{tn} = 0 \\
P_{tb} = 0 \\
P_{nt} = \dfrac{\partial \varepsilon_{tt}}{\partial R_b} - \dfrac{1}{(1-KR_n)} [T\ \varepsilon_{nt} + \dfrac{\partial \varepsilon_{bt}}{\partial S} + TR_b \dfrac{\partial \varepsilon_{bt}}{\partial R_n} - TR_n \dfrac{\partial \varepsilon_{bt}}{\partial R_b}] \\
P_{nn} = \dfrac{\partial \varepsilon_{tn}}{\partial R_b} - \dfrac{K\varepsilon_{bt}}{(1-KR_n)} \\
P_{nb} = \dfrac{\partial \varepsilon_{tb}}{\partial R_b} \\
P_{bt} = \dfrac{1}{(1-KR_n)} [K\ \varepsilon_{tt} + \dfrac{\partial \varepsilon_{nt}}{\partial S} + TR_b \dfrac{\partial \varepsilon_{nt}}{\partial R_n} - TR_n \dfrac{\partial \varepsilon_{nt}}{\partial R_b} - T\varepsilon_{bt}] - \dfrac{\partial \varepsilon_{tt}}{\partial R_n} \\
P_{bn} = \dfrac{1}{(1-KR_n)} [2K\ \varepsilon_{tn}] - \dfrac{\partial \varepsilon_{tn}}{\partial R_n} \\
P_{bb} = \dfrac{K\varepsilon_{tb}}{(1-KR_n)} - \dfrac{\partial \varepsilon_{tb}}{\partial R_n}
\end{cases}$$

....(4.3)

Taking the 'curl' of the tensor $\bar{\bar{P}}$ as a *post-factor*, i.e., $\bar{\bar{P}} \times \dfrac{\partial}{\partial \bar{R}}$, and again working symbolically on $\bar{\bar{P}}$, the tensor $\bar{\bar{S}}$ is obtained.

A typical calculation with the first component of tensor $\bar{\bar{P}}$ is shown

$$P_{tt}\ \bar{E}_t\bar{E}_t \times \frac{\partial}{\partial \bar{R}} = (P_{tt}\ \bar{E}_t\bar{E}_t) \times [\frac{\partial}{\partial S} + TR_b \frac{\partial}{\partial R_n} - TR_n \frac{\partial}{\partial R_b}]\bar{E}_t$$

$$+ (P_{tt}\ \bar{E}_t\bar{E}_t) \times \bar{E}_n \frac{\partial}{\partial R_n} + (P_{tt}\ \bar{E}_t\bar{E}_t) \times \bar{E}_b \frac{\partial}{\partial R_b}$$

Since the scalar operators can commute with the cross product and

$$\bar{E}_i \neq \bar{E}_i(R_j) \text{ for } i = t,n,b$$
$$j = n,b$$

$$P_{tt}\,\bar{E}_t\bar{E}_t \times \frac{\partial}{\partial\bar{R}} = [\frac{\partial P_{tt}}{\partial S}\,\bar{E}_t\bar{E}_t + P_{tt}\,\frac{\partial\bar{E}_t}{\partial S}\,\bar{E}_t + P_{tt}\,\bar{E}_t\,\frac{\partial\bar{E}_t}{\partial S}]\times\bar{E}_t$$

$$+ \frac{\partial P_{tt}}{\partial R_n}\,\bar{E}_t\bar{E}_t \times \bar{E}_n + \frac{\partial P_{tt}}{\partial R_b}\,\bar{E}_t\bar{E}_t \times \bar{E}_b$$

Kinematic Frenet-Serret Formulas (Appendix-A) lead to the following results:

$$P_{tt}\,\bar{E}_t\bar{E}_t \times \frac{\partial}{\partial\bar{R}} = -\frac{KP_{tt}}{(1-KR_n)}\,\bar{E}_t\bar{E}_b + \frac{\partial P_{tt}}{\partial R_n}\,\bar{E}_t\bar{E}_b - \frac{\partial P_{tt}}{\partial R_b}\,\bar{E}_t\bar{E}_n$$

Carrying out similar operations on all the remaining components of tensor $\bar{\bar{P}}$ and collecting the components of dyads $\bar{E}_i\bar{E}_j$, the tensor $\bar{\bar{S}}$ appears in the form:

$$\bar{\bar{S}} = \frac{\partial}{\partial\bar{R}} \times \bar{\bar{\varepsilon}} \times \frac{\partial}{\partial\bar{R}}$$

$$= S_{ij}\,\bar{E}_i\bar{E}_j = \begin{bmatrix} S_{tt}\,\bar{E}_t\bar{E}_t + S_{tn}\,\bar{E}_t\bar{E}_n + S_{tb}\,\bar{E}_t\bar{E}_b \\ +S_{nt}\,\bar{E}_n\bar{E}_t + S_{nn}\,\bar{E}_n\bar{E}_n + S_{nb}\,\bar{E}_n\bar{E}_b \\ +S_{bt}\,\bar{E}_b\bar{E}_t + S_{bn}\,\bar{E}_b\bar{E}_n + S_{bb}\,\bar{E}_b\bar{E}_b \end{bmatrix}$$

where the components of $S_{ij} = \bar{E}_i\bar{E}_j:\bar{\bar{S}}$ appear as functions of the coefficients of $P_{ij} = \bar{E}_i\bar{E}_j:\bar{\bar{P}}$

$$\begin{cases} S_{tt} = 0 \\ S_{tn} = -\dfrac{\partial P_{tt}}{\partial R_b} - \dfrac{K\, P_{nb}}{(1-KR_n)} \\ S_{tb} = [K\, P_{nn} - K\, P_{tt}]\,\dfrac{1}{(1-KR_n)} + \dfrac{\partial P_{tt}}{\partial R_n} \\ S_{nt} = \dfrac{\partial P_{nn}}{\partial R_b} - \dfrac{\partial P_{nb}}{\partial R_n} \\ S_{nn} = [T\, P_{nn} + \dfrac{\partial P_{nb}}{\partial S} + TR_b \dfrac{\partial P_{nb}}{\partial R_n} - TR_n \dfrac{\partial P_{nb}}{\partial R_b} - TP_{bb}]\,\dfrac{1}{(1-KR_n)} \\ \qquad - \dfrac{\partial P_{nt}}{\partial R_b} \\ S_{nb} = [T\, P_{nb} + T\, P_{bn} - K\, P_{nt} - \dfrac{\partial P_{nn}}{\partial S} - TR_b \dfrac{\partial P_{nn}}{\partial R_n} + TR_n \dfrac{\partial P_{nn}}{\partial R_n}]\,\dfrac{1}{(1-KR_n)} \\ \qquad + \dfrac{\partial P_{nt}}{\partial R_n} \\ S_{bt} = \dfrac{\partial P_{bn}}{\partial R_b} - \dfrac{\partial P_{bb}}{\partial R_n} \\ S_{bn} = [T\, P_{nb} + T\, P_{bn} + \dfrac{\partial P_{bb}}{\partial S} + TR_b \dfrac{\partial P_{bb}}{\partial R_n} - TR_n \dfrac{\partial P_{bb}}{\partial R_b}]\,\dfrac{1}{(1-KR_n)} \\ \qquad - \dfrac{\partial P_{bt}}{\partial R_b} \\ S_{bb} = [T\, P_{bb} - T\, P_{nn} - K\, P_{bt} - \dfrac{\partial P_{bn}}{\partial S} - TR_b \dfrac{\partial P_{bn}}{\partial R_n} + TR_n \dfrac{\partial P_{bn}}{\partial R_b}]\,\dfrac{1}{(1-KR_n)} \\ \qquad + \dfrac{\partial P_{bt}}{\partial R_n} \end{cases} \quad \ldots\ldots (4.4)$$

It is important to observe that the kinematic Saint-Venant compatibility tensor $\bar{\bar{S}}$ is symmetric, as $\bar{\bar{S}}_c = \bar{\bar{S}}$ for $\bar{\bar{\varepsilon}}_c = \bar{\bar{\varepsilon}}$.

Since tensor $\bar{\bar{S}}$ is a zero tensor, and since the tensor dyads $\bar{E}_i\bar{E}_j$ are unique, then each of the coefficients of $\bar{\bar{S}}$ (i.e. $S_{ij} = \bar{\bar{S}}:\bar{E}_i\bar{E}_j$) must vanish separately. Theoretically, nine scalar equations will result from the nine components of tensor $\bar{\bar{S}}$; but as tensor $\bar{\bar{S}}$ is symmetric only six of its scalar components are unique.

In order to evaluate components of $\bar{\bar{S}}$, i.e. S_{ij}, the following procedure is adopted: First, tensor components P_{ij} are evaluated by substituting the values of ε_{ij} from (3.13) in equations for P_{ij}, i.e. equations (4.3). The values of P_{ij} thus calculated are substituted in the equations for S_{ij}, equations (4.4). Finally it is observed that each of the components of $\bar{\bar{S}}$ (i.e., $S_{ij} = \bar{\bar{S}}:\bar{E}_i\bar{E}_j$) vanishes identically (see Appendix-E). Thus, it is concluded that 'compatibility' is rigorously satisfied and the displacement function $\bar{U}$ exists and is unique within any rigid body displacement. It should be noted that 'warping' has been neglected in the definition of strain components ε_{ij}.

CHAPTER 5

THE CAUCHY AXIOMS OF MOTION

§(5.1) Introduction

General equations specifying translational and rotational states of an arbitrary body as a continuum of volume v and bounded by an external surface Σ are given by the Cauchy Axioms of Motion (see Appendix C):

$$\int_v d\bar{F} = \int_v \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} dv + \int_v (-\rho \frac{d^2\bar{U}}{dt^2}) dv + \int_v \bar{f} dv = 0 \qquad \ldots.(5.1)$$

$$\int_v d\bar{M} = \int_v \bar{R} \times \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} dv + \int_v \bar{R} \times (-\rho \frac{d^2\bar{U}}{dt^2}) dv + \int_v \bar{R} \times \bar{f} dv = 0 \qquad \ldots.(5.1)$$

where,

$\bar{\bar{\sigma}}$ - stress tensor resulting from the stresses applied to the motion of the continuous body

$\rho = \frac{dm}{dv}$ - mass density.

$\frac{d^2\bar{U}}{dt^2}$ - material acceleration of the generic point $\bar{R}$ in the geometric space

$\bar{f}$ - body force per unit volume

v - total volume of the continuous body

§(5.2) Static Case

If the apparent force created by the motion of the body is absent, i.e., $\frac{d\bar{U}}{dt}$ = constant vector, equations (5.1) and (5.2) will reduce to the equilibrium equations

$$\int_V d\bar{F} = \int_V \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} dv + \int_V \bar{f}\, dv = 0 \qquad \ldots\ldots(5.3)$$

and

$$\int_V d\bar{M} = \int_V \bar{R} \times \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial R} dv + \int_V \bar{R}\times\bar{f}\, dv = 0 \qquad \ldots\ldots(5.4)$$

Figure (5.1) shows an infinitesimal free body of the slender member of length dS and volume dv. A characteristic parallel fibre of the slender member of length dS* has been located by the position vector $\bar{R}$ with respect to axes $\bar{E}_n,\bar{E}_b$ of the Euler Triad. The characteristic infinitesimal volume dv is given by

$$dv = dS^*\, dR_n\, dR_b$$

where,

dS^* - Length of the characteristic parallel fibre

dR_n - Infinitesimal length along the axis $\bar{E}_n$

dR_b - Infinitesimal length along the axis $\bar{E}_b$

As usual, in the theory of elasticity the body-force $\bar{f}$ is considered as an externally applied boundary stress (or neglecting it altogether as being small compared to the external

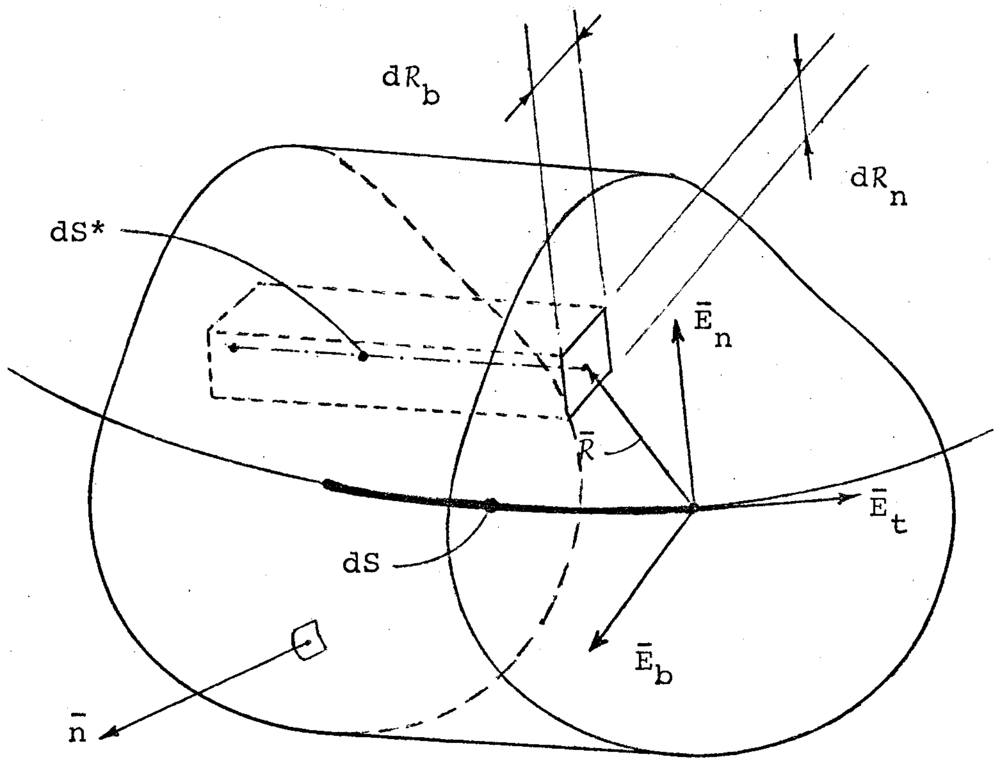


Figure (5.1)

applied forces).

Therefore, equation (5.3) can be further reduced to

$$\int_{R_b}\int_{R_n}\int_{S^*} \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} \, dS^* \, dR_n \, dR_b = 0$$

Substituting for the arc-length

$$dS^* = (1 - KR_n)dS$$

where the change in the value of torsion has been neglected over the infinitesimal distance dS.

$$\int_{S} \left[\int_{R_b}\int_{R_n} \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1 - KR_n) \, dR_n \, dR_b\right] dS = 0$$

Since the limits for R_n and R_b are definite, and the limit of integration over S is indefinite, above equation is satisfied for all S if,

$$\int_{R_b}\int_{R_n} \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1 - KR_n) \, dR_n \, dR_b = 0$$

or

$$\int_{A} \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-KR_n) \, dA = 0 \qquad \text{....(5.5)}$$

where, $dR_n \, dR_b = dA$ = infinitesimal normal cross-sectional area.

By advancing similar arguments, the equation of moment equilibrium can be written as

$$\int_A \bar{R} \times \{\frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-K\mathcal{R}_n)\} dA = 0 \qquad \ldots.(5.6)$$

§(5.3) Force Equilibrium Equation

Equation of force equilibrium, as given in the last article, appear as

$$\int_A \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-K\mathcal{R}_n) dA = 0$$

where A - normal cross-sectional area of the slender member. The directed derivative $\frac{\partial}{\partial \bar{R}}$ has already been calculated in §(3.2).

Introducing the stress tensor

$$\bar{\bar{\sigma}} = \bar{E}_t \bar{\sigma}_t + \bar{E}_n \bar{\sigma}_n + \bar{E}_b \bar{\sigma}_b$$

and taking cognizance of the directed derivative operator from equation (3.9)

$$(1-K\mathcal{R}_n) \frac{\partial}{\partial \bar{R}} \equiv \bar{E}_t \left(\frac{\partial}{\partial S} + T\mathcal{R}_b \frac{\partial}{\partial \mathcal{R}_n} - T\mathcal{R}_n \frac{\partial}{\partial \mathcal{R}_b}\right) + \bar{E}_n (1-K\mathcal{R}_n) \frac{\partial}{\partial \mathcal{R}_n} + \bar{E}_b (1-K\mathcal{R}_n) \frac{\partial}{\partial \mathcal{R}_b}$$

the integrand becomes

$$(1-K\mathcal{R}_n) \frac{\partial}{\partial \bar{R}} \cdot \bar{\bar{\sigma}} = [\bar{E}_t \left(\frac{\partial}{\partial S} + T\mathcal{R}_b \frac{\partial}{\partial \mathcal{R}_n} - T\mathcal{R}_n \frac{\partial}{\partial \mathcal{R}_b}\right) + \bar{E}_n (1-K\mathcal{R}_n) \frac{\partial}{\partial \mathcal{R}_n} + \bar{E}_b (1-K\mathcal{R}_n) \frac{\partial}{\partial \mathcal{R}_b}] \cdot (\bar{E}_t \bar{\sigma}_t + \bar{E}_n \bar{\sigma}_n + \bar{E}_b \bar{\sigma}_b)$$

Since the scalar operators can commute with dot products

$$(1-KR_n)\ \frac{\partial\cdot\bar{\bar{\sigma}}}{\partial\bar{R}} = \bar{E}_t\cdot[\frac{\partial}{\partial S}(\bar{E}_t\bar{\sigma}_t+\bar{E}_n\bar{\sigma}_n+\bar{E}_b\bar{\sigma}_b)+TR_b\ \frac{\partial}{\partial R_n}(\bar{E}_t\bar{\sigma}_t+\bar{E}_n\bar{\sigma}_n+\bar{E}_b\bar{\sigma}_b$$

$$- TR_n\ \frac{\partial}{\partial R_b}\ (\bar{E}_t\bar{\sigma}_t + \bar{E}_n\bar{\sigma}_n + \bar{E}_b\bar{\sigma}_b)]$$

$$+ (1-KR_n)\bar{E}_n\cdot\frac{\partial}{\partial R_n}\ (\bar{E}_t\bar{\sigma}_t + \bar{E}_n\bar{\sigma}_n + \bar{E}_b\bar{\sigma}_b)$$

$$+ (1-KR_n)\bar{E}_b\cdot\frac{\partial}{\partial R_b}\ (\bar{E}_t\bar{\sigma}_t + \bar{E}_n\bar{\sigma}_n + \bar{E}_b\bar{\sigma}_b)$$

It should be noted here that $\bar{E}_i \neq \bar{E}_i(R_j)$ for $i=t,n,b$ and $j=n,b$

Carrying out the indicated operations the following results are obtained:

$$(1-KR_n)\ \frac{\partial\cdot\bar{\bar{\sigma}}}{\partial\bar{R}} = [\frac{\partial\bar{\sigma}_t}{\partial S} - K\bar{\sigma}_n + TR_b\ \frac{\partial\bar{\sigma}_t}{\partial R_n} - TR_n\ \frac{\partial\bar{\sigma}_t}{\partial R_b} + (1-KR_n)\ \frac{\partial\bar{\sigma}_n}{\partial R_n} + (1-KR_n)\ \frac{\partial\bar{\sigma}_b}{\partial R_b}]$$

Hence equation (5.5) takes the following form

$$\int_A (1-KR_n)\ \frac{\partial\cdot\bar{\bar{\sigma}}}{\partial\bar{R}}\ dA = \int_A\ [\frac{\partial\bar{\sigma}_t}{\partial S} + TR_b\ \frac{\partial\bar{\sigma}_t}{\partial R_n} - TR_n\ \frac{\partial\bar{\sigma}_t}{\partial R_b} + \frac{\partial}{\partial R_n}\ \{(1-KR_n)\bar{\sigma}_n\} + \frac{\partial}{\partial R_b}\ \{(1-KR_n)\bar{\sigma}_b\}]dA = 0 \quad ....(5.7)$$

The equation above requires a closer examination in order to put it in a more useful form. Assuming $\frac{\partial}{\partial\bar{R}}$ to be a modified directed derivative restricted to the plane $\{\bar{E}_n,\bar{E}_b\}$ of the

Euler Triad, it can be specified as

$$\bar{E}_t \times \frac{\partial}{\partial \bar{R}} \times \bar{E}_t \equiv \frac{\partial}{\partial \bar{R}} \equiv \bar{E}_n \frac{\partial}{\partial R_n} + \bar{E}_b \frac{\partial}{\partial R_b}$$

and

$$\frac{\partial}{\partial \bar{R}} \cdot \{(1-KR_n)\bar{\bar{\sigma}}\} = \frac{\partial}{\partial R_n} \{(1-KR_n)\bar{\sigma}_n\} + \frac{\partial}{\partial R_b} \{(1-KR_n)\bar{\sigma}_b\}$$

Using the planar form of the Gauss-Divergence Theorem (as a special case of the general theorem) an integral over an area can be converted into an integral around the periphery of the planar surface:

$$\int_A [\frac{\partial}{\partial \bar{R}} \cdot (\ )]dA \equiv \oint_\Omega [\bar{n} \cdot (\ )]d\Omega$$

where

$\bar{n}$ - unit vector in the cross-section normal to the boundary of the cross-section

( ) - any admissible function

$d\Omega$ - infinitesimal arc length [see figure (5.2]

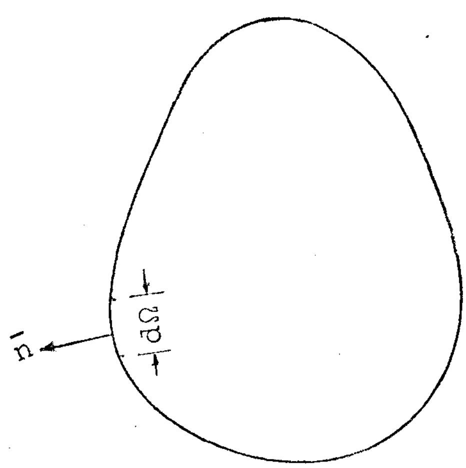


Figure (5.2)

Thus

$$\int_A [\frac{\partial}{\partial \bar{R}} \cdot \{(1-K\mathcal{R}_n)\bar{\bar{\sigma}}\}] dA = \oint_\Omega [\bar{n} \cdot \{(1-K\mathcal{R}_n)\bar{\bar{\sigma}}\}] d\Omega$$

and

$$\int_A \frac{\partial \bar{\sigma}_t}{\partial S} dA = \frac{\partial}{\partial S} \int_A \bar{\sigma}_t \, dA = \frac{\partial}{\partial S} \bar{F}_t(\sigma)$$

where $\bar{F}_t(\sigma)$ is the stress resultant on the surface whose normal is $\bar{E}_t$ and

$$T\mathcal{R}_b \frac{\partial \bar{\sigma}_t}{\partial \mathcal{R}_n} - T\mathcal{R}_n \frac{\partial \bar{\sigma}_t}{\partial \mathcal{R}_b} = (\bar{R} \times \bar{T}) \cdot \frac{\partial \bar{\sigma}_t}{\partial \bar{R}}$$

From the conversions above, equation (5.7) can finally be written

$$\frac{\partial \bar{F}_t(\sigma)}{\partial S} + \int_A [(\bar{R} \times \bar{T}) \cdot \frac{\partial \bar{\sigma}_t}{\partial \bar{R}}] dA + \oint_\Omega [\bar{n} \cdot \{(1-K\mathcal{R}_n)\bar{\bar{\sigma}}\}] d\Omega = 0$$

This equation represents the force equilibrium equation for the unidimensional field of the slender member.

The term $\oint_\Omega [\bar{n} \cdot \{(1-K\mathcal{R}_n)\bar{\bar{\sigma}}\}] d\Omega$ represents the applied loading function $\bar{P}(S)$ per unit length of elastica. Thus, the final direct form of the force equilibrium equation emerges in the form

$$\frac{\partial \bar{F}_t(\sigma)}{\partial S} + \int_A [(\bar{R} \times \bar{T}) \cdot \frac{\partial \bar{\sigma}_t}{\partial \bar{R}}] dA + \bar{P}(S) = 0 \qquad \ldots.(5.8)$$

where $\bar{P}(S)$ is defined by

$$\bar{P}(S) = \oint_{\Omega} [\bar{n}\cdot\{(1-KR_n)\bar{\bar{\sigma}}\}]d\Omega = \oint_{\Omega} \bar{\sigma}_n(1-KR_n)d\Omega$$

as $\bar{n}\cdot\bar{\bar{\sigma}} = \bar{\sigma}_n$.

Equation (5.8) is a field equation of unidimensional continuum. For linearly isotropic elasticity, the constitutive equation can be expressed as follows:

$$\bar{\bar{\sigma}} = 2\mu\bar{\bar{\varepsilon}} + \lambda(\bar{\bar{\varepsilon}}:\bar{\bar{I}})\bar{\bar{I}} \quad \text{(see Appendix D)}$$

where μ,λ - Cauchy-Lamé first and second constants respectively

$\bar{\bar{I}}$ - Identity tensor (referred to the geometric space in which $\bar{\bar{\sigma}}$ is evaluated).

$\bar{\bar{\varepsilon}}$ - Strain tensor

$(\bar{\bar{\varepsilon}}:\bar{\bar{I}})$ - First scalar invariant of the strain tensor.

From the constitutive equation above the stress vector can be obtained by

$$\bar{\sigma}_t = \bar{E}_t\cdot\bar{\bar{\sigma}}$$

or, after taking the dot-product

$$\bar{\sigma}_t = (2\mu+\lambda)\varepsilon_{tt}\,\bar{E}_t + 2\mu\,\varepsilon_{tn}\,\bar{E}_n + 2\mu\,\varepsilon_{tb}\,\bar{E}_b \quad \ldots.(5.9)$$

Therefore,

$$\frac{\partial\bar{\sigma}_t}{\partial\bar{R}} = [\bar{E}_n\frac{\partial}{\partial R_n} + \bar{E}_b\frac{\partial}{\partial R_b}][(2\mu+\lambda)\,\varepsilon_{tt}\,\bar{E}_t + 2\mu\,\varepsilon_{tn}\,\bar{E}_n + 2\mu\,\varepsilon_{tb}\,\bar{E}_b]$$

$$= (2\mu+\lambda)\frac{\partial\varepsilon_{tt}}{\partial R_n}\bar{E}_n\bar{E}_t + 2\mu\frac{\partial\varepsilon_{tn}}{\partial R_n}\bar{E}_n\bar{E}_n + 2\mu\frac{\partial\varepsilon_{tb}}{\partial R_n}\bar{E}_n\bar{E}_b$$

$$+ (2\mu+\lambda)\frac{\partial\varepsilon_{tt}}{\partial R_b}\bar{E}_b\bar{E}_t + 2\mu\frac{\partial\varepsilon_{tn}}{\partial R_b}\bar{E}_b\bar{E}_n + 2\mu\frac{\partial\varepsilon_{tb}}{\partial R_b}\bar{E}_b\bar{E}_b$$

The cross product

$$(\bar{R}\times\bar{T}) = (R_n\bar{E}_n + R_b\bar{E}_b)\times(T\bar{E}_t)$$

$$= TR_b\bar{E}_n - TR_n\bar{E}_b$$

and, therefore,

$$(\bar{R}\times\bar{T})\cdot\frac{\partial\bar{\sigma}_t}{\partial\bar{R}} = (TR_b\bar{E}_n - TR_n\bar{E}_b)\cdot[(2\mu+\lambda)\frac{\partial\varepsilon_{tt}}{\partial R_n}\bar{E}_n\bar{E}_t$$

$$+ 2\mu\frac{\partial\varepsilon_{tn}}{\partial R_n}\bar{E}_n\bar{E}_n + 2\mu\frac{\partial\varepsilon_{tb}}{\partial R_n}\bar{E}_n\bar{E}_b$$

$$+ (2\mu+\lambda)\frac{\partial\varepsilon_{tt}}{\partial R_b}\bar{E}_b\bar{E}_t + 2\mu\frac{\partial\varepsilon_{tn}}{\partial R_b}\bar{E}_b\bar{E}_n + 2\mu\frac{\partial\varepsilon_{tb}}{\partial R_b}\bar{E}_b\bar{E}_b]$$

$$= [(2\mu+\lambda)T(R_b\frac{\partial\varepsilon_{tt}}{\partial R_n} - R_n\frac{\partial\varepsilon_{tt}}{\partial R_b})]\bar{E}_t$$

$$+ [2\mu T(R_b\frac{\partial\varepsilon_{tn}}{\partial R_n} - R_n\frac{\partial\varepsilon_{tn}}{\partial R_b})]\bar{E}_n + [2\mu T(R_b\frac{\partial\varepsilon_{tb}}{\partial R_n}$$

$$- R_n\frac{\partial\varepsilon_{tb}}{\partial R_b})]\bar{E}_b$$

The stress resultant $\bar{F}_t(\sigma)$ can be referred to the mobile Euler directed base $\{\bar{E}_t,\bar{E}_n,\bar{E}_b\}$

$$\bar{F}_t(\sigma) = F_{tt}(\sigma)\bar{E}_t + F_{tn}(\sigma)\bar{E}_n + F_{tb}(\sigma)\bar{E}_b$$

Therefore,

$$\frac{\partial\bar{F}_t(\sigma)}{\partial S} = [\frac{\partial F_{tt}(\sigma)}{\partial S} - K F_{tn}(\sigma)]\bar{E}_t + [\frac{\partial F_{tn}(\sigma)}{\partial S} + K F_{tt}(\sigma) - T F_{tb}(\sigma)]\bar{E}_n$$

$$+ [\frac{\partial F_{tb}(\sigma)}{\partial S} + T F_{tn}(\sigma)]\bar{E}_b$$

Also, $\bar{P}(S)$, the external applied force can be referred to the

Euler directed base $\{\bar{E}_t, \bar{E}_n, \bar{E}_b\}$

$$\bar{P}(S) = P_t(S)\bar{E}_t + P_n(S)\bar{E}_n + P_b(S)\bar{E}_b$$

Substituting expressions for $\frac{\partial \bar{F}_t(\sigma)}{\partial S}$, $[(\bar{R}\times\bar{T}).\frac{\partial \bar{\sigma}_t}{\partial \bar{R}}]$ and $\bar{P}(S)$ in equation (5.8) and grouping the coefficients of the base vectors $\bar{E}_t$, $\bar{E}_n$, $\bar{E}_b$, the following equation is obtained:

$$[\frac{\partial F_{tt}(\sigma)}{\partial S} - K\,F_{tn}(\sigma) + (2\mu+\lambda)T\int_A (R_b \frac{\partial \varepsilon_{tt}}{\partial R_n} - R_n \frac{\partial \varepsilon_{tt}}{\partial R_b})dA + P_t(S)]\bar{E}_t$$

$$+[\frac{\partial F_{tn}(\sigma)}{\partial S} + K\,F_{tt}(\sigma) - T\,F_{tb}(\sigma) + 2\mu T\int_A (R_b \frac{\partial \varepsilon_{tn}}{\partial R_n} - R_n \frac{\partial \varepsilon_{tn}}{\partial R_b})dA + P_n(S)]\bar{E}_n$$

$$+(\frac{\partial F_{tb}(\sigma)}{\partial S} + T\,F_{tn}(\sigma) + 2\mu T\int_A (R_b \frac{\partial \varepsilon_{tb}}{\partial R_n} - R_n \frac{\partial \varepsilon_{tb}}{\partial R_b})dA + P_b(S)]\bar{E}_b = 0$$

Since the vectors $\{\bar{E}_t, \bar{E}_n, \bar{E}_b\}$ of the mobile Euler directed base constitute an independent vector set, each of the three coefficients must vanish separately in order to satisfy this equation. Thus, the following force equilibrium equations result:

$$\left.\begin{aligned}
&\frac{\partial F_{tt}(\sigma)}{\partial S} - K\,F_{tn}(\sigma) + (2\mu+\lambda)T\int_A (R_b \frac{\partial \varepsilon_{tt}}{\partial R_n} - R_n \frac{\partial \varepsilon_{tt}}{\partial R_b})dA + P_t(S) = 0 \\
&\frac{\partial F_{tn}(\sigma)}{\partial S} + K\,F_{tt}(\sigma) - T\,F_{tb}(\sigma) + 2\mu T\int_A (R_b \frac{\partial \varepsilon_{tn}}{\partial R_n} - R_n \frac{\partial \varepsilon_{tn}}{\partial R_b})dA + P_n(S) = 0 \\
&\frac{\partial F_{tb}(\sigma)}{\partial S} + T\,F_{tn}(\sigma) + 2\mu T\int_A (R_b \frac{\partial \varepsilon_{tb}}{\partial R_n} - R_n \frac{\partial \varepsilon_{tb}}{\partial R_b})dA + P_b(S) = 0
\end{aligned}\right\}$$

....(5.10)

These equations represent Force Equilibrium Equations applicable to all linearly elastic slender members. Since this investigation is concerned with linear, small-displacement theory, equations (5.10) are evaluated with reference to equations (3.14). It is important to observe that equations (3.14) are 'simplified' expressions for strain tensor components ε_{ij} where 'warping' has been assumed to be negligibly small and consequently was set equal to zero.

Substitution for the partial derivatives of ε_{ij} from equation (3.14) and integration result into the following Force Equilibrium equations:

$$\left\{\begin{aligned} &\frac{\partial F_{tt}(\sigma)}{\partial S} - K\,F_{tn}(\sigma) + (2\mu+\lambda)T\Big[A_{(R_n)}\Big(\frac{dm_1}{dS} + T\phi_b\Big) \\ &\qquad - A_{(R_b)}\Big(\theta K + Km_2 - \frac{d\phi_b}{dS} + Tm_1\Big)\Big] + P_t(S) = 0 \\ &\frac{\partial F_{tn}(\sigma)}{\partial S} + K\,F_{tt}(\sigma) - T\,F_{tb}(\sigma) + \mu T\,A_{(R_b)}\Big(\frac{d\theta}{dS} + K\phi_b + \frac{dm_2}{dS}\Big) + P_n(S) = 0 \\ &\frac{\partial F_{tb}(\sigma)}{\partial S} + T\,F_{tn}(\sigma) + \mu T\,A_{(R_n)}\Big(\frac{d\theta}{dS} + K\phi_b + \frac{dm_2}{dS}\Big) + P_b(S) = 0 \end{aligned}\right. \quad \ldots.(5.11)$$

where

$$A_{(R_n)} = \int_A R_b\,dA\;,\quad A_{(R_b)} = \int_A R_n\,dA$$

§(5.4) Moment Equilibrium Equation

Equations of moment equilibrium as given in §(5.2) appear as

$$\int_A \bar{R} \times \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-K\mathcal{R}_n)dA = 0$$

With reference to figure (3.1)

$$\bar{R} = \bar{R}^{\circ} + \bar{\mathcal{R}}$$

Substituting for $\bar{R}$

$$\int_A (\bar{R}^{\circ} + \bar{\mathcal{R}}) \times \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-K\mathcal{R}_n)dA = 0$$

or

$$\bar{R}^{\circ} \times \int_A \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-K\mathcal{R}_n)dA + \int_A \bar{\mathcal{R}} \times \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-K\mathcal{R}_n)dA = 0$$

where $\bar{R}^{\circ} \neq \bar{R}^{\circ}(A)$ denotes the position vector to the centroid of the cross-section.

The equation of force equilibrium (5.5) requires $\int_A \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-K\mathcal{R}_n)dA$ to vanish. Moreover, the term $(1-K\mathcal{R}_n) \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}}$ has already been evaluated on page (50). Therefore, the Equation of Moment Equilibrium takes the form

$$\int_A [\bar{\mathcal{R}} \times \frac{\partial \bar{\sigma}_t}{\partial S} - K(\bar{\mathcal{R}}\times\bar{\sigma}_n) + T\mathcal{R}_b \, \bar{\mathcal{R}} \times \frac{\partial \bar{\sigma}_t}{\partial \mathcal{R}_n} - T\mathcal{R}_n \, \bar{\mathcal{R}} \times \frac{\partial \bar{\sigma}_t}{\partial \mathcal{R}_b}$$

$$+ (1-K\mathcal{R}_n) \, \bar{\mathcal{R}} \times \frac{\partial \bar{\sigma}_n}{\partial \mathcal{R}_n} + (1-K\mathcal{R}_n) \, \bar{\mathcal{R}} \times \frac{\partial \bar{\sigma}_b}{\partial \mathcal{R}_b}] = 0$$

....(5.12)

In order to carry out integration of equation (5.12) over the cross-sectional area A the following identities are established:

$$\bar{R} \times \frac{\partial \bar{\sigma}_t}{\partial S} = \frac{\partial}{\partial S} (\bar{R}\times\bar{\sigma}_t) - [R_n(-K\bar{E}_t+T\bar{E}_b) + R_b(-T\bar{E}_t)]\times\bar{\sigma}_t$$

$$TR_b\bar{R} \times \frac{\partial \bar{\sigma}_t}{\partial R_n} = \frac{\partial}{\partial R_n} (TR_b\bar{R}\times\bar{\sigma}_t) - TR_b \bar{E}_n\times\bar{\sigma}_t$$

$$TR_n\bar{R}\times \frac{\partial \bar{\sigma}_t}{\partial R_b} = \frac{\partial}{\partial R_b} (TR_n \bar{R}\times\bar{\sigma}_t) - TR_n \bar{E}_b\times\bar{\sigma}_t$$

$$(1-KR_n)\bar{R} \times \frac{\partial \bar{\sigma}_n}{\partial R_n} + (1-KR_n)\bar{R}\times \frac{\partial \bar{\sigma}_b}{\partial R_b} = K\bar{R}\times\bar{\sigma}_t - (1-KR_n)(\bar{E}_n\times\bar{\sigma}_n+\bar{E}_b\times\bar{\sigma}_b)$$

$$- \frac{\partial}{\partial \bar{R}}\cdot[(1-KR_n)\bar{\bar{\sigma}}\times\bar{R}]$$

Substituting these results in the equation (5.12) and re-arranging the terms yields

$$\int_A [\frac{\partial}{\partial S} (\bar{R}\times\bar{\sigma}_t) + \bar{E}_t\times\bar{\sigma}_t + \frac{\partial}{\partial R_n} (TR_b \bar{R}\times\bar{\sigma}_t) - \frac{\partial}{\partial R_b} (TR_n \bar{R}\times\bar{\sigma}_t)$$

$$+ (1-KR_n)(\bar{E}_t\times\bar{\sigma}_t + \bar{E}_n\times\bar{\sigma}_n + \bar{E}_b\times\bar{\sigma}_b)]dA + \int_A (-) \frac{\partial}{\partial \bar{R}}\cdot[(1-KR_n)\bar{\bar{\sigma}}\times\bar{R}]dA=0$$

Expression

$$[\bar{E}_t\times\bar{\sigma}_t + \bar{E}_n\times\bar{\sigma}_n + \bar{E}_b\times\bar{\sigma}_b) = \bar{\bar{I}}\overset{\bullet}{\times}\bar{\bar{\sigma}}$$

represents the vector invariant of the symmetric stress tensor $\bar{\bar{\sigma}}$, i.e. $\bar{\bar{\sigma}}_c = \bar{\bar{\sigma}}$ and thus

$$\bar{\bar{I}}\overset{\bullet}{\times}\bar{\bar{\sigma}} = 0$$

Using the planar form of the Gauss Divergence Theorem as outlined in § (5.3)

$$\int_A \frac{\partial}{\partial \bar{R}} \cdot [(1-KR_n)\ \bar{\bar{\sigma}}\times\bar{R}]dA = \oint_\Omega [\bar{n}\cdot(1-KR_n)\ \bar{\bar{\sigma}}\times\bar{R}]d\Omega$$

This conversion leads the equation of moment equilibrium to the form

$$\frac{\partial}{\partial S}\int_A \bar{R}\times\bar{\sigma}_t\ dA + \bar{E}_t\times\int_A \bar{\sigma}_t\ dA + \int_A [\frac{\partial}{\partial R_n}(TR_b\ \bar{R}\times\bar{\sigma}_t)-\frac{\partial}{\partial R_b}(TR_n\ \bar{R}\times\bar{\sigma}_t)]dA$$

$$-\oint_\Omega \bar{n}\cdot\{(1-KR_n)\ \bar{\bar{\sigma}}\times\bar{R}\}d\Omega = 0$$

The Equation of Moment Equilibrium contains the couple of the applied boundary loads in the integral around the boundary. This applied couple per unit length of elastica is defined by

$$\bar{M}(S) = \int_\Omega -[\bar{n}.(1-KR_n)\ \bar{\bar{\sigma}}\times\bar{R}]d\Omega$$

The stress couple acting in the cross-section with the normal vector $\bar{E}_t$ is

$$\bar{E}_t\cdot\bar{\bar{M}}(\sigma) = \bar{M}_t(\sigma) = \int_A \bar{R}\times\bar{\sigma}_t\ dA \qquad ....(5.13)$$

and the stress resultant for the same cross-section is

$$\bar{E}_t\cdot\bar{\bar{F}}(\sigma) = \bar{F}_t(\sigma) = \int_A \bar{\sigma}_t\ dA$$

Employing these results in the moment equilibrium equation yields

$$\frac{\partial \bar{M}_t(\sigma)}{\partial S} + \bar{E}_t \times \bar{F}_t(\sigma) + \int_A [\frac{\partial}{\partial R_n} (TR_b \; \bar{R}\times\bar{\sigma}_t) - \frac{\partial}{\partial R_b} (TR_n \; \bar{R}\times\bar{\sigma}_t)]dA + \bar{M}(S) = 0$$

The integrand of this expression can be written as

$$(\bar{R}\times\bar{T}) \cdot \frac{\partial}{\partial \bar{R}} (\bar{R}\times\bar{\sigma}_t)$$

Hence the final direct form of the Moment Equilibrium Equation emerges as

$$\frac{\partial \bar{M}_t(\sigma)}{\partial S} + \bar{E}_t\times\bar{F}_t(\sigma) + \int_A [(\bar{R}\times\bar{T})\cdot \frac{\partial}{\partial \bar{R}} (\bar{R}\times\bar{\sigma}_t)]dA + \bar{M}(S) = 0$$

....(5.14)

In the manner similar to §(5.3), this equation can be resolved into its component form:

$$\left.\begin{aligned}
&\frac{\partial M_{tt}(\sigma)}{\partial S} - K\,M_{tn}(\sigma) + 2\mu T \int_A [R_b(\varepsilon_{tb} + R_n \frac{\partial \varepsilon_{tb}}{\partial R_n} - R_b \frac{\partial \varepsilon_{tn}}{\partial R_n}) \\
&-R_n(R_n \frac{\partial \varepsilon_{tb}}{\partial R_b} - \varepsilon_{tn} - R_b \frac{\partial \varepsilon_{tn}}{\partial R_b})]dA + M_t(S) = 0 \\
&\frac{\partial M_{tn}(\sigma)}{\partial S} + K\,M_{tt}(\sigma) - TM_{tb}(\sigma) - F_{tb}(\sigma) + (2\mu+\lambda)T\int_A [R_b^2 (\frac{\partial \varepsilon_{tt}}{\partial R_n}) \\
&-R_n(\varepsilon_{tt} + R_b \frac{\partial \varepsilon_{tt}}{\partial R_b})]dA + M_n(S) = 0 \\
&\frac{\partial M_{tb}(\sigma)}{\partial S} + T\,M_{tn}(\sigma) + F_{tn}(\sigma) - (2\mu+\lambda)T \int_A [R_b(\varepsilon_{tt} + R_n \frac{\partial \varepsilon_{tt}}{\partial R_n}) \\
&- R_n^2 (\frac{\partial \varepsilon_{tt}}{\partial R_b})]dA + M_b(S) = 0
\end{aligned}\right\}$$

....(5.15)

These equations represent Moment Equilibrium Equations applicable to all linearly elastic slender members. In conformity with the arguments mentioned on page (56), equations (5.15) are evaluated for strain tensor components given by equations (3.14). They are as follows:

$$\left.\begin{aligned}
&\frac{\partial M_{tt}(\sigma)}{\partial S} - KM_{tn}(\sigma) + M_t(S) = 0 \\
&\frac{\partial M_{tn}(\sigma)}{\partial S} + KM_{tt}(\sigma) - TM_{tb}(\sigma) - F_{tb}(\sigma) + (2\mu+\lambda)T[(\frac{dm_1}{dS} + T\phi_b)(I_{nn}-I_{bb}) \\
&+ 2I_{bn}(\theta K + Km_2 - \frac{d\phi_b}{dS} + Tm_1)] + M_n(S) = 0 \\
&\frac{\partial M_{tb}(\sigma)}{\partial S} + TM_{tn}(\sigma) + F_{tn}(\sigma) + (2\mu+\lambda)T\ [2I_{nb}\ (\frac{dm_1}{dS} + T\phi_b) \\
&- (\theta K + Km_2 - \frac{d\phi_b}{dS} + Tm_1)(I_{nn} - I_{bb})]+M_b(S) = 0
\end{aligned}\right\}$$

....(5.16)

where the

$$I_{nn} = \int_A R_b^{\,2}\, dA \ ; \ I_{bb} = \int_A R_n^{\,2}\, dA$$

$$I_{nb} = \int_A - R_n R_b\, dA = I_{bn}$$

represent the components of the Second Area Moment Tensor

$$\bar{\bar{I}} = \int_A (\bar{R}\cdot\bar{R}\ \bar{\bar{I}} - \bar{R}\bar{R})dA$$

§(5.5) Dynamic Case

Equations (5.1) and (5.2) of §(5.1) are general equations of motion valid for any body in space.

As mentioned in the static case, the body force $\bar{f}$ is considered to be added to the externally applied loads (or neglected altogether as being small compared to the external applied loading). Therefore,

$$\int_V \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} dv + \int_V (-\rho \frac{d^2\bar{U}}{dt^2}) dv = 0$$

and

$$\int_V \bar{R} \times \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} dv + \int_V \bar{R} \times (-\rho \frac{d^2\bar{U}}{dt^2}) dv = 0$$

In the manner similar to that of §(5.2), these two equations can be written in the form

$$\int_A [\frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-KR_n) - \rho \frac{d^2\bar{U}}{dt^2}] dA = 0 \quad ....(5.17)$$

and

$$\int_A [\bar{R} \times \frac{\partial \bar{\bar{\cdot}} \sigma}{\partial \bar{R}} (1-KR_n) - \rho \bar{R} \times \frac{d^2\bar{U}}{dt^2}] dA = 0 \quad ....(5.18)$$

Referred to figure (3.1)

$$\bar{R} = \bar{R}^{\circ} + \bar{R}$$

Substituting for $\bar{R}$ in equation (5.18)

$$\int [(\bar{R}^{\circ} + \bar{R}) \times \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-KR_n) - \rho (\bar{R}^{\circ} + \bar{R}) \times \frac{d^2\bar{U}}{dt^2}] dA = 0$$

or

$$\bar{R}^{\circ} \times \int_A [\frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-KR_n) - \rho \frac{d^2\bar{U}}{dt^2}] dA + \int_A [\bar{R} \times \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-KR_n) - \rho \bar{R} \times \frac{d^2\bar{U}}{dt^2}] dA = 0$$

because $\bar{R}^{\circ}$ is independent of A i.e., $\bar{R}^{\circ} \neq \bar{R}(A)$. Furthermore, equation (5.17) requires

$$\int_A [\frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-KR_n) - \rho \frac{d^2\bar{U}}{dt^2}] dA = 0$$

Hence the Equation of Moment Equilibrium assumes the form

$$\int_A [\bar{R} \times \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-KR_n) - \rho \bar{R} \times \frac{d^2\bar{U}}{dt^2}] dA = 0 \quad \text{....(5.19)}$$

The expression for displacement vector $\bar{U}$ of an arbitrary point in the geometric space of the slender member when it moves from the undeformed reference configuration to an arbitrary deformed configuration, has been obtained in §(3.4), equation (3.3). Since it is assumed that the ensuing displacement will be small, the linear acceleration of the arbitrary point of the elastica can be assumed to be the same throughout the cross-section. Hence, acceleration of arbitrary point in the cross-section can be represented by that of the centroid. Thus

$$\bar{U} = \bar{U}^{\circ} + \bar{\theta} \times \bar{R}$$

It should be noted that "warping effect" has beem assumed to be negligible, and the rotation $\bar{\theta} = \theta \bar{E}_t$

Therefore,

$$\frac{d^2\bar{U}}{dt^2} = \frac{d^2\bar{U}^\circ}{dt^2} + \frac{d^2}{dt^2}(\bar{\theta}\times\bar{R})$$

For the case of small displacements, the material derivatives of the Euler Triad can be assumed to be very small, and taken to be zero i.e.

$$\frac{d\bar{E}_i}{dt} = 0 \quad \text{for} \quad i = t,n,b \qquad \text{....(5.20)}$$

Since,

$$\frac{d^2\bar{U}}{dt^2} = \frac{d^2\bar{U}^\circ}{dt^2} + \frac{d^2\bar{\theta}}{dt^2} \times \bar{R} + 2\,\frac{d\bar{\theta}}{dt} \times \frac{d\bar{R}}{dt} + \bar{\theta} \times \frac{d^2\bar{R}}{dt^2}$$

and

$$\frac{d\bar{R}}{dt} = \frac{d}{dt}(R_n\bar{E}_n + R_b\bar{E}_b) = R_n\frac{d\bar{E}_n}{dt} + R_b\frac{d\bar{E}_b}{dt}$$

In consequence of equation (5.20), $\frac{d\bar{R}}{dt} = \frac{d^2\bar{R}}{dt^2} = 0$

Therefore,

$$\frac{d^2\bar{U}}{dt^2} = \frac{d^2\bar{U}^\circ}{dt^2} + \frac{d^2\bar{\theta}}{dt^2} \times \bar{R}$$

or

$$\int_A \frac{d^2\bar{U}}{dt^2}\,dA = \frac{d^2\bar{U}^\circ}{dt^2}\int_A dA + \frac{d^2\bar{\theta}}{dt^2} \times \int_A \bar{R}\,dA$$

or

$$\int_A \frac{d^2\bar{U}}{dt^2}\,dA = \frac{d^2\bar{U}^\circ}{dt^2}A + \frac{d^2\bar{\theta}}{dt^2} \times \int_A \bar{R}\,dA \qquad \text{....(5.21)}$$

And

$$\int_A \rho\bar{R} \times \frac{d^2\bar{U}}{dt^2}\, dA = \rho \int_A [\bar{R} \times \frac{d^2\bar{U}^\circ}{dt^2} + \bar{R} \times (\frac{d^2\bar{\theta}}{dt^2} \times \bar{R})]\, dA$$

$$= \rho \int_A [-(\frac{d^2\bar{U}^\circ}{dt^2} \times \bar{R}) + \frac{d^2\bar{\theta}}{dt^2} (\bar{R}\cdot\bar{R}) - (\frac{d^2\bar{\theta}}{dt^2}\cdot\bar{R})\bar{R}]\, dA$$

$$= \rho\ [-\frac{d^2\bar{U}^\circ}{dt^2} \times \int_A \bar{R}\, dA + \frac{d^2\bar{\theta}}{dt^2} \int_A (\bar{R}\cdot\bar{R})dA - \frac{d^2\bar{\theta}}{dt^2} \cdot \int_A \bar{R}\bar{R}\, dA]$$

$$= -\rho\ [\frac{d^2\bar{U}^\circ}{dt^2} \times \int_A \bar{R}\, dA - I_{tt} \frac{d^2\bar{\theta}}{dt^2}] \qquad \ldots.(5.22)$$

where

$$I_{tt} = \int_A (\bar{R}\cdot\bar{R})dA = \int_A (R_n^2 + R_b^2)dA$$

and $\frac{d^2\bar{\theta}}{dt^2} \cdot \int_A \bar{R}\bar{R}\ dA$ vanishes

§(5.6) <u>Force Axiom of Motion</u>

The integral $\int_A \frac{\partial\cdot\bar{\bar{\sigma}}}{\partial\bar{R}} (1-KR_n)dA$ has been investigated in detail in the static case §(5.2). This yields equation (5.8). Therefore, equation (5.17) in conjunction with equation (5.21) can be written in the direct form

$$\frac{\partial\bar{F}_t(\sigma)}{\partial S} + \int_A [(\bar{R}\times\bar{T})\cdot\frac{\partial\bar{\sigma}_t}{\partial\bar{R}}]dA - \rho\ [A \frac{d^2\bar{U}^\circ}{dt^2} + \frac{d^2\bar{\theta}}{dt^2} \times \int_A \bar{R}\ dA] + \bar{P}(S) = 0$$

$$\ldots.(5.23)$$

In order to refer (5.23) to the mobile directed base only

$$\rho[A \frac{d^2\bar{U}^\circ}{dt^2} + \frac{d^2\bar{\theta}}{dt^2} \times \int_A \bar{R}\ dA]$$

is required to be referred to the directed base, as the residual quantity has already been referred to the directed base (see §5.2). If

$$\bar{U}^{\circ} = U_t^{\circ}\bar{E}_t + U_n^{\circ}\bar{E}_n + U_b^{\circ}\bar{E}_b$$

then

$$\rho[A \frac{d^2\bar{U}^{\circ}}{dt^2} + \frac{d^2\bar{\theta}}{dt^2} \times \int_A \bar{R}\, dA] = \rho[A \frac{d^2}{dt^2} (U_t^{\circ}\bar{E}_t + U_n^{\circ}\bar{E}_n + U_b^{\circ}\bar{E}_b)]$$

$$+ \frac{d^2}{dt^2} (\theta\bar{E}_t) \times \int_A (R_n\bar{E}_n + R_b\bar{E}_b) dA]$$

$$= \rho[A(\frac{d^2U_t}{dt^2} \bar{E}_t + \frac{d^2U_n^{\circ}}{dt^2} \bar{E}_n + \frac{d^2U_b^{\circ}}{dt^2} \bar{E}_b)]$$

$$+ \frac{d^2\theta}{dt^2} \bar{E}_t \times \{A_{(R_b)}\bar{E}_n + A_{(R_n)}\bar{E}_b\}$$

upon the use of equation (5.20)

Hence,

$$\rho[A \frac{d^2\bar{U}^{\circ}}{dt^2} + \frac{d^2\bar{\theta}}{dt^2} \times \int \bar{R}\, dA] = \rho[\{ A \frac{d^2U_t^{\circ}}{dt^2}\} \bar{E}_t + \{A \frac{d^2U_n^{\circ}}{dt^2} - A_{(R_n)} \frac{d^2\theta}{dt^2}\}\bar{E}_n$$

$$+ \{A \frac{d^2U_b^{\circ}}{dt^2} + A_{(R_b)} \frac{d^2\theta}{dt^2}\} \bar{E}_b]$$

For an independent base vector set $\{\bar{E}_i\}(i=t,n,b)$ all the components of equation (5.23) vanish.

Therefore, from equation (5.10) and the equation above the following set of equations result:

$$\left.\begin{aligned}
&\frac{\partial F_{tt}(\sigma)}{\partial S} - K\, F_{tn}(\sigma) + (2\mu+\lambda)T \int_A (R_b \frac{\partial \varepsilon_{tt}}{\partial R_n} - R_n \frac{\partial \varepsilon_{tt}}{\partial R_b}) dA \\
&\qquad -\rho\, A \frac{d^2 U_t^\circ}{dt^2} + P_t(S) = 0 \\
&\frac{\partial F_{tn}(\sigma)}{\partial S} + K\, F_{tt}(\sigma) - T\, F_{tb}(\sigma) + 2\mu T \int_A (R_b \frac{\partial \varepsilon_{tn}}{\partial R_n} - R_n \frac{\partial \varepsilon_{tn}}{\partial R_b}) dA \\
&\qquad -\rho\, [A \frac{d^2 U_n^\circ}{dt^2} - A_{(R_n)} \frac{d^2\theta}{dt^2}] + P_n(S) = 0 \\
&\frac{\partial F_{tb}(\sigma)}{\partial S} + T\, F_{tn}(\sigma) + 2\mu T \int_A (R_b \frac{\partial \varepsilon_{tb}}{\partial R_n} - R_n \frac{\partial \varepsilon_{tb}}{\partial R_b}) dA \\
&\qquad -\rho\, [A \frac{d^2 U_b^\circ}{dt^2} + A_{(R_b)} \frac{d^2\theta}{dt^2}] + P_b(S) = 0
\end{aligned}\right\} \quad \ldots.(5.24)$$

These equations represent the Force Axiom of Motion applicable to all linearly elastic slender members. As mentioned on page (56 equations (5.24) are evaluated with reference to equations (3.14). They are as follows:

$$\begin{aligned}
&\frac{\partial F_{tt}(\sigma)}{\partial S} - K\, F_{tn}(\sigma) + (2\mu+\lambda)T[A_{(R_n)} (\frac{dm_1}{dS} + T\phi_n) - A_{(R_b)} (\theta K + Km_2 - \frac{d\phi_b}{dS} + Tm_1)] \\
&- \rho\, A \frac{d^2 U_t^\circ}{dt^2} + P_t(S) = 0 \\
&\frac{\partial F_{tn}(\sigma)}{\partial S} + K\, F_{tt}(\sigma) - T\, F_{tb}(\sigma) + \mu T\, A_{(R_b)} (\frac{d\theta}{dS} + K\phi_b + \frac{dm_2}{dS}) \\
&-\rho\, [A \frac{d^2 U_n^\circ}{dt^2} - A_{(R_n)} \frac{d^2\theta}{dt^2}] + P_n(S) = 0 \\
&\frac{\partial F_{tb}(\sigma)}{\partial S} + T\, F_{tn}(\sigma) + \mu T\, A_{(R_n)} (\frac{d\theta}{dS} + K\phi_b + \frac{dm_2}{dS}) \\
&-\rho [A \frac{d^2 U_b^\circ}{dt^2} + A_{(R_b)} \frac{d^2\theta}{dt^2}] + P_b(S) = 0
\end{aligned} \quad \ldots.(5.25)$$

§(5.7) Moment Axiom of Motion

The term $\int_A \bar{R} \times \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}} (1-KR_n)dA$ has been investigated in detail in connection with the static case §(5.2). This yields equation (5.14). Therefore, equation (5.18) in conjunction with (5.22) can be written in the direct form as follows:

$$\frac{\partial \bar{M}_t(\sigma)}{\partial S} + \bar{E}_t \times \bar{F}_t(\sigma) + \int_A [(\bar{R}\times\bar{T})\cdot\frac{\partial}{\partial \bar{R}} (\bar{R}\times\bar{\sigma}_t)]dA$$

$$+\rho \; [\frac{d^2\bar{U}^{\circ}}{dt^2} \times \int_A \bar{R} \, dA - I_{tt} \frac{d^2\bar{\theta}}{dt^2}] + \bar{M}(S) = 0$$

....(5.26)

Referring equation (5.26) to the directed base $\{\bar{E}_i\}$ only the term

$$\rho \; [\frac{d^2\bar{U}^{\circ}}{dt^2} \times \int_A \bar{R} \, dA - I_{tt} \frac{d^2\bar{\theta}}{dt^2}]$$

is required to be referred to the Euler directed base $\{\bar{E}_i\}$ $(i = t,n,b)$. Assuming the same form for displacement vector $\bar{U}^{\circ}$ as in §(5.6) and employing equation (5.20) for approximation yields

$$\rho[\frac{d^2\bar{U}^{\circ}}{dt^2} \times \int_A \bar{R} \, dA - I_{tt} \frac{d^2\bar{\theta}}{dt^2}]$$

$$= [(\frac{d^2U_t^{\circ}}{dt^2} \bar{E}_t + \frac{d^2U_n^{\circ}}{dt^2} \bar{E}_n + \frac{d^2U_b^{\circ}}{dt^2} \bar{E}_b) \times \{\bar{E}_n \, A_{(R_b)} + \bar{E}_b \, A_{(R_n)}\} - I_{tt} \frac{d^2\theta}{dt^2} \bar{E}_t]$$

$$= \rho [\frac{d^2 U_t^{\circ}}{dt^2} A_{(R_b)} \bar{E}_b - \frac{d^2 U_b^{\circ}}{dt^2} A_{(R_b)} \bar{E}_t - \frac{d^2 U_t^{\circ}}{dt^2} A_{(R_n)} \bar{E}_n + \frac{d^2 U_n^{\circ}}{dt^2} A_{(R_n)} \bar{E}_t - I_{tt} \frac{d^2\theta}{dt^2} \bar{E}_t]$$

$$= \rho [\{\frac{d^2 U_n^{\circ}}{dt^2} A_{(R_n)} - \frac{d^2 U_b^{\circ}}{dt^2} A_{(R_b)} - I_{tt} \frac{d^2\theta}{dt^2}\}\bar{E}_t - \{\frac{d^2 U_t^{\circ}}{dt^2} A_{(R_n)}\}\bar{E}_n + \{\frac{d^2 U_t^{\circ}}{dt^2} A_{(R_b)}\}\bar{E}_b]$$

Since the directed base vectors $\bar{E}_i$ represent an independent set of vectors, then all the components of Vectorial equation (5.26) have to vanish. Therefore, from equations (5.15) and the equation above

$$\left.\begin{aligned}
&\frac{\partial M_{tt}(\sigma)}{\partial S} - K M_{tn}(\sigma) + 2\mu T \int_A [R_b(\varepsilon_{tb} + R_n \frac{\partial \varepsilon_{tb}}{\partial R_n} - R_b \frac{\partial \varepsilon_{tn}}{\partial R_n}) \\
&- R_n(R_n \frac{\partial \varepsilon_{tb}}{\partial R_b} - \varepsilon_{tn} - R_b \frac{\partial \varepsilon_{tn}}{\partial R_b})]dA + \rho[\frac{d^2 U_n^{\circ}}{dt^2} A_{(R_n)} \\
&- \frac{d^2 U_b^{\circ}}{dt^2} A_{(R_b)} - I_{tt} \frac{d^2\theta}{dt^2}] + M_t(S) = 0 \\
&\frac{\partial M_{tn}(\sigma)}{\partial S} + K M_{tt}(\sigma) - T M_{tb}(\sigma) - F_{tb}(\sigma) + (2\mu+\lambda)T \int_A [R_b^2 (\frac{\partial \varepsilon_{tt}}{\partial R_n}) \\
&- R_n (\varepsilon_{tt} + R_b \frac{\partial \varepsilon_{tt}}{\partial R_b})]dA - \rho A_{(R_n)} \frac{d^2 U_t^{\circ}}{dt^2} + M_n(S) = 0 \\
&\frac{\partial M_{tb}(\sigma)}{\partial S} + T M_{tn}(\sigma) + F_{tn}(\sigma) - (2\mu+\lambda) T \int_A [R_b(\varepsilon_{tt} + R_n \frac{\partial \varepsilon_{tt}}{\partial R_n} \\
&- R_n^2 (\frac{\partial \varepsilon_{tt}}{\partial R_b})]dA + \rho A_{(R_b)} \frac{d^2 U_t^{\circ}}{dt^2} + M_b(S) = 0
\end{aligned}\right\} \quad \text{....(5.27)}$$

These equations represent the Moment Axiom of motion applicable to all linearly elastic slender members. As mentioned on page (56) equations (5.27) are evaluated with reference to equations (3.14). They are as follows:

$$\frac{\partial M_{tt}(\sigma)}{\partial S} - K\,M_{tn}(\sigma) + \rho[A_{(R_n)}\frac{d^2U_n^{\circ}}{dt^2} - A_{(R_b)}\frac{d^2U_b^{\circ}}{dt^2} - I_{tt}\frac{d^2\theta}{dt^2}] + M_t(S) = 0$$

$$\frac{\partial M_{tn}(\sigma)}{\partial S} + K\,M_{tt}(\sigma) - TM_{tb}(\sigma) - F_{tb}(\sigma) + (2\mu+\lambda)T[(\frac{dm_1}{dS} + T\phi_b)(I_{nn}-I_{bb}) + 2\,I_{bn}(\theta K + Km_2 - \frac{d\phi_b}{dS} + Tm_1)] - \rho A_{(R_n)}\frac{d^2U_t^{\circ}}{dt^2} + M_n(S) = 0$$

$$\frac{\partial M_{tb}(\sigma)}{\partial S} + TM_{tn}(\sigma) + F_{tn}(\sigma) + (2\mu+\lambda)[2I_{nb}(\frac{dm_1}{dS} + T\phi_b) - (I_{nn}-I_{bb})(\theta K+Km_2-\frac{d\phi_b}{dS} + Tm_1) + \rho A_{(R_b)}\frac{d^2U_t^{\circ}}{dt^2} + M_b(S) = 0$$

....(5.28)

CHAPTER 6

GENERALISED BERNOULLI-EULER EQUATION AND SPECIAL CASES

§ (6.1) Bernoulli Euler Equation

The stress couple acting on the characteristic cross-section of area A of the slender member has been established on page (59), equation (5.13). Re-writing the equation

$$\bar{M}_t(\sigma) = \int_A \bar{R}\times\bar{\sigma}_t \, dA \qquad \ldots\ldots (5.13)$$

Through the stress-strain relations as given in equation (5.9) the above equation can be converted into generalised Bernoulli-Euler equation of bending. The stress vector $\bar{\sigma}_t$ from equation (5.9) is

$$\bar{\sigma}_t = (2\mu+\lambda)\ \varepsilon_{tt}\ \bar{E}_t + 2\mu\ \varepsilon_{tn}\ \bar{E}_n + 2\mu\ \varepsilon_{tb}\ \bar{E}_b$$

where λ, μ are defined on page (53).

Substituting the values of strain tensor components ε_{tt}, ε_{tn}, ε_{tb} from equation (3.14) yields

$$\bar{\sigma}_t = (2\mu+\lambda)[\mathcal{R}_n(\frac{\partial m_1}{\partial S}+T\phi_b) + \mathcal{R}_b(m_2K + \theta K- \frac{\partial\phi_b}{\partial S} + Tm_1)]\bar{E}_t$$
$$+ (2\mu)[\frac{-\mathcal{R}_b}{2}(K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})]\bar{E}_n + 2\mu[\frac{\mathcal{R}_n}{2}(K\phi_b+\frac{dm_2}{dS} + \frac{d\theta}{dS})]\bar{E}_b$$

Now, in order to integrate equation (5.13) over the cross-sectional area A, let $\bar{\sigma}_t = \delta\bar{\pi}\times\bar{R}$. Hence variational vector $\delta\bar{\pi}$ appears as

$$\delta\bar{\pi} = \mu(K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})\bar{E}_t + (2\mu+\lambda)(m_2K + \theta K - \frac{d\phi_b}{dS} + Tm_1)\bar{E}_n - (2\mu+\lambda)\ (\frac{dm_1}{dS} + T\phi_b)\bar{E}_b$$

and represents a function of S only.

Therefore, (5.13) can be written as

$$\bar{M}_t(\sigma) = \int_A \bar{R} \times (\delta\bar{\pi} \times \bar{R})\ dA$$

$$= \int_A [(\bar{R}\cdot\bar{R})\delta\bar{\pi} - \bar{R}(\bar{R}\cdot\delta\bar{\pi})]dA$$

$$= \int_A [(\bar{R}\cdot\bar{R})\bar{\bar{1}}\cdot\delta\bar{\pi} - \bar{R}\bar{R}\cdot\delta\bar{\pi}]dA$$

$$= \int_A [(\bar{R}\cdot\bar{R})\bar{\bar{1}} - \bar{R}\bar{R}]dA\cdot\delta\bar{\pi}$$

$$= \bar{\bar{I}}\cdot\delta\bar{\pi}$$

where

$$\bar{\bar{I}} = \int_A (\bar{R}\cdot\bar{R})\bar{\bar{1}} - \bar{R}\bar{R}]\,dA$$

is a tensor entirely a function of the cross-sectional properties of the slender member. Tensor $\bar{\bar{I}}$ is called the *Second Area Moment Tensor.*

and $$\bar{\bar{I}}\cdot\delta\bar{\pi} = \delta\bar{\pi}$$

Identity tensor $\bar{\bar{I}} = \bar{E}_i\bar{E}_i$ for $i = t,n,b$.

Hence the Bernoulli-Euler equation in the generalised form appears as,

$$\bar{E}_t.\bar{\bar{M}}(\sigma) = \bar{M}_t(\sigma) = \bar{\bar{I}}\cdot\delta\bar{\pi} \qquad \ldots.(6.1)$$

The second area moment tensor can be referred to the mobile Euler directed base $\{E_i\}$

$$\bar{\bar{I}} = \begin{bmatrix} I_{tt}\bar{E}_t\bar{E}_t & + & 0 & + & 0 \\ + \quad 0 & + & I_{nn}\bar{E}_n\bar{E}_n & + & I_{nb}\bar{E}_n\bar{E}_b \\ + \quad 0 & + & I_{bn}\bar{E}_b\bar{E}_n & + & I_{bb}\bar{E}_b\bar{E}_b \end{bmatrix}$$

where

$$I_{tt} = \int_A (R_n^2 + R_b^2)\,dA$$

$$I_{nn} = \int_A R_b^2\,dA, \quad I_{bb} = \int_A R_n^2\,dA$$

and

$$I_{nb} = I_{bn} = \int_A -R_nR_b\,dA$$

I_{tt}, I_{nn}, I_{bb} are called the Second Area Moments about $\bar{E}_t,\bar{E}_n,\bar{E}_b$ axes respectively. I_{nb} is called the Second Area Product about $\bar{E}_n$ and $\bar{E}_b$ axes. It should be noted that I_{tt} is commonly known as the Polar Second Area Moment and often denoted by I_p.

The tensorial form of the Bernoulli-Euler Equation (6.1) can be expressed in the semidirect form:

$$\bar{M}_t(\sigma) = \bar{\bar{I}}\cdot\delta\bar{\pi}$$

$$= \begin{bmatrix} I_{tt}\bar{E}_t\bar{E}_t & + \quad 0 & + \quad 0 \\ + \quad 0 & + \; I_{nn}\bar{E}_n\bar{E}_b & + I_{nb}E_{nb} \\ + \quad 0 & + \; I_{bn}\bar{E}_b\bar{E}_n & + \; I_{bb}\bar{E}_b\bar{E}_b \end{bmatrix} \cdot \begin{bmatrix} [\mu(K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})\bar{E}_t \; + \\ (2\mu+\lambda)(m_2K + \theta K - \frac{d\phi_b}{dS} + Tm_1)\bar{E}_n \\ -(2\mu+\lambda)\;(\frac{dm_1}{dS} + T\phi_b)\bar{E}_b] \end{bmatrix}$$

By carrying out the dot product and noting that $\bar{E}_i\cdot\bar{E}_j = \delta_{ij}$, where δ_{ij} denotes the Kronecker Delta yields

$$\bar{M}_t(\sigma) = \mu\; I_{tt}\;(K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})\bar{E}_t$$
$$+ \;(2\mu+\lambda)[I_{nn}(m_2K+\theta K - \frac{d\phi_b}{dS} + Tm_1) - I_{nb}\;(\frac{dm_1}{dS} + T\phi_b)]\bar{E}_n$$
$$+ \;(2\mu+\lambda)\;[I_{bn}\;(m_2K+\theta K - \frac{d\phi_b}{dS} + Tm_1) - I_{bb}\;(\frac{dm_1}{dS} + T\phi_b)]\bar{E}_b$$

Therefore, components of the stress couple are as follows:

$$\left.\begin{aligned} M_{tt}(\sigma) &= \mu\; I_{tt}\;(K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS}) \\ M_{tn}(\sigma) &= (2\mu+\lambda)[I_{nn}(m_2K+\theta K - \frac{d\phi_b}{dS} + Tm_1) - I_{nb}\;(\frac{dm_1}{dS} + T\phi_b)] \\ M_{tb}(\sigma) &= (2\mu+\lambda)[I_{bn}(m_2K+\theta K - \frac{d\phi_b}{dS} + Tm_1) - I_{bb}\;(\frac{dm_1}{dS} + T\phi_b)] \end{aligned}\right\} \quad \ldots.(6.2)$$

§(6.2) Special Cases

Case (a): Equations (6.2) are the generalised Euler-Bernoulli force displacement relationships for the elastic, slender members of arbitrary spatial elastica (ie. the centre-line) and cross-sectional configuration. If the cross-section of the slender member is assumed circular, then, the warping is zero. Also,

$$I_{nb} = I_{bn} = 0$$

$$I_{nn} = I_{bb} = I$$

and

$$A_{(R_n)} = A_{(R_b)} = 0$$

Substituting for the kinematic parameters (m_1, m_2, ϕ_i) from page (17) in equations (6.2) results in

$$M_{tt}(\sigma) = \mu\, I_{tt}\, [K\, (\frac{dU_b^\circ}{dS} + T\, U_n^\circ) + \frac{d\theta}{dS} + \frac{dm_2}{dS}]$$

or

$$M_{tt}(\sigma) = \mu\, I_{tt}\, [\frac{d\theta}{dS} + K\, \frac{dU_b^\circ}{dS} + KTU_n^\circ + \frac{dm_2}{dS}] \qquad ....(6.3.a)$$

where,

$$\frac{dm_2}{dS} = \frac{d}{dS}\, [\frac{(\frac{d\phi_b}{dS} + T\phi_n)}{K + \delta K}]$$

$$= \frac{d}{dS}\, [\frac{1}{K}\, \{1 - \frac{\delta K}{K} + (\frac{\delta K}{K})^2 - \dots\}\, \{\frac{d\phi_b}{dS} + T\phi_n\}]$$

Similarly,

$$M_{tn}(\sigma) = (2\mu+\lambda)\ I_{nn}[-\frac{d}{dS}(\frac{d\overset{\circ}{U}_b}{dS} + T\overset{\circ}{U}_n) + \theta K + (m_2K + Tm_1)]$$

or

$$M_{tn}(\sigma) = (2\mu+\lambda)\ I\ \ [\theta K - \frac{d^2\overset{\circ}{U}_b}{dS^2} - \frac{dT}{dS}\overset{\circ}{U}_n - T\frac{d\overset{\circ}{U}_n}{dS} + (Km_2 + Tm_1)] \quad ....(6.3.b)$$

where

$$(Km_2 + Tm_1) = \frac{K}{\kappa}[\frac{d^2\overset{\circ}{U}_b}{dS^2} + \frac{dT}{dS}\overset{\circ}{U}_n + T\frac{d\overset{\circ}{U}_n}{dS}]$$

and

$$M_{tb}(\sigma) = -\ (2\mu+\lambda)\ I_{bb}\ [\frac{d}{dS}(-\frac{K}{\kappa}\phi_n) + T\phi_b]$$

$$= (2\mu+\lambda)\ I\ [\frac{d}{dS}(\frac{K}{\kappa})\phi_n + (\frac{K}{\kappa})\frac{d\phi_n}{dS} - T\phi_b]$$

$$= (2\mu+\lambda)\ I\ [\frac{d}{dS}(\frac{K}{\kappa})\phi_n + (\frac{K}{\kappa})\frac{d}{dS}(\frac{d\overset{\circ}{U}_n}{dS} + K\overset{\circ}{U}_t - T\overset{\circ}{U}_b)$$

$$- T(\frac{d\overset{\circ}{U}_b}{dS} + T\overset{\circ}{U}_n)]$$

or,

$$M_{tb}(\sigma) = (2\mu+\lambda)\ I[\frac{d}{dS}(\frac{K}{\kappa})\phi_n + (\frac{K}{\kappa})\{\frac{d^2\overset{\circ}{U}_n}{dS^2} + \frac{dK}{dS}\overset{\circ}{U}_t + K\frac{d\overset{\circ}{U}_t}{dS}$$

$$- \frac{dT}{dS}\overset{\circ}{U}_b - 2T\frac{d\overset{\circ}{U}_b}{dS} - T^2\overset{\circ}{U}_n\}] \quad ....(6.3.c)$$

Now, if it is assumed that $\frac{K}{\kappa}$ is approximately unity, which is very nearly true in the case of small-displacement field,

$$\frac{K}{\kappa} = \frac{K}{K+\delta K} = 1 - \frac{\delta K}{K} + (\frac{\delta K}{K})^2 - \ ...\ \doteq 1 \text{ for } \delta K<<K$$

equations 6.3(a),(b),(c) reduce to the following form:

$$M_{tt}(\sigma) = G\ I_{tt}\ [\frac{d\theta}{dS} + K\frac{d\overset{\circ}{U}_b}{dS} + KT\overset{\circ}{U}_n + \frac{dm_2}{dS}] \quad ....(6.4.a)$$

where,

$$\frac{dm_2}{dS} = \frac{1}{K}[\frac{d^3\overset{\circ}{U}_b}{dS^3} + \frac{d^2T}{dS^2}\overset{\circ}{U}_n + 3\frac{dT}{dS}\frac{d\overset{\circ}{U}_n}{dS} + 2T\frac{d^2\overset{\circ}{U}_n}{dS^2} - 2T\frac{dT}{dS}\overset{\circ}{U}_b$$

$$+ T \frac{dK}{dS} U_t^\circ - T^2 \frac{dU_b^\circ}{dS} + K \frac{dT}{dS} U_t^\circ + KT \frac{dU_t^\circ}{dS}]$$

$$- \frac{1}{K^2} \frac{dK}{dS} [\frac{d^2U_b^\circ}{dS^2} + \frac{dT}{dS} U_n^\circ + 2T \frac{dU_n^\circ}{dS} + TKU_t^\circ - T^2U_b^\circ]$$

$$M_{tn}(\sigma) = EI[\theta K] \qquad \ldots.(6.4.b)$$

$$M_{tb}(\sigma) = EI[\frac{d^2U_n^\circ}{dS^2} + \frac{dK}{dS} U_t^\circ + K \frac{dU_t^\circ}{dS} - \frac{dT}{dS} U_b^\circ - 2T \frac{dU_b^\circ}{dS} - T^2U_n^\circ] \ldots.(6.4.c)$$

It is consistent with the Bernoulli Hypothesis to use the approximation

$$(2\mu+\lambda) = 2\mu + \frac{2\nu}{1-2\nu}\mu = \frac{2(1-\nu)}{(1-2\nu)}\mu = \frac{(1-\nu)}{(1-2\nu)(1+\nu)} E \doteq E$$

where $\mu = G = \frac{E}{2(1+\nu)}$.

Under the assumption of circular cross-section the equations of motion can be further reduced.

Static Case:

From (5.11) the Force Equilibrium equations are:

$$\left.\begin{aligned} &\frac{\partial F_{tt}(\sigma)}{\partial S} - K F_{tn}(\sigma) + P_t(S) = 0 \\ &\frac{\partial F_{tn}(\sigma)}{\partial S} + K F_{tt}(\sigma) - T F_{tb}(\sigma) + P_n(S) = 0 \\ &\frac{\partial F_{tb}(\sigma)}{\partial S} + T F_{tn}(\sigma) + P_b(S) = 0 \end{aligned}\right\} \ldots.(6.5)$$

From (5.16) the Moment Equilibrium equations are:

$$\left.\begin{aligned} &\frac{\partial M_{tt}(\sigma)}{\partial S} - KM_{tn}(\sigma) + M_t(S) = 0 \\ &\frac{\partial M_{tn}(\sigma)}{\partial S} + KM_{tt}(\sigma) - TM_{tb}(\sigma) - F_{tb}(\sigma) + M_n(S) = 0 \\ &\frac{\partial M_{tb}(\sigma)}{\partial S} + TM_{tn}(\sigma) + F_{tn}(\sigma) + M_b(S) = 0 \end{aligned}\right\} \ldots.(6.6)$$

Dynamic Case:

From equation (5.25) the Force Equilibrium Equations of Motion are:

$$\left.\begin{aligned}
&\frac{\partial F_{tt}(\sigma)}{\partial S} - K\, F_{tn}(\sigma) - \rho A \frac{d^2 U_t^\circ}{dt^2} + P_t(S) = 0 \\
&\frac{\partial F_{tn}(\sigma)}{\partial S} + K\, F_{tt}(\sigma) - T\, F_{tb}(\sigma) - \rho A \frac{d^2 U_n^\circ}{dt^2} + P_n(S) = 0 \\
&\frac{\partial F_{tb}(\sigma)}{\partial S} + T\, F_{tn}(\sigma) - \rho A \frac{d^2 U_b^\circ}{dt^2} + P_b(S) = 0
\end{aligned}\right\} \quad \ldots.(6.7)$$

From (5.28) the Moment Equations of Motion are:

$$\left.\begin{aligned}
&\frac{\partial M_{tt}(\sigma)}{\partial S} - K\, M_{tn}(\sigma) - \rho I_{tt} \frac{d^2 \theta}{dt^2} + M_t(S) = 0 \\
&\frac{\partial M_{tn}(\sigma)}{\partial S} + K\, M_{tt}(\sigma) - T\, M_{tb}(\sigma) - F_{tb}(\sigma) + M_n(S) = 0 \\
&\frac{\partial M_{tb}(\sigma)}{\partial S} + T\, M_{tn}(\sigma) + F_{tn}(\sigma) + M_b(S) = 0
\end{aligned}\right\} \quad \ldots.(6.8)$$

Case (b): In this section equations of motion for an elastica in the form of a circular helix are deduced by further neglecting a few terms from equations obtained in case (a) above.

Dynamic-Kinematic Relations: Considering cases where $\frac{dm_2}{dS}$ and m_2K are small quantities compared to the terms retained in equations (6.3.a) and (6.3.b) respectively and making the same approximation as in case (a) above, i.e. $\frac{K}{\kappa} \doteq 1$, the dynamic-kinematic relations for a linearly elastic member appear as follows:

$$M_{tt}(\sigma) = GI_{tt}\left[\frac{d\theta}{dS} + K\frac{d\overset{\circ}{U}_b}{dS} + KT\overset{\circ}{U}_n\right] \quad ....(6.9)$$

$$M_{tn}(\sigma) = EI\left[\theta K - \frac{d^2\overset{\circ}{U}_b}{dS^2} - \frac{dT}{dS}\overset{\circ}{U}_n - 2T\frac{d\overset{\circ}{U}_n}{dS} - TK\overset{\circ}{U}_t + T^2\overset{\circ}{U}_b\right] ....(6.10)$$

$$M_{tb}(\sigma) = EI\left[\frac{d^2\overset{\circ}{U}_n}{dS^2} + \frac{dK}{dS}\overset{\circ}{U}_t + K\frac{d\overset{\circ}{U}_t}{dS} - \frac{dT}{dS}\overset{\circ}{U}_b - 2T\frac{d\overset{\circ}{U}_b}{dS} - T^2\overset{\circ}{U}_n\right] \quad ....(6.11)$$

The longitudinal strain equation (2.7) is no longer zero but serves to establish the longitudinal stress resultant $\sigma_{tt}A = E\,\overset{\circ}{\varepsilon}_{tt}A = E\phi_t A$

or

$$F_{tt}(\sigma) = EA\left(\frac{d\overset{\circ}{U}_t}{dS} - K\overset{\circ}{U}_n\right) \quad ....(6.12)$$

designated the presence of longitudinal motion.

Equation of Motion of a Circular Helix: For the case of circular helix the curvature K and the torsion T of the elastica are constant with respect to the arc-length parameter S. Therefore,

$$\frac{dK}{dS} = 0 \quad ; \quad \frac{dT}{dS} = 0 \quad ....(6.13)$$

From the Moment Equations of Motion as given in equations (6.8), the two transverse shear stress-resultants

$$F_{tb}(\sigma) = \frac{\partial M_{tn}(\sigma)}{\partial S} + KM_{tt}(\sigma) - TM_{tb}(\sigma) + M_n(S) \quad \ldots.(6.14)$$

and

$$F_{tn}(\sigma) = -\frac{\partial M_{tb}(\sigma)}{\partial S} - TM_{tn}(\sigma) - M_b(S) \quad \ldots.(6.15)$$

can be established.

Substituting (6.14) and (6.15) into the Force Equations of Motion (6.7) the following equations results:

$$\rho A \frac{d^2 U_t^{\circ}}{dt^2} = \frac{\partial F_{tt}(\sigma)}{\partial S} + K \frac{\partial M_{tb}(\sigma)}{\partial S} + KTM_{tn}(\sigma) + KM_b(S) + P_t(S) \quad \ldots.(6.16)$$

$$\rho A \frac{d^2 U_n^{\circ}}{dt^2} = KF_{tt}(\sigma) - TKM_{tt}(\sigma) - 2T \frac{\partial M_{tn}(\sigma)}{\partial S} - \frac{dT}{dS} M_{tn}(\sigma) + T^2 M_{tb}(\sigma)$$

$$- \frac{\partial^2 M_{tb}(\sigma)}{\partial S^2} - TM_n(S) - \frac{\partial M_n(S)}{\partial S} + P_n(S) \quad \ldots.(6.17)$$

$$\rho A \frac{d^2 U_b^{\circ}}{dt^2} = K \frac{\partial M_{tt}(\sigma)}{\partial S} + \frac{\partial K}{\partial S} M_{tt}(\sigma) + \frac{\partial^2 M_{tn}(\sigma)}{\partial S^2} - T^2 M_{tn}(\sigma) - 2T \frac{\partial M_{tb}(\sigma)}{\partial S}$$

$$- \frac{\partial T}{\partial S} M_{tb}(\sigma) + \frac{\partial M_n(S)}{\partial S} - TM_b(S) + P_b(S) \quad \ldots.(6.18)$$

Substituting equations (6.9) to (6.12) into equations (6.16), (6.17), (6.18) and the first equation of equations (6.8), four equations of motion in terms of the four displacement variables are obtained. The equations of motion for the free vibrations of a helical rod are obtained from these equations by setting the applied loading functions $P_t = P_n = P_b = 0$. The final equations are as follows:

$$\rho A \frac{d^2U_t^\circ}{dt^2} = E[(A + IK^2)\frac{d^2U_t^\circ}{dS^2} - IK^2T^2U_t^\circ] + EK[I\frac{d^3U_n^\circ}{dS^3} - (A + 3IT^2)\frac{dU_n^\circ}{dS}] + EIKT[T^2U_b^\circ - 3\frac{d^2U_b^\circ}{dS^2}] + EIK^2T(\theta) \quad \ldots.(6.19)$$

$$\rho A \frac{d^2U_n^\circ}{dt^2} = EK[(A + 3IT^2)\frac{dU_t^\circ}{dS} - I\frac{d^3U_t^\circ}{dS^3}] + E[- I\frac{d^4U_n^\circ}{dS^4} + 6IT^2\frac{d^2U_n^\circ}{dS^2} - (AK^2+K^2T^2\frac{G}{E}I_{tt}+IT^4)U_n^\circ] + ET[4I\frac{d^3U_b^\circ}{dS^3} - (\frac{G}{E}U_{tt}K^2 + 4IT^2)\frac{dU_b^\circ}{dS}] + EKT[-(2I + \frac{G}{E}I_{tt})\frac{d\theta}{dS}] \quad \ldots.(6.20)$$

$$\rho A \frac{d^2U_b^\circ}{dt^2} = 3IKT[T^2U_t^\circ - 3\frac{d^2U_b^\circ}{dS^2}] + ET[(4IT^2 + \frac{G}{E}I_{tt}K^2)\frac{dU_n^\circ}{dS} - 4I\frac{d^3U_n^\circ}{dS^3}] + E[- I\frac{d^4U_b^\circ}{dS^4} + (\frac{G}{E}I_{tt}K^2 + 6IT^2)\frac{d^2U_b^\circ}{dS^2} - IT^4U_b^\circ] + EK[(\frac{G}{E}I_{tt} + I)\frac{d^2\theta}{dS^2} - IT^2(\theta)] \quad \ldots.(6.21)$$

$$\rho I_{tt} \frac{d^2\theta}{dt^2} = EIK^2T\ U_t^\circ + EKT[2I + \frac{G}{E} I_{tt} \frac{dU_n^\circ}{dS}]$$

$$+ EK[(\frac{G}{E} I_{tt} + I) \frac{d^2U_b^\circ}{dS^2} - IT^2U_b^\circ]$$

$$+ E[\frac{G}{E} I_{tt} \frac{d^2\theta}{dS^2} - IK^2(\theta)] \qquad \ldots. (6.22)$$

Equations (6.19) to (6.22) are the same as those obtained by Tso[5]. Hence these equations can be solved by the method given in his manuscript.

CHAPTER 7

CONCLUSIONS

A fundamental approach for the analysis of thin, elastic members, which are curved and twisted in their natural configurations, has been presented by the direct kinematic method. Although, the analysis is restricted to the small-displacement theory, some of the results are for general applications.

The Force and Moment Equilibrium equations obtained by the author, viz. (6.5), (6.6) are in agreement with that given by Love[2] . The author has not taken cognizance of the orientation of the cross-sections of the slender member with respect to the elastica. It is tacitly assumed that the principal axes of the cross-section coincide with the normal and binormal axes of the mobile Euler base otherwise, the pristine orientation of the cross-section has to be prescribed as a function of the undeformed arc-length parameter S. For this reason the curvature vector has not been referred to the directions of the principal axes of the cross-section and, consequently, the component κ of Love[2] vanishes when comparing the two works.

There is a point of disagreement with the analysis of Love[2] over the Bernoulli-Euler force-displacement relationship. The present author obtained equations (6.4) which differ from those given by Love[2]. It is important to point out that the discrepancy in these equations has resulted from the manner of making approximations. The force-displacement relationships as obtained by Love[2] could have been obtained in this work if the author had neglected some of the quantities as small, but this would have made the analysis inconsistent by the present method of investigation.

The direct form of the force and moment equilibrium equations as given by DiPrima and Handelman[3] and later on by Massoud[4] are contained in the author's equations (5.8) and (5.14). The additional terms in these equations are the by-products of the tridimensional approach of analysis. It should be remembered that other degenerate cases of slender members, viz.circular helices, planar curved beams and linear members represent special cases of the present analysis.

Tso[5] in an independent investigation obtained the equations of motion similar to that obtained by the author in his equations (6.7) and (6.8).

The author recommends an extensive investigation of the case when the centre-line of the slender member no longer remains invariant in length in the process of deformation. This case is of interest when the slender members undergo large displacements. It would also be interesting to attempt to determine more precisely the limits of the slenderness of the members for the applications of above results.

REFERENCES

1. Truesdell, C., "Essays in the History of Mechanics", Springer-Verlag, New York, 1968.

2. Love, A.E.H., "A Treatise on the Mathematical Theory of Elasticity", Dover Publications, New York, 1944, Chapter (19-21).

3. DiPrima, R.C. and Handelman, G.H., "Vibration of Twisted Beams", Quarterly of Applied Mathematics Vol. 12, October 1954.

4. Massoud, M.P., 'Vectoral Derivation of the Equations for Small Vibrations of Twisted Curved Beams' Journal of Applied Mechanics, June 1965.

5. Tso, W.K., 'On the Motion of a Curved and Twisted Rod' Unpublished manuscript.

6. Struik, D.J. "Lectures on Classical Differential Geometry" 2nd Edition, Addison-Wesley, Reading, Mass. 1961.

7. McLean, L.C., "On the Direct Kinematic Theory of Thin Elastic Shells", M.Eng. Thesis, McMaster University, Sept. 1966.

8. Oravas, G. Æ, "Lecture Notes" Delivered at the McMaster University, Hamilton, Ontario, 1968-69.

APPENDIX-A

KINEMATIC SERRET-FRENET FORMULAS FOR SPACE CURVES

With reference to fig. (A-1),a space curve is completely defined by the position vector $\bar{R}(S)$, where $\bar{R}(S)$ is a function of the arc length parameter 'S' of the curve.

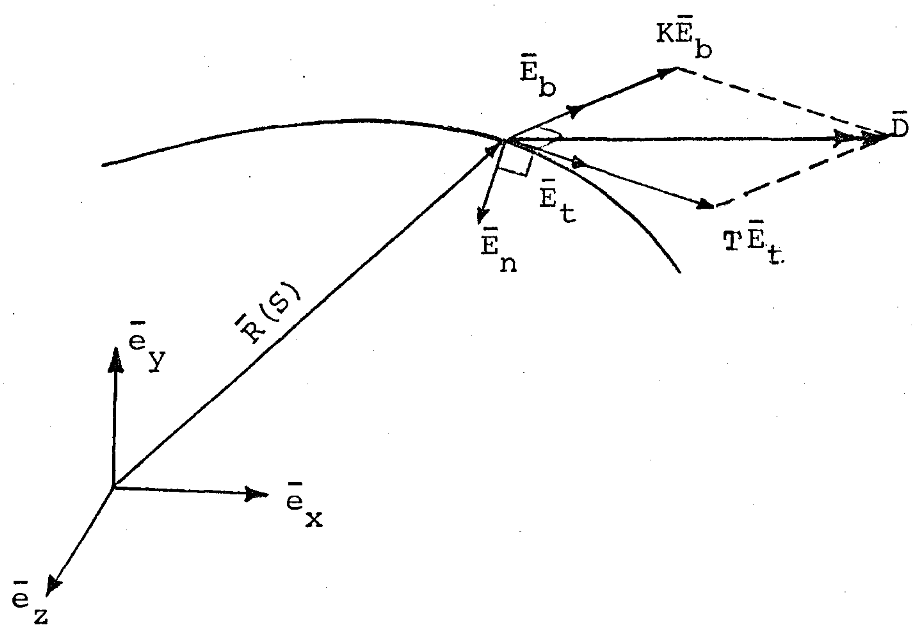


Figure A-1

The Frenet-Serret formulas of differential geometry applied to the reference state's centroidal curve are[6]

$$\frac{d\bar{E}_t}{dS} = \bar{D} \times \bar{E}_t = K\,\bar{E}_n \qquad \text{....(A.1)}$$

$$\frac{d\bar{E}_n}{dS} = \bar{D} \times \bar{E}_n = - K \bar{E}_t + T \bar{E}_n \qquad \text{....(A.2)}$$

$$\frac{d\bar{E}_b}{dS} = \bar{D} \times \bar{E}_b = - T \bar{E}_n \qquad \text{....(4.3)}$$

where the Darboux-Vector $\bar{D} = T\bar{E}_t + K\bar{E}_b$

$\bar{E}_t$ – unit tangent vector

$\bar{E}_n$ – unit normal vector

$\bar{E}_b$ – unit binormal vector

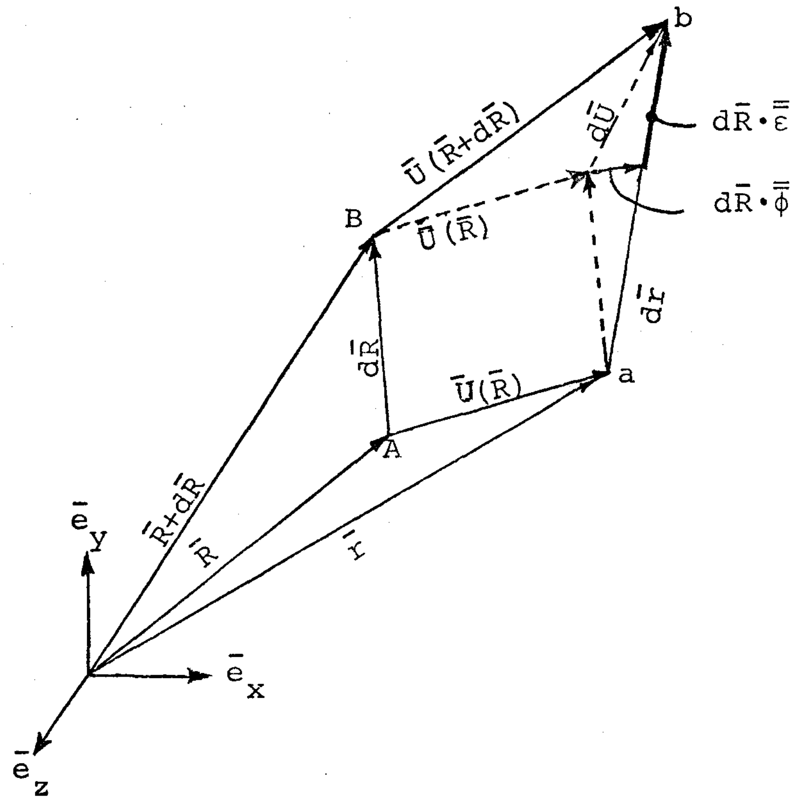
K – curvature of the elastica at the generic point

T – Torsion of the elastica at the generic point.

APPENDIX-B

LINEAR THEORY OF STRAIN

Assuming the deformation of the continuous medium to be homogeneous i.e. infinitesimal vectors $d\bar{R}$, deform into infinitesimal vectors $d\bar{r}$, but not to infinitesimal curves. Figure (B-1) depicts the geometric relations of this type of deformation field.



$(d\bar{R}\cdot\bar{\bar{\varepsilon}})$ and $(d\bar{R}\cdot\bar{\bar{\phi}})$ are components of $d\bar{U}$, parallel and perpendicular to $d\bar{r}$ and $d\bar{R}$ respectively.

Figure B-1

$\overline{AB} = d\bar{R}$ is a linear element in the continuum, $\bar{R}$ being the position vector to point A. After the deformation $\overline{ab} = d\bar{r}$ is the new linear element. Displacement at the point A is $\bar{U}(\bar{R})$, and at an infinitesimal distance away at point B, it is $\bar{U}(\bar{R}+d\bar{R})$. The line element $\overline{AB}$ is subjected to the most general linear displacement field which imparts rotation and stretching to $\overline{AB} = d\bar{R}$.

Expanding $\bar{U}(\bar{R}+d\bar{R})$ by Taylor's Series Expansion

$$\bar{U}(\bar{R}+d\bar{R}) = e^{d\bar{R}\cdot\frac{\partial}{\partial\bar{R}}}\,\bar{U}(\bar{R})$$

$$= [1 + d\bar{R}\cdot\frac{\partial}{\partial\bar{R}} + \frac{1}{2!}\,d\bar{R}d\bar{R}:\frac{\partial}{\partial\bar{R}}\,\frac{\partial}{\partial\bar{R}} + \ldots]\bar{U}(\bar{R})$$

Assuming first order approximations to be sufficiently accurate yields

$$\bar{U}(\bar{R} + d\bar{R}) \doteq \bar{U}(\bar{R}) + d\bar{R}\cdot\frac{\partial\bar{U}}{\partial\bar{R}} = \bar{U}(\bar{R}) + d\bar{U}(\bar{R})$$

or

$$d\bar{U}(\bar{R}) = \bar{U}(\bar{R} + d\bar{R}) - \bar{U}(\bar{R})$$

$$= d\bar{R}\cdot\frac{\partial\bar{U}}{\partial\bar{R}} = d\bar{R}\cdot\bar{\bar{U}}$$

where

$$\bar{\bar{U}} = \frac{\partial\bar{U}}{\partial\bar{R}}$$

is called the displacement gradient.

The relative terminal displacement of $d\bar{R}$ is then given by

$$d\bar{R}\cdot\frac{\partial\bar{U}}{\partial\bar{R}} = d\bar{R}\cdot\left[\frac{\partial\bar{U}}{\partial\bar{R}}^{(s)} + \frac{\partial\bar{U}}{\partial\bar{R}}^{(a)}\right]$$

$$= d\bar{R}\cdot\left[\frac{1}{2}\left(\frac{\partial\bar{U}}{\partial\bar{R}} + \frac{\bar{U}\partial}{\partial\bar{R}}\right) + \frac{1}{2}\left(\frac{\partial\bar{U}}{\partial\bar{R}} - \frac{\bar{U}\partial}{\partial\bar{R}}\right)\right] = d\bar{R}\cdot[\bar{\bar{\varepsilon}}+\bar{\bar{\phi}}]$$

where the first term in the above is called the *Strain Tensor*, $\bar{\bar{\varepsilon}}$; and the second term is called the *Mean Rotation Tensor* $\bar{\bar{\phi}}$, i.e.

$$\bar{\bar{\varepsilon}} = \frac{1}{2}\left(\frac{\partial\bar{U}}{\partial\bar{R}} + \frac{\bar{U}\partial}{\partial\bar{R}}\right) \quad \text{(symmetric part)}$$

and

$$\bar{\bar{\phi}} = \frac{1}{2}\left(\frac{\partial\bar{U}}{\partial\bar{R}} - \frac{\bar{U}\partial}{\partial\bar{R}}\right) \quad \text{(Antisymmetric part)}$$

Hence

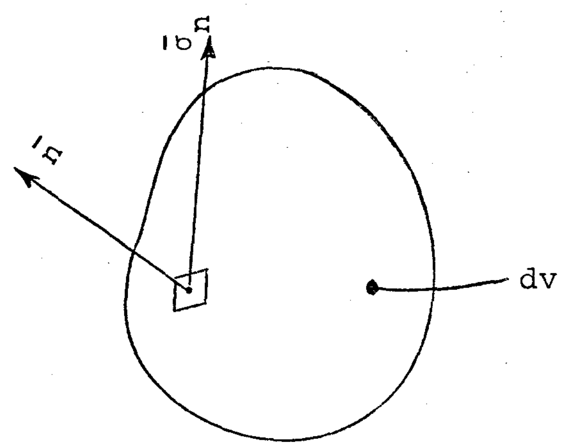
$$d\bar{R}\cdot(\bar{\bar{\varepsilon}}+\bar{\bar{\phi}}) = d\bar{R}\cdot\bar{\bar{\varepsilon}} + d\bar{R}\cdot\bar{\bar{\phi}} \quad \text{(see Figure B-1)}$$

$$= d\bar{R}\cdot\bar{\bar{\varepsilon}} + \frac{1}{2}\,\frac{\partial\times\bar{U}}{\partial\bar{R}} \times d\bar{R}$$

APPENDIX-C

CAUCHY'S FIRST AXIOM OF MOTION

If the surface stress $\bar{\sigma}_n$ is represented in terms of the Cauchy Stress Tensor Principle as $\bar{\sigma}_n = \bar{n}\cdot\bar{\bar{\sigma}}$ then the first axiom of motion (i.e. vanishing of all the forces in the inertial frame of reference) for a finite body of volume 'v' and bounding surface Σ assumes the form:



$$\int_v d\bar{F} = \int_v \left(-\frac{d^2\bar{U}}{dt^2}\right)\rho \, dv + \int_v \bar{f} \, dv + \int_\Sigma \bar{\sigma} \, dA$$

$$= \int_v \left[\bar{f} - \rho \frac{d^2\bar{U}}{dt^2}\right] dv + \int_A dA \, \bar{n}\cdot\bar{\bar{\sigma}}$$

$$= \int_v \left[\bar{f} - \rho \frac{d^2\bar{U}}{dt^2}\right] dv + \int_A d\bar{A}\cdot\bar{\bar{\sigma}} = 0. \quad \ldots.(C.1)$$

where, $d\bar{A} = dA \, \bar{n}$ – directed differential surface element

$\bar{U}$ – displacement of the arbitrary point $\bar{R}$

$\bar{f}$ - body force intensity per unit volume

$-\frac{d^2\bar{U}}{dt^2}\rho$ - apparent force created by motion per unit volume

$\frac{d^2\bar{U}}{dr^2}$ - material acceleration of the generic point $\bar{R}$ in the body

$\bar{\sigma}_n$ - stress applied at the external boundary point $\bar{R}_n$ of the moving body

ρ $= \frac{dm}{dv}$ - mass density.

The surface integral can be converted into corresponding volume integral by means of the Gauss Divergence Theorem as

$$\int_V \frac{\partial}{\partial \bar{R}} (\quad) dv \equiv \int_\Sigma d\bar{A} (\quad)$$

where ( ) denotes any admissible function.

Thus,

$$\int_V \frac{\partial}{\partial \bar{R}} \cdot \bar{\bar{\sigma}} \, dv = \int_\Sigma d\bar{A} \cdot \bar{\bar{\sigma}}$$

Substituting this conversion into eqs. (C.1), yields the field equation

$$\int_V d\bar{F} = \int_V [\bar{f} + \frac{\partial}{\partial \bar{R}} \cdot \bar{\bar{\sigma}} - \rho \frac{d^2\bar{U}}{dt^2}] \, dv = 0 \qquad \text{....(C-2)}$$

The <u>Second Axiom of Motion</u> requires the vanishing of the moments of all the forces, including the apparent forces brought about by the motion, acting on the body with respect to the inertial frame of reference.

Hence, in the case of the finite body the second axiom of motion assumes the form,

$$\int_V d\bar{M} = \int_V \frac{d\bar{M}}{dv}\, dv = \int_V \bar{R} \times \bar{f}\, dv + \int_V \bar{R} \times \left(-\rho \frac{d^2\bar{U}}{dt^2}\right) dv$$

$$+ \int_V \bar{R} \times \left(\frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{R}}\right) dv = 0 \qquad \ldots .(C\text{-}3)$$

where $\bar{R}$ is the position vector in the inertial frame to the generic point in the body.

APPENDIX-D

CONSTITUTIVE EQUATION FOR ISOTROPIC ELASTICITY

For the case of isotropic, homogeneous elastic body, the strain-stress relations are

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu \frac{\sigma_{yy}}{E} - \nu \frac{\sigma_{zz}}{E}$$

where E - Young's modulus for the material

ν - Poisson's ratio

or, in terms of the first stress invariant

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} + \nu \frac{\sigma_{xx}}{E} - \frac{\nu}{E} (\sigma_{xx}+\sigma_{yy}+\sigma_{zz})$$

$$= \frac{\sigma_{xx}}{E} (1+\nu) - \frac{\nu}{E} (\bar{\bar{\sigma}}:\bar{\bar{1}})$$

Similarly,

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E}(1+\nu) - \frac{\nu}{E} (\bar{\bar{\sigma}}:\bar{\bar{1}})$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} (1+\nu) - \frac{\nu}{E} (\bar{\bar{\sigma}}:\bar{\bar{1}})$$

The shear strain-stress relation

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{2\mu} = \frac{\sigma_{xy}}{2G}$$

where μ = G is called the Cauchy-Lamé first elastic coefficient.

$$\varepsilon_{ij} = \frac{\sigma_{ij}}{2\mu} \qquad \text{(D-1)}$$

where $i \neq j$, $i=x,y,z$

$j=x,y,z$

and for the principal components

$$\varepsilon_{ii} = \frac{\sigma_{ii}}{E}(1+\nu) - \frac{\nu}{E}(\bar{\bar{\sigma}}:\bar{\bar{I}}) \qquad \text{(D-2)}$$

and

$$\mu = \frac{E}{2(1+\nu)} \qquad \text{(D-3)}$$

Thus, equations (D-1) and (D-2) can be united through the use of equation (D-3). The result obtained is

$$\bar{\bar{\varepsilon}} = \frac{\bar{\bar{\sigma}}}{2\mu} - \frac{\nu}{E}(\bar{\bar{\sigma}}:\bar{\bar{I}})\bar{\bar{I}} \qquad \text{(D-4)}$$

where

$\bar{\bar{I}}$ – Identity tensor

$(\bar{\bar{\sigma}}:\bar{\bar{I}})$ – First scalar invariant of the stress tensor $\bar{\bar{\sigma}}$

Equation (D-4) can be inverted into stress-strain relation

$$\bar{\bar{\sigma}} = 2\mu\,\bar{\bar{\varepsilon}} + \lambda(\bar{\bar{\varepsilon}}:\bar{\bar{I}})\bar{\bar{I}} \qquad \text{(D-5)}$$

where the Cauchy-Lamé second elastic coefficient

$$\lambda = \frac{2\nu}{1-2\nu}\,\mu$$

APPENDIX E

In this appendix detailed calculations for the components of tensor $\bar{\bar{S}}$(i.e., S_{ij}) are carried out. For simplicity $(1-K\mathcal{R}_n)$ shall be written as $\chi = \chi(S,\mathcal{R}_n)$.

From equation (4.4)

$$S_{tn} = -\frac{\partial P_{tt}}{\partial \mathcal{R}_b} - \frac{K\,P_{nb}}{\chi}$$

Substituting for P_{tt}, P_{nb} from equation (4.3)

$$S_{tn} = -\frac{\partial}{\partial \mathcal{R}_b}\left(\frac{\partial \varepsilon_{bt}}{\partial \mathcal{R}_n} - \frac{\partial \varepsilon_{nt}}{\partial \mathcal{R}_b}\right) - \frac{K}{\chi}\frac{\partial \varepsilon_{tb}}{\partial \mathcal{R}_b}$$

Again, substituting for strains $\varepsilon_{bt}=\varepsilon_{tb}$, ε_{nt} from equation (3.13)

$$S_{tn} = -\frac{1}{2}\frac{\partial^2}{\partial \mathcal{R}_b \partial \mathcal{R}_n}\left[\frac{1}{\chi}\left\{\phi_b + \mathcal{R}_n\left(\frac{dm_2}{ds} + \frac{d\theta}{ds}\right)\right\} - \phi_b\right]$$

$$+\frac{1}{2}\frac{\partial^2}{\partial \mathcal{R}_b^2}\left[\frac{1}{\chi}\left\{\phi_n + \mathcal{R}_n m_1 K - \mathcal{R}_b\left(K\phi_b + \frac{dm_2}{ds} + \frac{d\theta}{ds}\right)\right\} + m_1\right]$$

$$-\frac{K}{\chi}\frac{\partial}{\partial \mathcal{R}_b}\left[\frac{1}{2\chi}\left\{\phi_b + \mathcal{R}_n\left(\frac{dm_2}{ds} + \frac{d\theta}{ds}\right)\right\} - \phi_b\right]$$

$$= -\frac{1}{2}\frac{\partial}{\partial \mathcal{R}_b}\left[\frac{K}{\chi^2}\left\{\phi_b + \mathcal{R}_n\left(\frac{dm_2}{ds} + \frac{d\theta}{ds}\right)\right\} + \frac{1}{\chi}\left(\frac{dm_2}{ds} + \frac{d\theta}{ds}\right)\right]$$

$$+\frac{1}{2}\frac{\partial}{\partial \mathcal{R}_b}\left[\frac{(-1)}{\chi}\left(K\phi_b + \frac{dm_2}{ds} + \frac{d\theta}{ds}\right)\right]$$

$= 0 + 0 + 0 = 0$ (Since variables inside the brackets are not functions of $\mathcal{R}_b$)

The component S_{tb} from (4.4) is given by

$$S_{tb} = (K P_{nn} - K P_{tt}) \frac{1}{\chi} + \frac{\partial P_{tt}}{\partial R_n} = (P_{nn} - P_{tt})\frac{K}{\chi} + \frac{\partial P_{tt}}{\partial R_n}$$

Substituting for P_{nn}, P_{tt} from (4.3)

$$S_{tb} = [\frac{\partial \varepsilon_{tn}}{\partial R_b} - \frac{K \varepsilon_{bt}}{\chi} - \frac{\partial \varepsilon_{bt}}{\partial R_n} + \frac{\partial \varepsilon_{nt}}{\partial R_b}] \frac{K}{\chi} + \frac{\partial}{\partial R_n} [\frac{\partial \varepsilon_{bt}}{\partial R_n} - \frac{\partial \varepsilon_{nt}}{\partial R_b}]$$

Again substituting for strain components from (3.13) and noting that $\varepsilon_{nt}=\varepsilon_{tn}$, $\varepsilon_{tb}=\varepsilon_{bt}$

$$\begin{aligned}
S_{tb} =& 2\frac{K}{\chi} \frac{1}{2} \frac{\partial}{\partial R_b} [\frac{1}{\chi} \{\phi_n + R_n m_1 K - R_b (K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})\} + m_1] \\
& - (\frac{K}{\chi})^2 \frac{1}{2} [\frac{1}{\chi} \{\phi_b + R_n (\frac{dm_2}{dS} + \frac{d\theta}{dS})\} - \phi_b] \\
& - \frac{K}{\chi} \frac{1}{2} \frac{\partial}{\partial R_n} [\frac{1}{\chi} \{\phi_b + R_n (\frac{dm_2}{dS} + \frac{d\theta}{dS})\} - \phi_b] \\
& + \frac{1}{2} \frac{\partial^2}{\partial R_n^2} [\frac{1}{\chi} \{\phi_b + R_n (\frac{dm_2}{dS} + \frac{d\theta}{dS})\} - \phi_b] \\
& - \frac{1}{2} \frac{\partial^2}{\partial R_n \partial R_b} [\frac{1}{\chi} \{\phi_n + R_n m_1 K - R_b (K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS})\} + m_1] \\
=& - \frac{K^2}{\chi^2} \phi_b - \frac{K}{\chi^2} (\frac{dm_2}{dS} + \frac{d\theta}{dS}) - \frac{K^2}{\chi^3} \{\phi_b + R_n (\frac{dm_2}{dS} + \frac{d\theta}{dS})\} \\
& + \frac{K^2 \phi_b}{2\chi^2} - \frac{K}{2\chi^2} (\frac{dm_2}{dS} + \frac{d\theta}{dS}) + \frac{K^2}{\chi^3} \{\phi_b + R_n (\frac{dm_2}{dS} + \frac{d\theta}{dS})\} \\
& + \frac{K}{\chi^2} (\frac{dm_2}{dS} + \frac{d\theta}{dS}) + \frac{K}{2\chi^2} \{K\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS}\} \\
=& 0
\end{aligned}$$

The Component S_{nn} from equation (4.4) is given by

$$S_{nn} = [TP_{nn} + \frac{\partial P_{nb}}{\partial S} + TR_b \frac{\partial P_{nb}}{\partial R_n} - TR_n \frac{\partial P_{nb}}{\partial R_b} - TP_{bb}]\frac{1}{\chi} - \frac{\partial P_{nt}}{\partial R_b}$$

Substituting from equation (4.3) for P_{nn}, P_{nb}, P_{bb}, P_{nt}

$$S_{nn} = \frac{1}{\chi} [T(\frac{\partial \varepsilon_{tn}}{\partial R_b} - \frac{K\varepsilon_{bt}}{\chi}) + \frac{\partial}{\partial S} (\frac{\partial \varepsilon_{tb}}{\partial R_b}) + TR_b \frac{\partial}{\partial R_n} (\frac{\partial \varepsilon_{tb}}{\partial R_b})$$

$$-TR_n \frac{\partial}{\partial R_b} (\frac{\partial \varepsilon_{tb}}{\partial R_b}) - T(\frac{K\varepsilon_{tb}}{\chi} - \frac{\partial \varepsilon_{tb}}{\partial R_n})]$$

$$- \frac{\partial}{\partial R_b} [\frac{\partial \varepsilon_{tt}}{\partial R_b} - \frac{1}{\chi} (T\varepsilon_{nt} + \frac{\partial \varepsilon_{bt}}{\partial S} + TR_b \frac{\partial \varepsilon_{bt}}{\partial R_n} - TR_n \frac{\partial \varepsilon_{bt}}{\partial R_b})]$$

$$= 2\frac{T}{\chi} \frac{\partial \varepsilon_{tn}}{\partial R_b} - 2 \frac{TK}{\chi^2} \varepsilon_{bt} + \frac{\partial^2 \varepsilon_{tt}}{\partial R_b^{\ 2}} + \frac{2T}{\chi} (\frac{\partial \varepsilon_{tb}}{\partial R_n} - R_n \frac{\partial^2 \varepsilon_{tb}}{\partial R_b^{\ 2}}$$

$$+ \frac{1}{\chi} [\frac{\partial}{\partial S}(\frac{\partial \varepsilon_{tb}}{\partial R_b}) + \frac{\partial}{\partial R_b} (\frac{\partial \varepsilon_{bt}}{\partial S})] + \frac{T}{\chi} R_b (\frac{\partial^2 \varepsilon_{tb}}{\partial R_n \partial R_b} + \frac{\partial^2 \varepsilon_{bt}}{\partial R_b \partial R_n}$$

Again substituting for strains ε_{ij} from equation (3.13) and carrying out the derivatives yields

$$S_{nn} = - \frac{T}{\chi^2} (\kappa\phi_b + \frac{dm_2}{dS} + \frac{d\theta}{dS}) - \frac{TK}{\chi^3} \{\phi_b + R_n (\frac{dm_2}{dS} + \frac{d\theta}{dS})\} + \frac{TK}{\chi^2} \phi_b$$

$$+ \frac{TK}{\chi^3} \{\phi_b + R_n (\frac{dm_2}{dS} + \frac{d\theta}{dS})\} + \frac{T}{\chi^2} (\frac{dm_2}{dS} + \frac{d\theta}{dS})$$

$$= 0$$

From equation (4.4) the component

$$S_{nb} = [TP_{nb} + TP_{bn} - KP_{nt} - \frac{\partial P_{nn}}{\partial S} - T\mathcal{R}_b \frac{\partial P_{nn}}{\partial \mathcal{R}_n} + T\mathcal{R}_n \frac{\partial P_{nn}}{\partial \mathcal{R}_n}]\frac{1}{\chi} + \frac{\partial P_{nt}}{\partial \mathcal{R}_n} .$$

Substituting for P_{ij} from (4.3) and carrying out some simplification results into

$$S_{nb} = -\frac{2T}{\chi}\frac{\partial \varepsilon_{tn}}{\partial \mathcal{R}_n} + \frac{2TK}{\chi^2}\varepsilon_{tn} + \frac{K^2T}{\chi^3}\mathcal{R}_b\,\varepsilon_{bt} - \frac{K}{\chi}\frac{\partial \varepsilon_{tt}}{\partial \mathcal{R}_b}$$

$$+ \frac{K}{\chi^2}\frac{\partial \varepsilon_{bt}}{\partial S} - \frac{1}{\chi}\frac{\partial}{\partial S}\left(\frac{\partial \varepsilon_{tn}}{\partial \mathcal{R}_b}\right) + \frac{1}{\chi}\left[\frac{\partial K}{\partial S}\frac{1}{\chi} + \frac{K^2}{\chi^2}\right]\varepsilon_{bt}$$

$$+ \frac{KT}{\chi^2}\mathcal{R}_b\frac{\partial \varepsilon_{bt}}{\partial \mathcal{R}_n} - \frac{T}{\chi}\mathcal{R}_b\frac{\partial^2 \varepsilon_{tn}}{\partial \mathcal{R}_n \partial \mathcal{R}_b} + \frac{\partial^2 \varepsilon_{tt}}{\partial \mathcal{R}_n \partial \mathcal{R}_b} - \frac{1}{\chi}\frac{\partial^2 \varepsilon_{bt}}{\partial \mathcal{R}_n \partial S}$$

$$- \frac{T}{\chi}\mathcal{R}_b\frac{\partial^2 \varepsilon_{bt}}{\partial^2 \mathcal{R}_n} .$$

Again substituting for the strain components ε_{ij} from equation (3.13) and carrying out indicated derivatives give the following results:

$$S_{nb} = \frac{TK}{\chi^3} Z_1 + \frac{TK\mathcal{R}_b}{\chi^3} Z_2 - \frac{T}{\chi^2} m_1 K + \frac{TK}{\chi^3} Z_1 - \frac{TK}{\chi^3}\mathcal{R}_b Z_2$$

$$+ \frac{K^2T}{2\chi^4}\mathcal{R}_b Z_5 - \frac{K^2T\mathcal{R}_b}{2\chi^3}\phi_b - \frac{K}{\chi^2} Z_4 + \frac{K}{2\chi^3}\frac{d}{dS}(Z_5)$$

$$+ \frac{K}{2\chi^2}\frac{\partial}{\partial S}\left(\frac{1}{\chi}\right) Z_5 - \frac{K}{2\chi^2}\frac{\partial \phi_b}{\partial S} + \frac{1}{2\chi}\frac{\partial}{\partial S}\left(\frac{1}{\chi}\right) Z_2 + \frac{1}{2\chi^2}\frac{\partial}{\partial S}(Z_2)$$

... (contd.)

$$+ \frac{1}{2\chi^3} (\frac{\partial K}{\partial S}) Z_5 - \frac{1}{2\chi^2} \frac{\partial K}{\partial S} \phi_b + \frac{R_n K}{2\chi^4} \frac{\partial K}{\partial S} Z_5$$

$$- \frac{R_n K}{2\chi^3} \frac{\partial K}{\partial S} \phi_b + \frac{K}{2\chi^3} \frac{\partial}{\partial S} (Z_5) + \frac{K}{2\chi^2} \frac{\partial}{\partial S} (\frac{1}{\chi}) Z_5 - \frac{K}{2\chi^2} \frac{\partial \phi_b}{\partial S}$$

$$+ \frac{TK^2 R_b}{2\chi^4} Z_5 + \frac{TK}{2\chi^3} R_b (\frac{\partial m_2}{\partial S} + \frac{\partial \theta}{\partial S}) + \frac{TKR_b}{2\chi^3} Z_2 + \frac{K}{\chi^2} Z_4$$

$$- \frac{KR_n}{\chi^4} (\frac{\partial K}{\partial S}) Z_5 - \frac{1}{2\chi^3} \frac{\partial K}{\partial S} Z_5 - \frac{1}{2\chi} \frac{\partial}{\partial S} (\frac{1}{\chi}) (\frac{\partial m_2}{\partial S} + \frac{\partial \theta}{\partial S})$$

$$- \frac{K}{2\chi^3} \frac{\partial}{\partial S} (Z_5) - \frac{1}{2\chi^2} \frac{\partial}{\partial S} [\frac{\partial m_2}{\partial S} + \frac{\partial \theta}{\partial S}] - \frac{TK^2}{\chi^4} R_b Z_5$$

$$- \frac{TK}{\chi^3} R_b (\frac{\partial m_2}{\partial S} + \frac{\partial \theta}{\partial S})$$

or

$$S_{nb} = 0$$

where for simplicity

$$Z_1 = \{\phi_n + R_n (m_1 K)\} \quad ; \quad Z_2 = (K\phi_b + \frac{\partial m_2}{\partial S} + \frac{\partial \theta}{\partial S})$$

$$Z_3 = (\frac{\partial m_1}{\partial S} + T\phi_b) \quad ; \quad Z_4 = (m_2 K + \theta K - \frac{\partial \phi_b}{\partial S} + Tm_1)$$

$$Z_5 = \{\phi_b + R_n (\frac{\partial m_2}{\partial S} + \frac{\partial \theta}{\partial S})\}$$

Similarly S_{nn} from equation (4.4) is

$$S_{bb} = [TP_{nn} + \frac{\partial P_{nb}}{\partial S} + T\mathcal{R}_b \frac{\partial P_{nb}}{\partial \mathcal{R}_n} - T\mathcal{R}_n \frac{\partial P_{nb}}{\partial \mathcal{R}_b} - TP_{bb}]\frac{1}{\chi} - \frac{\partial P_{nt}}{\partial \mathcal{R}_b}$$

Substituting for P_{nn}, P_{nb}, P_{nb}, P_{bb}, P_{nt} from (4.3) and some simplification leads to

$$S_{bb} = \frac{2KT}{\chi^2} \varepsilon_{tb} - \frac{2T}{\chi} \frac{\partial \varepsilon_{tb}}{\partial \mathcal{R}_n} - \frac{2T}{\chi} \frac{\partial \varepsilon_{tn}}{\partial \mathcal{R}_b} + \frac{2K}{\chi} \frac{\partial \varepsilon_{tt}}{\partial \mathcal{R}_n}$$

$$- \frac{2}{\chi^2} \frac{dK}{dS} \varepsilon_{tn} - \frac{2K}{\chi^2} \mathcal{R}_n \frac{\partial K}{\partial S} \varepsilon_{tn} - \frac{2K}{\chi^2} \frac{\partial \varepsilon_{tn}}{\partial S} + \frac{1}{\chi} \frac{\partial}{\partial S} (\frac{\partial \varepsilon_{tn}}{\partial \mathcal{R}_n})$$

$$- \frac{2TK^2}{\chi^3} \mathcal{R}_b \varepsilon_{tn} - \frac{2TK}{\chi^2} \mathcal{R}_b \frac{\partial \varepsilon_{tn}}{\partial \mathcal{R}_n} + \frac{2T}{\chi} \mathcal{R}_b \frac{\partial^2}{\partial \mathcal{R}_n^2} \varepsilon_{nt}$$

$$+ \frac{2TK\mathcal{R}_n}{\chi^2} (\frac{\partial \varepsilon_{tn}}{\partial \mathcal{R}_b}) - \frac{T\mathcal{R}_n}{\chi} \frac{\partial^2 \varepsilon_{tn}}{\partial \mathcal{R}_b \partial \mathcal{R}_n} + \frac{1}{\chi} \frac{\partial}{\partial \mathcal{R}_n} (\frac{\partial \varepsilon_{nt}}{\partial S})$$

$$- \frac{T\mathcal{R}_n}{\chi} \frac{\partial}{\partial \mathcal{R}_n} (\frac{\partial \varepsilon_{nt}}{\partial \mathcal{R}_b}) - \frac{\partial^2 \varepsilon_{tt}}{\partial \mathcal{R}_n^2}$$

Again substituting for the strain components ε_{ij} from equation (3.13) and carrying out indicated derivatives gives the following result:

$$S_{bb} = \frac{KT}{\chi^3} Z_5 - \frac{KT}{\chi^2} \phi_b - \frac{TK}{\chi^3} Z_5 - \frac{T}{\chi^2} (\frac{\partial m_2}{\partial S} + \frac{\partial \theta}{\partial S}) + \frac{T}{\chi^2} Z_2$$

$$+ \frac{2K^2}{\chi^3} \mathcal{R}_n Z_3 + \frac{2K^2}{\chi^3} \mathcal{R}_b Z_4 + \frac{2K}{\chi^2} Z_3 - \frac{1}{\chi^3} \frac{\partial K}{\partial S} Z_1$$

$$+ \frac{\mathcal{R}_b}{\chi^3} \frac{\partial K}{\partial S} Z_2 - \frac{1}{\chi^2} \frac{\partial K}{\partial S} m_1 - \frac{K\mathcal{R}_n}{\chi^4} (\frac{\partial K}{\partial S}) Z_1$$

... (contd)

$$+ \frac{K\mathcal{R}_n\mathcal{R}_b}{\chi^4}\frac{\partial K}{\partial S} Z_2 - \frac{Km_1\mathcal{R}_n}{\chi^3}\frac{\partial K}{\partial S} - \frac{K\mathcal{R}_n}{\chi^4}\frac{\partial K}{\partial S} Z_1$$

$$+ \frac{K\mathcal{R}_n\mathcal{R}_b}{\chi^4}\frac{\partial K}{\partial S} Z_2 - \frac{K}{\chi^3}\frac{\partial}{\partial S}(Z_1) + \frac{K\mathcal{R}_b}{\chi^3}\frac{\partial}{\partial S}(Z_2) - \frac{K}{\chi^2}\frac{\partial m_1}{\partial S}$$

$$- \frac{1}{2\chi^3}\frac{\partial K}{\partial S} Z_1 - \frac{1}{2\chi^3}\frac{\partial K}{\partial S}\mathcal{R}_b Z_2 + \frac{K\mathcal{R}_n}{\chi^4}\frac{\partial K}{\partial S} Z_1 - \frac{K\mathcal{R}_n\mathcal{R}_b}{\chi^4}\frac{\partial K}{\partial S} Z_2$$

$$+ \frac{K}{2\chi^3}\frac{\partial}{\partial S}(Z_1) - \frac{K}{2\chi^3}\mathcal{R}_b\frac{\partial}{\partial S}(Z_2) + \frac{\mathcal{R}_n m_1 K}{2\chi^3}\frac{\partial K}{\partial S} + \frac{1}{2\chi^2}\frac{\partial}{\partial S}(m_1 K)$$

$$+ \frac{TK^2\mathcal{R}_b^{\ 2}}{\chi^4} Z_2 - \frac{TK^2}{\chi^4}\mathcal{R}_b Z_1 - \frac{TK^2}{\chi^3}\mathcal{R}_b m_1 - \frac{TK^2}{\chi^4}\mathcal{R}_b Z_1 + \frac{TK^2}{\chi^4}\mathcal{R}_b^{\ 2} Z_2$$

$$- \frac{TK}{\chi^3} m_1 K\mathcal{R}_b + \frac{2TK^2\mathcal{R}_b}{\chi^4} Z_1 - \frac{2TK^2}{\chi^4}\mathcal{R}_b^{\ 2} Z_2 + \frac{2TK}{\chi^3} m_1 K\mathcal{R}_b$$

$$- \frac{TK}{\chi^3}\mathcal{R}_n Z_2 + \frac{TK\mathcal{R}_n}{2\chi^3} Z_2 + \frac{1}{2\chi^3}\left(\frac{\partial K}{\partial S}\right) Z_1 - \frac{1}{2\chi^3}\frac{\partial K}{\partial S}\mathcal{R}_b Z_2$$

$$+ \frac{K\mathcal{R}_n}{\chi^4}\frac{\partial K}{\partial S} Z_1 - \frac{K\mathcal{R}_n}{\chi^4}\frac{\partial K}{\partial S}\mathcal{R}_b Z_2 + \frac{K}{2\chi^3}\frac{\partial}{\partial S}(Z_1) + \frac{K}{2\chi^3}\mathcal{R}_b\frac{\partial}{\partial S}(Z_2)$$

$$+ \frac{1}{2\chi^2}\frac{\partial}{\partial S}(m_1 K) + \frac{K}{2\chi^3} m_1\mathcal{R}_n\frac{\partial K}{\partial S} + \frac{TK}{2\chi^3}\mathcal{R}_n Z_2 - \frac{2K^2}{\chi^3}\mathcal{R}_n Z_3$$

$$- \frac{2K}{\chi^3}\mathcal{R}_b Z_4 - \frac{2K}{\chi^2} Z_2$$

or

$$S_{bb} = 0$$

Since $\bar{\bar{S}} = S_{ij}\bar{E}_i\bar{E}_j$ is symmetric

$$S_{ij} = S_{ji}$$

and

thus all S_{ij} vanish.