CONFORMAL DIFFERENTIAL GEOMETRY
CONFORMAL DIFFERENTIAL GEOMETRY

By

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SCOPE AND CONTENTS:

This thesis is a study of some properties of arcs which remain invariant under certain types of conformal representations. The study is carried on first in the conformal plane, then in conformal 3-space, and finally in conformal n-space. It is comprised of most of the research on this subject which has been carried on to date by Professors N.D. Lane and Peter Scherk. I have assisted Dr. Lane in the creation of some of the material which this thesis contains.
I wish to express my sincere thanks to Dr. N.D. Lane, who was my supervisor in the creation of this thesis. The many hours which he spent on my behalf have greatly enriched my understanding of the subject of Mathematics.
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KEY TO NOTATIONS

In the notation of sections, the number of the chapter appears first, and this is followed by the number of the section in that chapter. If the section contains a sub-section, it is denoted by an additional number; e.g., section 8.5 is section 5 of chapter 8, while section 8.5.4 is the fourth sub-section of section 8.5.

In the notation of theorems and lemmas, the number of the chapter is given first and the number of the theorem or lemma in that section follows; e.g., Theorem 5.7 is the seventh theorem in chapter 5.
CHAPTER I

WHAT IS CONFORMAL GEOMETRY?

1.1. Stereographic Projection.

Consider a sphere in projective 3-space resting on a plane \( p \). Let \( N \) be the point on the sphere which is the most remote from the plane \( p \), and let \( P' \) be any other point on the sphere (cf. Fig. 1.1). The line \( NP' \) is extended to meet the plane \( p \) at the point \( P \). The mapping of the points \( P' \) of the sphere on the plane \( p \) in this manner is called a

![Diagram of Stereographic Projection](image-url)
1.2. The Notion of Angle.

We describe what we mean by the term angle by the following quotation from Sommerville's "An Introduction to the Geometry of N Dimensions" (Methuen 1929):

"Two linear spaces which have their highest degree of intersection determine an angle and this angle determines completely the shape of the figure consisting of the two spaces. For example, two straight lines in a plane determine a plane angle; two planes in 3-space determine a dihedral angle, which can be measured by means of a plane angle."

1.3. The Conformal Plane.

Let us now return to a consideration of the mapping described in section 1.1. It is not difficult to prove (cf. Snyder and Sisam, "Analytic Geometry of Space", Holt 1932, 1. This description applies to two, three, or n dimensions.
pp. 59-62) that under a stereographic projection, circles on the sphere which do not pass through $N$, map onto circles in the plane $p$, while circles on the sphere which pass through $N$, map onto straight lines in $p$. It is also true that angles are preserved under this mapping.

Obviously, there is a one-to-one correspondence between the points of the plane and the points of the sphere, with the exception of the point $N$. This point has no image in the plane $p$ if we think of $p$ as the Euclidean plane, while it has many images if $p$ is the projective plane. In order to preserve a 1-1 correspondence throughout, we postulate a single point at infinity for the plane $p$. The point $N$ then corresponds to this point at infinity; the plane $p$ is called the conformal plane. It is convenient now, for obvious reasons, to regard straight lines in $p$ as circles through the point at infinity (cf. Hilbert and Cohn-Vossen, "Geometry and the Imagination", Chelsea 1952, p 251).
It is clear from its definition that the conformal plane is identical with the Argand plane of complex numbers (cf. Copson, "Theory of Functions of a Complex Variable", Oxford 1935, pp 8-10). Consequently, some of the concepts and results stated in this thesis may be clarified by resorting to Complex Variable theory.

1.4. Definition of Conformal Geometry in the Conformal Plane.

Any mapping of the form

\[ w = \frac{az + b}{cz + d} \quad (a,b,c,d,z, \text{ complex, } ad - bc \neq 0), \]

in the Argand plane, maps circles into circles and preserves angles. A mapping of this form is called a Möbius Transformation, and is a conformal representation. Certain properties, then, of circles and arcs will remain invariant under

1. A conformal representation is an angle-preserving mapping. There are conformal representations in which circles are not necessarily transformed into circles, but we do not consider these.

2. For the definition of arc, see section 1.11.
these transformations. **Conformal geometry** is the study of the properties which remain invariant under such a conformal mapping.

### 1.5. Extension to Higher Dimensions.

The work of the previous sections may be generalized to three or more dimensions. Although we cannot make use of Complex Variable theory in these higher dimensions, we have another model which will be described presently, and which applies to any dimension.

Conformal 3-space may be represented on the surface of a hypersphere (or, as it is more explicitly termed, a 3-sphere) in projective 4-space; more generally, conformal n-space may be represented on the surface of an n-sphere in projective \((n - 1)\)-space. Accordingly, conformal n-space has a single point at infinity, so that a p-flat \((p = 1, 2, \ldots, n - 1)\) (cf. Sommerville "An Introduction to the Geometry of N Dimensions", p 8) is thought of as a p-sphere through the point at infinity. Any transformation which takes
place in conformal n-space, transforming p-spheres into p-spheres \((p=1,2,...,n-1)\), and leaving angles invariant is a conformal representation. **Conformal geometry** in \(n\)-dimensions, then, is the study of those properties which remain invariant under the transformations described.

Instead of making use of Complex Variable theory as a model when studying Conformal geometry in the conformal plane, we could make use of the following model:

The group of the projectivities (or one-to-one linear transformations of projective space) in 3-space which preserve a sphere is called the **orthogonal group** in three dimensions. Such linear transformations map planes into planes, and hence map the intersection of a plane and the sphere into another intersection of a plane and the sphere. Thus circles on the sphere are mapped into circles, and the orthogonal group in 3-space is equivalent to the conformal group of transformations in the conformal plane.

Similarly, the \((m+1)\)-dimensional orthogonal group
is equivalent to the conformal group of transformations in n dimensions (cf. Birkhoff and MacLane "A Survey of Modern Algebra", Macmillan 1953, chapter 9).

Remark. Conformal geometry has been set up axiomatically using established geometries as a model (cf. for example, A.J. Hoffman, "On the Foundations of Inversion Geometry", Trans. Am. Math. Soc., Vol 71, July-Dec. 1951). Presumably the axioms could be set up independently, using points and circles (spheres; (n-1)-spheres) as undefined elements, but it seems that this has never been done.

1.6. An Investigation of Angles.

1.6.1. We can get a clearer idea of what we mean by the term angle in the conformal plane by resorting to Complex Variable theory. It is not difficult to prove (cf. Carathéodory, "Theory of Functions of a Complex Variable", Chelsea 1954, pp 29-30) that under a Möbius transformation, the cross ratio of any four complex numbers, \(z_1, z_2, z_3, z_4\), is invariant; i.e.
This suggests that the cross-ratio is closely related to the notion of angle. Let \( z_1, z_2, z_3, \) and \( z_4 \) be respectively the complex numbers \( \lambda \) (with finite coefficients), \( 0, \infty, 1 \). Then

\[
(1.3) \quad \text{amp} \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)} = \text{amp} \lambda,
\]

i.e. the amplitude of the cross-ratio of these four complex numbers is equal to the angle \( \Theta \) between the line determined by the complex number \( \lambda \), and the positive real axis (cf. Fig 1.2). Note that relation (1.3) is unaltered if we interchange the values \( z_1, z_4, \) and \( z_2 \) and \( z_3 \).
Suppose two circles, $C_1$ and $C_2$ intersect (cf. Fig 1.3). Let $R$ and $Q$ be their points of intersection.

Let $S$ be a point on $C_1$ and $T$ a point on $C_2(S,T \neq R$ and $Q)$,

![Diagram of two circles intersecting](image)

*Fig 1.3*

and let $\theta$ be the angle at which $C_1$ and $C_2$ meet. By one or more Möbius transformations, we can let $Q = \infty$, $R = 0$, $S = 1$, $T = \lambda$. Then Fig 1.3 is transformed into Fig 1.2, i.e. circles and angles are preserved. Hence

$$
\theta = \text{amp} \frac{(T-R)(Q-S)}{(T-Q)(R-S)}
$$

It is also true that

$$
\theta = \text{amp} \frac{(S-Q)(R-T)}{(S-R)(Q-T)}
$$

1. Throughout this thesis, the symbol $\neq$ will mean "different from".
We therefore have an alternative definition for an angle in the conformal plane, viz.,

"If two circles, $C_1$ and $C_2$ intersect in $R$ and $Q$, and if $S$ and $T$ lie on $C_1$ and $C_2$ respectively, $S, T \neq R$ and $Q$, then the angle $\theta$ between $C_1$ and $C_2$ is the amplitude of the cross-ratio

$$\frac{(T-R)(Q-S)}{(T-Q)(R-S)},$$

where $Q, T, R, S$ are complex numbers".

If the two supplementary angles between $C_1$ and $C_2$ (see footnote) are equal, we say that $C_1$ and $C_2$ meet at right angles, or $C_1$ is orthogonal to $C_2$ ($C_2$ is orthogonal to $C_1$).

If the circles $C_1$ and $C_2$ in the conformal plane have only one point in common, say the point $R$, then the angle between $C_1$ and $C_2$ is zero, and we say that $C_1$ and $C_2$ touch at $R$.

1. Of course there are two angles between $C_1$ and $C_2$, one being the supplement of the other; the relative order of the points $Q, R, T, S$ governs the choice of the angle.
1.6.2. Proceeding to three dimensions, we let a circle $C$ intersect a sphere $S$ in two points, $R$ and $Q$. By one of the transformations described in section 1.5, we let $Q$ be carried into the point at infinity. Then $C$ becomes a straight line and $S$ becomes a plane (cf. Fig 1.4). Let $l$ be any line lying in $S$ and passing through $R$. Then $C$ and $l$ determine a plane.

The angle between $C$ and $l$ in 3-space is then the angle between $C$ and $l$ in the plane determined by these two lines.

If the angle between $C$ and $l$ is the same for all $l$, we say that $C$ and $S$ meet at right angles, or $C$ is orthogonal to $S$.

If $C$ meets $S$ at one point only, or if $C$ lies on $S$, then the angle between $C$ and $S$ is zero and we say that $C$ and $S$ touch.
If two circles $C_1$ and $C_2$ in conformal 3-space intersect twice, we let one point be carried to the point at infinity and thus are led to a clear definition of the angle between $C_1$ and $C_2$. If $C_1$ and $C_2$ meet in one point only, and lie on a common sphere, we have on the sphere a model of the conformal plane. Thus we have reduced this case to a case in section 1.6.1, and we see that $C_1$ and $C_2$ touch at their common point. If, however, $C_1$ and $C_2$ do not lie on a common sphere, they do not meet at angle zero.

Suppose that two spheres, $S_1$ and $S_2$, meet in a proper circle $C$. Let $Q$ be any point on $C$, and let $S_3$ be a sphere through $Q$ such that $C$ is orthogonal to $S_3$. Thus $C$ meets $S_3$ in another point, $R$, say. Let $Q$ be carried to infinity, so that $S_1$, $S_2$, and $S_3$ become planes, meeting in $R$ on the line $C$. The angle between $S_1$ and $S_2$ is equal to the angle on $S_3$ between the intersection of $S_1$ and $S_3$ and the intersection of $S_2$ and $S_3$.

If two spheres, $S_1$ and $S_2$ meet in a single point, $P$, 
then the angle between $S_1$ and $S_2$ is zero, and we say that $S_1$ and $S_2$ touch at $P$.

1.6.3. We now consider angles in conformal $n$-space. As in previous cases, $p$-spheres can be reduced to $p$-flats ($p = 1, 2, \ldots, n-1$) by the proper transformation. The angle between a $p$-sphere and a $q$-sphere is then the same as the angle between a $p$-flat and a $q$-flat. A discussion of this can be found in "An Introduction to the Geometry of $n$-Dimensions" by D.M.Y. Sommerville.

1.6.4. It should be evident by now that by a proper transformation, of the form described in sections 1.4 and 1.5, any proposition regarding angles can be greatly simplified. This method of attack will be used in some proofs.

1.7. The Closure Property of Conformal $n$-space.

As we have already noted, conformal $n$-space may be represented on the surface of an $n$-sphere in projective $(n+1)$-space ($n \neq 2, 3, \ldots$). Hence every infinite sequence of points in conformal $n$-space lies in an interval, and thus
possesses at least one accumulation point (cf. Hardy, "Pure Mathematics", Cambridge 1945, pp. 30-32).

Suppose now, for instance, that we have an infinite sequence of circles $C$ in the conformal plane. Then there exists a sub-sequence, $C' \subset C$, of circles which contains an infinite sequence of points possessing an accumulation point. Again, there is a sub-sequence $C'' \subset C'$ of circles which contains a different sequence of points possessing an accumulation point. Finally there is a sub-sequence $C''' \subset C''$ of circles which contains yet another sequence of points possessing an accumulation point. Thus we have a sequence $C'''$ of circles which possesses a limiting circle, the circle determined by the three accumulation points. This important result may be stated as follows:

1. The symbol $\subset$ means "contained in" ("is contained in") or "belonging to" ("belongs to"). The symbol $\in$ is reserved to mean "is a (single) element of". The symbol $\supset$ means "containing" ("contains"); i.e. if $A \subset B$, then $B \supset A$. 
Theorem 1.1. Every infinite sequence of circles in the conformal plane possesses at least one limit circle. We call such a limit circle an accumulation circle.

Obviously, the above result may be generalized. Thus we have the more general theorem, namely:

Theorem 1.2. Every infinite sequence of p-spheres \((p=1, 2, \ldots, n-1)\) in conformal n-space possesses at least one limit p-sphere (called an accumulation p-sphere).

1.8. Regions in Conformal n-space.

Any proper circle, \(C\) (i.e. \(C\) is not a point), divides the conformal plane into two open regions, the interior \(\text{int}\(C\)) of \(C\), and the exterior \(\text{ext}\(C\)) of \(C\). If we orient the circle \(C\), then the interior of \(C\) is the region lying to the left of the oriented circle (cf. Fig 1.5).

![Fig 1.5](image-url)
In general, any proper \((n-1)\)-sphere, \(S\), divides conformal \(n\)-space into two open regions, the interior \(\text{int } S\) of \(S\), and the exterior \(\overline{S}\) of \(S\). If \(P\) is a point not lying on \(S\), then the interior of \(S\) may be defined as the set of all points not lying on \(S\) and not separated from \(P\) by \(S\).

1.9. Convergence.

1.9.1. A sequence of points, \(P_1, P_2, \ldots\) in the conformal plane is said to be \textit{convergent} to a point \(P\), if, given any circle \(C\) with \(P \subset C\), there exists a number \(n = n(C)\) such that \(P \subset C\) for all \(\nu > n\).

In the same way, convergence of circles to a point is defined. Such a point is called a point-circle.

A sequence of circles, \(C_1, C_2, \ldots\) in the conformal plane is said to be \textit{convergent} to the proper circle \(C\) if, given any two points \(P\) and \(Q\) such that \(P \subset C\) and \(Q \subset \overline{C}\), there exists a number \(n = n(P, Q)\) such that \(P \subset C_\nu\) and \(Q \subset \overline{C_\nu}\) for all \(\nu > n\).

1.9.2. A sequence of points \(P_1, P_2, \ldots\) in conformal 3-space
is said to be **convergent** to a point P if, given any sphere S with P ⊂ S, there exists a number $n = n(S)$ such that $R_ν ⊂ S$ for all $ν > n$.

In the same way, convergence of circles and spheres to **point-circles** and **point-spheres** is defined.

A sequence of circles, $C_1, C_2, ...$ in conformal 3-space is said to be **convergent** to the circle C if, given any circle $C'$, which links $^1$ with C, there exists a number $n = n(C')$ such that $C_ν$ links with $C'$ for all $ν > n$.

A sequence of spheres, $S_1, S_2, ...$ in conformal 3-space is said to be **convergent** to the sphere S if, given any two points P and Q, where $P ⊂ S$ and $Q ⊂ S$, there exists a number $n = n(P, Q)$ such that $P ⊂ S_ν$ and $Q ⊂ S_ν$ for all $ν > n$.

1.9.3. A sequence of points $P_1, P_2, ...$ in conformal n-space is said to be **convergent** to a point P, if, given any (n-1)-sphere, $S$, with $P ⊂ S$, there exists a number $N = N(S)$ such that

1. $C'$ is said to link with C, if any sphere $S ⊃ C$ cuts any sphere $S' ⊃ C'$, while C and $C'$ have no common point.
$P_\nu \subset S$ for all $\nu > N$.

In the same way, convergence of $m$-spheres to \textbf{point-}$m$-spheres is defined ($m=1,2,\ldots,n-1$).

A sequence of $m$-spheres, $S_1^{(m)}, S_2^{(m)}, \ldots$ is said to be \textbf{convergent} to an $m$-sphere $S^{(m)}$, if to every $(n-m-1)$-sphere, $S^{(n-m-1)}$ which links\(^1\) with $S^{(m)}$, there exists a positive integer $n=n(S^{(n-m-1)})$ such that $S_\nu^{(m)}$ links with $S^{(n-m-1)}$ for all $\nu > n (m=1,2,\ldots,n-2)$.

Finally, a sequence of $(n-1)$-spheres, $S_1, S_2, \ldots$ in conformal $n$-space is said to be \textbf{convergent} to an $(m-1)$-sphere $S$, if, given any two points, $P$ and $Q$, where $P \subset S$ and $Q \subset \bar{S}$, there exists a number $n=n(P,Q)$ such that $P \subset S_\nu$ and $Q \subset \bar{S}_\nu$ for all $\nu > n$.

\textbf{1.10. Pencils of circles, spheres, and $m$-spheres.}

In the following, section 1.10.1 deals with the conformal plane, section 1.10.2 with conformal 3-space, and section 1.10.3 with conformal $n$-space.

\(^1\) cf. Seifert & Threlfall "Lehrbuch der Topologie", §77
1.10.1. The set of all circles that intersect two given circles at right angles is a linear pencil, \( \mathcal{P} \) of circles. A pencil \( \mathcal{P} \) of the first kind possesses two fundamental points, \( P \) and \( Q \) (cf. Fig. 1.6). A pencil \( \mathcal{P} \) of the second kind possesses one fundamental point, \( P \) (cf. Fig. 1.7) and is identical with the set of all circles that touch any circle of \( \mathcal{P} \) at \( P \). A pencil \( \mathcal{P} \) of the third kind possesses no funda-
To any pencil $\Pi$, and to every point $Q$, which is not a fundamental point of $\Pi$, there exists one and only one circle, $C(Q;\Pi)$ of $\Pi$ through $Q$. In the case of a pencil $\Pi$ of the second kind, the fundamental point is regarded as a point-circle belonging to the pencil $\Pi$.

1.10.2. The sphere through a proper circle $C$, and a point $P$, $P \not\subset C$ will be denoted by $S(P;C)$. We shall make use of pencils $\Pi$, of spheres and circles, determined by certain incidence and tangency conditions. A circle (point) which is common to

1. The symbol $\not\subset$ means "not lying on" ("does not lie on") or "not contained in" ("is not contained in"). The symbol $\notin$ means "is not an element of".
all the spheres (circles) of a pencil is called a fundamental circle (fundamental point) of the pencil. In the pencil $\mathcal{P}$ of spheres through a fundamental circle $C$, there exists one and only one sphere $S(P; \mathcal{P})$ of $\mathcal{P}$ through any point $P$ which does not lie on $C$. Similarly, in the pencil $\mathcal{P}$ of spheres (circles) which touch a given sphere (circle) at a given point $Q$, there is one and only one sphere $S(P; \mathcal{P})$ (circle $C(P; \mathcal{P})$) of $\mathcal{P}$ which passes through any point $P \neq Q$. The fundamental point $Q$ is regarded as a point-sphere (point-circle) belonging to $\mathcal{P}$.

1.10.3. An $m$-sphere through an $(m - 1)$-sphere $S^{(m-1)}$, and a point $P \notin S^{(m-1)}$ will be denoted by $S^{(m)}[p; S^{(m-1)}]$. The $m$-sphere through $m + 2$ points, $P_0, P_1, \ldots, P_{m+1}$, not all lying on the same $(m-1)$-sphere, will occasionally be denoted by $S(P_0, P_1, \ldots, P_{m+1})$. Such a set of points is said to be independent. We shall make use of pencils $\mathcal{P}^{(m)}$ of $m$-spheres.

---

1. cf. Sommerville, "An Introduction to the Geometry of N Dimensions", page 8. In two (cont'd on p 22 (bottom))
determined by certain incidence and tangency conditions. An (m-1)-sphere which is common to all the m-spheres of a pencil \( \Pi^{(m)} \) is called a fundamental (m-1)-sphere of \( \Pi^{(m)} \). In the pencil \( \Pi^{(m)} \) through a fundamental (m-1)-sphere \( S^{(m-1)} \), there is one and only one m-sphere \( S(P; \Pi^{(m)}) \) of \( \Pi^{(m)} \) through each point \( P \), which does not lie on \( S^{(m-1)} \). Similarly, in the pencil \( \Pi^{(m)} \) of all the m-spheres which touch a given m-sphere at a given point \( Q \), there is one and only one m-sphere \( S(P; \Pi^{(m)}) \) through each point \( P \neq Q \). The fundamental point \( Q \) is regarded as a point m-sphere belonging to \( \Pi^{(m)} \).

1.11. Arcs.

An arc \( A \) in conformal n-space \( (n = 2, 3, \ldots) \) is the continuous image of a real interval. The images of distinct points of this parameter interval are considered to be different dimensions, we speak of the circle \( C(P,Q,R) \) through the three independent (i.e. distinct) points \( P, Q, \) and \( R \), and in three dimensions we speak of the circle \( C(P,Q,R) \) and the sphere \( S(P,Q,R,T) \), where \( P, Q, R, \) and \( T \) are independent points (i.e. do not lie on the same circle).
ferent points of A even though they may coincide in the space. If a sequence of points of the parameter interval converges to a point p, we define the corresponding sequence of image points to be convergent to the image of p. The same small letters, p,t,... will denote both the points of the parameter interval, and their image points on A. The end- (interior-) points of A are the images of the end-(interior-) points of the parameter interval. If p is an interior point of A, this neighbourhood is decomposed by p into two (open) one-sided neighbourhoods.

1.12. Support and Intersection.

Let p be an interior point of an arc A in the conformal plane. Then we call p a point of support (intersection) with respect to a circle C, if a sufficiently small neighbourhood of p of A is decomposed by p into two one-sided neighbourhoods which lie in the same region (in different regions) bounded by C. C is then called a supporting (intersecting) circle of A at p. Thus C supports A at p if p ∉ C.
By definition, the point-circle p always supports A at p.

It is possible for a circle to have points different from p in common with every neighbourhood of p on A. In this case we say that C neither supports nor intersects A at p.

The above may be extended to three (n) dimensions simply by substituting the word sphere ((n-1)-sphere) for the word circle (and the letter S for the letter C).
CHAPTER II

DIFFERENTIABLE POINTS OF ARCS IN THE CONFORMAL PLANE

2.1. Introduction.

The goal of this chapter is a classification of the differentiable points of arcs in the conformal plane. The main tools are the intersection and support properties of families of circles through a differentiable point \( p \) of an arc \( A \). This chapter is the ground-work for chapters 3 and 4.

2.2. Differentiability.

Let \( p \) be a fixed point of an arc \( A \), and let \( t \) be a variable point of \( A \). If \( P, Q, \) and \( p \) are distinct points, the unique circle through these points will be denoted by \( C(P,Q,p) \).

The arc \( A \) is said to be differentiable at \( p \) if the following two conditions are satisfied:

Condition I: If the parameter \( t \) is sufficiently
close to, but different from, the parameter \( p \), the circle
\[ C(P,t,p) \] is uniquely defined, and converges as \( t \to p \).

Thus the limit circle, called a tangent circle, and denoted by \( C(P;\mathcal{T}) \) is independent of the way \( t \) converges to \( p \). The family of all such circles, together with the point-circle \( p \), will be denoted by the symbol \( \mathcal{T} \).

Condition II: If the parameter \( t \) is sufficiently close to, but different from the parameter \( p \), the circle
\[ C(t;\mathcal{T}) \] is uniquely defined, and converges as \( t \to p \).

This unique limit circle, called the osculating circle of \( A \) at \( p \), will be denoted by \( C(p) \).

2.3. Structure of the Families of Circles Through \( p \).

Theorem 2.1. Suppose that \( A \) satisfies Condition I at \( p \). Then \( t \) does not coincide with \( p \) if the parameter \( t \) is sufficiently close to, but different from, the parameter \( p \).

Proof: Let \( P \neq p \). Then by Condition I, \( C(P,t,p) \) is

\[ 1. \] The symbol \( \to \) means "converges to".
uniquely defined when the parameter \( t \) is close to, but different from, the parameter \( p \). Hence \( t \neq p \).

This theorem indicates a restriction which Condition I imposes on an arc; viz., the arc satisfying Condition I at the point \( p \), must have a neighbourhood of \( p \) which contains no point coincident with \( p \).

**Theorem 2.2.** Suppose that the parameter \( t \) is sufficiently close to, but different from, the parameter \( p \). If the circle \( C(P,t,p) \) converges as \( t \to p \) (\( t \in A \)), for a single point \( P \neq p \), then Condition I holds.

**Remark:** Theorem 2.2 shows that Condition I is stronger than necessary, and could be replaced by the condition laid down in the statement of this theorem.

**Proof of Theorem 2.2:** Let \( P, Q, R \) be three mutually distinct points. If the point \( R' \neq R \) converges to \( R \), then the angle between the circles \( C(R',R,P) \) and \( C(R',R,Q) \) con-
verges to zero. In particular, let \( R = p, R' = t \in A \). Then

\[
\lim_{t \to p} \kappa [C(P,t,p);C(Q,t,p)] = 0.
\]

Hence any accumulation circle \( C' \), of the circles \( C(Q,t,p) \) touches \( C(P;\mathcal{T}) \) at \( p \). Since \( C' \) also passes through the point \( Q \neq p \), it is uniquely determined. Hence \( C' = \lim_{t \to p} C(Q,t,p) = C(Q;\mathcal{T}) \).

**Theorem 2.3.** The set \( \mathcal{T} = \mathcal{T}(p) \) of all the tangent circles of \( A \) at \( p \) is a pencil of the second kind with fundamental point \( p \).

**Proof:** By Theorem 2.3, any two tangent circles, \( C(P;\mathcal{T}) \) and \( C(Q;\mathcal{T}) \) touch at \( p \).

Suppose that a circle \( C \) touches a circle of \( \mathcal{T} \) at \( p \), and let \( P \subset C, P \neq p \). Then \( C \) and \( C(P;\mathcal{T}) \) also touch at \( p \) and

1. This statement becomes trivial if we let \( R \) be carried into the point at infinity by a transformation as in section 1.4. Note that the circles themselves need not converge.

2. If \( C \) and \( C' \) are two circles, then \( \kappa [C;C'] \) means "the angle between \( C \) and \( C' \)."
have the point \( P \neq p \) in common. Hence \( C \) and \( C(P; \mathcal{T}) \) are identical, i.e. \( C \in \mathcal{T} \).

**Corollary 1.** If \( C(P; \mathcal{T}) \) and \( C(Q; \mathcal{T}) \) have another point in common, they are identical; thus there is one and only one circle of \( \mathcal{T} \) through each point \( P \neq p \).

While this is an immediate corollary of Theorem 2.3, it has a more basic proof which is worth noting, namely:

Suppose that \( C(P; \mathcal{T}) \) and \( C(Q; \mathcal{T}) \) have another point \( R \neq p \) in common. Then before the limit is reached, \( C(P, t, p) \) \((t \in A, t \neq p, t \to p)\) and \( C(Q, t, p) \) must have a point \( R' \) close to \( R \) in common. Since these two circles now have three points, \( t, p \) and \( R' \) in common, they are identical. Hence in the limit, \( C(P; \mathcal{T}) = C(Q; \mathcal{T}) \).

**Theorem 2.4.** Suppose \( A \) satisfies Condition I at \( p \).

Let \( \mathcal{W} \) be a pencil of the second kind with fundamental point \( p \). If \( t \to p (t \in A, t \neq p) \), and if \( \mathcal{W} \neq \mathcal{T} \), then

\[
\lim_{t \to p} C(t; \mathcal{W}) = p.
\]

**Proof:** If this statement were false, there would exist
a circle, C, such that $p \in C$, and a sequence of points $t \to p$, $t \in A$ such that $C(t; \Pi) \not\subset C$. Let $C'$ and $C''$ be the two circles of $\Pi$ which touch $C$ (cf. Fig 2.1). If we orient $C$ and $C'$ in such a way that $C$ lies in the closure of $\bar{C}' \cap C''$, then $C(t; \Pi) \subset (\bar{C}' \cap C'') \cup p$. Hence $t \in \bar{C}' \cap C''$.

![Diagram](image)

**Fig. 2.1**

Now let there be any sequence of points $Q \to p$, $Q \in (\bar{C}' \cap C'')$.

Let $C_Q$ be any accumulation circle of $C(P,Q,p)$ where $P \subset C', P \neq p$. If $\xi \left[ C_Q; C' \right] \neq 0$, then there is a small neighbourhood of the point $p$ in which $\bar{C}' \cap C''$ is void of any part of the circle $C_Q$.

1. If $X$ and $Y$ are two classes of elements, then $X \cap Y$ denotes the set of all elements in both $X$ and $Y$; $X \cup Y$ denotes the elements in either $X$ or $Y$ or both $X$ and $Y$. 
Therefore, if $C(P,Q,p)$ is very close to $C_Q$, it does not pass through $C' \cap C'$ in the immediate neighbourhood of the point $p$, and hence $Q \not\in C' \cap C'$. This contradiction leads us to the conclusion that $\mathfrak{L} [C_Q, C'] = 0$. Since $C_Q$ and $C'$ have the point $P \neq p$ in common, we see that $C_Q = C(P; \Pi)$, and is therefore unique. In particular, since $t \subset C' \cap C'$, $C(P,t;p) \rightarrow C(P;\Pi)$, i.e. $C(P;\Pi) = C(P;\Pi)$. This again is a contradiction. Thus if $\Pi \neq \Pi$, $C(t;\Pi) \rightarrow p$.

Theorem 2.5. Suppose A satisfies Condition I and Condition II at $p$. Then $C(p) \in \Pi$.

Proof: If $C(p) = p$, it belongs to $\Pi$ by definition. Suppose $C(p) \neq p$. Then $C(p)$, being the limit of a sequence of circles, each of which touches a given circle of $\Pi$, must itself touch this circle of $\Pi$ at $p$. Hence $C(p) \in \Pi$.

Corollary 1. If $P \subset C(p)$, $P \neq p$, then $C(p) = C(P;\Pi)$.

2.4. The Independence of the Differentiability Conditions

Condition I and Condition II are independent, as is shown by the following example. Introducing a rectangular
Cartesian coordinate system, we let the arc $A$ be defined by the equations

$$x = t, \quad y = \begin{cases} (1 - \sqrt{1-t^2}) \sin t^{-1}, & 0 < |t| \leq 1 \\ 0, & t = 0 \end{cases}$$

The curve lies between the two circles $x^2 + (y \pm 1)^2 = 1$ as shown in Fig 2.2. We examine the point $t = 0$ for differentiability.

Since we are only interested in values of $t$ close to zero, we may expand $\sqrt{1-t^2}$ by the Binomial Theorem,

i.e. $\sqrt{1-t^2} = (1-t^2)^{\frac{1}{2}} = [1 - \frac{1}{2} t^2 + o(t^3)]$. Let $P(x_1, y_1) \neq (0,0)$.

1. \[ \lim_{t \to 0} \frac{o(t^n)}{t^n} = 0 \]
Then the equation of the circle \( C(P,t,p) \) may be written
\[
\begin{vmatrix}
x^2 + y^2 & x & y \\
x_1^2 + y_1^2 & x_1 & y_1 \\
t^2 + o(t^3) & t & o(t)
\end{vmatrix} = 0
\]
We remove the common factor \( t \) and let \( t \to 0 \), obtaining
\[
\begin{vmatrix}
x^2 + y^2 & y \\
x_1^2 + y_1^2 & y_1
\end{vmatrix} = 0
\]
Thus Condition I holds.

Condition II, however, does not hold. The equation of \( C(t;\bar{t}) \) when \( t \) is close to 0 is
\[
\begin{vmatrix}
x^2 + y^2 & y \\
t^2 + o(t^3) & \frac{1}{2}t^2 \sin t^{-1} + o(t^3)
\end{vmatrix} = 0
\]
Removing the common factor \( t^2 \) and letting \( t \to 0 \), we obtain
\[
\lim_{t \to 0} (x^2 + y^2) \sin t^{-1} - 2y = 0.
\]
This circle does not converge. The fact that Condition II does not hold can also be seen from the fact that both of the circles \( x^2 + (y-1)^2 = 1 \) have points in common with any neighbourhood of \( t = 0 \). Thus the sequence \( C(t;\bar{t}) \) has two
accumulation circles, namely \( x^2 + (y \pm 1)^2 = 1 \).

2.5. Intersection and Support Properties of the Family of Non-Tangent Circles and the Family of Non-Osculating Tangent Circles.

Let \( p \) be a differentiable interior point of the arc \( A \).

**Theorem 2.6.** Every circle \( C \neq C(p) \) either supports or intersects \( A \) at \( p \).

**Proof:** If \( C \) neither supports nor intersects \( A \) at \( p \), then \( p \in C \), and there exists a sequence of points \( t \to p \), such that \( t \in A \cap C \) and \( t \neq p \). Let \( P \in C, P \neq p \). Then \( C = C(P, t, p) \) for each \( t \) in the sequence, and Condition I implies that \( C = C(P; \mathcal{T}) \).

Now \( C \in \mathcal{T} \) and still contains the above sequence of points, \( t \). Thus \( C = C(t; \mathcal{T}) \) for each \( t \) in the sequence, and Condition II implies that \( C = C(p) \).

**Theorem 2.7.** Non-tangent circles through \( p \) all intersect or all support \( A \) at \( p \).

**Proof:** Let \( C' \) and \( C'' \) be two non-tangent circles
through p. Suppose that \( C' \) and \( C'' \) intersect each other in two points (cf. Fig 2.3), and let their other point of intersection be \( P \). Suppose further that \( C' \) supports, 

![Diagram showing intersection of \( C' \) and \( C'' \) at points \( P \).]

while \( C'' \) intersects \( A \) at \( p \). No generality is lost when \( C' \) is oriented so that \( A \subseteq C' \). Thus the region \( A \cap C' \) is not void. Let \( t \in A \cap C' \cap C'' \). Then

\[
C(P, t, p) \subseteq (C' \cap C'') \cup (C' \cap C'') \cup P \cup P.
\]

If we allow \( t \) to approach \( p \), we obtain in the limit,

(2.1) \( C(P; t) \subseteq (C' \cap C'') \cup (C' \cap C'') \cup C \cup C'' \).

Considering now a sequence of points, \( t' \to p \), where \( t' \in A \cap C' \cap C'' \), we obtain symmetrically the relation

(2.2) \( C(P; t) \subseteq (C' \cap C'') \cup (C' \cap C'') \cup C' \cup C'' \).

Comparing relations (2.1) and (2.2), we are led to one of the
contradictions, \( C(P; \ell) = C' \) or \( C(P; \ell) = C'' \).

If \( C' \cap C'' = p \), we choose a third non-tangent circle, \( C''' \), which intersects \( C' \) in two points. Then \( C''' \) also intersects \( C'' \) in two points. Applying the above to \( C' \) and \( C''' \), and again to \( C''' \) and \( C'' \), we find that \( C', C'', \) and \( C''' \) either all support or all intersect \( A \) at \( p \).

**Theorem 2.8.** If \( C(p) \neq p \), every non-osculating tangent circle supports \( A \) at \( p \).

**Proof:** Let \( C \) be a non-osculating tangent circle of \( A \) at \( p \), and suppose that \( C \) intersects \( A \) at \( p \). Then \( A \cap C \) and \( A \cap \bar{C} \) are not void. If \( t \in A \cap C \), then by Theorem 2.3,

\[
C(t; \ell) \subset C \cup p.
\]

Hence, if \( t \to p \),

\[
(2.3) \quad C(p) \subset C \cup C.
\]

Letting \( t' \to p \), \( t' \in A \cap \bar{C} \), we obtain symmetrically,

\[
(2.4) \quad C(p) \subset \bar{C} \cup C.
\]

A comparison of relations (2.3) and (2.4) leads to the conclusion \( C(p) = C \), which is false. Therefore \( C \) supports \( A \) at \( p \).
Theorem 2.9. If \( C(p) = p \), the non-osculating tangent circles at \( p \) all support \( A \) at \( p \), or they all intersect \( A \) at \( p \).

Proof: Let \( C' \) and \( C'' \) be two non-osculating tangent circles at \( p \). For the sake of argument, we shall assume that \( C' \) supports \( A \) at \( p \) while \( C'' \) intersects \( A \) at \( p \). We orient \( C' \) and \( C'' \) so that \( C'' \subseteq \overline{C'} \) and \( C' \subseteq C'' \) (cf. Fig 2.4).

Then \( A \cap C' \) is not void. Let \( t \in A \cap C' \cap C'' \), so that \( C(t; \mathcal{L}) \subseteq (C' \cap C'') \). As \( t \to p \), the circle \( C(t; \mathcal{L}) \) will lie in the latter region bounded by the two proper circles \( C' \) and \( C'' \). Consequently, \( C(t; \mathcal{L}) \) cannot converge to \( p \).

2.6. A Classification of the Differentiable Points

The preceding section yields a classification of the differentiable points of plane curves (cf. Table 2.1). The
first four and last four examples refer to the curves

\[ x = t^n, \quad y = t^{n+m}, \]

while the middle two examples refer to the curves

\[ x = t^n, \quad y = \begin{cases} \frac{t^{n+m}}{\sin t^{-1}}, & 0 < |t| < 1 \\ 0, & t = 0 \end{cases} \]

In each case we examine the differentiability properties of the point \( t = 0 \). By the method used in section 2.4, we find that the point \( t = 0 \) is differentiable in each case. This method also gives us the pencil \( \mathcal{T} \), and tells us whether or not \( C(p) \) is a point-circle. Support and intersection properties can be determined in many cases simply by a consideration of symmetry.

We introduce the characteristic \((a_0, a_1, a_2; i)\), with the following properties:

- \( i = 1 \) or \( 2 \).
- \( a_0 = 1 \) or \( 2 \).
- \( a_1 = 1 \) or \( 2 \).
- \( a_2 = 1, 2, \) or \( \infty \).

\( i = 1 \) if \( C(p) \neq p \); \( i = 2 \) if \( C(p) = p \).

\( a_0 \) is even or odd, according as the non-tangent circles
support or intersect.

\[ a_0 + a_1 \] is even or odd, according as the non-osculating tangent circles support or intersect.

\[ a_0 + a_1 + a_2 \] is even if \( C(p) \) supports and odd if \( C(p) \) intersects, while \( a_2 = \infty \) if \( C(p) \) neither supports nor intersects.

Theorem 2.8 imposes a restriction on the characteristic, namely; if \( i=1 \), \( a_0 + a_1 \) is even. The convention that the point-circle \( p \) always supports yields a further restriction, that is; if \( i=2 \), then \( a_0 + a_1 + a_2 \) is even.

With the above restrictions in mind, we see that when \( i=1 \) we have two choices for \( a_0 \), one choice for \( a_1 \), and three choices for \( a_2 \), a total of six different choices for the characteristic. If \( i=2 \), we have two choices for \( a_0 \), two for \( a_1 \), and one choice for \( a_2 \), four choices in all. Thus we have ten different types of differentiable points.

All congruences in Table 2.1 are mod 2.

It is interesting to note that all the tangent
circles (including \( C(p) \)) support if and only if \( a_2 = 2 \).

Curves containing the various types of differentiable points are illustrated in Figs. 2.5 to 2.10 inclusive. The curves are identified by the characteristic of the point \((0,0)\) through which they pass. The relationship of each arc to a tangent circle is depicted by superimposing a non-osculating tangent circle upon each diagram.
<table>
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<th>i</th>
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<td>support</td>
<td>$\neq p$</td>
<td>intersects</td>
<td>$a_0 + a_1 = 0$</td>
</tr>
<tr>
<td></td>
<td>$(1,1,2;1)$</td>
<td></td>
<td></td>
<td>supports</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(2,2,1;1)$</td>
<td></td>
<td></td>
<td>intersects</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(2,2,2;1)$</td>
<td></td>
<td></td>
<td>supports</td>
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</tr>
<tr>
<td></td>
<td>$(1,1,\infty;1)$</td>
<td></td>
<td></td>
<td>neither supports nor intersects</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(2,2,\infty;1)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$(1,1,2;2)$</td>
<td>support</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(1,2,1;2)$</td>
<td>intersect</td>
<td>$= p$</td>
<td>supports</td>
<td>$a_0 + a_1 + a_2 = 0$</td>
</tr>
<tr>
<td></td>
<td>$(2,1,1;2)$</td>
<td>intersect</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(2,2,2;2)$</td>
<td>support</td>
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Table 2.1
CHAPTER III

CHARACTERISTIC AND ORDER OF DIFFERENTIABLE POINTS IN THE
CONFORMAL PLANE

3.1. Introduction.

In this chapter, various theorems dealing with the cyclic orders of points and arcs will be discussed. The close connection between the characteristic of a differentiable point and the order of that point will be brought out. It will then become evident why the particular form, \((a_0, a_1, a_2; i)\), for the characteristic was chosen.

3.2 Arcs of Finite and Bounded Cyclic Order

An arc \(A\) is said to be of finite cyclic order if it has only a finite number of points in common with any circle. If some circle meets \(A\) \(n\) times and no circle meets \(A\) more than \(n\) times, where \(n\) is some specific integer,
then A is said to be of bounded cyclic order, and n is called the (cyclic) order of A. If p is any point on A, the order of p is the minimum of the orders of all the neighbourhoods of p on A.

Lemma 3.1. Let A be an arc of finite order, and let a circle C intersect A at a point p. Then any circle C', sufficiently close to C, also intersects A, and does so in an odd number of points close to p.

Proof: Since C intersects A at p, the end-points of a sufficiently small neighbourhood M, of p, lie in opposite regions with respect to C. Hence they lie on opposite sides.

1. It should be noted that there is a difference between an arc of bounded cyclic order and one of only finite cyclic order. It is possible, in the case of an arc of finite cyclic order, to find for each circle through a finite number of points on the arc, another circle through a still greater, but finite, number of points. An arc with such a property would not be of bounded order.
of \( C' \). Since \( C' \) meets \( M \) a finite number of times, it must intersect \( M \) an odd number of times.

3.3. Characteristic and Order.

The following theorem illustrates in part the reason for choosing the characteristic in the form given in Chapter II. Theorem 3.5 will sharpen this theorem and complete the investigation of characteristic and order.

Theorem 3.1. Let \( p \) be a differentiable interior point of an arc \( A \). Suppose that \( p \) has the characteristic \((a_0, a_1, a_2; i)\). Then the order of \( p \) is not less than \( a_0 + a_1 + a_2 \).

This theorem is trivially true if \( a_2 = \infty \) (cf. \( \S \) 2.6), for then every neighbourhood of \( p \) on \( A \) has an infinite number of points in common with \( C(p) \). For this reason we confine our ensuing proof to the case \( a_2 < \infty \). The proof follows after the discussion in section 3.3.1.

3.3.1. Let \( \Pi_2 = \mathcal{T} \) be the pencil of tangent circles through \( p \), where \( C(p; \Pi_2) = C(p) \). Let \( \Pi_1 \) be a pencil of the first kind with \( p \) as one of its fundamental points, and let
\( C(p; \pi_1) \), which is a member of \( \mathcal{T} \), be different from \( C(p) \).

Finally, let \( \mathcal{P}_0 \) be a pencil of the first kind where 
\( C(p; \mathcal{P}_0) \notin \mathcal{T} \). Then \( p \) is not one of the fundamental points of \( \mathcal{P}_0 \).

**Lemma 3.2.** The pencil \( \mathcal{P}_j \) \((j = 0, 1, 2)\) contains circles arbitrarily close to, but different from, \( C_j \), \( C(p; \mathcal{P}_j) \), which meet a neighbourhood \( M \) of \( p \) in not less than \( a_j \) points outside \( p \). If the order of \( p \) is finite, and if \( M \) is small enough, \( C \) can be chosen so that the number of intersections of \( M \) with \( C \) exceeds \( a_j \) by a non-negative even integer.

**Proof:** Let \( D_j \in \mathcal{P}_j \), \( D_j \neq C_{j+1} \). If \( j = 2 \) and \( C(p) = p \), we make the convention that \( C_3 \) does not exist, and that \( C_3 \) is the whole plane with the exception of the point \( p \) itself.

We now define the regions

\[
E_j = (C_{j+1} \cap D_j) \cup (\bar{C}_{j+1} \cap \bar{D}_j),
\]

and

\[
\bar{E}_j = (C_{j+1} \cap D_j) \cup (\bar{C}_{j+1} \cap \bar{D}_j)
\]
(cf. Fig. 3.1).

Let $\Pi_j$ ($\overline{\Pi}_j$) denote the set of those circles of $\Pi_j$ that pass through $E_j$ ($\overline{E}_j$). Then every circle of $\Pi_j$, with

\begin{align*}
(a) \quad & j = 2, \quad C(p) \neq p \\
(b) \quad & j = 2, \quad C(p) = p \\
(c) \quad & j = 1 \\
(d) \quad & j = 0
\end{align*}
the exception of $C_{j+1}$ and $D_j$, belongs either to $\overline{\Pi}_j$ or to $\overline{\Pi}_j$.

If we intersect $\overline{\Pi}_j$ with an orthogonal circle, we establish a 1-1 correspondence between the circles of $\overline{\Pi}_j$ and the points of the orthogonal circle, and thus we can speak of a "betweenness" relation in $\overline{\Pi}_j$ ($\overline{\Pi}_j$).

The neighbourhood $M$ of $p$ is decomposed by $p$ into two one-sided neighbourhoods $N$ and $N'$. We can choose our $M$ so small that neither of the neighbourhoods $N$ and $N'$ have points in common with $C_{j+1}$ or with $D_j$. Thus $N$ ($N'$) lies entirely in one of the two regions $E_j$ and $\overline{E}_j$. Let $t$ and $t'$ denote the points of $N$ and $N'$ respectively. Thus all the circles $C(t;\overline{\Pi}_j)$ belong to $\overline{\Pi}_j$ or else to $\overline{\Pi}_j$. We lose no generality in supposing that $N \subseteq \overline{C}_{j+1} \cap \overline{D}_j \subseteq \overline{E}_j$. Then $C(t;\overline{\Pi}_j) \subseteq \overline{\Pi}_j$ for every $t$.

Let $e$ be the end-point of $N$ distinct from $p$. Then $C(e;\overline{\Pi}_j)$ is the end-circle of a one-sided neighbourhood $\nu$ of $C_{j+1}$ in $\overline{\Pi}_j$. If $t$ moves from $e$ to $p$, then $C(t;\overline{\Pi}_j)$ moves con-
tinuously in $\Pi_j$ from $C(e; \Pi_j)$ to $C_{j+1}$; hence every circle of $\nu$ meets $N$.

Let $C$ be some fixed circle belonging to $\nu$. Then, if $t$ is sufficiently close to $p$, $C(t; \Pi_j)$ is so close to $C_{j+1}$ that $C$ lies between $C(t; \Pi_j)$ and $C(e; \Pi_j)$. Thus the points $t$ and $e$ are separated by $C$, and since the sub-arc $N$ is of finite order, $C$ must intersect $N$ at least once. By Lemma 3.1, $C$ intersects $A$ an odd number of times.

Similarly, if $t' \in N'$, the circles $C(t'; \Pi_j)$ comprise a one-sided family of circles $\nu'$, bounded by $C_{j+1}$ and $C(e; \Pi_j)$, where $p$ and $e$ are the end-points of $N'$. There is a circle $C' \in \nu'$ which intersects $N'$ an odd number of times.

Now if $a_j = 1$, one of the circles $C_{j+1}$ and $D_j$ supports $A$ at $p$, while the other one intersects $A$ at $p$ (cf. §2.6); hence $N' \in E_j$. Thus $C$ does not meet $N'$, and meets $N$ an odd number of times. On the other hand, if $a_j = 2$, both of the circles $C_{j+1}$ and $D_j$ intersect $A$ at $p$, or they both support
A at p; hence \( N' \in \overline{E_j} \). Thus if \( C \) is sufficiently close to \( C_{j+1} \), it will meet both \( N \) and \( N' \), an odd number of times each. This completes the proof of Lemma 3.2.

3.3.2. We are now in a position to prove Theorem 3.1. We proceed by first approximating \( C(p) \) by another tangent circle, then the latter by a non-tangent circle through \( p \), and finally that circle by one which does not contain \( p \).

Let \( M_2 \subset M \) be a neighbourhood of \( p \) on \( A \). By Lemma 3.2, there exists a non-osculating tangent circle \( C_2 \) which is close to \( C(p) \) and intersects \( M_2 \) at least \( a_2 \) times outside \( p \). Now let \( M_1 \subset M_2 \) be a neighbourhood of \( p \) which contains none of the points of intersection of \( C_2 \) with \( M_2 \) (except \( p \), if it is a point of intersection). Again by Lemma 3.2, there exists a non-tangent circle \( C \), which intersects \( M_1 \) in at least \( a_1 \) points outside \( p \). Finally, let \( M_0 \subset M_1 \) be a neighbourhood of \( p \) which contains none of the points of intersection of \( C_1 \) with \( M_1 \) (except \( p \), if it is a point of intersection).
Using Lemma 3.2 once more, we find that there exists a circle \( C_0 \), not passing through \( p \), which intersects \( M_0 \) in at least \( a_0 \) points. Altogether, \( C_0 \) meets \( M \) at least \( a_0 + a_1 + a_2 \) times.

As a consequence of the proof of Theorem 3.1, we have

**Corollary 1.** If the order of the differentiable point \( p \) is bounded, then there exists to every neighbourhood of \( p \) a circle arbitrarily close to \( C(p) \) which does not pass through \( p \), and which intersects that neighbourhood in not less than \( a_0 + a_1 + a_2 \) points.

3.4. **Two Lemmas On Arcs of Finite Cyclic Order.**

**Lemma 3.3.** Let \( A \) be an arc of finite cyclic order.

If the parameter \( t_n \) tends to one of the end-points of the parameter interval, then the sequence of points \( t_n \) converges.

**Proof:** Let \( \lim_{\nu \to \infty} t_{2\nu} = p \) and \( \lim_{\nu \to \infty} t_{2\nu+1} = q \) be any two accumulation points of the sequence \( t_n \). We may assume that \( t_{n+1} \) lies between \( t_n \) and \( t_{n+2} \) for all \( n \). If \( p \neq q \),
let \( C \) be a circle separating these two points. Thus there is a number \( N = N(C) \) such that \( t_{2\nu} \) and \( t_{2\nu+1} \) are separated for all \( \nu > N \). But this implies that the arc \( A \) meets \( C \) an infinite number of times, which is not true. Hence \( p = q \).

Lemma 3.4. Let \( p \) be an end-point of an arc \( A \) of finite cyclic order. Then the arc \( A \) is differentiable at \( p \).

Proof: Suppose Condition I of section 2.2 is not satisfied. Let \( t_{2\nu} \) and \( t_{2\nu+1} \) be two sequences of points converging to \( p \) such that \( C(P,t_{2\nu},p) \to C_o \) and \( C(P,t_{2\nu+1},p) \to C_1 \neq C_o \) (\( P \neq p \)). We may assume that \( t_{n+1} \) lies between \( t_n \) and \( t_{n+2} \) on \( A \). Let \( C' \) and \( C'' \) be two circles through \( P \) and \( p \) which separate \( C_o \) and \( C_1 \) (cf. Fig. 3.2). Then, for each \( n \) sufficiently large, \( C' \) and \( C'' \) separate \( C(P,t_n,p) \) and \( C(P,t_{n+1},p) \). Hence at least one of the circles \( C' \) and \( C'' \) meets the arc \( A \) an infinite number of times, contrary to our hypothesis. Thus Condition I holds.

Now let us suppose that Condition II of section 2.2 does not hold. Let \( t_{2\nu} \) and \( t_{2\nu+1} \) be two sequences of points
As before, we assume that \( t_{n+1} \) lies between \( t_n \) and \( t_{n+2} \) on \( A \). Both of the circles \( C_0 \) and \( C_1 \), being the limit of sequences of tangent circles, are themselves tangent circles, and by Theorem 2.3, they touch at \( p \).

Suppose first of all, that \( C_0 \) and \( C_1 \) are both pro-
per circles (cf. Fig. 3.3). We may orient \( C_0 \) and \( C_1 \) in such a way that \( C_1 \subset C_0 \cup p \) and \( C_0 \subset \overline{C}_1 \cup p \). Consider a circle \( C \in \mathcal{C} \) \( (C \subset (C_0 \cap \overline{C}_1) \cup p) \) oriented so that \( C_1 \subset C \cup p \) and \( C_0 \subset \overline{C} \cup p \). Then, for sufficiently large \( v \), \( C(t_{2v+1}; \mathcal{C}) \subset C \cup p \), and \( C(t_{2v}, \mathcal{C}) \subset \overline{C} \cup p \). Here again the arc \( A \) crosses \( C \) an infinite number of times, which is impossible.

If now, \( C_1 \) for instance is the point-circle \( p \), consider two circles of \( \mathcal{T}, C \) and \( C' \) \( (C \subset C_0 \cup p, C' \subset \overline{C}_0 \cup p) \), oriented in such a way that \( C_0 \subset (\overline{C} \cap C') \cup p \) (cf. Fig. 3.4).

![Fig. 3.4](image)

Then for sufficiently large \( v \), \( C(t_{2v}; \mathcal{C}) \subset (\overline{C} \cap C') \cup p \), while \( C(t_{2v+1}; \mathcal{C}) \subset C \cup \overline{C}' \cup p \). Since these two regions are
separated by $C$ and $C'$, one or both of these circles will meet $A$ an infinite number of times as $v \to \infty$. Since this too is impossible by our hypothesis, Condition II holds, and the point $p$ is differentiable.

3.5. Arrows of Order Three.

Since any three distinct points define a circle, the cyclic order of any arc is at least three. The remainder of this chapter is directly concerned with arcs of order three. We shall denote such an arc by the symbol $A_3$. The two lemmas of the previous section are true in particular of arcs of order three.

3.6. General Tangent Circles.

Let $A_3$ be an arc of order three with end-point $p$, and let $q \in A_3 \cup p$. We call a circle $C$ a general tangent circle at the point $q$, if there exists a sequence of triplets of mutually distinct points, $q_v, q'_v, Q_v$, such that $q_v$ and $q'_v$ converge on $A_3$ to $q$, and that

$$\lim C(Q_v, q_v, q'_v) = C.$$
If, in addition, $Q_\nu$ converges on $A_3$ to $q$, then we call $C$ a **general osculating circle** at $q$. If we let the sequence $q'_\nu$ be the single point $q$, and let $Q_\nu$ be a single point $Q \neq q$, $C$ is then an ordinary tangent circle of $A$ at $q$. Hence an ordinary tangent circle is a general tangent circle. Let $Q_\nu \rightarrow Q \neq q$, and let $q_\nu$ and $q'_\nu \rightarrow q$. Choose any neighbourhood of $q$ on $A_3$. Then if $C(q_\nu,q'_\nu,Q_\nu) \rightarrow C$, a non-osculating general tangent circle of $A_3$ at $q$, and if $q_\nu$ and $q'_\nu$ are sufficiently close to $q$, the end-points of the above neighbourhood will lie in the same region with respect to $C(q_\nu,q'_\nu,Q_\nu)$, and hence will lie in the same region with respect to $C$. Hence $C$ supports $A_3$ at $q$. By similar reasoning, we find that a general osculating circle intersects $A_3$ at $q$.

3.7. An Important Property of Arcs of Order Three.

We now introduce **multiplicities**; that is, we count the end-point $p$ of $A_3$ three times on $C(p)$ and twice on any other tangent circle at $p$, while we count a point $q \in A_3 \cup p$
three times on a general osculating circle at \( q \) and twice on a non-osculating general tangent circle at \( q \). We wish to prove the following theorem:

**Theorem 3.2.** No circle meets \( A_3 \cup p \) more than three times; i.e., the inclusion of \( p \) and the introduction of multiplicities does not alter the order of \( A_3 \).

The proof of Theorem 3.2 results from the lemmas proved in the remainder of section 3.7.

3.7.1. **Lemma 3.5.** If a circle \( C \) meets \( A_3 \) in two points, then at least one of these points is an intersection.

**Proof:** Let \( C \) meet \( A_3 \) in \( q_1 \) and \( q_2 \), and let \( M_1 \) and \( M_2 \) be small neighbourhoods of \( q_1 \) and \( q_2 \) respectively. If \( q_1 \) \((i = 1, 2)\) is a point of support, then there is a circle close to \( C \) which meets \( M_i \) in two points. Now if \( M_1 \) and \( M_2 \) are both in \( C \), say, then there is a circle \( C' \subset C \) so close to \( C \) that it intersects \( M_1 \) and \( M_2 \) twice each. This is impossible since \( A_3 \) is of order three.

On the other hand, if \( M_1 \subset C \) and \( M_2 \subset C \), then since
A₃ has points on either side of C, it must intersect C in some point q₃. Let C' be a circle through one point of M₁ and two points of M₂, where C' is close to C (cf. Fig. 3.5).

![Figure 3.5](image)

Now the end-points of a small neighbourhood M₃ of q₃ (M₃ is so small that it has no points in common with M₁ or with M₂) lie in opposite regions with respect to C. By section 1.9.1, they also lie in opposite regions with respect to C' when C' is sufficiently close to C. Hence C' intersects M₃.

Thus we have another contradiction, since C' can only meet A₃ three times at most.

Lemma 3.6. A circle C through three points of A₃
does not support \( A_3 \) at any of these points.

Proof: Lemma 3.5 implies that \( A_3 \cap C \) has at most one point of support. If \( C \) supports \( A_3 \) at one point of contact, and intersects \( A_3 \) in two other points, then there is a circle close to \( C \) which meets \( A_3 \) in at least four points, which cannot be true.

3.7.2. Suppose that a circle \( C \) through \( p \) meets \( A_3 \) in three points, \( q_1, q_2, \) and \( q_3 \). By Lemma 3.6 they are all intersections. Choose disjoint neighbourhoods \( N \) of \( p \) and \( M \) of \( q_1 \) on \( A \). If \( t \to p, t \in N \), then \( C(t, q_2, q_3) \to C \). By section 1.9.1, \( C(t, q_2, q_3) \) separates the end-points of \( M \) if \( t \) is sufficiently close to \( p \). Thus \( C(t, q_2, q_3) \) meets \( A_3 \) again in the neighbourhood of \( q_1 \). This contradiction yields

**Lemma 3.7.** No circle meets \( A_3 \cup p \) in four points.

Now suppose that a circle through \( p \) meets \( A_3 \) in two points, \( q_1 \) and \( q_2 \), and suppose further that \( q_2 \) is a point of support. By Lemma 3.5, \( q_1 \) is a point of intersection. Let
$M_1$ and $M_2$ be small neighbourhoods of $q_1$ and $q_2$ respectively.

Let $C'$ be a circle through $p$ and two points of $M_2$. Then if $C'$ is sufficiently close to $C$, it will intersect $M_1$, thus meeting $A_3 \cup p$ in four points. This again is not true.

Combining this result with Lemma 3.6, we generalize the latter lemma, obtaining

**Lemma 3.8.** A circle through three points of $A_3 \cup p$

does not support $A_3$ at any of these points.

3.7.3. Suppose that a circle $C \in \mathcal{T}$ meets $A_3$ in two points, $q_1$ and $q_2$. By Lemma 3.8 these points are both intersections.

Let $N$ and $M$ be disjoint neighbourhoods of $p$ and $q_1$ respectively. Let $t \in N$, $t \rightarrow p$. Then $C' = C(q_2, t, p)$, when it is close enough to $C$, meets $M$ in a point near $q_1$. Thus $C'$ meets $A_3 \cup p$ at least four times, contrary to Lemma 3.7.

This yields

**Lemma 3.9.** No circle of $\mathcal{T}$ meets $A_3$ in two points.

Suppose a circle $C$ of $\mathcal{T}$ supports $A_3$ at $q$. Then
Then there is a circle of \( \mathcal{U} \) close to \( C \) which intersects \( A_3 \) in at least two points. This contradicts Lemma 3.9, and we have

**Lemma 3.10.** If a circle of \( \mathcal{U} \) meets \( A_3 \), it does so in a point of intersection.

### 3.7.4. Suppose that \( C(p) \) meets \( A_3 \) at a point \( q \). By Theorem 2.5 and Lemma 3.10, \( q \) is a point of intersection. Let \( N \) and \( M \) be disjoint neighbourhoods of \( p \) and \( q \) respectively, and let \( t \in N, t \to p \). Then \( C(t;\mathcal{U}) \), when it is close to \( C(p) \), will meet \( M \), contradicting Lemma 3.9. Thus we have

**Lemma 3.11.** \( C(p) \) does not meet \( A_3 \).

### 3.7.5. Multiplicities Relative to General Tangent Circles.

In the following we shall not consider general tangent and osculating circles at \( p \), the end-point of \( A_3 \), since we shall later discover that they are identical with the ordinary tangent and osculating circles already discussed.

**Lemma 3.12.** Let \( C \) be a general non-osculating tan-
gent circle of \( A_3 \) at \( q \). Then \( C \) meets \( A_3 \cup p \) elsewhere in at most one point and that point is not a point of support.

**Proof:** By section 3.6, \( C \) supports \( A_3 \) at \( q \). Hence, by Lemma 3.8, \( C \) meets \( A_3 \cup p \) at most once outside \( q \). By Lemma 3.5, this point is an intersection if it is on \( A_3 \). If the point is \( p \) itself, Lemmas 3.10 and 3.11 prohibit multiplicities at \( p \).

**Lemma 3.13.** Let \( C \) be a general osculating circle of \( A_3 \) at \( q \). Then \( C \) does not meet \( A_3 \cup p \) elsewhere.

**Proof:** Being a general osculating circle, \( C \)

\[ \text{lim } C(q_v,q'_v,q''_v), \] where \( q_v, q'_v, \) and \( q''_v \) converge on \( A_3 \) to \( q \). Suppose \( C \) meets \( A_3 \cup p \) in another point \( r \neq q \). Then \( C(q_v,q'_v,q''_v) \) intersects the orthogonal circle to \( C \) through \( q \) and \( r \) in a point \( r_v \) converging to \( r \) (cf. Fig. 3.6). Thus

\[ C(q_v,q'_v,q''_v) = C(q_v,q'_v,r_v). \]

Let \( Q_1, Q_2, S, T \) be variable points, and let \( Q_1 \) and \( Q_2 \) converge to the same point \( Q; Q_1 \neq Q_2 \). Suppose there is
a circle separating $Q$ from both $S$ and $T$. Then

$$\lim x[C(Q_1, Q_2, S); C(Q_1, Q_2, T)] = 0$$

whether the circles $C(Q_1, Q_2, S)$ and $C(Q_1, Q_2, T)$ themselves converge or not. In particular,

$$\lim x[C(q_v, q_{v'}, r_v); C(q_v, q_{v'}, r_v)] = 0,$$

and since any accumulation circle of $C(q_v, q_{v'}, r_v)$ contains the point $r$ in common with $C = \lim C(q_v, q_{v'}, r_v)$,

$$\lim C(q_v, q_{v'}, r_v) = C.$$
But $C(q_\nu,q_\nu',r)$, if it is sufficiently close to $C$, does not separate the end-points of any small neighbourhood of $q$.

This follows from the fact that $A_3$ has order three. Thus in the limit, $C$ supports $A_3$ at $q$, contradicting the last sentence of section 3.6. Hence $C$ does not meet $A_3 \cup p$ outside $q$.

3.8. Strong Differentiability.

An arc $A$ is said to be strongly differentiable at a point $p$, if the following two conditions are satisfied:

**Condition I':** Let $R \neq p$, $R' \rightarrow R$. If the two distinct points $u$ and $v$ converge on $A$ to $p$, then the circle $C(R',u,v)$ always converges.

**Condition II':** If the three distinct points $u$, $v$, and $w$ converge on $A$ to $p$, then the circle $C(u,v,w)$ always converges.

Suppose that $R' = R$, $u = p$. Then $C(R',u,v)$ becomes $C(R,v,p)$, which converges to $C(R;\mathcal{L})$. Since the limit circle $C(R',u,v)$ does not depend on the choice of the sequences
u, v, and R', we see that

\[ \lim C(R', u, v) = C(R; L). \]

Similarly, we find that \( \lim C(u, v, w) = C(p) \), since

\[
C(p) = \lim_{v \to p} C(v; L) = \lim_{u \to p} \lim_{v \to p} C(u, v, p).
\]

Thus strong differentiability implies ordinary differentiability.

We now prove another important theorem, namely

**Theorem 3.3.** Let \( p \) be the end-point of an open arc \( A_3 \) of order three. Then \( A_3 \cup p \) is strongly differentiable at \( p \).

**Proof:** According to Lemma 3.4, \( A_3 \cup p \) is differentiable at \( p \). Let \( B \) be a sub-arc of \( A_3 \) bounded by \( p \) and \( e \), and let \( p, t, u, v, d, e, f \) lie on \( A_3 \cup p \) in the indicated order. We orient all circles \( C \), where \( f \not\in C \), in such a way that \( f \subset \overline{C} \).

The above conditions indicate that

\[
(1) \quad u \subset C(p, t, e) \cap \overline{C}(t, d, e)
\]

(cf. Fig. 3.7). Consequently,
Let $I$ denote the region in relation (3.3). From (3.3) we obtain

\[(3.4) \quad \lim_{t,u \to p} C(t,u,e) \subset \left[ \overline{C}(e;\ell) \cap \overline{C}(p,d,e) \right] \\
\cup \left[ \overline{C}(e;\ell) \cap C(p,d,e) \right] \cup C(e;\ell) \cup C(p,d,e).\]

By II, we shall mean the limit of $I$ as $t \to p$. Let $C$ be any accumulation circle of $C(t,u,e)$. As a point $r$ runs continuously on $B$ from $d$ to $p$, $C(p,r,e)$ runs continuously through the region II from $C(p,d,e)$ to $C(e;\ell)$. Conversely,
every circle through II and the points p and e meets B.

Hence if C passes through II \cup C(p,d,e), it intersects B at some point r, where r = d if C = C(p,d,e). But then C(t,u,e), when it is close to C, intersects B again near r, contrary to Theorem 3.2. Thus C = C(e; t).

Now let P be any point \neq p, and let C' be any accumulation circle of C(P,t,u). As in the proof of Lemma 3.13,

\[ \lim \mathcal{K}[C(P,t,u); C(t,u,e)] = 0, \]

that is,

\[ \mathcal{K}[C'; C(e; t)] = 0. \]

Thus, by Theorem 2.3,

\[ C' = C(P; t). \]

We now prove simultaneously that C(p,u,v) \rightarrow C(p), and assuming this, that C(t,u,v) \rightarrow C(p). Proceeding as we did previously, we note

(ii) \quad v \subset \overline{C}(u; t) \cap C(p,u,e)

(ii') \quad v \subset \overline{C}(p,t,u) \cap C(t,u,e).
Relations (ii) and (ii') yield

\[(3.5) \quad C(p, u, v) \subseteq \left( \overline{C}(u; t) \cap C(p, u, e) \right) \cup \left[ C(u; t) \cap \overline{C}(p, u, e) \right] \cup p \cup u, \]

and

\[(3.5') \quad C(t, u, v) \subseteq \left( \overline{C}(p, t, u) \cap C(t, u, e) \right) \cup \left[ C(p, t, u) \cap \overline{C}(t, u, e) \right] \cup t \cup u \]

respectively. Let \(III\) denote either the region in relation (3.5), or that in (3.5'). Relations (3.5) and (3.5') yield

\[(3.6) \quad \lim C(p, u, v) \subseteq \left( \overline{C}(p) \cap C(e; t) \right) \cup \left[ C(p) \cap \overline{C}(e; t) \right] \cup C(p) \cup C(e; t) \]

and

\[(3.6') \quad \lim C(t, u, v) \subseteq \left( \overline{C}(p) \cap C(e; t) \right) \cup \left[ C(p) \cap \overline{C}(e; t) \right] \cup C(p) \cup C(e; t) \]

respectively. By IV we shall mean the limit of III as \(u \to p\) (as \(t, u, \to p\)). Let \(C\) be any accumulation circle of \(C(p, u, v)\) (of \(C(t, u, v)\)). Since \(C\) is the limit of a sequence of tangent circles of \(A_3\) at \(p\), we see that \(C \in T\). As a point \(r \ldots\)
runs continuously on $B$ from $e$ to $p$, $C(r;\mathcal{L})$ runs continuously through the region IV from $C(e;\mathcal{L})$ to $C(p)$. Conversely, every tangent circle through IV meets $B$. Hence if $C$ passes through $IV \cup C(e;\mathcal{L})$, it intersects $B$ at some point $r$, where $r = e$ if $C = C(e;\mathcal{L})$. But then $C(p,u,v)$ ($C(t,u,v)$), when it is close to $C$, intersects $B$ again near $r$, contrary to Theorem 3.2. Thus $C = C(p)$.

**Corollary 1.** Let two distinct points $u$ and $v$ converge on $A_3 \cup p$ to $p$, and let $R' \to R$, $R \neq p$. Let $C_1$ ($C_2$) be a general tangent circle of $A_3 \cup p$ at $u$ through $R'$ (through $v$). Let $C'_2$ be a general osculating circle of $A_3$ at $u$. Then

(3.7) \[ \lim C_1 = C(R;\mathcal{L}) \]

(3.8) \[ \lim C_2 = \lim C'_2 = C(p). \]

**Proof:** We may assume that each of the above sequences of circles possesses an accumulation circle. $C_1$ can be replaced by a circle $C(R',u_1,u_2)$ close to $C_1$ such that $u_1$
and \( u_2 \) are distinct and converge with \( u \) to \( p \). Thus by Theorem 3.3,

\[
\lim C_1 = \lim C(R', u_1, u_2) = C(R; U).
\]

Similarly, \( C_2 \) and \( C'_2 \) can be replaced by circles \( C(v, u_1, u_2) \) and \( C(u_1, u_2, u_3) \) close to \( C_1 \) and \( C_2 \) respectively, such that \( u_1, u_2, \) and \( u_3 \) are distinct and converge with \( u \) to \( p \). Hence, by Theorem 3.3,

\[
\begin{align*}
\lim C_2 &= \lim C(v, u_1, u_2) \\
\lim C'_2 &= \lim C(u_1, u_2, u_3)
\end{align*}
\]

\[ C(p). \]

3.9. Differentiability Properties of Interior Points of \( A_3 \).

Theorem 3.4. Let \( u \) be a point of an open arc \( A_3 \) of order three. Then

(3.9) The two one-sided tangent circles of \( u \) through a fixed point \( R \neq u \) coincide. This implies that the points of \( A_3 \) all satisfy Condition I of section 2.2.

(3.10) The set of general tangent circles of \( u \) coincides with the pencil of tangent circles of \( u \). The set of general
osculating circles of \( u \) forms a closed interval in the pencil of all the tangent circles of \( u \), bounded by the two non-degenerate one-sided osculating circles of \( A_3 \) at \( u \).

In particular, there is one and only one general tangent circle of \( u \) through each point \( R \neq u \). This implies that the points of \( A_3 \) all satisfy Condition I' of section 3.8.

Proof of (3.9): We first consider the case \( R \in A_3 \).

Let \( B = B_1 \cup u \cup B_2 \) be a sub-arc of \( A_3 \) bounded by \( R = e \) and \( f \). Let \( C_1 = \lim_{t_1 \to u} C(t_1, u, e) \), \( t_1 \in B_1 \) (cf. Fig. 3.8) be distinct one-sided tangent circles of \( u \) through \( e \) (cf. Lemma 3.4). By section 3.6 and Lemma 3.5, \( C_1 \) supports \( A_3 \) at \( u \),
intersects $A_3$ at $e$, and has no other point in common with $A_3$. Hence we may assume that $B_1 \cup B_2 \subset C_1 \cap C_2$. Let $C$ be any circle through $u$ and $e$ which passes through the region $(C_1 \cap C_2) \cup (\overline{C_1} \cap \overline{C_2})$. Thus $C$ supports $B$ at $u$. Let $\Pi$ be the pencil of the second kind of the circles which touch $C$ at $u$. By Theorem 2.4, applied to $B_1$,

$$\lim_{t_1 \to u} C(t_1; \Pi) = u \quad \text{for } t_1 \in B_1.$$ 

Conversely, every sufficiently small circle of $\Pi$ which meets $B_1$ meets $B_2$, and does not separate $e$ and $f$. Hence this circle meets $B$ on one hand three times, and on the other hand, with an even multiplicity, i.e., it meets $B \subset A_3$ at least four times, contrary to Theorem 3.2. Thus the two one-sided tangent circles coincide in the circle $C(e; \Pi)$. Let $C'(R; \Pi)$ and $C''(R; \Pi)$ be the two one-sided tangent circles of $A_3$ at $u$ through a point $R \not\in A_3$. Since

$$\vartheta[C'(R; \Pi); C(e; \Pi)] = 0,$$

and

$$\vartheta[C''(R; \Pi); C(e; \Pi)] = 0,$$
it is true that

\[ \xi [\xi (R; R); \xi''(R; R)] = 0, \]

and since these two circles have the point \( R \neq u \) in common, they coincide. This completes the proof of relation (3.9).

**Proof of (3.10):** Let \( C_1 = \lim_{t_1 \to u} C(t_1; \tau) \), \( t_1 \in B_1 \), be the two one-sided osculating circles of \( A_3 \) at \( u \). Since \( C(t_1; \tau) \) supports \( A_3 \) at \( u \), intersects \( A_3 \) at \( t_1 \), and does not meet \( A_3 \) elsewhere, \( C_1 \) intersects \( A_3 \) at \( u \). Thus \( C_1 \neq u \).

We may assume that \( B_1 \cup B_2 \) lies in \( \xi(e; \tau) \). By Theorem 3.2, \( C_1 \), considered as a general osculating circle of \( B \) at \( u \), has no point in common with \( A_3 \) except \( u \). Thus \( C_1 \subseteq \xi(e; \tau) \cup u \), and we may assume that \( C(e; \tau) \subseteq \overline{C}_1 \cup u \);

thus \( B_1 \subseteq \overline{C}_1 \cap \xi(e; \tau) \). Since \( C_1 \) intersects \( A_3 \) at \( u \), \( B_2 \subseteq \xi_1 \) (cf. Fig. 3.9). Since \( C(f; \tau) \) supports \( A_3 \) at \( u \), \( B_1 \cup B_2 \subseteq \overline{C}(f; \tau) \). Hence \( C_2 = \lim_{t_2 \to u} C(t_2; \tau) \), \( t_2 \in B_2 \), lies in the closure of \( \overline{C}(f; \tau) \cap \xi_1 \). Since \( C_2 \) does not meet \( A_3 \) outside \( u \), it either coincides with \( C_1 \), or it lies in \( (\xi_1 \cap \overline{C}(f; \tau)) \cup \xi \xi_1 \). The circles of the family \( \xi \) fall into
one of two classes: (i) Those tangent circles through a
point \( R \in \overline{C_1} \cup C_2 \) (together with the point-circle \( u \)) which
support \( A_3 \) at \( u \), and are therefore non-osculating general
tangent circles of \( u \); (ii) Those tangent circles through a
point \( R \in (C_1 \cap C_2) \cup C_2 \), \( R \neq u \), which intersect \( A_3 \) at \( u \), and are
therefore general osculating circles.

Conversely, every non-osculating general tangent
circle (every general osculating circle) of \( u \) is an ordi-
nary tangent circle of \( u \) lying in \( \overline{C_1} \cup C_2 \cup u \) (in \( (C_1 \cap C_2) \)).
\[ \mathcal{U} \cup ( \mathcal{C}_1 \cup \mathcal{C}_2 ) \]. We prove this statement as follows:

First, let \( \mathcal{C} \) be any non-osculating general tangent circle of \( \mathcal{U} \), and suppose that \( \mathcal{C} \not\in \mathcal{T} \). We know that \( \mathcal{C} \) supports \( \mathcal{A}_3 \) at \( \mathcal{U} \). Hence we may assume that a sufficiently small neighbourhood \( \mathcal{M} \) of \( \mathcal{U} \) on \( \mathcal{A}_3 \) lies in \( \mathcal{G} \), and even in \( \mathcal{G}(f;\mathcal{T}) \cap \mathcal{G}(e;\mathcal{T}) \cap \mathcal{G} \) (cf. Fig. 3.10). Let \( \mathcal{P} \) be the pencil of the second kind of the circles which touch \( \mathcal{C} \) at \( \mathcal{U} \). Since

\[
\lim_{t_1 \to \mathcal{U}} \mathcal{C}(t_1;\mathcal{P}) = \mathcal{U}, \quad t_1 \in B_1,
\]

every small circle of \( \mathcal{P} \) in \( \mathcal{G} \) meets both \( B_1 \) and \( B_2 \) and does not separate the end-points of \( \mathcal{M} \). Hence such a circle meets
M at least three times and with an even multiplicity, i.e.,
it meets $M$ at least four times. This again contradicts
Theorem 3.2. Thus every non-osculating general tangent cir-
cle of $u$ is an ordinary tangent circle of $u$. By (i), such
a circle lies in $\overline{C}_1 \cup C_2 \cup u$.

Next, let $C$ be a general osculating circle of $u$.

Let $C = \lim_{u_1,v_1,u_2 \to u} C'$, where $C' = C(u_1,v_1,u_2)$, $u_1,v_1 \in B_1$,
$u_2 \in B_2$. Obviously, $C$ cannot be the point-circle $u$, since
$C$ intersects $A_3$ at $u$. Let $R \subset C$, $R \neq u$, and suppose that
$C'$ intersects the orthogonal circle of $C$ through $u$ and $R$ at
$R'$. Thus $C' = C(u_1,v_1,R')$ and

$$C = \lim_{u_1,v_1,R' \to u} C(u_1,v_1,R') = C(R; \tau).$$

From (ii), $R \subset (\overline{C}_1 \cap \overline{C}_2) \cup C_1 \cup C_2$. Thus every general os-
culating circle is a (non-degenerate) tangent circle of $u$
lying in the closure of $\overline{C}_1 \cap \overline{C}_2$.

**Corollary 1.** If an interior point of an arc of order
three is differentiable, it is strongly differentiable.
Proof: By Theorem 3.4, Condition I' is satisfied for all interior points of $A_3$.

If the point $u \in A_3$ is differentiable, the two one-sided osculating circle coincide. Thus Theorem 3.4 implies that Condition II' also holds for the point $u$.


In this section we collect additional material on arcs of order three, needed for the proof of the final theorem in this chapter. Let $p$ be an end-point of $A_3$. The arc $B$ and the points $t, u, v, e, f$ are the same as in section 3.8.

3.10.1. We first extend formulas (3.5) and (3.5') to certain limit cases in which some of the points involved coincide. The circle $C(t, u, v)$ separates the regions

$$(3.11) \quad C(p, t, v) \cap C(t, v, e)$$

and

$$(3.12) \quad \bar{C}(p, t, v) \cap \bar{C}(t, v, e)$$

(cf. relation (3.5')).
Suppose that the distinct points \( t_0, v_0, e \) lie on \( B \cup e \) in the indicated order. Let \( C_0 \) be the general tangent circle of \( B \) at \( t_0 \) through \( v_0 \). Then \( C_0 \) can be obtained as the limit of circles \( C(t,u,v) \), if the triplets \( t,u,v \) converge to \( t_0,t_0,v_0 \). Since \( C(t,u,v) \) and the regions of \((3.5')\) depend continuously on \( t,u, \) and \( v \), \((3.5')\) implies that \( C_0 \) lies in the closure of the region

\[
R = \left[ C(p,t_0,v_0) \cap \overline{C}(t_0,v_0,e) \right] \\
\cup \left[ \overline{C}(p,t_0,v_0) \cap C(t_0,v_0,e) \right].
\]

As \( C_0 \) meets \( C(p,t_0,v_0) \) and \( C(t_0,v_0,e) \) only at \( t_0 \) and \( v_0 \), this implies that \( C_0 \subset R \cup t_0 \cup v_0 \). Replacing \( t_0 \) again by \( t \), and \( v_0 \) by \( v \), we thus have: the relation \((3.5')\) remains valid for \( u = t \) if \( C(t,t,v) \) is interpreted to mean the tangent circle of \( B \) at \( t \) through \( v \).

Similarly, \((3.5)\) and \((3.5')\) remain valid for \( u = v \), with the corresponding interpretation of \( C(t,v,v) \). Finally, these formulas remain valid for \( v = e \) if \( C(t,e,e) \) and \( C(p,e,e) \)
stand for the tangent circles of $A_3$ at $e$ through $t$ and $p$ respectively (cf. §3.9).

Let $v_1 \in B$, and let $C_1$ denote any general osculating circle of $B$ at $v_1$. Thus $C_1$ will be the limit of $C(t,u,v)$ if $t,u,v$ converge to $v_1$ in a suitable fashion. By section 3.9, the circles $C(p,t,v)$ and $C(t,v,e)$ are also convergent to the tangent circles $C_2$ and $C_3$ of $B$ at $v_1$ through $p$ and $e$ respectively. Furthermore, $p \in C_3$ and $e \in \overline{C}_2$ because of our orientation convention. This implies

$$C_2 \subseteq C_3 \cup v_1$$

and

$$C_3 \subseteq \overline{C}_2 \cup v_1.$$
From (3.5'), \( C_1 \) lies in the closure of
\[
(C_2 \cap \overline{C}_3) \cup (\overline{C}_2 \cap C_3)
\]
(cf. Fig. 3.11). Since \( C_2 \cap \overline{C}_3 \) is empty, and \( C_1 \neq C_2, C_3 \), this implies
\[
(3.13) \quad C_1 \subset (\overline{C}_2 \cap C_3) \cup \nu_1.
\]
Since each \( C(t,u,v) \) separates the regions (3.11) and (3.12), \( C_1 \) will separate \( C_2 \cap C_3 = \overline{C}_2 \) and \( \overline{C}_2 \cap C_3 = \overline{C}_3 \). Replacing \( \nu_1 \) by \( \nu \), we obtain: relation (3.5') remains valid, and
\( C(t,u,v) \) separates the regions (3.11) and (3.12) for \( t = u = v \), if \( C(v,v,v) \) is interpreted to mean any general osculating circle of \( B \) at \( v \), provided \( C(p,v,v) \) and \( C(v,v,e) \) stand for the tangent circles of \( B \) at \( v \) through \( p \) and \( e \) respectively.

3.10.2. Considering again relation (3.5') we observe that one of the regions (3.11) and (3.12) will lie in \( C(t,u,v) \), the other one in \( C(t,u,v) \). Since
\[
f \subset \overline{C}(p,t,v) \cap \overline{C}(t,v,e) \cap \overline{C}(t,u,v),
\]
this relation implies
(3.14) \[ C(p,t,v) \cap \overline{C}(t,v,e) \subseteq \overline{C}(t,u,v), \]
and therefore

(3.15) \[ C(p,t,v) \cap \overline{C}(t,v,e) \subseteq \overline{C}(t,u,v). \]

Specializing by letting \( t = p \), we obtain

(3.14') \[ \overline{C}(v;t) \cap \overline{C}(p,v,e) \subseteq \overline{C}(p,u,v), \]
and

(3.15') \[ \overline{C}(v;t) \cap \overline{C}(p,v,e) \subseteq \overline{C}(p,u,v). \]

Applying the case \( v = e \) of (3.14') and (3.15'), and replacing \( u \) by \( v \) afterwards, we obtain,

(3.16) \[ \overline{C}(e;t) \cap \overline{C}(p,e,e) \subseteq \overline{C}(p,v,e), \]
and

(3.17) \[ \overline{C}(e;t) \cap \overline{C}(p,e,e) \subseteq \overline{C}(p,v,e). \]

Now \( \overline{C}(e;t) \subseteq \overline{C}(v;t) \), since \( e \subseteq \overline{C}(v;t) \). Therefore, applying relations (3.16) and (3.14'), we have

(3.18) \[ \overline{C}(e;t) \cap \overline{C}(p,e,e) \subseteq \overline{C}(v;t) \cap \left[ \overline{C}(e;t) \cap \overline{C}(p,e,e) \right] \]
\[ \subseteq \overline{C}(v;t) \cap \overline{C}(p,v,e) \]
\[ \subseteq \overline{C}(p,u,v). \]
Similarly, $\mathcal{C}(t; T) \subseteq \mathcal{C}(v; T)$ when $t$ is close to $p$. Therefore, in the limit, $\mathcal{C}(p) \subseteq \mathcal{C}(v; T)$. Also, $\mathcal{C}(p) \subseteq \mathcal{C}(e; T)$. Hence, applying relations (3.17) and (3.15'), we have

$$\mathcal{C}(p) \cap \mathcal{C}(p, e, e) \subseteq \mathcal{C}(v; T) \cap \mathcal{C}(e; T) \cap \mathcal{C}(p, e, e)$$

$$\subseteq \mathcal{C}(v; T) \cap \mathcal{C}(p, v, e)$$

$$\subseteq \mathcal{C}(p, u, v).$$

3.10.3. Assume for the moment that $p, t, u, v$, are mutually distinct. The region

$$\mathcal{C}(p, t, u) \cap \mathcal{C}(p, u, v)$$

is bounded by two arcs of the circles $C(p, t, u)$ and $C(p, u, v)$ with the common end-points $p$ and $u$. Since $v \in \mathcal{C}(p, t, u)$ and

\[ \text{Fig. 3.12} \]
t \subset \overline{C}(p,u,v) \ (cf. \ Fig. \ 3.12), \ these \ arcs \ do \ not \ contain \ v
or \ t. \ Hence \ they \ meet \ C(t,u,v) \ only \ at \ u, \ and \ the \ region
(3.20) \ is \ contained \ in \ one \ of \ the \ two \ regions \ bounded \ by
C(t,u,v). \ Since \ the \ boundary \ point, \ p \ of \ the \ region \ (3.20)
lies \ in \ C(t,u,v), \ this \ implies

(3.21) \quad \overline{C}(p,t,u) \cap \overline{C}(p,u,v) \subset C(t,u,v).

The \ arguments \ of \ section \ 3.10.1 \ now \ show \ that \ relation
(3.21) \ remains \ valid \ if \ C(t,u,v) \ is \ any \ general \ tangent
circle, \ provided \ C(p,u,u) \ then \ stands \ for \ the \ tangent \ circle
at \ u \ through \ p.

By \ relations \ (3.15), \ (3.21), \ and \ (3.15'),

\begin{align*}
\overline{C}(t,u,v) & \supset \overline{C}(p,t,v) \cap \overline{C}(t,v,e) \\
& \supset \overline{C}(p,t,v) \cap [\overline{C}(p,t,v) \cap \overline{C}(p,v,e)] \\
& = \overline{C}(p,t,v) \cap \overline{C}(p,v,e) \\
& \supset [\overline{C}(v; \tau) \cap \overline{C}(p,v,e)] \cap \overline{C}(p,v,e) \\
& = \overline{C}(v; \tau) \cap \overline{C}(p,v,e).
\end{align*}

In \ particular, \ the \ above \ yields

(3.22) \quad \overline{C}(t,u,v) \supset \overline{C}(v; \tau) \cap \overline{C}(p,v,v).

3.10.4. \ Let \ \theta \ denote \ the \ pencil \ of \ the \ orthogonal \ circles \ of \ \tau.
On account of Theorem 3.3, B can be chosen so small that no circle of \( \Theta \) meets \( B \cup e \) more than once (otherwise this circle would approach a circle of \( \mathcal{L} \)). By Theorem 2.4,

\[
(3.23) \quad C(p;\Theta) = \lim_{t \to p} C(t;\Theta) = p.
\]

Thus, making B small enough, we may also assume that \( C(f;\Theta) \) does not meet B.

Since \( C(v;\mathcal{L}) \) meets the circle \( C_0 = C(t,u,v) \), the pencil \( \mathcal{L} \) contains a circle lying in \( C(v;\mathcal{L}) \cup C(v;\mathcal{L}) \) and touching \( C_0 \) from within, say at \( R \). Thus

\[
(3.24) \quad C(R;\mathcal{L}) \cap C_0 = R; \quad C(R;\mathcal{L}) \subseteq C_0 \cap C(v;\mathcal{L}).
\]

The circle \( C(R;\Theta) \) can be characterized as the unique circle of \( \Theta \) normal to \( C_0 \) (cf. Fig 3.13). We wish to prove the following

**Lemma 3.14.** \( C(R;\Theta) \) intersects \( B \).

**Proof:** Our proof derives from relation (3.23) and the fact that

\[
(3.25) \quad R \subseteq C(v;\Theta)
\]

**Proof of (3.25):** If the point \( t \) moves on \( B \cup p \)
from \( p \) to \( v \), \( C(p,t,v) \) moves from \( C(v; \mathcal{L}) \) to \( C(p,v,v) \) and passes through the closure of \( \mathcal{Q}(v; \mathcal{L}) \cap \mathcal{Q}(v; \theta) \). Hence \( C(p,t,v) \) does not pass through \( \mathcal{Q}(v; \mathcal{L}) \cap \overline{\mathcal{Q}}(v; \theta) \). Since \( \mathcal{Q}(v; \mathcal{L}) \) contains this region, so does \( C(p,t,v) \) and \( C(p,v,v) \). Hence, by relation (3.22),

\[
\mathcal{Q}(t,u,v) \supseteq \mathcal{Q}(v; \mathcal{L}) \cap \mathcal{Q}(p,v,v)
= \mathcal{Q}(v; \mathcal{L}) \cap [\mathcal{Q}(v; \mathcal{L}) \cap \overline{\mathcal{Q}}(v; \theta)]
= \mathcal{Q}(v; \mathcal{L}) \cap \overline{\mathcal{Q}}(v; \theta).
\]
Thus if \( R \neq v \), \( R \) does not lie in the above region. However, \( R \subset \mathcal{C}(v; t) \) in this case, and so
\[
R \subset \mathcal{C}(v; t) \cap \mathcal{C}(v; \theta),
\]
which proves relation (3.25). If \( R = v \), \( B \) can be made small enough to ensure that \( C(R; \theta) = C(v; \theta) \) intersects \( A_3 \) at \( v \).

If \( t = v \), \( C_0 \) is a general osculating circle of \( B \) at this point. Approximating \( C_0 \) by circles through three distinct points, and making use of the above, we observe that relation (3.25) remains valid unless \( t = u = v = R \). But in that case, \( C(v; t) \) touches \( C_0 \) at \( v \) and therefore is a tangent circle of \( B \) at \( v \), since \( C_0 \) is a tangent circle at \( v \). This is excluded by Theorem 3.2.

3.10.5. Any point \( Q \) induces an orientation of all the circles \( C \) with \( Q \notin C \), if \( C \) is defined through \( Q \subset C \).

Let \( Q \notin C(p) \), \( R \notin C(p) \), and let \( \overline{C} (\overline{C}) \) denote the orientation induced by \( Q \) (\( R \)). Suppose that \( \overline{C}(p) = \overline{C}(p) \).

Thus \( Q \subset \overline{C}(p) \) and \( R \subset \overline{C}(p) \). We vary \( C \) continuously, star-
ting from \( C(p) \). As long as \( C \) does not pass through \( Q \) or \( R \), \( \tilde{C} \) and \( \tilde{\tilde{C}} \) will depend continuously on \( C \). Thus we shall still have \( \tilde{C} = \tilde{\tilde{C}} \).

By Theorem 3.3, a circle \( C \) which meets \( A_3 \) three times in \( p \cup B \cup e \) lies close to \( C(p) \) if \( B \) is sufficiently small. Hence \( \tilde{C} = \tilde{\tilde{C}} \) for every such \( C \).

We specialize, letting \( Q = f \in A_3 \). Since \( f \not\in B \cup e \), the formulas of section 3.10 hold true. Thus they remain valid with respect to the orientation induced by \( R \), provided \( B \) is small enough. Since \( f \subset \overline{C}(p) \) is equivalent to \( A_3 \subset \overline{C}(p) \), this yields the following

**Lemma 3.15.** Suppose the point \( R \not\in C(p) \) induces an orientation with \( A_3 \subset \overline{C}(p) \). Then the formulas of section 3.10 remain valid for this orientation if \( B \) is small enough.

If \( A_3 \subset C(p) \) for the orientation induced by \( R \), then the above argument shows: replace each \( C \) and \( \tilde{C} \) in these formulas by \( \tilde{C} \) and \( \tilde{\tilde{C}} \) respectively. Then the resulting for-
mulas hold true if $B$ is small enough.

3.11. Conformally Elementary Points.

A point $p$ of an arc $A$ is said to be a conformally elementary point if there exists a neighbourhood of $p$ on $A$ which is decomposed by $p$ into two one-sided neighbourhoods of order three. By Theorem 3.3 their closures are strongly differentiable at $p$. The following theorem sharpens Theorem 3.1 in the case of conformally elementary points.

Theorem 3.5. Let $p$ be a differentiable conformally elementary point of an arc $A$, and let $(a_0, a_1, a_2; i)$ be the characteristic of $p$. Then $p$ has cyclic order $a_0 + a_1 + a_2$.

This theorem remains valid if a point $q \neq p$ is counted twice (three times) on any general tangent (osculating) circle of $q$, and if $p$ itself is counted $a_0 (a_0 + a_1; a_0 + a_1 + a_2)$ times on any circle through $p$ (on any tangent circle of $p$; on $C(p)$).

We may assume that $A$ itself is decomposed by $p$ into
two open arcs, \( A_3 \) and \( A_3' \), of order three. Hence the order of \( A \), and therefore that of \( p \), is not greater than six.

3.11.1. Let \( M \) be a neighbourhood of \( p \) on \( A \). For any circle \( D \), let \( M(D) = \mathcal{M}(D, M) \) denote the multiplicity with which \( D \) meets \( M \).

Lemma 3.16. Suppose the circle \( C \) does not pass through the end-points of \( M \). Then

\[
M(D) \equiv M(C) \pmod{2}
\]

for every \( D \) sufficiently close to \( C \).

Proof: Suppose \( C \) meets \( M \) at the points \( t \) with the multiplicities \( \sigma(t) \) and nowhere else. Thus

\[
M(C) = \sum_t \sigma(t).
\]

Construct disjoint neighbourhoods \( M_t \) in \( M \) about the points \( t \). The end-points of \( M_t \) lie on the same side or on opposite sides of \( C \) depending on whether \( \sigma(t) \) is even or odd. If \( D \) is sufficiently close to \( C \), then \( D \) will not pass through the end-points of \( M_t \), and these end-points will lie
on the same side of D if and only if they lie on the same side of C. On the other hand, D will meet \( M_t \) with an even or odd multiplicity according as its end-points lie on the same side or on opposite sides of D. Thus D will meet \( M_t \) with a multiplicity \( \rho(t) \equiv \sigma(t) \pmod{2} \) if D lies sufficiently close to C.

If each \( M_t \) is omitted from the closure of M, we obtain a closed set which has no points in common with C. Hence if D is sufficiently close to C this set does not meet D, and we have

\[
\mathcal{M}(D) = \sum_t \rho(t) \equiv \sum_t \sigma(t) = \mathcal{M}(C) \pmod{2}.
\]

3.11.2. We continue the discussion of section 3.11.1.

**Lemma 3.17.** Let \( C \neq C(p) \). Then

\[
(3.27) \quad \mathcal{M}(D) \leq \mathcal{M}(C)
\]

for every circle D sufficiently close to C, unless \( a_0 = a_1 = 1 \), \( C \in \mathcal{T} \), and \( p \notin D \).

**Proof:** Let \( t \in C \cap M; t \neq p \). Suppose that there is
a sequence of circles $D_\lambda$ converging to $C$, and a sequence of
eighbourhoods $M_\lambda$ of $t$ converging to $t$ such that each $D_\lambda$
meets $M_\lambda$ at least $\sigma_\lambda$ times ($\sigma_\lambda \leq 3$). Then each $D$ can be
replaced by another circle which meets $M_\lambda$ in not less than
$\sigma_\lambda$ distinct points, and such that the sequence of the new
circles also converges to $C$. Thus $C$ will meet $M$ at least $\sigma_\lambda$
times at $t$; i.e., $\sigma_\lambda \leq \sigma(t)$. Hence we have: there exists a
neighbourhood of $t$ on $M$ which is met not more than $\sigma(t)$ times
by every $D$ sufficiently close to $C$.

Let $p \subset C$, $C \not\subset \mathcal{T}$. Then $C$ meets $M$ at $p$ with a mul-
tiplicity $\equiv a_0 \pmod{2}$. On the other hand, by Theorem 3.3,
there exists a neighbourhood of $p$ which is met not more
than twice by any circle sufficiently close to $C$. But $p \subset C$;
hence $0 < \mathcal{M}(C) \leq 2$, and therefore $\mathcal{M}(C) = a_0$. Thus, by Lem-
ma 3.16, any circle $D$ which is sufficiently close to $C$ meets
a neighbourhood of $p$ with a multiplicity $\equiv a_0 \pmod{2}$.
Hence this multiplicity is $\leq a_0$, and we have relation (3.27)
Now let $C \in \mathcal{T}$, $C \neq C(p)$, and let $M_0 = N_0 \cup p \cup N'_0$ be a sufficiently small neighbourhood of $p$. Let $D$ be sufficiently close to $C$. If $p \in D$, $D \notin \mathcal{T}$, then $D$ will meet $N_0$ and $N'_0$ not more than once each. Hence $D$ meets $M_0$ with a multiplicity $\leq a_0 + 2$ and $\equiv a_0 + a_1 \pmod{2}$. Thus this multiplicity is $\leq a_0 + a_1$, and again we have relation (3.27).

Suppose now that $p \not\in D$. Then $D$ will meet $N_0$ and $N'_0$ not more than twice each. Hence $D$ meets $M_0$ with a multiplicity $\leq 4$ and $\equiv a_0 + a_1 \pmod{2}$. This again yields relation (3.27) unless $a_0 = a_1 = 1$.

3.11.3. Lemma 3.18. Let $A = A_3 \cup p \cup A'_3$. There exists a neighbourhood $M_3 = N_3 \cup p \cup N'_3$ ($N_3 \subseteq A_3$, $N'_3 \subseteq A'_3$) such that every tangent circle of $p$ which meets $N_3 \cup N'_3$ meets $A_3 \cup A'_3$ exactly $a_2$ times. In particular, no tangent circle of $p$ meets $M_3$ more than $a_2$ times outside $p$.

Proof: A circle of $\mathcal{T}$ meets $A_3$ or $A'_3$ not more than
once each. Thus it meets $A_3 \cup A_3'$ not more than twice. By Lemma 3.16, a circle will meet $A$ with a multiplicity $\equiv a_0 + a_1 + a_2$ (mod 2) if it is sufficiently close to $C(p)$. Hence $C(t; \mathcal{L})$ will meet $A_3 \cup A_3'$ with a multiplicity $\equiv a_2$ if it is close enough to $p$. Such a circle will therefore meet $A_3 \cup A_3'$ exactly $a_2$ times.

3.10.4. Lemma 3.19. There exists a neighbourhood $M_2 \subset M_3$ which is met at most $a_0 + a_1 + a_2$ times by every circle through $p$.

Proof: On account of Lemma 3.18, it suffices to consider non-tangent circles through $p$. Hence it suffices to construct a one-sided neighbourhood $N_2' \subset N_3'$ of $p$ such that any circle $D$ through $p$ that meets $N_2'$ twice will meet $M_3$ at most $a_0 + a_1 + a_2$ times.

By Lemma 3.16, $N_2'$ can be chosen so small that any such circle $D = C(u_1', u_2', p)$ ($u_1', u_2' \subset N_2'$) is so close to $C(p)$ that it meets $M_3$ with a multiplicity $\equiv a_0 + a_1 + a_2$ (mod 2).
Since $D$ meets $N_3$ and $N'_3$ not more than twice each, it will meet $M_3$ at most $a_0 + 4$ times. This yields our statement if $a_1 + a_2 > 2$.

Let $a_1 + a_2 = 2$, i.e., $a_1 = a_2 = 1$. Let $e$ denote the end-point of $N_3$, and suppose that the points $u, v, e$ lie on $N_3 \cup e$ in the indicated order. Making $N_2'$ still smaller, we may assume that it does not meet $C(p, e, e)$ (cf. §3.10). Obviously $N_2'$ has no points in common with $C(p)$ and $C(e; \mathcal{U})$.

We have

$$N_3 \subset \overline{C(p)} \cap \overline{C(e; \mathcal{U})} \cap \overline{C(p, e, e)}$$

(cf. Fig. 3.14). Since $a_1 = a_2 = 1$, it follows that
\[ N_2' \subset \mathcal{C}(p) \cap \mathcal{C}(p_e,e,e), \]

or else

\[ N_2' \subset \overline{\mathcal{C}}(e;e) \cap \overline{\mathcal{C}}(p,e,e). \]

Hence relations (3.19) and (3.18) imply that \( N_2' \) lies either in \( \mathcal{C}(p,u,v) \) or in \( \overline{\mathcal{C}}(p,u,v) \). Thus \( N_2' \) does not meet \( \mathcal{C}(p,u,v) \).

Any circle \( D \) through \( p \) and two points of \( N_2' \) meets \( M_3 \) with a multiplicity \( \equiv a_0 + 1 + 1 \pmod{2} \); i.e., it meets \( N_3 \cup N_3' \) an even number of times. It meets \( N_3' \) exactly twice.

From the above, \( D \) cannot meet \( N_3 \) twice. Hence \( D \) and \( N_3 \) are disjoint and \( D \) meets \( M_3 \) with the total multiplicity \( a_0 + 2 = a_0 + a_1 + a_2 \).

3.11.5. We can now prove Theorem 3.5 if \( a_0 + a_1 + a_2 > 4 \). It suffices to show that there is a one-sided neighbourhood \( N_1' \subset N_2' \) of \( p \) such that no circle \( D \) through three points of \( N_1' \cup p \) meets \( M_3 \) more than \( a_0 + a_1 + a_2 \) times. On account of Lemma 3.19, we need only consider circles \( D \) which do not pass through \( p \).
By Theorem 3.3 and Lemma 3.16, \( N_1 \) can be chosen such that any circle \( D \) meets \( M_3 \) with a multiplicity \( \equiv a_0 + a_1 + a_2 \pmod{2} \). Since \( p \not\in D \), and since \( D \) meets \( N_3 \) and \( N'_3 \) at most three times each, it will meet \( M_3 \) at most six times. This yields our assertion.

3.11.6. The case \( a_0 + a_1 + a_2 = 4; a_0 = 1 \). Let \( M_1 \subseteq M_2 \) be so small that the material in sections 3.10.4 and 3.10.5 can be applied to \( N_1 = M_1 \cap N_2 \) and \( N'_1 = M_1 \cap N'_2 \). Thus some circle of \( \Theta \) does not meet \( N_1 \cup N'_1 \). Since \( a_0 = 1 \), this circle will intersect \( M_1 \) at \( p \). Hence no circle of \( \Theta \) can meet both \( N_1 \) and \( N'_1 \). Thus if the circle \( C_0 \) meets \( N_1 \) in three points, the circle \( C(R; \Theta) \) intersects \( N_1 \) (cf. Lemma 3.14). However, \( C(R; \Theta) \) does not meet \( N'_1 \) and hence Lemma 3.14 implies that \( C_0 \) does not meet \( N'_1 \) three times. Taking section 3.10.5 into account, we can state: no circle meets \( M_1 \) more than five times.

By Theorem 3.3 and Lemma 3.16, a neighbourhood
$M_0 \subset M_1$ of $p$ exists such that every circle through three
points of $N_0 = M_0 \cap N_1$ or of $N'_0 = M_0 \cap N'_1$ meets $M_1$ with an
even multiplicity i.e. four times. By Lemma 3.19, any cir-
cle through more than four points of $M_0$ does not go through
$p$, and hence meets either $N_0$ or $N'_0$ three times. Hence, by
the above result, $M_0$ has the order four.

3.11.7. The case (2,1,1;2). Let $e \in N_2$, $e' \in N'_1$. Let $M_e$
denote the neighbourhood of $p$ with the end-points $e$ and $e'$;
$N_e = M_e \cap N_2$, $N'_e = M_e \cap N'_2$. By Lemmas 3.18 and 3.19, $C(e;\Gamma)$
($C(e';\Gamma)$) meets $M_2$ exactly three times at $p$, exactly once
at $e$ ($e'$), and nowhere else. Thus, by Theorem 3.3 and Lem-
mas 3.16 and 3.17, there is a one-sided neighbourhood $N_1 \subset N_e$
such that every circle through $e$ ($e'$) and two points of $N_1$

1. It may be that a short proof for this case, of the
nature of the proof for the case $a_0 + a_1 + a_2 = 4$ exists.
This proof, however, has value in itself, for it is an ex-
ample of a topological proof rather than a geometrical one.
We shall see more proofs of this type in Chapter 8.
meets $M_2$ exactly three times outside $e$ ($e'$). Since these three points converge to $p$ with $N_1$, $N_1$ may be chosen such that all these circles meet $M_e$ exactly three times each.

Now let $u \in N_1$, $u' \in N'_e$ be arbitrary, and let $\Pi$ denote the pencil of circles through $u$ and $u'$ (cf. Fig. 3.15).

![Diagram with labeled points and circles](image)

**Fig. 3.15**

By Lemma 3.19, $C(p;\Pi)$ meets $M_e$ only four times. Thus $C(p;\Pi)$ meets $M_e$ exactly twice at $p$, once each at $u$ and $u'$, and nowhere else. If $t$ lies on $N_1$ and is sufficiently close to $p$, then $C(t;\Pi)$ continues to meet $M_e$ exactly four times (Lemmas 3.16 and 3.17). Since $C(p;\Pi) \notin \mathcal{L}$, the fourth point $t'$ lies on $N'_e$ (Lemma 3.18 and Theorem 3.3). From the above,
C(t;\Omega) passes neither through e nor through e'. Thus

C = C(t;\Omega) has the following properties:

1. p, e, e' lie on the same side of C;

2. C meets \Omega exactly twice.

The circles C(p;\Omega) and C(e';\Omega) decompose \Omega into two open intervals. Let \Omega_1 denote that interval which contains the above circles C(t;\Omega). We orient \Omega_1 in the direction from C(p;\Omega) to C(e';\Omega). The circle C satisfies (i) and (ii) if it lies in \Omega_1 sufficiently close to C(p;\Omega). Put the circle D equal to C(e';\Omega) if (i) and (ii) hold true for every circle of \Omega_1; otherwise, let D denote the greatest lower bound of the set of all the circles of \Omega_1 for which at least one of these conditions is not satisfied. Thus D \neq C(p;\Omega). Let \Omega_2 denote the sub-interval of \Omega_1 bounded by C(p;\Omega) and D.

Every circle C \in \Omega_2 satisfies (i) and (ii). Thus C meets \Omega_e (\Omega'_e) in exactly one more point t (t'), and t lies in \Omega_1. The point t (t') depends continuously on C. For
t ≠ u (t' ≠ u'), the correspondence C → t (C → t') is 1-1. Hence it is strictly monotonic, even for t = u (t' = u') (cf. Theorem 3.4). Thus the limits r = \lim_{C \to D} t and r' = \lim_{C \to D} t' exist. The point r (r') lies on the intersection of D with the closure of N₁ (N₁'). It is different from p.

If r' = e', the points t' cover the whole of N₁'. In particular, N₂ contains all the circles C(t'; N), including the case t' = u'. Thus every circle through u and u' that meets N₁' at least twice, meets N₁' and N₁ - and even N₁ - exactly twice each.

Let r' ≠ e'. Thus r' ∈ N₁'. From the above, e ≠ D and e' ≠ D. Hence D lies in N₁ and still satisfies condition (i). Hence, (i) will remain valid for all circles of N₁ sufficiently close to D. In particular, these circles will meet N₁ exactly twice. Thus r \notin N₁, by the definition of D, and r will be the end-point of N₁ different from p.
Thus the points t will cover the whole of $N_1$. Repeating
the argument of the preceding paragraph, we obtain: every
circle through u and u' that meets $N_1$ at least twice, meets
$N_e$ and $N'_e$ exactly twice each.

The last two paragraphs imply: any circle that meets
$N_1$ and $N'_e$ at least twice each, meets $N_e$ and $N'_e$ exactly twice
each. Hence such a circle meets $N_e$ exactly four times out-
side p. Combining this result with Lemma 3.19, we find
that the neighbourhood $N_1 \cup p \cup N'_e$ has the order four.

3.11.8. The case $a_0 + a_1 + a_2 = 3$. Suppose that the points
$p,t,u,v$ lie on $N_2 \cup p$ in the indicated order. The points
t,u,v need not be mutually distinct. By Lemma 3.19, the
circles $C(p,t,u)$ and $C(p,u,v)$ do not meet $N_3$. They inter-
sect $M_3$ at each of these points. Hence

\[(3.28) \quad N'_3 \subset \overline{C(p,t,u)} \cap \overline{C(p,u,v)}\]

(cf. Fig. 3.16). Hence relation (3.21) implies that

$N'_3 \subset \overline{C(t,u,v)}$. In particular, $C(t,u,v)$ does not meet $N'_3$. 


Symmetrically, any circle through three points of $N_2'$ does not meet $N_2$.

Let $e \in N_2$, $e' \in N_2'$. Let $M_e$ denote the neighbourhood of $p$ bounded by $e$ and $e'$. Let $M_1$ be a neighbourhood of $p$ whose end-points lie in $M_e$. By Lemma 3.18, $C(e';\mathbb{L})$ meets $M_1$ exactly twice at $p$, and nowhere else. Thus, by Theorem 3.3 and Lemma 3.16, there is a one-sided neighbourhood $N_0 \subset N_1$ ($N_1 = M_1 \cap N_2$) of $p$, such that every circle through $e'$ and two points of $N_0$ meets $M_1$ with an even multiplicity.

Let $t,u, e \in N_0$. As we have seen, $C(t,u,e)$ does not
meet \( N_1^1 = M_1 \cap N_2^1 \). Hence

\[ (3.29) \quad e \not\subset C(t,u,u') \]

for every \( t,u \in N_0 \), \( u' \in N_1^1 \). Furthermore, \( C(t,u,e) \) meets \( M_1 \) an even number of times. By the above, this circle meets \( N_2 \) and \( N_2^1 \) not more than twice each. Hence it meets \( N_1^1 \) exactly twice. By Lemma 3.19 it does not pass through \( p \). Thus it meets \( N_1^1 \) with an even multiplicity. Since \( e' \in N_2^1 \), \( e' \not\in N_1^1 \), this multiplicity is less than two. Hence \( C(t,u,e') \) does not meet \( N_1^1 \), and we have

\[ (3.30) \quad e' \not\subset C(t,u,u') \]

for every \( t,u \in N_0 \), \( u' \in N_1^1 \).

Let \( u \in N_0 \), \( u' \in N_1^1 \). By Lemma 3.19, \( C(p,u,u') \) meets \( M_2 \) exactly three times. Thus it separates \( e \) and \( e' \). If \( t \) moves on \( N_0 \) between \( p \) and \( u \), the circle \( C(t,u,u') \) depends continuously on \( t \). By relations \( (3.29) \) and \( (3.30) \), it never passes through \( e \) or \( e' \). Thus every such circle \( C(t,u,u') \) also separates \( e \) and \( e' \). Hence it meets \( M_e \) an odd number of
times. The beginning of this sub-section implies that it meets \( M_2 \) less than five times. Hence it meets \( M_8 \) exactly three times. Thus any circle through two points of \( N_0 \) and a point of \( N_1 \) meets \( N_0 \cup p \cup N_1 \) nowhere else.

Combining the above results with Lemma 3.19, we see that \( N_0 \cup p \cup N_1 \) has order three. This completes the proof of Theorem 3.5.


Let \( p \in A \) decompose \( A \) into two arcs of order three. Then

(i) \( p \) satisfies Condition I' if and only if \( p \) satisfies Condition I and \( a_0 = 1 \).

(ii) \( A \) is strongly differentiable at \( p \) if and only if \( p \) is differentiable and \( a_0 = a_1 = 1 \).

Proof: (i) Let \( p \) satisfy Condition I'. Then \( p \) satisfies Condition I, and \( a_0 \) is defined. If \( a_0 = 2 \), every non-tangent circle through \( p \) supports \( A \) at \( p \). Thus there
are sequences of circles through two points of $A$ converging to $p$, whose limit circles are not tangent circles. Since this contradicts Condition I', $a_o$ must be 1.

Let $p$ satisfy Condition I and suppose $a_o = 1$. Any circle which converges to a non-tangent circle through $p$ meets a small neighbourhood $M = N \cup p \cup N'$ with an odd multiplicity, and does not meet $N \cup p$ or $N' \cup p$ more than once each. Thus it meets $M$ exactly once. Hence any limit circle of a sequence through two points of $A$ converging to $p$ is a tangent circle of $p$. Thus $A$ satisfies Condition I' at $p$.

(ii) Suppose $A$ is strongly differentiable at $p$.

Then $A$ is also differentiable at $p$. By (i), $a_o = 1$. If $a_1 = 2$, section 3.3.2 implies that there are circles which meet an arbitrarily small neighbourhood of $p$ three times, and which converge to a non-osculating tangent circle. Since every circle through three points converging to $p$ converges to $C(p)$, $a_1$ must be 1.
Next, suppose $A$ is differentiable at $p$ and $a_0 = a_1 = 1$.

From (i) $A$ satisfies Condition I'. Thus we must show that any circle through three points of $A$ converging to $p$ converges to $C(p)$.

If $a_2 = 1$, section 3.11.8 implies that there is a small neighbourhood of $p$ which is met at most three times by any circle. Thus the limit circle of a sequence through three points of $A$ converging to $p$ is an intersecting tangent circle, and is therefore $C(p)$.

If $a_2 = 2$, $p$ has the characteristic $(1,1,2;i)$, where $i = 1$ or 2. Let $M_2 = N_2 \cup p \cup N'_2$ be so small that no circle meets $M_2$ more than four times (cf. §3.11.6). Let $M_1 \subseteq M_2$ be so small that if $e \in N_1$, $C(e; T)$ meets $N'_2$.

Now choose $M \subseteq M_1$ so small that $C(e,t,t')$ meets $N'_2 - N'_1$.

---

1. Given two sets of elements, $X$ and $Y$, where $Y \subseteq X$, the set $X - Y$ is made up of all the elements of $X$ except those that are in the set $Y$. 
Thus \( C(e,t,t') \) does not meet \( M_0 \) outside \( t \) and \( t' \).

\( C(t,t',p) \) is close to a tangent circle of \( p \), and meets \( M_1 \) with an even multiplicity. Thus \( C(t,t',p) \) meets \( N_1 \cup p \) or \( N_1 \cup p \) three times, and hence \( C(t,t',p) \) converges to \( C(p) \) with \( t \) and \( t' \).

Let \( D \) be any circle through four points \( t,u,e \in N_0 \), \( t',u',e \in N_0 \) converging to \( p \). Now \( u \) and \( u' \in C(e,t,t') \).

Since \( u \) and \( u' \notin C(t,t',p) \), at least one (and hence both) of the points \( u \) and \( u' \) lies in the region,

\[ C(e,t,t') \cap \overline{C(t,t',p)} \]

Thus

\[ C(u,t,t') \subseteq [C(e,t,t') \cap \overline{C(t,t',p)}] \cup [\overline{C(e,t,t')} \cap \overline{C(t,t',p)}] \cup u \cup u' \]

as \( t,t' \to p \), any limit circle of \( C(u,t,t') \) will be a tangent circle of \( p \).

\[ C_0 = \lim C(u,t,t') \]

\[ = [C(e;\tau) \cap \overline{C(p)}] \cup [\overline{C(e;\tau)} \cap \overline{C(p)}] \cup C(e;\tau) \cup C(p) \]

Since \( D \) cannot meet \( M_2 \) more than four times, \( C_0 \) cannot
intersect A outside p. Hence

\[ C_0 \not\in \left[ \mathcal{C}(e; \mathcal{T}) \cap \overline{C}(p) \right] \cup \mathcal{C}(e; \mathcal{T}), \]

and therefore \( C_0 = \mathcal{C}(p) \).
CHAPTER IV

VERTICES OF CLOSED CURVES IN THE CONFORMAL PLANE

4.1. Introduction.

A closed curve in the conformal plane is one for which the two points whose parameters are end-points of the parameter interval, coincide.

One of the reasons for the subsequent investigation is to obtain a strictly conformal proof of the Four Vertex Theorem.¹ This goal has not yet been reached; the purpose of this chapter is only to indicate some of the steps likely to lead to a proof of this theorem.

4.2. \(L\)-vertices.

Let \(p\) be a differentiable point of a closed curve

¹ For a statement and proof of this theorem, see Blaschke, "Vorlesungen Über Differential Geometrie", Dover 1945, page 31.
A, and let \( p \) have the characteristic \((a_0, a_1, a_2; i)\). We shall assume that \( p \) is not a multiple point of \( A \). Let \( \mathcal{T} \) be the pencil of tangent circles of \( A \) at \( p \). Suppose that \( A \) has finite \( \mathcal{T} \)-order, i.e., every circle of \( \mathcal{T} \) meets \( A \) in a finite number of points. We call a point \( u \neq p \) a \( \mathcal{T} \)-vertex if \( C(u; \mathcal{T}) \) supports \( A \) at \( u \). We call \( p \) a \( \mathcal{T} \)-vertex if every tangent circle of \( p \) supports \( A \) at \( p \).  

4.3 Preliminary Material.

The following remarks will be useful in our discussion.

4.3.1. Suppose that a circle \( C \) meets \( A \) in a finite number of points, and that \( C \) intersects \( A \) at \( u \). Then the end-points \( e \) and \( f \) of a suitable neighbourhood \( M \) of \( t \) on \( A \) lie in opposite regions with respect to the circle \( C \). Hence the complement \( M' \) of \( M \) in the arc \( A \) has its end-points \( e \) and \( f \) in opposite regions with respect to the circle \( C \). But \( C \) meets \( A \) in only a finite number of points, and therefore \( C \) must intersect \( M' \) in an odd number of points. Thus

---

1. In this case, \( a_2 = 2 \). (cf. §2.16)
every circle $C$ intersects $A$ an even number of times.

4.3.2. Suppose the tangent circle $C$ supports $A$ at $t \neq p$. Then $C \neq p$, and there is a neighbourhood $M$ of $t$ on $A$ whose closure lies in $C \cup t$, say. In particular, the end-points of $M$ will lie on the same side of $C$. Let $C'$ be a tangent circle in $C \cup p$, and let it be close enough to $C$ that these end-points will still lie on the same side of $C'$ (cf. Fig 4.1).

Since $C'$ separates them from $t$, $C'$ will intersect $M$ in two points. On the other hand, a tangent circle in $C \cup p$ will not meet $M$.

4.3.3. If all the tangent circles support $A$ at $p$, then $a_2 = 2$, and for every circle $C \in \mathcal{L}$ there is a neighbourhood
M of p and two one-sided neighbourhoods of C in \( \mathcal{T} \) such that each circle of one of them intersects \( M \) twice, while each circle of the other one does not meet \( M \) outside \( p \).

4.3.4. If the circle \( C \) of \( \mathcal{T} \) intersects \( A \) at \( t \), there is a neighbourhood \( M \) of \( t \) on \( A \) whose end-points lie on opposite sides of \( C \). Let \( C' \) be a tangent circle close enough to \( C \) that these end-points are separated by \( C' \). Then \( C' \) intersects \( M \) in at least one point.

4.4. Curves With a Finite Number of \( \mathcal{T} \)-vertices.

**Theorem 4.1.** If the number of \( \mathcal{T} \)-vertices of \( A \) is finite, it is even.

**Proof:** Let \( C_0 \neq p \) be an arbitrary circle orthogonal to \( \mathcal{T} \). (Thus \( p \subset C_0 \)). If \( t \neq p \), then \( C(t;\mathcal{T}) \) intersects \( C_0 \) at exactly one point \( P(t) \neq p \). If \( t = p \), then define \( C(p;\mathcal{T}) \) to be \( C(p) \). If \( C(p) = p \), define \( P(p) \) to be \( p \); if \( C(p) \neq p \), define \( P(p) \) to be the intersection \( \neq p \) of \( C_0 \) with \( C(p) \). Then \( C(t;\mathcal{T}) \) and \( P(t) \) depend continuously on \( t \) over the whole of \( A \).
If \( t \) runs through \( A \), then the point \( P(t) \) changes its direction if and only if \( t \) passes through a \( \mathcal{T} \)-vertex. This follows for \( t \neq p \) from the definition (§ 4.2) of \( \mathcal{T} \)-vertices and from section 4.3.2; for \( t = p \), it follows from section 4.2 and section 4.3.3. Thus \( P(t) \) changes its sense only a finite number of times. The mapping \( P(t) \) of \( A \) on \( C_0 \) being periodical, this number must be even.

Theorem 4.2. \( A \) has no \( \mathcal{T} \)-vertices if and only if every proper circle of \( \mathcal{T} \) meets \( A \) exactly once outside \( p \).

Proof: Suppose that every proper circle of \( \mathcal{T} \) meets \( A \) exactly once outside \( p \). If \( C \subset \mathcal{T} \) supports \( A \) at a point \( u \neq p \), then there exist circles of \( \mathcal{T} \) close to \( C \) which intersect \( A \) in at least two points close to \( u \). Hence there are no \( \mathcal{T} \)-vertices at points \( u \neq p \). Since \( C \) intersects \( A \) exactly once outside \( p \), it must intersect \( A \) at \( p \). Thus \( p \) itself is not a \( \mathcal{T} \)-vertex.

On the other hand, suppose that \( A \) has no \( \mathcal{T} \)-vertices. If the point \( t \) runs through \( A \) as in the proof of Theorem 4.1,
the point \( P(t) \) does not change its direction on \( C_0 \). Hence \( P(t) \) makes at least one complete circuit of \( C_0 \); in particular, \( P(t) \) passes through \( p \). This happens only when \( t = p \) and \( C(p) = p \). Thus \( P(t) \) makes exactly one circuit of \( C_0 \), and hence every proper circle of \( C \) is met by \( A \) exactly once outside \( p \).

**Corollary 1.** If \( A \) has no \( T \)-vertices, then \( C(p) = p \).

**Corollary 2.** If every proper circle of \( C \) meets \( A \) exactly once outside \( p \), then \( C(p) = p \).

**Corollary 3.** If a circle \( C \) of \( C \) (\( C \neq p \) if \( C(p) = p \)) does not meet \( A \) outside \( p \), then \( A \) has at least two \( T \)-vertices.

**Corollary 4.** If a circle \( C \) of \( C \) (\( C \neq p \) if \( C(p) = p \)) supports \( A \) at \( p \), then \( A \) has at least two \( T \)-vertices.

**Proof:** If \( C \) does not meet \( A \) outside \( p \), then \( A \) has at least two \( T \)-vertices. If \( C \) intersects \( A \) outside \( p \), it intersects \( A \) once more (cf. §4.3.1), and again \( A \) has at least two \( T \)-vertices. If \( C \) supports \( A \) outside \( p \), \( A \) has one, and hence at least two, \( T \)-vertices.
4.4.1. By definition, \( A \) has \( \mathcal{T} \)-order \( n \) if no circle meets \( A \) outside \( p \) in more than \( n \) points, and some circle meets \( A \) \( n \) times outside \( p \).

Suppose that \( A \) has \( \mathcal{T} \)-order \( n \), and let \( C \) be a circle of \( \mathcal{T} \) which meets \( A \) in \( n \) points \( \neq p \). If \( C \cap (A - p) \) is composed of \( m \) points of support which have neighbourhoods lying in \( C \), \( k \) points of support which have neighbourhoods lying in \( \mathcal{C} \), and \( r \) points of intersection, then \( m + k + r = n \).

Consider two circles, \( C' \) and \( C'' \), of \( \mathcal{T} \). If \( C' \subset C \cup p \) is sufficiently close to \( C \), it meets \( A \) in exactly \( 2m + r \) points (cf. §4.3.2), while if \( C'' \subset C \cup p \) is sufficiently close to \( C \), it meets \( A \) in exactly \( 2k + r \) points. Since \( A \) has \( \mathcal{T} \)-order \( n \), we see that

\[
2m + r \leq n = m + k + r
\]

and

\[
2k + r \leq n = m + k + r,
\]

whence

\[
m \leq k \leq m.
\]

Thus \( m = k \). Since \( 2m + r = 2k + r = n \), we see in addition that

\[
r \equiv n \pmod{2}.
\]
4.4.2. If $A$ has $\mathcal{T}$-order $2n+1$, then a circle $C \in \mathcal{T}$, which meets $A$ in $2n+1$ points outside $p$ must intersect $A$ at $p$ (cf. §4.4.1 and §4.3.1). Again by section 4.4.1, there is a circle $C'$ of $\mathcal{T}$ sufficiently close to $C$ which intersects $A$ in exactly $2n+1$ points outside $p$. Hence the non-osculating circles of $\mathcal{T}$ intersect $A$ at $p$ and therefore $C(p) = p$ (cf. §2.6). These remarks enable us to extend Theorem 4.2, Corollary 2 to Corollary 5. If $A$ has $\mathcal{T}$-order $2n+1$, then $C(p) = p$, and the non-osculating circles of $\mathcal{T}$ intersect $A$ at $p$.

4.4.3. Theorem 4.3. Suppose that a tangent circle, $C$ of $A$ at $p$ meets $A$ in exactly $n$ points $\neq p$. Then $A$ has at least $n-1$ $\mathcal{T}$-vertices. If, in addition, the non-osculating tangent circles support $A$ at $p$, then $A$ has at least $n \mathcal{T}$-vertices.

1. This condition is, of course, automatically satisfied if $C(p) \neq p$.

2. Sections 4.4.1 and 4.3.1, together with Theorem 4.2 Corollary 5, imply the (cont'd on Page 117 (bottom))
Proof: The points \( \neq p \) of \( A \cap C \) decompose \( A \) into \( n \) closed arcs \( B \) such that no interior point \( \neq p \) of an arc \( B \) lies on \( C \). Let \( B_0 \) denote that arc \( B \) which contains \( p \). It is sufficient to prove

(i) Each \( B \neq B_0 \) contains at least one interior \( \tau \)-vertex;

and, under our additional assumption,

(ii) \( B_0 \) also contains an interior \( \tau \)-vertex.

For each \( B \) we define the subset \( \mathcal{V} = \mathcal{V}(B) \) of \( \mathcal{T} \) as follows: if \( B \neq B_0 \), then \( \mathcal{V} \) shall be the set of those circles of \( \mathcal{T} \) that meet \( B \); if \( B = B_0 \), then \( \mathcal{V} \) is the union of \( C(p) \) with the set of all the tangent circles which meet \( B \) outside \( p \).

In either case, \( \mathcal{V} \) will be a connected, closed subset of \( \mathcal{T} \). If \( B \neq B_0 \), or if \( B = B_0 \) and \( C(p) \neq p \), then \( \mathcal{V} \) does following remark: If \( n \) is positive and even, then the number of intersections of \( C \) with \( A \) outside \( p \) is even. Hence \( C \) supports \( A \) at \( p \) and the additional assumption is automatically satisfied. On the other hand, if \( n \) is odd, this condition cannot hold.
not contain the point-circle $p$. If $B = B_0$ and $C(p) = p$, then, from our additional assumption, all the tangent circles support, and some tangent circles near $p$ will not belong to $\mathcal{V}$. Hence $\mathcal{V}$ is a proper subset (i.e. a closed sub-interval) of $\mathcal{L}$. At least one of the end-circles of $\mathcal{V}$, say $C'$, is different from $C$. Thus $C' \cap B$ does not contain the end-points of $B$. Since $C' \in \mathcal{V}$, this circle actually has at least one point in common with $B$. If $C'$ intersects $B$ outside $p$, every circle of $\mathcal{L}$ close to $C'$ also intersects $B$.

Thus any point $\neq p$ of $C' \cap B$ is a point of support, i.e., a $\mathcal{L}$-vertex. Suppose that $C' \cap B = p$. Then $B = B_o$. In this case, $\mathcal{V} = C(p) \cup C(t; \mathcal{L})$, where $t \in B_o$, $t \neq p$, and since $C' \in \mathcal{V}$, it follows that $C' = C(p)$. Hence $C(p) = C'$ supports $B_o$ at $p$. By our additional assumption, $p$ is a $\mathcal{L}$-vertex.

We can write the proof of Theorem 4.3 in a different way. By definition, $p$ is a $\mathcal{L}$-vertex in this case.
way, using the orthogonal circle $C_0$ of Theorem 4.1.

The circle $C \in \mathcal{T}$, which meets $A$ in $n$ points outside $p$, again divides $A$ into $n$ closed sub-arcs $B$; $B_0$ is that sub-arc $B$ which contains $p$.

If $t$ moves through $B \neq B_0$, $P(t)$ moves on $C_0$, and returns to its initial position without passing through $p$. Thus $P(t)$ must reverse its direction on $C_0$, and therefore have an interior $\mathcal{T}$-vertex. Since $P(t) \neq p$ when $C(p) \neq p$, this even holds true when $B = B_0$, provided $C(p) \neq p$.

If $C(p) = p$, $p$ is a $\mathcal{T}$-vertex by definition provided the other circles of $\mathcal{T}$ support $A$ at $p$.

If, in this theorem, $C \cap A$ contains $m$ points of support different from $p$, our proof shows that we have at least $m + n - 1 \mathcal{T}$-vertices, and at least $m + n$ under the additional assumption.

From Theorem 4.1, we obtain

**Corollary 1. If a circle of $\mathcal{T}$ meets $A$ in $2n$ points**
different from \( p \), then \( A \) has at least \( 2n \) \( \tau \)-vertices.

**Theorem 4.4.** Suppose that there is a circle of \( \mathcal{L} \) (different from \( p \) if \( C(p) = p \)) which meets \( A \) only at \( p \).

Then \( A \) has exactly two \( \tau \)-vertices if and only if no circle of \( \mathcal{L} \) meets \( A \) in more than two points different from \( p \).

**Proof:** Our assumption implies that the non-osculating circles of \( \mathcal{L} \) support \( A \) at \( p \) (cf. §4.3.1 and Theorem 2.8). Suppose there exists a circle \( C \) of \( \mathcal{L} \) which meets \( A \) at more than two points \( \neq p \). By Theorem 4.3, \( A \) has at least three \( \tau \)-vertices.

Now suppose that no circle of \( \mathcal{L} \) meets \( A \) in more than two points \( \neq p \). Let \( \mathcal{V} \) be the closed interval consisting of \( C(p) \) and all those circles of \( \mathcal{L} \) which meet \( A \) outside \( p \). By our assumptions, \( \mathcal{V} \) is a closed, connected, proper sub-interval of \( \mathcal{T} \). As \( t \) moves over \( A \), \( P(t) \) moves over a proper sub-arc of \( C_0 \) (cf. Theorem 4.1), and returns to its starting point. Hence every interior circle of \( \mathcal{V} \) meets \( A \) at least
twice outside \( p \). If \( A \) had another \( T \)-vertex, \( P(t) \) would cover an arc of \( C_0 \) at least three times, and then some circle of \( T \) would meet \( A \) at least three times. Thus \( A \) has exactly two \( T \)-vertices, which belong to the end-circles of \( V \), where \( P(t) \) reverses its direction.

**Theorem 4.5.** Suppose that there is a circle of \( T \) (different from \( p \) if \( C(p) = p \)) which supports \( A \) at \( p \). If \( A \) has exactly four \( T \)-vertices, then \( A \) has \( T \)-order four.

**Proof:** Our first assumption implies that the non-osculating circles of \( T \) support, and hence there exist circles of \( T \) which do not meet \( A \) outside \( p \).

If \( A \) has exactly four \( T \)-vertices, Theorem 4.3 implies that the \( T \)-order of \( A \) does not exceed 4. By Theorem 4.4, the \( T \)-order is at least 3. Section 4.4.2 implies that \( A \) has \( T \)-order 4.

**4.4.4.** The following are examples of curves with no \( T \)-vertices, two \( T \)-vertices, and four \( T \)-vertices respectively. The
first three examples refer to $\Gamma$-vertices relative to the origin as fundamental point of the pencil $\Gamma$.

(a) $x = t^3, y = t^5$. The non-osculating tangent circles at the origin intersect. They therefore intersect at their only other point of contact with the arc, and we have no $\Gamma$-vertices.

(b) $x = t, y = t^2$. The non-osculating tangent circles at the origin all support. The $x$-axis meets the arc only at the origin and at $\infty$, and since it supports at the origin, it must also support at $\infty$. Thus we have a $\Gamma$-vertex at $\infty$.

The osculating circle also supports at the origin. Hence we have a $\Gamma$-vertex at the origin. The non-osculating tangent circles which do not go through $\infty$ intersect the arc in two points outside $p$ and do not meet the arc elsewhere except at the origin. Thus we have only two $\Gamma$-vertices.

(c) $r = a \tan \theta \sec \theta, 0 \leq \theta < \pi/4, \ 3\pi/4 \leq \theta < \pi; \ r = a \csc \theta, \ \pi/4 \leq \theta < 3\pi/4$. 


All the circles of $\mathcal{C}$ support at the origin, and hence give rise to a $\mathcal{C}$-vertex there. The circle $r = \sqrt{2}a \sin \theta$ of $\mathcal{C}$ supports the arc at $r = \sqrt{2}a$, $\theta = \pi/4$, and at $r = \sqrt{2}a$, $\theta = 3\pi/4$. Our final $\mathcal{C}$-vertex is found at $r = a$, $\theta = \pi/2$, for the circle $r = a \sin \theta$ supports the arc there.

(d) This example again illustrates an arc with no $\mathcal{C}$-vertices. Its particular interest lies in the fact that the same arc was used as an example of an arc having two $\mathcal{C}$-vertices, where the fundamental point of $\mathcal{C}$ was the origin.

Let $A$ be the parabola $x = t$, $y = t^2$. The tangent circles of $A$ at $t = \infty$ are straight lines parallel to the $y$-axis. Each of them intersects the parabola exactly once for a finite value of $t$, and therefore must intersect $A$ again at $t = \infty$.

4.5. $\Pi$-vertices.

Let $p$ and $s$ be differentiable points of a closed curve $A$, and let $p$ ($s$) have the characteristic $(a_0, a_1, a_2; i)$. 
We shall assume that $p$ and $s$ are not multiple points of $A$.

Let $\mathcal{P}$ be the pencil of circles through the fundamental points $p$ and $s$. Suppose that $A$ has finite $\mathcal{P}$-order, i.e., every circles of $\mathcal{P}$ meets $A$ in a finite number of points. We call the point $t \neq p, s$, a $\mathcal{P}$-vertex if $C(t; \mathcal{P})$ supports $A$ at $t$. We call $p$ ($s$) a $\mathcal{P}$-vertex if the non-tangent circles of $\mathcal{P}$ at $p$ ($s$) support $A$ at $p$ ($s$) when $C(s; T_p) (C(p; T_s))$ supports, or intersect $A$ at $p$ ($s$) when $C(s; T_p) (C(p; T_s))$ intersects.

4.6. Remarks Useful in the Development of the Theory of $\mathcal{P}$-vertices.

4.6.1. Suppose that the circle $C(t; \mathcal{P})$ supports $A$ at $t \neq p, s$.

Then there is a neighbourhood $M$ of $t$ on $A$ whose closure lies in $C(t; \mathcal{P}) \cup t$, say. In particular, the end-points of $M$ will lie in $C(t; \mathcal{P})$. Let $C'$ lie in the region

$$\left[ C(t; \mathcal{P}) \cap \overline{C} \right] \cup \left[ C(t; \mathcal{P}) \cap \overline{C} \right] \cup p \cup s,$$

1. The symbol $C(R; \mathcal{L}_q)$ means "the tangent circle of $A$ at $q$ through $R^\mathcal{P}$, for all points $q \in A$, and $R \neq q."
say, where $C$ is any fixed circle of $\mathcal{P}$ such that

$$M \subset \left[ \overline{C(t;\mathcal{P})} \cap \overline{C} \right] \cup \mathcal{P}.$$

Suppose that $C'$ is close enough to $C(t;\mathcal{P})$ that the end-points of $M$ will lie on the same side of $C'$. Since $C'$ separates them from $t$, $C'$ will intersect $M$ in two points. On the other hand, a circle of $\mathcal{P}$ in the region

$$\left[ \overline{C(t;\mathcal{P})} \cap \overline{C} \right] \cup \left[ \overline{C(t;\mathcal{P})} \cap \overline{C} \right] \cup p \cup s$$

will not meet $M$.

4.6.2. If the circle $C$ of $\mathcal{P}$ intersects $A$ at $t$, there is a neighbourhood $M$ of $t$ on $A$ whose end-points lie on opposite sides of $C$. Let $C'$ be a circle of $\mathcal{P}$ close enough to $C$ that these end-points are separated by $C'$. Then $C'$ intersects $M$ in at least one point.

4.6.3. Suppose that $C(s;\mathcal{P}_p)$ supports (intersects) $A$ at $p$. Then the end-points of a sufficiently small neighbourhood $M = B_1 \cup p \cup B_2$ of $p$ on $A$ will lie in the same region (in different regions) with respect to $C(s;\mathcal{P}_p)$. 
Suppose that the non-tangent circles through \( p \) support (intersect) \( A \) at \( p \). Let \( C' \) be the orthogonal circle to \( C(s; \tau_p) \) through \( p \) and \( s' \) (cf. Fig. 4.2). Since \( C' \) supports (intersects) \( A \) at \( p \), \( B_1 \) and \( B_2 \) must lie in the same region (in different regions) with respect to \( C' \). We lose no generality in assuming that \( B_1 \cup B_2 \) lies in the region

\[
C' \cap C(s; \tau_p) \quad \left( [(C' \cap C(s; \tau_p)) \cup (\bar{C}' \cap \bar{C}(s; \tau_p))] \right).
\]

Now let \( C_0 \) be any circle which is orthogonal to the family \( \Pi \), and let \( R \) be any point of \( C_0 \) which lies in the region \( \bar{C}' \cap C(s; \tau_p) \). Thus...
\[ C(R; \mathcal{P}) \subset \left[ \overline{C} \cap C(s; T_p) \right] \cup \left[ \overline{C} \cap \overline{C}(s; T_p) \right] \cup p \cup s, \]

and therefore \( C(R; \mathcal{P}) \) meets \( M \) only at \( p \). Hence this circle (which can be as close to \( C(s; T_p) \) as we please) does not meet \( M \) outside \( p \).

On the other hand, if the non-tangent circles of \( \mathcal{P} \) at \( p \) intersect (support) \( A \) at \( p \), then \( B_1 \cup B_2 \) lies in the region
\[ \left[ \overline{C} \cap C(s; T_p) \right] \cup \left[ \overline{C} \cap \overline{C}(s; T_p) \right] \left( \left[ \overline{C} \cap C(s; T_p) \right] \cup \left[ \overline{C} \cap \overline{C}(s; T_p) \right] \right), \]

say. Let \( t_1 \in B_1 \) (i = 1, 2). Then
\[ C(t_1; \mathcal{P}) \subset \left[ \overline{C} \cap C(s; T_p) \right] \cup \left[ \overline{C} \cap \overline{C}(s; T_p) \right] \cup p \cup s, \]
say, while
\[ C(t_2; \mathcal{P}) \subset \left[ \overline{C} \cap C(s; T_p) \right] \cup \left[ \overline{C} \cap \overline{C}(s; T_p) \right] \cup p \cup s. \]

Hence all circles of \( \mathcal{P} \) close to \( C(s; T_p) \) meet \( M \) at least once outside \( p \).

Obviously, the above is also true when we interchange the roles of \( p \) and \( s \).

4.7. **Curves with a Finite Number of \( \mathcal{P} \)-vertices**.
Theorem 4.6. If the number of $\Pi$-vertices of $A$ is finite, it is even.

Proof: Let $C_o \neq p, s$, be an arbitrary circle orthogonal to $\Pi$. $C(t; \Pi)$ intersects $C_o$ in exactly two points, $P(t)$, and $P'(t)$. Thus $P(t), P'(t)$ and $C(t; \Pi)$ depend continuously on $t$ over the whole of $A$.

If $t$ runs through $A$, then the points $P(t)$ and $P'(t)$ change their direction if and only if $t$ passes through a $\Pi$-vertex. This follows for $t \neq p, s$, from the definition (§ 4.5) of $\Pi$-vertices, and sections 4.6.1 and 4.6.2; for $t = p$ or $s$, it follows from section 4.5 and section 4.6.3. Thus $P(t)$ and $P'(t)$ change their direction only a finite number of times. Since the direction of motion of $P(t)$ and $P'(t)$ on $C_o$ must be the same when a circuit of $A$ is completed as when it began, the number of changes of direction must be even.

Theorem 4.7. If every circle of $\Pi$ except $C(s; \overline{t}_p)$
and $C(p; I_s)$ meets $A$ exactly once outside $p$ and $s$, then

(i) $A$ has no $\mathcal{N}$-vertices outside $p$ and $s$,

(ii) $C(s; I_p)$ and $C(p; I_s)$ do not meet $A$ outside $p$ and $s$.

(iii) $A$ has no $\mathcal{N}$-vertices.

(iv) $C(s; I_p) \neq C(p; I_s)$.

Proof: (i) Suppose that $C(u; \mathcal{N})$ supports $A$ at a point $u \neq p, s$. Then there exist circles of $\mathcal{N}$ close to $C(u; \mathcal{N})$ which intersect $A$ in at least two points close to $u$, contrary to our assumption. Hence there are no $\mathcal{N}$-vertices at points $u \neq p, s$.

(ii) If $C(s; I_p)(C(p; I_s))$ meets $A$ at a point $u \neq p, s$, then by (i) and section 4.6.2, there are circles of $\mathcal{N}$ close to $C(s; I_p)(C(p; I_s))$ which meet $A$ in at least two points $\neq p$ and $s$, one being near $u$, and the other near $p; (s)$. Hence $C(s; I_p)$ and $C(p; I_s)$ do not meet $A$ outside $p$ and $s$.

(iii) In view of (i), we need only consider the
points p and s. If \( C(s; T_p) \) supports (intersects) \( A \) at \( p \), then by (ii) and section 4.3.1, it supports (intersects) \( A \) at \( s \). We have a \( T \)-vertex at \( p \) if and only if the non-tangent circles at \( p \) also support (intersect) \( A \) at \( p \). If this is the case, \( C(u; T) \), \( u \neq p, s \), will intersect \( A \) at \( u \) by (i); by section 4.3.1 and our assumption it will intersect (support) \( A \) at \( s \). But since \( C(p; T_s) \) supports (intersects) \( A \) at \( s \), \( s \) is not a \( T \)-vertex. Thus we have only one \( T \)-vertex on \( A \), which contradicts Theorem 4.6. Hence \( A \) has no \( T \)-vertices.

(iv) If \( C(s; T_p) = C(p; T_s) = 0 \), then a circle \( C' \) of \( T \) close to \( C \) will meet \( A \) at least twice outside \( p \) and \( s \), once near \( p \), and once near \( s \). This follows from (iii).

Hence \( C(s; T_p) \neq C(p; T_s) \).

4.8. The Relation Between \( T \)-vertices and \( T \)-vertices.

If we allow the point \( s \in A \) approach \( p \in A \) along \( A \), then the pencil \( T \) through \( p \) and \( s \) becomes the pencil \( T \) through \( p \). We obtained a \( T \)-vertex at \( p \) if the non-tangent
circles of $\pi$ supported (intersected) $A$ at $p$ when $C(s; \ell_p)$ supported (intersected) $A$ at $p$; we now get a $\ell$-vertex at $p$ if the non-osculating circles of $\ell$ support $A$ at $p$ when $C(p)$ supports $A$ at $p$ (it is impossible for all the circles of $\ell$ to intersect $A$ at $p$). A $\pi$-vertex at a point $u \in A$, $u \neq p$ when $p$ and $s$ coincide is simply a $\ell$-vertex at that point. Therefore, if appropriate minor changes are made, any theorem that is true for $\pi$-vertices is also true for $\ell$-vertices. The converse of this statement is not true; the study of $\pi$-vertices is more complex, as can be seen from the small part of that theory which has been presented here.
CHAPTER V

DIFFERENTIABLE POINTS OF ARCS IN CONFORMAL 3-SPACE

5.1. Introduction.

We now begin an investigation in conformal 3-space which parallels the work in two dimensions. The change from two to three dimensions is of considerable note, chiefly because of the fact that instead of dealing with one continuous entity (the circle) and the discrete point-pair, we now must consider two continuous entities, the circle and the sphere.

5.2. Differentiability.

Let p be a fixed point of an arc A, and let t be a variable point of A. If P, Q, and p are mutually distinct points, the unique circle through these points will be denoted by \( C(P,Q;Y_o) \). The symbol \( Y_o \) itself will denote the
family of all circles through \( p \), including the point-circle \( p \).

A is called **once-differentiable** at \( p \) if the following condition \( \Gamma_1 \) is satisfied:

\[
\Gamma_1: \text{If the parameter } t \text{ is sufficiently close to, but different from, the parameter } p, \text{ the circle } C(P,t;\gamma_0) \text{ is uniquely defined, and converges if } t \text{ tends to } p.
\]

Thus the limit circle, which will be denoted by \( C(P;\gamma_1) \), is independent of the way \( t \) converges to \( p \). The family of all such circles, together with the point-circle \( p \), will be denoted by the symbol \( \gamma_1 \).

A is called **twice-differentiable** at \( p \) if, in addition to the condition \( \Gamma_1 \), the following condition is also satisfied:

\[
\Gamma_2: \text{If the parameter } t \text{ is sufficiently close to, but different from, the parameter } p, \text{ the circle } C(t;\gamma_1) \text{ is uniquely defined, and converges if } t \text{ tends to } p.
\]
The limit circle of the sequence \( C(t; \gamma) \) will be denoted by \( C(\gamma_2) \) the osculating circle of \( A \) at \( p \), and occasionally by the symbol \( \gamma_2 \) alone.

5.3. Structure of the Families of Circles Through \( p \).

In this section, relations among the families of circles \( \gamma_0, \gamma_1, \gamma_2 \), are discussed.

Theorem 5.1. Suppose \( A \) satisfies condition \( \Gamma_1 \) at \( p \). Then \( t \) does not coincide with \( p \) if the parameter \( t \) is sufficiently close to, but different from, the parameter \( p \).

Proof: Let \( P \) be any point different from \( p \). By condition \( \Gamma_1 \), \( C(P,t;\gamma_0) \) is defined when the parameter \( t \) is close to, but different from, the parameter \( p \). Thus \( t \neq p \).

Theorem 5.2. Suppose that \( A \) satisfies condition \( \Gamma_1 \) at \( p \). Then the angle at \( p \) between any two circles of \( \gamma_1 \) is 0.

Proof: Let \( P, Q, R_1, R_2 \), be variable points, and let \( R_1 \) and \( R_2 \) converge to the same point \( R \). Suppose there is a sphere separating \( R \) from both \( P \) and \( Q \). Then
whether or not the circles themselves converge. In particular, the angle between $C(P; \lambda_1)$ and $C(Q; \lambda_1)$ is equal to 0.

**Corollary 1.** If $C(P; \lambda_1)$ and $C(Q; \lambda_1)$ have another point in common, they are identical; thus there is one and only one circle of $\lambda_1$ through each point $P \neq p$.

**Corollary 2.** $\lambda_1$ consists of those circles $C$ which meet a given circle of $\lambda_1$ at $p$ at the angle 0.

**Proof:** Let $P \subset C$, $P \neq p$. Suppose that $C$ meets some circle of $\lambda_1$ at angle 0 at $p$. Then $C$ and $C(P; \lambda_1)$ also meet at angle 0 at $p$ and have the point $P$ in common. Hence they are identical.

**Corollary 3.** If $\lambda_1$ holds for a single point $P \neq p$, then it holds for all such points.

**Proof:** If $Q \neq p$, by relation (3.1),

1. This becomes obvious if we let $P$ or $Q$ be the fixed point at infinity.
\[
\lim_{\theta \to 0} \left[ C(Q,t;\gamma_0); C(P,t;\gamma_o) \right] = 0.
\]
Hence \(C(Q,t;\gamma_0)\) converges to the unique circle through Q which touches \(C(P;\gamma_1)\) at \(p\).

**Theorem 5.3.** Suppose \(A\) satisfies the conditions \(\Gamma_1\) and \(\Gamma_2\) at \(p\). Then

\[
\gamma_0 \supseteq \gamma_1 \supseteq \gamma_2.
\]

**Proof:** It is clear that \(\gamma_0 \supseteq \gamma_1\). If \(C(\gamma_2) = p\), it belongs to \(\gamma_1\) by definition. Suppose \(C(\gamma_2) \neq p\). Then \(C(\gamma_2)\), being the limit of a sequence of circles \(C(t;\gamma_1)\) each of which touches a given circle \(C(P;\gamma_1)\) of \(\gamma_1\), must itself touch \(C(P;\gamma_1)\) at \(p\). Thus \(C(\gamma_2) \subseteq \gamma_1\).

**Corollary 1.** If \(P \subseteq C(\gamma_2); P \neq p\), then \(C(\gamma_2) = C(P;\gamma_1)\).

The conditions \(\Gamma_1\) and \(\Gamma_2\) are independent. Consider for example, the arc

\[
x = t, \ y = t^2, \ z = \begin{cases} (1 - \sqrt{1-t^2-t^4})\sin t^{-1}, & 0 < |t| \leq \frac{1}{2} \\ 0, & t = 0 \end{cases}
\]

Considering the vector \(\mathbf{t} = xi + yj + zk\), we let \(\theta\) be the angle between \(t\) and the \(x\)-axis. The vector \(t\) represents the
circle of $\gamma_0$ through the point at infinity and the point $t$.

As $t \to 0$,

$$\cos \theta = \frac{\mathbf{1} \cdot \mathbf{t}}{||\mathbf{t}||} = \frac{t}{\sqrt{t^2 + t^4 + [t^2/2(1+t^2) + o(t^3)]^2 \sin^2 t - 1}}$$

$$= \frac{1}{\sqrt{1 + t^2 + o(t)}}$$

$$\to 1 \quad \text{as} \quad t \to 0.$$  

Thus Condition $\Gamma_1$ holds at $t = 0$ for the point $\infty$, and therefore by Theorem 5.2 Corollary 3, it holds for all points $P \neq p$.

However, condition $\Gamma_2$ is not satisfied at $t = 0$. The plane through the $x$-axis (which by the above $\in \gamma_1$) and the point $x(t), y(t), z(t)$ contains circles which pass through $t$ and which touch the $x$-axis; i.e., it contains the circles $C(t; \gamma_1)$. This is also true of the sphere through $t$ which touches the $xy$-plane. Thus $C(t; \gamma_1)$ is the intersection of the former plane and the sphere. But as $t \to 0$, neither the sphere nor the plane, nor the intersection of the sphere
and the plane, converges. (The method of determining this is similar to that used in §2.4.) Hence \( \Pi_2 \) does not hold.

§4. Differentiable Points of Arcs

In addition to the conditions \( \Pi_1 \) and \( \Pi_2 \), three more conditions, involving spheres, are introduced. Suppose \( P, Q, \) and \( R \) are any three fixed points such that \( P, Q, R, \) and \( p \) do not all lie on the same circle. It will be convenient to denote the unique sphere through \( p \) and the points \( P, Q, \) and \( R, \) by the symbol \( S(P, Q, R; \sigma_0) \). \( \sigma_0 \) will denote the family of all spheres through \( p \), including the point-sphere \( p \).

A is called thrice-differentiable at \( p \) if the following three conditions are satisfied:

\[ \Sigma_1: \text{ If the parameter } t \text{ is sufficiently close to, but different from, the parameter } p, \text{ the sphere } S(P, Q, t; \sigma_0) \text{ is uniquely defined, and converges as } t \to p \text{ to a limit sphere which will be denoted by } S(P, Q; \sigma_1). \]
\[ \Sigma_2: \text{If the parameter } t \text{ is sufficiently close to, but different from, the parameter } p, \text{ the sphere } S(P,t;\sigma_1) \text{ is uniquely defined, and converges as } t \to p \text{ to a limit sphere which will be denoted by } S(P;\sigma_2). \]

\[ \Sigma_3: \text{If the parameter } t \text{ is sufficiently close to, but different from, the parameter } p, \text{ the sphere } S(t;\sigma_2) \text{ is uniquely defined, and converges as } t \to p \text{ to a limit sphere which will be denoted by } S(\sigma_3). \]

The family of all the spheres \( S(P,Q;\sigma_1) \), together with the point sphere \( p \), will be denoted by the symbol \( \sigma_1 \). The family of all the spheres \( S(P;\sigma_2) \) will be denoted by the symbol \( \sigma_2 \); if \( C(\gamma_2) = p \), this family will also include the point-sphere, \( p \). The members of \( \sigma_1 \) and \( \sigma_2 \) will sometimes be called \textit{singly tangent} (or \textit{1-tangent}) and \textit{doubly tangent} (or \textit{2-tangent}) spheres, respectively. The unique osculating sphere, \( S(\sigma_3) \) will occasionally be denoted by the symbol \( \sigma_3 \) alone.

The point \( p \) is called a \textit{differentiable point} of \( A \).
if $A$ is thrice-differentiable at $p$.

5.5 Structure of the Families of Spheres Through $p$.

Although the conditions $\Gamma_1$ and $\Gamma_2$ are independent, not all the conditions $\Gamma_1, \Gamma_2, \Sigma_1, \Sigma_2$ and $\Sigma_3$ are independent. In addition, the families of spheres $\sigma_0, \sigma_1, \sigma_2$, and $\sigma_3$ are closely connected with the families of circles $\gamma_0, \gamma_1$, and $\gamma_2$.

**Theorem 5.4** Suppose $A$ satisfies condition $\Sigma_1$ at $p$. Let $C$ be any circle. Then $t \notin C$ if the parameter $t$ is sufficiently close to, but different from the parameter $p$.

**Proof:** The assertion is clearly true if $p \notin C$.

Suppose $p \subset C$, and let $P, Q, p$ be mutually distinct points on $C$. By condition $\Sigma_1$, $S(P, Q, t; \sigma_0)$ is defined when $t$ is sufficiently close to $p$. Thus $t \notin C(P, Q, p) = C$.

The following example shows that $\Gamma_1$ does not imply $\Sigma_1$ in general (cf., however, Theorem §.5). Consider the arc
in the neighbourhood of $t = 0$. If $P = \infty$, $Q = (1, 0, 0)$, and $p = (0, 0, 0)$, the sphere $S(P, Q, t; \sigma_0)$ does not converge, while for example, $C(P, t; \gamma_0)$ converges to the x-axis; by Theorem 5.2, Corollary 3, \( \Sigma_1 \) is satisfied.

**Theorem 5.5.** If \( A \) satisfies \( \Sigma_1 \) at \( p \), then \( \Gamma_1 \) holds there, and

\[
C(Q; \gamma_1) = \prod_P S(P, Q; \sigma_1). 
\]

Conversely, let \( A \) satisfy \( \Gamma_1 \) at \( p \). Then \( \Sigma_1 \) holds at \( p \) for all pairs \( P, Q \), such that \( P \notin C(Q; \gamma_1) \); then

\[
S(P, Q; \sigma_1) = S[P; C(Q; \gamma_1)].
\]

**Proof:** Suppose that \( \Sigma_1 \) holds at \( p \). If \( Q \neq p \),

\[
\lim_{t \to p} C(Q, t; \gamma_0) = \lim_{t \to p} \prod_P S(P, Q, t; \sigma_0) = \prod_P S(P, Q; \sigma_1).
\]

1. Given a family, \( \Pi \), of spheres (or m-spheres in higher dimensions), by the symbol \( \prod_P S(P; \Pi) \) we mean the common intersection of all the spheres belonging to \( \Pi \).
Hence $C(Q,t;\mathcal{Y}_0)$ converges, and

$$C(Q;\mathcal{Y}_1) = \prod_{P} S(P, Q; \sigma_1).$$

Conversely, suppose that $\Pi_1$ holds. If $P \notin C(Q;\mathcal{Y}_1)$, then $P \notin C(Q,t;\mathcal{Y}_0)$ when $t$ is sufficiently close to $p$, and

$$S(P; C(Q;\mathcal{Y}_1)) = \lim_{t \to p} S(P; C(Q,t;\mathcal{Y}_1))$$

$$= \lim_{t \to p} S(P, Q, t; \sigma_0).$$

Thus for all pairs of points $P$ and $Q$ such that $P \notin C(Q;\mathcal{Y}_1)$, $S(P,Q,t;\sigma_0)$ converges, $\Sigma_1$ is satisfied, and $S(P,Q;\sigma_1)$ is the sphere through $P$ and $C(Q;\mathcal{Y}_1)$.

**Corollary 1.** There is only one sphere of $\sigma_1$ which contains two points not on the same circle of $\mathcal{Y}_1$.

**Remark:** Condition $\Pi_1$ is still satisfied when $\Sigma_1$ is replaced by a weaker assumption, namely:

Suppose $S'_1 = S(P_1, Q_1, t; \sigma_0) \to S_1$, $S'_2 = S(P_2, Q_2, t; \sigma_0) \to S_2$, and suppose further that $S_1 \cap S_2 = C \neq p$. Then $\Pi_1$ holds at $p$.

**Proof:** Let $S'_1 \cap S'_2 = C'$. Then $C' \to C$, and $C' \not\supset p$ and $t$. As in relation (5.1),
Thus, \( C(P_1, t; \gamma_0) \) converges to the unique circle through \( P_1 \) which touches \( C \) at \( p \). By Theorem 5.2, Corollary 3, \( \Gamma_1 \) holds at \( p \).

If, however, \( S_1 \cap S_2 = p \), \( \Gamma_1 \) need not hold; for example, take \( P_1 = \infty \), \( Q = (1,0,0) \), \( P_2 = (0,0,2) \), \( Q_2 = (1,0,1) \), \( p = (0,0,0) \), and let \( A \) be the arc

\[
x = \begin{cases} 
  t \sin t^{-1}, & 0 < |t| \leq 1 \\
  0, & t = 0 
\end{cases}, \quad y = t, \quad z = t^2.
\]

\( S_1 \) converges to the \( xy \)-plane, \( S_2 \) converges to the sphere

\[ x^2 + y^2 + z^2 - 2z = 0, \]

but \( \Gamma_1 \) does not hold.

**Theorem 5.6.** Suppose that \( \Sigma_1 \) holds at \( p \). Choose \( C \in \gamma_1, C \neq p \). Then \( \sigma_1 \) is the set of all spheres which touch \( C \) at \( p \).

**Proof:** Suppose that a sphere \( S(P, Q; \sigma_1) \) of \( \sigma_1 \) meets \( C \) in a point \( R \neq p \). If \( R = C(Q; \gamma_1) \), then by Theorem 5.5 and Theorem 5.2, Corollary 1,

\[ S(P, Q; \sigma_1) \supseteq C(Q; \gamma_1) = C, \]
while if $R \not\in C(Q; \gamma_1)$,

$$S(P,Q;\sigma_1) = S[R;C(Q;\gamma_1)] = S(R,Q;\sigma_1)$$

$$S[Q;C(R;\gamma_1)] = S(Q;C) \supset C.$$ 

Conversely, suppose that a sphere $S$ touches $C$ at $p$.

If $S \supset C$, then $S \in \sigma_1$ (Theorem 5.5). If $S \cap C = p$, choose a point $Q \subset S$, $Q \neq p$. Let $C_o = S(Q;C) \cap S$. Then $C_o$ touches $C$ at $p$. By Theorem 5.2, Corollary 2, $C_o \in \gamma_1$. Since $S \supset C_o$ and $C_o \in \gamma_1$, it follows from Theorem 5.5 that $S \in \sigma_1$.

**Theorem 5.7.** If $A$ satisfies $\Sigma_1$ and $\Sigma_2$ at $p$, then $\Gamma_1$ and $\Gamma_2$ will also hold there, and equations (5.2) and (5.4)

$$C(\gamma_2) = \prod_p S(P;\sigma_2)$$

will be satisfied there. Conversely, let $A$ satisfy $\Gamma_1$ and $\Gamma_2$ at $p$, and let $C(\gamma_2) \neq p$. If $P \notin C(\gamma_2)$, then $\Sigma_2$ will hold at $p$ for $P$, and $S(P;\sigma_2)$ will be the sphere through $P$ and $C(\gamma_2)$.

**Proof:** Suppose that $\Sigma_1$ and $\Sigma_2$ hold at $p$. In view of Theorem 5.5, we have only to show that $\Sigma_2$ implies $\Gamma_2$,
and that relation (5.4) holds. By relation (5.2),

\[
\lim_{t \to p} C(t; \gamma_1) = \lim_{t \to p} \prod_p S(P, t; \sigma_1) = \prod_p S(P; \sigma_2).
\]

Hence \( C(t; \gamma_1) \) converges, and \( C(\gamma_2) = \prod_p S(P; \sigma_2) \). Thus \( \Sigma_2 \) implies \( \Gamma_2 \) and relation (5.4) holds.

Conversely, suppose that \( \Gamma_1 \) and \( \Gamma_2 \) hold and that \( C(\gamma_2) \notin p \). If \( P \notin C(\gamma_2) \), then \( P \notin C(t; \gamma_1) \) when \( t \) is sufficiently close to \( p \), and by Theorem 5.5,

\[
S[P; C(\gamma_2)] = \lim_{t \to p} S[P; C(t; \gamma_1)] = \lim_{t \to p} S(P, t; \sigma_1).
\]

Hence \( S(P, t; \sigma_1) \) exists and converges. Thus \( S(P; \sigma_2) = S[P; C(\gamma_2)] \).

**Corollary 1.** If \( A \) satisfies \( \Sigma_1 \) (\( \Sigma_1 \) and \( \Sigma_2 \)) at \( p \), then \( A \) is once- (twice-) differentiable there.

In particular, this implies

**Corollary 2.** If \( p \) is a differentiable point of \( A \),

then \( \Gamma_1 \) and \( \Gamma_2 \) hold there.

**Corollary 3.** \( S(\sigma_3) \Rightarrow C(\gamma_2) \).
Proof: By relation (5.4),

$$S(t;\sigma_2) \supset \prod_p S(p;\sigma_2)$$

$$= C(\gamma_2).$$

Hence $S(\sigma_3) \supset C(\gamma_2)$.

This implies

Corollary 4. If $S(\sigma_3) = p$, then $C(\gamma_2) = p$.

Corollary 5. If $C(\gamma_2) \neq p$, $\sigma_2$ consists of the spheres through $C(\gamma_2)$.

The conditions $\Gamma_1$ and $\Gamma_2$ by themselves do not imply $\Sigma_1$ in general, whether or not $C(\gamma_2) = p$. Consider, for example, the arc

$$x = t, \quad y = \begin{cases} t^3 \sin t^{-1}, & 0 \leq |t| \leq 1 \\ 0, & t = 0 \end{cases}, \quad z = \begin{cases} t^3 \cos t^{-1}, & 0 \leq |t| \leq 1 \\ 0, & t = 0 \end{cases},$$

which satisfies $\Gamma_1$ and $\Gamma_2$ at $t = 0$, $C(\gamma_2)$ being the x-axis.

When $P = \infty$, $Q = (1,0,0)$, the sphere $S(P,Q,t;\sigma_0)$ is a plane through the x-axis, and this plane does not converge when $t \to 0$. Thus $\Sigma_1$ is not satisfied.

Condition $\Sigma_1$ is a very strong one, for it implies
not only $\Gamma_1$, but, as the following theorems show, $\Sigma_2$ and $\Gamma_2$ as well, and even $\Sigma_3$ in the case $C(\gamma_2) \neq p$.

**Theorem 5.8.** Suppose that $A$ satisfies $\Sigma_1$ at $p$. Then $A$ also satisfies $\Sigma_2$ at $p$.

**Proof:** Let $P$ be any point $\neq p$. Theorem 5.4 implies that $t$ does not lie on $C(P;\gamma_1^*)$ if $t$ is close to $p$. Hence by Theorem 5.5, $S(P,t;\sigma_1^*) = S[t;C(P;\gamma_1^*)]$. Let $Q \subset C(P;\gamma_1^*)$, $Q \neq P,p$. Then $C(P;\gamma_1^*) = C(P,Q;\gamma_0)$. Thus

\[ S(P,t;\sigma_1^*) = S[t;C(P,Q;\gamma_0)] = S(P,Q,t;\sigma_0), \]

and $\Sigma_1$ now implies that

\[ \lim_{t \to p} S(P,t;\sigma_1^*) = S(P,Q;\sigma_1^*). \]

Since $S(P;\sigma_2^*)$ exists for each point $P \neq p$, $\Sigma_2$ is satisfied.

**Corollary 1.** If $A$ satisfies $\Sigma_1$ at $p$, it also satisfies $\Gamma_2$ there.

**Proof:** By Theorem 5.7, condition $\Sigma_2$ implies $\Gamma_2$.

**Corollary 2.** If $A$ satisfies $\Sigma_1$ at $p$, then $p$ is a
differentiable point of $A$ if and only if $S(t;\sigma_2)$ converges as $t \to p$.

Relation (5.5) implies

**Corollary 3.** $S(P;\sigma_2) \in \sigma_1$.

**Theorem 5.9.** Suppose that $A$ satisfies $\Sigma_1$ (and hence $\Sigma_2$, $\Gamma_1$, and $\Gamma_2$) at $p$, and suppose that $C(y_2) \neq p$. Then $A$ also satisfies $\Sigma_3$ at $p$.

**Proof:** If $t$ is close to, but different from, $p$, $S(t;\sigma_2)$ is defined. By Theorem 5.4, $t \notin C(y_2)$, and by Theorem 5.7, $S(t;\sigma_2) = S[t;C(y_2)]$. Let $P \subset C(y_2)$, $P \neq p$. Then by Theorem 5.3, Corollary 1, $C(y_2) = C(P;y_1)$ and hence

$$S(t;\sigma_2) = S[t;C(P;y_1)] = S(P,t;\sigma_1).$$

Condition $\Sigma_2$ now implies that

$$(5.6) \lim_{t \to p} S(t;\sigma_2) = \lim_{t \to p} S(P,t;\sigma_1) = S(P;\sigma_2).$$

Thus $S(t;\sigma_2)$ converges, and $\Sigma_3$ holds.
Corollary 1. If A satisfies condition $\Sigma_1$ at $p$, and if $C(\vec{y}_2) \neq p$, then $p$ is a differentiable point of $A$.

The following example shows that $p$ need not be a differentiable point of $A$ when $\Sigma_1$ is satisfied and $C(\vec{y}_2) = p$.

Consider the arc defined by

$$x = t^2, \quad y = t^3, \quad z = \begin{cases} t^4 \sin t^{-1}, & 0 < |t| < 1 \\ 0, & t = 0 \end{cases}.$$ 

It can readily be verified that $A$ satisfies $\Sigma_1$ at $t = 0$, and that the spheres of $\sigma_2$ touch the xy-plane at the origin. Thus $C(\vec{y}_2)$ is a point circle. However, as $t \to 0$, $S(t; \sigma_2)$ oscillates, and $x^2 + y^2 + z^2 \pm z = 0$ are two accumulation spheres of the sequence $S(t; \sigma_2)$. Thus $\Sigma_3$ does not hold at $t = 0$.

Theorem 5.10. Let $\Sigma_1$ hold at $p$, and let $C(\vec{y}_2) = p$.

Then $\sigma_2$ is the set of spheres which touch a given proper sphere of $\sigma_2$ at $p$.

Proof: Let $P$ and $Q$ be variable points, and let $C$ be a variable circle converging to a fixed point. Suppose there
is a sphere which separates this point from $P$ and $Q$. Then

$$\lim x[S(P;C);S(Q;C)] = 0$$

whether or not the spheres $S(P;C)$ and $S(Q;C)$ themselves converge. In particular, let $P$ and $Q$ be fixed points $\neq p$, and let $C = C(t;\gamma_1) \to p$, as $t \to p$, $t \in A$, $t \neq p$. Then

$$(5.7) x[S(P;\sigma_2);S(Q;\sigma_2)] = \lim_{t \to p} x[S(P,t;\sigma_1);S(Q,t;\sigma_1)] = 0.$$ 

Hence any two spheres of $\sigma_2$ touch at $p$.

Conversely, let $S$ be a sphere which touches $S(P;\sigma_2)$. Choose a point $Q \subset S$, $Q \neq p$. Then $S(Q;\sigma_2)$ also touches $S(P;\sigma_2)$ at $p$, and $S(Q;\sigma_2) = S$. Thus $S \in \sigma_2$.

**Corollary 1.** $\sigma_2$ is the family of spheres, the intersection of any two of which is $C(\gamma_2)$ (cf. Theorem 5.7, Cor. 5).

**Corollary 2.** There is one and only one sphere of $\sigma_2$ through each point $\not\in C(\gamma_2)$; i.e., if $Q \subset S(P;\sigma_2)$,

1. This statement becomes obvious if we let $P$ or $Q$ be the fixed point at infinity.
$Q \not\in C(\gamma_2)$, then $S(P;\sigma_2) = S(Q;\sigma_2)$.

**Theorem 5.11.** If $p$ is a differentiable point of $A$, then

$$\sigma_0 \supset \sigma_1 \supset \sigma_2 \supset \sigma_3.$$  \hspace{1cm} (5.8)

**Proof:** Evidently $\sigma_0 \supset \sigma_1$. Theorem 5.8, Corollary 3 shows that $\sigma_1 \supset \sigma_2$. This can also be seen as follows: let $P \neq p$. By Theorem 5.6, any sphere $S(P;\sigma_2)$ of $\sigma_2$ is the limit of a sequence of spheres $S(P,t;\sigma_1)$, each of which touches a proper circle $C \in \gamma_1$ at $p$. Thus $S(P;\sigma_2)$ also touches $C$ at $p$, and $S(P;\sigma_2) \in \sigma_1$.

Let $C(\gamma_2) \neq p$. By Theorem 5.7, $\sigma_2$ is the set of all the spheres through $C(\gamma_2)$. Hence $S(\sigma_3)$, being the limit of a sequence of spheres through $C(\gamma_2)$, is itself a sphere through $C(\gamma_2)$, and thus a sphere of $\sigma_2$. Relation (5.6) also implies that $\sigma_2 \supset \sigma_3$ when $C(\gamma_2) \neq p$. Suppose $C(\gamma_2) = p$. By Theorem 5.10, $\sigma_2$ is the set of all the spheres which touch a given sphere $\neq p$ of $\sigma_2$ at $p$. Hence $S(\sigma_3)$,
being the limit of a sequence of such tangent spheres, is itself a sphere of $\sigma_2$.

This section can be summarized by the following remark: let $p$ be a differentiable point of an arc $A$. Let $P \neq p$. In addition, if $S(\sigma_3) \neq p$, let $P \subset S(\sigma_3)$. Let

$$C = \begin{cases} C(\gamma_2) \text{ if } C(\gamma_2) \neq p, \\ C(P; \gamma_1) \text{ if } C(\gamma_2) = p \end{cases}, \quad S = \begin{cases} S(\sigma_3) \text{ if } S(\sigma_3) \neq p, \\ S(P; \sigma_2) \text{ if } S(\sigma_3) = p \end{cases}.$$  

Then $C \subset S$, and the structures of $\gamma_1$, $\sigma_1$, and $\sigma_2$ are completely determined by $C$ and $S$.

5.6 Intersection and Support Properties of the Families $\sigma_0 - \sigma_1$, $\sigma_1 - \sigma_2$, and $\sigma_2 - \sigma_3$.

Let $p$ be a differentiable interior point of $A$.

**Theorem 5.12.** Every sphere $S \neq S(\sigma_3)$ either supports or intersects $A$ at $p$.

**Proof:** If $S$ neither supports nor intersects $A$ at $p$, then $p \subset S$, and there exists a sequence of points $t \to p$, $t \in A \cap S$, $t \neq p$. We may assume that conditions $\Sigma_1$, $\Sigma_2$, and $\Sigma_3$ hold for this sequence since they hold for any
sequence $t \rightarrow p$, $t \in A$, $t \neq p$. Choose points $P$ and $Q$ on $S$ such that $P, Q,$ and $p$ are mutually distinct. Then condition \( \Sigma_1 \) implies that $S = S(P, Q, t; \sigma_0)$ for each $t$, and hence $S = S(P, Q, \sigma_1)$.

By Theorem 5.5, $S = S(P, Q; \sigma_1)$ $\supseteq C(P; \gamma_1)$, and by Theorem 5.4, $t \not\in C(P; \gamma_1)$, and again by Theorem 5.5,

$$S = S[t; C(P; \gamma_1)] = S(P, t; \sigma_1).$$

Condition \( \Sigma_2 \) now implies that $S = S(P; \sigma_2)$.

Finally, by Theorem 5.7, $S \supseteq C(\gamma_2)$, and by Theorem 5.4, $t \not\in C(\gamma_2)$. If $C(\gamma_2) \neq p$, Theorem 5.7 implies that

$$S = S[t; C(\gamma_2)] = S(t; \sigma_2),$$

while if $C(\gamma_2) = p$, Theorem 5.10 implies that $S = S(t; \sigma_2)$. Applying the condition \( \Sigma_2 \), we are led to the contradiction $S = S(\sigma_3)$.

**Theorem 5.13.** If $S(\sigma_3) = p$, then the spheres of $\sigma_2 - \sigma_3$ all intersect $A$ at $p$, or they all support.

**Proof:** Let $S'$ and $S''$ be two distinct spheres of $\sigma_2 - \sigma_3$. Since $S(\sigma_3) = p$, Theorem 5.7, Corollary 4 implies...
that \( S' \) and \( S'' \) touch at \( p \). Thus we may assume that
\[
S'' \subset (p \cup \overline{S'}) \text{ and } S' \subset (p \cup \overline{S''}).
\]
Suppose now, for example, that \( S' \) supports \( A \) at \( p \) while \( S'' \) intersects (cf. Fig 5.1).

Then \( A \cap \overline{S''} \) is not void, and hence \( A \subset (p \cup \overline{S'}) \).
Let \( t \to p \)

\[
\text{Fig 5.1}
\]

in \( A \cap \overline{S''} \); thus \( t \subset \overline{S''} \cap \overline{S'} \). Hence

\[
S(t;\sigma_2) \subset (\overline{S''} \cap \overline{S'}) \cup p.
\]

Consequently \( S(t;\sigma_2) \) cannot converge to \( S(\sigma_3) = p \) as \( t \to p \).

Thus \( S' \) and \( S'' \) must both support or both intersect \( A \) at \( p \).

**Theorem 5.14.** If \( S(\sigma_3) \neq p \) and \( C(\sigma_2) = p \), then every

sphere of \( \sigma_2-\sigma_3 \) supports \( A \) at \( p \).
Proof: Suppose that \( C(\gamma_2) = p \), so that the spheres of \( \sigma_2 \) all touch at \( p \) (Theorem 5.10). Let \( S \in \sigma_2, S \neq S(\sigma_3), \) \( S \neq p \). If a sequence of points \( t \) exists such that \( t \in A \cap \bar{S} \) \( t \rightarrow p \), then each \( S(t;\sigma_2) \) lies in the closure of \( \bar{S} \). Hence \( S(\sigma_3) \) will lie in the same domain, and therefore even in \( pu\bar{S} \).

Similarly, the existence of a sequence \( t' \in A \cap \bar{S}, t' \rightarrow p \), implies that \( S(\sigma_3) \subset p \cup \bar{S} \). Thus if \( S \) intersects \( A \) at \( p \), \( S(\sigma_3) \subset (p \cup \bar{S}) \cap (p \cup \bar{S}) = p \); in other words, \( S(\sigma_3) = p \).

**Theorem 5.15.** All the spheres of \( \sigma_0 - \sigma_1 (\sigma_1 - \sigma_2; \sigma_2 - \sigma_3) \) support \( A \) at \( p \), or they all intersect.

Proof: Let \( S' \) and \( S'' \) be two distinct spheres of \( \sigma_0 - \sigma_1 (\sigma_1 - \sigma_2; \sigma_2 - \sigma_3) \). Suppose for the moment that the intersection \( S' \cap S'' \) is a proper circle \( C_0 = C(P,C;\gamma_0) \) \( (C_1 = C(P;\gamma_1); C_2 = C(\gamma_2)) \). Suppose, for example that \( S' \) intersects while \( S'' \) supports \( A \) at \( p \). With no loss in generality, we may assume that \( A \subset \bar{S}'' \cup p \). Thus \( A \cap \bar{S}' \) and \( A \cap \bar{S}' \) are not void (cf Fig 5.2). If \( t \in A \cap \bar{S}' \) by Theorems 5.4, 5.5, and 5.7, \( S(P,Q,t;\sigma_0) = S(t;C_0) \) \( (S(P,t;\sigma_1) = S(t;C_1); \)
\( S(t;\sigma_2) = S(t;C_2) \) lies in the closure of 
\[
(\bar{S}' \cap \bar{S}'' \cap \bar{S}''') \cup (\bar{S}' \cap \bar{S}''').
\]

Letting \( t \to p \) on \( A \), we conclude that \( S(P,Q;\sigma_1) \) \((S(P;\sigma_2);S(\sigma_3))\) lies in the same closed domain. By letting \( t' \) converge to \( p \) through \( \bar{S}' \cap A \), we obtain symmetrically that \( S(P,Q;\sigma_1) \) \((S(P;\sigma_2);S(\sigma_3))\) also lies in the closure of 
\[
(\bar{S}' \cap \bar{S}'' \cap \bar{S}''') \cup (\bar{S}' \cap \bar{S}''').
\]

Hence \( S(P,Q;\sigma_1) \) \((S(P;\sigma_2);S(\sigma_3))\) lies in the intersection \( S' \cup S'' \), of these two domains, i.e., \( S(P,Q;\sigma_1) \) \((S(P;\sigma_2);S(\sigma_3))\) is either \( S' \) or \( S'' \), contrary to our assumptions. Thus \( S' \) and \( S'' \) both support or they both intersect in this case.

Suppose now that \( S' \cap S'' = p \). In view of Theorems
5.13 and 5.14, there remain to be considered only the cases where \( S' \) and \( S'' \) both belong to \( \sigma_0 - \sigma_1 \), or both belong to \( \sigma_1 - \sigma_2 \). By Theorem 5.6, any sphere \( S \) through \( p \), which does not touch a circle \( C \) of \( \gamma_1 \), belongs to \( \sigma_0 - \sigma_1 \); by Theorem 5.6, Theorem 5.7, Corollary 5, and Theorem 5.10, any sphere \( S \) which touches a circle \( C \) of \( \gamma_1 \) but does not contain \( C(\gamma_2) \) in case \( C(\gamma_2) \neq p \), or does not touch a proper sphere of \( \sigma_2 \) in case \( C(\gamma_2) = p \), belongs to \( \sigma_1 - \sigma_2 \).

Hence there exists a sphere \( S \) of \( \sigma_0 - \sigma_1 \) (\( \sigma_1 - \sigma_2 \)) which intersects \( S' \) and \( S'' \) respectively in a proper circle.

From the above, \( S \) and \( S' \), and also \( S \) and \( S'' \), both support or both intersect \( A \) at \( p \). Thus \( S' \) and \( S'' \) both support or both intersect \( A \) at \( p \).

**Theorem 5.16.** If \( C(\gamma_2) \neq p \), every sphere of \( \sigma_1 - \sigma_2 \) supports \( A \) at \( p \).

**Proof:** Suppose \( S \in \sigma_1 - \sigma_2 \) intersects \( A \) at \( p \). Let \( t \rightarrow p, t \in A \cap S, t \neq p \). By Theorem 5.6, \( C(t; \gamma_1) \) touches \( S \) at \( p \) and hence \( C(t; \gamma_1) \subseteq S \cup p \). Since \( C(t; \gamma_1) \rightarrow C(\gamma_2) \), it
follows that \( C(\mathcal{V}_2) \subseteq \mathcal{S} \cup \mathcal{S} \). If \( t' \) converges to \( p \) through \( A \cap \mathcal{S} \), it follows symmetrically that \( C(\mathcal{V}_2) \subseteq \mathcal{S} \cup \mathcal{S} \). Thus \( C(\mathcal{V}_2) \subseteq \mathcal{S} \). Since \( \mathcal{S} \notin \sigma_2 \), however, Theorem 5.7 implies that \( C(\mathcal{V}_2) = p \).

5.7 A Classification of the Differentiable Points.

The characteristic, \((a_0, a_1, a_2, a_3; i)\), of a differentiable point \( p \) of an arc \( A \) is defined as follows:

- \( i = 1, 2, \text{ or } 3 \).
- \( a_0 = 1 \) or \( 2 \).
- \( a_1 = 1 \) or \( 2 \).
- \( a_2 = 1 \) or \( 2 \).
- \( a_3 = 1, 2, \text{ or } \infty \).

- \( i = 1 \) if \( C(\mathcal{V}_2) \neq p \); \( i = 2 \) if \( C(\mathcal{V}_2) = p \), \( S(\sigma_3) \neq p \);
- \( i = 3 \) if \( S(\sigma_3) = p \).

- \( a_0 \) is even or odd according as the spheres of \( \sigma_0 - \sigma_1 \) support or intersect.

- \( a_0 + a_1 \) is even or odd according as the spheres of \( \sigma_1 - \sigma_2 \) support or intersect.

- \( a_0 + a_1 + a_2 \) is even or odd according as the spheres
of \( \sigma_2-\sigma_3 \) support or intersect.

\[ a_0+a_1+a_2+a_3 \] is even if \( S(\sigma_3) \) supports, odd if \( S(\sigma_3) \) intersects, while \( a_3=\infty \) if \( S(\sigma_3) \) neither supports nor intersects.

Theorems 5.16, 5.14, and the convention that \( S(\sigma_3) \) supports when it is the point-sphere, lead to the following restrictions on the characteristic:

If \( i=1 \), then \( a_0+a_1 \) must be even;

if \( i=2 \), then \( a_0+a_1+a_2 \) must be even;

if \( i=3 \), then \( a_0+a_1+a_2+a_3 \) must be even.

As a result of these restrictions, there are just 32 types of differentiable points; there are 12 when \( i=1 \), 12 when \( i=2 \), and 8 when \( i=3 \).

Examples of each of the 32 types are given by the curves

(I) \[ x = t^m, \ y = t^n, \ z = t^r, \]

for the cases \( a_3=1 \) or 2, and

(II) \[ x = t^m, \ y = t^n, \ z = \begin{cases} t^r \sin t^{-1}, & \text{if } 0 < |t| \leq 1 \\ 0, & t = 0 \end{cases} \]
for the cases $a_2 = \infty$, all relative to the point $t = 0$.

The indices $m, n,$ and $r$ are positive integers and $m < n < r$.

The different types are determined by the parities of the indices $m, n,$ and $r$, and the relative magnitudes of $m, n, r,$ and $2m$. In each of these examples the circles of $\gamma_1$ and the spheres of $\sigma_2$ touch the $x$-axis at the origin. In the case $i = 1$, $\sigma_2$ is the family of planes through the $x$-axis, while in each of the cases $i = 2$ or $3$, $\sigma_2$ is the family of spheres which touch the $xy$-plane at the origin (cf. remark at the end of §5.5).

Table 5.1 lists examples of all the types of differentiable points, together with their characteristics; table 5.2 summarizes properties of these types. Congruences are mod 2. Figures 5.3 to 5.14 inclusive illustrate the various types of curves, differentiable at the origin, having the indicated characteristics at this point.
<table>
<thead>
<tr>
<th>Equation</th>
<th>(i = 1)</th>
<th>(m &lt; 2m &lt; n &lt; r)</th>
<th>(i = 2)</th>
<th>(m &lt; n &lt; 2m &lt; r)</th>
<th>(i = 3)</th>
<th>(m &lt; n &lt; r &lt; 2m)</th>
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</thead>
<tbody>
<tr>
<td>((1,1,1,1;1))</td>
<td>((1,1,2,1;2))</td>
<td>((1,1,1,1;3))</td>
<td>((1,1,2,2;3))</td>
<td>((1,1,1,3))</td>
<td>((1,1,2,3))</td>
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<tr>
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<td>((1,2,1,3))</td>
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<tr>
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<tr>
<td>((2,2,1,1;1))</td>
<td>((2,2,1,2;2))</td>
<td>((2,2,1,3))</td>
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<td>((2,2,2,2;2))</td>
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<td>((2,2,2,3))</td>
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</tr>
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</table>

**Table 5.1**
<table>
<thead>
<tr>
<th>No.</th>
<th>( C(\gamma_2) )</th>
<th>( S(\sigma_3) )</th>
<th>Characteristic ( (a_0, a_1, a_2, a_3; 1) )</th>
<th>Restrictions</th>
<th>Examples: (I) or (II)</th>
<th>No. of Types</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \neq p )</td>
<td>( \neq p )</td>
<td>( a_3 = 1 \text{ or } 2 ) ( (a_0, a_1, a_2, a_3; 1) )</td>
<td>( a_0 + a_1 \equiv 0 )</td>
<td>( m &lt; 2m &lt; n &lt; r ) ( x )-axis ( xy )-plane</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>( = p )</td>
<td>( \neq p )</td>
<td>( a_1 = 1 \text{ or } 2 ) ( (a_0, a_1, a_2, \infty; 1) )</td>
<td>( a_0 + a_1 + a_2 \equiv 0 )</td>
<td>( m &lt; n &lt; 2m &lt; r ) ( x = y = z = 0 ) ( xy )-plane</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>( p )</td>
<td>( p )</td>
<td>( a_3 = 1 \text{ or } 2 ) ( (a_0, a_1, a_2, a_3; 3) )</td>
<td>( a_0 + a_1 + a_2 + a_3 \equiv 0 )</td>
<td>( m &lt; n &lt; r &lt; 2m ) ( x = y = z = 0 ) ( x = y = z = 0 )</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 5.2
CHAPTER VI

CHARACTERISTIC AND ORDER OF DIFFERENTIABLE POINTS IN CONFORMAL 3-SPACE

6.1. Introduction.

The goal of this chapter is the proof of the following theorem, which is analogous to Theorem 3.1.

Theorem 6.1. Let p be a differentiable point of an arc A in conformal 3-space. Suppose that p has characteristic \((a_0, a_1, a_2, a_3; i)\). Then the conformal order of p is not less than \(a_0 + a_1 + a_2 + a_3\).

This theorem implies

Corollary 1. If the order of p is bounded, then to every neighbourhood of p there corresponds a sphere arbitrarily close to \(S(\sigma_3)\) which does not pass through p, and which intersects that neighbourhood in not less than \(a_0 + a_1\)
An arc $A$ is said to be of finite spherical order if it has only a finite number of points in common with any sphere. If some sphere meets $A$ $n$ times, and no sphere meets $A$ more than $n$ times, where $n$ is some specific integer, then $A$ is said to be of bounded spherical order, and $n$ is called the (spherical) order of $A$. If $p$ is any point on $A$, the order of $p$ is the minimum of the orders of all the neighbourhoods of $p$ on $A$.

Lemma 6.1. Let $B$ be an arc of finite order. If a sphere $S$ intersects $B$ at $t$, then every sphere sufficiently close to $S$ intersects $B$ in at least one point.

Proof: The end-points of some neighbourhood $M \subseteq B$ of $t$ lie in different regions with respect to $S$. Hence they also lie in different regions with respect to any sphere $S'$ sufficiently close to $S$. Since $M$ and $S'$ have only a finite
number of points in common, one of them must be an intersection.

It is clear that $S'$ will intersect $M$ in an odd number of points.


The ensuing discussion simplifies the proof of Theorem 6.1. As in section 5.4, $\sigma_3$, $\sigma_2$, $\sigma_1$, and $\sigma_0$ will denote the families of tangent spheres $S(\sigma_3)$, $S(P;\sigma_2)$, $S(P,Q;\sigma_1)$, and $S(P,Q,R;\sigma_0)$ respectively. Now suppose that $P$, $Q$, and $R$ are fixed points such that $p \notin C(P,Q,R)$. The symbol $\Pi_4$ will denote $S(\sigma_3)$; $\Pi_3(t)$, $\Pi_2(t)$, and $\Pi_1(t)$ will denote the linear families of spheres $S(t;\sigma_2)$, $S(P,t;\sigma_1)$, and $S(P,Q,t;\sigma_0)$ respectively. $\Pi_0(t)$ will denote the linear family of spheres $S(P,Q,R,t)$.

6.3.1. Let $M$ be any neighbourhood of $p$ on $A$. We wish to show that to every sphere $S_{r-1} \in \Pi_{r-1}$ there corresponds a sphere of $\Pi_r$ arbitrarily close to, but different from, $S_{r-1}$.
which meets \( M \) outside \( p \) in not less than \( a_r \) points. If \( p \) has finite order, and if \( \tilde{M} \) is small enough, we can even say that there are spheres of \( \tilde{\Pi}_r \) close to \( S_{r-1} \) which meet \( M \) outside \( p \) in \( a_{r+2n} \) points, \( n \geq 0 \) (\( r = 0, 1, 2, 3; \) we assume that \( a_3 < \infty \) when \( r = 3 \)).

6.3.2 Let \( T_r \) be a sphere of the family \( \tilde{\Pi}_r - \tilde{\Pi}_{r+1}, r = 0, 1, 2, 3 \). If \( r = 3 \) and \( C(\mathcal{V}_2) = p \), let \( E_3 = S(\sigma_3) \) when \( S(\sigma_3) \neq p \), but let \( E_3 = T_3 \) when \( S(\sigma_3) = p \). If \( r < 3 \), or if \( r = 3 \) and \( C(\mathcal{V}_2) \neq p \), \( E_r \) will not be defined. In any case, we define the regions

\[
E_r = \left[ \overline{T_r \cap S_{r+1}} \right] \cup \left[ \overline{T_r \cap \overline{S}_{r+1}} \right]
\]

and

\[
\overline{E_r} = \left[ \overline{T_r \cap S_{r+1}} \right] \cup \left[ \overline{T_r \cap S_{r-1}} \right]
\]

(cf. Fig. 6.1).

Let \( \overline{\Pi_r} \) (\( \overline{\Pi_r} \)) denote the set of those spheres of \( \overline{\Pi_r} \)

1. If \( S_{r+1} \) is the point-sphere \( p \), \( S_{r+1} \) is void, and \( \overline{S}_{r+1} \) is the whole plane with the exception of the point \( p \).
(a) $0 \leq r \leq 3$, $C(Y_2) \neq p$ when $r = 3$

(b) $r = 3$, $C(Y_2) = p$, $S(\sigma_3) \neq p$

(c) $r = 3$, $C(Y_2) = p$, $S(\sigma_3) = p$

Fig. 6.1
that pass through $E_R$ ($\overline{E}_R$). Then every sphere of $\overline{\Pi}_R$ except $T_R$ and $S_{r+1}$ belongs either to $\overline{\Pi}_R$ or to $\overline{E}_R$. By intersecting $\overline{\Pi}_R$ with an orthogonal circle $C_o$, we can construct a 1-1 correspondence between the spheres of $\overline{\Pi}_R$ ($\overline{E}_R$) and the points of $C_o$, and hence an ordering of the spheres in $\overline{\Pi}_R$ ($\overline{E}_R$).

We can choose our neighbourhood $M$ so small that $T_R$ and $S_{r+1}$ have no points in common with the two one-sided neighbourhoods $N$ and $N'$ into which $M$ is decomposed by $p$. This follows for $S = S(\sigma_3^J)$ from our assumption $a_3 < \infty$, and for the other spheres it follows from Theorem 5.12. Thus $N (N')$ lies entirely in the region $E_R$ or else entirely in the region $\overline{E}_R$. Let $t$ and $t'$ denote the points of $N$ and $N'$ respectively; thus either all the spheres of $\Pi_R(t)$ belong to $\Pi_R$ or all of them are in $\overline{\Pi}_R$. Without restriction of generality, let $N \subset \overline{T_R \cap S_{r+1}} \subset \overline{E}_R$. Then $\Pi_R(t)$ belongs to $\overline{\Pi}_R$ for every $t$. Let $e \in N$. Then $\Pi_R(e)$ is the end-sphere of a one-sided neighbourhood $\delta$ of $S_{r+1}$ in $\overline{\Pi}_R$. If $t$ moves from $e$ to $p$, then $\Pi_R(t)$ moves in $\overline{\Pi}_R$ from $\Pi_R(e)$ to $S_{r+1}$. Hence the
spheres $\pi_r(t)$ omit none of the spheres of $\mathcal{S}$, i.e., every sphere of $\mathcal{S}$ meets $N$. Let $S \in \mathcal{S}$. Thus $S$ lies between $\pi_r(e)$ and $S_{r+1} = \lim_{t \to p} \pi_r(t)$. If $t$ is sufficiently close to $p$, then $t$ does not belong to $S$, and $S$ will also lie between $\pi_r(e)$ and $\pi_r(t)$. Since $e \not\in S$, and since the points $t$ and $e$ lie in $\overline{T_r} \cap S_{r+1}$, they will also be separated by $S$.

Let the order of $p$ be finite. Then we may assume that $M$ is also of finite order. In addition, $S$ will meet $N$ in a finite number of points only, and at least one of them will be an intersection. Replacing $N$ by the one-sided neighbourhood of $p$ with end-point $e$, we can even state that $S$ will intersect $N$ in an odd number of points.

Similarly, there exists a one-sided neighbourhood $\mathcal{S}'$ of $S_{r+1}$ in $\overline{T_r}$ such that each of its spheres meets $N'$.

If $p$ has finite order, and if $N'$ is sufficiently small, then $\mathcal{S}'$ can be chosen such that each sphere of $\mathcal{S}'$ intersects $N'$ in an odd number of points.
6.3.3. If \( a_r = 1 \), then one of the spheres \( T_r \) and \( S_{r+1} \) intersects, while the other one supports \( M \) at \( p \); therefore \( N' \in E_r \).

If \( a_r = 2 \), then \( T_r \) and \( S_{r+1} \) either both intersect or both support; hence \( N' \in E_r \). Thus the spheres \( \Pi_r(t') \) belong to \( \overline{\Pi_r} \) or to \( \overline{\Pi_r} \), according as \( a_r = 1 \) or \( 2 \). This holds true, in particular, of the spheres of the neighbourhoods \( \delta \) and \( \delta' \).

Since \( \delta \in \overline{\Pi_r} \), it follows that \( \delta \) and \( \delta' \) lie on opposite sides of \( S_{r+1} \) or on the same side, depending on whether \( a_r = 1 \) or \( a_r = 2 \). This implies our statements in section 6.3.1.

6.3.4. The proof of Theorem 6.1 now follows readily. Obviously, we may assume that the order of \( p \) is finite, and in particular, that \( a_3 < \infty \).

We prove our theorem by first approximating \( S_4 = S(\sigma_3) \) = \( \Pi_4 \) by a sphere \( S_3 \) of \( \Pi_3 \), \( S_3 \) by a sphere \( S_2 \) of \( \Pi_2 \), \( S_2 \) by a sphere \( S_1 \) of \( \Pi_1 \), and finally we approximate \( S_1 \) by a sphere which does not contain \( p \).

Let \( M_3 \) be a neighbourhood of finite order of \( p \) on \( A \).
From section 6.3.1, there exists a sphere $S_3 \in \Pi_3$ close to, but different from, $S(\sigma_3)$ which intersects $M_3$ in not less than $a_3$ points $t_3$ outside $p$.

In $M_3$ we construct mutually disjoint neighbourhoods $B_3$ of the $t_3$ and $M_2$ of $p$. Choose a point $P$ on $S_3$, $P \notin C(\gamma_2)$. Then $S_3 = S(P; \sigma_3)$. Let $\Pi_2$ be the pencil of spheres of $\sigma_2$ through $P$; thus $\Pi_2(t) = S(P, t; \sigma_2)$ and $S_3 = \lim_{t \to P} \Pi_2(t)$. By section 6.3.1, there exists a sphere $S_2 \in \Pi_2$ close to, but different from, $S_3$, which intersects $M_2$ in not less than $a_2$ points $t_2$ outside $p$, and which intersects each $B_3$.

In $M_2$, we construct mutually disjoint neighbourhoods $B_2$ of the $t_2$, and $M_1$ of $p$. Choose a point $Q$ on $S_2$, $Q \notin C(P; \gamma_1)$. Then $S_2 = S(P, Q; \sigma_1)$. Let $\Pi_1$ be the pencil of spheres of $\sigma_1$ through $P$ and $Q$. Thus $\Pi_1(t) = S(P, Q, t; \sigma_0)$, and $S_2 = \lim_{t \to P} \Pi_1(t)$. By section 6.3.1, there exists a sphere $S_1$ of $\Pi_1$ close to, but different from, $S_2$, which intersects $M_1$ in not less than $a_1$ points $t_1$ outside $p$, and which intersects
each of the $a_2 + a_3$ arcs $B_2$ and $B_3$.

In $M_1$, we construct mutually disjoint neighbourhoods $B_1$ of $t_1$ and $M_0$ of $p$. Choose a point $R \subset S$, $R \notin C(P,Q;Y_0)$. Let $\Pi_0$ be the pencil of spheres through $P, Q,$ and $R$, and let

$\Pi_0(t) = S(P,Q,R,t)$. Then $S_1 = \lim_{t \to p} \Pi_0(t)$. By section 6.3.1, there exists a sphere $S_0$ of $\Pi_0$ close to, but different from, $S_1$, which intersects $M_0$ in not less than $a_0$ points $t_0$ outside $p$, and which intersects each of the $a_1 + a_2 + a_3$ arcs, $B_1, B_2,$ and $B_3$. Altogether, $S_0$ will be close to $S(\sigma_3)$ and will intersect $M_3$ in not less than $a_0 + a_1 + a_2 + a_3$ points, all of which are different from $p$. 
CHAPTER VII

ARCS OF SPHERICAL ORDER FOUR IN CONFORMAL 3-SPACE

7.1. Introduction.

This chapter extends to three dimensions the work of sections 3.4 to 3.8 inclusive. The fact already noted in section 5.1, that we are now dealing with two continuous entities, the circle and the sphere, will make this work considerably more delicate than that of Chapter III.

We denote an arc of order four (cf. § 6.2) by the symbol $A_4$.

7.2. Two Lemmas on Arrows of Finite Spherical Order.

Lemma 7.1. A point of an arc $A$ of finite order converges if its parameter tends to one of the end-points of the parameter interval.

In particular, this is true of an arc of order four.
Proof: Let $t_{2i}$ and $t_{2i+1}$ ($i = 0, 1, 2, \ldots$) be two sequences of points whose parameters tend to the same endpoint of the parameter interval. Suppose that $\lim_{i \to \infty} t_{2i} = p$, and $\lim_{i \to \infty} t_{2i+1} = q$, where $p$ and $q$ are accumulation points, and $p \neq q$. We may assume that $t_{n+1}$ lies between $t_n$ and $t_{n+2}$ on the arc.

Let $S$ be a sphere which separates $p$ and $q$. Then for sufficiently large $n$, $S$ separates $t_n$ and $t_{n+1}$. Thus $S$ meets $A$ in an infinite number of points, contrary to our assumption.

By the above lemma, we see that $A_4$ has two well-defined end-points.

Lemma 7.2. An end-point of an arc $A$ of finite order is automatically differentiable (in the sense of section 5.4).

Proof: Obviously, $A$ has only a finite number of points in common with any circle $C(P, Q, p)$ through mutually
distinct points $P, A,$ and $p$. Thus, when $t \in A$ is sufficiently close to $p$, the sphere $S(P, Q, t; \sigma_0)$ is defined.

Suppose that there are two sequences of points, $t_{2i}$ and $t_{2i+1}$, different from $p$, converging on $A$ to $p$ such that the spheres $S_{2i} = S(P, Q, t_{2i}; \sigma_0)$ and $S_{2i+1} = S(P, Q, t_{2i+1}; \sigma_0)$ converge to different limit spheres, $S_0$ and $S_1$ respectively. We may assume that $t_{n+1}$ lies between $p$ and $t_n$. If $i$ is large, $S_{2i}$ ($S_{2i+1}$) will be close to $S_0$ ($S_1$). Let $S$ and $S'$ be two spheres through $S_0 \cap S_1$ which separate $S_0$ and $S_1$. Then $S$ and $S'$ will separate $S_n$ and $S_{n+1}$, and therefore $t_n$ and $t_{n+1}$ for every large $n$. Hence the subarc of $A$ bounded by $t_n$ and $t_{n+1}$ will meet $S \cup S'$. Thus $A$ will meet $S \cup S'$ an infinite number of times. This is impossible. Hence condition $\Sigma_1$ holds at $p$.

The above discussion shows that $\Sigma_2$ and $\Sigma_3$ also are satisfied at $p$ (cf. Theorems 5.8 and 5.9), where in the latter case, $C(\gamma_2) \neq p$. 
If \( C(y_2) = p \), then by Theorem 5.10, the spheres of \( \sigma_2 \) all touch at \( p \). Let \( t_{2i} \) and \( t_{2i+1} \) be two sequences of points converging to \( p \) in such a way that \( S(t_{2i}; \sigma_2) \) and \( S(t_{2i+1}; \sigma_2) \) approach two different limit spheres, \( S_0 \) and \( S_1 \) respectively. Each of these limit spheres, being the limit of a sequence of spheres that touch any sphere of \( \sigma_2 \) at \( p \), also touches any sphere of \( \sigma_2 \) at \( p \).

Suppose, to begin, that \( S_0 \) and \( S_1 \) are both proper spheres. Suppose further, that \( S_1 \subset S_0 \cup p \) and \( S_0 \subset S_1 \cup p \).

Consider a sphere \( S \subset \sigma_2 \), \( S \subset (S_0 \cap S_1) \cup p \). Then \( S \) separates \( S_0 \) and \( S_1 \) except at the point \( p \), i.e., \( S_1 \subset S = p \) and \( S_0 \subset S \cup p \), say. Hence, for sufficiently large \( i \), \( S(t_{2i+1}; \sigma_2) \subset S \cup p \) and \( S(t_{2i}; \sigma_2) \subset S \cup p \). Here again, the arc \( A \) crosses \( C \) an infinite number of times, which by our hypothesis is impossible.

If now, \( S_1 \) for instance, is the point-sphere \( p \), consider two proper spheres of \( \sigma_2 \), \( S \) and \( S' \), where \( S \subset S_0 \cup p \), \( S' \subset S_0 \cup p \), and \( S_0 \subset S \cap S' \cup p \). Then for sufficiently large \( i \),
$S(t_{2};\sigma_{2}) \subseteq (S \cap S') \cup p$, while $S(t_{21+1};\sigma_{2}) \subseteq S \cup S' \cup p$. Since these two regions are separated by $S \cup S'$, one or both of these spheres will meet $A$ an infinite number of times.

Since this again is impossible by our hypothesis, condition $\Sigma_{3}$ holds, and the point $p$ is differentiable.

7.3. Multiplicities:

7.3.1. We call a sphere $S$ a general $(r-1)$-tangent sphere of order of contact $r-1$ ($r \geq 2, 3,$ or $4$) at a point $t$ of an arc $A$ if there exists a sequence of $r$-tuples, $t_{1}, t_{2}, \ldots, t_{r}$, of points which converge on $A$ to $t$ such that $S$ is the limit of a sequence of spheres $S'$ through the $t_{1}$. Let $t \in A_{4}$. Any sphere through $t$ will intersect or support $A_{4}$ there. A general $(r-1)$-tangent sphere intersects $A_{4}$ at $t$ if $r$ is odd, and supports $A_{4}$ at $t$ if $r$ is even.

We usually call a general $3$-tangent sphere a general osculating sphere.

Let $p$ be an end-point of $A_{4}$. As in section 3.7, we introduce multiplicities and count $p$ $r$-times on any
sphere of \( \sigma_{r-1}-\sigma_r \) \((r=2,3)\), and four times on \( S(\sigma_3) \). A point \( t \in A_4 \cup p \) is counted \( r \) times on a general \((r-1)\)-tangent sphere \((r=2,3,4)\). We wish to prove the following theorem:

**Theorem 7.1.** No sphere meets \( A_4 \cup p \) more than four times; i.e., the inclusion of \( p \) and the introduction of multiplicities does not alter the order of \( A_4 \).

The proof of Theorem 7.1 results from the discussion in the remainder of section 7.3.

7.3.2. **Lemma 7.3.** If a sphere \( S \) meets \( A_4 \) in three points, then at least two of these points are intersections.

**Proof:** Let \( S \) meet \( A_4 \) in \( q_1, q_2, \) and \( q_3 \), and let \( M_i \) be sufficiently small neighbourhoods of \( q_i \) \((i=1,2,3)\). If \( q_i \) is a point of support, then there is a sphere close to \( S \) which meets \( M_i \) in two points.

Suppose that \( q_1, q_2, \) and \( q_3 \) are all points of support. If \( M_1, M_2, \) and \( M_3 \) all lie in \( S \), say, then there
exists a sphere close to $S$ which will meet $A_4$ at least six times. On the other hand, if $M_1, M_2 \subset S$ and $M_3 \subset \overline{S}$, then $S$ must intersect $A_4$ in a fourth point $q_4$; hence there is a sphere close to $S$ which meets $A_4$ at least five times (cf. Lemma 6.1). Both of these cases are impossible since $A_4$ is of order four.

We note that by the latter argument, if $M_1 \subset S$ and $M_3 \subset \overline{S}$, then $S$ intersects $A_4$ at some point.

Suppose that $q_1$ and $q_2$ are points of support, while $S$ is a point of intersection. If $M_1, M_2 \subset S$, then, as before, some sphere close to $S$ will intersect $A_4$ five times. If $M_1 \subset S$ and $M_2 \subset \overline{S}$, then let $r$ be the necessary point of intersection on $A_4 \cap S$, and let $S_0$ be a sphere which separates $q_1$ and $q_2$. Hence $S_0 \cap S = C$ is a proper circle. Without loss of generality, we may assume that $M_1 \subset (S \cap S_0) \cup q_1$ and $M_2 \subset (\overline{S} \cap \overline{S}_0) \cup q_2$. Then a sphere $S' \subset (S \cap S_0) \cup (\overline{S} \cap \overline{S}_0) \cup C$, which is sufficiently close to $S$, will meet $M_1$ and $M_2$ twice each. Since $r$ is a point of
intersection, \( S' \) will also meet \( A_4 \) near \( r \) (cf. Lemma 6.1).

This again is a contradiction.

Lemma 7.3 implies

**Lemma 7.4.** A sphere \( S \) through four points of \( A_4 \)

**Proof:** By Lemma 7.3, \( A_4 \cap S \) has at most one point

of support. If \( A_4 \cap S = q_1, q_2, q_3 \) and \( q_4 \), where \( q_4 \) is a point

of support, then there is a sphere close to \( S \) which meets

\( A_4 \) five times, once each near \( q_1, q_2, \) and \( q_3 \), and twice

near \( q_4 \). This is impossible.

7.3.3. Suppose that a sphere \( S \) through \( p \) meets \( A_4 \) in four

points, \( q_1, q_2, q_3, \) and \( q_4 \). By Lemma 7.4 they are all

intersections. Choose disjoint neighbourhoods \( N \) of \( p \) and

\( M \) of \( q \) which do not contain \( q_2, q_3, \) or \( q_4 \). If \( t \) converges

to \( p \) in \( N \), then \( S' = S(q_2, q_3, q_4, t) \) converges to \( S \). By

Lemma 6.1, \( S' \) will intersect \( M \) if \( t \) is sufficiently close to

\( p \). Hence this sphere meets \( A_4 \) in no fewer than five points,

contrary to the definition of \( A_4 \). This yields
Lemma 7.5. No sphere meets $A_4 \cup p$ in five points.

Lemmas 7.3, 7.4, and 7.5 imply

Lemma 7.6. A sphere $S$ through four points of $A_4 \cup p$ does not support $A_4$ at any of these points.

Proof: By Lemma 7.3, $A_4 \cap S$ has at most one point of support. If there is one point of support, Lemma 7.4 implies that $S$ goes through $p$. Hence a suitable sphere through $p$ which is close to $S$, will meet $A_4 \cup p$ five times, contrary to Lemma 7.5.

Note that Lemma 7.6 is a generalization of Lemma 7.4.

7.3.4. Suppose that a sphere $S$ of $\sigma_1$ meets $A_4$ in three points $q_1, q_2,$ and $q_3$. By Lemma 7.6, they are all intersections. Choose disjoint neighbourhoods $N$ of $p$ and $M$ of $q_1$, which do not contain $q_2$ or $q_3$. If $t$ converges to $p$ in $N$, then $S' = S(q_2, q_3, t, \sigma_0)$ converges to $S$. By Lemma 6.1, $S'$ will intersect $M$ if $t$ is sufficiently close to $p$. Hence this sphere meets $A_4 \cup p$ in no fewer than five points, con-
trary to Lemma 7.5. This yields

Lemma 7.7. No sphere of $\sigma_1$ meets $A_4$ in three points.

Suppose a sphere $S$ of $\sigma_1$ supports $A_4$ at $q$. Then some sphere of $\sigma_1$ close to $S$ will intersect a neighbourhood of $q$ in two points. This, with Lemma 7.7 yields

Lemma 7.8. If a sphere of $\sigma_1$ supports $A_4$ at some point, then it does not meet $A_4$ again.

7.3.5. Suppose that a sphere $S$ of $\sigma_2$ meets $A_4$ in two points $q_1$ and $q_2$. By Lemma 7.8, both points are intersections.

Choose disjoint neighbourhoods $N$ of $p$ and $M$ of $q_1$ which do not contain $q_2$. If $t$ converges to $p$ in $N$, then $S' = S(q_2, t; \sigma_1)$ converges to $S$. By Lemma 6.1, $S'$ will intersect $M$ if $t$ is sufficiently close to $p$. Hence this sphere meets $A_4$ in no fewer than three points, contrary to Lemma 7.7. This yields

Lemma 7.9. No sphere of $\sigma_2$ meets $A_4$ in two points.

Suppose that a sphere $S$ of $\sigma_2$ supports $A_4$ at $q$, and let $M$ be a small neighbourhood of $q$ on $A_4$. We consider two
cases:

(i) \( C(\mathcal{Y}_2) \neq p \). In this case, the spheres of \( \sigma_2 \) all contain \( C(\mathcal{Y}_2) \). However, \( q \not\in C(\mathcal{Y}_2) \), for if it did, we could find spheres of \( \sigma_2 \) through two points of \( A_4 \), contrary to Lemma 7.9. Let \( S_0 \neq S \) be a sphere of \( \sigma_2 \), and let \( M \subset S_0 \cap S_1 \). Then there is a sphere \( S' \in \sigma_2 \) passing through \( (S_0 \cap S) \cup (S_0 \cap S) \cup C(\mathcal{Y}_2) \), which is so close to \( S \) that it intersects \( M \) in two points.

(ii) \( C(\mathcal{Y}_2) = p \). In this case, the spheres of \( \sigma_2 \) all touch at \( p \). Let \( M \subset S \). Then there is a sphere \( S' \in S \cup p, S' \in \sigma_2 \), which is so close to \( S \) that it intersects \( M \) in two points.

Thus in either case, we have a sphere of \( \sigma_2 \) which meets \( A_4 \) in at least two points, contrary to Lemma 7.9.

Hence we have

**Lemma 7.10.** No sphere of \( \sigma_2 \) through a point \( q \in A_4 \) supports at that point.
7.3.6. Suppose that \( S(\sigma_3) \) meets \( A_4 \) in one point \( q \). By Lemma 7.10, it is an intersection. Choose disjoint neighbourhoods \( N \) of \( p \) and \( M \) of \( q \). If \( t \) converges to \( p \) in \( N \), then 
\[ S' = S(t;\sigma_2) \] converges to \( S(\sigma_3) \). By Lemma 6.1, \( S' \) will intersect \( M \) if \( t \) is sufficiently close to \( p \). Hence this sphere meets \( A_4 \) in no fewer than two points, contrary to Lemma 7.9. This yields

**Lemma 7.11.** \( S(\sigma_3) \) does not meet \( A_4 \).

7.3.7. **Multiplicities Relative to General Tangent Spheres.**

In the following we shall not consider general tangent spheres at the point \( p \), since we shall learn in section 7.4 that such spheres are members of the families \( \sigma_1, \sigma_2, \) or \( \sigma_3 \), depending on their order of contact.

**Lemma 7.12.** Let \( q_1, q_2 \rightarrow q \) on \( A_4 \), and let \( t \in A_4 \), \( t \neq q \). Let \( C(t,q_1,q_2) \rightarrow C_0 \). Then \( C_0 \) does not meet \( A_4 \cup p \) outside \( q \) and \( t \).

**Proof:** Suppose that \( C_0 \ni u, u \in A_4 \cup p, u \neq t, q \). Let
$v \in A_4 \cup p$, $v \not\in C_0$. Then $S(v;C_0) = \lim S[v;C(t,q_1,q_2)]$

$= \lim S(v,t,q_1,q_2)$ does not meet $A_4 \cup p$ elsewhere. Hence the end-points of a small neighbourhood of $q$ on $A_4$ are not separated by $q_1$ and $q_2$. Thus $S(v;C_0) = \lim S[v;C(t,u,q)]$

$= \lim S(v,t,u,q)$ intersects $A_4$ at $q$. Thus we have the proof by contradiction.

**Lemma 7.13.** Let $q_1,q_2,q_3 \rightarrow q$ on $A_4$ such that $C(q_1,q_2,q_3) \rightarrow C_0$. Then $C_0$ does not meet $A_4 \cup p$ outside $q$.

**Proof:** Suppose that $t \in C_0$, $t \in A_4 \cup p$, $t \neq q$. Thus $C_0 \neq q$. Choose a point $u \in A_4 \cup p$, $u \not\in C_0$, and let $S = S(u;C_0)$. Then $S(u,q_1,q_2,q_3) = S[u;C(q_1,q_2,q_3)] \rightarrow S_0$. Since $S(u,q_1,q_2,q_3)$ does not meet $A_4 \cup p$ elsewhere, the end-points of a small neighbourhood of $q$ on $A_4$ are separated by this sphere; hence its limit sphere $S$ must intersect $A_4$ at $q$. Since

$$S[t;C(t,q_1,q_2);C(q_1,q_2,q_3)] \rightarrow 0,$$

any accumulation circle, $C_1$, of $C(t,q_1,q_2)$ passes through $t$ and touches $C_0$ at $q$. Thus $C_1 = C_0$, and the sphere $S(u,t,q_1,q_2)$
not meet \( A_4 \cup p \) elsewhere, the end-points of a small neighbourhood of \( q \) on \( A_4 \) are not separated by this sphere, hence the limit sphere \( S \) must support \( A_4 \) at \( q \), and we have a contradiction.

A general 1-tangent sphere at a point \( q \in A_4 \) that is not a general 2-tangent sphere, supports \( A_4 \) at \( q \). By Lemma 7.6 it does not meet \( A_4 \cup p \) in three other points.

**Lemma 7.14.** A general 2-tangent sphere of \( A_4 \) at a point \( q \) does not meet \( A_4 \cup p \) at two other points. It does not support \( A_4 \) at any of these points of contact.

**Proof:** Let \( S \) be the limit sphere of a sequence of spheres \( S' \) through three mutually distinct points, \( q_1, q_2, q_3 \), which converge on \( A_4 \) to \( q \). Let \( u \) and \( t \) lie on \( S \), \( u \neq t \neq q \neq u \). Let \( S'' = S(t, q_1, q_2, q_3) \), let \( C_0 \) be any limit circle of \( G(q_1, q_2, q_3) \), and let \( S_0 \) be any limit sphere of \( S'' \). If \( C_0 = p \), then \( S \) and \( S_0 \) touch at \( q \), and if \( C_0 \neq p \), \( S_0 \cap S = C_0 \).
Since in addition, \( S \) and \( S_0 \) have the point \( t \) in common (\( t \notin S_0 \) by Lemma 7.13), it follows that \( S = S_0 \). Since \( S^n \) does not meet \( A_4 \cup p \) again, \( S_0 \) intersects \( A_4 \) at \( q \).

Let \( S'' = S(t,u,q_1,q_2) \), and let \( C_1 \) be any limit circle of \( C(t,q_1,q_2) \). By Lemma 7.12, \( u \notin C_1 \), and since any limit sphere \( S_1 \) of \( S'' \) contains \( u \) and \( C_1 \), the fact that \( S_0 = \lim S^n \) also contains \( u \) and \( C_1 \) implies that \( S_1 = S_0 = S \).

Since \( S'' = S(t,u,q_1,q_2) \) does not meet \( A_4 \cup p \) elsewhere, \( S_1 \) supports \( A_4 \) at \( q \). \( S_0 \), however, intersects \( A_4 \) at \( q \). Thus \( S \) does not meet \( A_4 \cup p \) at two points \( \neq q \). If \( t \neq p \), \( S^n \) intersects \( A_4 \) at \( t \). Hence the end-points of a small neighbourhood of \( t \) lie on opposite sides of \( S^n \), and thus they also lie on opposite sides of \( S \). Thus no general 2-tangent sphere of \( A_4 \) at \( q \) supports \( A_4 \) at another point.

Lemma 7.15. A general 3-tangent sphere of \( A_4 \) does not meet \( A_4 \cup p \) again.

Proof: Let \( S \) be a general 3-tangent sphere of \( A_4 \) at
q, and suppose that $S$ meets $A_4 \cup p$ again at $t$. Suppose that $S$ is the limit of a sequence of spheres $S' = S(q_1, q_2, q_3, q_4)$, where the $q_i$ are mutually distinct points converging on $A_4$ to $q$. Thus $S$ supports $A_4$ at $q$. Let $S'' = S(t, q_1, q_2, q_3)$.

Then any limit sphere $S_0$ of $S''$ contains $t$ and intersects $A_4$ at $q$.

Let $C_0$ be any accumulation circle of $C(q_1, q_2, q_3)$. Then $S \cap S_0 \supset C_0$. If $C_0 = q$, then $S$ and $S_0$ touch at $q$ and have the point $t \neq q$ in common. If $C_0 \neq q$, then $C_0 \supset t$ by Lemma 7.12, and hence $S \cap S_0 \supset C_0 \cup t$. In either case, we have the contradiction $S_0 = S$.

7.3.8. Theorem 7.1 yields several interesting results concerning the families of circles $\gamma_0$, $\gamma_1$, and $\gamma_2$.

Lemma 7.5 implies

**Corollary 1.** No circle meets $A_4 \cup p$ in more than three points.

**Corollary 2.** No circle of $\gamma_1$ meets $A_4$ more than once.
Proof: Let \( u \in A_4 \cap C(t; \gamma_1) \), and let \( v \in A_4 \), \( v \notin C(t; \gamma_1) \). Then \( S[v; C(t; \gamma_1)] = S(v, t; \sigma_1) \) meets \( A_4 \) in three distinct points, contrary to Lemma 7.7.

**Corollary 3.** \( C(\gamma_2) \) does not meet \( A_4 \).

Proof: We are only concerned with the case \( C(\gamma_2) \neq p \).

Let \( u \in A_4 \cap C(\gamma_2) \), and let \( v \in A_4 \), \( v \notin C(\gamma_2) \). Then \( S[v; C(\gamma_2)] = S(v; \sigma_2) \) meets \( A_4 \) in two distinct points, contrary to Lemma 7.9.

### 7.4. Strong Differentiability.

We call an arc \( A \) strongly differentiable at a point \( p \) if the arc is differentiable at that point and if, in addition, the following three conditions hold:

\[ \Sigma_1: \text{Let } P, Q, P \text{ be mutually distinct points, where } P \notin C(Q; \gamma_1) \text{ and let } P' \to P, Q' \to Q. \text{ If the two distinct points } t \text{ and } u \text{ converge on } A \text{ to } p, \text{ then } S(P; Q; t, u) \text{ always converges.} \]

\[ \Sigma_2: \text{Let } P \notin C(\gamma_2), P' \to P. \text{ If the three mutually distinct points } t, u, v, \text{ converge on } A \text{ to } p, \text{ then } S(P; t, u, v) \]
converges.

\[ \Sigma_3': S(t, u, v, w) \] converges if the four mutually distinct points \( t, u, v, w \), converge on \( A \) to \( p \).

7.4.1. It is clear that the limit of the spheres \( S(P; Q; t, u) \) depends only on \( P, Q, \) and \( p \), and not on the choice of the sequences \( u \) and \( v \). In particular, if \( P' = P, \) \( Q' = Q, \) and \( u = p \), we see that \( \Sigma_3' \) implies \( \Sigma_1 \), except where \( P \in C(Q; \gamma_1) \), and

\[ \lim S(P; Q; t, u) = S(P, Q; \sigma_1). \]

Similarly, the limit of the spheres \( S(P; t, u, v) \) depends only on \( P \) and \( p \). Since

\[ S(P; \sigma_2) = \lim_{v \to p} S(P, v; \sigma_1) = \lim_{u \to p} \lim_{v \to p} S(P, u, v; \sigma_0), \]

we see that \( \lim S(P; t, u, v) = S(P; \sigma_2) \), if \( P \not\in C(\gamma_2) \).

Finally, the limit of the spheres \( S(t, u, v, w) \) depends only on \( p \). Since

\[ S(\sigma_3) = \lim_{w \to p} S(w; \sigma_2) \lim_{v \to p} \lim_{w \to p} S(v, w; \sigma_1), \]

we verify that \( \lim S(t, u, v, w) = S(\sigma_3) \). Thus \( \Sigma_3' \) implies \( \Sigma_3 \).
7.4.2. We now prove the following important theorem, which generalizes Lemma 7.2:

**Theorem 7.2.** Let \( p \) be an end-point of an open arc \( A_4 \) of order four. Then \( A_4 \cup p \) is strongly differentiable at \( p \).

Before verifying Theorem 7.2, we prove an interesting corollary, which asserts that Theorem 7.2 extends itself automatically to include the cases where \( \Sigma'_1, \Sigma'_2, \) and \( \Sigma'_3 \) are weakened so as to permit multiplicities as defined in section 7.3.

**Corollary 1.** Let three distinct points \( t, u, \) and \( v \) converge on \( A_4 \cup p \) to \( p \), and let \( P' \to P, Q' \to Q \), where \( P \) and \( Q \) are mutually distinct, and where \( P \notin C(Q; \Sigma'_1) \) and \( P \notin C(\Sigma'_2) \).

Let \( S_1' (S_2'; S_3'; S_3') \) be a general 1-tangent sphere of \( A_4 \cup p \) at \( t \) through \( P' \) and \( Q' \) (\( P' \) and \( u, u \) and \( v \); a point of support, \( u \)).

Let \( S_2' (S_3') \) be a general 2-tangent sphere at \( t \) through \( P' (u) \).

Finally, let \( S_3'' \) be a general osculating sphere at \( t \). Then

\[
(7.1) \quad \lim S_1 = S(P, Q; \Sigma'_1).
\]
Proof of the Corollary: We may assume that each of the above sequences of spheres possesses an accumulation sphere. $S_1$ can be replaced by a sphere $S(P;Q;t_1,t_2)$ close to $S_1$ and such that $t_1$ and $t_2$ are distinct, and converge with $t$ to $p$. Thus

$$\lim S_1 = \lim S(P;Q;t_1,t_2) = S(P,Q;\sigma_1).$$

Similarly, $S_2$ and $S'_2$ can be replaced by spheres $S(P;u,t_1,t_2)$, and $S(P;t_1,t_2,t_3)$ respectively, close to $S_2$ and $S'_2$ such that $t_1,t_2$, and $t_3$ are distinct, and converge with $t_2$ to $p$. Again

$$\lim S_2 = \lim S(P;u,t_1,t_2) = S(P;\sigma_2).$$

Finally, $S_3, S'_3, S''_3$, and $S'''_3$ can be replaced by spheres $S(u,v,t_1,t_2)$, $S(u_1,u_2,t_1,t_2)$, $S(u,t_1,t_2,t_3)$, and $S(t_1,t_2,t_3,t_4)$ respectively. Hence

(7.2) \[ \lim S_2 = \lim S'_2 = S(P;\sigma_2). \]

(7.3) \[ \lim S_3 = \lim S'_3 = \lim S''_3 = \lim S'''_3 = S(\sigma_3). \]
Thus Theorem 7.2 implies our corollary.

7.4.3. We prove Theorem 7.2 in the remaining sub-sections of section 7.4. We shall let $B$ be an open sub-arc of $A_4$ bounded by $p$ and an interior point $f$ of $A_4$. Let $g$ be any point of $A_4$ outside $B$. We orient those spheres $S$ for which $g \notin S$ so that $g \subseteq S$. In particular, the set of such spheres contains all the spheres which meet $B \cup p \cup f$ four times. Their orientation is continuous. The points $t, u, v, w, d, e, f$ are assumed to be mutually distinct, and to lie on $B \cup f$ in the indicated order.

7.4.4. It is therefore evident that

\[(1) \quad u \subset \overline{S}(p, t, e, f) \cap \overline{S}(t, d, e, f).\]

Consequently,
(7.4) \[ S(t,u,e,f) \subseteq \left[ \overline{S}(p,t,e,f) \cap \overline{S}(t,d,e,f) \right] \\
\quad \cup \left[ \overline{S}(p,t,e,f) \cap \overline{S}(t,d,e,f) \right] \cup S(t,e,f). \]

Let I denote the region in (7.4). From (7.4) we obtain

(7.5) \[ \lim_{u,t \to p} S(t,u,e,f) \subseteq \left[ \overline{S}(e,f;\sigma_1) \cap \overline{S}(p,d,e,f) \right] \\
\quad \cup \left[ \overline{S}(e,f;\sigma_1) \cap \overline{S}(p,d,e,f) \right] \cup S(e,f;\sigma_1) \cup S(p,d,e,f). \]

By II we shall mean the limit of I as \( t \to p \). Let S be any limit sphere of \( S(t,u,e,f) \). As a point r runs continuously on B from d to p, \( S(p,r,e,f) \) runs continuously through the region II from \( S(p,d,e,f) \) to \( S(e,f;\sigma_1) \). Conversely, every sphere through II and \( C(p,e,f) \) meets B. Hence, if S passes through II \( \cup S(p,d,e,f) \), it intersects B at some point r, where \( r = d \) if \( S=S(p,d,e,f) \) (otherwise r lies between p and d). But then \( S(t,u,e,f) \), when it is close to S, intersects B again near r, contrary to Theorem 7.1. Thus \( S=S(e,f;\sigma_1) \).

**Corollary 2.** \( \lim_{t,u \to p} C(t,u,e) = C(e;\sigma_1) \).

**Proof:** \( \lim_{t,u \to p} C(t,u,e) = \lim_{t,u \to p} \prod_{f \in A_4} S(t,u,e,f) = \prod S(e,f;\sigma_1) \).
7.4.5. We now prove simultaneously that $S(p,u,v,f) \rightarrow S(f;\sigma_2)$, and assuming this, that $S(t,u,v,f) \rightarrow S(f;\sigma_2)$. We first note that

(ii) \[ u \subset \bar{S}(v,f;\sigma_1) \cap \bar{S}(p,v,e,f), \]

and correspondingly,

(ii') \[ u \subset \bar{S}(p,t,v,f) \cap \bar{S}(t,v,e,f). \]

Relations (ii) and (ii') yield

(7.6) \[ S(p,u,v,f) \subset \left[ \bar{S}(v,f;\sigma_1) \cap \bar{S}(p,v,e,f) \right] \]
\[ \cup \left[ \bar{S}(v,f;\sigma_1) \cap \bar{S}(p,v,e,f) \right] \cup C(p,v,f) \]

and

(7.6') \[ S(t,u,v,f) \subset \left[ \bar{S}(p,t,v,f) \cap \bar{S}(t,v,e,f) \right] \]
\[ \cup \left[ \bar{S}(p,t,v,f) \cap \bar{S}(t,v,e,f) \right] \cup C(t,v,f) \]

respectively. Let III denote either the region in (7.3) or the region in (7.3'). From (7.3) we obtain

(7.4) \[ \lim_{u,v \rightarrow p} S(p,u,v,f) \subset \left[ \bar{S}(f;\sigma_2) \cap \bar{S}(e,f;\sigma_1) \right] \]
\[ \cup \left[ \bar{S}(f;\sigma_2) \cap \bar{S}(e,f;\sigma_1) \right] \cup S(f;\sigma_2) \cup S(e,f;\sigma_1), \]
while (7.3') yields

$$(7.4') \lim_{t,u,v \to p} S(t,u,v,f) = \left[\overline{S}(f;\sigma_2) \cap \overline{S}(e,f;\sigma_1)\right] \cup \left[\overline{S}(f;\sigma_2) \cap \overline{S}(e,f;\sigma_1)\right] \cup S(f;\sigma_2) \cup S(e,f;\sigma_1).$$

By IV we shall mean the limit of III as $v \to p$ (as $t,v \to p$).

Let $S$ be any limit sphere of $S(p,u,v,f)$ ($S(t,u,v,f)$). Since $S \subseteq \lim C(p,v,f) \left(\lim C(t,v,f)\right) = C(f;\gamma_1)$, we see that $S \in \sigma_1$.

As a point $r$ runs continuously on B from $e$ to $p$, $S(r,f;\sigma_1)$ runs continuously through the region IV from $S(e,f;\sigma_1)$ to $S(f;\sigma_2)$. Conversely, every sphere through IV and $C(f;\gamma_1)$ meets B. Hence if S passes through IV $\cup S(e,f;\sigma_1)$, it intersects B at some point $r$, where $r=e$ if $S=S(e,f;\sigma_1)$ (otherwise $r$ lies between $p$ and $e$). But then $S(p,u,v,f)$ ($S(t,u,v,f)$), when it is close to $S$, intersects B again near $r$, contrary to Theorem 7.1. Thus $S=S(f;\sigma_2)$.

**Corollary 3.** \(\lim_{t,u,v \to p} C(t,u,v) = \lim_{u,v \to p} C(p,u,v) = C(\gamma_2).\)

**Proof:** \(\lim_{t,u,v \to p} C(t,u,v) = \lim_{f \in A_4} \bigcap_{f \in A_4} S(t,u,v,f)\)
\[ \lim_{u,v \to p} C(p,u,v) = \lim_{f \in A_4} \prod S(p,u,v,f) = \prod S(f;\sigma_2) = C(\gamma_2). \]

7.4.6. Here we prove, again simultaneously, that \( S(u,v;\sigma_1) \rightarrow S(\sigma_3), \) \( S(p,t,u,v) \rightarrow S(\sigma_3), \) and \( S(t,u,v,w) \rightarrow S(\sigma_3), \) each proposition being assumed to prove the following one. Proceeding in the previous manner, we note

(iii) \[ u \subset \S_2(v;\sigma_2) \cap \S_1(v,f;\sigma_1) \]

(iii') \[ u \subset \S_1(t,v;\sigma_1) \cap \S_1(p,t,v,f) \]

(iii'') \[ u \subset \S_1(p,t,v,w) \cap \S_1(t,v,w,f). \]

Relations (iii), (iii'), and (iii'') yield

(7.8) \[ S(u,v;\sigma_1) \subset \big[ \S_2(v;\sigma_2) \cap \S_1(v,f;\sigma_1) \big] \]

\[ \cup \big[ \S_1(v,o_2) \cap \S_1(v,f;\sigma_1) \big] \cup C(v;\gamma_1), \]

(7.8') \[ S(p,t,u,v) \subset \big[ \S_1(t,v;\sigma_1) \cap \S_1(p,t,v,f) \big] \]

\[ \cup \big[ \S_1(t,v;\sigma_1) \cap \S_1(p,t,v,f) \big] \cup C(p,t,v), \]

and

(7.8'') \[ S(t,u,v,w) \subset \big[ \S_1(p,t,v,w) \cap \S_1(t,v,w,f) \big] \]

\[ \cup \big[ \S_1(p,t,v,w) \cap \S_1(t,v,w,f) \big] \cup C(t,v,w), \]
respectively. Let \( V \) denote either the region of (7.8), or that of (7.8'), or that of (7.8''). Relations (7.8), (7.8'), and (7.8'') yield

\[
\text{(7.9) } \lim_{u,v \to p} S(u,v;\sigma_1) \subseteq \left[ \overline{S}(\sigma_3) \cap \overline{S}(f;\sigma_2) \right] \\
\subseteq \overline{S}(\sigma_3) \cup S(f;\sigma_2),
\]

\[
\text{(7.9') } \lim_{t,u,v \to p} S(p,t,u,v) \subseteq \left[ \overline{S}(\sigma_3) \cap \overline{S}(f;\sigma_2) \right] \\
\subseteq \overline{S}(\sigma_3) \cup S(f;\sigma_2),
\]

and

\[
\text{(7.9'') } \lim_{t,u,v,w \to p} S(t,u,v,w) \subseteq \left[ \overline{S}(\sigma_3) \cup S(f;\sigma_2) \right] \\
\subseteq \overline{S}(\sigma_3) \cup S(f;\sigma_2),
\]

respectively. By VI, we shall mean the limit of \( V \) as \( v \to p \) (as \( t,v \to p \); as \( t,v,w \to p \)). Let \( S \) be any limit sphere of \( S(u,v;\sigma_1) \) (\( S(p,t,u,v); S(t,u,v,w) \)). Since \( S \) contains

\[
\lim C(v;\gamma_1) \left( \lim C(p,t,v); \lim C(t,v,w) \right) = C(\gamma_2),
\]

we see that \( S \not\subseteq \sigma_2 \) unless \( C(\gamma_2) = p \). If \( C(\gamma_2) = p \), \( S(\sigma_3) \) and \( S(f;\sigma_2) \) touch at \( p \); hence \( S \subseteq \left[ \overline{S}(\sigma_3) \cap \overline{S}(f;\sigma_2) \right] \cup S(\sigma_3) \cup S(f;\sigma_2) \), and since \( S \supseteq p \), it must touch \( S(f;\sigma_2) \) at \( p \); hence \( S \subseteq \sigma_2 \). As a
point \( r \) runs continuously on \( B \) from \( f \) to \( p \), \( S(r;\sigma_2) \) runs continuously through the region \( VI \) from \( S(f;\sigma_2) \) to \( S(\sigma_3) \).

Conversely, every sphere through \( VI \) and \( C(\gamma_2) \) meets \( B \). Hence if \( S \) passes through \( VI \cup S(f;\sigma_2) \), it intersects \( B \cup f \) at \( r \), where \( r = f \) if \( S = S(f;\sigma_2) \). But then \( S(u,v;\sigma_1) \) \( (S(p,t,u,v); S(t,u,v,w)) \), when it is close to \( S \), intersects \( B \) again near \( r \), contrary to Theorem 7.1. Thus \( S \supset S(\sigma_3) \). If \( C(\gamma_2) \neq p \), there is no difficulty in seeing that \( S = S(\sigma_3) \). The same is true if \( S(\sigma_3) = p \). Suppose \( C(\gamma_2) = p \) and \( S(\sigma_3) \neq p \).

By a consideration of the method in which \( S(u,v,\sigma_1) \)
\( (S(p,t,u,v); S(t,u,v,w)) \) converges to \( S \) through the region \( V \), we find that \( S \) must be a proper sphere of \( \sigma_2 \); hence \( S = S(\sigma_3) \).

7.4.7. We now generalize section 7.4.4. Let \( P \not\in C(e;\gamma_1) \), and let \( P' \rightarrow P, P \neq p \). Then, by Corollary 2 (§7.4.4)

\[
\lim_{t,u \rightarrow p} S(P',t,u,e) = \lim_{P' \rightarrow P} S[P',C(t,u,e)] = S[P,C(e;\gamma_1)] = S(P,e;\sigma_1).
\]
Corollary 4. \( \lim_{t,u \to p} C(P',t,u) = C(P;Y_1) \).

Proof: \( \lim_{t,u \to p} C(P',t,u) = \lim_{P' \to P} \prod_{e \in A_4} S(P',t,u,e) = \prod_{e \in A_4} S(P,e;\sigma_1) = C(P;\sigma_1) \).

With this corollary in mind, let \( Q \not\in C(P;\sigma_1) \), and let \( Q' \to Q, Q \neq p \). Then

\[
\lim_{P' \to P, Q' \to Q} S(P',Q',t,u) = \lim_{P' \to P, Q' \to Q} S(Q';C(P',t,u)) = S(Q;C(P;\sigma_1)) = S(P,Q;\sigma_1).
\]

Remark: If \( Q \subset C(P;\sigma_1) \), let \( S \) be any accumulation sphere of \( S(P',Q',t,u) \). Since \( S \supset \lim C(P',t,u) = C(P;\sigma_1) \), \( S \in \sigma_1 \).

7.4.8. Finally, we generalize section 7.4.5. Let \( P \not\in C(\sigma_2) \), let \( C(\sigma_2) \neq p \), and let \( P' \to P \). Then by Corollary 3 (§ 7.4.5),

\[
\lim_{P' \to P} S(P',t,u,v) = \lim_{P' \to P} S[P';C(t,u,v)] = S[P;C(\sigma_2)] = S(P;\sigma_2).
\]

If \( C(\sigma_2) = p \), then

\[
\lim \left[ S(P',t,u,v) \cap S(t,u,v,f) \right] = \lim C(t,u,v) = p, \quad P \neq p,
\]
i.e. any accumulation sphere, $S$, of $S(P', t, u, v)$ touches $\lim S(t, u, v, f) \in \sigma_2$ at $p$. Hence $S \in \sigma_2$. Since $S$ touches any sphere of $\sigma_2$ at $p$ and goes through a point $P \neq p$, $S = S(P; \sigma_2)$.

**Remark:** If $P \in C(\mathcal{Y}_2)$ and $C(\mathcal{Y}_2) \neq p$, let $S$ be any accumulation sphere of $S(P', t, u, v)$. Since $S \supset \lim C(t, u, v) = C(\mathcal{Y}_2)$, $S \in \sigma_2$.

All the results in section 7.4.8 also hold if $t = p$. 
CHAPTER VIII

CONFORMALLY ELEMENTARY POINTS OF ARCS IN CONFORMAL 3-SPACE

8.1. Introduction.

A point \( p \) of an arc \( A \) is said to be conformally elementary if there exists a neighbourhood of \( p \) on \( A \) which is decomposed by \( p \) into two one-sided neighbourhoods of spherical order four. As a result of Theorem 7.2, these two one-sided neighbourhoods are strongly differentiable at \( p \).

I conjecture that the statement in Theorem 8.1 is universally true; the discussion in Chapter 8 is a partial proof of this theorem, proving twelve out of the twenty-four cases.

Theorem 8.1. Let \( p \) be a differentiable conformally elementary point of an arc \( A \), and let \((a_0, a_1, a_2, a_3; i)\) be the characteristic of \( p \). Then \( p \) has the spherical order

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This theorem remains valid if \( p \) is counted \( a_0 \) times on any sphere of \( \sigma_0 \), \( a_0+a_1 \) times on any sphere of \( \sigma_1 \), \( a_0+a_1+a_2 \) times on any sphere of \( \sigma_2 \), and finally, \( a_0+a_1+a_2+a_3 \) times on \( S(\sigma_3) \). The theorem also remains valid if a point \( u \in A \) is counted twice on any sphere which supports \( A \) at \( u \), three times on any general 2-tangent sphere, and four times on any general 3-tangent sphere at \( u \). There is no loss in generality if we assume that \( A \) itself is decomposed by \( p \) into two open arcs \( A_4 \) and \( A'_4 \) of order four. Thus the order of \( A \), and therefore that of \( p \), is not greater than 8.

8.2. Some Necessary Formulas.

Before beginning the proof of Theorem 8.1, we prove several helpful relations involving regions. As in section 7.4, we let \( A_4 \) be an arc of order four, and let \( p \) be an end-point of \( A_4 \). We assume that the points \( p, t, u, v, d, e, f, g \) lie on \( A_4 \cup p \) in the indicated order. If any sphere \( S \) does not contain the point \( g \), then \( g \in \overline{S} \). We consider spheres.
through the point $p$, which by Theorem 7.2 is strongly differentiable.

To begin, we observe that

$$v \in \mathcal{S}(e; \sigma_2) \cap \mathcal{S}(e, f; \sigma_1).$$

Hence

$$\mathcal{S}(v, e; \sigma_1) \subset \left[ \mathcal{S}(e; \sigma_2) \cap \mathcal{S}(e, f; \sigma_1) \right] \cup \left[ \mathcal{S}(e, f; \sigma_1) \cap \mathcal{S}(e, f; \sigma_1) \right] \cup \mathcal{C}(e, \gamma_1).$$

Therefore $\mathcal{S}(v, e; \sigma_1)$ separates the regions

$$\mathcal{S}(e; \sigma_2) \cap \mathcal{S}(e, f; \sigma_1)$$

and

$$\mathcal{S}(e, f; \sigma_1) \cap \mathcal{S}(e, f; \sigma_1).$$

Since

$$g \subset \mathcal{S}(e; \sigma_2) \cap \mathcal{S}(e, f; \sigma_1) \cap \mathcal{S}(v, e; \sigma_1),$$

we find that

$$\mathcal{S}(v, e; \sigma_1) \supset \mathcal{S}(e; \sigma_2) \cap \mathcal{S}(e, f; \sigma_1)$$

and consequently,

$$\mathcal{S}(v, e; \sigma_1) \supset \mathcal{S}(e, f; \sigma_1).$$

In exactly the same manner, we verify that

$$\mathcal{S}(u, v; \sigma_1) \supset \mathcal{S}(v; \sigma_2) \cap \mathcal{S}(v, e; \sigma_1)$$

and

$$\mathcal{S}(u, v; \sigma_1) \supset \mathcal{S}(v; \sigma_2) \cap \mathcal{S}(v, e; \sigma_1)$$

and again,

$$\mathcal{S}(v; \sigma_2) \supset \mathcal{S}(\sigma_2) \cap \mathcal{S}(e; \sigma_2)$$
and

\[(8.6)\]
\[S(v; \sigma_2) \supseteq S(\sigma_3) \cap S(e; \sigma_2)\]

Results (8.4), (8.6), and (8.2) yield

\[(8.7)\]
\[S(u, v; \sigma_1) \supseteq S(v; \sigma_2) \cap S(v, e; \sigma_1)\]

\[\supseteq [S(\sigma_3) \cap S(e; \sigma_2)] \cap [S(e; \sigma_2) \cap S(e, f; \sigma_1)]\]

\[= S(\sigma_3) \cap S(e; \sigma_2) \cap S(e, f; \sigma_1),\]

while results (8.3), (8.5), and (8.1) yield

\[(8.8)\]
\[S(u, v; \sigma_0) \supseteq S(\sigma_3) \cap S(e; \sigma_2) \cap S(e, f; \sigma_1).\]

Using the same methods, we can prove the following two relations

\[(8.9)\]
\[S(t, u, v; \sigma_0) \supseteq S(u, v, d; \sigma_0) \cap S(u, v; \sigma_1)\]

\[\supseteq [S(v, d, e; \sigma_0) \cap S(v, d; \sigma_1)] \cap [S(v, d; \sigma_1) \cap S(v; \sigma_2)]\]

\[= S(v, d, e; \sigma_0) \cap S(v, d; \sigma_1) \cap S(v; \sigma_2)\]

\[\supseteq [S(d, e, f; \sigma_0) \cap S(d, e; \sigma_1)] \cap [S(d, e; \sigma_1) \cap S(d; \sigma_2)]\]

\[\cap [S(d; \sigma_2) \cap S(\sigma_3)]\]

\[= S(d, e, f; \sigma_0) \cap S(d, e; \sigma_1) \cap S(d; \sigma_2) \cap S(\sigma_3)\]

\[(8.10)\]
\[S(t, u, v; \sigma_0) \supseteq S(d, e, f; \sigma_0) \cap S(d, e; \sigma_1)\]

\[\cap S(d; \sigma_2) \cap S(\sigma_3).\]
8.3. **Multiplicities on the arc $A=A_{4}\cup p\cup A_{4}$.**

8.3.1. **Lemma 8.1.** Let $M$ be a neighbourhood of $p$ on $A$. Let $\mathcal{M}(S) = \mathcal{M}(S, M)$ denote the multiplicity with which a sphere $S$ meets $M$. Suppose that $S$ does not pass through the end-points of $M$. Then for every sphere $S'$ sufficiently close to $S$,

$$\mathcal{M}(S') \equiv \mathcal{M}(S) \pmod{2}.$$ 

**Proof:** Suppose that $S$ meets $M$ at the points $t$ with the multiplicities $\rho(t)$, and nowhere else. Then

$$\mathcal{M}(S) = \sum_t \rho(t).$$

Construct disjoint neighbourhoods $B$ in $M$ about the points $t$. The end-points of $B$ lie on the same side or on opposite sides of $S$ according as $\rho(t)$ is even or odd. If $S'$ is sufficiently close to $S$, then $S'$ will not pass through the end-points of $B$, and they will lie on the same side of $S'$ if and only if they lie on the same side of $S$. The multiplicity with which $S'$ meets $B$, however, will also be even or odd, depending on whether the end-points of $B$ lie on the same side or on opposite sides of $S'$. Thus $S'$ will meet $B$ with a
multiplicity \( \rho'(t) \equiv \rho(t) \pmod{2} \) if \( S' \) is sufficiently close to \( S \). If each \( B \) is omitted from the closure of \( M \), there is left a closed set, which has no points in common with \( S \). If \( S' \) is sufficiently close to \( S \), this set does not meet \( S' \) either, and therefore

\[
\mathcal{M}(S') = \sum_t \rho'(t) \equiv \sum_t \rho(t) \pmod{2} = \mathcal{M}(S).
\]

**8.3.2.** Let \( S \) be any sphere, and let \( t \in S \cap M, t \neq p \).

Let \( p \) be the multiplicity with which \( S \) meets \( M \). Suppose there is a sequence of spheres \( S' \) converging to \( S \), and a corresponding sequence of neighbourhoods \( B' \) of \( t \) converging to \( t \) such that each \( S' \) meets \( B' \) at least \( \rho' \) times, where \( \rho' \leq 4 \). Then each \( S' \) can be replaced by another sphere \( S'' \) which meets \( B' \) in not less than \( \rho' \) distinct points, and such that the sequence \( S'' \) also converges to \( S \). Hence \( S \) will meet \( M \) at least \( \rho' \) times at \( t \); therefore \( \rho' \leq \rho \). This proves

**Lemma 8.2.** If \( S' \) is sufficiently close to \( S \), and \( t \in M, t \neq p \), and finally, if \( S \) has multiplicity \( \rho(t) \) at \( t \), there is a neighbourhood of \( t \) on \( M \) which is met not
more than \( \mathcal{O}(t) \) times by \( S' \).

8.4. \textbf{Multiplicities of the Families} \( \sigma_0 - \sigma_1, \sigma_1 - \sigma_2, \) \text{and} \( \sigma_2 - \sigma_3. \)

8.4.1. We now consider a sphere \( S \) of \( \sigma_0 - \sigma_1 \). By section 5.7, \( S \) meets \( M \) at \( p \) with a multiplicity \( \equiv a_0 \pmod{2} \). Let \( S' \to S \), and suppose that each \( S' \) meets \( M = N \cup p \cup N' \) in more than two points converging with \( S' \) to \( p \). Then \( S' \) meets \( N \cup p \), say, in at least two points \( t \) and \( u \) which converge with \( S' \) to \( p \).

Let \( e \in N \). Then by Theorem 7.2, Corollary 2, \( C(e,t,u) \to C(e;Y_1) \). Suppose \( C' \to C \neq p \), \( C' \subset S' \), \( C' \to t',u \). Then

\[
\lim \varphi \left[ C';C(e,t,u) \right] = 0,
\]

and by Theorem 5.6, \( S \in \sigma_1 \). Thus there exists a neighbourhood of \( p \) which is met not more than twice by any sphere \( S' \) close to \( S \). This leads to

\[
\text{Lemma 8.3. Every sphere of } \sigma_0 - \sigma_1 \text{ meets } M \text{ at } p \text{ with the multiplicity } a_0. \text{ Thus if } S' \to S \in \sigma_0 - \sigma_1, \text{ there is a neighbourhood } M_0 \text{ of } p \text{ which is met by every } S' \text{ sufficiently close to } S \text{ not more than } a_0 \text{ times.}
\]
8.4.2. Let \( S \in \sigma_1 - \sigma_2 \). Let \( S' \) be close to \( S \), \( S' \not\in \sigma_1 \), and let \( M_1 = N_1 \cup p \cup N'_1 \) be a small neighbourhood of \( p \).

By Theorem 7.2, Cor 3, \( C(u,t,p) \to C(\gamma_2) \) as \( u, t \to p \). Thus if \( S' \supseteq C(u,t,p) \), then \( S \in \sigma_2 \). Hence if \( p \in S' \), then \( S' \) will meet \( N_1(N'_1) \) not more than once. Thus \( S' \) meets \( M_1 \) with a multiplicity \( \equiv a_0 + a_1 \pmod{2} \), and it also meets \( M_1 \) with a multiplicity \( \leq a_0 + 2 \). Hence the multiplicity with which \( S' \) meets \( M_1 \) is \( \leq a_0 + a_1 \).

Next suppose that \( p \not\in S' \). Then \( S' \) will meet \( N_1(N'_1) \) at most twice. Hence \( S' \) meets \( M_1 \) with a multiplicity \( \leq 4 \) and \( \equiv a_0 + a_1 \pmod{2} \). Thus we have

\textbf{Lemma 8.4. Every sphere of } \sigma_1 - \sigma_2 \textbf{ meets a small neighbourhood } M_1 \textbf{ of } p \textbf{ with a multiplicity } \leq a_0 + a_1 \textbf{ unless } a_0 = a_1 = 1.\]

8.4.3. Let \( S \in \sigma_2 - \sigma_3 \). Let \( S' \to S \), \( S' \not\in \sigma_2 \). Let \( M_2 = N_2 \cup p \cup N'_2 \) be a small neighbourhood of \( p \).

Case (i). If \( S' \subseteq \sigma_1 \), and \( a_0 + a_1 \neq 2 \), \( S' \) does not meet \( N_2(N'_2) \) more than once. Therefore \( S' \) meets \( M_2 \) with a
multiplicity \( a_0 + a_1 + 2 \) and \( \equiv a_0 + a_1 + a_2 \pmod{2} \). Hence 

S' meets \( M_2 \) with a multiplicity \( \leq a_0 + a_1 + a_2 \).

Case (ii). If \( S' \ni p, S' \notin \sigma_1 \), then S' meets \( N_2(N_2) \) at most twice. Then S' meets \( M_2 \) with a multiplicity \( \leq a_0 + a_1 + a_2 \) (mod 2). Thus S' meets \( M_2 \) with a multiplicity \( \leq a_0 + a_1 + a_2 \) unless \( a_1 = a_2 = 1 \).

Case (iii). If \( p \notin S' \), then S' meets \( N_2(N_2) \) at most three times. Thus S' meets \( M_2 \) with a multiplicity \( \leq 6 \) and \( \equiv a_0 + a_1 + a_2 \pmod{2} \). Thus S' meets \( M_2 \) with a multiplicity \( \leq a_0 + a_1 + a_2 \) unless \( a_0 + a_1 + a_2 \leq 4 \).

Thus we have

**Lemma 8.5.** Let \( S \in \sigma_2 - \sigma_3 \) and let \( S' \rightarrow S, S' \notin \sigma_2 \). If

(i) \( a_0 + a_1 \neq 2 \) and \( S' \in \sigma_1 \),

or if

(ii) \( a_1 + a_2 \neq 2 \) and \( S' \ni p, S' \notin \sigma_1 \),

or finally, if

(iii) \( a_0 + a_1 + a_2 > 4, S' \notin p \).

then there exists a neighbourhood \( M_2 \) of \( p \) which is met by
at most $a_0 + a_1 + a_2$ times.

8.5. Properties of the Families $\sigma_3$, $\sigma_2$, $\sigma_1$, and $\sigma_0$.

Let $\Lambda = A'_4 \cup p \cup A'_4$.

8.5.1. By Theorem 7.1, $S(\sigma_3)$ does not meet $\Lambda$ outside $p$. Thus we can assign to $p$ the multiplicity $a_0 + a_1 + a_2 + a_3$ on $S(\sigma_3)$.

8.5.2. Lemma 8.6. There exists a neighbourhood $M_4 = N_4 \cup p \cup N'_4$ such that every sphere of $\sigma_2$ which meets $N_4 \cup N'_4$, meets $A'_4 \cup A'_4$ exactly $a_3$ times.

In particular, no sphere of $\sigma_2$ meets $M_4$ more than $a_3$ times.

Proof: By Theorem 7.2, a sphere of $\sigma_2$ meets $A'_4(A'_4)$ at most once. Thus it meets $A'_4 \cup A'_4$ at most twice. By Lemma 8.1, every sphere close to $S(\sigma_3)$ will meet $\Lambda$ with a multiplicity $\equiv a_0 + a_1 + a_2 + a_3 \pmod{2}$. Hence $S(t; \sigma_2)$ will meet $A'_4 \cup A'_4$ with a multiplicity $\equiv a_3$ if $t$ is sufficiently close to $p$. Such a sphere will therefore meet $A'_4 \cup A'_4$ exactly $a_3$ times. Thus we can assign to $p$ the multiplicity
\( a_0 + a_1 + a_2 \) with respect to any circle of \( \sigma_2 - \sigma_3 \).

8.5.3. Lemma 8.7. There exists a neighbourhood \( N_j \) of \( M_4 \) which is met at most \( a_2 + a_3 \) times outside \( p \) by every sphere of \( \sigma_1 \).

**Proof:** Because of Lemma 8.6, we have only to consider the spheres of \( \sigma_1 - \sigma_2 \). If suffices to construct a one-sided neighbourhood \( N'_3 \subset N'_4 \) of \( p \) such that any sphere of \( \sigma_1 - \sigma_2 \) that meets \( N'_3 \) twice will meet \( M_4 \) at most \( a_2 + a_3 \) times outside \( p \).

By Lemma 8.1, \( N'_4 \) can be chosen so small that a sphere \( S \) of \( \sigma_1 - \sigma_2 \) through two points of \( N'_3 \) is close to \( S(\sigma_3) \) and meets \( M_4 \) with a multiplicity \( \equiv a_0 + a_1 + a_2 + a_3 \pmod 2 \). Since \( S \) meets \( N_4 \) and \( N'_4 \) not more than twice each, it will meet \( M_4 \) outside \( p \) at most four times. Since \( S \) meets \( M_4 \) at \( p \) with a multiplicity \( \equiv a_0 + a_1 \pmod 2 \), it meets \( M_4 \) outside \( p \) with a multiplicity \( \equiv a_2 + a_3 \pmod 2 \). Hence Lemma 8.7 holds if \( a_2 + a_3 > 2 \).

Let \( a_2 + a_3 = 2 \) so that \( a_2 = a_3 = 1 \). Let \( f \) denote the end-point of \( N_4 \). Suppose the points \( u, v, e, f \), lie on \( N_4 \cup f \).
in the indicated order. Choose a small neighbourhood

\( N'_3 \subset N'_4 \) so that \( N'_3 \) has no points in common with \( S(\sigma'_3) \),

\( S(e;\sigma_2) \), or \( S(e,f;\sigma'_1) \). We then have, as in section 8.2,

\[ N'_3 \subset S(\sigma'_3) \cap S(e;\sigma_2) \cap S(e,f;\sigma'_1). \]

Now \( a_2 = a_3 = 1 \). Thus if \( S(\sigma'_3) \) intersects \( A \) at \( p \), then

\( S(e;\sigma_2) \) supports and \( S(e,f;\sigma'_1) \) intersects, while if \( S(\sigma'_3) \)

supports, then \( S(e;\sigma_2) \) intersects and \( S(e,f;\sigma'_1) \) supports.

Hence

\[ N'_3 \subset S(\sigma'_3) \cap S(e;\sigma_2) \cap S(e,f;\sigma'_1), \]

or

\[ N'_3 \subset S(\sigma'_3) \cap S(e;\sigma_2) \cap S(e,f;\sigma'_1). \]

By relations (6.7) and (8.8), \( N'_3 \) lies either in \( S(u,v;\sigma'_1) \)

or in \( S(u,v;\sigma'_1) \). Thus \( N'_3 \) does not meet \( S(u,v;\sigma'_1) \). By

Lemma 8.1, any sphere \( S \) of \( \sigma'_1-\sigma_2 \) through two points of \( N'_3 \)

meets \( M_4 \) with a multiplicity \( \equiv a_0 + a_1 + l + l \) (mod 2). Thus it

meets \( N_4 \cup N'_4 \) an even number of times. It meets \( N'_4 \) exactly
twice. From the above, \( S \) then cannot meet \( N_4 \) twice. Hence

\( S \) and \( N'_4 \) are disjoint, and \( S \) meets \( M_4 \) with the total

multiplicity \( a_0 + a_1 + 2 = a_0 + a_1 + a_2 + a_3 \).
8.5.4. Lemma 8.8. There exists a neighbourhood \( M_2 \) of \( M_3 \) which is met at most \( a_1 + a_2 + a_3 \) times outside \( p \) by every sphere of \( \sigma_0 \) unless \( a_1 + a_2 + a_3 = 4, a_2 = 2 \).

**Proof:** In view of Lemma 8.7, we have only to consider the spheres of \( \sigma_0 - \sigma_1 \).

(i) By Lemma 8.1, \( N_2 \) can be chosen so small that a sphere \( S \) through \( p \) and three points of \( N_2 \) is close to \( S(\sigma_3) \) and meets \( M_3 \) with a multiplicity \( \equiv a_0 + a_1 + a_2 + a_3 \pmod{2} \). Since \( S \) meets \( N_2 \) and \( N'_2 \) not more than three times each, it will meet \( M_3 \) outside \( p \) at most six times. Since \( S \) meets \( M_3 \) at \( p \) with a multiplicity \( \equiv a_0 \), it meets \( M_3 \) outside \( p \) with a multiplicity \( \equiv a_1 + a_2 + a_3 \pmod{2} \). Thus Lemma 8.8 holds if \( a_1 + a_2 + a_3 > 4 \).

(ii) Let \( a_1 + a_2 + a_3 = 3 \), so that \( a_1 = a_2 = a_3 = 1 \). Suppose that \( M_3 \) is so small that it has no points outside \( p \) in common with \( S(\sigma_3), S(d; \sigma_2), S(d,e; \sigma_1) \) and \( S(d,e,f; \sigma_0) \). Then

\[
N_3 \subseteq S(\sigma_3) \cap S(d, \sigma_2) \cap S(d,e; \sigma_1) \cap S(d,e,f; \sigma_0)
\]
while $N'_3$ lies either in

$$S(\sigma_3) \cap S(d;\sigma_2) \cap S(d,e;\sigma_1) \cap S(d,e,f;\sigma_0),$$

or else in

$$S(\sigma_3) \cap S(d;\sigma_2) \cap S(d,e;\sigma_1) \cap S(d,e,f;\sigma_0),$$

according as $a_0 = 1$ or $2$. By relation (8.9) and (8.10), $N'_3$ lies either in $S(t,u,v;\sigma_0)$ or in $S(t,u,v;\sigma_0)$. Thus $N'_3$ does not meet $S(t,u,v;\sigma_0)$.

(ii') To complete case (ii) we show that $M_2$ may be chosen so that a sphere of $\sigma_0$ through two points $u$ and $t$ of $N_2$, and a point $u'$ of $N'_2$ does not meet $N_2 \cup N'_2$ elsewhere. Let $h \in N_3$, $h' \in N'_3$. Let $M_h$ denote the neighbourhood of $p$ bounded by $h$ and $h'$. Let $M_2$ be a neighbourhood of $p$ whose end-points lie in $M_h$. By the above, $S(t,u,h;\sigma_0)$ does not meet $N'_3$. Thus $h \notin S(u,u',t;\sigma_0)$.

$S(h';\sigma_2)$ intersects $M_2$ at $p$, and does not meet $M_2$ elsewhere (cf. Lemma 8.6). Hence there is a neighbourhood $N_1 \subseteq N_2$ such that $S(h',u,t;\sigma_0)$ meets $N_2 \cup N'_2$ with an even multiplicity when $u$ and $t \in N_1$. By (ii), $S(h',u,t;\sigma_0)$ cannot
meet $N_2 \cup N'_2$ four times, hence it meets $M_2$ only at $u$ and $t$
outside $p$, and intersects $M_2$ at these points. Thus

$$S(h',u,t;\sigma_0)$$

does not meet $N'_2$. Hence if $u' \in N'_2$, $h' \not\in S(u',u,t;\sigma)$

By section (8.5.3), $S(u,u';\sigma_1)$ intersects $M_3$ at $u$ and $u'$, and

does not meet $M_3$ elsewhere outside $p$. Hence $S(u,u';\sigma_1)$

separates or does not separate $h$ and $h'$ according as $a_0 = 2$ or

If $t$ is close to $p$, the same holds for $S(u,u',t;\sigma_0)$. But $S(u,u',t;\sigma_0)$ does not pass through $h$ or $h'$ for

any $u,t \in N_1$, $u' \in N'_2$. Then as $t$ moves in $N_1$, $S(u,u',t;\sigma_0)$

meets $N_h \cup N'_h$ an odd number of times. By (ii) it meets $N_h$

and $N'_h$ at most twice each, hence it must meet $M_h$ exactly

three times outside $p$. Thus any sphere of $\sigma_0$ through two

points of $N_1$ and a point of $N'_2$ meets $N_h \cup N'_h$ nowhere else.

The fifteen cases for which $a_1 + a_2 + a_3 \neq 4$ are now

disposed of. There remain the six cases for which $a_1 + a_2 + a_3 = 4$,

$a_2 \neq 2$.

(iii) The cases $a_1 = 2$, $a_2 = a_3 = 1$. Let $e,e' \in M_3$,

$e \in N_3$, $e' \in N'_3$. By Lemma 8.5, there is a neighbourhood
of \( p \) such that no sphere of \( \sigma_o \) through \( e \) or \( e' \) meets \( N_2 \cup N'_1 \) in four points. We shall prove that a sphere \( S \) of \( \sigma_o \) through two points \( v \) and \( u \) of \( N_2 \), and two points \( v' \) and \( u' \) of \( N'_1 \) does not meet \( M_2 \) elsewhere. By Lemma 8.7, \( S(v,v';\sigma_1) \) does not meet \( M_3 \) elsewhere. It intersects \( N_2 \) at \( v \) (\( N'_1 \) at \( v' \)) and meets \( M_2 \) at \( p \) with a multiplicity \( \equiv a_0 \).

Let \( M_1 = N_1 \cup p \cup N'_1 \subset M_2 \) be so small that

(a) no sphere of \( \sigma_o \) through four points of \( N_1 \cup N'_1 \) passes through \( v \) or \( v' \);

(b) no sphere of \( \sigma_o \) through two points of \( N_1 \) or \( N'_1 \) passes through both \( v \) and \( v' \).

(cf. Lemma 8.5 and Theorem 7.2). Thus \( v \) and \( v' \) do not lie in \( M_1 \). By Lemma 8.1, there exists a neighbourhood \( M \subset M_1 \) of \( p \) such that \( S(v,v',t;\sigma_o) \) meets \( M_1 \) with a multiplicity \( \equiv a_0 \) if \( t \in M \). From the above, \( S(v,v',t;\sigma_o) \) meets \( N_1 \cup N'_1 \) with an even multiplicity, and it meets \( N_1 \) and \( N'_1 \) at most once each. Thus \( S(v,v',t;\sigma_o) \) intersects \( N_1 \) only at \( t \), and intersects \( N'_1 \) at one point \( t' \) only. Let \( t \) move
on $N_2$ towards $u$. Then $S(v,v',t;\sigma_o^-)$ does not pass through $e$ or $e'$, $t'$ does not converge to $p$, and $S(v,v',t;\sigma_o^-)$ continues to meet $N_e$ and $N_e'$ with an even multiplicity, i.e., exactly twice each. Thus when $t = u$ or $t' = u'$, $S(v,v',t;\sigma_o^-)$ coincides with $S$.

(iv) The cases $a_1 = a_2 = 1$, $a_3 = 2$. Choose $g$, $f$, $e,e',f',g'$ on $N_3 \cup N_3'$ in the indicated order so that no sphere of $\sigma_1$ through any two of these points is a sphere of $\sigma_2^-$. Given $g$ and $g'$, we can choose $f$ and $f'$ so that $S(f;\sigma_2^-)$ ($S(f'\sigma_2^-)$) meets $N_g$ ($N_g$). Now choose $e$ ($e'$) between $p$ and $f$ ($f'$) such that $S(f;\sigma_2^-)$ ($S(f';\sigma_2^-)$) does not meet $N_e \cup e'$ ($N_e \cup e'$).

By section 8.4.2, there is a neighbourhood $M_2 \subset M_e$ of $p$ such that a sphere through $p$ and any two of the points $g,f,e,e',f',g'$ meets $N_2 \cup N_2'$ at most once. Let $v \in N_2$, $v' \in N_2'$ so that the sphere $S(f,v';\sigma_1)$ converges to $S(f;\sigma_2^-)$ if $v'$ converges to $p$.

If $t$ is sufficiently close to $p$, $S(f,v',t;\sigma_o^-)$ is close to $S(f,v';\sigma_1)$, which in turn is close to $S(f;\sigma_2^-)$. 
Thus by section 8.4.2, $S(f,v',t;\sigma_o)$ meets $N_g$ and $N'_g$ twice each, and meets $N'_e$ only once. From the above, $S(f,v',t;\sigma_o)$ cannot pass through $g,g'$, or $e'$. Let $M_1 \subset M_2$ be a neighbourhood of $p$ which does not contain $t$. By Lemma 8.7, there exists a still smaller neighbourhood $M_0 \subset M_1$ such that $S(f,v',t;\sigma_o)$ meets $M_0$ at $p$ only. Thus $S(f,v',t;\sigma_o)$ meets $N_g-N_0$ and $N'_g-N'_0$ exactly twice each, and it meets $N'_e-N_0$ exactly once. As $t$ moves in $M_2$, $S(f,v',t;\sigma_o)$ meets $N_g-N_0$ and $N'_g-N'_0$ with an even multiplicity, i.e. exactly twice each, and it meets $N'_e-N'_1$ with an odd multiplicity $\leq 2$, i.e., exactly once.

Hence $S(f,v',t;\sigma_o)$ meets $N_2$ and $N'_2$ exactly once each. Similarly $S(f',v,t;\sigma_o)$ meets $N_2$ and $N'_2$ exactly once each.

Consider a small neighbourhood $M \subset M_f$ of $p$, and any sphere of $\sigma_1$ which meets $N \cup N'$ twice. This sphere is close to a sphere of $\sigma_2$, and meets some neighbourhood of $p$ in $M_f$ with a multiplicity $\equiv a_0+1+1$. But a sphere of $\sigma_1$ meets $M_f$ at $p$ with a multiplicity $\equiv a_0+1$; hence it meets some
neighbourhood of \( p \) outside \( p \) with an odd multiplicity, i.e., at least three times. Thus there exists a neighbourhood \( M \subset M_f \) such that any sphere of \( \sigma_1 \) that meets \( N \cup N' \) twice meets \( M_f \) at least three times outside \( p \). By Lemma 8.7, it does not meet \( M_f \) outside \( p \) more than three times.

Thus \( S(v,v';\sigma_1) \) meets \( M_f \) in exactly one more point say \( u' \). If \( t \) is close to \( p \), \( S(v,v',t;\sigma_0) \) meets \( M_f \) exactly \( a_0 \) times at \( p \), once each at \( v,v',t \) and near \( u' \) and nowhere else. By Lemma 8.7, there is a small neighbourhood \( M_0 \subset M_2 \) of \( p \) which does not contain \( t \), and which is not met by \( S(v,v',t;\sigma_0) \) outside \( p \). As \( t \) moves in \( N_2 \), \( S(v,v',t;\sigma_0) \) meets \( N_f-N_0 \) and \( N'_f-N'_0 \) each with an even multiplicity, i.e. twice each. Thus no sphere of \( \sigma_0 \) meets \( M_2 \) more than four times outside \( p \).

8.6. Proof of Theorem 8.1 When \( a_0+a_1+a_2+a_3 > 6 \).

If is sufficient to show that there is a one-sided neighbourhood \( N'_0 \subset N'_2 \) of \( p \) such that no sphere \( S \) through four points of \( N'_0 \cup p \) meets \( M_4 \) more than \( a_0+a_1+a_2+a_3 \).
times. On account of Lemma 8.8, we need only consider spheres $S$ which do not pass through $p$. By Lemma 8.1, $N'_o$ can be chosen such that any $S$ close to $S(\sigma^3)$ meets $M_4$ with a multiplicity $\equiv a_0 + a_1 + a_2 + a_3 \pmod{2}$. Since $p \not\in S$, and since $S$ meets $N_4$ ($N'_4$) at most four times, it will meet $M_4$ at most eight times. This yields Theorem 8.1 in this case.

8.7. The case $a_0 + a_1 + a_2 + a_3 = 4$.

On account of Lemma 8.8, we need only consider the spheres which do not contain $p$.

8.7.1. A small neighbourhood $N'_o$ of $N'_1$ does not meet $S(d,e,f,g)$. Let the points $p,t,u,v,w,d,e,f,g,h$ lie on $A_4 \cup p$ ($A_4$ is a one-sided neighbourhood of $A$ and is of order four) in the indicated order. For any sphere $S$ with $h \not\in S$, we make the convention that $h \subset S$. Hence, if $N'_1 \subset N'_d$,

$$N'_1 \subset S(d,e,f,g) \cap S(d,e,f;\sigma^0) \cap S(d,e;\sigma_1) \cap S(d;\sigma_2) \cap S(\sigma^3),$$

and since $a_0 = a_1 = a_2 = a_3 = 1$,

$$N'_o \subset S(d,e,f,g) \cap S(d,e,f;\sigma^0) \cap S(d,e;\sigma_1) \cap S(d;\sigma_2) \cap S(\sigma^3).$$

Let $t,u,v,w$ lie on $N'_1$. By a method similar to that
employed in section 8.2, we find that

\[ \bar{S}(t,u,v,w) \supset \bar{S}(u,v,w,d) \cap \bar{S}(p,u,v,w) \]

\[ = \bar{S}(v,w,d,e) \cap \bar{S}(p,v,w,d) \cap \bar{S}(v,w;\sigma_1) \]

\[ \supset \bar{S}(w,d,e,f) \cap \bar{S}(p,w,d,e) \cap \bar{S}(w,d;\sigma_1) \]

\[ \cap \bar{S}(w,d;\sigma_1) \cap \bar{S}(w;\sigma_2) \]

\[ \supset \bar{S}(d,e,f,g) \cap \bar{S}(p,d,e,f) \cap \bar{S}(d,e;\sigma_0) \cap \bar{S}(d,e;\sigma_1) \]

\[ \cap \bar{S}(d,e;\sigma_1) \cap \bar{S}(d;\sigma_2) \cap \bar{S}(\sigma_2) \]

Thus a sphere through four points of \( N_1 \) does not meet \( N_0 \).

Symmetrically, there is a neighbourhood \( N_0 \) such that a sphere through four points of \( N_1 \) does not meet \( N_0 \).

8.7.2. Let \( M_0 = N_0 \cup p \cup N_0 \). Let \( h, k, l \in N_0, h', k', l' \in N_1 \).

By Lemmas 8.2, 8.3, and 8.8, there is a neighbourhood \( M \subset M_0 \) of \( p \) such that a sphere through any three of the points \( h, k, l, h', k', l' \) and a point of \( M \) does not meet \( M_0 \) elsewhere.

Thus if \( u, t, \in N \) and \( u', t' \in N' \), the spheres \( S(k,h,u',t') \), \( S(k,h',u',t') \), \( S(k',h,u,t') \) and \( S(k',h',u,t) \) do not pass through \( l \) or \( l' \). By section 8.7.1, these spheres do not
support $M_o$ at any other point and, by Lemma 8.8, they do not pass through $p$. Since $S(k,h,u',p), S(k',h,u,p)$ and $S(k',h',u,p)$ do not meet $M_o$ elsewhere, there is a neighbourhood $M_\infty = N_\infty \cup p \cup N_0'$ of $p$, $M_\infty \subset M$ such that $S(k,h,u',t'), S(k',h',u,t')$, $S(k,h',u',t)$ and $S(k',h',u,t)$ do not meet $M_o$ elsewhere if $t \in N_\infty$, and $t' \in N_\infty'$. Thus each of these spheres meets $N_\infty$ ($N_\infty'$) exactly twice. Letting $t$ and $t'$ move on $N$ and $N'$ respectively, we find that $S(k,h,u',t'), S(k',h,u,t')$, $S(k',h',u',t)$ and $S(k',h',u,t)$ also meet $N_\infty$ and $N_\infty'$ with an even multiplicity, i.e. exactly twice each. Thus the spheres $S(h,u,u',t')$ and $S(h',u,u',t)$ do not pass through $k, k'$, or $p$ when $u, t \in N$, and $u', t' \in N'$. Since $S(h',u,u',p)$ and $S(h,u,u',p)$ do not meet $M_o$ elsewhere, there is a small neighbourhood $M_\varphi = N_\varphi \cup p \cup N_\varphi'$, $M_\varphi \subset M_o$, such that $S(h',u,u',t)$ and $S(h,u,u',t')$ do not meet $M_o$ again if $t \in N_\varphi$ and $t' \in N_\varphi'$. Thus $k, p$, and $k'$ lie on the same side of these spheres.

As $t$ and $t'$ range on $N$ and $N'$ respectively, $S(h,u,u',t')$ and $S(h',u,u',t)$ do not pass through $p$ and
continue to meet \( N_k \) and \( N'_k \) with an even multiplicity, i.e., exactly four times. Thus \( S(u, u', t, t') \) does not pass through \( h \) or \( h' \) if \( u, t, \in N, u', t' \in N' \). Since \( S(u, u', t, p) \) does not meet \( M \) elsewhere, the same holds for \( S(u, u', t, t') \) if \( t' \) is sufficiently close to \( p, t' \in N' \). Thus \( S(u, u', t, t') \) does not separate \( h, p, \) or \( h' \). As \( t' \) moves on \( N' \), \( S(u, u', t, t') \) does not pass through \( h, h', \) or \( p \), and by section 8.7.1 it does not support \( M \) at any point. Thus \( S(u, u', t, t') \) does not meet \( M_h \) elsewhere if \( u, t, \in N \) and \( u', t' \in N' \).

8.8. The Cases, \((2,2,1,1;1)\) and \((2,2,1,1;3)\).

Let \( d \) and \( d' \) belong to the neighbourhood \( M \) of Lemma 8.8 (iii). On account of Lemma 8.8, we need only consider spheres which do not pass through \( p \). By Lemma 8.5, we can choose in \( M \) a neighbourhood \( M_0 = N_0 \cup p \cup N'_0 \) of \( p \) such that no sphere through six points of \( M_0 \) passes through \( d \) or \( d' \).

We shall prove that any sphere which passes through three points \( w, v, u, \) of \( N_0 \), and three points \( w', v', u', \) of \( N'_0 \), does not meet \( M_0 \) again. By Lemma 8.7, \( S(w, w'; \sigma_1) \) does not meet
$M$ outside $p, w,$ and $w'$. It intersects $M$ at $w$ and $w'$, and supports $M$ at $p$. By Lemma 8.6 there is a neighbourhood $N_\alpha = N_\alpha \cup p \cup N_\alpha'$ of $p$ such that $S(w, w', t; \sigma_0)$ meets $N_\alpha (N_\alpha')$ at most once. There is a neighbourhood $M_\beta \subset M_\alpha$ of $p$ such that $S(w, w; t; \sigma_0)$ meets $M_\alpha$ with an even multiplicity if $t \in N_\beta$.

Since $S(w, w; t; \sigma_0)$ supports $M_\alpha$ at $p$, it meets $N_\alpha \cup N_\alpha'$ with an even multiplicity, i.e., it meets $N_\alpha$ once at $t$ and $N_\alpha'$ once at some point $t'$. Let $M_\gamma$ be a neighbourhood of $p$ which does not contain $t$. Then there is a neighbourhood $M_\delta$, with end-points $\delta$ and $\delta'$, such that $S(w, w; t; \sigma_0)$ does not pass through $N_\delta \cup N_\delta'$ if $t \in N_\delta - N_\delta'$. As $t$ moves on $N_\delta - N_\delta'$, $S(w, w; t; \sigma_0)$ does not pass through $d, d', \delta$ or $\delta'$. Thus $S(w, w; t; \sigma_0)$ meets $N_d - N_\delta (N_d' - N_\delta')$ with an even multiplicity, i.e. twice. Suppose that $t$ reaches $v$ while $t'$ meets $N_\delta'$ between $\delta'$ and $v'$. Thus $S(w, w; v; \sigma_0)$ meets $N_\delta'$ at a point $t'$ between $\delta'$ and $v'$ and does not meet $d$ elsewhere. It intersects $M$ at $w, w', v$, and supports $M$ at $p$.

Thus $S(w, w; v, t')$ supports $M$ at $p$ and does not meet
M elsewhere. Let \( t \) move in \( N \) towards \( v' \). Then at first, \( S(w, w; v, t') \) meets a small neighbourhood of \( p \) exactly twice, does not pass through \( p \), and does not meet this neighbourhood twice on one side of \( p \); i.e., it meets this neighbourhood once on each side of \( p \). Thus \( S(w, w; v, t') \) meets \( N \) (\( N' \)) with an odd multiplicity. As \( t' \) moves on \( N \) towards \( v' \), \( S(w, w; v, t') \) does not pass through \( d, d' \), or \( p \), hence it does not when \( t' = v' \), and \( S(w, w; v, t') \) meets \( N \) (\( N' \)) with an odd multiplicity, i.e., three times each.

### 8.9. The Cases (2,1,1;2;2) and (2,1,1,2;3).

By Lemma 8.8, we need only consider spheres which do not contain \( p \). Aside from \( p \) and the neighbourhood \( M_2 \) of Lemma 8.6 (iv), the points and neighbourhoods described in this section do not refer to those previously mentioned.

Choose \( e, f, g, e', f', g' \) on \( M_2 \) such that any sphere of \( \sigma_1 \) through two of these points does not belong to \( \sigma_2 \). By Lemma 8.4 there is a neighbourhood \( M_1 = N_1 \cup p \cup N_1' \) of \( p \) such
that every sphere through two of the points $g, f, g', f'$ meets $M_1$ at most three times. For a given $g$ and $g'$, choose $f$ ($f'$) such that $S(f; \sigma_2) (S(f'; \sigma_2))$ meets $N'_g (N'_g)$. Next choose $e$ ($e'$) between $p$ and $f$ ($f'$) such that $S(f; \sigma_2) (S(f'; \sigma_2))$ does not meet $e' \cup N'_e (e \cup N'_e)$.

If $u'$ is close to $p$, $u' \in N'_e$, $S(f, u'; \sigma_1)$ is close to $S(f; \sigma_2)$ and meets $N'_g$ at $f$ only, $N'_e$ at $u'$ only, and $N'_g$ once outside $N'_e$. If $t$ is sufficiently close to $p$, $S(f, u'; t; \sigma_1)$ will meet $N'_g$ and $N'_g$ twice each, but $N'_e$ only once. By Lemma 8.4, as $t$ moves in $N_1$, $S(f, u'; t; \sigma_1)$ remains close to $S(f; \sigma_2)$ and continues to meet $N'_g$ outside $N'_e$. Thus $S(f, u'; t; \sigma_1)$ meets $N'_g (N'_g)$ exactly twice. By Lemmas 8.1 and 8.4, there is a neighbourhood $M_0$ such that $S(f, u'; t, r)$ meets $M_0$ with an even multiplicity $\leq 3$, i.e., twice. Since $S(f, v; t; \sigma_1) \not\in \sigma_1$, $S(f, u'; t, r)$ must meet $N_0$ ($N'_0$) once each. Thus $S(f, u'; t, r)$ meets $N'_g$ and $N'_g$ three times each. From the above, if $r$ moves in $N_1$, $S(f, u'; t, r)$ does not pass through $p, g, g'$, or $e'$,
and \( p \) and \( g \) \((g')\) will lie on opposite sides while \( p \) and \( e' \) will lie on the same side of \( S(f,u; t,r) \). Thus \( S(f,u; t,r) \) will meet \( N_g \) \((N'g)\) exactly three times and \( N_e \) an even number of times, i.e., exactly twice. Hence \( S(f,u; t,r) \) will meet \( N_1 \) and \( N'_1 \) only twice each, i.e., it will meet \( M_1 \) only four times. Similarly, \( S(f'u; t,r) \) meets \( M_1 \) only four times.

Thus any sphere through five points of \( M_1 \) does not pass through \( f \) or \( f' \).

We shall prove that a sphere through three points \( w, v, u \), of \( N_1 \), and three points of \( N'_1 \) does not meet \( M_f \) elsewhere.

Starting with \( S(t; \sigma_2) \), we note that it meets \( N_f \) again if \( t \) is sufficiently close to \( p \). Letting \( t \) move in \( N_1 \) towards \( w \), we can assume that \( S(w; \sigma_2) \) meets \( N'_1 \) at a point \( t' \) between \( p \) and \( w' \). If \( t \) is close to \( p \), \( S(w,t; \sigma_1) \) meets \( N'_1 \) at \( t' \) and does not meet \( M_2 \) elsewhere. Let \( M_\infty \) be a small neighbourhood of \( p \) which does not contain \( t \). There is a
still smaller neighbourhood \( M_\delta \subset M_\varepsilon \) such that \( N_\delta \cup N_\delta \) is not met by \( S(w,t;\sigma_1) \). Thus as \( t \) moves on \( N_1 - N_\delta \) towards \( v \), \( S(w,t;\sigma_1) \) meets \( N_1 - N_\delta \) at \( w \) and \( t \) only, and meets \( N_1' - N_\delta' \) once. Eventually we get either (i) \( S(w,v;\sigma_1) \) meeting \( N_1' - N_\delta' \) between \( p \) and \( w' \), or (ii) \( S(w,w';\sigma_1) \) meeting \( N_1 \) between \( p \) and \( v \).

Case (i): If \( r' \) is close to \( p \), \( S(w,v,r';\sigma_0) \) meets \( M_\varepsilon \) exactly four times outside \( p \). Let \( M_\delta \) be a small neighbourhood of \( p \) not including \( r' \). Thus as \( r' \) moves towards \( w' \), \( S(w,v,r';\sigma_0) \) meets \( N_1' \) an even number of times, i.e., twice. Thus we obtain \( S(w,v,w';\sigma_0) \) when a point reaches \( w' \) and this sphere meets \( N_1' \) again at a point \( t' \) between \( p \) and \( w' \).

Case (ii): If \( r' \) is close to \( p \), \( S(w,w;r';\sigma_0) \) meets \( M_2 \) again only at one other point of \( N_1 \) between \( p \) and \( v \). As above, as \( r' \) moves towards \( v' \), there is a neighbourhood of \( p \) which is not met outside \( p \) by \( S(w,w;r';\sigma_0) \). Thus, as above, we can obtain \( S(w,w;v';\sigma_0) \) meeting \( N_1 \) between \( p \) and \( v \),
or $S(w,w;v;\sigma_0)$ meeting $N_1$ between $p$ and $v'$.

Let $r'$ move towards $v'$. Then $S(w,v,w;r')$ does not pass through $p$ if $r' \neq r$, and it does not pass through $f$ or $f'$ as long as it contains more than four points of $M_1$. It meets $M_1$ near $p$ with an even multiplicity and it cannot meet $N_1 (N'_1)$ twice arbitrarily close to $p$. Thus it meets $N_1 (N'_1)$ once. When $r'$ reaches $v$, we get $S(w,v,w;v')$ which meets $N_f (N'_f)$ an odd number of times $\geq 2$, i.e., three times.

8.10. Conjecture.

Let $p$ decompose $A$ into two arcs of order four. Then $p$ is strongly differentiable if and only if $p$ is a differentiable point, and $a_0 = a_1 = a_2 = 1$. 
CHAPTER IX

DIFFERENTIABLE POINTS OF ARCS IN CONFORMAL N-SPACE

9.1. Introduction.

In this chapter we generalize to n dimensions the work of Chapters 2 and 5. The change from three to n dimensions is not as pronounced as the change from two to three dimensions, although the necessarily complicated notation, and the absence of any visual aid, make it appear quite difficult.

9.2. Differentiability.

Let p be a fixed point of an arc A, and let t be a variable point of A. Let 1 ≤ m < n. If p, p₁, ..., p_{m+1} do not lie on the same (m-1)-sphere, then there exists a unique m-sphere \( S^{(m)}_o = S^{(m)}(p_1, ..., p_{m+1}; T_o) \) through these points. We call A \((m+1)\)-times differentiable at p if the following sequence of conditions is satisfied:

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\( \Gamma_r^{(m)} \): If the parameter \( t \) is sufficiently close to, but different from, the parameter \( p \), then the \( m \)-sphere

\[ S^{(m)}(P_1, \ldots, P_{m+1-r}; t; \Gamma_{r-1}) \]

is uniquely defined. It converges if \( t \) tends to \( p \). Thus the limit sphere

\[ S_r^{(m)} = S^{(m)}(P_1, \ldots, P_{m+1-r}; \Gamma_r) \]

will be independent of the way \( t \) converges to \( p \) \((r = 1, 2, \ldots, m+1)\); condition \( \Gamma_{m+1}^{(m)} \) reads: \( S^{(m)}(t; \Gamma_m) \) exists and converges to \( S_{m+1}^{(m)} = S^{(m)}(\Gamma_{m+1}) \).

It is convenient to use the symbols \( S^{(o)} \) to denote pairs of points \( P, p \), and \( S^{(o)}_1 \) to denote the point pair \( p, p \) (i.e., the point \( p \)).

We call \( A \) once-differentiable at \( p \) if \( \Gamma_1^{(1)} \) is satisfied. The point \( p \) is called a differentiable point of \( A \) if \( A \) is \( n \)-times differentiable at \( p \).

Let \( \mathcal{L}_r^{(m)} \) denote the family of all the \( S_r^{(m)} \)'s. Thus

\[ \mathcal{L}_{m+1}^{(m)} \]

consists only of \( S_{m+1}^{(m)} \), the osculating \( m \)-sphere of \( A \) at \( p \).

9.3. The Structure of the Families \( \mathcal{L}_r^{(m)} \) of \( m \)-spheres through \( p \).
Theorem 9.1. Suppose A satisfies condition $I_l^{(m)}$ at $p$. Let $S^{(m-1)}$ be any $(m-1)$-sphere. Then there is a neighbourhood $N$ of $p$ on $A$ such that if $t \in N, t \neq p$, then $t \not\in S^{(m-1)}$, $(m = 1, 2, \ldots, n-1)$.

Proof: The assertion is evidently true if $p \not\in S^{(m-1)}$. Suppose $p \in S^{(m-1)}$. Choose points $P_1, \ldots, P_m$ on $S^{(m-1)}$ such that $p, P_1, \ldots, P_m$ are independent. If the parameter $t$ is sufficiently close to, but different from, the parameter $p$, condition $I_l^{(m)}$ implies that $S^{(m)}(P_1, \ldots, P_m, t; \Omega_0)$ is uniquely defined. Thus $t \not\in S^{(m-1)}(P_1, \ldots, P_m; \Omega_0) = S^{(m-1)}$.

Corollary 1. If $A$ satisfies condition $I_l^{(m)}$ at $p$, and $S^{(k)}$ is any $k$-sphere, then $t \not\in S^{(k)}$ when the parameter $t$ is sufficiently close to, but different from, the parameter $p$ $(k = 0, 1, \ldots, m-1)$.

In particular, this holds when $m = n-1$.

Theorem 9.2. Let $l < m < n; 1 \leq k \leq m$. If $A$ satisfies $I_l^{(m)}, \ldots, I_k^{(m)}$ at $p$, then $I_l^{(m-1)}, \ldots, I_k^{(m-1)}$ will hold
there and

\[(9.1) \ S^{(m-1)}(p_1, \ldots, p_{m-r}; \tau_r) = \prod_p S^{(m)}(p_1, \ldots, p_{m-r}, p; \tau_r). \]

Conversely, let \( A \) satisfy \( \tau_1^{(m-1)}, \ldots, \tau_k^{(m-1)} \) at \( p \), and let \( S^{(m-1)} \neq p \) if \( k = m \). If \( p_{m-r+1} \notin S^{(m-1)}(p_1, \ldots, p_{m-r}; \tau_r) \), then \( \tau_r^{(m)} \) will hold for the points \( p_1, \ldots, p_{m-r+1} \) and

\[(9.2) \ S^{(m)}(p_1, \ldots, p_{m-r+1}; \tau_r) = S^{(m)}(p_{m-r+1}; S^{(m-1)}(p_1, \ldots, p_{m-r}; \tau_r)) \]

\((r = 1, \ldots, k)\).

Remark: In general, \( \tau_1^{(m-1)}, \ldots, \tau_k^{(m-1)} \) do not imply \( \tau_1^{(m)}, \ldots, \tau_k^{(m)} \) (cf. e.g., \( \S \) 5.5).

Proof: (by induction with respect to \( k \)): Suppose \( k = 1 \); \( l < m < n \). Let \( \tau_1^{(m)} \) hold. If \( p_1, \ldots, p_{m-1}, P, p \) are independent points, \( S^{(m)}(p_1, \ldots, p_{m-1}, P, t; \tau_o) \) exists when \( t \) is sufficiently close to \( p, t \neq p, t \in A \). Thus \( p_1, \ldots, p_{m-1}, P, t,p \), are also independent, \( S^{(m-1)}(p_1, \ldots, p_{m-1}, t; \tau_o) \) exists, and

\[ S^{(m-1)}(p_1, \ldots, p_{m-1}, t; \tau_o) = \prod_p S^{(m)}(p_1, \ldots, p_{m-1}, P, t; \tau_o). \]

If \( t \to p \), \( S^{(m)}(p_1, \ldots, p_{m-1}, P, t; \tau_o) \) converges, and hence
$S^{(m-1)}(P_1, \ldots, P_{m-1}, t; T_0)$ also converges, $\Gamma_1^{(m-1)}$ is satisfied, and
\[
S^{(m-1)}(P_1, \ldots, P_{m-1}; T_1) = \prod_{P} S^{(m)}(P_1, \ldots, P_{m-1}, P; T_1).
\]

Next, suppose that $\Gamma_1^{(m-1)}$ is satisfied, and
\[P_m \not\in S^{(m-1)}(P_1, \ldots, P_{m-1}; T_1).\]
Then $P_m \not\in S^{(m-1)}(P_1, \ldots, P_{m-1}, t; T_0)$ when $t$ is sufficiently close to $p$, $t \in A$, $t \neq p$, and
\[S^{(m)}(P_1, \ldots, P_{m}; T_{0}) = S^{(m)} \left[ P_{m}; S^{(m-1)}(P_1, \ldots, P_{m-1}, t; T_{0}) \right]
\]
exists. Hence when $t \to p$, $S^{(m)}(P_1, \ldots, P_{m}; T_{0})$ converges, $\Gamma_1^{(m)}$ is satisfied relative to the points $P_1, \ldots, P_m$, and
\[S^{(m)}(P_1, \ldots, P_{m}; T_{1}) = S^{(m)} \left[ P_{m}; S^{(m-1)}(P_1, \ldots, P_{m-1}; T_{1}) \right].
\]

Thus Theorem 9.2 is true when $k=1$.

Assume that Theorem 9.2 holds when $k$ is replaced by $1, 2, \ldots, h$, where $1 \leq h < k \leq m$.

Let $\Gamma_1^{(m)}, \ldots, \Gamma_{h+1}^{(m)}$ hold. Then $S^{(m)}(P_1, \ldots, P_{m-h}; P; t; T_h)$ exists when $t$ is sufficiently close to $p$, $t \neq p$, $t \in A$. Now $\Gamma_1^{(m)}, \ldots, \Gamma_{h}^{(m)}$ imply $\Gamma_1^{(m-1)}, \ldots, \Gamma_{h}^{(m-1)}$. If $h=m-1$,
\[\Gamma_{m-1}^{(m-1)} = \Gamma_{m-1}^{(m-1)}\]
implies that $S_{m-1}^{(m-1)}(t; T_{m-1})$ exists,
if $t \neq p$. If $h < m-1$, $\Gamma_l^{(m-1)}, \ldots, \Gamma_h^{(m-1)}$ imply $\Gamma_1^{(m-2)}, \ldots, \Gamma_h^{(m-2)}$. Thus $S^{(m-2)}(P_1, \ldots, P_{m-h-1}; \tau_h)$ exists. Furthermore, $\Gamma_1^{(m-1)}$ and Theorem 9.1 imply that $t \neq S^{(m-2)}(P_1, \ldots, P_{m-h-1}; \tau_h)$. But then Theorem 9.2, with $k$ replaced by $h$, implies that

$$S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h) = S^{(m-1)}[t; S^{(m-2)}(P_1, \ldots, P_{m-h-1}; \tau_h)]$$

exists. By Theorem 9.2 again, with $k$ replaced by $h$,

$$S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h) = \prod_P S^{(m)}(P_1, \ldots, P_{m-h-1}, P; t; \tau_h).$$

When $t \rightarrow p$, $S^{(m)}(P_1, \ldots, P_{m-h-1}, P; t; \tau_h)$ converges, hence $S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h)$ also converges, $\Gamma_{h+1}^{(m-1)}$ is satisfied, and

$$S^{(m-1)}(P_1, \ldots, P_{m-h-1}; \tau_{h+1}) = \prod_P S^{(m)}(P_1, \ldots, P_{m-h-1}, P; \tau_{h+1}).$$

Next suppose $\Gamma_1^{(m-1)}, \ldots, \Gamma_{h+1}^{(m-1)}$ hold, and let $P_m \neq S^{(m-1)}(P_1, \ldots, P_{m-h-1}; \tau_{h+1})$. Then $P_{m-h} \neq S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h)$ if $t$ is sufficiently close to $p$, $t \in A$, $t \neq p$. But Theorem 9.2, with $k$ replaced by $h$, then implies that
exists. Hence when \( t \to p \), \( S^{(m)}(p_1, \ldots, P_{m-h-1}, p_{m-h}, t; \tau_h) \) converges, \( \tau_{h+1}^{(m)} \) is satisfied for \( P_1, \ldots, P_{m-h} \), and

\[
S^{(m)}(P_1, \ldots, P_{m-h}; \tau_{h+1}) = S^{(m)}\left[P_{m-h}; S^{(m-1)}(P_1, \ldots, P_{m-h+1}; \tau_{h+1})\right].
\]

**Corollary 1.** Let \( 1 \leq m < n \). If \( A \) is \((m+1)\)-times differentiable at \( p \), then it is \( m \)-times differentiable there.

**Corollary 2.** If \( A \) satisfies \( \tau_1^{(n-1)}, \ldots, \tau_{m+1}^{(n-1)} \) at \( p \), then it is \((m+1)\)-times differentiable there \((0 \leq m < n)\).

**Corollary 3.**

\[
S^{(m-1)}_m \subset S^{(m)}_{m+1} \quad (m = 1, 2, \ldots, n-1).
\]

**Proof:** By relation (9.1)

\[
S^{(m)}(t; \tau_m) \supset \bigcup_P S^{(m)}(P; \tau_m) = S^{(m-1)}_m.
\]

Hence \( S^{(m)}_{m+1} \supset S^{(m-1)}_m \).

The last remark implies

**Corollary 4.** Let \( 1 \leq m < n \). If \( S^{(m)}_{m+1} = p \), then \( S^{(r)}_{r+1} = p \) \((r = 0, 1, \ldots, m-1)\).

Thus there is an index \( i \), where \( 1 \leq i \leq n \) such that \( S^{(r)}_{r+1} = p \).
for \( r = 0, 1, \ldots, i-1 \), but \( S^{(r)}_{r+1} \neq p \) if \( r \geq i \).

**Corollary 5.** Let \( 1 \leq m < n; 1 \leq r \leq m \). Then

\[
S^{(m)}(P_1, \ldots, P_{m+1-r}; \mathcal{T}_r) \supset S^{(m-1)}(P_1, \ldots, P_{m+1-r}; \mathcal{T}_{r-1}).
\]

**Proof:**

\[
S^{(m)}(P_1, \ldots, P_{m+1-r}; \mathcal{T}_r) = \lim_{t \to p} S^{(m)}(P_1, \ldots, P_{m+1-r}, t; \mathcal{T}_{r-1})
\]

\[
\supset S^{(m-1)}(P_1, \ldots, P_{m+1-r}; \mathcal{T}_{r-1}).
\]

Form Corollary 5, we get

**Corollary 6.** Let \( 1 \leq m < n; 1 \leq r \leq m \). If \( P_{m+2-r} \)

\[
\subseteq S^{(m)}(P_1, \ldots, P_{m+1-r}; \mathcal{T}_r)
\]

and \( P_{m+2-r} \neq S^{(m-1)}(P_1, \ldots, P_{m+2-r}; \mathcal{T}_{r-1}) \) then

\[
S^{(m)}(P_1, \ldots, P_{m+1-r}; \mathcal{T}_r) = S^{(m)}(P_1, \ldots, P_{m+2-r}; \mathcal{T}_{r-1}).
\]

**Theorem 9.3.** Let \( 1 < r \leq m < n \). Suppose \( \mathcal{T}^{(m)}_r \) are satisfied at \( p \).

(i) If \( S^{(r-1)}_r \neq p, \mathcal{T}^{(m)}_r \) consists of all the \( m \)-spheres through \( S^{(r-1)}_r \).

(ii) Let \( S^{(r-1)}_r = p \). Choose any \( S^{(r)}_r \in \mathcal{T}^{(r)}_r \). Then

\( \mathcal{T}^{(m)}_r \) is the set of all the \( m \)-spheres which touch \( S^{(r)}_r \) at \( p \).
Proof of (i): By Theorem 9.2, equation (9.1),
\[ S^{(m)}(P_1, \ldots, P_{m+r-1}; T_r) \supset S^{(m-1)}(P_1, \ldots, P_{m-r}; T_r) \]
\[ \supset \cdots \supset S^{(r)}(P_1; T_r) \supset S^{(r-1)}(T_r) . \]

Let \( S^{(m)} \) be any \( m \)-sphere through \( S^{(r-1)}(T_r) \). By Theorem 9.2, if \( P_1 \in S^{(m)} \), \( P_1 \notin S^{(r-1)} \),
\[ S^{(r)}(P_1; S^{(r-1)}(T_r)) = S^{(r)}(P_1; T_r) \subseteq S^{(m)} . \]
Suppose \( S^{(k)}(P_1, \ldots, P_{k+1-r}; T_r) \subseteq S^{(m)} \), \( (r \leq k < m) \). Choose \( P_{k+2-r} \in S^{(m)} \), \( P_{k+2-r} \notin S^{(k)}(P_1, \ldots, P_{k+1-r}; T_r) \). Then by Theorem 9.2
\[ S^{(k+1)}(P_1, \ldots, P_{k+2-r}; T_r) \]
\[ = S^{(k+1)}[P_{k+2-r}; S^{(k)}(P_1, \ldots, P_{k+1-r}; T_r)] \subseteq S^{(m)} . \]
For \( k = m-1 \), this yields \( S^{(m)}(P_1, \ldots, P_{m+1-r}; T_r) = S^{(m)} \). Thus \( S^{(m)} \in T^{(m)}_r \).

Proof of (ii): Suppose \( S^{(r-1)}(T_r) = p \). As above, we have
\[ S^{(m)}_r = S^{(m)}(P_1, \ldots, P_{m+r-1}; T_r) \supset \cdots \supset S^{(r)}(P_1; T_r) \]
Let \( S^{(r)}(Q; T_r) \) be any \( S^{(r)} \in T^{(r)}_r \). By Theorem 9.2, equation (9.1),
\[ S^{(r)}(P, t; T_{r-1}) \cap S^{(r)}(Q, t; T_{r-1}) \supset S^{(r-1)}(t; T_{r-1}) . \]
Let \( P \) and \( Q \) be variable points and let \( S^{(r-1)} \) be a variable
(r-1)-sphere converging to a fixed point. Suppose there is an (n-1)-sphere which separates this point from P and Q. Then
\[ \lim_{t \to p} \left[ S^{(r)}(P;S^{(r-1)}), S^{(r)}(Q;S^{(r-1)}) \right] = 0 \]
whether or not the spheres \( S^{(r)}(P;S^{(r-1)}) \) and \( S^{(r)}(Q;S^{(r-1)}) \) themselves converge.

In particular,

\[ \lim_{t \to p} \left[ S^{(r)}(P,t;\tau_{r-1}), S^{(r)}(Q,t;\tau_{r-1}) \right] = 0. \]

Thus \( S^{(r)}(P;\tau_r) \) touches \( S^{(r)}(Q;\tau_r) \) at \( p \). Furthermore, if \( S^{(r)}(P;\tau_r) \) and \( S^{(r)}(Q;\tau_r) \) have a point \( \neq p \) in common, they coincide. Thus \( \tau_r^{(r)} \) consists of the family of \( r \)-spheres which touch \( S^{(r)}(Q;\tau_r) \) at \( p \).

Suppose \( r < m \) and an \( m \)-sphere \( S^{(m)}_r = S^{(m)}(P_1,\ldots,P_{m+1-r};\tau_r) \) of \( \tau_r^{(m)} \) has a point \( R \neq p \) in common with \( S^{(r)}(Q;\tau_r) \).

From the above, \( S^{(r)}(R;\tau_r) = S^{(r)}(Q;\tau_r) \). If \( R \in S^{(r)}(P_1;\tau_r) \) we have
\[ S^{(m)} \supset S^{(r)}(P_1;\tau_r) = S^{(r)}(R;\tau_r) = S^{(r)}(Q;\tau_r) \]
while if \( R \notin S^{(r)}(P_1;\tau_r) \), we have by Theorem 9.2
\[ S^{(m)}_r \supset S^{(r+1)}[R;S^{(r)}(P_1;\tau_r)] = S^{(r+1)}(P_1,R;\tau_r) = S^{(r+1)}[P_1,S^{(r)}(R;\tau_r)] \supset S^{(r)}(R;\tau_r) = S^{(r)}(Q;\tau_r). \]
On the other hand, suppose an $m$-sphere $S^{(m)}$ touches $S^{(r)} = S^{(r)}(Q; r)$ at $p$. If $S^{(m)} \subseteq S^{(r)}$ it follows, as in the proof of part (i), that $S^{(m)} \in \mathcal{T}^{(m)}$. Suppose $S^{(m)} \cap S^{(r)} = p$. Choose an $S^{(r)} \subseteq S^{(m)}$ such that $S^{(r)}$ touches $S^{(r)}(Q; r)$ at $p$. Thus $S^{(r)} \in \mathcal{T}^{(r)}$. It again follows that $S^{(m)} \in \mathcal{T}^{(m)}$.

**Corollary 1.** Let $\Gamma^{(r-1)}_1, \ldots, \Gamma^{(r-1)}_r$ hold and let $S^{(r-1)}_r = p$. Suppose $\lim_{t \to p} S^{(r)}(P, t; r-1)$ exists for a single point $P \neq p$. Then $\Gamma^{(r)}_r$ holds at $p$ ($1 < r < n$).

**Proof:** This follows from equation (9.3).

**Corollary 2.** There is only one $S^{(m)}_r$ of the pencil $\mathcal{T}^{(m)}_r$ which contains $(m+1-r)$ points which do not lie on the same $S^{(m-1)}$.

**Proof:** Such an $S^{(r)}_r$ can be uniquely constructed as in the proof of (i), Theorem 9.3.

**Corollary 3.** If two $S^{(m)}_r$'s intersect in an $S^{(m-1)}$, then this $S^{(m-1)} \in \mathcal{T}^{(m-1)}_r$.

**Proof:** The $S^{(m)}_r$'s, and hence also $S^{(m-1)}$, contain $S^{(r-1)}_r$. In case $S^{(r-1)}_r = p$, let $P \subset S^{(m-1)}$, $P \neq p$. 
Then the $S^r_m$'s and hence also $S^{(m-1)}$ contains $S^{(r)}(P; T_r)$.

**Corollary 4.**

$$T^0_m \supset T^1_m \supset \ldots \supset T^r_m.$$  

**Proof:** When $k < m$, or when $k = m$ and $S^{(m-1)}_m \neq p$, Theorem 9.3 implies that $T^k_m$ is the set of all the $m$-spheres through $S^{(k-1)}_k$. Hence $S^k_m$, being the limit of a sequence of such $m$-spheres, must itself contain $S^{(k-1)}_k$, and by Theorem 9.3, $S^{(m)}_m \in T^k_m$. Suppose $k = m$ and $S^{(m-1)}_m \neq p$. By Theorem 9.3, $T^m_m$ is the set of all the $m$-spheres which touch a given $m$-sphere $S^m_m \neq p$ of $T^m_m$ at $p$. Hence $S^{(m)}_m$, being the limit of a sequence of such $m$-spheres, must itself touch $S^m_m$ at $p$, and, again by Theorem 9.3, $S^{(m)}_m \in T^m_m$.

**Theorem 9.4.** Let $1 < m < n$: $1 < k < m$, and suppose that $S^{(m-1)}_m \neq p$ if $k = m$. If the conditions $T^1_m, \ldots, T^k_m$ hold at $p$, then $T^{(m)}_{k+1}$ also holds there.

**Proof:** By Theorem 9.2, $T^{(m-1)}_1$ holds at $p$. Hence if $p, P_1, \ldots, P_{m-k}$ are independent points $S^{(m-1)}(P_1, \ldots, P_{m-k}; T_k)$ is defined. Furthermore, by Theorem 9.1, we can assume
that \( t \not\in S^{(m-1)}(P_1, \ldots, P_{m-k}; T_k) \) and by Theorem 9.2 again,
\[
S^{(m)}(P_1, \ldots, P_{m-k}, t; T_k) = S^{(m)}\left[ t; S^{(m-1)}(P_1, \ldots, P_{m-k}; T_k) \right].
\]
Thus \( S^{(m)}(P_1, \ldots, P_{m-k}, t; T_k) \) exists when \( t \) is close to \( p \), 
\( t \in A \), \( t \neq p \). Choose a point \( P_{m+1-k} \in S^{(m-1)}(P_1, \ldots, P_{m-k}; T_k) \),
\( P_{m+1-k} \not\in S^{(m-2)}(P_1, \ldots, P_{m-k}; T_{k-1}) \).

Then Theorem 9.2, Corollary 6 implies that
\[
S^{(m-1)}(P_1, \ldots, P_{m-k}; T_k) = S^{(m-1)}(P_1, \ldots, P_{m+1-k}; T_{k-1}),
\]
when \( k < m \), or \( k = m \), and \( S^{(m-2)}_{m-1} \neq p \); if \( k = m \) and \( S^{(m-2)}_{m-1} = p \),
this equation follows from Theorem 9.3, Corollary 4. Hence
\[
\lim_{t \to p} S^{(m)}(P_1, \ldots, P_{m-k}, t; T_k) \\
= \lim_{t \to p} S^{(m)}\left[ t; S^{(m-1)}(P_1, \ldots, P_{m+1-k}; T_{k-1}) \right] \\
= \lim_{t \to p} S^{(m)}(P_1, \ldots, P_{m+1-k}, t; T_{k-1}) \\
= S^{(m)}(P_1, \ldots, P_{m+1-k}; T_k).
\]
Thus \( \Gamma_{k+1}^{(m)} \) holds at \( p \) and
\[
(9.4) \quad S^{(m)}(P_1, \ldots, P_{m-k}; T_{k+1}) = S^{(m)}(P_1, \ldots, P_{m+1-k}; T_k) \in T^{(m)}_k.
\]

**Corollary 1.** If \( \Gamma_1^{(m)} \) holds at \( p \), then \( \Gamma_r^{(m)} \) holds there, \( r = 1, 2, \ldots, m \). Furthermore, if \( S^{(m-1)}_m \neq p \), \( A \) is
Corollary 2. If $\Gamma_{n-1}^{(n-1)}$ holds at $p$, then $p$ is a differentiable point of $A$ if and only if
\[ \lim_{t \to p} S_{n-1}^{(n-1)}(t; \tau_{n-1}) \]
exists and converges if $t$ tends to $p$.

Corollary 3. If $\Gamma_{0}^{(n-1)}$ holds at $p$, and $S_{n-1}^{(n-2)} \neq 0$, then $p$ is a differentiable point of $A$.

Corollary 4. If $\Gamma_{1}^{(m)}$ holds at $p$, all the conditions
\[ \Gamma_{k}^{(r)} \text{ except } \Gamma_{m+1}^{(m)}, \text{ automatically hold at } p \quad (1 \leq k \leq r+1 \leq m+1). \]

Let $p$ be a differentiable point of $A$. We define the index $i$ of $p$ as in Theorem 9.2, Corollary 4. Let $P \subset S_{i+1}^{(i)}$, $P \neq p$. Let $S_{m}^{(m)} = S_{m}^{(m)}(P; I_{m})$, $m = 0, 1, \ldots, m-1$. Then the set of $I_{i}^{(m)}$'s is completely determined by the sequence
\[ S_{0}^{(0)} \subset S_{1}^{(1)} \subset \cdots \subset S_{1}^{(i)} \subset S_{i+1}^{(i+1)} \subset \cdots \subset S_{n}^{(n-1)}. \]
Its structure is determined by the single index $i$.

9.4. **Support and Intersection Properties of $\tau_{r}^{(n-1)} - \tau_{r+1}^{(n-1)}$.**

Let $p$ be a differentiable interior point of $A$.

Our classification of the differentiable points $p$ of $A$ will be based on the index $i$ of $p$, and on the support and
intersection properties of $S^{(n-1)}_n$ and the families $T^{(n-1)}_r - T^{(n-1)}_{r+1}$, $r = 0, 1, \ldots, n-1$. We shall omit the superscript $(n-1)$ of $T^{(n-1)}_r$ when there is no ambiguity; thus $T^{(n-1)}_r = T^{(n-1)}_r$.

Theorem 9.5. Every $(n-1)$-sphere $\neq S^{(n-1)}_n$ either supports or intersects $A$ at $p$.

Proof: If an $(n-1)$-sphere $S$ neither supports nor intersects $A$ at $p$, then $p \subset S$ and there exists a sequence of points $t \to p$, $t \in A \cap S$, $t \neq p$. Suppose $p, P_1, \ldots, P_n$ are independent points on $S$. Suppose that for $0 \leq r < n-1$,

$$S = S^{(n-1)}(P_1, \ldots, P_{n-r}; T^r_r).$$ By equation (9.1),

$$S^{(n-1)}(P_1, \ldots, P_{n-r}; T^r_r) = S^{(n-2)}(P_1, \ldots, P_{n-r-1}; T^r_r).$$

By Theorem 9.1, $t \not\subset S^{(n-2)}(P_1, \ldots, P_{n-r-1}; T^r_r)$ and by equation (9.2),

$$S = S^{(n-1)}(t; S^{(n-2)}(P_1, \ldots, P_{n-r-1}; T^r_r)) = S^{(n-1)}(P_1, \ldots, P_{n-r-1}, t; T^r_r)$$

for each $t$. Condition $T^{(n-1)}_{r+1}$ now implies that

$$S = S^{(n-1)}(P_1, \ldots, P_{n-r-1}; T^r_{r+1}).$$

Thus we get, in this way,

$$S = S^{(n-1)}(P_1; T^{(n-1)}_{n-1}).$$
By Theorem 9.2, \( S \supseteq S_{n-1}^{(n-2)} \), and by Theorem 9.1, \( t \not\in S_{n-1}^{(n-2)} \)
when the parameter \( t \) is close to, but different from, the
parameter \( p \). If \( S_{n-1}^{(n-2)} \neq p \), equation (9.2) implies that
\[ S = S_{n-1}^{(n-1)} \left[ t; S_{n-1}^{(n-2)} \right] = S_{n-1}^{(n-1)}(t; \mathcal{T}_{n-1}), \]
while if \( S_{n-1}^{(n-2)} = p \), Theorem 9.3 implies that \( S = S_{n-1}^{(n-1)}(t; \mathcal{T}_{n-1}) \). Applying con-
dition \( \Gamma_n^{(n-1)} \), we are led to the conclusion \( S = S_n^{(n-1)} \).

**Theorem 9.6.** If \( S_n^{(n-1)} = p \), then the \((n-1)\)-spheres
of \( \mathcal{L}_{n-1} - \mathcal{L}_n \) all intersect \( A \) at \( p \), or they all support.

**Proof:** Let \( S' \) and \( S'' \) be two distinct \((n-1)\)-spheres
of \( \mathcal{L}_{n-1} - \mathcal{L}_n \). Since \( S_n^{(n-1)} = p \), Theorem 9.2, Corollary 4
implies that \( S_{n-1}^{(n-2)} = p \), and Theorem 9.3 implies that \( S' \) and
\( S'' \) touch at \( p \). Thus we may assume that \( S'' \subset (p \cup S') \) and
\( S' \subset (p \cup S'') \). Suppose now, for example, that \( S' \) supports \( A \)
at \( p \) while \( S'' \) intersects. Then \( A \cap S'' \) is not void and
\( A \subset (p \cup S') \). Let \( t \to p \) in \( A \cap S'' \). Hence \( S_{n-1}^{(n-1)}(t; \mathcal{T}_{n-1}) \)
\( \subset (S'' \cap S') \cup p \). Consequently, \( S(t; \mathcal{T}_{n-1}) \) cannot converge to
\( S_{n-1}^{(n-1)} = p \), as \( t \) tends to \( p \). Thus \( S' \) and \( S'' \) must both support,
Theorem 9.7. If \( S^{(r)}_{r+1} \neq p \) while \( S^{(r-1)}_r = p \), then every \((n-1)\)-sphere of \( \overline{L}_r - \overline{L}_{r+1} \) supports \( A \) at \( p \) (1 \( \leq r \leq n-1 \)).

Proof: Suppose \( S^{(r-1)}_r = p \), so that by Theorem 9.3, the \( r \)-spheres of \( L_r \) all touch any \((n-1)\)-sphere of \( L_r \). Let \( S \in L_r - L_{r+1} \), \( S \neq p \). If a sequence of points \( t \) exists such that \( t \in A \cap \overline{S} \), \( t \to p \), then each \( S^{(r)}_r(t;L_r) \) lies in the closure of \( \overline{S} \). Hence \( S^{(r)}_r \) will also lie in the same closed domain.

Since \( S^{(r)}_{r+1} \in L_r \), either \( S^{(r)}_{r+1} = p \), or it touches \( S \) at \( p \). Since \( S \neq L_{r+1} \), \( S^{(r)}_{r+1} \) must lie in \( p \cup \overline{S} \). Similarly, the existence of a sequence \( t' \in \overline{S} \cap A \), \( t' \to p \), implies that \( S^{(r)}_{r+1} \subseteq p \cup \overline{S} \).

Thus if \( S \) intersects \( A \) at \( p \), \( S^{(r)}_{r+1} \subseteq (p \cup \overline{S}) \cap (p \cup \overline{S}) = p \); i.e., \( S^{(r)}_{r+1} = p \).

Theorem 9.8. All the \((n-1)\)-spheres of \( L_r - L_{r+1} \) support \( A \) at \( p \), or they all intersect; \( r = 0, 1, \ldots, n-1 \).

Proof: Let \( S' \) and \( S'' \) be two distinct \((n-1)\)-spheres of \( L_r \). Suppose, for the moment, that the intersection \( S' \cap S'' \)
is a proper \((n-2)\)-sphere \(S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \mathcal{T}_r)\). Suppose for example that \(S'\) intersects, while \(S''\) supports \(A\) at \(p\).

Thus \(A \cap S'\) and \(A \cap S''\) are not void. With no loss in generality, we may assume that \(A \subseteq \overline{S''} \cup \{p\}\). If \(t\) is close to \(p\), \(t \neq p\), Theorem 9.1 implies that \(t \notin S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \mathcal{T}_r)\), and equation \((9.2)\) implies that

\[
S^{(n-1)}(t; S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \mathcal{T}_r)) = S^{(n-1)}(P_1, \ldots, P_{n-r-1}, t; \mathcal{T}_r).
\]

If \(t \in A \cap S'\), then \(S^{(n-1)}(P_1, \ldots, P_{n-r-1}, t; \mathcal{T}_r)\) lies in the closure of

\[
(S' \cap \overline{S''}) \cup (S'' \cap S''').
\]

Letting \(t\) tend to \(p\), we conclude that \(S^{(n-1)}(P_1, \ldots, P_{n-r-1}; \mathcal{T}_{r+1})\) lies in the same closed domain. By letting \(t\) converge to \(p\) through \(S' \cap A\), we obtain symmetrically that

\[
S^{(n-1)}(P_1, \ldots, P_{n-r-1}; \mathcal{T}_{r+1})\text{ also lies in the closure of }\]

\[
(S' \cap \overline{S''}) \cup (S'' \cap S'').
\]

Hence \(S^{(n-1)}(P_1, \ldots, P_{n-r-1}; \mathcal{T}_{r+1})\) lies in the intersection \(S' \cup S''\) of these two domains, i.e., \(S^{(n-1)}(P_1, \ldots, P_{n-r-1}; \mathcal{T}_{r+1})\).
is either $S'$ or $S''$. In other words, one of the $(n-1)$-spheres $S'$ and $S''$ belongs to $\mathcal{T}_{r+1}$. Thus if $S'$ and $S''$ belong to $\mathcal{T}_r - \mathcal{T}_{r+1}$, and have a proper $S^{(n-2)}$ in common, they both support or both intersect $A$ at $p$.

Suppose now that $S' \cap S'' = p$. Theorem 9.3 implies that $S^{(r-1)}_r = p$. In view of Theorems 9.6 and 9.7, there remain to be considered only the cases where $r < n-1$, and indeed, when $r \leq n-2$, we have only to consider those cases for which $S^{(r)}_{r+1} = p$.

By Theorem 9.3, any $S^{(n-1)}$ which touches an $S^{(r)}_r$, but which does not touch an $S^{(r+1)}_{r+1}$ belongs to $\mathcal{T}_r - \mathcal{T}_{r+1}$. Hence there exists an $(n-1)$-sphere $S$ of $\mathcal{T}_r - \mathcal{T}_{r+1}$ which intersects $S'$ and $S''$ respectively in a proper $(n-2)$-sphere. From the above, $S$ and $S'$, and also $S$ and $S''$ both support or both intersect $A$ at $p$. Thus $S'$ and $S''$ both support or both intersect $A$ at $p$ in this case also.

9.5. Characteristics and a Classification of the Differen-
The characteristic, \((a_0, a_1, \ldots, a_n; i)\) of a differentiable point \(p\) of an arc \(A\) is defined as follows:

\[ a_r = 1 \text{ or } 2 \text{ when } r < n; \quad a_n = 1, 2, \text{ or } \infty. \]

The index \(i = 1, 2, \ldots, n\).

\[ a_0 + \ldots + a_r \text{ is even or odd according as every } \]
\[ S_{r}^{(n-1)} \text{ of } \tau_r - \tau_{r+1} \text{ supports or intersects } A \text{ at } p; r = 0, 1, \ldots, n-1. \]

\[ a_0 + \ldots + a_n \text{ is even if } S_{n}^{(n-1)} \text{ supports, odd if } \]
\[ S_{n}^{(n-1)} \text{ intersects, while } a = \infty \text{ if } S_{n}^{(n-1)} \text{ neither supports nor intersects } A \text{ at } p. \]

Finally the characteristic of \(p\) has index \(i\) if and only if \(S_i^{(i-1)} = p\), while \(S_i^{(i)} \neq p\).

Theorem 9.7, and the convention that \(S_n^{(n-1)}\) supports \(A\) at \(p\) when \(S_n^{(n-1)} = p\), lead to the following restriction on the characteristic \((a_0, a_1, \ldots, a_n; i)\): 

\[ \sum_{k=0}^{i} a_k \equiv 0 \pmod{2}. \]

As a result of this restriction, the number of types
of differentiable points corresponding to each value of $i < n$ is $3(2)^{n-1}$, and there are $2^n$ types when $i = n$. Thus there are $(3n - 1)2^{n-1}$ types altogether.

If we introduce a rectangular Cartesian coordinate system into the conformal n-space, examples of each of the $(3n - 1)2^{n-1}$ types are given by the curves

(I) \[ x_1 = t^{m_1}, \ x_2 = t^{m_2}, \ldots, \ x_n = t^{m_n} \]

in the cases $a_n = 1$ or 2, and

(II) \[ x_1 = t^{m_1}, \ x_2 = t^{m_2}, \ldots, \ x_n = \begin{cases} t^{m_n} \sin t^{-1}, & 0 < |t| < 1 \\ 0, & t = 0 \end{cases} \]

for the cases in which $a_n = \infty$, all relative to the point $t = 0$. The $m_r$ are positive integers, and $m_1 < m_2 < \ldots < m_n$.

The different types are determined by the parities of the $m_i$, and by the relative magnitudes of the $m_r$ and $2m_1$. In each of these examples, the $S_1^{(m)}$ touch the $x_1$-axis at the origin; $m = 1, 2, \ldots, n-1$.

When $m_1 < 2m_1 < m_{i+1}$, the point $t = 0$ has a characteristic of the form $(a_0, a_1, \ldots, a_n; i)$ where $a_n$ can be $1,$
2, or $\infty$, and $i < n$.

When $m_n < 2m_1$, the point $t = 0$ has a characteristic of the form $(a_0, a_1, \ldots, a_n; n)$ where $a_n$ is either 1 or 2.

Table 9.1 lists some of the properties of a differentiable point $p$ having the characteristic $(a_0, a_1, \ldots, a_n; i)$. 
<table>
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<tr>
<td>$a_n = \infty$</td>
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<td></td>
<td></td>
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</table>
| $i = n$ | $a_n = 1 \text{ or } 2$ | | | $\sum_{r 
eq 0}^{n} a_r = 0 \pmod{2}$ | $m_n < 2m_1$ |

Table 9.1
BIBLIOGRAPHY


Lane, N.D., **Differentiable Points of arcs in Conformal n-space.** To appear in Pacific J. Math.


Sommerville, D.M.Y., **The Elements of Non-Euclidean Geometry.** London: G. Bell and Sons, 1914.

Sommerville, D.M.Y., **An Introduction to the Geometry of N Dimensions.** London: Methuen, 1929.