CONFORMAL DIFFERENTIAL GEOMETRY

CONFORMAL DIFFERENTIAL GEOMETRY

By

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SCOPE AND CONTENTS:

This thesis is a study of some properties of arcs which remain invariant under certain types of conformal representations. The study is carried on first in the conformal plane, then in conformal 3-space, and finally in conformal n-space. It is comprised of most of the research on this subject which has been carried on to date by Professors N.D. Lane and Peter Scherk. I have assisted Dr. Lane in the creation of some of the material which this thesis contains.

ACKNOWLEDGEMENT

I wish to express my sincere thanks to Dr. N.D. Lane, who was my supervisor in the creation of this thesis. The many hours which he spent on my behalf have greatly enriched my understanding of the subject of Mathematics.

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KEY TO NOTATIONS

In the notation of sections, the number of the chapter appears first, and this is followed by the number of the section in that chapter. If the section contains a sub-section, it is denoted by an additional number; e.g., section 8.5 is section 5 of chapter 8, while section 8.5.4 is the fourth sub-section of section 8.5.

In the notation of theorems and lemmas, the number of the chapter is given first and the number of the theorem or lemma in that section follows; e.g., Theorem 5.7 is the seventh theorem in chapter 5.

CHAPTER I

WHAT IS CONFORMAL GEOMETRY?

1.1. Stereographic Projection.

Consider a sphere in projective 3-space resting on a plane p. Let N be the point on the sphere which is the most remote from the plane p, and let P' be any other point on the sphere (cf. Fig. 1.1). The line NP' is extended to meet the plane p at the point P. The mapping of the points P' of the sphere on the plane p in this manner is called a

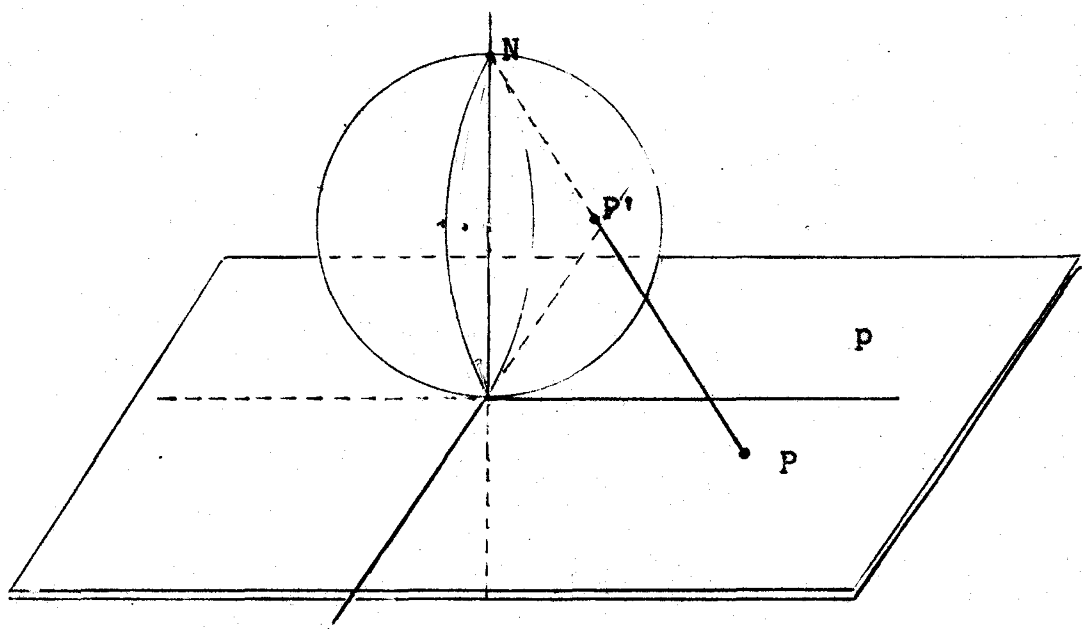

Fig. 1.1

stereographic projection.

1.2. The Notion of Angle.

We describe what we mean by the term angle[1] by the following quotation from Sommerville's "An Introduction to the Geometry of N Dimensions" (Methuen 1929):

"Two linear spaces which have their highest degree of intersection determine an angle and this angle determines completely the shape of the figure consisting of the two spaces. For example, two straight lines in a plane determine a plane angle; two planes in 3-space determine a dihedral angle, which can be measured by means of a plane angle."

1.3. The Conformal Plane.

Let us now return to a consideration of the mapping described in section 1.1. It is not difficult to prove (cf. Snyder and Sisam, "Analytic Geometry of Space", Holt 1932,

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1. This description applies to two, three, or n dimensions.

pp. 59-62) that under a stereographic projection, circles on the sphere which do not pass through N, map onto circles in the plane p, while circles on the sphere which pass through N, map onto straight lines in p. It is also true that angles are preserved under this mapping.

Obviously, there is a one-to-one correspondence between the points of the plane and the points of the sphere, with the exception of the point N. This point has no image in the plane p if we think of p as the Euclidean plane, while it has many images if p is the projective plane. In order to preserve a 1-1 correspondence throughout, we postulate a single _point at infinity_ for the plane p. The point N then corresponds to this point at infinity; the plane p is called the _conformal plane_. It is convenient now, for obvious reasons, to regard straight lines in p as circles through the point at infinity (cf. Hilbert and Cohn-Vossen, "Geometry and the Imagination", Chelsea 1952, p 251).

It is clear from its definition that the conformal plane is identical with the Argand plane of complex numbers (cf. Copson, "Theory of Functions of a Complex Variable", Oxford 1935, pp 8-10). Consequently, some of the concepts and results stated in this thesis may be clarified by resorting to Complex Variable theory.

1.4. Definition of Conformal Geometry in the Conformal Plane.

Any mapping of the form

(1.1) $w = \frac{az + b}{cz + d}$ (a,b,c,d,z, complex, $ad - bc \neq 0$),

in the Argand plane, maps circles into circles and preserves angles. A mapping of this form is called a Möbius Transformation, and is a conformal representation.[1] Certain properties, then, of circles and arcs[2] will remain invariant under

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1. A conformal representation is an angle-preserving mapping. There are conformal representations in which circles are not necessarily transformed into circles, but we do not consider these.

2. For the definition of arc, see section 1.11.

these transformations. Conformal geometry is the study of the properties which remain invariant under such a conformal mapping.

1.5. Extension to Higher Dimensions.

The work of the previous sections may be generalized to three or more dimensions. Although we cannot make use of Complex Variable theory in these higher dimensions, we have another model which will be described presently, and which applies to any dimension.

Conformal 3-space may be represented on the surface of a hypersphere (or, as it is more explicitly termed, a 3-sphere) in projective 4-space; more generally, conformal n-space may be represented on the surface of an n-sphere in projective (n - 1)-space. Accordingly, conformal n-space has a single point at infinity, so that a p-flat (p = 1,2, . . . n - 1) (cf. Sommerville "An Introduction to the Geometry of N Dimensions", p 8) is thought of as a p-sphere through the point at infinity. Any transformation which takes place in conformal n-space, transforming p-spheres into p-spheres (p=1,2,...n-1), and leaving angles invariant is a conformal representation. <u>Conformal geometry</u> in n-dimentions, then, is the study of those properties which remain invariant under the transformations described.

Instead of making use of Complex Variable theory as a model when studying Conformal geometry in the conformal plane, we could make use of the following model:

The group of the projectivities (or one-to-one linear transformatfons of projective space) in 3-space which preserve a sphere is called the <u>orthogonal group</u> in three dimensions. Such linear transformations map planes into planes, and hence map the intersection of a plane and the sphere into another intersection of a plane and the sphere. Thus circles on the sphere are mapped into circles, and the orthogonal group in 3-space is equivalent to the conformal group of transformations in the conformal plane.

Similarly, the (n+1)-dimensional orthogonal group is equivalent to the conformal group of transformations in n dimensions (cf. Birkhoff and MacLane "A Survey of Modern Algebra", Macmillan 1953, chapter 9).

Remark. Conformal geometry has been set up axiomatically using established geometries as a model (cf. for example, A.J. Hoffman, "On the Foundations of Inversion Geometry", Trans. Am. Math. Soc., Vol 71, July-Dec. 1951). Presumably the axioms could be set up independently, using points and circles (spheres; (n-1)-spheres) as undefined elements, but it seems that this has never been done.

1.6. An Investigation of Angles.

1.6.1. We can get a clearer idea of what we mean by the term angle in the conformal plane by resorting to Complex Variable theory. It is not difficult to prove (cf. Carathéodory, "Theory of Functions of a Complex Variable", Chelsea 1954, pp 29-30) that under a Möbius transformation, the cross ratio of any four complex numbers, z_1, z_2, z_3, z_4, is invariant; i.e.

$$(1.2) \qquad \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_3)(w_2-w_4)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)} .$$

This suggests that the cross-ratio is closely related to the notion of angle. Let z_1, z_2, z_3, and z_4 be respectively the complex numbers λ (with finite coefficients), $0, \infty$, and 1. Then

$$(1.3) \qquad \text{amp} \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)} = \text{amp}\,\lambda,$$

i.e. the amplitude of the cross-ratio of these four complex numbers is equal to the angle θ between the line determined by the complex number λ, and the positive real axis (cf. Fig 1.2). Note that relation (1.3) is unaltered

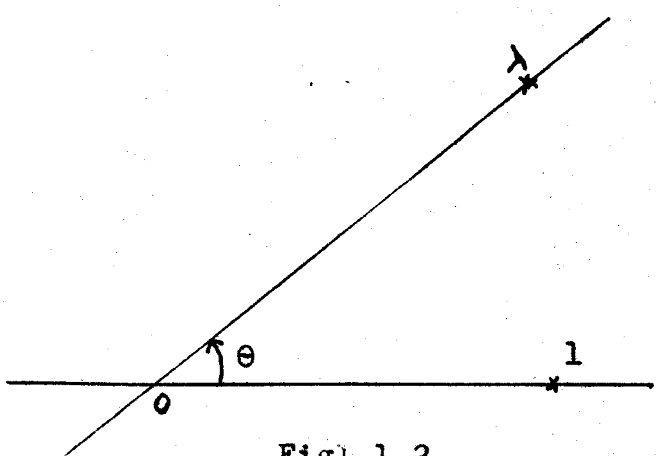

Fig. 1.2

if we interchange the values z_1, and z_4, and z_2 and z_3.

Suppose two circles, C_1 and C_2 intersect (cf. Fig 1.3). Let R and Q be their points of intersection. Let S be a point on C_1 and T a point on C_2 ($S,T \neq R$ and Q),

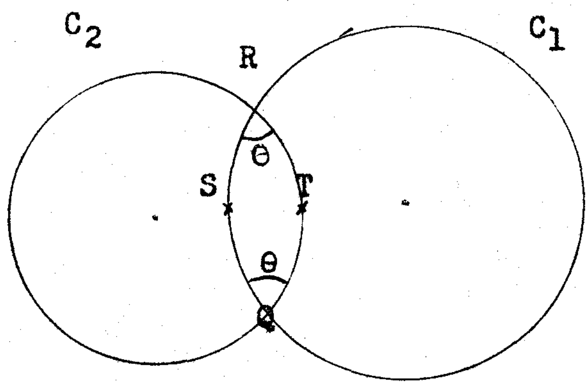

Fig 1.3

and let θ be the angle at which C_1 and C_2 meet.[1] By one or more Möbius transformations, we can let $Q = \infty$, $R = 0$, $S = 1$, $T = \lambda$. Then Fig 1.3 is transformed into Fig 1.2, i.e. circles and angles are preserved. Hence

$$(1.4) \qquad \theta = \text{amp} \; \frac{(T-R)(Q-S)}{(T-Q)(R-S)} \; .$$

It is also true that

$$\theta = \text{amp} \; \frac{(S-Q)(R-T)}{(S-R)(Q-T)} \; .$$

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1. Throughout this thesis, the symbol $\neq$ will mean "different from".

We therefore have an alternative definition for an angle in the conformal plane, viz.,

"If two circles, C_1 and C_2 intersect in R and Q, and if S and T lie on C_1 and C_2 respectively, $S,T \neq R$ and Q, then the angle [1] between C_1 and C_2 is the amplitude of the cross-ratio

$$\frac{(T-R)(Q-S)}{(T-Q)(R-S)} ,$$

where Q,T,R, and S are complex numbers".

If the two supplementary angles between C_1 and C_2 (see footnote) are equal, we say that C1 and C_2 <u>meet at right angles</u>, or C_1 is <u>orthogonal</u> to C_2 (C_2 is orthogonal to C1).

If the circles C_1 and C_2 in the conformal plane have only one point in common, say the point R, then the angle between C_1 and C_2 is zero, and we say that C_1 and C_2 <u>touch</u> at R.

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1. Of course there are two angles between C_1 and C_2, one being the supplement of the other; the relative order of the points Q,R,T, and S governs the choice of the angle.

1.6.2. Proceeding to three dimensions, we let a circle C intersect a sphere S in two points, R and Q. By one of the transformations described in section 1.5, we let Q be carried into the point at infinity. Then C becomes a straight line and S becomes a plane (cf. Fig 1.4). Let ℓ be any line lying

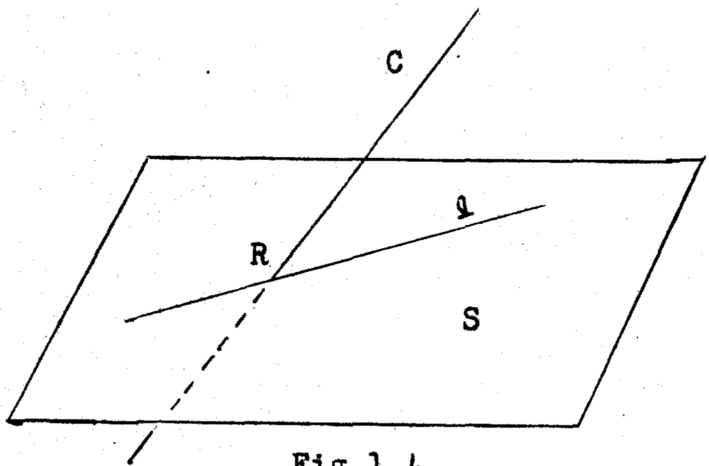

Fig 1.4.

in S and passing through R. Then C and ℓ determine a plane. The angle between C and ℓ in 3-space is then the angle between C and ℓ in the plane determined by these two lines. If the angle between C and ℓ is the same for all ℓ, we say that C and S meet at right angles, or C is orthogonal to S.

If C meets S at one point only, or if C lies on S, then the angle between C and S is zero and we say that C and S touch.

If two circles C_1 and C_2 in conformal 3-space intersect twice, we let one point be carried to the point at infinity and thus are led to a clear definition of the angle between C_1 and C_2. If C_1 and C_2 meet in one point only, and lie on a common sphere, we have on the sphere a model of the conformal plane. Thus we have reduced this case to a case in section 1.6.1, and we see that C_1 and C_2 <u>touch</u> at their common point. If, however, C_1 and C_2 do not lie on a common sphere, they do not meet at angle zero.

Suppose that two spheres, S_1 and S_2, meet in a proper circle C. Let Q be any point on C, and let S_3 be a sphere through Q such that C is orthogonal to S_3. Thus C meets S_3 in another point, R, say. Let Q be carried to infinity, so that S_1, S_2, and S_3 become planes, meeting in R on the line C. The angle between S_1 and S_2 is equal to the angle on S_3 between the intersection of S_1 and S_3 and the intersection of S_2 and S_3.

If two spheres, S_1 and S_2 meet in a single point, P, then the angle between S_1 and S_2 is zero, and we say that S_1 and S_2 touch at P.

1.6.3. We now consider angles in conformal n-space. As in previous cases, p-spheres can be reduced to p-flats (p= 1, 2,...n-1) by the proper transformation. The angle between a p-sphere and a q-sphere is then the same as the angle between a p-flat and a q-flat. A discussion of this can be found in "An Introduction to the Geometry of n-Dimensions" by D.M.Y. Sommerville.

1.6.4. It should be evident by now that by a proper transformation, of the form described in sections 1.4 and 1.5, any proposition regarding angles can be greatly simplified. This method of attack will be used in some proofs.

1.7. The Closure Property of Conformal n-space.

As we have already noted, conformal n-space may be represented on the surface of an n-sphere in projective (n+1)-space(n=2,3,...). Hence every infinite sequence of points in conformal n-space lies in an interval, and thus possesses at least one <u>accumulation point</u> (cf. Hardy, "Pure Mathematics", Cambridge 1945, pp. 30-32).

Suppose now, for instance, that we have an infinite sequence of circles C in the conformal plane. Then there exists a sub-sequence,[1] $C' \subset C$, of circles which contains an infinite sequence of points possessing an accumulation point. Again, there is a sub-sequence $C'' \subset C'$ of circles which contains a different sequence of points possessing an accumulation point. Finally there is a sub-sequence $C''' \subset C''$ of circles which contains yet another sequence of points possessing an accumulation point. Thus we have a sequence C''' of circles which possesses a limiting circle, the circle determined by the three accumulation points. This important result may be stated as follows:

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1. The symbol $\subset$ means "contained in" ("is contained in") or "belonging to" ("belongs to"). The symbol $\in$ is reserved to mean "is a (single) element of". The symbol $\supset$ means "containing" ("contains"); i.e. if $A \subset B$, then $B \supset A$.

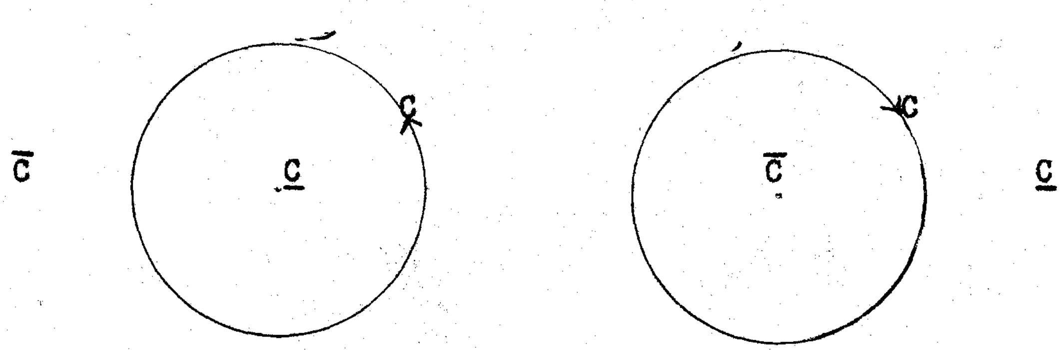
Theorem 1.1. *Every infinite sequence of circles in the conformal plane possesses at least one limit circle.* We call such a limit circle an *accumulation circle*.

Obviously, the above result may be generalized. Thus we have the more general theorem, namely:

Theorem 1.2. *Every infinite sequence of p-spheres (p=1,2,...,n-1) in conformal n-space possesses at least one limit p-sphere* (called an *accumulation p-sphere*).

1.8. Regions in Conformal n-space.

Any proper circle, C(i.e. C is not a point), divides the conformal plane into two open regions, the *interior* $\underline{C}$ of C, and the *exterior* $\bar{C}$ of C. If we orient the circle C, then the interior of C is the region lying to the left of the oriented circle (cf. Fig 1.5).



(a) (b)

Fig 1.5

In general, any proper (n-1)-sphere, S, divides conformal n-space into two open regions, the interior $\underline{S}$ of S, and the exterior $\overline{S}$ of S. If P is a point not lying on S, then the interior of S may be defined as the set of all points not lying on S and not separated from P by S.

1.9. Convergence.

1.9.1. A sequence of points, P_1, P_2,... in the conformal plane is said to be convergent to a point P, if, given any circle C with $P \subset \underline{C}$, there exists a number $n = n(C)$ such that $P_\nu \subset \underline{C}$ for all $\nu > n$.

In the same way, convergence of circles to a point is defined. Such a point is called a point-circle.

A sequence of circles, C_1, C_2,... in the conformal plane is said to be convergent to the proper circle C if, given any two points P and Q such that $P \subset \underline{C}$ and $Q \subset \overline{C}$, there exists a number $n=n(P,Q)$ such that $P \subset \underline{C}_\nu$ and $Q \subset \overline{C}_\nu$ for all $\nu > n$.

1.9.2. A sequence of points $P_1, P_2, \ldots$ in conformal 3-space is said to be convergent to a point P if, given any sphere S with $P \subset \underline{S}$, there exists a number $n=n(S)$ such that $P_\nu \subset \underline{S}$ for all $\nu > n$.

In the same way, convergence of circles and spheres to point-circles and point-spheres is defined.

A sequence of circles, C_1, C_2,... in conformal 3-space is said to be convergent to the circle C if, given any circle C', which links[1] with C, there exists a number $n=n(C')$ such that C_ν links with C' for all $\nu > n$.

A sequence of spheres, S_1, S_2,... in conformal 3-space is said to be convergent to the sphere S if, given any two points P and Q, where $P \subset \underline{S}$ and $Q \subset \overline{S}$, there exists a number $n=n(P,Q)$ such that $P \subset \underline{S}_\nu$ and $Q \subset \overline{S}_\nu$ for all $\nu > n$.

1.9.3. A sequence of points P_1, P_2,... in conformal n-space is said to be convergent to a point P, if, given any (n-1)-sphere, S , with $P \subset \underline{S}$, there exists a number $N=N(S)$ such that

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1. C' is said to link with C, if any sphere $S \supset C$ cuts any sphere $S' \supset C'$, while C and C' have no common point.

$P_\nu \subset \underline{S}$ for all $\nu > N$.

In the same way, convergence of m-spheres to <u>point-m-spheres</u> is defined $(m = 1,2,\ldots,n-1)$.

A sequence of m-spheres, $S_1^{(m)} S_2^{(m)},\ldots$ is said to be <u>convergent</u> to an m-sphere $S^{(m)}$, if to every (n-m-1)-sphere, $S^{(n-m-1)}$ which links[1] with $S^{(m)}$, there exists a positive integer $n = n(S^{(n-m-1)})$ such that $S_\nu^{(m)}$ links with $S^{(n-m-1)}$ for all $\nu > n (m = 1,2,\ldots,n-2)$.

Finally, a sequence of (n-1)-spheres, $S_1, S_2, \ldots$ in conformal n-space is said to be <u>convergent</u> to an (n-1)-sphere S, if, given any two points, P and Q, where $P \subset \underline{S}$ and $Q \subset \overline{S}$, there exists a number $n = n(P,Q)$ such that $P \subset \underline{S}_\nu$ and $Q \subset \overline{S}_\nu$ for all $\nu > n$.

<u>1.10. Pencils of circles, spheres, and m-spheres.</u>

In the following, section 1.10.1 deals with the conformal plane, section 1.10.2 with conformal 3-space, and section 1.10.3 with conformal n-space.

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1. cf. Seifert & Threlfall "Lehrbuch der Topologie" §77

1.10.1. The set of all circles that intersect two given circles at right angles is a linear pencil, π of circles. A pencil π of the first kind possesses two fundamental points, P and Q (cf. Fig. 1.6). A pencil π of the second kind

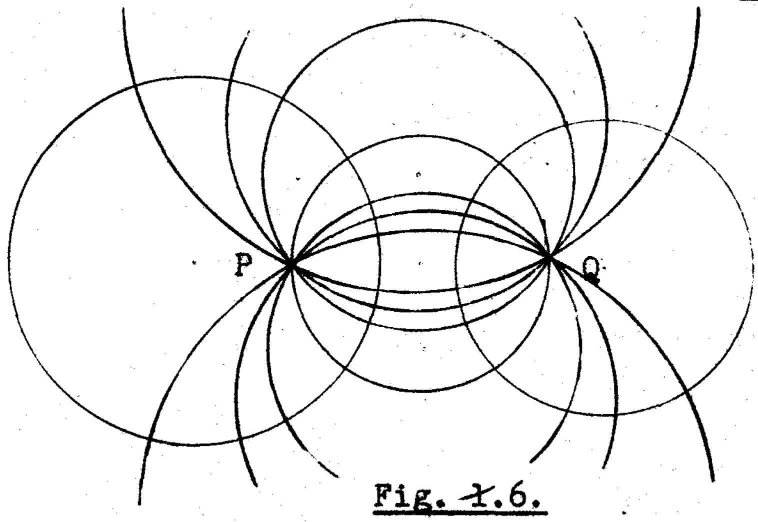

Fig. 1.6.

possesses one fundamental point, P (cf. Fig. 1.7) and is identical with the set of all circles that touch any circle

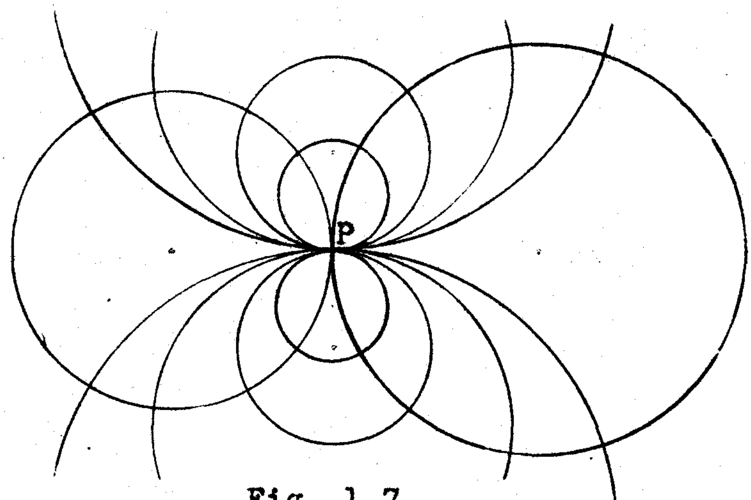

Fig. 1.7.

of π at P. A pencil π of the third kind possesses no fundamental point; any two circles of Π are disjoint (cf.Fig.1.8).

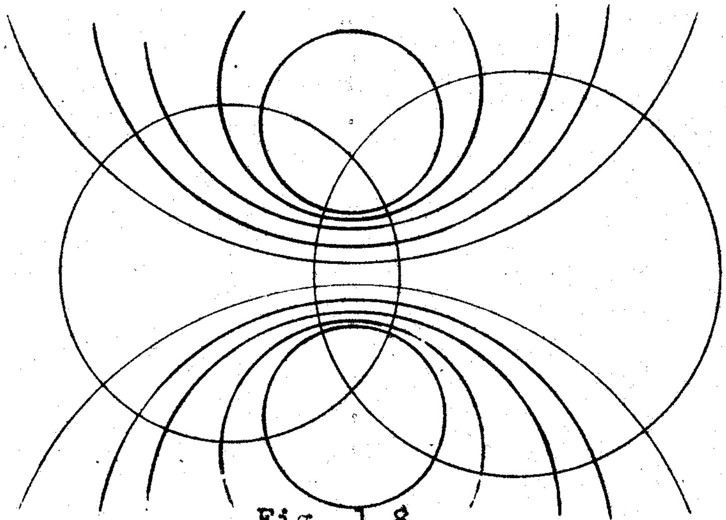

Fig. 1.8.

To any pencil Π, and to every point Q, which is not a fundamental point of Π, there exists one and only one circle, $C(Q;\Pi)$ of Π through Q. In the case of a pencil Π of the second kind, the fundamental point is regarded as a point-circle belonging to the pencil Π.

1.10.2. The sphere through a proper circle C, and[1] a point P, $P \not\subset C$ will be denoted by S(P;C). We shall make use of pencils Π, of spheres and circles, determined by certain incidence and tangency conditions. A circle (point) which is common to

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1. The symbol $\not\subset$ means "not lying on" ("does not lie on") or "not contained in" ("is not contained in"). The symbol $\notin$ means "is not an element of".

all the spheres (circles) of a pencil is called a <u>fundamental circle</u> (<u>fundamental point</u>) of the pencil. In the pencil Π of spheres through a fundamental circle C, there exists one and only one sphere $S(P;\Pi)$ of Π through any point P which does not lie on C. Similarly, in the pencil Π of spheres (circles) which touch a given sphere (circle) at a given point Q, there is one and only one sphere $S(P;\Pi)$ (circle $C(P;\Pi)$) of Π which passes through any point $P \neq Q$. The fundamental point Q is regarded as a point-sphere (point-circle) belonging to Π.

<u>1.10.3.</u> An m-sphere through an (m - 1)-sphere $S^{(m-1)}$, and a point $P \not\subset S^{(m-1)}$ will be denoted by $S^{(m)}[P;S^{(m-1)}]$. The m-sphere through m + 2 points, $P_o, P_1, \ldots P_{m+1}$, not all lying on the same m-1)-sphere, will occasionally be denoted by $S(P_o, P_1, \ldots, P_{m+1})$. Such a set of points is said to be <u>independent</u>.[1] We shall make use of pencils $\Pi^{(m)}$ of m-spheres

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1. cf, Sommerville, "An Introduction to the Geometry of N Dimensions", page 8. In two (cont'd on p 22 (bottom))

determined by certain incidence and tangency conditions. An $(m-1)$-sphere which is common to all the m-spheres of a pencil $\pi^{(m)}$ is called a fundamental $(m-1)$-sphere of $\pi^{(m)}$. In the pencil $\pi^{(m)}$ through a fundamental $(m-1)$-sphere $S^{(m-1)}$, there is one and only one m-sphere $S(P;\pi^{(m)})$ of $\pi^{(m)}$ through each point P, which does not lie on $S^{(m-1)}$. Similarly, in the pencil $\pi^{(m)}$ of all the m-spheres which touch a given m-sphere at a given point Q, there is one and only one m-sphere $S(P;\pi^{(m)})$ through each point $P \neq Q$. The fundamental point Q is regarded as a point m-sphere belonging to $\pi^{(m)}$.

1.11. Arcs.

An arc A in conformal n-space $(n = 2,3,\ldots)$ is the continuous image of a real interval. The images of distinct points of this parameter interval are considered to be dif-

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dimensions, we speak of the circle $C(P,Q,R)$ through the three independent (i.e. distinct) points P,Q, and R, and in three dimensions we speak of the circle $C(P,Q,R)$ and the sphere $S(P,Q,R,T)$, where P,Q,R, and T are independent points (i.e. do not lie on the same circle).

ferent points of A even though they may coincide in the space. If a sequence of points of the parameter interval converges to a point p, we define the corresponding sequence of image points to be convergent to the image of p. The same small letters, p,t,... will denote both the points of the parameter interval, and their image points on A. The end- (interior-) points of A are the images of the end-(interior-) points of the parameter interval. If p is an interior point of A, this neighbourhood is decomposed by p into two (open) one-sided neighbourhoods.

1.12. Support and Intersection.

Let p be an interior point of an arc A in the conformal plane. Then we call p a point of support (intersection) with respect to a circle C, if a sufficiently small neighbourhood of p on A is decomposed by p into two one-sided neighbourhoods which lie in the same region (in different regions) bounded by C. C is then called a supporting (intersecting) circle of A at p. Thus C supports A at p if $p \not\subset C$.

By definition, the point-circle p always supports A at p.

It is possible for a circle to have points different from p in common with every neighbourhood of p on A. In this case we say that C <u>neither supports nor intersects</u> A at p.

The above may be extended to three (n) dimensions simply by substituting the word sphere ((n-1)-sphere) for the word circle (and the letter S for the letter C).

CHAPTER II

DIFFERENTIABLE POINTS OF ARCS IN THE CONFORMAL PLANE

2.1. Introduction.

The goal of this chapter is a classification of the differentiable points of arcs in the conformal plane. The main tools are the intersection and support properties of families of circles through a differentiable point p of an arc A. This chapter is the ground-work for chapters 3 and 4.

2.2. Differentiability.

Let p be a fixed point of an arc A, and let t be a variable point of A. If P, Q, and p are distinct points, the unique circle through these points will be denoted by C(P,Q,p).

The arc A is said to be differentiable at p if the following two conditions are satisfied:

Condition I: If the parameter t is sufficiently close to, but different from, the parameter p, the circle C(P,t,p) is uniquely defined, and converges[1] as $t \to p$.

Thus the limit circle, called a <u>tangent</u> <u>circle</u>, and denoted by $C(P;\tau)$ is independent of the way t converges to p. The family of all such circles, together with the point-circle p, will be denoted by the symbol τ.

Condition II: If the parameter t is sufficiently close to, but different from the parameter p, the circle $C(t;\tau)$ is uniquely defined, and converges as $t \to p$.

This unique limit circle, called the <u>osculating</u> <u>circle</u> of A at p, will be denoted by C(p).

<u>2.3.</u> <u>Structure of the Families of Circles Through p</u>.

<u>Theorem 2.1.</u> <u>Suppose that A satisfies Condition I at p. Then t does not coincide with p if the parameter t is sufficiently close to, but different from, the parameter p</u>.

<u>Proof</u>: Let $P \neq p$. Then by Condition I, C(P,t,p) is

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1. The symbol $\to$ means "converges to".

uniquely defined when the parameter t is close to, but different from, the parameter p. Hence $t \neq p$.

This theorem indicates a restriction which Condition I imposes on an arc; viz., the arc satisfying Condition I at the point p, must have a neighbourhood of p which contains no point coincident with p.

Theorem 2.2. Suppose that the parameter t is sufficiently close to, but different from, the parameter p. If the circle $C(P,t,p)$ converges as $t \rightarrow p$ $(t \in A)$, for a single point $P \neq p$, then Condition I holds.

Remark: Theorem 2.2 shows that Condition I is stronger than necessary, and could be replaced by the condition laid down in the statement of this theorem.

Proof of Theorem 2.2: Let P, Q, R be three mutually distinct points. If the point $R' \neq R$ converges to R, then the angle between the circles $C(R',R,P)$ and $C(R',R,Q)$ converges to zero.[1] In particular, let $R = p$, $R' = t \in A$. Then[2]

$$\lim_{t \to p} \sphericalangle\left[C(P,t,p);C(Q,t,p)\right] = 0.$$

Hence any accumulation circle C', of the circles $C(Q,t,p)$ touches $C(P;\tau)$ at p. Since C' also passes through the point $Q \neq p$, it is uniquely determined. Hence $C' = \lim_{t \to p} C(Q,t,p) = C(Q;\tau)$.

<u>Theorem 2.3.</u> <u>The set $\tau = \tau(p)$ of all the tangent circles of A at p is a pencil of the second kind with fundamental point p.</u>

<u>Proof</u>: By Theorem 2.3, any two tangent circles, $C(P;\tau)$ and $C(Q;\tau)$ touch at p.

Suppose that a circle C touches a circle of τ at p, and let $P \subset C$, $P \neq p$. Then C and $C(P;\tau)$ also touch at p and

---

1. This statement becomes trivial if we let R be carried into the point at infinity by a transformation as in section 1.4. Note that the circles themselves need not converge.

2. If C and C' are two circles, then $\sphericalangle[C;C']$ means "the angle between C and C'"

have the point $P \neq p$ in common. Hence C and $C(P;\tau)$ are identical, i.e. $C \in \tau$.

<u>Corollary 1.</u> <u>If $C(P;\tau)$ and $C(Q;\tau)$ have another point in common, they are identical; thus there is one and only one circle of τ through each point $P \neq p$.</u>

While this is an immediate corollary of Theorem 2.3, it has a more basic proof which is worth noting, namely:

Suppose that $C(P;\tau)$ and $C(Q;\tau)$ have another point $R \neq p$ in common. Then before the limit is reached, $C(P,t,p)$ $(t \in A, t \neq p,\ t \to p)$ and $C(Q,t,p)$ must have a point R' close to R in common. Since these two circles now have three points, t,p and R' in common, they are identical. Hence in the limit, $C(P;\tau) = C(Q;\tau)$.

<u>Theorem 2.4.</u> <u>Suppose A satisfies Condition I at p. Let π be a pencil of the second kind with fundamental point p. If $t \to p (t \in A, t \neq p)$, and if $\pi \neq \tau$, then</u>

$$\lim_{t \to p} C(t;\pi) = p.$$

<u>Proof</u>: If this statement were false, there would exist

a circle, C, such that $p \subset \underline{C}$, and a sequence of points $t \to p$, $t \in A$ such that $C(t;\Pi) \not\subset \underline{C}$. Let C' and C" be the two circles of Π which touch C (cf. Fig 2.1). If we orient C and C' in such a way that C lies in the closure of[1] $\bar{C}' \cap \underline{C}''$, then $C(t;\Pi) \subset (\bar{C}' \cap \underline{C}'') \cup p$. Hence $t \subset \bar{C}' \cap \underline{C}''$.

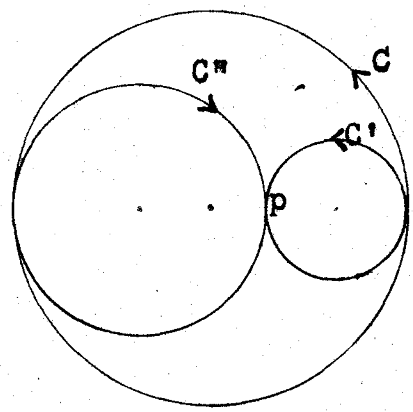


Fig. 2.1

Now let there be any sequence of points $Q \to p$, $Q \subset (\bar{C}' \cap \underline{C}'')$. Let C_Q be any accumulation circle of $C(P,Q,p)$ where $P \subset C'$, $P \neq p$. If $\sphericalangle\left[C_Q;C'\right] \neq 0$, then there is a small neighbourhood of the point p in which $\bar{C}' \cap \underline{C}''$ is void of any part of the circle C_Q.

1. If X and Y are two classes of elements, then $X \cap Y$ denotes the set of all elements in both X and Y; $X \cup Y$ denotes the elements in either X or Y or both X and Y.

Therefore, if $C(P,Q,p)$ is very close to C_Q, it does not pass through $\overline{C}' \cap \underline{C}''$ in the immediate neighbourhood of the point p, and hence $Q \notin \overline{C}' \cap \underline{C}''$. This contradiction leads us to the conclusion that $\measuredangle [C_Q, C'] = 0$. Since C_Q and C' have the point $P \neq p$ in common, we see that $C_Q = C(P;\pi)$, and is therefore unique. In particular, since $t \subset \overline{C}' \cap \underline{C}''$, $C(P,t;p) \rightarrow C(P;\pi)$, i.e. $C(P;\tau) = C(P;\pi)$. This again is a contradiction. Thus if $\pi \neq \tau$, $C(t;\pi) \rightarrow p$.

<u>Theorem 2.5.</u> <u>Suppose A satisfies Condition I and Condition II at p. Then $C(p) \in \tau$.</u>

<u>Proof:</u> If $C(p) = p$, it belongs to τ by definition. Suppose $C(p) \neq p$. Then $C(p)$, being the limit of a sequence of circles, each of which touches a given circle of τ, must itself touch this circle of τ at p. Hence $C(p) \in \tau$.

<u>Corollary 1.</u> <u>If $P \subset C(p)$, $P \neq p$, then $C(p) = C(P;\tau)$.</u>

2.4. The Independence of the Differentiability Conditions

Condition I and Condition II are independent, as is shown by the following example. Introducing a rectangular

Cartesian coordinate system, we let the arc A be defined by the equations

$$x = t,\ y = \begin{cases} (1-\sqrt{1-t^2}\,)\sin t^{-1}, & 0 < |t| \leq 1 \\ 0 & , t = 0 \end{cases}$$

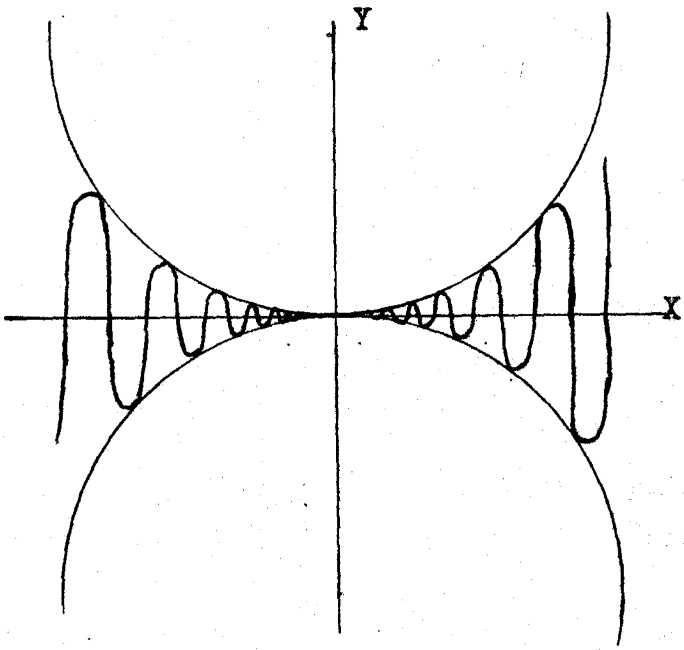


Fig 2.2

The curve lies between the two circles $x^2+(y\pm 1)^2 = 1$ as shown in Fig 2.2. We examine the point $t=0$ for differentiability.

Since we are only interested in values of t close to zero,[1] we may expand $\sqrt{1-t^2}$ by the Binomial Theorem, i.e. $\sqrt{1-t^2} = (1-t^2)^{\frac{1}{2}} = \left[1-\frac{1}{2}t^2\ o(t^3)\right]$. Let $P(x_1,y_1) \neq (0,0)$.

---

1. $\lim_{t \to 0} \frac{o(t^n)}{t^n} = 0$

Then the equation of the circle C(P,t,p) may be written

$$\begin{vmatrix} x^2+y^2 & x & y \\ x_1^2+y_1^2 & x_1 & y_1 \\ t^2+o(t^3) & t & o(t) \end{vmatrix} = 0$$

We remove the common factor t and let $t \to 0$, obtaining

$$\begin{vmatrix} x^2+y^2 & y \\ x_1^2+y_1^2 & y_1 \end{vmatrix} = 0$$

Thus Condition I holds.

Condition II, however, does not hold. The equation of $C(t;\tau)$ when t is close to 0 is

$$\begin{vmatrix} x^2+y^2 & y \\ t^2+o(t^3) & \tfrac{1}{2}t^2\sin t^{-1}+o(t^3) \end{vmatrix} = 0$$

Removing the common factor t^2 and letting $t \to 0$, we obtain

$$\lim_{t \to 0} (x^2+y^2)\sin t^{-1} - 2y = 0.$$

This circle does not converge. The fact that Condition II does not hold can also be seen from the fact that both of the circles $x^2 + (y\pm 1)^2 = 1$ have points in common with any neighbourhood of $t=0$. Thus the sequence $C(t;\tau)$ has two accumulation circles, namely $x^2 + (y \pm 1)^2 = 1$.

2.5. Intersection and Support Properties of the Family of Non-Tangent Circles and the Family of Non-Osculating Tangent Circles.

Let p be a differentiable interior point of the arc A.

Theorem 2.6. Every circle $C \neq C(p)$ either supports or intersects A at p.

Proof: If C neither supports nor intersects A at p, then $p \subset C$, and there exists a sequence of points $t \to p$, such that $t \in A \cap C$ and $t \neq p$. Let $P \subset C, P \neq p$. Then $C = C(P,t,p)$ for each t in the sequence, and Condition I implies that $C = C(P; \mathcal{T})$.

Now $C \in \mathcal{T}$ and still contains the above sequence of points, t. Thus $C = C(t;\mathcal{T})$ for each t in the sequence, and Condition II implies that $C = C(p)$.

Theorem 2.7. Non-tangent circles through p all intersect or all support A at p.

Proof: Let C' and C" be two non-tangent circles through p. Suppose that C' and C" intersect each other in two points (cf. Fig 2.3), and let their other point of intersection be P. Suppose further that C' supports,

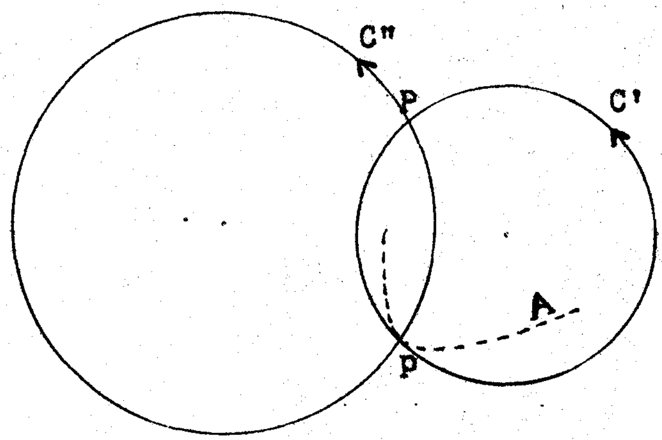


Fig 2.3

while C" intersects A at p. No generality is lost when C' is oriented so that $A \subset \underline{C}'$. Thus the region $A \cap \underline{C}'$ is not void. Let $t \in A \cap \underline{C}' \cap \underline{C}''$. Then

$$C(P,t,p) \subset (\underline{C}' \cap \underline{C}'') \cup (\bar{C}' \cap \bar{C}'') \cup P \cup p.$$

If we allow t to approach p, we obtain in the limit,

(2.1) $C(P;\tau) \subset (\underline{C}' \cap \underline{C}'') \cup (\bar{C}' \cap \bar{C}'') \cup C' \cup C''$.

Considering now a sequence of points, $t' \to p$, where $t' \in A \cap \underline{C}' \cap C''$, we obtain symmetrically the relation

(2.2) $C(P;\tau) \subset (\underline{C}' \cap \bar{C}'') \cup (\bar{C}' \cap \underline{C}'') \cup C' \cup C''$.

Comparing relations (2.1) and (2.2), we are led to one of the contradictions, $C(P;\tau) = C'$ or $C(P;\tau) = C''$.

If $C' \cap C'' = p$, we choose a third non-tangent circle, C''', which intersects C' in two points. Then C''' also intersects C'' in two points. Applying the above to C' and C''', and again to C''' and C'', we find that C', C'', and C''' either all support or all intersect A at p.

<u>Theorem 2.8. If $C(p) \neq p$, every non-osculating tangent circle supports A at p.</u>

<u>Proof:</u> Let C be a non-osculating tangent circle of A at p, and suppose that C intersects A at p, Then $A \cap \underline{C}$ and $A \cap \overline{C}$ are not void. If $t \in A \cap \underline{C}$, then by Theorem 2.3, $C(t;\tau) \subset \underline{C} \cup p$. Hence, if $t \to p$,

(2.3) $$C(p) \subset \underline{C} \cup C.$$

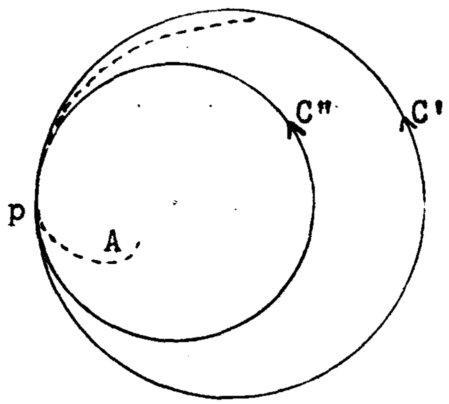
Letting $t' \to p$, $t' \in A \cap \overline{C}$, we obtain symmetrically,

(2.4) $$C(p) \subset \overline{C} \cup C.$$

A comparison of relations (2.3) and (2.4) leads to the conclusion $C(p) = C$, which is false. Therefore C supports A at p.

<u>Theorem 2.9.</u> <u>If $C(p) = p$, the non-osculating tangent circles at p all support A at p, or they all intersect A at p</u>.

<u>Proof:</u> Let C' and C'' be two non osculating tangent circles at p. For the sake of argument, we shall assume that C' supports A at p while C'' intersects A at p. We orient C' and C'' so that $C'' \subset \underline{C}' \cup p$ and $C' \subset C'' \cup p$ (cf. Fig 2.4).



<u>Fig 2.4</u>

Then $A \cap \underline{C}'$ is not void. Let $t \in A \cap \underline{C}' \cap \bar{C}''$, so that $C(t;\tau) \subset (\underline{C}' \cap \bar{C}'') \cup p$. As $t \to p$, the circle $C(t,\tau)$ will lie in the latter region bounded by the two proper circles C' and C''. Consequently, $C(t;\tau)$ cannot converge to p.

2.6. A Classification of the Differentiable Points

The preceding section yields a classification of the differentiable points of plane curves (cf. Table 2.1). The first four and last four examples refer to the curves

$$x = t^n \qquad y = t^{n+m},$$

while the middle two examples refer to the curves

$$x = t^n, \qquad y = \begin{cases} t^{n+m} \sin t^{-1}, & 0 < |t| \leq 1 \\ 0 & , t = 0 \end{cases} .$$

In each case we examine the differentiability properties of the point $t = 0$. By the method used in section 2.4, we find that the point $t = 0$ is differentiable in each case. This method also gives us the pencil $\mathcal{T}$, and tells us whether or not $C(p)$ is a point-circle. Support and intersection properties can be determined in many cases simply by a consideration of symmetry.

We introduce the characteristic $(a_o, a_1, a_2; i)$, with the following properties:

$$i = 1 \text{ or } 2.$$
$$a_o = 1 \text{ or } 2.$$
$$a_1 = 1 \text{ or } 2.$$
$$a_2 = 1, 2, \text{ or } \infty.$$

$i = 1$ if $C(p) \neq p$; $i = 2$ if $C(p) = p$.

a_o is even or odd, according as the non-tangent circles support or intersect.

$a_o + a_1$ is even or odd, according as the non-osculating tangent circles support or intersect.

$a_o + a_1 + a_2$ is even if C(p) supports and odd if C(p) intersects, while $a_2 = \infty$ if C(p) neither supports nor intersects.

Theorem 2.8 imposes a restriction on the characteristic, namely; if $i = 1$, $a_o + a_1$ is even. The convention that the point-circle p always supports yields a further restriction, that is; if $i = 2$, then $a_o + a_1 + a_2$ is even.

With the above restrictions in mind, we see that when $i = 1$ we have two choices for a_o, one choice for a_1, and three choices for a_2, a total of six different choices for the characteristic. If $i = 2$, we have two choices for a_o, two for a_1, and one choice for a_2, four choices in all. Thus we have ten different types of differentiable points.

All congruences in Table 2.1 are mod 2.

It is interesting to note that all the tangent circles (including C(p)) support if and only if $a_2 = 2$.

Curves containing the various types of differentiable points are illustrated in Figs. 2.5 to 2.10 inclusive. The curves are identified by the characteristic of the point (0,0) through which they pass. The relationship of each arc to a tangent circle is depicted by superimposing a non-osculating tangent circle upon each diagram.

i	Characteristic	Tangent circles $\neq C(p)$	$C(p)$		Restriction on characteristic	Examples			
								Tangent circles	$C(p)$
1	$(1,1,1;1)$	support	$\neq p$	intersects	$a_0+a_1 \equiv 0$	$0 < n < m$	$n \equiv 1$, $m \equiv 0$	touch x-axis	x-axis
	$(1,1,2;1)$			supports			$n \equiv m \equiv 1$		
	$(2,2,1;1)$			intersects			$n \equiv 0$, $m \equiv 1$		
	$(2,2,2;1)$			supports			$n \equiv m \equiv 0$		
	$(1,1,\infty;1)$			neither supports nor intersects			$n \equiv 1$		
	$(2,2,\infty;1)$						$n \equiv 0$		
2	$(1,1,2;2)$	support	$=p$	supports	$a_0+a_1+a_2 \equiv 0$	$n > m > 0$	$n \equiv m \equiv 1$		$x = 0$, $y = 0$
	$(1,2,1;2)$	intersect					$n \equiv 1$, $m \equiv 0$		
	$(2,1,1;2)$	intersect					$n \equiv 0$, $m \equiv 1$		
	$(2,2,2;2)$	support					$n \equiv m \equiv 0$		

Table 2.1

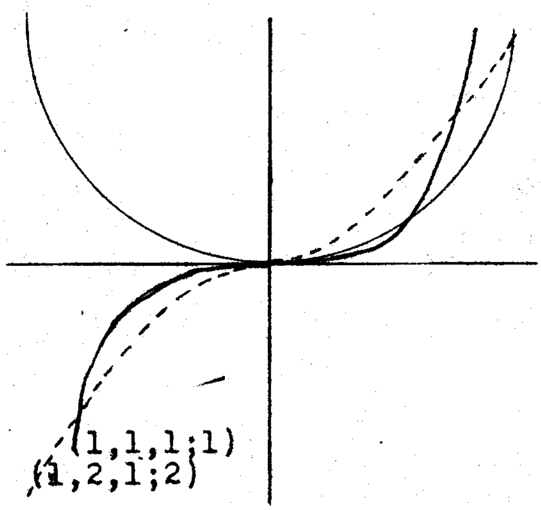


Fig. 2.5

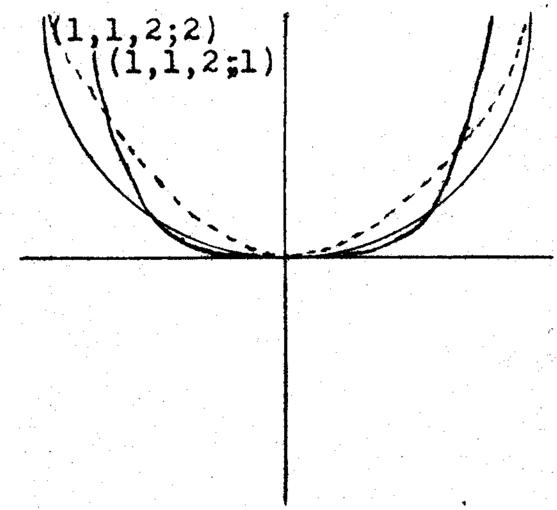


Fig. 2.6

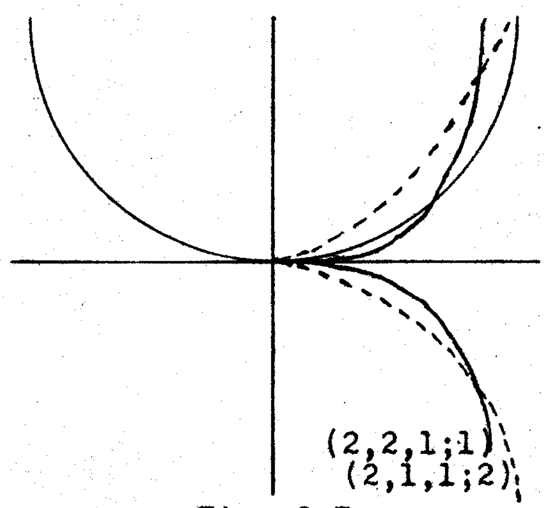


Fig. 2.7

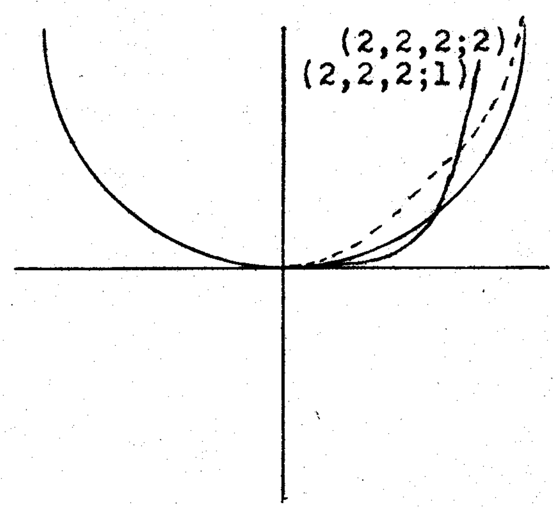


Fig. 2.8

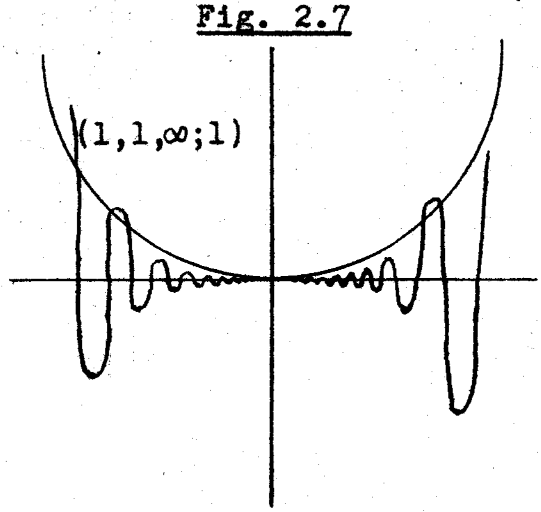


Fig. 2.9

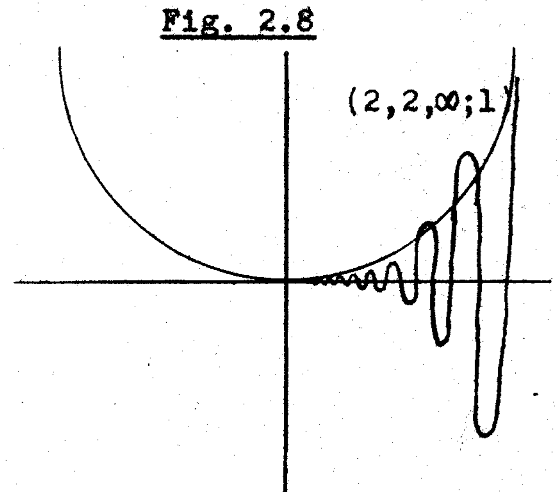


Fig. 2.10

CHAPTER III

CHARACTERISTIC AND ORDER OF DIFFERENTIABLE POINTS IN THE CONFORMAL PLANE

3.1. Introduction.

In this chapter, various theorems dealing with the cyclic orders of points and arcs will be discussed. The close connection between the characteristic of a differentiable point and the order of that point will be brought out. It will then become evident why the particular form, $(a_o, a_1, a_2; i)$, for the characteristic was chosen.

3.2 Arcs of Finite and Bounded Cyclic Order

An arc A is said to be of finite cyclic order if it has only a finite number of points in common with any circle. If some circle meets A n times and no circle meets A more than n times, where n is some specific integer, then A is said to be of bounded cyclic order,[1] and n is called the (cyclic) order of A. If p is any point on A, the order of p is the minimum of the orders of all the neighbourhoods of p on A.

Lemma 3.1. Let A be an arc of finite order, and let a circle C intersect A at a point p. Then any circle C', sufficiently close to C, also intersects A, and does so in an odd number of points close to p.

Proof: Since C intersects A at p, the end-points of a sufficiently small neighbourhood M, of p, lie in opposite regions with respect to C. Hence they lie on opposite sides

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1. It should be noted that there is a difference between an arc of bounded cyclic order and one of only finite cyclic order. It is possible, in the case of an arc of finite cyclic order, to find for each circle through a finite number of points on the arc, another circle through a still greater, but finite, number of points. An arc with such a property would not be of bounded order.

of C'. Since C' meets M a finite number of times, it must intersect M an odd number of times.

3.3. Characteristic and Order.

The following theorem illustrates in part the reason for choosing the characteristic in the form given in Chapter II. Theorem 3.5 will sharpen this theorem and complete the investigation of characteristic and order.

Theorem 3.1. Let p be a differentiable interior point of an arc A. Suppose that p has the characteristic $(a_0, a_1, a_2; i)$. Then the order of p is not less than $a_0 + a_1 + a_2$.

This theorem is trivially true if $a_2 = \infty$ (cf. § 2.6), for then every neighbourhood of p on A has an infinite number of points in common with C(p). For this reason we confine our ensuing proof to the case $a_2 < \infty$. The proof follows after the discussion in section 3.3.1.

3.3.1. Let $\Pi_2 = \tau$ be the pencil of tangent circles through p, where $C(p; \Pi_2) = C(p)$. Let Π_1 be a pencil of the first kind with p as one of its fundamental points, and let

$C(p;\Pi_1)$, which is a member of $\mathcal{T}$, be different from $C(p)$. Finally, let Π_o be a pencil of the first kind where $C(p;\Pi_o) \notin \mathcal{T}$. Then p is not one of the fundamental points of Π_o.

Lemma 3.2. The pencil Π_j $(j = 0,1,2)$ contains circles arbitrarily close to, but different from, $C_{j+1} = C(p;\Pi_j)$, which meet a neighbourhood M of p in not less than a_j points outside p. If the order of p is finite, and if M is small enough, C can be chosen so that the number of intersections of M with C exceeds a_j by a non-negative even integer.

Proof: Let $D_j \in \Pi_j$, $D_j \neq C_{j+1}$. If $j = 2$ and $C(p) = p$, we make the convention that $\underline{C}_3$ does not exist, and that $\bar{C}_3$ is the whole plane with the exception of the point p itself. We now define the regions

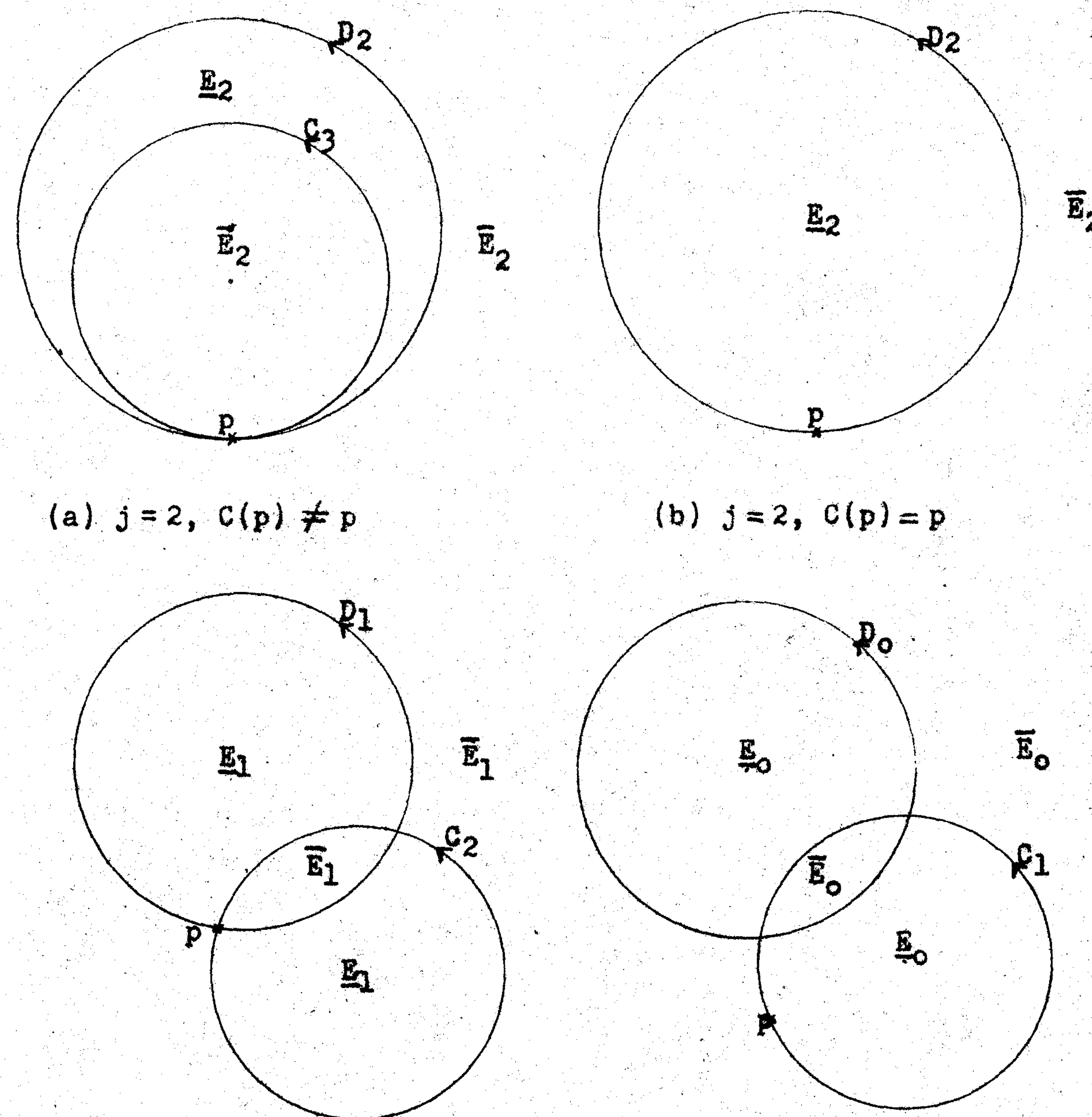
$$(3.1) \qquad \underline{E}_j = (\underline{C}_{j+1} \cap \bar{D}_j) \cup (\bar{C}_{j+1} \cap \underline{D}_j),$$

and

$$(3.2) \qquad \bar{E}_j = (\underline{C}_{j+1} \cap \underline{D}_j) \cup (\bar{C}_{j+1} \cap \bar{D}_j)$$

(cf. Fig. 3.1).

Let $\underline{\Pi}_j$ ($\overline{\Pi}_j$) denote the set of those circles of Π_j that pass through $\underline{E}_j$ ($\overline{E}_j$). Then every circle of Π_j, with



(a) $j = 2$, $C(p) \neq p$ (b) $j = 2$, $C(p) = p$

(c) $j = 1$ (d) $j = 0$

Fig. 3.1

the exception of C_{j+1} and D_j, belongs either to $\underline{\Pi}_j$ or to $\overline{\Pi}_j$. If we intersect Π_j with an ortogonal circle, we establish a 1-1 correspondence between the circles of Π_j and the points of the orthogonal circle, and thus we can speak of a "betweenness" relation in $\underline{\Pi}_j$ ($\overline{\Pi}_j$).

The neighbourhood M of p is decomposed by p into two one-sided neighbourhoods N and N'. We can choose our M so small that neither of the neighbourhoods N and N' have points in common with C_{j+1} or with D_j. Thus N (N') lies entirely in one of the two regions $\underline{E}_j$ and $\overline{E}_j$. Let t and t' denote the points of N and N' respectively. Thus all the circles $C(t;\Pi_j)$ belong to $\underline{\Pi}_j$ or else to $\overline{\Pi}_j$. We lose no generality in supposing that $N \subset \overline{C}_{j+1} \cap \overline{D}_j \subset \overline{E}_j$. Then $C(t;\Pi_j) \in \overline{\Pi}_j$ for every t.

Let e be the end-point of N distinct from p. Then $C(e;\Pi_j)$ is the end-circle of a one-sided neighbourhood ν of C_{j+1} in Π_j. If t moves from e to p, then $C(t;\Pi_j)$ moves continuously in Π_j from $C(e;\Pi_j)$ to C_{j+1}; hence every circle of ν meets N.

Let C be some fixed circle belonging to ν. Then, if t is sufficiently close to p, $C(t;\Pi_j)$ is so close to C_{j+1} that C lies between $C(t;\Pi_j)$ and $C(e;\Pi_j)$. Thus the points t and e are separated by C, and since the sub-arc N is of finite order, C must intersect N at least once. By Lemma 3.1, C intersects A an odd number of times.

Similarly, if $t' \in N'$, the circles $C(t';\Pi_j)$ comprise a one-sided family of circles ν', bounded by C_{j+1} and $C(e;\Pi_j)$, where p and e are the end-points of N'. There is a circle $C' \in \nu'$ which intersects N' an odd number of times.

Now if $a_j = 1$, one of the circles C_{j+1} and D_j supports A at p, while the other one intersects A at p (cf.§2.6); hence $N' \in \underline{E}_j$. Thus C does not meet N', and meets N an odd number of times. On the other hand, if $a_j = 2$, both of the circles C_{j+1} and D_j intersect A at p, or they both support

A at p; hence $N' \in \bar{E}_j$. Thus if C is sufficiently close to C_{j+1}, it will meet both N and N', an odd number of times each. This completes the proof of Lemma 3.2.

3.3.2. We are now in a position to prove Theorem 3.1. We proceed by first approximating C(p) by another tangent circle, then the latter by a non-tangent circle through p, and finally that circle by one which does not contain p.

Let $M_2 \subset M$ be a neighbourhood of p on A. By Lemma 3.2, there exists a non-osculating tangent circle C_2 which is close to C(p) and intersects M_2 at least a_2 times outside p. Now let $M_1 \subset M_2$ be a neighbourhood of p which contains none of the points of intersection of C_2 with M_2 (except p, if it is a point of intersection). Again by Lemma 3.2, there exists a non-tangent circle C, which intersects M_1 in at least a_1 points outside p. Finally, let $M_0 \subset M_1$ be a neighbourhood of p which contains none of the points of intersection of C_1 with M_1 (except p, if it is a point of intersection).

Using Lemma 3.2 once more, we find that there exists a circle C_o, not passing through p, which intersects M_o in at least a_o points. Altogether, C_o meets M at least $a_o + a_1+a_2$ times.

As a consequence of the proof of Theorem 3.1, we have

Corollary 1. If the order of the differentiable point p is bounded, then there exists to every neighbourhood of p a circle arbitrarily close to C(p) which does not pass through p, and which intersects that neighbourhood in not less than $a_o + a_1 + a_2$ points.

3.4. Two Lemmas On Arcs of Finite Cyclic Order.

Lemma 3.3. Let A be an arc of finite cyclic order. If the parameter t_n tends to one of the end-points of the parameter interval, then the sequence of points t_n converges.

Proof: Let $\lim_{\nu\to\infty} t_{2\nu} = p$ and $\lim_{\nu\to\infty} t_{2\nu+1} = q$ be any two accumulation points of the sequence t_n. We may assume that t_{n+1} lies between t_n and t_{n+2} for all n. If $p \neq q$, let C be a circle separating these two points. Thus there is a number $N = N(C)$ such that $t_{2\nu}$ and $t_{2\nu+1}$ are separated for all $\nu > N$. But this implies that the arc A meets C an infinite number of times, which is not true. Hence $p = q$.

Lemma 3.4. Let p be an end-point of an arc A of finite cyclic order. Then the arc A is differentiable at p.

Proof: Suppose Condition I of section 2.2 is not satisfied. Let $t_{2\nu}$ and $t_{2\nu+1}$ be two sequences of points converging to p such that $C(P,t_{2\nu},p) \rightarrow C_o$ and $C(P,t_{2\nu+1},p) \rightarrow C_1 \neq C_o$ $(P \neq p)$. We may assume that t_{n+1} lies between t_n and t_{n+2} on A. Let C' and C" be two circles through P and p which separate C_o and C_1 (cf. Fig. 3.2). Then, for each n sufficiently large, C' and C" separate $C(P,t_n,p)$ and $C(P,t_{n+1},p)$. Hence at least one of the circles C' and C" meets the arc A an infinite number of times, contrary to our hypothesis. Thus Condition I holds.

Now let us suppose that Condition II of section 2.2 does not hold. Let $t_{2\nu}$ and $t_{2\nu+1}$ be two sequences of points

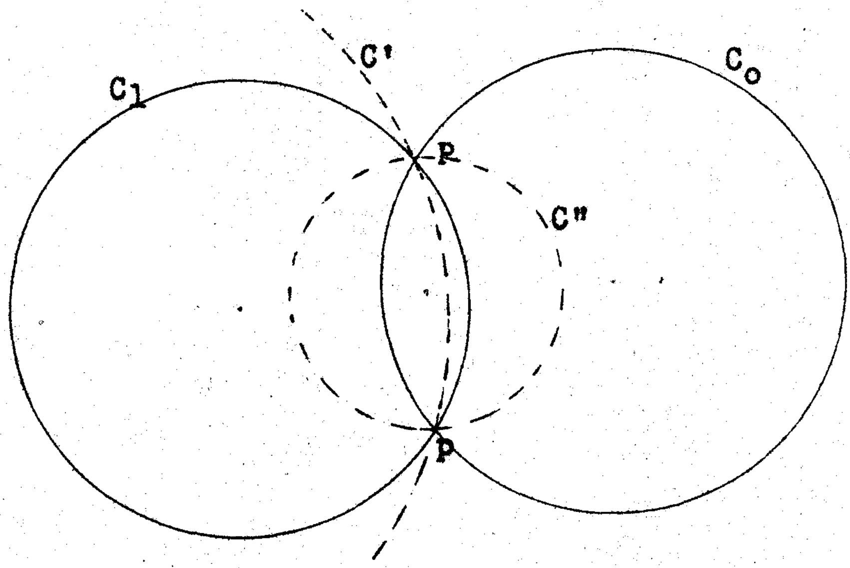


Fig. 3.2

converging to p such that $C(t_{2\nu};\tau) \rightarrow C_o$ and $C(t_{2\nu+1};\tau) \rightarrow C_1 \neq C_o$. As before, we assume that t_{n+1} lies between t_n and t_{n+2} on A. Both of the circles C_o and C_1, being the limit of sequences of tangent circles, are themselves tangent circles, and by Theorem 2.3, they touch at p.

Suppose first of all, that C_o and C_1 are both pro-

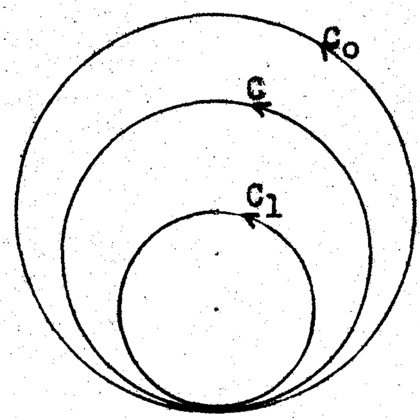


Fig. 3.3

per circles (cf. Fig. 3.3). We may orient C_o and C_1 in such a way that $C_1 \subset \underline{C}_o \cup p$ and $C_o \subset \overline{C}_1 \cup p$. Consider a circle $C \in \mathcal{T}$ $(C \subset (\underline{C}_o \cap \overline{C}_1) \cup p)$ oriented so that $C_1 \subset \underline{C} \cup p$ and $C_o \subset \overline{C} \cup p$. Then, for sufficiently large ν, $C(t_{2\nu+1};\mathcal{T}) \subset \underline{C} \cup p$, and $C(t_{2\nu},\mathcal{T}) \subset \overline{C} \cup p$. Here again the arc A crosses C an infinite number of times, which is impossible.

If now, C_1 for instance is the point-circle p, consider two circles of $\mathcal{T}$, C and C' $(C \subset \underline{C}_o \cup p,\ C' \subset \overline{C}_o \cup p)$, oriented in such a way that $C_o \subset (\overline{C} \cap \underline{C}') \cup p$ (cf. Fig. 3.4).

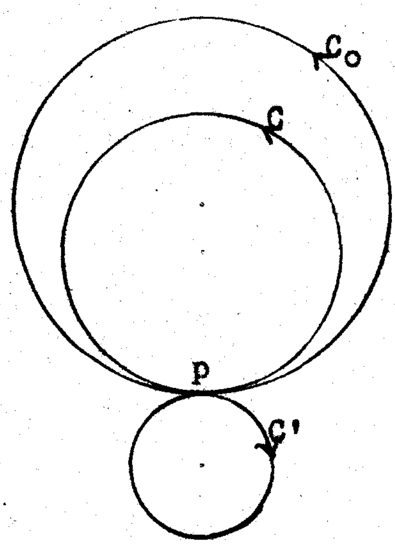


Fig. 3.4

Then for sufficiently large ν, $C(t_{2\nu};\mathcal{T}) \subset (\overline{C} \cap \underline{C}') \cup p$, while $C(t_{2\nu+1};\mathcal{T}) \subset \underline{C} \cup \overline{C}' \cup p$. Since these two regions are separated by C and C', one or both of these circles will meet A an infinite number of times as $\nu \to \infty$. Since this too is impossible by our hypothesis, Condition II holds, and the point p is differentiable.

3.5. Arcs of Order Three.

Since any three distinct points define a circle, the cyclic order of any arc is at least three. The remainder of this chapter is directly concerned with arcs of order three. We shall denote such an arc by the symbol A_3. The two lemmas of the previous section are true in particular of arcs of order three.

3.6. General Tangent Circles.

Let A_3 be an arc of order three with end-point p, and let $q \in A_3 \cup p$. We call a circle C a general tangent circle at the point q, if there exists a sequence of triplets of mutually distinct points, q_ν, q'_ν, Q_ν, such that q_ν and q'_ν converge on A_3 to q, and that

$$\lim C(Q_\nu, q_\nu, q'_\nu) = C.$$

If, in addition, Q_ν converges on A_3 to q, then we call C a <u>general osculating circle</u> at q. If we let the sequence q'_ν be the single point q, and let Q_ν be a single point $Q \neq q$, C is then an ordinary tangent circle of A at q. Hence an ordinary tangent circle is a general tangent circle. Let $Q_\nu \to Q \neq q$, and let q_ν and $q'_\nu \to q$. Choose any neighbourhood of q on A_3. Then if $C(q_\nu, q'_\nu, Q_\nu) \to C$, a non-osculating general tangent circle of A_3 at q, and if q_ν and q'_ν are sufficiently close to q, the end-points of the above neighbourhood will lie in the same region with respect to $C(q_\nu, q'_\nu, Q_\nu)$, and hence will lie in the same region with respect to C. Hence C supports A_3 at q. By similar reasoning, we find that a general osculating circle intersects A_3 at q.

3.7. An Important Property of Arcs of Order Three.

We now introduce <u>multiplicities</u>; that is, we count the end-point p of A_3 three times on C(p) and twice on any other tangent circle at p, while we count a point $q \in A_3 \cup p$

three times on a general osculating circle at q and twice on a non-osculating general tangent circle at q. We wish to prove the following theorem:

Theorem 3.2. No circle meets $A_3 \cup p$ more than three times; i.e., the inclusion of p and the introduction of multiplicities does not alter the order of A_3.

The proof of Theorem 3.2 results from the lemmas proved in the remainder of section 3.7.

3.7.1. Lemma 3.5. If a circle C meets A_3 in two points, then at least one of these points is an intersection.

Proof: Let C meet A_3 in q_1 and q_2, and let M_1 and M_2 be small neighbourhoods of q_1 and q_2 respectively. If q_i $(i = 1,2)$ is a point of support, then there is a circle close to C which meets M_i in two points. Now if M_1 and M_2 are both in $\underline{C}$, say, then there is a circle $C' \subset \underline{C}$ so close to C that it intersects M_1 and M_2 twice each. This is impossible since A_3 is of order three.

On the other hand, if $M_1 \subset \underline{C}$ and $M_2 \subset \overline{C}$, then since

A_3 has points on either side of C, it must intersect C in some point q_3. Let C' be a circle through one point of M_1 and two points of M_2, where C' is close to C (cf. Fig. 3.5).

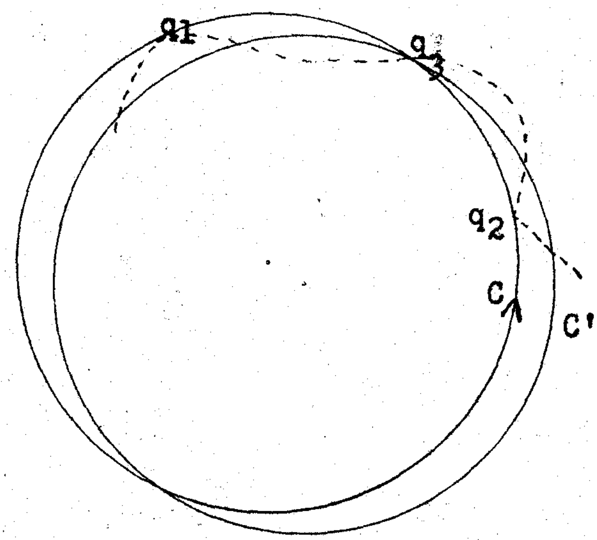


Fig. 3.5

Now the end-points of a small neighbourhood M_3 of q_3 (M_3 is so small that it has no points in common with M_1 or with M_2) lie in opposite regions with respect to C. By section 1.9.1, they also lie in opposite regions with respect to C' when C' is sufficiently close to C. Hence C' intersects M_3. Thus we have another contradiction, since C' can only meet A_3 three times at most.

Lemma 3.6. A circle C through three points of A_3

<u>does not support A_3 at any of these points</u>.

<u>Proof</u>: Lemma 3.5 implies that $A_3 \cap C$ has at most one point of support. If C supports A_3 at one point of contact, and intersects A_3 in two other points, then there is a circle close to C which meets A_3 in at least four points, which cannot be true.

<u>3.7.2.</u> Suppose that a circle C through p meets A_3 in three points, q_1, q_2, and q_3. By Lemma 3.6 they are all intersections. Choose disjoint neighbourhoods N of p and M of q_1 on A. If $t \rightarrow p$, $t \in N$, then $C(t,q_2,q_3) \rightarrow C$. By section 1.9.1, $C(t,q_2,q_3)$ separates the end-points of M if t is sufficiently close to p. Thus $C(t,q_2,q_3)$ meets A_3 again in the neighbourhood of q_1. This contradiction yields

<u>Lemma 3.7.C</u> <u>No circle meets $A_3 \cup p$ in four points</u>.

Now suppose that a circle through p meets A_3 in two points, q_1 and q_2, and suppose further that q_2 is a point of support. By Lemma 3.5, q_1 is a point of intersection. Let

M_1 and M_2 be small neighbourhoods of q_1 and q_2 respectively. Let C' be a circle through p and two points of M_2. Then if C' is sufficiently close to C, it will intersect M_1, thus meeting $A_3 \cup p$ in four points. This again is not true. Combining this result with Lemma 3.6, we generalize the latter lemma, obtaining

<u>Lemma 3.8. A circle through three points of $A_3 \cup p$ does not support A_3 at any of these points.</u>

<u>3.7.3.</u> Suppose that a circle $C \in \mathcal{T}$ meets A_3 in two points, q_1 and q_2. By Lemma 3.8 these points are both intersections. Let N and M be disjoint neighbourhoods of p and q_1 respectively. Let $t \in N$, $t \to p$. Then $C' = C(q_2,t,p)$, when it is close enough to C, meets M in a point near q_1. Thus C' meets $A_3 \cup p$ at least four times, contrary to Lemma 3.7. This yields

<u>Lemma 3.9. No circle of $\mathcal{T}$ meets A_3 in two points.</u>

Suppose a circle C of $\mathcal{T}$ supports A_3 at q. Then

Then there is a circle of $\mathfrak{T}$ close to C which intersects A_3 in at least two points. This contradicts Lemma 3.9, and we have

Lemma 3.10. If a circle of $\mathfrak{T}$ meets A_3, it does so in a point of intersection.

3.7.4. Suppose that C(p) meets A_3 at a point q. By Theorem 2.5 and Lemma 3.10, q is a point of intersection. Let N and M be disjoint neighbourhoods of p and q respectively, and let $t \in N$, $t \to p$. Then $C(t;\mathfrak{T})$, when it is close to C(p), will meet M, contradicting Lemma 3.9. Thus we have

Lemma 3.11. C(p) does not meet A_3.

3.7.5. Multiplicities Relative to General Tangent Circles.

In the following we shall not consider general tangent and osculating circles at p, the end-point of A_3, since we shall later discover that they are identical with the ordinary tangent and osculating circles already discussed.

Lemma 3.12. Let C be a general non-osculating tangent circle of A_3 at q. Then C meets $A_3 \cup p$ elsewhere in at most one point and that point is not a point of support.

Proof: By section 3.6, C supports A_3 at q. Hence, by Lemma 3.8, C meets $A_3 \cup p$ at most once outside q. By Lemma 3.5, this point is an intersection if it is on A_3. If the point is p itself, Lemmas 3.10 and 3.11 prohibit multiplicities at p.

Lemma 3.13. Let C be a general osculating circle of A_3 at q. Then C does not meet $A_3 \cup p$ elsewhere.

Proof: Being a general osculating circle, C $= \lim C(q_\nu, q'_\nu, q''_\nu)$, where q_ν, q'_ν, and q''_ν converge on A_3 to q. Suppose C meets $A_3 \cup p$ in another point $r \neq q$. Then $C(q_\nu, q'_\nu, q''_\nu)$ intersects the orthogonal circle to C through q and r in a point r_ν converging to r (cf. Fig. 3.6). Thus $C(q_\nu, q'_\nu, q''_\nu) = C(q_\nu, q'_\nu, r_\nu)$.

Let Q_1, Q_2, S, T be variable points, and let Q_1 and Q_2 converge to the same point Q; $Q_1 \neq Q_2$. Suppose there is a circle separating Q from both S and T. Then

$$\lim \sphericalangle\left[C(Q_1,Q_2,S);C(Q_1,Q_2,T)\right] = 0$$

whether the circles $C(Q_1,Q_2,S)$ and $C(Q_1,Q_2,T)$ themselves

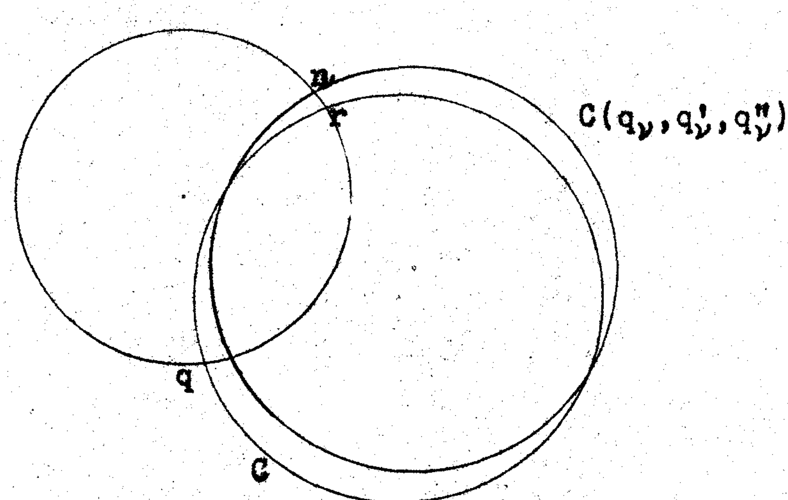


Fig. 3.6

converge or not.[1] In particular,

$$\lim \sphericalangle\left[C(q_\nu,q'_\nu,r_\nu);C(q_\nu,q'_\nu,r)\right] = 0,$$

and since any accumulation circle of $C(q_\nu,q'_\nu,r)$ contains the point r in common with $C = \lim C(q_\nu,q'_\nu,r_\nu)$,

$$\lim C(q_\nu,q'_\nu,r) = C.$$

---

1. This becomes evident if we let the point S or the point T be the point at infinity. This, of course, makes that point no longer a variable point, but the generality is sufficient for our needs.

But $C(q_\nu, q'_\nu, r)$, if it is sufficiently close to C, does not separate the end-points of any small neighbourhood of q. This follows from the fact that A_3 has order three. Thus in the limit, C supports A_3 at q, contradicting the last sentence of section 3.6. Hence C does not meet $A_3 \cup p$ outside q.

3.8. Strong Differentiability.

An arc A is said to be *strongly differentiable* at a point p, if the following two conditions are satisfied:

Condition I': Let $R \neq p$, $R' \to R$. If the two distinct points u and v converge on A to p, then the circle $C(R', u, v)$ always converges.

Condition II': If the three distinct points u, v, and w converge on A to p, then the circle $C(u, v, w)$ always converges.

Suppose that $R' = R$, $u = p$. Then $C(R', u, v)$ becomes $C(R, v, p)$, which converges to $C(R; \tau)$. Since the limit circle $C(R', u, v)$ does not depend on the choice of the sequences u,v, and R', we see that

$$\lim C(R',u,v) = C(R;\tau).$$

Similarily, we find that lim C(u,v,w) = C(p), since

$$C(p) = \lim_{v \to p} C(v;\tau) = \lim_{u \to p} \lim_{v \to p} C(u,v,p).$$

Thus strong differentiability implies ordinary differentiability.

We now prove another important theorem, namely

<u>Theorem 3.3. Let p be the end-point of an open arc A_3 of order three. Then $A_3 \cup p$ is strongly differentiable at p.</u>

<u>Proof</u>: According to Lemma 3.4, $A_3 \cup p$ is differentiable at p. Let B be a sub-arc of A_3 bounded by p and e, and let p,t,u,v,d,e,f lie on $A_3 \cup p$ in the indicated order. We orient all circles C, where $f \not\subset C$, in such a way that $f \subset \overline{C}$.

The above conditions indicate that

(i) $$u \subset \underline{C}(p,t,e) \cap \overline{C}(t,d,e)$$

(cf. Fig. 3.7). Consequently,

$$(3.3) \qquad C(t,u,e) \subset \left[\underline{C}(p,t,e) \cap \overline{C}(t,d,e)\right]$$
$$\cup \left[\overline{C}(p,t,e) \cap \underline{C}(t,d,e)\right] \cup t \cup e.$$

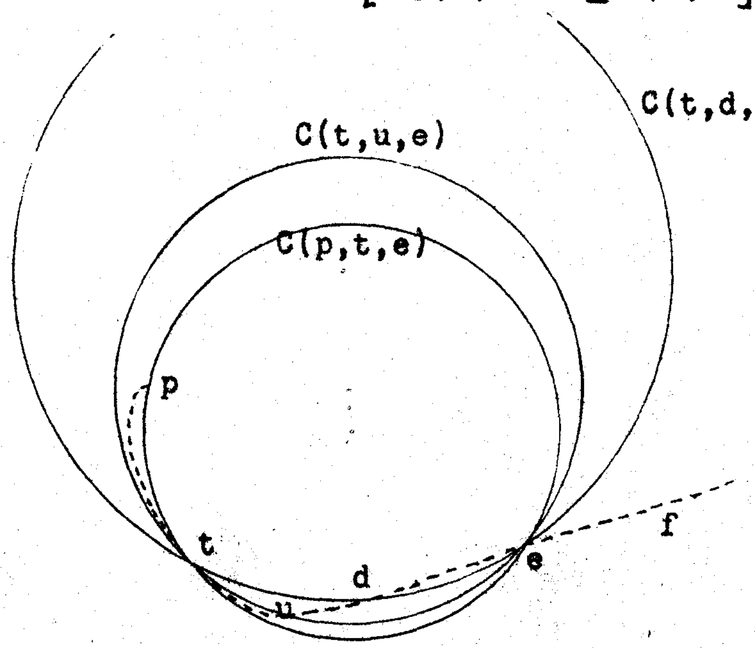


Fig. 3.7

Let I denote the region in relation (3.3). From (3.3) we obtain

$$(3.4) \qquad \lim_{t,u \to p} C(t,u,e) \subset \left[\underline{C}(e;\tau) \cap \overline{C}(p,d,e)\right]$$
$$\left[\overline{C}(e;\tau) \cap \underline{C}(p,d,e)\right] \cup C(e;\tau) \cup C(p,d,e).$$

By II, we shall mean the limit of I as $t \to p$. Let C be any accumulation circle of $C(t,u,e)$. As a point r runs continuously on B from d to p, $C(p,r,e)$ runs continuously through the region II from $C(p,d,e)$ to $C(e;\tau)$. Conversely, every circle through II and the points p and e meets B. Hence if C passes through II $\cup$ C(p,d,e), it intersects B at some point r, where r = d if C = C(p,d,e). But then C(t,u,e), when it is close to C, intersects B again near r, contrary to Theorem 3.2. Thus C = C(e;τ).

Now let P be any point $\neq$ p, and let C' be any accumulation circle of C(P,t,u). As in the proof of Lemma 3.13,

$$\lim \measuredangle\left[C(P,t,u);C(t,u,e)\right] = 0,$$

that is,

$$\measuredangle\left[C';C(e;\tau)\right] = 0.$$

Thus, by Theorem 2.3,

$$C' = C(P;\tau).$$

We now prove simultaneously that C(p,u,v) $\rightarrow$ C(p), and assuming this, that C(t,u,v) $\rightarrow$ C(p). Proceeding as we did previously, we note

(ii) $$v \subset \overline{C}(u;\tau) \cap \underline{C}(p,u,e)$$

(ii') $$v \subset \overline{C}(p,t,u) \cap \underline{C}(t,u,e).$$

Relations (ii) and (ii') yield

$$(3.5) \qquad C(p,u,v) \subset \left[\overline{C}(u;\tau) \cap \underline{C}(p,u,e)\right]$$
$$\left[\underline{C}(u;\tau) \cap \overline{C}(p,u,e)\right] \cup p \cup u,$$

and

$$(3.5') \qquad C(t,u,v) \subset \left[\overline{C}(p,t,u) \cap \underline{C}(t,u,e)\right]$$
$$\cup \left[\underline{C}(p,t,u) \cap \overline{C}(t,u,e)\right] \cup t \cup u$$

respectively. Let III denote either the region in relation (3.5), or that in (3.5'). Relations (3.5) and (3.5') yield

$$(3.6) \qquad \lim C(p,u,v) \subset \left[\overline{C}(p) \cap \underline{C}(e;\tau)\right]$$
$$\cup \left[\underline{C}(p) \cap \overline{C}(e;\tau)\right] \cup C(p) \cup C(e;\tau)$$

and

$$(3.6') \qquad \lim C(t,u,v) \subset \left[\overline{C}(p) \cap \underline{C}(e;\tau)\right]$$
$$\cup \left[\underline{C}(p) \cap \overline{C}(e;\tau)\right] \cup C(p) \cup C(e;\tau)$$

respectively. By IV we shall mean the limit of III as $u \to p$ (as $t,u, \to p$). Let C be any accumulation circle of $C(p,u,v)$ (of $C(t,u,v)$). Since C is the limit of a sequence of tangent circles of A_3 at p, we see that $C \in \mathcal{T}$. As a point r runs continuously on B from e to p, $C(r;\tau)$ runs continuously through the region IV from $C(e;\tau)$ to $C(p)$. Conversely, every tangent circle through IV meets B. Hence if C passes through IV $\cup$ $C(e;\tau)$, it intersects B e at some point r, where $r = e$ if $C = C(e;\tau)$. But then $C(p,u,v)$ $\big(C(t,u,v)\big)$, when it is close to C, intersects B again near r, contrary to Theorem 3.2. Thus $C = C(p)$.

Corollary 1. Let two distinct points u and v converge on $A_3 \cup p$ to p, and let $R' \to R$, $R \neq p$. Let C_1 (C_2) be a general tangent circle of $A_3 \cup p$ at u through R' (through v). Let C_2' be a general osculating circle of A_3 at u. Then

(3.7) $\lim C_1 = C(R;\tau)$

(3.8) $\lim C_2 = \lim C_2' = C(p)$.

Proof: We may assume that each of the above sequences of circles possesses an accumulation circle. C_1 can be replaced by a circle $C(R',u_1,u_2)$ close to C_1 such that u_1

and u_2 are distinct and converge with u to p. Thus by Theorem 3.3,

$$\lim C_1 = \lim C(R', u_1, u_2) = C(R; \mathfrak{T}).$$

Similarily, C_2 and C_2' can be replaced by circles $C(v, u_1, u_2)$ and $C(u_1, u_2, u_3)$ close to C_1 and C_2 respectively, such that u_1, u_2, and u_3 are distinct and converge with u to p. Hence, by Theorem 3.3,

$$\left.\begin{array}{l} \lim C_2 = \lim C(v, u_1, u_2) \\ \lim C_2' = \lim C(u_1, u_2, u_3) \end{array}\right\} = C(p).$$

3.9. Differentiability Properties of Interior Points of A_3.

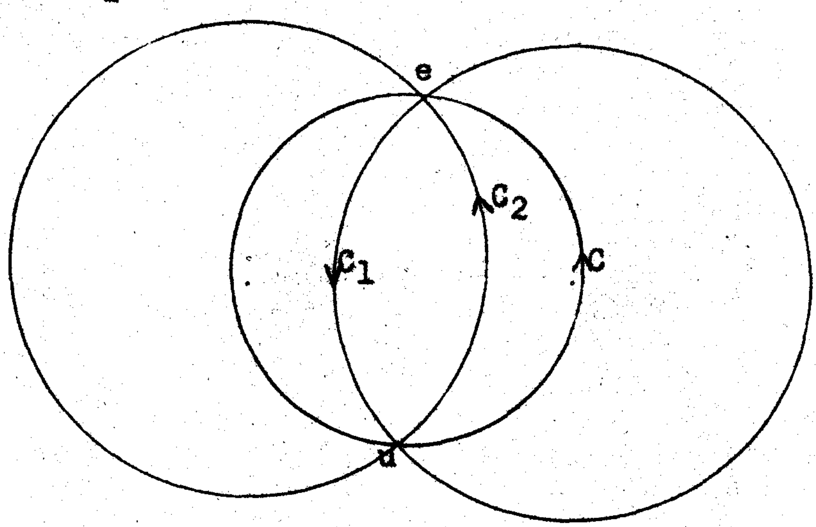
Theorem 3.4. Let u be a point of an open arc A_3 of order three. Then

(3.9) The two one-sided tangent circles of u through a fixed point $R \neq u$ coincide. This implies that the points of A_3 all satisfy Condition I of section 2.2.

(3.10) The set of general tangent circles of u coincides with the pencil of tangent circles of u. The set of general

<u>osculating circles of u forms a closed interval in the pencil of all the tangent circles of u, bounded by the two non-degenerate one-sided osculating circles of A_3 at u</u>. In particular, there is one and only one general tangent circle of u through each point $R \neq u$. This implies that the points of A_3 all satisfy Condition I' of section 3.8.

<u>Proof of (3.9)</u>: We first consider the case $R \in A_3$. Let $B = B_1 \cup u \cup B_2$ be a sub-arc of A_3 bounded by $R = e$ and f. Let $C_i = \lim_{t_i \to u} C(t_i,u,e)$, $t_i \in B_i$ (cf. Fig. 3.8) be dis-



<u>Fig. 3.8</u>

tinct one-sided tangent circles of u through e (cf. Lemma 3.4). By section 3.6 and Lemma 3.5, C_i supports A_3 at u, intersects A_3 at e, and has no other point in common with A_3. Hence we may assume that $B_1 \cup B_2 \subset \underline{C}_1 \cap \underline{C}_2$. Let C be any circle through u and e which passes through the region $(\underline{C}_1 \cap \bar{C}_2) \cup (\bar{C}_1 \cap \underline{C}_2)$. Thus C supports B at u. Let π be the pencil of the second kind of the circles which touch C at u. By Theorem 2.4, applied to B_i,

$$\lim_{t_i \to u} C(t_i;\pi) = u \qquad t_i \in B_i.$$

Conversely, every sufficiently small circle of π which meets B_1 meets B_2, and does not separate e and f. Hence this circle meets B on one hand three times, and on the other hand, with an even multiplicity, i.e., it meets $B \subset A_3$ at least four times, contrary to Theorem 3.2. Thus the two one-sided tangent circles coincide in the circle $C(e;\tau)$. Let $C'(R;\tau)$ and $C''(R;\tau)$ be the two one-sided tangent circles of A_3 at u through a point $R \notin A_3$. Since

$$\sphericalangle\left[C'(R;\tau);C(e;\tau)\right] = 0,$$

and

$$\sphericalangle\left[C''(R;\tau);C(e;\tau)\right] = 0,$$

it is true that

$$\sphericalangle\left[C'(R;\tau);C''(R;\tau)\right] = 0,$$

and since these two circles have the point $R \neq u$ in common, they coincide. This completes the proof of relation (3.9).

Proof of (3.10): Let $C_i = \lim_{t_i \to u} C(t_i;\tau)$, $t_i \in B_i$, be the two one-sided osculating circles of A_3 at u. Since $C(t_i;\tau)$ supports A_3 at u, intersects A_3 at t_i, and does not meet A_3 elsewhere, C_i intersects A_3 at u. Thus $C_i \neq u$.

We may assume that $B_1 \cup B_2$ lies in $\underline{C}(e;\tau)$. By Theorem 3.2, C_1, considered as a general osculating circle of B at u, has no point in common with A_3 except u. Thus $C_1 \subset \underline{C}(e;\tau) \cup u$, and we may assume that $C(e;\tau) \subset \overline{C}_1 \cup u$; thus $B_1 \subset \overline{C}_1 \cap \underline{C}(e;\tau)$. Since C_1 intersects A_3 at u, $B_2 \subset \underline{C}_1$ (cf. Fig. 3.9). Since $C(f;\tau)$ supports A_3 at u, $B_1 \cup B_2 \subset \overline{C}(f;\tau)$. Hence $C_2 = \lim_{t_2 \to u} C(t_2;\tau)$, $t_2 \in B_2$, lies in the closure of $\overline{C}(f;\tau) \cap \underline{C}_1$. Since C_2 does not meet A_3 outside u, it either coincides with C_1, or it lies in $\left(\underline{C}_1 \cap \overline{C}(f;\tau)\right) \cup u$. The circles of the family τ fall into

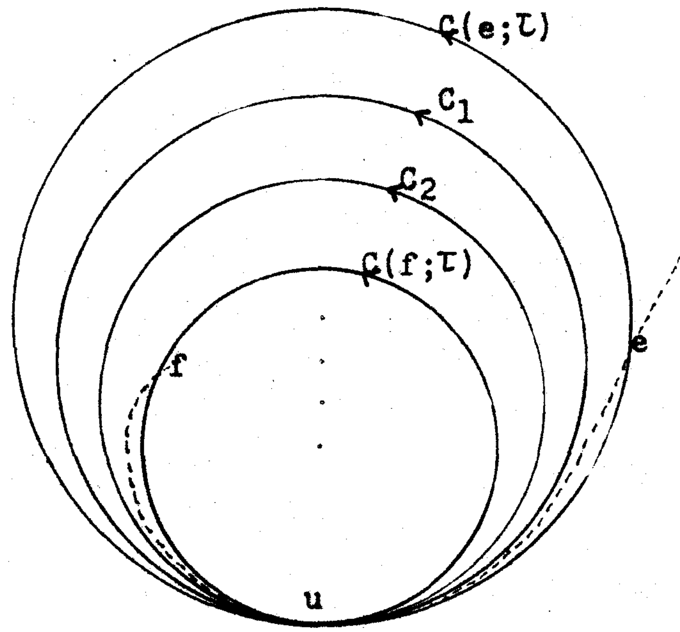


Fig. 3.9

one of two classes: (i) Those tangent circles through a point $R \subset \overline{C}_1 \cup \underline{C}_2$ (together with the point-circle u) which support A_3 at u, and are therefore non-osculating general tangent circles of u; (ii) Those tangent circles through a point $R \subset (\underline{C}_1 \cap \overline{C}_2) \cup C_2$, $R \neq u$, which intersect A_3 at u, and are therefore general osculating circles.

Conversely, every non-osculating general tangent circle (every general osculating circle) of u is an ordinary tangent circle of u lying in $\overline{C}_1 \cup \underline{C}_2 \cup u$ (in $(\underline{C}_1 \cap \overline{C}_2)$

$\cup C_1 \cup C_2$). We prove this statement as follows:

First, let C be any non-osculating general tangent circle of u, and suppose that $C \notin \mathfrak{T}$. We know that C supports A_3 at u. Hence we may assume that a sufficiently small neighbourhood M of u on A_3 lies in $\underline{C}$, and even in $\overline{C}(f;\mathfrak{T}) \cap \underline{C}(e;\mathfrak{T}) \cap \underline{C}$ (cf. Fig. 3.10). Let π be the pencil

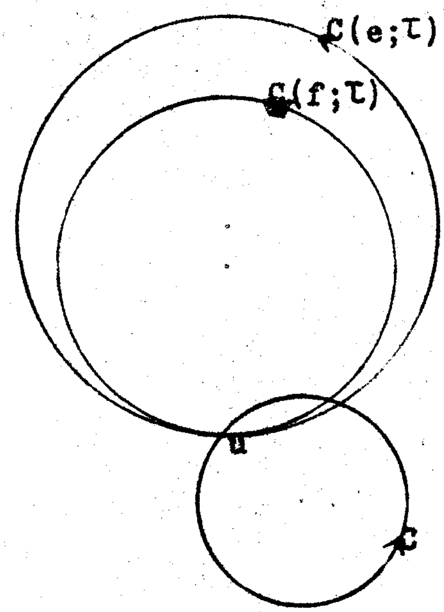


Fig. 3.10

of the second kind of the circles which touch C at u. Since

$$\lim_{t_i \to u} C(t_i;\pi) = u, \qquad t_i \in B_i,$$

every small circle of π in $\underline{C}$ meets both B_1 and B_2 and does not separate the end-points of M. Hence such a circle meets

M at least three times and with an even multiplicity, i.e., it meets M at least four times. This again contradicts Theorem 3.2. Thus every non-osculating general tangent circle of u is an ordinary tangent circle of u. By (i), such a circle lies in $\bar{C}_1 \cup \underline{C}_2 \cup u$.

Next, let C be a general osculating circle of u. Let $C = \lim_{u_1,v_1,u_2 \to u} C'$, where $C' = C(u_1,v_1,u_2)$, $u_1,v_1 \in B_1$, $u_2 \in B_2$. Obviously, C cannot be the point-circle u, since C intersects A_3 at u. Let $R \subset C$, $R \neq u$, and suppose that C' intersects the orthogonal circle of C through u and R at R'. Thus $C' = C(u_1,v_1,R')$ and

$$C = \lim_{\substack{u_1,v_1 \to u \\ R' \to R}} C(u_1,v_1,R') = C(R;\tau).$$

From (ii), $R \subset (\underline{C}_1 \cap \bar{C}_2) \cup C_1 \cup C_2$. Thus every general osculating circle is a (non-degenerate) tangent circle of u lying in the closure of $\underline{C}_1 \cap \bar{C}_2$.

<u>Corollary 1.</u> <u>If an interior point of an arc of order three is differentiable, it is strongly differentiable.</u>

Proof: By Theorem 3.4, Condition I' is satisfied for all interior points of A_3.

If the point $u \in A_3$ is differentiable, the two one-sided osculating circle coincide. Thus Theorem 3.4 implies that Condition II' also, holds for the point u.

3.10. More Properties of Arcs of Order Three.

In this section we collect additional material on arcs of order three, needed for the proof of the final theorem in this chapter. Let p be an end-point of A_3. The arc B and the points t,u,v,e,f, are the same as in section 3.8.

3.10.1. We first extend formulas (3.5) and (3.5') to certain limit cases in which some of the points involved coincide. The circle C(t,u,v) separates the regions

$$\underline{C}(p,t,v) \cap \underline{C}(t,v,e) \tag{3.11}$$

and

$$\overline{C}(p,t,v) \cap \overline{C}(t,v,e) \tag{3.12}$$

(cf. relation (3.5')).

Suppose that the distinct points t_o, v_o, e lie on $B \cup e$ in the indicated order. Let C_o be the general tangent circle of B at t_o through v_o. Then C_o can be obtained as the limit of circles $C(t,u,v)$, if the triplets t,u,v converge to t_o, t_o, v_o. Since $C(t,u,v)$ and the regions of (3.5') depend continuously on t, u, and v, (3.5') implies that C_o lies in the closure of the region

$$R = \left[\underline{C}(p,t_o,v_o) \cap \overline{C}(t_o,v_o,e)\right] \cup \left[\overline{C}(p,t_o,v_o) \cap \underline{C}(t_o,v_o,e)\right].$$

As C_o meets $C(p,t_o,v_o)$ and $C(t_o,v_o,e)$ only at t_o and v_o, this implies that $C_o \subset R \cup t_o \cup v_o$. Replacing t_o again by t, and v_o by v, we thus have: the relation (3.5') remains valid for $u = t$ if $C(t,t,v)$ is interpreted to mean the tangent circle of B at t through v.

Similarly, (3.5) and (3.5') remain valid for $u = v$, with the corresponding interpretation of $C(t,v,v)$. Finally, these formulas remain valid for $v = e$ if $C(t,e,e)$ and $C(p,e,e)$

stand for the tangent circles of A_3 at e through t and p respectively (cf. § 3.9).

Let $v_1 \in B$, and let C_1 denote any general osculating circle of B at v_1. Thus C_1 will be the limit of $C(t,u,v)$ if t,u,v converge to v_1 in a suitable fashion. By section 3.9, the circles $C(p,t,v)$ and $C(t,v,e)$ are also convergent to the tangent circles C_2 and C_3 of B at v_1 through p and e respectively. Furthermore, $p \subset \underline{C}_3$ and $e \subset \overline{C}_2$ because of our orientation convention. This implies

$$C_2 \subset \underline{C}_3 \cup v_1$$

and

$$C_3 \subset \overline{C}_2 \cup v_1.$$

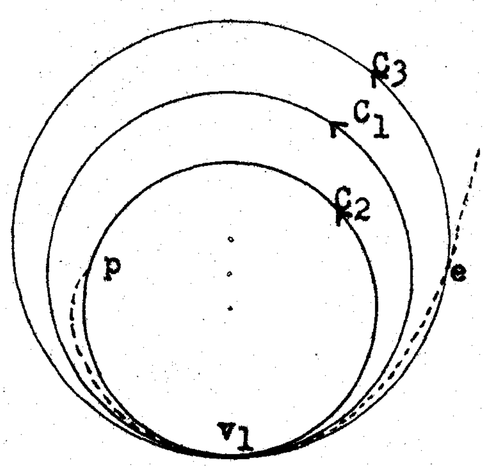


Fig. 3.11

From (3.5'), C_1 lies in the closure of

$$(\underline{C}_2 \cap \bar{C}_3) \cup (\bar{C}_2 \cap \underline{C}_3)$$

(cf. Fig. 3.11). Since $\underline{C}_2 \cap \bar{C}_3$ is empty, and $C_1 \neq C_2, C_3$, this implies

$$(3.13) \qquad C_1 \subset (\bar{C}_2 \cap \underline{C}_3) \cup v_1.$$

Since each $C(t,u,v)$ separates the regions (3.11) and (3.12), C_1 will separate $\underline{C}_2 \cap \underline{C}_3 = \underline{C}_2$ and $\bar{C}_2 \cap \bar{C}_3 = \bar{C}_3$. Replacing v_1 by v, we obtain: relation (3.5') remains valid, and $C(t,u,v)$ separates the regions (3.11) and (3.12) for $t = u = v$, if $C(v,v,v)$ is interpreted to mean any general osculating circle of B at v, provided $C(p,v,v)$ and $C(v,v,e)$ stand for the tangent circles of B at v through p and e respectively.

<u>3.10.2.</u> Considering again relation (3.5') we observe that one of the regions (3.11) and (3.12) will lie in $\underline{C}(t,u,v)$, the other one in $C(t,u,v)$. Since

$$f \subset \bar{C}(p,t,v) \cap \bar{C}(t,v,e) \cap \bar{C}(t,u,v),$$

this relation implies

$$(3.14) \qquad \overline{C}(p,t,v) \cap \overline{C}(t,v,e) \subset \overline{C}(t,u,v),$$

and therefore

$$(3.15) \qquad \underline{C}(p,t,v) \cap \underline{C}(t,v,e) \subset \underline{C}(t,u,v).$$

Specializing by letting t = p, we obtain

$$(3.14') \qquad \overline{C}(v;\tau) \cap \overline{C}(p,v,e) \subset \overline{C}(p,u,v),$$

and

$$(3.15') \qquad \underline{C}(v;\tau) \cap \underline{C}(p,v,e) \subset \underline{C}(p,u,v).$$

Applying the case $v = e$ of (3.14') and (3.15'), and replacing u by v afterwards, we obtain,

$$(3.16) \qquad \overline{C}(e;\tau) \cap \overline{C}(p,e,e) \subset \overline{C}(p,v,e),$$

and

$$(3.17) \qquad \underline{C}(e;\tau) \cap \underline{C}(p,e,e) \subset \underline{C}(p,v,e).$$

Now $\overline{C}(e;\tau) \subset \overline{C}(v;\tau)$, since $e \subset \overline{C}(v;\tau)$. Therefore, applying relations (3.16) and (3.14'), we have

$$(3.18) \qquad \overline{C}(e;\tau) \cap \overline{C}(p,e,e) \subset \overline{C}(v;\tau) \cap \left[\overline{C}(e;\tau) \cap \overline{C}(p,e,e)\right]$$
$$\subset \overline{C}(v;\tau) \cap \overline{C}(p,v,e)$$
$$\subset \overline{C}(p,u,v).$$

Similarly, $\underline{C}(t;\tau) \subset \underline{C}(v;\tau)$ when t is close to p. Therefore, in the limit, $\underline{C}(p) \subset \underline{C}(v;\tau)$. Also, $\underline{C}(p) \subset \underline{C}(e;\tau)$. Hence, applying relations (3.17) and (3.15'), we have

$$(3.19) \qquad \underline{C}(p) \cap \underline{C}(p,e,e) \subset \underline{C}(v;\tau) \cap \underline{C}(e;\tau) \cap \underline{C}(p,e,e)$$

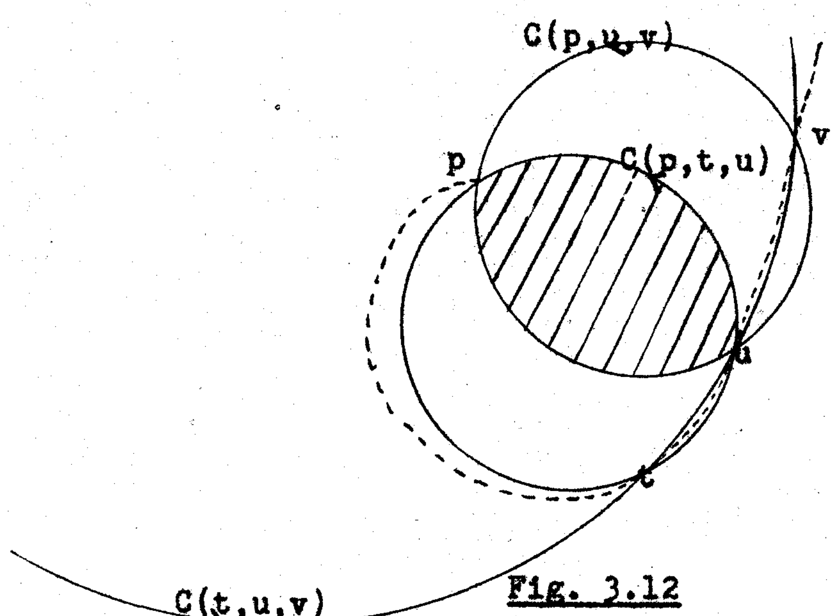
$$\subset \underline{C}(v;\tau) \cap \underline{C}(p,v,e)$$

$$\subset \underline{C}(p,u,v).$$

<u>3.10.3.</u> Assume for the moment that p,t,u,v, are mutually distinct. The region

$$(3.20) \qquad \underline{C}(p,t,u) \cap \underline{C}(p,u,v)$$

is bounded by two arcs of the circles C(p,t,u) and C(p,u,v) with the common end-points p and u. Since $v \subset \overline{C}(p,t,u)$ and



<u>Fig. 3.12</u>

$t \subset \overline{C}(p,u,v)$ (cf. Fig. 3.12), these arcs do not contain v or t. Hence they meet C(t,u,v) only at u, and the region (3.20) is contained in one of the two regions bounded by C(t,u,v). Since the boundary point, p of the region (3.20) lies in $\underline{C}(t,u,v)$, this implies

$$(3.21) \qquad \underline{C}(p,t,u) \cap \underline{C}(p,u,v) \subset \underline{C}(t,u,v).$$

The arguments of section 3.10.1 now show that relation (3.21) remains valid if C(t,u,v) is any general tangent circle, provided C(p,u,u) then stands for the tangent circle at u through p.

By relations (3.15), (3.21), and (3.15'),

$$\begin{aligned} \underline{C}(t,u,v) &\supset \underline{C}(p,t,v) \cap \underline{C}(t,v,e) \\ &\supset \underline{C}(p,t,v) \cap \left[\underline{C}(p,t,v) \cap \underline{C}(p,v,e)\right] \\ &= \underline{C}(p,t,v) \cap \underline{C}(p;v,e) \\ &\supset \left[\underline{C}(v;\tau) \cap \underline{C}(p,v,e)\right] \cap \underline{C}(p,v,e) \\ &= \underline{C}(v;\tau) \cap \underline{C}(p,v,e). \end{aligned}$$

In particular, the above yields

$$(3.22) \qquad \underline{C}(t,u,v) \supset \underline{C}(v;\tau) \cap \underline{C}(p,v,v).$$

<u>3.10.4</u>. Let Θ denote the pencil of the orthogonal circles of τ.

On account of Theorem 3.3, B can be chosen so small that no circle of θ meets $B \cup e$ more than once (otherwise this circle would approach a circle of τ). By Theorem 2.4,

(3.23) $$C(p;\theta) = \lim_{t \to p} C(t;\theta) = p.$$

Thus, making B small enough, we may also assume that $C(f;\theta)$ does not meet B.

Since $C(v;\tau)$ meets the circle $C_o = C(t,u,v)$, the pencil τ contains a circle lying in $\underline{C}(v;\tau) \cup C(v;\tau)$ and touching C_o from within, say at R. Thus

(3.24) $$C(R;\tau) \cap C_o = R; \qquad \underline{C}(R;\tau) \subset \underline{C}_o \cap \underline{C}(v;\tau).$$

The circle $C(R;\theta)$ can be characterized as the unique circle of θ normal to C_o (cf. Fig 3.13). We wish to prove the following

<u>Lemma 3.14</u>. <u>$C(R;\theta)$ intersects B.</u>

<u>Proof:</u> Our proof derives from relation (3.23) and the fact that

(3.25) $$R \subset \underline{C}(v;\theta)$$

<u>Proof of (3.25)</u>: If the point t moves on $B \cup p$

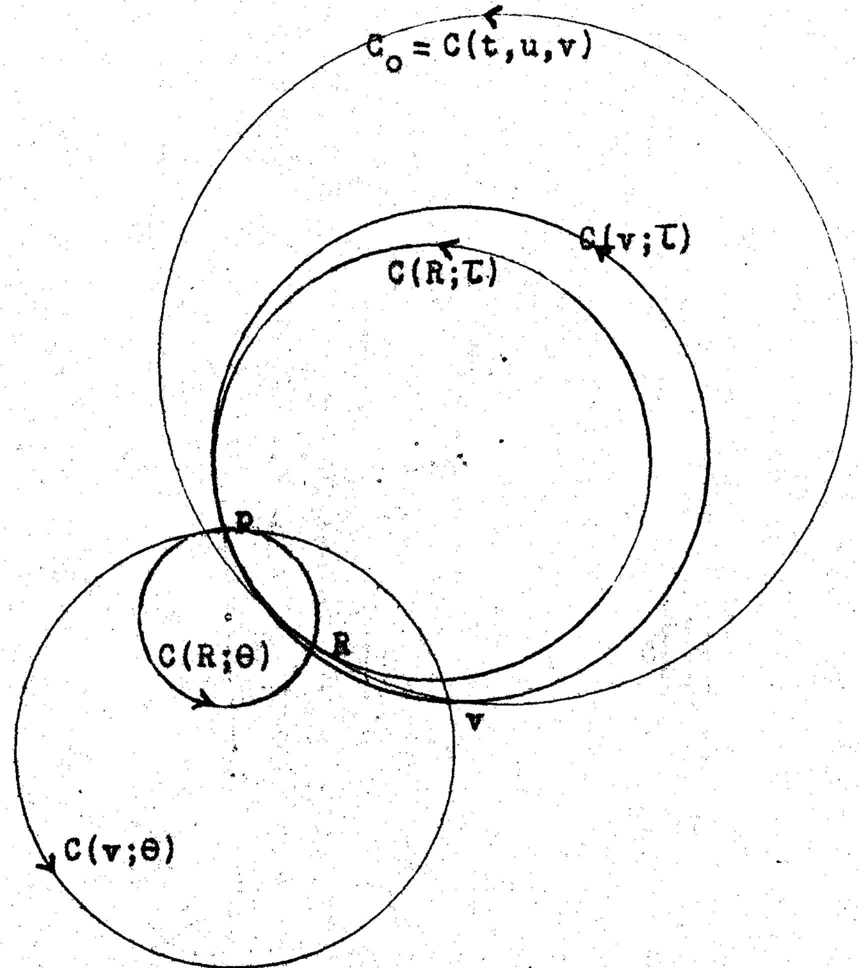


Fig 3.13

from p to v, C(p,t,v) moves from C(v; τ) to C(p,v,v) and passes through the closure of $\underline{C}(v;\tau) \cap \underline{C}(v;\theta)$. Hence C(p,t,v) does not pass through $\underline{C}(v;\tau) \cap \overline{C}(v;\theta)$. Since $\underline{C}(v;\tau)$ contains this region, so does $\underline{C}(p,t,v)$ and $\underline{C}(p,v,v)$. Hence, by relation (3.22),

$$\begin{aligned}\underline{C}(t,u,v) &\supset \underline{C}(v;\tau) \cap \underline{C}(p,v,v)\\ &\supset \underline{C}(v;\tau) \cap [\underline{C}(v;\tau) \cap \overline{C}(v;\theta)]\\ &= \underline{C}(v;\tau) \cap \overline{C}(v;\theta).\end{aligned}$$

Thus if $R \neq v$, R does not lie in the above region. However, $R \subset \underline{C}(v;\tau)$ in this case, and so

$$R \subset \underline{C}(v;\tau) \cap \underline{C}(v;\theta),$$

which proves relation (3.25). If $R = v$, B can be made small enough to ensure that $C(R;\theta) = C(v;\theta)$ intersects A_3 at v.

If $t = v$, C_o is a general osculating circle of B at this point. Approximating C_o by circles through three distinct points, and making use of the above, we observe that relation (3.25) remains valid unless $t = u = v = R$. But in that case, $C(v;\tau)$ touches C_o at v and therefore is a tangent circle of B at v, since C_o is a tangent circle at v. This is excluded by Theorem 3.2.

3.10.5. Any point Q induces an orientation of all the circles C with $Q \not\subset C$, if C is defined through $Q \subset \overline{C}$.

Let $Q \not\subset C(p)$, $R \not\subset C(p)$, and let $\overline{C}$ ($\widetilde{C}$) denote the orientation induced by Q (R). Suppose that $\overline{C}(p) = \widetilde{C}(p)$. Thus $Q \subset \widetilde{C}(p)$ and $R \subset \overline{C}(p)$. We vary C continuously, starting from C(p). As long as C does not pass through Q or R, $\overline{C}$ and $\widetilde{C}$ will depend continuously on C. Thus we shall still have $\overline{C} = \widetilde{C}$.

By Theorem 3.3, a circle C which meets A_3 three times in $p \cup B \cup e$ lies close to C(p) if B is sufficiently small. Hence $\overline{C} = \widetilde{C}$ for every such C.

We specialize, letting $Q = f \in A_3$. Since $f \notin B \cup e$, the formulas of section 3.10 hold true. Thus they remain valid with respect to the orientation induced by R, provided B is small enough. Since $f \subset \overline{C}(p)$ is equivalent to $A_3 \subset \overline{C}(p)$, this yields the following

<u>Lemma 3.15. Suppose the point $R \not\subset C(p)$ induces an orientation with $A_3 \subset \overline{C}(p)$. Then the formulas of section 3.10 remain valid for this orientation if B is small enough</u>.

If $A_3 \subset \underline{C}(p)$ for the orientation induced by R, then the above argument shows: replace each $\underline{C}$ and $\overline{C}$ in these formulas by $\overline{C}$ and $\underline{C}$ respectively. Then the resulting formulas hold true if B is small enough.

3.11. Conformally Elementary Points.

A point p of an arc A is said to be a conformally elementary point if there exists a neighbourhood of p on A which is decomposed by p into two one-sided neighbourhoods of order three. By Theorem 3.3 their closures are strongly differentiable at p. The following theorem sharpens Theorem 3.1 in the case of conformally elementary points.

Theorem 3.5. Let p be a differentiable conformally elementary point of an arc A, and let $(a_o, a_1, a_2; i)$ be the characteristic of p. Then p has cyclic order $a_o + a_1 + a_2$.

This theorem remains valid if a point $q \neq p$ is counted twice (three times) on any general tangent (osculating) circle of q, and if p itself is counted a_o ($a_o + a_1$; $a_o + a_1 + a_2$) times on any circle through p (on any tangent circle of p; on C(p)).

We may assume that A itself is decomposed by p into two open arcs, A_3 and A_3', of order three. Hence the order of A, and therefore that of p, is not greater than six.

3.11.1. Let M be a neighbourhood of p on A. For any circle D, let $\mu(D) = \mu(D,M)$ denote the multiplicity with which D meets M.

Lemma 3.16. Suppose the circle C does not pass through the end-points of M. Then

(3.26) $$\mu(D) \equiv \mu(C) \pmod 2$$

for every D sufficiently close to C.

Proof: Suppose C meets M at the points t with the multiplicities $\sigma(t)$ and nowhere else. Thus

$$\mu(C) = \sum_t \sigma(t).$$

Construct disjoint neighbourhoods M_t in M about the points t. The end-points of M_t lie on the same side or on opposite sides of C depending on whether $\sigma(t)$ is even or odd. If D is sufficiently close to C, then D will not pass through the end-points of M_t, and these end-points will lie on the same side of D if and only if they lie on the same side of C. On the other hand, D will meet M_t with an even or odd multiplicity according as its end-points lie on the same side or on opposite sides of D. Thus D will meet M_t with a multiplicity $\rho(t) \equiv \sigma(t) \pmod 2$ if D lies sufficiently close to C.

If each M_t is omitted from the closure of M, we obtain a closed set which has no points in common with C. Hence if D is sufficiently close to C this set does not meet D, and we have

$$\mu(D) = \sum_t \rho(t) \equiv \sum_t \sigma(t) = \mu(C) \pmod 2.$$

<u>3.11.2.</u> We continue the discussion of section 3.11.1.

<u>Lemma 3.17.</u> <u>Let $C \neq C(p)$. Then</u>

(3.27) $$\mu(D) \leq \mu(C)$$

<u>for every circle D sufficiently close to C, unless $a_o = a_1 = 1$, $C \in \mathcal{T}$, and $p \notin D$.</u>

<u>Proof</u>: Let $t \in C \cap M$; $t \neq p$. Suppose that there is

a sequence of circles D_λ converging to C, and a sequence of neighbourhoods M_λ of t converging to t such that each D_λ meets M_λ at least σ_λ times ($\sigma_\lambda \leqslant 3$). Then each D can be replaced by another cicle which meets M_λ in not less than σ_λ distinct points, and such that the sequence of the new circles also converges to C. Thus C will meet M at least σ_λ times at t; i.e., $\sigma_\lambda \leqslant \sigma(t)$. Hence we have: there exists a neighbourhood of t on M which is met not more than $\sigma(t)$ times by every D sufficiently close to C.

Let $p \subset C$, $C \notin \mathcal{T}$. Then C meets M at p with a multiplicity $\equiv a_o$ (mod 2). On the other hand, by Theorem 3.3, there exists a neighbourhood of p which is met not more than twice by any circle sufficiently close to C. But $p \subset C$; hence $0 < \mu(C) \leq 2$, and therefore $\mu(C) = a_o$. Thus, by Lemma 3.16, any circle D which is sufficiently close to C meets a neighbourhood of p with a multiplicity $\equiv a_o$ (mod 2). Hence this multiplicity is $\leqslant a_o$, and we have relation (3.27) in this case.

Now let $C \in \mathcal{T}$, $C \neq C(p)$, and let $M_o = N_o \cup p \cup N_o'$ be a sufficiently small neighbourhood of p. Let D be sufficiently close to C. If $p \subset D$, $D \notin \mathcal{T}$, then D will meet N_o and N_o' not more than once each. Hence D meets M_o with a multiplicity $\leq a_o + 2$ and $\equiv a_o + a_1$ (mod 2). Thus this multiplicity is $\leq a_o + a_1$, and again we have relation (3.27).

Suppose now that $p \not\subset D$. Then D will meet N_o and N_o' not more than twice each. Hence D meets M_o with a multiplicity ≤ 4 and $\equiv a_o + a_1$ (mod 2). This again yields relation (3.27) unless $a_o = a_1 = 1$.

3.11.3. Lemma 3.18. Let $A = A_3 \cup p \cup A_3'$. There exists a neighbourhood $M_3 = N_3 \cup p \cup N_3'$ ($N_3 \subset A_3$, $N_3' \subset A_3'$) such that every tangent circle of p which meets $N_3 \cup N_3'$ meets $A_3 \cup A_3'$ exactly a_2 times. In particular, no tangent circle of p meets M_3 more than a_2 times outside p.

Proof: A circle of $\mathcal{T}$ meets A_3 or A_3' not more than once each. Thus it meets $A_3 \cup A_3'$ not more than twice. By Lemma 3.16, a circle will meet A with a multiplicity $\equiv a_0 + a_1 + a_2 \pmod 2$ if it is sufficiently close to C(p). Hence $C(t;\tau)$ will meet $A_3 \cup A_3'$ with a multiplicity $\equiv a_2$ if t is close enough to p. Such a circle will therefore meet $A_3 \cup A_3'$ exactly a_2 times.

3.10.4. Lemma 3.19. There exists a neighbourhood $M_2 \subseteq M_3$ which is met at most $a_0 + a_1 + a_2$ times by every circle through p.

Proof: On account of Lemma 3.18, it suffices to consider non-tangent circles through p. Hence it suffices to construct a one-sided neighbourhood $N_2' \subset N_3'$ of p such that any circle D through p that meets N_2' twice will meet M_3 at most $a_0 + a_1 + a_2$ times.

By Lemma 3.16, N_2' can be chosen so small that any such circle $D = C(u_1', u_2', p)$ $(u_1', u_2' \in N_2')$ is so close to C(p) that it meets M_3 with a multiplicity $\equiv a_0 + a_1 + a_2 \pmod 2$.

Since D meets N_3 and N_3' not more than twice each, it will meet M_3 at most $a_o + 4$ times. This yields our statement if $a_1 + a_2 > 2$.

Let $a_1 + a_2 = 2$, i.e., $a_1 = a_2 = 1$. Let e denote the end-point of N_3, and suppose that the points u,v,e lie on $N_3 \cup e$ in the indicated order. Making N_2' still smaller, we may assume that it does not meet C(p,e,e) (cf. § 3.10). Obviously N_2' has no points in common with C(p) and C(e;τ). We have

$$N_3 \subset \overline{C}(p) \cap \underline{C}(e;\tau) \cap \overline{C}(p,e,e)$$

(cf. Fig. 3.14). Since $a_1 = a_2 = 1$, it follows that

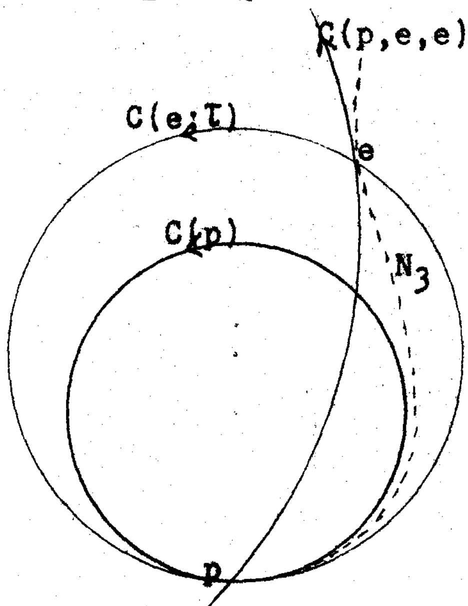


Fig. 3.14

$$N_2' \subset \underline{C}(p) \cap \underline{C}(p,e,e),$$

or else

$$N_2' \subset \overline{C}(e;\tau) \cap \overline{C}(p,e,e).$$

Hence relations (3.19) and (3.18) imply that N_2' lies either in $\underline{C}(p,u,v)$ or in $\overline{C}(p,u,v)$. Thus N_2' does not meet $C(p,u,v)$.

Any circle D through p and two points of N_2' meets M_3 with a multiplicity $\equiv a_o + 1 + 1 \pmod 2$; i.e., it meets $N_3 \cup N_3'$ an even number of times. It meets N_3' exactly twice. From the above, D cannot meet N_3 twice. Hence D and N_3 are disjoint and D meets M_3 with the total multiplicity $a_o + 2 = a_o + a_1 + a_2$.

<u>3.11.5.</u> We can now prove Theorem 3.5 if $a_o + a_1 + a_2 > 4$. It suffices to show that there is a one-sided neighbourhood $N_1' \subset N_2'$ of p such that no circle D through three points of $N_1' \cup p$ meets M_3 more than $a_o + a_1 + a_2$ times. On account of Lemma 3.19, we need only consider circles D which do not pass through p.

By Theorem 3.3 and Lemma 3.16, N_1' can be chosen such that any circle D meets M_3 with a multiplicity $\equiv a_o + a_1 + a_2 \pmod 2$. Since $p \not\subset D$, and since D meets N_3 and N_3' at most three times each, it will meet M_3 at most six times. This yields our assertion.

3.11.6. The case $a_o + a_1 + a_2 = 4$; $a_o = 1$. Let $M_1 \subset M_2$ be so small that the material in sections 3.10.4 and 3.10.5 can be applied to $N_1 = M_1 \cap N_2$ and $N_1' = M_1 \cap N_2'$. Thus some circle of θ does not meet $N_1 \cup N_1'$. Since $a_o = 1$, this circle will intersect M_1 at p. Hence no circle of θ can meet both N_1 and N_1'. Thus if the circle C_o meets N_1 in three points, the circle $C(R;\theta)$ intersects N_1 (cf. Lemma 3.14). However, $C(R;\theta)$ does not meet N_1' and hence Lemma 3.14 implies that C_o does not meet N_1' three times. Taking section 3.10.5 into account, we can state: no circle meets M_1 more than five times.

By Theorem 3.3 and Lemma 3.16, a neighbourhood

$M_o \subset M_1$ of p exists such that every circle through three points of $N_o = M_o \cap N_1$ or of $N'_o = M_o \cap N'_1$ meets M_1 with an even multiplicity i.e. four times. By Lemma 3.19, any circle through more than four points of M_o does not go through p, and hence meets either N_o or N'_o three times. Hence, by the above result, M_o has the order four.

<u>3.11.7.</u> <u>The case[1] (2,1,1;2).</u> Let $e \subset N_2$, $e' \subset N'_2$. Let M_e denote the neighbourhood of p with the end-points e and e'; $N_e = M_e \cap N_2$, $N'_e = M_e \cap N'_2$. By Lemmas 3.18 and 3.19, $C(e;\tau)$ $(C(e';\tau))$ meets M_2 exactly three times at p, exactly once at e (e'), and nowhere else. Thus, by Theorem 3.3 and Lemmas 3.16 and 3.17, there is a one-sided neighbourhood $N_1 \subset N_e$ such that every circle through e (e') and two points of N_1

- -

1. It may be that a short proof for this case, of the nature of the proof for the case $a_o + a_1 + a_2 = 4$ ($a_o = 1$) exists. This proof, however, has value in itself, for it is an example of a topological proof rather than a geometrical one. We shall see more proofs of this type in Chapter 8.

meets M_2 exactly three times outside e (e'). Since these three points converge to p with N_1, N_1 may be chosen such that all these circles meet M_e exactly three times each.

Now let $u \in N_1$, $u' \in N'_e$ be arbitrary, and let π denote the pencil of circles through u and u' (cf. Fig. 3.15).

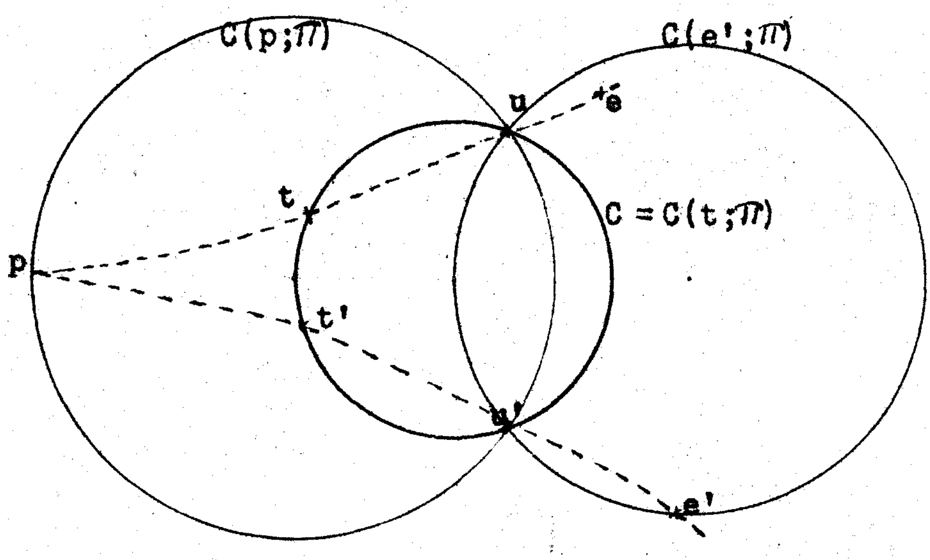


Fig. 3.15

By Lemma 3.19, C(p;π) meets M_e only four times. Thus C(p;π) meets M_e exactly twice at p, once each at u and u', and nowhere else. If t lies on N_1 and is sufficiently close to p, then C(t;π) continues to meet M_e exactly four times (Lemmas 3.16 and 3.17). Since C(p;π) $\notin \mathcal{T}$, the fourth point t' lies on N'_e (Lemma 3.18 and Theorem 3.3). From the above,

$C(t;\Pi)$ passes neither through e nor through e'. Thus $C = C(t;\Pi)$ has the following properties:

(i) p,e,e' lie on the same side of C;

(ii) C meets N_1 exactly twice.

The circles $C(p;\Pi)$ and $C(e';\Pi)$ decompose Π into two open intervals. Let Π_1 denote that interval which contains the above circles $C(t;\Pi)$. We orient Π_1 in the direction from $C(p;\Pi)$ to $C(e';\Pi)$. The circle C satisfies (i) and (ii) if it lies in Π_1 sufficiently close to $C(p;\Pi)$. Put the circle D equal to $C(e';\Pi)$ if (i) and (ii) hold true for every circle of Π_1; otherwise, let D denote the greatest lower bound of the set of all the circles of Π_1 for which at least one of these conditions is not satisfied. Thus $D \neq C(p;\Pi)$. Let Π_2 denote the sub-interval of Π_1 bounded by $C(p;\Pi)$ and D.

Every circle $C \in \Pi_2$ satisfies (i) and (ii). Thus C meets N_e (N_e') in exactly one more point t (t'), and t lies in N_1. The point t (t') depends continuously on C. For

$t \neq u$ $(t' \neq u')$, the correspondence $C \to t$ $(C \to t')$ is 1-1. Hence it is strictly monotonic, even for $t = u$ $(t' = u')$ (cf. Theorem 3.4). Thus the limits $r = \lim_{C \to D} t$ and $r' = \lim_{C \to D} t'$ exist. The point r (r') lies on the intersection of D with the closure of N_1 (N'_e). It is different from p.

If $r' = e'$, the points t' cover the whole of N'_e. In particular, Π_2 contains all the circles $C(t';\Pi)$, including the case $t' = u'$. Thus every circle through u and u' that meets N'_e at least twice, meets N'_e and N_1 - and even N_e - exactly twice each.

Let $r' \neq e'$. Thus $r' \in N'_e$. From the above, $e \not\subset D$ and $e' \not\subset D$. Hence D lies in Π_1 and still satisfies condition (i). Hence, (i) will remain valid for all circles of Π_1 sufficiently close to D. In particular, these circles will meet N_e exactly twice. Thus $r \notin N_1$, by the definition of D, and r will be the end-point of N_1 different from p.

Thus the points t will cover the whole of N_1. Repeating the argument of the preceeding paragraph, we obtain: every circle through u and u' that meets N_1 at least twice, meets N_e and N'_e exactly twice each.

The last two paragraphs imply: any circle that meets N_1 and N'_e at least twice each, meets N_e and N'_e exactly twice each. Hence such a circle meets M_e exactly four times outside p. Combining this result with Lemma 3.19, we find that the neighbourhood $N_1 \cup p \cup N'_e$ has the order four.

3.11.8. The case $a_0 + a_1 + a_2 = 3$. Suppose that the points p,t,u,v lie on $N_2 \cup p$ in the indicated order. The points t,u,v need not be mutually distinct. By Lemma 3.19, the circles C(p,t,u) and C(p,u,v) do not meet N'_3. They intersect M_3 at each of these points. Hence

$$N'_3 \subset \underline{C}(p,t,u) \cap \underline{C}(p,u,v) \tag{3.28}$$

(cf. Fig. 3.16). Hence relation (3.21) implies that $N'_3 \subset \underline{C}(t,u,v)$. In particular, C(t,u,v) does not meet N'_2.

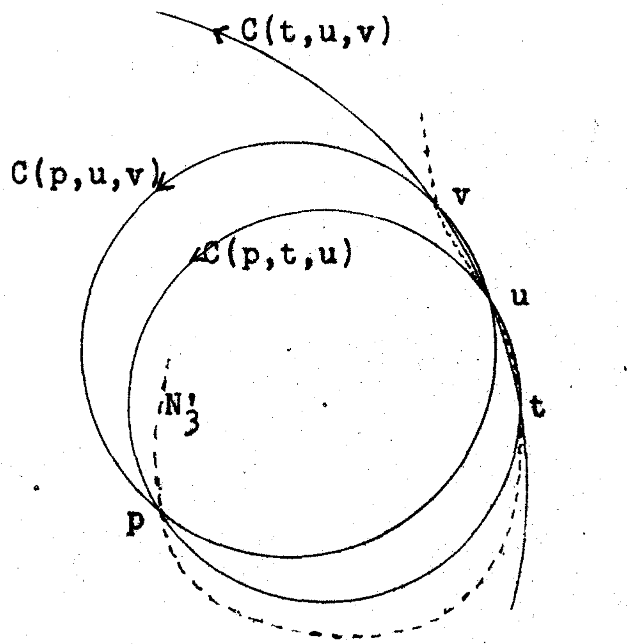


Fig. 3.16

Symmetrically, any circle through three points of N_2' does not meet N_2.

Let $e \in N_2$, $e' \in N_2'$. Let M_e denote the neighbourhood of p bounded by e and e'. Let M_1 be a neighbourhood of p whose end-points lie in M_e. By Lemma 3.18, $C(e';T)$ meets M_1 exactly twice at p, and nowhere else. Thus, by Theorem 3.3 and Lemma 3.16, there is a one-sided neighbourhood $N_o \subset N_1$ $(N_1 = M_1 \cap N_2)$ of p, such that every circle through e' and two points of N_o meets M_1 with an even multiplicity.

Let $t,u, \in N_o$. As we have seen, $C(t,u,e)$ does not meet $N_1' = M_1 \cap N_2'$. Hence

(3.29) $$e \not\subset C(t,u,u')$$

for every $t,u \in N_o$, $u' \in N_1'$. Furthermore, $C(t,u,e)$ meets M_1 an even number of times. By the above, this circle meets N_2 and N_2' not more than twice each. Hence it meets N_1 exactly twice. By Lemma 3.19 it does not pass through p. Thus it meets N_1' with an even multiplicity. Since $e' \in N_2'$, $e' \not\in N_1'$, this multiplicity is less than two. Hence $C(t,u,e')$ does not meet N_1', and we have

(3.30) $$e' \not\subset C(t,u,u')$$

for every $t,u \in N_o$, $u' \in N_1'$.

Let $u \in N_o$, $u' \in N_1'$. By Lemma 3.19, $C(p,u,u')$ meets M_2 exactly three times. Thus it separates e and e'. If t moves on N_o between p and u, the circle $C(t,u,u')$ depends continuously on t. By relations (3.29) and (3.30), it never passes through e or e'. Thus every such circle $C(t,u,u')$ also separates e and e'. Hence it meets M_e an odd number of times. The beginning of this sub-section implies that it meets M_2 less than five times. Hence it meets M_e exactly three times. Thus any circle through two points of N_o and a point of N_1' meets $N_o \cup p \cup N_1'$ nowhere else.

Combining the above results with Lemma 3.19, we see that $N_o \cup p \cup N_1'$ has order three. This completes the proof of Theorem 3.5.

<u>3.12.</u> <u>Remark.</u>

Let $p \in A$ decompose A into two arcs of order three. Then

(i) p satisfies Condition I' if and only if p satisfies Condition I and $a_o = 1$.

(ii) A is strongly differentiable at p if and only if p is differentiable and $a_o = a_1 = 1$.

<u>Proof</u>: (i) Let p satisfy Condition I'. Then p satisfies Condition I, and a_o is defined. If $a_o = 2$, every non-tangent circle through p supports A at p. Thus there

are sequences of circles through two points of A converging to p, whose limit circles are not tangent circles. Since this contradicts Condition I', a_o must be 1.

Let p satisfy Condition I and suppose $a_o = 1$. Any circle which converges to a non-tangent circle through p meets a small neighbourhood $M = N \cup p \cup N'$ with an odd multiplicity, and does not meet $N \cup p$ or $N' \cup p$ more than once each. Thus it meets M exactly once. Hence any limit circle of a sequence through two points of A converging to p is a tangent circle of p. Thus A satisfies Condition I' at p.

(ii) Suppose A is strongly differentiable at p. Then A is also differentiable at p. By (i), $a_o = 1$. If $a_1 = 2$, section 3.3.2 implies that there are circles which meet an arbitrarily small neighbourhood of p three times, and which converge to a non-osculating tangent circle. Since every circle through three points converging to p converges to C(p), a_1 must be 1.

Next, suppose A is differentiable at p and $a_0 = a_1 = 1$. From (i) A satisfies Condition I'. Thus we must show that any circle through three points of A converging to p converges to C(p).

If $a_2 = 1$, section 3.11.8 implies that there is a small neighbourhood of p which is met at most three times by any circle. Thus the limit circle of a sequence through three points of A converging to p is an intersecting tangent circle, and is therefore C(p).

If $a_2 = 2$, p has the characteristic (1,1,2;i), where $i = 1$ or 2. Let $M_2 = N_2 \cup p \cup N_2'$ be so small that no circle meets M_2 more than four times (cf. § 3.11.6). Let $M_1 \subset M_2$ be so small that if $e \in N_1$, $C(e;\tau)$ meets N_2'. Now choose $M_0 \subset M_1$ so small that C(e,t,t') meets[1] $N_2' - N_1'$.

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1. Given two sets of elements, X and Y, where $Y \subset X$, the set X-Y is made up of all the elements of X except those that are in the set Y.

Thus $C(e,t,t')$ does not meet M_o outside t and t'.

$C(t,t',p)$ is close to a tangent circle of p, and meets M_1 with an even multiplicity. Thus $C(t,t',p)$ meets $N_1 \cup p$ or $N_1' \cup p$ three times, and hence $C(t,t',p)$ converges to $C(p)$ with t and t'.

Let D be any circle through four points $t,u, \in N_o$, $t',u', \in N_o'$ converging to p. Now u and $u' \subset \underline{C}(e,t,t')$. Since u and $u' \not\subset C(t,t',p)$, at least one (and hence both) of the points u and u' lies in the region,

$$\underline{C}(e,t,t') \cap \overline{C}(t,t',p)$$

Thus

$$C(u,t,t') \subset [\underline{C}(e,t,t') \cap \overline{C}(t,t',p)] \cup [\overline{C}(e,t,t') \cap \underline{C}(t,t',p)] \cup t \cup t'$$

as $t,t' \rightarrow p$, any limit circle of $C(u,t,t')$ will be a tangent circle of p.

$$C_o = \lim C(u,t,t')$$

$$\subset [\underline{C}(e;\tau) \cap \overline{C}(p)] \cup [\overline{C}(e;\tau) \cap \underline{C}(p)] \cup C(e;\tau) \cup C(p)$$

$$= [\underline{C}(e;\tau) \cap \overline{C}(p)] \cup C(e;\tau) \cup C(p).$$

Since D cannot meet M_2 more than four times, C_o cannot intersect A outside p. Hence

$$C_o \not\subset \left[\underline{C}(e;\tau) \cap \overline{C}(p)\right] \cup C(e;\tau),$$

and therefore $C_o = C(p)$.

CHAPTER IV

VERTICES OF CLOSED CURVES IN THE CONFORMAL PLANE

4.1. Introduction.

A closed curve in the conformal plane is one for which the two points whose parameters are end-points of the parameter interval, coincide.

One of the reasons for the subsequent investigation is to obtain a strictly conformal proof of the Four Vertex Theorem.[1] This goal has not yet been reached; the purpose of this chapter is only to indicate some of the steps likely to lead to a proof of this theorem.

4.2. τ-vertices.

Let p be a differentiable point of a closed curve

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1. For a statement and proof of this theorem, see Blaschke, "Vorlesungen Über Differential Geometrie", Dover 1945, page 31.

A, and let p have the characteristic $(a_o, a_1, a_2; i)$. We shall assume that p is not a multiple point of A. Let τ be the pencil of tangent circles of A at p. Suppose that A has finite τ-order, i.e, every circle of τ meets A in a finite number of points. We call a point $u \neq p$ a <u>τ-vertex</u> if $C(u; \tau)$ supports A at u. We call p a <u>τ-vertex</u> if every tangent circle of p supports A at p.[1]

<u>4.3</u> <u>Preliminary Material.</u>

The following remarks will be useful in our discussion.

<u>4.3.1</u>. Suppose that a circle C meets A in a finite number of points, and that C intersects A at u. Then the end-points e and f of a suitable neighbourhood M of t on A lie in opposite regions with respect to the circle C. Hence the complement M' of M in the arc A has its end-points e and f in opposite regions with respect to the circle C. But C meets A in only a finite number of points, and therefore C must intersect M' in an odd number of points. Thus

---

1. In this case, $a_2 = 2$. (cf. §2.6)

every circle C intersects A an even number of times.

4.3.2. Suppose the tangent circle C supports A at $t \neq p$. Then $C \neq p$, and there is a neighbourhood M of t on A whose closure lies in $\underline{C} \cup t$, say. In particular, the end-points of M will lie on the same side of C. Let C' be a tangent circle in $\underline{C} \cup p$, and let it be close enough to C that these end-points will still lie oh the same side of C' (cf. Fig 4.1).

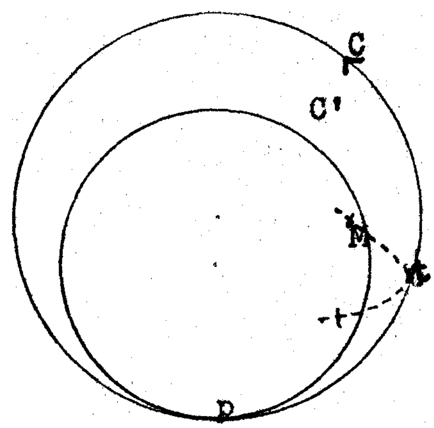

Fig 4.1

Since C' separates them from t, C' will intersect M in two points. On the other hand, a tangent circle in $\overline{C} \cup p$ will not meet M.

4.3.3. If all the tangent circles support A at p, then $a_2 = 2$, and for every circle $C \in \mathcal{T}$ there is a neighbourhood

M of p and two one-sided neighbourhoods of C in τ such that each circle of one of them intersects M twice, while each circle of the other one does not meet M outside p.

4.3.4. If the circle C of τ intersects A at t, there is a neighbourhood M of t on A whose end-points lie on opposite sides of C. Let C' be a tangent circle close enough to C that these end-points are separated by C'. Then C' intersects M in at least one point.

4.4. Curves With a Finite Number of τ-vertices.

Theorem 4.1. If the number of τ-vertices of A is finite, it is even.

Proof: Let $C_o \neq p$ be an arbitrary circle orthogonal to τ. (Thus $p \subset C_o$). If $t \neq p$, then $C(t;\tau)$ intersects C_o at exactly one point $P(t) \neq p$. If $t = p$, then define $C(p;\tau)$ to be $C(p)$. If $C(p) = p$, define $P(p)$ to be p; if $C(p) \neq p$, define $P(p)$ to be the intersection $\neq p$ of C_o with $C(p)$. Then $C(t;\tau)$ and $P(t)$ depend continuously on t over the whole of A.

If t runs through A, then the point P(t) changes its direction if and only if t passes through a τ-vertex. This follows for $t \neq p$ from the definition (§ 4.2) of τ-vertices and from section 4.3.2; for $t = p$, it follows from section 4.2 and section 4.3.3. Thus P(t) changes its sense only a finite number of times. The mapping P(t) of A on C_o being periodical, this number must be even.

Theorem 4.2. A has no τ-vertices if and only if every proper circle of τ meets A exactly onee outside p.

Proof: Suppose that every proper circle of τ meets A exactly once outside p. If $C \in \tau$ supports A at a point $u \neq p$, then there exist circles of τ close to C which intersect A in at least two points close to u. Hence there are no τ-vertices at points $u \neq p$. Since C intersects A exactly once outside p, it must intersect A at p. Thus p itself is not a τ-vertex.

On the other hand, suppose that A has no τ-vertices. If the point t runs through A as in the proof of Theorem 4.1, the point $P(t)$ does not change its direction on C_o. Hence $P(t)$ makes at least one complete circuit of C_o; in particular, $P(t)$ passes through p. This happens only when $t = p$ and $C(p) = p$. Thus $P(t)$ makes exactly one circuit of C_o, and hence every proper circle of τ is met by A exactly once outside p.

Corollary 1. If A has no τ-vertices, then $C(p) = p$.

Corollary 2. If every proper circle of τ meets A exactly once outside p, then $C(p) = p$.

Corollary 3. If a circle C of τ ($C \neq p$ if $C(p) = p$) does not meet A outside p, then A has at least two τ-vertices.

Corollary 4. If a circle C of τ ($C \neq p$ if $C(p) = p$) supports A at p, then A has at least two τ-vertices.

Proof: If C does not meet A outside p, then A has at least two τ-vertices. If C intersects A outside p, it intersects A once more (cf. § 4.3.1), and again A has at least two τ-vertices. If C supports A outside p, A has one, and hence at least two, τ-vertices.

4.4.1. By definition, A has τ-order n if no circle meets A outside p in more than n points, and some circle meets A n times outside p.

Suppose that A has τ-order n, and let C be a circle of τ which meets A in n points $\neq$ p. If $C \cap (A-p)$ is composed of m poihts of support which have neighbourhoods lying in $\overline{C}$, k points of support which have neighbourhoods lying in $\underline{C}$, and r points of intersection, then $m+k+r = n$. Consider two circles, C' and C", of τ. If $C' \subset \overline{C} \cup p$ is sufficiently close to C, it meets A in exactly 2m+r points (cf.§4.3.2), while if $C'' \subset \underline{C} \cup p$ is sufficiently close to C, it meets A in exactly 2k+r poihts. Since A has τ-order n, we see that

$$2m+r \leq n = m+k+r$$

and

$$2k+r \leq n = m+k+r,$$

whence

$$m \leq k \leq m.$$

Thus $m = k$. Since $2m+r = 2k+r = n$, we see in addition that $r \equiv n \pmod 2$.

4.4.2. If A has τ-order 2n+1, then a circle $C \in \tau$, which meets A in 2n+1 points outside p must intersect A at p (cf. § 4.4.1 and § 4.3.1). Again by section 4.4.1, there is a circle C' of τ sufficiently close to C which intersects A in exactly 2n+1 points outside p. Hence the non-osculating circles of τ intersect A at p and therefore C(p) = p (cf. § 2.6). These remarks enable us to extend Theorem 4.2, Corollary 2 to

Corollary 5. *If A has τ-order 2n+1, then C(p) = p, and the non-osculating circles of τ intersect A at p.*

4.4.3. Theorem 4.3. Suppose that a tangent circle, C of A at p meets A in exactly n points $\neq$ p. Then A has at least n-1 τ-vertices. If, in addition,[1] the non-osculating tangent circles support A at p, then A has at least n τ-vertices.[2]

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1. This condition is, of course, automatically satisfied if C(p) $\neq$ p.

2. Sections 4.4.1 and 4.3.1, together with Theorem 4.2 Corollary 5, imply the (cont'd on Page 117 (bottom))

Proof: The points $\neq p$ of $A \cap C$ decompose A into n closed arcs B such that no interior point $\neq p$ of an arc B lies on C. Let B_o denote that arc B which contains p. It is sufficient to prove

(i) Each $B \neq B_o$ contains at least one interior τ-vertex;

and, under our additional assumption,

(ii) B_o also contains an interior τ-vertex.

For each B we define the subset $\nu = \nu(B)$ of τ as follows: if $B \neq B_o$, then ν shall be the set of those circles of τ that meet B; if $B = B_o$, then ν is the union of C(p) with the set of all the tangent circles which meet B outside p.

In either case, ν will be a connected, closed subset of τ. If $B \neq B_o$, or if $B = B_o$ and $C(p) \neq p$, then ν does

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following remark: If n is positive and even, then the number of intersections of C with A outside p is even. Hence C supports A at p and the additional assumption is automatically satisfied. On the other hand, if n is odd, this condition cannot hold.

not contain the point-circle p. If $B = B_o$ and $C(p) = p$, then, from our additional assumption, all the tangent circles support,[1] and some tangent circles near p will not belong to ν. Hence ν is a proper subset (i.e. a closed subinterval) of τ. At least one of the end-circles of ν, say C', is different from C. Thus $C' \cap B$ does not contain the end-points of B. Since $C' \in \nu$, this circle actually has at least one point in common with B. If C' intersects B outside p, every circle of τ close to C' also intersects B. Thus any point $\neq p$ of $C' \cap B$ is a point of support, i.e., a τ-vertex. Suppose that $C' \cap B = p$. Then $B = B_o$. In this case, $\nu = C(p) \cup C(t;\tau)$, where $t \in B_o$, $t \neq p$, and since $C' \in \nu$, it follows that $C' = C(p)$. Hence $C(p) = C'$ supports B_o at p. By our additional assumption, p is a τ-vertex.

We can write the proof of Theorem 4.3 in a different

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1. By definition, p is a τ-vertex in this case.

way, using the orthogonal circle C_o of Theorem 4.1.

The circle $C \in \mathcal{T}$, which meets A in n points outside p, again divides A into n closed sub-arcs B; B_o is that sub-arc B which contains p.

If t moves through $B \neq B_o$, P(t) moves on C_o, and returns to its initial position without passing through p. Thus P(t) must reverse its direction on C_o, and B must therefore have an interior $\mathcal{T}$-vertex. Since $P(t) \neq p$ when $C(p) \neq p$, this even holds true when $B = B_o$, provided $C(p) \neq p$. If $C(p) = p$, p is a $\mathcal{T}$-vertex by definition provided the other circles of $\mathcal{T}$ support A at p.

If, in this theorem, $C \cap A$ contains m points of support different from p, our proof shows that we have at least $m + n - 1$ $\mathcal{T}$-vertices, and at least $m + n$ under the additional assumption.

From Theorem 4.1, we obtain

<u>Corollary 1.</u> <u>If a circle of $\mathcal{T}$ meets A in 2n points</u>

different from p, then A has at least 2n τ-vertices.

Theorem 4.4. Suppose that there is a circle of τ (different from p if $C(p) = p$) which meets A only at p. Then A has exactly two τ-vertices if and only if no circle of τ meets A in more than two points different from p.

Proof: Our assumption implies that the non-osculating circles of τ support A at p (cf. § 4.3.1 and Theorem 2.8). Suppose there exists a circle C of τ which meets A at more than two points $\neq$ p. By Theorem 4.3, A has at least three τ-vertices.

Now suppose that no circle of τ meets A in more than two points $\neq$ p. Let ν be the closed interval consisting of C(p) and all those circles of τ which meet A outside p. By our assumptions, ν is a closed, connected, proper sub-interval of τ. As t moves over A, P(t) moves over a proper sub-arc of C_o (cf. Theorem 4.1), and returns to its starting point. Hence every interior circle of ν meets A at least twice outside p. If A had another τ-vertex, P(t) would cover an arc of C_o at least three times, and then some circle of τ would meet A at least three times. Thus A has exactly two τ-vertices, which belong to the end-circles of ν, where P(t) reverses its direction.

Theorem 4.5. Suppose that there is a circle of τ (different from p if C(p) = p) which supports A at p. If A has exactly four τ-vertices, then A has τ-order four.

Proof: Our first assumption implies that the non-osculating circles of τ support, and hence there exist circles of τ which do not meet A outside p.

If A has exactly four τ-vertices, Theorem 4.3 implies that the τ-order of A does not exceed 4. By Theorem 4.4, the τ-order is at least 3. Section 4.4.2 implies that A has τ-order 4.

4.4.4. The following are examples of curves with no τ-vertices, two τ-verteces, and four τ-vertices respectively. The first three examples refer to τ-vertices relative to the origin as fundamental point of the pencil τ.

(a) $x = t^3$, $y = t^5$. The non-osculating tangent circles at the origin intersect. They therefore intersect at their only other point of contact with the arc, and we have no τ-vertices.

(b) $x = t$, $y = t^2$. The non-osculating tangent circles at the origin all support. The x-axis meets the arc only at the origin and at ∞, and since it supports at the origin, it must also support at ∞. Thus we have a τ-vertex at ∞. The osculating circle also supports at the origin. Hence we have a τ-vertex at the origin. The non-osculating tangent circles which do not go through ∞ intersect the arc in two points outside p and do not meet the arc elsewhere except at the origin. Thus we have only two τ-vertices.

(c) $r = a \tan \theta \sec \theta$, $0 \leq \theta < \pi/4$, $3\pi/4 \leq \theta < \pi$;

$r = a \operatorname{cosec} \theta$, $\pi/4 \leq \theta < 3\pi/4$.

All the circles of τ support at the origin, and hence give rise to a τ-vertex there. The circle $r = \sqrt{2}a \sin \theta$ of τ supports the arc at $r = \sqrt{2}a$, $\theta = \pi/4$, and at $r = \sqrt{2}a$, $\theta = 3\pi/4$. Our final τ-vertex is found at $r = a$, $\theta = \pi/2$, for the circle $r = a \sin \theta$ supports the arc there.

(d) This example again illustrares an arc with no τ-vertices. Its particular interest lies in the fact that the same arc was used as an example of an arc having two τ-vertices, where the fundamental point of τ was the origin.

Let A be the parabola $x = t$, $y = t^2$. The tangent circles of A at $t = \infty$ are straight lines parallel to the y-axis. Each of them intersects the parabola exactly once for a finite value of t, and therefore must intersect A again at $t = \infty$.

4.5. π-vertices.

Let p and s be differentiable points of a closed curve A, and let p (s) have the characteristic $(a_0, a_1, a_2; i)$.

We shall assume that p and s are not multiple points of A. Let Π be the pencil of circles through the fundamental points p and s. Suppose that A has finite Π-order, i.e., every circles of Π meets A in a finite number of points. We call the point $t \neq p,s$, a <u>Π-vertex</u> if $C(t;\Pi)$ supports A at t. We call p (s) a <u>Π-vertex</u> if the non-tangent circles of Π at p (s) support A at p (s) when $C(s;\tau_p)$ $\big(C(p;\tau_s)\big)$ supports, or intersect A at p (s) when $C(s;\tau_p)$ $\big(C(p;\tau_s)\big)$ intersects[1].

<u>4.6. Remarks Useful in the Development of the Theory of Π-vertices.</u>

<u>4.6.1.</u> Suppose that the circle $C(t;\Pi)$ supports A at $t \neq p,s$. Then there is a neighbourhood M of t on A whose closure lies in $\underline{C}(t;\Pi) \cup t$, say. In particular, the end-points of M will lie in $\underline{C}(t;\Pi)$. Let C' lie in the region

$$\left[\underline{C}(t;\Pi) \cap \overline{C}\right] \cup \left[\overline{C}(t;\Pi) \cap \underline{C}\right] \cup p \cup s,$$

---

1. The symbol $C(R;\tau_q)$ means "the tangent circle of A at q through R", for all points $q \in A$, and $R \neq q$.

say, where C is any fixed circle of Π such that

$$M \subset [\underline{C}(t;\Pi) \cap \bar{C}] \cup t.$$

Suppose that C' is close enough to C(t;Π) that the end-points of M will lie on the same side of C'. Since C' separates them from t, C' will intersect M in two points. On the other hand, a circle of Π in the region

$$[\underline{C}(t;\Pi) \cap \underline{C}] \cup [\bar{C}(t;\Pi) \cap \bar{C}] \cup p \cup s$$

will not meet M.

4.6.2. If the circle C of Π intersects A at t, there is a neighbourhood M of t on A whose end-points lie on opposite sides of C. Let C' be a circle of Π close enough to C that these end-points are separated by C'. Then C' intersects M in at least one point.

4.6.3. Suppose that $C(s;\mathcal{T}_p)$ supports (intersects) A at p. Then the end-points of a sufficiently small neighbourhood $M = B_1 \cup p \cup B_2$ of p on A will lie in the same region (in different regions) with respect to $C(s;\mathcal{T}_p)$.

Suppose that the non-tangent circles through p support (intersect) A at p. Let C' be the orthogonal circle to $C(s;\tau_p)$ through p and s (cf. Fig. 4.2). Since C' sup-

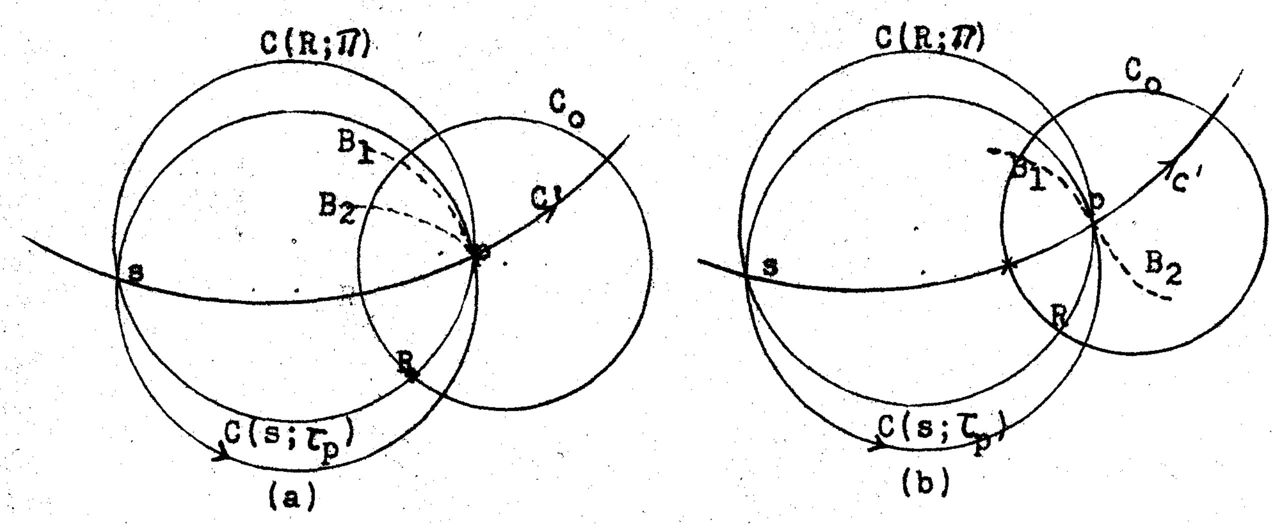


Fig. 4.2

ports (intersects) A at p, B_1 and B_2 must lie in the same region (in different regions) with respect to C'. We lose no generality in assuming that $B_1 \cup B_2$ lies in the region

$$\underline{C}' \cap \underline{C}(s;\tau_p) \qquad \left(\left[\underline{C}' \cap \underline{C}(s;\tau_p)\right] \cup \left[\bar{C}' \cap \bar{C}(s;\tau_p)\right]\right).$$

Now let C_o be any circle which is orthogonal to the family Π, and let R be any point of C_o which lies in the region $\bar{C}' \cap \underline{C}(s;\tau_p)$. Thus

$$C(R;\pi) \subset [\bar{C}' \cap \underline{C}(s;\tau_p)] \cup [\underline{C}' \cap \bar{C}(s;\tau_p)] \cup p \cup s,$$

and therefore C(R;) meets M only at p. Hence this circle (which can be as close to $C(s;\tau_p)$ as we please) does not meet M outside p.

On the other hand, if the non-tangent circles of π at p intersect (support) A at p, then $B_1 \cup B_2$ lies in the region

$$[\underline{C}' \cap \underline{C}(s;\tau_p)] \cup [\bar{C}' \cap \underline{C}(s;\tau_p)] \ \left([\underline{C}' \cap \underline{C}(s;\tau_p)] \cup [\underline{C}' \cap \bar{C}(s;\ _p)]\right),$$

say. Let $t_i \in B_i$ $(i = 1,2)$. Then

$$C(t_1;\pi) \subset [\underline{C}' \cap \underline{C}(s;\tau_p)] \cup [\bar{C}' \cap \bar{C}(s;\tau_p)] \cup p \cup s,$$

say, while

$$C(t_2;\pi) \subset [\bar{C}' \cap \underline{C}(s;\tau_p)] \cup [\underline{C}' \cap \bar{C}(s;\tau_p)] \cup p \cup s.$$

Hence all circles of π close to $C(s;\tau_p)$ meet M at least once outside p.

Obviously, the above is also true when we interchange the roles of p and s.

4.7. Curves with a Finite Number of π-vertices.

Theorem 4.6. If the number of Π-vertices of A is finite, it is even.

Proof: Let $C_o \neq p,s$, be an arbitrary circle orthogonal to Π. $C(t;\Pi)$ intersects C_o in exactly two points, $P(t)$, and $P'(t)$. Thus $P(t), P'(t)$ and $C(t;\Pi)$ depend continuously on t over the whole of A.

If t runs through A, then the points $P(t)$ and $P'(t)$ change their direction if and only if t passes through a Π-vertex. This follows for $t \neq p,s$ from the definition (§ 4.5) of Π-vertices, and sections 4.6.1 and 4.6.2; for $t = p$ or s, it follows from section 4.5 and section 4.6.3. Thus $P(t)$ and $P'(t)$ change their direction only a finite number of times. Since the direction of motion of $P(t)$ and $P'(t)$ on C_o must be the same when a circuit of A is completed as when it began, the number of changes of direction must be even.

Theorem 4.7. If every circle of Π except $C(s;\tau_p)$

and $C(p;\tau_s)$ meets A exactly once outside p and s, then

(i) A has no π-vertices outside p and s,

(ii) $C(s;\tau_p)$ and $C(p;\tau_s)$ do not meet A outside p and s,

(iii) A has no π-vertices,

(iv) $C(s;\tau_p) \neq C(p;\tau_s)$.

Proof: (i) Suppose that $C(u;\pi)$ supports A at a point $u \neq p,s$. Then there exist circles of π close to $C(u;\pi)$ which intersect A in at least two points close to u, contrary to our assumption. Hence there are no π-vertices at points $u \neq p,s$.

(ii) If $C(s,\tau_p)\big(C(p,\tau_s)\big)$ meets A at a point $u \neq p,s$, then by (i) and section 4.6.2, there are circles of π close to $C(s,\tau_p)\big(C(p,\tau_s)\big)$ which meet A in at least two points $\neq$ p and s, one being near u, and the other near p (s). Hence $C(s;\tau_p)$ and $C(p;\tau_s)$ do not meet A outside p and s.

(iii) In view of (i), we need only consider the points p and s. If $C(s;\tau_p)$ supports (intersects) A at p, then by (ii) and section 4.3.1, it supports (intersects) A at s. We have a π-vertex at p if and only if the non-tangent circles at p also support (intersect) A at p. If this is the case, $C(u;\pi)$, $u \neq p,s$, will intersect A at u by (i); by section 4.3.1 and our assumption it will intersect (support) A at s. But since $C(p,\tau_s)$ supports (intersects) A at s, s is not a π-vertex. Thus we have only one π-vertex on A, which contradicts Theorem 4.6. Hence A has no π-vertices.

(iv) If $C(s;\tau_p)=C(p;\tau_s)=C$, then a circle C' of π close to C will meet A at least twice outside p and s, once near p, and once near s. This follows from (iii). Hence $C(s;\tau_p) \neq C(p;\tau_s)$.

4.8. The Relation Between π-vertices and τ-vertices.

If we allow the point $s \in A$ approach $p \in A$ along A, then the pencil π through p and s becomes the pencil τ through p. We obtained a π-vertex at p if the non-tangent circles of Π supported (intersected) A at p when $C(s;\tau_p)$ supported (intersected) A at p; we now get a τ-vertex at p if the non-osculating circles of τ support A at p when C(p) supports A at p (it is impossible for all the circles of τ to intersect A at p). A Π-vertex at a point $u \in A$, $u \neq p$ when p and s coincide is simply a τ-vertex at that point. Therefore, if appropriate minor changes are made, any theorem that is true for Π-vertices is also true for τ-vertices. The converse of this statement is not true; the study of Π-vertices is more complex, as can be seen from the small part of that theory which has been presented here.

CHAPTER V

DIFFERENTIABLE POINTS OF ARCS IN CONFORMAL 3-SPACE

5.1. Introduction.

We now begin an investigation in conformal 3-space which parallels the work in two dimensions. The change from two to three dimensions is of considerable note, chiefly because of the fact that instead of dealing with one continuous entity (the circle) and the discrete point-pair, we now must consider two continuous entities, the circle and the sphere.

5.2. Differentiability.

Let p be a fixed point of an arc A, and let t be a variable point of A. If P, Q, and p are mutually distinct points, the unique circle through these points will be denoted by $C(P,Q;\gamma_0)$. The symbol γ_0 itself will denote the family of all circles through p, including the point-circle p.

A is called <u>once-differentiable</u> at p if the following condition Γ_1 is satisfied:

Γ_1: If the parameter t is sufficiently close to, but different from, the parameter p, the circle $C(P,t;\gamma_o)$ is uniquely defined, and converges if t tends to p.

Thus the limit circle, which will be denoted by $C(P;\gamma_1)$, is independent of the way t converges to p. The family of all such circles, together with the point-circle p, will be denoted by the symbol γ_1.

A is called <u>twice-differentiable</u> at p if, in addition to the condition Γ_1, the following condition is also satisfied:

Γ_2: If the parameter t is sufficiently close to, but different from, the parameter p, the circle $C(t;\gamma_1)$ is uniquely defined, and converges if t tends to p.

The limit circle of the sequence $C(t;\ _1)$ will be denoted by $C(\gamma_2)$ the <u>osculating circle</u> of A at p, and occasionally by the symbol γ_2 alone.

5.3. Structure of the Families of Circles Through p.

In this section, relations among the families of circles γ_o, γ_1, γ_2, are discussed.

<u>Theorem 5.1. Suppose A satisfies condition Γ_1 at p. Then t does not coincide with p if the parameter t is sufficiently close to, but different from, the parameter p.</u>

<u>Proof</u>: Let P be any point different from p. By condition Γ_1, $C(P,t;\gamma_o)$ is defined when the parameter t is close to, but different from, the parameter p. Thus $t \neq p$.

<u>Theorem 5.2. Suppose that A satisfies condition Γ_1 at p. Then the angle at p between any two circles of γ_1 is 0.</u>

<u>Proof</u>: Let P, Q, R_1, R_2, be variable points, and let R_1 and R_2 converge to the same point R. Suppose there is a sphere separating R from both P and Q. Then

$$(5.1) \qquad \lim \sphericalangle\left[C(P,R_1,R_2);C(Q,R_1,R_2)\right] = 0,$$

whether or not the circles themselves converge.[1] In particular, the angle between $C(P;\gamma_1)$ and $C(Q;\gamma_1)$ is equal to 0.

Corollary 1. *If $C(P;\gamma_1)$ and $C(Q;\gamma_1)$ have another point in common, they are identical; thus there is one and only one circle of γ_1 through each point $P \neq p$.*

Corollary 2. *γ_1 consists of those circles C which meet a given circle of γ_1 at p at the angle 0.*

Proof: Let $P \subset C$, $P \neq p$. Suppose that C meets some circle of γ_1 at angle 0 at p. Then C and $C(P;\gamma_1)$ also meet at angle 0 at p and have the point P in common. Hence they are identical.

Corollary 3. *If Γ_1 holds for a single point $P \neq p$, then it holds for all such points.*

Proof: If $Q \neq p$, by relation (3.1),

---

1. This becomes obvious if we let P or Q be the fixed point at infinity.

$$\lim \measuredangle\left[C(Q,t;\gamma_o);C(P,t;\gamma_o)\right] = 0.$$

Hence $C(Q,t;\gamma_o)$ converges to the unique circle through Q which touches $C(P;\gamma_1)$ at p.

Theorem 5.3. Suppose A satisfies the conditions Γ_1 and Γ_2 at p. Then

$$\gamma_o \supset \gamma_1 \supset \gamma_2.$$

Proof: It is clear that $\gamma_o \supset \gamma_1$. If $C(\gamma_2) = p$, it belongs to γ_1 by definition. Suppose $C(\gamma_2) \neq p$. Then $C(\gamma_2)$, being the limit of a sequence of circles $C(t;\gamma_1)$ each of which touches a given circle $C(P;\gamma_1)$ of γ_1, must itself touch $C(P;\gamma_1)$ at p. Thus $C(\gamma_2) \in \gamma_1$.

Corollary 1. If $P \subset C(\gamma_2), P \neq p$, then $C(\gamma_2) = C(P;\gamma_1)$.

The conditions Γ_1 and Γ_2 are independent. Consider for example, the arc

$$x = t,\ y = t^2,\ z = \begin{cases} (1-\sqrt{1-t^2-t^4})\sin t^{-1}, & 0<|t|\leq\frac{1}{2}. \\ 0 & , t = 0 \end{cases}$$

Considering the vector $\vec{t} = x\vec{i}+y\vec{j}+z\vec{k}$, we let θ be the angle between t and the x-axis. The vector t represents the circle of γ_o through the point at infinity and the point t.

As $t \to 0$,

$$\cos \theta = \frac{\vec{i}.\vec{t}}{|\vec{i}||\vec{t}|} = \frac{t}{\sqrt{t^2+t^4+[t^2/2(1+t^2)+o(t^3)]^2\sin^2 t^{-1}}}$$

$$= \frac{1}{\sqrt{1+t^2+o(t)}}$$

$$\to 1 \qquad , \qquad \text{as } t \to 0.$$

Thus Condition Γ_1 holds at $t=0$ for the point ∞, and therefore by Theorem 5.2 Corollary 3, it holds for all points $P \neq p$.

However, condition Γ_2 is not satisfied at $t=0$. The plane through the x-axis (which by the above $\in \gamma_1$) and the point $x(t),y(t),z(t)$ contains circles which pass through t and which touch the x-axis; i.e., it contains the circles $C(t;\gamma_1)$. This is also true of the sphere through t which touches the xy-plane. Thus $C(t;\gamma_1)$ is the intersection of the former plane and the sphere. But as $t \to 0$, neither the sphere nor the plane, nor the intersection of the sphere and the plane, converges. (The method of determining this is similar to that used in §2.4.) Hence Γ_2 does not hold.

§.4. Differentiable Points of Arcs

In addition to the conditions Γ_1 and Γ_2, three more conditions, involving spheres, are introduced. Suppose P,Q, and R are any three fixed points such that P,Q,R, and p do not all lie on the same circle. It will be convenient to denote the unique sphere through p and the points P,Q, and R, by the symbol $S(P,Q,R;\sigma_0)$. σ_Φ will denote the family of all spheres through p, including the point-sphere p.

A is called thrice-differentiable at p if the following three conditions are satisfied:

Σ_1: If the parameter t is sufficiently close to, but different from, the parameter p, the sphere $S(P,Q,t;\sigma_0)$ is uniquely defined, and converges as $t \to p$ to a limit sphere which will be denoted by $S(P,Q;\sigma_1)$.

Σ_2: If the parameter t is sufficiently close to, but different from, the parameter p, the sphere $S(P,t;\sigma_1)$ is uniquely defined, and converges as $t \to p$ to a limit sphere which will be denoted by $S(P;\sigma_2)$.

Σ_3: If the parameter t is sufficiently close to, but different from, the parameter p, the sphere $S(t;\sigma_2)$ is uniquely defined, and converges as $t \to p$ to a limit sphere which will be denoted by $S(\sigma_3)$.

The family of all the spheres $S(P,Q;\sigma_1)$, together with the point sphere p, will be denoted by the symbol σ_1. The family of all the spheres $S(P;\sigma_2)$ will be denoted by the symbol σ_2; if $C(\gamma_2)=p$, this family will also include the point-sphere, p. The members of σ_1 and σ_2 will sometimes be called <u>singly tangent</u> (or <u>1-tangent</u>) and <u>doubly tangent</u> (or <u>2-tangent</u>) <u>spheres</u>, respectively. The unique <u>osculating sphere</u>, $S(\sigma_3)$ will occasionally be denoted by the symbol σ_3 alone.

The point p is called a <u>differentiable point</u> of A if A is thrice-differentiable at p.

5.5 Structure of the Families of Spheres Through p.

Although the conditions Γ_1 and Γ_2 are independent, not all the conditions $\Gamma_1, \Gamma_2, \Sigma_1, \Sigma_2$ and Σ_3 are independent. In addition, the families of spheres $\sigma_0, \sigma_1, \sigma_2$, and σ_3 are closely connected with the families of circles γ_0, γ_1, and γ_2.

Theorem 5.4 Suppose A satisfies condition Σ_1 at p. Let C be any circle. Then $t \not\subset C$ if the parameter t is sufficiently close to, but different from the parameter p.

Proof: The assertion is clearly true if $p \not\subset C$. Suppose $p \subset C$, and let P,Q,p be mutually distinct points on C. By condition Σ_1, $S(P,Q,t;\sigma_0)$ is defined when t is sufficiently close to p. Thus $t \not\subset C(P,Q,p) = C$.

The following example shows that Γ_1 does not imply Σ_1 in general (cf., however, Theorem 5.5). Consider the arc

$$x = t,\ y = \begin{cases} t^2\cos t^{-1}, & 0<|t|\leq 1 \\ 0, & t=0 \end{cases},\ z = \begin{cases} t^2\sin t^{-1}, & 0<|t|\leq 1 \\ 0, & t=0 \end{cases},$$

in the neighbourhood of $t = 0$. If $P=\infty$, $Q=(1,0,0)$, and $p=(0,0,0)$, the sphere $S(P,Q,t;\sigma_0)$ does not converge, while for example, $C(P,t;\gamma_0)$ converges to the x-axis; by Theorem 5.2, Corollary 3, Γ_1 is satisfied.

<u>Theorem 5.5.</u> <u>If A satisfies Σ_1 at p, then Γ_1 holds there, and</u>

$$C(Q;\gamma_1) = \prod_P{}^1 S(P,Q;\sigma_1). \tag{5.2}$$

<u>Conversely, let A satisfy Γ_1 at p. Then Σ_1 holds at p for all pairs P,Q, such that $P \not\subset C(Q;\gamma_1)$; then $S(P,Q;\sigma_1) = S[P;C(Q;\gamma_1)]$.</u>

<u>Proof:</u> Suppose that Σ_1 holds at p. If $Q \neq p$,

$$\lim_{t\to p} C(Q,t;\gamma_0) = \lim_{t\to p} \prod_P S(P,Q,t;\sigma_0)$$

$$= \prod_P S(P,Q;\sigma_1).$$

---

1. Given a family, Π, of spheres (or m-spheres in higher dimensions), by the symbol $\prod_P S(P;\Pi)$ we mean the common intersection of all the spheres belonging to Π.

Hence $C(Q,t;\gamma_o)$ converges, and

$$C(Q;\gamma_1) = \prod_P S(P,Q;\sigma_1).$$

Conversely, suppose that Γ_1 holds. If $P \not\subset C(Q;\gamma_1)$, then $P \not\subset C(Q,t;\gamma_o)$ when t is sufficiently close to p, and

$$(5.3) \qquad S[P;C(Q;\gamma_1)] = \lim_{t \to p} S[P;C(Q,t;\gamma_o)]$$

$$= \lim_{t \to p} S(P,Q,t;\sigma_o).$$

Thus for all pairs of points P and Q such that $P \not\subset C(Q;\gamma_1)$, $S(P,Q,t;\sigma_o)$ converges, Σ_1 is satisfied, and $S(P,Q;\sigma_1)$ is the sphere through P and $C(Q;\gamma_1)$.

<u>Corollary 1.</u> <u>There is only one sphere of σ_1 which contains two points not on the same circle of γ_1.</u>

<u>Remark</u>: Condition Γ_1 is still satisfied when Σ_1 is replaced by a weaker assumption, namely:

Suppose $S_1' = S(P_1,Q_1,t;\sigma_o) \to S_1$, $S_2' = S(P_2,Q_2,t;\sigma_o) \to S_2$, and suppose further that $S_1 \cap S_2 = C \neq p$. Then Γ_1 holds at p.

<u>Proof</u>: Let $S_1' \cap S_2' = C'$. Then $C' \to C$, and $C' \supset p$ and t. As in relation (5.1),

$$\lim \sphericalangle\left[C(P_1,t;\gamma_0);C'\right] = 0.$$

Thus, $C(P_1,t;\gamma_0)$ converges to the unique circle through P_1 which touches C at p. By Theorem 5.2, Corollary 3, Γ_1 holds at p.

If, however, $S_1 \cap S_2 = p$, Γ_1 need not hold; for example, take $P_1 = \infty$, $Q = (1,0,0)$, $P_2 = (0,0,2)$, $Q_2 = (1,0,1)$, $p = (0,0,0)$, and let A be the arc

$$x = \begin{cases} t \sin t^{-1}, & 0 < |t| \leq 1 \\ 0 & , t = 0 \end{cases}, \quad y = t, \quad z = t^2.$$

S_1' converges to the xy-plane, S_2' converges to the sphere $x^2 + y^2 + z^2 - 2z = 0$, but Γ_1 does not hold.

<u>Theorem 5.6.</u> <u>Suppose that Σ_1 holds at p. Choose $C \in \gamma_1$, $C \neq p$. Then σ_1 is the set of all spheres which touch C at p.</u>

<u>Proof</u>: Suppose that a sphere $S(P,Q;\sigma_1)$ of σ_1 meets C in a point $R \neq p$. If $R \subset C(Q;\gamma_1)$, then by Theorem 5.5 and Theorem 5.2, Corollary 1,

$$S(P,Q;\sigma_1) \supset C(Q;\gamma_1) = C,$$

while if $R \not\subset C(Q;\gamma_1)$,

$$S(P,Q;\sigma_1) = S[R;C(Q;\gamma_1)] = S(R,Q;\sigma_1)$$

$$S[Q;C(R;\gamma_1)] = S(Q;C) \supset C.$$

Conversely, suppose that a sphere S touches C at p. If $S \supset C$, then $S \in \sigma_1$ (Theorem 5.5). If $S \cap C = p$, choose a point $Q \subset S$, $Q \neq p$. Let $C_o = S(Q;C) \cap S$. Then C_o touches C at p. By Theorem 5.2, Corollary 2, $C_o \in \gamma_1$. Since $S \supset C_o$ and $C_o \in \gamma_1$, it follows from Theorem 5.5 that $S \in \sigma_1$.

Theorem 5.7. *If A satisfies Σ_1 and Σ_2 at p, then Γ_1 and Γ_2 will also hold there, and equations (5.2) and*

$$C(\gamma_2) = \prod_P S(P;\sigma_2) \tag{5.4}$$

will be satisfied there. Conversely, let A satisfy Γ_1 and Γ_2 at p, and let $C(\gamma_2) \neq p$. If $P \not\subset C(\gamma_2)$, then Σ_2 will hold at p for P, and $S(P;\sigma_2)$ will be the sphere through P and $C(\gamma_2)$.

Proof: Suppose that Σ_1 and Σ_2 hold at p. In view of Theorem 5.5, we have only to show that Σ_2 implies Γ_2, and that relation (5.4) holds. By relation (5.2),

$$\lim_{t \to p} C(t;\gamma_1) = \lim_{t \to p} \prod_P S(P,t;\sigma_1)$$
$$= \prod_P S(P;\sigma_2).$$

Hence $C(t;\gamma_1)$ converges, and $C(\gamma_2) = \prod_P S(P;\sigma_2)$. Thus Σ_2 implies Γ_2 and relation (5.4) holds.

Conversely, suppose that Γ_1 and Γ_2 hold and that $C(\gamma_2) \neq p$. If $P \not\subset C(\gamma_2)$, then $P \not\subset C(t;\gamma_1)$ when t is sufficiently close to p, and by Theorem 5.5,

$$S\left[P;C(\gamma_2)\right] = \lim_{t \to p} S\left[P;C(t;\gamma_1)\right]$$
$$= \lim_{t \to p} S(P,t;\sigma_1).$$

Hence $S(P,t;\sigma_1)$ exists and converges. Thus $S(P;\sigma_2) = S\left[P;C(\gamma_2)\right]$.

<u>Corollary 1.</u> <u>If A satisfies Σ_1 (Σ_1 and Σ_2) at p, then A is once- (twice-) differentiable there</u>.

In particular, this implies

<u>Corollary 2.</u> <u>If p is a differentiable point of A, then Γ_1 and Γ_2 hold there</u>.

<u>Corollary 3.</u> <u>$S(\sigma_3) \supset C(\gamma_2)$.</u>

Proof: By relation (5.4),

$$S(t;\sigma_2) \supset \prod_P S(P;\sigma_2)$$

$$= C(\ '\ _2).$$

Hence $S(\sigma_3) \supset C(\gamma_2)$.

This implies

Corollary 4. If $S(\sigma_3) = p$, then $C(\gamma_2) = p$.

Corollary 5. If $C(\gamma_2) \neq p$, σ_2 consists of the spheres through $C(\gamma_2)$.

The conditions Γ_1 and Γ_2 by themselves do not imply Σ_1 in general, whether or not $C(\gamma_2) = p$. Consider, for example, the arc

$$x = t,\ y = \begin{cases} t^3 \sin t^{-1}, & 0<|t|\leqslant 1 \\ 0, & t=0 \end{cases},\ z = \begin{cases} t^3 \cos t^{-1}, & 0<|t|\leqslant 1 \\ 0, & t=0 \end{cases},$$

which satisfies Γ_1 and Γ_2 at $t=0$, $C(\gamma_2)$ being the x-axis. When $P=\infty$, $Q = (1,0,0)$, the sphere $S(P,Q,t;\sigma_0)$ is a plane through the x-axis, and this plane does not converge when $t \to 0$. Thus Σ_1 is not satisfied.

Condition Σ_1 is a very strong one, for it implies not only Γ_1, but, as the following theorems show, Σ_2 and Γ_2 as well, and even Σ_3 in the case $C(\gamma_2) \neq p$.

Theorem 5.8. *Suppose that A satisfies Σ_1 at p. Then A also satisfies Σ_2 at p.*

Proof: Let P be any point $\neq p$. Theorem 5.4 implies that t does not lie on $C(P;\gamma_1)$ if t is close to p. Hence by Theorem 5.5, $S(P,t;\sigma_1) = S[t;C(P;\gamma_1)]$. Let $Q \subset C(P;\gamma_1)$, $Q \neq P,p$. Then $C(P;\gamma_1) = C(P,Q;\gamma_0)$. Thus

$$S(P,t;\sigma_1) = S[t;C(P,Q;\gamma_0)]$$

$$= S(P,Q,t;\sigma_0),$$

and Σ_1 now implies that

(5.5) $$\lim_{t \to p} S(P,t;\sigma_1) \quad S(P,Q;\sigma_1).$$

Since $S(P;\sigma_2)$ exists for each point $P \neq p$, Σ_2 is satisfied.

Corollary 1. *If A satisfies Σ_1 at p, it also satisfies Γ_2 there.*

Proof: By Theorem 5.7, condition Σ_2 implies Γ_2.

Corollary 2. *If A satisfies Σ_1 at p, then p is a*

<u>differentiable point of A if and only if $S(t;\sigma_2)$ converges as $t \to p$.</u>

Relation (5.5) implies

<u>Corollary 3.</u> <u>$S(P;\sigma_2) \in \sigma_1$.</u>

<u>Theorem 5.9.</u> <u>Suppose that A satisfies Σ_1 (and hence Σ_2, Γ_1, and Γ_2) at p, and suppose that $C(\gamma_2) \neq p$. Then A also satisfies Σ_3 at p.</u>

<u>Proof</u>: If t is close to, but different from, p, $S(t;\sigma_2)$ is defined. By Theorem 5.4, $t \not\subset C(\gamma_2)$, and by Theorem 5.7, $S(t;\sigma_2) = S[t;C(\gamma_2)]$. Let $P \subset C(\gamma_2)$, $P \neq p$. Then by Theorem 5.3, Corollary 1, $C(\gamma_2) = C(P;\gamma_1)$ and hence

$$S(t;\sigma_2) = S[t;C(P;\gamma_1)]$$

$$= S(P,t;\sigma_1).$$

Condition Σ_2 now implies that

$$\lim_{t\to p} S(t;\sigma_2) = \lim_{t\to p} S(P,t;\sigma_1) \tag{5.6}$$

$$= S(P;\sigma_2).$$

Thus $S(t;\sigma_2)$ converges, and Σ_3 holds.

Corollary 1. If A satisfies condition Σ_1 at p, and if $C(\gamma_2) \neq p$, then p is a differentiable point of A.

The following example shows that p need not be a differentiable point of A when Σ_1 is satisfied and $C(\gamma_2) = p$. Consider the arc defined by

$$x = t^2, \quad y = t^3, \quad z = \begin{cases} t^4 \sin t^{-1}, & 0 < |t| \leq 1 \\ 0, & t = 0 \end{cases}.$$

It can readily be verified that A satisfies Σ_1 at $t = 0$, and that the spheres of σ_2 touch the xy-plane at the origin. Thus $C(\gamma_2)$ is a point circle. However, as $t \to 0$, $S(t;\sigma_2)$ oscillates, and $x^2 + y^2 + z^2 \pm z = 0$ are two accumulation spheres of the sequence $S(t;\sigma_2)$. Thus Σ_3 does not hold at $t = 0$.

Theorem 5.10. Let Σ_1 hold at p, and let $C(\gamma_2) = p$. Then σ_2 is the set of spheres which touch a given proper sphere of σ_2 at p.

Proof: Let P and Q be variable points, and let C be a variable circle converging to a fixed point. Suppose there is a sphere which separates this point from P and Q. Then

$$\lim \sphericalangle\left[S(P;C);S(Q;C)\right] = 0$$

whether or not the spheres S(P;C) and S(Q;C) themselves converge.[1] In particular, let P and Q be fixed points $\neq p$, and let $C = C(t;\gamma_1) \to p$, as $t \to p$, $t \in A$, $t \neq p$. Then

$$(5.7)\quad \sphericalangle\left[S(P;\sigma_2);S(Q;\sigma_2)\right] = \lim_{t \to p} \sphericalangle\left[S(P,t;\sigma_1);S(Q,t;\sigma_1)\right]$$
$$= 0.$$

Hence any two spheres of σ_2 touch at p.

Conversely, let S be a sphere which touches $S(P;\sigma_2)$. Choose a point $Q \subset S$, $Q \neq p$. Then $S(Q;\sigma_2)$ also touches $S(P;\sigma_2)$ at p, and $S(Q;\sigma_2) = S$. Thus $S \in \sigma_2$.

Corollary 1. σ_2 is the family of spheres, the intersection of any two of which is $C(\gamma_2)$ (cf. Theorem 5.7, Cor. 5).

Corollary 2. There is one and only one sphere of σ_2 through each point $\not\subset C(\gamma_2)$; i.e., if $Q \subset S(P;\sigma_2)$,

- -

1. This statement becomes obvious if we let P or Q be the fixed point at infinity.

<u>$Q \not\subset C(\gamma_2)$, then $S(P;\sigma_2) = S(Q;\sigma_2)$.</u>

<u>Theorem 5.11.</u> <u>If p is a differentiable point of A, then</u>

(5.8) $$\sigma_0 \supset \sigma_1 \supset \sigma_2 \supset \sigma_3.$$

<u>Proof</u>: Evidently $\sigma_0 \supset \sigma_1$. Theorem 5.8, Corollary 3 shows that $\sigma_1 \supset \sigma_2$. This can also be seen as follows: let $P \neq p$. By Theorem 5.6, any sphere $S(P;\sigma_2)$ of σ_2 is the limit of a sequence of spheres $S(P,t;\sigma_1)$, each of which touches a proper circle $C \in \gamma_1$ at p. Thus $S(P;\sigma_2)$ also touches C at p, and $S(P;\sigma_2) \in \sigma_1$.

Let $C(\gamma_2) \neq p$. By Theorem 5.7, σ_2 is the set of all the spheres through $C(\gamma_2)$. Hence $S(\sigma_3)$, being the limit of a sequence of spheres through $C(\gamma_2)$, is itself a sphere through $C(\gamma_2)$, and thus a sphere of σ_2. Relation (5.6) also implies that $\sigma_2 \supset \sigma_3$ when $C(\gamma_2) \neq p$. Suppose $C(\gamma_2) = p$. By Theorem 5.10, σ_2 is the set of all the spheres which touch a given sphere $\neq p$ of σ_2 at p. Hence $S(\sigma_3)$, being the limit of a sequence of such tangent spheres, is itself a sphere of σ_2.

This section can be summarized by the following remark: let p be a differentiable point of an arc A. Let $P \neq p$. In addition, if $S(\sigma_3) \neq p$, let $P \subset S(\sigma_3)$. Let

$$C = \begin{cases} C(\gamma_2) \text{ if } C(\gamma_2) \neq p \\ C(P;\gamma_1) \text{ if } C(\gamma_2) = p \end{cases}, \quad S = \begin{cases} S(\sigma_3) \text{ if } S(\sigma_3) \neq p \\ S(P;\sigma_2) \text{ if } S(\sigma_3) = p \end{cases}.$$

Then $C \subset S$, and the structures of γ_1, σ_1, and σ_2 are completely determined by C and S.

5.6 Intersection and Support Properties of the Families $\sigma_0 - \sigma_1$, $\sigma_1 - \sigma_2$, and $\sigma_2 - \sigma_3$.

Let p be a differentiable interior point of A.

Theorem 5.12. Every sphere $S \neq S(\sigma_3)$ either supports or intersects A at p.

Proof: If S niether supports nor intersects A at p, then $p \subset S$, and there exists a sequence of points $t \to p$, $t \in A \cap S$, $t \neq p$. We may assume that conditions Σ_1, Σ_2, and Σ_3 hold for this sequence since they hold for any sequence $t \to p$, $t \in A$, $t \neq p$. Choose points P and Q on S such that P, Q, and p are mutually distinct. Then condition Σ_1 implies that $S = S(P,Q,t;\sigma_0)$ for each t, and hence $S = S(P,Q,\sigma_1)$.

By Theorem 5.5, $S = S(P,Q;\sigma_1) \supset C(P;\gamma_1)$. By Theorem 5.4, $t \not\subset C(P;\gamma_1)$, and again by Theorem 5.5,

$$S = S[t;C(P;\gamma_1)] = S(P,t;\sigma_1).$$

Condition Σ_2 now implies that $S = S(P;\sigma_2)$.

Finally, by Theorem 5.7, $S \supset C(\gamma_2)$, and by Theorem 5.4, $t \not\subset C(\gamma_2)$. If $C(\gamma_2) \neq p$, Theorem 5.7 implies that

$$S = S[t;C(\gamma_2)] = S(t;\sigma_2),$$

while if $C(\gamma_2) = p$, Theorem 5.10 implies that $S = S(t;\sigma_2)$. Applying the condition Σ_3, we are led to the contradiction $S = S(\sigma_3)$.

<u>Theorem 5.13</u>. <u>If $S(\sigma_3) = p$, then the spheres of σ_2-σ_3 all intersect A at p, or they all support.</u>

<u>Proof:</u> Let S' and S" be two distinct spheres of σ_2-σ_3. Since $S(\sigma_3)$ p, Theorem 5.7, Corollary 4 implies

that S' and S'' touch at p. Thus we may assume that

$S'' \subset (p \cup \overline{S}')$ and $S' \subset (p \cup \underline{S}'')$. Suppose now, for example,

that S' supports A at p while S'' intersects (cf. Fig 5.1).

Then $A \cap \overline{S}''$ is not void, and hence $A \subset (p \cup \overline{S}')$. Let $t \to p$

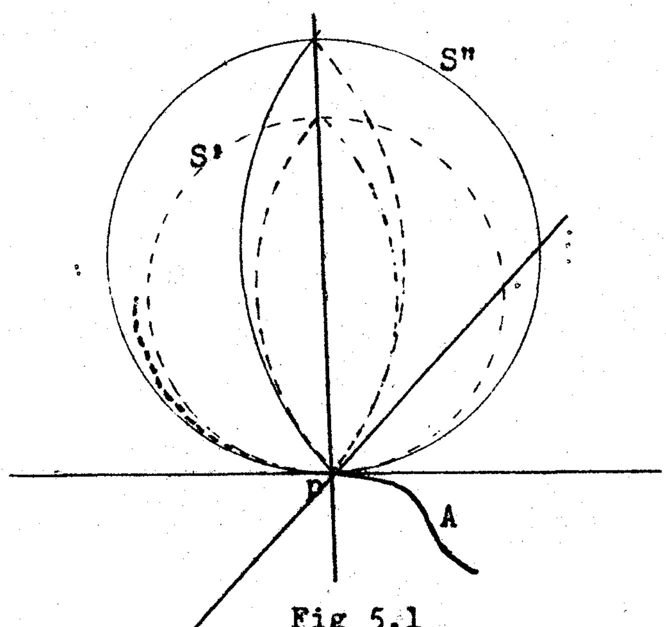


Fig 5.1

in $A \cap \underline{S}''$; thus $t \subset \underline{S}'' \cap \overline{S}'$. Hence

$$S(t;\sigma_2) \subset (\underline{S}'' \cap \overline{S}') \cup p.$$

Consequently $S(t;\sigma_2)$ cannot converge to $S(\sigma_3) = p$ as $t \to p$.

Thus S' and S'' must both support or both intersect A at p.

Theorem 5.14. If $S(\sigma_3) \neq p$ and $C(\gamma_2) = p$, then every sphere of σ_2-σ_3 supports A at p.

Proof: Suppose that $C(\gamma_2) = p$, so that the spheres of σ_2 all touch at p (Theorem 5.10). Let $S \in \sigma_2, S \neq S(\sigma_3)$, $S \neq p$. If a sequence of points t exists such that $t \in A \cap \bar{S}$, $t \to p$, then each $S(t;\sigma_2)$ lies in the closure of $\bar{S}$. Hence $S(\sigma_3)$ will lie in the same domain, and therefore even in $p \cup \bar{S}$. Similarly, the existence of a sequence $t' \in A \cap \underline{S}$, $t' \to p$, implies that $S(\sigma_3) \subset p \cup \underline{S}$. Thus if S intersects A at p, $S(\sigma_3) \subset (p \cup \bar{S}) \cap (p \cup \underline{S}) = p$; in other words, $S(\sigma_3) = p$.

Theorem 5.15. All the spheres of σ_0-σ_1 (σ_1-σ_2; σ_2-σ_3) support A at p, or they all intersect.

Proof: Let S' and S" be two distinct spheres of σ_0-σ_1 (σ_1-σ_2; σ_2-σ_3). Suppose for the moment that the intersection $S' \cap S''$ is a proper circle $C_0 = C(P,Q;\gamma_0)$ $\left(C_1 = C(P;\gamma_1);\ C_2 = C(\gamma_2)\right)$. Suppose, for example that S' intersects while S" supports A at p. With no loss in generality, we may assume that $A \subset \bar{S}'' \cup p$. Thus $A \cap \underline{S}'$ and $A \cap \bar{S}'$ are not void (cf Fig 5.2). If $t \in A \cap \underline{S}'$ by Theorems 5.4, 5.5, and 5.7, $S(P,Q,t;\sigma_0) = S(t;C_0)$ $\big(S(P,t;\sigma_1) = S(t;C_1)$;

$S(t;\sigma_2) = S(t;C_2)\big)$ lies in the closure of

$$(\underline{S}' \cap \bar{S}'') \cup (\bar{S}' \cap \underline{S}'').$$

Letting $t \to p$ on A, we conclude that $S(P,Q;\sigma_1)\ \big(S(P;\sigma_2);S(\sigma_3)\big)$

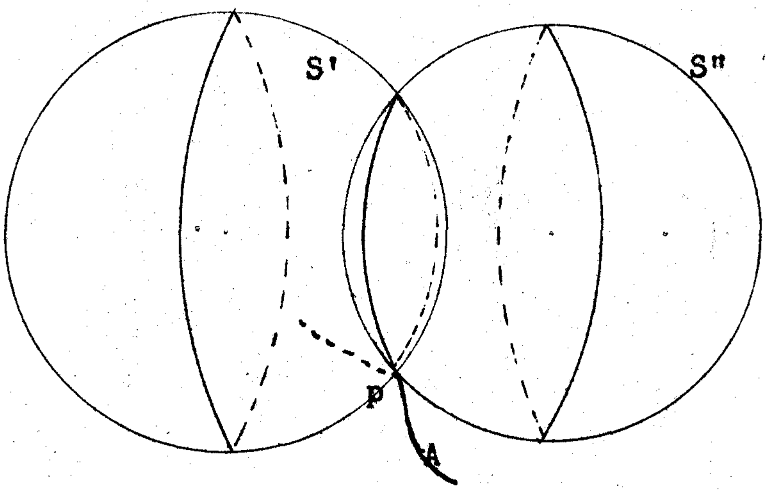


Fig 5.2

lies in the same closed domain. By letting t' converge to p through $\bar{S}' \cap A$, we obtain symmetrically that $S(P,Q;\sigma_1)$ $\big(S(P;\sigma_2);\ S(\sigma_3)\big)$ also lies in the closure of

$$(\underline{S}' \cap \underline{S}'') \cup (\bar{S}' \cap \bar{S}'').$$

Hence $S(P,Q;\sigma_1)\ \big(S(P;\sigma_2);S(\sigma_3)\big)$ lies in the intersection $S' \cup S''$, of these two domains, i.e., $S(P,Q;\sigma_1)\ \big(S(P;\sigma_2);S(\sigma_3)\big)$ is either S' or S", contrary to our assumptions. Thus S' and S" both support or they both intersect in this case.

Suppose now that $S' \cap S'' = p$. In view of Theorems

5.13 and 5.14, there remain to be considered only the cases where S' and S" both belong to $\sigma_0 - \sigma_1$, or both belong to $\sigma_1 - \sigma_2$. By Theorem 5.6, any sphere S through p, which does not touch a circle C of γ_1, belongs to $\sigma_0 - \sigma_1$; by Theorem 5.6, Theorem 5.7, Corollary 5, and Theorem 5.10, any sphere S which touches a circle C of γ_1 but does not contain $C(\gamma_2)$ in case $C(\gamma_2) \neq p$, or does not touch a proper sphere of σ_2 in case $C(\gamma_2) = p$, belongs to $\sigma_1 - \sigma_2$. Hence there exists a sphere S of $\sigma_0 - \sigma_1$ ($\sigma_1 - \sigma_2$) which intersects S' andS" respectively in a proper circle. From the above, S and S', and also S and S", both support or both intersect A at p. Thus S' and S" both support or both intersect A at p.

<u>Theorem 5.16.</u> <u>If $C(\gamma_2) \neq p$, every sphere of $\sigma_1 - \sigma_2$ supports A at p.</u>

<u>Proof</u>: Suppose $S \in \sigma_1 - \sigma_2$ intersects A at p. Let $t \to p$, $t \in A \cap \underline{S}$, $t \neq p$. By Theorem 5.6, $C(t;\gamma_1)$ touches S at p and hence $C(t;\gamma_1) \subset \underline{S} \cup p$. Since $C(t;\gamma_1) \to C(\gamma_2)$, it

follows that $C(\gamma_2) \subset \underline{S} \cup S$. If t' converges to p through $A \cap \overline{S}$, it follows symmetrically that $C(\gamma_2) \subset \overline{S} \cup S$. Thus $C(\gamma_2) \subset S$. Since $S \notin \sigma_2$, however, Theorem 5.7 implies that $C(\gamma_2) = p$.

5.7 A Classification of the Differentiable Points.

The **characteristic**, $(a_o, a_1, a_2, a_3; i)$, of a differentiable point p of an arc A is defined as follows:

$i = 1, 2,$ or 3.
$a_o = 1$ or 2.
$a_1 = 1$ or 2.
$a_2 = 1$ or 2.
$a_3 = 1, 2,$ or ∞.

$i = 1$ if $C(\gamma_2) \neq p$; $i = 2$ if $C(\gamma_2) = p$, $S(\sigma_3) \neq p$; $i = 3$ if $S(\sigma_3) = p$.

a_o is even or odd according as the spheres of $\sigma_o - \sigma_1$ support or intersect.

$a_o + a_1$ is even or odd according as the spheres of $\sigma_1 - \sigma_2$ support or intersect.

$a_o + a_1 + a_2$ is even or odd according as the spheres of σ_2-σ_3 support or intersect.

$a_0+a_1+a_2+a_3$ is even if $S(\sigma_3)$ supports, odd if $S(\sigma_3)$ intersects, while $a_3=\infty$ if $S(\sigma_3)$ neither supports nor intersects.

Theorems 5.16, 5.14, and the convention that $S(\sigma_3)$ supports when it is the point-sphere, lead to the following restrictions on the characteristic:

If $i=1$, then a_0+a_1 must be even;

if $i=2$, then $a_0+a_1+a_2$ must be even;

if $i=3$, then $a_0+a_1+a_2+a_3$ must be even.

As a result of these restrictions, there are just 32 types of differentiable points; there are 12 when $i=1$, 12 when $i=2$, and 8 when $i=3$.

Examples of each of the 32 types are given by the curves

(I) $$x=t^m,\ y=t^n,\ z=t^r,$$

for the cases $a_3=1$ or 2, and

(II) $$x=t^m,\ y=t^n,\ z=\begin{cases} t^r \sin t^{-1}, & \text{if } 0<|t|\leqslant 1 \\ 0 & , t=0 \end{cases},$$

for the cases $a_3 = \infty$, all relative to the point $t = 0$. The indices m,n, and r are positive integers and $m<n<r$. The different types are determined by the parities of the indices m,n, and r, and the relative magnitudes of m,n,r, and 2m. In each of these examples the circles of γ_1 and the spheres of σ_2 touch the x-axis at the origin. In the case $i = 1$, σ_2 is the family of planes through the x-axis, while in each of the cases $i = 2$ or 3, σ_2 is the family of spheres which touch the xy-plane at the origin (cf. remark at the end of § 5.5).

Table 5.1 lists examples of all the types of differentiable poihts, together with their characteristics; table 5.2 summarizes properties of these types. Congruences are mod 2. Figures 5.3 to 5.14 inclusive illustrate the various types of curves, differentiable at the origin, having the indicated characteristics at this point.

Equation	$i = 1$	$m<2m<n<r$			$i = 2$	$m<n<2m<r$			$i = 3$	$m<n<r<2m$		
I	$(1,1,1,1;1)$	$m \equiv 1$	$n \equiv 1$	$r \equiv 0$	$(1,1,2,1;2)$	$m \equiv 1$	$n \equiv 0$	$r \equiv 1$	$(1,1,1,1;3)$	$m \equiv 1$	$n \equiv 0$	$r \equiv 1$
	$(1,1,1,2;1)$			$r \equiv 1$	$(1,1,2,2;2)$			$r \equiv 0$	$(1,1,2,2;3)$			$r \equiv 0$
	$(1,1,2,1;1)$		$n \equiv 0$	$r \equiv 1$	$(1,2,1,1;2)$		$n \equiv 1$	$r \equiv 1$	$(1,2,2,1;3)$		$n \equiv 1$	$r \equiv 1$
	$(1,1,2,2;1)$			$r \equiv 0$	$(1,2,1,2;2)$			$r \equiv 0$	$(1,2,1,2;3)$			$r \equiv 0$
	$(2,2,1,1;1)$	$m \equiv 0$	$n \equiv 1$	$r \equiv 0$	$(2,1,1,1;2)$	$m \equiv 0$	$n \equiv 1$	$r \equiv 1$	$(2,1,1,2;3)$	$m \equiv 0$	$n \equiv 1$	$r \equiv 0$
	$(2,2,1,2;1)$			$r \equiv 1$	$(2,1,1,2;2)$			$r \equiv 0$	$(2,1,2,1;3)$			$r \equiv 1$
	$(2,2,2,1;1)$		$n \equiv 0$	$r \equiv 1$	$(2,2,2,1;2)$		$n \equiv 0$	$r \equiv 1$	$(2,2,1,1;3)$		$n \equiv 0$	$r \equiv 1$
	$(2,2,2,2;1)$			$r \equiv 0$	$(2,2,2,2;2)$			$r \equiv 0$	$(2,2,2,2;3)$			$r \equiv 0$
II	$(1,1,1,\infty;1)$	$m \equiv 1$	$n \equiv 1$		$(1,1,2,\infty;2)$	$m \equiv 1$	$n \equiv 0$					
	$(1,1,2,\infty;1)$		$n \equiv 0$		$(1,2,1,\infty;2)$		$n \equiv 1$					
	$(2,2,1,\infty;1)$	$m \equiv 0$	$n \equiv 1$		$(2,1,1,\infty;2)$	$m \equiv 0$	$n \equiv 1$					
	$(2,2,2,\infty;1)$		$n \equiv 0$		$(2,2,2,\infty;2)$		$n \equiv 0$					

<u>Table 5.1</u>

i	$C(\gamma_2)$	$S(\sigma_3)$	Characteristic $(a_o,a_1,a_2,a_3;i)$	Restrictions		Examples: (I) or (II)				No. of Types
								$C(\gamma_2)$	$S(\sigma_3)$	
1	$\neq p$	$\neq p$	$(a_o,a_1,a_2,a_3;1)$ $a_3=1$ or 2	σ_1-σ_2 supports	$a_o + a_1 \equiv 0$	$m<2m<n<r$	I	x-axis	xy-plane	8
			$(a_o,a_1,a_2,\infty;1)$				II			4
2	$=p$	$\neq p$	$(a_o,a_1,a_2,a_3;2)$ $a_3=1$ or 2	σ_2-σ_3 supports	$a_o+a_1+a_2 \equiv 0$	$m<n<2m<r$	I	$x=y=z=0$	xy-plane	8
			$(a_o,a_1,a_2,\infty;2)$				II			4
3	p	p	$(a_o,a_1,a_2,a_3;3)$ $a_3=1$ or 2	σ_3 supports	$a_o+a_1+a_2+a_3 \equiv 0$	$m<n<r<2m$	I	$x=y=z=0$	$x=y=z=0$	8

Table 5.2

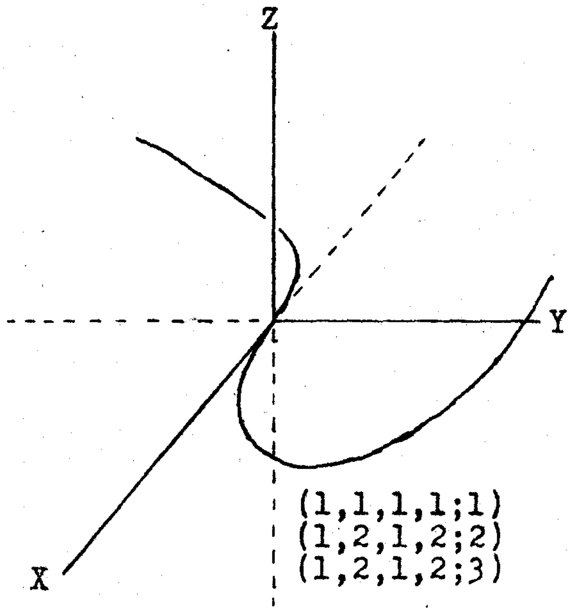


Fig. 5.3

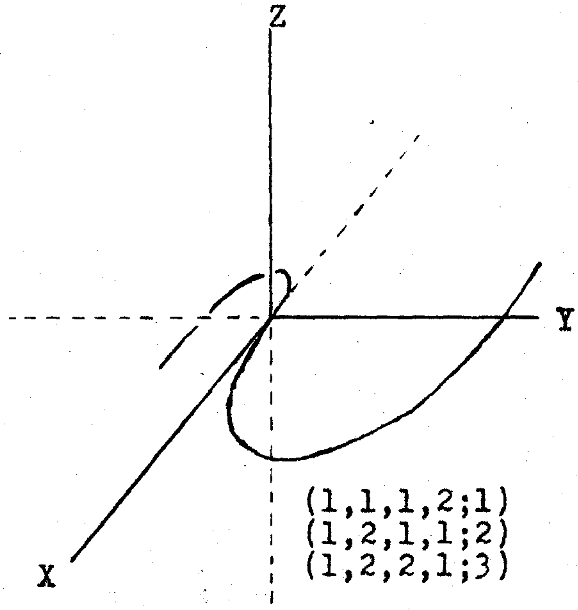


Fig. 5.4

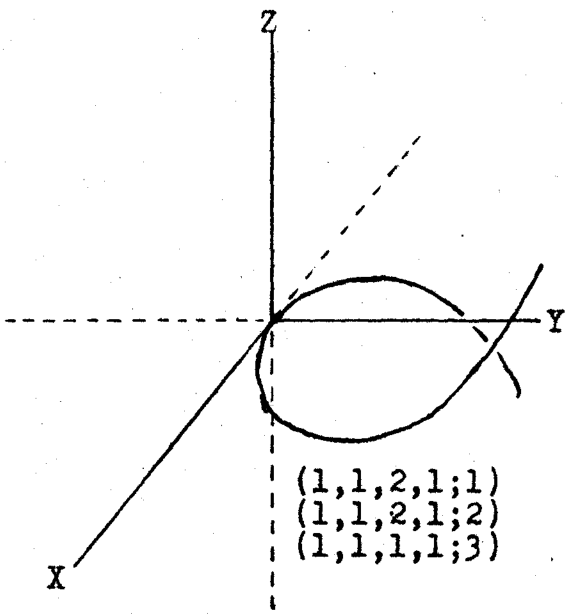


Fig. 5.5

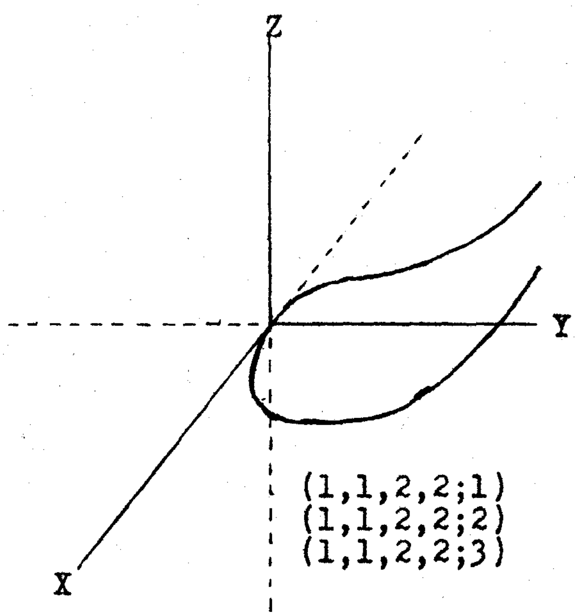


Fig. 5.6

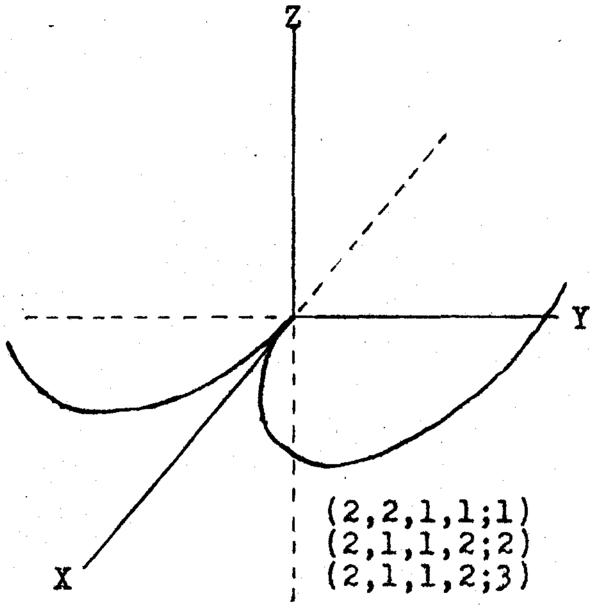


Fig. 5.7

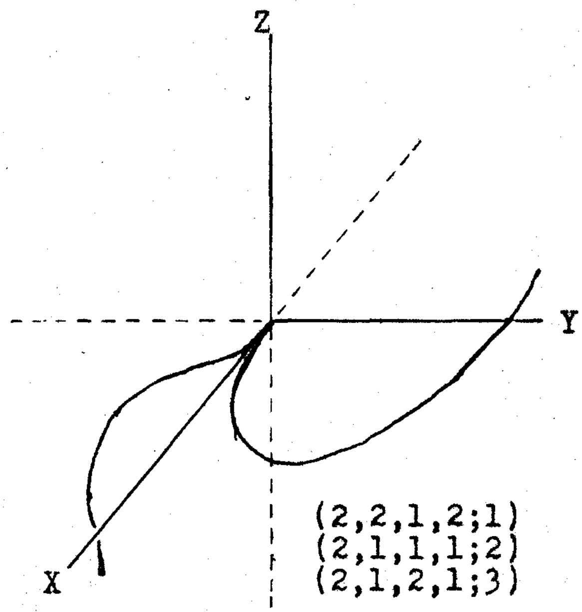


Fig. 5.8

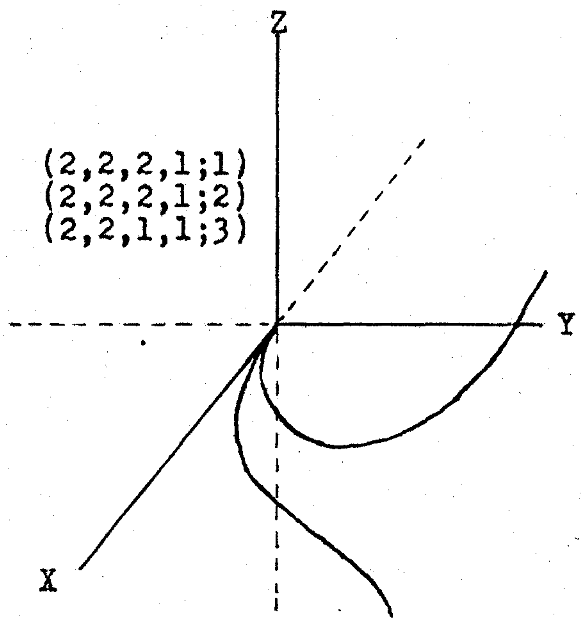


Fig. 5.9

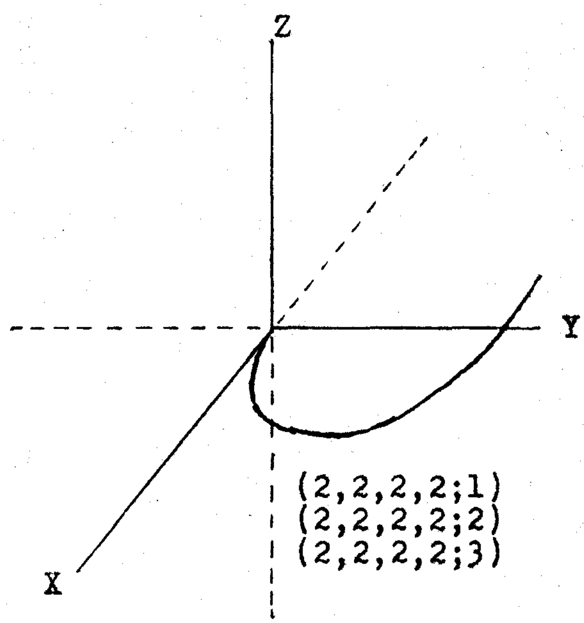


Fig. 5.10

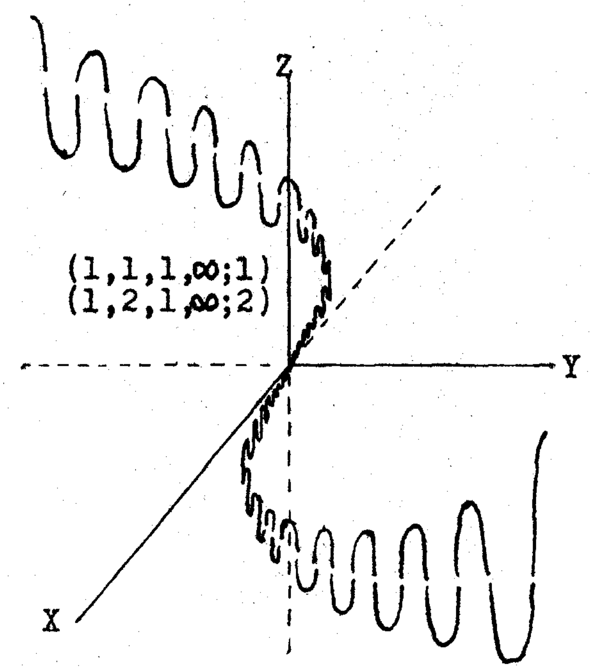


Fig. 5.11

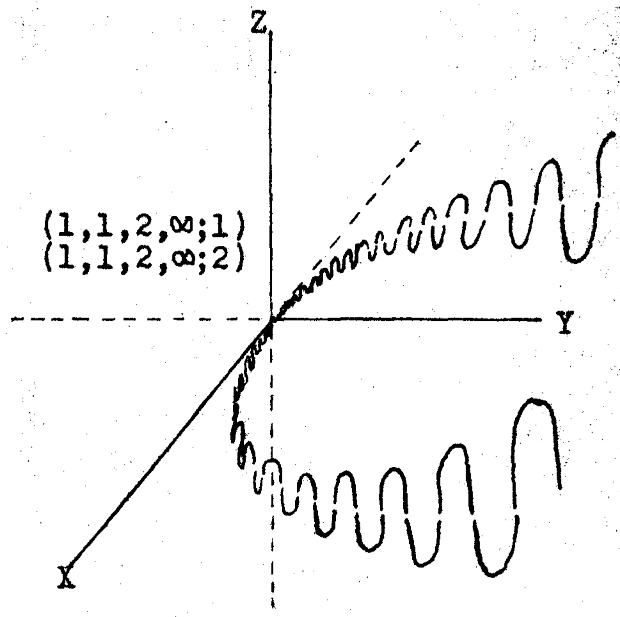


Fig. 5.12

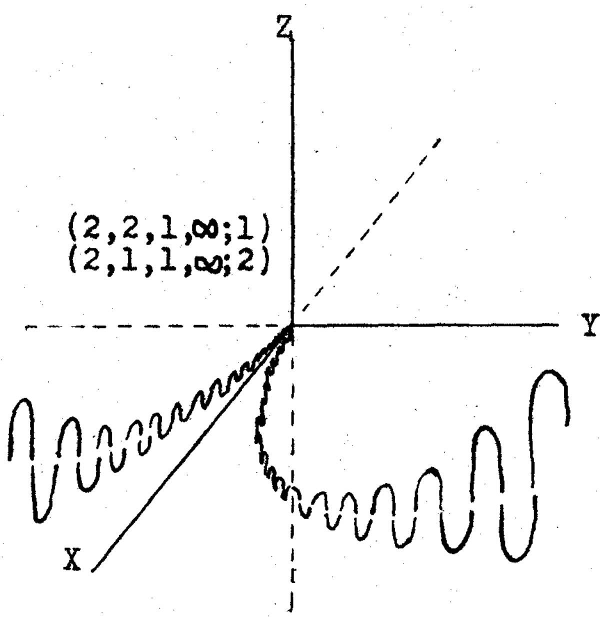


Fig. 5.13

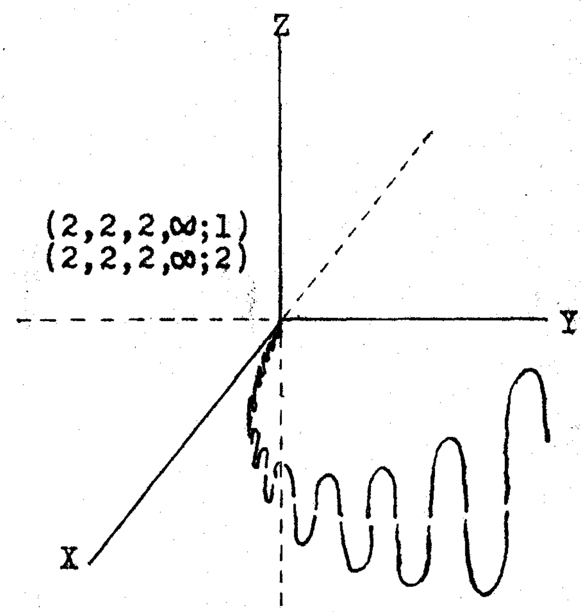


Fig. 5.14

CHAPTER VI

CHARACTERISTIC AND ORDER OF DIFFERENTIABLE POINTS IN CONFORMAL 3-SPACE

6.1. Introduction.

The goal of this chapter is the proof of the following theorem, which is analogous to Theorem 3.1.

Theorem 6.1. *Let p be a differentiable point of an arc A in conformal 3-space. Suppose that p has characteristic $(a_0, a_1, a_2, a_3; i)$. Then the conformal order of p is not less than $a_0 + a_1 + a_2 + a_3$.*

This theorem implies

Corollary 1. *If the order of p is bounded, then to every neighbourhood of p there corresponds a sphere arbitrarily close to $S(\sigma_3)$ which does not pass through p, and which intersects that neighbourhood in not less than $a_0 + a_1$*

$+a_2 + a_3$ points.

6.2. Arcs of Finite and Bounded Spherical Order.

An arc A is said to be of finite spherical order if it has only a finite number of points in common with any sphere. If some sphere meets A n times, and no sphere meets A more than n times, where n is some specific integer, then A is said to be of bounded spherical order, and n is called the (spherical) order of A. If p is any point on A, the order of p is the minimum of the orders of all the neighbourhoods of p on A.

Lemma 6.1. Let B be an arc of finite order. If a sphere S intersects B at t, then every sphere sufficiently close to S intersects B in at least one point.

Proof: The end-points of some neighbourhood $M \subset B$ of t lie in different regions with respect to S. Hence they also lie in different regions with respect to any sphere S' sufficiently close to S. Since M and S' have only a finite number of points in common, one of them must be an intersection.

It is clear that S' will intersect M in an odd number of points.

6.3. Proof of Theorem 6.1.

The ensuing discussion simplifies the proof of Theorem 6.1. As in section 5.4, σ_3, σ_2, σ_1, and σ_0 will denote the families of tangent spheres $S(\sigma_3)$, $S(P;\sigma_2)$, $S(P,Q;\sigma_1)$, and $S(P,Q,R;\sigma_0)$ respectively. Now suppose that P, Q, and R are fixed points such that $p \not\subset C(P,Q,R)$. The symbol Π_4 will denote $S(\sigma_3)$; $\Pi_3(t)$, $\Pi_2(t)$, and $\Pi_1(t)$ will denote the linear families of spheres $S(t;\sigma_2)$, $S(P,t;\sigma_1)$, and $S(P,Q,t;\sigma_0)$ respectively. $\Pi_0(t)$ will denote the linear family of spheres $S(P,Q,R,t)$.

6.3.1. Let M be any neighbourhood of p on A. We wish to show that to every sphere $S_{r-1} \in \Pi_{r-1}$ there corresponds a sphere of Π_r arbitrarily close to, but different from, S_{r-1}, which meets M outside p in not less than a_r points. If p has finite order, and if M is small enough, we can even say that there are spheres of Π_r close to S_{r-1} which meet M outside p in a_{r+2n} points, $n \geq 0$ ($r = 0,1,2,3$; we assume that $a_3 < \infty$ when $r = 3$).

6.3.2. Let T_r be a sphere of the family $\Pi_r - \Pi_{r+1}$, $r = 0,1,2,3$. If $r = 3$ and $C(\gamma_2) = p$, let $E_3 = S(\sigma_3)$ when $S(\sigma_3) \neq p$, but let $E_3 = T_3$ when $S(\sigma_3) = p$. If $r < 3$, or if $r = 3$ and $C(\gamma_2) \neq p$, E_r will not be defined. In any case, we define the regions[1]

$$\underline{E}_r = [\overline{T}_r \cap \underline{S}_{r+1}] \cup [\underline{T}_r \cap \overline{S}_{r+1}]$$

and

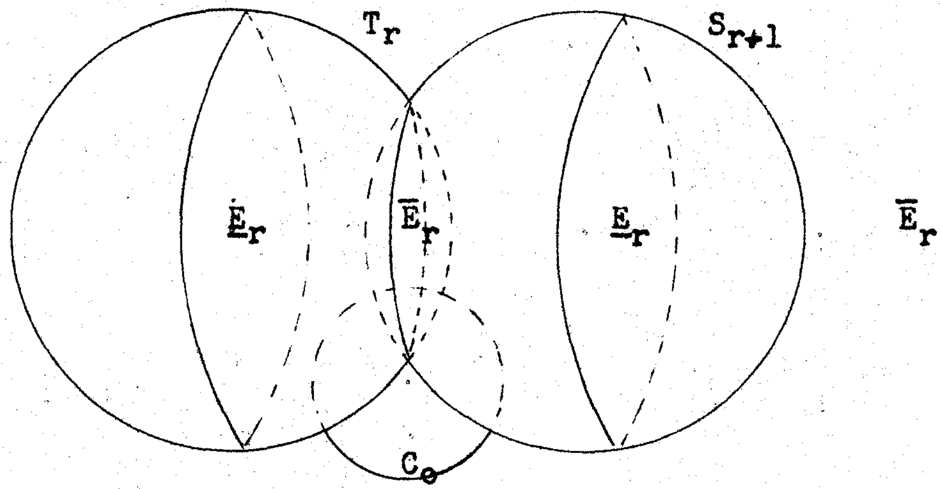
$$\overline{E}_r = [\overline{T}_r \cap \overline{S}_{r+1}] \cup [\underline{T}_r \cap \underline{S}_{r-1}]$$

(cf. Fig. 6.1).

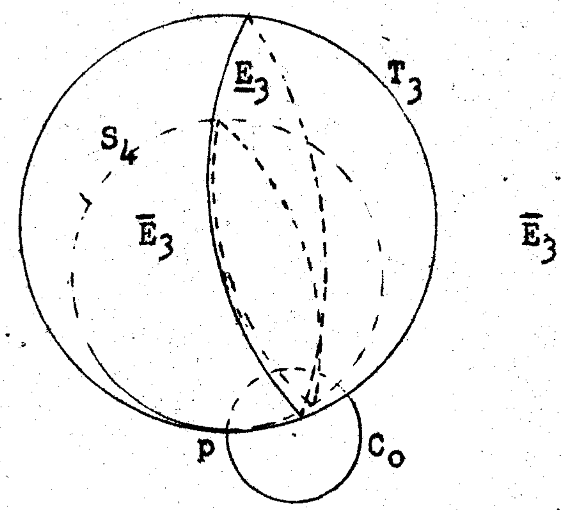
Let $\underline{\Pi}_r$ ($\overline{\Pi}_r$) denote the set of those spheres of Π_r

- -

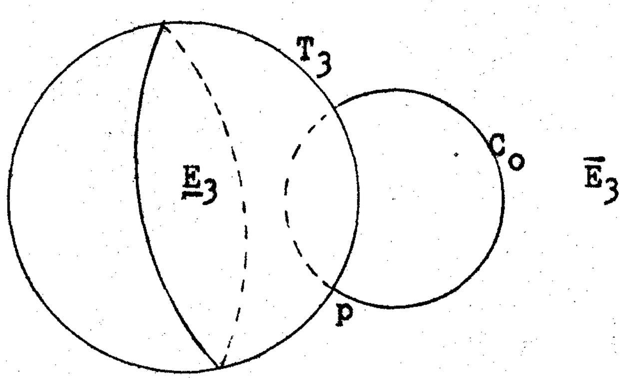
1. If S_{r+1} is the point-sphere p, $\underline{S}_{r+1}$ is void, and $\overline{S}_{r+1}$ is the whole plane with the exception of the point p.



(a) $0 \leq r \leq 3$, $C(\gamma_2) \neq p$ when $r = 3$



(b) $r = 3$, $C(\gamma_2) = p$, $S(\sigma_3) \neq p$



(c) $r = 3$, $C(\gamma_2) = p$, $S(\sigma_3) = p$

<u>Fig. 6.1</u>

that pass through $\underline{E}_r$ ($\overline{E}_r$). Then every sphere of Π_r except T_r and S_{r+1} belongs either to $\underline{\Pi}_r$ or to $\overline{\Pi}_r$. By intersecting Π_r with an orthogonal circle C_o, we can construct a 1-1 correspondence between the spheres of $\underline{\Pi}_r$ ($\overline{\Pi}_r$) and the points of C_o, and hence an ordering of the spheres in $\underline{\Pi}_r$ ($\overline{\Pi}_r$).

We can choose our neighbourhood M so small that T_r and S_{r+1} have no points in common with the two one-sided neighbourhoods N and N' into which M is decomposed by p. This follows for $S = S(\sigma_3)$ from our assumption $a_3 < \infty$, and for the other spheres it follows from Theorem 5.12. Thus N (N') lies intirely in the region $\underline{E}_r$ or else entirely in the region $\overline{E}_r$. Let t and t' denote the points of N and N' respectively; thus either all the spheres of $\Pi_r(t)$ belong to $\underline{\Pi}_r$ or all of them are in $\overline{\Pi}_r$. Without restriction of generality, let $N \subset [\overline{T}_r \cap \overline{S}_{r+1}] \subset \overline{E}_r$. Then $\Pi_r(t)$ belongs to $\overline{\Pi}_r$ for every t. Let $e \in N$. Then $\Pi_r(e)$ is the end-sphere of a one-sided neighbourhood δ of S_{r+1} in Π_r. If t moves from e to p, then $\Pi_r(t)$ moves in $\overline{\Pi}_r$ from $\Pi_r(e)$ to S_{r+1}. Hence the spheres $\Pi_r(t)$ omit none of the spheres of δ, i.e., every sphere of δ meets N. Let $S \in \delta$. Thus S lies between $\Pi_r(e)$ and $S_{r+1} = \lim_{t \to p} \Pi_r(t)$. If t is sufficiently close to p, then t does not belong to S, and S will also lie between $\Pi_r(e)$ and $\Pi_r(t)$. Since $e \not\subset S$, and since the points t and e lie in $\overline{T}_r \cap \underline{S}_{r+1}$, they will also be separated by S.

Let the order of p be finite. Then we may assume that M is also of finite order. In addition, S will meet N in a finite number of points only, and at least one of them will be an intersection. Replacing N by the one-sided neighbourhood of p with end-point e, we can even state that S will intersect N in an odd number of points.

Similarly, there exists a one-sided neighbourhood δ' of S_{r+1} in Π_r such that each of its spheres meets N'. If p has finite order, and if N' is sufficiently small, then δ' can be chosen such that each sphere of δ' intersects N' in an odd number of points.

6.3.3. If $a_r = 1$, then one of the spheres T_r and S_{r+1} intersects, while the other one supports M at p; therefore $N' \in \underline{E}_r$. If $a_r = 2$, then T_r and S_{r+1} either both intersect or both support; hence $N' \in \overline{E}_r$. Thus the spheres $\Pi_r(t')$ belong to $\underline{\Pi}_r$ or to $\overline{\Pi}_r$, according as $a_r = 1$ or 2. This holds true, in particular, of the spheres of the neighbourhoods δ and δ'. Since $\delta \in \overline{\Pi}_r$, it follows that δ and δ' lie on opposite sides of S_{r+1} or on the same side, depending on whether $a_r = 1$ or $a_r = 2$. This implies our statements in section 6.3.1.

6.3.4. The proof of Theorem 6.1 now follows readily. Obviously, we may assume that the order of p is finite, and in particular, that $a_3 < \infty$.

We prove our theorem by first approximating $S_4 = S(\sigma_3) = \Pi_4$ by a sphere S_3 of Π_3, S_3 by a sphere S_2 of Π_2, S_2 by a sphere S_1 of Π_1, and finally we approximate S_1 by a sphere which does not contain p.

Let M_3 be a neighbourhood of finite order of p on A.

From section 6.3.1, there exists a sphere $S_3 \in \Pi_3$ close to, but different from, $S(\sigma_3)$ which intersects M_3 in not less than a_3 points t_3 outside p.

In M_3 we construct mutually disjoint neighbourhoods B_3 of the t_3 and M_2 of p. Choose a point P on S_3, $P \notin C(\gamma_2)$. Then $S_3 = S(P;\sigma_2)$. Let Π_2 be the pencil of spheres of σ_1 through P; thus $\Pi_2(t) = S(P,t;\sigma_1)$ and $S_3 = \lim_{t \to p} \Pi_2(t)$. By section 6.3.1, there exists a sphere $S_2 \in \Pi_2$ close to, but different from, S_3, which intersects M_2 in not less than a_2 points t_2 outside p, and which intersects each B_3.

In M_2, we construct mutually disjoint neighbourhoods B_2 of the t_2, and M_1 of p. Choose a point Q on S_2, $Q \notin C(P;\gamma_1)$. Then $S_2 = S(P,Q;\sigma_1)$. Let Π_1 be the pencil of spheres of σ_1 through P and Q. Thus $\Pi_1(t) = S(P,Q,t;\sigma_0)$, and $S_2 = \lim_{t \to p} \Pi_1(t)$. By section 6.3.1, there exists a sphere S_1 of Π_1 close to, but different from, S_2, which intersects M_1 in not less than a_1 points t_1 outside p, and which intersects each of the $a_2 + a_3$ arcs B_2 and B_3.

In M_1, we construct mutually disjoint neighbourhoods B_1 of t_1 and M_o of p. Choose a point $R \subset S$, $R \not\subset C(P,Q;\gamma_o)$. Let Π_o be the pencil of spheres through P,Q, and R, and let $\Pi_o(t) = S(P,Q,R,t)$. Then $S_1 = \lim_{t \to p} \Pi_o(t)$. By section 6.3.1, there exists a sphere S_o of Π_o close to, but different from, S_1, which intersects M_o in not less than a_o points t_o outside p, and which intersects each of the $a_1 + a_2 + a_3$ arcs, B_1, B_2, and B_3. Altogether, S_o will be close to $S(\sigma_3)$ and will intersect M_3 in not less than $a_o + a_1 + a_2 + a_3$ points, all of which are different from p.

CHAPTER VII

ARCS OF SPHERICAL ORDER FOUR IN CONFORMAL 3-SPACE

7.1. Introduction.

This chapter extends to three dimensions the work of sections 3.4 to 3.8 inclusive. The fact already noted in section 5.1, that we are now dealing with two continuous entities, the circle and the sphere, will make this work considerably more delicate than that of Chapter III.

We denote an arc of order four (cf. § 6.2) by the symbol A_4.

7.2. Two Lemmas on Arcs of Finite Spherical Order.

Lemma 7.1. A point of an arc A of finite order converges if its parameter tends to one of the end-points of the parameter interval.

In particular, this is true of an arc of order four.

Proof: Let t_{2i} and t_{2i+1} $(i=0,1,2,\ldots)$ be two sequences of points whose parameters tend to the same end-point of the parameter interval. Suppose that $\lim_{i\to\infty} t_{2i}=p$, and $\lim_{i\to\infty} t_{2i+1}=q$, where p and q are accumulation points, and $p\neq q$. We may assume that t_{n+1} lies between t_n and t_{n+2} on the arc.

Let S be a sphere which separates p and q. Then for sufficiently large n, S separates t_n and t_{n+1}. Thus S meets A in an infinite number of points, contrary to our assumption.

By the above lemma, we see that A_4 has two well-defined end-points.

Lemma 7.2. An end-point of an arc A of finite order is automatically differentiable (in the sense of section 5.4).

Proof: Obviously, A has only a finite number of points in common with any circle C(P,Q,p) through mutually distinct points P,A, and p. Thus, when $t \in A$ is sufficiently close to p, the sphere $S(P,Q,t;\sigma_o)$ is defined.

Suppose that there are two sequences of points, t_{2i} and t_{2i+1}, different from p, converging on A to p such that the spheres $S_{2i} = S(P,Q,t_{2i};\sigma_o)$ and $S_{2i+1} = S(P,Q,t_{2i+1};\sigma_o)$ converge to different limit spheres, S_o and S_1 respectively. We may assume that t_{n+1} lies between p and t_n. If i is large, S_{2i} (S_{2i+1}) will be close to S_o (S_1). Let S and S' be two spheres through $S_o \cap S_1$ which separate S_o and S_1. Then S and S' will separate S_n and S_{n+1}, and therefore t_n and t_{n+1} for every large n. Hence the subarc of A bounded by t_n and t_{n+1} will meet $S \cup S'$. Thus A will meet $S \cup S'$ an infinite number of times. This is impossible. Hence condition Σ_1 holds at p.

The above discussion shows that Σ_2 and Σ_3 also are satisfied at p (cf. Theorems 5.8 and 5.9), where in the latter case, $C(\gamma_2) \neq p$.

If $C(\gamma_2) = p$, then by Theorem 5.10, the spheres of σ_2 all touch at p. Let t_{2i} and t_{2i+1} be two sequences of points converging to p in such a way that $S(t_{2i};\sigma_2)$ and $S(t_{2i+1};\sigma_2)$ approach two different limit spheres, S_o and S_1 respectively. Each of these limit spheres, being the limit of a sequence of spheres that touch any sphere of σ_2 at p, also touches any sphere of σ_2 at p.

Suppose, to begin, that S_o and S_1 are both proper spheres. Suppose further, that $S_1 \subset \underline{S}_o \cup p$ and $S_o \subset \overline{S}_1 \cup p$. Consider a sphere $S \in \sigma_2$, $S \subset (\underline{S}_o \cap \overline{S}_1) \cup p$. Then S separates S_o and S_1 except at the point p, i.e., $S_1 \subset \underline{S} = p$ and $S_o \subset \overline{S} \cup p$, say. Hence, for sufficiently large i, $S(t_{2i+1};\sigma_2) \subset \underline{S} \cup p$ and $S(t_{2i};\sigma_2) \subset \overline{S} \cup p$. Here again, the arc A crosses C an infinite number of times, which by our hypothesis is impossible.

If now, S_1 for instance, is the point-sphere p, consider two proper spheres of σ_2, S and S', where $S \subset \underline{S}_o \cup p$, $S' \subset \overline{S}_o \cup p$, and $S_o \subset \overline{S} \cap \underline{S}' \cup p$. Then for sufficiently large i,

$S(t_{2i};\sigma_2) \subset (\bar{S} \cap \underline{S}') \cup p$, while $S(t_{2i+1};\sigma_2) \subset \underline{S} \cap \bar{S}' \cup p$. Since these two regions are separated by $S \cup S'$, one or both of these spheres will meet A an infinite number of times. Since this again is impossible by our hypothesis, condition Σ_3 holds, and the point p is differentiable.

7.3. Multiplicities;

7.3.1. We call a sphere S a general (r-1)-tangent sphere of order of contact r-1 (r 2,3,or 4) at a point t of an arc A if there exists a sequence of r-tuples, $t_1, t_2, \ldots t_r$, of points which converge on A to t such that S is the limit of a sequence of spheres S' through the t_i. Let $t \in A_4$. Any sphere through t will intersect or support A_4 there. A general (r-1)-tangent sphere intersects A_4 at t if r is odd, and supports A_4 at t if r is even.

We usually call a general 3-tangent sphere a general osculating sphere.

Let p be an end-point of A_4. As in section 3.7, we introduce multiplicities and count p r-times on any sphere of $\sigma_{r-1}-\sigma_r$ $(r=2,3)$, and four times on $S(\sigma_3)$. A point $t \in A_4 \cup p$ is counted r times on a general (r-1)-tangent sphere $(r=2,3,4)$. We wish to prove the following theorem:

<u>Theorem 7.1.</u> <u>No sphere meets $A_4 \cup p$ more than four times; i.e., the inclusion of p and the introduction of multiplicities does not alter the order of A_4.</u>

The proof of Theorem 7.1 results from the discussion in the remainder of section 7.3.

<u>7.3.2</u>. <u>Lemma 7.3.</u> <u>If a sphere S meets A_4 in three points, then at least two of these points are intersections.</u>

<u>Proof</u>: Let S meet A_4 in q_1, q_2, and q_3, and let M_i be sufficiently small neighbourhoods of q_i $(i=1,2,3)$. If q_i is a point of support, then there is a sphere close to S which meets M_i in two points.

Suppose that q_1, q_2, and q_3 are all points of support. If M_1, M_2, and M_3 all lie in <u>S</u>, say, then there exists a sphere close to S which will meet A_4 at least six times. On the other hand, if $M_1, M_2 \subset \underline{S}$ and $M_3 \subset \overline{S}$, then S must intersect A_4 in a fourth point q_4; hence there is a sphere close to S which meets A_4 at least five times (cf. Lemma 6.1). Both of these cases are impossible since A_4 is of order four.

We note that by the latter argument, if $M_1 \subset \underline{S}$ and $M_3 \subset \overline{S}$, then S intersects A_4 at some point.

Suppose that q_1 and q_2 are points of support, while S is a point of intersection. If $M_1, M_2 \subset \underline{S}$, then, as before, some sphere close to S will intersect A_4 five times. If $M_1 \subset \underline{S}$ and $M_2 \subset \overline{S}$, then let r be the necessary point of intersection on $A_4 \cap S$, and let S_o be a sphere which separates q_1 and q_2. Hence $S_o \cap S = C$ is a proper circle. Without loss of generality, we may assume that $M_1 \subset (\underline{S} \cap \underline{S}_o) \cup q_1$ and $M_2 \subset (\overline{S} \cap \overline{S}_o) \cup q_2$. Then a sphere $S' \subset (\underline{S} \cap \underline{S}_o) \cup (\overline{S} \cap \overline{S}_o) \cup C$, which is sufficiently close to S, will meet M_1 and M_2 twice each. Since r is a point of intersection, S' will also meet A_4 near r (cf. Lemma 6.1). This again is a contradiction.

Lemma 7.3 implies

Lemma 7.4. A sphere S through four points of A_4 does not support A_4 at any of these points.

Proof: By Lemma 7.3, $A_4 \cap S$ has at most one point of support. If $A_4 \cap S = q_1, q_2, q_3$ and q_4, where q_4 is a point of support, then there is a sphere close to S which meets A_4 five times,- once each near q_1, q_2, and q_3, and twice near q_4. This is impossible.

7.3.3. Suppose that a sphere S through p meets A_4 in four points, q_1, q_2, q_3, and q_4. By Lemma 7.4 they are all intersections. Choose disjoint neighbourhoods N of p and M of q which do not contain q_2, q_3, or q_4. If t converges to p in N, then S' S(q_2, q_3, q_4, t) converges to S. By Lemma 6.1, S' will intersect M if t is sufficiently close to p. Hence this sphere meets A_4 in no fewer than five points, contrary to the definition of A_4. This yields

Lemma 7.5. No sphere meets $A_4 \cup p$ in five points.

Lemmas 7.3, 7.4, and 7.5 imply

Lemma 7.6. A sphere S through four points of $A_4 \cup p$ does not support A_4 at any of these points.

Proof: By Lemma 7.3, $A_4 \cap S$ has at most one point of support. If there is one point of support, Lemma 7.4 implies that S goes through p. Hence a suitable sphere through p which is close to S, will meet $A_4 \cup p$ five times, contrary to Lemma 7.5.

Note that Lemma 7.6 is a generalization of Lemma 7.4.

7.3.4. Suppose that a sphere S of σ_1 meets A_4 in three points q_1, q_2, and q_3. By Lemma 7.6, they are all intersections. Choose disjoint neighbourhoods N of p and M of q_1, which do not contain q_2 or q_3. If t converges to p in N, then $S' = S(q_2, q_3, t, \sigma_0)$ converges to S. By Lemma 6.1, S' will intersect M if t is sufficiently close to p. Hence this sphere meets $A_4 \cup p$ in no fewer than five points, contrary to Lemma 7.5. This yields

<u>Lemma 7.7.</u> <u>No sphere of σ_1 meets A_4 in three points</u>.

Suppose a sphere S of σ_1 supports A_4 at q. Then some sphere of σ_1 close to S will intersect a neighbourhood of q in two points. This, with Lemma 7.7 yields

<u>Lemma 7.8.</u> <u>If a sphere of σ_1 supports A_4 at some point, then it does not meet A_4 again.</u>

<u>7.3.5.</u> Suppose that a sphere S of σ_2 meets A_4 in two points q_1 and q_2. By Lemma 7.8, both points are intersections. Choose disjoint neighbourhoods N of p and M of q_1 which do not contain q_2. If t converges to p in N, then $S' = S(q_2, t; \sigma_1)$ converges to S. By Lemma 6.1, S' will intersect M if t is sufficiently close to p. Hence this sphere meets A_4 in no fewer than three points, contrary to Lemma 7.7. This yields

<u>Lemma 7.9.</u> <u>No sphere of σ_2 meets A_4 in two points.</u>

Suppose that a sphere S of σ_2 supports A_4 at q, and let M be a small neighbourhood of q on A_4. We consider two cases:

(i) $C(\gamma_2) \neq p$. In this case, the spheres of σ_2 all contain $C(\gamma_2)$. However, $q \not\subset C(\gamma_2)$, for if it did, we could find spheres of σ_2 through two points of A_4, contrary to Lemma 7.9. Let $S_o \neq S$ be a sphere of σ_2, and let $M \subset \underline{S}_o \cap \underline{S}_1$. Then there is a sphere $S' \in \sigma_2$ passing through $(\underline{S}_o \cap \underline{S}) \cup (\bar{S}_o \cap \bar{S}) \cup C(\gamma_2)$, which is so close to S that it intersects M in two points.

(ii) $C(\gamma_2) = p$. In this case, the spheres of σ_2 all touch at p. Let $M \subset \underline{S}$. Then there is a sphere $S' \subset \underline{S} \cup p$, $S' \in \sigma_2$, which is so close to S that it intersects M in two points.

Thus in either case, we have a sphere of σ_2 which meets A_4 in at least two points, contrary to Lemma 7.9. Hence we have

<u>Lemma 7.10.</u> <u>No sphere of σ_2 through a point $q \in A_4$ supports at that point.</u>

7.3.6. Suppose that $S(\sigma_3)$ meets A_4 in one point q. By Lemma 7.10, it is an intersection. Choose disjoint neighbourhoods N of p and M of q. If t converges to p in N, then $S' = S(t;\sigma_2)$ converges to $S(\sigma_3)$. By Lemma 6.1, S' will intersect M if t is sufficiently close to p. Hence this sphere meets A_4 in no fewer than two points, contrary to Lemma 7.9. This yields

Lemma 7.11. $S(\sigma_3)$ does not meet A_4.

7.3.7. Multiplicities Relative to General Tangent Spheres.

In the following we shall not consider general tangent spheres at the point p, since we shall learn in section 7.4 that such spheres are members of the families σ_1, σ_2, or σ_3, depending on their order of contact.

Lemma 7.12. Let $q_1, q_2 \rightarrow q$ on A_4, and let $t \in A_4$, $t \neq q$. Let $C(t, q_1, q_2) \rightarrow C_o$. Then C_o does not meet $A_4 \cup p$ outside q and t.

Proof: Suppose that $C_o \supset u$, $u \in A_4 \cup p$, $u \neq t, q$. Let

$v \in A_4 \cup p$, $v \not\subset C_o$. Then $S(v;C_o) = \lim S[v;C(t,q_1,q_2)]$ $= \lim S(v,t,q_1,q_2)$ does not meet $A_4 \cup p$ elsewhere. Hence the end-points of a small neighbourhood of q on A_4 are not separated by q_1 and q_2. Thus $S(v;C_o) = \lim S[v;C(t,u,q)]$ $= \lim S(v,t,u,q)$ intersects A_4 at q. Thus we have the proof by contradiction.

<u>Lemma 7.13. Let $q_1, q_2, q_3 \to q$ on A_4 such that $C(q_1,q_2,q_3) \to C_o$. Then C_o does not meet $A_4 \cup p$ outside q.</u>

<u>Proof</u>: Suppose that $t \subset C_o$, $t \in A_4 \cup p$, $t \neq q$. Thus $C_o \neq q$. Choose a point $u \in A_4 \cup p$, $u \not\subset C_o$, and let $S = S(u;C_o)$. Then $S(u,q_1,q_2,q_3) = S[u;C(q_1,q_2,q_3)] \to S_o$. Since $S(u,q_1,q_2,q_3)$ does not meet $A_4 \cup p$ elsewhere, the end-points of a small neighbourhood of q on A_4 are separated by this sphere; hence its limit sphere S must intersect A_4 at q. Since

$$\sphericalangle [C(t,q_1,q_2);C(q_1,q_2,q_3)] \to 0,$$

any accumulation circle, C_1, of $C(t,q_1,q_2)$ passes through t and touches C_o at q. Thus $C_1 = C_o$, and the sphere $S(u,t,q_1,q_2)$

$= S[u;C(t,q_1,q_2)] \rightarrow S(u;C_o) = S$. Since $S(u,t,q_1,q_2)$ does not meet $A_4 \cup p$ elsewhere, the end-points of a small neighbourhood of q on A_4 are not separated by this sphere, hence the limit sphere S must support A_4 at q, and we have a contradiction.

A general 1-tangent sphere at a point $q \in A_4$ that is not a general 2-tangent sphere, supports A_4 at q. By Lemma 7.6 it does not meet $A_4 \cup p$ in three other points.

Lemma 7.14. A general 2-tangent sphere of A_4 at a point q does not meet $A_4 \cup p$ at two other points. It does not support A_4 at any of these points of contact.

Proof: Let S be the limit sphere of a sequence of spheres S' through three mutually distinct points, q_1, q_2, q_3, which converge on A_4 to q. Let u and t lie on S, $u \neq t \neq q \neq u$. Let $S'' = S(t,q_1,q_2,q_3)$, let C_o be any limit circle of $C(q_1,q_2,q_3)$, and let S_o be any limit sphere of S". If $C_o = p$, then S and S_o touch at q, and if $C_o \neq p$, $S_o \cap S = C_o$.

Since in addition, S and S_o have the point t in common ($t \not\subset C_o$ by Lemma 7.13), it follows that $S = S_o$. Since S" does not meet $A_4 \cup p$ again, S_o intersects A_4 at q.

Let $S''' = S(t,u,q_1,q_2)$, and let C_1 be any limit circle of $C(t,q_1,q_2)$. By Lemma 7.12, $u \not\subset C_1$, and since any limit sphere S_1 of S"' contains u and C_1, the fact that $S_o = \lim S''$ also contains u and C_1 implies that $S_1 = S_o = S$. Since $S''' = S(t,u,q_1,q_2)$ does not meet $A_4 \cup p$ elsewhere, S_1 supports A_4 at q. S_o, however, intersects A_4 at q. Thus S does not meet $A_4 \cup p$ at two points $\neq q$. If $t \neq p$, S" intersects A_4 at t. Hence the end-points of a small neighbourhood of t lie on opposite sides of S", and thus they also lie on opposite sides of S. Thus no general 2-tangent sphere of A_4 at q supports A_4 at another point.

<u>Lemma 7.15.</u> <u>A general 3-tangent sphere of A_4 does not meet $A_4 \cup p$ again.</u>

<u>Proof</u>: Let S be a general 3-tangent sphere of A_4 at

q, and suppose that S meets $A_4 \cup p$ again at t. Suppose that S is the limit of a sequence of spheres $S' = S(q_1, q_2, q_3, q_4)$, where the q_i are mutually distinct points converging on A_4 to q. Thus S supports A_4 at q. Let $S'' = S(t, q_1, q_2, q_3)$. Then any limit sphere S_o of S'' contains t and intersects A_4 at q.

Let C_o be any accumulation circle of $C(q_1, q_2, q_3)$. Then $S \cap S_o \supset C_o$. If $C_o = q$, then S and S_o touch at q and have the point $t \neq q$ in common. If $C_o \neq q$, then $C_o \not\ni t$ by Lemma 7.12, and hence $S \cap S_o \supset C_o \cup t$. In either case, we have the contradiction $S_o = S$.

7.3.8. Theorem 7.1 yields several interesting results concerning the families of circles γ_o, γ_1, and γ_2.

Lemma 7.5 implies

Corollary 1. No circle meets $A_4 \cup p$ in more than three points.

Corollary 2. No circle of γ_1 meets A_4 more than once.

Proof: Let $u \in A_4 \cap C(t;\gamma_1)$, and let $v \in A_4$, $v \notin C(t;\gamma_1)$. Then $S[v;C(t;\gamma_1)] = S(v,t;\sigma_1)$ meets A_4 in three distinct points, contrary to Lemma 7.7.

Corollary 3. $C(\gamma_2)$ does not meet A_4.

Proof: We are only concerned with the case $C(\gamma_2) \neq p$. Let $u \in A_4 \cap C(\gamma_2)$, and let $v \in A_4$, $v \not\subset C(\gamma_2)$. Then $S[v;C(\gamma_2)] = S(v;\sigma_2)$ meets A_4 in two distinct points, contrary to Lemma 7.9.

7.4. Strong Differentiability.

We call an arc A strongly differentiable at a point p if the arc is differentiable at that point and if, in addition, the following three conditions hold:

Σ_1': Let P,Q,p be mutually distinct points, where $P \not\subset C(Q;\gamma_1)$ and let $P' \to P$, $Q' \to Q$. If the two distinct points t and u converge on A to p, then $S(P',Q', t,u)$ always converges.

Σ_2': Let $P \not\subset C(\gamma_2)$, $P' \to P$. If the three mutually distinct points t,u,v, converge on A to p, then $S(P',t,u,v)$

converges.

Σ_3': $S(t,u,v,w)$ converges if the four mutually distinct points t,u,v,w, converge on A to p.

<u>7.4.1.</u> It is clear that the limit of the spheres $S(P';Q';t,u)$ depends only on P,Q, and p, and not on the choice of the sequences u and v. In particular, if $P' = P$, $Q' = Q$, and $u = p$, we see that Σ_1' implies Σ_1, except where $P \subset C(Q;\gamma_1)$, and

$$\lim S(P';Q';t,u) = S(P,Q;\sigma_1).$$

Similarly, the limit of the spheres $S(P';t,u,v)$ depends only on P and p. Since

$$S(P;\sigma_2) = \lim_{v \to p} S(P,v;\sigma_1) = \lim_{u \to p} \lim_{v \to p} S(P,u,v;\sigma_0),$$

we see that $\lim S(P';t,u,v) = S(P;\sigma_2)$, if $P \not\subset C(\gamma_2)$.

Finally, the limit of the spheres $S(t,u,v,w)$ depends only on p. Since

$$S(\sigma_3) = \lim_{w \to p} S(w;\sigma_2) \quad \lim_{v \to p} \lim_{w \to p} S(v,w;\sigma_1)$$

$$\lim_{u \to p} \lim_{v \to p} \lim_{w \to p} S(u,v,w;\sigma_0),$$

we verify that $\lim S(t,u,v,w) = S(\sigma_3)$. Thus Σ_3' implies Σ_3.

7.4.2. We now prove the following important theorem, which generalizes Lemma 7.2:

Theorem 7.2. Let p be an end-point of an open arc A_4 of order four. Then $A_4 \cup p$ is strongly differentiable at p.

Before verifying Theorem 7.2, we prove an interesting corollary, which asserts that Theorem 7.2 extends itself automatically to include the cases where Σ_1', Σ_2', and Σ_3' are weakened so as to permit multiplicities as defined in section 7.3.

Corollary 1. Let three distinct points t, u, and v converge on $A_4 \cup p$ to p, and let $P' \to P$, $Q' \to Q$, where P and Q are mutually distinct, and where $P \not\subset C(Q;\gamma_1)$ and $P \not\subset C(\gamma_2)$. Let S_1 $(S_2;S_3;S_3')$ be a general 1-tangent sphere of $A_4 \cup p$ at t through P' and Q' (P' and u;u and v; a point of support, u). Let S_2' (S_3'') be a general 2-tangent sphere at t through P' (u). Finally, let S_3''' be a general osculating sphere at t. Then

(7.1) $\quad \lim S_1 = S(P,Q;\sigma_1)$.

(7.2) $\underline{\lim S_2 = \lim S_2' = S(P;\sigma_2)}$.

(7.3) $\underline{\lim S_3 = \lim S_3' = \lim S_3'' = \lim S_3''' = S(\sigma_3)}$.

Proof of the Corollary: We may assume that each of the above sequences of spheres possesses an accumulation sphere. S_1 can be replaced by a sphere $S(P!Q!t_1,t_2)$ close to S_1 and such that t_1 and t_2 are distinct, and converge with t to p. Thus

$$\lim S_1 = \lim S(P!Q!t_1,t_2) = S(P,Q;\sigma_1).$$

Similarly, S_2 and S_2' can be replaced by spheres $S(P!u,t_1,t_2)$, and $S(P!t_1,t_2,t_3)$ respectively, close to S_2 and S_2' such that t_1,t_2, and t_3 are distinct, and converge with t_2 to p. Again

$$\left.\begin{array}{l}\lim S_2 = \lim S(P!u,t_1,t_2)\\ \lim S_2' = \lim S(P!t_1,t_2,t_3)\end{array}\right\} = S(P;\sigma_2).$$

Finally, S_3, S_3', S_3'', and S_3''' can be replaced by spheres $S(u,v,t_1,t_2)$, $S(u_1,u_2,t_1,t_2)$, $S(u,t_1,t_2,t_3)$, and $S(t_1,t_2,t_3,t_4)$ respectively. Hence

$$\left.\begin{aligned} \lim S_3 &= \lim S(u,v,t_1,t_2) \\ \lim S_3' &= \lim S(u_1,u_2,t_1,t_2) \\ \lim S_3'' &= \lim S(u,t_1,t_2,t_3) \\ \lim S_3''' &= \lim S(t_1,t_2,t_3,t_4) \end{aligned}\right\} = S(\sigma_3).$$

Thus Theorem 7.2 implies our corollary.

7.4.3. We prove Theorem 7.2 in the remaining sub-sections of section 7.4. We shall let B be an open sub-arc of A_4 bounded by p and an interior point f of A_4. Let g be any point of A_4 outside B f. We orient those spheres S for which $g \not\subset S$ so that $g \subset \overline{S}$. In particular, the set of such spheres contains all the spheres which meet $B \cup p \cup f$ four times. Their orientation is continuous. The points t,u,v, w,d,e,f are assumed to be mutually distinct, and to lie on $B \cup f$ in the indicated order.

7.4.4. It is therefore evident that

(i) $$u \subset \overline{S}(p,t,e,f) \cap \underline{S}(t,d,e,f).$$

Consequently,

$$(7.4) \quad S(t,u,e,f) \subset \left[\overline{S}(p,t,e,f) \cap \underline{S}(t,d,e,f)\right] \cup \left[\underline{S}(p,t,e,f) \cap \overline{S}(t,d,e,f)\right] \cup C(t,e,f).$$

Let I denote the region in (7.4). From (7.4) we obtain

$$(7.5) \quad \lim_{u,t \to p} S(t,u,e,f) \subset \left[\overline{S}(e,f;\sigma_1) \cap \underline{S}(p,d,e,f)\right] \cup \left[\underline{S}(e,f;\sigma_1) \cap \overline{S}(p,d,e,f)\right] \cup S(e,f;\sigma_1) \cup S(p,d,e,f).$$

By II we shall mean the limit of I as $t \to p$. Let S be any limit sphere of S(t,u,e,f). As a point r runs continuously on B from d to p, S(p,r,e,f) runs continuously through the region II from S(p,d,e,f) to $S(e,f;\sigma_1)$. Conversely, every sphere through II and C(p,e,f) meets B. Hence, if S passes through II $\cup$ S(p,d,e,f), it intersects B at some point r, where $r = d$ if $S = S(p,d,e,f)$ (otherwise r lies between p and d). But then S(t,u,e,f), when it is close to S, intersects B again near r, contrary to Theorem 7.1. Thus $S = S(e,f;\sigma_1)$.

<u>Corollary 2.</u> $\underline{\operatorname{Lim}_{t,u \to p} C(t,u,e) = C(e;\gamma_1).}$

<u>Proof</u>: $$\operatorname{Lim}_{t,u \to p} C(t,u,e) = \lim \prod_{f \in A_4} S(t,u,e,f) = \prod S(e,f;\sigma_1)$$

$$= C(e;\gamma_1).$$

7.4.5. We now prove simultaneously that $S(p,u,v,f) \to S(f;\sigma_2)$, and assuming this, that $S(t,u,v,f) \to S(f;\sigma_2)$. We first note that

(ii) $$u \subset \overline{S}(v,f;\sigma_1) \cap \underline{S}(p,v,e,f),$$

and correspondingly,

(ii') $$u \subset \overline{S}(p,t,v,f) \cap \underline{S}(t,v,e,f).$$

Relations (ii) and (ii') yield

(7.6) $$S(p,u,v,f) \subset \left[\overline{S}(v,f;\sigma_1) \cap \underline{S}(p,v,e,f)\right] \cup \left[\underline{S}(v,f;\sigma_1) \cap \overline{S}(p,v,e,f)\right] \cup C(p,v,f)$$

and

(7.6') $$S(t,u,v,f) \subset \left[\overline{S}(p,t,v,f) \cap \underline{S}(t,v,e,f)\right] \cup \left[\underline{S}(p,t,v,f) \cap \overline{S}(t,v,e,f)\right] \cup C(t,v,f)$$

respectively. Let III denote either the region in (7.3) or the region in (7.3'). From (7.3) we obtain

(7.4) $$\lim_{u,v \to p} S(p,u,v,f) \subset \left[\overline{S}(f;\sigma_2) \cap \underline{S}(e,f;\sigma_1)\right] \cup \left[\underline{S}(f;\sigma_2) \cap \overline{S}(e,f;\sigma_1)\right] \cup S(f;\sigma_2) \cup S(e,f;\sigma_1),$$

while (7.3') yields

$$(7.4')\quad \lim_{t,u,v\to p} S(t,u,v,f) \subset \left[\bar{S}(f;\sigma_2) \cap \underline{S}(e,f;\sigma_1)\right] \cup \left[\underline{S}(f;\sigma_2) \cap \bar{S}(e,f;\sigma_1)\right] \cup S(f;\sigma_2) \cup S(e,f;\sigma_1).$$

By IV we shall mean the limit of III as $v \to p$ (as $t,v \to p$). Let S be any limit sphere of $S(p,u,v,f)$ $\big(S(t,u,v,f)\big)$. Since $S \supset \lim C(p,v,f)$ $\big(\lim C(t,v,f)\big) = C(f;\gamma_1)$, we see that $S \in \sigma_1$. As a point r runs continuously on B from e to p, $S(r,f;\sigma_1)$ runs continuously through the region IV from $S(e,f;\sigma_1)$ to $S(f;\sigma_2)$. Conversely, every sphere through IV and $C(f;\gamma_1)$ meets B. Hence if S passes through IV $\cup S(e,f;\sigma_1)$, it intersects B at some point r, where $r = e$ if $S = S(e,f;\sigma_1)$ (otherwise r lies between p and e). But then $S(p,u,v,f)$ $\big(S(t,u,v,f)\big)$, when it is close to S, intersects B again near r, contrary to Theorem 7.1. Thus $S = S(f;\sigma_2)$.

<u>Corollary 3.</u> $\underline{\operatorname{Lim}_{t,u,v\to p} C(t,u,v) = \lim_{u,v\to p} C(p,u,v) = C(\gamma_2).}$

<u>Proof</u>: $\operatorname{Lim}_{t,u,v\to p} C(t,u,v) = \lim \prod_{f\in A_4} S(t,u,v,f)$

$$= \prod S(f;\sigma_2) = C(\gamma_2).$$

$$\lim_{u,v \to p} C(p,u,v) = \lim \prod_{f \in A_4} S(p,u,v,f)$$

$$= \prod S(f;\sigma_2) = C(\gamma_2).$$

7.4.6. Here we prove, again simultaneously, that $S(u,v;\sigma_1) \to S(\sigma_3)$, $S(p,t,u,v) \to S(\sigma_3)$, and $S(t,u,v,w) \to S(o_3)$, each proposition being assumed to prove the following one. Proceeding in the previous manner, we note

(iii) $\quad u \subset \underline{S}(v;\sigma_2) \cap \overline{S}(v,f;\sigma_1)$

(iii') $\quad u \subset \underline{S}(t,v;\sigma_1) \cap \overline{S}(p,t,v,f)$

(iii") $\quad u \subset \overline{S}(p,t,v,w) \cap \underline{S}(t,v,w,f).$

Relations (iii), (iii'), and (iii") yield

(7.8) $$S(u,v;\sigma_1) \subset \left[\underline{S}(v;\sigma_2) \cap \overline{S}(v,f;\sigma_1)\right] \cup \left[\overline{S}(v;o_2) \cap \underline{S}(v,f;\sigma_1)\right] \cup C(v;\gamma_1),$$

(7.8') $$S(p,t,u,v) \subset \left[\underline{S}(t,v;\sigma_1) \cap \overline{S}(p,t,v,f)\right] \cup \left[\overline{S}(t,v;\sigma_1) \cap \underline{S}(p,t,v,f)\right] \cup C(p,t,v),$$

and

(7.8") $$S(t,u,v,w) \subset \left[\overline{S}(p,t,v,w) \cap \underline{S}(t,v,w,f)\right] \cup \left[\underline{S}(p,t,v,w) \cap \overline{S}(t,v,w,f)\right] \cup C(t,v,w),$$

respectively. Let V denote either the region of (7.8), or that of (7.8'), or that of (7.8"). Relations (7.8), (7.8'), and (7.8") yield

$$(7.9) \qquad \lim_{u,v \to p} S(u,v;\sigma_1) \subset \left[\underline{S}(\sigma_3) \cap \overline{S}(f;\sigma_2)\right] \cup \left[\overline{S}(\sigma_3) \cap \underline{S}(f;\sigma_2)\right] \cup S(\sigma_3) \cup S(f;\sigma_2),$$

$$(7.9') \qquad \lim_{t,u,v \to p} S(p,t,u,v) \subset \left[\underline{S}(\sigma_3) \cap \overline{S}(f;\sigma_2)\right] \cup \left[\overline{S}(\sigma_3) \cap \underline{S}(f;\sigma_2)\right] \cup S(\sigma_3) \cup S(f;\sigma_2),$$

and

$$(7.9'') \qquad \lim_{t,u,v,w \to p} S(t,u,v,w) \subset \left[\underline{S}(\sigma_3) \cap S(f;\sigma_2)\right] \cup \left[\overline{S}(\sigma_3) \cap \underline{S}(f;o_2)\right] \cup S(\sigma_3) \cup S(f;\sigma_2),$$

respectively. By VI, we shall mean the limit of V as $v \to p$ (as $t,v \to p$; as $t,v,w \to p$). Let S be any limit sphere of $S(u,v;\sigma_1)$ $\big(S(p,t,u,v);\ S(t,u,v,w)\big)$. Since S contains $\lim C(v;\gamma_1)$ $\big(\lim C(p,t,v);\ \lim C(t,v,w)\big) = C(\gamma_2)$, we see that $S \in \sigma_2$ unless $C(\gamma_2) = p$. If $C(\gamma_2) = p$, $S(\sigma_3)$ and $S(f;\sigma_2)$ touch at p; hence $S \subset \left[\overline{S}(\sigma_3) \cap \underline{S}(f;\sigma_2)\right] \cup S(\sigma_3) \cup S(f;\sigma_2)$, and since $S \supset p$, it must touch $S(f;\sigma_2)$ at p; hence $S \in \sigma_2$. As a point r runs continuously on B from f to p, $S(r;\sigma_2)$ runs continuously through the region VI from $S(f;\sigma_2)$ to $S(\sigma_3)$. Conversely, every sphere through VI and $C(\gamma_2)$ meets B. Hence if S passes through $VI \cup S(f;\sigma_2)$, it intersects $B \cup f$ at r, where $r = f$ if $S = S(f;\sigma_2)$. But then $S(u,v;\sigma_1)$ $\big(S(p,t,u,v);$ $S(t,u,v,w)\big)$, when it is close to S, intersects B again near r, contrary to Theorem 7.1. Thus $S \supset S(\sigma_3)$. If $C(\gamma_2) \neq p$, there is no difficulty in seeing that $S = S(\sigma_3)$. The same is true if $S(\sigma_3) = p$. Suppose $C(\gamma_2) = p$ and $S(\sigma_3) \neq p$. By a consideration of the method in which $S(u,v,\sigma_1)$ $\big(S(p,t,u,v);\ S(t,u,v,w)\big)$ converges to S through the region V, we find that S must be a proper sphere of σ_2; hence $S = S(\sigma_3)$.

<u>7.4.7.</u> We now generalize section 7.4.4. Let $P \not\subset C(e;\gamma_1)$, and let $P' \to P$, $P \neq p$. Then, by Corollary 2 (§7.4.4)

$$\lim_{\substack{t,u \to p \\ P' \to P}} S(P',t,u,e) = \lim S\left[P',C(t,u,e)\right]$$

$$= S\left[P;C(e;\gamma_1)\right] = S(P,e;\sigma_1).$$

<u>Corollary 4.</u> $\lim_{\substack{t,u\to p \\ P'\to P}} C(P',t,u) = C(P;\gamma_1)$.

<u>Proof</u>: $\lim_{\substack{t,u\to p \\ P'\to P}} C(P',t,u) = \lim \prod_{e\in A_4} S(P',t,u,e)$

$$= \prod S(P,e;\sigma_1) = C(P;\gamma_1).$$

With this corollary in mind, let $Q \not\subset C(P;\gamma_1)$, and let $Q' \to Q$, $Q \neq p$. Then

$$\lim_{\substack{t,u\to p \\ P'\to P, Q'\to Q}} S(P',Q',t,u) = \lim S[Q';C(P',t,u)]$$

$$= S[Q;C(P;\gamma_1)]$$

$$= S(P,Q;\sigma_1).$$

<u>Remark:</u> If $Q \subset C(P;\gamma_1)$, let S be any accumulation sphere of $S(P',Q',t,u)$. Since $S \supset \lim C(P',t,u) = C(P;\gamma_1)$, $S \in \sigma_1$.

<u>7.4.8.</u> Finally, we generalize section 7.4.5. Let $P \not\subset C(\gamma_2)$, let $C(\gamma_2) \neq p$, and let $P' \to P$. Then by Corollary 3 (§ 7.4.5),

$$\lim_{\substack{t,u,v\to p \\ P'\to P}} S(P',t,u,v) = \lim S[P';C(t,u,v)]$$

$$= S[P;C(\gamma_2)] = S(P;\sigma_2).$$

If $C(\gamma_2) = p$, then

$$\lim [S(P',t,u,v) \cap S(t,u,v,f)]$$

$$= \lim C(t,u,v) = p, \qquad P \neq p,$$

i.e. any accumulation sphere, S, of $S(P',t,u,v)$ touches $\lim S(t,u,v,f) \in \sigma_2$ at p. Hence $S \in \sigma_2$. Since S touches any sphere of σ_2 at p and goes through a point $P \neq p$, $S = S(P;\sigma_2)$.

<u>Remark</u>: If $P \subset C(\gamma_2)$ and $C(\gamma_2) \neq p$, let S be any accumulation sphere of $S(P',t,u,v)$. Since $S \supset \lim C(t,u,v) = C(\gamma_2)$, $S \in \sigma_2$.

All the results in section 7.4.8 also hold if $t = p$.

CHAPTER VIII

CONFORMALLY ELEMENTARY POINTS OF ARCS IN CONFORMAL 3-SPACE

8.1. Introduction.

A point p of an arc A is said to be *conformally elementary* if there exists a neighbourhood of p on A which is decomposed by p into two one-sided neighbourhoods of spherical order four. As a result of Theorem 7.2, these two one-sided neighbourhoods are strongly differentiable at p.

I conjecture that the statement in Theorem 8.1 is universally true; the discussion in Chapter 8 is a partial proof of this theorem, proving twelve out of the twenty-four cases.

Theorem 8.1. *Let p be a differentiable conformally elementary point of an arc A, and let $(a_0, a_1, a_2, a_3; i)$ be the characteristic of p. Then p has the spherical order*

$\underline{a_0 + a_1 + a_2 + a_3}$.

This theorem remains valid if p is counted a_0 times on any sphere of σ_0, a_0+a_1 times on any sphere of σ_1, $a_0+a_1+a_2$ times on any sphere of σ_2, and finally, $a_0+a_1+a_2+a_3$ times on $S(\sigma_3)$. The theorem also remains valid if a point $u \in A$ is counted twice on any sphere which supports A at u, three times on any general 2-tangent sphere, and four times on any general 3-tangent sphere at u. There is no loss in generality if we assume that A itself is decomposed by p into two open arcs A_4 and A_4' of order four. Thus the order of A, and therefore that of p, is not greater than 8.

8.2. Some Necessary Formulas.

Before beginning the proof of Theorem 8.1, we prove several helpful relations involving regions. As in section 7.4, we let A_4 be an arc of order four, and let p be an end-point of A_4. We assume that the points p,t,u,v,d,e,f,g lie on $A_4 \cup p$ in the indicated order. If any sphere S does not contain the point g, then $g \subset \overline{S}$. We consider spheres through the point p, which by Theorem 7.2 is strongly differentiable.

To begin, we observe that

$$v \subset \underline{S}(e;\sigma_2) \cap \overline{S}(e,f;\sigma_1).$$

Hence

$$S(v,e;\sigma_1) \subset [\underline{S}(e;\sigma_2) \cap \overline{S}(e,f;\sigma_1)] \cup [\overline{S}(e;\sigma_2) \cap \underline{S}(e,f;\sigma_1)] \cup C(e;\gamma_1).$$

Therefore $S(v,e;\sigma_1)$ separates the regions

$$\underline{S}(e;\sigma_2) \cap \underline{S}(e,f;\sigma_1)$$

and

$$\overline{S}(e;\sigma_2) \cap \overline{S}(e,f;\sigma_1).$$

Since

$$g \subset \overline{S}(e;\sigma_2) \cap \overline{S}(e,f;\sigma_1) \cap \overline{S}(v,e;\sigma_1),$$

we find that

(8.1) $$\overline{S}(v,e;\sigma_1) \supset \overline{S}(e;\sigma_2) \cap \overline{S}(e,f;\sigma_1)$$

and consequently,

(8.2) $$\underline{S}(v,e;\sigma_1) \supset \underline{S}(e;\sigma_2) \cap \underline{S}(e,f;\sigma_1).$$

In exactly the same manner, we verify that

(8.3) $$\overline{S}(u,v;\sigma_1) \supset \overline{S}(v;\sigma_2) \cap \overline{S}(v,e;\sigma_1)$$

and

(8.4) $$\underline{S}(u,v;\sigma_1) \supset \underline{S}(v;\sigma_2) \cap \underline{S}(v,e;\sigma_1)$$

and again,

(8.5) $$\overline{S}(v;\sigma_2) \supset \overline{S}(\sigma_3) \cap \overline{S}(e;\sigma_2)$$

and

(8.6) $$\underline{S}(v;\sigma_2) \supset \underline{S}(\sigma_3) \cap \underline{S}(e;\sigma_2)$$

Results (8.4), (8.6), and (8.2) yield

(8.7) $$\begin{aligned}\underline{S}(u,v;\sigma_1) &\supset \underline{S}(v;\sigma_2) \cap \underline{S}(v,e;\sigma_1) \\ &\supset \left[\underline{S}(\sigma_3) \cap \underline{S}(e;\sigma_2)\right] \cap \left[\underline{S}(e;\sigma_2) \cap \underline{S}(e,f;\sigma_1)\right] \\ &= \underline{S}(\sigma_3) \cap \underline{S}(e;\sigma_2) \cap \underline{S}(e,f;\sigma_1),\end{aligned}$$

while results (8.3), (8.5), and (8.1) yield

(8.8) $$\bar{S}(u,v;o_1) \supset \bar{S}(\sigma_3) \cap \bar{S}(e;\sigma_2) \cap \bar{S}(e,f;\sigma_1).$$

Using the same methods, we can prove the following two relations

(8.9) $$\begin{aligned}\underline{S}(t,u,v;\sigma_0) &\supset \underline{S}(u,v,d;\sigma_0) \cap \underline{S}(u,v;\sigma_1) \\ &\supset \left[\underline{S}(v,d,e;\sigma_0) \cap \underline{S}(v,d;\sigma_1)\right] \cap \left[\underline{S}(v,d;\sigma_1) \cap \underline{S}(v;\sigma_2)\right] \\ &= \underline{S}(v,d,e;\sigma_0) \cap \underline{S}(v,d;\sigma_1) \cap \underline{S}(v;\sigma_2) \\ &\supset \left[\underline{S}(d,e,f;\sigma_0) \cap \underline{S}(d,e;\sigma_1)\right] \cap \left[\underline{S}(d,e;\sigma_1) \cap \underline{S}(d;\sigma_2)\right] \\ &\qquad \cap \left[\underline{S}(d;\sigma_2) \cap \underline{S}(\sigma_3)\right] \\ &= \underline{S}(d,e,f;\sigma_0) \cap \underline{S}(d,e;\sigma_1) \cap \underline{S}(d;\sigma_2) \cap \underline{S}(\sigma_3)\end{aligned}$$

(8.10) $$\begin{aligned}\bar{S}(t,u,v;\sigma_0) &\supset \bar{S}(d,e,f;\sigma_0) \cap \bar{S}(d,e;\sigma_1) \\ &\qquad \cap \bar{S}(d;\sigma_2) \cap \bar{S}(\sigma_3).\end{aligned}$$

8.3. Multiplicities on the arc $A = A_4 \cup p \cup A_4'$.

8.3.1. Lemma 8.1. Let M be a neighbourhood of p on A. Let $\mu(S)=\mu(S,M)$ denote the multiplicity with which a sphere S meets M. Suppose that S does not pass through the end-points of M. Then for every sphere S' sufficiently close to S,

$$\mu(S') \equiv \mu(S) \pmod 2.$$

Proof: Suppose that S meets M at the points t with the multiplicities $\rho(t)$, and nowhere else. Then

$$\mu(S) = \sum_t \rho(t).$$

Construct disjoint neighbourhoods B in M about the points t. The end-points of B lie on the same side or on opposite sides of S according as $\rho(t)$ is even or odd. If S' is sufficiently close to S, then S' will not pass through the end-points of B, and they will lie on the same side of S' if and only if they lie on the same side of S. The multiplicity with which S' meets B, however, will also be even or odd, depending on whether the end-points of B lie on the same side or on opposite sides of S'. Thus S' will meet B with a multiplicity $\rho'(t) \equiv \rho(t)$ (mod 2) if S' is sufficiently close to S. If each B is omitted from the closure of M, there is left a closed set, which has no poihts in common with S. If S' is sufficiently close to S, this set does not meet S' either, and therefore

$$\mathcal{M}(S') = \sum_t \rho'(t) \equiv \sum_t \rho(t) \pmod 2 \equiv \mathcal{M}(S).$$

8.3.2. Let S be any sphere, and let $t \in S \cap M$, $t \neq p$. Let p be the multiplicity with which S meets M. Suppose there is a sequence of spheres S' converging to S, and a corresponding sequence of neighbourhoods B' of t converging to t such that each S' meets B' at least ρ' times, where $\rho' \leq 4$. Then each S' can be replaced by another sphere S" which meets B' in not less than ρ' distinct points, and such that the sequence S" also converges to S. Hence S will meet M at least ρ' times at t; therefore $\rho' \leq \rho$. This proves

Lemma 8.2. If S' is sufficiently close to S, and $t \in M$, $t \neq p$, and finally, if S has multiplicity $\rho(t)$ at t, there is a neighbourhood of t on M which is met not more than $\rho(t)$ times by S'.

8.4. Multiplicities of the Families $\sigma_0-\sigma_1$, $\sigma_1-\sigma_2$, and $\sigma_2-\sigma_3$.

8.4.1. We now consider a sphere S of $\sigma_0-\sigma_1$. By section 5.7, S meets M at p with a multiplicity $\equiv a_0$ (mod 2). Let $S' \to S$, and suppose that each S' meets $M = N \cup p \cup N'$ in more than two points converging with S' to p. Then S' meets $N \cup p$, say, in at least two points t and u which converge with S' to p. Let $e \in N$. Then by Theorem 7.2, Corollary 2, $C(e,t,u) \to C(e;\gamma_1)$. Suppose $C' \to C \neq p$, $C' \subset S'$, $C' \supset t,u$. Then

$$\lim \sphericalangle \left[C';C(e,t,u)\right] = 0,$$

and by Theorem 5.6, $S \in \sigma_1$. Thus there exists a neighbourhood of p which is met not more than twice by any sphere S' close to S. This leads to

Lemma 8.3. Every sphere of $\sigma_0-\sigma_1$ meets M at p with the multiplicity a_0. Thus if $S' \to S \in \sigma_0-\sigma_1$, there is a neighbourhood M_0 of p which is met by every S' sufficiently close to S not more than a_0 times.

8.4.2. Let $S \in \sigma_1 - \sigma_2$. Let S' be close to S, $S' \notin \sigma_1$, and let $M_1 = N_1 \cup p \cup N_1'$ be a small neighbourhood of p.

By Theorem 7.2, Cor 3, $C(u,t,p) \to C(\gamma_2)$ as $u,t \to p$. Thus if $S' \supset C(u,t,p)$, then $S \in \sigma_2$. Hence if $p \subset S'$, then S' will meet $N_1(N_1')$ not more than once. Thus S' meets M_1 with a multiplicity $\equiv a_0+a_1 \pmod 2$, and it also meets M_1 with a multiplicity $\leqslant a_0+2$. Hence the multiplicity with which S' meets M_1 is $\leqslant a_0+a_1$.

Next suppose that $p \not\subset S'$. Then S' will meet $N_1(N_1')$ at most twice. Hence S' meets M_1 with a multiplicity $\leqslant 4$ and $\equiv a_0+a_1 \pmod 2$. Thus we have

Lemma 8.4. Every sphere of $\sigma_1-\sigma_2$ meets a small neighbourhood M_1 of p with a multiplicity $\leqslant a_0+a_1$ unless $a_0 = a_1 = 1$.

8.4.3. Let $S \in \sigma_2 - \sigma_3$. Let $S' \to S$, $S' \notin \sigma_2$. Let $M_2 = N_2 \cup p \cup N_2'$ be a small neighbourhood of p.

Case (i). If $S' \in \sigma_1$, and $a_0+a_1 \neq 2$, S' does not meet $N_2(N_2')$ more than once. Therefore S' meets M_2 with a multiplicity a_o+a_1+2 and $\equiv a_o+a_1+a_2 \pmod 2$. Hence S' meets M_2 with a multiplicity $\leqslant a_o+a_1+a_2$.

Case (ii). If $S' \supset p$, $S' \notin \sigma_1$, than S' meets $N_2(N_2')$ at most twice. Then S' meets M_2 with a multiplicity $\leqslant a_o+4$ and $\equiv a_o+a_1+a_2 \pmod 2$. Thus S' meets M_2 with a multiplicity $\leqslant a_o+a_1+a_2$ unless $a_1 = a_2 = 1$.

Case (iii). If $p \not\subset S'$, then S' meets $N_2(N_2')$ at most three times. Thus S' meets M_2 with a multiplicity $\leqslant 6$ and $\equiv a_o+a_1+a_2 \pmod 2$. Thus S' meets M_2 with a multiplicity $\leqslant a_o+a_1+a_2$ unless $a_o+a_1+a_2 \leqslant 4$.

Thus we have

<u>Lemma 8.5.</u> <u>Let $S \in \sigma_2 - \sigma_3$ and let $S' \to S$, $S' \notin \sigma_2$. If</u>

<u>(i) $a_o+a_1 \neq 2$ and $S' \in \sigma_1$,</u>

<u>or if</u>

<u>(ii) $a_1+a_2 \neq 2$ and $S' \supset p$, $S' \notin \sigma_1$,</u>

<u>or finally, if</u>

<u>(iii) $a_o+a_1+a_2 > 4$, $S' \not\supset p$,</u>

<u>then there exists a neighbourhood M_2 of p which is met by</u>

S' at most $a_o+a_1+a_2$ times.

8.5. Properties of the Families $\sigma_3, \sigma_2, \sigma_1$, and σ_o.

Let $A = A_4 \cup p \cup A_4'$.

8.5.1. By Theorem 7.1, $S(\sigma_3)$ does not meet A outside p. Thus we can assign to p the multiplicity $a_o+a_1+a_2+a_3$ on $S(\sigma_3)$.

8.5.2. Lemma 8.6. There exists a neighbourhood $M_4 = N_4 \cup p \cup N_4'$ such that every sphere of σ_2 which meets $N_4 \cup N_4'$, meets $A_4 \cup A_4'$ exactly a_3 times.

In particular, no sphere of σ_2 meets M_4 more than a_3 times.

Proof: By Theorem 7.2, a sphere of σ_2 meets $A_4 (A_4')$ at most once. Thus it meets $A_4 \cup A_4'$ at most twice. By Lemma 8.1, every sphere close to $S(\sigma_3)$ will meet A with a multiplicity $\equiv a_o+a_1+a_2+a_3$ (mod 2). Hence $S(t;\sigma_2)$ will meet $A_4 \cup A_4'$ with a multiplicity $\equiv a_3$ if t is sufficiently close to p. Such a sphere will therefore meet $A_4 \cup A_4'$ exactly a_3 times. Thus we can assign to p the multiplicity

$a_o+a_1+a_2$ with respect to any circle of $\sigma_2-\sigma_3$.

8.5.3. Lemma 8.7. There exists a neighbourhood M_3 of M_4 which is met at most a_2+a_3 times outside p by every sphere of σ_1.

Proof: Because of Lemma 8.6, we have only to consider the spheres of $\sigma_1-\sigma_2$. If suffices to construct a one-sided neighbourhood $N_3' \subset N_4'$ of p such that any sphere of $\sigma_1-\sigma_2$ that meets N_3' twice will meet M_4 at most a_2+a_3 times outside p. By Lemma8.1, N_3' can be chosen so small that a sphere S of $\sigma_1-\sigma_2$ through two points of N_3' is close to $S(\sigma_3)$ and meets M_4 with a multiplicity $\equiv a_o+a_1+a_2+a_3$ (mod 2). Since S meets N_4 and N_4' not more than twice each, it will meet M_4 outside p at most four times. Since S meets M_4 at p with a multiplicity $\equiv a_o+a_1$ (mod 2), it meets M_4 outside p with a multiplicity $\equiv a_2+a_3$ (mod 2). Hence Lemma 8.7 holds if $a_2+a_3 > 2$.

Let $a_2+a_3 = 2$ so that $a_2 = a_3 = 1$. Let f denote the end-point of N_4. Suppose the points u,v,e,f, lie on $N_4 \cup f$

in the indicated order. Choose a small neighbourhood $N_3' \subset N_4'$ so that N_3' has no points in common with $S(\sigma_3)$, $S(e;\sigma_2)$, or $S(e,f;\sigma_1)$. We then have, as in section 8.2,

$$N_3 \subset \overline{S}(\sigma_3) \cap \underline{S}(e;\sigma_2) \cap \overline{S}(e,f,\sigma_1).$$

Now $a_2 = a_3 = 1$. Thus if $S(\sigma_3)$ intersects A at p, then $S(e;\sigma_2)$ supports and $S(e,f;\sigma_1)$ intersects, while if $S(\sigma_3)$ supports, then $S(e;\sigma_2)$ intersects and $S(e,f;\sigma_1)$ supports. Hence

$$N_3' \subset \underline{S}(\sigma_3) \cap \underline{S}(e;\sigma_2) \cap \underline{S}(e,f;\sigma_1),$$

or

$$N_3' \subset \overline{S}(\sigma_3) \cap \overline{S}(e;o_2) \cap \overline{S}(e,f;\sigma_1).$$

By relations (8.7) and (8.8), N_3' lies either in $\underline{S}(u,v;\sigma_1)$ or in $\overline{S}(u,v;\sigma_1)$. Thus N_3' does not meet $S(u,v;\sigma_1)$. By Lemma 8.1, any sphere S of σ_1-σ_2 through two points of N_3' meets M_4 with a multiplicity $\equiv a_o+a_1+1+1$ (mod 2). Thus it meets $N_4 \cup N_4'$ an even number of times. It meets N_4' exactly twice. From the above, S then cannot meet N_4 twice. Hence S and N_4 are disjoint, and S meets M_4 with the total multiplicity $a_o+a_1+2 = a_o+a_1+a_2+a_3$.

<u>8.5.4. Lemma 8.8. There exists a neighbourhood M_2 of M_3 which is met at most $a_1+a_2+a_3$ times outside p by every sphere of σ_0 unless $a_1+a_2+a_3 = 4$, $a_2 = 2$.</u>

<u>Proof</u>: In view of Lemma 8.7, we have only to consider the spheres of $\sigma_0 - \sigma_1$.

(i) By Lemma 8.1, N_2 can be chosen so small that a sphere S through p and three points of N_2 is close to $S(\sigma_3)$ and meets M_3 with a multiplicity $\equiv a_0+a_1+a_2+a_3 \pmod 2$. Since S meets N_2 and N_2' not more than three times each, it will meet M_3 outside p at most six times, Since S meets M_3 at p with a multiplicity $\equiv a_0$, it meets M_3 outside p with a multiplicity $\equiv a_1+a_2+a_3 \pmod 2$. Thus Lemma 8.8 holds if $a_1+a_2+a_3 > 4$.

(ii) Let $a_1+a_2+a_3 = 3$, so that $a_1 = a_2 = a_3 = 1$. Suppose that M_3 is so small that it has no points outside p in common with $S(\sigma_3)$, $S(d;\sigma_2)$, $S(d,e;\sigma_1)$ and $S(d,e,f;\sigma_0)$. Then

$$N_3 \subset \overline{S}(\sigma_3) \cap \underline{S}(d,\sigma_2) \cap \overline{S}(d,e;\sigma_1) \cap \underline{S}(d,e,f;\sigma_0)$$

while N_3' lies either in

$$\overline{S}(\sigma_3)\cap\overline{S}(d;\sigma_2)\cap\overline{S}(d,e;\sigma_1)\cap\overline{S}(d,e,f;\sigma_0),$$

or else in

$$\underline{S}(\sigma_3)\cap\underline{S}(d;\sigma_2)\cap\underline{S}(d,e;\sigma_1)\cap S(d,e,f;\sigma_0),$$

according as $a_o = 1$ or 2. By relation (8.9) and (8.10), N_3' lies either in $\overline{S}(t,u,v;\sigma_o)$ or in $\underline{S}(t,u,v;o_o)$. Thus N_3' does not meet $S(t,u,v;o_o)$.

(ii') To complete case (ii) we show that M_2 may be chosen so that a sphere of σ_o through two points u and t of N_2, and a point u' of N_2' does not meet $N_2 \cup N_2'$ elsewhere. Let $h \in N_3$, $h' \in N_3'$. Let M_h denote the heighbourhood of p bounded by h and h'. Let M_2 be a neighbourhood of p whose end-points lie in M_h. By the above, $S(t,u,h;\sigma_o)$ does not meet N_3'. Thus $h \not\subset S(u,u',t;\sigma_o)$.

$S(h';\sigma_2)$ intersects M_2 at p, and does not meet M_2 elsewhere (cf. Lemma 8.6). Hence there is a neighbourhood $N_1 \subset N_2$ such that $S(h',u,t;\sigma_o)$ meets $N_2 \cup N_2'$ with an even multiplicity when u and $t \in N_1$. By (ii), $S(h',u,t;\sigma_o)$ cannot meet $N_2 \cup N_2'$ four times, hence it meets M_2 only at u and t outside p, and intersects M_2 at these points. Thus $S(h',u,t;\sigma_0)$ does not meet N_2'. Hence if $u' \in N_2'$, $h' \not\subset S(u',u,t;\sigma_0)$. By section (8.5.3), $S(u,u';\sigma_1)$ intersects M_3 at u and u', and does not meet M_3 elsewhere outside p. Hence $S(u,u';\sigma_1)$ separates or does not separate h and H' according as $a_0 = 2$ or 1. If t is close to p, the same holds for $S(u,u',t;\sigma_0)$.

But $S(u,u',t;\sigma_0)$ does not pass through h or h' for any $u,t \in N_1$, $u' \in N_2'$. Then as t moves in N_1, $S(u,u',t;\sigma_0)$ meets $N_h \cup N_h'$ an odd number of times. By (ii) it meets N_h and N_h' at most twice each, hence it must meet M_h exactly three times outside p. Thus any sphere of σ_0 through two points of N_1 and a point of N_2' meets $N_1 \cup p \cup N_2'$ nowhere else.

The fifteen cases for which $a_1+a_2+a_3 \neq 4$ are now disposed of. There remain the six cases for which $a_1+a_2+a_3 = 4$, $a_2 \neq 2$.

(iii) The cases $a_1 = 2$, $a_2 = a_3 = 1$. Let $e,e' \in M_3$, $e \in N_3$, $e' \in N_3'$. By Lemma 8.5, there is a neighbourhood

$M_2 \subset M_3$ of p such that no sphere of σ_o through e or e' meets $N_2 \cup N_2'$ in four points. We shall prove that a sphere S of σ_o through two points v and u of N_2, and two points v' and u' of N_2' does not meet M_2 elsewhere. By Lemma 8.7, $S(v,v';\sigma_1)$ does not meet M_3 elsewhere. It intersects N_2 at v (N_2' at v') and meets M_2 at p with a multiplicity $\equiv a_o$. Let $M_1 = N_1 \cup p \cup N_1' \subset M_2$ be so small that

(a) no sphere of σ_o through four points of $N_1 \cup N_1'$ passes through v or v';

(b) no sphere of σ_o through two points of N_1 or N_1' passes through both v and v'.

(cf. Lemma 8.5 and Theorem 7.2). Thus v and v' do not lie in M_1. By Lemma 8.1, there exists a neighbourhood $M \subset M_1$ of p such that $S(v,v',t;\sigma_o)$ meets M_1 with a multiplicity $\equiv a_o$ if $t \in M$. From the above, $S(v,v',t;\sigma_o)$ meets $N_1 \cup N_1'$ with an even multiplicity, and it meets N_1 and N_1' at most once each. Thus $S(v,v',t;\sigma_o)$ intersects N_1 only at t, and intersects N_1' at one point t' only. Let t move on N_2 towards u. Then $S(v,v',t;\sigma_o)$ does not pass through e or e', t' does not converge to p, and $S(v,v',t;\sigma_o)$ continues to meet N_e and N_e' with an even multiplicity, i.e., exactly twice each. Thus when $t=u$ or $t'=u'$, $S(v,v',t;\sigma_o)$ coincides with S.

(iv) The cases $a_1=a_2=1$, $a_3=2$. Choose g, f, e,e',f',g' on $N_3 \cup N_3'$ in the indicated order so that no sphere of σ_1 through any two of these points is a sphere of σ_2. Given g and g', we can choose f and f' so that $S(f;\sigma_2)$ $\left(S(f'\sigma_2)\right)$ meets N_g' (N_g). Now choose e (e') between p and f (f') such that $S(f;\sigma_2)$ $\left(S(f';\sigma_2)\right)$does not meet $N_e' \cup e'$ $(N_e \cup e)$.

By section 8.4.2, there is a neighbourhood $M_2 \subset M_e$ of p such that a sphere through p and any two of the points g,f,e,e',f',g' meets $N_2 \cup N_2'$ at most once. Let $v \in N_2$, $v' \in N_2'$ so that the sphere $S(f,v';\sigma_1)$ converges to $S(f;\sigma_2)$ if v' converges to p.

If t is sufficiently close to p, $S(f,v',t;\sigma_o)$ is close to $S(f,v';\sigma_1)$, which in turn is close to $S(f;\sigma_2)$.

Thus by section 8.4.2, $S(f,v',t;\sigma_o)$ meets N_g and N'_g twice each, and meets N'_e only once. From the above, $S(f,v',t;\sigma_o)$ cannot pass through g,g', or e'. Let $M_1 \subset M_2$ be a neighbourhood of p which does not contain t. By Lemma 8.7, there exists a still smaller neighbourhood $M_o \subset M_1$ such that $S(f,v',t;\sigma_o)$ meets M_o at p only. Thus $S(f,v',t;\sigma_o)$ meets N_g-N_o and $N'_g-N'_o$ exactly twice each, and it meets N'_e-N_o exactly once. As t moves in M_2, $S(f,v',t;\sigma_o)$ meets N_g-N_o and $N'_g-N'_o$ with an even multiplicity, i.e. exactly twice each, and it meets $N'_e-N'_1$ with an odd multiplicity ≤ 2, i.e., exactly once.

Hence $S(f,v',t;\sigma_o)$ meets N_2 and N'_2 exactly once each. Similarly $S(f',v,t;\sigma_o)$ meets N_2 and N'_2 exactly once each.

Consider a small neighbourhood $M \subset M_f$ of p, and any sphere of σ_1 which meets $N \cup N'$ twice. This sphere is close to a sphere of σ_2, and meets some neighbourhood of p in M_f with a multiplicity $\equiv a_o+1+1$. But a sphere of σ_1 meets M_f at p with a multiplicity $\equiv a_o+1$; hence it meets some neighbourhood of p outside p with an odd multiplicity, i.e., at least three times. Thus there exists a neighbourhood $M \subset M_f$ such that any sphere of σ_1 that meets $N \cup N'$ twice meets M_f at least three times outside p. By Lemma 8.7, it does not meet M_f outside p more than three times.

Thus $S(v,v';\sigma_1)$ meets M_f in exactly one more point say u'. If t is close to p, $S(v,v',t;\sigma_0)$ meets M_f exactly a_0 times at p, once each at v,v',t and near u' and nowhere else. By Lemma 8.7, there is a small neighbourhood $M_\alpha \subset M_2$ of p which does not contain t, and which is not met by $S(v,v',t;\sigma_0)$ outside p. As t moves in N_2, $S(v,v',t;\sigma_0)$ meets $N_f - N_\alpha$ and $N'_f - N'_\alpha$ each with an even multiplicity, i.e. twice each. Thus no sphere of σ_0 meets M_2 more than four times outside p.

8.6. Proof of Theorem 8.1 When $a_0+a_1+a_2+a_3 \geq 6$.

If is sufficient to show that there is a one-sided neighbourhood $N'_0 \subset N'_2$ of p such that no sphere S through four points of $N'_0 \cup p$ meets M_4 more than $a_0+a_1+a_2+a_3$

times. On account of Lemma 8.8, we need only consider spheres S which do not pass through p. By Lemma 8.1, N'_o can be chosen such that any S close to $S(\sigma_3)$ meets M_4 with a multiplicity $\equiv a_o+a_1+a_2+a_3 \pmod 2$. Since $p \not\subset S$, and since S meets N_4 (N'_4) at most four times, it will meet M_4 at most eight times. This yields Theorem 8.1 in this case.

8.7. The case $a_o+a_1+a_2+a_3=4$.

On account of Lemma 8.8, we need only consider the spheres which do not contain p.

8.7.1. A small neighbourhood N'_o of N'_1 does not meet S(d,e,f,g). Let the points p,t,u,v,w,d,e,f,g,h, lie on $A_4 \cup p$ (A_4 is a one-sided neighbourhood of A and is of order four) in the indicated order. For any sphere S with $h \not\subset S$, we make the convention that $h \subset \bar{S}$. Hence, if $N_1 \subset N_d$,

$$N_1 \subset \bar{S}(d,e,f,g) \cap \underline{S}(d,e,f;\sigma_o) \cap \bar{S}(d,e;\sigma_1) \cap \underline{S}(d;\sigma_2) \cap \bar{S}(\sigma_3),$$

and since $a_o = a_1 = a_2 = a_3 = 1$,

$$N'_o \subset \bar{S}(d,e,f,g) \cap \bar{S}(d,e,f;\sigma_o) \cap \bar{S}(d,e;\sigma_1) \cap \bar{S}(d;\sigma_2) \cap \bar{S}(\sigma_3).$$

Let t,u,v,w lie on N_1. By a method similar to that employed in section 8.2, we find that

$$\bar{S}(t,u,v,w) \supset \bar{S}(u,v,w,d) \cap \bar{S}(p,u,v,w)$$

$$\supset \left[\bar{S}(v,w,d,e) \cap \bar{S}(p,v,w,d)\right] \cap \left[\bar{S}(p,v,w,d) \cap \bar{S}(v,w;\sigma_1)\right]$$

$$\supset \left[\bar{S}(w,d,e,f) \cap \bar{S}(p,w,d,e)\right] \cap \left[\bar{S}(p,w,d,e) \cap \bar{S}(w,d;\sigma_1)\right]$$

$$\cap \left[\bar{S}(w,d;\sigma_1) \cap \bar{S}(w;\sigma_2)\right]$$

$$\supset \left[\bar{S}(d,e,f,g) \cap \bar{S}(p,d,e,f)\right] \cap \left[\bar{S}(d,e,f;\sigma_0) \cap \bar{S}(d,e;\sigma_1)\right]$$

$$\cap \left[\bar{S}(d,e;\sigma_1) \cap \bar{S}(d;\sigma_2)\right] \cap \left[\bar{S}(d;\sigma_2) \cap \bar{S}(\sigma_3)\right]$$

$$= \bar{S}(d,e,f,g) \cap \bar{S}(d,e,f;\sigma_0) \cap \bar{S}(d,e;\sigma_1) \cap \bar{S}(d;\sigma_2) \cap \bar{S}(\sigma_3).$$

Thus a sphere through four points of N_1 does not meet N_0'. Symmetrically, there is a neighbourhood N_0 such that a sphere through four points of N_1' does not meet N_0.

<u>8.7.2.</u> Let $M_0 = N_0 \cup p \cup N_0'$. Let $h,k,\ell \in N_0, h',k',\ell' \in N_0'$. By Lemmas 8.2, 8.3, and 8.8, there is a neighbourhood $M \subset M_0$ of p such that a sphere through any three of the points h,k,ℓ,h',k',ℓ' and a point of M does not meet M_0 elsewhere. Thus if $u,t, \in N$ and $u',t' \in N'$, the spheres $S(k,h,u',t')$, $S(k,h',u',t)$, $S(k',h,u,t')$ and $S(k',h',u,t)$ do not pass through ℓ or ℓ'. By section 8.7.1, these spheres do not

support M_o at any other point and, by Lemma 8.8, they do not pass through p. Since $S(k,h,u',p)$, $S(k',h,u,p)$ and $S(k',h',u,p)$ do not meet M_o elsewhere, there is a neighbourhood $M_\alpha = N_\alpha \cup p \cup N'_\alpha$ of p, $M_\alpha \subset M$ such that $S(k,h,u',t')$, $S(k',h,u,t')$, $S(k,h',u',t)$ and $S(k',h',u,t)$ do not meet M_o elsewhere if $t \in N_\alpha$, and $t' \in N'_\alpha$. Thus each of these spheres meets N_{ℓ} (N'_{ℓ}) exactly twice. Letting t and t' move on N and N' respectively, we find that $S(k,h,u',t')$, $S(k',h,u,t')$, $S(k,h',u',t)$ and $S(k',h',u,t)$ also meet N_{ℓ} and N'_{ℓ} with an even multiplicity, i.e. exactly twice each. Thus the spheres $S(h,u,u',t')$ and $S(h',u,u',t)$ do not pass through k, k', or p when $u,t \in N$, and $u',t' \in N'$. Since $S(h',u,u',p)$ and $S(h,u,u',p)$ do not meet M_o elsewhere, there is a small neighbourhood $M_\beta = N_\beta \cup p \cup N'_\beta$, $M_\beta \subset M_o$, such that $S(h',u,u',t)$ and $S(h,u,u',t')$ do not meet M_o again if $t \in N_\beta$ and $t' \in N'_\beta$. Thus k, p, and k' lie on the same side of these spheres.

As t and t' range on N and N' respectively, $S(h,u,u',t')$ and $S(h',u,u',t)$ do not pass through p and continue to meet N_k and N_k' with an even multiplicity, i.e., exactly four times. Thus $S(u,u',t,t')$ does not pass through h or h' if $u,t, \in N$, $u',t' \in N'$. Since $S(u,u',t,p)$ does not meet M elsewhere, the same holds for $S(u,u',t,t')$ if t' is sufficiently close to p, $t' \in N'$. Thus $S(u,u',t,t')$ does not separate h, p, or h'. As t' moves on N', $S(u,u',t,t')$ does not pass through h, h', or p, and by section 8.7.1 it does not support M at any point. Thus $S(u,u',t,t')$ does not meet M_h elsewhere if $u,t, \in N$ and $u',t' \in N'$.

<u>8.8. The Cases, (2,2,1,1;1) and (2,2,1,1;3)</u>.

Let d and d' belong to the neighbourhood M of Lemma 8.8 (iii). On account of Lemma 8.8, we need only consider spheres which do not pass through p. By Lemma 8.5, we can choose in M a neighbourhood $M_o = N_o \cup p \cup N_o'$ of p such that no sphere through six points of M_o passes through d or d'. We shall prove that any sphere which passes through three points w, v, u, of N_o, and three points w', v', u', of N_o', does not meet M_o again. By Lemma 8.7, $S(w,w';\sigma_1)$ does not meet

M outside p,w, and w'. It intersects M at w and w', and supports M at p. By Lemma 8.6 there is a neighbourhood $M_\alpha = N_\alpha \cup p \cup N'_\alpha$ of p such that $S(w,w',t;\sigma_0)$ meets N_α (N'_α) at most once. There is a neighbourhood $M_\beta \subset M_\alpha$ of p such that $S(w,w',t;\sigma_0)$ meets M_α with an even multiplicity if $t \in N_\beta$. Since $S(w,w',t;\sigma_0)$ supports M_0 at p, it meets $N_\alpha \cup N'_\alpha$ with an even multiplicity, i.e., it meets N_α once at t and N'_α once at some point t'. Let M_γ be a neighbourhood of p which does not contain t. Then there is a neighbourhood M_δ, with end-points δ and δ', such that $S(w,w',t;\sigma_0)$ does not pass through $N_\delta \cup N'_\delta$ if $t \in N_0 - N_\delta$. As t moves on $N_0 - N_\delta$, $S(w,w', t;\sigma_0)$ does not pass through d, d', δ or δ'. Thus $S(w,w',t;\sigma_0)$ meets $N_d - N_\delta$ ($N'_d - N'_\delta$) with an even multiplicity, i.e. twice. Suppose that t reaches v while t' meets N'_0 between δ' and v'. Thus $S(w,w',v;\sigma_0)$ meets N'_0 at a point t' between δ' and v' and does not meet d elsewhere. It intersects M at w,w',v, and supports M at p.

Thus $S(w,w',v,t')$ supports M at p and does not meet

M elsewhere. Let t move in N towards v'. Then at first, S(w,w!v,t') meets a small neighbourhood of p exactly twice, does not pass through p, and does not meet this neighbourhood twice on one side of p; i,e., it meets this neighbourhood once on each side of p. Thus S(w,w!v,t') meets N (N') with an odd multiplicity. As t' moves on N towards v', S(w,w!v,t') does not pass through d,d', or p, hence it does not when t'= v', and S(w,w!v,t') meets N (N') with an odd multiplicity, i.e., three times each.

8.9. The Cases (2,1,1,2;2) and (2,1,1,2;3).

By Lemma 8.8, we need only consider spheres which do not contain p. Aside from p and the neighbourhood M_2 of Lemma 8.8 (iv), the points and neighbourhoods described in this section do not refer to those previously mentioned.

Choose e,f,g,e',f',g' on M_2 such that any sphere of σ_1 through two of these points does not belong to σ_2. By Lemma 8.4 there is a neighbourhood $M_1 = N_1 \cup p \cup N_1'$ of p such that every sphere through two of the points g, f, g', f' meets M_1 at most three times. For a given g and g', choose f (f') such that $S(f;\sigma_2)$ $\big(S(f';\sigma_2)\big)$ meets N'_g (N_g). Next choose e (e') between p and f (f') such that $S(f;\sigma_2)$ $\big(S(f';o_2)\big)$ does not meet $e' \cup N'_e$ ($e \cup N_e$).

If u' is close to p, $u' \in N'_e$, $S(f,u';\sigma_1)$ is close to $S(f;\sigma_2)$ and meets N_g at f only, N'_e at u' only, and N'_g once outside N'_e. If t is sufficiently close to p, $S(f,u';t;\sigma_0)$ will meet N_g and N'_g twice each, but N'_e only once. By Lemma 8.4, as t moves in N_1, $S(f,u';t;\sigma_0)$ remains close to $S(f;\sigma_2)$ and continues to meet N'_g outside N'_e. Thus $S(f,u';t;\sigma_0)$ meets N_g (N'_g) exactly twice. By Lemmas 8.1 and 8.4, there is a neighbourhood M_o such that $S(f,u';t,r)$ meets M_o with an even multiplicity ≤ 3, i.e., twice. Since $S(f,v';t;\sigma_0) \notin \sigma_1$, $S(f,u';t,r)$ must meet N_o (N'_o) once each. Thus $S(f,u';t,r)$ meets N_g and N'_g three times each. From the above, if r moves in N_1, $S(f,u';t,r)$ does not pass through p, g, g', or e', and p and g (g') will lie on opposite sides while p and e' will lie on the same side of S(f,u!t,r). Thus S(f,u!t,r) will meet N_g (N'_g) exactly three times and N'_e an even number of times, i.e., exactly twice. Hence S(f,u!t,r) will meet N_1 and N'_1 only twice each, i.e., it will meet M_1 only four times. Similarly, S(f!u!t,r) meets M_1 only four times. Thus any sphere through five points of M_1 does not pass through f or f'.

We shall prove that a sphere through three points w,v,u, of N_1, and three points of N'_1 does not meet M_f elsewhere.

Starting with $S(t;\sigma_2)$, we note that it meets N'_f again if t is sufficiently close to p. Letting t move in N_1 towards w, we can assume that $S(w;\sigma_2)$ meets N'_1 at a point t' between p and w'. If t is close to p, $S(w,t;\sigma_1)$ meets N'_1 at t' and does not meet M_2 elsewhere. Let M_α be a small neighbourhood of p which does not contain t. There is a still smaller neighbourhood $M_\beta \subset M_\alpha$ such that $N_\beta \cup N'_\beta$ is not met by $S(w,t;\sigma_1^-)$. Thus as t moves on $N_1 - N_\beta$ towards v, $S(w,t;\sigma_1^-)$ meets $N_1 - N_\beta$ at w and t only, and meets $N'_1 - N'_\beta$ once. Eventually we get either (i) $S(w,v;\sigma_1^-)$ meeting $N'_1 - N'_\beta$ between p and w', or (ii) $S(w,w';\sigma_1^-)$ meeting N_1 between p and v.

Case (i): If r' is close to p, $S(w,v,r';\sigma_o^-)$ meets M_f exactly four times outside p. Let M_γ be a small neighbourhood of p not including r'. Thus as r' moves towards w', $S(w,v,r';\sigma_o^-)$ meets N'_f an even number of times, i.e., twice. Thus we obtain $S(w,v,w';\sigma_o^-)$ when a point reaches w', and this sphere meets N'_1 again at a point t' between p and w'.

Case (ii): If r' is close to p, $S(w,w',r';\sigma_o^-)$ meets M_2 again only at one other point of N_1 between p and v. As above, as r' moves towards v', there is a neighbourhood of p which is not met outside p by $S(w,w',r';\sigma_o^-)$. Thus, as above, we can obtain $S(w,w',v';\sigma_o^-)$ meeting N_1 between p and v, or $S(w,w';v;\sigma_o)$ meeting N_1' between p and v'.

Let r' move towards v'. Then $S(w,v,w';r')$ does not pass through p if $r' \neq r$, and it does not pass through f or f' as long as it contains more than four points of M_1. It meets M_1 near p with an even multiplicity and it cannot meet N_1 (N_1') twice arbitrarily close to p. Thus it meets N_1 (N_1') once. When r' reaches v, we get $S(w,v,w';v')$ which meets N_f (N_f') an odd number of times ≥ 2, i.e., three times.

<u>8.10. Conjecture.</u>

<u>Let p decompose A into two arcs of order four. Then p is strongly differentiable if and only if p is a differentiable point, and $a_o = a_1 = a_2 = 1$.</u>

CHAPTER IX

DIFFERENTIABLE POINTS OF ARCS IN CONFORMAL N-SPACE

9.1. Introduction.

In this chapter we generalize to n dimensions the work of Chapters 2 and 5. The change from three to n dimensions is not as pronounced as the change from two to three dimensions, although the necessarily complicated notation, and the absence of any visual aid, make it appear quite difficult.

9.2. Differentiability.

Let p be a fixed point of an arc A, and let t be a variable point of A. Let $1 \leq m < n$. If $p, P_1, \ldots, P_{m+1}$ do not lie on the same (m-1)-sphere, then there exists a unique m-sphere $S_o^{(m)} = S^{(m)}(P_1, \ldots, P_{m+1}; \tau_o)$ through these points. We call A <u>(m+1)-times differentiable</u> at p if the following sequence of conditions is satisfied:

$\Gamma_r^{(m)}$: If the parameter t is sufficiently close to, but different from, the parameter p, then the m-sphere $S^{(m)}(P_1,\ldots,P_{m+1-r},t;\tau_{r-1})$ is uniquely defined. It converges if t tends to p. Thus the limit sphere

$$S_r^{(m)} = S^{(m)}(P_1,\ldots,P_{m+1-r};\tau_r)$$

will be independent of the way t converges to p $\big(r=1,2,\ldots,m+1$; condition $\Gamma_{m+1}^{(m)}$ reads: $S^{(m)}(t;\tau_m)$ exists and converges to $S_{m+1}^{(m)} = S^{(m)}(\tau_{m+1})\big)$.

It is convenient to use the symbols $S_o^{(o)}$ to denote pairs of points P,p, and $S_1^{(o)}$ to denote the point pair p,p (i.e., the point p).

We call A <u>once-differentiable</u> at p if $\Gamma_1^{(1)}$ is satisfied. The point p is called a <u>differentiable point</u> of A if A is n-times differentiable at p.

Let $\tau_r^{(m)}$ denote the family of all the $S_r^{(m)}$'s. Thus $\tau_{m+1}^{(m)}$ consists only of $S_{m+1}^{(m)}$, the <u>osculating m-sphere</u> of A at p.

<u>9.3. The Structure of the Families $\tau_r^{(m)}$ of m-spheres through p.</u>

<u>Theorem 9.1. Suppose A satisfies condition $\Gamma_1^{(m)}$ at p. Let $S^{(m-1)}$ be any (m-1)-sphere. Then there is a neighbourhood N of p on A such that if $t \in N$, $t \neq p$, then $t \not\subset S^{(m-1)}$, $(m=1,2,\ldots,n-1)$.</u>

<u>Proof</u>: The assertion is evidently true if $p \not\subset S^{(m-1)}$. Suppose $p \subset S^{(m-1)}$. Choose points $P_1,\ldots,P_m$ on $S^{(m-1)}$ such that $p,P_1,\ldots P_m$ are independent. If the parameter t is sufficiently close to, but different from, the parameter p, condition $\Gamma_1^{(m)}$ implies that $S^{(m)}(P_1,\ldots,P_m,t;\tau_o)$ is uniquely defined. Thus $t \not\subset S^{(m-1)}(P_1,\ldots,P_m;\tau_o) = S^{(m-1)}$.

<u>Corollary 1. If A satisfies condition $\Gamma_1^{(m)}$ at p, and $S^{(k)}$ is any k-sphere, then $t \not\subset S^{(k)}$ when the parameter t is sufficiently close to, but different from, the parameter p $(k=0,1,\ldots,m-1)$.</u>

In particular, this holds when $m = n-1$.

<u>Theorem 9.2. Let $1 < m < n$; $1 \leq k \leq m$. If A satisfies $\Gamma_1^{(m)},\ldots,\Gamma_k^{(m)}$ at p, then $\Gamma_1^{(m-1)},\ldots,\Gamma_k^{(m-1)}$ will hold</u>

there and

(9.1) $S^{(m-1)}(P_1,\ldots P_{m-r};\tau_r) = \prod_P S^{(m)}(P_1,\ldots,P_{m-r},P;\tau_r)$.

Conversely, let A satisfy $\Gamma_1^{(m-1)},\ldots,\Gamma_k^{(m-1)}$ at p, and let $S_m^{(m-1)} \neq p$ if $k=m$. If $P_{m-r+1} \not\subset S^{(m-1)}(P_1,\ldots,P_{m-r};\tau_r)$, then $\Gamma_r^{(m)}$ will hold for the points $P_1,\ldots,P_{m-r+1}$ and

(9.2) $S^{(m)}(P_1,\ldots,P_{m-r+1};\tau_r)=S^{(m)}\left[P_{m-r+1};S^{(m-1)}(P_1,\ldots,P_{m-r};\tau_r)\right]$

$(r=1,\ldots k)$.

Remark: In general, $\Gamma_1^{(m-1)},\ldots,\Gamma_k^{(m-1)}$ do not imply $\Gamma_1^{(m)},\ldots,\Gamma_k^{(m)}$ (cf. e.g., § 5.5).

Proof: (by induction with respect to k): Suppose $k=1$; $1<m<n$. Let $\Gamma_1^{(m)}$ hold. If $P_1,\ldots,P_{m-1},P,p$ are independent points, $S^{(m)}(P_1,\ldots,P_{m-1},P,t;\tau_o)$ exists when t is sufficiently close to p, $t \neq p$, $t \in A$. Thus $P_1,\ldots,P_{m-1},P$, t,p, are also independent, $S^{(m-1)}(P_1,\ldots,P_{m-1},t;\tau_o)$ exists, and

$$S^{(m-1)}(P_1,\ldots,P_{m-1},t;\tau_o) = \prod_P S^{(m)}(P_1,\ldots,P_{m-1},P,t;\tau_o).$$

If $t \to p$, $S^{(m)}(P_1,\ldots,P_{m-1},P,t;\tau_o)$ converges, and hence

$S^{(m-1)}(P_1,\ldots,P_{m-1},t;\tau_o)$ also converges, $\Gamma_1^{(m-1)}$ is satisfied, and

$$S^{(m-1)}(P_1,\ldots,P_{m-1};\tau_1) = \prod_P S^{(m)}(P_1,\ldots,P_{m-1},P;\tau_1).$$

Next, suppose that $\Gamma_1^{(m-1)}$ is satisfied, and $P_m \not\subset S^{(m-1)}(P_1,\ldots,P_{m-1};\tau_1)$. Then $P_m \not\subset S^{(m-1)}(P_1,\ldots,P_{m-1},t;\tau_o)$ when t is sufficiently close to p, $t \in A$, $t \neq p$, and

$$S^{(m)}(P_1,\ldots,P_m,t;\tau_o) = S^{(m)}\left[P_m;S^{(m-1)}(P_1,\ldots,P_{m-1},t;\tau_o)\right]$$

exists. Hence when $t \to p$, $S^{(m)}(P_1,\ldots,P_m,t;\tau_o)$ converges, $\Gamma_1^{(m)}$ is satisfied relative to the points $P_1,\ldots,P_m$, and

$$S^{(m)}(P_1,\ldots,P_m;\tau_1) = S^{(m)}\left[P_m;S^{(m-1)}(P_1,\ldots,P_{m-1};\tau_1)\right].$$

Thus Theorem 9.2 is true when $k=1$.

Assume that Theorem 9.2 holds when k is replaced by $1,2,\ldots,h$, where $1 \leq h < k \leq m$.

Let $\Gamma_1^{(m)},\ldots,\Gamma_{h+1}^{(m)}$ hold. Then $S^{(m)}(P_1,\ldots,P_{m-h-1},P,t;\tau_h)$ exists when t is sufficiently close to p, $t \neq p$, $t \in A$. Now $\Gamma_1^{(m)},\ldots,\Gamma_h^{(m)}$ imply $\Gamma_1^{(m-1)},\ldots,\Gamma_h^{(m-1)}$. If $h=m-1$, $\Gamma_h^{(m-1)} = \Gamma_{m-1}^{(m-1)}$ implies that $S_h^{(m-1)} = S^{(m-1)}(t;\tau_{m-1})$ exists, if $t \neq p$. If $h < m-1$, $\Gamma_1^{(m-1)}, \ldots, \Gamma_h^{(m-1)}$ imply $\Gamma_1^{(m-2)}, \ldots, \Gamma_h^{(m-2)}$. Thus $S^{(m-2)}(P_1, \ldots, P_{m-h-1}; \tau_h)$ exists. Furthermore, $\Gamma_1^{(m-1)}$ and Theorem 9.1 imply that $t \not\subset S^{(m-2)}(P_1, \ldots, P_{m-h-1}; \tau_h)$. But then Theorem 9.2, with k replaced by h, implies that

$$S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h) = S^{(m-1)}\left[t; S^{(m-2)}(P_1, \ldots, P_{m-h-1}; \tau_h)\right]$$

exists. By Theorem 9.2 again, with k replaced by h,

$$S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h) = \prod_P S^{(m)}(P_1, \ldots, P_{m-h-1}, P, t; \tau_h).$$

When $t \to p$, $S^{(m)}(P_1, \ldots, P_{m-h-1}, P, t; \tau_h)$ converges, hence $S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h)$ also converges, $\Gamma_{h+1}^{(m-1)}$ is satisfied, and

$$S^{(m-1)}(P_1, \ldots, P_{m-h-1}; \tau_{h+1}) = \prod_P S^{(m)}(P_1, \ldots, P_{m-h-1}, P; \tau_{h+1}).$$

Next suppose $\Gamma_1^{(m-1)}, \ldots, \Gamma_{h+1}^{(m-1)}$ hold, and let $P_{m-h} \not\subset S^{(m-1)}(P_1, \ldots, P_{m-h-1}; \tau_{h+1})$. Then $P_{m-h} \not\subset S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h)$ if t is sufficiently close to p, $t \in A$, $t \neq p$. But Theorem 9.2, with k replaced by h, then implies that

$$S^{(m)}(P_1,\ldots,P_{m-h-1},P_{m-h},t;\tau_h)=S^{(m)}\left[P_{m-h};S^{(m-1)}(P_1,\ldots,P_{m-h-1},t;\tau_h)\right]$$

exists. Hence when $t \to p$, $S^{(m)}(P_1,\ldots,P_{m-h},t;\tau_h)$ converges, $\Gamma_{h+1}^{(m)}$ is satisfied for $P_1,\ldots,P_{m-h}$, and

$$S^{(m)}(P_1,\ldots,P_{m-h};\tau_{h+1}) = S^{(m)}\left[P_{m-h};S^{(m-1)}(P_1,\ldots,P_{m-h+1};\tau_{h+1})\right].$$

Corollary 1. Let $1 \leq m < n$. If A is $(m+1)$-times differentiable at p, then it is m-times differentiable there.

Corollary 2. If A satisfies $\Gamma_1^{(n-1)},\ldots,\Gamma_{m+1}^{(n-1)}$ at P, then it is $(m+1)$-times differentiable there $(0 \leq m < n)$.

Corollary 3.

$$S_m^{(m-1)} \subset S_{m+1}^{(m)} \qquad (m = 1,2,\ldots,n-1).$$

Proof: By relation (9.1)

$$S^{(m)}(t;\tau_m) \supset \prod_{P} S^{(m)}(P;\tau_m) = S_m^{(m-1)}.$$

Hence $S_{m+1}^{(m)} \supset S_m^{(m-1)}$.

The last remark implies

Corollary 4. Let $1 \leq m < n$. If $S_{m+1}^{(m)} = p$, then $S_{r+1}^{(r)} = p$ $(r = 0,1,\ldots, m-1)$.

Thus there is an index i, where $1 \leq i \leq n$ such that $S_{r+1}^{(r)} = p$

for $r = 0,1,\ldots, i-1$, but $S_{r+1}^{(r)} \neq p$ if $r \geq i$.

Corollary 5. Let $1 \leq m < n$; $1 \leq r \leq m$. Then

$$S^{(m)}(P_1,\ldots,P_{m+1-r};\tau_r) \supset S^{(m-1)}(P_1,\ldots,P_{m+1-r};\tau_{r-1}).$$

Proof:

$$S^{(m)}(P_1,\ldots,P_{m+1-r};\tau_r) = \lim_{t \to p} S^{(m)}(P_1,\ldots,P_{m+1-r},t;\tau_{r-1})$$

$$\supset S^{(m-1)}(P_1,\ldots,P_{m+1-r};\tau_{r-1}).$$

Form Corollary 5, we get

Corollary 6. Let $1 \leq m < n$; $1 \leq r \leq m$. If P_{m+2-r} $\subset S^{(m)}(P_1,\ldots, P_{m+1-r};\tau_r)$ and $P_{m+2-r} \not\subset S^{(m-1)}(P_1,\ldots,$ $P_{m+1-r};\tau_{r-1})$ then

$$S^{(m)}(P_1,\ldots,P_{m+1-r};\tau_r) = S^{(m)}(P_1,\ldots,P_{m+2-r};\tau_{r-1}).$$

Theorem 9.3. Let $1 < r \leq m < n$. Suppose $\Gamma_1^{(m)},\ldots,$ $\Gamma_r^{(m)}$ are satisfied at p.

(i) If $S_r^{(r-1)} \neq p$, $\tau_r^{(m)}$ consists of all the m-spheres through $S_r^{(r-1)}$.

(ii) Let $S_r^{(r-1)} = p$. Choose any $S_r^{(r)} \in \tau_r^{(r)}$. Then $\tau_r^{(m)}$ is the set of all the m-spheres which touch $S_r^{(r)}$ at p.

Proof of (i): By Theorem 9.2, equation (9.1),

$$S^{(m)}(P_1,\ldots,P_{m+1-r};\mathcal{T}_r)\supset S^{(m-1)}(P_1,\ldots,P_{m-r};\mathcal{T}_r)$$

$$\supset\ldots\supset S^{(r)}(P_1;\mathcal{T}_r)\supset S_r^{(r-1)}.$$

Let $S^{(m)}$ be any m-sphere through $S_r^{(r-1)}$. By Theorem 9.2, if

$P_1\subset S^{(m)}$, $P_1\not\subset S_r^{(r-1)}$,

$$S^{(r)}(P_1;S_r^{(r-1)})=S^{(r)}(P_1;\mathcal{T}_r)\subset S^{(m)}.$$

Suppose $S^{(k)}(P_1,\ldots,P_{k+1-r};\mathcal{T}_r)\subset S^{(m)}$, $(r\leq k<m)$. Choose

$P_{k+2-r}\subset S^{(m)}$, $P_{k+2-r}\not\subset S^{(k)}(P_1,\ldots,P_{k+1-r};\mathcal{T}_r)$. Then by

Theorem 9.2

$$S^{(k+1)}(P_1,\ldots,P_{k+2-r};\mathcal{T}_r)$$

$$=S^{(k+1)}\left[P_{k+2-r};S^{(k)}(P_1,\ldots,P_{k+1-r};\mathcal{T}_r)\right]\subset S^{(m)}.$$

For $k=m-1$, this yields $S^{(m)}(P_1,\ldots,P_{m+1-r};\mathcal{T}_r)=S^{(m)}$. Thus

$S^{(m)}\in\mathcal{T}_r^{(m)}$.

Proof of (ii): Suppose $S_r^{(r-1)}=p$. As above, we have

$$S_r^{(m)}=S^{(m)}(P_1,\ldots,P_{m+1-r};\mathcal{T}_r)\supset\ldots\supset S^{(r)}(P_1;\mathcal{T}_r).$$

Let $S^{(r)}(Q;\mathcal{T}_r)$ be any $S_r^{(r)}\in\mathcal{T}_r^{(r)}$. By Theorem 9.2, equation (9.1),

$$S^{(r)}(P,t;\mathcal{T}_{r-1})\cap S^{(r)}(Q,t;\mathcal{T}_{r-1})\supset S^{(r-1)}(t;\mathcal{T}_{r-1}).$$

Let P and Q be variable points and let $S^{(r-1)}$ be a variable (r-1)-sphere converging to a fixed point. Suppose there is an (n-1)-sphere which separates this point from P and Q. Then

$$\lim \measuredangle \left[S^{(r)}\left(P;S^{(r-1)}\right);\ S^{(r)}\left(Q;S^{(r-1)}\right)\right] = 0$$

whether or not the spheres $S^{(r)}\left[P;S^{(r-1)}\right]$ and $S^{(r)}\left(Q;S^{(r-1)}\right)$ themselves converge.

In particular,

$$(9.3) \qquad \lim_{t \to p} \measuredangle\left[S^{(r)}(P,t;\tau_{r-1}),S^{(r)}(Q,t;\tau_{r-1})\right] = 0.$$

Thus $S^{(r)}(P;\tau_r)$ touches $S^{(r)}(Q;\tau_r)$ at p. Furthermore, if $S^{(r)}(P;\tau_r)$ and $S^{(r)}(Q;\tau_r)$ have a point $\neq p$ in common, they coincide. Thus $\tau_r^{(r)}$ consists of the family of r-spheres which touch $S^{(r)}(Q;\tau_r)$ at p.

Suppose $r < m$ and an m-sphere $S_r^{(m)} = S^{(m)}(P_1,\ldots,P_{m+1-r};\tau_r)$ of $\tau_r^{(m)}$ has a point $R \neq p$ in common with $S^{(r)}(Q;\tau_r)$. From the above, $S^{(r)}(R;\tau_r) = S^{(r)}(Q;\tau_r)$. If $R \subset S^{(r)}(P_1;\tau_r)$ we have

$$S^{(m)} \supset S^{(r)}(P_1;\tau_r) = S^{(r)}(R;\tau_r) = S^{(r)}(Q;\tau_r)$$

while if $R \not\subset S^{(r)}(P_1;\tau_r)$, we have by Theorem 9.2

$$S_r^{(m)} \supset S^{(r+1)}\left[R;S^{(r)}(P_1;\tau_r)\right]$$

$$= S^{(r+1)}(P_1,R;\tau_r) = S^{(r+1)}\left[P_1;S^{(r)}(R;\tau_r)\right] \supset S^{(r)}(R;\tau_r) = S^{(r)}(Q;\tau_r).$$

On the other hand, suppose an m-sphere $S^{(m)}$ touches $S_r^{(r)} = S^{(r)}(Q;\tau_r)$ at p. If $S^{(m)} \supset S_r^{(r)}$ it follows, as in the proof of part (i), that $S^{(m)} \in \tau_r^{(m)}$. Suppose $S^{(m)} \cap S_r^{(r)} = p$. Choose an $S^{(r)} \subset S^{(m)}$ such that $S^{(r)}$ touches $S^{(r)}(Q;\tau_r)$ at p. Thus $S^{(r)} \in \tau_r^{(r)}$. It again follows that $S^{(m)} \in \tau_r^{(m)}$.

Corollary 1. Let $\Gamma_1^{(r-1)}, \ldots, \Gamma_r^{(r-1)}$ hold and let $S_r^{(r-1)} = p$. Suppose $\lim_{t \to p} S^{(r)}(P,t;\tau_{r-1})$ exists for a single point $P \neq p$. Then $\Gamma_r^{(r)}$ holds at p $(1<r<n)$.

Proof: This follows from equation (9.3).

Corollary 2. There is only one $S_r^{(m)}$ of the pencil $\tau_r^{(m)}$ which contains $(m+1-r)$ points which do not lie on the same $S^{(m-1)}$.

Proof: Such an $S_r^{(r)}$ can be uniquely constructed as in the proof of (i), Theorem 9.3.

Corollary 3. If two $S_r^{(m)}$'s intersect in an $S^{(m-1)}$ then this $S^{(m-1)} \in \tau_r^{(m-1)}$.

Proof: The $S_r^{(m)}$'s, and hence also $S^{(m-1)}$, contain $S_r^{(r-1)}$. In case $S_r^{(r-1)} = p$, let $P \subset S^{(m-1)}$, $P \neq p$.

Then the $S_r^{(m)}$'s and hence also $S^{(m-1)}$ contains $S^{(r)}(P;\mathcal{T}_r)$.

Corollary 4.

$$\mathcal{T}_o^{(m)} \supset \mathcal{T}_1^{(m)} \supset \ldots \supset \mathcal{T}_{m+1}^{(m)}.$$

Proof: When $k < m$, or when $k = m$ and $S_m^{(m-1)} \neq p$, Theorem 9.3 implies that $\mathcal{T}_k^{(m)}$ is the set of all the m-spheres through $S_k^{(k-1)}$. Hence $S_{k+1}^{(m)}$, being the limit of a sequence of such m-spheres, must itself contain $S_k^{(k-1)}$, and by Theorem 9.3, $S_{k+1}^{(m)} \in \mathcal{T}_k^{(m)}$. Suppose $k=m$ and $S_m^{(m-1)} = p$. By Theorem 9.3, $\mathcal{T}_m^{(m)}$ is the set of all the m-spheres which touch a given m-sphere $S_m^{(m)} \neq p$ of $\mathcal{T}_m^{(m)}$ at p. Hence $S_{m+1}^{(m)}$, being the limit of a sequence of such m-spheres, must itself touch $S^{(m)}$ at p, and, again by Theorem 9.3, $S_{m+1}^{(m)} \in \mathcal{T}_m^{(m)}$.

Theorem 9.4. Let $1<m<n$; $1 \leq k \leq m$, and suppose that $S_m^{(m-1)} \neq p$ if $k = m$. If the conditions $\Gamma_1^{(m)}, \ldots, \Gamma_k^{(m)}$ hold at p, then $\Gamma_{k+1}^{(m)}$ also holds there.

Proof: By Theorem 9.2, $\Gamma_1^{(m-1)}$ holds at p. Hence if $p, P_1, \ldots, P_{m-k}$ are independent points $S^{(m-1)}(P_1, \ldots, P_{m-k};\mathcal{T}_k)$ is defined. Furthermore, by Theorem 9.1, we can assume that $t \not\subset S^{(m-1)}(P_1,\ldots,P_{m-k};\tau_k)$ and by Theorem 9.2 again,

$$S^{(m)}(P_1,\ldots,P_{m-k},t;\tau_k) = S^{(m)}\left[t;S^{(m-1)}(P_1,\ldots,P_{m-k};\tau_k)\right].$$

Thus $S^{(m)}(P_1,\ldots,P_{m-k},t;\tau_k)$ exists when t is close to p, $t \in A$, $t \neq p$. Choose a point $P_{m+1-k} \subset S^{(m-1)}(P_1,\ldots,P_{m-k};\tau_k)$, $P_{m+1-k} \not\subset S^{(m-2)}(P_1,\ldots,P_{m-k};\tau_{k-1})$.

Then Theorem 9.2, Corollary 6 implies that

$$S^{(m-1)}(P_1,\ldots,P_{m-k};\tau_k) = S^{(m-1)}(P_1,\ldots,P_{m+1-k};\tau_{k-1}),$$

when $k < m$, or $k = m$, and $S_{m-1}^{(m-2)} \neq p$; if $k = m$ and $S_{m-1}^{(m-2)} = p$, this equation follows from Theorem 9.3, Corollary 4. Hence

$$\lim_{t \to p} S^{(m)}(P_1,\ldots,P_{m-k},t;\tau_k)$$

$$= \lim_{t \to p} S^{(m)}\left[t, S^{(m-1)}(P_1,\ldots,P_{m+1-k};\tau_{k-1})\right]$$

$$= \lim_{t \to p} S^{(m)}(P_1,\ldots,P_{m+1-k},t;\tau_{k-1})$$

$$= S^{(m)}(P_1,\ldots,P_{m+1-k};\tau_k).$$

Thus $\Gamma_{k+1}^{(m)}$ holds at p and

(9.4) $S^{(m)}(P_1,\ldots,P_{m-k};\tau_{k+1}) = S^{(m)}(P_1,\ldots,P_{m+1-k};\tau_k) \in \tau_k^{(m)}$.

<u>Corollary 1.</u> <u>If $\Gamma_1^{(m)}$ holds at p, then $\Gamma_r^{(m)}$ holds there, $r = 1, 2, \ldots, m$. Furthermore, if $S_m^{(m-1)} \neq p$, A is</u>

m+1 times differentiable at p.

Corollary 2. *If $\Gamma_1^{(n-1)}$ holds at p, then p is a differentiable point of A if and only if $\lim_{t \to p} S^{(n-1)}(t;\tau_{n-1})$ exists and converges if t tends to p.*

Corollary 3. *If $\Gamma_1^{(n-1)}$ holds at p, and $S_{n-1}^{(n-2)} \neq p$, then p is a differentiable point of A.*

Corollary 4. *If $\Gamma_1^{(m)}$ holds at p, all the conditions $\Gamma_k^{(r)}$, except $\Gamma_{m+1}^{(m)}$, automatically hold at p ($1 \leq k \leq r+1 \leq m+1$).*

Let p be a differentiable point of A. We define the index i of p as in Theorem 9.2, Corollary 4. Let $P \subset S_{i+1}^{(i)}$, $P \neq p$. Let $S_m^{(m)} = S^{(m)}(P;\tau_m)$, $m = 0,1,\ldots, m-1$. Then the set of $\tau_r^{(m)}$'s is completely determined by the sequence

$$S_0^{(o)} \subset S_1^{(1)} \subset \ldots \subset S_i^{(i)} = S_{i+1}^{(i)} \subset S_{i+2}^{(i+1)} \subset \ldots \subset S_n^{(n-1)}.$$

Its structure is determined by the single index i.

9.4. Support and Intersection Properties of $\tau_r^{(n-1)} - \tau_{r+1}^{(n-1)}$.

Let p be a differentiable interior point of A.

Our classification of the differentiable points p of A will be based on the index i of p, and on the support and intersection properties of $S_n^{(n-1)}$ and the families $\mathcal{T}_r^{(n-1)} - \mathcal{T}_{r+1}^{(n-1)}$, $r = 0,1,\ldots,n-1$. We shall omit the superscript (n-1) of $\mathcal{T}_r^{(n-1)}$ when there is no ambiguity; thus $\mathcal{T}_r = \mathcal{T}_r^{(n-1)}$.

<u>Theorem 9.5. Every (n-1)-sphere $\neq S_n^{(n-1)}$ either supports or intersects A at p.</u>

<u>Proof</u>: If an (n-1)-sphere S neither supports nor intersects A at p, then $p \subset S$ and there exists a sequence of points $t \to p$, $t \in A \cap S$, $t \neq p$. Suppose $p, P_1, \ldots, P_n$ are independent points on S. Suppose that for $0 \leq r < n-1$, $S = S^{(n-1)}(P_1, \ldots, P_{n-r}; \mathcal{T}_r)$. By equation (9.1),

$$S^{(n-1)}(P_1, \ldots, P_{n-r}; \mathcal{T}_r) \supset S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \mathcal{T}_r).$$

By Theorem 9.1, $t \not\subset S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \mathcal{T}_r)$ and by equation (9.2),

$$S = S^{(n-1)}\left[t; S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \mathcal{T}_r)\right] = S^{(n-1)}(P_1, \ldots, P_{n-r-1}, t; \mathcal{T}_r)$$

for each t. Condition $\Gamma_{r+1}^{(n-1)}$ now implies that

$$S = S^{(n-1)}(P_1, \ldots, P_{n-r-1}; \mathcal{T}_{r+1}).$$

Thus we get, in this way,

$$S = S^{(n-1)}(P_1; \mathcal{T}_{n-1}).$$

By Theorem 9.2, $S \supset S_{n-1}^{(n-2)}$, and by Theorem 9.1, $t \not\subset S_{n-1}^{(n-2)}$ when the parameter t is close to, but different from, the parameter p. If $S_{n-1}^{(n-2)} \neq p$, equation (9.2) implies that $S = S^{(n-1)}\left[t; S_{n-1}^{(n-2)}\right] = S^{(n-1)}(t; \mathcal{T}_{n-1})$, while if $S_{n-1}^{(n-2)} = p$, Theorem 9.3 implies that $S = S^{(n-1)}(t; \mathcal{T}_{n-1})$. Applying condition $\Gamma_n^{(n-1)}$, we are led to the conclusion $S = S_n^{(n-1)}$.

<u>Theorem 9.6.</u> <u>If $S_n^{(n-1)} = p$, then the (n-1)-spheres of $\mathcal{T}_{n-1} - \mathcal{T}_n$ all intersect A at p, or they all support.</u>

<u>Proof</u>: Let S' and S" be two distinct (n-1)-spheres of $\mathcal{T}_{n-1} - \mathcal{T}_n$. Since $S_n^{(n-1)} = p$, Theorem 9.2, Corollary 4 implies that $S_{n-1}^{(n-2)} = p$, and Theorem 9.3 implies that S' and S" touch at p. Thus we may assume that $S'' \subset (p \cup \bar{S}')$ and $S' \subset (p \cup \underline{S}'')$. Suppose now, for example, that S' supports A at p while S" intersects. Then $A \cap \bar{S}''$ is not void and $A \subset (p \cup \bar{S}')$. Let $t \to p$ in $A \cap \underline{S}''$. Hence $S^{(n-1)}(t; \mathcal{T}_{n-1}) \subset (\underline{S}'' \cap \bar{S}') \cup p$. Consequently, $S(t; \mathcal{T}_{n-1})$ cannot converge to $S_n^{(n-1)} = p$, as t tends to p. Thus S' and S" must both support,

or both intersect A at p.

<u>Theorem 9.7. If $S_{r+1}^{(r)} \neq p$ while $S_r^{(r-1)} = p$, then every (n-1)-sphere of $\mathcal{T}_r - \mathcal{T}_{r+1}$ supports A at p $(1 \leq r \leq n-1)$.</u>

<u>Proof</u>: Suppose $S_r^{(r-1)} = p$, so that by Theorem 9.3, the r-spheres of $\mathcal{T}_r^{(r)}$ all touch any (n-1)-sphere of $\mathcal{T}_r$. Let $S \in \mathcal{T}_r - \mathcal{T}_{r+1}$, $S \neq p$. If a sequence of points t exists such that $t \in A \cap \overline{S}$, $t \to p$, then each $S^{(r)}(t;\mathcal{T}_r)$ lies in the closure of $\overline{S}$. Hence $S_{r+1}^{(r)}$ will also lie in the same closed domain. Since $S_{r+1}^{(r)} \in \mathcal{T}_r^{(r)}$, either $S_{r+1}^{(r)} = p$, or it touches S at p. Since $S \notin \mathcal{T}_{r+1}$, $S_{r+1}^{(r)}$ must lie in $p \cup \overline{S}$. Similarly, the existence of a sequence $t' \in \underline{S} \cap A$, $t' \to p$, implies that $S_{r+1}^{(r)} \subset p \cup \underline{S}$. Thus if S intersects A at p, $S_{r+1}^{(r)} \subset (p \cup \overline{S}) \cap (p \cup \underline{S}) = p$; i.e., $S_{r+1}^{(r)} = p$.

<u>Theorem 9.8. All the (n-1)-spheres of $\mathcal{T}_r - \mathcal{T}_{r+1}$ support A at p, or they all intersect; $r = 0,1,\ldots,n-1$.</u>

<u>Proof</u>: Let S' and S" be two distinct (n-1)-spheres of $\mathcal{T}_r$. Suppose, for the moment, that the intersection $S' \cap S''$

is a proper (n-2)-sphere $S^{(n-2)}(P_1,\ldots,P_{n-r-1};\tau_r)$. Suppose for example that S' intersects, while S" supports A at p. Thus $A \cap \underline{S}'$ and $A \cap \overline{S}'$ are not void. With no loss in generality, we may assume that $A \subset \overline{S}'' \cup p$. If t is close to p, $t \neq p$, Theorem 9.1 implies that $t \notin S^{(n-2)}(P_1,\ldots,P_{n-r-1};\tau_r)$, and equation (9.2) implies that

$$S^{(n-1)}\left[t;S^{(n-2)}(P_1,\ldots,P_{n-r-1};\tau_r)\right] = S^{(n-1)}(P_1,\ldots,P_{n-r-1},t;\tau_r).$$

If $t \in A \cap \underline{S}'$, then $S^{(n-1)}(P_1,\ldots,P_{n-r-1},t;\tau_r)$ lies in the closure of

$$(\underline{S}' \cap \overline{S}'') \cup (\overline{S}' \cap \underline{S}'').$$

Letting t tend to p, we conclude that $S^{(n-1)}(P_1,\ldots,P_{n-r-1};\tau_{r+1})$ lies in the same closed domain. By letting t converge to p through $\overline{S}' \cap A$, we obtain symmetrically that $S^{(n-1)}(P_1,\ldots,P_{n-r-1};\tau_{r+1})$ also lies in the closure of

$$(\overline{S}' \cap \overline{S}'') \cup (\underline{S}' \cap \underline{S}'').$$

Hence $S^{(n-1)}(P_1,\ldots,P_{n-r-1};\tau_{r+1})$ lies in the intersection $S' \cup S''$ of these two domains, i.e., $S^{(n-1)}(P_1,\ldots,P_{n-r-1};\tau_{r+1})$

is either S' or S". In other words, one of the (n-1)-spheres S' and S" belongs to $\mathcal{T}_{r+1}$. Thus if S' and S" belong to $\mathcal{T}_r - \mathcal{T}_{r+1}$, and have a proper $S^{(n-2)}$ in common, they both support or both intersect A at p.

Suppose now that $S' \cap S'' = p$. Theorem 9.3 implies that $S_r^{(r-1)} = p$. In view of Theorems 9.6 and 9.7, there remain to be considered only the cases where $r < n-1$, and indeed, when $r \leq n-2$, we have only to consider those cases for which $S_{r+1}^{(r)} = p$.

By Theorem 9.3, any $S^{(n-1)}$ which touches an $S_r^{(r)}$, but which does not touch an $S_{r+1}^{(r+1)}$ belongs to $\mathcal{T}_r - \mathcal{T}_{r+1}$. Hence there exists an (n-1)-sphere S of $\mathcal{T}_r - \mathcal{T}_{r+1}$ which intersects S' and S" respectively in a proper (n-2)-sphere. From the above, S and S', and also S and S" both support or both intersect A at p. Thus S' and S" both support or both intersect A at p in this case also.

<u>9.5. Characteristics and a Classification of the Differen-</u>

tiable Points.

The characteristic, $(a_o, a_1, \ldots, a_n; i)$ of a differentiable point p of an arc A is defined as follows:

$a_r = 1$ or 2 when $r < n$; $a_n = 1, 2$, or ∞. The index $i = 1, 2, \ldots, n$.

$a_o + \ldots + a_r$ is even or odd according as every $S_r^{(n-1)}$ of $\mathcal{T}_r - \mathcal{T}_{r+1}$ supports or intersects A at $p; r = 0, 1, \ldots n-1$.

$a_o + \ldots + a_n$ is even if $S_n^{(n-1)}$ supports, odd if $S_n^{(n-1)}$ intersects, while $a_n = \infty$ if $S_n^{(n-1)}$ neither supports nor intersects A at p.

Finally the characteristic of p has index i if and only if $S_i^{(i-1)} = p$, while $S_{i+1}^{(i)} \neq p$.

Theorem 9.7, and the convention that $S_n^{(n-1)}$ supports A at p when $S_n^{(n-1)} = p$, lead to the following restriction on the characteristic $(a_o, a_1, \ldots, a_n; i)$:

$$\sum_{k=0}^{i} a_k \equiv 0 \pmod 2.$$

As a result of this restriction, the number of types of differentiable points corresponding to each value of $i < n$ is $3(2)^{n-1}$, and there are 2^n types when $i = n$. Thus there are $(3n - 1)2^{n-1}$ types altogether.

If we introduce a rectangular Cartesian coordinate system into the conformal n-space, examples of each of the $(3n - 1)2^{n-1}$ types are given by the curves

$$\text{(I)} \qquad x_1 = t^{m_1},\ x_2 = t^{m_2},\ \ldots,\ x_n = t^{m_n}$$

in the cases a_n 1 or 2, and

$$\text{(II)} \qquad x_1 = t^{m_1},\ x_2 = t^{m_2}, \ldots,\ x_n = \begin{cases} t^{m_n} \sin t^{-1}, & 0 < |t| \leq 1 \\ 0, & t = 0 \end{cases}$$

for the cases in which $a_n = \infty$, all relative to the point $t = 0$. The m_r are positive integers, and $m_1 < m_2 < \ldots < m_n$. The different types are determined by the parities of the m_i, and by the relative magnitudes of the m_r and $2m_1$. In each of these examples, the $S_1^{(m)}$ touch the x_1-axis at the origin; $m = 1, 2, \ldots, n-1$.

When $m_i < 2m_1 < m_{i+1}$, the point $t = 0$ has a characteristic of the form $(a_0, a_1, \ldots, a_n; i)$ where a_n can be 1, 2, or ∞, and $i < n$.

When $m_n < 2m_1$, the point $t=0$ has a characteristic of the form $(a_o,a_1,\ldots,a_n;n)$ where a_n is either 1 or 2. Table 9.1 lists some of the properties of a differentiable point p having the characteristic $(a_o,a_1,\ldots,a_n;i)$.

<table>
<tr><th rowspan="2">Index</th><th rowspan="2">a_n</th><th colspan="2">Osculating</th><th rowspan="2">Supporting family</th><th rowspan="2">Restriction</th><th rowspan="2" colspan="2">Example</th></tr>
<tr><th>(i-1)-sphere</th><th>i-sphere</th></tr>
<tr><td rowspan="2">$i<n$</td><td>$a_n=1$ or 2</td><td rowspan="3">$S_i^{(i-1)}=p$</td><td rowspan="2">$S_{i+1}^{(i)} \neq p$</td><td rowspan="2">$\tau_i - \tau_{i+1}$</td><td rowspan="2">$\sum_{r=0}^{i} a_r \equiv 0 \pmod 2$</td><td>I</td><td rowspan="2">$m_i < 2m_1 < m_{i+1}$</td></tr>
<tr><td>$a_n=\infty$</td><td>II</td></tr>
<tr><td>$i=n$</td><td>$a_n=1$ or 2</td><td></td><td>τ_n</td><td>$\sum_{r=0}^{n} a_r \equiv 0 \pmod 2$</td><td>I</td><td>$m_n < 2m_1$</td></tr>
</table>

Table 9.1

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