TRANSVERSE VIBRATIONS OF A BEAM HAVING NONLINEAR CONSTRAINT

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HAVING NONLINEAR CONSTRAINT

BY

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SCOPE AND CONTENT:

Transverse vibrations of a beam with one end fixed and the other supported on nonlinear spring have been studied. Theoretical analysis has been carried out for two different cases of springs, viz.; cubic nonlinear and bilinear types.

Theoretical results for bilinear case have been compared with those obtained experimentally. The effect of end mass has also been considered in theoretical analysis.

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ABSTRACT

The free and forced, undamped, transverse vibrations of a beam, with one end clamped and the other supported on a nonlinear spring, have been studied. Theoretical analysis has been carried out for two different cases of springs, viz.; cubic nonlinear and bilinear types. For the study of forced vibrations the exciting sinusoidal force has been considered to act at the spring-supported end of the beam. The analysis is an approximate one since it involves the solution of nonlinear boundary value problems. Theoretical results for the bilinear case have been compared with those obtained experimentally.

It has been shown that free vibrations can occur in an infinite number of frequency ranges and each of the frequencies of free vibration corresponds to a definite modal configuration. The results of forced harmonic response reveal the possibility of multiplicity of jump phenomena in the frequency ranges of free vibrations. However, in the case of bilinear spring jump phenomena may not occur if the amplitude of exciting force is above a certain value. Furthermore, in the case of cubic nonlinear spring it has been demonstrated that subharmonic vibrations can occur in an infinite number of frequency ranges.

NOMENCLATURE

x = Distance along the beam from fixed end, inch

y = Lateral deflection of any point on the beam, inch

E = Young's Modulus of Elasticity, lbf/in.^{2}

ρ = Mass density, lbm/in.^{3}

L = Length of beam, inch

a = Cross sectional area of the beam, in.^{2}

I = Area moment of inertia, in.^{4}

M = Mass of beam, lbm

m = End mass, lbm

c = $(EI/\rho a)^{\frac{1}{2}}$, in.^{2}/sec.

K_ℓ = Constant of a linear spring, lbf/in.

K_n = Nonlinearity constant of a cubic nonlinear spring, lbf/in.^{3}

y_o = Deflection of bilinear spring at which spring constant changes, inch

K_1 = Constant of bilinear spring for deflection $\leqslant y_o$ lbf/in.

K_2 = Constant of bilinear spring for deflection $> y_o$ lbf/in.

F = Amplitude of sinusoidal exciting force, lbf

F_s = Restoring force of spring, lbf

ω = Frequency of excitation, rad./sec.

λ = $(\omega/c)^{\frac{1}{2}}$, inch^{-1}

x_ℓ = x/L

C_h = Amplitude coefficient of harmonic component of response

C_t = Amplitude coefficient of third order harmonic component of response

C_s = Amplitude coefficient of one-third order subharmonic component of response

X_x = Dimensionless amplitude of vibration at x

X_a = Dimensionless harmonic component of deflection of spring supported end

X_{a3} = Dimensionless third order harmonic component of deflection of spring-supported end

$X_{a1/3}$ = Dimensionless one-third order subharmonic component of deflection of spring supported end

X_o = y_o/L

Ω = λL

α = $K_\ell L^3/EI$

β = $K_n L^5/EI$

α_1 = $K_1 L^3/EI$

α_2 = $K_2 L^3/EI$

μ = m/M

P = $F L^2/EI$

P_s = $F_s L^2/EI$

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INTRODUCTION

1. INTRODUCTION

1.1 GENERAL INTRODUCTION:

The beam vibrations involving linear boundary conditions have been dealt with in a number of references [1,2,3,4]*. Solution in closed form can be obtained for problems with homogeneous boundary conditions. Techniques have already been developed to solve the problems involving time dependent boundary conditions [5,6,7]. Saito [9] analysed the forced lateral vibrations of a beam, with a concentrated mass, mounted on parallel elastic supports at each end. Similar work has been done by Miller [10] considering damped flexible end supports. Springfield and Raney [11] made theoretical and experimental investigations to find the optimum parameters of end supports. Lee and Saibel [8] developed a general expression to find the frequency equation for the vibration of a constrained beam with any combination of intermediate elastic or rigid supports, concentrated masses and sprung masses.

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* Numbers in square brackets indicate references given in bibliography.

In all the above cases the boundary conditions are linear, i.e. they can be expressed by linear equations. However, there may be some systems for which the boundary conditions are nonlinear, for example, a beam on nonlinear supports. The behaviour of such systems can still be described by linear partial differential equations, but the nonlinear boundary conditions cause difficulties in the analysis.

Porter and Billet[12] have made an approximate analysis for the vibrations of a uniform bar in longitudinal motion. One end of the bar was fixed and the other was anchored by means of a cubic nonlinear spring. Their results show that the system exhibits nonlinear oscillations if the boundary conditions are nonlinear.

1.2 OBJECT AND SCOPE:

The object of the present investigation is to study the vibration of a beam with one end clamped and the other supported on a nonlinear spring. It is proposed to analyse the system with two different types of springs:

(i) cubic nonlinear ($F_s = K_\ell y + K_n y^3$) and

(ii) bilinear springs.

The forced vibration response is to be determined with sinusoidal excitation provided at the end to which the spring is attached.

The survey of literature indicates that no work has been done on the above problem. It is expected that this study will be useful from both academic and practical points of view.

The theoretical analysis carried out for this problem is given in chapter 2. An experimental study has been made for the case of bilinear spring, the details of which can be found in chapter 3. The results and conclusions of this work are given in chapter 4. For a cantilever beam with linear spring support at its free end, the variation of natural frequencies with the spring stiffness is reported in appendix I.

1.3 A BRIEF HISTORY OF NONLINEAR VIBRATIONS:

Basically all the problems in mechanics are nonlinear from the outset. The linearizations commonly practised are approximating devices that are quite satisfactory for the practical purposes. However, there are certain cases in which linear treatment may not be applicable at all. The phenomena of nonlinear vibration have long been recognized, but the recent developments in the theory and methods of nonlinear analysis have been stimulated by the works of Duffing and Van der Pol.

Since exact solutions in terms of known functions can be found for only a few nonlinear equations, most of the available references [13, 14, 15, 16] are concerned with obtaining approximate solutions. Although a large number of approximate analytic methods exist, most are applicable only to a small class of problems, and in general they require that the nonlinear parameter be small.

For systems in which the nonlinearity is relatively large, approximate analytic methods are in general inadequate. Ergin [17] developed a line segment approximation for nonlinear systems, and found that for some problems involving even large nonlinearities, only two line segments were enough to give sufficient accuracy. Application of this method was made to a single degree of freedom system having a cubic nonlinearity in the spring force and subjected to various transient excitations.

Den Hartog and Mikina [18] have found the solution of single degree mass - spring system with initial set in the spring. A general case of bilinear spring has been considered by Den Hartog and Heils [19] . The solution was found on the assumption that the motion curve is symmetrical every quarter wave. Some investigators [20, 21, 22] who considered hysteretic, bilinear, single degree of freedom systems, found electronic analog computers extremely useful for response prediction. Brock [25] has presented a simple iterative procedure employing numerical integrations for the analysis of single degree of freedom systems having nonlinear elasticity. Soroka [24] considered the free vibrations of two degree freedom system with nonlinear unsymmetrical elasticity. Such a system is characteristic of aircraft propeller super-charger installation. The results show that one mass may oscillate several times while the other mass is going through one oscillation. The ratio of amplititude of one mass with respect to the other changes with amplitude. Ehrich [23, 26] has indicated that the clearance between shaft and rotor provides bilinear elasticity which can cause subharmonic vibration.

Rosenberg [27] has defined the concept of normal modes in nonlinear multi - degree of freedom system. The problem of finding the modes reduces to a geometrical maximum - minimum problem in an n - space of known metric. The solution of the geometrical problem reduces the coupled equations of motion to n uncoupled equations whose natural frequencies can always be found by a single quadrature. Paslay and Gurtin [28] have found the vibration response of a linear undamped system resting on a nonlinear spring. Caughey [29] has analysed the forced oscillations of a semi - infinite rod exhibiting weak bilinear hysteresis. Tauchert and Ayre [30] have found the shock response of a simple beam on nonlinear supports. The transient response was obtained by considering lumped mass system and applying numerical methods.

The dynamic analysis of non-linear continuous systems and multi-degree freedom systems has received less attention than single degree of freedom systems although several specific problems of this type have been investigated as mentioned above. The main reason for the lack of literature on non-linear continuous systems seems to be the difficulty of analysing them.

THEORETICAL ANALYSIS

2. THEORETICAL ANALYSIS

Figure 2.1 shows the schematic diagram of the system considered for the present investigation. It consists of a uniform beam which is clamped at one end and attached to a nonlinear spring at the other end. The beam is of length L, cross sectional area a, mass density ρ ,and flexural rigity EI . The sinusoidal exciting force (F sin ωt) is acting at the spring - supported end which also carries a concentrated mass m .

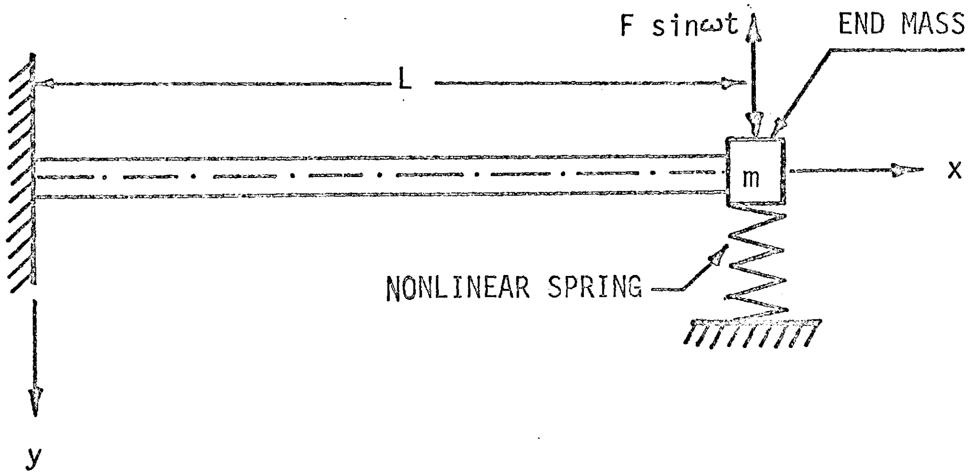


FIGURE 2.1 : SCHEMATIC DIAGRAM OF THE SYSTEM

Neglecting the effects of shearing forces and rotory inertia, the differential equation for the transverse motion of the beam can be written as follows:

$$\frac{\partial^4 y}{\partial x^4} + \frac{\rho a}{EI} \cdot \frac{\partial^2 y}{\partial t^2} = 0$$

or $$\frac{\partial^4 y}{\partial x^4} + \frac{1}{c^2} \cdot \frac{\partial^2 y}{\partial t^2} = 0 \quad \ldots\ldots\ldots (2.1)$$

where, $c^2 = EI/\rho a$

The four boundary conditions are:

1. at $x = 0$, $y = 0$ .... (2.2)

2. at $x = 0$, $\dfrac{\partial y}{\partial x} = 0$ .... (2.3)

3. at $x = L$, $\dfrac{\partial^2 y}{\partial x^2} = 0$ .... (2.4)

4. at $x = L$,

$$EI \frac{\partial^3 y}{\partial x^3} = F_s + m \frac{\partial^2 y}{\partial t^2} - F \sin \omega t$$

.... (2.5)

where, F_s = Restoring force of spring at time t

Any solution of the problem must satisfy these four boundary conditions and equation (2.1).

2.1 CASE OF CUBIC NONLINEAR SPRING:

In this section the spring considered has a cubic nonlinearity. The restoring force of such spring can be expressed by the relation:

$$F_s = K_\ell \, y + K_n \, y^3 \qquad \text{........(2.6)}$$

where K_ℓ and K_n are positive constants of the spring.

The spring described by (2.6) has symmetrical odd characteristic , i.e., $F_s(y) = - F_s(-y)$, as shown in figure 2.2 .

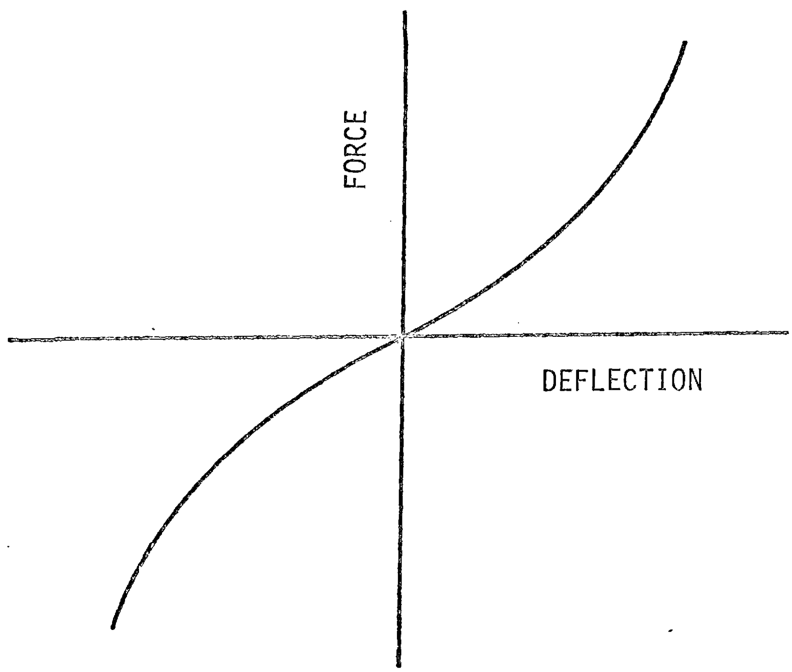


FIGURE 2.2 : RESTORING FORCE CHARACTERISTIC OF CUBIC NONLINEAR SPRING

For this case equation (2.5) becomes;

at $x = L$

$$EI \frac{\partial^3 y}{\partial x^3} = K_l y + K_n y^3 + m \frac{\partial^2 y}{\partial t^2} - F \sin \omega t \qquad \text{........(2.7)}$$

2.1.1 Harmonic Response:-

For the steady state motion of the system, the solution of equation (2.1) has to satisfy the boundary conditions (2.2), (2.3), (2.4) and (2.7) . To obtain a harmonic solution of (2.1), let

$$y(x,t) = X(x) \sin \omega t \qquad \text{........(2.8)}$$

If (2.8) is to be the solution of (2.1), it follows that X(x) must satisfy the equation:

$$\frac{d^4 X}{d x^4} - \frac{\omega^2}{c^2} X = 0$$

or

$$\frac{d^4 X}{d x^4} - \lambda^4 X = 0$$

where $\lambda^4 = \omega^2/c^2$

and

$$X(x) = C_1 \cosh \lambda x + C_2 \sinh \lambda x + C_3 \cos \lambda x + C_4 \sin \lambda x$$

From the conditions (2.2) and (2.3)

$$C_3 = -C_1$$

$$C_4 = -C_2$$

hence,

$$X(x) = C_1(\cosh\lambda x - \cos\lambda x) + C_2(\sinh\lambda x - \sin\lambda x)$$

The condition (2.4) requires that

$$C_2 = -\frac{\cosh\lambda L + \cos\lambda L}{\sinh\lambda L + \sin\lambda L} \cdot C_1$$

For simplicity, let

$$A = \cosh\lambda L + \cos\lambda L$$

$$B = \sinh\lambda L + \sin\lambda L$$

$$C = \cosh\lambda L - \cos\lambda L$$

$$D = \sinh\lambda L - \sin\lambda L$$

$$\therefore \quad X(x) = C_1\left[(\cosh\lambda x - \cos\lambda x) - \frac{A}{B}(\sinh\lambda x - \sin\lambda x)\right]$$

........(2.9 a)

Dividing this by L

$$X_x = C_R\left[(\cosh\lambda x - \cos\lambda x) - \frac{A}{B}(\sinh\lambda x - \sin\lambda x)\right]$$

........(2.9 b)

where, $X_x = \dfrac{X(x)}{L}$, dimensionless amplitude of vibration at x

$C_R = \dfrac{C_1}{L}$, amplitude coefficient

In (2.9 b) taking x = L ;

$$X_a = C_R\left(C - \frac{AD}{B}\right) \qquad \text{........(2.10)}$$

where, X_a = Dimensionless amplitude of deflection of spring supported end

The condition (2.7) will determine the remaining constant c_1. Substituting (2.9 a) in (2.8) and then in (2.7);

$$EI\, c_1 \lambda^3 \left(D - \frac{A^2}{B} \right) \sin \omega t = K_\ell c_1 \left(C - \frac{AD}{B} \right) \sin \omega t + K_n c_1^3 \left(C - \frac{AD}{B} \right)^3 \sin^3 \omega t - m \omega^2 c_1 \left(C - \frac{AD}{B} \right) \sin \omega t - F \sin \omega t$$

Putting this equation in dimensionless form:

$$\Omega^3 X_a \frac{BD - A^2}{BC - AD} \sin \omega t = \alpha X_a \sin \omega t + \beta X_a^3 \sin^3 \omega t - \mu X_a \Omega^4 \sin \omega t - P \sin \omega t$$

where,

$\Omega = \lambda L$, dimensionless frequency parameter

$\alpha = K_\ell \dfrac{L^3}{EI}$, constant of spring (dimensionless stiffness at zero deflection)

$\beta = K_n \dfrac{L^5}{EI}$, constant of spring (dimensionless nonlinearity parameter)

$\mu = \dfrac{m}{M}$, mass ratio

$P = F \dfrac{L^2}{EI}$, dimensionless exciting force

Now using the relation

$$\sin^3 \omega t = \frac{3}{4} \sin \omega t - \frac{1}{4} \sin 3 \omega t$$

in the above equation;

$$\left[\Omega^3 \frac{BD - A^2}{BC - AD} X_a - \alpha X_a - \frac{3\beta}{4} X_a^3 + \Omega^4 \mu X_a + P\right] \sin\omega t$$
$$+ \frac{1}{4} \beta X_a^3 \sin 3\omega t = 0 \qquad \text{........(2.11)}$$

For the assumed solution (2.8) to be exact, the coefficients of $\sin\omega t$ and $\sin 3\omega t$ in (2.11) should be separately zero. However, the coefficient of $\sin 3\omega t$ will vanish at all frequencies only in the linear case $\beta = 0$ (i.e., $K_n = 0$), so the assumed solution (2.8) is only an approximate one for the nonlinear case $\beta \neq 0$. In fact an exact solution will contain harmonics of higher order. This has been analysed in the next section. To ensure the approximate validity of (2.8), the coefficient of $\sin\omega t$ in (2.11) should vanish, i.e., X_a must satisfy the equation:

$$X_a^3 + \left[\frac{4\alpha}{3\beta} - \frac{4\Omega^3}{3\beta} \frac{BD - A^2}{BC - AD} - \frac{4\Omega^4}{3\beta}\mu\right] X_a - \frac{4P}{3\beta} = 0$$
$$\text{........(2.12)}$$

Equations (2.9), (2.10) and (2.12) determine approximately the frequency response of the cantilever beam supported on a nonlinear spring at the free end and carrying an end mass.

In the linear case ($\beta = 0$), the equation (2.12) yields:

$$X_a = \frac{P}{\alpha - \Omega^3 \dfrac{BD - A^2}{BC - AD} - \Omega^4 \mu} \qquad \text{......(2.13)}$$

The natural frequencies of free vibration of the beam with linear spring and an end mass can be obtained by solving the equation;

$$\alpha - \Omega^3 \cdot \frac{BD - A^2}{BC - AD} - \Omega^4 \mu = 0 \qquad \text{......(2.14)}$$

For the case with no end mass, the frequency equation is

$$\alpha - \Omega^3 \cdot \frac{BD - A^2}{BC - AD} = 0 \; .$$

However, in nonlinear case $\beta \neq 0$, (2.12) is a cubic equation and, therefore, yields either one or three real values of X_a, depending upon the value of frequency parameter Ω.

The backbone curves for the free oscillations in the nonlinear case can be obtained by putting P = 0 in equation (2.12).

Therefore,

$$X_a^2 = \frac{3}{4\beta} \cdot \left[\Omega^3 \cdot \frac{BD - A^2}{BC - AD} + \Omega^4 \mu - \alpha \right] \qquad \text{......(2.15)}$$

The free vibrations will exist only for the frequencies for which X_a^2 is positive.

For $|X_a| = 0$, from equation (2.15)

$$\Omega^3 \frac{BD - A^2}{BC - AD} + \Omega^4 \mu - \alpha = 0 \qquad \text{......(2.16)}$$

For $|X_a| = \infty$, from equation (2.15)

$$BC - AD = 0$$

or $\tanh \Omega = \tan \Omega$ ....(2.17)

Equation (2.16) gives the natural frequencies of the beam with a linear spring and an end mass. Equation (2.17) gives the natural frequencies of clamped-simply supported beam. These two equations determine the frequency ranges in which free vibrations can exist.

For the case where $\mu = \alpha = 0$, equation (2.16) becomes:

$$BD - A^2 = 0$$

or $1 + \cosh \Omega \cos \Omega = 0$ ......(2.18)

and (2.17) remains unchanged.

Equation (2.18) gives the natural frequencies of simple cantilever beam.

To determine the loci of the points of vertical tangency on the forced response curves, $\dfrac{\partial \Omega}{\partial X_a} = 0$.

Therefore, from equation (2.12)

$$3 X_a^2 + \frac{4\alpha}{3\beta} - \frac{4\Omega^3}{3\beta} \cdot \frac{BD - A^2}{BC - AD} - \frac{4\Omega^4}{3\beta}\mu = 0$$

......(2.19)

For $X_a = 0$

$$\Omega^3 \frac{BD - A^2}{BC - AD} + \Omega^4 \mu - \alpha = 0$$

and for $X_a = \infty$

$$BC - AD = 0$$

or $\tanh \Omega = \tan \Omega$

These two equations are the same as (2.16) and (2.17), hence, the points of vertical tangency can lie only in the frequency ranges for which free vibrations exist.

2.1.2 Superharmonic Response:-

The fact that the assumed harmonic solution (2.8) does not exactly satisfy the last boundary condition, given by (2.7), indicates that the required solution contains odd, higher order harmonics and has the following form:

$$y = \sum_{\gamma=1,3,5,\cdots} X_\gamma(x) \sin \gamma \omega t \qquad \text{........(2.20)}$$

If this is to be the solution of (2.1), it follows that

$$\frac{d^4 X_\gamma}{d x^4} - \frac{\gamma^2 \omega^2}{c^2} X_\gamma = 0$$

or
$$X_\gamma = d_{1\gamma} \cosh(\sqrt{\gamma}\,\lambda x) + d_{2\gamma} \sinh(\sqrt{\gamma}\,\lambda x) + d_{3\gamma} \cos(\sqrt{\gamma}\,\lambda x) + d_{4\gamma} \sin(\sqrt{\gamma}\,\lambda x)$$

From first three boundary conditions (2.2), (2.3) and (2.4)

$$d_{3\gamma} = -d_{1\gamma} \quad , \quad d_{4\gamma} = -d_{2\gamma}$$

and $$d_{2r} = -\frac{A_r}{B_r}\cdot d_{1r}$$

where, $$A_r = \cosh(\sqrt{r}\,\lambda L) + \cos(\sqrt{r}\,\lambda L)$$

$$B_r = \sinh(\sqrt{r}\,\lambda L) + \sin(\sqrt{r}\,\lambda L)$$

Further, let

$$C_r = \cosh(\sqrt{r}\,\lambda L) - \cos(\sqrt{r}\,\lambda L)$$

$$D_r = \sinh(\sqrt{r}\,\lambda L) - \sin(\sqrt{r}\,\lambda L)$$

and writing d_r instead of d_{1r}

$$X_r(x) = d_r\cdot\Big[\big(\cosh(\sqrt{r}\,\lambda x) - \cos(\sqrt{r}\,\lambda x)\big) - \frac{A_r}{B_r}\big(\sinh(\sqrt{r}\,\lambda x) - \sin(\sqrt{r}\,\lambda x)\big)\Big] \quad \text{......(2.21)}$$

Therefore,

$$y(x,t) = \sum_{r=1,3,\cdots} d_r\Big[\big(\cosh(\sqrt{r}\,\lambda x) - \cos(\sqrt{r}\,\lambda x)\big) - \frac{A_r}{B_r}\big(\sinh(\sqrt{r}\,\lambda x) - \sin(\sqrt{r}\,\lambda x)\big)\Big]\sin r\omega t$$

......(2.22)

The constants d_r's can be determined by requiring the solution to satisfy the last boundary condition (2.7). However, if large number of terms are taken in (2.22) , the calculations become lengthy and tedious.

For second order approximation let us include the third order harmonic in the solution, i.e., taking

$$y = d_1\left[(\cosh\lambda x - \cos\lambda x) - \frac{A}{B}(\sinh\lambda x - \sin\lambda x)\right]\cdot\sin\omega t$$
$$+ d_3\left[(\cosh(\sqrt{3}\lambda x) - \cos(\sqrt{3}\lambda x)) - \frac{A_3}{B_3}(\sinh(\sqrt{3}\lambda x) - \sin(\sqrt{3}\lambda x))\right]\sin 3\omega t$$

and using the condition (2.7)

$$\Omega^3 X_a \frac{BD - A^2}{BC - AD}\sin\omega t + 3\sqrt{3}\,\Omega^3 X_{a3}\frac{B_3D_3 - A_3^2}{B_3C_3 - A_3D_3}\sin 3\omega t$$
$$= \alpha(X_a\sin\omega t + X_{a3}\sin 3\omega t) - \Omega^4\mu(X_a\sin\omega t + 9X_{a3}\sin 3\omega t) + \beta(X_a\sin\omega t + X_{a3}\sin 3\omega t)^3 - P\sin\omega t \qquad \text{........(2.23)}$$

In this equation X_a and X_{a3} are obtained by taking $x = L$ and dividing the equation (2.21) by L, therefore,

$$X_a = C_h\left(C - \frac{AD}{B}\right) \qquad \text{........(2.24 a)}$$

$$X_{a3} = C_t\left(C_3 - \frac{A_3D_3}{B_3}\right) \qquad \text{........(2.24 b)}$$

Where,

$C_h = \dfrac{d_1}{L}$, amplitude coefficient of harmonic component

$C_t = \dfrac{d_3}{L}$, amplitude coefficient of third order harmonic component

In equation (2.23), using the following relations;

$$(X_a \sin\omega t + X_{a3} \sin 3\omega t)^3 = X_a^3 \sin^3\omega t + X_{a3}^3 \sin^3 3\omega t + 3X_a^2 X_{a3} \sin^2\omega t \sin 3\omega t + 3X_a X_{a3}^2 \sin\omega t \sin^2 3\omega t$$

$$\sin^3\omega t = \frac{3}{4}\sin\omega t - \frac{1}{4}\sin 3\omega t$$

$$\sin^3 3\omega t = \frac{3}{4}\sin 3\omega t - \frac{1}{4}\sin 9\omega t$$

$$\sin^2\omega t \sin 3\omega t = -\frac{1}{4}\sin\omega t + \frac{1}{2}\sin 3\omega t - \frac{1}{4}\sin 5\omega t$$

$$\sin\omega t \sin^2 3\omega t = \frac{1}{2}\sin\omega t + \frac{1}{4}\sin 5\omega t - \frac{1}{4}\sin 7\omega t$$

and collecting the terms in $\sin\omega t$ and $\sin(3\omega t)$ and insisting that their coefficients must be separately zero, one gets the following two nonlinear simultaneous equations in X_a and X_{a3} .

$$\frac{3}{4}\beta X_a^3 - \frac{3}{4}\beta X_a^2 + \left(\alpha - \Omega^4\mu - \Omega^3 \frac{BD - A^2}{BC - AD} + \frac{3}{2}\beta X_{a3}^2\right) X_a - P = 0$$

$$\frac{3}{4}\beta X_{a3}^3 + \left(\alpha - 9\Omega^4\mu - 3\sqrt{3}\,\Omega^3 \frac{B_3 D_3 - A_3^2}{B_3 C_3 - A_3 D_3} + \frac{3}{2}\beta X_a^2\right) X_{a3} - \frac{1}{4}\beta X_a^3 = 0$$

Taking $\mu = 0$ (i.e., no end mass) and simplifying, the above two equations are reduced to:

$$X_a^3 - X_{a3} X_a^2 + \left[\frac{4}{3\beta}\left(\alpha - \Omega^3 \frac{BD - A^2}{BC - AD}\right) + 2 X_{a3}^2\right] X_a - \frac{4P}{3\beta} = 0 \quad \text{........(2.25 a)}$$

$$X_{a3}^3 + \left[\frac{4}{3\beta}\left(\alpha - 3\sqrt{3}\,\Omega^3 \frac{B_3 D_3 - A_3^2}{B_3 C_3 - A_3 D_3}\right) + 2 X_a^2\right] X_{a3} - \frac{1}{3} X_a^3 = 0 \quad \text{........(2.25 b)}$$

With the help of equations (2.24) and (2.25), the response including third order harmonic can be determined.

2.1.3 Subharmonic Response:-

The existance of subharmonic vibrations in nonlinear single degree of freedom systems is well known. The nonlinear continuous systems can also be expected to exhibit similar motions. In this analysis only one third subharmonic vibrations have been investigated. A similar approach can be followed to analyse subharmonic motions of other orders.

To investigate the one third order subharmonics, let

$$y(x,t) = X_{1/3}(x) \sin\frac{\omega t}{3} + X_1(x) \sin\omega t \quad \text{......(2.26)}$$

be the approximate expression for the motion of the beam. For any value of x, the fundamental component of vibration now has a frequency equal to one third to that of the excitation. If (2.26) is to be the solution of (2.1), $X_{1/3}$ and X_1 must satisfy the following equations;

$$\frac{d^4 X_{1/3}}{d x^4} - \frac{\omega^2}{9c^2} X_{1/3} = 0$$

$$\frac{d^4 X_1}{d x^4} - \frac{\omega^2}{c^2} X = 0$$

So that

$$X_{1/3}(x) = d_1 \cosh\frac{\lambda x}{\sqrt{3}} + d_2 \sinh\frac{\lambda x}{\sqrt{3}} + d_3 \cos\frac{\lambda x}{\sqrt{3}} + d_4 \sin\frac{\lambda x}{\sqrt{3}}$$

$$X_1(x) = C_1 \cosh \lambda x + C_2 \sinh \lambda x + C_3 \cos \lambda x + C_3 \sin \lambda x$$

after satisfying the conditions (2.2), (2.3) and (2.4);

$$d_3 = - d_1 \qquad C_3 = - C_1$$

$$d_4 = - d_2 \qquad C_4 = - C_2$$

$$d_2 = - \frac{A_{1/3}}{B_{1/3}} d_1 \qquad C_2 = - \frac{A}{B} C_1$$

where ,

$$A_{1/3} = \cosh\frac{\lambda L}{\sqrt{3}} + \cos\frac{\lambda L}{\sqrt{3}}$$

$$B_{1/3} = \sinh\frac{\lambda L}{\sqrt{3}} + \sin\frac{\lambda L}{\sqrt{3}}$$

Further, let $C_{1/3} = \cosh \frac{\lambda L}{\sqrt{3}} - \cos \frac{\lambda L}{\sqrt{3}}$

$$D_{1/3} = \sinh \frac{\lambda L}{\sqrt{3}} - \sin \frac{\lambda L}{\sqrt{3}}$$

Therefore,

$$X_{1/3}(x) = d_1 \left[\left(\cosh \frac{\lambda x}{\sqrt{3}} - \cos \frac{\lambda x}{\sqrt{3}} \right) - \frac{A_{1/3}}{B_{1/3}} \left(\sinh \frac{\lambda x}{\sqrt{3}} - \sin \frac{\lambda x}{\sqrt{3}} \right) \right] \quad \text{......(2.27 a)}$$

$$X_1(x) = c_1 \left[(\cosh \lambda x - \cos \lambda x) - \frac{A}{B} (\sinh \lambda x - \sin \lambda x) \right] \quad \text{......(2.27 b)}$$

or

$$\frac{X_{1/3}(x)}{L} = C_s \left[\left(\cosh \frac{\lambda x}{\sqrt{3}} - \cos \frac{\lambda x}{\sqrt{3}} \right) - \frac{A_{1/3}}{B_{1/3}} \left(\sinh \frac{\lambda x}{\sqrt{3}} - \sin \frac{\lambda x}{\sqrt{3}} \right) \right] \quad \text{......(2.28 a)}$$

$$\frac{X_1(x)}{L} = C_h \left[(\cosh \lambda x - \cos \lambda x) - \frac{A}{B} (\sinh \lambda x - \sin \lambda x) \right] \quad \text{......(2.28 b)}$$

where, $C_s = \frac{d_1}{L}$, amplitude coefficient of subharmonic component

$C_h = \frac{c_1}{L}$, amplitude coefficient of harmonic component and

$$X_{a1/3} = C_s \left(C_{1/3} - \frac{A_{1/3} D_{1/3}}{B_{1/3}} \right) \quad \text{........(2.29 a)}$$

$$X_a = C_R \left(C - \frac{AD}{B} \right) \quad \text{........(2.29 b)}$$

where, $X_{a1/3}$ is dimensionless amplitude of one third harmonic component of end deflection

and X_a is dimensionless amplitude of harmonic component of end deflection

Substituting equations (2.27) in (2.26) and then making use of (2.7)

$$EI \cdot \left[\frac{d_1 \lambda^3}{3\sqrt{3}} \left(D_{1/3} - \frac{A_{1/3}^2}{B_{1/3}} \right) \sin\frac{\omega t}{3} + c_1 \lambda^3 \left(D - \frac{A^2}{B} \right) \sin \omega t \right]$$

$$= K_l \cdot \left[d_1 \left(C_{1/3} - \frac{A_{1/3} D_{1/3}}{B_{1/3}} \right) \sin\frac{\omega t}{3} + c_1 \left(C - \frac{AD}{B} \right) \sin \omega t \right] +$$

$$K_n \left[d_1 \left(C_{1/3} - \frac{A_{1/3} D_{1/3}}{B_{1/3}} \right) \sin\frac{\omega t}{3} + c_1 \left(C - \frac{AD}{B} \right) \sin \omega t \right]^3 -$$

$$m\omega^2 \left[\frac{d_1}{9} \left(C_{1/3} - \frac{A_{1/3} D_{1/3}}{B_{1/3}} \right) \sin\frac{\omega t}{3} + c_1 \left(C - \frac{AD}{B} \right) \sin \omega t \right] -$$

$$F \sin \omega t$$

or putting it in dimensionless form

$$\frac{\Omega^3}{3\sqrt{3}} \left(\frac{B_{1/3} D_{1/3} - A_{1/3}^2}{B_{1/3} C_{1/3} - A_{1/3} D_{1/3}} \right) X_{a1/3} \sin\frac{\omega t}{3} + \Omega^3 \left(\frac{BD - A^2}{BC - AD} \right) X_a \sin \omega t$$

$$= \alpha\left(Xa_{1/3}\sin\frac{\omega t}{3} + Xa\sin\omega t\right) + \beta\left(Xa_{1/3}\sin\frac{\omega t}{3} + Xa\sin\omega t\right)^3$$

$$- \Omega^4\mu\left(\frac{Xa_{1/3}}{9}\sin\frac{\omega t}{3} + Xa\sin\omega t\right) - P\sin\omega t \quad \ldots\ldots(2.30)$$

But,

$$\left(Xa_{1/3}\sin\frac{\omega t}{3} + Xa\sin\omega t\right)^3 = Xa_{1/3}^3\sin^3\frac{\omega t}{3} + Xa^3\sin^3\omega t + 3Xa_{1/3}^2 Xa\sin^2\frac{\omega t}{3}\sin\omega t + 3Xa_{1/3}Xa^2\sin\frac{\omega t}{3}\sin^2\omega t$$

and

$$\sin^3\frac{\omega t}{3} = \frac{3}{4}\sin\frac{\omega t}{3} - \frac{1}{4}\sin\omega t$$

$$\sin^3\omega t = \frac{3}{4}\sin\omega t - \frac{1}{4}\sin 3\omega t$$

$$\sin^2\frac{\omega t}{3}\sin\omega t = -\frac{1}{4}\sin\frac{\omega t}{3} + \frac{1}{2}\sin\omega t - \frac{1}{4}\sin\frac{5\omega t}{3}$$

$$\sin\frac{\omega t}{3}\sin^2\omega t = \frac{1}{2}\sin\frac{\omega t}{3} + \frac{1}{4}\sin\frac{5\omega t}{3} - \frac{1}{4}\sin\frac{7\omega t}{3}$$

These relations show that (2.30) can be exactly satisfied in the linear case when $\beta = Xa_{1/3} = 0$. This dictates that solution (2.20) is an approximate one and the exact solution will contain harmonics of other orders. Nevertheless, the quantities c_1 and d_1 which define the approximate solution (2.26) can be found by insisting that equation (2.30) be satisfied for terms in $\sin\frac{\omega t}{3}$ and $\sin\omega t$. This gives the following two equations:

$$X_{a_{1/3}}^{3} - X_a X_{a_{1/3}}^{2} - \frac{4\,\Omega^3}{9\sqrt{3}\,\beta} \cdot \frac{B_{1/3} D_{1/3} - A_{1/3}^{2}}{B_{1/3} C_{1/3} - A_{1/3} D_{1/3}} X_{a_{1/3}}$$

$$- \frac{4\,\Omega^4}{27\,\beta}\,\mu\, X_{a_{1/3}} + \frac{4\alpha}{3\beta} X_{a_{1/3}} + 2 X_a^2\, X_{a_{1/3}} = 0 \quad \text{......(2.31 a)}$$

$$X_a^3 + \left[\frac{4\alpha}{3\beta} - \frac{4\,\Omega^3}{3\,\beta}\;\frac{BD - A^2}{BC - AD} - \frac{4\,\Omega^4}{3\beta}\mu + 2X_{a_{1/3}}^{2}\right] X_a$$

$$- \left(\tfrac{1}{3}\, X_{a_{1/3}}^{3} + \frac{4P}{3\beta} \right) = 0 \quad \text{......(2.31 b)}$$

From (2.31 a), either $X_{a_{1/3}} = 0$ or this equation has two roots. For $X_{a_{1/3}}$ to exist, the condition to be satisfied is

$$- \frac{7X_a^2}{4} - \frac{4\alpha}{3\beta} + \frac{4\,\Omega^4}{27\beta}\,\mu + \frac{4\,\Omega^3}{9\sqrt{3}\,\beta}\;\frac{B_{1/3} D_{1/3} - A_{1/3}^{2}}{B_{1/3} C_{1/3} - A_{1/3} D_{1/3}} \geqslant 0$$

......(2.32)

For the case for which $\alpha = \mu = 0$, (2.32) reduces to;

$$- \frac{7X_a^2}{4} + \frac{4\,\Omega^3}{9\sqrt{3}\,\beta}\;\frac{B_{1/3} D_{1/3} - A_{1/3}^{2}}{B_{1/3} C_{1/3} - A_{1/3} D_{1/3}} \geqslant 0$$

But,

$$\frac{B_{1/3} D_{1/3} - A_{1/3}^{2}}{B_{1/3} C_{1/3} - A_{1/3} D_{1/3}} = - \frac{1 + \cosh\frac{\Omega}{\sqrt{3}}\,\cos\frac{\Omega}{\sqrt{3}}}{\cosh\frac{\Omega}{\sqrt{3}}\,\sin\frac{\Omega}{\sqrt{3}} - \sinh\frac{\Omega}{\sqrt{3}}\,\cos\frac{\Omega}{\sqrt{3}}}$$

Therefore,

$$\frac{7X_a^2}{4} + \frac{4\,\Omega^3}{9\sqrt{3}\beta}\;\frac{1 + \cosh\frac{\Omega}{\sqrt{3}} \cdot \cos\frac{\Omega}{\sqrt{3}}}{\cosh\frac{\Omega}{\sqrt{3}}\,\sin\frac{\Omega}{\sqrt{3}} - \sinh\frac{\Omega}{\sqrt{3}}\,\cos\frac{\Omega}{\sqrt{3}}} \leqslant 0$$

First term in this expression is a non-negative quantity, hence, for subharmonic vibration to exist;

$$\frac{1 + \cosh\frac{\Omega}{\sqrt{3}}\cos\frac{\Omega}{\sqrt{3}}}{\cosh\frac{\Omega}{\sqrt{3}}\sin\frac{\Omega}{\sqrt{3}} - \sinh\frac{\Omega}{\sqrt{3}}\cos\frac{\Omega}{\sqrt{3}}} \leqslant 0 \qquad \text{......(2.33)}$$

This condition will determine the frequency ranges in which subharmonic vibration can exist. The response in these ranges can be obtained with the help of equations (2.26), (2.28), (2.29) and the following two nonlinear simultaneous equations:

$$X_{a1/3}^2 - X_a X_{a1/3} - \frac{4\Omega^3}{9\sqrt{3}\beta} \cdot \frac{B_{1/3} D_{1/3} - A_{1/3}^2}{B_{1/3} C_{1/3} - A_{1/3} D_{1/3}} + 2X_a^2 = 0 \qquad \text{......(2.34 a)}$$

$$X_a^3 + \left[- \frac{4\Omega^3}{3\beta} \cdot \frac{BD - A^2}{BC - AD} + 2X_{a1/3}^2 \right] X_a - \left(\frac{1}{3} X_{a1/3}^3 + \frac{4P}{3\beta} \right) = 0 \qquad \text{......(2.34 b)}$$

2.2 CASE OF BILINEAR SPRING:

In this section the response of the system, shown in figure 2.1, has been determined for the case of bilinear spring. The restoring force characteristic in a genaral case of bilinear spring is as shown in figure 2.3 .

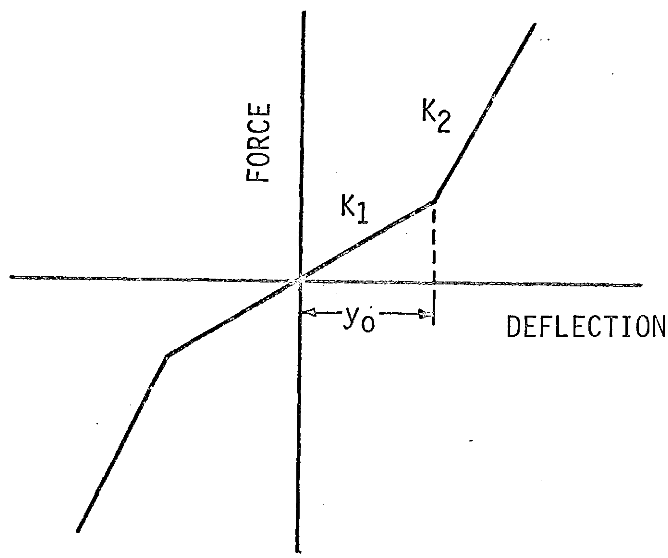


FIGURE 2.3 : RESTORING FORCE CHARACTERISTIC OF A BILINEAR SPRING

The restoring force of such spring is expressed by two equations which are linear within their corresponding ranges.

For $|y| \leq y_o$, $\quad F_S = K_1 y$

and For $|y| > y_o$, $\quad F_S = K_2 y + (K_1 - K_2) y_o$

where, y_0 is the deflection of the spring at which the spring constant changes from K_1 to K_2 .

For this case the first three boundary conditions are given by (2.2), (2.3) and (2.4), but the fourth boundary condition is expressed by different equations for two ranges of deflection;

For $|y| \leqslant y_0$, at $x = L$

$$EI \frac{\partial^3 y}{\partial x^3} = K_1 y + m \frac{\partial^2 y}{\partial t^2} - F \sin \omega t \qquad \text{........(2.35)}$$

For $|y| > y_0$, at $x = L$

$$EI \frac{\partial^3 y}{\partial x^3} = (K_1 - K_2) y_0 + K_2 y + m \frac{\partial^2 y}{\partial t^2} - F \sin \omega t$$

........(2.36)

To determine the harmonic response with bilinear spring, it will be assumed that the response is symmetrical every quarter wave. With this, let the response be

$$y(x,t) = X(x) \sin \omega t \qquad \text{........(2.37)}$$

where X(x) should satisfy

$$\frac{d^4 X}{dx^4} - \frac{\omega^2}{c^2} X = 0 .$$

Therefore,

$$X(x) = C_1 \cosh \lambda x + C_2 \sinh \lambda x + C_3 \cos \lambda x + C_4 \sin \lambda x$$

Using equation (2.2), (2.3) and (2.4)

$$X(x) = C_1 \left[(\cosh \lambda x - \cos \lambda x) - \frac{A}{B} (\sinh \lambda x - \sin \lambda x) \right]$$

Dividing the above equation by L

$$X_x = C_h \left[(\cosh\lambda x - \cos\lambda x) - \frac{A}{B} (\sinh\lambda x - \sin\lambda x) \right]$$

........(2.38)

Where X_x and C_h are similar to as defined for cubic nonlinear case.

and $$X_a = C_h \left(C - \frac{AD}{B} \right)$$ ....(2.39)

The fourth boundary condition given by equation (2.35) in the linear case can be satisfied exactly by the assumed harmonic solution. The main interest is to obtain the response when the amplitude of end deflection is greater than y_o. It can be seen that equation (2.36) cannot be exactly satisfied by the assumed solution (2.37). In such a case an approximate solution will be obtained by satisfying the fourth boundary condition at zero deflection and at its peak value. A similar procedure has been indicated in reference 2 for obtaining the response of nonlinear single degree of freedom systems. At zero deflection the boundary condition is indentically satisfied and for the peak value, equation (2.36) becomes;

at $x = L$

$$EI \frac{d^3 X(x)}{dx^3} = F_s - m\omega^2 X(x) - F$$

Here F_s is the spring force at the peak value of end deflection.

Substituting (2.38) in the above equation:

$$EI\,\lambda^3 C_1 \left(D - \frac{A^2}{B}\right) = F_s - m\,\omega^2 C_1 \left(C - \frac{AD}{B}\right) - F$$

or

$$\left(\Omega^3 \frac{BD - A^2}{BC - AD} + \Omega^4 \mu\right) X_a + P = P_s \qquad \text{........(2.40)}$$

where, $P_s = \dfrac{F_s L^2}{EI}$, dimensionless spring force.

For $|y| \leq y_o$, $P_s = \alpha_1 X_a$

and for $|y| > y_o$, $P_s = \alpha_2 X_a + (\alpha_1 - \alpha_2) X_o$

where, $\alpha_1 = \dfrac{K_1 L^3}{EI}$, dimensionless spring constant for end deflection $\leq y_0$

$\alpha_2 = \dfrac{K_2 L^3}{EI}$, dimensionless spring constant for end deflection $> y_0$

$X_o = \dfrac{y_o}{L}$, dimensionless end deflection at which the spring constant changes

The value of X_a determined by (2.40) ensures the approximate validity of (2.37). Equation (2.40) can be conveniently solved graphically as illustrated in figure 2.4 . The right hand side of (2.40) is the spring force, while the left side expresses a straight line on force-deflection diagram with ordinate intercept P and the slope S equal to $\tan^{-1}\left(-\Omega^3 \dfrac{BD - A^2}{BC - AD} + \Omega^4 \mu\right)$. The intersection of this line with the spring characteristics gives the value of X_a which satisfies (2.40).

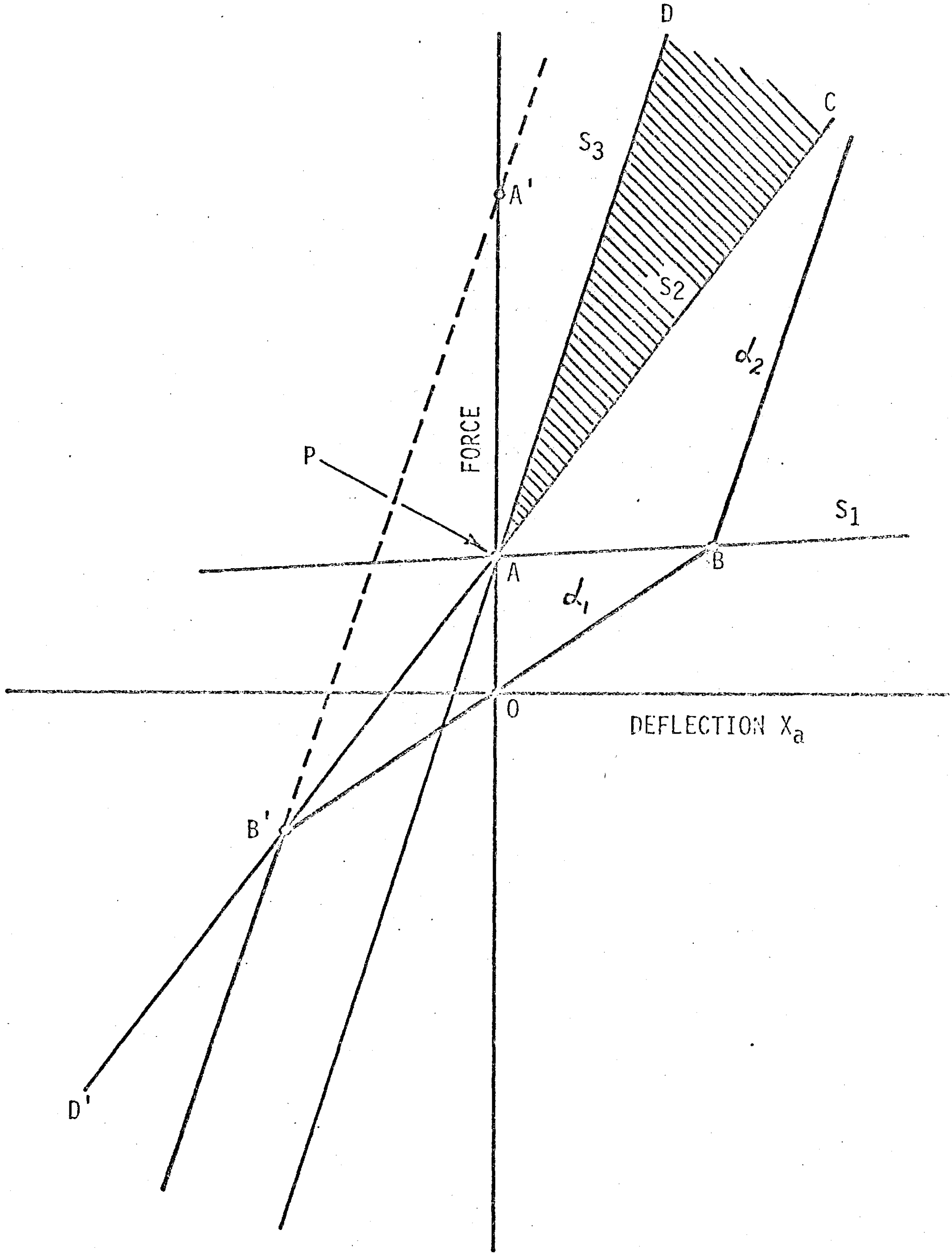


FIGURE 2.4 : GRAPHICAL CONSTRUCTION TO OBTAIN RESPONSE OF THE SYSTEM IN CASE OF BILINEAR SPRING

From figure (2.4), it can be observed that if the slope S falls between the slopes of line AD (S_3) and AC (S_2), there will be three values of X_a which can satisfy equation (2.40). The value of S will be between S_2 and S_3 for an infinite number of frequency ranges. These frequency ranges are bounded by the natural frequencies of a cantilever beam supported on a linear spring and carrying a concentrated mass at the free end. The lower bound of each range corresponds to spring of stiffness α_1 and the upper bound corresponds to spring of stiffness α_2.

The slopes of lines AB, AC and AD; i.e., S_1, S_2 and S_3 respectively, are given by the following relations:

$$S_1 = \alpha_1 - \frac{P}{X_0} \qquad \text{........(2.41 a)}$$

$$S_2 = \alpha_1 + \frac{P}{X_0} \qquad \text{........(2.41 b)}$$

$$S_3 = \alpha_2 \qquad \text{........(2.41 c)}$$

Considering figure (2.4), the end deflection for different ranges of slope S can be computed as follows:

(1) $S \leqslant S_1$: one solution

$$X_a = \frac{P}{\alpha_1 - S} \qquad \text{......(2.42 a)}$$

(2) $S_1 < S < S_2$: one solution

$$X_a = \frac{P + (\alpha_2 - \alpha_1) X_0}{(\alpha_2 - S)} \qquad \text{......(2.42 b)}$$

(3) $S_3 \geqslant S \geqslant S_2$: three solutions

$$\left.\begin{aligned} X_a &= \frac{P}{(\alpha_1 - S)} \\ X_a &= \frac{P + (\alpha_2 - \alpha_1) X_0}{(\alpha_2 - S)} \\ X_a &= \frac{P - (\alpha_2 - \alpha_1) X_0}{(\alpha_2 - S)} \end{aligned}\right] \qquad \text{...(2.42 c)}$$

It can be noticed from figure 2.4 that only one solution will exist for all values of S if the force P is equal to or more than indicated by point A'. Point A' is obtained by the intersection of line D'B' with the force axis.

Equations (2.38), (2.39) and (2.42) determine the harmonic response with bilinear spring and an end mass.

For free vibrations of the beam P is zero and in figure 2.4, point A will coincide with point O. In this case it can be seen that free vibrations will exist for the frequency ranges for which $\alpha_1 \leqslant S \leqslant \alpha_2$. Outside this range the free vibrations will not exist. For $S = \alpha_1$, the amplitude of end deflection will be zero to y_0 , and $S = \alpha_2$ will correspond to

X_a equal to infinity. For $\alpha_1 < S < \alpha_2$, the amplitude will vary between y_0 and infinity. The modal configuration corresponding to any frequency of free vibration is to be determined by equation (2.38).

An alternative approach to obtain the free vibration curves is to assume end deflection and find the slope S. The natural frequencies of cantilever beam,supported on a linear spring of stiffness S at the free end, will correspond to the frequencies of free vibration for the assumed end deflection. In this way,by changing the value of end deflection,all the frequency ranges of free vibration can be determined.

The computer programmes to determine free and forced vibration response of the system,with both types of springs, are given in appendix II.

EXPERIMENTAL ANALYSIS

3. EXPERIMENTAL ANALYSIS

In order to verify the validity of the theoretical results, an experimental study of the problem has been done for the case of bilinear spring.

3.1 EXPERIMENTAL SET-UP:

Figure 3.1 shows the over-all picture of the experimental set-up. A mild steel beam of 45" length was chosen for experimental purpose. The properties of the beam are given in appendix III. One end of the beam was fixed while the other was attached to a rigid support through a bilinear spring. An electromagnetic shaker was used to provide the sinusoidal excitation. The core of the shaker was attached to the beam through a linear spring so as to cause no damage to the shaker and allow sufficient deflection of the beam. The beam was excited in the horizontal direction in order to avoid the gravitational effects. The shaker was so positioned that the line of action of the exciting force coincided with the axis of the bilinear spring. The details of excitation are shown in figure 3.2. The various equipments used in the experiment are listed in appendix IV.

The shaker was powered through a power amplifier which in turn was connected to R. C. generator. An ammeter was provided in the circuit to ensure that the exciting force is kept constant for different frequencies.

A capacitive type of displacement transducer was used to measure the amplitude at the end of the beam whereas the displacements at other points of the beam were measured by means of proximity vibration transducer. Details of mounting and positioning of the displacement and the proximity vibration transducers are shown in figures 3.2 and 3.3 respectively. The output of these transducers was fed to a cathode ray oscilloscope through oscillators and reactance converters. This system of measurement converts the movement of beam at point of measurement into a voltage signal which is displayed on the screen of cathode ray oscilloscope.

3.2 DESIGN OF BILINEAR SPRING:

The bilinear spring used in the experiment was obtained by combining two different linear springs in series as shown in figure 3.4 . In this arrangement one end of the spring K_1' is fixed to the beam and the other to stopper (3), whereas the spring K_2' connects the stopper to a rigid support. Two discs (1 and 2) are fixed to a threaded bar which is attached to the end of the beam such that it is perpendicular to the axis of the beam and coaxial with the springs K_1' and K_2'. The clearance provided between the threaded bar and the stopper allows the free axial motion of the bar. Provision is made to adjust the gap δ between the discs and the stopper. All parts, except the springs, of this arrangement were made of aluminium in order to reduce the weight.

If springs K_1' and K_2' have symmetrical characteristics, then this arrangement would give bilinear characteristics as shown in figure 2.3. Consider that the springs are being compressed, then upto a certain deflection y_0 both K_1' and K_2' act, but at deflection y_0 disc (1) touches the stopper which stops further compression of K_1' and only spring K_2' is compressed. Similarly when the springs are being stretched, at deflection y_0, disc (2) touches the stopper and stops further action of K_1' and only K_2' will be stretched beyond the deflection y_0. In this way action of a symmetrical bilinear spring is obtained.

The deflection y_0 at which the stiffness of the spring system changes depends upon the initial gap δ and the stiffness K_1' and K_2'. The relation to find y_0 is:

$$y_0 = \left(1 + \frac{K_1'}{K_2'} \right) \delta$$

and the constants of the bilinear spring are:

(i) For Deflection $\leqslant y_o$

$$K_1 = \frac{K_1' K_2'}{K_1' + K_2'}$$

(ii) For deflection $> y_o$

$$K_2 = K_2'$$

The bilinear spring used in the experiment was obtained with following specifications:

$$K_1 = 7.24 \text{ lbf/in.}$$

$$K_2 = 35.20 \text{ lbf/in.}$$

$$y_o = 0.052 \text{ inch}$$

3.3 TEST PROGRAMME:

The test programme included the measurement of amplitudes of vibration with varying frequency at different levels of excitation.

Before the actual experimentation the displacement transducers were calibrated. In order to determine the frequency response of the system, the amplitudes were measured at five points with $x = 15, 20, 25, 35$ and 45 inches. Two different sinusoidal excitations of 0.5 lbf and

1.5 lbf were used in the experiments. In both the cases the amplitudes of vibrations were recorded by varying the frequency and keeping the exciting force constant.

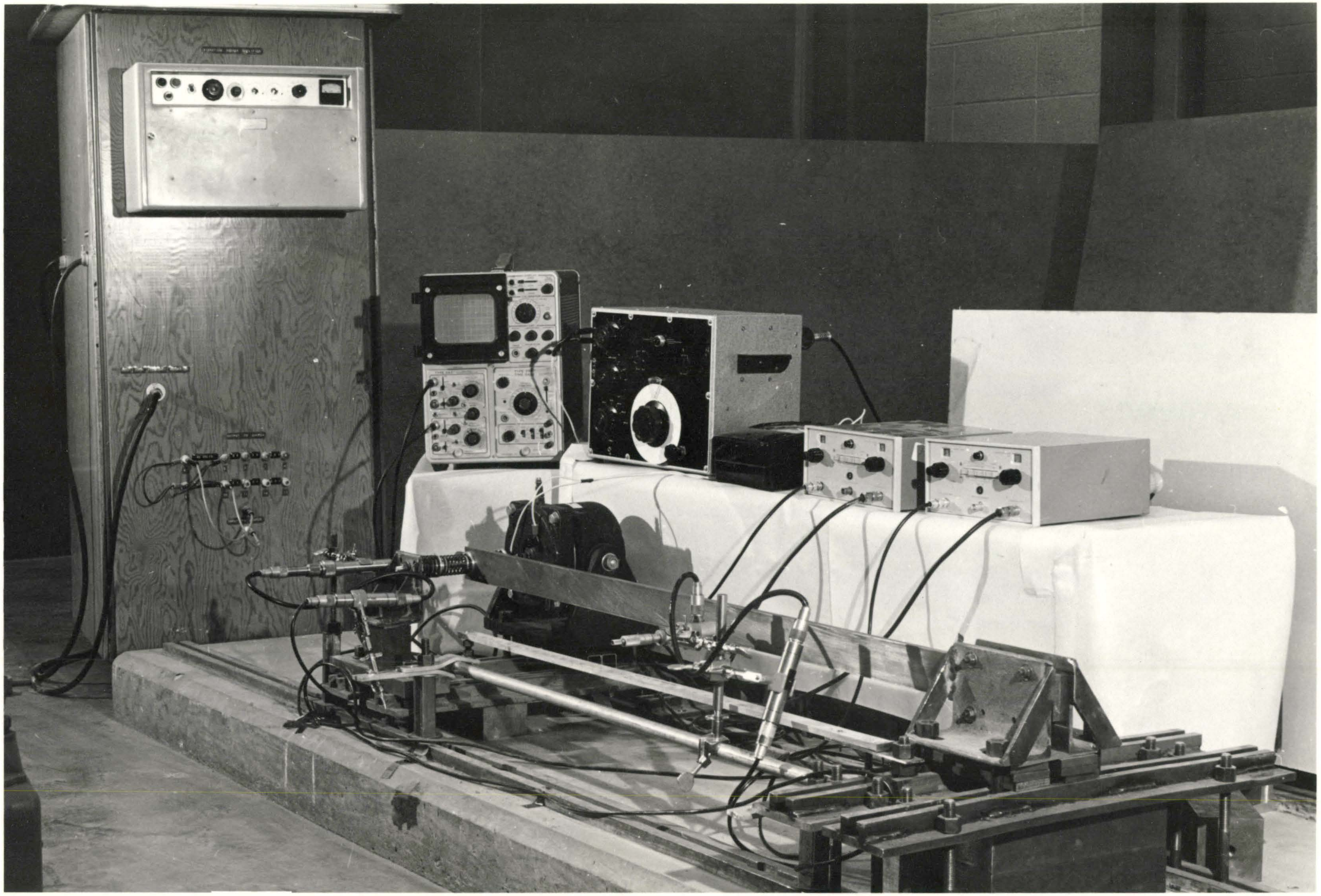


FIGURE 3.1 : OVERALL PICTURE OF EXPERIMENTAL SET-UP

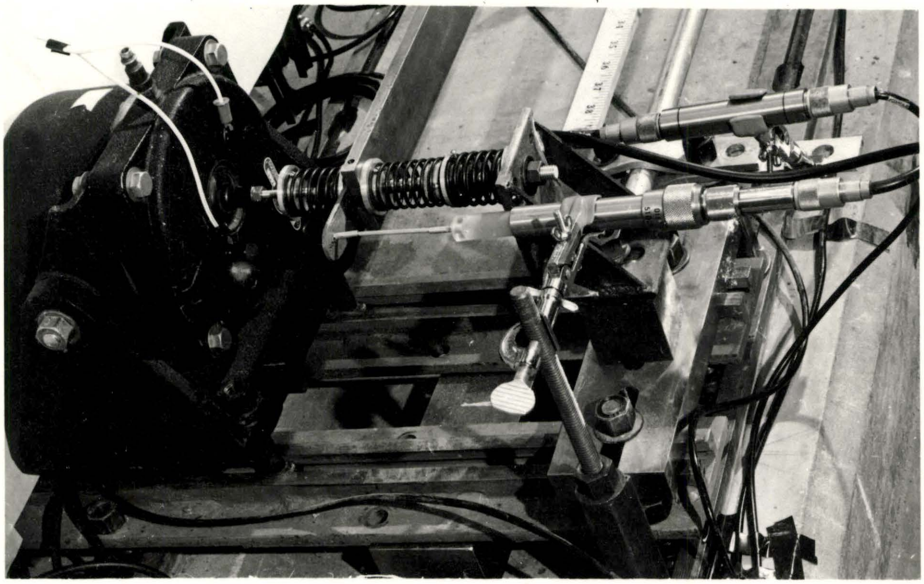


FIGURE 3.2 : DETAILS OF EXCITATION AND MEASUREMENT OF END DISPLACEMENT

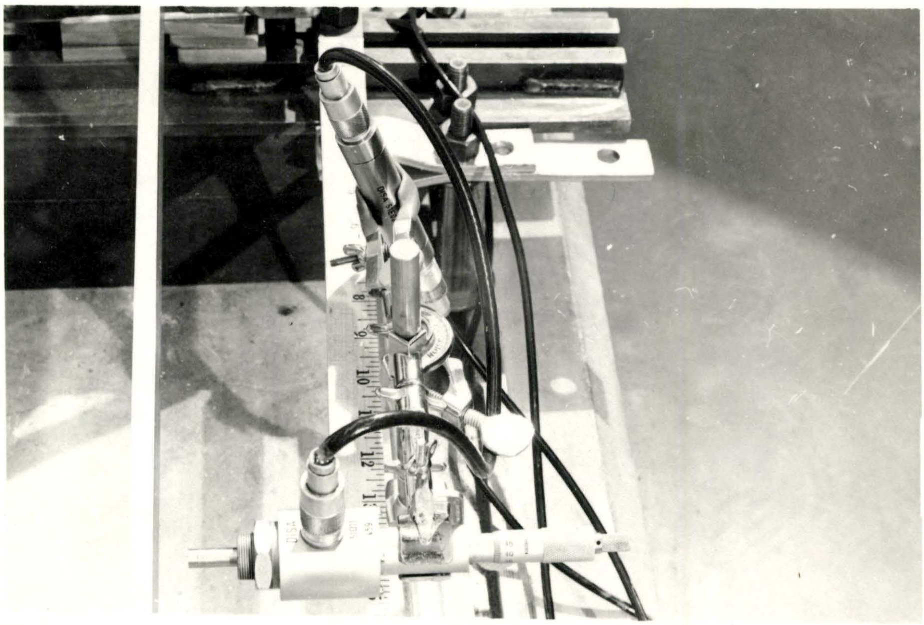


FIGURE 3.3 : SETTING OF PROXIMITY VIBRATION TRANSDUCER TO MEASURE DISPLACEMENT ALONG THE BEAM

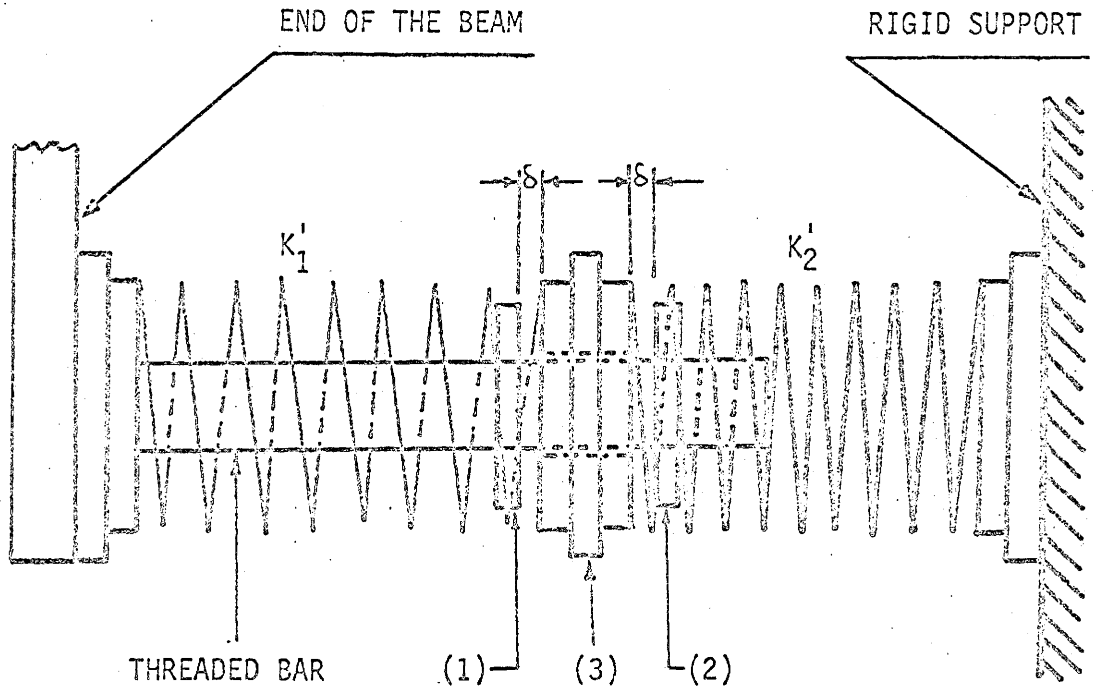


FIGURE 3.4 : SCHEMATIC DIAGRAM OF BILINEAR SPRING

RESULTS AND CONCLUSIONS

4. RESULTS AND CONCLUSIONS

4.1 RESULTS:

4.1.1 Beam With Cubic Nonlinear Spring:-

The results of theoretical analysis show that free vibrations of beam,with cubic nonlinear spring, exist in an infinite number of frequency ranges. Each of the possible frequencies of free vibration corresponds to a specific value of amplitude coefficient C_h and to a definite modal configuration as given by equation (2.9). The free vibration curves for different values of α with $\beta = 0, 10, 10^2, 10^3, 10^4, 10^5, \infty$ are given in figures 4.1(a), 4.1(b), 4.2, and 4.3 . All these curves are for $\mu = 0$. For $\alpha = 0$, the amplitude coefficient and the amplitude of end deflection have been plotted in figures 4.1(a) and 4.1(b) respectively. It is apparent from figure 4.1(b) that frequency of free vibration increases as the amplitude is increased. The frequency ranges of free vibrations for $\alpha = 0$ are specified by the natural frequencies of simple cantilever beam and those of clamped - simply supported beam. In figures 4.2 and 4.3 amplitude coefficient C_h has been plotted for $\alpha = 2$ and 10 respectively. The nature of these curves is similar to those of figure 4.1(a) although the starting points of the curves shift towards higher values of frequencies as α is increased.

Figures 4.4 to 4.9 show the harmonic response of the system for $\alpha = 0$ and different values of nonlinearity parameter β and the external force P. The free vibration curves divide the forced harmonic response in a number of separate sections. Further, the coefficient C_h changes its sign whenever a free vibration curve is crossed. Therefore, in each section the system behaves essentially in the same manner as some of the nonlinear single degree of freedom systems.

Equation (2.19) indicates that the point of vertical tangency can lie only in the frequency ranges for which the free vibrations exist. The points of vertical tangency correspond to values of Ω at which equation (2.12) changes from having one real root to having three real roots. It can, therefore, be inferred that the frequency ranges within which the solution may become unstable are the same as for free vibrations.

The harmonic response obtained with the help of equations (2.9), (2.10) and (2.12) is only an approximate solution of the problem. An exact solution will contain higher order harmonics. With the help of equations (2.24) and (2.25), response including third order harmonic component can be found which is a better approximation. In figure 4.10 response including third order harmonic component has been shown for $\mu = 0$, $\alpha = 0$, $\beta = 10^4$ and $P = 0.1$. It has been found that the response becomes unstable for: (a) the frequencies for which the harmonic response is unstable, and (b) the frequencies which are $(1/3)^{\frac{1}{2}}$ times the frequencies in (a).

Equation (2.33) gives the necessary condition for one-third order subharmonic response to exist. This condition will be satisfied only when the values of both numerator and denominator in equation (2.33) do not have the same sign. This means that subharmonic response can occur in the frequency ranges specified by $\sqrt{3}\,\Omega_{1n}$ and $\sqrt{3}\,\Omega_{2n}$, ($n = 1,2,3, \ldots\ldots, \infty$) , where, Ω_1 and Ω_2 correspond to the natural frequencies of simple cantilever beam and clamped - simply supported beam respectively. Figure 4.11 shows the subharmonic response of the system only in the first frequency range for $\alpha = 0$, $\beta = 10^4$, $P = 0.1$ and $\mu = 0$ It can be noticed that subharmonic curves arise through the bifurcation from the harmonic response at the point on the C_h curve corresponding to $C_s = 0$.

4.1.2 Beam With Bilinear Spring:-

Figures 4.12(a), 4.12(b) and 4.13 give theoretical free vibration curves for different values of α_1 with α_2 = 10, 100, 1000 and μ = 0 . In figure 4.12(a), the amplitude coefficient C_h has been plotted for the case in which there is a clearance between the beam and the linear spring of stiffness α_2, whereas in figure 4.12(b) the amplitude of end deflection has been plotted for the same values of clearance and α_2 . The free vibration curves in figure 4.13 are for amplitude coefficient C_h for α_1 = 2.0 and varying α_2 . The nature of these curves is essentially the same as in figure 4.12(a) , although the starting points of free vibrations have been shifted towards higher frequencies.

In this case also, the free vibration curves exist in an infinite number of frequency ranges. These frequency ranges are bounded by the natural frequencies of a cantilever beam supported on a linear spring at its free end. The lower bound of each range corresponds to a spring of stiffness α_1 and the upper bound corresponds to a spring of stiffness α_2 .

It can be observed from figure 4.12(b) that the rate of increase of frequency of free vibration decreases with the end deflection. Contrary to this, in cubic nonlinear spring, the rate of increase of frequency first increases and then decreases (see Figure 4.1(b)) . This may be due to the fact that the rate of increase of stiffness in bilinear case decreases with the deflection whereas in case of cubic nonlinear spring it increases.

The experimental and theoretical values of response have been found out at five points along the beam and they are given in tables 4.1 to 4.4 . The theoretical free vibration values are given in table 4.5 . Figures 4.14 and 4.15 show the theoretical and experimental response at $x_\ell = 1.0$ and 0.555 respectively for $P = 3.425 \times 10^{-3}$. In these figures jump phenomena has been observed in the first free vibration zone. Jump phenomena can also be seen in figure 4.20 which shows qualitative response observed experimentally. Figures 4.16 and 4.17 show the response at $P = 10.275 \times 10^{-3}$. No jump has been observed for this value of the force.

It can be seen from figures 4.14 to 4.17 that at large amplitudes the experimental and theoretical values differ significantly, but at low amplitudes the two values are in good agreement. Probably, damping has appreciable effect at large amplitudes whereas it can be neglected at low amplitudes. Moreover, the theoretical solution is an approximate one .

Furthermore, the second peak in the experimental values is observed at lower frequency of excitation than corresponding to the theoretical value. This may be attributed to the difficulty of obtaining a perfectly clamped end of the beam in the experimental set-up.

Figures 4.18 and 4.19 give the typical time - motion curves observed experimentally at the spring supported end of the beam. In these figures it can be seen that every quarter wave is symmetrical and the vibrations are periodic. At large amplitudes, the response is not exactly sinusoidal indicating the presence of higher order harmonics.

4.2 CONCLUSIONS:

From this study the following conclusions have been drawn:

(1) Free vibrations of the system can exist in an infinite number of frequency ranges and each of the frequencies of free vibration depends on the amplitude of end deflection and corresponds to a definite modal configuration.

(2) For the case of bilinear spring, the rate of increase of frequency of free vibration decreases with the end deflection as the upper bound of each frequency range of free vibration is approached. However, for the case of cubic nonlinear spring this rate first increases and then decreases.

(3) Harmonic response can exhibit jump phenomena in the frequency ranges corresponding to free vibrations. However, in case of bilinear spring, jump phenomena may not occur if the amplitude of exciting force is above a certain value, which depends on the restoring force characteristic of the spring.

(4) Subharmonic vibrations can occur in the case of cubic nonlinear spring in a number of specified frequency ranges.

(5) The free vibration curves divide the forced harmonic response in a number of separate sections. In each of these sections the system behaves essentially in the same manner as some of the nonlinear single degree of freedom systems.

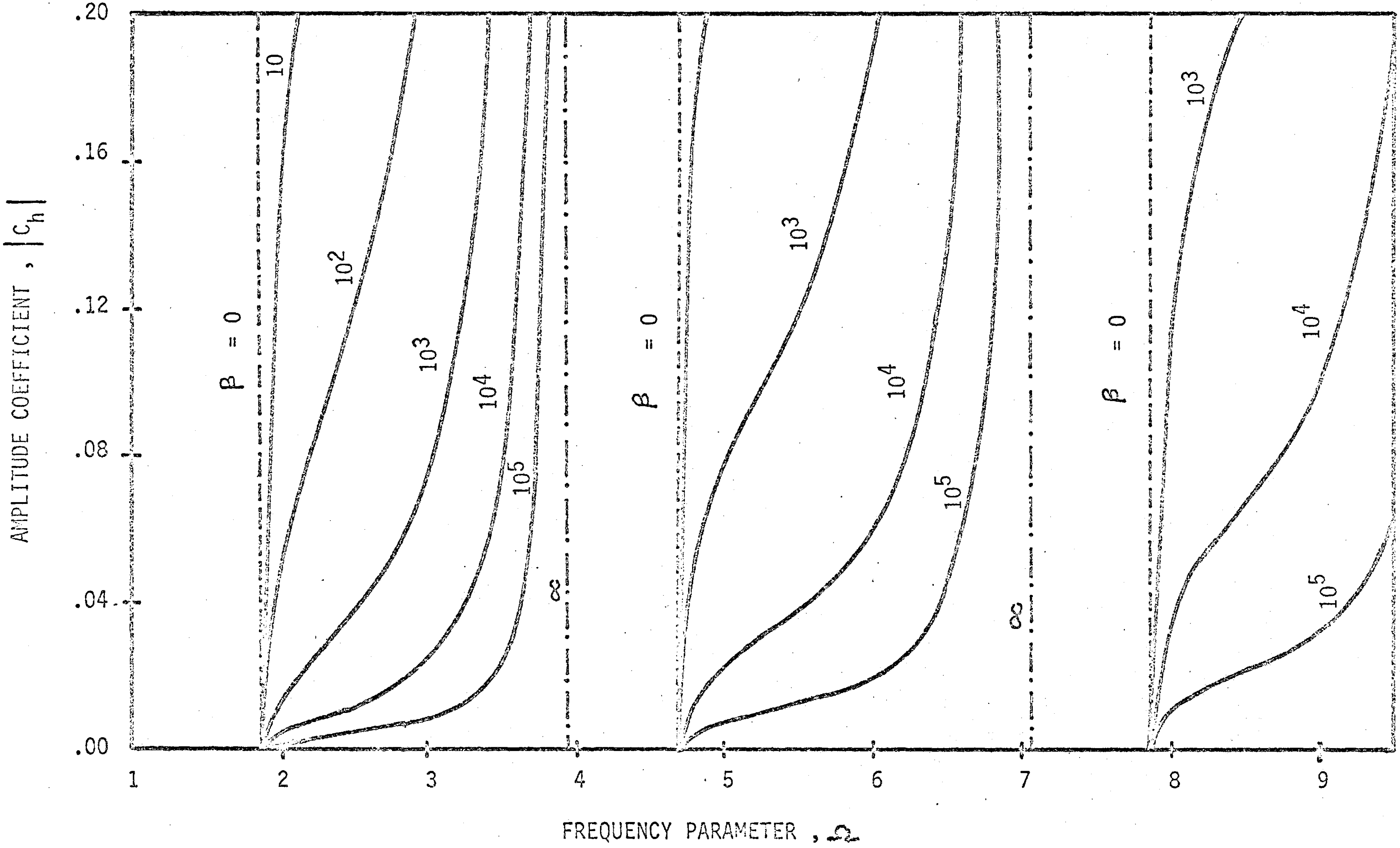


FIGURE 4.1(a) : FREE VIBRATION CURVES WITH CUBIC NONLINEAR SPRING

$\alpha = 0, \beta = 0, 10^1, 10^2, 10^3, 10^4, 10^5, \infty, \mu = 0$

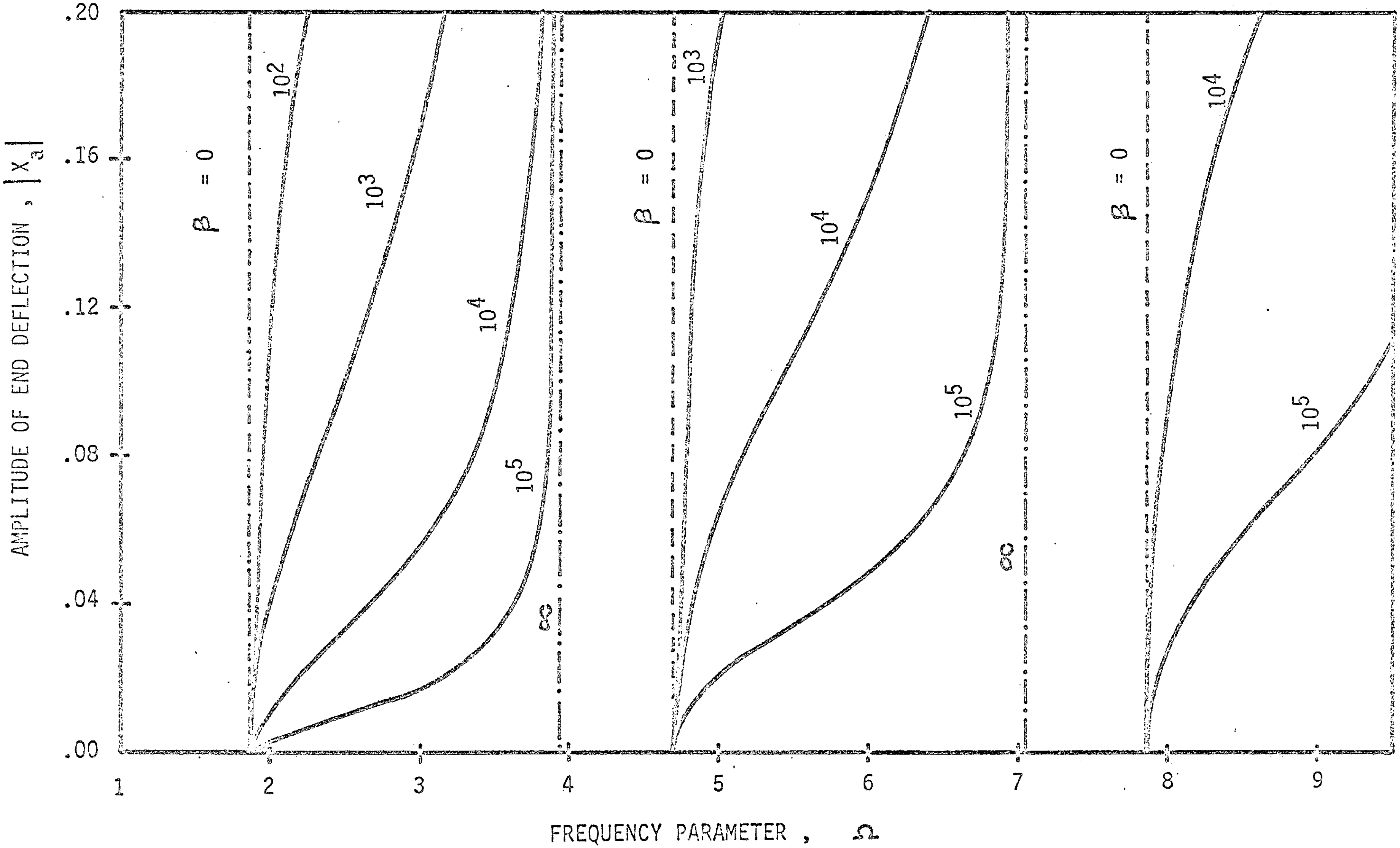


FIGURE 4.1(b) : FREE VIBRATION CURVES WITH CUBIC NONLINEAR SPRING

$\alpha = 0, \ \beta = 0, 10^1, \ 10^2, \ 10^3, 10^4, \ 10^5, \infty \ , \mu = 0$

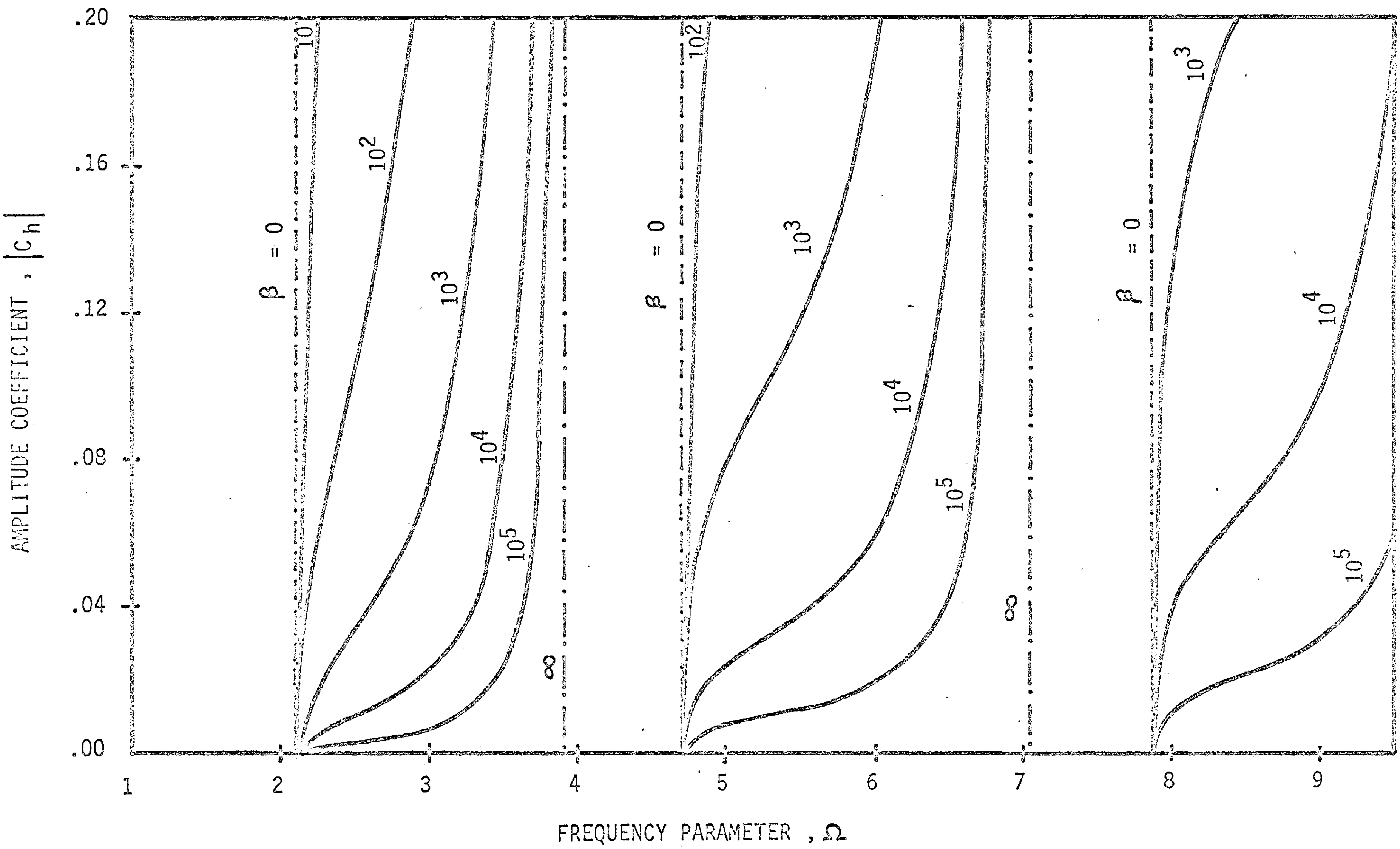


FIGURE 4.2 : FREE VIBRATION CURVES WITH CUBIC NONLINEAR SPRING

$\alpha = 2$, $\beta = 0, 10^1, 10^2, 10^3, 10^4, 10^5, \infty$, $\mu = 0$

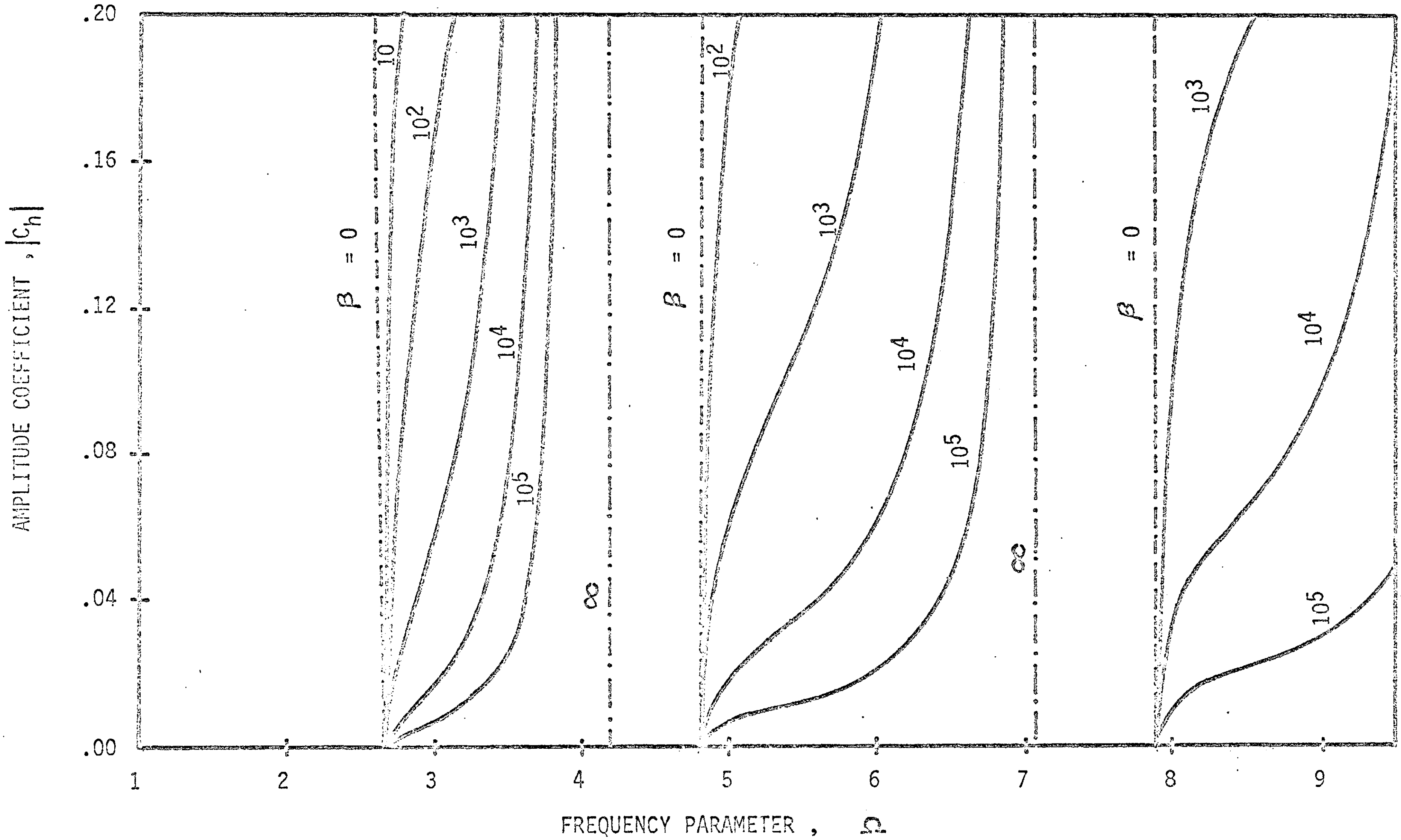


FIGURE 4.3 : FREE VIBRATION CURVES WITH CUBIC NONLINEAR SPRING

$\alpha = 10$, $\beta = 0, 10^1, 10^2, 10^3, 10^4, 10^5$, $\mu = 0$

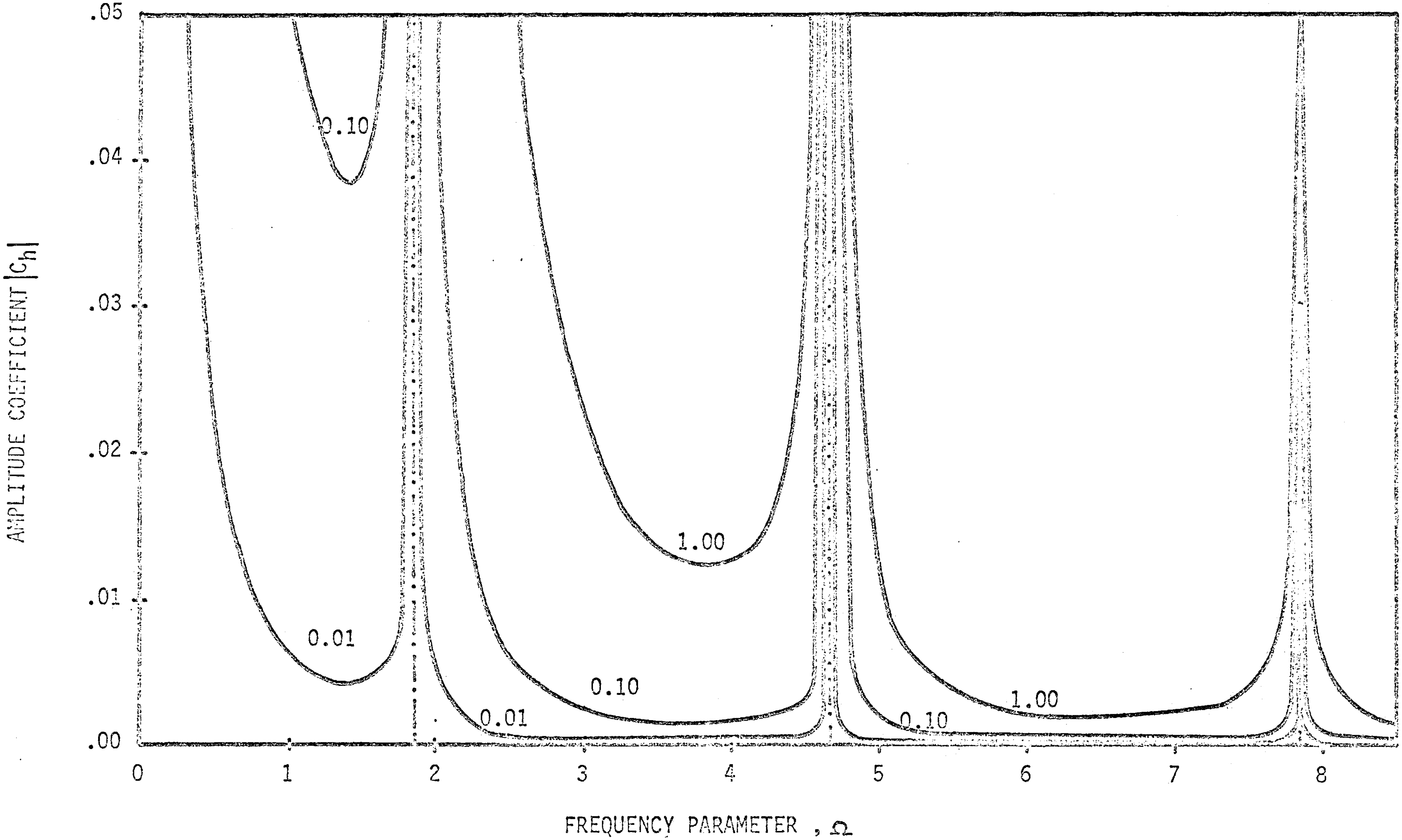


FIGURE 4.4 : HARMONIC RESPONSE OF SIMPLE CANTILEVER BEAM (α = 0, β = 0)

FOR P = 0.01, 0.10, 1.00

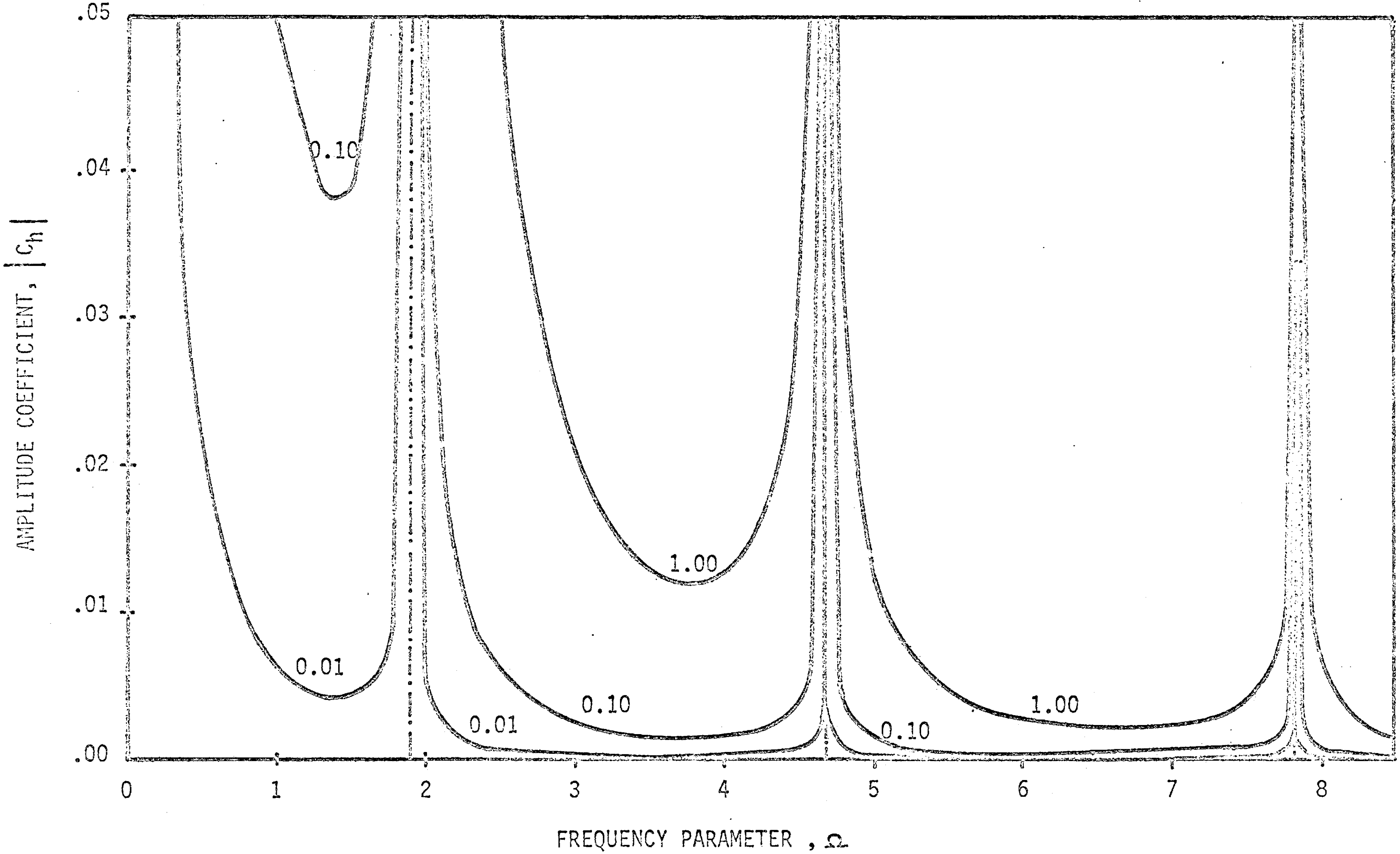


FIGURE 4.5 : HARMONIC RESPONSE WITH CUBIC NONLINEAR SPRING

$\alpha = 0$, $\beta = 10$, FOR P = 0.01, 0.10, 1.00

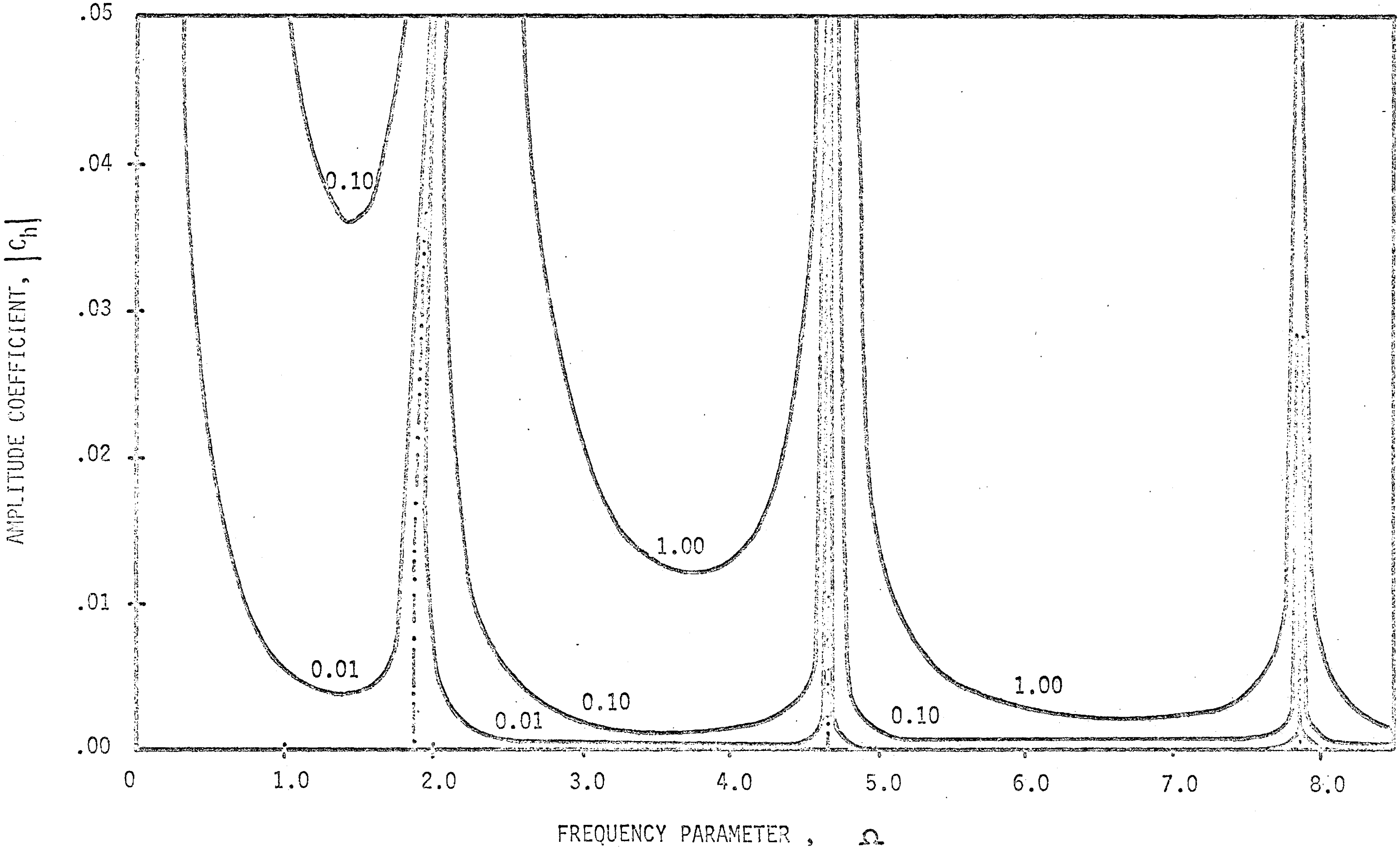


FIGURE 4.6 : HARMONIC RESPONSE WITH CUBIC NONLINEAR SPRING

$\alpha = 0$, $\beta = 10^2$, FOR P = 0.01, 0.10, 1.00

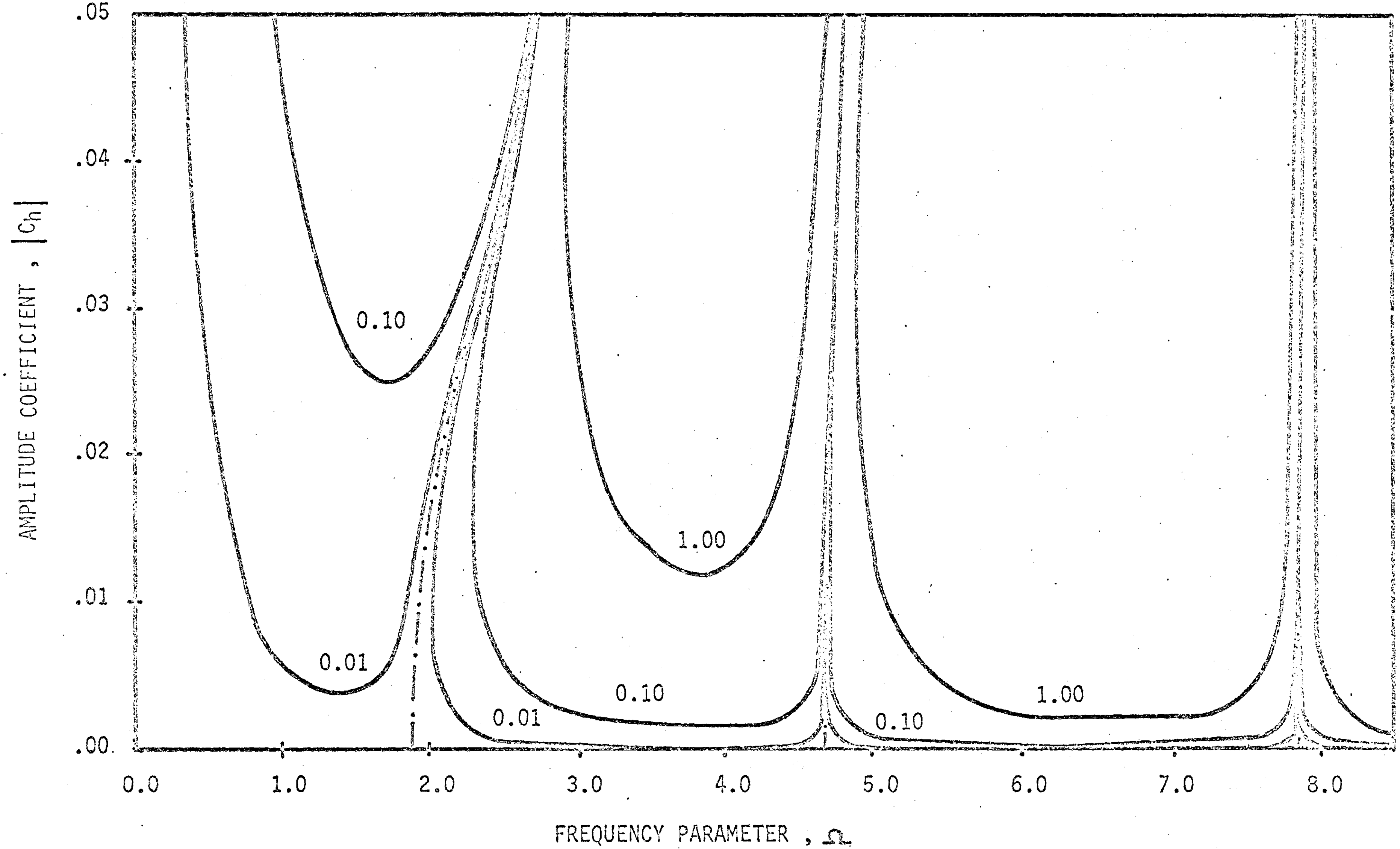


FIGURE 4.7 : HARMONIC RESPONSE WITH CUBIC NONLINEAR SPRING

$\alpha = 0$, $\beta = 10^3$, FOR P = 0.01, 0.10, 1.00

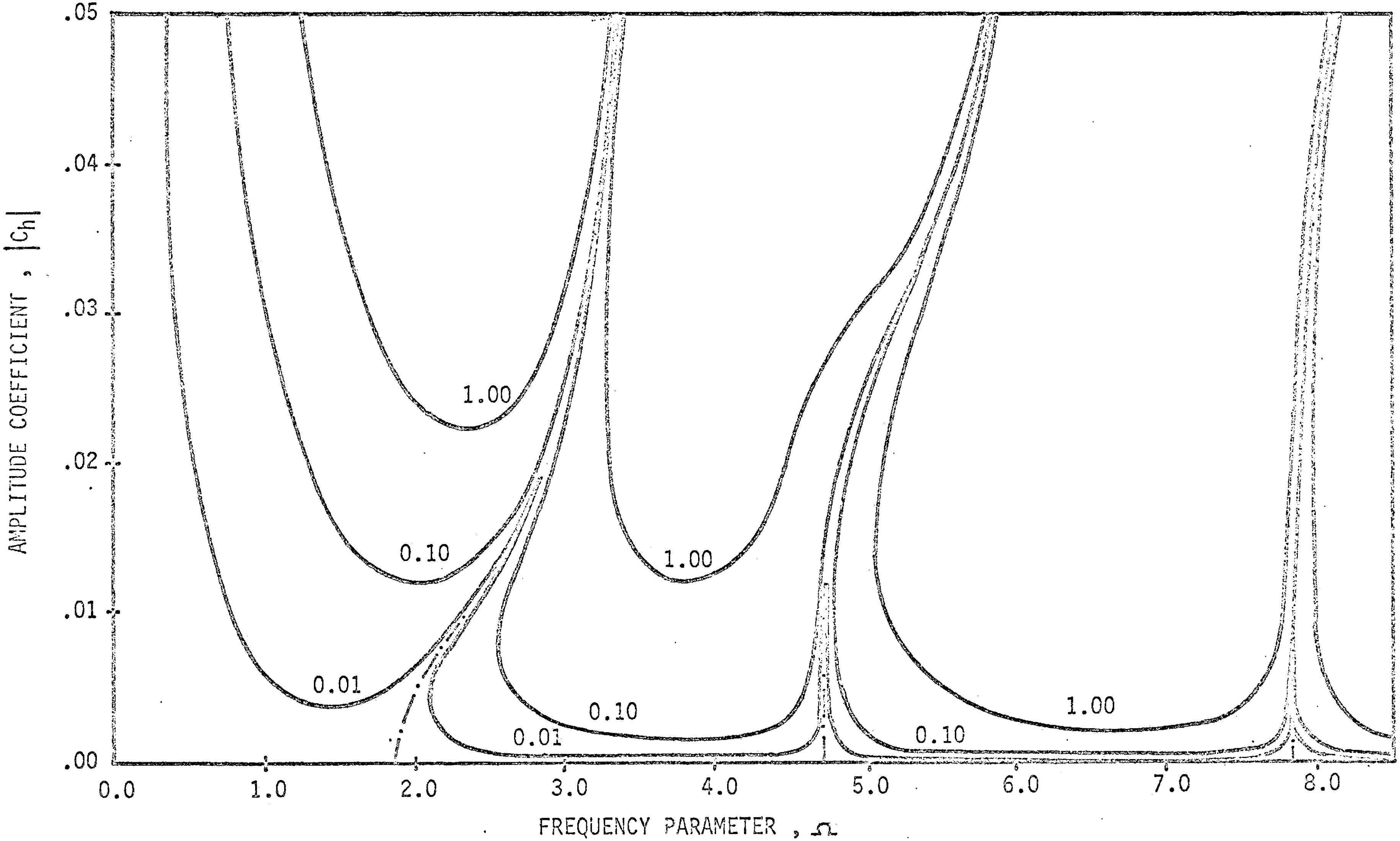


FIGURE 4.8 : HARMONIC RESPONSE WITH CUBIC NONLINEAR SPRING

$\alpha = 0$, $\beta = 10^4$, FOR P = 0.01, 0.10, 1.00

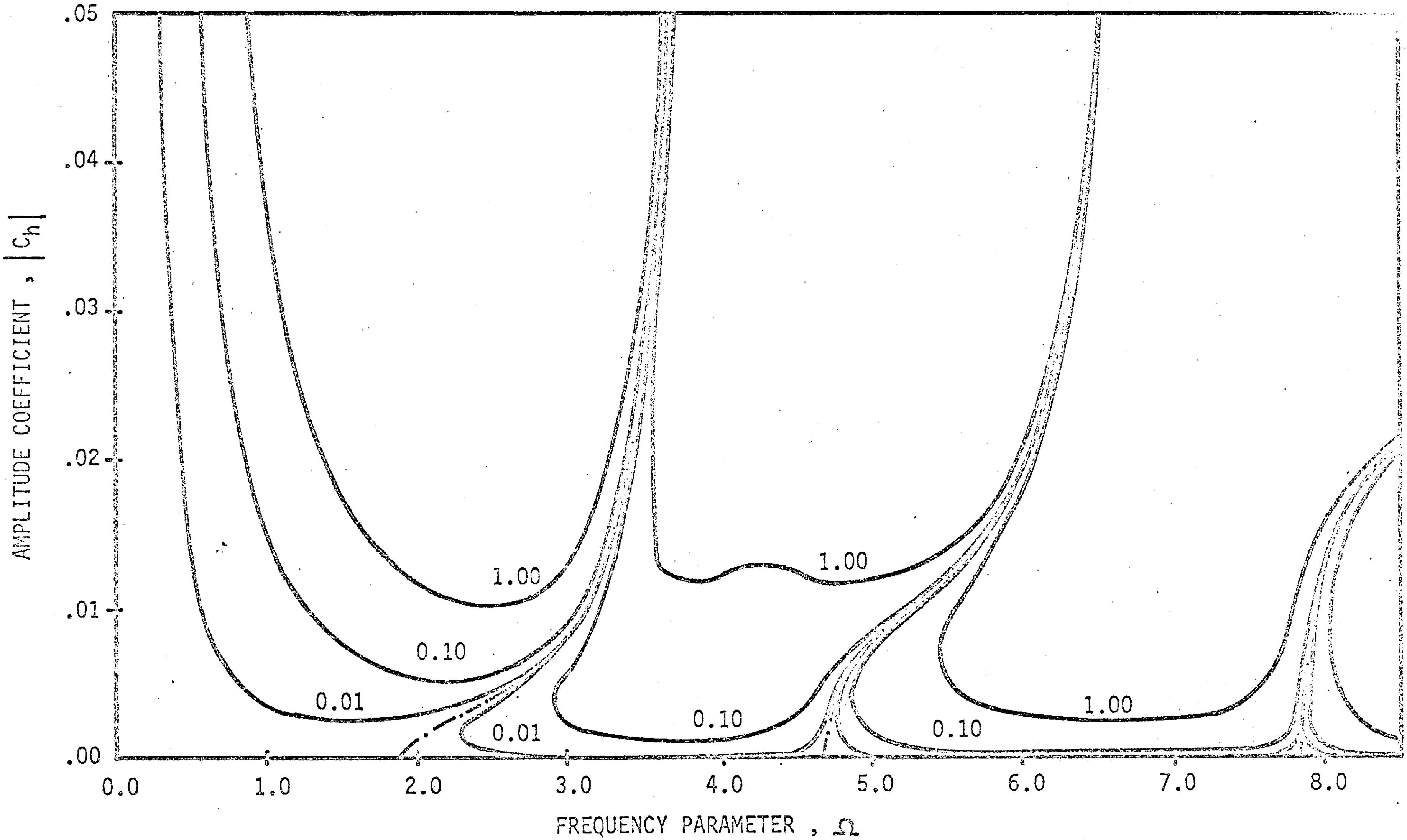


FIGURE 4.9 : HARMONIC RESPONSE WITH CUBIC NONLINEAR SPRING

$\alpha = 0$, $\beta = 10^5$, FOR P = 0.01, 0.10, 1.00

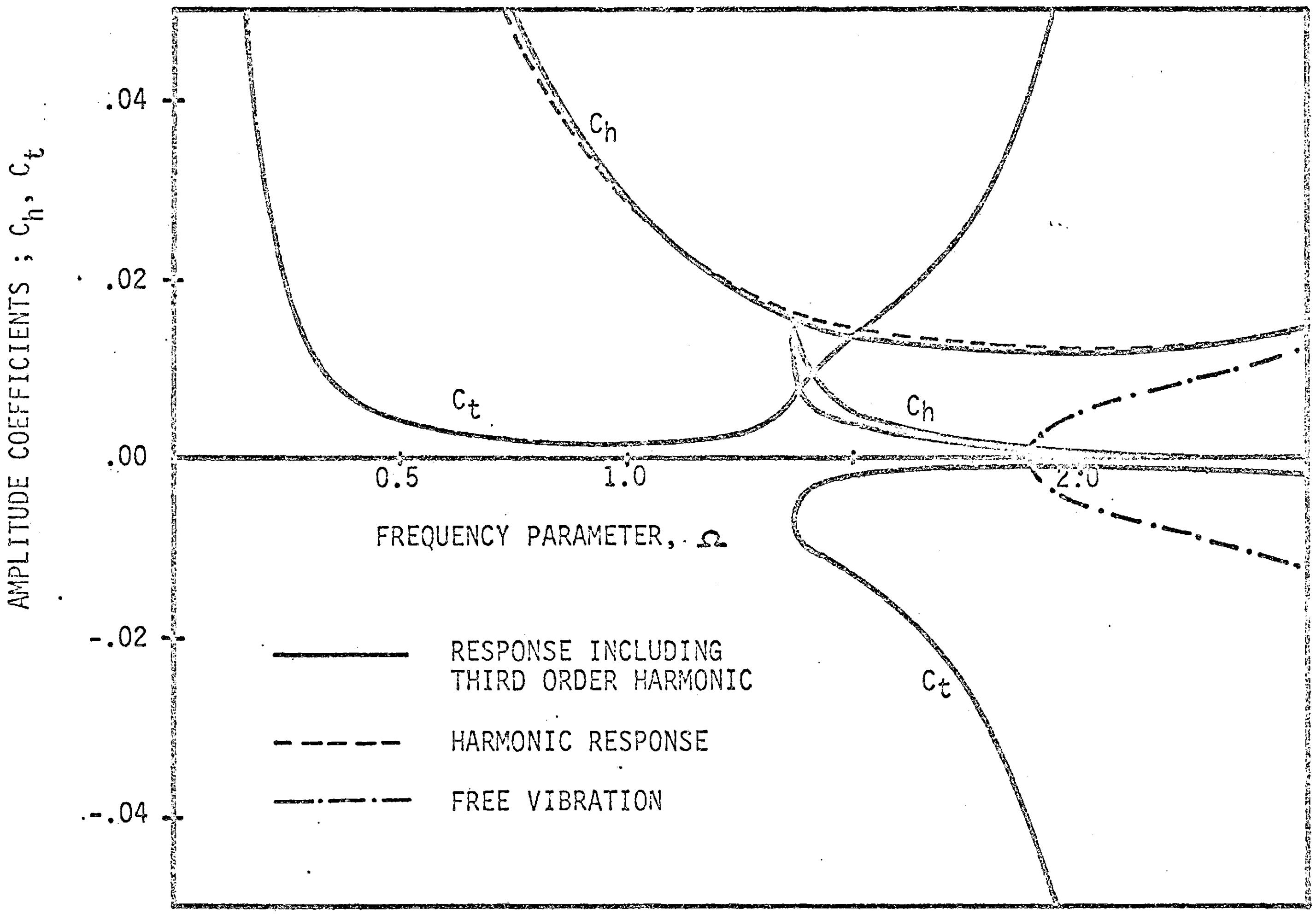


FIGURE 4.10 : RESPONSE, INCLUDING THIRD ORDER HARMONIC COMPONENT, WITH CUBIC NONLINEAR SPRING, $\alpha = 0$, $\beta = 10^4$, P = 0.1

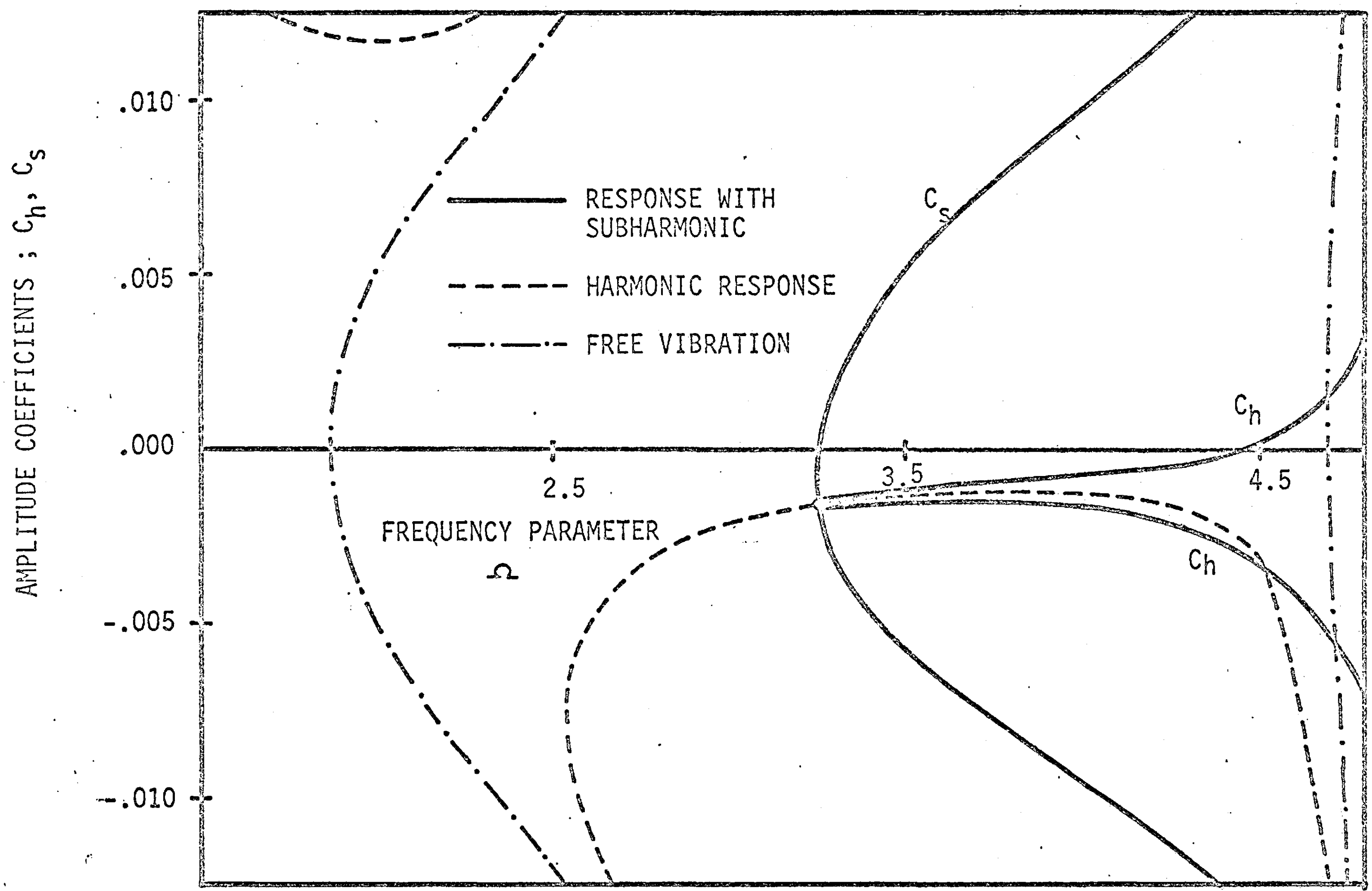


FIGURE 4.11 : RESPONSE, INCLUDING ONE-THIRD ORDER SUBHARMONIC, WITH CUBIC NONLINEAR SPRING, $\alpha = 0$, $\beta = 10^4$, $P = 0.1$

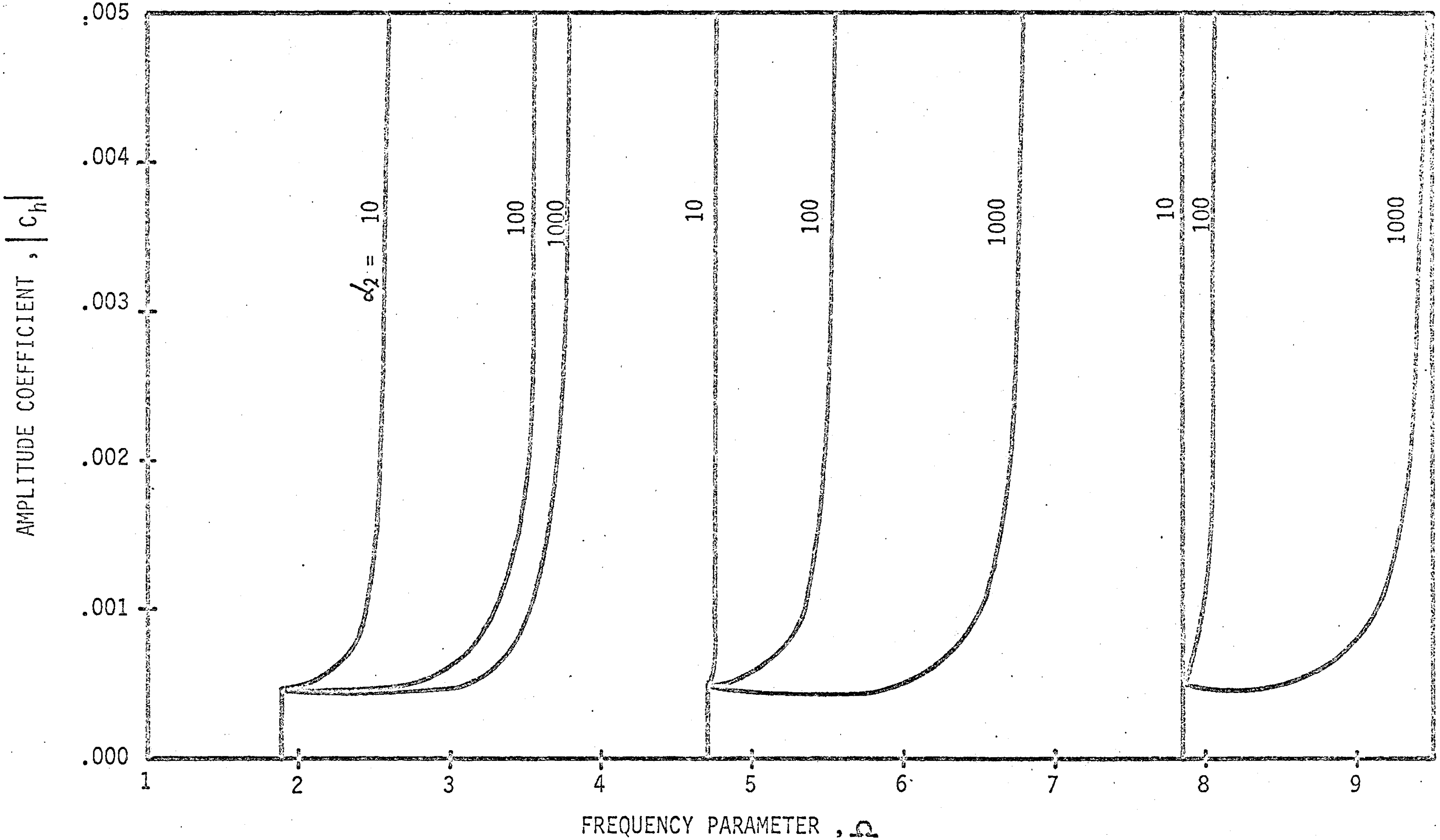


FIGURE 4.12(a) : FREE VIBRATION CURVES WITH BILINEAR SPRING

$\alpha_1 = 0$, $\alpha_2 = 10, 100, 1000$, $\mu = 0$, $X_0 = 0.001$

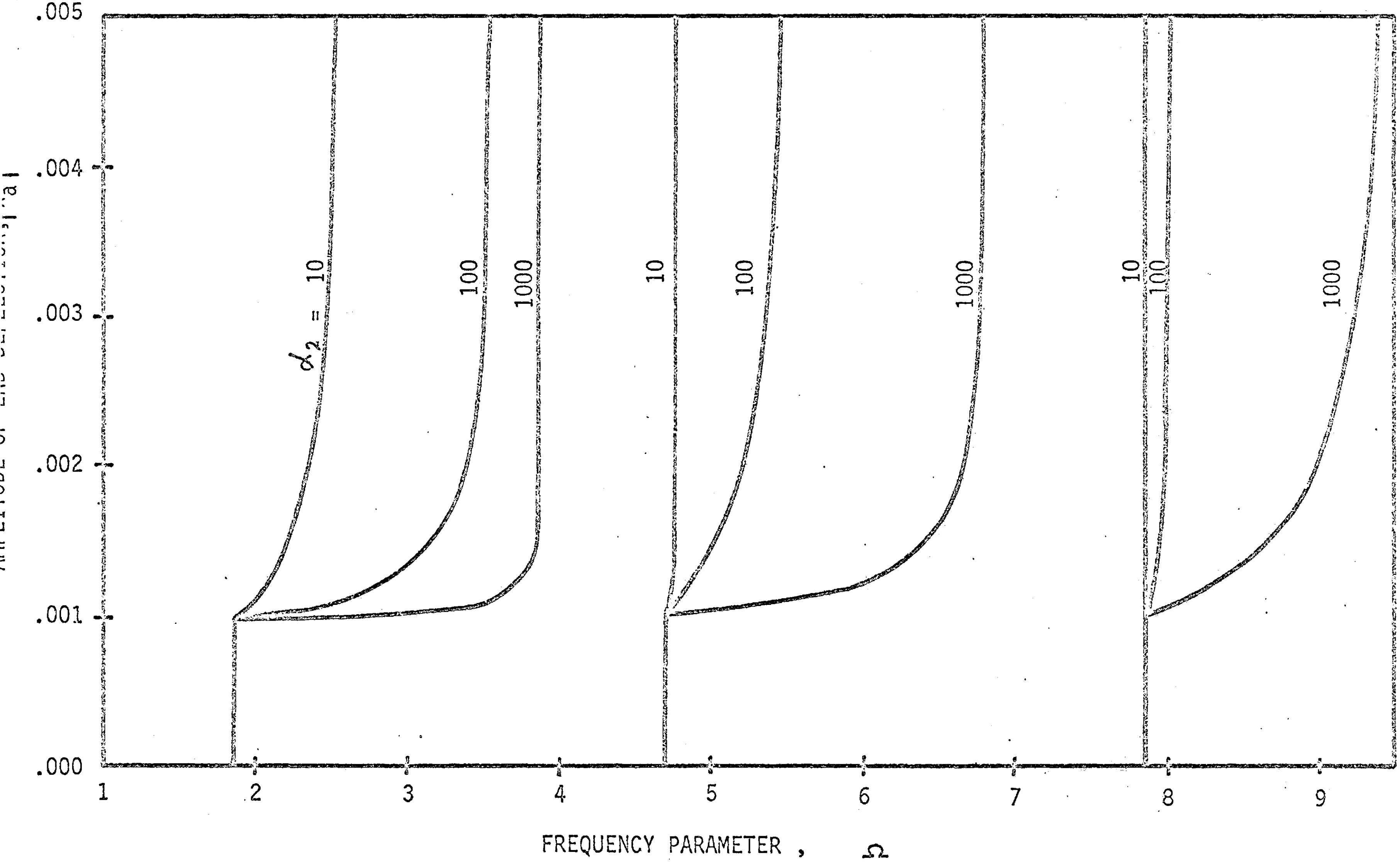


FIGURE 4.12(b) : FREE VIBRATION CURVES WITH BILINEAR SPRING

$\alpha_1 = 0$, $\alpha_2 = 10, 100, 1000$, $\mu = 0$, $X_0 = 0.001$

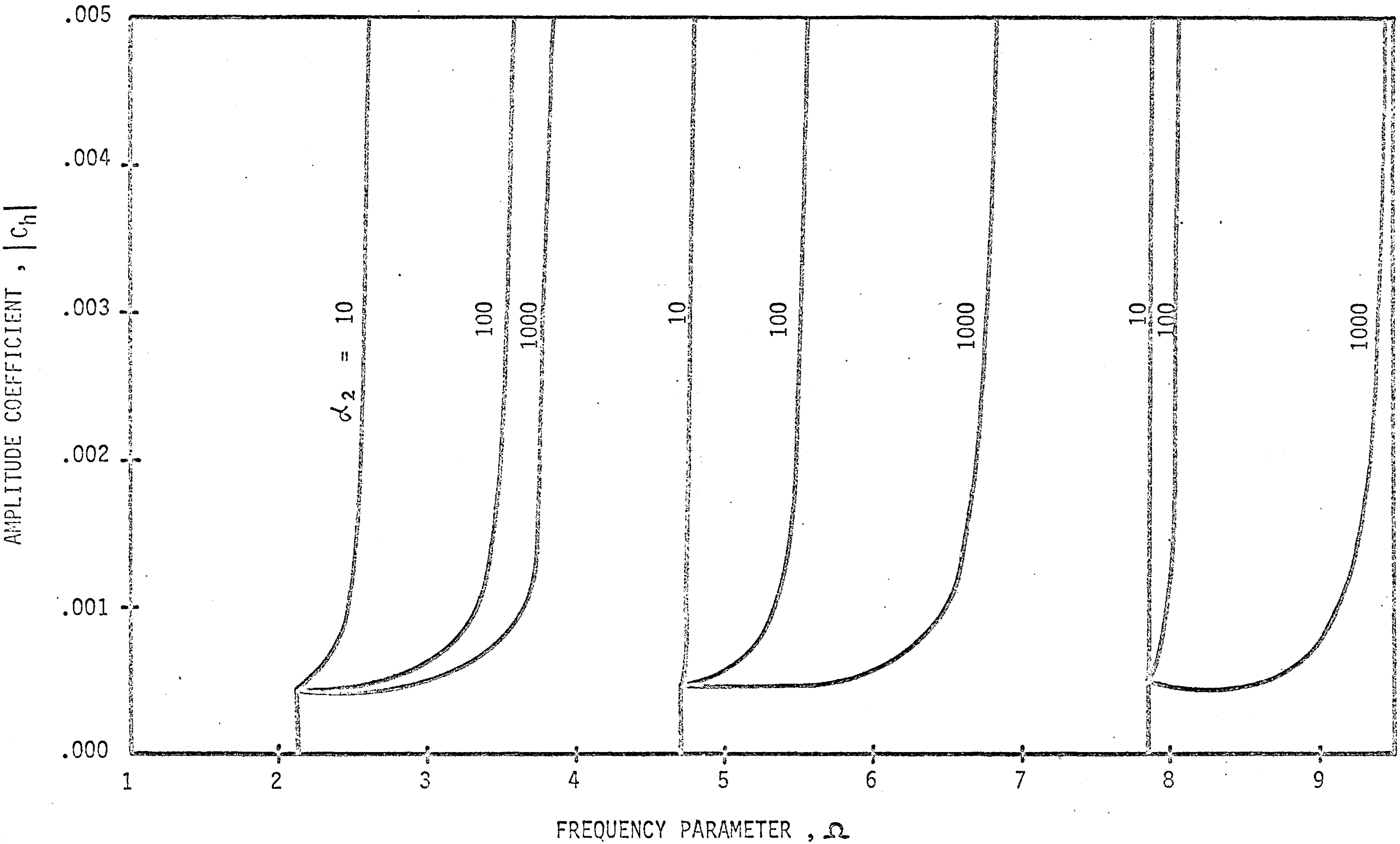


FIGURE 4.13 : FREE VIBRATION CURVES WITH BILINEAR SPRING

$\alpha_1 = 2$, $\alpha_2 = 10, 100, 1000$, $\mu = 0.$, $X_0 = 0.001$

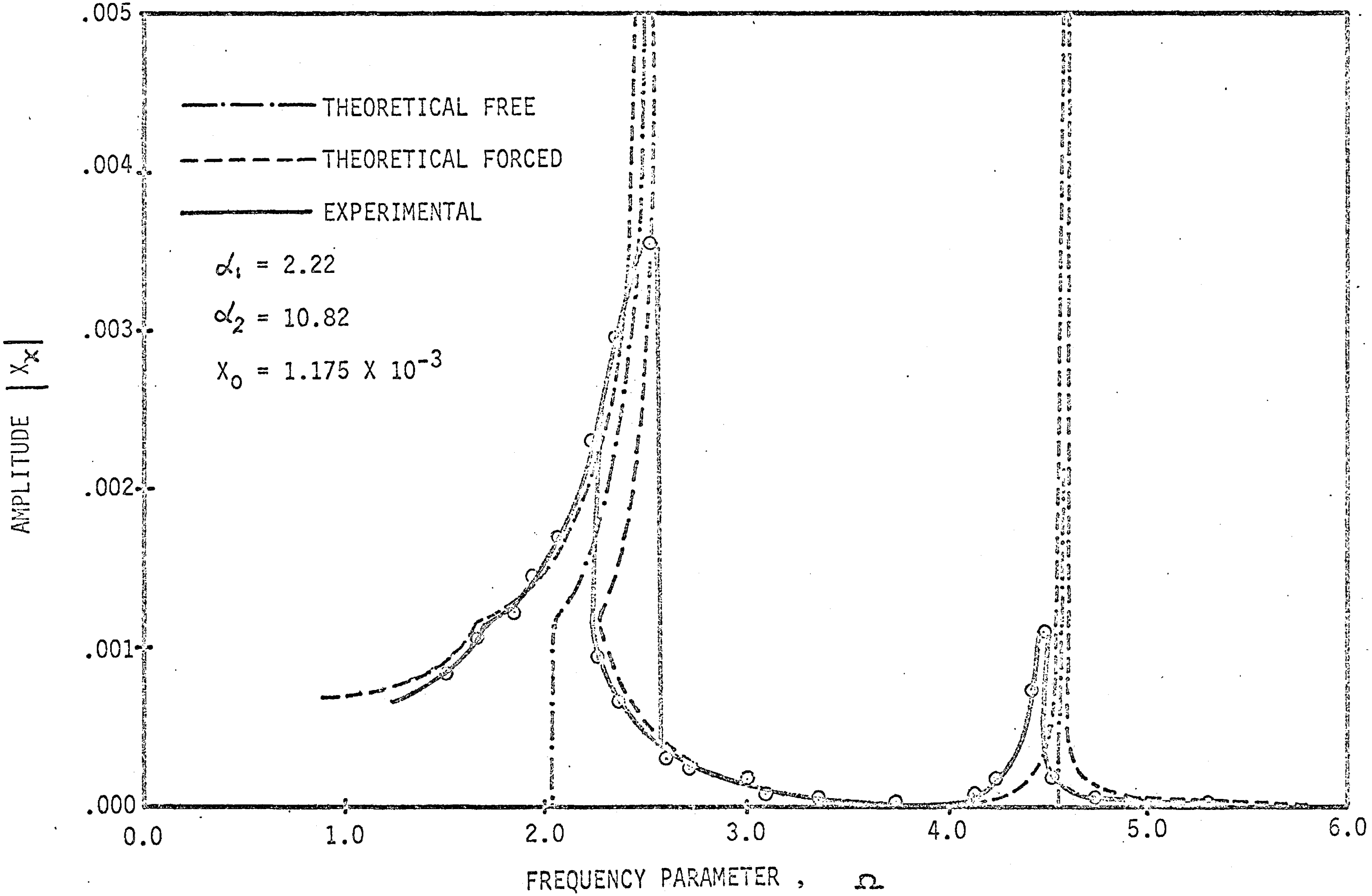


FIGURE 4.14 : RESPONSE WITH BILINEAR SPRING AT x_ℓ = 1.000

P = 3.425 X 10^{-3}

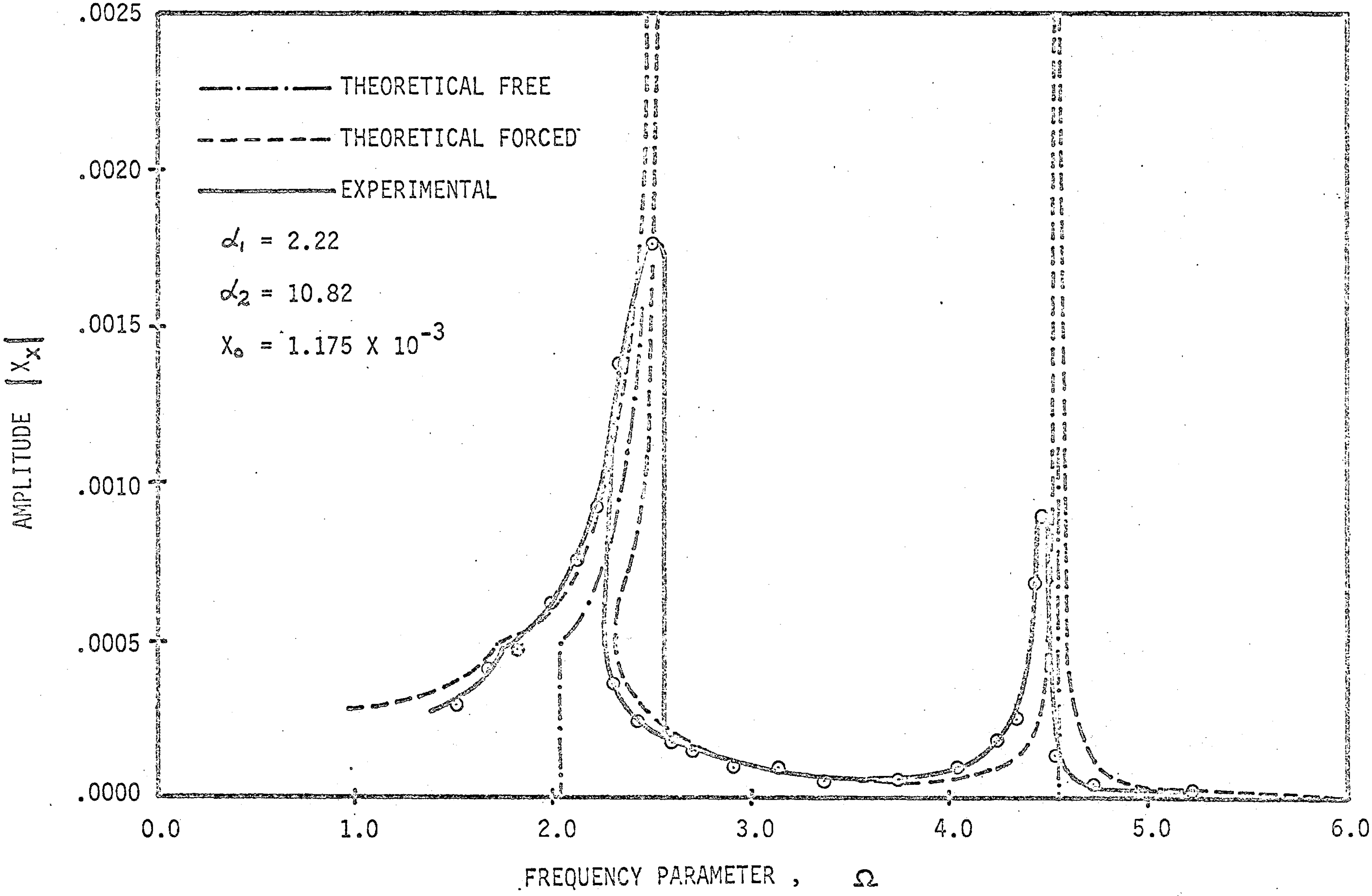


FIGURE 4.15 : RESPONSE WITH BILINEAR SPRING AT $x_\ell = 0.555$

$P = 3.425 \times 10^{-3}$

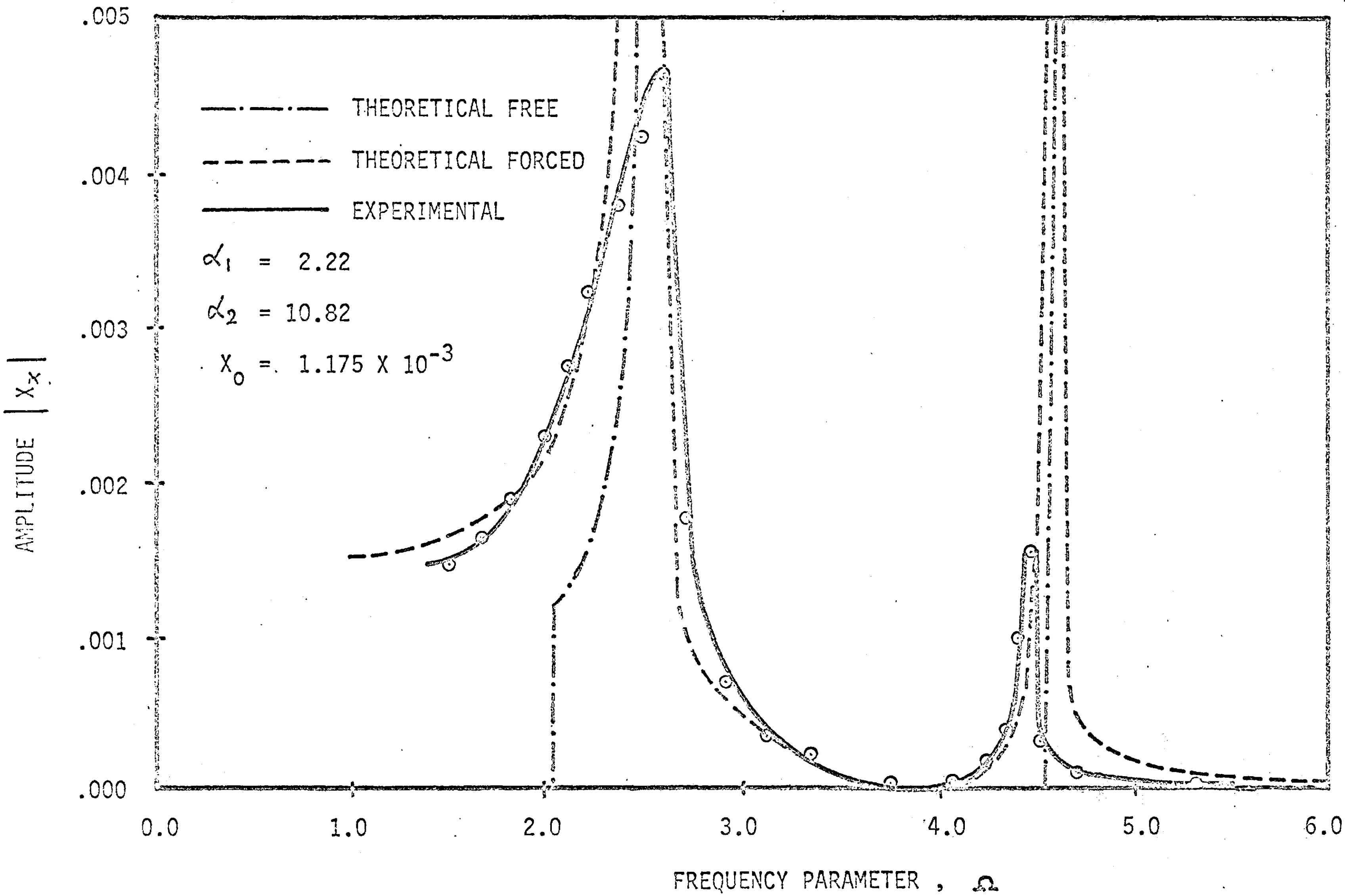


FIGURE 4.16 : RESPONSE WITH BILINEAR SPRING AT $x_l = 1.000$

$P = 10.275 \times 10^{-3}$

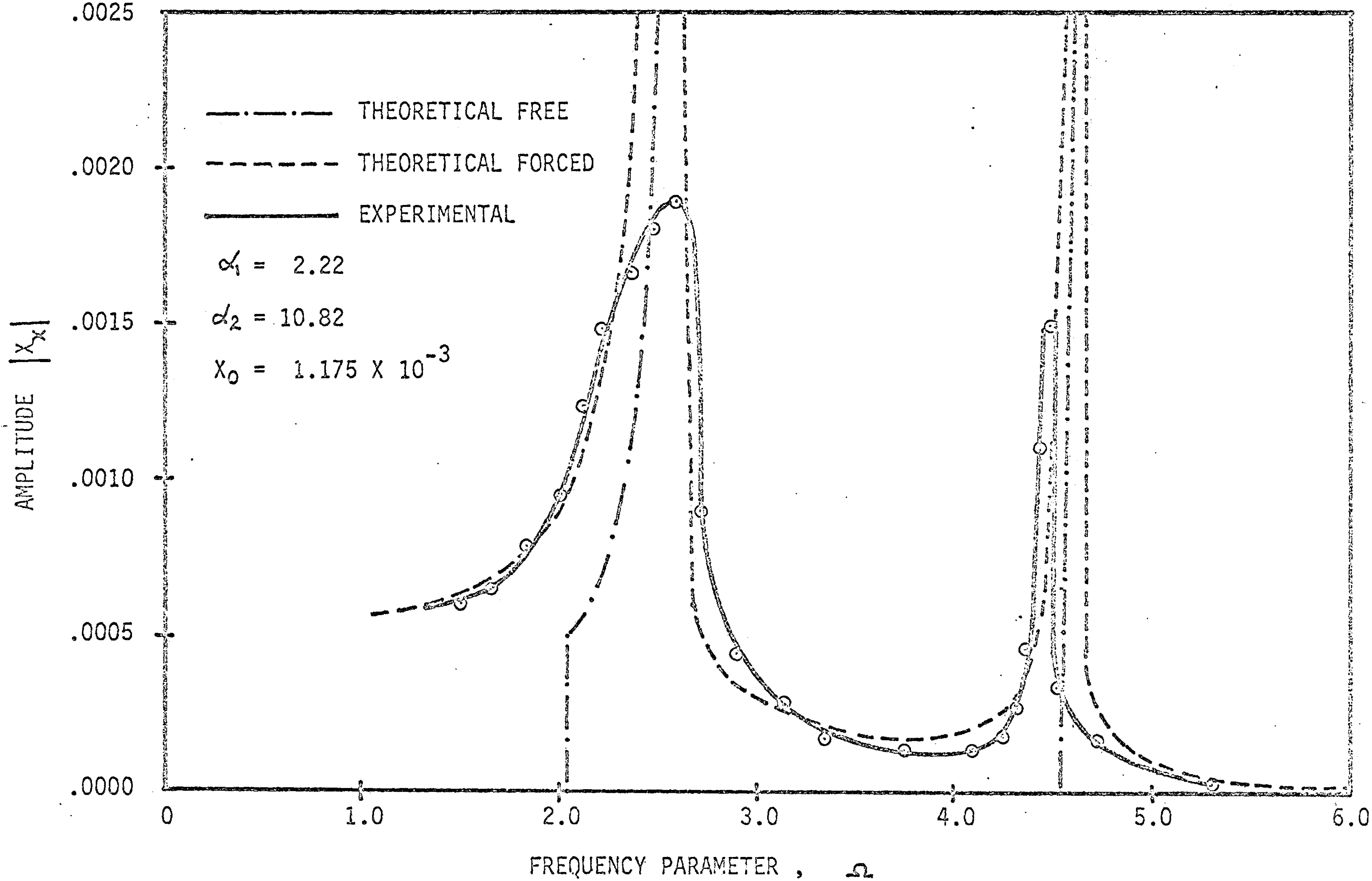


FIGURE 4.17 : RESPONSE WITH BILINEAR SPRING AT $x_\xi = 0.555$

$P = 10.275 \times 10^{-3}$

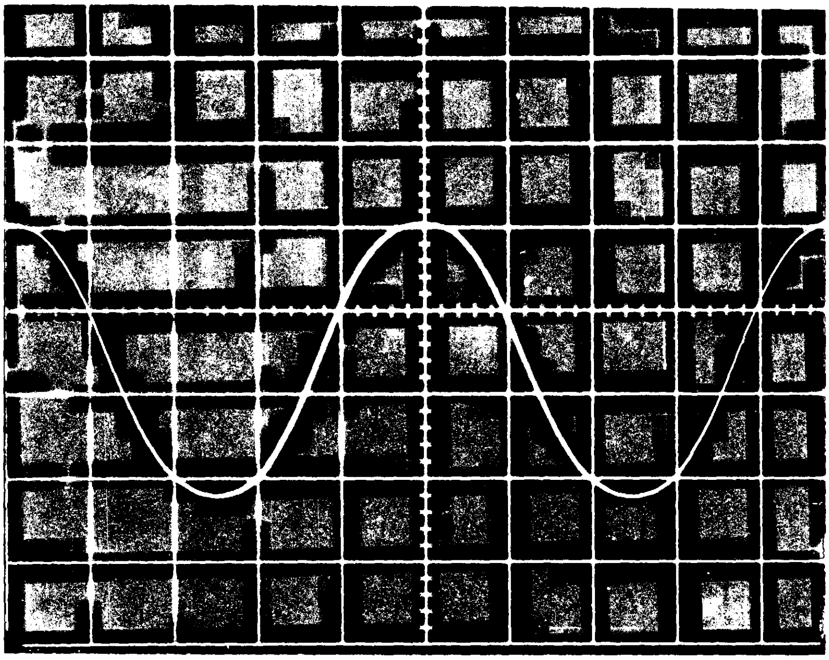


FIGURE 4.18 : TIME MOTION CURVE OBSERVED EXPERIMENTALLY WITH BILINEAR SPRING AT LOW AMPLITUDE , FOR $P = 3.425 \times 10^{-3}$

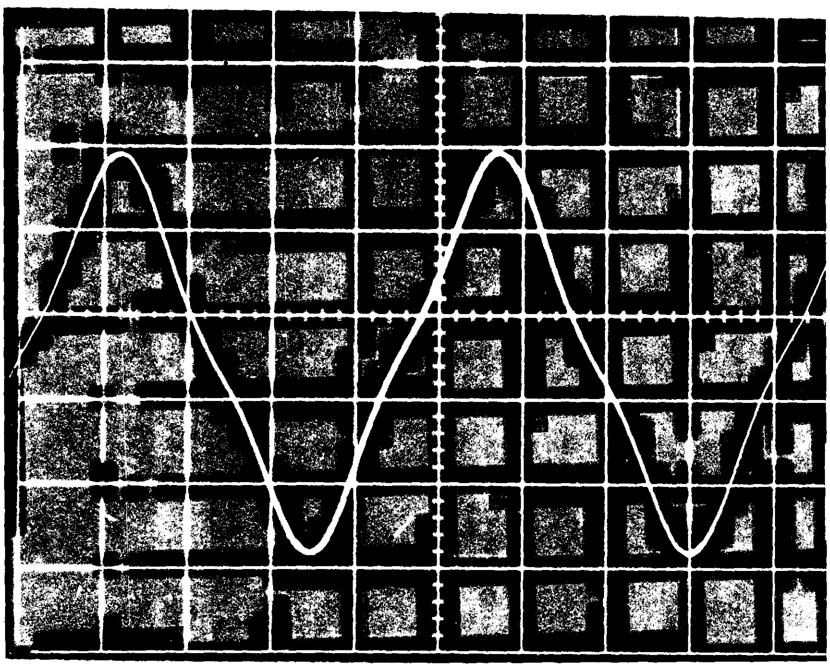
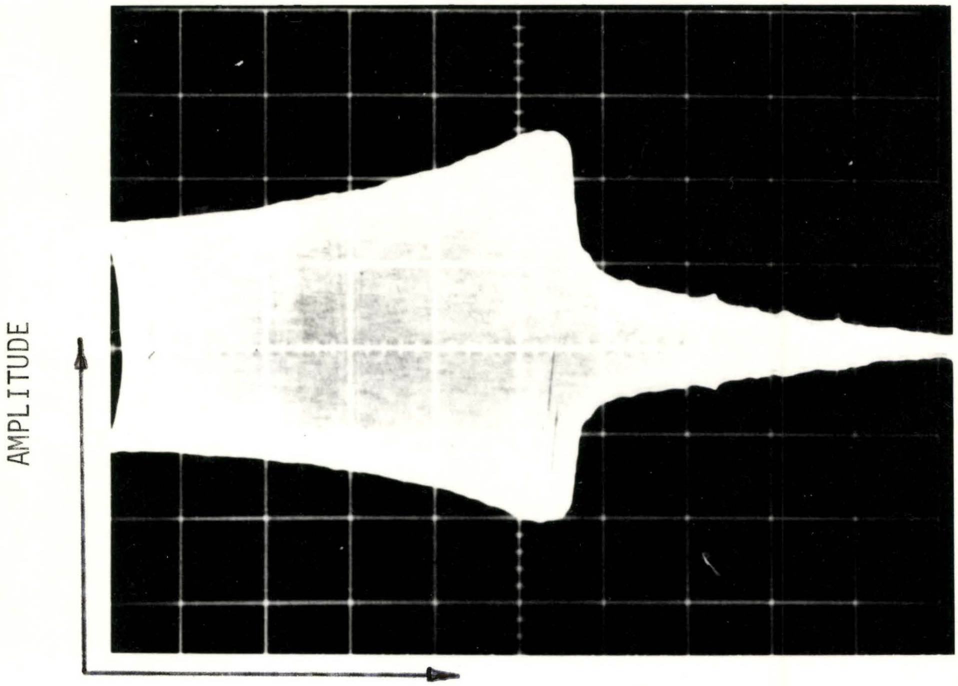
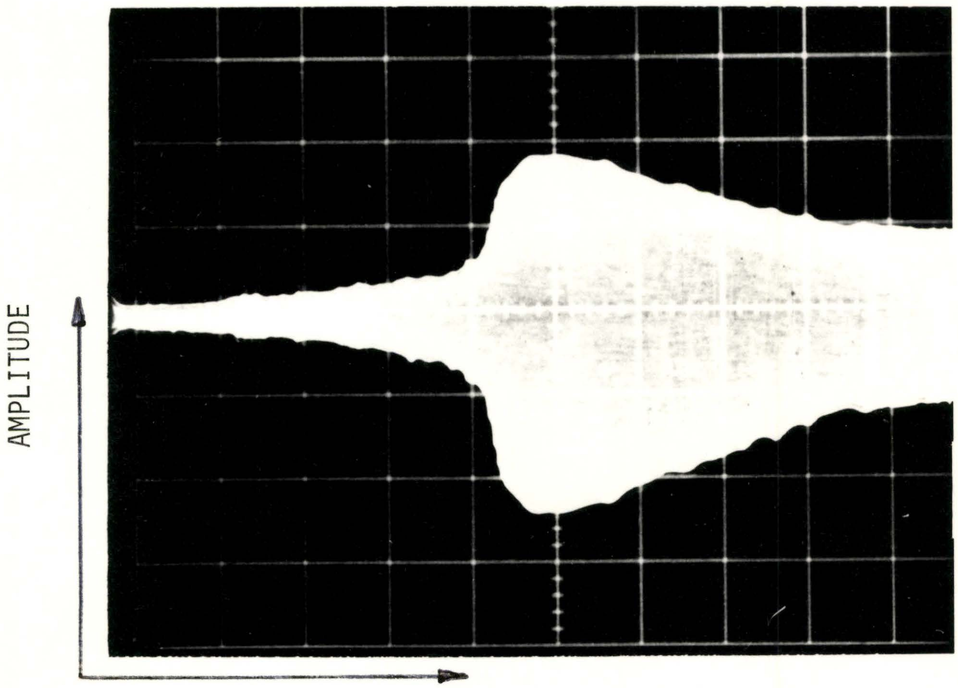


FIGURE 4.19 : TIME MOTION CURVE OBSERVED EXPERIMENTALLY WITH BILINEAR SPRING AT HIGHER AMPLITUDE, FOR $P = 3.425 \times 10^{-3}$



FREQUENCY INCREASING
(a)



FREQUENCY DECREASING
(b)

FIGURE 4.20 : QUALITATIVE RESPONSE OBTAINED EXPERIMENTALLY AT THE END OF THE BEAM WITH BILINEAR SPRING FOR $P = 3.425 \times 10^{-3}$

TABLE 4.1 THEORETICAL VALUES OF X_x FOR THE CASE OF BILINEAR SPRING

$\alpha_1 = 2.22$ $\alpha_2 = 10.82$ $X_0 = 1.175E-03$

FOR P = 3.425E-03

Ω \ x_ℓ	1.000	0.777	0.555	0.444	0.333
1.06	7.029E-04	4.740E-04	2.671E-04	1.791E-04	1.053E-04
1.29	7.720E-04	5.228E-04	2.960E-04	1.990E-04	1.173E-04
1.49	8.964E-04	6.106E-04	3.482E-04	2.348E-04	1.389E-04
1.67	1.134E-03	7.782E-04	4.477E-04	3.032E-04	1.801E-04
1.83	1.273E-03	8.820E-04	5.129E-04	3.493E-04	2.085E-04
1.98	1.433E-03	1.004E-03	5.918E-04	4.055E-04	2.434E-04
2.11	1.682E-03	1.195E-03	7.147E-04	4.932E-04	2.980E-04
2.24	2.106E-03	1.520E-03	9.248E-04	6.432E-04	3.915E-04
2.30	2.448E-03	1.783E-03	1.095E-03	7.648E-04	4.673E-04
	-1.114E-03	-8.118E-04	-4.984E-04	-3.481E-04	-2.127E-04
	-1.209E-03	-8.805E-04	-5.406E-04	-3.776E-04	-2.307E-04
2.36	2.964E-03	2.180E-03	1.351E-03	9.480E-04	5.815E-04
	-8.489E-04	-6.243E-04	-3.871E-04	-2.715E-04	-1.666E-04
	-1.463E-03	-1.076E-03	-6.672E-04	-4.681E-04	-2.871E-04
2.42	3.824E-03	2.841E-03	1.780E-03	1.254E-03	7.725E-04
	-6.767E-04	-5.027E-04	-3.149E-04	-2.220E-04	-1.367E-04
	-1.888E-03	-1.402E-03	-8.786E-04	-6.192E-04	-3.814E-04
2.48	5.538E-03	4.159E-03	2.634E-03	1.865E-03	1.154E-03
	-5.563E-04	-4.178E-04	-2.646E-04	-1.874E-04	-1.159E-04
	-2.734E-03	-2.053E-03	-1.300E-03	-9.209E-04	-5.696E-04
2.53	1.061E-02	8.059E-03	5.162E-03	3.674E-03	2.282E-03
	-4.676E-04	-3.553E-04	-2.276E-04	-1.620E-04	-1.006E-04
	-5.237E-03	-3.979E-03	-2.549E-03	-1.814E-03	-1.127E-03
2.59	4.257E-01	3.275E-01	2.123E-01	1.519E-01	9.478E-02
	-3.997E-04	-3.075E-04	-1.994E-04	-1.426E-04	-8.899E-05
	-2.102E-01	-1.617E-01	-1.048E-01	-7.499E-02	-4.679E-02
2.69	-3.030E-04	-2.396E-04	-1.593E-04	-1.152E-04	-7.255E-05
2.89	-1.908E-04	-1.613E-04	-1.137E-04	-8.411E-05	-5.401E-05
3.13	-1.168E-04	-1.108E-04	-8.506E-05	-6.496E-05	-4.280E-05
3.34	-7.359E-05	-8.252E-05	-7.022E-05	-5.549E-05	-3.754E-05
3.74	-2.174E-05	-5.381E-05	-5.982E-05	-5.089E-05	-3.630E-05
3.92	-1.093E-06	-4.647E-05	-6.158E-05	-5.445E-05	-3.988E-05
4.09	2.251E-05	-4.229E-05	-6.968E-05	-6.411E-05	-4.817E-05
4.23	5.028E-05	-4.158E-05	-8.533E-05	-8.107E-05	-6.216E-05
4.29	7.191E-05	-4.271E-05	-9.994E-05	-9.652E-05	-7.475E-05
4.36	1.057E-04	-4.580E-05	-1.247E-04	-1.225E-04	-9.579E-05
4.42	1.695E-04	-5.342E-05	-1.741E-04	-1.737E-04	-1.372E-04
4.48	3.445E-04	-7.741E-05	-3.141E-04	-3.187E-04	-2.542E-04
4.54	1.420E-03	-2.170E-04	-1.166E-03	-1.204E-03	-9.694E-04

4.60	9.985E-03	-9.380E-04	-7.476E-03	-7.847E-03	-6.380E-03
	-4.727E-04	4.441E-05	3.539E-04	3.715E-04	3.021E-04
	-4.930E-03	4.631E-04	3.691E-03	3.874E-03	3.150E-03
4.67	-2.309E-04	1.032E-05	1.591E-04	1.699E-04	1.395E-04
4.72	-1.555E-04	4.114E-07	9.946E-05	1.080E-04	8.954E-05
4.78	-1.183E-04	-4.011E-06	7.073E-05	7.819E-05	6.541E-05
4.84	-9.597E-05	-6.348E-06	5.394E-05	6.068E-05	5.125E-05
5.01	-6.172E-05	-8.959E-06	2.968E-05	3.524E-05	3.061E-05
5.28	-3.830E-05	-9.382E-06	1.521E-05	1.987E-05	1.809E-05
5.79	-1.993E-05	-8.364E-06	6.303E-06	1.032E-05	1.035E-05
6.25	-1.143E-05	-7.698E-06	3.246E-06	7.314E-06	8.156E-06

TABLE 4.2 THEORETICAL VALUES OF X_x FOR THE CASE OF BILINEAR SPRING

$\alpha_1 = 2.22$ $\alpha_2 = 10.82$ $X_0 = 1.175E-03$

FOR P = 10.275E-03

Ω \ x_ℓ	1.000	0.777	0.555	0.444	0.333
1.06	1.513E-03	1.020E-03	5.748E-04	3.853E-04	2.266E-04
1.29	1.563E-03	1.059E-03	5.995E-04	4.029E-04	2.375E-04
1.49	1.641E-03	1.118E-03	6.373E-04	4.298E-04	2.542E-04
1.67	1.754E-03	1.204E-03	6.925E-04	4.691E-04	2.786E-04
1.83	1.917E-03	1.328E-03	7.726E-04	5.261E-04	3.140E-04
1.98	2.158E-03	1.513E-03	8.914E-04	6.108E-04	3.667E-04
2.11	2.533E-03	1.800E-03	1.077E-03	7.429E-04	4.489E-04
2.24	3.172E-03	2.290E-03	1.393E-03	9.689E-04	5.897E-04
2.30	3.688E-03	2.686E-03	1.649E-03	1.152E-03	7.038E-04
2.36	4.464E-03	3.283E-03	2.036E-03	1.428E-03	8.760E-04
2.42	5.760E-03	4.279E-03	2.681E-03	1.889E-03	1.164E-03
2.48	8.342E-03	6.265E-03	3.968E-03	2.810E-03	1.738E-03
2.53	1.598E-02	1.214E-02	7.776E-03	5.534E-03	3.438E-03
2.59	6.413E-01	4.934E-01	3.198E-01	2.288E-01	1.428E-01
2.69	-9.089E-04	-7.187E-04	-4.780E-04	-3.456E-04	-2.176E-04
2.89	-5.724E-04	-4.840E-04	-3.410E-04	-2.523E-04	-1.620E-04
3.13	-3.505E-04	-3.323E-04	-2.552E-04	-1.949E-04	-1.284E-04
3.34	-2.208E-04	-2.476E-04	-2.107E-04	-1.665E-04	-1.126E-04
3.74	-6.523E-05	-1.614E-04	-1.795E-04	-1.527E-04	-1.089E-04
3.92	-3.279E-06	-1.394E-04	-1.847E-04	-1.634E-04	-1.196E-04
4.09	6.753E-05	-1.269E-04	-2.090E-04	-1.923E-04	-1.445E-04
4.23	1.508E-04	-1.248E-04	-2.560E-04	-2.432E-04	-1.865E-04
4.29	2.157E-04	-1.281E-04	-2.998E-04	-2.896E-04	-2.242E-04
4.36	3.172E-04	-1.374E-04	-3.742E-04	-3.674E-04	-2.874E-04
4.42	5.084E-04	-1.603E-04	-5.222E-04	-5.212E-04	-4.117E-04
4.48	1.033E-03	-2.322E-04	-9.422E-04	-9.562E-04	-7.626E-04
4.54	2.139E-03	-3.268E-04	-1.757E-03	-1.813E-03	-1.460E-03
4.60	1.504E-02	-1.413E-03	-1.126E-02	-1.182E-02	-9.610E-03
4.67	-6.926E-04	3.096E-05	4.773E-04	5.097E-04	4.184E-04
4.72	-4.665E-04	1.234E-06	2.984E-04	3.241E-04	2.686E-04
4.78	-3.550E-04	-1.203E-05	2.122E-04	2.346E-04	1.962E-04
4.84	-2.879E-04	-1.904E-05	1.618E-04	1.820E-04	1.537E-04
5.01	-1.852E-04	-2.688E-05	8.903E-05	1.057E-04	9.183E-05
5.28	-1.149E-04	-2.815E-05	4.563E-05	5.962E-05	5.428E-05
5.79	-5.980E-05	-2.509E-05	1.891E-05	3.096E-05	3.105E-05
6.25	-3.428E-05	-2.309E-05	9.738E-06	2.194E-05	2.447E-05

TABLE 4.3 : EXPERIMENTAL VALUES OF $|X_x| \times 10^3$ FOR THE CASE OF BILINEAR SPRING

$\alpha_1 = 2.22$ $\alpha_2 = 10.82$ $X_0 = 1.175 \times 10^{-3}$

FOR $P = 3.425 \times 10^{-3}$

Ω \ x_l	1.000	0.777	0.555	0.444	0.333
1.49	0.850	0.522	0.276	0.222	0.132
1.67	1.040	0.710	0.400	0.276	0.178
1.83	1.240	0.890	0.490	0.355	0.178
1.98	1.510	1.060	0.620	0.444	0.222
2.11	1.740	1.240	0.755	0.490	0.312
2.24	2.290	1.550	0.930	0.740	0.400
2.36	3.020	2.220	1.380	0.934	0.622
	0.710	0.520	0.310	0.276	0.132
2.48	3.550	2.640	1.780	1.240	0.756
	0.530	0.440	0.220	0.178	0.089
2.59	0.380	0.310	0.178	0.132	0.089
2.69	0.244	0.178	0.155	0.111	0.066
2.89	0.190	0.133	0.111	0.089	0.066
3.13	0.095	0.089	0.089	0.066	0.044
3.34	0.047	0.066	0.066	0.044	0.044
3.74	0.000	0.044	0.066	0.044	0.044
4.09	0.047	0.044	0.089	0.066	0.044
4.23	0.190	0.066	0.178	0.089	0.066
4.36	0.285	0.132	0.247	0.245	0.178
4.42	0.755	0.198	0.710	0.710	0.444
4.48	1.050	0.198	0.930	0.935	0.666
4.54	0.190	0.044	0.089	0.111	0.089
4.60	0.133	0.022	0.044	0.044	0.022
4.72	0.047	0.022	0.044	0.022	0.022
5.28	0.022	0.000	0.022	0.022	0.022

TABLE 4.4 : EXPERIMENTAL VALUES OF $|X_X| \times 10^3$ FOR THE CASE OF BILINEAR SPRING

$\alpha_1 = 2.22$ $\alpha_2 = 10.82$ $X_0 = 1.175 \times 10^{-3}$

FOR $P = 10.275 \times 10^{-3}$

Ω \ x_ℓ	1.000	0.777	0.555	0.444	0.333
1.49	1.470	0.970	0.620	0.400	0.222
1.67	1.660	1.150	0.665	0.444	0.274
1.83	1.890	1.280	0.754	0.532	0.310
1.98	2.220	1.550	0.930	0.620	0.355
2.11	2.750	1.860	1.110	0.754	0.444
2.24	3.330	2.430	1.460	1.020	0.532
2.36	3.780	2.870	1.660	1.100	0.666
2.48	4.230	3.100	1.780	1.440	0.775
2.59	4.650	3.330	1.880	1.550	0.775
2.69	1.780	1.420	0.930	0.660	0.444
2.89	0.690	0.565	0.440	0.310	0.222
3.13	0.333	0.310	0.272	0.222	0.133
3.34	0.222	0.222	0.178	0.155	0.089
3.74	0.044	0.133	0.133	0.111	0.066
4.09	0.000	0.111	0.133	0.133	0.066
4.23	0.089	0.089	0.155	0.133	0.089
4.36	0.333	0.178	0.444	0.333	0.222
4.42	0.890	0.222	1.110	0.666	0.333
4.48	1.550	0.666	1.500	0.800	0.666
4.54	0.312	0.111	0.310	0.310	0.222
4.60	0.222	0.089	0.222	0.222	0.178
4.72	0.133	0.111	0.178	0.178	0.133
5.28	0.044	0.022	0.044	0.066	0.044

TABLE 4.5 THEORETICAL VALUES OF X_x FOR FREE VIBRATION FOR THE CASE OF BILINEAR SPRING

$\alpha_1 = 2.22$ $\alpha_2 = 10.82$ $X_0 = 1.175E-03$

x_ℓ / Ω	1.000	0.777	0.555	0.444	0.333
2.08	1.201E-03	8.507E-04	5.068E-04	3.491E-04	2.106E-04
2.10	1.234E-03	8.754E-04	5.227E-04	3.604E-04	2.177E-04
2.12	1.269E-03	9.024E-04	5.401E-04	3.729E-04	2.254E-04
2.14	1.307E-03	9.319E-04	5.592E-04	3.865E-04	2.339E-04
2.16	1.349E-03	9.643E-04	5.801E-04	4.014E-04	2.432E-04
2.18	1.396E-03	1.000E-03	6.032E-04	4.179E-04	2.534E-04
2.20	1.448E-03	1.040E-03	6.287E-04	4.361E-04	2.648E-04
2.22	1.505E-03	1.084E-03	6.571E-04	4.564E-04	2.774E-04
2.24	1.569E-03	1.133E-03	6.889E-04	4.791E-04	2.916E-04
2.26	1.641E-03	1.188E-03	7.247E-04	5.047E-04	3.075E-04
2.28	1.723E-03	1.251E-03	7.653E-04	5.336E-04	3.256E-04
2.30	1.816E-03	1.322E-03	8.116E-04	5.667E-04	3.462E-04
2.32	1.923E-03	1.405E-03	8.649E-04	6.048E-04	3.700E-04
2.34	2.048E-03	1.501E-03	9.270E-04	6.492E-04	3.976E-04
2.36	2.195E-03	1.614E-03	1.000E-03	7.014E-04	4.302E-04
2.38	2.370E-03	1.748E-03	1.087E-03	7.638E-04	4.691E-04
2.40	2.583E-03	1.912E-03	1.193E-03	8.395E-04	5.163E-04
2.42	2.846E-03	2.114E-03	1.324E-03	9.334E-04	5.749E-04
2.44	3.181E-03	2.372E-03	1.491E-03	1.053E-03	6.492E-04
2.46	3.620E-03	2.709E-03	1.710E-03	1.209E-03	7.468E-04
2.48	4.220E-03	3.171E-03	2.009E-03	1.423E-03	8.803E-04
2.50	5.090E-03	3.840E-03	2.443E-03	1.733E-03	1.074E-03
2.52	6.464E-03	4.897E-03	3.128E-03	2.224E-03	1.380E-03
2.54	8.959E-03	6.818E-03	4.374E-03	3.114E-03	1.936E-03
2.56	1.489E-02	1.138E-02	7.331E-03	5.230E-03	3.256E-03
2.58	4.688E-02	3.600E-02	2.330E-02	1.665E-02	1.038E-02
4.56	1.380E-03	-1.881E-04	-1.105E-03	-1.146E-03	-9.251E-04
4.57	1.694E-03	-2.141E-04	-1.335E-03	-1.388E-03	-1.122E-03
4.58	2.183E-03	-2.549E-04	-1.695E-03	-1.767E-03	-1.431E-03
4.59	3.055E-03	-3.283E-04	-2.338E-03	-2.444E-03	-1.982E-03
4.60	5.054E-03	-4.971E-04	-3.811E-03	-3.995E-03	-3.246E-03
4.61	1.434E-02	-1.284E-03	-1.066E-02	-1.121E-02	-9.120E-03

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APPENDIX I

NATURAL FREQUENCIES OF A CANTILEVER BEAM SUPPORTED ON A LINEAR SPRING AT THE FREE END

APPENDIX I

NATURAL FREQUENCIES OF A CANTILEVER BEAM SUPPORTED ON A LINEAR SPRING AT THE FREE END

The natural frequencies of a cantilever beam supported on a linear spring at the free end can be obtained by solving the equation:

$$\alpha + \Omega^3 \cdot \frac{1 + \cosh\Omega \cos\Omega}{\cosh\Omega \sin\Omega - \sinh\Omega \cos\Omega} = 0$$

Figures 5.1(a) and 5.1(b) show the variation in the natural frequencies with the spring stiffness α. It can be observed that all the natural frequencies increase as the spring stiffness increases. However, the lower natural frequencies are affected significantly at lower values of spring stiffness, whereas the higher natural frequencies are affected for larger values of stiffness. This may be attributed to the fact that lower mode shapes change even with the addition of a weak spring, but the higher mode shapes remain almost unchanged until the spring is strong enough.

Further, it can be noticed from figures 5.1(a) and 5.1(b) that as the stiffness increases from zero to infinity, all the natural frequencies become asymptotic to the nearest corresponding natural frequency of clamped-simply supported beam. In the figures the natural frequencies of clamped-simply supported beam are indicated by broken lines.

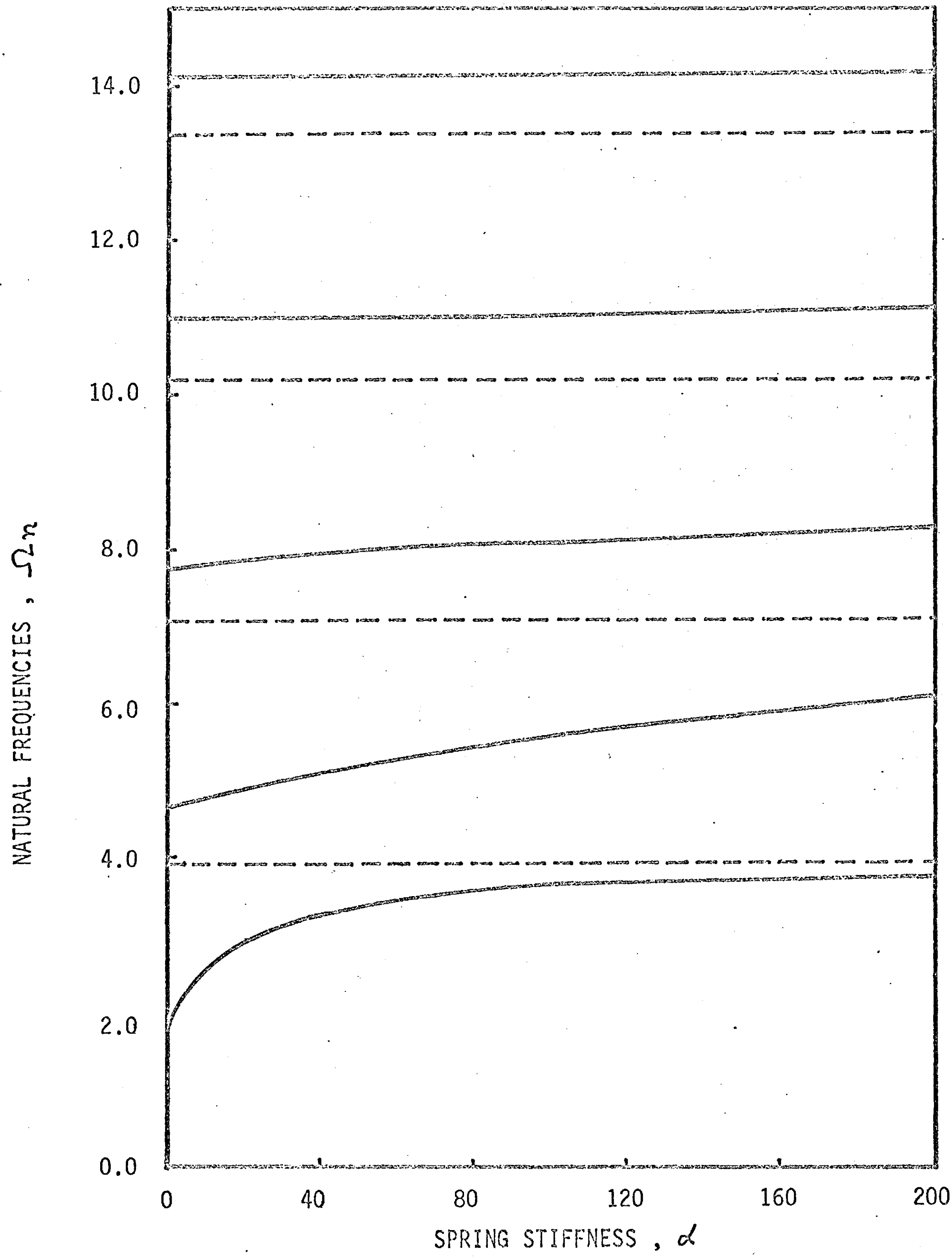

FIGURE 5.1(a) : VARIATION OF NATURAL FREQUENCIES WITH THE STIFFNESS OF LINEAR SPRING SUPPORT

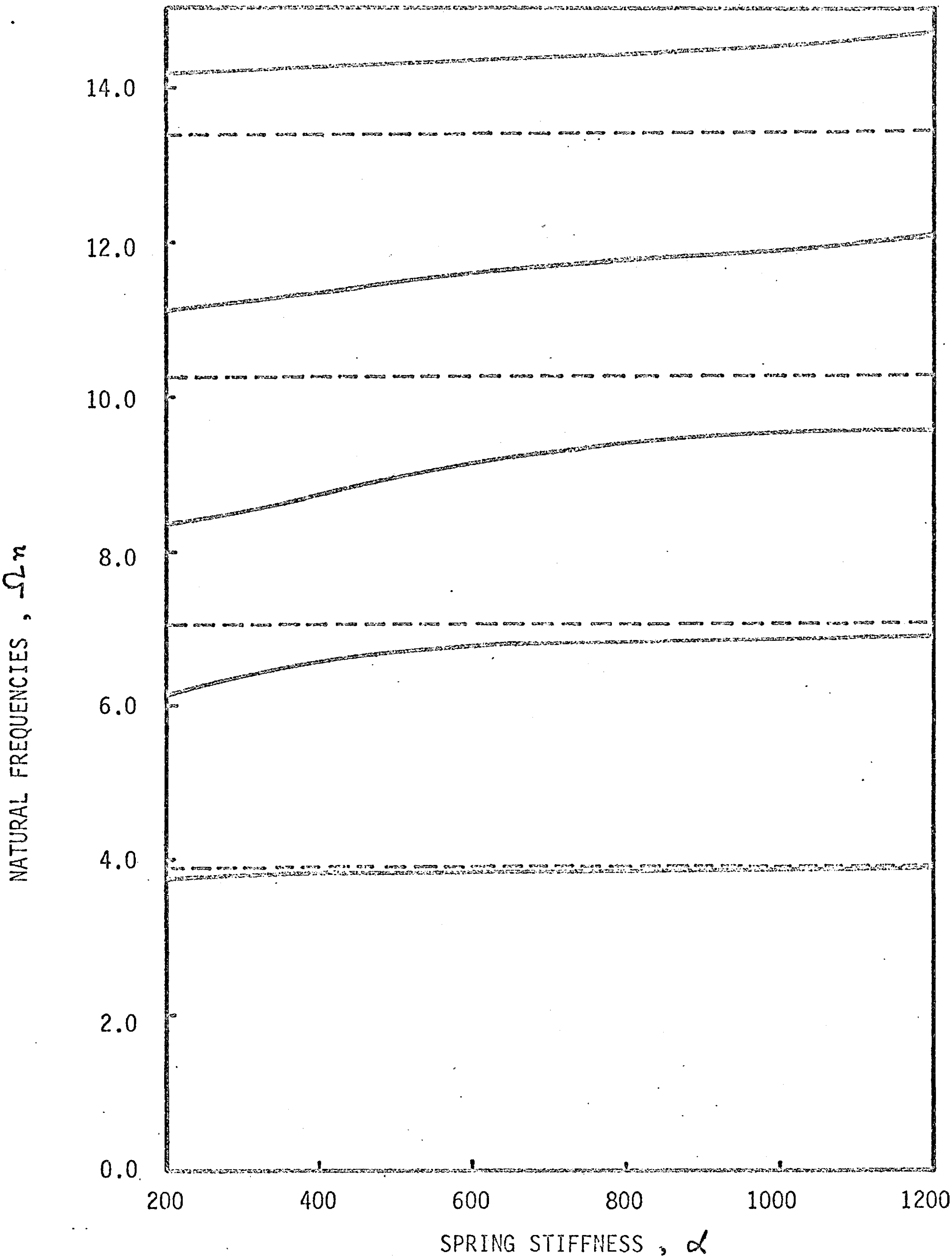

FIGURE 5.1(b) : VARIATION OF NATURAL FREQUENCIES WITH THE STIFFNESS OF LINEAR SPRING SUPPORT

APPENDIX II

COMPUTER PROGRAMMES

```
A4203,T100,LC10000.
RUN(S,,,,,,10000)
REDUCE.
LGO.
'          6400 END OF RECORD
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C       THIS PROGRAM IS TO OBTAIN THE FREE VIBRATION CURVES WITH
C       CUBIC NONLINEAR SPRING, VARYING THE VALUE OF BETA
C
C       AL=ALPHA
C       BT=BETA
C
        COSH(W)=(EXP(W)+EXP(-W))/2.0
        SINH(W)=(EXP(W)-EXP(-W))/2.0
        AL=0.0
  7     BT=10.0
  4     WRITE(6,5) AL,BT
  5     FORMAT(/10X,7HALPHA =,F8.1,10X,6HBETA =,F9.1/)
        W=1.5
  1     A=COSH(W)+COS(W)
        B=SINH(W)+SIN(W)
        C=COSH(W)-COS(W)
        D=SINH(W)-SIN(W)
        AA=3.0*BT/4.0
        AX=-(AL-(W**3)*(B*D-A*A)/(B*C-A*D))/AA
        IF(AX.LE.0.0) GO TO 3
        XA=SQRT(AX)
        CH=XA/(C-A*D/B)
        WRITE(6,2) W,CH,XA
  2     FORMAT(10X,5E15.5)
  3     W=W+0.1
        IF(W.LT.16.0) GO TO 1
        BT=BT*10.0
        IF(BT.LE.1.0001E5) GO TO 4
        AL=AL+2.0
        IF(AL.GT.2.0) AL=AL+6.0
        IF(AL.LE.10.0) GO TO 7
        STOP
        END
'          6400 END OF RECORD
'          6400 END FILE
```

```
A4203.
RUN(S,,,,,10000)
REDUCE.
LGO.
0          6400 END OF RECORD
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C       THIS PROGRAM IS TO DETERMINE THE RESPONSE OF SIMPLE
C       CANTILEVER BEAM (ALPHA= 0, BETA= 0)
C
        COSH(W)=(EXP(W)+EXP(-W))/2.0
        SINH(W)=(EXP(W)-EXP(-W))/2.0
        P=0.01
  1     WRITE(6,2) P
  2     FORMAT(/10X,3HP =,F8.3/)
        W=0.1
  3     A=COSH(W)+COS(W)
        B=SINH(W)+SIN(W)
        C=COSH(W)-COS(W)
        D=SINH(W)-SIN(W)
        XA=-P*(B*C-A*D)/((W**3)*(B*D-A*A))
        CH=XA/(C-A*D/B)
        WRITE(6,4) W,CH,XA
  4     FORMAT(10X,F5.2,2E15.3)
        W=W+0.1
        IF(W.LE.10.0) GO TO 3
        P=P*10.0
        IF(P.LE.1.00) GO TO 1
        STOP
        END
0          6400 END OF RECORD
0          6400 END FILE
```

```
A4203,T200.
RUN(S,,,,,,10000)
REDUCE.
LGO.
'        6400 END OF RECORD
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C       THIS PROGRAM IS TO DETERMINE THE HARMONIC RESPONSE IN
C       CASE OF CUBIC NONLINEAR SPRING, WITH ALPHA= 0,AND
C       VARYING BETA, FOR P= 0.01,0.10,1.00
C
C       C1,C2,C3 REPRESENT AMPLITUDE COEFFICIENTS, AND XA,XI
C       AND XJ REPRESENT THE END DEFLECTIONS
C
        COSH(W)=(EXP(W)+EXP(-W))/2.0
        SINH(W)=(EXP(W)-EXP(-W))/2.0
        AL=0.0
        P=0.01
  18    BT=10.0
  17    WRITE(6,1) AL,BT,P
   1    FORMAT(/10X,7HALPHA =,F6.3,5X,6HBETA =,F8.1,5X,3HP =,
     1 F7.3/)
        W=0.1
   2    A=COSH(W)+COS(W)
        B=SINH(W)+SIN(W)
        C=COSH(W)-COS(W)
        D=SINH(W)-SIN(W)
        AAA=3.0*BT/4.0
        AI=(AL-(W**3)*(B*D-A*A)/(B*C-A*D))/AAA
        AJ=-P/AAA
        XA=-AJ/AI
        N=1
   5    FX=XA**3+AI*XA+AJ
        DFX=3.0*XA*XA+AI
        DXA=-FX/DFX
        XA=XA+DXA
        N=N+1
        IF(N.GT.500) GO TO 11
        IF(ABS(DXA).GE.ABS(XA)*1.0E-6) GO TO 5
        T=C-A*D/B
        C1=XA/T
        E=-XA/2.0
        F=E*E-AI-XA*XA
        IF(F.GE.0.0) GO TO 10
        WRITE(6,9) W,C1,XA
   9    FORMAT(10X,3E15.3)
        GO TO 11
  10    XI=E+SQRT(F)
        XJ=E-SQRT(F)
        C2=XI/T
        C3=XJ/T
        WRITE(6,15) W,C1,C2,C3,XA,XI,XJ
  15    FORMAT(10X,7E15.3)
```

```
 11     W=W+0.1
        IF(W.LT.10.0) GO TO 2
        BT=BT*10.0
        IF(BT.LE. 1.0001E5) GO TO 17
        P=P*10.0
        IF(P.LE. 1.0001) GO TO 18
 14     STOP
        END
          6400 END OF RECORD
          6400 END FILE
```

```
A4203,LC10000.
RUN(S,,,,,,10000)
REDUCE.
LGO.
'          6400 END OF RECORD
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C       THIS PROGRAM IS TO DETERMINE THE RESPONSE, TAKING FIRST
C       AND THIRD ORDER HARMONIC COMPONENTS, FOR BEAM WITH CUBIC
C       NON-LINEAR SPRING
C
C       X = AMP. OF END DEFLECTION CORRESPONDING TO FIRST ORDER
C           HARMONIC COMPONENT
C       Y = AMP. OF END DEFLECTION CORRESPONDING TO THIRD ORDER
C           HARMONIC COMPONENT
C
       DIMENSION X1(3),YY(3),ZI(27),ZJ(27)
       COSH(W)=(EXP(W)+EXP(-W))/2.0
       SINH(W)=(EXP(W)-EXP(-W))/2.0
       P=0.1
       AL=0.0
       BT=10000.0
       W=0.1
       YY(1)=0.0
       YY(2)=0.0
       YY(3)=0.0
   1   W3=W*SQRT(3.0)
       A=COSH(W)+COS(W)
       B=SINH(W)+SIN(W)
       C=COSH(W)-COS(W)
       D=SINH(W)-SIN(W)
       A3=COSH(W3)+COS(W3)
       B3=SINH(W3)+SIN(W3)
       C3=COSH(W3)-COS(W3)
       D3=SINH(W3)-SIN(W3)
       DI=C-A*D/B
       DJ=C3-A3*D3/B3
       P1=-1.0
       Q1=4.0*(AL-(W**3)*(B*D-A*A)/(B*C-A*D))/(3.0*BT)
       R1=2.0
       S1=-4.0*P/(3.0*BT)
       Q2=4.0*(AL-(W3**3)*(B3*D3-A3*A3)/(B3*C3-A3*D3))/(3.0*BT)
       R2=2.0
       T2=-1.0/3.0
       CALL APPROX(AL,BT,P,W,A,B,C,D,X1,M)
       L=0
       DO 4 I=1,M
       Y=YY(I)
       DO 4 J=1,2
       X=X1(I)
       N=0
   2   F=X**3+P1*Y*X*X+(Q1+R1*Y*Y)*X+S1
       G=Y**3+(Q2+R2*X*X)*Y+T2*X**3
```

```
      FX=3.0*X*X+2.0*P1*X*Y+Q1+R1*Y*Y
      FY=P1*X*X+2.0*R1*X*Y
      GX=2.0*R2*X*Y+3.0*T2*X*X
      GY=3.0*Y*Y+Q2+R2*X*X
      DX=(F*GY-G*FY)/(GX*FY-FX*GY)
      DY=(F*GX-G*FX)/(FX*GY-GX*FY)
      X=X+DX
      Y=Y+DY
      N=N+1
      IF(N.GT.1000.OR.X.EQ.0.0.OR.Y.EQ.0.0) GO TO 3
      IF(ABS(DX/X).GT.1.0E-5.OR.ABS(DY/Y).GT.1.0E-5) GO TO 2
      IF(J.EQ.1) YY(I)=Y
      L=L+1
      ZI(L)=X
      ZJ(L)=Y
      QQ=-(3.0*Y*Y/4. +Q2+R2*X*X)
      IF(QQ.LT.0.0) GO TO 3
      YI=-Y/2.0+SQRT(QQ)
      YJ=-Y/2.0-SQRT(QQ)
      K=1
      Y=YI
    9 X=0.0
   10 F=X**3+P1*Y*X*X+(Q1+R1*Y*Y)*X+S1
      G=Y**3+(Q2+R2*X*X)*Y+T2*X**3
      FX=3.0*X*X+2.0*P1*X*Y+Q1+R1*Y*Y
      FY=P1*X*X+2.0*R1*X*Y
      GX=2.0*R2*X*Y+3.0*T2*X*X
      GY=3.0*Y*Y+Q2+R2*X*X
      DX=(F*GY-G*FY)/(GX*FY-FX*GY)
      DY=(F*GX-G*FX)/(FX*GY-GX*FY)
      X=X+DX
      Y=Y+DY
      N=N+1
      IF(N.GT.1000.OR.X.EQ.0.0.OR.Y.EQ.0.0) GO TO 12
      IF(ABS(DX/X).GT.1.0E-5.OR.ABS(DY/Y).GT.1.0E-5) GO TO 10
      L=L+1
      ZI(L)=X
      ZJ(L)=Y
   12 Y=YJ
      K=K+1
      IF(K.LE.2) GO TO 9
    3 Y=0.0
    4 CONTINUE
      IF(L.LT.1) GO TO 6
      DO 11 IJ=1,L
      IF(IJ.EQ.1) GO TO 14
      IK=IJ-1
      DO 13 II=1,IK
      IF(ABS(1.0-ZI(II)/ZI(IJ)).LE.1.0E-5.AND.ABS(1.0-ZJ(II)/
     1 ZJ(IJ)).LE.1.0E-5) GO TO 11
   13 CONTINUE
   14 X=ZI(IJ)
      Y=ZJ(IJ)
```

```
      C1=X/DI
      C3=Y/DJ
      WRITE(6,5) W,C1,C3,X,Y
    5 FORMAT(/10X,F9.3,6E14.3)
   11 CONTINUE
      GO TO 8
    6 WRITE(6,7)
    7 FORMAT(10X,17HSHECK CONVERGENCE)
    8 W=W+0.1
      IF(W.GT.1.3.AND.W.LE.1.49) W=W-0.09
      IF(W.LE.10.0) GO TO 1
      STOP
      END
      SUBROUTINE APPROX(AL,BT,P,W,A,B,C,D,X1,M)
C
C     SUBROUTINE APPROX IS TO DETERMINE THE STARTING VALUE OF
C     HARMONIC COMPONENT OF RESPONSE REQUIRED IN THE MAIN
C     PROGRAM
C
      DIMENSION X1(3)
      AAA=3.0*BT/4.0
      AI=(AL-(W**3)*(B*D-A*A)/(B*C-A*D))/AAA
      AJ=-P/AAA
      XA=-AJ/AI
      N=1
    5 FX=XA**3+AI*XA+AJ
      DFX=3.0*XA*XA+AI
      DXA=-FX/DFX
      XA=XA+DXA
      N=N+1
      IF(N.GT.500) GO TO 11
      IF(ABS(DXA).GE.ABS(XA)*1.0E-6) GO TO 5
      M=1
      X1(1)=XA
      E=-XA/2.0
      F=E*E-AI-XA*XA
      IF(F.LT.0.0) GO TO 11
      M=3
      X1(2)=E+SQRT(F)
      X1(3)=E-SQRT(F)
   11 RETURN
      END
'       6400 END OF RECORD
'       6400 END FILE
```

```
A4203.
RUN(S,,,,,,10000)
REDUCE.
LGO.
'         6400 END OF RECORD
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C       THIS PROGRAM IS TO FIND THE SUB-HARMONIC RESPONSE WITH
C       CUBIC NON-LINEAR SPRING
C
C       C1 IS AMP. COEFF. AND XX IS END DEF. CORRESPONDING TO
C       HARMONIC COMPONENT. C31 AND C32 ARE AMP. COEFF. AND X3
C       AND X32 ARE END DEF. CORRESPONDING TO SUBHARMONIC
C       COMPONENT
C
        COSH(W)=(EXP(W)+EXP(-W))/2.0
        SINH(W)=(EXP(W)-EXP(-W))/2.0
        DIMENSION XX(6),YY(6),U(7)
        AL=0.0
        P=0.1
        BT=10000.0
  5     BT=10000.
  7     W=3.2
        WRITE(6,9) AL,BT,P
  9     FORMAT(/9X,7HALPHA =,F4.1,5X,6HBETA =,F9.1,5X,3HP =,
     1  F5.2/)
  1     W3=W/SQRT(3.0)
        A=COSH(W)+COS(W)
        B=SINH(W)+SIN(W)
        C=COSH(W)-COS(W)
        D=SINH(W)-SIN(W)
        A3=COSH(W3)+COS(W3)
        B3=SINH(W3)+SIN(W3)
        C3=COSH(W3)-COS(W3)
        D3=SINH(W3)-SIN(W3)
        AI=-1.0
        BI=2.0
        CI=4.0*AL/(3.*BT) -4.*(W**3)*(B3*D3-A3*A3)/((B3*C3-A3*
     1  D3)*9.0*SQRT(3.0)*BT)
        DI=4.*AL/(3.*BT) -4.*(W**3)*(B*D-A*A)/((B*C-A*D)*3.*BT)
        EI=2.0
        FI=-4.*P/(3.*BT)
        GI=-1./3.
        HI=AI*AI-4.*BI
        PI=1.0+2.*EI*AI*AI/4.0 -EI*BI -GI*(AI**3)/8. -3.0*AI*GI
     1  *HI/8.0
        QI=DI-EI*CI +3.*AI*GI*CI/2.0
        SI=EI*AI-GI*HI/4.0 -3.*AI*AI*GI/4.0
```

```
      U(1)=PI*PI-SI*SI*HI/4.0
      U(2)=0.0
      U(3)=2.*PI*QI +SI*SI*CI-SI*GI*CI*HI/2.
      U(4)=2.*PI*FI
      U(5)=QI*QI+2.*SI*GI*CI*CI-GI*GI*CI*CI*HI/4.0
      U(6)=2.*QI*FI
      U(7)=FI*FI +GI*GI*(CI**3)
      N=6
      DO 2 I=1,N
      CALL BAIRST(U,XX,YY,N)
      C1=XX(I)/(C-A*D/B)
      WRITE(6,3)W,C1,XX(I),YY(I)
 3    FORMAT(10X,4E13.3)
      IF(YY(I).NE.0.0) GO TO 2
      PP=-AI*XX(I)/2.0
      QQ=(PP*PP-(XX(I)**2) *BI-CI)
      IF(QQ.LT.0.0) GO TO 2
      Q=SQRT(QQ)
      X3=PP+Q
      X32=PP-Q
      C31=X3/(C3-A3*D3/B3)
      C32=X32/(C3-A3*D3/B3)
      WRITE(6,4) W ,C31,C32,X3,X32
 4    FORMAT(55X,5E12.3)
 2    CONTINUE
      W=W+0.1
      IF(W.LE.7.0) GO TO 1
      BT=BT*10.0
      IF(BT.LE.1.001E05) GO TO 7
      P=P*10.0
      IF(P.LE.1.0) GO TO 5
34    STOP
      END
        6400 END OF RECORD
        6400 END FILE
```

```
A4203,T100,LC10000.
RUN(S,,,,,,10000)
LGO.
'          6400 END OF RECORD
       PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C       THIS PROGRAM IS TO DETERMINE THE FREE VIBRATIONS FOR
C       THE SYSTEM WITH BILINEAR SPRING, TAKING END MASS = 0
C
C       W  = OMEGA
C       AI = ALPHA ONE
C       AJ = ALPHA TWO
C
        COSH(W)=(EXP(W)+EXP(-W))/2.0
        SINH(W)=(EXP(W)-EXP(-W))/2.0
        XO=0.001
        AI=0.0
  1     AJ=10.0
 21     WRITE(6,2) AI,AJ
  2     FORMAT(/10X,11HALPHA ONE =,F5.1,5X,11HALPHA TWO =,F9.1/)
        W=1.5
  3     A=COSH(W)+COS(W)
        B=SINH(W)+SIN(W)
        C=COSH(W)-COS(W)
        D=SINH(W)-SIN(W)
        S=(W**3)*(B*D-A*A)/(B*C-A*D)
        IF(S.LT.AI.OR.S.GT.AJ) GO TO 5
        XA=XO*(AJ-AI)/(AJ-S)
        CH=XA*B/(B*C-A*D)
        WRITE(6,4) W,CH,XA
  4     FORMAT(10X,F6.2,2E14.3)
  5     W=W+0.05
        IF(W.LE.10.0) GO TO 3
        AJ=AJ*10.0
        IF(AJ.LE.1000.0) GO TO 21
        AI=AI+2.0
        IF(AI.LE.2.0) GO TO 1
  6     STOP
        END
'          6400 END OF RECORD
'          6400 END FILE
```

```
A4203.
RUN(S)
REDUCE.
LGO.
'          6400 END OF RECORD
      PROGRAM TST (INPUT,PUNCH,TAPE5=INPUT,TAPE7=PUNCH)
C
C     THIS PROGRAM IS TO DETERMINE THE FREE VIBRATIONS FOR
C     THE SYSTEM WITH BILINEAR SPRING, CONSIDERING END
C     MASS
C
C     AMU= END MASS/BEAM MASS
C
      DIMENSION VV(7)
      COSH(W)=(EXP(W)+EXP(-W))/2.0
      SINH(W)=(EXP(W)-EXP(-W))/2.0
      XO=0.001175
      AMU=0.0424
      AI=2.22
      AJ=10.82
      W=1.5
    1 A=COSH(W)+COS(W)
      B=SINH(W)+SIN(W)
      C=COSH(W)-COS(W)
      D=SINH(W)-SIN(W)
      S=(W**3)*(B*D-A*A)/(B*C-A*D) +(W**4)*AMU
      IF(S.LT.AI.OR.S.GT.AJ) GO TO 5
      XA=XO*(AJ-AI)/(AJ-S)
      CH=XA*B/(B*C-A*D)
      XL=15.0
      DO 2 I=1,5
      L=5-I+1
      WX=W*XL/45.0
      VV(L)=CH*(COSH(WX)-COS(WX)-A*(SINH(WX)-SIN(WX))/B)
      XL=XL+5.0
      IF(I.GE.3) XL=XL+5.0
    2 CONTINUE
      WRITE(7,4) W,(VV(I),I=1,5)
    4 FORMAT(10X,F6.2,5E11.3)
    5 W=W+0.02
      IF(W.GT.4.0) W=W-0.01
      IF(W.LE.6.0) GO TO 1
    6 STOP
      END
'          6400 END OF RECORD
'          6400 END FILE
```

```
A4203.
RUN(S)
REDUCE.
LGO.
'         6400 END OF RECORD
        PROGRAM TST (INPUT,PUNCH,TAPE5=INPUT,TAPE7=PUNCH)
C
C       PROGRAM TO FIND THE RESPONSE WITH BILINEAR SPRING.
C       THEORETICAL ANALYSIS
C
C       C1,C2,C3 ARE THE AMPLITUDE COEFFICIENTS AND XA1,XA2,XA3
C       ARE THE END DEFLECTIONS IN THE UNSTABLE RANGE
C
        DIMENSION VV(6),UV(6),UU(6)
        COSH(W)=(EXP(W)+EXP(-W))/2.0
        SINH(W)=(EXP(W)-EXP(-W))/2.0
        XO=0.001175
        AMU=0.0424
        AI=2.22
        AJ=10.82
        P=0.003425
        K=1
 25     S1=AI-P/XO
        S2=AI+P/XO
        S3=AJ
        DO 24 J=1,36
        READ(5,28) FREQ
 28     FORMAT(F10.0)
        W=SQRT(6.284*FREQ)*0.298
        A=COSH(W)+COS(W)
        B=SINH(W)+SIN(W)
        C=COSH(W)-COS(W)
        D=SINH(W)-SIN(W)
        C1=B*P/(AI*(B*C-A*D)-(W**3)*(B*D-A*A)-(W**4)*AMU*
      1 (B*C-A*D))
        XA=C1*(C-A*D/B)
        S=(W**3)*(B*D-A*A)/(B*C-A*D)+(W**4)*AMU
        IF(S3.GE.S.AND.S.GE.S2) GO TO 200
        IF(S.GT.S3.OR.S.LE.S1) GO TO 100
        XA=(XO*(AI-AJ)-P)/(S-AJ)
        C1=XA*B/(B*C-A*D)
100     XL=15.0
        DO 2 I=1,5
        L=5-I+1
        WX=W*XL/45.0
        VV(L)=C1*(COSH(WX)-COS(WX)-A*(SINH(WX)-SIN(WX))/B)
        XL=XL+5.0
        IF(I.GE.3) XL=XL+5.0
  2     CONTINUE
```

```
      WRITE(7,4)W,(VV(I),I=1,5)
    4 FORMAT(10X,F6.2,5E11.3)
      GO TO 24
  200 XA1=(XO*(AI-AJ)-P)/(S-AJ)
      XA2=-P/(S-AI)
      XA3=(XO*(AJ-AI)-P)/(S-AJ)
      T=C-A*D/B
      C1=XA1/T
      C2=XA2/T
      C3=XA3/T
      XL=15.0
      DO 3 I=1,5
      L=5-I+1
      WX=W*XL/45.0
      VV(L)=C1*(COSH(WX)-COS(WX)-A*(SINH(WX)-SIN(WX))/B)
      UV(L)=C2*(COSH(WX)-COS(WX)-A*(SINH(WX)-SIN(WX))/B)
      UU(L)=C3*(COSH(WX)-COS(WX)-A*(SINH(WX)-SIN(WX))/B)
      XL=XL+5.0
      IF(I.GE.3) XL=XL+5.0
    3 CONTINUE
      WRITE(7,5) W,(VV(I),I=1,5),(UV(I),I=1,5),(UU(I),I=1,5)
    5 FORMAT(10X,F6.2,5E11.3,2(/16X,5E11.3))
   24 CONTINUE
   22 P=0.010275
      K=K+1
      IF(K.LE.2) GO TO 25
      STOP
      END
        6400 END OF RECORD
```

```
A4203.
RUN(S)
REDUCE.
LGO.
'          6400 END OF RECORD
      PROGRAM TST (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C
C       NATURAL FREQUENCIES OF CANTILEVER BEAM WITH LINEAR SPRING.
C       VARYING THE STIFFNESS,ALPHA
C
C       AL = ALPHA
C       W  = OMEGA
C
        COSH(W)=(EXP(W)+EXP(-W))/2.0
        SINH(W)=(EXP(W)-EXP(-W))/2.0
        F(W) =(W**3)*(1.0+COSH(W)*COS(W))+ AL*(COSH(W)*SIN(W)
      1 -SINH(W)*COS(W))
C
C       NATURAL FREQUENCIES ARE THE ROOTS OF THE EQUATION
C       F(W) = 0
C
        DF(W)=(W**3)*(SINH(W)*COS(W)-COSH(W)*SIN(W)) +3.0*W*W*
      1 (1.0+COSH(W)*COS(W)) +2.0*AL*SINH(W)*SIN(W)
        WM=15.0
        H=0.2
        AL=0.0
 11     WRITE(6,12) AL
 12     FORMAT(//10X,4HAL =,F8.1)
        W=0.1
 10     FI=F(W)
  2     W=W+H
        IF(W.GT.WM) GO TO 13
        FJ=F(W)
        IF(FI*FJ.LE.0.0) GO TO 3
        FI=FJ
        GO TO 2
  3     N=1
        W=W-H/2.0
4       DW=-FI/DF(W)
        W=W+DW
        FI=F(W)
        N=N+1
        IF(N.GT.100) GO TO 9
        IF(ABS(DW).GE.W*1.0E-5) GO TO 4
        WRITE(6,6) W
  6     FORMAT(10X,F12.3)
        W=W+H
```

```
      IF(W.LE.WM) GO TO 10
 13   AL=AL+10.0
      IF(AL.GT.200.0) AL=AL+90.0
      IF(AL.LE.2000.) GO TO 11
  9   STOP
      END
        6400 END OF RECORD
        6400 END FILE
```

APPENDIX III

PROPERTIES OF BEAM

APPENDIX III

PROPERTIES OF BEAM

Material	Mild Steel
Length	45"
Width	2"
Thickness	$^3/_8$"
Weight of the beam, M	9.45 lbs
End mass, m (shaker core + effective mass of the spring)	0.40 lb
Mass ratio, μ (= $^m/_M$)	0.0424
EI	29.61×10^4 lbf in^2
$(EI/\rho_a)^{\frac{1}{2}}$	2.28×10^4 in^2 sec^{-1}

APPENDIX IV

LIST OF EQUIPMENT

APPENDIX IV

LIST OF EQUIPMENT

1. Goodman's Vibration Shaker (Model V. 390/200)
2. Vibration Shaker Amplifier
3. R. C. Generator (Philips Z9.060.69)
4. Ammeter
5. Micrometer Proximity Transducer (DISA 51D11)
6. Displacement Transducer (DISA 51D05)
7. Oscillators (DISA 51E02)
8. Tuning Plug (DISA 51E03)
9. Reactance Converters (DISA 51E01)
10. Storage Oscilloscope (Tectronics 564)