QUASIANALYTIC ILYASHENKO ALGEBRAS

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Abstract

A recent result (see [12]) was the construction of a quasianalytic class containing all transition maps at hyperbolic singularities with logarithmic monomials in their series expansions. The end goal being obtaining o-minimality of this structure, we need an extension to several variables stable under certain operations (such as blow-up substitutions). As a first step towards the several variable extension, we construct a quasianalytic Hardy field extending the previous class where the monomials are now allowed to be any definable function in $\mathbb{R}_{\mathrm{an,exp}}$.

Contents

1 History and Motivation

"This is the question as to the maximum number and position of Poincaré's boundary cycles (cycles limites) for a differential equation of the first order and degree of the form $\frac{dy}{dx} = \frac{Y}{X}$ where X and Y are rational integral functions of the n-th degree in x and y." (Hilbert, [4]). The second part of Hilbert's 16th problem remains unsolved to this day and a number of questions arise from Hilbert's initial statement:

Question 1. *Is it true that a polynomial vector field on the real plane has a finite number of limit cycles?*

In 1923, Dulac claimed a solution of the first problem but it was found later that his proof contained a gap. It was only in the early 1990's that Écalle and Ilyashenko proved independently that planar polynomial vector fields have finitely many limit cycles. The idea of Dulac is that if a polynomial vector field has infinitely many limit cycles, then they must converge to a polycycle (a finite collection of singular points and orbits forming a Jordan curve); so the problem was reduced to proving the non-accumulation of limit cycles on polycycles.

Question 2. *Is it true that the number of limit cycles of a polynomial vector field of degree d on the real plane is bounded by a constant depending on d only?*

If the second problem has an affirmative answer, the bound is called the *Hilbert number* and is denoted $H(d)$. Only the case $d = 1$ is known and $H(1) = 0$, as linear vector fields have no limit cycles.

Question 3. *Find an upper bound for $H(d)$.*

The second question can be reduced to studying analytic families of real analytic vector fields (see [6] for details):

Question 4. *Is there a uniform bound on the number of limit cycles in analytic families of real analytic planar vector fields?*

A positive answer was given in [8] if we restrict ourselves to vector fields with compact parameter space having only isolated non-resonant hyperbolic singularities. In order to obtain locally uniform bounds on the number of limit cycles, the approach was using quasianalyticity and its connection to o-minimality:
Let ξ be a real analytic vector field in $\mathbb{R}^2$ and Γ a polycycle of ξ with hyperbolic non-resonant singular points $p_0, p_1, \ldots, p_k$ and trajectories $\gamma_0, \ldots, \gamma_k$ connecting the p_i's in the order following the flow. For each i, it is possible to choose transverse segments Λ_i^-, Λ_i^+ sufficiently close to p_i such that:

- there exists real analytic maps $h_i : \Lambda_i^+ \to \Lambda_{i+1}^-$ representing the flow of ξ from Λ_i^+ to Λ_{i+1}^-,
- each trajectory starting on Λ_i^- sufficiently close to p_i crosses Λ_i^+ near p_i. We can therefore define *correspondence maps* $\sigma_i : \Lambda_i^- \to \Lambda_i^+$.

The Poincaré map σ is then represented by the finite composition $\sigma = h_k \circ \sigma_k \circ \cdots \circ \sigma_1 \circ h_0 \circ \sigma_0$ where each $\sigma_i \circ e^{-x}$ is an *almost regular map*:

Definition 1.1 (see §0.3 in [6])**.** The germ of a map f is said to be *almost regular* if it has a holomorphic extension $\mathfrak{f}$ to some standard quadratic domain Ω (see 2.3) and can be expanded in an asymptotic *Dulac exponential series* in this domain, i.e., if:

$$\forall N \in \mathbb{N}, \left| \mathfrak{f}(z) - \sum_{j=0}^{N} P_j(z) e^{-\nu_j z} \right| = o(e^{-\nu_N z}) \text{ as } |z| \to +\infty \text{ in } \Omega$$

where $P_j \in \mathbb{R}[x]$ with $P_0 \in \mathbb{R}^{>0}$ and $0 < \nu_0 < \nu_1 < \ldots$ with $\lim\limits_{n \to +\infty} \nu_n = +\infty$. We denote by $\mathcal{I}$, the class of all almost regular maps.

In the case where the singularities are non-resonant, all the P_j's are actually real numbers and it is proven in [8] that the h_i's and the σ_i's (and hence σ) are definable in a same o-minimal structure which leads to uniform bounds on the number of

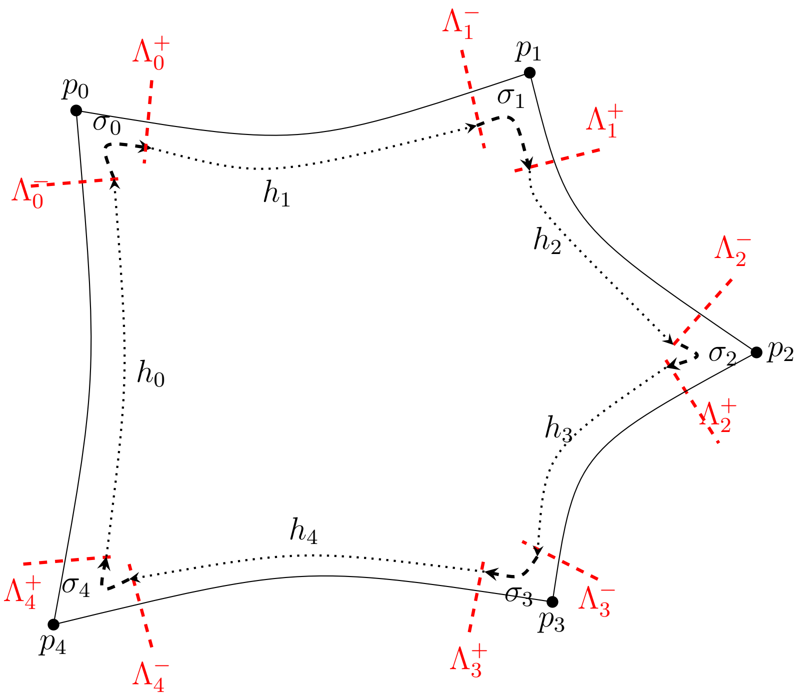

Figure 1: Example of a polycycle for ξ

limit cycles in this case.

This thesis is part of an ongoing project with P. Speissegger and T. Kaiser aiming at modifying the procedure presented above in order to settle the general hyperbolic case where one of the main difficulties is that the P_j's are not constant.

Objective. *Given an analytic family ξ of real analytic planar polynomial vector fields with hyperbolic singularities, we want to construct a multivariable quasi-analytic algebra (stable under the operations needed for o-minimality) containing all the corresponding transition maps and prove that the obtained stucture is o-minimal.*

A recent result (see [12]) was the construction of a quasianalytic field with logarithmic generalized power series as asymptotic expansions, containing $\mathcal{I}$. The monomials in the construction were in the set $\{e^{-\log_k} | k \in \mathbb{N}\}$ ($\log_k := \log \circ \cdots \circ \log$ (k times) and $\log_0 := \mathrm{id}$).

In order to obtain o-minimality of this class, we need an extension to several variables stable under certain operations (such as blow-up substitutions). We want, for example, stability under substitutions such as $x \mapsto xy$ which will map $\log_2 x$ to $\log(\log x + \log y)$ which is not an element of $(e^{-x}, e^{-y}, \frac{1}{x}, \frac{1}{y}, \dots)$. The idea is instead to first replace $(x, \log, \log_2, \dots)$ by any definable curve $(f_0, f_1, f_2, \dots)$ in $\mathbb{R}_{\text{an,exp}}$ and then extend to several variables. As a first step towards the several variable extension, we construct in this thesis a quasianalytic Hardy field extending the class obtained in [12] where the monomials are now allowed to be any definable function in $\mathbb{R}_{\text{an,exp}}$.

2 Setup and Definitions

2.1 Series

2.1.1 Generalized Series

Definition 2.1. A *set of monomials* is an ordered set $(\mathfrak{M}, \leq)$ of germs at $+\infty$ of real one-variable functions. A subset $S \subset \mathfrak{M}$ is said to be *Noetherian* or *anti-well ordered* if there is no strictly increasing infinite sequence of elements in S.

In our case, the monomials will be elements of $\mathcal{H}_{\text{an,exp}}$ which has a total linear order (further details are given in section 2.4).

Definition 2.2. Let K be a field of *coefficients* and $\mathfrak{M}$ be a group of monomials. For a function $f : \mathfrak{M} \to K$, we define its *support* to be the set $\text{supp}(f) := \{m \in \mathfrak{M} \mid f(m) \neq 0\}$. If $\text{supp}(f)$ is Noetherian, we call f a *generalized series*, and the set of all generalized series with coefficients in K and monomials in $\mathfrak{M}$ is denoted by $K((\mathfrak{M}))$. We also denote f by $\sum_{m \in \mathfrak{M}} f_m m$ where $f_m := f(m)$ is the *coefficient* of the *term* $m \in \text{supp}(f)$.

Remark. $K((\mathfrak{M}))$ is an abelian group where addition of two elements f and g is defined as follows:

$$f + g := \sum_{m \in \mathfrak{M}} (f_m + g_m) m$$

Definition 2.3. Given a non-zero element $f \in K((\mathfrak{M}))$, the maximal element in $\mathrm{supp}(f)$ is called the *leading monomial* of f and is denoted $lm(f)$.

Fact 2.4 (see [15]). *$K((\mathfrak{M}))$ is a field where multiplication of two elements f ang g is defined by:*

$$fg := \sum_{m \in \mathfrak{M}} \left(\sum_{pq=m} f_p g_q \right) m.$$

We assume for the rest of the section that $\mathfrak{M}$ is equipped with a total linear order.

Definition 2.5. A subset $S \subset \mathfrak{M}$ is said to be *natural* if for all $m \in \mathfrak{M}$, $S \cap [m, +\infty[$ is finite.

Remark. If a set is natural, then it is Noetherian. Naturality ensures that the support doesn't have accumulation points. For example, $S := \{\cdots \prec e^{\frac{1}{k}x} \prec \cdots \prec e^{\frac{1}{2}x} \prec e^x\}$ is Noetherian but not natural.

Lemma 2.6. *Let $S, S' \subset \mathfrak{M}$ be two natural subsets and assume that $\mathcal{M}$ is closed under multiplication. Then, the following holds:*

(1) *Every subset of S is natural with the induced ordering*

(2) *$S \cup S'$ is natural*

(3) *$S \cdot S'$ is natural*

(4) *If $S \prec 1$ and for all $k \in \mathbb{N}$, S^k is coinitial in $\mathcal{M}$, then $S^* := \bigcup_{k \in \mathbb{N}} S^k$ is natural.*

Proof. (1) Follows immediately from definitions.

(2) Follows from $(S \cup S') \cap [m, +\infty[\subseteq (S \cap [m, +\infty[) \cup (S' \cap [m, +\infty[)$.

(3) We first assume that $S, S' \leq 1$. Let $m \in \mathcal{M}$, then $S \cdot S' \cap [m, +\infty[= \{n_j = s_j s'_j\}_{j \in J}$ for some index set J. Since $S, S' \leq 1$, $n_j = s_j s'_j \leq s_j, s'_j \leq 1$ for all $j \in J$ so $\{s_j\}_{j \in J}$ and $\{s'_j\}_{j \in J}$ are both subsets of $[m, +\infty[$. By naturality of S and S', they are both finite so $\{n_j\}_{j \in J}$ must be finite as well. Now, for the general case, if S is natural, then for any $m \in \mathcal{M}$, $S \cap [m, +\infty[$ has a maximal element s_M. Then, for all $s \in S$, $\frac{s}{s_M} \leq 1$ and we can work with the set $\frac{S}{s_M}$ instead (if $S \cap [m, +\infty[= \emptyset$, we just take $s_M := m$).

(4) Let $m \in \mathcal{M}$, then $S \cap [m, +\infty[$ is finite, say it is equal to $\{s_0, \dots, s_j\}$ with $s_0 < \cdots < s_j$. Then, for all $k \in \mathbb{N}$, the maximal element of S^k is s_j^k. Since the powers of elements of S are coinitial in $\mathcal{M}$, there exists $k_0 \in \mathbb{N}$ such that $s_j^{k_0} < m$ which implies that $S^{k_0} \cap [m, +\infty[= \emptyset$. Now, $(s_j^k)_{k \in \mathbb{N}}$ is a strictly decreasing sequence so for all $k \geq k_0$, $S^k \cap [m, +\infty[= \emptyset$. Hence, $S^k \cap [m, +\infty[= \bigcup_{0<k<k_0} \left(S^k \cap [m, +\infty[\right)$, which is finite by naturality of each S^k.

$\square$

2.1.2 Generalized Power Series

Definition 2.7. Let R be a ring and $X = (X_1, \dots, X_k)$ be a k-tuple of indeterminates. We consider formal power series of the form:

$$f(X) = \sum_{\alpha} a_\alpha X^\alpha$$

where $\alpha = (\alpha_1, \dots, \alpha_k) \in (\mathbb{R}^{\geq 0})^k$ and $X^\alpha = X_1^{\alpha_1} \dots X_k^{\alpha_k}$. Given f as before, we define the *support of* f as $\operatorname{supp}(f) := \{\alpha |\, a_\alpha \neq 0\} \subset (\mathbb{R}^{\geq 0})^k$. We define the set of *generalized power series*, denoted by $R[[X^*]]$, to be the set of formal power series with Noetherian support.

Definition 2.8. A set $S \subset (\mathbb{R}^{\geq 0})^k$ is said to be *natural* if for every compact box $B \subset \mathbb{R}^k$, $A \cap B$ is finite.

2.2 Hardy Fields

The contents of this section are taken from [2].

2.2.1 Germs

Definition 2.9. We say that two functions $f, g : \mathbb{R} \to \mathbb{R}$ *ultimately agree* if there exists $a \in \mathbb{R}$ such that $f_{[a,+\infty[} = g_{[a,+\infty[}$. It is an equivalence relation and the equivalence class of a function f is called the *germ of* f *at* $+\infty$ and is denoted $germ(f)$.

Remark. The set of germs forms a ring with addition $germ(f) + germ(g) := germ(f+g)$ and multiplication $germ(f) \cdot germ(g) := germ(f \cdot g)$. If f is ultimately differentiable, we define $germ(f)' := germ(f')$.

Definition 2.10. A set of germs is called a *Hardy field* if it is a field closed under differentiation.

From now on, we will denote a function and its germ by the same symbol.

Facts 2.11 (see §1 in [2])**.** *Let K be a Hardy field, and $f \in K$, then:*

(1) *ultimately, either $f(x) = 0$, $f(x) > 0$ or $f(x) < 0$ so K can be made into an ordered differential field by defining $f > 0$ if $f(x)$ is ultimately positive,*

(2) *ultimately, either f is constant, strictly increasing or strictly decreasing.*

2.2.2 Dominance Relations

Definition 2.12. Let $(K, 0, 1, +, -)$ be a field. A *dominance relation* on K is a binary relation $\preceq$ on K such that for all $f, g, h \in K$:

(1) $0 \prec 1$

(2) $f \preceq f$

(3) $f \preceq g$ and $g \preceq h \Rightarrow f \preceq h$

(4) $f \preceq g$ or $g \preceq f$

(5) If $h \neq 0$, $(f \preceq g \Leftrightarrow fh \preceq gh)$

(6) $(f \preceq h$ and $g \preceq h) \Rightarrow f + g \preceq h$

If $f \preceq g$, we say that f is *dominated by* g. If $f \preceq g$ and $g \preceq f$, we say that f and g are *asymptotic* and write $f \asymp g$. If $f - g \prec f$, we say that f and g are *equivalent* and write $f \sim g$ (note that $f \sim g$ implies $f \asymp g$).
Given $f \in K$, we say that f is *bounded* if $f \preceq 1$, *infinitesimal* if $f \prec 1$ and *infinite* if $f \succ 1$.

Fact 2.13 (see §1 in [2]). *If $(K, \leq)$ is an ordered field, then $0 \leq f \leq g$ implies $f \preceq g$. If $(K, \leq, \delta)$ is a differential ordered field with constant field C, then $f \preceq g :\Leftrightarrow \exists c \in C^{>0}, |f| \leq c\,|g|$ is a dominance relation so every ordered differential field and in particular every Hardy field can be equipped with a dominance relation.*

Fact 2.14 (see §1 in [2]). *There is a bijective correspondence between dominance relations on K and valuations on K. In other words, if v is a valuation on K, then the following relation is a dominance relation:*

$$f \preceq g :\Leftrightarrow v(f) \geq v(g)$$

Conversely, if $\preceq$ is a dominance relation on K, then $K^{\preceq 1} := \{f \in K \mid f \preceq 1\}$ is a valuation ring of K with maximal ideal $K^{\prec 1} := \{f \in K \mid f \prec 1\}$.

In a Hardy field, the dominance relation in fact 2.13 can be interpreted in terms of limits at $+\infty$:

Facts 2.15 (see lemma 1.3 in [2]). *Let K be a Hardy field and $f, g \in K$, then the following holds:*

(1) $f \preceq g \Leftrightarrow \lim\limits_{x \to +\infty} \frac{f(x)}{g(x)} \in \mathbb{R}$

(2) $f \prec g \Leftrightarrow \lim\limits_{x \to +\infty} \frac{f(x)}{g(x)} = 0$

(3) $f \asymp g \Leftrightarrow \lim\limits_{x \to +\infty} \frac{f(x)}{g(x)} \in \mathbb{R}^*$

(4) $f \sim g \Leftrightarrow \lim\limits_{x \to +\infty} \frac{f(x)}{g(x)} = 1$

Some useful properties of Hardy fields:

Facts 2.16 (see proposition 1.4 in [2]). *Let K be a Hardy field and $f, g \in K^\times$, then the following holds:*

(1) $f, g \not\asymp 1 \Rightarrow (f \preceq g \Leftrightarrow f' \preceq g')$

(2) $f \preceq 1 \Leftrightarrow f' \prec 1$

2.2.3 Archimedean Classes and Comparability Classes

Definition 2.17. Let $(G, \leq)$ be an ordered abelian group and $a \in G$. The *archimedean class* of a is the set $\{g \in G \mid \exists n \geq 1, |a| \leq n\,|g| \text{ and } |g| \leq n\,|a|\}$.

Definition 2.18. Let K be an ordered differential field with field of constants C. For $f, g \in K$ with $f, g \succ 1$, we say that f is *comparable* to g if there exists $n \geq 1$ such that $|f| \leq |g|^n$ and $|g| \leq |f|^n$. Comparability is an equivalence relation and the equivalence class of f is called the *comparability class* of f and is denoted $[f]$.

Lemma 2.19. *Let K be as above and $f, g \in K$ be in the same archimedean class. Then, e^f and e^g are in the same comparability class.*

Proof. For simplicity of the notation, assume that $f, g > 0$. By definition, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} g \leq f \leq ng$, then $(e^g)^{\frac{1}{n}} = e^{\frac{1}{n}g} \leq e^f \leq e^{ng} = (e^g)^n$. $\square$

2.2.4 Quasianalytic Algebras

We assume in this section that G is a multiplicative subgroup of some Hardy field $\mathcal{H}$ of $\mathcal{C}^\infty$ germs at $+\infty$.

Definition 2.20. Let $F = \sum\limits_m c_m m$ be an element of $\mathbb{R}((G))$ where the valuation map is injective on G. For $q \in G$, we say denote by $F_g := \sum\limits_{m \geq g} c_m m$, the *truncation of F above g*. Note that since $\mathrm{supp}(F)$ is Noetherian, F_g is finite.

Definition 2.21. A tuple (K, G, T) is called a *quasianalytic asymptotic algebra* if:

(1) K is an $\mathbb{R}$-algebra of $\mathcal{C}^\infty$ germs at $+\infty$.

(2) G is a multiplicative subgroup of some Hardy field $\mathcal{H}$ of $\mathcal{C}^\infty$ germs at $+\infty$ and the valuation map is injective on G.

(3) T is an injective $\mathbb{R}$-algebra homomorphism from K to $\mathbb{R}((G))$ such that:

- $T(K)$ is truncation closed, i.e., for every $f \in K$ and $g \in G$, there exists $h \in K$ such that $T(h) = T(f)_g$,
- $\forall f \in K, \forall g \in G, |f(x) - T^{-1}(T(f)_g)(x)| = o(g(x))$ as $x \to +\infty$.

In our case, the valuation map on G will not be injective so we need to generalize the definition for truncation in a series. The idea is to work with equivalence classes of series and to group all monomials with the same valuation together when we do the truncation:

Definition 2.22. Let $F = \sum\limits_m c_m m$ be an element of $\mathbb{R}((G))$. For $q \in G$, we denote by $F_g := \sum\limits_{m \succeq g} c_m m$, the *truncation of F above g*. Note that F_g is finite for this case as well.

Definition 2.23. A tuple (K, G, O, T) is called a *generalized quasianalytic asymptotic algebra* if:

(1) and (2) hold as in definition 2.21 without the valuation independence assumption

(3) T is an injective $\mathbb{R}$-algebra homomorphism from K to $\mathbb{R}((G))/O$ where O is a prime ideal of $\mathbb{R}((G))$ and the following holds:

- $T(K)$ is truncation closed
- for all $f \in K, g \in G$ and F such that $T(f) = F+O$, $|f(x)-\Phi\left(F_g\right)(x)| = o(g(x))$ as $x \to +\infty$ where Φ is defined as follows:
  Let $\mathcal{R} := \{F \in \mathbb{R}((G)) \,|\, \exists f \in K, T(f) = F+O\}$. We define $\Phi : \mathcal{R} \to K$ to be the surjective map $F \mapsto f$ so that for all $f \in K$ with $T(f) = F+O$, $T(\Phi(F)) = T(f) = F+O$. Note that since $T(K)$ is truncation closed, for every $F \in \mathcal{R}$ and $g \in G$, $F_g \in \mathcal{R}$.

2.3 Standard Quadratic Domains and the Phragmén-Lindelöf Principle

As mentioned in the introduction, transition maps at hyperbolic singularities have holomorphic extension to standard quadratic domains and for the rest of the thesis, we will work with germs of functions having asymptotic expansions holding in standard quadratic domains.

Definition 2.24. A subset of $\mathbb{C}$ is called a *standard quadratic domain* if it is of the form $\Omega_C := \{z + C\sqrt{1+z} \mid Re(z) > 0\}$ for some $C > 0$.

Fact 2.25 (Phragmén-Lindelöf principle, Lemma 24.37 in [7])**.** *Let $\Omega \subset \mathbb{C}$ be a standard quadratic domain and $f : \overline{\Omega} \to \mathbb{C}$ be a holomorphic function. If f is bounded and for all $n \in \mathbb{N}$ and $z \in \Omega$, $f(z) = o(e^{-nz})$ as $|z| \to +\infty$ then f is the* 0 *function.*

Definition 2.26. Let $\mathcal{U} \subset \mathbb{C}$ be an open set. A function $\varphi : \mathcal{U} \to \mathbb{C}$ is said to be *conformal* if it is holomorphic and injective.

Fact 2.27 (see section 14.7 in [11])**.** *If $\mathcal{U}$ is a domain, then $\varphi : \mathcal{U} \to \mathbb{C}$ is conformal if and only if it is biholomorphic (i.e. both φ and φ^{-1} are holomorphic).*

Lemma 2.28. *For all $C > 0$, $\varphi_C : \mathbb{C}^{>0} \to \mathbb{C}$, with $\varphi_C(z) := z + C\sqrt{1+z}$ is conformal.*

Proof. The principal square root is holomorphic on $\mathbb{C} \setminus \mathbb{R}^{\leq 0}$. Hence, $z \mapsto z + C\sqrt{1+z}$ is holomorphic on $\mathbb{C}^{>0}$ so φ_C is holomorphic on $\mathbb{C}^{>0}$ as well. For injectivity, we prove that φ_C has a compositional inverse:

$$\begin{aligned}
&\omega = z + C\sqrt{1+z}\\
\Rightarrow &(\omega - z)^2 = C^2(1+z)\\
\Leftrightarrow &\omega^2 - 2\omega z + z^2 = C^2 + C^2 z\\
\Leftrightarrow &z^2 - 2\left(\omega + \frac{C^2}{2}\right) z = C^2 - \omega^2\\
\Leftrightarrow &z^2 - 2\left(\omega + \frac{C^2}{2}\right) z + \left(\omega + \frac{C^2}{2}\right)^2 = C^2 - \omega^2 + \left(\omega + \frac{C^2}{2}\right)^2\\
\Leftrightarrow &\left(z - \left(\omega + \frac{C^2}{2}\right)\right)^2 = C^2 - \omega^2 + \omega^2 + C^2\omega + \frac{C^4}{4}\\
\Leftrightarrow &\left(\omega + \frac{C^2}{2} - z\right)^2 = C^2\left(1 + \omega + \frac{C^2}{4}\right)\\
\Rightarrow &\omega + \frac{C^2}{2} - z = C\sqrt{1 + \omega + \frac{C^2}{4}}(*)\\
\Rightarrow &z = \omega + \frac{C^2}{2} - C\sqrt{1 + \omega + \frac{C^2}{4}}
\end{aligned}$$

$(*)$ follows from the fact that $\mathrm{Re}(z) > 0$ implies $\mathrm{Re}\left(\left(\omega + \frac{C^2}{2} - z\right)^2\right) > 0$ and $\mathrm{Re}(1+\omega+\frac{C^2}{4}) > 0$. Hence, φ_C^{-1} is the holomorphic map $z \mapsto z+\frac{C^2}{2}-C\sqrt{1+z+\frac{C^2}{4}}$ so φ_C is injective. □

Facts 2.29. *For all $\alpha, C, D > 0$, the following holds:*

(1)

$$\delta(\Omega_C) = \varphi_C(i\mathbb{R}) = \left\{ C\sqrt{\frac{\sqrt{1+r^2}+1}{2}} + i\left(r + C\,\mathrm{sgn}(r)\sqrt{\frac{\sqrt{1+r^2}-1}{2}}\right) \mid r \in \mathbb{R} \right\}$$

(2) *There exists a continuous function* $f_C : [C, +\infty[\to \mathbb{R}$ *such that* $\mathrm{Im}(\varphi_C(ir)) = f_C(\mathrm{Re}(\varphi_C(ir)))$ *for all* $r \in \mathbb{R}$ *and* $f_C(r) \sim 2\left(\frac{r}{C}\right)^2$. *In particular, for all* $C > 0$, $z \in \Omega_C$ *if and only if* $\mathrm{Re}(z) > C$ *and* $|\mathrm{Im}(z)| < f_C(\mathrm{Re}(z))$.

(3) $\Omega_C + \alpha\Omega_C \subset \Omega_{\min(1,\alpha)C}$

(4) $\mathrm{Log}(\Omega_C) \subset \Omega_D$

(5) *If* $D > C$, *then there exists* $\varepsilon > 0$ *such that* $\mathcal{V}_\varepsilon(\Omega_D) \subset \Omega_C$ *where* $\mathcal{V}_\varepsilon(A) = \{z \in \mathbb{C} |\ d(z, A) < \varepsilon\}$

(6) *There exist* k, K *such that* $ke^{K\sqrt{|z|}} \leq |e^z| \leq e^{|z|}$ *for* $|z|$ *large enough in* Ω_C

Proof. (1) By the open mapping theorem, since φ_C is biholomorphic, it is an open and closed map. Let $\mathcal{U} := \mathbb{C}^{>0}$, then $\delta(\mathcal{U}) = i\mathbb{R}$ so the following holds:

$$\begin{aligned}
\varphi_C(i\mathbb{R}) &= \varphi_C(\delta(\mathcal{U})) \\
&= \varphi_C(\overline{\mathcal{U}}) \setminus \varphi_C(\mathcal{U}^\circ) \ \ (\varphi_C \text{ is injective}) \\
&\subset \overline{(\varphi_C(\mathcal{U}))} \setminus \varphi_C(\mathcal{U})^\circ \ \ (\varphi_C \text{ is open and closed}) \\
&= \delta(\Omega_C)
\end{aligned}$$

Likewise, $\varphi_C^{-1}(\delta(\Omega_C)) \subset \delta(\mathcal{U})$. Hence, the first equality holds. The second equality follows directly from the formula of the principal square root in cartesian coordinates:

$$\sqrt{z} = \sqrt{\frac{|z| + \mathrm{Re}(z)}{2}} + i\,\mathrm{sgn}(\mathrm{Im}(z))\sqrt{\frac{|z| - \mathrm{Re}(z)}{2}} \text{ for } z \in \mathbb{C} \setminus \mathbb{R}^{\leq 0}$$

(2) Let $g_C : [C, +\infty[\to \mathbb{R}$ be the map $x \mapsto \sqrt{\left(2\left(\frac{x}{C}\right)^2 - 1\right)^2 - 1}$. It is easy to check that for $r \in \mathbb{R}$, $g_c(\mathrm{Re}(\varphi_C(ir))) = |r|$ and that $g_c(x) \sim 2\left(\frac{x}{C}\right)^2$. Now, define $f_C(x) := g_C(x) + C\sqrt{\frac{\sqrt{1+g_C(x)^2}-1}{2}}$ for $x \in [C, +\infty[$. Then, for all $r \geq 0$, $f_c(\mathrm{Re}(\varphi_C(ir))) = \mathrm{Im}(\varphi_C(ir))$.

(3) , (4), (5) and (6): see [12].

□

Definition 2.30. A set $\Omega \subset \mathbb{C}^{>0}$ is said to be a *standard domain* if there exists $a > 0$ and $f :]a, +\infty[\to]0, +\infty[$ such that:

$$\Omega = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > a \text{ and } |\mathrm{Im}(z)| < f(\mathrm{Re}(z))\}.$$

In particular, every standard quadratic domain is standard domain.

2.4 $\mathcal{H}_{\mathrm{an,exp}}$ and Geometrically Pure Functions

The content of this section comes from work in progress by P. Speissegger and T. Kaiser (see [9]). Recall that the objective is to generalize the construction done in [12] by allowing any germ in $\mathcal{H}_{\mathrm{an,exp}}$ (the Hardy field of all germs at $+\infty$ of unary functions defined in $\mathbb{R}_{\mathrm{an,exp}}$, see [14] and [15] for more details) as monomials. The main issue is that these germs do not in general verify the desired holomorphic extension properties. We need each $f_i \circ f_j^{-1}$ $(j < i)$ to have a holomorphic extension $\mathfrak{f}_i \circ \mathfrak{f}_j^{-1} : \overline{\Omega_{ij}} \to \mathbb{C}$ where Ω_{ij} is a standard quadratic domain and such that $\mathfrak{f}_i \circ \mathfrak{f}_j^{-1}$ maps standard quadratic domains into standard domains. It turns out that there is a subset of $\mathcal{H}_{\mathrm{an,exp}}$, called *geometrically pure functions of level at most* 0, that verify these properties. Moreover, every germ in $\mathcal{H}_{\mathrm{an,exp}}$ can be decomposed as a finite sum of geometrically pure functions.

More precisely, in [10], it is shown that $\mathbb{R}_{\mathrm{an,exp}}$ is *levelled*, i.e. that every definable infinitely increasing function has *level*.

Definition 2.31. Let f be an element of $\mathcal{H}_{\text{an,exp}}$. We say that f has *level* $s \in \mathbb{Z}$ if there exists $k \in \mathbb{N}$ such that $\log_k(f) \sim \log_{k-s}$. Then, s is unique and we write $\text{level}(f) = s$.

Without giving the precise definition, we can also associate to every element f in $\mathcal{H}_{\text{an,exp}}$ a complexity corresponding to the number of times *exp* is used in the construction of the term in $\mathcal{L}_{\text{an,exp,log}}$ (see introduction of [14]). We call this complexity the *exponential height* of f and denote it by $\text{eh}(f)$.

Example. For all $k \in \mathbb{N}$, we have:

- $\text{eh}(\log_k) = \text{level}(\log_k) = -k$, in particular $\text{eh}(x) = \text{level}(x) = 0$
- $\text{eh}(exp_k(x)) = \text{level}(exp_k(x)) = k$
- $\text{eh}(x + exp(-x)) = 1$ but $\text{level}(x + exp(-x)) = 0$
- for all $f \in \mathcal{H}_{\text{an,exp}}$, $\text{level}(f) \leq \text{eh}(f)$.

Facts 2.32. *The set* $\mathcal{E}_0 := \{f \in \mathcal{H}_{\text{an,exp}} \ | eh(f) \leq 0\}$ *verifies the following:*

(1) $\mathcal{E}_0$ *is a differential subfield of* $\mathcal{H}_{\text{an,exp}}$

(2) $\mathcal{E}_0$ *is stable under composition, i.e. for* $f, g \in \mathcal{E}_0$ *with* $g \succ 1$, $f \circ g \in \mathcal{E}_0$.

Definition 2.33. An infinite element f is said to be *geometrically pure* if $\text{eh}(f) = \text{level}(f)$.

Facts 2.34. *Let* f *and* g *be such that* $x \geq f > g$*. Then the following holds:*

(1) *if both* f *and* g *are geometrically pure, then* $\text{eh}(g \circ f^{-1}) \leq 0$,

(2) *if* $\text{eh}(f) \leq 0$, *then* f *has a holomorphic extension* $\mathfrak{f}$ *on the right half-plane that maps standard quadratic domains into standard domains.*

Facts 2.35. (1) *For all $f \in \mathcal{H}_{\mathrm{an,exp}}$, there exist geometrically pure $g_1, \ldots, g_k$ such that $f = g_1 + \cdots + g_k$.*

(2) *For all $f \in \mathcal{H}_{\mathrm{an,exp}}$, there exist unique $h \in \mathcal{H}_{\mathrm{an,exp}}$, $c \in \mathbb{R}$ and g geometrically pure such that $f = g + h + c$ with $\mathrm{level}(g) = \mathrm{level}(f)$ and h is either 0 or $\mathrm{level}(h) > \mathrm{level}(f)$.*

The holomorphic extensions of pure functions verify the following properties that will be important when doing the construction.

Facts 2.36. *Let $x > f, g \succ 1$ with holomorphic extensions $\mathfrak{f}$ and $\mathfrak{g}$ be such that $\mathrm{eh}(f), \mathrm{eh}(g) \leq 0$. Then the following holds:*

(1) *if $e^{-f} = o(e^{-g})$, then $\lim\limits_{|z| \to +\infty} \left| \frac{e^{-\mathfrak{f}(z)}}{e^{-\mathfrak{g}(z)}} \right| = 0$ on any standard quadratic domain Ω,*

(2) *if $\lim\limits_{x \to +\infty} f(x) = c \in \mathbb{R}$ and $\mathrm{level}(f) \leq 0$, then $\lim\limits_{|z| \to +\infty} \mathfrak{f}(z) = c$ in the right half plane and $\mathrm{level}(e^{-f}) \leq 0$.*

3 Construction of a Quasianalytic Asymptotic Algebra

Notation. Let Ω be a standard quadratic domain and $\mathfrak{f}, \mathfrak{g} : \Omega \to \mathbb{C}$ be two holomorphic functions. We write:

- $|\mathfrak{f}| = o(|\mathfrak{g}|)$ if $\lim\limits_{|z| \to +\infty \text{ in } \Omega} \left| \frac{\mathfrak{f}(z)}{\mathfrak{g}(z)} \right| = 0$
- $|\mathfrak{f}| \asymp |\mathfrak{g}|$ if $\lim\limits_{|z| \to +\infty \text{ in } \Omega} \left| \frac{\mathfrak{f}(z)}{\mathfrak{g}(z)} \right|$ is a non-zero real number.

3.1 Valuation Independent case

Let $f_0, f_1, \dots, f_k$ be elements of $\mathcal{H}_{\text{an,exp}}$ verifying:

P1. $x = f_0 > f_1 > \dots > f_k \succ 1$

P2. $f_0, \dots, f_k$ are in distinct archimedean classes

P3. for all $0 \le i < j \le k$, $\text{eh}(f_j \circ f_i^{-1}) \le 0$.

Since the f_i's are in distinct archimedean classes, the monomials e^{-f_i} belong to distinct comparability classes; in particular, they have $\mathbb{R}$-independent valuation, i.e. for all $r_1, \dots r_k \in \mathbb{R}$, if $r_1 v(e^{-f_1}) + \dots + r_k v(e^{-f_k}) = 0$, then $r_1 = \dots = r_k = 0$.

Notation. Given the tuple $(f_0, \dots, f_k)$ we introduce the following inductive notation:

- $f^{<k>} := (f_0, f_1, \dots, f_k)$,
- $f^{<k-1>} = (f_1 \circ f_1^{-1}, \dots, f_k \circ f_1^{-1})$,

  $\vdots$
- $f^{<1>} = (f_{k-1} \circ f_{k-1}^{-1}, f_k \circ f_{k-1}^{-1})$,
- $f^{<0>} = (f_k \circ f_k^{-1}) = (f_0)$.

3.1.1 Base case: $k = 0$

In the base case, the coefficients in the series expansion are real numbers and the monomials are powers of e^{-x}. We let $M(f_0) :=< e^{-f_0} >$ be the $\mathbb{R}$-multiplicative vector space generated by $\{e^{-f_0}\}$ so each element $n \in M(f_0)$ is of the form $e^{-\alpha x}$ for some $\alpha \in \mathbb{R}$. Each monomial n has a holomorphic extension $\mathfrak{n} := e^{-\alpha z}$ to the right half plane and $|\mathfrak{n}(z)| = n(\mathrm{Re}(z))$. If $\alpha \geq 0$, then n and $\mathfrak{n}$ are bounded.

Definition 3.1. Let $\mathcal{A}_{f<0>}$ be the set of germs at $+\infty$ of functions $f : \mathbb{R} \to \mathbb{R}$ such that:

(1) f has a bounded holomorphic extension $\mathfrak{f} : \overline{\Omega} \to \mathbb{C}$ where Ω is a standard quadratic domain

(2) there exists a series $F = \sum\limits_{m \in M(f_0)} a_n n$ with natural support included in $M(f_0)^{\leq 1}$ and $a_n \in \mathbb{R}$ such that:

$$\forall m \in M(f_0), \left| \mathfrak{f} - \sum_{n \geq m} a_n \mathfrak{n} \right| = o(|\mathfrak{m}|) \text{ as } |z| \to +\infty \text{ in } \Omega.$$

In that case, we say that F is an *asymptotic expansion* of f and write $f \sim F$.

Remark. The set $\mathcal{A}_{f<0>} \circ (-\log)$ contains all analytic functions near 0^+ and all correspondence maps near non-resonant hyperbolic singularities of planar real analytic vector fields (see [8] for details).

Lemma 3.2. *Let m and n be elements of $M(f_0)$, then $m = o(n) \Leftrightarrow |\mathfrak{m}| = o(|\mathfrak{n}|)$ on any standard quadratic domain.*

Proof. It directly follows from fact 2.29(6). $\square$

Lemma 3.3. *Let f and g be elements of $\mathcal{A}_{f<0>}$ and let Ω be a standard quadratic domain on which both f and g have bounded holomorphic extensions. Then $f = o(g) \Leftrightarrow |\mathfrak{f}| = o(|\mathfrak{g}|)$ on Ω.*

Proof. Assume that $f \sim F = \sum a_n n$ and $g \sim G = \sum b_n n$ with leading monomials m_0 and n_0 respectively. Then, the following holds:

$$\lim_{|z|\to+\infty \text{ in } \Omega} \left| \frac{\mathfrak{f}(z) - a_0\mathfrak{m}_0(z)}{\mathfrak{m}_0(z)} \right| = 0$$
$$\Rightarrow \lim_{|z|\to+\infty \text{ in } \Omega} \left| \frac{\mathfrak{f}(z)}{a_0\mathfrak{m}_0(z)} - 1 \right| = 0$$
$$\Rightarrow |\mathfrak{f}(z)| \asymp a_0 |\mathfrak{m}_0(z)| \asymp |\mathfrak{m}_0(z)| \ (a_0 \in \mathbb{R}^*)$$

Likewise, we obtain that $|\mathfrak{g}(z)| \asymp |\mathfrak{n}_0(z)|$. Since a standard quadratic domain contains the positive real line, we also have:

$$\lim_{x\to+\infty} \left| \frac{f}{a_0 m_0} - 1 \right| = 0$$

Hence, $f \asymp m_0$ and $g \asymp n_0$. Now, using lemma 3.2, we obtain that $m_0 = o(n_0) \Leftrightarrow |\mathfrak{m}_0| = o(|\mathfrak{n}_0|)$ on Ω so the desired equivalence follows.

□

In order to show that $\mathcal{A}_{f^{<0>}}$ is an $\mathbb{R}$-algebra, we need the following lemma for the multiplication of two series.

Lemma 3.4. *Let* $f \sim F = \sum_{n\in M(f_0)} a_n n$ *and* $g \sim G = \sum_{n\in M(f_0)} b_n n$ *be elements of* $\mathcal{A}_{f^{<0>}}$. *Then,*

(1) *For all* $n \in M(f_0)$, *there are finitely many elements* $p \in \operatorname{supp}(F)$ *and* $q \in \operatorname{supp}(G)$ *such that* $pq = n$.

(2) *The set* $\operatorname{supp}(FG) = \{n \in M(f_0) \mid \sum_{pq=n} a_p b_q \neq 0\}$ *is natural and included in* $M(f_0)^{\preceq 1}$.

Proof. (1) Let $P_n := \{p \in \operatorname{supp}(F) \mid \exists q \in \operatorname{supp}(G), pq = n\}$ and $Q_n := \{q \in \operatorname{supp}(G) \mid \exists p \in \operatorname{supp}(F), pq = n\}$. Then, since both $\operatorname{supp}(F)$ and $\operatorname{supp}(G)$ are subsets of $M(f_0)^{\preceq 1}$, $n = pq \preceq p$ and $n = pq \preceq q$ for all $p \in P_n$ and $q \in Q_n$.

Hence, P_n and Q_n are subsets of $[n, +\infty[$ and by naturality of $\mathrm{supp}(F)$ and $\mathrm{supp}(G)$, they must both be finite.

(2) $\mathrm{supp}(FG)$ is natural by lemma 2.6(3). Also, since both $\mathrm{supp}(F)$ and $\mathrm{supp}(G)$ are subsets of $M(f_0)^{\leq 1}$ and $\mathrm{supp}(FG) \subset \mathrm{supp}(F) \cdot \mathrm{supp}(G)$, we obtain that $\mathrm{supp}(FG) \subset M(f_0)^{\leq 1}$ as well.

□

Lemma 3.5. *$\mathcal{A}_{f^{<0>}}$ is an $\mathbb{R}$-algebra.*

Proof. We just need to check that $\mathcal{A}_{f^{<0>}}$ is a subalgebra of the algebra of germs of functions at $+\infty$. Let $f, g \in \mathcal{A}_{f^{<0>}}$, then there exists $C, D > 0$ such that f and g have a bounded holomorphic extension to the closure of standard quadratic domains Ω_C and Ω_D respectively. Also, there exist series $F = \sum_{n \in M(f_0)} a_n n$, $G = \sum_{n \in M(f_0)} b_n n$ with real coefficients such that $f \sim F$ on Ω_C, $g \sim G$ on Ω_D and $\mathrm{supp}(F), \mathrm{supp}(G) \subset M(f_0)^{\leq 1}$ are natural.

- Let $r \in \mathbb{R}$, then rf also has a bounded holomorphic extension to Ω_C and for all $m \in M(f_0)$, $\left| r\mathfrak{f} - \sum_{n \geq m} (ra_n)\mathfrak{n} \right| = o(|\mathfrak{m}|)$ and $\mathrm{supp}(rF) = \mathrm{supp}(F) \subset M(f_0)^{\leq 1}$ is natural.

- Let $E := \min\{C, D\}$, then $\Omega_C, \Omega_D \subset \Omega_E$ so $f+g$ has a bounded holomorphic extension to Ω_E and for all $m \in M(f_0)$, $\left| \mathfrak{f} + \mathfrak{g} - \sum_{n \geq m} (a_n + b_n)\mathfrak{n} \right| = o(|\mathfrak{m}|)$ and $\mathrm{supp}(F+G) \subset \mathrm{supp}(F) \cup \mathrm{supp}(G) \subset M(f_0)^{\leq 1}$ is natural.

- By the same argument as above, fg has a bounded holomorphic extension to Ω_E. Now, for all $m \in M(f_0)$, we have:

$$\left(\sum_{n \geq m} a_n n\right)\left(\sum_{n \geq m} b_n n\right) = \sum_{n \geq m}\left(\sum_{pq=n} a_p b_q\right) n + \varepsilon$$

where $\sum_{pq=n} a_p b_q$ and $\varepsilon = \sum_{m > n \geq m^2} \left(\sum_{pq=n} a_p b_q\right) n$ are finite sums (by lemma 3.4) and $|\boldsymbol{\varepsilon}| = o(|\mathfrak{m}|)$ (since $\mathrm{supp}(\varepsilon) < m$). Hence, the following holds:

$$\begin{aligned}\left|\mathfrak{f}\mathfrak{g} - \sum_{n\geq m}\left(\sum_{pq=n} a_p b_q\right)\mathfrak{n}\right| &= \left|\mathfrak{f}\mathfrak{g} - \mathfrak{f}\sum_{n\geq m} b_n\mathfrak{n} + \mathfrak{f}\sum_{n\geq m} b_n\mathfrak{n} - \left(\sum_{n\geq m} a_n\mathfrak{n}\right)\left(\sum_{n\geq m} b_n\mathfrak{n}\right) + \boldsymbol{\varepsilon}\right| \\ &\leq |\mathfrak{f}|\,|\mathfrak{g} - \sum_{n\geq m} b_n\mathfrak{n}| + |\sum_{n\geq m} b_n\mathfrak{n}||\mathfrak{f} - \sum_{n\geq m} a_n\mathfrak{n}| + |\boldsymbol{\varepsilon}| \\ &= o(|\mathfrak{m}|)\end{aligned}$$

The last equality follows from the boundedness of $|\mathfrak{f}|$ and $\left|\sum_{n\geq m} b_n\mathfrak{n}\right|$ (as a finite sum of bounded elements). Hence, $fg \in \mathcal{A}(f_0)$ and $fg \sim FG$.

$\square$

Lemma 3.6. *If* $0 \sim F = \sum_{n\in M(f_0)} a_n n$, *then* $a_n = 0$ *for all* $n \in M(f_0)$.

Proof. Assume that $\mathrm{supp}(F) \neq \emptyset$ and let m_0 be the leading monomial of F. Then, $|0 - \sum_{n\geq m_0} a_n\mathfrak{n}| = o(|\mathfrak{m}_\mathfrak{o}|)$ is equivalent to $|a_0\mathfrak{m}_\mathfrak{o}| = o(|\mathfrak{m}_\mathfrak{o}|)$ and since $a_0 \in \mathbb{R}$, we must have $a_0 = 0$.

$\square$

Corollary 3.7. *Each* $f \in \mathcal{A}_{f<0>}$ *has a unique asymptotic expansion.*

Proof. Assume that $f \sim \sum a_n n$ and $f \sim \sum b_n n$, then $F := \sum(a_n - b_n)n$ is an asymptotic expansion of 0. By lemma 3.6, $(a_n - b_n) = 0$ for all $n \in M(f_0)$.

$\square$

Definition 3.8. Let $f \in \mathcal{A}_{f<0>}$ with $f \sim F$, we define $T_{f<0>} : \mathcal{A}_{f<0>} \to \mathbb{R}((M(f_0)))$ to be the map $f \mapsto F$.

Lemma 3.9. *Let* $f \neq g$ *be elements of* $\mathcal{A}_{f<0>}$. *Then,* f *and* g *have distinct asymptotic expansions.*

Proof. Assume that f and g have the same asymptotic expansion, then $|\mathfrak{f} - \mathfrak{g}| = o(|\mathfrak{m}|)$ for all $m \in M(f_0)$. In particular, it holds for $m_j := e^{-jx} \in M(f_0)$ $(j \in \mathcal{N})$

so by the Phragmén-Lindelöf principle (fact 2.25), $|\mathfrak{f} - \mathfrak{g}| = 0$ on Ω so $f - g = 0$ i.e. $f = g$.

$\square$

Corollary 3.10. *$T_{f^{<0>}}$ is a well-defined and injective $\mathbb{R}$-algebra homomorphism.*

Proof. Follows directly from corollary 3.7 and lemma 3.9.

$\square$

Lemma 3.11. (1) *$T_{f^{<0>}}(\mathcal{A}_{f^{<0>}})$ is truncation closed.*

(2) *For all $f \in \mathcal{A}_{f^{<0>}}$ and $m \in M(f_0)$, $|\mathfrak{f} - \boldsymbol{T^{-1}_{f^{<0>}}}\left([T_{f^{<0>}}(f)]_m\right)| = o(|\mathfrak{m}|)$ as $|z| \to +\infty$ in Ω.*

Proof. Let $f \in \mathcal{A}_{f^{<0>}}$ with $f \sim F$ where $F := \sum\limits_{n \in M(f_0)} a_n n$ and fix $m \in M(f_0)$.

(1) We need to show there exists $g \in \mathcal{A}_{f^{<0>}}$ such that $T_{f^{<0>}}(g) = [T_{f^{<0>}}(f)]_m = F_m$. If $m < \operatorname{supp}(F)$, $F_m = \sum\limits_{n \geq m} a_n n = F$ so we can just take $g := f$. Otherwise, $F_m = \sum\limits_{n \geq m} a_n n$ is a finite sum of bounded elements (since $\operatorname{supp}(F)$ is natural and included in $M(f_0)^{\leq 1}$) so $\mathfrak{F}_m := \sum\limits_{n \geq m} a_n \mathfrak{n}$ is a bounded holomorphic extension of F_m to any standard quadratic domain.

We want to show that we can take $g := F_m$ with itself as asymptotic expansion so we need to prove that for all $t \in M(f_0)$, $|\mathfrak{F}_\mathfrak{m} - (\mathfrak{F}_\mathfrak{m})_t| = o(|\mathfrak{t}|)$ (where $(\mathfrak{F}_\mathfrak{m})_t$ is the truncation of the finite sum $\mathfrak{F}_\mathfrak{m}$ above t). If $t < m$, then $(\mathfrak{F}_\mathfrak{m})_t = \sum\limits_{n \geq m \text{ and } n \geq t} a_n \mathfrak{n} = \mathfrak{F}_\mathfrak{m}$ so $|\mathfrak{F}_\mathfrak{m} - (\mathfrak{F}_\mathfrak{m})_t| = 0 = o(|\mathfrak{t}|)$. If not, $|\mathfrak{F}_\mathfrak{m} - (\mathfrak{F}_\mathfrak{m})_t| = \left|\sum\limits_{t > n \geq m} a_n \mathfrak{n}\right| = o(|\mathfrak{t}|)$. Hence, $g = F_m \in \mathcal{A}(f_0)$ as desired.

(2) Now, $T^{-1}_{f^{<0>}}\left([T_{f^{<0>}}(f)]_m\right) = g$ has a bounded holomorphic extension $\mathfrak{g}$ and we obtain by definition of $f \sim F$:

$$|\mathfrak{f} - \mathfrak{g}| = \begin{cases} |\mathfrak{f} - \mathfrak{f}| = 0 = o(\mathfrak{m}) \text{ if } m < \operatorname{supp}(F) \\ |\mathfrak{f} - \sum\limits_{n \geq m} a_n \mathfrak{n}| = o(\mathfrak{m}) \text{ otherwise} \end{cases}$$

□

Corollary 3.12. *$(\mathcal{A}_{f^{<0>}}, M(f_0), T_{f^{<0>}})$ is a quasianalytic asymptotic algebra.*

Proof. It directly follows from lemmas 3.5, 3.6, 3.7, 3.9 and 3.11.

□

We can formally extend $\mathcal{A}_{f^{<0>}}$ to its fraction field but we also want the multiplicative inverses to have compatible asymptotic expansions.

Lemma 3.13. *Let $f \in \mathcal{A}_{f^{<0>}} \setminus \{0\}$ with $f \sim F$, then $\frac{1}{f} \sim \frac{1}{F}$.*

Proof. Let f be a nonzero element of $\mathcal{A}_{f^{<0>}}$ with asymptotic expansion $F = \sum a_n n$. Let m_0 be the leading monomial of F i.e.:

$$|\mathfrak{f} - a_0\mathfrak{m}_\mathfrak{o}| = o(|\mathfrak{m}_\mathfrak{o}|)$$

Let $\varepsilon := \frac{f}{a_0 m_0} - 1$ and $E := \frac{F}{a_0 m_0} - 1$. Since $m_0 = e^{-\alpha_0 f_0}$ for some $\alpha_0 \geq 0$, it has a bounded holomorphic extension $\mathfrak{m}_\mathfrak{o} : \overline{\Omega_0} \to \mathbb{C} \setminus \{0\}$ to the closure of some standard quadratic domain Ω_0. Since $\varepsilon \prec 1$, it also has holomorphic extension $\boldsymbol{\varepsilon} = \frac{\mathfrak{f}}{a_0\mathfrak{m}_\mathfrak{o}} - 1$ to some standard quadratic domain Ω_ε. To show that it is an element of $\mathcal{A}_{f^{<0>}}$, we need $\varepsilon \sim E$ (which implies that ε is bounded). Let $m \in M(f_0)$ and let $\mathfrak{E}_m$ be the holomorphic extension of E_m on Ω. then:

$$\begin{aligned}
&|\mathfrak{f} - \sum_{n \geq m_0 m} a_n \mathfrak{n}| = o(|\mathfrak{m}_\mathfrak{o}\mathfrak{m}|) \\
&\Rightarrow \lim_{|z| \to +\infty} \left| \frac{\mathfrak{f} - \sum\limits_{n \geq m_0 m} a_n \mathfrak{n}}{\mathfrak{m}_\mathfrak{o}\mathfrak{m}} \right| = 0 \\
&\Rightarrow \lim_{|z| \to +\infty} \left| \frac{\mathfrak{f} - \sum\limits_{n \geq m_0 m} a_n \mathfrak{n}}{a_0 \mathfrak{m}_\mathfrak{o}\mathfrak{m}} \right| = 0 \\
&\Rightarrow \lim_{|z| \to +\infty} \left| \frac{1}{\mathfrak{m}} \left[\left(\frac{\mathfrak{f}}{a_0 \mathfrak{m}_\mathfrak{o}} - 1 \right) - \left(\sum_{n \geq m_0 m} \frac{a_n}{a_0} \frac{\mathfrak{n}}{\mathfrak{m}_\mathfrak{o}} - 1 \right) \right] \right| = 0 \\
&\Rightarrow \lim_{|z| \to +\infty} \left| \frac{\boldsymbol{\varepsilon} - \mathfrak{E}_m}{\mathfrak{m}} \right| = 0 \text{ (since } E_m = \frac{F_{m_0 m}}{a_0 m_0} - 1) \\
&\Rightarrow |\boldsymbol{\varepsilon} - \mathfrak{E}_m| = o(|\mathfrak{m}|)
\end{aligned}$$

Since $\mathrm{supp}(E) \prec 1$, $(\mathrm{supp}(E))^*$ is natural by lemma 2.6. Let $m \in M(f_0)$, we need to prove that:

$$\left| \frac{1}{1 + \boldsymbol{\varepsilon}} - \left(\sum_{k \in \mathbb{N}} (-1)^k \mathfrak{E}^k \right)_m \right| = o(|\mathfrak{m}|)$$

Since $(\mathrm{supp}(E))^*$ is natural, there are finitely many powers of E whose support contain the monomial m. In other words, there exists $k_m \in \mathbb{N}$ such that $supp \left(\sum_{k > k_m} (-1)^k E^k \right) \cap [m, +\infty[= \emptyset$, hence:

$$\left(\sum_{k \in \mathbb{N}} (-1)^k \mathfrak{E}^k \right)_m = \sum_{k \leq k_m} (-1)^k (\mathfrak{E}^k)_m$$

Hence, the following holds:

$$
\begin{aligned}
&\left|\frac{1}{1+\varepsilon}-\left(\sum_{k\in\mathbb{N}}(-1)^k\mathfrak{E}^k\right)_m\right| \\
=&\left|\frac{1}{1+\varepsilon}-\sum_{k\leq k_m+1}(-1)^k\varepsilon^k+\sum_{k\leq k_m+1}(-1)^k\varepsilon^k-\sum_{k\leq k_m}(-1)^k(\mathfrak{E}^k)_m\right| \\
=&\left|\left(\frac{1}{1+\varepsilon}-\sum_{k\leq k_m+1}(-1)^k\varepsilon^k\right)+(-1)^{k_m+1}\varepsilon^{k_m+1}+\sum_{k\leq k_m}(-1)^k(\varepsilon^k-(\mathfrak{E}^k)_m)\right|
\end{aligned}
$$

We use the following facts:

(1) $\left|\frac{1}{1+\varepsilon}-\sum\limits_{k\leq k_m+1}(-1)^k\varepsilon^k\right| = o(\left|\varepsilon^{k_m+1}\right|) = o(|\mathfrak{m}|)$ since $|\varepsilon^{k_m+1}-(\mathfrak{E}^{k_m+1})_m| = o(|\mathfrak{m}|)$ and $|(\mathfrak{E}^{k_m+1})_m| = o(|\mathfrak{m}|)$ (since $\operatorname{supp}((E^{k_m+1})_m) \prec m$)

(2) $|(-1)^{k_m+1}\varepsilon^{k_m+1}| = o(|\mathfrak{m}|)$ for the same reason as above

(3) for all $k \in \mathbb{N}$, $|\varepsilon^k-(\mathfrak{E}^k)_m| = o(|\mathfrak{m}|)$ and since $\left|\sum\limits_{k\leq k_m}(-1)^k(\varepsilon^k-(\mathfrak{E}^k)_m)\right|$ is a finite sum, it is equal to $o(|\mathfrak{m}|)$ as well.

Hence, $\left|\frac{1}{1+\varepsilon}-\sum\limits_{k\leq k_m+1}(-1)^k\varepsilon^k\right| = o(|\mathfrak{m}|)$ for all m as desired i.e. $\frac{1}{1+\varepsilon} \sim \frac{1}{1+E}$.
Let $F^{-1} := \frac{1}{a_0m_0}\frac{1}{1+E}$, then $\frac{1}{f} = \frac{1}{a_0m_0}\frac{1}{1+\varepsilon} \sim F^{-1}$. $\square$

Definition 3.14. Let $\mathcal{F}_{f^{<0>}}$ denote the fraction field of $\mathcal{A}_{f^{<0>}}$. After extending $T_{f^{<0>}}$ to $\mathcal{F}_{f^{<0>}}$ accordingly, we obtain that $(\mathcal{F}_{f^{<0>}}, M(f_0), T_{f^{<0>}})$ is a qaa field.

Remark. Elements of $\mathcal{F}_{f^{<0>}}$ are not necessarily bounded or have a bounded holomorphic extension and the support of the corresponding series may contain (finitely many) infinite monomials (i.e. the condition $\operatorname{supp}(F) \subset M(f_0)^{\leq 1}$ doesn't hold any more).

3.1.2 Inductive step: $k > 0$

Definition 3.15. Let $\mathcal{A}_{f^{<k>}}$ be the set of germs at $+\infty$ of functions $f : \mathbb{R} \to \mathbb{R}$ such that:

(1) f has a bounded holomorphic extension $\mathfrak{f} : \overline{\Omega} \to \mathbb{C}$ where Ω is a standard quadratic domain

(2) there exists a series $F = \sum_{m \in M(f_0)} (a_n \circ f_1) n$ with natural support included in $M(f_0)^{\leq 1}$ and $a_n \in \mathcal{F}_{f^{<k-1>}}$ such that:

$$\forall m \in M(f_0), \left| \mathfrak{f} - \sum_{n \geq m} (\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1)\mathfrak{n} \right| = o(|\mathfrak{m}|) \text{ as } |z| \to +\infty \text{ in } \Omega.$$

In that case, we say that F is an *asymptotic expansion* of f and write $f \sim F$.

The construction of $\mathcal{A}_{f^{<k>}}$ is inductive. The first step is to construct $\mathcal{A}_{f^{<0>}} = \mathcal{A}_{(f_k \circ f_k^{-1})}$ (which corresponds to the base case $k = 0$ described in the previous section). In the second step, the coefficients are elements of $\mathcal{F}_{f^{<0>}}$ composed with $f_k \circ f_{k-1}^{-1}$ and so on. In the last step, described in definition 3.15, to obtain the coefficients, we compose elements of $\mathcal{F}_{f^{<k-1>}}$ with $f_1 \circ f_0^{-1}$ $(= f_1)$. For all these steps, the monomials are always elements of $M(f_0)$ i.e. real powers of e^{-x}.

$$x = f_0 > f_1 > \cdots > f_k$$

step k: construction of $\mathcal{A}_{f^{<k>}}$

$$f^{<k>} := (f_0, f_1, \ldots, f_k)$$

$$\circ(f_1 \circ f_0^{-1})^{-1} \Big\downarrow \equiv \circ f_1^{-1}$$

$$f \sim \sum_{n \in M(f_0)} (a_n \circ f_1 \circ f_0^{-1})n, \ \text{where } a_n \in \mathcal{F}_{f^{<k-1>}}$$

$$x = f_1 \circ f_1^{-1} > \cdots > f_k \circ f_1^{-1}$$

step $k-1$: construction of $\mathcal{A}_{f^{<k-1>}}$

$$f^{<k-1>} = (f_1 \circ f_1^{-1}, \ldots, f_k \circ f_1^{-1})$$

$$\circ(f_2 \circ f_1^{-1})^{-1} \Big\downarrow \equiv \circ f_1 \circ f_2^{-1}$$

$$f \sim \sum_{n \in M(f_0)} (a_n \circ f_2 \circ f_1^{-1})n, \ \text{where } a_n \in \mathcal{F}_{f^{<k-2>}}$$

$$x = f_2 \circ f_2^{-1} > f_3 \circ f_2^{-1} > \cdots > f_k \circ f_2^{-1}$$

step $k-2$: construction of $\mathcal{A}_{f^{<k-2>}}$

$$\vdots \qquad\qquad \vdots$$

$$x = f_{k-1} \circ f_{k-1}^{-1} > f_k \circ f_{k-1}^{-1}$$

step 1 construction of $\mathcal{A}_{f^{<1>}}$

$$f^{<1>} = (f_{k-1} \circ f_{k-1}^{-1}, f_k \circ f_{k-1}^{-1})$$

$$\circ(f_k \circ f_{k-1}^{-1})^{-1} \Big\downarrow \equiv \circ f_{k-1} \circ f_k^{-1}$$

$$f \sim \sum_{n \in M(f_0)} (a_n \circ f_k \circ f_{k-1}^{-1})n, \ \text{where } a_n \in \mathcal{F}_{f^{<0>}}$$

$$x = f_k \circ f_k^{-1}$$

step 0: construction of $\mathcal{A}_{f^{<0>}}$

$$f^{<0>} = (f_k \circ f_k^{-1}) = (f_0)$$

$$f \sim \sum_{n \in M(f_0)} a_n n, \ \text{where } a_n \in \mathbb{R}$$

Example. If we take $f_0 > f_1 := \log$, then $\mathcal{A}_{f^{<1>}} \circ (-\log)$ contains all $f \sim \sum_{n \in \mathbb{N}} P_n(\log x) x^{-\nu_n x}$ where $P_n \in \mathbb{R}[x]$ and $(\nu_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence of positive real numbers with $\lim_{n \to +\infty} \nu_n = +\infty$. In particular, it will contain all correspondence maps near hyperbolic singularities of planar real analytic vector fields.

A key point that we will use several times is that $|\mathfrak{a}_n \circ \mathfrak{f}_1(z)|$ is larger than any positive power of $e^{-\operatorname{Re}(z)}$ and smaller than any positive power $e^{\operatorname{Re}(z)}$. A consequence of this lemma is that even if $a_n \circ f_1$ is infinite (elements of the fraction field are not always bounded), $|\mathfrak{a}_n \circ \mathfrak{f}_1(z)|$ is still bounded.

Lemma 3.16. *Let $f \sim F = \sum(a_n \circ f_1)n$ be as in definition 3.15, then for all $n \in \operatorname{supp}(F)$ and $\alpha > 0$, $e^{-\alpha \operatorname{Re}(z)} = o(|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1|)$ and $|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1| = o(e^{\alpha \operatorname{Re}(z)})$ on Ω.*

Proof. Let $n \in M(f_0)$. Since, by definition, $a_n \in \mathcal{F}_{f^{<k-1>}}$, $a_n \sim G$ for some $G = \sum(b_q \circ f_2 \circ f_1^{-1})q$ where $q \in M(f_0)$, $b_q \in \mathcal{F}_{f^{<k-2>}}$. Let q_0 be the leading monomial of G, then by induction:

$$e^{-\beta \operatorname{Re}(z)} = o(\left|\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2 \circ \mathfrak{f}_1^{-1}\right|) \text{ and } \left|\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2 \circ \mathfrak{f}_1^{-1}\right| = o(e^{\beta \operatorname{Re}(z)}) \text{ for all } \beta > 0$$

which implies:

(1) $|\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2 \circ \mathfrak{f}_1^{-1}| \geq |\mathfrak{q}_\mathfrak{o}|$ and

(2) $e^{-\beta \operatorname{Re}(\mathfrak{f}_1(z))} = o(|\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2|)$ and $|\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2| = o(e^{\beta \operatorname{Re}(f_1(z))})$ for all $\beta > 0$

Since $f_0 \succ f_1$, $e^{-\beta f_1} \succ e^{-\alpha f_0}$ and $e^{\beta f_1} \prec e^{\alpha f_0}$ for all $\alpha > 0$. Hence, $e^{-\alpha \operatorname{Re}(z)} = o(|\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2(z)|)$ and $|\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2(z)| = o(e^{\alpha \operatorname{Re}(z)})$ for all $\alpha > 0$. The first point implies $o(\mathfrak{q}_\mathfrak{o}) = o((\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2 \circ \mathfrak{f}_1^{-1})\mathfrak{q}_\mathfrak{o})$ so the following holds:

$$\begin{aligned}
&\left|\mathfrak{a}_\mathfrak{n} - (\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2 \circ \mathfrak{f}_1^{-1})\mathfrak{q}_\mathfrak{o}\right| = o(\mathfrak{q}_\mathfrak{o}) = o((\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2 \circ \mathfrak{f}_1^{-1})\mathfrak{q}_\mathfrak{o}) \\
\Rightarrow &\lim_{|z| \to +\infty} \left| \frac{\mathfrak{a}_\mathfrak{n} - (\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2 \circ \mathfrak{f}_1^{-1})\mathfrak{q}_\mathfrak{o}}{(\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2 \circ \mathfrak{f}_1^{-1})\mathfrak{q}_\mathfrak{o}} \right| = 0 \\
\Rightarrow &|\mathfrak{a}_\mathfrak{n}| \asymp \left|(\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2 \circ \mathfrak{f}_1^{-1})\mathfrak{q}_\mathfrak{o}\right| \\
\Rightarrow &|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1| \asymp |(\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_2)(\mathfrak{q}_\mathfrak{o} \circ \mathfrak{f}_1)| \\
\Rightarrow &e^{-\alpha \operatorname{Re}(z)} |\mathfrak{q}_\mathfrak{o} \circ \mathfrak{f}_1| = o(|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1|) \text{ and } |\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1| = o(e^{\alpha \operatorname{Re}(z)} |\mathfrak{q}_\mathfrak{o} \circ \mathfrak{f}_1|) \text{ for all } \alpha > 0
\end{aligned}$$

We have two cases: $q_0 = 1$ or $q_0 = e^{-\alpha_0 x}$ for some $\alpha_0 > 0$. In the first case, $|(\mathfrak{q}_\mathfrak{o} \circ \mathfrak{f}_1)| = 1$ and we are done. In the second case, $|(\mathfrak{q}_\mathfrak{o} \circ \mathfrak{f}_1)| = \left|e^{-\alpha_0 \mathfrak{f}_1}\right| = o(1)$ so $|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1| = o(e^{\alpha \operatorname{Re}(z)})$. Since $f_0 \succ f_1$, $e^{-\alpha_0 f_1} \succ e^{-\alpha f_0}$ for all $\alpha > 0$ so we obtain $e^{-\alpha \operatorname{Re}(z)} = o(|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1|)$ as desired.

□

Corollary 3.17. *Let $f \sim F = \sum(a_n \circ f_1)n$ be as in definition 3.15, then for all $n \in \mathrm{supp}(F)$:*

(1) *$o(\mathfrak{n})$ implies $o((\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1)\mathfrak{n})$*

(2) *$|(\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1)\mathfrak{n}|$ is bounded.*

Proof. (1) Since $\mathrm{supp}(F) \subset M(f_0)^{\leq 1}$, $n = e^{-\alpha_n x}$ $(\alpha_n \geq 0)$. Lemma 3.48 implies that $|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1| \geq e^{-\alpha_n x} = |\mathfrak{n}|$ so $|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1 \mathfrak{n}| \geq |\mathfrak{n}|$. Hence, $o(\mathfrak{n})$ implies $o((\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1)\mathfrak{n})$.

(2) If $n = 1$ and $a_1 \neq 0$, then it is the leading monomial of F and we have $|\mathfrak{f} - \mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1| = o(1)$. Since $|\mathfrak{f}|$ is bounded, $|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1|$ is bounded as well so $|(\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1)\mathfrak{n}|$ is bounded as desired. Otherwise, $n = e^{-\alpha_n x}$ $(\alpha_n > 0)$ and the following holds:

$$
\begin{aligned}
|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1| &= o(e^{\alpha_n \mathrm{Re}(z)}) \text{ by lemma 3.48} \\
\Rightarrow |(\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1)\mathfrak{n}| &= o(e^{\alpha_n \mathrm{Re}(z)} e^{-\alpha_n \mathrm{Re}(z)}) = o(1)
\end{aligned}
$$

□

Lemma 3.18. *Let f and g be elements of $\mathcal{A}_{f<k>}$, then $f = o(g) \Leftrightarrow |\mathfrak{f}| = o(|\mathfrak{g}|)$.*

Proof. Assume that $f \sim F = \sum(a_n \circ f_1)n$ and $g \sim G = \sum(b_n \circ f_1)n$ with leading monomials m_0 and n_0 respectively. By lemma 3.48, $o(\mathfrak{m}_\mathfrak{o}) = o((\mathfrak{a}_\mathfrak{o} \circ \mathfrak{f}_1)\mathfrak{m}_\mathfrak{o})$ and $o(\mathfrak{n}_\mathfrak{o}) = o((\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_1)\mathfrak{n}_\mathfrak{o})$. Hence, the following holds:

$$
\begin{aligned}
&|\mathfrak{f} - (\mathfrak{a}_\mathfrak{o} \circ \mathfrak{f}_1)\mathfrak{m}_\mathfrak{o}| = o(|\mathfrak{m}_\mathfrak{o}|) = o(|(\mathfrak{a}_\mathfrak{o} \circ \mathfrak{f}_1)\mathfrak{m}_\mathfrak{o}|) \\
\Rightarrow &\lim_{|z| \to +\infty} \left| \frac{\mathfrak{f} - (\mathfrak{a}_\mathfrak{o} \circ \mathfrak{f}_1)\mathfrak{m}_\mathfrak{o}}{(\mathfrak{a}_\mathfrak{o} \circ \mathfrak{f}_1)\mathfrak{m}_\mathfrak{o}} \right| = 0 \\
\Rightarrow &\lim_{|z| \to +\infty} \left| \frac{\mathfrak{f}}{(\mathfrak{a}_\mathfrak{o} \circ \mathfrak{f}_1)\mathfrak{m}_\mathfrak{o}} - 1 \right| = 0 \\
\Rightarrow &|\mathfrak{f}| \asymp |(\mathfrak{a}_\mathfrak{o} \circ \mathfrak{f}_1)\mathfrak{m}_\mathfrak{o}| = |\mathfrak{a}_\mathfrak{o} \circ \mathfrak{f}_1|\, m_0(\mathrm{Re}(z))
\end{aligned}
$$

Likewise $|\mathfrak{g}| \asymp |\mathfrak{b}_\mathfrak{o} \circ \mathfrak{f}_1|\, n_0(\mathrm{Re}(z))$. In a standard quadratic domain, $|z| \to +\infty$

implies that $Re(z) \to +\infty$ so we also have:

$$\lim_{\mathrm{Re}(z)\to+\infty}\left|\frac{f(\mathrm{Re}(z))}{(a_0\circ f_1)(\mathrm{Re}(z))m_0(\mathrm{Re}(z))}-1\right|=0 \text{ and}$$
$$\lim_{\mathrm{Re}(z)\to+\infty}\left|\frac{g(\mathrm{Re}(z))}{(b_0\circ f_1)(\mathrm{Re}(z))n_0(\mathrm{Re}(z))}-1\right|=0$$
$$\Rightarrow f(\mathrm{Re}(z)) \asymp (a_0\circ f_1)(\mathrm{Re}(z))m_0(\mathrm{Re}(z)) \text{ and } g(\mathrm{Re}(z)) \asymp (b_0\circ f_1)(\mathrm{Re}(z))n_0(\mathrm{Re}(z))$$

We have two cases, either $m_0 = n_0 = 1$ or at least one of following equalities hold: $m_0 = e^{-\alpha_0 x}$ $(\alpha_0 > 0)$ or $n_0 = e^{-\beta_0 x}$ $(\beta_0 > 0)$. In the second case, we obtain by lemma 3.48 that $m_0 \prec n_0 \Rightarrow |\mathfrak{a}_0 \circ \mathfrak{f}_1|\, m_0(\mathrm{Re}(z)) = o(|\mathfrak{b}_0 \circ \mathfrak{f}_1|\, n_0(\mathrm{Re}(z)))$ and $m_0 \prec n_0 \Rightarrow a_0 \circ f_1 m_0 \prec b_0 \circ f_1 n_0$. Hence, the following holds:

$$f = o(g) \Leftrightarrow \begin{cases} m_0 = o(n_0) \text{ if } m_0 \text{ or } n_0 \prec 1 \\ a_0 \circ f_1 = o(b_0 \circ f_1) \text{ if } m_0 = n_0 = 1 \end{cases}$$

and

$$|\mathfrak{f}| = o(|\mathfrak{g}|) \Leftrightarrow \begin{cases} m_0 = o(n_0) \text{ if } m_0 \text{ or } n_0 \prec 1 \\ |\mathfrak{a}_0 \circ \mathfrak{f}_1| = o(|\mathfrak{b}_0 \circ \mathfrak{f}_1|) \text{ if } m_0 = n_0 = 1 \end{cases}$$

By induction, $a_0 \circ f_1 = o(b_0 \circ f_1) \Leftrightarrow |\mathfrak{a}_0 \circ \mathfrak{f}_1| = o(|\mathfrak{b}_0 \circ \mathfrak{f}_1|)$ so we obtain that $f = o(g) \Leftrightarrow |\mathfrak{f}| = o(|\mathfrak{g}|)$ as desired. □

Lemma 3.19. *Each $f \in \mathcal{A}_{f<k>}$ has a unique asymptotic expansion.*

Proof. It suffices to show that if $0 \sim F = \sum(a_n \circ f_1)n$, then $a_n \circ f_1 = 0$ for all $n \in M(f_0)$. Assume that $\mathrm{supp}(F) \neq \emptyset$ and let $m_0 = e^{-\alpha_0 x}$ $(\alpha_0 \geq 0)$ be the leading monomial of F. Take $m_1 = e^{-\alpha_1 x} \in M(f_0)$ be such that $\alpha_1 > \alpha_0$ and $\mathrm{supp}(F) \cap [m_1, m_0[= \emptyset$. Then, the following holds:

$$\begin{aligned}
&\left|0 - \sum_{n \geq m_1} (\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1)\mathfrak{n}\right| = o(|\mathfrak{m}_1|) \\
\Rightarrow &\lim_{|z| \to +\infty} \left|\frac{(\mathfrak{a}_\mathfrak{o} \circ \mathfrak{f}_1)\mathfrak{m}_\mathfrak{o}}{\mathfrak{m}_1}\right| = 0 \\
\Rightarrow &\lim_{|z| \to +\infty} \left|(\mathfrak{a}_\mathfrak{o} \circ \mathfrak{f}_1) e^{(\alpha_1 - \alpha_0)z}\right| = 0
\end{aligned}$$

Since $\alpha_1 > \alpha_0$, $\lim\limits_{|z| \to +\infty} \left|e^{(\alpha_1 - \alpha_0)z}\right| = +\infty$. Also, by lemma 3.48, $|\mathfrak{a}_\mathfrak{o} \circ \mathfrak{f}_1| = o(|e^{(\alpha_1 - \alpha_0)z}|)$. Hence, we must have $\mathfrak{a}_\mathfrak{o} \circ \mathfrak{f}_1 = 0$ which implies $a_0 \circ f_1 = 0$.

□

Lemma 3.20. *Let $f \neq g$ be elements of $\mathcal{A}_{f^{<k>}}$. Then, f and g have distinct asymptotic expansions.*

Proof. Assume that f and g have the same asymptotic expansion, then $|\mathfrak{f} - \mathfrak{g}| = o(|\mathfrak{m}|)$ for all $m \in M(f_0)$. In particular, it holds for $m_j := e^{-jx} \in M(f_0)$ $(j \in \mathbb{N})$ so by the Phragmén-Lindelöf principle (theorem 2.25), $|\mathfrak{f} - \mathfrak{g}| = 0$ in Ω so $f - g = 0$.

□

Definition 3.21. We define the function $T_{f^{<k>}}$ from $\mathcal{A}_{f^{<k>}}$ to $\mathbb{R}((M(f_0, \ldots, f_k)))$ inductively by:

$$f \mapsto \sum_{n \in M(f_0)} \left(\left(T_{f^{<k-1>}}(a_n)\right) \circ f_1\right) n \text{ where } f \sim \sum_{n \in M(f_0)} (a_n \circ f_1) n$$

So $T_{f^{<k>}}(f)$ is a series with support of order-type at most ω^{k+1} and real coefficients.

Lemma 3.22. *$T_{f^{<k>}}$ is well-defined and injective.*

Proof. Follows directly from lemmas 3.19 and 3.20.

□

Lemma 3.23. (1) *$T_{f^{<k>}}(\mathcal{A}_{f^{<k>}})$ is truncation closed.*

(2) *For all* $f \in \mathcal{A}_{f<k>}$ *and* $m \in M(f_0, \ldots, f_k)$,

$$\left|\mathfrak{f} - \boldsymbol{T^{-1}_{f<k>}}\left(\left[T_{f<k>}(f)\right]_m\right)\right| = o(|\mathfrak{m}|) \text{ as } |z| \to +\infty \text{ in } \Omega.$$

Proof. Let $f \in \mathcal{A}_{f<k>}$ with $f \sim F = \sum_{n\in M(f_0)} (a_n \circ f_1)n$ and fix $m \in M(f_0, \ldots, f_k)$.

(1) We need to show that there exists $g \in \mathcal{A}_{f<k>}$ such that $T_{f<k>}(g) = \left[T_{f<k>}(f)\right]_m$. Since $m \in M(f_0, \ldots, f_k)$, $m = m_0 m_r$ for some $m_0 = e^{-\alpha_0 f_0} \in M(f_0)$ and $m_r \in M(f_1, \ldots, f_k)$. Hence, we have:

$$\begin{aligned}\left[T_{f<k>}(f)\right]_m &= \left[\sum_{n\in M(f_0)} \left(T_{f<k-1>}(a_n) \circ f_1\right) n\right]_m \\ &= \sum_{n>m_0} \left(T_{f<k-1>}(a_n) \circ f_1\right) n + \left[\left(T_{f<k-1>}(a_{m_0}) \circ f_1\right)\right]_{m_r} m_0\end{aligned}$$

By induction, there exists $h \in \mathcal{F}_{f<k-1>}$ such that:

$$T_{f<k-1>}(h) \circ f_1 = \left[\left(T_{f<k-1>}(a_{m_0}) \circ f_1\right)\right]_{m_r} \text{ and } |\mathfrak{a}_{m_0} \circ \mathfrak{f}_1 - \mathfrak{h} \circ \mathfrak{f}_1| = o(|\mathfrak{m}_r|)$$

Now, we want to prove that we can take $g := \sum_{n>m_0} (a_n \circ f_1)n + (h \circ f_1)m_0$ so we need g to have a bounded holomorphic extension to a standard quadratic domain. $\left|\sum_{n>m_0} (\mathfrak{a}_n \circ \mathfrak{f}_1)\mathfrak{n}\right|$ is a finite sum of bounded elements by corollary 3.17 and $|(\mathfrak{h} \circ \mathfrak{f}_1)\mathfrak{m}_0|$ is also bounded by corollary 3.17. Hence, $g \in \mathcal{A}_{f<k>}$ with itself as asymptotic expansion and $T_{f<k>}(\mathcal{A}_{f<k>})$ is truncation closed as desired.

(2) We need to show that $|\mathfrak{f} - \mathfrak{g}| = o(|\mathfrak{m}|)$:

$$\begin{aligned}|\mathfrak{f} - \mathfrak{g}| &= \left|\mathfrak{f} - \left(\sum_{n>m_0} (\mathfrak{a}_n \circ \mathfrak{f}_1)\mathfrak{n} + (\mathfrak{h} \circ \mathfrak{f}_1)\mathfrak{m}_0\right)\right| \\ &\leq \left|\mathfrak{f} - \sum_{n\geq m_0} (\mathfrak{a}_n \circ \mathfrak{f}_1)\mathfrak{n}\right| + |(\mathfrak{a}_{m_0} \circ \mathfrak{f}_1)\mathfrak{m}_0 - (\mathfrak{h} \circ \mathfrak{f}_1)\mathfrak{m}_0|\end{aligned}$$

Let $m_1 \in M(f_0)$ be such that $m_1 \prec m_0$ and $\mathrm{supp}(F) \cap [m_1, m_0[= \emptyset$, then we have:

$$\left|\mathfrak{f} - \sum_{n \geq m_1} (\mathfrak{a}_n \circ \mathfrak{f}_1)\mathfrak{n}\right| = o(|\mathfrak{m}_1|)$$

$$\Rightarrow \left|\mathfrak{f} - \sum_{n \geq m_0} (\mathfrak{a}_n \circ \mathfrak{f}_1)\mathfrak{n}\right| = o(|\mathfrak{m}_1|) = o(|\mathfrak{m}|)$$

Also, since $|\mathfrak{a}_{m_0} \circ \mathfrak{f}_1 - \mathfrak{h} \circ \mathfrak{f}_1| = o(|\mathfrak{m}_r|)$, we obtain $|(\mathfrak{a}_{m_0} \circ \mathfrak{f}_1)\mathfrak{m}_0 - (\mathfrak{h} \circ \mathfrak{f}_1)\mathfrak{m}_0| = o(|\mathfrak{m}|)$. Hence, $|\mathfrak{f} - \mathfrak{g}| = o(|\mathfrak{m}|)$ as desired.

□

Corollary 3.24. *$(\mathcal{A}_{f<k>}, M(f_0, \ldots, f_k), T_{f<k>})$ is a qaa algebra.*

Lemma 3.25. *Let $f \in \mathcal{A}_{f<k>} \setminus \{0\}$ with $f \sim F$, then $\frac{1}{f} \sim \frac{1}{F}$.*

Proof. Let f be a nonzero element of $\mathcal{A}_{f<k>}$ with asymptotic expansion $F = \sum (a_n \circ f_1)n$ with $a_n \in \mathcal{F}_{f<k-1>}$. Let m_0 be the leading monomial of F so:

$$|\mathfrak{f} - (\mathfrak{a}_0 \circ \mathfrak{f}_1)\mathfrak{m}_0| = o(|\mathfrak{m}_0|)$$

Let $\varepsilon := \frac{f}{(a_0 \circ f_1)m_0} - 1$ and $E := \frac{F}{(a_0 \circ f_1)m_0} - 1$. By induction, $\frac{1}{a_0} \in \mathcal{F}_{f<k-1>}$. Also, ε has a holomorphic extension $\boldsymbol{\varepsilon} = \frac{\mathfrak{f}}{(\mathfrak{a}_0 \circ \mathfrak{f}_1)\mathfrak{m}_0} - 1$ to the closure of some standard quadratic domain Ω_ε. To show that it is an element of $\mathcal{A}_{f<k>}$, we need $\varepsilon \sim E$ (which implies that ε is bounded). Let $m \in M(f_0)$, then:

$$|\mathfrak{f} - \sum_{n \geq m_0 m} (\mathfrak{a}_n \circ \mathfrak{f}_1)\mathfrak{n}| = o(|\mathfrak{m}_0\mathfrak{m}|) = o(|(\mathfrak{a}_0 \circ \mathfrak{f}_1)\mathfrak{m}_0\mathfrak{m}|) \text{ (By corollary 3.17)}$$

$$\Rightarrow \lim_{|z| \to +\infty} \left| \frac{\mathfrak{f} - \sum_{n \geq m_0 m} (\mathfrak{a}_n \circ \mathfrak{f}_1)\mathfrak{n}}{(\mathfrak{a}_0 \circ \mathfrak{f}_1)\mathfrak{m}_0\mathfrak{m}} \right| = 0$$

$$\Rightarrow \lim_{|z| \to +\infty} \left| \frac{1}{\mathfrak{m}} \left[\left(\frac{\mathfrak{f}}{(\mathfrak{a}_0 \circ \mathfrak{f}_1)\mathfrak{m}_0} - 1 \right) - \left(\sum_{n \geq m_0 m} \frac{(\mathfrak{a}_n \circ \mathfrak{f}_1)}{(\mathfrak{a}_0 \circ \mathfrak{f}_1)} \frac{\mathfrak{n}}{\mathfrak{m}_0} - 1 \right) \right] \right| = 0$$

$$\Rightarrow \lim_{|z| \to +\infty} \left| \frac{\boldsymbol{\varepsilon} - \mathfrak{E}_m}{\mathfrak{m}} \right| = 0$$

$$\Rightarrow |\boldsymbol{\varepsilon} - \mathfrak{E}_m| = o(|\mathfrak{m}|)$$

The rest of the proof is similar to lemma 3.13. □

Lemma 3.26. *For all $f \in \mathcal{F}_{f^{<k>}}$, there exists $g \in \mathcal{A}_{f^{<k>}}$ and $m \in M(f_0, \dots, f_k)$ such that $f = \frac{g}{m}$ i.e.:*

$$\mathcal{F}_{f^{<k>}} = \frac{\mathcal{A}_{f^{<k>}}}{M(f_0, \dots, f_k)}$$

Hence, every element of $\mathcal{A}_{f^{<k>}}$ can be written as a monomial times a unit.

Proof. Let $f \in \mathcal{F}_{f^{<k>}} \setminus \{0\}$ with asymptotic expansion $F = \sum_{n \in M(f_0)} (a_n \circ f_1)n$ where $a_n \in \mathcal{F}_{f^{<k-1>}}$. An element of $\mathcal{F}_{f^{<k>}}$ is in $\mathcal{A}_{f^{<k>}}$ if it is bounded by lemma 3.25. Let m_0 be the leading monomial of F, then by induction, $a_0 = \frac{g_0}{q_0}$ for some $g_0 \in \mathcal{A}_{f^{<k-1>}}$ and $q_0 \in M(f_1 \circ f_1^{-1}, \dots, f_k \circ f_1^{-1})$. Hence, $a_0 \circ f_1 = \frac{g_0 \circ f_1}{q_0 \circ f_1}$ where $g_0 \circ f_1 \in \mathcal{A}_{f^{<k>}}$ and $q_0 \circ f_1 \in M(f_1, \dots, f_k)$. Since $f \asymp (a_0 \circ f_1)m_0$, $\frac{f}{(a_0 \circ f_1)m_0}$ is bounded so it is an element of $\mathcal{A}_{f^{<k>}}$. Now, let:

$$g := \frac{f}{(a_0 \circ f_1)m_0} \in \mathcal{A}_{f^{<k>}} \text{ and } m := \frac{q_0 \circ f_1}{m_0} \in M(f_0, \dots, f_k).$$

We obtain that $f = \frac{g}{m}$ as desired. □

Since we want to take the direct limit of all such algebras, we need to know that by doing the construction on a subsequence of $f_0 > f_1 > \dots > f_k$, we obtain a subalgebra of $\mathcal{F}_{f^{<k>}}$

Lemma 3.27. *For all $i \leq k$ and strictly increasing $\varphi : \{0, 1, \dots, i\} \to \{0, 1, \dots, k\}$ with $\varphi(0) = 0$:*

(1) $\mathcal{F}_{f_\varphi^{<i>}} \subset \mathcal{F}_{f^{<k>}}$ *where* $f_\varphi^{<i>} := (f_{\varphi(0)}, f_{\varphi(1)} \dots, f_{\varphi(i)})$

(2) $T_{f_\varphi^{<i>}} = T_{f^{<k>}}|_{\mathcal{F}_{f_\varphi^{<i>}}}$

Proof. Let $f \in \mathcal{F}_{f_\varphi^{<i>}}$, then $f \sim F$ for some $F = \sum_{n \in M(f_0)} (a_n \circ f_{\varphi(1)})n$ with $a_n \in \mathcal{F}_{f_\varphi^{<i-1>}}$ where $f_\varphi^{<i-1>} = (f_{\varphi(1)} \circ f_{\varphi(1)}^{-1}, f_{\varphi(2)} \circ f_{\varphi(1)}^{-1}, \dots, f_{\varphi(i)} \circ f_{\varphi(1)}^{-1})$.

(1) Let $b_n := a_n \circ f_{\varphi(1)} \circ f_1^{-1}$ so that $F = \sum_{n \in M(f_0)} (b_n \circ f_1)n$. To show that $f \in \mathcal{F}_{f^{<k>}}$, it suffices to prove that for all $n \in \mathrm{supp}(F), b_n \in \mathcal{F}_{f^{<k-1>}}$. We will start by showing that $\mathcal{F}_{f_\varphi^{<i-1>}} \subset \mathcal{F}_{f^{<k-\varphi(1)>}}$, where:

$$f^{<k-\varphi(1)>} = (f_{\varphi(1)} \circ f_{\varphi(1)}^{-1}, f_{\varphi(1)+1} \circ f_{\varphi(1)}^{-1}, \dots, f_k \circ f_{\varphi(1)}^{-1})$$

We define the following:

$$g_j := f_{\varphi(1)+j} \circ f_{\varphi(1)}^{-1} \text{ for } j \in \{0, 1, \dots, k - \varphi(1)\}$$

so that the following holds:

$$g_0 = f_{\varphi(1)} \circ f_{\varphi(1)}^{-1} > g_1 = f_{\varphi(1)+1} \circ f_{\varphi(1)}^{-1} > \cdots > g_{k-\varphi(1)} = f_k \circ f_{\varphi(1)}^{-1}.$$

Let $\psi : \{0, 1, \dots, i-1\} \to \{0, 1, \dots, k-1\}$ be the strictly increasing map $j \mapsto \varphi(j+1)$ for $j \geq 1$ and $\psi(0) := 0$ so that:

$$g_{\psi(0)} = f_{\varphi(1)} \circ f_{\varphi(1)}^{-1} > g_{\psi(1)} = f_{\varphi(2)} \circ f_{\varphi(1)}^{-1} > g_{\psi(2)} = f_{\varphi(3)} \circ f_{\varphi(1)}^{-1} > \cdots > g_{\psi(i-1)} = f_{\varphi(i)} \circ f_{\varphi(1)}^{-1}.$$

Hence, with $g_\psi^{<i-1>} = (g_{\psi(0)}, g_{\psi(1)}, \dots, g_{\psi(i-1)})$ and $g^{<k-\varphi(1)>} = (g_0, g_1, \dots, g_{k-\varphi(1)})$, we obtain by induction that:

$$\mathcal{F}_{g_\psi^{<i-1>}} \subset \mathcal{F}_{g^{<k-\varphi(1)>}}$$
$$\Leftrightarrow \mathcal{F}_{f_\varphi^{<i-1>}} \subset \mathcal{F}_{f^{<k-\varphi(1)>}}$$

Hence, $a_n \in \mathcal{F}_{f^{<k-\varphi(1)>}}$. Now, by definition, we have:

$$a_n \circ f_{\varphi(1)} \circ f_{\varphi(1)-1}^{-1} \in \mathcal{F}_{f^{<k-(\varphi(1)-1)>}}$$

where $f^{<k-(\varphi(1)-1)>} = (f_{\varphi(1)-1} \circ f_{\varphi(1)-1}^{-1}, f_{\varphi(1)} \circ f_{\varphi(1)-1}^{-1}, f_{\varphi(1)+1} \circ f_{\varphi(1)-1}^{-1}, \dots, f_k \circ f_{\varphi(1)-1}^{-1})$.

Iterating this process, we obtain that:

$$b_n = a_n \circ f_{\varphi(1)} \circ f_1^{-1} = \left(a_n \circ f_{\varphi(1)} \circ f_{\varphi(1)-1}^{-1}\right) \circ \left(f_{\varphi(1)-1} \circ f_{\varphi(1)-2}^{-1}\right) \circ \cdots \circ \left(f_2 \circ f_1^{-1}\right)$$
$$\in \mathcal{F}_{f^{<k-1>}}$$

(2) Given f and F as above, we have by definition:

$$T_{f_\varphi^{<i>}}(f) = \sum_{n \in M(f_0)} \left(\left(T_{f_\varphi^{<i-1>}}(a_n)\right) \circ f_{\varphi(1)}\right) n$$

and

$$T_{f^{<k>}}(f) = \sum_{n \in M(f_0)} \left(\left(T_{f^{<k-1>}}(b_n)\right) \circ f_1\right) n$$

With the same setting as above, we obtain by induction:

$$T_{g_\psi^{<i-1>}} = T_{g^{<k-\varphi(1)>}}|_{\mathcal{F}_{g_\varphi^{<i-1>}}}$$
$$\Leftrightarrow T_{f_\varphi^{<i-1>}} = T_{f^{<k-\varphi(1)>}}|_{\mathcal{F}_{f_\varphi^{<i-1>}}}$$

Hence, $T_{f_\varphi^{<i-1>}}(a_n) = T_{f^{<k-\varphi(1)>}}(a_n)$. Now, by definition,

$$\left[T_{f^{<k-\varphi(1)>}}(a_n)\right] \circ f_{\varphi(1)}$$
$$= \left[T_{f^{<k-(\varphi(1)-1)>}}(a_n \circ f_{\varphi(1)} \circ f_{\varphi(1)-1}^{-1})\right] \circ f_{\varphi(1)-1}$$

Iterating this process, we obtain that:

$$\left[T_{f^{<k-1>}}(b_n)\right] \circ f_1 = \left[T_{f^{<k-1>}}(a_n \circ f_{\varphi(1)} \circ f_1^{-1})\right] \circ f_1 = \cdots = \left[T_{f^{<k-\varphi(1)>}}(a_n)\right] \circ f_{\varphi(1)}$$

□

By the previous lemma, the set of all fields $\mathcal{F}_{f^{<k>}}$ is a directed set (with respect to inclusion on the set of all finite tuples $(f_0, ..., f_k)$) so taking the direct limit, we obtain a field $\mathcal{F}$ and a common extension T such that $(\mathcal{F}, \mathcal{M}, T)$ is a qaa field where $\mathcal{M} := \{\prod_{i=0}^{k} e^{\alpha_i f_i} \mid f_0 > \cdots > f_k$ verify $P1, P2, P3$ and $\alpha_i \in \mathbb{R}\}$.

Remark. In particular, $(\mathcal{F}, \mathcal{M}, T)$ is an extension of the qaa field constructed in [12] where the construction is done with $f_0 > \log > \log_2 > \dots$.

3.2 General case

In general, the e^{-f_i}'s do not have independent valuations and we need to modify the definition of asymptotic expansions to preserve stability under addition and multiplication. Let $f_0, f_1, \dots, f_k$ be elements of $\mathcal{H}_{\mathrm{an,exp}}$ verifying:

P1. $x = f_0 > f_1 > \cdots > f_k \succ 1$

P2. the f_i's are in $l+1$ distinct archimedean classes:

$$\underline{f_0 > \cdots > f_{k_1-1}} > \underline{f_{k_1} > \cdots > f_{k_2-1}} > \cdots > \underline{f_{k_{(l-1)}} > \cdots > f_{k_l-1}} > \underline{f_{k_l} > \cdots > f_k}$$

P3. for all $0 \le i < j \le k$ such that f_i and f_j are in distinct archimedean classes, $\mathrm{eh}(f_j \circ f_i^{-1}) \le 0$.

Lemma 3.28. *Let $\mathfrak{f} : \overline{\Omega} \to \mathbb{C}$ be a holomorphic map with Ω a standard quadratic domain. Then, for all $m \in M(f_0, \dots, f_{k_1-1})$, $|\mathfrak{f}| = o(|\mathfrak{m}|)$ implies $|\mathfrak{f}'| = o(|\mathfrak{m}|)$.*

Proof. Since $m \in M(f_0, \dots, f_{k_1-1})$, there exist $\alpha_i \in \mathbb{R}$ such that $m = e^{-\sum\limits_i \alpha_i f_i}$. Let $D > C$ be such that $\mathcal{V}_1(\Omega_D) \subset \Omega_C$ (see facts 2.29). Since all the $\mathfrak{f}_i$ are holomorphic, $\sum\limits_i \alpha_i \mathfrak{f}_i$ is continuous so for all $z \in \Omega_D$. Since each f_i is in the same archimedean class as f_0 and is an element of a Hardy field, there exists $c > 0$ such that $f_i \sim c f_0$. By facts 2.36, $\lim\limits_{|z| \to +\infty} \frac{\mathfrak{f}_i}{\mathfrak{f}_0} = c$ so each $\mathfrak{f}_i$ (and hence $\sum\limits_i \alpha_i \mathfrak{f}_i$) is in fact uniformly continuous. Hence, there exists $\delta > 0$ such that for all $\omega \in \mathbb{C}$ with $|z - \omega| < \delta$, $\left| \sum\limits_i \alpha_i \mathfrak{f}_i(z) - \sum\limits_i \alpha_i \mathfrak{f}_i(\omega) \right| < 1$. Let $\gamma := \min(1, \frac{\delta}{2})$ so that for $z \in \Omega_D$, $|z - \omega| = \gamma$ implies that $\omega \in \Omega_C$ and $|z - \omega| < \delta$. Hence, for $|z - \omega| = \gamma$, the following holds:

$$\begin{aligned}
&\left| \sum_i \alpha_i \mathfrak{f}_i(z) - \sum_i \alpha_i \mathfrak{f}_i(\omega) \right| < 1 \\
\Rightarrow\ & \mathrm{Re}\Big(\sum_i \alpha_i \mathfrak{f}_i(z)\Big) < \mathrm{Re}\Big(\sum_i \alpha_i \mathfrak{f}_i(\omega)\Big) + 1 \\
\Rightarrow\ & e^{-\mathrm{Re}(\sum_i \alpha_i \mathfrak{f}_i(z))} > e^{-1} e^{-\mathrm{Re}(\sum_i \alpha_i \mathfrak{f}_i(\omega))} \\
\Rightarrow\ & |\mathfrak{m}(z)| > e^{-1} |\mathfrak{m}(\omega)|
\end{aligned}$$

Using Cauchy's formula, we obtain:

$$\begin{aligned}
|\mathfrak{f}'(z)| &\leq \frac{\sup_{|z-\omega|=\gamma_z} |\mathfrak{f}(\omega)|}{\gamma} \\
\Rightarrow \frac{|\mathfrak{f}'(z)|}{|\mathfrak{m}(z)|} &\leq \frac{\sup_{|z-\omega|=\gamma} |\mathfrak{f}(\omega)|}{\gamma\, |\mathfrak{m}(z)|} \\
\Rightarrow \frac{|\mathfrak{f}'(z)|}{|\mathfrak{m}(z)|} &\leq \frac{\sup_{|z-\omega|=\gamma} |\mathfrak{f}(\omega)|}{\gamma e^{-1}\, |\mathfrak{m}(\omega)|}
\end{aligned}$$

As $|z| \to +\infty$ in Ω_D, $|\omega| \to +\infty$ in Ω_C and since $|\mathfrak{f}| = o(|\mathfrak{m}|)$ in Ω_C, the LHS goes to 0 and we obtain that $|\mathfrak{f}'| = o(|\mathfrak{m}|)$ in Ω_D as desired. □

Lemma 3.29. *Let $f \in \mathcal{H}_{\text{an,exp}}$ be in the same archimedean class as f_0 with* $\text{eh}(f) \leq 0$*, then the following holds:*

(1) *f^{-1} is in the same archimedean class as f_0.*

(2) *there exists $c > 0$ such that $\lim_{x\to+\infty} f'(x) = c$ and $\lim_{|z|\to+\infty \text{ in } \Omega} \mathfrak{f}'(z) = c$.*

Proof. (1) f is in the same archimedean class as f_0 implies that there exists $n \in \mathbb{N}$ such that:

$$\frac{1}{n} f \leq f_0 \leq nf \Rightarrow \frac{1}{n} f_0 \leq f^{-1} \leq nf_0$$

(2) Since f and f_0 are infinite elements of the Hardy field, the following holds:

$$\begin{aligned}
\frac{1}{n} f_0 \leq f \leq nf_0 &\Rightarrow \frac{1}{n} f_0 \preceq f \preceq nf_0 \\
&\Rightarrow \frac{1}{n} \preceq f' \preceq n \\
&\Rightarrow f' \asymp 1
\end{aligned}$$

Since f' is in a Hardy field, it is ultimately increasing, decreasing or constant. Hence, there exists $c > 0$ such that $\lim_{x\to+\infty} f'(x) = c$. By fact 2.34, we also obtain that $\lim_{|z|\to+\infty \text{ in } \Omega} \mathfrak{f}'(z) = c$.

□

Lemma 3.30. *Let $f_0 > f_1 > \cdots > f_k$ verify $P1, P2$ and $P3$. For all $h \in \mathcal{H}_{\text{an,exp}}$ geometrically pure of level 0 and $i \leq l$, if we insert h in $f^{<i>}$ and obtain the new tuple $h^{<i>} := (\underline{f_{k_{(l-i)}} \circ f^{-1}_{k_{(l-i)}}, \dots, h, \dots, f_{k_{(l-i+1)-1}} \circ f^{-1}_{k_{(l-i)}}}, \dots, f_k \circ f^{-1}_{k_{(l-i)}})$, then the tuple $h^{<l>} := (f_0, \dots, \underline{f_{k_{(l-i)}}, \dots, h \circ f_{k_{(l-i)}}, \dots, f_{k_{(l-i+1)-1}}}, \dots, f_k)$ also verifies $P1, P2$ and $P3$.*
In particular this holds for $h = f_i - ln(f_i')$.

Proof. Assume that $i = l$, then we just need to check that for all $j \geq k_1$, $\text{eh}(f_j \circ h^{-1}) \leq 0$. Since both f_j ($\text{level}(f_j) = 0$ and $P3$ implies that $\text{eh}(f_j) \leq 0$) and h are geometrically pure and less than f_0, it follows directly from fact 2.34(1). Assume that $i = l - 1$ (the rest of the proof will follow by induction), then we need to check that:

(1) $\text{eh}((h \circ f_{k_1}) \circ f_j^{-1}) \leq 0$ for all $0 \leq j \leq k_1 - 1$

(2) $\text{eh}(f_j \circ (h \circ f_{k_1})^{-1}) \leq 0$ for all $j \geq k_2$.

(1) By assumption, $\text{eh}(f_{k_1} \circ f_j^{-1}) \leq 0$ and $\text{eh}(h) \leq 0$ (it is geometrically pure of level 0) so by fact 2.32(2), we obtain the desired result.

(2) Since $\text{eh}(f_j \circ (h \circ f_{k_1})^{-1}) = \text{eh}((f_j \circ f_{k_1}^{-1}) \circ h^{-1})$, the conclusion follows from the base case ($\text{eh}(h^{-1}) \leq 0$ by fact 2.32(2)).

□

3.2.1 Base case: $l = 0$

We assume here that $f_0, \dots, f_k$ are in the same archimedean class and we let $M(f_0, \dots, f_k) :=< e^{-f_0}, \dots, e^{-f_k} >$.

Lemma 3.31. *Each $n \in M(f_0, \dots, f_k)$ has a holomorphic extension $\mathfrak{n} : \overline{\Omega_{C_n}} \to \mathbb{C}$ where Ω_{C_n} is a standard quadratic domain and for $n, m \in M(f_0, \dots, f_k)$, $n = o(m)$ implies $|\mathfrak{n}| = o(|\mathfrak{m}|)$ as $|z| \to +\infty$ in $\Omega_{\max(C_n, C_m)}$.*

Proof. For each $n \in M(f_0, \ldots, f_k)$, there exists $\alpha_i \in \mathbb{R}$ such that $n = e^{-\sum_{i=0}^{k} \alpha_i f_i}$. The conclusion follows directly from $P3$ and fact 2.34(2). $\square$

Definition 3.32. We define $\mathcal{A}_{f<0>}$ to be the set of germs at $+\infty$ of functions $f : \mathbb{R} \to \mathbb{R}$ such that:

(1) f has a bounded holomorphic extension $\mathfrak{f} : \overline{\Omega} \to \mathbb{C}$ where Ω is a standard quadratic domain

(2) there exists a series $F := \sum_{n \in M(f_0,\ldots,f_k)} a_n n$ with natural support included in $M(f_0, \ldots, f_k)^{\leq 1}$ where we only use positive coefficients for the monomials and $a_n \in \mathbb{R}$ such that:

$$\forall m \in M(f_0, \ldots, f_k), \left| \mathfrak{f} - \sum_{n \succeq m} a_n \mathfrak{n} \right| = o(|\mathfrak{m}|) \text{ as } |z| \to +\infty \text{ in } \Omega$$

In that case, we say that F is an *asymptotic expansion* of f and write $f \sim F$.

Lemma 3.33. *Let* $f \sim F = \sum_{n \in M} a_n n$ *and* $g \sim G = \sum_{n \in M} b_n n$ *be elements of* $\mathcal{A}_{f<0>}$. *Then, the following holds:*

(1) *For all* $n \in M$*, there are finitely many elements* $p \in \operatorname{supp}(F)$ *and* $q \in \operatorname{supp}(G)$ *such that* $pq = n$

(2) *The set* $\operatorname{supp}(FG) = \{n \mid \sum_{\substack{pq=n \\ p \in \operatorname{supp}(F) \\ q \in \operatorname{supp}(G)}} a_p b_q \neq 0\}$ *is natural and included in* $M^{\leq 1}$.

(3) *For all* $m \in M$*, there are finitely many* $n \in \operatorname{supp}(FG)$ *such that* $n \asymp m$.

Proof. The proofs of 1. and 2. are similar to lemma 3.4. For 3., if we take any $s \in M$ such that $s \prec m$, we obtain by naturality of $\operatorname{supp}(FG)$ that $\operatorname{supp}(FG) \cap [s, +\infty[\supset \{n \in \operatorname{supp}(FG) \mid n \asymp m\}$ is finite. $\square$

Proposition 3.34. $\mathcal{A}_{f<0>}$ *is an* $\mathbb{R}$*-algebra.*

Proof. The proofs for stability under scalar multiplication and addition are similar to lemma 3.4. For multiplication, we obtain that for all $m \in M$:

$$\left(\sum_{n \succeq m} a_n n\right)\left(\sum_{n \succeq m} b_n n\right) = \sum_{n \succeq m}\left(\sum_{pq=n} a_p b_q\right) n + \varepsilon$$

where $\sum_{pq=n} a_p b_q$ and $\varepsilon = \sum_{m \succ n \succeq m^2}\left(\sum_{pq=n} a_p b_q\right) n$ are finite sums (by lemma 3.33) and $|\boldsymbol{\varepsilon}| = o(|\mathfrak{m}|)$ (since $\mathrm{supp}(\varepsilon) \prec m$). The rest of the proof is similar.

□

Remark. Without valuation independence, elements of $\mathcal{A}_{f^{<0>}}$ do not have unique asymptotic expansions in general but distinct elements still have distinct asymptotic expansions by the Phragmén-Lindelöf principle.

Example 3.35. Let $f_0 = x > f_1 := x - 1$, then f_0 and f_1 verify properties $P_1, P2, P3$ and we have:

$$M(f_0, f_1) = < e^{-f_0}, e^{-f_1} > = < e^{-x}, ee^{-x} > = < e^{-x} > .$$

Consider $F := ee^{-f_0} - e^{-f_1} = 0$, then for all $m \prec e^{-x}, F_m = 0$ so $|0 - \mathfrak{F}_m| = 0 = o(|\mathfrak{m}|)$ and for all $m \succeq e^{-x}, F_m = F = 0$ so $|0 - \mathfrak{F}_m| = 0 = o(|\mathfrak{m}|)$ as well. Hence, $0 \sim F$ but $\mathrm{supp}(F) = \{e^{-f_0}, e^{-f_1}\} \neq \emptyset$.

Definition 3.36. Let $\mathcal{R}_{f^{<0>}}$ be the subring of $\mathbb{R}((M(f_0, \dots, f_k)^{\leq 1}))$ defined by:

$$\mathcal{R}_{f^{<0>}} := \{F \in \mathbb{R}((M(f_0, \dots, f_k)^{\leq 1})) \mid \exists f \in \mathcal{A}_{f^{<0>}}, f \sim F\}$$

and $\mathcal{O}_{f^{<0>}}$ be the set of asymptotic expansions of 0, i.e.

$$\mathcal{O}_{f^{<0>}} := \{F \in \mathcal{R}_{f^{<0>}} \mid 0 \sim F\}.$$

Lemma 3.37. *$\mathcal{O}_{f^{<0>}}$ is a prime ideal of the ring $\mathcal{R}_{f^{<0>}}$.*

Proof. Clearly, $\mathcal{O}_{f<0>}$ is a subring of $\mathcal{R}_{f<0>}$. Now, given $F \in \mathcal{O}_{f<0>}$ and $G \in \mathcal{R}_{f<0>}$ with $g \sim G$ for some $g \in \mathcal{A}_{f<0>}$, we obtain by proposition 3.34 that FG is an asymptotic expansion of $0g = 0$ so $FG \in \mathcal{O}_{f<0>}$. To prove that $\mathcal{O}_{f<0>}$ is a prime ideal, let $F, G \in \mathcal{R}_{f<0>}$ with $FG \in \mathcal{O}_{f<0>}$. Then, there exist $f, g \in \mathcal{A}_{f<0>}$ such that $f \sim F$ and $g \sim G$ and $fg \sim FG$. Since $0 \sim FG$ and distinct germs have distinct asymptotic expansions, we must have $fg = 0$. Since the field of germs of functions at $+\infty$ is an integral domain, $f = 0$ or $g = 0$, so $F \in \mathcal{O}_{f<0>}$ or $G \in \mathcal{O}_{f<0>}$ as desired. □

Lemma 3.38. *Let $f, g \in \mathcal{A}_{f<0>}$ with $f \sim F$ and $g \sim G$, then $f = g$ if and only if $F - G \in \mathcal{O}_{f<0>}$.*

Proof. One direction follows directly from the definition of $\mathcal{O}_{f<0>}$. For the other direction, assume that $F - G \in \mathcal{O}_{f<0>}$, then for all $m \in M(f_0, \ldots, f_k)$, $|\mathfrak{F}_m - \mathfrak{G}_m| = o(|\mathfrak{m}|)$. Hence, the following holds:

$$\begin{aligned} |\mathfrak{f} - \mathfrak{g}| &\leq |\mathfrak{f} - \mathfrak{F}_m| + |\mathfrak{F}_m - \mathfrak{G}_m| + |\mathfrak{g} - \mathfrak{G}_m| \\ &= o(|\mathfrak{m}|) + o(|\mathfrak{m}|) + o(|\mathfrak{m}|) = o(|\mathfrak{m}|) \end{aligned}$$

In particular, it holds for $m_j := e^{-jf_0} \in M(f_0, \ldots, f_k)$ ($j \in \mathbb{N}$) so by the Phragmén-Lindelöf principle (theorem 2.25), $|\mathfrak{f} - \mathfrak{g}| = 0$ in Ω so $f - g = 0$ as desired. □

Definition 3.39. We now consider equivalence classes of asymptotic expansions and we have the following bijection:

$$\begin{aligned} \tau_{f<0>} : \mathcal{A}_{f<0>} &\to \mathcal{R}_{f<0>}/\mathcal{O}_{f<0>} \\ f &\mapsto F + \mathcal{O}_{f<0>} \end{aligned}$$

We define $T_{f<0>} : \mathcal{A}_{f<0>} \to \mathbb{R}((M(f_0, \ldots, f_k)))/O_{f<0>}$ to be the map $f \mapsto F + O_{f<0>}$ where $O_{f<0>} := \mathcal{O}_{f<0>} \subset \mathbb{R}((M(f_0, \ldots, f_k)))$.

We also let $\Phi_{f<0>} : \mathcal{R}_{f<0>} \to \mathcal{A}_{f<0>}$ be the surjective map $F \mapsto f$ (it is well defined by lemma 3.38) where $T_{f^{<0>}}(f) = F + O_{f^{<0>}}$. Note that for all $F \in \mathcal{R}_{f^{<0>}}$, $T_{f^{<0>}}(\Phi_{f^{<0>}}(F)) = F + O_{f^{<0>}}$.

Remark. In the base case, since the coefficients of the series are just real numbers, $O_{f^{<0>}}$ and $\mathcal{O}_{f^{<0>}}$ are the same and $\tau_{f^{<0>}}$ is just a restriction of $T_{f^{<0>}}$.

Corollary 3.40. *$T_{f^{<0>}}$ is well-defined and injective.*

Proof. Follows directly from lemma 3.38. □

Lemma 3.41. (1) *$T_{f^{<0>}}(\mathcal{A}_{f^{<0>}})$ is truncation closed.*

(2) *For all $f \in \mathcal{A}_{f^{<0>}}, m \in M(f_0)$ and $F \in \mathbb{R}((M(f_0, \ldots, f_{k_1-1})))$ with $T_{f^{<0>}}(f) = F + O_{f^{<0>}}$,*

$$|\mathfrak{f} - \boldsymbol{\Phi}_{\boldsymbol{f}^{<0>}}(F_m)| = o(|\mathfrak{m}|) \text{ as } |z| \to +\infty \text{ in } \Omega.$$

Proof. Let $f \in \mathcal{A}_{f^{<0>}}$ with $f \sim F + \mathcal{O}_{f^{<0>}}$ where $F = \sum\limits_{n \in M(f_0,\ldots,f_k)} a_n n$ and fix $m \in M(f_0, \ldots, f_k)$.

- We need to show that there exists $g \in \mathcal{A}_{f^{<0>}}$ such that $T_{f^{<0>}}(g) = [T_{f^{<0>}}(f)]_m = \sum\limits_{n \succeq m} a_n n = F_m$. If $m \prec \operatorname{supp}(F)$, $F_m = F$ so we can just take $g := f \in \mathcal{A}_{f^{<0>}}$ and we are done.

  Otherwise, $F_m = \sum\limits_{n \succeq m} a_n n$ is a finite sum of bounded elements (since $\operatorname{supp}(F)$ is natural and included in $M(f_0, \ldots, f_k)^{\preceq 1}$) so $\mathfrak{F}_\mathfrak{m} := \sum\limits_{n \succeq m} a_n \mathfrak{n}$ is a bounded holomorphic extension to the closure of a standard quadratic domain.

  We want to show that we can take $g := F_m$ with itself as asymptotic expansion so we need to prove that for all $t \in M(f_0), |\mathfrak{F}_m - (\mathfrak{F}_m)_t| = o(|\mathfrak{t}|)$. If $t \preceq m$, then $(F_m)_t = F_m$ so the previous equality is trivial. If $t \succ m$, then $(F_m)_t = F_t$ and:

$$|\mathfrak{F}_m - (\mathfrak{F}_m)_t| \leq |\mathfrak{f} - \mathfrak{F}_m| + |\mathfrak{f} - \mathfrak{F}_t| = o(|\mathfrak{m}|) + o(|\mathfrak{t}|) = o(|\mathfrak{t}|)$$

Hence $g = F_m \in \mathcal{A}_{f^{<0>}}$ as desired.

- The proof follows directly from the definition of $\Phi_{f^{<0>}}$ and by a reasoning similar to lemma 3.11.

$\square$

Corollary 3.42. $(\mathcal{A}_{f^{<0>}}, M(f_0, \ldots, f_k), T_{f^{<0>}})$ *is a generalized quasianalytic asymptotic algebra.*

Lemma 3.43. *For all* $i \leq k$ *and strictly increasing* $\varphi : \{0, 1, \ldots, i\} \to \{0, 1, \ldots, k\}$ *with* $\varphi(0) = 0$*, the following holds:*

(1) $\mathcal{A}_{f_\varphi^{<0>}} \subset \mathcal{A}_{f^{<0>}}$

(2) $T_{f_\varphi^{<0>}} = T_{f^{<0>}}|_{\mathcal{A}_{f_\varphi^{<0>}}}$

Proof. Let $f \in \mathcal{A}_{f_\varphi^{<0>}}$, then $f \sim F$ for some $F = \sum\limits_{n \in M(f_0, f_{\varphi(1)} \ldots, f_{\varphi(i)})} a_n n$.

(1) To show that $f \in \mathcal{A}_{f^{<0>}}$, it suffices to prove that:

- For all $n \in \mathrm{supp}(F)$, $n \in M(f_0, \ldots, f_k)$
- For all $m \in M(f_0, \ldots, f_k)$, $\left|\mathfrak{f} - \sum\limits_{n \succeq m} a_n \mathfrak{n}\right| = o(|\mathfrak{m}|)$

The first point follows directly from $M(f_0, f_{\varphi(1)}, \ldots, f_{\varphi(i)}) \subset M(f_0, \ldots, f_k)$. For the second, we know that the equality holds for $m \in M(f_0, f_{\varphi(1)} \ldots, f_{\varphi(i)})$ and we need to show that it holds for all monomials in $M(f_0, f_1, \ldots, f_k)$ as well. Let $m \in M(f_0, f_1, \ldots, f_k) \setminus M(f_0, f_{\varphi(1)}, \ldots, f_{\varphi(i)})$. Then, since $\mathrm{supp}(F) \subset M(f_0, f_{\varphi(1)}, \ldots, f_{\varphi(i)})$, there exists $m' \in M(f_0, f_{\varphi(1)}, \ldots, f_{\varphi(i)})$ such that $m' \succ m$ and $]m, m'] \cap \mathrm{supp}(F) = \emptyset$ (which implies $F_m = F_{m'}$).

Hence, the following holds:

$$\begin{aligned}
&\left|\mathfrak{f} - \sum_{n \succeq m'} a_n \mathfrak{n}\right| = o(|\mathfrak{m}'|) \\
\Rightarrow &\left|\mathfrak{f} - \sum_{n \succeq m} a_n \mathfrak{n}\right| = o(|\mathfrak{m}'|) \text{ (since } F_m = F_{m'}) \\
\Rightarrow &\left|\mathfrak{f} - \sum_{n \succeq m} a_n \mathfrak{n}\right| = o(|\mathfrak{m}|) \text{ (since } m' \succ m \text{ implies } |\mathfrak{m}| = o(|\mathfrak{m}'|))
\end{aligned}$$

(2) By definition, we have:

$$T_{f_\varphi^{<0>}}(f) = F + O_{f_\varphi^{<0>}} \text{ and } T_{f^{<0>}}(f) = F + O_{f^{<0>}}$$

Since $\mathcal{O}_{f^{<0>}}$ lies over $\mathcal{O}_{f_\varphi^{<0>}}$ $(\mathcal{O}_{f^{<0>}} \cap \mathcal{R}_{f_\varphi^{<0>}} = \mathcal{O}_{f_\varphi^{<0>}})$, we obtain that $\mathcal{R}_{f_\varphi^{<0>}}/\mathcal{O}_{f_\varphi^{<0>}} \hookrightarrow \mathcal{R}_{f^{<0>}}/\mathcal{O}_{f^{<0>}}$. Indeed, it directly follows from the second isomorphism theorem:

$$\begin{aligned}
\mathcal{R}_{f_\varphi^{<0>}}/\mathcal{O}_{f_\varphi^{<0>}} &= \mathcal{R}_{f_\varphi^{<0>}}/\left(\mathcal{O}_{f^{<0>}} \cap \mathcal{R}_{f_\varphi^{<0>}}\right) \\
&\approx \left(\mathcal{R}_{f_\varphi^{<0>}} + \mathcal{O}_{f^{<0>}}\right)/\mathcal{O}_{f^{<0>}} \\
&\subset \mathcal{R}_{f^{<0>}}/\mathcal{O}_{f^{<0>}}
\end{aligned}$$

Hence, $\tau_{f_\varphi^{<0>}} = \tau_{f^{<0>}}|_{\mathcal{A}_{f_\varphi^{<0>}}}$ which implies that $T_{f_\varphi^{<0>}} = T_{f^{<0>}}|_{\mathcal{A}_{f_\varphi^{<0>}}}$ as desired.

□

Proposition 3.44. *Let $f \in \mathcal{A}_{f^{<0>}}$ such that $f \sim F = \sum_{n \in M(f_0,\ldots,f_k)} a_n n$, then $f' \in \mathcal{A}_{h^{<0>}}$ where $h^{<0>} = (f_0, f_1, \ldots, f_k, h_1, \ldots, h_k)$ for $h_i := \beta_i f_i - ln(f_i')$ with $\beta_i \in \mathbb{R}$ chosen to such that $f_k > h_1 > \cdots > h_k$ and $f' \sim F' = \sum_{n \in M(f_0,\ldots,f_k,h_1,\ldots,h_k)} a_n n'$.*

Proof. By lemma 3.29(2), for each $i \in \{1, \ldots, k\}$, $f_i - \ln(f_i')$ is in the same archimedean class as f_i. Hence, we can chose real numbers $\beta_i \geq 0$ such that

$f_k > \beta_1 f_1 - ln(f_1') > \cdots > \beta_k f_k - ln(f_k') \succ 1$. We now do a new construction with the functions $f_0 > f_1 > \cdots > f_k > h_1 > \cdots > h_k$ where $h_i := \beta_i f_i - ln(f_i')$ and obtain the generalized qaa algebra $\mathcal{A}_{h^{<0>}} \supset \mathcal{A}_{f^{<0>}}$ ($h^{<0>}$ verifies $P3$ by lemma 3.30).

For each $n \in M(f_0, \ldots, f_k)$, there exist $\alpha_i^n \geq 0$ such that $n = e^{-\sum_i \alpha_i^n f_i}$ so $n' = \left(-\sum\limits_i \alpha_i^n f_i'\right) n$ and:

$$F' = \sum_{n \in M(f_0,\ldots,f_k)} a_n \left(-\sum_i \alpha_i^n f_i'\right) n = \sum_{q \in M(f_0,\ldots,f_k,h_1,\ldots,h_k)} b_q q$$

where $q = f_i' n = e^{ln(f_i')} e^{-\sum\limits_{i=0}^{k} \alpha_i^n f_i} = e^{-\left(\sum\limits_{i=0}^{k} \alpha_i^n f_i - ln(f_i')\right)}$ and $b_q = -a_n \alpha_i^n$. Note that $\text{supp}(F') = \bigcup\limits_{0 \leq i \leq k} f_i' \, \text{supp}(F)$ is natural as well.

For all $m \in M(f_0, \ldots, f_k)$, we have $|\mathfrak{f} - \mathfrak{F}_{\mathfrak{m}}| = \left|\mathfrak{f} - \sum\limits_{n \succeq m} a_n \mathfrak{n}\right| = o(|\mathfrak{m}|)$ and we want to prove that:

$$\forall s \in M(f_0, \ldots, f_k, h_1, \ldots, h_k), |\mathfrak{f}' - (\mathfrak{F}')_s| = \left|\mathfrak{f}' - \sum_{q \succeq s} b_q \mathfrak{q}\right| = o(|\mathfrak{s}|)$$

For any $s \in M(f_0, \ldots, f_k, h_1, \ldots, h_k)$, there exists $\alpha_i^s, \gamma_i^s \geq 0$ such that $s = e^{-\left(\sum \alpha_i^s f_i - \sum \gamma_i^s ln(f_i')\right)} = e^{-\sum \alpha_i^s f_i} e^{\sum \gamma_i^s ln(f_i')}$. Since each $f_i' \asymp 1$, we also have $e^{\sum \gamma_i^s ln(f_i')} \asymp 1$. Hence, if we let $m := e^{-\sum \alpha_i^s f_i} \in M(f_0, \ldots, f_k)$, we have $s \asymp m$ so the following holds:

$$\sum_{q \succeq s} b_q q = \sum_{n \succeq m} \left(\sum_{i=0}^{k} b_{f_i' n} f_i' n\right) = \sum_{n \succeq m} \left(-\sum_{i=0}^{k} a_n \alpha_i^n f_i' n\right) = \left(\sum_{n \succeq m} a_n n\right)'$$

In other words, $(F')_s = (F_m)'$. By lemma 3.31, $s \asymp m$ implies $|\mathfrak{s}| \asymp |\mathfrak{m}|$ so we also obtain $(\mathfrak{F}')_s = (\mathfrak{F}_m)'$. Now, applying lemma 3.28, we have:

$$\begin{aligned} |\mathfrak{f} - \mathfrak{F}_m| = o(\mathfrak{m}) &\Rightarrow |\mathfrak{f}' - (\mathfrak{F}_m)'| = o(\mathfrak{m}) \\ &\Rightarrow |\mathfrak{f}' - (\mathfrak{F}')_s| = o(\mathfrak{s}) \end{aligned}$$

□

We cannot mimic the proof of the multiplicative inverse for the general case but we define the following

Definition 3.45. Let $\widetilde{\mathcal{F}}_{f<0>}$ be the set $\{\frac{g}{m} \mid g \in \mathcal{A}_{f<0>} \text{ and } m \in M(f_0, \ldots, f_k)\}$ and we extend $T_{f<0>}$ as follows:

$$\begin{aligned} \widetilde{T}_{f<0>} : \widetilde{\mathcal{F}}_{f<0>} &\to \mathbb{R}((M(f_0, \ldots, f_k)))/O_{f<0>} \\ f = \frac{g}{m} &\mapsto \frac{T_{f<0>}(g)}{m} + O_{f<0>} \end{aligned}$$

We also extend $\Phi_{f<0>}$ to $\widetilde{\Phi}_{f<0>} : \frac{\mathbb{R}((M(f_0,\ldots,f_{k_1})))}{M(f_0,\ldots,f_{k_1})} \to \widetilde{\mathcal{F}}_{f<0>}$ that maps $\frac{G}{m}$ to $\frac{g}{m}$ for g and G such that $T_{f<0>}(g) = G + O_{f<0>}$.

If the f_i's are valuation independent, $\widetilde{\mathcal{F}}_{f<0>}$ is a field but in the general case, it is an integral domain (it follows directly from the fact that $\mathcal{O}_{f<0>}$ is a prime ideal).

Lemma 3.46. $(\widetilde{\mathcal{F}}_{f<0>}, M(f_0, \ldots, f_k), \widetilde{T}_{f<0>})$ *is a generalized qaa integral domain.*

Proof. It is easy to check that $\widetilde{T}_{f<0>}$ is a well-defined injective $\mathbb{R}$-algebra homomorphism. Let $f \in \widetilde{\mathcal{F}}_{f<0>}$, then $f = \frac{g}{m_0}$ for some $g \in \mathcal{A}_{f<0>}$ and $m_0 \in M(f_0, \ldots, f_k)$. Since $\mathfrak{f} := \frac{\mathfrak{g}}{m_0}$ is a holomorphic extension of f to a standard quadratic domain, it only remains to show that for all $m \in M(f_0, \ldots, f_k)$ and $F \in \mathbb{R}((M(f_0, \ldots, f_k)))$ with $\widetilde{T}_{f<0>}(f) = F + O_{f<0>}$, the following holds:

$$\left| \mathfrak{f} - \widetilde{\boldsymbol{\Phi}}_{\boldsymbol{f}<0>}(F_m) \right| = o(|\mathfrak{m}|)$$

By definition of $\widetilde{T}_{f^{<0>}}$, $F = \frac{G}{m_0}$ where $g \sim G$. Using a similar argument to that of lemma 3.13, we obtain that $\left[\frac{G}{m_0}\right]_m = \frac{G_{m_0 m}}{m_0}$ which implies that:

$$\widetilde{\Phi}_{f^{<0>}}\left(\left[\frac{G}{m_0}\right]_m\right) = \frac{\Phi_{f^{<0>}}\left(G_{m_0 m}\right)}{m_0}$$

Hence, the following holds:

$$\begin{aligned}
&\left|\mathfrak{g} - \boldsymbol{\Phi}_{\boldsymbol{f}^{<0>}}\left(G_{m_0 m}\right)\right| = o(|\mathfrak{m}_0 \mathfrak{m}|) \\
\Rightarrow &\left|\frac{\mathfrak{g}}{\mathfrak{m}_0} - \frac{\boldsymbol{\Phi}_{\boldsymbol{f}^{<0>}}\left(G_{m_0 m}\right)}{\mathfrak{m}_0}\right| = o(|\mathfrak{m}|) \\
\Rightarrow &\left|\mathfrak{f} - \widetilde{\boldsymbol{\Phi}}_{\boldsymbol{f}^{<0>}}\left(F_m\right)\right| = o(|\mathfrak{m}|)
\end{aligned}$$

□

Notation. From now on, we will denote $\widetilde{T}_{f^{<0>}}$ by $T_{f^{<0>}}$.

3.2.2 Inductive step: $l > 0$

Definition 3.47. Let $\mathcal{A}_{f^{<l>}}$ be the set of germs at $+\infty$ of functions $f : \mathbb{R} \to \mathbb{R}$ such that:

(1) f has a bounded holomorphic extension $\mathfrak{f} : \overline{\Omega} \to \mathbb{C}$ where Ω is a standard quadratic domain

(2) there exists a series $F := \sum_{n \in M(f_0, \ldots, f_{k_1 - 1})} (a_n \circ f_{k_1}) n$ with natural support included in $M(f_0, \ldots, f_{k_1-1})^{\leq 1}$ where we only use positive coefficients for the monomials and $a_n \in \widetilde{\mathcal{F}}_{f^{<l-1>}}$ such that:

$$\forall m \in M(f_0, \ldots, f_{k_1-1}), \left| \mathfrak{f} - \sum_{n \succeq m} (\mathfrak{a}_{\mathfrak{n}} \circ \mathfrak{f}_{k_1}) \mathfrak{n} \right| = o(|\mathfrak{m}|)$$

In that case, we say that F is an *asymptotic expansion* of f and write $f \sim F$.

The construction of $\mathcal{A}_{f^{<l>}}$ is similar to the valuation independent case, except that we only shift when the element is in a different archimedean class. The first step is to construct $\mathcal{A}_{f^{<0>}}$ where $f^{<0>} = \underline{(f_{k_l} \circ f_{k_l}^{-1} > \cdots > f_k \circ f_{k_l}^{-1})}$ (which corresponds to the base case $k = 0$ described in the previous section). In the second step, the coefficients are elements of $\widetilde{\mathcal{F}}_{f^{<0>}}$ composed with $f_{k_l} \circ f_{k_l - 1}^{-1}$ and so on. In the last step, described in definition 3.47, to obtain the coefficients, we compose elements of $\widetilde{\mathcal{F}}_{f^{<k_l-1>}}$ with $f_{k_1} \circ f_0^{-1}$ $(= f_{k_1})$. For all these steps, the monomials are always comparable to e^{-x}.

step l: construction of $\mathcal{A}_{f^{<l>}}$

$$f \sim \sum_{n \in M(f_0, f_1, \ldots, f_{k_1 - 1})} (a_n \circ f_{k_1} \circ f_0^{-1})n = \sum (a_n \circ f_{k_1})n, \text{ where } a_n \in \widetilde{\mathcal{F}}_{f^{<l-1>}}$$

$$f^{<l>} = (\underline{f_0, f_1, \ldots, f_{k_1 - 1}}, \underline{f_{k_1}, \ldots, f_{k_2 - 1}}, \ldots, \underline{f_{k_l}, \ldots, f_k})$$

$$\circ(f_{k_1} \circ f_0^{-1})^{-1} \Big\downarrow \equiv \circ f_{k_1}^{-1}$$

step $l-1$: construction of $\mathcal{A}_{f^{<l-1>}}$

$$f \sim \sum_{n \in M(f_{k_1} \circ f_{k_1}^{-1}, \ldots, f_{k_2 - 1} \circ f_{k_1}^{-1})} (a_n \circ f_{k_2} \circ f_{k_1}^{-1})n, \text{ where } a_n \in \widetilde{\mathcal{F}}_{f^{<l-2>}}$$

$$f^{<l-1>} = (\underline{f_{k_1} \circ f_{k_1}^{-1} > \cdots > f_{k_2 - 1} \circ f_{k_1}^{-1}} > \cdots > \underline{f_{k_l} \circ f_{k_1}^{-1} > \cdots > f_k \circ f_{k_1}^{-1}})$$

$$\circ(f_{k_2} \circ f_{k_1}^{-1})^{-1} \Big\downarrow \equiv \circ f_{k_1} \circ f_{k_2}^{-1}$$

step $l-2$: construction of $\mathcal{A}_{f^{<l-2>}}$

$$f^{<l-1>} = (\underline{f_{k_2} \circ f_{k_2}^{-1} > \cdots > f_{k_3 - 1} \circ f_{k_2}^{-1}} > \cdots > \underline{f_{k_l} \circ f_{k_2}^{-1} > \cdots > f_k \circ f_{k_2}^{-1}})$$

$$\vdots$$

step 1: construction of $\mathcal{A}_{f^{<1>}}$

$$f \sim \sum_{n \in M(f_{k_{(l-1)}} \circ f_{k_{(l-1)}}^{-1}, \ldots, f_{k_l - 1} \circ f_{k_{(l-1)}}^{-1})} (a_n \circ f_{k_l} \circ f_{k_{(l-1)}}^{-1})n, \text{ where } a_n \in \widetilde{\mathcal{F}}_{f^{<0>}}$$

$$f^{<1>} = (\underline{f_{k_{(l-1)}} \circ f_{k_{(l-1)}}^{-1} > \cdots > f_{k_l - 1} \circ f_{k_{(l-1)}}^{-1}} > \underline{f_{k_l} \circ f_{k_{(l-1)}}^{-1} > \cdots > f_k \circ f_{k_{(l-1)}}^{-1}})$$

$$\circ(f_{k_l} \circ f_{k_{(l-1)}}^{-1})^{-1} \Big\downarrow \equiv \circ f_{k_{(l-1)}} \circ f_{k_l}^{-1}$$

step 0: construction of $\mathcal{A}_{f^{<0>}}$

$$f \sim \sum_{n \in M(f_{k_l} \circ f_{k_l}^{-1}, \ldots, f_k \circ f_{k_l}^{-1})} a_n n, \text{ where } a_n \in \mathbb{R}$$

$$f^{<0>} = (\underline{f_{k_l} \circ f_{k_l}^{-1} > \cdots > f_k \circ f_{k_l}^{-1}})$$

Similarly to the valuation independent case, we need the coefficients to be incomparable to the monomials.

Lemma 3.48. *Let $f \sim F = \sum(a_n \circ f_{k_1})n$ be as in definition 3.47, then for all $n \in \operatorname{supp}(F)$ and $\alpha > 0$, $e^{-\alpha \operatorname{Re}(z)} = o(|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_{k_1}(z)|)$ and $|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_{k_1}(z)| = o(e^{\alpha \operatorname{Re}(z)})$.*

Proof. Let $n \in M(f_0, \dots, f_{k_1-1})$. Since $a_n \in \widetilde{\mathcal{F}}_{f^{<l-1>}}$, $a_n = \frac{g_n}{m_n}$ for some $g_n \in \mathcal{A}_{f^{<l-1>}}$ and $m_n \in M(f_{k_1} \circ f_{k_1}^{-1}, \dots, f_{k_2-1} \circ f_{k_1}^{-1})$. Hence, $g_n \sim G$ for some $G = \sum(b_q \circ f_{k_2} \circ f_{k_1}^{-1})q$ where $b_q \in \widetilde{\mathcal{F}}_{f^{<l-2>}}$. Let q_0 be the leading monomial of G, then by induction, for all $q \asymp q_0$:

$$e^{-\beta \operatorname{Re}(z)} = o(\left|\mathfrak{b}_\mathfrak{q} \circ \mathfrak{f}_{k_2} \circ \mathfrak{f}_{k_1}^{-1}(z)\right|) \text{ and } \left|\mathfrak{b}_\mathfrak{q} \circ \mathfrak{f}_{k_2} \circ \mathfrak{f}_{k_1}^{-1}(z)\right| = o(e^{\beta \operatorname{Re}(z)}) \text{ for all } \beta > 0$$

which implies:

(1) $\left|\mathfrak{b}_\mathfrak{q} \circ \mathfrak{f}_{k_2} \circ \mathfrak{f}_{k_1}^{-1}(z)\right| \geq |\mathfrak{q}_\mathfrak{o}(z)|$ and

(2) $e^{-\beta \operatorname{Re}(\mathfrak{f}_{k_1}(z))} = o(\left|\mathfrak{b}_\mathfrak{q} \circ \mathfrak{f}_{k_2}(z)\right|)$ and $\left|\mathfrak{b}_\mathfrak{q} \circ \mathfrak{f}_{k_2}(z)\right| = o(e^{\beta \operatorname{Re}(\mathfrak{f}_{k_1}(z))})$ for all $\beta > 0$

Since $f_0 \succ f_{k_1}$, $e^{-\beta f_{k_1}} \succ e^{-\alpha f_0}$ and $e^{\beta f_{k_1}} \prec e^{\alpha f_0}$ for all $\alpha > 0$ so $e^{-\alpha \operatorname{Re}(z)} = o(|\mathfrak{b}_\mathfrak{q} \circ \mathfrak{f}_{\mathfrak{k}_2}(z)|)$ and $|\mathfrak{b}_\mathfrak{q} \circ \mathfrak{f}_{\mathfrak{k}_2}(z)| = o(e^{\alpha \operatorname{Re}(z)})$ for all $\alpha > 0$. Hence, the following holds:

$$|\mathfrak{a}_\mathfrak{n}| \asymp \left|\sum_{q \asymp q_0} (\mathfrak{b}_\mathfrak{q} \circ \mathfrak{f}_{k_2} \circ \mathfrak{f}_{k_1}^{-1})\mathfrak{q}\right|$$

$$\Rightarrow |\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1| \asymp \left|\sum_{q \asymp q_0} (\mathfrak{b}_\mathfrak{q} \circ \mathfrak{f}_{k_2})(\mathfrak{q} \circ \mathfrak{f}_{k_1})\right|$$

$$\Rightarrow e^{-\alpha \operatorname{Re}(z)} \left|\sum_{q \asymp q_0} (\mathfrak{q} \circ \mathfrak{f}_{k_1}(z))\right| = o(|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1(z)|) \text{ and}$$

$$|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_1(z)| = o\left(e^{\alpha \operatorname{Re}(z)} \left|\sum_{q \asymp q_0} (\mathfrak{q} \circ \mathfrak{f}_{k_1}(z))\right|\right) \text{ for all } \alpha > 0$$

If $q_0 = 1$,the rest of the proof is similar to lemma 3.48. Otherwise, each q is comparable to e^{-x} (and by lemma 3.31 so are each $|\mathfrak{q}|$) so $|\mathfrak{q} \circ \mathfrak{f}_{k_1}(z)| = o(e^{-\alpha \operatorname{Re}(z)})$

for all $\alpha > 0$. In both cases, we obtain $e^{-\alpha \operatorname{Re}(z)} = o(|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_{k_1}(z)|)$ and $|\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_{k_1}(z)| = o(e^{\alpha \operatorname{Re}(z)})$ as desired. $\square$

Corollary 3.49. *Let $f \sim F = \sum(a_n \circ f_{k_1})n$ be as in definition 3.47, then for all $n \in \operatorname{supp}(F)$:*

(1) $o(|\mathfrak{n}|)$ *implies* $o(|(\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_{k_1}|)\mathfrak{n})$,

(2) $|(\mathfrak{a}_\mathfrak{n} \circ \mathfrak{f}_{k_1})\mathfrak{n}|$ *is bounded.*

Proof. The proof is similar to lemma 3.17.

$\square$

Lemma 3.50. *Let $f \sim F = \sum(a_n \circ f_1)n$ and $g \sim G = \sum(b_n \circ f_1)n$ be elements of $\mathcal{A}_{f^{<l>}}$. Then, the following holds:*

(1) *For all $n \in M(f_0, \dots, f_{k_1-1})$, there are finitely many elements $p \in \operatorname{supp}(F)$ and $q \in \operatorname{supp}(G)$ such that $pq = n$*

(2) *The set* $\operatorname{supp}(FG) = \{n \mid \sum_{\substack{pq=n \\ p\in\operatorname{supp}(F) \\ q\in\operatorname{supp}(G)}} a_p b_q \neq 0\}$ *is natural and included in* $M(f_0, \dots, f_k)^{\preceq 1}$

(3) *For all $m \in M(f_0, \dots, f_k)$, there are finitely many $n \in \operatorname{supp}(FG)$ such that $n \asymp m$.*

Corollary 3.51. $\mathcal{A}_{f^{<l>}}$ *is an $\mathbb{R}$-algebra.*

Definition 3.52. Let $\mathcal{R}_{f^{<l>}}$ be the ring:

$$\mathcal{R}_{f^{<l>}} := \{F \in \widetilde{\mathcal{F}}_{f^{<l-1>}} \circ f_{k_1}((M(f_0, \dots, f_{k_1-1})^{\preceq 1})) \mid \exists f \in \mathcal{A}_{f^{<l>}}, f \sim F\}$$

and $\mathcal{O}_{f^{<l>}}$ be the set of asymptotic expansions of 0, i.e.

$$\mathcal{O}_{f^{<l>}} := \{F \in \mathcal{R}_{f^{<l>}} \mid 0 \sim F\}.$$

We obtain similarly that $\mathcal{O}_{f<l>}$ is a prime ideal of the ring $\mathcal{R}_{f<l>}$.

Lemma 3.53. *Let $f, g \in \mathcal{A}_{f<l>}$ with $f \sim F$ and $g \sim G$, then $f = g$ if and only if $F - G \in \mathcal{O}_{f<l>}$.*

Proof. The proof is similar to 3.38. $\square$

Definition 3.54. We now consider equivalence classes of asymptotic expansions and we have the following bijection:

$$\begin{aligned} \tau_{f<l>} : \mathcal{A}_{f<l>} &\to \mathcal{R}_{f<l>}/\mathcal{O}_{f<l>} \\ f &\mapsto F + \mathcal{O}_{f<l>} \end{aligned}$$

We define the map $T_{f<l>}$ as follows:

$$\begin{aligned} T_{f<l>} : \mathcal{A}_{f<l>} &\to \mathbb{R}((M(f_0, \dots, f_k)))/O_{f<l>} \\ f &\mapsto \sum_{n \in M(f_0, \dots, f_{k_1 - 1})} \left(T_{f<l-1>}(a_n) \circ f_{k_1}\right) n + O_{f<l>} \end{aligned}$$

where $O_{f<l>}$ is defined inductively in a similar way:

$$O_{f<l>} := \{ \sum_{n \in M(f_0, \dots, f_{k_1 - 1})} \left(T_{f<l-1>}(a_n) \circ f_{k_1}\right) n \mid 0 \sim \sum (a_n \circ f_{k_1}) n \}$$

We also let $\widetilde{\mathcal{R}}_{f<l>} := \{F \in \mathbb{R}((M(f_0, \dots, f_k))) \mid \exists f \in \mathcal{A}_{f<l>}, T_{f<l>}(f) = F\}$ and define $\Phi_{f<l>} : \widetilde{\mathcal{R}}_{f<l>} \to \mathcal{A}_{f<l>}$ be the surjective map $F \mapsto f$ for $T_{f<l>}(f) = F + O_{f<l>}$. Note that it is well defined by lemma 3.53 and for all $F \in \widetilde{\mathcal{R}}_{f<l>}$, $T_{f<l>}(\Phi_{f<l>}(F)) = F + O_{f<l>}$.

Remark. $\tau_{f<l>}(f)$ is an equivalence class of series with support of order ω where the coefficients are in $\widetilde{\mathcal{F}}_{f<l-1>} \circ f_{k_1}$ and the monomials are elements of $M(f_0, \dots, f_{k_1 - 1})$ (comparable to e^{-x}) whereas $\mathcal{T}_{f<l>}(f)$ is an equivalence class of series with support of order ω^{l+1} where the coefficients are real numbers and the monomials are in all of $M(f_0, \dots, f_k)$.

Corollary 3.55. $T_{f^{<l>}}$ *is well-defined and injective.*

Proof. Follows directly from lemma 3.53. $\square$

Lemma 3.56. (1) $T_{f^{<l>}}(A_{f^{<l>}})$ *is truncation closed.*

(2) *For all* $f \in \mathcal{A}_{f^{<l>}}$, $m \in M(f_0, \dots, f_k)$ *and* $F \in \mathbb{R}((M(f_0, \dots, f_k)))$ *with* $T_{f^{<l>}}(f) = F + O_{f^{<l>}}$,

$$\left|\mathfrak{f} - \mathbf{\Phi}_{\boldsymbol{f}^{<l>}}(F_m)\right| = o(|\mathfrak{m}|) \text{ as } |z| \to +\infty \text{ in } \Omega.$$

Proof. Let $f \in \mathcal{A}_{f^{<l>}}$ with $f \sim F = \sum\limits_{n \in M(f_0, \dots, f_{k_1 - 1})} (a_n \circ f_{k_1}) n$ and fix $m \in M(f_0, \dots, f_k)$.

(1) We need to show that there exists $g \in \mathcal{A}_{f^{<l>}}$ such that $T_{f^{<l>}}(g) = \left[T_{f^{<l>}}(f)\right]_m$. Since $m \in M(f_0, \dots, f_k)$, $m = m_0 m_r$ for some $m_0 \in M(f_0, \dots, f_{k_1-1})$ (comparable to e^{-x})and $m_r \in M(f_{k_1}, \dots, f_k)$ (in a larger comparability class than e^{-x}). Hence, we have:

$$\begin{aligned}\left[T_{f^{<l>}}(f)\right]_m &= \left[\sum_{n \in M(f_0 \dots, f_{k_1-1})} \left(T_{f^{<l-1>}}(a_n) \circ f_{k_1}\right) n\right]_m \\ &= \sum_{n \succ m_0} \left(T_{f^{<l-1>}}(a_n) \circ f_{k_1}\right) n + \sum_{n \asymp m_0} \left[\left(T_{f^{<l-1>}}(a_n) \circ f_{k_1}\right)\right]_{m_r} n\end{aligned}$$

The set $\{n \in M(f_0 \dots, f_{k_1-1}) \,| n \asymp m_0\}$ is finite by lemma 3.50, say it is equal to $\{n_0, \dots, n_q\}$. By induction, for each n_j, there exists $h_j \in \widetilde{\mathcal{F}}_{f^{<l-1>}}$ such that:

$$T_{f^{<l-1>}}(h_j) \circ f_{k_1} = \left[\left(T_{f^{<l-1>}}(a_j) \circ f_{k_1}\right)\right]_{m_r} \text{ and } |\mathfrak{a}_j \circ \mathfrak{f}_{k_1} - \mathfrak{h}_j \circ \mathfrak{f}_{k_1}| = o(|\mathfrak{m}_r|)$$

Now, we want to prove that we can take $g := \sum\limits_{n \succ m_0} (a_n \circ f_1) n + \sum\limits_{0 \le j \le q} (h_j \circ f_{k_1}) n_j$ so we need g to have a bounded holomorphic extension to a standard quadratic domain. $\left|\sum_{n \succ m_0} (\mathfrak{a}_n \circ \mathfrak{f}_1)\mathfrak{n}\right|$ is a finite sum of bounded elements by corollary 3.17 and each $|(\mathfrak{h}_j \circ \mathfrak{f}_{k_1})\mathfrak{n}_j|$ is also bounded by corollary 3.17. Hence, $g \in \mathcal{A}_{f^{<l>}}$ with itself as asymptotic expansion and $T_{f^{<l>}}(\mathcal{A}_{f^{<l>}})$ is truncation closed as desired.

(2) The proof showing that $|\mathfrak{f} - \mathfrak{g}| = o(|\mathfrak{m}|)$ is similar to lemma 3.23.

$\square$

Lemma 3.57. *For all $i \leq k$ and strictly increasing $\varphi : \{0, 1, \ldots, i\} \to \{0, 1, \ldots, k\}$ with $\varphi(0) = 0$, the following holds:*

(1) $\mathcal{A}_{f_\varphi^{<l'>}} \subset \mathcal{A}_{f^{<l>}}$

(2) $T_{f_\varphi^{<l'>}} = T_{f^{<l>}}|_{\mathcal{A}_{f_\varphi^{<l'>}}}$

where $l' \leq l$ is the number of distinct archimedean classes of $\{f_{\varphi(0)}, f_{\varphi(1)}, \ldots, f_{\varphi(i)}\}$.

Proof. Let $f \in \mathcal{A}_{f_\varphi^{<l>}}$. Assume that the $f_{\varphi(j)}$'s are in the following l' archimedean classes:

$$\underline{f_{\varphi(0)} > f_{\varphi(1)} > \cdots > f_{\varphi(l_1-1)}} > \underline{f_{\varphi(l_1)} > \cdots > f_{\varphi(l_2-1)}} > \cdots > \underline{f_{\varphi(l')} > \cdots > f_{\varphi(i)}}$$

Then, $f \sim F$ for some $F = \sum_{n \in M(f_{\varphi(0)}, \ldots, f_{\varphi(l_1-1)})} (a_n \circ f_{\varphi(l_1)})n$ with $a_n \in \widetilde{\mathcal{F}}_{f_\varphi^{<l'-1>}}$ where $f_\varphi^{<l'-1>} = (f_{\varphi(l_1)} \circ f^{-1}_{\varphi(l_1)}, f_{\varphi(l_1+1)} \circ f^{-1}_{\varphi(l_1)} \ldots, f_{\varphi(i)} \circ f^{-1}_{\varphi(l_1)})$.

(1) $f_{\varphi(l_1)}$ may not be the first element of its archimedean class in the original sequence, i.e. the sequence could be of the form

$$\cdots > f_{k_j-1} > \underline{f_{k_j} > f_{k_j+1} > \cdots > f_{\varphi(l_1)} > \cdots > f_{k_{j+1}-1}} > f_{k_{j+1}} \cdots$$

We first compose a_n by $\circ f_{\varphi(l_1)} \circ f_{k_j}^{-1}$ to obtain that $a_n \circ f_{\varphi(l_1)} \circ f_{k_j}^{-1}$ is an element of $\widetilde{\mathcal{F}}_{f_t^{<l-j>}}$ where

$$f_t^{<l-j>} = (f_{k_j} \circ f_{k_j}^{-1}, f_{\varphi(l_1)} \circ f_{k_j}^{-1}, \ldots, f_{\varphi(l_2-1)} \circ f_{k_j}^{-1}).$$

In order to obtain the rest of the elements in the same archimedean class of $f_{\varphi(l_1)}$, we use the same reasoning as in lemma 3.43 and obtain that $a_n \circ f_{\varphi(l_1)} \circ f_{k_j}^{-1}$ is an element of $\widetilde{\mathcal{F}}_{f^{<l-j>}}$ where

$$f^{<l-j>} = (f_{k_j} \circ f_{k_j}^{-1}, \dots, f_{\varphi(l_1)} \circ f_{k_j}^{-1}, \dots, f_k \circ f_{k_j}^{-1}).$$

We now make successive compositions $\left(a_n \circ f_{\varphi(l_1)} \circ f_{k_j}^{-1}\right) \circ \left(f_{k_j} \circ f_{k_{j-1}}^{-1}\right) \circ \cdots \circ \left(f_{k_2} \circ f_{k_1}^{-1}\right)$ and at each composition, we repeat the argument of lemma 3.43 to obtain all the elements in the same archimedean class. We then obtain that $a_n \circ f_{\varphi(l_1)} \circ f_{k_1}^{-1}$ is an element of $\widetilde{\mathcal{F}}_{f^{<l>}}$ as desired.

(2) The proof is similar.

□

Lemma 3.58. *Let $f \in \mathcal{A}_{f^{<l>}}$ be such that $f \sim F = \sum\limits_{n \in M(f_0, \dots, f_{k_1 - 1})} (a_n \circ f_{k_1}) n$, where $a_n \in \widetilde{\mathcal{F}}_{f^{<l-1>}}$. Then, $f' \in \mathcal{A}_{h^{<l>}}$ for some $h^{<l>} \supset f^{<l>}$ and the following holds:*

$$f' \sim F' = \sum_{n \in M(f_0, \dots, f_{k_1 - 1})} \left[f'_{k_1} (a'_n \circ f_{k_1}) n + (a_n \circ f_{k_1}) n'\right]$$

Proof. For each $n \in M(f_0, \dots, f_{k_1 - 1})$, there exist $\alpha_i^n \geq 0$ such that $n = e^{-\sum\limits_{i=0}^{k_1-1} \alpha_i^n f_i}$ so $n' = \left(-\sum\limits_{i=0}^{k_1-1} \alpha_i^n f'_i\right) n$ and:

$$F' = \sum_{n \in M(f_0, \dots, f_{k_1 - 1})} \left[f'_{k_1} (a'_n \circ f_{k_1}) n + (a_n \circ f_{k_1}) \left(-\sum_{i=0}^{k_1-1} \alpha_i^n f'_i\right) n\right]$$

Hence, $\operatorname{supp}(F') = \operatorname{supp}(F) \cup \bigcup\limits_{i=1}^{k_1-1} f'_i n$ (note that it is natural as well).
In order to obtain the desired result, we need:

(1) Each $f'_i n$ to be a monomial

(2) Each $f'_{k_1}(a'_n \circ f_{k_1})$ to be of the form $b_n \circ f_{k_1}$ where b_n is in the set of coefficients i.e. we want $(f'_{k_1} \circ f_{k_1}^{-1})a'_n$ to be in the set of coefficients.

- **Step 1: Enlarge the monomial set to contain $f'_i n$**

  For $i \in \{1, \ldots, k_1 - 1\}$, $f_i - ln(f'_i)$ is in the same archimedean class as f_0 and in a larger archimedean class than f_{k_1} so we can choose real numbers $\beta_i > 0$ such that:

$$\underline{f_0 > \cdots > f_{k_1-1} > \beta_1 f_1 - ln(f'_1) > \cdots > \beta_{k_1-1} f_{k_1-1} - ln(f'_{k_1-1})} > f_{k_1} > \cdots > f_k$$

  Then, if we do the construction for $\mathcal{A}_{h_1^{<l>}}$ where

$$h_1^{<l>} := (\underline{f_0, \ldots, f_{k_1-1}, h_1, \ldots, h_{k_1-1}}, f_{k_1}, \ldots, f_k)$$

  with $h_i := \beta_i f_i - ln(f'_i)$ for $i \in \{1, \ldots, k_1 - 1\}$, the new monomial set will be $M(f_0, \ldots, f_{k_1-1}, h_1, \ldots, h_{k_1-1})$ and will contain supp(F').

- **Step 2: Enlarge the set of coefficients to contain $(f'_{k_1} \circ f_{k_1}^{-1})a'_n$**

  - *Step 2a: Enlarge the set of coefficients to contain a'_n*

    $a_n \in \widetilde{\mathcal{F}}_{f^{<l-1>}}$ means that $a_n = \frac{g_n}{m_n}$ for some $g_n \in \mathcal{A}_{f^{<l-1>}}$ and $m_n \in M(f_{k_1} \circ f_{k_1}^{-1}, \ldots, f_k \circ f_{k_1}^{-1})$. By induction, $g'_n \in \mathcal{A}_{h_2^{<l-1>}}$ for some $h_2^{<l-1>}$ equal to:

$$(\underline{f_{k_1} \circ f_{k_1}^{-1}, \ldots, f_{k_2-1} \circ f_{k_1}^{-1}, h_{k_1+1} \circ f_{k_1}^{-1}, \ldots, h_{k_2-1} \circ f_{k_1}^{-1}}, f_{k_2} \circ f_{k_1}^{-1}, \ldots, f_k \circ f_{k_1}^{-1})$$

    where $h_j := \beta_j f_j - ln\left(\frac{f'_j}{f'_{k_1}}\right)$ for $j \in \{k_1 + 1, \ldots, k_2 - 1\}$ and β_j is a positive real number chosen such that:

$$\underline{f_{k_1} > \cdots > f_{k_2-1} > h_{k_1+1} > \cdots > h_{k_2-1}} > f_{k_2} > \cdots > f_k$$

It is possible to choose such a β_j because $f_j - ln\left(\frac{f'_j}{f'_{k_1}}\right)$ is in the same archimedean class as f_{k_1} and in a larger archimedean class than f_{k_2}. Then, the following holds:

$$\begin{aligned} h_j \circ f_{k_1}^{-1} =& \beta_j(f_j \circ f_{k_1}^{-1}) - ln\left(\frac{f'_j \circ f_{k_1}^{-1}}{f'_{k_1} \circ f_{k_1}^{-1}}\right) \\ =& \beta_j(f_j \circ f_{k_1}^{-1}) - ln\left((f_{k_1}^{-1})'(f'_j \circ f_{k_1}^{-1})\right) \\ =& \beta_j(f_j \circ f_{k_1}^{-1}) - ln\left((f_j \circ f_{k_1}^{-1})'\right) \text{ as desired} \end{aligned}$$

Now, $a'_n = \frac{g'_n \cdot m_n - m'_n \cdot g_n}{m_n^2}$. Since g_n, g'_n, m_n and m'_n are elements of $\mathcal{A}_{h_2^{<l-1>}}$, $g'_n \cdot m_n - m'_n \cdot g_n$ is also an element of this algebra and $m_n^2 \in M(f_{k_1} \circ f_{k_1^{-1}}, \ldots, f_k \circ f_{k_1^{-1}})$ which is a subset of

$$M(f_{k_1} \circ f_{k_1^{-1}}, \ldots, f_{k_2-1} \circ f_{k_1^{-1}}, h_{k_1+1} \circ f_{k_1^{-1}}, \ldots, h_{k_2-1} \circ f_{k_1^{-1}}, \ldots, f_k \circ f_{k-1}^{-1}).$$

– *Step 2b: Enlarge the set of coefficients to contain* $(f'_{k_1} \circ f_{k_1}^{-1})$

Since we want to consider $(f'_{k_1} \circ f_{k_1}^{-1})a'_n$, we can introduce another function $h_{k_1} := \beta_{k_1} f_{k_1} - ln(f'_{k_1})$ (also in the same archimedean class as f_{k_1}) where β_{k_1} is chosen such that $f_{k_2-1} > h_{k_1} > h_{k_1+1}$. Then,

$$\begin{aligned} h_{k_1} \circ f_{k_1}^{-1} =& \beta_{k_1}(f_{k_1} \circ f_{k_1}^{-1}) - ln(f'_{k_1} \circ f_{k_1}^{-1}) \\ =& \beta_{k_1} f_0 - ln(f'_{k_1} \circ f_{k_1}^{-1}) \end{aligned}$$

so we can write $f'_{k_1} \circ f_{k_1}^{-1}$ as an element of the algebra divided by a monomial:

$$f'_{k_1} \circ f_{k_1}^{-1} = \frac{e^{-\left(\beta_{k_1} f_0 - ln(f'_{k_1} \circ f_{k_1}^{-1})\right)}}{e^{-\beta_{k_1} f_0}}$$

Now we do the construction for:

$$\underline{f_0 > \cdots > f_{k_1-1} > h_1 > \cdots > h_{k_1-1}} > \underline{f_{k_1} > \cdots > f_{k_2-1} > h_{k_1} > h_{k_1+1} > \dots}$$
$$\underline{> h_{k_2-1}} > f_{k_2} > \cdots > f_k$$

and we have:

$$\begin{aligned} F' &= \sum_{n \in M(f_0,\dots,f_{k_1-1})} \left[f'_{k_1}(a'_n \circ f_{k_1})n + (a_n \circ f_{k_1}) \left(-\sum_i \alpha_i^n f'_i \right) n \right] \\ &= \sum_{q \in M(f_0,\dots,f_{k_1-1},h_1,\dots,h_{k_1-1})} (b_q \circ f_{k_1})q \end{aligned}$$

where b_q is an element of $\mathcal{A}_{h^{<l>}}$ ($h^{<l>}$ verifies $P3$ by lemma 3.30) with

$h^{<l>} := (f_{k_1} \circ f_{k_1}^{-1}, \dots, f_{k_2-1} \circ f_{k_1}^{-1}, h_{k_1+1} \circ f_{k_1}^{-1}, \dots, h_{k_2-1} \circ f_{k_1}^{-1}, f_{k_2} \circ f_{k_1}^{-1}, \dots, f_k \circ f_{k_1}^{-1})$ and

$$b_q = \begin{cases} (f'_{k_1} \circ f_{k_1}^{-1})a'_n & \text{if } q = n \text{ for some } n \in \operatorname{supp}(F) \\ -\alpha_i^n a_n & \text{if } q = f'_i n \text{ for some } n \in \operatorname{supp}(F) \text{ and } i \in \{1,\dots,k_1-1\} \end{cases}$$

For all $m \in M(f_0,\dots,f_{k_1-1})$, we have $|\mathfrak{f} - \mathfrak{F}_m| = \left| \mathfrak{f} - \sum_{n \succeq m} (\mathfrak{a}_n \circ \mathfrak{f}_{k_1})\imath\mathfrak{n} \right| = o(|\mathfrak{m}|)$ and we want to prove that for all $s \in M(f_0,\dots,f_{k_1-1},h_1,\dots,h_{k_1-1})$,

$$|\mathfrak{f}' - (\mathfrak{F}')_s| = \left| \mathfrak{f}' - \sum_{q \succeq s} (\mathfrak{b}_q \circ \mathfrak{f}_{k_1})\mathfrak{q} \right| = o(|\mathfrak{s}|)$$

.

By definition, there exists $\alpha_i^s, \gamma_i^s \geq 0$ such that $s = e^{-\sum_{i=0}^{k_1-1} \alpha_i^s f_i} e^{\sum_{i=1}^{k_1-1} \gamma_i^s ln(f'_i)} \asymp m$ where $m := e^{-\sum_{i=0}^{k_1-1} \alpha_i^s f_i} \in M(f_0,\dots,f_{k_1-1})$. Now, for any $q \in \operatorname{supp}(F')$, there exists $n \in \operatorname{supp}(F)$ such that $q = n$ or $q = f'_i n \asymp n$ for some $i \in \{1,\dots,k_1-1\}$. In both cases, $q \succeq s$ implies $n \succeq m$ so $(F')_s = (F_m)'$. Hence, $(\mathfrak{F}')_s = (\mathfrak{F}_m)'$ so applying lemma 3.28, we obtain the desired result:

$$\begin{aligned}|\mathfrak{f} - \mathfrak{F}_m| = o(|\mathfrak{m}|) &\Rightarrow |\mathfrak{f}' - (\mathfrak{F}_m)'| = o(|\mathfrak{m}|)\\ &\Rightarrow |\mathfrak{f}' - (\mathfrak{F}')_s| = o(|\mathfrak{m}|) = o(|\mathfrak{s}|)\end{aligned}$$

□

Definition 3.59. Let $\widetilde{\mathcal{F}}_{f^{<l>}}$ be the set $\{\frac{g}{m} \,|g \in \mathcal{A}_{f^{<l>}}$ and $m \in M(f_0, \dots, f_k)\}$ and we extend $T_{f^{<l>}}$ as follows:

$$\begin{aligned}\widetilde{T}_{f^{<l>}} : \widetilde{\mathcal{F}}_{f^{<l>}} &\to \mathbb{R}((M(f_0, \dots, f_k)))/O_{f^{<l>}}\\ f = \frac{g}{m} &\mapsto \frac{T_{f^{<l>}}(g)}{m} + O_{f^{<l>}}\end{aligned}$$

We obtain similarly that $(\widetilde{\mathcal{F}}_{f^{<l>}}, M(f_0, \dots, f_k), \widetilde{T}_{f^{<l>}})$ is a generalized qaa integral domain. Our objective is to obtain a field after taking the direct limit at the end and even if $\widetilde{\mathcal{F}}_{f^{<l>}}$ is just an integral domain, the multiplicative inverse will live in some larger algebra (obtained by enlarging the monomial set).

Lemma 3.60. *Let $f \in \mathcal{A}_{f^{<l>}} \setminus \mathcal{O}_{f^{<l>}}$, then there exists $h^{<l>} \supset f^{<l>}$ such that $\frac{1}{f} \in \widetilde{\mathcal{F}}_{h^{<l>}}$.*

Proof. For simplicity of the notation, we assume that $l = 1$ (the general case follows the same idea) i.e.

$$f^{<1>} = (\underline{f_0, \dots, f_{k_1-1}}, \underline{f_{k_1}, \dots, f_{k_2}}).$$

Let $f \in \mathcal{A}_{f^{<l>}} \setminus \mathcal{O}_{f^{<l>}}$, then $f \sim F$ for some $F = \sum\limits_{n \in M(f_0, \dots, f_{k_1-1})} (a_n \circ f_{k_1})n$ where $a_n \in \widetilde{\mathcal{F}}_{f^{<l-1>}}$. Let m_0 be the leading monomial of F, then:

$$\left|\mathfrak{f} - \sum_{n \preceq m_0} (\mathfrak{a}_n \circ \mathfrak{f}_{k_1})\mathfrak{n}\right| = o(|\mathfrak{m}_0|)$$

Now, for all n, $a_n \sim G_n$ for some $G_n = \sum\limits_{q \in M(f_{k_1} \circ f_{k_1}^{-1}, \ldots, f_{k_2 - 1} \circ f_{k_1}^{-1})} b_q q$, where $b_q \in \mathbb{R}$. Let q_n be the leading monomial of G_n so that the following holds:

$$\begin{aligned} &\forall n \asymp m_0, \left| \mathfrak{a}_n - \sum_{q \asymp q_n} b_q \mathfrak{q} \right| = o(|\mathfrak{q}_n|) \\ \Rightarrow &\forall n \asymp m_0, \left| \mathfrak{a}_n \circ \mathfrak{f}_{k_1} - \sum_{q \asymp q_n} b_q (\mathfrak{q} \circ \mathfrak{f}_{k_1}) \right| = o(|\mathfrak{q}_n \circ \mathfrak{f}_{k_1}|) \end{aligned}$$

Let $\varepsilon_n := a_n \circ f_{k_1} - \sum\limits_{q \asymp q_n} b_q(q \circ f_{k_1})$, then ε_n has a holomorphic extension to a standard quadratic domain $\boldsymbol{\varepsilon_n} := \mathfrak{a}_n \circ \mathfrak{f}_{k_1} - \sum\limits_{q \asymp q_n} b_q(\mathfrak{q} \circ \mathfrak{f}_{k_1})$ and the following holds:

$$\begin{aligned} &|\boldsymbol{\varepsilon_n}| = o(|\mathfrak{q}_n \circ \mathfrak{f}_{k_1}|) \\ \Rightarrow &|\boldsymbol{\varepsilon_n} \mathfrak{n}| = o(|(\mathfrak{q}_n \circ \mathfrak{f}_{k_1})\mathfrak{n}|) = o(|\mathfrak{n}|) = o(|\mathfrak{m}_0|) \text{ (by corollary 3.49 and since } n \asymp m_0) \end{aligned}$$

Now, we obtain:

$$\begin{aligned} &\left| \mathfrak{f} - \sum_{n \asymp m_0} (\mathfrak{a}_n \circ \mathfrak{f}_{k_1}) \mathfrak{n} \right| = o(|\mathfrak{m}_0|) \\ \Rightarrow &\left| \mathfrak{f} - \sum_{n \asymp m_0} (b_q(\mathfrak{q} \circ \mathfrak{f}_{k_1})\mathfrak{n} + \boldsymbol{\varepsilon_n} \mathfrak{n}) \right| = o(|\mathfrak{m}_0|) \\ \Rightarrow &\left| \mathfrak{f} - \sum_{n \asymp m_0} b_q(\mathfrak{q} \circ \mathfrak{f}_{k_1})\mathfrak{n} \right| = o(|\mathfrak{m}_0|) \text{ (since } |\boldsymbol{\varepsilon_n} \mathfrak{n}| = o(|\mathfrak{m}_0|)) \end{aligned}$$

Let $M := \sum\limits_{n \asymp m_0} b_q(q \circ f_{k_1})n$ (we can assume that $M \neq 0$ since we are working modulo $\mathcal{O}_{f^{<l>}}$). Since each monomial in the finite (by lemma 3.50) sum has the same valuation as m_0 (comparable to e^{-f_0} by definition), there exists $k \in \mathbb{N}$ such that $M \leq e^{-kf_0}$. On the other hand, M verifies the hypotheses of the Phragmén-Lindelöf principle so if $M = o(e^{-jf_0})$ for all $j \in \mathbb{N}$, M must be equal to 0 which contradicts our initial assumption. Hence, there exists $j \in \mathbb{N}$ such that $M \geq e^{-nf_0}$. Combining the two inequalities, we obtain that M is comparable to e^{-f_0}. Let $m' \prec m_0$ be such that $]m_0, m'] \cap \operatorname{supp}(F) = \emptyset$ so that the following implications hold:

$$\begin{aligned}\left|\mathfrak{f}-\sum_{n\succeq m'}(\mathfrak{a}_n\circ\mathfrak{f}_{k_1})\mathfrak{n}\right|=o(|\mathfrak{m}'|)&\Rightarrow\left|\mathfrak{f}-\sum_{n\asymp m_0}(\mathfrak{a}_n\circ\mathfrak{f}_{k_1})\mathfrak{n}\right|=o(|\mathfrak{m}'|)\\&\Rightarrow\left|\mathfrak{f}-\sum_{n\asymp m_0}b_q(\mathfrak{q}\circ\mathfrak{f}_{k_1})\mathfrak{n}\right|=o(|\mathfrak{m}'|)\\&\Rightarrow|\mathfrak{f}-\boldsymbol{M}|=o(|\mathfrak{m}'|)\end{aligned}$$

Now, $\frac{M}{m'}=\frac{m_0}{m'}\left(\sum_{n\asymp m_0}b_q(q\circ f_{k_1})\frac{n}{m_0}\right)$. Each $\frac{n}{m_0}\asymp c_n$ for some non-zero real number c_n and each $q\circ f_{k_1}$ is of smaller comparability class than e^{f_0}. Even if $\left(\sum_{n\asymp m_0}b_q(q\circ f_{k_1})\frac{n}{m_0}\right)$ tends to 0, it level at most 0 by facts 2.36(2) so in either case, $\sum_{n\asymp m_0}b_q(q\circ f_{k_1})\frac{n}{m_0}$ has a smaller comparability class than e^{f_0}. Since $\frac{m_0}{m'}\succ 1$ and is in the same comparability class as e^{f_0}, we obtain that $\frac{M}{m'}\succ 1$. Hence, $\frac{f}{M}-1\prec 1$ and $\operatorname{supp}(\frac{F}{M}-1)\prec 1$. Now, $f_M:=\ln(\frac{1}{M})$ is in the same archimedean class as f_0 and by fact 2.35(2), $f_M=g+h+c$ where $c\in\mathbb{R}$, g is geometrically pure with $\operatorname{level}(g)=\operatorname{level}(f_M)=0$ (and therefore $\operatorname{eh}(g)=0$), $h=0$ or $\operatorname{level}(h)>0$ in which case, $\lim e^{-h}=c_h$ for some $c_h>0$.

To simplify the notation, we write $M=\sum_{0\leq i\leq j}e^{-h_i}$ for some $j\in\mathbb{R}$ and h_i such that $\operatorname{eh}(h_i)\leq 0$. Then, the following holds:

$$\begin{aligned}M=e^{-f_M}&=e^{-g}\left(\sum_i e^{g-h_i}\right)\\ \Leftrightarrow e^{-h-c}&=\sum_i e^{g-h_i}\end{aligned}$$

Along the real line, $\lim_{x\to+\infty}e^{g-h_i}=c_i$ for some $c_i\in\mathbb{R}^{\geq 0}$ and $\lim_{x\to+\infty}e^{-h-c}=d$ for some $d\in\mathbb{R}^{>0}$. Hence, at least one the $c_i's$ is positive. By fact 2.36(2), $\lim_{|z|\to+\infty}e^{\mathfrak{g}-\mathfrak{h}_i}=c_i$ on Ω (since $\operatorname{eh}(g-h_i)\leq 0$). Since $\frac{\boldsymbol{M}}{e^{-\mathfrak{g}}}=\sum_i e^{\mathfrak{g}-\mathfrak{h}_i}$, we obtain that $|\boldsymbol{M}|\asymp|e^{-\mathfrak{g}}|$. Now, since g is geometrically pure of level 0, we can do the construction for $g^{<1>}:=(\underline{f_0,\ldots,f_{k_1-1},g},\underline{f_{k_1},\ldots,f_{k_2}})\supset f^{<1>}$ (the tuple $g^{<1>}$ verifies $P3$ by lemma 3.30 and we assumed without loss of generality that $f_{k_1-1}>$

g). e^{-g} is now a monomial in $M(f_0, \ldots, f_{k_1-1}, g)$ and the following holds in $\mathcal{A}_{g^{<1>}}$:

$$\begin{aligned} |\mathfrak{f} - \boldsymbol{M}| &= o(\left|e^{-\mathfrak{g}}\right|) \\ \Rightarrow \lim_{|z| \to +\infty} \left| \frac{\mathfrak{f} - \boldsymbol{M}}{e^{-\mathfrak{g}}} \right| &= 0 \end{aligned}$$

Since $|\boldsymbol{M}| \asymp |e^{-\mathfrak{g}}|$, there exists $a \in \mathbb{R} \setminus \{0\}$ such that $\lim\limits_{|z| \to +\infty} \left|\frac{\mathfrak{f}}{ae^{-\mathfrak{g}}} - 1\right| = 0$ on Ω. Let $\varepsilon := \frac{f}{ae^{-g}} - 1$, then $\varepsilon \prec 1$ and by repeating the same reasoning as in lemma 3.13, we obtain that $\varepsilon \sim E := \frac{F}{ae^{-g}} - 1$ and then $\frac{1}{1+\varepsilon} \sim \sum\limits_{k \in \mathbb{N}} E^k$ which implies that $\frac{1}{f} = \frac{ae^{-g}}{1+\varepsilon} \sim \frac{1}{F}$ as desired. $\square$

4 General Construction

We take any infinite elements $f_0 > f_1 > \cdots > f_k$ of $\mathcal{H}_{an,\exp}$.

(1) Using the results of theorem 2.35, we can decompose uniquely each f_i as a finite sum of geometrically pure functions $f_i = g_{i1} + \cdots + g_{ik_i}$ where each g_{ij} is either infinite, infinitesimal or equal to a constant.

(2) We remove constants, possible duplicates and if $g_{ij} \prec 1$, we replace it by $\frac{1}{g_{ij}} \succ 1$.

(3) Now, we order the set $\{g_{ij}\}_{1\leq i\leq k, 1\leq j\leq k_i}$ and obtain a longer sequence $g_0 > g_1 > \cdots > g_{k'} \succ 1$ (if $g_0 > x$, we compose the whole sequence by g_0^{-1}) which verifies $P1, P2$ and $P3$ by fact 2.34(2).

(4) Assume that the g_i's are in l distinct archimedean classes, then we construct the generalized qaa algebra $(\mathcal{A}_{g^{<l>}}, M(g_0, \ldots, g_{k'}), T_{g^{<l>}})$ where $g^{<l>} = (g_0, \ldots, g_{k'})$ as in definition 3.47.

Remark. $M(f_0, \ldots, f_k)$ is the multiplicative group generated by monomials of the form $e^{-\alpha_i f_i} = \prod_{1\leq j\leq k_i} e^{-\alpha_i g_{ij}}$ and is a subset of $M(g_0, \ldots, g_{k'})$. Elements of $\mathcal{A}_{g^{<l>}}$ have asymptotic expansions involving monomials in $M(g_0, \ldots, g_{k'})$ and the idea is to restrict ourselves to series whose support is included in $M(f_0, \ldots, f_k)$. Note that they form an $\mathbb{R}$-algebra because such series are stable under addition and multiplication.

Definition 4.1.

- We define the algebra $\mathcal{A}(f_0, \ldots, f_k)$ to be the elements of $\mathcal{A}_{g^{<l>}}$ that have at least one asymptotic expansion (modulo the ideal $O_{g^{<l>}}$) involving only monomials in $M(f_0, \ldots, f_k)$.

- Similarly, we define $O(f_0, \ldots, f_k)$ to be the elements of $O_{g^{<l>}}$ whose support is included in $M(f_0, \ldots, f_k)$. It is easy to check that it is a prime ideal of $\mathbb{R}((M(f_0, \ldots, f_k)))$

- We define $T_{(f_0,\dots,f_k)} : \mathcal{A}(f_0,\dots,f_k) \to \mathbb{R}((M(f_0,\dots,f_k)))/O(f_0,\dots,f_k)$ to be $T_{g<l>}|_{\mathcal{A}(f_0,\dots,f_k)}$ and we obtain that $T_{(f_0,\dots,f_k)}$ is an injective $\mathbb{R}-$algebra homomorphism.

- $(\mathcal{A}(f_0,\dots,f_k), M(f_0,\dots,f_k), T_{(f_0,\dots,f_k)})$ is now a generalized qaa algebra and we similarly define $\widetilde{\mathcal{F}}(f_0,\dots,f_k)$ as follows:

$$\widetilde{\mathcal{F}}(f_0,\dots,f_k) := \frac{\mathcal{A}(f_0,\dots,f_k)}{M(f_0,\dots,f_k)}$$

Lemma 4.2. $(\widetilde{\mathcal{F}}(f_0,\dots,f_k), M(f_0,\dots,f_k), T_{(f_0,\dots,f_k)})$ *is a generalized qaa integral domain.*

Proof. The proof is similar to lemma 3.46.

□

Theorem 4.3. *The direct limit of the integral domains described above,* $(\mathcal{F}, \mathcal{M}, \mathcal{T})$*, is a generalized qaa Hardy field.*

Proof. It follows from lemmas 3.60 and 3.58.

□

Conclusion

As mentioned in the introduction, the end goal is to extend the class $\mathcal{I}$ into a quasianalytic algebra and obtain o-minimality using a similar procedure to the one in [8]. One of the main challenges of the extension to a multivariable class was to determine what multivariable logarithmic asymptotic series are. The approach taken is to first enlarge the set of monomials to the set of all functions definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$ and then extend the construction to any curve $(f_1, f_2, \dots)$ where the f_i's have the same number of variables.

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