NEWTON-OKOUNKOV BODIES OF BOTT-SAMELSON & PETERSON VARIETIES

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Abstract

The theory of Newton-Okounkov bodies can be viewed as a generalization of the theory of toric varieties; it associates a convex body to an arbitrary variety (equipped with auxiliary data). Although initial steps have been taken for formulating geometric situations under which the Newton-Okounkov body is a rational polytope, there is much that is still unknown. In particular, very few concrete and explicit examples have been computed thus far.

In this thesis, we explicitly compute Newton-Okounkov bodies of some cases of Bott-Samelson and Peterson varieties (for certain classes of auxiliary data on these varieties). Both of these varieties arise, for instance, in the geometric study of representation theory.

Background on the theory of Newton-Okounkov bodies and the geometry of flag and Grassmannian varieties is provided, and well as background on Bott-Samelson varieties, Hessenberg varieties, and Peterson varieties. In the last chapter we also discuss how certain techniques developed in this thesis can be generalized. In particular, a generalization of the flat family of Hessenberg varieties constructed in Chapter 6, which may allow us to compute Newton-Okounkov bodies of more general Peterson varieties, is an ongoing collaboration with H. Abe and M. Harada.

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List of Abbreviations and Symbols

$\mathbb{N}$	Natural numbers (including 0): $0, 1, 2, \ldots$
$\mathbb{P}^n$	Complex projective space $\mathbb{P}^n := \mathbb{P}(\mathbb{C}^n)$
$O(1)$	Anti-tautological line bundle over projective space
T^k	Compact torus $T := T^k \cong (S')^k$
G	Complex, connected, reductive, algebraic group, with Lie algebra $\mathfrak{g}$
B	Borel subgroup of G, with Lie algebra $\mathfrak{b}$
H	Maximal torus of G, with Lie algebra $\mathfrak{h}$
P	Parabolic subgroup of G
N	Normalizer of H in G
W	Weyl group; $W = N/H$
G/B	General flag variety
X_w	Schubert variety corresponding to w, i.e. the closure of $B\tilde{w}B/B$ in G/B, where $w \in W$ and $\tilde{w}$ is a representative of w in N

r	Rank of G
Λ	the weight lattice of G
$\langle \cdot, \cdot \rangle$	Killing form on $\mathfrak{h}^*$
$\{\alpha_1, \ldots, \alpha_r\}$	the set of positive simple roots (with an ordering) with respect to the choices $H \subset B \subset G$
$\{\alpha_1^V, \ldots, \alpha_r^V\}$	Coroots corresponding to the set of simple roots; $\alpha^V := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$
s_α	Simple reflection corresponding to the simple root α; $s_\alpha : \Lambda_w \to \Lambda_w, \lambda \mapsto \langle \lambda, \alpha^V \rangle$. These simple reflections generate W, and they satisfy $s_k(\alpha_k) = -\alpha_k$
$\{\omega_1, \ldots, \omega_r\}$	Fundamental weights; the weights that satisfy $\langle \omega_i, \alpha_j^V \rangle = \delta_{i,j}$. Geometrically, these are the first weights met along the edges of the Weyl chamber
P_α	Minimal parabolic subgroup of G associated to the positive simple root α; $P_\alpha = B \cup B s_\alpha B$
F_α	Chevalley generator of the root subspace $\mathfrak{g}_{-\alpha}$
$Fl(\mathbb{C}^n)$	Full flag variety in Lie type A; $Fl(\mathbb{C}^n) \cong GL_n\mathbb{C}/B$
$Gr(k, n)$	Grassmannian variety of k-planes in $\mathbb{C}^n$
I	ℓ-tuple of positive integers $I := (i_1, \ldots, i_\ell)$
Z_I	Bott-Samelson variety corresponding to $I = (i_1, \ldots, i_\ell)$
m	ℓ-tuple of nonnegative integers $m := (m_1, \ldots, m_\ell)$
$O(m)$	Line bundle over Z_I corresponding to m

$\mathcal{T}(I,m)$	Set of standard tableaux of shape (I,m)
$\Theta(\mathcal{T}(I,m))$	Set of standard monomials of shape (I,m)
Pet_n	Peterson variety sitting inside of $Fl(\mathbb{C}^n)$

Chapter 1

Background

In this chapter we set the stage for the remainder of the thesis. First, we recall very briefly the theory of toric varieties, which partly motivated the development of the theory of Newton-Okounkov bodies. Second, we give a precise definition and the construction of Newton-Okounkov bodies for general algebraic varieties. Thirdly, because the two main classes of varieties studied in this thesis – Bott-Samelson varieties and Peterson varieties – are both intimately related to the flag variety, we recall some basic definitions and constructions related to the variety of flags in $\mathbb{C}^n$.

1.1 Motivation: the Theory of Toric Varieties

A central theme in algebraic geometry is to associate combinatorial objects to algebraic varieties. The combinatorics generally encodes certain geometric or topological data of the corresponding varieties. Developing such bridges between combinatorics and geometry can allow researchers to answer geometric questions by looking at the combinatorics, or vice versa. For example, in the study of toric varieties, this correspondence is “perfect” in the sense that there is a precise one-to-one correspondence between toric varieties and fans, or, in the symplectic-geometric version of this correspondence we use below, a one-to-one correspondence between symplectic toric manifolds (a special case of toric varieties) and so-called Delzant polytopes. In particular, given the combinatorial data of a fan (or Delzant polytope), there is an explicit way to construct its corresponding toric variety (or symplectic toric manifold). Thus, the geometry of the toric variety (e.g. its cohomology ring, Betti numbers, etc.) are completely encoded in the combinatorics of the corresponding fan. The study of toric varieties has deep connections with polyhedral geometry, commutative algebra, combinatorics, and symplectic geometry, among many other fields. Its elegant structure also makes it an invaluable tool in other areas of research such as coding theory, physics, and algebraic statistics.

Since the main topic of this thesis is not toric varieties, we do not give precise definitions. We refer the reader to [Ful93] and [CdS03] for more detailed treatments of what follows. To give the reader a sense of the objects under consideration, however, we give below a concrete example of a toric variety and its corresponding Delzant polytope.

Example 1.1. The complex projective plane $\mathbb{P}^2$ is a smooth projective toric variety and a symplectic toric manifold. The torus $T^2 = (\mathbb{C}^*)^2$ can act on $\mathbb{P}^2$ as follows: $t \cdot z = (t_1, t_2) \cdot [z_0 : z_1 : z_2] := [z_0 : t_1 z_1 : t_2 z_2]$. The T^2-fixed points $(\{z \in \mathbb{P}^2 | t \cdot z = z\})$ are $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$. There exists a function, $\mu : \mathbb{P}^2 \to \mathbb{R}^2$, called the *moment map* with special geometric properties [CdS03, §1.4 and §1.6]; in particular, the image of the moment map is precisely the Delzant polytope associated to $\mathbb{P}^2$ under the correspondence referenced above. In this example, it turns out that $\mu : \mathbb{P}^2 \to \mathbb{R}^2$ is given by

$\mu([z_0 : z_1 : z_2]) = \frac{1}{2|z|^2}(|z_1|^2, |z_2|^2)$, up to a constant. The image $\Delta := \mu(\mathbb{P}^2)$ is the Delzant polytope corresponding to $\mathbb{P}^2$ (see Figure 1.1). The vertices of Δ correspond to the T^2-fixed points: $\mu([1:0:0]) = (0,0), \mu([0:1:0]) = (1,0), \mu([0:0:1]) = (0,1)$. This is not a coincidence. In general, for X an arbitrary symplectic toric manifold with respect to a torus T and $\mu : X \to \mathfrak{t}^*$ a moment map for this T-action, the moment map μ takes points with a k-dimensional stabilizer to codimension-k faces of the moment map image $\mu(X) =: \Delta$, and the inverse of a point on the polytope consists of exactly one torus orbit.

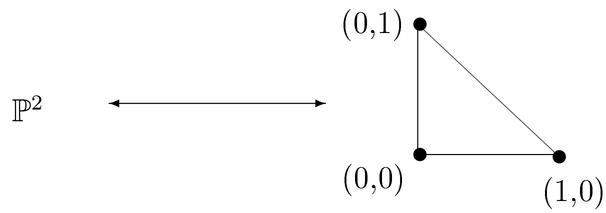


Figure 1.1: Delzant polytope of $\mathbb{P}^2$.

On the other hand, given a Delzant polytope Δ, we can reconstruct the symplectic toric manifold X using the so-called Delzant construction (see [Ham08] or [CdS03] for details), and in the case of $\mathbb{P}^2$ one can check that this Delzant construction is essentially the usual construction of projective space as the quotient of $S^5 \subseteq \mathbb{C}^3$ by scalar multiplication by a circle S^1.

As indicated above, in the setting of a toric variety, the geometry of X is fully encoded in its associated polytope $\Delta = \mu(X)$. Given a general projective variety X which is *not* a toric variety, one might ask:

Is it possible to associate to X some combinatorial object which encodes important geometric/ topological data of X (e.g. its degree, Betti numbers, cohomology ring, orbit types)? (1.1)

There exist some special situations where this is possible: for instance, in the *theory of Hamiltonian torus actions and moment maps*, and *Goresky-Kottwitz-MacPherson (GKM) theory*. However, these theories have some disadvantages. In particular, in both cases, they only apply to special types of varieties, and the correspondence between the varieties and associated combinatorial objects is not one-to-one. In the case of the theory of Hamiltonian torus actions and moment maps, this has to do with the fact that when a variety is not toric, its corresponding moment polytope is not *maximal-dimensional* (the (real) dimension of the torus is less than half of the (real) dimension of the manifold). An example which will be central to this thesis is given below.

Example 1.2. Let $Fl(\mathbb{C}^3)$ denote the variety of flags in $\mathbb{C}^3$, i.e. the set of nested sequences of subspaces $\{0 \subset V_1 \subset V_2 \subset \mathbb{C}^3 \mid \dim_{\mathbb{C}}(V_i) = i\}$. The flag $Fl(\mathbb{C}^3)$ is an example of a symplectic manifold which is *not* toric. Indeed, viewing the torus T^2 as the subgroup of the diagonal matrices in $SL_3\mathbb{C}$, we can define an action of T^2 on $Fl(\mathbb{C}^3)$ by acting by linear transformations on the set of flags in $\mathbb{C}^3$. Using the theory of Hamiltonian torus actions and moment maps, one can compute a moment map μ associated to this action, and it turns out that the corresponding moment polytope $\mu(Fl(\mathbb{C}^3))$ is as follows:

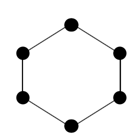


Figure 1.2: Moment polytope of $Fl(\mathbb{C}^3)$.

However, this polytope is *not* maximal-dimensional, since $\dim_{\mathbb{R}} Fl(\mathbb{C}^3) = 6$, but the moment polytope has $\dim_{\mathbb{R}} = 2$. Indeed, in this situation the torus T^2 is 2-dimensional, whereas the flag variety $Fl(\mathbb{C}^3)$ is (real) 6-dimensional.

In general, a partial answer to Question 1.1 is given by the theory of *Newton-Okounkov bodies*. Suppose X is an arbitrary projective algebraic variety over $\mathbb{C}$, and let $\dim_{\mathbb{C}} = n$. Building on the work of Okounkov [Oko96, Oko98], Kaveh-Khovanskii [KK12] and Lazarsfeld-Mustata [LM09] construct a convex body Δ in $\mathbb{R}^n$ associated to X equipped with the auxiliary data of a divisor D and a choice of valuation ν on the space of rational functions $\mathbb{C}(X)$. There are several advantages of this theory over other theories (such as Hamiltonian torus actions, etc.). First, it applies to an arbitrary projective algebraic variety. Second, under a mild hypothesis on the auxiliary data, the construction guarantees that the associated convex body Δ is maximal-dimensional. Thirdly, the construction also guarantees that the Euclidean volume of Δ in $\mathbb{R}^n$ gives the degree of the image of X under the Kodaira map corresponding to D; i.e. $\text{vol}\Delta = \frac{1}{n!}\text{deg}X$. Dave Anderson [And13] took Kaveh-Khovanskii and Lazarsfeld-Mustata's work a step further and developed a criterion for when a certain semigroup is finitely generated, which implies that the corresponding Okounkov body is a rational polytope. The facts above suggest that, in the situation when Δ is a rational polytope, the combinatorics of Δ should encode some important geometric data.

There are many open questions in Newton-Okounkov body theory. In particular, very few explicit examples of Newton-Okounkov bodies have been computed thus far. Therefore, it is an interesting problem to compute new concrete examples. In this thesis, we will explicitly compute the Newton-Okounkov bodies of certain Bott-Samelson and Peterson varieties, for certain choices of auxiliary data D and ν.

1.2 Newton-Okounkov Bodies

As we indicated above, the main results of this thesis are explicit computations of Newton-Okounkov bodies for certain varieties. In this section, we record our conventions and recall the basic construction of Newton-Okounkov bodies.

In what follows, we fix a projective irreducible variety X over $\mathbb{C}$, and set $n := \dim_{\mathbb{C}} X$. We also equip $\mathbb{Z}^n$ with a total order as follows.

Definition 1.3. *We define* ***lexicographic order*** *on $\mathbb{Z}^n$ as follows. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$. We say $\alpha > \beta$, with respect to lexicographic order, if the leftmost non-zero entry in $\alpha - \beta \in \mathbb{Z}^n$ is positive.*

Example 1.4. We have $(2, 1, 6) > (2, 1, 5)$, since $\alpha - \beta = (0, 0, 1)$. Also we have $(1, 4, 2) > (0, 5, 3)$, since $\alpha - \beta = (1, -1, -1)$.

In what follows, we always equip $\mathbb{Z}^n$ with lexicographic order. The essential insight in the theory of Newton-Okounkov bodies is that valuations (valued in $\mathbb{Z}^n$) can be used to translate problems concerning algebra into those about semigroups in $\mathbb{Z}^n$. By considering algebras naturally associated to varieties (e.g. homogeneous coordinate rings, or more generally, algebras of sections of a line bundle) we can therefore obtain a method to associate a semigroup to geometric data. To explain this in more detail, we now introduce the notion of a pre-valuation (a weaker version of a valuation, defined on a vector space) and then define the notion of a valuation (which, in our context, is defined on a $\mathbb{C}$-algebra).

Definition 1.5. *Let V be a vector space over $\mathbb{C}$. We say that a function $\nu : V \setminus \{0\} \to \mathbb{Z}^n$ is a* ***pre-valuation*** *with values in $\mathbb{Z}^n$ if for all $f, g \in V \setminus \{0\}$:*

(i) $\nu(f + g) \geq min\{\nu(f), \nu(g)\}$ for all $f, g \in V$ with f, g and $f + g$ all non-zero,

(ii) $\nu(kf) = \nu(f)$, *for all* $k \in \mathbb{C}^*$ *and* $f \neq 0$ *in* V.

Moreover, a pre-valuation is said to have ***one-dimensional leaves*** *if it satisfies:*

(iii) if $\nu(f) = \nu(g)$, *then* $\exists\ k \in \mathbb{C}$ *such that* $\nu(g - kf) > \nu(g)$ *OR* $g - kf = 0$.

The condition (iii) above can also be formulated as follows. For $\alpha \in \mathbb{Z}^n$, define $V_\alpha := \{f \in V \mid \nu(f) \geq \alpha \text{ or } f = 0\}$. This is a subspace of V. Also define the **leaf** $\hat{V}_\alpha$ **above** α **in** $\mathbb{Z}^n$ as the quotient vector space $\hat{V}_\alpha := V_\alpha / \cup_{\alpha<\beta} V_\beta$. Then condition (iii) is equivalent to the requirement that $\dim_{\mathbb{C}}(\hat{V}_\alpha) \leq 1$ for all α. This explains the terminology.

An elementary but important property of a pre-valuation with one-dimensional leaves is the following (see e.g. [Kav15, Proposition 1.9]) :

Proposition 1.6. *Let* V *be a* $\mathbb{C}$*-vector space and* $\nu : V \setminus \{0\} \to \mathbb{Z}^n$ *a pre-valuation (with values in* $\mathbb{Z}^n$*) with one-dimensional leaves. Let* $W \subseteq V$ *be a finite-dimensional subspace of* V*. Then* $\dim_{\mathbb{C}}(W) = \#\nu(W \setminus \{0\})$.

We now define a valuation on a $\mathbb{C}$-algebra, which is a pre-valuation which additionally behaves well with respect to the algebra structure.

Definition 1.7. *Let* A *be a* $\mathbb{C}$*-algebra. A pre-valuation on* A, $\nu : A \setminus \{0\} \to \mathbb{Z}^n$, *is said to be a* ***valuation*** *if it satisfies the additional property:*

(iv) $\nu(fg) = \nu(f) + \nu(g)$ *for all* $f, g \in A \setminus \{0\}$.

The image of A *in* $\mathbb{Z}^n$ *is a semigroup and is called the* ***value semigroup*** *of* (A, ν).

As mentioned above, in our geometric setting we will consider the algebra of sections of line bundles over our variety X. Moreover, we will deal with valuations on such algebras constructed in a specific and geometric way, as we now describe. Let

$$\{pt\} = Y_n \subset Y_{n-1} \subset \cdots \subset Y_1 \subset Y_0 = X$$

be a flag of irreducible subvarieties where $\dim(Y_i) = n - i$ and each Y_i is non-singular at the point Y_n. This is called an *admissible flag* [LM09, §1]. We denote such a flag by $Y_\bullet = \{Y_n \subset Y_{n-1} \subset \cdots \subset Y_1 \subset Y_0\}$. A *system of parameters* [Kav15] with respect to such a flag is a collection $y_1, \ldots, y_n$ of rational functions on X such that $y_k \mid_{Y_{k-1}}$ is a well-defined, not identically zero, rational function on Y_{k-1}, which has a zero of first order on Y_k.

Let L be a line bundle on X and k a positive integer. Given an admissible flag $Y_\bullet$ and a system of parameters $y_1, \ldots, y_n$ with respect to $Y_\bullet$, we can define a pre-valuation $\nu : H^0(X, L^{\otimes k}) \setminus \{0\} \to \mathbb{Z}^n$ as follows. For a non-zero section $s \in H^0(X, L^{\otimes k})$ we define $\nu(s) := (k_1, \ldots, k_n)$, where the k_i are inductively defined in the following way. First we define

$$k_1 := \mathrm{ord}_{Y_1}(s),$$

i.e. k_1 is the order of vanishing of s along Y_1. Define $s_1 := sy_1^{-k_1}\Big|_{Y_1}$. Note that s_1 is not identically zero on Y_1 by the definition of k_1. Then we define $k_2 :=\mathrm{ord}_{Y_2}(s_1)$, and so on.

Theorem 1.8. *[LM09, Lemma 1.3] The pre-valuation $\nu : H^0(X, L^{\otimes k}) \setminus \{0\} \to \mathbb{Z}^n$ constructed above has one-dimensional leaves.*

Let $\mathcal{R}_k$ denote the span of the image of the k-fold product $H^0(X, L) \times \cdots \times H^0(X, L)$ in $H^0(X, L^{\otimes k})$, under the natural map given by taking the tensor product of sections. Let $R = \bigoplus_{k \geq 0} \mathcal{R}_k$ denote the graded $\mathbb{C}$-algebra of sections of L. It is not difficult to see that the pre-valuation ν defined above specifies a valuation, by slight abuse of notation also denoted ν, on R [Kav15, Prop 1.11]. Given our valuation ν, we may define the additive semigroup

$$S(R,\nu) := \bigcup_{k>0} \{(k,\nu(s)) | s \in \mathcal{R}_k \setminus \{0\}\} \subset \mathbb{N} \times \mathbb{Z}^n.$$

We then define the convex body Δ by

$$\Delta := \overline{\operatorname{conv}\left(\bigcup_{k>0} \left\{ \frac{x}{k} \mid (k,x) \in S(R,\nu) \right\} \right)}.$$

The convex body Δ can also be described as the slice at $\{k = 1\}$ of the cone $C \subset \mathbb{R} \times \mathbb{R}^n$ generated by the semigroup (i.e. the smallest closed convex cone centered at the origin containing $S(R,\nu)$), projected to $\mathbb{R}^n$ via the projection to the second factor $(k,x) \mapsto x$.

Definition 1.9. *The convex body $\Delta = \Delta(X,R,\nu)$ above is called the* ***Newton-Okounkov body*** *of R with respect to ν.*

Remark 1.10. This definition can naturally be extended to allow for a choice of subspace $V \subset H^0(X,L)$ and $R(V) := \bigoplus_{k\geq 0} V^k \subset R$, where V^k is the image of the k-fold product $V \times \cdots \times V$ in $H^0(X, L^{\otimes k})$. In this setup, we may denote the Newton-Okounkov body by $\Delta(X, R(V), \nu)$. In Chapter 4, where we compute the Newton-Okounkov bodies of Bott-Samelson varieties, we take $V = H^0(X,L)$. In this context we use the notation $\Delta = \Delta(X,L,\nu)$ instead of $\Delta(X,R,\nu)$, so that the line bundle L is explicit in the notation. In Chapter 7, where we compute the Newton-Okounkov bodies of Peterson varieties, we use a subset $V \subseteq H^0(X,L)$, and use the notation $\Delta(X, R(V), \nu)$.

When computing Newton-Okounkov bodies, the following language will be useful.

Definition 1.11. *We define the* ***level-j piece*** *of a semigroup $S(R,\nu)$ to be $\nu(R_j \setminus \{0\})$. Moreover, we say that the Newton-Okounkov body is* ***generated in level-j*** *if*

$$\Delta := \overline{\operatorname{conv}\left(\bigcup_{k=1}^{j}\left\{\frac{x}{k} \mid (k,x) \in S(R,\nu)\right\}\right)}. \tag{1.1}$$

Remark 1.12. We note that if a Newton-Okounkov body Δ is generated in level-j, then Δ is a rational polytope (i.e. its vertices have rational coordinates). Indeed, if Δ is generated in level-j, then by (1.1) we know that in order to compute Δ it suffices to look at the vertices of each level-k piece $\nu(\mathcal{R}_k \setminus \{0\})$ for $1 \leq k \leq j$. More precisely, any point in Δ can be written in the form $\frac{1}{k}p$ for some $1 \leq k \leq j$ and for some point p in the level-k piece $\nu(\mathcal{R}_k \setminus \{0\})$. Moreover, p is in the convex hull of the vertices v_i of the level-k piece, i.e. $p = \sum c_i v_i$ for some $c_i \in \mathbb{R}$ such that $\sum c_i \leq 1$ and vertices v_i. It follows that $\frac{1}{k}p$ is in the convex hull of $\frac{1}{k}$ times the vertices v_i of the level-k piece. Therefore, Δ is exactly the convex hull of $\frac{1}{k}$ times the vertices of the level-k pieces for $1 \leq k \leq j$. Each vertex v_i in the level-k piece has rational entries, and therefore, $\frac{1}{k}v_i$ has rational entries too. Therefore, Δ is a rational polytope.

Remark 1.13. In general, the semigroup $S(R,\nu)$ may not be finitely generated. For example, in Example 5.10 of [And13], the Newton-Okounkov body of an elliptic curve and a certain choice of valuation yields the semigroup $\{(0,0)\} \cup \{(m,r) \mid 0 \leq r \leq 3m-1\} \subset \mathbb{N} \times \mathbb{Z}$, which is not finitely generated, since every lattice point on the line $r = 3m-1$ is needed to generate the semigroup. In general, the question of when the semigroup $S(R,\nu)$ is finitely generated is subtle and is sensitive to the choice of valuation. When the semigroup is not finitely generated, there is no guarantee that the associated Newton-Okounkov body is a rational polytope.

The main goal of this thesis is to explicitly compute Newton-Okounkov bodies of some special cases of Bott-Samelson varieties and Peterson varieties, with respect to valuations defined with respect to geometrically natural flags of subvarieties. In the examples we consider, it turns out that the semigroup is in fact generated in level one, i.e. $\Delta = \mathrm{conv}(\nu(H^0(X, L)))$. This fact is critical for us, since it reduces the problem to a finite computation (since $\dim H^0(X, L) < \infty$). It is worth noting here that the proof of the fact that the semigroup is generated in level one is different in the Bott-Samelson case and the Peterson case. In the setting of Bott-Samelson varieties, for certain choices of parameters, it turns out we have some a priori knowledge of the image $\nu(\mathcal{R}_k)$ at the k-th level, by using a special basis for $\mathcal{R}_k$. We are therefore able to show directly that for any $(k, x) \in S(R, \nu)$, we have $\frac{x}{k} \in \mathrm{conv}(\nu(H^0(X, L)))$.

In the case of Peterson varieties, we rely on a different approach, for which we recall some generalities. Suppose a variety X of complex dimension n is embedded in a projective space $\mathbb{P}^N$ and L is the restriction to X of the antitautological bundle $O(1)$ on $\mathbb{P}^N$. Then the homogeneous coordinate ring of X in $\mathbb{P}^N$ is a graded $\mathbb{C}$-subalgebra of the graded ring of sections $\oplus_k \mathcal{R}_k$, so we may take it to be the $R(V)$ in Remark 1.10. Suppose ν is a valuation with one-dimensional leaves as above. It is a general fact that the volume of $\Delta(X, R(V), \nu)$ exactly encodes the growth coefficient of the Hilbert function $H_{R(V)}(k) = \dim_{\mathbb{C}}(V^k)$ of $R(V)$ (see e.g. [Kav15, Theorem 1.15]), that is:

$$\lim_{k \to \infty} \frac{H_{R(V)}(k)}{k^n} = \mathrm{vol}(\Delta(X, R(V), \nu)).$$

Moreover, in the case at hand, it is a classical result that the LHS also agrees (up to a factor of $n!$) with the degree of the embedding $X \hookrightarrow \mathbb{P}^N$, i.e.,

$$\frac{1}{n!}\deg(X \subseteq \mathbb{P}^N) = \lim_{k \to \infty} \frac{H_{R(V)}(k)}{k^n}$$

which implies

$$\frac{1}{n!}\deg(X \subseteq \mathbb{P}^N) = \mathrm{vol}(\Delta(X, R(V), \nu)) \tag{1.2}$$

in our case.

The discussion above implies that if we can compute the degree of X in $\mathbb{P}(H^0(X, L))^*$ by some other means, then we know the volume of $\Delta(X, R(V), \nu)$, independent of any properties of the semigroup or of the $\nu(\mathcal{R}_k)$. In particular, if we are able to obtain - via direct computations - a set of points in $\nu(R_1)$ whose convex hull has volume equal to $\frac{1}{n!}$ times the degree, then we may immediately conclude that this convex hull is in fact equal to $\Delta(X, R(V), \nu)$. This is the approach we take in the case of the Peterson variety.

We close this introduction with one concrete and complete example of a computation of a Newton-Okounkov body.

Example 1.14. Let $X = \mathbb{P}^2$ be the complex projective plane and let $O(1)$ be the dual of the tautological bundle. Let x, y, z be the usual homogeneous coordinates for $\mathbb{P}^2 = \{[x : y : z]\}$. There is a natural flag of subvarieties in $\mathbb{P}^2$ specified by $[0 : 0 : 1] = \{x = y = 0\} \subset \{x = 0\} \subset X = \mathbb{P}^2$. Note that $\{x = y = 0\} = \{[0 : 0 : 1]\}$ is a single point, while $\{x = 0\} = \{[0 : y : z]\}$ is isomorphic to the projective line $\mathbb{P}^1$. On the affine chart $U = \{z \neq 0\} = \{[x : y : 1]\} \cong \mathbb{C}^2$ in $\mathbb{P}^2$, we may view x and y as a system of parameters for this flag of subvarieties.

This flag and system of parameters defines a valuation ν, as described above. More precisely, given a non-zero polynomial $f = f(x, y)$ in the variables x and y (so a holomorphic function on the affine chart U), $\nu(f)$ is given by (k_1, k_2), where $c_k x^{k_1} y^{k_2}$ is the lowest term in f with respect to lexicographic order. This valuation is often called a *lowest term valuation.*

Recall that the space of sections $H^0(\mathbb{P}^2, O(1)^{\otimes k})$ can be interpreted as the degree-k homogeneous polynomials in x, y, and z. With respect to the above system of parameters, $\mathcal{R}_1$ can then be identified with $H^0(\mathbb{P}^2, O(1)) = \text{span}\{x, y, 1\}$, and $\mathcal{R}_k$ is the vector space of polynomials in x and y with degree at most k.

In particular, for any element $(k, t_1, t_2) \subset S(R, \nu) \subset \mathbb{N} \times \mathbb{Z}^2$ in the semi-group, we have, by definition, that (t_1, t_2) is in the level-k piece, i.e. $(t_1, t_2) \in \nu(\mathcal{R}_k \setminus \{0\})$. This, in turn, implies that $t_1 + t_2 \leq k$ and so $\frac{t_1}{k} + \frac{t_2}{k} \leq 1$. This in turn means that $(\frac{t_1}{k}, \frac{t_2}{k}) \in \text{conv}\,(\nu(H^0(\mathbb{P}^2, O(1))))$ since $(1,0) = \nu(x)$ and $(0,1) = \nu(y)$ are in the image of $H^0(\mathbb{P}^2, O(1))$. From this we may conclude that the Newton-Okounkov body is $\text{conv}(\nu(H^0(\mathbb{P}^2, O(1))))$, i.e., it suffices to compute just the "level-one piece" $\nu(\mathcal{R}_1)$. In the language of Definition 1.11 we say that the Newton-Okounkov body is generated in level-one. Since $\mathcal{R}_1$ consists of the polynomials of degree at most 1 in the variables x and y, we have $\dim\mathcal{R}_1 = \dim(\text{span}\{x, y, 1\}) = 3$ and we can explicitly compute that $\nu(x) = (1,0), \nu(y) = (0,1)$ and $\nu(1) = (0,0)$. Therefore, the Newton-Okounkov body $\Delta(\mathbb{P}^2, O(1), \nu)$ is a triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

Alternatively, we could have used the fact that the degree of $\mathbb{P}^2$ in $\mathbb{P}(H^0(\mathbb{P}^2, O(1))^*)$ is 1, and therefore (from the discussion above) the area of the Newton-Okounkov body is $\frac{1}{2}$. The convex hull of the image of x, y, and 1 has area $\frac{1}{2}$, and so this convex hull must be the entire Newton-Okounkov body.

1.3 Flags and Grassmannians

The main results of this thesis are computations of Newton-Okounkov bodies of varieties which are closely related to the variety of flags in $\mathbb{C}^n$, or more generally, of flag varieties associated to complex reductive algebraic groups. As such, we dedicate this section to establishing notation and definitions concerning flag varieties and Grassmannian varieties. To set the stage, we first describe the standard flag variety in $\mathbb{C}^n$. We then generalize the notion to include partial flag varieties in $\mathbb{C}^n$ (in particular, the Grassmannian), and explain how these are special cases of a more general construction using (complex, connected, reductive) algebraic groups G and their parabolic subgroups P. In this general context, the standard flag variety in $\mathbb{C}^n$ corresponds to the "Lie type A" case $G = GL_n\mathbb{C}$. While we focus almost exclusively on the $G = GL_n\mathbb{C}$ case in this thesis, it is useful to set up the general terminology since much of the literature (and in particular, many of our references) on the subject uses the more general set-up. We conclude by defining important maps and line bundles over these varieties which arise in later chapters. For a more detailed treatment of what follows, see [Bri05] and [SKKT00].

The (full) flag variety in $\mathbb{C}^n$ is defined to be the collection of sequences of nested linear subspaces of $\mathbb{C}^n$:

$$Fl(\mathbb{C}^n) := \{V_\bullet = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\} \mid \dim_{\mathbb{C}} V_k = k \text{ for all } k\}.$$

Slightly more generally, given a sequence of increasing positive integers $0 < d_1 < \cdots < d_m < n$, a **flag of type** $(\mathbf{d_1}, \ldots, \mathbf{d_m})$ is an increasing sequence of linear subspaces $V_\bullet = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_m \subset \mathbb{C}^n \mid \dim_{\mathbb{C}} V_k = d_k$ for all $k = 1, 2, \ldots, m\}$. We then define the **(partial) flag variety** $\mathbf{Fl_{d_1,\ldots,d_m}(\mathbb{C}^n)}$, to be the set of flags of type $(d_1, \ldots, d_m)$. A special case of a partial flag variety is of particular importance and is called a *Grassmannian variety*; it corresponds to the case $m = 1$ and $d_1 = k$. Thus

$$Gr(k, n) := \{0 \subset V_k \subset \mathbb{C}^n\}$$

and so $Gr(k,n)$ is the variety of k-dimensional subspaces of $\mathbb{C}^n$.

It is well-known that the full flag variety $Fl(\mathbb{C}^n)$ and the partial flag varieties can also be realized as homogeneous spaces $GL_n\mathbb{C}/P$ where P is a parabolic subgroup of $GL_n\mathbb{C}$ as we now quickly recall.

Let $\{e_1,\ldots,e_n\}$ denote the standard ordered basis in $\mathbb{C}^n$, where e_i is the vector with a 1 in the i-th spot and 0's elsewhere. For any d with $1 \leq d \leq n$, denote by E_d the span of the first d standard basis vectors, $E_d := \mathrm{span}_{\mathbb{C}}\{e_1,\ldots,e_d\}$. For any sequence $(d_1,\ldots,d_m)$ as above, we call the flag of standard coordinate subspaces $E_{\bullet(d_1,\ldots,d_m)} = \{0 \subset E_{d_1} \subset E_{d_2} \subset \cdots \subset E_{d_m} \subset \mathbb{C}^n\}$ the *standard flag* (of type $(d_1,\ldots,d_m)$). Notice that for any element $V_\bullet = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_m \subset \mathbb{C}^n\} \in Fl_{d_1,\ldots,d_m}(\mathbb{C}^n)$, we can realize V_1 as the span of some vectors $\{v_1,\ldots,v_{d_1}\}$, V_2 as the span of $\{v_1,\ldots,v_{d_1}\}$ plus $d_2 - d_1$ additional vectors $\{v_{d_1+1},\ldots,v_{d_2}\}$, and so on. At the final step, $\mathbb{C}^n$ is the span of n vectors $\{v_1 \ldots, v_{d_1}, v_{d_1+1}, \ldots, v_{d_2}, \ldots, v_{d_m+1}, \ldots, v_n\}$. Let g be the invertible matrix whose i-th column is the vector v_i. It is not hard to see that g, interpreted as a linear transformation of $\mathbb{C}^n$, takes the standard flag to $V_\bullet$, i.e. $V_\bullet = gE_{\bullet(d_1,\ldots,d_m)} = \{0 \subset gE_{d_1} \subset gE_{d_2} \subset \cdots \subset gE_{d_m} \subset g\mathbb{C}^n\}$. In particular, the group $GL_n\mathbb{C}$ acts transitively on the flag variety $Fl_{d_1,\ldots,d_m}(\mathbb{C}^n)$ for any $(d_1,\ldots,d_m)$. Moreover, it is not hard to see that the isotropy group $P(d_1,\ldots,d_m)$ of the standard flag $E_{\bullet(d_1,\ldots,d_m)}$ consists of block-upper triangular invertible matrices with invertible diagonal blocks of sizes $d_1, d_2 - d_1, d_3 - d_2, \ldots, n - d_m$ (see Example 1.15 below). Moreover,

$$Fl_{d_1,\ldots,d_m}(\mathbb{C}^n) \cong GL_n\mathbb{C}/P(d_1,\ldots,d_m),$$

via the isomorphism which sends an element $[g] \in GL_n\mathbb{C}/P(d_1,\ldots,d_m)$ to $gE_{\bullet(d_1,\ldots,d_m)} \in Fl_{d_1,\ldots,d_m}(\mathbb{C}^n)$. In particular, $P(1,\ldots,n-1) = B$, the Borel subgroup of upper triangular invertible matrices, so

$$Fl(\mathbb{C}^n) \cong GL_n\mathbb{C}/B,$$

and we also have

$$Gr(k,n) \cong GL_n\mathbb{C}/P(k)$$

where by definition, $P(k)$ is the subgroup of block-upper triangular invertible matrices with two diagonal blocks of sizes k and $n-k$.

Example 1.15. The isotropy group, $P(1,4,5)$, of the standard flag $E_{\bullet(1,4,5)} \subset Fl_{1,4,5}(\mathbb{C}^7)$ is the group of block upper triangular invertible matrices with invertible diagonal blocks of sizes 1, $4-1=3$, $5-4=1$, and $7-5=2$ (here $n=7$ and $m=3$):

$$P(1,4,5) = \begin{pmatrix} * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \end{pmatrix}.$$

Here and below, we will denote an element $[g] \in GL_n\mathbb{C}/P(d_1,\ldots,d_m) = G/P$ by using square brackets:

$$[g] = gP = \left[\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\right].$$

From the discussion above, such a coset $[g] = gP$ can be interpreted as a flag of type $(d_1,\ldots,d_m)$ by taking the span of the leftmost d_i vectors in the matrix above, for each i with $1 \le i \le m$.

As mentioned above, we focus on the $GL_n\mathbb{C}$ case in this thesis, but the constructions above naturally generalize to other Lie types and much of the literature (including those which we include as references) state their results in the more general language. Thus we now take a moment to record the general definitions.

Let G be a complex, connected, reductive, algebraic group. Recall that a parabolic subgroup P of G has the property that $B \subseteq P \subset G$, where B is a Borel subgroup of G.

Definition 1.16. *A (generalized)* ***flag variety*** *of G is a homogeneous space of G of the form G/P, where P is a parabolic subgroup of G. In the special case $P = B$, we call G/B a* ***full flag variety****.*

We will now define an important algebraic variety which has close ties to the flag variety, namely the *Schubert variety*. Consider an element of the symmetric group $w \in S_n$, where we think about w as an $n \times n$ permutation matrix (each row and column contains a single 1 with 0's elsewhere). Consider the element of the full flag variety $wB \in Fl(\mathbb{C}^n)$. For $B \subset GL_n\mathbb{C}$ the Borel subgroup of invertible upper triangular matrices, consider the orbit $BwB \subset Fl(\mathbb{C}^n)$. It it well known that $Fl(\mathbb{C}^n)$ can be written as the disjoint union of these orbits BwB (see [Bri05, Proposition 1.1.1]). The orbits BwB are called **Schubert cells**. The closure of a Schubert cell $\overline{BwB}$ in $Fl(\mathbb{C}^n)$ under the Zariski topology is called a **Schubert variety** X_w.

In the remainder of this section, we record some standard algebraic-geometric constructions related to the Grassmannians $Gr(k, n)$ and the flag varieties more generally: the Plücker maps and the resulting (pullback) line bundles. Before describing the Plücker maps, it is useful to first recall two standard maps from algebraic geometry: the *Veronese mapping* and the *Segre embed-*

ding (see e.g. [SKKT00] for more details).

Definition 1.17. *Let $\mathbb{P}^r$ be the r-dimensional complex projective space, with homogeneous coordinates $x_0, x_1, \ldots, x_r$. Notice that there are precisely $\binom{d+r}{d}$ distinct monomials in the variables $x_0, x_1, \ldots, x_r$ of total degree equal to d. Let $z = \binom{d+r}{d} - 1$. The **d-th Veronese mapping** is the map given by the list of all such monomials:*

$$\begin{aligned} \nu_d : \mathbb{P}^r &\to \mathbb{P}^z \\ [x_0 : \cdots : x_r] &\mapsto \underbrace{[x_0^d : x_0^{d-1}x_1 : \cdots : x_r^d]}_{\text{all monomials of degree } d}, \end{aligned}$$

where $z = \binom{d+r}{d} - 1$ as above.

The next map is defined in a similar fashion.

Definition 1.18. *Let $\mathbb{P}^r$ be the r-dimensional complex projective space, considered with homogeneous coordinates $x_0, x_1, \ldots, x_r$, and similarly $\mathbb{P}^s$ with homogeneous coordinates $y_0, y_1, \ldots, y_s$. Notice that there are precisely $(r+1)(s+1)$ many distinct monomials of the form $x_i y_j$ for some i, j with $0 \le i \le r$ and $0 \le j \le s$. The **Segre embedding**, $\Sigma_{r,s} = \Sigma$ is given by the list of all such degree-two (or "degree $(1,1)$") monomials:*

$$\begin{aligned} \Sigma_{r,s} : \mathbb{P}^r \times \mathbb{P}^s &\to \mathbb{P}^{(r+1)(s+1)-1} \\ ([x_0 : \cdots : x_r], [y_0 : \cdots : y_s]) &\mapsto [x_0 y_0 : x_0 y_1 : \cdots : x_i y_i : \cdots : x_r y_s]. \end{aligned}$$

We now define the Plücker mapping of $Gr(k,n)$ into $\mathbb{P}^{\binom{n}{k}-1}$. Given an element $[g] \in Gr(k,n)$ (written in matrix form), first note that the k-dimensional subspace V_k of $\mathbb{C}^n$ specified by $[g]$ is given by the span of the leftmost k rows in the matrix $[g]$. For any choice of $1 \le c_1 < c_2 < \cdots < c_k \le n$, let us denote by $P_{c_1 \cdots c_k}$ the determinant of the $k \times k$ submatrix given by the rows

$1 \leq c_1 < \cdots < c_k \leq n$ in the leftmost k columns of $[g]$. Notice that there are precisely $\binom{n}{k}$ choices of such square submatrices. The **Plücker embedding** of $Gr(k,n)$ into $\mathbb{P}^{\binom{n}{k}-1}$ is then defined by the list of all such determinants:

$$\begin{aligned} p_k : Gr(k,n) &\to \mathbb{P}^{\binom{n}{k}-1} \\ [g] &\mapsto [\cdots : P_{c_1\cdots c_k} : \cdots]. \end{aligned} \tag{1.3}$$

This map is well-defined because $[g] = [h]$ implies that $g = hA$ for an invertible block-upper triangular matrix A, and in particular the value of $P_{c_1\cdots c_k}$ associated to $[g]$ differs from that of $[h]$ by a fixed non-zero scalar (namely, the determinant of the upper $k \times k$ submatrix of A) for any values of $c_1 < c_2 < \cdots < c_k$. The following is an example of the Plücker embedding when $n = 3$.

Example 1.19. The Plücker embedding of $Gr(2,3)$ into $\mathbb{P}^2$ is given by:

$$\begin{aligned} p_2 : Gr(2,3) &\to \mathbb{P}^2 \\ \left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right] &\mapsto [P_{12} : P_{13} : P_{23}] = \left[\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} : \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} : \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right]. \end{aligned}$$

We can also embed the flag variety $Fl_{d_1,\ldots,d_m}(\mathbb{C}^n)$ into projective space as follows. First we embed the flag variety into a product of Grassmannian varieties. Second, we apply the Plücker embedding to each coordinate in this product to further embed into a product of projective spaces. Then, we apply the Segre embedding to the product of projective spaces to embed into a single (large) projective space. More precisely, first we map $Fl_{d_1,\ldots,d_m}(\mathbb{C}^n)$ into a product of Grassmannians $Gr(d_1,n) \times \cdots \times Gr(d_m,n)$ via a map ι which essentially "reads off" each of the k-dimensional subspaces in a given flag $V_\bullet$:

$$\begin{aligned} \iota : Fl_{d_1,\ldots,d_m}(\mathbb{C}^n) &\to Gr(d_1,n) \times \cdots \times Gr(d_m,n) \\ \{V_1 \subset \cdots \subset V_m\} &\mapsto (V_1, \ldots, V_m). \end{aligned} \tag{1.4}$$

Then we apply the Plücker embedding for a Grassmannian (as described above) to each coordinate:

$$\begin{aligned} p : Gr(d_1, n) \times \cdots \times Gr(d_m, n) \to \quad & \mathbb{P}^{\binom{n}{d_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{d_m}-1} \\ (V_1, \ldots, V_m) \mapsto \quad & [p_1(V_1) : \cdots : p_m(V_m)], \end{aligned} \tag{1.5}$$

where p_k is the Plücker embedding of $Gr(d_k, n)$. Finally, we map the product of projective spaces into a large projective space using the Segre embedding, Σ, described above.

The **(standard) Plücker embedding** of $Fl_{d_1,\ldots,d_m}(\mathbb{C}^n)$ into projective space is given by composing equations (1.4) and (1.5), and then applying the Segre embedding, i.e. $\Sigma \circ p \circ \iota$.

Given a sequence of integers $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0)$, one can also define the Plücker mapping of $Fl(\mathbb{C}^n)$ corresponding to λ. This map is the same as the standard Plücker embedding given above, except that we apply the $(\lambda_k - \lambda_{k+1})$-th Veronese mapping, $\nu_{\lambda_k - \lambda_{k+1}}$, to the k-th coordinate in $\mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$ before applying the Segre embedding Σ.

We denote this Veronese mapping as follows:

$$\begin{aligned} \nu : \mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1} \to \quad & \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_{n-1}} \\ [x_1 : \cdots : x_{n-1}] \mapsto \quad & [\nu_{\lambda_1}(x_1) : \cdots : \nu_{\lambda_{n-1}}(x_{n-1})], \end{aligned} \tag{1.6}$$

where $\nu_{\lambda_k - \lambda_{k+1}}$ is the $(\lambda_k - \lambda_{k+1})$-th Veronese mapping, and
$r_i := \binom{\binom{n}{k}-1+(\lambda_k-\lambda_{k+1})}{\binom{n}{k}-1} - 1$.

Definition 1.20. *The **Plücker mapping associated to** λ, φ_λ, of $Fl(\mathbb{C}^n)$ into projective space is given by composing equations (1.4), (1.5), and (1.6), and then applying the Segre embedding, i.e. $\varphi_\lambda = \Sigma \circ \nu \circ p \circ \iota$.*

Note that the standard Plücker embedding is the Plücker mapping corresponding to

$$\lambda = (n-1, n-2, \ldots, 1, 0).$$

Finally, using these maps, we can define certain pullback line bundles over the Grassmannian variety and flag variety which play an important role in our discussion. In order to define the pullback bundle over $Gr(k,n)$, we first embed $Gr(k,n)$ into projective space using the Plücker mapping, p. Let $O(1)$ denote the dual of the tautological bundle over any projective space.

Definition 1.21. *The* ***Plücker bundle*** *over the Grassmannian variety $Gr(k,n)$ is the pullback line bundle $p^*(O(1))$.*

Similarly, we define a line bundle over $Fl(\mathbb{C}^n)$ by first mapping $Fl(\mathbb{C}^n)$ into projective space, using the Plücker mapping, φ_λ.

Definition 1.22. *The* ***Plücker (line) bundle corresponding to*** *λ over the full flag variety $Fl(\mathbb{C}^n)$ is the pullback line bundle $\mathcal{L}^\lambda := \varphi_\lambda^*(O(1))$.*

We conclude this section with a concrete example.

Example 1.23. Let $\lambda = (2,1,0)$. We embed $Fl(\mathbb{C}^3)$ into $\mathbb{P}^8$ via the Plücker mapping corresponding to λ. First, embed $Fl(\mathbb{C}^3)$ into $Gr(1,3) \times Gr(2,3)$ by $V_\bullet = (0 \subseteq V_1 \subseteq V_2 \subseteq \mathbb{C}^3) \mapsto (V_1, V_2)$. Second, map the image of this into $\mathbb{P}^2 \times \mathbb{P}^2$ using the corresponding Grassmannian Plücker mappings. Since $\lambda_1 - \lambda_2 = \lambda_2 - \lambda_3 = 1$, we do not apply any Veronese mappings to $\mathbb{P}^2 \times \mathbb{P}^2$. Finally, the Segre embedding takes $\mathbb{P}^2 \times \mathbb{P}^2$ to $\mathbb{P}^8$. We pull back the $O(1)$ bundle over $\mathbb{P}^8$ to $Fl(\mathbb{C}^3)$. This pullback bundle is our $\mathcal{L}^\lambda$. The following diagram illustrates this map:

$$\begin{tikzcd}
\mathcal{L}_\lambda := (\Sigma \circ p \circ \iota)^*(\mathcal{O}(1)) \arrow[d] & & & & \mathcal{O}(1) \arrow[d] \\
Fl(\mathbb{C}^3) \arrow[r, "\iota"'] & Gr(1,3) \times Gr(2,3) \arrow[r, "Plücker", "p"'] & \mathbb{P}^2 \times \mathbb{P}^2 \arrow[r, "Segre", "\Sigma"'] & & \mathbb{P}^8
\end{tikzcd}$$

Concretely, for an element $V_\bullet = \{0 \subset V_1 \subset V_2 \subset \mathbb{C}^3\} \in Fl(\mathbb{C}^3)$, the composition $\Sigma \circ p \circ \iota$ is given by:

$$\begin{aligned}
&V_\bullet \mapsto (V_1, V_2) \mapsto ([P_1 : P_2 : P_3], [P_{12} : P_{13} : P_{23}]) \\
&\mapsto [P_1P_{12} : P_1P_{13} : P_1P_{23} : P_2P_{12} : P_2P_{13} : P_2P_{23} : P_3P_{12} : P_3P_{13} : P_3P_{23}].
\end{aligned}$$

Part I

Bott-Samelson Varieties

Chapter 2

Background on Bott-Samelson Varieties and Generalized Demazure Modules

Bott-Samelson varieties are closely related to flag varieties and Schubert varieties, and as such, they are important in the study of representation theory. In recent years, they have also been studied in the context of Newton-Okounkov bodies. The first explicit computation of Newton-Okounkov bodies of a flag variety is due to Okounkov [Oko98]. Using a geometric valuation, Okounkov identified the Newton-Okounkov bodies of symplectic flag varieties with symplectic Gelfand-Zetlin polytopes. Kaveh [Kav15] computed the Newton-Okounkov bodies of general flag varieties G/B of complex algebraic groups G using a highest term valuation corresponding to a certain coordinate system on a Bott-Samelson variety. He showed that these Newton-Okounkov bodies can be realized as Berenstein-Littlemann string polytopes associated to a choice of reduced-word decomposition of the longest element of the Weyl group and a choice of an irreducible representation. In the process, Kaveh computed the Newton-Okounkov bodies, $\Delta(Z_I, L)$, of Bott-Samelson varieties Z_I (to be defined precisely below), where L is a line bundle which can be realized as a pullback line bundle from the flag G/B. Using the same valuation as Kaveh, Fujita [Fuj15] generalized Kaveh's results to include line bundles which do not necessarily have to be pullback bundles; he also has analogous results for another valuation with respect to different coordinates. Similarly, with respect to coordinates different from Kaveh (also used in Section 6.4 in [And13]), Kiritchenko [Kir15] has computed the Newton-Okounkov bodies of line bundles on the flag variety G/B. Recently, Harada and Yang [HY16] have also explicitly computed the Newton-Okounkov bodies of Bott-Samelson varieties using yet another valuation; they found that the polytopes that arise are special cases of the Grossberg-Karshon twisted cubes. Seppänen and Schmitz [SS14] also study the so-called global Newton-Okounkov body of Bott-Samelson varieties using the same valuation as ours, and they show it is rational polyhedral and give an inductive description of it.

In this thesis, we use the same coordinate system as in [Kav15], but unlike Kaveh we use a lowest term valuation associated to these coordinates. In general, the Newton-Okounkov body depends very much on the choice of valuation and understanding the nature of this dependence seems to be a rather subtle problem (see e.g. [Kav15, Remark 2.3]).

2.1 Bott-Samelson Varieties

Bott-Samelson varieties are central in the study of the geometry of flag varieties and in the study of Schubert calculus. They were originally introduced in the context of differential geometry by Bott and Samelson [BS58]. This construction was then adapted by Demazure [Dem74] and Hansen [Han73] to apply to an algebraic-geometric context. They showed that Bott-Samelson varieties could be realized as desingularizations of Schubert varieties and therefore could be used in Schubert calculus. For instance, Bott-Samelson varieties have been used to study the Chow ring and projective coordinate ring of G/B, to find character formulas for irreducible representations of G, and to determine properties about Schubert varieties, such as being normal and Cohen-Macauley with rational singularities [BK07]. Bott-Samelson varieties are useful partly because they are birationally isomorphic to Schubert varieties and can also be factored into iterated $\mathbb{P}^1$ fibrations, each with a birational map to G/B.

In this chapter, we will define Bott-Samelson varieties, describe the line bundles over them, and give a basis for the space of sections of these line bundles. Although the results of this thesis are for Lie type A, we follow the standards of the literature in this subject and give the definition of these varieties in general Lie type.

Let G be a complex, connected, reductive, algebraic Lie group of rank r, B a Borel subgroup, and $\{\alpha_1, \dots, \alpha_r\}$ the set of positive simple roots associated to this choice of B. Let W be the Weyl group generated by simple reflections $s_1, \dots s_r$ corresponding to the roots α_i. Let P_{α_k} be the minimal parabolic subgroup associated to the simple reflection s_k. We give an example to illustrate these definitions in the type A case.

Example 2.1. Let $G = GL_3\mathbb{C}$. Let B be the Borel subgroup of invertible upper triangular matrices and H the Cartan subgroup of invertible diagonal matrices. We denote a diagonal matrix with diagonal entries h_1, h_2 and h_3 by $[h_1, h_2, h_3]$. We have two simple roots $\{\alpha_1, \alpha_2\}$. Concretely, we can think of these roots as $\alpha_1 = L_1 - L_2$ and $\alpha_2 = L_2 - L_3$, where $L_i(h) = h_i$, for

$h = [h_1, h_2, h_3] \in \mathfrak{h}$. Then $W \cong S_3$, the symmetric group generated by simple reflections $s_1 = 213$ and $s_2 = 132$, where we have written permutations in standard one-line notation. The two minimal parabolic subgroups, P_{α_1} and P_{α_2}, have the form

$$P_{\alpha_1} = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \text{ and } P_{\alpha_2} = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix},$$

where $*$ denotes an arbitrary parameter (subject to the condition that the matrices in question are invertible).

Given this set-up, we can now define a *Bott-Samelson variety.*

Definition 2.2. *Fix a sequence $I = (i_1, \ldots, i_\ell) \in \{1, 2, \ldots, r\}^\ell$, which specifies a sequence of positive roots $(\alpha_{i_1}, \ldots, \alpha_{i_\ell})$. Let P_I denote the product of the corresponding minimal parabolics $P_{\alpha_{i_1}} \times \ldots \times P_{\alpha_{i_\ell}}$. We define a right action of B^ℓ on P_I by*

$$(p_1, \ldots, p_\ell) \cdot (b_1, \ldots, b_\ell) = (p_1 b_1, {b_1}^{-1} p_2 b_2, \ldots, {b_{\ell-1}}^{-1} p_\ell b_\ell). \tag{2.1}$$

Then the ***Bott-Samelson variety,*** *Z_I, corresponding to I, is defined to be $Z_I := P_I / B^\ell$.*

Theorem 2.3. *[Dem74, §3] Z_I is a smooth projective algebraic variety of dimension ℓ.*

In order to discuss the relationship between Bott-Samelson varieties and Schubert varieties, we need the following definition.

Definition 2.4. *The* ***length*** *of an element of the Weyl group, $w \in W$, is the smallest positive integer k so that w can be written as the product of k simple reflections. The length of w is denoted $\ell(w)$. A (word) decomposition $w = s_{i_1} \ldots s_{i_\ell}$ is called* ***reduced*** *when $\ell = \ell(w)$.*

For a reduced word $w = s_{i_1} \dots s_{i_\ell}$ and $I = (i_1, \dots, i_\ell)$, the multiplication map

$$\begin{aligned} \pi : Z_I &\to X_w \subset G/B \\ [(p_1, \dots, p_\ell)] &\mapsto [p_1 \cdots p_\ell] \end{aligned} \tag{2.2}$$

is a birational morphism between Z_I and X_w, and in particular, if $w = w_0$ is the longest element of the Weyl group, then the image of Z_I is the full flag variety G/B ([Dem74, §3],[Jan87, §13.5]).

In Lie type A, we also have a map taking Z_I to a product of Grassmannian varieties as follows. Let $Gr(k, n)$ denote the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$. First, we can map Z_I into $Gr(i_j, n)$ for $1 \leq j \leq \ell$ as follows:

$$\begin{aligned} \mu_j : Z_I &\to Gr(i_j, n) \cong GL_n\mathbb{C}/P(i_j) \\ [(p_1, \dots, p_\ell)] &\mapsto [p_1 \cdots p_k]. \end{aligned} \tag{2.3}$$

It is clear from the definition of the action (2.1) and of Z_I that the above μ_j is well-defined. Let

$$Gr(I) := Gr(i_1, n) \times \cdots \times Gr(i_\ell, n) \tag{2.4}$$

denote the product of Grassmannians corresponding to the sequence $I = (i_1, \dots, i_\ell)$. We now define a map μ_I from Z_I to $Gr(I)$ by letting the j-th coordinate be the image of μ_j:

$$\begin{aligned} \mu_I : Z_I &\to Gr(I) \\ [(p_1, \dots, p_\ell)] &\mapsto ([p_1], [p_1 p_2], \dots, [p_1 \cdots p_\ell]). \end{aligned} \tag{2.5}$$

The above maps will be important in the sections that follow.

We denote the (set of) homogeneous coordinates on the j-th factor of $Gr(I)$ by

$$x^{(j)}, \tag{2.6}$$

e.g. the 4-th homogeneous coordinate of $\mu_I([(p_1,\ldots,p_\ell)]) \in Gr(I)$ is $x^{(4)} = [p_1p_2p_3p_4]$.

We close this section by giving an alternative useful way to construct and interpret Bott-Samelson varieties, as an iterated fibre product [Mag98, §1]. Namely, it turns out that the Bott-Samelson variety can also be described as

$$Z_I = eB \times_{G/P_{\alpha_{i_1}}} G/B \times_{G/P_{\alpha_{i_2}}} \ldots \times_{G/P_{\alpha_{i_\ell}}} G/B \subset \underbrace{G/B \times \ldots \times G/B}_{\ell+1}$$

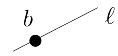
where we recall that in general, given maps $f: X \to Z$ and $g: Y \to Z$, the fibre product $X \times_Z Y$ is $\{(x,y) \in X \times Y \mid f(x) = g(y) \in Z\} \subseteq X \times Y$. Thus we can build the Bott-Samelson variety through the following iterative process. Define $Z_{\mathbf{O}} := eB$ and $I^\ell = (i_1, \ldots i_{\ell-1})$ if $\ell > 1$, $I^\ell = \mathbf{O}$ if $\ell = 1$. Then the Bott-Samelson variety Z_I is realized as $Z_{I^\ell} \times_{G/P_{\alpha_{i_\ell}}} G/B = \{(x,y) \in Z_{I^\ell} \times G/B \mid f(x) = g(y)\}$, where $f: Z_{I^\ell} \xrightarrow{\pi} G/B \to G/P_{\alpha_{i_\ell}}$ is the composition of the above map π with the natural projection $G/B \to G/P_{\alpha_{i_\ell}}$ and $g: G/B \to G/P_{\alpha_{i_\ell}}$ is the natural projection. Note that by definition of the fibre product we have a (Cartesian) commutative diagram

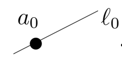
$$\begin{tikzcd} Z_{I^\ell} \times_{G/P_{\alpha_{i_\ell}}} G/B \arrow[r] \arrow[d] & G/B \arrow[d, "g"] \\ Z_{I^\ell} \arrow[r, "f"'] & G/P_{\alpha_{i_\ell}}. \end{tikzcd}$$

This iterative process builds a tower of $\mathbb{P}^1$-bundles with Z_I as the last step:

$$Z_{\mathbf{O}} \leftarrow Z_{(i_1)} \leftarrow Z_{(i_1,i_2)} \leftarrow \ldots \leftarrow Z_I.$$

We illustrate these definitions concretely in the following specific example.

Example 2.5. Let $G = GL_3\mathbb{C}$ and $I = (1,2,1)$. An element of the flag variety $GL_3(\mathbb{C})/B \cong Fl(\mathbb{C}^3)$ is a sequence $V_\bullet = \{0 \subset V_1 \subset V_2 \subset \mathbb{C}^3\}$. In the projectivization $\mathbb{P}^2 = \mathbb{P}(\mathbb{C}^3)$ of $\mathbb{C}^3$, the line V_1 corresponds to a point $b \in \mathbb{P}^2$ and the 2-dimensional subspace V_2 corresponds to a 1-dimensional line ℓ in $\mathbb{P}^2$. Thus, in this example we will interpret a flag $[g] \in GL_3(\mathbb{C})/B$ as $\{b \in \ell \subset \mathbb{P}^2\}$, a "point in a line in $\mathbb{P}^2$", illustrated schematically as 

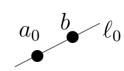
. We also let a_0 denote the point in $\mathbb{P}^2$ specified by the span of e_1 (the first standard basis vector) and ℓ_0 the line in $\mathbb{P}^2$ specified by span$\{e_1, e_2\}$. Then the "standard flag" $Z_{\mathbf{O}} = eB$ clearly corresponds to $\{a_0 \in \ell_0 \subset \mathbb{P}^2\} =$ 

. Notice also that since $G/P_{\alpha_1} = Gr(2,3)$ and $G/P_{\alpha_2} = Gr(1,3)$, the projection maps from $Z_{\mathbf{O}}$ are given by reading off the "ℓ coordinate" and the "b coordinate", respectively:

$$\begin{array}{cc} f : Z_{\mathbf{O}} \to G/P_{\alpha_1}, & g : G/B \to G/P_{\alpha_1} \\ \{a_0 \in \ell_0 \subset \mathbb{P}^2\} \mapsto \{\ell_0 \subset \mathbb{P}^2\} & \{b \in \ell \subset \mathbb{P}^2\} \mapsto \{\ell \subset \mathbb{P}^2\}. \end{array}$$

From this it is not difficult to see that the fibre product $Z_{(1)} = Z_{\mathbf{O}} \times_{G/P_{\alpha_1}} G/B$ can be described as follows:

$$Z_{(1)} = \big\{(x,y) \in Z_{\mathbf{O}} \times G/B \mid x = \{a_0 \in \ell_0 \subset \mathbb{P}^2\}, y = \{b \in \ell_0 \subset \mathbb{P}^2\}\big\} =$$

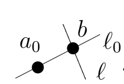


and in particular, $Z_{(1)}$ is the set of points in ℓ_0, hence is isomorphic to $\mathbb{P}^1$. For the next step, we have by definition $Z_{(1,2)} = Z_{(1)} \times_{G/P_{\alpha_2}} G/B$, where the maps f and g are given by

$$\begin{array}{cc} f : Z_{(1)} \to G/P_{\alpha_2}, & g : G/B \to G/P_{\alpha_2} \\ (\{a_0 \in \ell_0 \subset \mathbb{P}^2\}, \{b \in \ell_0 \subset \mathbb{P}^2\}) \mapsto \{b \subset \mathbb{P}^2\} & \{a \in \ell \subset \mathbb{P}^2\} \mapsto \{a \subset \mathbb{P}^2\} \end{array}$$

where f is the composition of $\pi : Z_{(1)} \to G/B$ with the natural projection $G/B \to G/P_{\alpha_2}$. Therefore, at the next step of our construction we have that $Z_{(1,2)}$ consists of certain pairs $((x,y),z) \in Z_{(1)} \times G/B$ as follows:

$$Z_{(1,2)} = \big\{(x,y) = (\{a_0 \in \ell_0 \subset \mathbb{P}^2\}, \{b \in \ell_0 \subset \mathbb{P}^2\}), z = \{b \in \ell \subset \mathbb{P}^2\}\big\} =$$



.

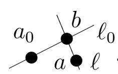
Finally, we know that, by definition, $Z_{(1,2,1)} = Z_{(1,2)} \times_{G/P_{\alpha_1}}$, where the maps f and g are given by:

$$f : Z_{(1,2)} \to G/P_{\alpha_1}$$
$$\big((\{a_0 \in \ell_0 \subset \mathbb{P}^2\}, \{b \in \ell_0 \subset \mathbb{P}^2\}), \{b \in \ell \subset \mathbb{P}^2\}\big) \mapsto \{\ell \subset \mathbb{P}^2\},$$

$$g : G/B \to G/P_{\alpha_1}$$
$$\{a \in \tilde{\ell} \subset \mathbb{P}^2\} \mapsto \{\tilde{\ell} \subset \mathbb{P}^2\}.$$

Therefore, the Bott-Samelson variety $Z_{(1,2,1)}$ can be described as

$$Z_{(1,2,1)} = \{\big((\{a_0 \in \ell_0 \subset \mathbb{P}^2\}, \{b \in \ell_0 \subset \mathbb{P}^2\}), \{b \in \ell \subset \mathbb{P}^2\}\big), \{a \in \ell \subset \mathbb{P}^2\}\}$$

$=$ 

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2.2 A Coordinate System on Z_I and its Associated Valuation

Let $G = GL_n\mathbb{C}$ and $I = (i_1, \ldots, i_\ell)$ as in the previous section, and let Z_I denote the associated Bott-Samelson variety. We now describe a particular coordinate system on Z_I (following Kaveh [Kav15, §2.2]) and define a valuation on spaces of rational functions of Z_I using these coordinates. This is the valuation considered in the next two chapters.

Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be the set of simple positive roots corresponding to I. For each α_i, let F_{α_i} denote the corresponding Chevalley generator. Let $U^-_{\alpha_i} = \{exp(tF_{\alpha_i}) \mid t \in \mathbb{C}\} \subseteq GL_n\mathbb{C}$. We consider the map

$$\Phi : \mathbb{C}^\ell \to U^-_{\alpha_{i_1}} \times \cdots \times U^-_{\alpha_{i_\ell}} \to Z_I$$
$$(t_1, \ldots, t_\ell) \mapsto [(exp(t_1 F_{\alpha_{i_1}}), \ldots, exp(t_\ell F_{\alpha_{i_\ell}}))]. \tag{2.7}$$

It will be frequently useful to consider the composition of Φ with the projection $\pi : Z_I \to G/B$, as defined in (2.2). It is straightforward to see that the composition is given by:

$$\begin{aligned} \pi \circ \Phi : \mathbb{C}^\ell &\to U^-_{\alpha_{i_1}} \times \cdots \times U^-_{\alpha_{i_\ell}} \to GL_n\mathbb{C}/B \cong Fl(\mathbb{C}^n) \\ (t_1, \ldots, t_\ell) &\mapsto [(exp(t_1 F_{\alpha_{i_1}}) \cdots exp(t_\ell F_{\alpha_{i_\ell}})]. \end{aligned} \tag{2.8}$$

Example 2.6. Let $n = 3$ and $I = (1, 2, 1)$. Then

$$F_{\alpha_1} = F_{\alpha_3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, F_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$\begin{aligned} \pi(\Phi(t_1, t_2, t_3)) &= \pi \left(\left[\begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ t_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \right) \\ &= \left[\begin{pmatrix} 1 & 0 & 0 \\ t_1 + t_3 & 1 & 0 \\ t_2 t_3 & t_2 & 1 \end{pmatrix} \right]. \end{aligned}$$

The map $\Phi : \mathbb{C}^\ell \to Z_I$ gives coordinates on Z_I near the point $[e, e, \ldots, e] \in Z_I$ [Kav15, Proposition 2.2]. With respect to these coordinates, consider the flag of subvarieties

$$Y_\ell = \overline{\{t_1 = \ldots = t_\ell = 0\}} \subset \cdots \subset Y_2 = \overline{\{t_1 = t_2 = 0\}} \subset Y_1 = \overline{\{t_1 = 0\}} \subset Y_0 = Z_I.$$

We know Z_I is smooth and $\dim_{\mathbb{C}} Z_I = \ell$ (see Theorem 2.3). Therefore, this flag is an admissible flag, since $\dim_{\mathbb{C}} Y_i = \ell - i$ and each Y_i is nonsingular at the point Y_ℓ. Moreover, $t_1, \ldots, t_\ell$ is a system of parameters about this flag, since $t_{k|Y_{k-1}}$ is a well-defined, not identically zero, rational function on Y_{k-1} and has a zero of first order on Y_k.

For the following, we focus on computations which are restricted to the (open dense) affine coordinate chart on Z_I given by the coordinate system Φ above, with variables $t_1, \ldots, t_\ell$. Recall that holomorphic sections of line bundles may be represented by polynomials in the t_i in this coordinate chart; such identifications are described in the following sections. Thus we may ask how the geometric (pre)valuation ν defined by the admissible flag above (following the method in Section 1.2) behaves on polynomials in the t_i. The lemma below shows that, in fact, ν corresponds to taking the lowest term with respect to lexicographic order on $\mathbb{Z}^\ell$ (cf. Definition 1.3). This is in contrast to Kaveh's choice in [Kav15], where he uses the *highest term* valuation.

Lemma 2.7. *Given a polynomial $f = \sum a_\alpha t^\alpha$, where $\alpha = (a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell$,*

$$\nu(f) = \min_{\text{lex}} \left\{ \alpha \in \mathbb{Z}^\ell \mid a_\alpha \neq 0 \right\}.$$

Proof. By definition, $\nu(f) = (k_1, \ldots, k_\ell)$, where $k_1 = \operatorname{ord}_{Y_1}(f)$, $k_2 = \operatorname{ord}_{Y_2}\left(f t_1^{-k_1} \Big|_{Y_1} \right)$, and so on (see §1.2). With respect to our coordinates, the order of vanishing of f along $Y_1 = \{t_1 = 0\}$ is precisely the lowest exponent of t_1 which occurs in the terms which appear in f. Denote this value by a_1, so $k_1 = a_1$. Next, the definition of ν states that we must divide f by $t_1^{k_1}$ and then restrict to $Y_1 = \{t_1 = 0\}$, where t_1 is the parameter corresponding to Y_1. In our case this is $t_1^{a_1}$. Note that in the notation of §1.2, $t_1 = y_1$. Dividing f by $t_1^{a_1}$ and restricting to $\{t_1 = 0\}$ yields monomials in the variables $t_2, \ldots, t_\ell$, corresponding to those monomials appearing in f whose t_1 exponent is a_1. (The other monomials must vanish once we restrict to $\{t_1 = 0\}$.) At the next step, the same reasoning shows that k_2 is the lowest exponent of t_2 which occurs in these remaining monomials (i.e. $k_2 = a_2$), and so on. Thus the valuation ν picks out the exponent vector which is minimal in lexicographic order. □

Example 2.8. The lowest term valuation, ν, maps the sections in $H^0(Z_I, L)$ to $\mathbb{Z}^\ell$ as follows:

$$\nu(t_1^2t_2t_3^3t_5 + t_1t_3^2 + t_1t_4^2) = (1, 0, 0, 2, 0, \ldots, 0),$$

$$\nu(t_1t_2^4t_3^2t_4 + t_2^4t_3t_5 + t_3t_6 + t_1t_6) = (0, 0, 1, 0, 0, 1, 0 \ldots, 0).$$

It will be useful to set the following notation.

Definition 2.9. *The **lowest term** of a polynomial $f = \sum a_\alpha t^\alpha$ is*

$$LT(f) := a_{\nu(f)}t^{\nu(f)}.$$

2.3 Line Bundles on Bott-Samelson Varieties

Let $G = GL_n\mathbb{C}$ and $I = (i_1, \ldots, i_\ell)$. Let Z_I be the corresponding Bott-Samelson variety as in the previous sections. We will now define certain line bundles over Z_I. Let $\{\alpha_{i_1}, \ldots, \alpha_{i_\ell}\}, \{s_{i_1}, \ldots, s_{i_\ell}\}$ and $\{\omega_{i_1}, \ldots, \omega_{i_\ell}\}$ be the corresponding sequences of positive simple roots, simple reflections, and fundamental weights, respectively. In type A, a fundamental weight $\omega_j \in \mathfrak{h}^*$ is of the form $\omega_j = L_1 + \ldots + L_j$, where L_k is the linear functional defined by $L_k(h_1, \ldots, h_n) := h_k$ for $(h_1, \ldots h_n) \in \mathfrak{h}$. Let $m = (m_1, \ldots, m_\ell) \in \mathbb{N}^\ell$. We now define the line bundle over Z_I corresponding to m by a quotient construction as follows:

$$O(m) = \mathcal{O}(m_1, \ldots, m_\ell) := Z_I \times_{B^\ell} \mathbb{C}$$

where the action of B^ℓ on the product $Z_I \times \mathbb{C}$ is defined as

$$(b_1, \ldots, b_\ell) \cdot (p, k) := (b \cdot p, e^{m_1\omega_{i_1}}(b_1^{-1}) \cdots e^{m_\ell\omega_{i_\ell}}(b_\ell^{-1})k)$$

for $p \in Z_I$ and $k \in \mathbb{C}$.

It turns out that any line bundle over a Bott-Samelson variety can be written in this form.

Theorem 2.10. *[LT04, §3.1] Any line bundle over Z_I is isomorphic to $O(m_1, \ldots, m_\ell)$, for some $(m_1, \ldots, m_\ell) \in \mathbb{N}^\ell$.*

Remark 2.11. The notation above differs slightly from that in [LT04], where the m_i's are written in reverse order.

Another way to think about the line bundle $O(m)$ is as follows. Consider the Plücker bundle $p^*(O(1))$ over the Grassmannian $Gr(i_j, n)$ (Definition 1.21) and its pullback to Z_I under the map $\mu_j : Z_I \to Gr(i_j, n)$ (cf. equation (2.3)). This pullback bundle is isomorphic to $O(\hat{m}_j)$, where $\hat{m}_j$ has a 1 in the j-th position and 0's elsewhere. In other words:

$$O(\hat{m}_j) = O(0, \ldots, 0, \underbrace{1}_{j-\text{th spot}}, 0, \ldots, 0) \cong \mu_j^* \circ p^*(\mathcal{O}(1)).$$

More generally, if $\tilde{m}_j = (0, \ldots, 0, \underbrace{m_j}_{j-\text{th spot}}, 0, \ldots, 0)$ for some positive integer m_j, then $O(\tilde{m}_j) \cong \mu_j^* \circ (p^*(\mathcal{O}(1))^{\otimes m_j})$. Finally, for $m = (m_1, \ldots, m_\ell)$, we have

$$O(m) \cong O(\tilde{m}_1) \otimes \cdots \otimes O(\tilde{m}_\ell). \tag{2.9}$$

2.4 Standard Monomial Basis

In modern representation theory, combinatorial models have been used to construct bases for representations. For example, the famous theory of crystal bases and string polytopes gives a combinatorial model for constructing bases of irreducible representations, V_λ, of connected reductive algebraic groups G using directed graphs. Spaces of global sections of Bott-Samelson varieties, $H^0(Z_I, O(m))$, also appear in representation theory as so-called *generalized Demazure modules*; special cases of $H^0(Z_I, O(m))$ yield the irreducible representations V_λ mentioned above. The goal of Chapters 3 and 4 is to compute Newton-Okounkov bodies of a Bott-Samelson variety Z_I with respect to special cases of the line bundles $O(m)$ introduced in the previous section. Such a computation necessarily involves analyzing the vector spaces $H^0(Z_I, O(m))$. In the cases when $H^0(Z_I, O(m))$ can be identified with an irreducible representation V_λ as above, Kaveh used the theory of crystal bases to compute an associated Newton-Okounkov body [Kav15]. However, for the cases we consider in this thesis (for which we do not necessarily have such an isomorphism $H^0(Z_I, O(m)) \cong V_\lambda$), we use a different basis, namely the *standard-monomial basis* developed by Lakshmibai, Littelmann, and Magyar [LM97, LM98, LLM02]. In this section, we summarize the explicit algorithm for computing the standard monomial basis for $H^0(Z_I, O(m))$ in the case of $GL_n\mathbb{C}$, as described in [LM98].

2.4.1 Definition of Standard Tableaux

Throughout this discussion, we let $G = GL_n\mathbb{C}$ and fix a sequence $I = (i_1, \ldots, i_\ell)$ and an integer vector $m = (m_1, \ldots, m_\ell) \in \mathbb{N}^\ell$. In order to construct the standard monomial basis of $H^0(Z_I, O(m))$, we need some notation and definitions.

Definition 2.12. *A* ***tableau*** *is a sequence of integers $\tau = (r_1, \ldots r_N)$ with $r_j \in \{1, 2, \ldots, n\}$ for $N \geq 0$. We let $\boldsymbol{O}$ represent the empty tableau.*

Definition 2.13. *The* ***concatenation*** *$\tau \star \tau'$ of two tableaux $\tau = (r_1, \ldots r_N)$ and $\tau' = (r'_1, \ldots r'_N)$, is defined as*

$$\tau \star \tau' = (r_1, \ldots, r_N, r'_1, \ldots, r'_N).$$

We also define $\boldsymbol{O} \star \tau = \tau \star \boldsymbol{O} = \tau$ for any tableau τ, and define $\tau^{\star j} := \tau \star \cdots \star \tau$ (i.e. a concatenation of j many copies of τ). For $j = 0$, we define $\tau^{\star 0} = \boldsymbol{O}$. For τ a tableau and T a set of tableaux, we denote by $\tau \star T$ the set of tableaux *obtained by concatenating each element of T with τ, i.e.*

$$\tau \star T := \{\tau \star \sigma \mid \sigma \in T\}.$$

Definition 2.14. *A **column** of size j is a tableau $\kappa = (r_1, \ldots, r_j)$ such that the entries are strictly increasing, i.e. $1 \leq r_1 < \cdots < r_j \leq n$. Let $\bar{w}_i$ denote the column $(1, 2, \ldots, i)$. Then $\bar{w}_i^{\star j} = (1, 2, \ldots i, \ldots, 1, 2, \ldots, i)$ (i.e. j copies of $\bar{w}_i$).*

Definition 2.15. *We define a **tableau of shape** (I, m) to be the concatenation of m_1 columns of size i_1, m_2 columns of size i_2, etc.:*

$$\tau = \underbrace{\kappa_{11} \star \kappa_{12} \star \cdots \star \kappa_{1m_1}}_{m_1 \text{ of size } i_1} \star \underbrace{\kappa_{21} \star \cdots \star \kappa_{2m_2}}_{m_2 \text{ of size } i_2} \star \cdots \star \underbrace{\kappa_{\ell 1} \star \cdots \star \kappa_{\ell m_\ell}}_{m_\ell \text{ of size } i_\ell},$$

where each κ_{kj} is a column of size i_k for each k and j. If $m_k = 0$ for any k, then there is the empty tableau $\boldsymbol{O}$ in the corresponding position of τ.

Definition 2.16. *For $i \in \{1, \ldots, n-1\}$, we now define the **lowering operator** f_i. The operator f_i takes a tableau $\tau = (r_1, \ldots, r_N)$ to either the formal null symbol $\boldsymbol{O}$ or to a new tableau $\tau' = (r'_1, \ldots, r'_N)$, by changing a single entry $r_j = i$ to $r'_j = i + 1$ and leaving all other entries alone, according to the following algorithm:*

1. *Ignore all entries of τ except those equal to i or $i + 1$.*

2. *Considering the string consisting of i's and $i + 1$'s, if an i in the string is immediately followed by an $i + 1$, then ignore that pair of entries.*

3. *Upon completing Step 2, if the new (sub)string is of the form*

$$i + 1, i + 1, \ldots, i + 1, i, i, \ldots, i$$

(i.e. no $i + 1$ appears to the right of any i), stop and proceed to Step 4. Otherwise, return to and repeat Step 2.

4. *If the remaining (sub)string does not contain any i's, then return the null symbol, i.e.* $f_i(\tau) = \mathbf{O}$. *Otherwise, change the leftmost* i *to* $i+1$. *This new string (along with all other original entries of* τ*) is* $f_i(\tau)$.

Example 2.17. Below, we schematically illustrate the steps in the computation of $f_2^k(\tau)$ for different values of k, for $\tau = 124123123113221$.

$$\begin{array}{lcccccccccccccccc}
\tau & = & 1 & 2 & 4 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 1 & 3 & 2 & 2 & 1 \\
 & & \cdot & 2 & \cdot & \cdot & 2 & 3 & \cdot & 2 & 3 & \cdot & \cdot & 3 & 2 & 2 & \cdot \\
 & & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & 2 & 2 & \cdot \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{2} & \mathbf{2} & \cdot \\
f_2(\tau) & = & 1 & 2 & 4 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 1 & 3 & \mathbf{3} & \mathbf{2} & 1 \\
(f_2)^2(\tau) & = & 1 & 2 & 4 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 1 & 3 & \mathbf{3} & \mathbf{3} & 1 \\
(f_2)^3(\tau) & = & \mathbf{O} & & & & & & & & & & & & & &
\end{array}$$

Next, we define the Demazure operator Λ_i which takes a tableau τ to a set of tableaux. More precisely, we define

$$\Lambda_i(\tau) = \{\tau, f_i(\tau), f_i^2(\tau), \ldots\} \setminus \{\mathbf{O}\} \text{ if } \tau \neq \mathbf{O} \tag{2.10}$$

and $\Lambda_i(\mathbf{O}) = \{\mathbf{O}\}$. We also define Λ_i acting on a set of tableaux T by applying Λ_i to each element and taking the union:

$$\Lambda_i(\mathrm{T}) := \bigcup_{\tau \in \mathrm{T}} \Lambda_i(\tau). \tag{2.11}$$

Example 2.18. We apply Λ_1 to the set of tableaux $T = \{123121, 13\}$:

$$\begin{aligned}
\Lambda_1(T) &= \{123121, f_1(123121), f_1^2(123121), \ldots\} \cup \{13, f_1(13), f_1^2(13), \ldots\} \setminus \{\mathbf{O}\} \\
&= \{123121, \quad 123122, \qquad\qquad\qquad\qquad 13, \quad 23\}.
\end{aligned}$$

Definition 2.19. *We call a tableau of shape* (I, m) ***standard*** *if it is an element of the set*

$$\mathcal{T}(I, m) := \Lambda_{i_1}(\bar{w}_{i_1}^{\star m_1} \star \Lambda_{i_2}(\bar{w}_{i_2}^{\star m_2} \star \cdots \Lambda_{i_\ell}(\bar{w}_{i_\ell}^{\star m_\ell}) \cdots)).$$

In particular, the above definition gives an explicit algorithm for computing $\mathcal{T}(I,m)$ by a sequence of Demazure operators and concatenation operators. We illustrate this in the following explicit example.

Example 2.20. If $I=(1,2,1)$ and $m=(0,1,1)$, then the set of all standard tableaux of shape (I,m) is produced as follows:

$$\begin{aligned}&\{\mathbf{O}\} \overset{1\star}{\to} \{1\} \overset{\Lambda_1}{\to} \{1,2\} \overset{12\star}{\to} \{121,122\} \overset{\Lambda_2}{\to} \{121,131,122,132,133\}\\ &\overset{\mathbf{O}\star}{\to} \{121,131,122,132,133\} \overset{\Lambda_1}{\to} \{121,131,231,232,122,132,133,233\}.\end{aligned}$$

2.4.2 The Standard Monomial Basis

We will now show how to associate an element of $H^0(Z_I, O(m))$ to a standard tableau.

Suppose we have a point in the Grassmannian represented by an $n\times k$ matrix as follows:

$$x=\left[\begin{pmatrix} x_{11} & \cdots & x_{1k}\\ \vdots & \ddots & \vdots\\ x_{n1} & \cdots & x_{nk}\end{pmatrix}\right]\in Gr(k,n)\cong G/P(k).$$

Suppose $\kappa=(r_1,\ldots,r_k)$ is a column in the sense of Definition 2.14. We denote by $\Theta_\kappa(x)$ the homogeneous coordinate in the Plücker mapping p_k of $Gr(k,n)$ (see §1.3, (1.3)) which corresponds to the $k\times k$ minor given by the rows $(r_1,\ldots,r_k)$ in κ. Specifically,

$$\Theta_\kappa(x)=\det_{k\times k}\begin{pmatrix} x_{r_11} & \cdots & x_{r_1k}\\ \vdots & \ddots & \vdots\\ x_{r_k1} & \cdots & x_{r_kk}\end{pmatrix}. \tag{2.12}$$

Recalling that a basis of sections of the dual of the tautological bundle over any projective space may be given by its homogeneous coordinates, we see that

$\Theta_\kappa(x)$ may be viewed as an element of $H^0(Gr(k,n), p_k^*(O(1)))$ for any column κ. Similarly, given a tableau $\tau = \kappa_{11} \star \cdots \star \kappa_{\ell m_\ell}$ of shape (I, m), let

$$\Theta_\tau := \prod_{j=1}^{\ell} \prod_{m=1}^{m_j} \Theta_{\kappa_{jm}}(x^{(j)}) \in H^0(Gr(I), p^*(O(m))), \tag{2.13}$$

where $x^{(j)}$ denotes the homogeneous coordinate on the j-th factor of $Gr(I)$, as defined in (2.6) and (2.4), and $p^*(O(m))$ denotes the Plücker bundle over $Gr(I)$, as defined in Definition 1.21. Note that we use the product notation in (2.13) to be consistent with the notation used in [LM98], but strictly speaking we mean a tensor product of sections. By slight abuse of notation we also denote by Θ_τ the pullback to Z_I of the section Θ_τ via the map μ_I (see equation (2.5)). By construction, this new Θ_τ is a section of the bundle $O(m)$ in (2.9).

Definition 2.21. *If τ is a standard tableau in the sense of Definition 2.19, then we call Θ_τ a **standard monomial**.*

We denote the set of standard monomials by

$$\Theta(\mathcal{T}(I, m)). \tag{2.14}$$

The following theorem explains why standard monomials are useful in the study of generalized Demazure modules.

Theorem 2.22 ([LM98])**.** *The set of standard monomials of shape (I, m) forms a basis for the space of sections $H^0(Z_I, O(m))$.*

We close the section with some concrete computations.

Example 2.23. Let $I = (1, 2, 1)$ and $m = (0, 1, 1)$. In what follows, we describe the basis of standard monomials $\Theta(\mathcal{T}(I, m))$ concretely as polynomials in the coordinates t_1, t_2, t_3 (note $\ell = 3$ in this case) on the coordinate neighborhood of Z_I as constructed in Section 2.2, where we take

$$\Phi(t_1,t_2,t_3)=\left[\begin{pmatrix}1&0&0\\t_1&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&t_2&1\end{pmatrix},\begin{pmatrix}1&0&0\\t_3&1&0\\0&0&1\end{pmatrix}\right]\in Z_I.$$

The composition with μ_I then yields

$$\mu_I(\Phi(t_1,t_2,t_3))=\left[\begin{pmatrix}1\\t_1\\0\end{pmatrix},\begin{pmatrix}1&0\\t_1&1\\0&t_2\end{pmatrix},\begin{pmatrix}1\\t_1+t_3\\t_2t_3\end{pmatrix}\right]\in Gr(1)\times Gr(2)\times Gr(1).$$

Here and in all of the discussions in the next chapter, it is implicit that we have trivialized the line bundle $O(m)$ over this open neighborhood using as a base section the standard monomial corresponding to the standard tableau $\bar{w}_{i_1}^{\star m_1}\star\bar{w}_{i_2}^{\star m_2}\star\cdots\star\bar{w}_{i_\ell}^{\star m_\ell}$, which in the special case under consideration is $12\star 1=121$ (since $m_1=0, m_2=1$ and $m_3=1$), i.e. we view each section as a rational function by dividing by the base section $\Theta_{121}=1$. From Example 2.20 we know the set of all standard tableaux of shape (I,m) is given by $\{121,131,231,232,122,132,133,233\}$. We can now explicitly compute $\Theta_{121}=\Theta_{12}x^{(2)}\Theta_1x^{(3)}=1$, $\Theta_{131}=t_2$, $\Theta_{231}=t_1t_2$, $\Theta_{232}=t_1t_2(t_1+t_3)$, $\Theta_{122}=t_1+t_3$, $\Theta_{132}=t_2(t_1+t_3)$, $\Theta_{133}=t_2(t_2t_3)$, and $\Theta_{233}=t_1t_2(t_2t_3)$. By Theorem 2.22, these 8 polynomials describe the restrictions to this coordinate chart of a basis for the space of sections $H^0(Z_{(1,2,1)},O(0,1,1))$.

Chapter 3

Injectivity of ν on the Standard Monomial Basis

We will show in this chapter that our lowest term valuation ν (§2.2) is injective on the standard monomial basis $\Theta(\mathcal{T}(I, m))$ for $I = (1, 2, \ldots, n-1, 1, 2, \ldots, n-2, \ldots, 1, 2, 1) \in \mathbb{N}^\ell$ and line bundles $O(m)$ with $m = (m_1, m_2, \ldots, m_{2n-3}, 0, \ldots, 0) \in \mathbb{N}^\ell$. In other words, distinct basis elements map under ν to distinct values in $\mathbb{Z}^\ell$. Since ν has one-dimensional leaves, this fact will help us compute the Newton-Okounkov body $\Delta(Z_I, O(m), \nu)$ in Chapter 4.

This chapter is technical and purely combinatorial. The arguments rely heavily on the structure of a standard tableau $\tau \in \mathcal{T}(I, m)$, which by definition is produced by successive application of concatenation operators and Demazure operators.

3.1 Notation

We first establish some notation. Recall that a tableau of shape (I, m) has the form

$$\tau = \kappa_{11} \star \kappa_{12} \star \cdots \star \kappa_{1m_1} \star \kappa_{21} \star \cdots \star \kappa_{2m_2} \star \cdots \star \kappa_{\ell 1} \star \cdots \star \kappa_{\ell m_\ell},$$

where each κ_{js} is a column of size i_j for each j and s (see Definition 2.15). The following example illustrates this definition.

Example 3.1. If $I = (1, 2, 1)$ and $m = (1, 2, 2)$, then a tableau, τ, of shape (I, m) has the form $\tau = \kappa_{11} \star \kappa_{21} \star \kappa_{22} \star \kappa_{31} \star \kappa_{32}$, where κ_{11} has length $i_1 = 1$, κ_{21} and κ_{22} each have length $i_2 = 2$, and κ_{31} and κ_{32} each have length $i_3 = 1$. Note there is only one word κ_{11} with first index equal to 1 since $m_1 = 1$, two words κ_{21} and κ_{22} with first index equal to 2 since $m_2 = 2$, and two words κ_{31} and κ_{32} with first index equal to 3 since $m_3 = 2$. More specifically, an example of a τ of shape (I, m) is $1 \star 13 \star 12 \star 2 \star 1$ where $\kappa_{11} = 1$, $\kappa_{21} = 13$, $\kappa_{22} = 12$, $\kappa_{31} = 2$ and $\kappa_{32} = 1$.

In order to refer to certain parts of τ, we make the following definition.

Definition 3.2. *Let $I = (i_1, \ldots, i_\ell)$, $m = (m_1, \ldots, m_\ell)$. Let τ be a standard tableau, $\tau \in \mathcal{T}(I, m)$. Fix $j \in \{1, \ldots, \ell\}$. We define the **$j$-sector**, denoted $[\mathbf{M_j}]$, to be the subsequence of τ consisting of the κ_{js}, for $1 \leq s \leq m_j$, i.e. $[M_j]$ consists of the concatenation of the words κ_{js} with first index equal to j. More precisely, $[M_j] := \kappa_{j1} \star \kappa_{j2} \star \cdots \star \kappa_{jm_j}$. Intuitively, the j-sector is the part of the tableau τ corresponding to the j-th component i_j in the word $I = (i_1, \ldots, i_\ell)$.*

By definition of the j-sector, we may write τ as a concatenation of the $[M_j]$'s:

$$\tau = [M_1] \star \cdots \star [M_\ell].$$

Example 3.3. In the setting of Example 3.1, $\tau = 1 \star 13 \star 12 \star 2 \star 1$ can be decomposed as $[M_1]\star[M_2]\star[M_3]$, where $[M_1] = \kappa_{11} = 1$, $[M_2] = \kappa_{21}\star\kappa_{22} = 1312$ and $[M_3] = \kappa_{31} \star \kappa_{32} = 21$.

Remark 3.4. For notational convenience, we will sometimes drop the $\star$ concatenation notation, and simply right the concatenation of two sectors $[M_i]$ and $[M_j]$ as $[M_i][M_j]$.

In the arguments below, we always use a specific choice of sequence I which depends on a positive integer n and is defined as follows.

Definition 3.5. *Fix a positive integer n. We define a sequence $I(n)$ of length $\ell = \frac{n(n-1)}{2}$ to be*

$$I(n) = (1, 2, \ldots, n-1, 1, 2, \ldots, n-2, \ldots, 1, 2, 1) = \bar{w}_{n-1} \star \bar{w}_{n-2} \star \cdots \star \bar{w}_2 \star \bar{w}_1. \tag{3.1}$$

It may be useful for the reader to remember that the sequence $I(n)$ corresponds to the reduced word decomposition

$$(s_1 s_2 s_3 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2)(s_1) \tag{3.2}$$

of the longest element, w_0, in the Weyl group S_n.

Thus far in our discussion, the word I and the vector m have had indexing set $\{1, 2, \ldots, \ell\}$, ordered via the usual total ordering on $\mathbb{N}$. When working with the word $I(n) = (1, 2, \ldots, n-1, 1, 2, \ldots, n-2, \ldots)$ as above, for notational purposes, it will be convenient to allow other finite totally ordered sets as indexing sets. In particular, let

$$\mathcal{S} := \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq a \leq n-1, 1 \leq b \leq n-a\},$$

and equip $\mathcal{S}$ with lexicographic order. Thus, listed in increasing order with respect to this total order, we have

$$\mathcal{S} = \{(1,1),(1,2),\dots,(1,n-1),(2,1),(2,2),\dots,(2,n-2),\dots,\\ (n-2,1),(n-2,2),(n-1,1)\}. \tag{3.3}$$

Note that $|\mathcal{S}| = \ell = \frac{n(n-1)}{2}$. Define $i_{(a,b)} := b$. Then $I(n)$ is the word $I(n) = \left(i_{(a,b)}\right)_{(a,b)\in\mathcal{S}}$ where $\mathcal{S}$ is listed in increasing order with respect to lexicographic order. Similarly, the vector $(m_{(a,b)})_{(a,b)\in\mathcal{S}}$ can be defined with indexing set $\mathcal{S}$, and for $(a,b) \in \mathcal{S}$, the (a,b)-sector of a standard tableau τ of shape $(I(n), m)$ is the concatenation

$$[M_{a,b}] := \kappa_{(a,b),1} \star \kappa_{(a,b),2} \star \cdots \star \kappa_{(a,b),m_{(a,b)}}.$$

In the injectivity results below, we will be concerned with certain groups of sectors, motivating the following definition.

Definition 3.6. *Let $\mathcal{S}$ and $I(n)$ be as above. Let $m = (m_{(a,b)})_{(a,b)\in\mathcal{S}} \in \mathbb{N}^\ell$. Suppose τ is a standard tableau of shape $(I(n), m)$. We define the* ***k-chain*** *of τ to be the concatenation of the (k,j)-sectors for $1 \le j \le n-k$, i.e. $[M_{k,1}] \star [M_{k,2}] \star \cdots \star [M_{k,n-k}]$.*

Intuitively, the k-chain of τ is the part which corresponds to the segment $(s_1 s_2 \cdots s_{n-k})$ in the reduced word decomposition (3.2) above, so for example the 1-chain corresponds to the first segment $(s_1 s_2 \cdots s_{n-1})$, the 2-chain to the second segment $(s_1 s_2 \cdots s_{n-2})$, etc.

Remark 3.7. In the proofs in the sections that follow, we in fact only use the 1-chain and the 2-chain of a tableau τ.

3.2 Properties of Lowering Operators

The purpose of this section is to record some observations, to be used in the arguments below, which relate the behaviour of the lowering operators f_i of Definition 2.16 with the behaviour of certain projection operators which we define below.

Recall that a general tableau as in Definition 2.12 is simply a sequence of integers $\tau = (r_1, \ldots, r_N)$ for some non-negative integer N. For any sequence of positive integers $a_1, \ldots, a_k$ with $\sum_{j=1}^{k} a_j = N$, we can naturally break up τ into "pieces" of lengths specified by the a_j, i.e.

$$\tau = (r_1, r_2, \ldots, r_N) = \underbrace{(r_1, \ldots, r_{a_1})}_{a_1 \text{ entries}} \star \underbrace{(r_{a_1+1}, \ldots, r_{a_1+a_2})}_{a_2 \text{ entries}} \star \cdots \star \underbrace{(r_{\sum_{j=1}^{k-1} a_j+1}, \ldots, r_N)}_{a_k \text{ entries}}$$

and we may evidently define projections from the set of tableaux of length N to the set of tableaux of length a_j for any j by

$$\tau = (r_1, r_2, \ldots, r_N) \mapsto \underbrace{(r_{a_1+\cdots+a_{j-1}+1}, \ldots, r_{a_1+\ldots+a_j})}_{a_j \text{ entries}}.$$

More specifically, we will be interested in looking at projections to certain j-sectors and k-chains as in Definitions 3.2 and 3.6. As a special case of the above discussion, observe that for $I = (i_1, i_2, \ldots, i_\ell)$ and $m = (m_1, m_2, \ldots, m_\ell)$, a tableau of shape (I, m) as in Definition 2.15

$$\tau = \kappa_{11} \star \kappa_{12} \star \cdots \star \kappa_{1m_1} \star \kappa_{21} \star \cdots \star \kappa_{2m_2} \star \cdots \star \kappa_{\ell 1} \star \cdots \star \kappa_{\ell m_\ell} = [M_1] \star \cdots [M_\ell]$$

can be projected to any of the κ_{js}'s, $[M_j]$'s, or concatenations thereof. Thus for instance the projection

$$\tau \mapsto \kappa_{js} \tag{3.4}$$

yields a projection from the set of tableaux of shape (I, m) to the set of tableaux of length i_j, which we note can also be viewed as the set of tableaux of shape (I, m) where $I = (i_1, \ldots, i_\ell)$ and $m' = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the j-th coordinate. Similarly, the projection

$$\tau \mapsto [M_j] \tag{3.5}$$

is a projection to the set of tableaux of shape (I, m'') where $I = (i_1, \ldots, i_\ell)$, $m'' = (0, \ldots, 0, m_j, 0, \ldots, 0)$, and the projection

$$\tau \mapsto [M_1][M_2] \cdots [M_j] \tag{3.6}$$

is a projection to the set of tableaux of shape (I, m''') where $I = (i_1, \ldots, i_\ell)$ and $m''' = (m_1, m_2, \ldots, m_j, 0, \ldots, 0)$.

As we have just seen, projection from a tableau to a smaller tableau is quite straightforward. For our later arguments however, we in fact wish to restrict attention to the set of standard tableaux of a given shape. Given the discussion of the projections (3.4), (3.5), and (3.6) above, it becomes natural to ask whether the projection of a standard tableau $\tau \in \mathcal{T}(I, m)$ is also standard, i.e. for example, if $\tau \in \mathcal{T}(I, m)$ is standard, then is it true that $[M_j]$ is standard, i.e., is it an element in $\mathcal{T}(I, (0, \ldots, 0, m_j, 0, \ldots, 0))$?

To address this question, the following lemma will be useful. Here f_i is the lowering operator defined in Definition 2.16.

Lemma 3.8. *Let τ and τ' be tableaux with entries in $\{1, \ldots, n\}$. Then one of the following holds:*

1. $f_i(\tau \star \tau') = f_i(\tau) \star \tau'$, $f_i(\tau) \neq \boldsymbol{O}$
2. $f_i(\tau \star \tau') = \tau \star f_i(\tau')$, $f_i(\tau') \neq \boldsymbol{O}$
3. $f_i(\tau \star \tau') = \boldsymbol{O}$.

Proof. To see this, we go back to the definition of the lowering operator f_i in Definition 2.16, which is given by an explicit algorithm which involves "cancelling" (ignoring) a pair $(i, i+1)$ in the string when the $i+1$ appears immediately to the right of the i.

Although the algorithm stated in Definition 2.16 specifies a certain order in which to make these cancellations, it is useful to note that the result is in fact independent of this ordering, in the following sense: we can replace the instructions in Step 2 of the algorithm of Definition 2.16 with the instructions "Considering the string consisting only of i's and $i+1$'s, if there exists an i in the string immediately followed by an $i+1$, then *choose* one such i, and then ignore that pair $(i, i+1)$ of entries." It is not hard to see that the final result of the algorithm is independent of the choices made in a repeated implementation of this altered Step 2, so $f_i(\sigma)$ for any σ is well-defined by such an algorithm. In particular, in applying the algorithm to a concatenation $\tau \star \tau'$, we may *first* apply the algorithm separately to both τ and τ' and obtain, as an intermediate step, strings of the form $i+1, i+1, \ldots, i+1, i, i, \ldots, i$ (as in Step 3 of the algorithm) for each τ and τ', resulting in the string (for $\tau \star \tau'$)

$$\underbrace{i+1, i+1, \ldots, i+1, i, i, \ldots, i}_{\substack{\text{obtained from } \tau, \\ r = \text{ number of } i+1\text{'s}, \\ s = \text{ number of } i\text{'s appearing}}} \underbrace{i+1, i+1, \ldots, i+1, i, i, \ldots, i}_{\substack{\text{obtained from } \tau', \\ t = \text{ number of } i+1\text{'s}, \\ u = \text{ number of } i\text{'s appearing}}}. \tag{3.7}$$

In order to complete the algorithm for $\tau \star \tau'$, we now need to "cancel" the pairs of i's (occurring in the string from τ) with $i+1$'s (occurring in the string from τ'). There are three possible cases.

Case 1: $\mathbf{u = 0}$ and $\mathbf{t \geq s}$. In this case, all of the i's appearing in the string coming from τ cancel with the $i+1$'s in the string coming from τ', and there are no i's in the final string from $\tau \star \tau'$. Thus, by Step 4 in the algorithm of Definition 2.16, $f_i(\tau \star \tau') = \mathbf{O}$ in this case.

Case 2: $\mathbf{u > 0}$ and $\mathbf{t \geq s}$. As in the case above, in this case all of the i's in the string from τ cancel with the $i+1$'s in the string from τ'. But $u > 0$, so the final string from $\tau \star \tau'$ is of the form $i+1, i+1, \ldots, i+1, i, \ldots, i$ where the leftmost i is an entry that is coming from τ', not τ. Thus the entries in τ remain unchanged. Moreover, this leftmost i (which gets changed to an $i+1$) is the same as the i that gets changed when computing $f_i(\tau')$. Thus $f_i(\tau \star \tau') = \tau \star f_i(\tau')$ in this case.

Case 3: $\mathbf{t < s}$. In this case, not all i's in the string from τ cancel with $i+1$'s in the string from τ', hence the final string for $\tau \star \tau'$ has a leftmost i which was originally from τ. Since the elements of τ' remained unchanged, we conclude $f_i(\tau \star \tau') = f_i(\tau) \star \tau'$.

□

The following is immediate from the above lemma.

Corollary 3.9. *Let τ_1 and τ_2 be tableaux with entries in $\{1, \ldots, n\}$. Then for a fixed $i \in \{1, \ldots, n\}$ and a non-negative integer $a \in \mathbb{N}$ we have that either $f_i^a(\tau_1 \star \tau_2) = \boldsymbol{O}$ or there exists $a_1, a_2 \in \mathbb{N}$ with $a_1 + a_2 = a$ such that the following are true:*

- $f_i^a(\tau_1 \star \tau_2) = f_i^{a_1}(\tau_1) \star f_i^{a_2}(\tau_2)$
- $f_i^{a_1}(\tau_1) \neq \boldsymbol{O}$
- $f_i^{a_2}(\tau_2) \neq \boldsymbol{O}$.

As we already mentioned, the above results will help us analyze certain subtableaux of a standard tableau. We now refine our understanding further by analyzing the relationship between projections and Demazure operators. Specifically, we show below that certain types of "projections to subtableaux preserve standardness", in a sense we make precise below.

For this discussion we fix $n > 0$ and $N > 0$ positive integers. Consider the set of tableaux $\tau = (r_1, r_2, \dots, r_N)$ of length N with entries in $\{1, 2, \dots, n\}$. Let j, k be positive integers with $1 \le j \le k \le N$ and denote by pr the projection to the subtableau given by reading off consecutive entries in τ starting at the j-th and ending at the k-th entry, i.e.,

$$\mathrm{pr}(\tau = (r_1, \dots, r_N)) = (r_j, r_{j+1}, \dots, r_k). \tag{3.8}$$

We may also naturally extend the definition of pr to sets of tableaux of length N by applying pr to each element of the set, e.g. if $\mathcal{S} = \{\tau_1, \tau_2, \dots, \tau_s\}$ is a set of tableaux of length N, then

$$\mathrm{pr}(\mathcal{S}) := \{\mathrm{pr}(\tau_1), \mathrm{pr}(\tau_2), \dots, \mathrm{pr}(\tau_s)\}. \tag{3.9}$$

Recall that $\Lambda_i(\tau) = \{\tau, f_i(\tau), f_i^2(\tau), \dots\} \setminus \{\mathbf{O}\}$ is the Demazure operator taking a tableau τ to a set of tableaux (see Definition 2.10). We have the following.

Lemma 3.10. *Let τ be a tableau of length $N > 0$ with entries in $\{1, 2, \dots, n\}$. Let j, k be positive integers such that $1 \le j \le k \le N$ and let* pr *denote the projection operator in* (3.8) *and* (3.9). *Let $i \in \{1, 2, \dots, n\}$ and let Λ_i denote the corresponding Demazure operator. Then*

$$\mathrm{pr}(\Lambda_i(\tau)) \subseteq \Lambda_i(\mathrm{pr}(\tau)).$$

Proof. An element of $\Lambda_i(\tau)$ is of the form $f_i^a(\tau)$ for some integer $a \ge 0$ by definition of Λ_i (here $f_i^a(\tau) \neq \mathbf{O}$). By definition of pr, we may write

$$\tau = \tau' \star \mathrm{pr}(\tau) \star \tau''$$

for $\tau' = (r_1, \dots, r_{j-1})$ and $\tau'' = (r_{k+1}, \dots, r_N)$. By Corollary 3.9 applied to $\tau_1 = \tau'$ and $\tau_2 = \mathrm{pr}(\tau) \star \tau''$ and since $f_i^a(\tau) \neq \mathbf{O}$ by assumption, there must exist $a_1, a_2 \in \mathbb{N}$ with $a_1 + a_2 = a$ with $f_i^a(\tau) = f_i^{a_1}(\tau') \star f_i^{a_2}(\mathrm{pr}(\tau) \star \tau'')$ where $f_i^{a_2}(\mathrm{pr}(\tau) \star \tau'') \neq \mathbf{O}$. Applying Corollary 3.9 again to $\tau_1 = \mathrm{pr}(\tau)$ and $\tau_2 = \tau''$, there must exist $b_1, b_2 \in \mathbb{N}$ with $b_1 + b_2 = a_2$ and with $f_i^{a_2}(\mathrm{pr}(\tau) \star \tau'') = f_i^{b_1}(\mathrm{pr}(\tau)) \star f_i^{b_2}(\tau'')$ and $f_i^{b_1}(\mathrm{pr}(\tau)) \neq \mathbf{O}$. But then

$$f_i^a(\tau) = f_i^{a_1}(\tau') \star f_i^{b_1}(\mathrm{pr}(\tau)) \star f_i^{b_2}(\tau'')$$

and $\mathrm{pr}(f_i^a(\tau)) = f_i^{b_1}(\mathrm{pr}(\tau)) \in \Lambda_i(\mathrm{pr}(\tau))$, as desired. □

We next analyze how the Demazure operators and certain projection operators interact with concatenation operators. Since we deal only with a special type of projection operator, it will be useful to introduce some notation. We used the notation pr above for a generic projection to a subtableau of consecutive entries. In the arguments below we project to subtableaux corresponding to certain subsets of columns within each sector. Let $I = I(n)$ as in Definition 3.5 and let $m = (m_1, m_2, \ldots, m_\ell) \in \mathbb{N}^\ell$ where $\ell = \frac{n(n-1)}{2}$. Recall that a tableau τ of shape (I, m) as in Definition 2.15 is a concatenation of m_1 columns of size i_1, m_2 columns of size i_2, etc., so that

$$\tau = \underbrace{\kappa_{11} \star \kappa_{12} \star \cdots \star \kappa_{1m_1}}_{m_1 \text{ of size } i_1} \star \underbrace{\kappa_{21} \star \cdots \star \kappa_{2m_2}}_{m_2 \text{ of size } i_2} \star \cdots \star \underbrace{\kappa_{\ell 1} \star \cdots \star \kappa_{\ell m_\ell}}_{m_\ell \text{ of size } i_\ell},$$

where each κ_{js} is a column of size i_j for each j and s. Recall from Definition 3.2 that the j-sector $[M_j]$ of τ consists of the concatenation of the columns corresponding to the j-th component i_j in the word I, so $[M_j] := \kappa_{j1} \star \kappa_{j2} \star \cdots \star \kappa_{jm_j}$. For each j with $1 \leq j \leq \ell$, let S_j denote a consecutive subset of the indices $\{1, 2, \ldots, m_j\}$ (indexing the columns in the j-sector), i.e. S_j is of the form $\{p, p+1, \ldots, q\}$ for some p, q with integers $1 \leq p \leq q \leq m_j$. Let $\mathrm{pr}_{S_1, S_2, \ldots, S_\ell}$ denote the projection operator which takes the subtableau of τ given by reading off (in sequence) exactly the columns of each j-sector which are contained in S_j. So for example if $S_j = \{1, 2, \ldots, m_j\}$ for all $j \leq k$ and $S_j = \emptyset$ for all $j > k$ then $\mathrm{pr}_{S_1,\ldots,S_\ell}$ would correspond to reading off the first k sectors of τ. In the rest of this section we always take sets S_j with the property that $\mathrm{pr}_{S_1,\ldots,S_\ell}$ is a projection of the form described in (3.8), namely, that it reads off a subtableau of consecutive entries in τ.

Example 3.11. For example, let $I = (1, 2, 1)$ and $m = (2, 6, 1)$. Consider the tableau $\tau = 21 \star 232323131312 \star 1$. Let $S_1 = \{2\}$, $S_2 = \{1, 2, 3, 4\}$ and $S_3 = \emptyset$. Then $\mathrm{pr}_{S_1,S_2,S_3} = 1 \star 23232313$.

Before stating the next lemma we must set some conventions. First suppose $I = (i_1, \ldots, i_\ell)$ and $m = (m_1, \ldots, m_\ell)$ are fixed. It follows from the definition that the concatenation operator $\bar{w}_{i_1}^{\star m_1}$ sends tableaux of shape $((i_2, \ldots, i_\ell), (m_2, \ldots, m_\ell))$ to those of shape (I, m). Now in addition suppose that for each k with $1 \leq k \leq \ell$, we have a subset $S_k \subseteq \{1, 2, \ldots, m_k\}$ such that $\mathrm{pr}_{S_1,\ldots,S_\ell}$ is a projection of tableaux of shape (I, m) to subtableaux as in the previous discussion. We may also use the same subsets S_k for $2 \leq k \leq \ell$ to define a projection of tableaux of shape $((i_2, \ldots, i_\ell), (m_2, \ldots, m_\ell))$. In the lemma below we denote this projection by $\mathrm{pr}_{S_2,\ldots,S_\ell}$. With these conventions we can now state the following, which is immediate from the definitions.

Lemma 3.12. *Let $I = (i_1, \ldots, i_\ell)$ and $m = (m_1, \ldots, m_\ell)$ and $S_1, S_2, \ldots, S_\ell$ be as above. Let τ be a tableau of shape $((i_2, \ldots, i_\ell), (m_2, \ldots, m_\ell))$. Then*

$$\mathrm{pr}_{S_1,S_2,\ldots,S_\ell}(\bar{w}_{i_1}^{\star m_1} \star \tau) = \bar{w}_{i_1}^{\star |S_1|} \star \mathrm{pr}_{S_2,\ldots,S_\ell}(\tau).$$

We now use Lemmas 3.10 and 3.12 to show that projections of the form $\mathrm{pr}_{S_1,\ldots,S_\ell}$ (where again we assume that this projection reads off consecutive entries in τ) "preserves standard tableaux". In the statement of the lemma below we use that such an operator $\mathrm{pr}_{S_1,\ldots,S_\ell}$ sends a tableau of shape (I,m) to a tableau of shape $(I,(|S_1|,|S_2|,\ldots,|S_\ell|))$.

Lemma 3.13. *Let $I=(i_1,\ldots,i_\ell)$ and $m=(m_1,\ldots,m_\ell)$ and $S_1,S_2,\ldots,S_\ell$ be as above. Then the projection operator $\mathrm{pr}_{S_1,S_2,\ldots,S_\ell}$ maps $\mathcal{T}(I,m)$ to $\mathcal{T}(I,(|S_1|,|S_2|,\ldots,|S_\ell|))$, i.e. if τ is a standard tableau of shape (I,m), then $\mathrm{pr}_{S_1,\ldots,S_\ell}(\tau)$ is a standard tableau of shape $(I,(|S_1|,|S_2|,\ldots,|S_\ell|))$.*

Proof. Recall that $\mathcal{T}(I,m)$ is defined by a sequence of Demazure operators and concatenation operators

$$\mathcal{T}(I,m):=\Lambda_{i_1}(\bar{w}_{i_1}^{\star m_1}\star\Lambda_{i_2}(\bar{w}_{i_2}^{\star m_2}\star\cdots\Lambda_{i_\ell}(\bar{w}_{i_\ell}^{\star m_\ell})\cdots)).$$

We wish to apply $\mathrm{pr}_{S_1,\ldots,S_\ell}$ to the set $\mathcal{T}(I,m)$ and see that it is contained in $\mathcal{T}(I,(|S_1|,|S_2|,\ldots,|S_\ell|))$. We have

$$\begin{aligned}
\mathrm{pr}_{S_1,\ldots,S_\ell}(\mathcal{T}(I,m)) &= \mathrm{pr}_{S_1,\ldots,S_\ell}(\Lambda_{i_1}(\bar{w}_{i_1}^{\star m_1}\star\Lambda_{i_2}(\bar{w}_{i_2}^{\star m_2}\star\cdots\Lambda_{i_\ell}(\bar{w}_{i_\ell}^{\star m_\ell})\cdots)))\\
&\subseteq \Lambda_{i_1}(\mathrm{pr}_{S_1,\ldots,S_\ell}(\bar{w}_{i_1}^{\star m_1}\star\Lambda_{i_2}(\bar{w}_{i_2}^{\star m_2}\star\cdots\Lambda_{i_\ell}(\bar{w}_{i_\ell}^{\star m_\ell})\cdots))) \text{ by Lemma 3.10}\\
&= \Lambda_{i_1}\bar{w}_{i_1}^{\star|S_1|}\mathrm{pr}_{S_2,\ldots,S_\ell}(\Lambda_{i_2}((\bar{w}_{i_2}^{\star m_2}\star\cdots\Lambda_{i_\ell}(\bar{w}_{i_\ell}^{\star m_\ell})\cdots))) \text{ by Lemma 3.12.}
\end{aligned}$$

Repeating the above argument for $\mathrm{pr}_{S_2,\ldots,S_\ell}$ relative to $\mathrm{pr}_{S_3,\ldots,S_\ell}$, and so on, it is straightforward to see that

$$\begin{aligned}
\mathrm{pr}_{S_1,\ldots,S_\ell}(\mathcal{T}(I,m)) &\subseteq \Lambda_{i_1}(\bar{w}_{i_1}^{\star|S_1|}\star\Lambda_{i_2}(\bar{w}_{i_2}^{\star|S_2|}\star\cdots\Lambda_{i_\ell}(\bar{w}_{i_\ell}^{\star|S_\ell|})\cdots))\\
&= \mathcal{T}(I,(|S_1|,|S_2|,\ldots,|S_\ell|))
\end{aligned}$$

as desired. □

In the later arguments in this section, we only use Lemma 3.13 for the cases which lead to a projection of τ to its 1-chain and to its 2-chain. More specifically, using now the indexing set $\mathcal{S} = \{(a, b)\}$ as in Section 3.1, these correspond to the choices

$$S_{1,p} = \{1, 2, \ldots, m_{(1,p)}\} \text{ for } 1 \leq p \leq n-1 \text{ and } S_{a,b} = \emptyset \text{ for } a \neq 1 \tag{3.10}$$

and

$$S_{2,p} = \{1, 2, \ldots, m_{(2,p)}\} \text{ for } 1 \leq p \leq n-2 \text{ and } S_{a,b} = \emptyset \text{ for } a \neq 2 \tag{3.11}$$

respectively. We denote by $\mathrm{pr}_{1-chain}$ (respectively $\mathrm{pr}_{2-chain}$) the projection corresponding to the choices of $S_{(a,b)}$ as in (3.10) (respectively (3.11)).

We record the following Corollary.

Corollary 3.14. *Let $I = I(n)$ and $m = (m_{(a,b)})_{(a,b)\in\mathcal{S}} \in \mathbb{N}^{\ell}$.*

(a) Let $\mathrm{pr}_{1-chain}$ denote, as above, the projection of a tableau τ of shape $(I(n), m)$ to its 1-chain, i.e.,

$$\mathrm{pr}_{1-chain}(\tau) = [M_{1,1}][M_{1,2}]\cdots[M_{1,n-1}].$$

If τ is standard, then $\mathrm{pr}_{1-chain}(\tau)$ is standard of shape $(I(n), (m_{(1,1)}, \ldots, m_{(1,n-1)}, 0, \ldots, 0))$. Thus there is a well-defined map

$$\mathrm{pr}_{1-chain} : \mathcal{T}(I(n), m) \to \mathcal{T}(I(n), (m_{(1,1)}, \ldots, m_{(1,n-1)}, , 0, \ldots, 0)).$$

(b) Let $\mathrm{pr}_{2-chain}$ denote, as above, the projection of a tableau τ of shape $(I(n), m)$ to its 2-chain, i.e.,

$$\mathrm{pr}_{2-chain}(\tau) = [M_{2,1}][M_{2,2}]\cdots[M_{2,n-1}].$$

If τ is standard, then $\mathrm{pr}_{2-chain}(\tau)$ is standard of shape $(I(n), (0, \ldots, 0, m_{(2,1)}, \ldots, m_{(2,n-2)}, 0, \ldots, 0))$. Thus there is a well-defined map

$$\mathrm{pr}_{2-chain} : \mathcal{T}(I(n), m) \to \mathcal{T}(I(n), (0, \ldots, 0, m_{(2,1)}, \ldots, m_{(2,n-1)}, 0, \ldots, 0)).$$

Example 3.15. Let $I = I(3) = (1, 2, 1)$ and $m = (0, 1, 1)$. From Example 2.20 we know $\mathcal{T}(I, m) = \{121, 131, 231, 232, 122, 132, 133, 233\}$. For $\tau = 132$, $[M_1] = \emptyset$, $[M_2] = 13$, and $[M_3] = 2$. Therefore, $\mathrm{pr}_{1-chain}(\tau) = 13$ and $\mathrm{pr}_{2-chain}(\tau) = 2$. We can compute the standard monomials $\Theta_{\mathrm{pr}_{1-chain}}(\tau) = t_2$ and $\Theta_{\mathrm{pr}_{2-chain}}(\tau) = t_1 + t_3$, using the coordinates on $Gr(1) \times Gr(2) \times Gr(1)$ from Example 2.23.

3.3 Injectivity for the Case $n = 3$

In this section, we explore the case where $n = 3$ and $I = I(3)$, so throughout this section we fix $I = (1, 2, 1)$ and $m = (m_1, m_2, m_3)$. Although we prove the general result in the next section (Proposition 3.21), the concrete computations for this special case serve as a warm-up for the general case.

We first recall the algorithm which produces the set of standard tableaux of shape $(I(3), m) = ((1, 2, 1), (m_1, m_2, m_3))$. By Definition 2.19, $\mathcal{T}((1, 2, 1), (m_1, m_2, m_3))$ is the set

$$\Lambda_1(1^{\star m_1} \star \Lambda_2((12)^{\star m_2} \star \Lambda_1(1^{\star m_3}))) \tag{3.12}$$

where for any $m_i \in \mathbb{N}$, $1^{\star m_i}$ denotes the word $11 \cdots 1$ with m_i copies of 1, and $(12)^{\star m_i}$ denotes the word $1212 \cdots 12$ with m_i copies of 12. To describe this set explicitly we proceed step by step. First, from the definition of the Demazure operator Λ_1, we have

$$\Lambda_1(1^{\star m_3}) = \{\underbrace{11 \cdots 1}_{m_3 \text{ times}}, f_1(11 \cdots 1), f_1^2(11 \cdots 1), \ldots\} \setminus \{\mathbf{O}\}.$$

From the definition of the operator f_1, which successively changes certain 1's into 2's, it is not difficult to see that

$$\Lambda_1(1^{\star m_3}) = \{\tau_0 := \underbrace{1\cdots 1}_{\substack{m_3 \text{ copies}\\ \text{of } 1}}, \tau_1 := 2\underbrace{1\cdots 1}_{\substack{m_3-1 \text{ copies}\\ \text{of } 1}}, \tau_2 := 22\underbrace{1\cdots 1}_{\substack{m_3-2 \text{ copies}\\ \text{of } 1}}, \ldots, \tau_{m_3} := \underbrace{2\cdots 2}_{\substack{m_3 \text{ copies}\\ \text{of } 2}}\},$$

where we use τ_k to denote the word with k 2's on the left and $m_3 - k$ 1's on the right.

Next we must apply the Demazure operator Λ_2 to the set

$$\begin{aligned}(12)^{\star m_2} \star \{\tau_0, \ldots, \tau_{m_3}\} &= \{(12)^{\star m_2} \star \tau_0, (12)^{\star m_2} \star \tau_1, \ldots, (12)^{\star m_2} \star \tau_{m_3}\}\\ &= \{\underbrace{1212\cdots 12}_{\substack{m_2 \text{ copies}\\ \text{of } 12}}\underbrace{11\cdots 1}_{\substack{m_3 \text{ copies}\\ \text{of } 1}}, \underbrace{1212\cdots 12}_{\substack{m_2 \text{ copies}\\ \text{of } 12}}2\underbrace{1\cdots 1}_{\substack{m_3-1 \text{ copies}\\ \text{of } 1}}, \ldots, \underbrace{1212\cdots 12}_{\substack{m_2 \text{ copies}\\ \text{of } 12}}\underbrace{22\cdots 2}_{\substack{m_3 \text{ copies}\\ \text{of } 2}}\}.\end{aligned}$$

We analyze each $(12)^{\star m_2} \star \tau_k$ separately. Let $k \in \{0, 1, \ldots, m_3\}$. Then by definition of Demazure operators, Λ_2 successively changes 2's to 3's starting from the left, so we have

$$\begin{aligned}\Lambda_2((12)^{\star m_2} \star \tau_k) &= \Lambda_2(\underbrace{1212\cdots 12}_{\substack{m_2 \text{ copies}\\ \text{of } 12}}\underbrace{22\cdots 2}_{\substack{k \text{ copies}\\ \text{of } 2}}\underbrace{1\cdots 1}_{\substack{m_3-k \text{ copies}\\ \text{of } 1}})\\
&= \{\underbrace{1212\cdots 12}_{\substack{m_2 \text{ copies}\\ \text{of } 12}}\underbrace{2\cdots 2}_{\substack{k \text{ copies}\\ \text{of } 2}}\underbrace{1\cdots 1}_{\substack{m_3-k \text{ copies}\\ \text{of } 1}}, 13\underbrace{12\cdots 12}_{\substack{m_2-1 \text{ copies}\\ \text{of } 12}}\underbrace{2\cdots 2}_{\substack{k \text{ copies}\\ \text{of } 2}}\underbrace{1\cdots 1}_{\substack{m_3-k \text{ copies}\\ \text{of } 1}},\\
&\quad 1313\underbrace{12\cdots 12}_{\substack{m_2-2 \text{ copies}\\ \text{of } 12}}\underbrace{2\cdots 2}_{\substack{k \text{ copies}\\ \text{of } 2}}\underbrace{1\cdots 1}_{\substack{m_3-k \text{ copies}\\ \text{of } 1}}, \ldots, \underbrace{1313\cdots 13}_{\substack{m_2 \text{ copies}\\ \text{of } 13}}\underbrace{2\cdots 2}_{\substack{k \text{ copies}\\ \text{of } 2}}\underbrace{1\cdots 1}_{\substack{m_3-k \text{ copies}\\ \text{of } 1}},\\
&\quad \underbrace{1313\cdots 13}_{\substack{m_2 \text{ copies}\\ \text{of } 13}}3\underbrace{2\cdots 2}_{\substack{k-1 \text{ copies}\\ \text{of } 2}}\underbrace{1\cdots 1}_{\substack{m_3-k \text{ copies}\\ \text{of } 1}}, \underbrace{1313\cdots 13}_{\substack{m_2 \text{ copies}\\ \text{of } 13}}33\underbrace{2\cdots 2}_{\substack{k-2 \text{ copies}\\ \text{of } 2}}\underbrace{1\cdots 1}_{\substack{m_3-k \text{ copies}\\ \text{of } 1}},\\
&\quad \ldots, \underbrace{1313\cdots 13}_{\substack{m_2 \text{ copies}\\ \text{of } 13}}\underbrace{3\cdots 3}_{\substack{k \text{ copies}\\ \text{of } 3}}\underbrace{1\cdots 1}_{\substack{m_3-k \text{ copies}\\ \text{of } 1}}\},\end{aligned} \tag{3.13}$$

so there are m_2+k+1 distinct words contained in $\Lambda_2((12)^{\star m_2}\star\tau_k)$ corresponding to the number (ranging between 0 and m_2+k) of left-most 2's that have been changed to 3's. The set $\Lambda_2((12)^{\star m_2} \star \Lambda_1(1^{\star m_3})) = \Lambda_2((12)^{\star m_2} \star \{\tau_0, \ldots, \tau_{m_3}\})$

is then obtained by taking the union over all $k \in \{0, 1, \ldots, m_3\}$ of the sets $\Lambda_2((12)^{\star m_2} \star \tau_k)$ as described in (3.13).

Finally, the concatenation with $1^{\star m_1}$ and the Demazure operator Λ_1 adds a word with all 1's and changes certain 1's to 2's. We note that both of these operations do not alter the 3's which are present in the word.

The discussion so far leads to the following.

Lemma 3.16. *Let $\tau \in \mathcal{T}(I(3) = (1, 2, 1), m = (m_1, m_2, m_3))$. Let $[M_2]$ and $[M_3]$ denote the 2-sector and 3-sector of τ, respectively. If 12 appears as a subword of $[M_2]$, then 3 does not appear as a subword of $[M_3]$.*

Proof. As already observed above, the last step in producing the set of standard tableaux (3.12) is the concatenation with $1^{\star m_1}$ and the Demazure operator Λ_1, but these do not affect the presence or absence of 3's in the 3-sector of $[M_3]$. Moreover, for $\tau \in \mathcal{T}(I(3), m)$, by definition (3.12) there must exist $a \in \mathbb{N}$ and $\tau' \in \Lambda_2((12)^{\star m_2} \star \Lambda_1(1^{\star m_3}))$ such that $\tau = f_1^a(1^{\star m_1} \star \tau')$, and the 2-sector $[M_2]$ of τ' corresponds to the (location of) the subwords

$$\underbrace{1313 \cdots 13}_{\substack{s \text{ copies} \\ \text{of } 13}} \; \underbrace{12 \cdots 12}_{\substack{m_2 - s \text{ copies} \\ \text{of } 12}} \tag{3.14}$$

for $s \in \{0, \ldots, m_2\}$ in the words explicitly listed in (3.13).

In particular, it follows from the definition of f_1 (which changes 1's to 2's) that if a 12 appears as a subword in the 2-sector $[M_2]$ of τ, then a 12 had to have already appeared in the subword (3.14) of τ'. Therefore, in order to prove the claim of the lemma it suffices to prove that for any $k \in \{0, 1, \ldots, m_3\}$ and any word τ' in $\Lambda_2((12)^{\star m_2} \star \tau_k)$ as listed in (3.13), if a 12 appears in the subword of the form (3.14), then a 3 does not appear in the last m_3 places of τ' (corresponding to the 3-sector $[M_3]$ of τ). This can be seen directly by examining the words listed in (3.13).

□

Remark 3.17. Intuitively, the lemma can be explained by noting that because Λ_2 changes 2's to 3's "starting from the left", if there are some 12's remaining in the left part of a word, then the 2's in the right part of the word cannot have already been changed to 3's.

We also have the following.

Lemma 3.18. *Let $\tau \in \mathcal{T}(I(3) = (1,2,1), m = (m_1, m_2, m_3))$. Let $[M_1]$ and $[M_2]$ denote the 1-sector and 2-sector of τ, respectively. Then if 23 appears as a subword of $[M_2]$ then 1 does not appear in $[M_1]$, and in particular $[M_1] = 222\cdots 2$ (m_1 times).*

Proof. By definition, a standard tableau $\tau \in \mathcal{T}(I(3), m)$ is in the set

$$\Lambda_1(\underbrace{11\cdots 1}_{m_1 \text{ times}} \star \Lambda_2(\underbrace{1212\cdots 12}_{m_2 \text{ times}} \star \Lambda_3(\underbrace{11\cdots 1}_{m_3 \text{ times}}))).$$

In particular, the 2-sector of a standard tableau τ is of the form

$$\mathrm{pr}(f_1^{\alpha_1}(11\cdots 1 \star f_2^{\alpha_2}(12\cdots 12))$$

for some $\alpha_1, \alpha_2 \in \mathbb{N}$, where pr denotes the projection to the last $2m_2$ entries in this word of length $m_1 + 2m_2$. Since f_2 successively raises a 2 to a 3, starting from the left, it follows that $f_2^{\alpha_2}(12\cdots 12) = \underbrace{13\cdots 13}_{\alpha_2 \text{ times}}\ \underbrace{12\cdots 12}_{m_2-\alpha_2 \text{ times}}$, i.e. the leftmost α_2 copies of 12 get changed to 13's.

The next steps are a concatenation with $11\cdots 1 = 1^{\star m_1}$ and an application of $f_1^{\alpha_1}$ to $11\cdots 1 \star f_2^{\alpha_2}(12\cdots 12) = 11\cdots 1 \star 13\cdots 13 \star 12\cdots 12$. The only way for a 23 to appear in the 2-sector is for a 13 to be changed to a 23 due to the fact that f_1 raises 1's to 2's. However, by the definition of f_1, a 1 appearing in the 2-sector cannot be raised unless the leftmost m_1 copies of 1 have already been raised to a 2. Thus, if a 23 appears in $[M_2]$, then a 1 cannot appear in $[M_1]$, as claimed. $\square$

In Example 2.23 we saw that for $I = (1, 2, 1)$, standard monomials can be realized as products of minors of the coordinates

$$\left[\begin{pmatrix} 1 \\ t_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t_1 & 1 \\ 0 & t_2 \end{pmatrix}, \begin{pmatrix} 1 \\ t_1 + t_3 \\ t_2 t_3 \end{pmatrix}\right] \in Gr(1) \times Gr(2) \times Gr(1), \tag{3.15}$$

where $\Theta_1 x^{(1)} = 1$, $\Theta_2 x^{(1)} = t_1$, $\Theta_{12} x^{(2)} = 1$, $\Theta_{13} x^{(2)} = t_2$, $\Theta_{23} x^{(2)} = t_1 t_2$, $\Theta_1 x^{(3)} = 1$, $\Theta_2 x^{(3)} = t_1 + t_3$, and $\Theta_3 x^{(3)} = t_2 t_3$.

Before proving the main proposition of this section, namely Proposition 3.20, we need to prove the following lemma. Let W denote the set of polynomials in the variables obtained by taking products of minors in the matrices shown in (3.15). More precisely, a polynomial is in W exactly if it is of the form $(A_1(t_1, t_2, t_3))(A_2(t_1, t_2, t_3))(A_3(t_1, t_2, t_3))$, where $A_1(t_1, t_2, t_3)$ is a product of 1×1 minors of $\begin{pmatrix} 1 \\ t_1 \\ 0 \end{pmatrix}$, $A_2(t_1, t_2, t_3)$ is a product of 2×2 minors of $\begin{pmatrix} 1 & 0 \\ t_1 & 1 \\ 0 & t_2 \end{pmatrix}$, and $A_3(t_1, t_2, t_3)$ is a product of 1×1 minors of $\begin{pmatrix} 1 \\ t_1 + t_3 \\ t_2 t_3 \end{pmatrix}$. The possible minors that arise are listed after (3.15), so in particular any such product in W must be of the form

$$t_1^{k_1} t_2^{k_2} t_3^{k_3} (t_1 + t_3)^{k_4} \tag{3.16}$$

for some non-negative integers k_i. Let $I(3) = (1, 2, 1)$ and fix $m = (m_1, m_2, m_3) \in \mathbb{N}^3$.

Lemma 3.19. *For any polynomial $f \in W$, there exists at most one standard tableau $\tau \in \mathcal{T}(I(3), m)$ such that $f = \Theta_\tau$.*

Proof. Suppose $f = t_1^{k_1} t_2^{k_2} t_3^{k_3} (t_1 + t_3)^{k_4}$ as in (3.16) for a fixed (unique) set of $k_1, k_2, k_3, k_4 \in \mathbb{N}$ and suppose there exists $\tau \in \mathcal{T}(I(3), m)$ with $f = \Theta_\tau$. We wish to show τ is unique. Let a_1 and a_2 denote the number of 1's and 2's in $[M_1]$ respectively, b_{23}, b_{13}, b_{12} the number of 23's, 13's, and 12's in $[M_2]$ respectively, and c_1, c_2 and c_3 the number of 1's, 2's, and 3's in $[M_3]$ respectively. Then from the definition of Θ_τ and the list of minors of matrices in (3.15) as above we conclude

$$\Theta_\tau = t_1^{a_2} t_2^{b_{13}} (t_1 t_2)^{b_{23}} (t_1 + t_3)^{c_2} (t_2 t_3)^{c_3} = t_1^{a_2 + b_{23}} t_2^{b_{13} + b_{23} + c_3} t_3^{c_3} (t_1 + t_3)^{c_2}.$$

We are assuming $f = t_1^{k_1} t_2^{k_2} t_3^{k_3} (t_1 + t_3)^{k_4}$ is equal to Θ_τ, so we immediately see that

$$a_2 + b_{23} = k_1 \tag{3.17}$$

$$b_{13} + b_{23} + c_3 = k_2 \tag{3.18}$$

$$c_3 = k_3 \tag{3.19}$$

$$c_2 = k_4. \tag{3.20}$$

We would like to assert that a_2, b_{13}, b_{23}, c_2 and c_3 are uniquely determined by the constraints above. Being a system of 4 equations in 5 variables, this may seem impossible, but we are aided by Lemma 3.18 as follows. Suppose $b_{23} \neq 0$. Then by Lemma 3.18 we know that $a_2 = m_1$ and hence $b_{23} = k_1 - m_1 > 0$ by the first equation; this uniquely determines a_2 and b_{23}. Evidently c_3 and c_4 are already uniquely determined by the last 2 equations, and now b_{13} is uniquely determined by the second equation (and the fixed values of b_{23} and c_3). On the other hand, if $b_{23} = 0$, then evidently the given equations uniquely determine the other variables c_2, c_3, b_{13}, and a_2. Now by definition of these parameters we have

$$a_1 + a_2 = m_1$$
$$b_{12} + b_{13} + b_{23} = m_2$$
$$c_1 + c_2 + c_3 = m_3$$

from which we see that the determination of a_2, b_{13}, b_{23}, c_2 and c_3 also determines a_1, b_{12}, and c_1. From this it follows that the composition of each sector $[M_1], [M_2]$ and $[M_3]$ is uniquely determined by f, i.e., there exists at most one $\tau = [M_1] \star [M_2] \star [M_3]$ with the property that $\Theta_\tau = f$, as desired. $\square$

Proposition 3.20. *Let $m = (m_1, m_2, m_3) \in \mathbb{N}^3$. Then $\nu : \Theta(\mathcal{T}(I(3), m)) \longrightarrow \mathbb{Z}^3$ is injective, i.e. the valuation ν takes distinct values on distinct standard monomials in $\Theta(\mathcal{T}(I(3), m))$.*

Proof. We begin by noting that $\Theta(\mathcal{T}(I(3), m)) \subset W$ by definition. Let us fix a $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}^3$. To prove the proposition, it suffices to show that there exists at most one $f \in W$ such that $\nu(f) = \gamma$ and $f \in \Theta(\mathcal{T}(I(3), m))$.

Suppose $f \in W$ and $\nu(f) = \gamma$. Recall that ν is the lowest term valuation with respect to the variables t_1, t_2, t_3, so $\nu(t_1^{k_1} t_2^{k_2} t_3^{k_3} (t_1 + t_3)^{k_4}) = t_1^{k_1} t_2^{k_2} t_3^{k_3+k_4}$. Thus in order to have $\nu(f) = \gamma = (\gamma_1, \gamma_2, \gamma_3)$, there must exist some $i \in \mathbb{Z}$, $0 \le i \le \gamma_3$, such that $f = t_1^{\gamma_1} t_2^{\gamma_2} t_3^{i} (t_1 + t_3)^{\gamma_3 - i}$. For such an i we let f_i denote the polynomial

$$t_1^{\gamma_1} t_2^{\gamma_2} t_3^{i} (t_1 + t_3)^{\gamma_3 - i}. \tag{3.21}$$

We begin by considering the $\gamma_2 \le m_2$ case. If $f_i = t_1^{\gamma_1} t_2^{\gamma_2} t_3^{i} (t_1 + t_3)^{\gamma_3 - i}$ is a standard monomial in $\Theta(\mathcal{T}(I(3), m))$ for some m, then $f_i = \Theta_\tau$ for some $\tau \in \mathcal{T}(I(3), m)$. If $\gamma_2 \le m_2$, then there must be a 12 in $[M_2]$, since the other two possible minors in $[M_2]$ contain a power of t_2. By Lemma 3.16 this means that there are no 3's in $[M_3]$. Therefore, there must exist a monomial in f_i

which does not contain a positive power of t_3, since the only minor with a t_3 in every term is the minor corresponding to 3 in $[M_3]$. This implies that $i = 0$.

Next we consider the case in which $\gamma_2 > m_2$. Again suppose there exists $\tau \in \mathcal{T}(I(3), m)$ such that $f_i = \Theta_\tau$. Note that there are no minors in $[M_1]$ which contain a positive power of t_2. Moreover, we note that the only minor in $[M_3]$ which contains a positive power of t_2 is the minor corresponding to 3.

Since $\gamma_2 - m_2 > 0$, the above observations imply that there must be a 3 in $[M_3]$. By Lemma 3.16 this means that there are no 12's in $[M_2]$. Therefore, the columns of $[M_2]$ together contribute a power of $t_2^{m_2}$ to all monomials in f_i. This means that $[M_3]$ must contribute a $t_2^{\gamma_2 - m_2}$. Therefore, there are exactly $\gamma_2 - m_2$ 3's in $[M_3]$. This implies that $i = \gamma_2 - m_2$.

We can now prove that for a fixed γ and m there exists at most one $f \in \Theta(\mathcal{T}(I(3), m))$ with $\nu(f) = \gamma$. Indeed, if $m_2 \geq \gamma_2$ then from above we see that f must be of the form $f_0 = t_1^{\gamma_1} t_2^{\gamma_2}(t_1 + t_3)^{\gamma_3}$, and by Lemma 3.19 there exists at most one $\tau \in \mathcal{T}(I(3), m)$ such that $\Theta_\tau = f_0 = f$. Similarly if $m_2 < \gamma_2$ then we must have $f = f_{\gamma_2 - m_2}$ and again by Lemma 3.19 there is at most one τ with $\Theta_\tau = f_{\gamma_2 - m_2} = f$. This proves the claim.

□

3.4 Injectivity for the General Case

We now make the injectivity argument in the general case, as follows. Let $n \in \mathbb{N}$ with $n \geq 3$. Let $I = I(n)$ be the corresponding sequence as in Definition 3.5 and recall that for $I(n)$ we will find it convenient to use the indexing set $\mathcal{S} = \{(a, b) \mid 1 \leq a \leq n-1, 1 \leq b \leq n-a\}$ as in (3.3). Recall also from

Definition 3.6 that for $\tau \in \mathcal{T}(I(n), m)$ we call

$$[M_{1,1}][M_{1,2}] \cdots [M_{1,n-1}] \text{ and } [M_{2,1}][M_{2,2}] \cdots [M_{2,n-2}]$$

the 1-chain and the 2-chain of τ, respectively. Remembering that $I(n)$ is associated to the reduced word decomposition

$$(s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2)(s_1)$$

of the longest element in the permutation (Weyl) group S_n, the 1-chain and the 2-chain can be interpreted as the parts of τ associated to the leftmost two subwords $(s_1 s_2 \cdots s_{n-1})$ and $(s_1 s_2 \cdots s_{n-2})$, respectively, of this word decomposition. Now let $m = (m_{(a,b)})_{(a,b)\in\mathcal{S}} \in \mathbb{N}^\ell$ where $\ell = \frac{n(n-1)}{2}$ and we use the same indexing set $\mathcal{S}$ as for $I(n)$, so e.g. the (a,b)-sector $[M_{a,b}]$ of τ contains $m_{(a,b)}$ columns of size $i_{(a,b)}$.

Throughout this section we will assume that $\mathbf{m_{(a,b)} = 0}$ for $\mathbf{a > 2}$, or in other words, that $\tau \in \mathcal{T}(\mathbf{I(n), m})$ consists only of a 1-chain and a 2-chain, and all k-chains for higher k are empty. We now fix such an $\mathbf{m} \in \mathbb{N}^{\boldsymbol{\ell}}$.

The main result of this section, and this chapter, is the next proposition.

Proposition 3.21. *Let $I = I(n)$ and $m = (m_{(1,1)}, \ldots, m_{(2,n-2)}, 0, \ldots, 0) \in \mathbb{N}^\ell$ be as above and let $\mathcal{T}(I(n), m)$ and $\Theta(\mathcal{T}(I(n), m))$ be the corresponding sets of standard tableaux and standard monomial basis elements as in Section 2.4.2. Let $\nu : \Theta(\mathcal{T}(I(n), m)) \to \mathbb{Z}^\ell$ denote the lowest term valuation defined in Section 2.2 (restricted to $\Theta(\mathcal{T}(I(n), m))$). Then ν is injective on $\Theta(\mathcal{T}(I(n), m))$.*

Remark 3.22. We chose $I = I(n)$ and the lowest-term valuation ν because they happened to behave well with respect to the standard monomial basis. See Remark 3.36 and Example 3.35 at the end of this section for a discussion about why these restrictive conditions were placed on m.

Before proving Proposition 3.21, we must establish some notation and lemmas.

Given a column $\kappa = (r_1, \ldots, r_j)$ in the (s,j)-th sector, recall that by definition (see (2.12))

$$\Theta_\kappa(x^{(s,j)}) = \det_{j\times j}\begin{pmatrix} x_{r_1 1} & \cdots & x_{r_1 j} \\ \vdots & \ddots & \vdots \\ x_{r_j 1} & \cdots & x_{r_j j}\end{pmatrix} \in H^0(Gr(j,n), p_j^*(O(1))) \qquad (3.22)$$

where $x^{(s,j)} \in Gr(j,n)$ denotes the coordinate entries of $\mu_{I(n)}(Z_{I(n)}) \subset Gr(I(n)) = Gr(1,n) \times Gr(2,n) \times \cdots \times Gr(n-1,n) \times Gr(1,n) \times \cdots \times Gr(n-2,n)$, as defined in (2.6) and the matrix entries in the right hand side of (3.22) refers to the $j \times j$ submatrix of $Gr(j,n)$ whose minor gives us the Plücker coordinate $P_{r_1,\ldots,r_j}$ (see §1.3, (1.3)).

In what follows it will be helpful to work out an analogue of Example 2.23 in our general case. More specifically, since our standard monomials $\Theta(\mathcal{T}(I(n),m))$ are defined by pullbacks under $\mu_{I(n)} \circ \Phi$ it will be useful to have explicit matrix representations of the elements of $Gr(i_{(a,b)}, n)$ corresponding to a point $(t_{a,b})_{(a,b)\in\mathcal{S}} \in \mathbb{C}^\ell$ under the composition

$$\mathbb{C}^\ell \xrightarrow{\Phi} Z_{I(n)} \xrightarrow{\mu_{I(n)}} Gr(I(n)) = \prod_{(a,b)\in\mathcal{S}} Gr(i_{(a,b)}, n).$$

Since $\mu_{I(n)}$ is defined by successive products in (2.5) and because Φ sends each $t_{a,b}$ to

$$exp(t_{a,b}F_{\alpha_{i_{(a,b)}}}) = exp(t_{a,b}F_{\alpha_b}) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & t_{a,b} & \ddots & \\ & & & & 1 \end{pmatrix}$$

(where the $t_{a,b}$ appears immediately below the main diagonal in the b-th column), it is not difficult to see that in the first $n-1$ factors of $Gr(I(n))$

corresponding to the 1-chain, these matrices are as follows:

$$\begin{pmatrix} 1 \\ t_{1,1} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ t_{1,1} & 1 \\ 0 & t_{1,2} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ t_{1,1} & 1 & 0 & \cdots & 0 \\ 0 & t_{1,2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & t_{1,n-1} \end{pmatrix}.$$

Similarly it is not difficult to see that the next $n-2$ components of $\mu_{I(n)}(Z_{I(n)})$ are of the form:

$$\begin{pmatrix} 1 \\ t_{1,1}+t_{2,1} \\ t_{2,1}t_{1,2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ t_{1,1}+t_{2,1} & 1 \\ t_{2,1}t_{1,2} & t_{1,2}+t_{2,2} \\ 0 & t_{2,2}t_{1,3} \\ \vdots & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ t_{1,1}+t_{2,1} & 1 & 0 & \cdots & 0 \\ t_{2,1}t_{1,2} & t_{1,2}+t_{2,2} & \ddots & \ddots & \vdots \\ 0 & t_{2,2}t_{1,3} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & t_{2,n-2}+t_{1,n-2} \\ 0 & \cdots & \cdots & 0 & t_{2,n-2}t_{1,n-1} \end{pmatrix}.$$

We note that strictly speaking, following the conventions of Section 1.3, we should represent elements of $Gr(j,n)$ by $n \times n$ matrices, not $n \times j$ matrices. But because an element in $Gr(j,n)$ is independent of the rightmost $n-j$ columns in such a matrix representation, we have cut off from the notation the $n \times (n-j)$ matrix consisting of the right $n-j$ columns.

We will frequently refer back to these matrices and the terms in each entry. The following lemma summarizes this information and also computes lowest terms.

Lemma 3.23.

a) Let $x^{(1,j)}$ denote the $n \times j$ matrix representing the component in $Gr(i_{1,j} = j, n)$ of $\mu_{I(n)} \circ \Phi((t_{a,b})_{(a,b)\in\mathcal{S}})$ corresponding to the index $(1,j)$. The (i,k)-th *entry of* $x^{(1,j)}$ *equals 1 if* $i = k$, $t_{1,k}$ *if* $i = k+1$, *and 0 otherwise. In particular, with respect to the usual lexicographic order (Definition 1.3), its lowest term equals 1 if* $i = k$, $t_{1,k}$ *if* $i = k+1$, *and 0 otherwise.*

b) Let $x^{(2,j)}$ *denote the* $n \times j$ *matrix representing the component in* $Gr(i_{(2,j)} = j, n)$ *of* $\mu_{I(n)} \circ \Phi((t_{a,b})_{(a,b)\in\mathcal{S}})$ *corresponding to the index* $(2,j)$. *The* (i,k)*-th entry of* $x^{(2,j)}$ *equals 1 if* $i = k$, $t_{1,k} + t_{2,k}$ *if* $i = k+1$, $t_{1,k+1}t_{2,k}$ *if* $i = k+2$, *and 0 otherwise. In particular, with respect to the usual lexicographic order, its lowest term equals 1 if* $i = k$, $t_{2,k}$ *if* $i = k+1$, $t_{1,k+1}t_{2,k}$ *if* $i = k+2$, *and 0 otherwise.*

Using the above, we next compute the lowest term of a minor of $x^{(s,b)}$ with respect to the same lexicographic order.

Lemma 3.24. *Let* $(s,b) \in \mathcal{S}$ *with* $s \in \{1,2\}$ *and let* $x^{(s,b)}$ *be the* $n \times b$ *matrix representing a component of* $\mu_{I(n)} \circ \Phi$ *as in Lemma 3.23. Let* $\kappa = (r_1, r_2, \ldots, r_b) \in \mathbb{N}^b$ *with* $r_j \in \{1, 2, \ldots, n\}$ *for all* j *and* $r_1 < r_2 < \cdots < r_b$, *and let* $A_\kappa = [a_{ij}]$ *denote the* $b \times b$ *submatrix of* $x^{(s,b)}$ *obtained by taking the* r_1*-th,* r_2*-th, ... and* r_b*-th rows of* $x^{(s,b)}$. *Then the lowest term with respect to lexicographic order of the determinant* $\det A_\kappa$ *of* A_κ *is the product of the lowest terms of the diagonal entries of* A_κ.

Proof. We begin with some preliminary arguments, for which we treat the $s = 1$ and $s = 2$ cases separately. First suppose $s = 1$, so that $x^{(s,b)}$ represents a component of $Gr(I(n))$ associated to the 1-chain. By Lemma 3.23 *a)*, we know that the (i,k)-th entry in $x^{(1,j)}$ is 1 if $i = k$, $t_{1,k}$ if $k = i-1$, and 0 otherwise. Recall that with respect to our lexicographic order, we have $\nu(1) < \nu(t_{1,n-1}) < \nu(t_{1,n-2}) < \cdots < \nu(t_{1,1})$. Therefore, the non-zero entries of $x^{(1,j)}$ strictly increase from right to left along rows, and strictly increase down columns. Hence the submatrix A_κ also has this property.

Recall that the determinant of a $b \times b$ matrix can be expressed as a sum over the index set S_b, the set of permutations on b letters, i.e. $\det A_\kappa = \sum_{w \in S_b} \mathrm{sgn}(w) a_{w(1),1} a_{w(2),2} \cdots a_{w(b),b}$. We claim that in the case of $\det A_\kappa$, the summands corresponding to distinct permutations are distinct monomials. The fact that they are monomials follows immediately from the fact that each matrix entry in A_κ is a monomial by Lemma 3.23. Next suppose σ, σ' are distinct permutations in S_b. Since we are comparing summands in the determinant, we may assume without loss of generality that the terms in $\det A_\kappa$ corresponding to σ and σ' are both non-zero. Suppose further, without loss of generality, that $\sigma < \sigma'$ with respect to standard lexicographic order when σ and σ' are written in one-line notation. Let k denote the first (leftmost) entry where $\sigma(k) < \sigma'(k)$. Since the terms corresponding to σ and σ' are non-zero, we must have $a_{\sigma(k),k} = 1$ and $a_{\sigma'(k),k} = t_{1,k}$. Since the variable $t_{1,k}$ does not appear in any other column of A_κ, this implies $t_{1,k}$ does not appear with a positive exponent in the term corresponding to σ, while it does appear in that of σ'. In particular, the terms are distinct, as claimed.

Now we consider the case $s = 2$, so $x^{(s,b)}$ corresponds to a component in the 2-chain. By Lemma 3.23 *b)*, the (i,k)-th entry in $x^{(2,j)}$ is 1 if $i = k$, $t_{1,k} + t_{2,k}$ if $k = i - 1$, $t_{1,k+1}t_{2,k}$ if $k = i - 2$, and 0 otherwise, with lowest terms 1, $t_{2,k}$, $t_{1,k+1}t_{2,k}$ and 0 respectively. In particular the non-constant entries of $x^{(2,j)}$ are all distinct, and as in the $s = 1$ case, the (valuations of the) lowest terms strictly increase down columns and strictly increase from right to left along rows. Hence the submatrix A_κ also has this property.

Similarly to the $s = 1$ case, we now claim that each non-zero term in $\det(A_\kappa)$ has a distinct lowest term (with respect to lexicographic order). Following the arguments for the $s = 1$ case above, suppose without loss of generality that $\sigma, \sigma' \in S_b$ are distinct, $\sigma < \sigma'$ in lexicographic order, and both correspond to non-zero terms in $\det A_\kappa$. The lowest term of the summand corresponding to σ (respectively σ') will be the product of the lowest terms of the matrix entries $a_{\sigma(k),k}$ (respectively $a_{\sigma'(k),k}$) for $1 \leq k \leq b$. Let k denote the first index where $\sigma(k) < \sigma'(k)$. Since the only non-zero entries in the k-th column occur in rows k, $k+1$ and $k+2$, and because $\sigma(k) < \sigma'(k)$ by assumption, we conclude $\sigma'(k)$ must equal either $k+1$ or $k+2$, so $a_{\sigma'(k),k}$ is either $t_{1,k} + t_{2,k}$ or $t_{1,k+1}t_{2,k}$.

Note that $t_{1,k+1}$ only appears in a lowest term in the k-th column. Therefore, if $a_{\sigma'(k),k} = t_{1,k+1}t_{2,k}$, then $a_{\sigma(k),k} \neq t_{1,k+1}t_{2,k}$ and so $t_{1,k+1}$ does not appear in the lowest term of the term corresponding to σ. Similarly, note that $t_{2,k}$ only appears in the k-th column. Therefore, if $a_{\sigma'(k),k} = t_{1,k} + t_{2,k}$, then $a_{\sigma(k),k} = 1$, and hence $t_{2,k}$ does not appear in the lowest term of the term corresponding to σ. Therefore, the summands corresponding to σ and σ' must have distinct lowest terms.

From the above arguments, it follows that there are no cancellations of lowest terms in the computation of $\det A_\kappa$ as a sum over the set of permutations S_b. Hence the lowest term of $\det A_\kappa$ is simply the lowest among the lowest terms of the summands corresponding to σ, as σ ranges over S_b.

We now argue that this lowest term (for $s = 1$ or 2) is that of the summand corresponding to the identity permutation. To see this, consider a non-identity permutation $\sigma \in S_b$ corresponding to a non-zero summand in $\det A_\kappa$. Let k be the largest integer such that $\sigma(k) < k$ (which exists since $\sigma \neq \mathrm{id}$). Since σ is a permutation, there must exist q such that $\sigma(q) = k$, and by the maximality of k we must have $q < k$. Schematically we get the diagram of matrix entries:

$$
\begin{array}{rc}
 & \begin{array}{cccccc} & q & & & k & \end{array} \\
\begin{array}{r} \\ q \\ \\ \\ k=\sigma(q) \\ \\ \end{array} &
\left(\begin{array}{cccccc}
\ddots & \vdots & & & \vdots & \\
\cdots & a_{q,q} & \cdots & \cdots & \vdots & \cdots \\
 & \vdots & \ddots & & a_{\sigma(k),k} & \\
 & \vdots & & \ddots & \vdots & \\
\cdots & a_{\sigma(q)=k,q} & \cdots & \cdots & a_{k,k} & \cdots \\
 & \vdots & & & \vdots & \ddots
\end{array}\right).
\end{array}
$$

Next we claim that the lowest term of $a_{\sigma(q)=k,q}$ contains a positive power of a variable which is strictly larger than all of the variables which appear in the lowest terms of $a_{q,q}$ and $a_{k,k}$. For this we take cases. Suppose $s = 1$. Then, as in the argument above, since there are only two non-zero entries in any given column of A_κ, we must have $a_{q,q} = 1$ and $a_{\sigma(q)=k,q} = t_{1,q}$. Similarly, there are only at most two non-zero entries in any given row, so $a_{k,k} = 1$. The variable $t_{1,q}$ appears in the q-th column of A_κ and nowhere else, so this implies $t_{1,q}$ appears in the lowest term of $a_{\sigma(q)=k,q}$ but not in either $a_{q,q}$ or $a_{k,k}$. Now suppose $s = 2$. By Lemma 3.23 and reasoning similar to that above, we know $a_{\sigma(q)=k,q}$ must equal either $t_{2,q}$ or $t_{1,q+1}t_{2,q}$. If $a_{\sigma(q)=k,q} = t_{2,q}$ then by Lemma 3.23 we must have $a_{q,q} = 1 = a_{k,k}$, hence the claim holds. If $a_{\sigma(q)=k,q} = t_{1,q+1}t_{2,q}$ then the lowest term of $a_{q,q}$ is either 1 or $t_{2,q}$ and the lowest term of $a_{k,k}$ is either 1 or $t_{2,q+1}$. In either case, the variable $t_{1,q+1}$ does not appear, and the claim holds.

We may inductively repeat the above argument for the next largest k' such that $\sigma(k') < k'$, and so on. In this way we see that the lowest term of the product $a_{\sigma(1),1}a_{\sigma(2),2}\cdots a_{\sigma(b),b}$ must be larger than the lowest term of $a_{1,1}a_{2,2}\cdots a_{b,b}$. Thus the lowest of the possible lowest terms occurs in the summand corresponding to the identity permutation, as desired.

□

We next observe some useful properties of the entries in a standard tableau of shape $(I(n), m)$, where $I(n)$ and m are as above. For the discussion below it will be useful to visualize a tableau in a certain way. Recall that for our special case of m we have

$$\tau = [M_{1,1}] \star [M_{1,2}] \star \cdots \star [M_{1,n-1}] \star [M_{2,1}] \star [M_{2,2}] \star \cdots \star [M_{2,n-2}]$$

(so τ consists of only a 1-chain and a 2-chain) where each $[M_{s,j}]$ is of the form

$$[M_{s,j}] = \kappa_{(s,j),1} \star \cdots \kappa_{(s,j),m_{(s,j)}}$$

where each $\kappa_{(s,j),k}$ is a column of length j. For one such column suppose we have $\kappa_{(s,j),k} = (r^1_{(s,j),k}, r^2_{(s,j),k}, \cdots, r^j_{(s,j),k})$ where the entries are in $\{1, 2, \dots, n\}$. In what follows we visualize such a column as

$r^1_{(s,j),k}$
$r^2_{(s,j),k}$
$\vdots$
$r^j_{(s,j),k}$

and similarly we visualize the (s,j)-sector $[M_{s,j}]$ as a sequence of $m_{(s,j)}$ many columns of length j

$r^1_{(s,j),1}$	$r^1_{(s,j),2}$	$\cdots$	$r^1_{(s,j),m_{(s,j)}}$
$r^2_{(s,j),1}$	$r^2_{(s,j),2}$	$\cdots$	$r^2_{(s,j),m_{(s,j)}}$
$\vdots$	$\vdots$	$\cdots$	$\vdots$
$r^j_{(s,j),1}$	$r^j_{(s,j),2}$	$\cdots$	$r^j_{(s,j),m_{(s,j)}}$

(so it looks like a $j \times m_{(s,j)}$ box). Putting these together, we can visualize the entire 2-chain as a sequence of columns that look like a staircase, starting with $m_{(2,1)}$ many columns of length 1, $m_{(2,2)}$ many columns of length 2, etc.

Example 3.25. Suppose $n = 5$ and $m_{(2,1)} = 2, m_{(2,2)} = 1$ and $m_{(2,3)} = 2$. Then a 2-chain $2 \star 2 \star 13 \star 124 \star 123$ can be represented as:
$$\begin{array}{|c|c|c|c|c|} \hline 2 & 2 & 1 & 1 & 1 \\ \hline \multicolumn{2}{c|}{} & 3 & 2 & 2 \\ \cline{3-5} \multicolumn{3}{c|}{} & 4 & 3 \\ \cline{4-5} \end{array}.$$

The following general lemma follows easily from the definition of Demazure operators.

Lemma 3.26. *Let τ be a tableau with entries in $\{1, 2, \ldots, n\}$, represented as a diagram as above. Assume that the length of the columns in this representation weakly increase from left to right. Suppose in addition that the entries of τ (in its diagram representation as above)*

(a) strictly increase down columns and

(b) weakly decrease from left to right along rows.

Let $i \in \{1, 2, \ldots, n\}$. Then $f_i(\tau)$ (represented as a diagram in the same way as τ) also has properties (a) and (b).

Proof. The Demazure operator f_i only changes one entry of τ by definition, so it suffices to consider the relationship between it and the entries immediately below and to its left in the diagram of τ (if they exist). Suppose we have $\begin{array}{|c|} \hline x \\ \hline y \\ \hline \end{array}$ in the original tableau τ where x is the entry that gets changed from x to $x+1$. By assumption $x < y$. The only way for $x+1$ to fail to be less than y is if $y = x+1$. But then the pair $x, x+1$ would be "cancelled" in the definition/construction of f_i and x cannot change to $x+1$, so we get a contradiction. Thus we must have $y > x + 1$ and hence $\begin{array}{|c|}\hline x{+}1 \\ \hline y \\ \hline \end{array}$ is still strictly increasing down columns, as desired. Next suppose $\begin{array}{|c|c|}\hline z & x \\ \hline \end{array}$ is the original tableau τ where x gets changed to $x + 1$. By assumption $z \geq x$, so the only way for $x + 1$ to fail to be less than or equal to z is if $z = x$. In order to derive a contradiction, suppose $z = x$. We know that x was raised by f_i and z was not. Since z is to the left of x, this means that the box directly below z must equal $x+1$ before f_i is applied: $\begin{array}{|c|c|}\hline z & x \\ \hline x{+}1 & \\ \hline \end{array}$. But then by properties *(a)* and *(b)* the box directly below x must also equal $x + 1$: $\begin{array}{|c|c|}\hline z & x \\ \hline x{+}1 & x{+}1 \\ \hline \end{array}$. This means that the pair x, $x + 1$ would be "cancelled" in the application of f_i, and hence x cannot change to $x+1$, which is a contradiction. This completes the proof. $\square$

The following is now straightforward.

Lemma 3.27. *Let* $\tau \in \mathcal{T}(I(n), m)$ *and let* $\mathrm{pr}_{1-chain}(\tau) = [M_{1,1}] \star [M_{1,2}] \star \cdots \star [M_{1,n-1}]$ *and* $\mathrm{pr}_{2-chain}(\tau) = [M_{2,1}] \star [M_{2,2}] \star \cdots \star [M_{2,n-2}]$ *be its (projection to the)* 1*-chain and* 2*-chain respectively. Represent both chains as diagrams as in the discussion preceding Example 3.25. Then the entries of both the* 1*-chain and the* 2*-chain satisfy properties (a) and (b) of Lemma 3.26.*

Proof. We argue for the case of the 2-chain. The 1-chain is similar (and even simpler) and we leave it to the reader. We have already seen that $\mathrm{pr}_{2-chain}(\tau)$ is standard by Lemma 3.13, and a standard tableau is produced by a sequence of concatenation and Demazure operators. Since we know from Lemma 3.26 that any f_i preserves properties (a) and (b) in the statement of Lemma 3.26, it suffices to check that the concatenation operator also preserves these properties during the process of building a standard tableau of shape

$(I(n), (0, \ldots, 0, m_{(2,1)}, m_{(2,2)}, \ldots, m_{(2,n-2)}, 0, \ldots))$. By definition, standard tableau of this shape are given by

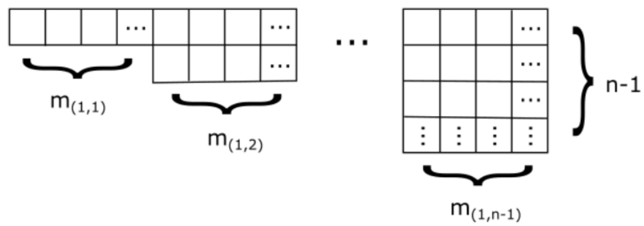
$$\Lambda_1\Lambda_2\cdots\Lambda_{n-1}(\bar{w}_1^{\star m_{(2,1)}}\star\Lambda_2(\bar{w}_2^{\star m_{(2,2)}}\star\cdots\star\Lambda_{n-2}(\bar{w}_{n-2}^{\star m_{(2,n-2)}})\cdots)). \tag{3.23}$$

Also recall that, in the diagrammatic representation of a 2-chain discussed above, $\bar{w}_i^{\star m_{(2,i)}}$ is represented as a block of the form:

$$\begin{array}{|c|c|c|c|}
\hline
1 & 1 & \cdots & 1 \\ \hline
2 & 2 & \cdots & 2 \\ \hline
\vdots & \vdots & \cdots & \vdots \\ \hline
i & i & \cdots & i \\ \hline
\end{array}$$

Now notice that, in the equation (3.23), the only Demazure operators that have been applied before a concatenation by $\bar{w}_i^{\star m_{(2,i)}}$ are the f_j's associated to j strictly larger than i, which means that they can only affect rows which are strictly below any entry in $\bar{w}_i^{\star m_{(2,i)}}$. From this it follows that concatenation with $\bar{w}_i^{\star m_{(2,i)}}$ preserves properties (a) and (b), since $\bar{w}_i^{\star m_{(2,i)}}$ also consists of columns which strictly increase going down and weakly decrease from left to right along rows. This proves the claim. $\square$

In the following discussion, we will refer to a representation of a word in a diagram as above as a **staircase**. Here it is assumed that a staircase consists of a sequence of top-justified columns whose lengths weakly increase from left to right. When we represent a word as a staircase, we occasionally refer to a **box** of the staircase (indicating the location of an entry of the word, and occasionally - by slight abuse of language - also the entry itself). It should be noted that under our restrictions on $I(n)$ and m, a standard tableau τ of shape $(I(n), m)$ is a concatenation of two separate staircases, one corresponding to its 1-chain, and the other corresponding to its 2-chain, so the staircase of the 1-chain would be



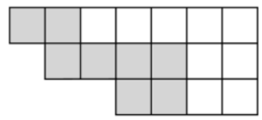
and similarly for the 2-chain. (The above figure only indicates the shape of the staircase; the entries are omitted for visual simplicity.) We will refer to these as the 1-chain staircase and the 2-chain staircase.

Next we introduce a simplification of the notation which further clarifies matters. Let $i \in \{1, 2, \ldots, n-1\}$. Notice that by the definition of $\mathcal{T}(I(n), m)$, the construction of a 1-chain of a standard tableau $\tau \in \mathcal{T}(I(n), m)$ applies the Demazure operator Λ_i only once. Moreover, the only Demazure operators Λ_j which are applied after Λ_i are those with $j < i$. This implies that for any fixed box in the 1-chain, that box can be changed at most once in the entire algorithm. And since any Λ_i can only increase a box by 1, this means that for any given box in the 1-chain which is in the p-th row, its entry can only be either a $p+1$ or a p, in accordance with whether it was changed by a (single) Demazure operator or not, respectively. Given this binary situation, for the purposes of further analysis it suffices to only record the data of whether or not a box in the p-th row is, or is not, raised by 1. Below, we choose to indicate by shading in *grey* a box which was increased, and using a white box to indicate one which was not.

Example 3.28. With the above conventions, we may notate a 1-chain such as

2	2	1	1	1	1	1
	3	3	3	3	2	2
			4	4	3	3

by



.

Recall that, by Lemma 3.27, a 1-chain staircase must have the property that its entries are strictly increasing down columns and weakly decreasing from left to right along rows. This implies that if any given box is grey, then all of the boxes to its "southwest" must also be grey, where by "southwest" we mean any box below and to its left, i.e., if a box is in the p-th row and q-th column, then a box in the i-th row and j-th column is to its southwest if $i \geq p$ and $j \leq q$. Notice that this means that if we remove the grey boxes in such a staircase, then what remains (i.e. the white boxes) still form a (connected) staircase.

Next we consider the 2-chain staircase of τ. Here, the only difference between the 1-chain and the 2-chain is that a box in the 2-chain can be acted upon by *two* Demazure operators, due to the fact that the Demazure operators applied in the algorithm for creating $\mathcal{T}(I(n), m)$ are, in sequence, (reading left to right)

$$\Lambda_{n-2}, \Lambda_{n-3}, \ldots, \Lambda_2, \Lambda_1, \Lambda_{n-1}, \Lambda_{n-2}, \ldots, \Lambda_2, \Lambda_1$$

so it is possible, for example, for a box to be first raised by Λ_{n-2} and then by the Λ_{n-1}. Otherwise, the reasoning for the 1-chain staircase can be used in a similar fashion and it is straightforward to see that a given box in the p-th row of a 2-chain staircase must be p, $p+1$, or $p+2$, depending on whether it was acted upon by 0, 1, or 2 Demazure operators respectively. Following the scheme introduced above, we choose to indicate these as follows:

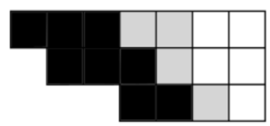
- a box which has been increased by 2 is *black*,

- a box which has been increased by 1 is *grey*,
- a box which has not been increased is *white*.

Example 3.29. In this scheme, we can visualize a 2-chain staircase such as

3	3	3	2	2	1	1
	4	4	4	3	2	2
			5	5	4	3

as



.

Moreover, by the same reasoning as for the 1-chain, it follows from Lemma 3.27 that if any given box in the 2-chain is black, then any box to its southwest must be black, and if any box is grey, then any box to its southwest must be either grey or black. Thus, if we remove the black boxes, what remains is still a staircase, and if we remove both grey and black boxes, what remains is still a staircase.

We refer to coloured diagrams such as the above as **shaded staircases**. We record the above discussion in the following lemma.

Lemma 3.30. *Let $\tau \in \mathcal{T}(I(n), m)$.*

1. *The 1-chain shaded staircase has the property that if we remove the grey boxes, then what remains is still a (connected) staircase.*
2. *The 2-chain shaded staircase has the property that if we remove the black boxes, what remains is still a staircase, and if we remove both grey and black boxes, what remains is still a staircase.*

We now introduce the following notation. Let $\tau \in \mathcal{T}(I(n), m)$. For $p \in \{1, 2, \ldots, n-1\}$ we let b_p denote the number of grey boxes in the p-th row of the 1-chain shaded staircase. For $p \in \{1, 2, \ldots, n-2\}$, we let a_p and c_p denote the number of grey and black boxes, respectively, in the p-th row of the 2-chain shaded staircase. It will be useful later on to note that, by definition and by Lemma 3.30, the b_p, a_p and c_p are integers satisfying

- $0 \leq b_p \leq m_{(1,p)} + b_{p+1}$,
- $a_p \geq 0$, $c_p \geq 0$, and $a_p + c_p \leq m_{(2,p)}$.

The following immediately follows from Lemma 3.30.

Lemma 3.31. *Let $\tau \in \mathcal{T}(I(n), m)$. Then τ is uniquely determined by the b_p, a_p and c_p as defined above.*

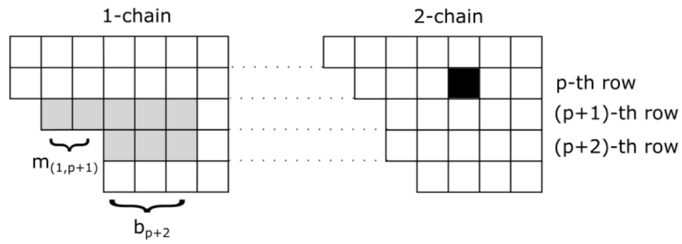
The next statement is an elementary but crucial technical observation that allows us to prove Proposition 3.21.

Lemma 3.32. *Let $\tau \in \mathcal{T}(I(n), m)$ and b_p, a_p and c_p the corresponding integers defined above. Let $p \in \{1, 2, \ldots, n-2\}$. Then:*

$$\text{if } c_p \neq 0 \text{ then } b_{p+1} = b_{p+2} + m_{(1,p+1)}$$

where we take the convention $b_n = 0$. In particular, if $c_p \neq 0$ then $b_{p+1} \geq m_{(1,p+1)}$.

In terms of our staircase diagrams, the above lemma states that if there is a black box in p-th row of the 2-chain shaded staircase, then the grey boxes in the $(p+1)$-th and $(p+2)$-th rows of the 1-chain shaded staircase have to "line up", as in the schematic example below:



Proof. Suppose $\tau \in \mathcal{T}(I(n), m)$ and that $c_p \neq 0$, i.e., there is a black box in the p-th row of the 2-chain shaded staircase of τ. As discussed when we defined the shaded staircases, this precisely means that the box in question was acted upon by exactly two lowering operators, namely, f_p and f_{p+1} (in that order), associated to $\Lambda_{2,p}$ and $\Lambda_{1,p+1}$ respectively. The remainder of this argument is similar in spirit to those in the previous section (see e.g. Remark 3.17). Specifically, at the step in the algorithm when $\Lambda_{1,p+1}$ is applied, we know from Lemma 3.27 that the boxes strictly below and at or to the left of the box in question are already black. This means that the only boxes which can contain a $p+2$ in the 2-chain are those in columns strictly to the right of the box in question.

To prove the statement of the lemma, it suffices to show the following. At the step in the algorithm when $\Lambda_{1,p+1}$ is applied, any $p+1$ contained in a white box in the $(p+1)$-th row of the 1-chain which sits immediately above a grey box in the $(p+2)$-th row (in the same column) is not cancelled by a $p+2$ anywhere in the 2-chain. By the argument above, and because $\Lambda_{1,p+1}$ has not yet been applied, there cannot be such a $p+2$ anywhere in the 2-chain at or to the left of the column containing the box in question. On the other hand, such a $p+2$ cannot be contained in any column to the right of the box in question, since otherwise the box in question (which contains a $p+1$) would be cancelled by this $p+2$ and hence cannot become black, contracting the assumption. Therefore, when $\Lambda_{1,p+1}$ is applied the entries in the leftmost $m_{(1,p+1)} + b_{p+2}$ boxes of the $(p+1)$-th row of the 1-chain will raise to a $p+2$, which is precisely what the statement of the lemma asserts. $\square$

Our next lemma computes the lowest term of (and hence the valuation evaluated on) the standard monomial Θ_τ corresponding to a standard tableau $\tau \in \mathcal{T}(I(n), m)$.

Lemma 3.33. *Let $\tau \in \mathcal{T}(I(n), m)$, let Θ_τ denote its corresponding standard monomial, and let b_p, a_p and c_p be the parameters defined above. Then the lowest term $LT(\Theta_\tau)$ of Θ_τ is given by*

$$LT(\Theta_\tau) = t_{1,1}^{b_1} t_{1,2}^{b_2+c_1} \cdots t_{1,n-1}^{b_{n-1}+c_{n-2}} t_{2,1}^{c_1+a_1} t_{2,2}^{c_2+a_2} \cdots t_{2,n-2}^{c_{n-2}+a_{n-2}}. \tag{3.24}$$

In particular, $\nu(\Theta_\tau) = (b_1, b_2 + c_1, \ldots, b_{n-1} + c_{n-2}, c_1 + a_1, c_2 + a_2, \ldots, c_{n-2} + a_{n-2}, 0, \ldots, 0)$.

In other words:

- the exponent of $t_{1,1}$ in $LT(\Theta_\tau)$ is the number of grey boxes in the 1st row of the 1-chain shaded staircase,
- for $p \in \{2, \ldots, n-1\}$, the exponent of $t_{1,p}$ in $LT(\Theta_\tau)$ is the sum of the number of grey boxes in the p-th row of the 1-chain shaded staircase and the number of black boxes in the $(p-1)$-th row of the 2-chain shaded staircase, and
- for $p \in \{1, 2, \ldots, n-2\}$, the exponent of $t_{2,p}$ is the number of shaded boxes (either grey or black) in the p-th row of the 2-chain shaded staircase.

Proof. By definition, Θ_τ is a product of determinants with one factor corresponding to each column of τ. Since the lowest term of a product is the product of the lowest terms of each factor, in order to prove the claim of the lemma it now suffices to prove:

- for a column κ in the 1-chain, the lowest term of $\det A_\kappa$ is the product of the $t_{1,p}$ where the p-th box in κ is grey, i.e.

$$\prod_{\substack{p\in\{1,2,\ldots,n-1\}\\ \text{the } p\text{-th box in } \kappa \text{ is grey}}} t_{1,p}.$$

- for a column κ in the 2-chain, the lowest term of $\det A_\kappa$ is the product

$$\prod_{\substack{p\in\{1,2,\ldots,n-2\}\\ \text{the } p\text{-th box in } \kappa \text{ is black}}} (t_{2,p}t_{1,p+1}) \cdot \prod_{\substack{p\in\{1,2,\ldots,n-2\}\\ \text{the } p\text{-th box in } \kappa \text{ is grey}}} t_{2,p}.$$

  We have already seen in Lemma 3.24 that the lowest term of $\det A_\kappa$ is the product of the lowest terms of the entries along the main diagonal of A_κ. Recalling that a grey (respectively black) box in the p-th row contains the entry $p+1$ (respectively $p+2$), the claim now follows from Lemma 3.23. The assertion about $\nu(\Theta_\tau)$ follows since ν simply reads off the exponents of the lowest term.

□

Our last lemma shows that the values of the b_p as defined above are uniquely determined by $\nu(\Theta_\tau)$.

Lemma 3.34. *Let $\tau \in \mathcal{T}(I(n), m)$ and suppose*

$$\nu(\Theta_\tau) = (k_{1,1}, \ldots, k_{1,n-1}, k_{2,1}, \ldots, k_{2,n-2}, 0, \ldots, 0) \in \mathbb{N}^\ell$$

for some $k_{s,j} \in \mathbb{N}$. Then for any $p \in \{2, \ldots, n-1\}$, we have

$$b_p := \min\{k_{1,p}, m_{(1,p)} + b_{p+1}\}.$$

Proof. For each $3 \leq p \leq n$ we wish to prove that if $k_{1,p-1} > m_{(1,p-1)} + b_p$ then $b_{p-1} = m_{(1,p-1)} + b_p$, and if $k_{1,p-1} \leq m_{(1,p-1)} + b_p$ then $b_{p-1} = k_{1,p-1}$. So, suppose $k_{1,p-1} > m_{(1,p-1)} + b_p$. In the discussion following Lemma 3.30 we recorded that $b_{p-1} \leq m_{(1,p-1)} + b_p$. Moreover, by Lemma 3.33 we know that $k_{1,p-1} = b_{p-1} + c_{p-2}$ for $p \geq 3$. Putting our initial assumption and these two facts together, we have that $m_{(1,p-1)} + b_p < k_{1,p-1} = b_{p-1} + c_{p-2} \leq m_{(1,p-1)} + b_p + c_{p-2}$, which implies that $c_{p-2} > 0$. By Lemma 3.32 this implies that $b_{p-1} = b_p + m_{(1,p-1)}$, as desired.

Now, suppose $k_{1,p-1} \leq m_{(1,p-1)} + b_p$. Since $k_{1,p-1} = b_{p-1} + c_{p-2}$ (Lemma 3.33) it suffices to show that $c_{p-2} = 0$. In order to derive a contradiction, suppose $c_{p-2} > 0$. Then by Lemma 3.32 this implies that $b_{p-1} = b_p + m_{(1,p-1)}$. By our assumption, this implies that $b_{p-1} \geq k_{1,p-1} = b_{p-1} + c_{p-2}$, which means that $0 \geq c_{p-2}$, which is a contradiction. This completes the proof. $\square$

Using this lemma, we can see that $b_{n-1} = \min\{k_{1,n-1}, m_{(1,n-1)}\}$ (using our convention that $b_n = 0$). In particular, b_{n-1} is uniquely determined by $\nu(\Theta_\tau)$. Similarly, using this lemma, b_{n-2} is uniquely determined by $\nu(\Theta_\tau)$, since b_{n-1} is, and so on.

We are finally ready to prove Proposition 3.21.

Proof of Proposition 3.21. Suppose

$$\nu(\Theta_\tau) = (k_{1,1}, k_{1,2}, \ldots, k_{1,n-1}, k_{2,1}, k_{2,2}, \ldots, k_{2,n-2}, 0, \ldots, 0) \in \mathbb{N}^\ell$$

for some $k_{a,b} \in \mathbb{N}$. We just saw in Lemma 3.34 that for $p \in \{2, \ldots, n-1\}$, we have

$$b_p = \min\{k_{1,p}, m_{(1,p)} + b_{p+1}\}.$$

Moreover, by Lemma 3.33 we know that $b_1 = k_{1,1}$. In particular, the b_p are uniquely determined by $\nu(\Theta_\tau)$. But then for $p \in \{2, \ldots, n-1\}$ we have that

$$c_{p-1} = k_{1,p} - b_p$$

by Lemma 3.33, hence the c_p's are also uniquely determined, and in turn

$$a_p = k_{2,p} - c_p$$

also by Lemma 3.33, which means the a_p's are also uniquely determined. By Lemma 3.31 this means there exists a unique τ such that

$$\nu(\Theta_\tau) = (k_{1,1}, k_{1,2}, \ldots, k_{1,n-1}, k_{2,1}, \ldots, k_{2,n-2}, 0, \ldots, 0),$$

i.e., ν is injective. This completes the proof. $\square$

We conclude this section with an example where the standard monomial basis is not mapped injectively under ν and the m does not satisfy the restrictive hypothesis placed at the beginning of this section (i.e. that $m_{(a,b)} = 0$ for $a > 2$).

Example 3.35. Let $I = I(5) = (1, 2, 3, 4, 1, 2, 3, 1, 2, 1)$,

$m = (0, 0, 0, 0, 0, 0, 1, 1, 0, 0)$. We will first show that the tableaux $134 \star 2$ and $124 \star 3$ are contained in the set of standard tableaux $\mathcal{T}(I(5), m)$. We will trace through the algorithm used for creating the set of standard tableaux (see Definition 2.19 and Example 2.20). Recall that each Demazure operator $\Lambda_{j,k}$ takes a set of tableaux to another set of tableaux. At each step, we only write down the tableaux we need to consider, and omit the other tableaux in the set.

$$\begin{aligned}
&\{\mathbf{O}\} \overset{\mathbf{O}\star}{\to} \{\mathbf{O}\} \overset{\Lambda_{4,1}}{\to} \{\mathbf{O}\} \overset{\mathbf{O}\star}{\to} \{\mathbf{O}\} \overset{\Lambda_{3,2}}{\to} \{\mathbf{O}\} \overset{1\star}{\to} \{1\} \overset{\Lambda_{3,1}}{\to} \{2\} \overset{123\star}{\to} \{123\star 2\}\\
&\overset{\Lambda_{2,3}}{\to} \{123\star 2\} \overset{\mathbf{O}\star}{\to} \{123\star 2\} \overset{\Lambda_{2,2}}{\to} \{123\star 2, 123\star 3\} \overset{\mathbf{O}\star}{\to} \{123\star 2, 123\star 3\}\\
&\overset{\Lambda_{2,1}}{\to} \{123\star 2, 123\star 3\} \overset{\mathbf{O}\star}{\to} \{123\star 2, 123\star 3\} \overset{\Lambda_{1,4}}{\to} \{123\star 2, 123\star 3\}\\
&\overset{\mathbf{O}\star}{\to} \{123\star 2, 123\star 3\} \overset{\Lambda_{1,3}}{\to} \{124\star 2, 124\star 3\} \overset{\mathbf{O}\star}{\to} \{124\star 2, 124\star 3\}\\
&\overset{\Lambda_{1,2}}{\to} \{134\star 2, 124\star 3\} \overset{\mathbf{O}\star}{\to} \{134\star 2, 124\star 3\} \overset{\Lambda_{1,1}}{\to} \{134\star 2, 124\star 3\}.
\end{aligned}$$

Therefore $134\star 2$ and $124\star 3$ are in $\mathcal{T}(I(5), m)$. We will now compute the corresponding standard monomials $\Theta_{134\star 2}$ and $\Theta_{124\star 3}$. To do this, it will be helpful to consider the entries in $\mu_{I(5)}(Z_{(I(5))})$. The first four components of $\mu_{I(5)}(Z_{(I(5))})$ are of the form:

$$\underbrace{\begin{pmatrix} 1 \\ t_{1,1} \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{x^{(1)}} \times \underbrace{\begin{pmatrix} 1 & 0 \\ t_{1,1} & 1 \\ 0 & t_{1,2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{x^{(2)}} \times \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ t_{1,1} & 1 & 0 \\ 0 & t_{1,2} & 1 \\ 0 & 0 & t_{1,3} \\ 0 & 0 & 0 \end{pmatrix}}_{x^{(3)}} \times \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ t_{1,1} & 1 & 0 & 0 \\ 0 & t_{1,2} & 1 & 0 \\ 0 & 0 & t_{1,3} & 1 \\ 0 & 0 & 0 & t_{1,4} \end{pmatrix}}_{x^{(4)}}.$$

The next three components of $\mu_{I(5)}(Z_{(I(5))})$ are of the form:

$$\underbrace{\begin{pmatrix} 1 \\ t_{1,1}+t_{2,1} \\ t_{1,2}t_{2,1} \\ 0 \\ 0 \end{pmatrix}}_{x^{(5)}} \times \underbrace{\begin{pmatrix} 1 & 0 \\ t_{1,1}+t_{2,1} & 1 \\ t_{1,2}t_{2,1} & t_{1,2}+t_{2,2} \\ 0 & t_{1,3}t_{2,2} \\ 0 & 0 \end{pmatrix}}_{x^{(6)}} \times \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ t_{1,1}+t_{2,1} & 1 & 0 \\ t_{1,2}t_{2,1} & t_{1,2}+t_{2,2} & 1 \\ 0 & t_{1,3}t_{2,2} & t_{1,3}+t_{2,3} \\ 0 & 0 & t_{1,4}t_{2,3} \end{pmatrix}}_{x^{(7)}}.$$

The eighth component of $\mu_{I(5)}(Z_{(I(5))})$ is of the form:

$$\underbrace{\begin{pmatrix} 1 \\ t_{1,1}+t_{2,1}+t_{3,1} \\ t_{1,2}t_{2,1}+t_{3,1}(t_{1,2}+t_{2,2}) \\ t_{1,3}t_{2,2}t_{3,1} \\ 0 \end{pmatrix}}_{x^{(8)}}.$$

Notice that

$$\Theta_2(x^{(8)}) = t_{1,1}+t_{2,1}+t_{3,1}$$

and

$$\Theta_3(x^{(8)}) = t_{1,2}t_{2,1}+t_{3,1}(t_{1,2}+t_{2,2}) = t_{1,2}t_{2,1}+t_{1,2}t_{3,1}+t_{2,2}t_{3,1}.$$

Also,

$$\Theta_{124}(x^{(7)}) = \begin{vmatrix} 1 & 0 & 0 \\ t_{1,1}+t_{2,1} & 1 & 0 \\ 0 & t_{1,3}t_{2,2} & t_{1,3}+t_{2,3} \end{vmatrix} = t_{1,3}+t_{2,3},$$

and

$$\begin{aligned}\Theta_{134}(x^{(7)}) = \begin{vmatrix} 1 & 0 & 0 \\ t_{1,2}t_{2,1} & t_{1,2}+t_{2,2} & 1 \\ 0 & t_{1,3}t_{2,2} & t_{1,3}+t_{2,3} \end{vmatrix} &= (t_{1,2}+t_{2,2})(t_{1,3}+t_{2,3}) - t_{1,3}t_{2,2} \\ &= t_{1,2}t_{1,3}+t_{1,2}t_{2,3}+t_{2,2}t_{2,3}.\end{aligned}$$

By definition,

$$\Theta_{124\star 3} = \Theta_{124}(x^{(7)})\Theta_3(x^{(8)}) = (t_{1,3}+t_{2,3})(t_{1,2}t_{2,1}+t_{1,2}t_{3,1}+t_{2,2}t_{3,1}).$$

So, $\nu(\Theta_{124\star 3}) = \nu(t_{2,2}t_{2,3}t_{3,1}) = (0,0,0,0,0,1,1,1,0,0)$. Similarly,

$$\Theta_{134\star 2} = \Theta_{134}(x^{(7)})\Theta_2(x^{(8)}) = (t_{1,2}t_{1,3}+t_{1,2}t_{2,3}+t_{2,2}t_{2,3})(t_{1,1}+t_{2,1}+t_{3,1}).$$

So, $\nu(\Theta_{134\star 2}) = \nu(t_{2,2}t_{2,3}t_{3,1}) = (0,0,0,0,0,1,1,1,0,0)$. In particular, both $\Theta_{134\star 2}$ and $\Theta_{124\star 3}$ have the same image under ν.

Remark 3.36. When we only consider standard monomials with $m = (m_{(1,1)}, \ldots, m_{(2,n-2)}, 0, \ldots, 0) \in \mathbb{N}^{\ell}$ (i.e. we only consider the 1-chain and 2-chain) then Lemma 3.32 tells us that the 1-chain and 2-chain behave "distinctly" in the algorithm for creating standard tableaux. More precisely, if a $t_{1,p}$ appears as a lowest term in a column in the 2-chain of τ, then it must have appeared a "maximal" number of times as a lowest term in columns in the 1-chain. However, if we allow a 3-chain for example (i.e. allow m to have nonzero terms after m_{2n-3}), then the 2-chain and 3-chain do not necessarily behave "distinctly". In particular, in Example 3.35, we obtained a standard monomial $\Theta_{124\star 3}$ whose 3-chain contributes a lowest term $t_{2,2}$, but whose 2-chain does not. Such things do not happen when we only consider 1-chains and 2-chains, which allowed us to prove injectivity in these cases.

Chapter 4

Newton-Okounkov Bodies of Bott-Samelson Varieties

In this chapter, we compute the Newton-Okounkov bodies of Bott-Samelson varieties Z_I with respect to the lowest term valuation ν defined in §2.2, for $I = I(n)$ and line bundles $O(m)$ with $m = (m_{(1,1)}, \ldots, m_{(2,n-2)}, 0, \ldots, 0) \in \mathbb{N}^{\ell}$. First, for these special cases we show that the corresponding semigroup $S(R, \nu)$ is generated in level one, so in particular it is finitely generated. It then follows from our results that, for these cases, the Newton-Okounkov body of $Z_{I(n)}$ (with respect to $O(m)$ and ν) is the convex hull of the image under ν of the standard monomial basis of $H^0(Z_{I(n)}, O(m))$. We apply these results in Theorem 4.3 to give an explicit description of the Newton-Okounkov bodies of $Z_{(1,2,1)}$ by a concrete list of inequalities and by giving a list of its vertices.

4.1 The Semigroup is Generated in Level 1

Let $I(n) = (i_1, \ldots, i_\ell) = (1, 2, \ldots, n-1, 1, 2, \ldots, n-2, \ldots, 1, 2, 1)$ and $m = (m_{(1,1)}, \ldots, m_{(2,n-2)}, 0, \ldots, 0) \in \mathbb{N}^\ell$ be as in Definition 3.5. For a positive integer k, let km denote the vector

$$km := (km_{(1,1)}, km_{(1,2)}, \ldots, km_{(2,n-2)}, 0, \ldots, 0) \in \mathbb{N}^\ell.$$

Let $\tilde{m}_{(s,j)}$ denote the vector in $\mathbb{N}^\ell$ consisting of an $m_{(s,j)}$ in the (s,j)-th coordinate and 0's elsewhere.

Before computing the Newton-Okounkov body of $Z_{I(n)}$, we need to establish the following lemma. Recall that $\Theta(\mathcal{T}(I(n), km)$ is the set of standard monomial basis vectors for $H^0(Z_{I(n)}, O(km))$, as in (2.14).

Lemma 4.1. *Let $\ell = \frac{n(n-1)}{2}$ and let $I(n) = (i_1, \ldots, i_\ell)$ be as defined in Definition 3.5. Let $m = (m_{(1,1)}, \ldots, m_{(2,n-2)}, 0, \ldots, 0) \in \mathbb{N}^\ell$ and let $k \in \mathbb{N}$. Then for any $(z_1, z_2, \ldots, z_\ell) \in \nu(\Theta(\mathcal{T}(I(n), km)))$ we have* $\left(\frac{1}{k}z_1, \frac{1}{k}z_2, \ldots, \frac{1}{k}z_\ell\right) \in \mathrm{conv}(\nu(H^0(Z_{I(n)}, O(m))))$.

Proof. Let $(z_1, z_2, \ldots, z_\ell) \in \nu(\Theta(\mathcal{T}(I(n), km)))$ and let $\tau' \in \mathcal{T}(I(n), km)$ with $\nu(\Theta_{\tau'}) = (z_1, \ldots, z_\ell)$. Recall that the standard tableau τ' may be decomposed into pieces as follows (Definitions 2.15 and 3.2):

$$\tau' = \underbrace{\kappa_{(1,1)1} \cdots \kappa_{(1,1)(km_{(1,1)})}}_{[M'_{1,1}]} \underbrace{\kappa_{(1,2)1} \cdots \kappa_{(1,2)(km_{(1,2)})}}_{[M'_{1,2}]} \cdots \underbrace{\kappa_{(2,n-2)1} \cdots \kappa_{(2,n-2)(km_{(2,n-2)})}}_{[M'_{2,n-2}]}.$$

Note that there are only $2n-3$ sectors $[M'_{s,j}]$ (in the sense of Definition 3.2) in τ', since in the case under consideration $km = (km_{(1,1)}, \ldots, km_{(2,n-2)}, 0, \ldots, 0)$, all coordinates after the $(2, n-2)$-coordinate are equal to zero. We can further divide each sector, $[M'_{s,j}]$, into k pieces which we label as $[M^q_{s,j}]$ for $1 \leq q \leq k$ as follows:

$$[M'_{s,j}] = \underbrace{\kappa_{(s,j)1} \dots \kappa_{(s,j)m_{(s,j)}}}_{[M^1_{s,j}]} \cdot \underbrace{\kappa_{(s,j)(m_{(s,j)}+1)} \dots \kappa_{(s,j)(2m_{(s,j)})}}_{[M^2_{s,j}]}$$

$$\cdot \dots \cdot \underbrace{\kappa_{(s,j)((k-1)m_{(s,j)}+1)} \dots \kappa_{(s,j)(km_{(s,j)})}}_{[M^k_{s,j}]} .$$

i.e.

$$\tau' = [M'_{1,1}] \cdots [M'_{2,n-2}] = \underbrace{[M^1_{1,1}] \dots [M^k_{1,1}]}_{[M'_{1,1}]} \cdot \underbrace{[M^1_{1,2}] \dots [M^k_{1,2}]}_{[M'_{1,2}]} \cdots \underbrace{[M^1_{2,n-2}] \dots [M^k_{2,n-2}]}_{[M'_{2,n-2}]} .$$

By Lemma 3.13 we know that $[M'_{s,j}] \in \mathcal{T}(I(n), (0, \dots, 0, \underbrace{km_{(s,j)}}_{(s,j)-\text{th spot}}, 0, \dots, 0))$. Moreover, by Lemma 3.13 we know that $[M^q_{s,j}] \in \mathcal{T}(I(n), \tilde{m}_{(s,j)})$ for any $1 \leq q \leq k$. Therefore $\Theta_{[M^q_{s,j}]} \in \Theta(\mathcal{T}(I(n), \tilde{m}_{(s,j)})) \subset H^0(Z_{I(n)}, O(\tilde{m}_{(s,j)}))$.

Now, define

$$\tau_q := [M^q_{1,1}][M^q_{1,2}] \cdots [M^q_{2,n-2}], \text{ for } 1 \leq q \leq k.$$

Since each $\Theta_{[M^q_{s,j}]}$ is a section of $O(\tilde{m}_{(s,j)})$ this implies that the product $\Theta_{\tau_q} = \Theta_{[M^q_{1,1}]} \cdots \Theta_{[M^q_{2,n-2}]}$ may be considered as a section of the tensor product $O(\tilde{m}_{(1,1)}) \otimes \cdots \otimes O(\tilde{m}_{(2,n-2)}) \cong O(m)$ (see §2.3, (2.9)). Therefore, for each q with $1 \leq q \leq k$ we have a section $\Theta_{\tau_q} \in H^0(Z_{I(n)}, O(m))$.

Since $\Theta_{\tau'}$ is a product of minors, we have $\Theta_{\tau'} = \Theta_{\tau_1} \cdot \Theta_{\tau_2} \cdot \dots \cdot \Theta_{\tau_k}$. Therefore,

$$\left(\frac{1}{k} z_1, \frac{1}{k} z_2, \dots, \frac{1}{k} z_\ell\right) = \frac{1}{k}\nu(\Theta_{\tau'}) = \frac{1}{k}\nu(\Theta_{\tau_1} \dots \Theta_{\tau_k}) = \frac{1}{k}(\nu(\Theta_{\tau_1}) + \dots + \nu(\Theta_{\tau_k}))$$

is contained in the convex hull of $\nu(H^0(Z_{I(n)}, O(m)))$, as desired.

$\square$

The following theorem tells us that the Newton-Okounkov body of $Z_{I(n)}$, with respect to the above choices of $O(m)$ and ν, is the convex hull of the (finite) set of integer lattice points determined by the standard monomial basis vectors.

Theorem 4.2. *Let $I(n) = (i_1, \ldots, i_\ell)$ and $m = (m_{(1,1)}, \ldots, m_{(2,n-2)}, 0, \ldots, 0) \in \mathbb{N}^\ell$ be as defined in Definition 3.5. Then the Newton-Okounkov body $\Delta(Z_{I(n)}, O(m), \nu)$ is the convex hull of the image of $\Theta(\mathcal{T}(I(n), m))$ under the valuation ν, i.e.*

$$\Delta(Z_{I(n)}, O(m), \nu) = \mathrm{conv}(\nu(\Theta(\mathcal{T}(I(n), m)))).$$

Proof. By Proposition 3.21, ν is injective on both $\Theta(\mathcal{T}(I(n), m))$ and $\Theta(\mathcal{T}(I(n), km))$. Since ν additionally has one-dimensional leaves, $\nu(H^0(Z_{I(n)}, O(km))) = \nu(\Theta(\mathcal{T}(I(n), km)))$. On the other hand, we have already seen in Lemma 4.1 that any pair $(k, z = (z_1, \ldots, z_\ell)) \in \mathbb{N} \times \mathbb{Z}^\ell$ with $z \in \nu(\Theta(\mathcal{T}(I(n), km))) = \nu(H^0(Z_{I(n)}, O(km)))$ has the property that $\frac{1}{k}(k, z) = (1, \frac{1}{k}z)$ is contained in $\{1\} \times \mathrm{conv}(\nu(H^0(Z_{I(n)}, O(m))))$. This means that the cone C in $\mathbb{R}_{\geq 0} \times \mathbb{R}^\ell$ generated by

$$\bigcup_{k>0, k\in\mathbb{Z}} \left\{(k, \nu(\sigma)) \in \mathbb{N} \times \mathbb{Z}^\ell \,\middle|\, \sigma \in \mathcal{R}_k = H^0(Z_{I(n)}, O(km))\right\}$$

is contained in the cone generated by $\left\{(1, \nu(\sigma)) \,\middle|\, \sigma \in H^0(Z_{I(n)}, O(m))\right\}$. Thus the Newton-Okounkov body is generated in level 1 in the sense of Definition 1.11. In particular, the intersection of C with $\{1\} \times \mathbb{R}^\ell$ is exactly the convex hull of $\nu(H^0(Z_{I(n)}, O(m))) = \nu(\Theta(\mathcal{T}(I(n), m)))$ and hence the Newton-Okounkov body is exactly this convex hull, as claimed. $\square$

4.2 Explicit Computation of Newton-Okounkov Body of $Z_{(1,2,1)}$

In the previous section, we proved that the Newton-Okounkov body $\Delta(Z_{I(n)}, O(m), \nu)$ is the convex hull of the images under ν of the standard monomial basis vectors. In this section, we will explicitly compute this Newton-Okounkov body in terms of inequalities and give a list of its vertices for the special case $n = 3$ and $\ell = \frac{n(n-1)}{2} = 3$.

Theorem 4.3. *The Newton-Okounkov body* $\Delta := \Delta(Z_{(1,2,1)}, \mathcal{O}(m_1, m_2, m_3), \nu)$ *can be described in any of the following ways:*

(a) $\Delta = \mathrm{conv}(\nu(\Theta(\mathcal{T}(1,2,1), (m_1, m_2, m_3))))$,

(b) Δ *is the convex hull of the following:*

$$(0,0,0), \ (0,0,m_3), \ (0, m_2+m_3, m_3), \ (0, m_2, 0), \ (m_1, 0, 0), \ (m_1, 0, m_3),$$
$$(m_1+m_2, m_2, 0), \ (m_1+m_2, m_2+m_3, m_3), \ (m_1+m_2, m_2, m_3)$$

and the above points are precisely the vertices of Δ,

(c) Using standard coordinates x, y, z *for* $\mathbb{R}^3$*, the polytope* Δ *is cut out as a subset of* $\mathbb{R}^3$ *by the following inequalities:*

$$0 \leq x \leq m_1 + m_2, \quad 0 \leq z \leq m_3, \quad x - m_1 \leq y \leq z + m_2, \quad y \geq 0.$$

In particular, we have $\mathrm{vol}\Delta = \frac{1}{2}m_1 m_3^2 + m_1 m_2 m_3 + \frac{1}{2}m_2 m_3^2 + \frac{1}{2}m_2^2 m_3$.

Before proving the above theorem, we first need to establish the following lemma.

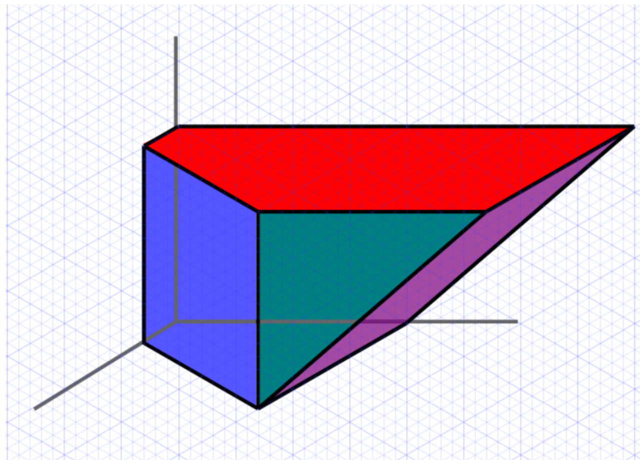


Figure 4.1: Newton-Okounkov body $\Delta(Z_{(1,2,1)}, O(m_1, m_2, m_3), \nu)$.

Lemma 4.4. *Elements in the set of standard monomials $\mathcal{T}((1,2,1),(m_1,m_2,m_3))$ must have one of the following four forms:*

(i) $\underbrace{2\cdots2}_{\alpha_1}\underbrace{1\cdots1}_{m_1-\alpha_1}\underbrace{13\cdots13}_{\beta_1}\underbrace{12\cdots12}_{m_2-\beta_1}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1}$

(ii) $\underbrace{2\cdots2}_{\alpha_1}\underbrace{23\cdots23}_{\beta_2}\underbrace{13\cdots13}_{\beta_1}\underbrace{12\cdots12}_{m_2-\beta_1-\beta_2}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1}$

(iii) $\underbrace{2\cdots2}_{\alpha_1}\underbrace{1\cdots1}_{m_1-\alpha_1}\underbrace{13\cdots13}_{\beta_1}\underbrace{3\cdots3}_{\gamma_2}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1-\gamma_2}$

(iv) $\underbrace{2\cdots2}_{\alpha_1}\underbrace{23\cdots23}_{\beta_2}\underbrace{13\cdots13}_{\beta_1}\underbrace{3\cdots3}_{\gamma_2}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1-\gamma_2}$,

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \geq 0$, $\alpha_1 \leq m_1$, $\beta_1 + \beta_2 \leq m_2$, and $\gamma_1 + \gamma_2 \leq m_3$. Moreover, if β_2 is non-zero, then $\alpha_1 = m_1$.

Proof. By Definition 2.19, an element of the set

$\mathcal{T}((1,2,1),(m_1,m_2,m_3))$ is formed by first beginning with a string $\underbrace{1\cdots1}_{m_3}$ and applying the Demazure operator Λ_1 to it (see (2.11)). This produces a set of tableaux with elements of the form $\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1}$, where $0 \leq \gamma_1 \leq m_3$.

Next, we concatenate each tableau in this set with m_2 many 12's, producing a set containing tableaux that look like $\underbrace{12\cdots12}_{m_2}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1}$. We then apply the Demazure operator Λ_2 to this set, producing tableaux that look like $\underbrace{13\cdots13}_{\beta_1}\underbrace{12\cdots12}_{m_2-\beta_1}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1}$ and $\underbrace{13\cdots13}_{\beta_1=m_2}\underbrace{3\cdots3}_{\gamma_2}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1-\gamma_2}$, where $0 \leq \beta_1 \leq m_2$, $0 \leq \gamma_2$, and $\gamma_1+\gamma_2 \leq m_3$. Next, we concatenate each tableau in this set with m_1 many 1's, producing a set containing tableaux that look like $\underbrace{1\cdots1}_{m_1}\underbrace{13\cdots13}_{\beta_1}\underbrace{12\cdots12}_{m_2-\beta_1}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1}$ and $\underbrace{1\cdots1}_{m_1}\underbrace{13\cdots13}_{\beta_1=m_2}\underbrace{3\cdots3}_{\gamma_2}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1-\gamma_2}$. Finally, we apply the Demazure operator Λ_1 to each tableau in this set. Applying Λ_1 to tableaux of the form $\underbrace{1\cdots1}_{m_1}\underbrace{13\cdots13}_{\beta_1}\underbrace{12\cdots12}_{m_2-\beta_1}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1}$ produces tableaux that look like

$$\underbrace{2\cdots2}_{\alpha_1}\underbrace{1\cdots1}_{m_1-\alpha_1}\underbrace{13\cdots13}_{\beta_1}\underbrace{12\cdots12}_{m_2-\beta_1}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1} \text{ and}$$

$$\underbrace{2\cdots2}_{\alpha_1=m_2}\underbrace{23\cdots23}_{\beta_2}\underbrace{13\cdots13}_{\beta_1}\underbrace{12\cdots12}_{m_2-\beta_1-\beta_2}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1},$$

where $\alpha_1, \beta_2 \geq 0$ and $\beta_1+\beta_2 \leq m_2$ (see *(i)* and *(ii)* in the statement of the lemma). Similarly, applying Λ_1 to tableaux of the form

$\underbrace{1\cdots1}_{m_1}\underbrace{13\cdots13}_{\beta_1=m_2}\underbrace{3\cdots3}_{\gamma_2}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1-\gamma_2}$ produces tableaux that look like

$$\underbrace{2\cdots2}_{\alpha_1}\underbrace{1\cdots1}_{m_1-\alpha_1}\underbrace{13\cdots13}_{\beta_1=m_2}\underbrace{3\cdots3}_{\gamma_2}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1-\gamma_2} \text{ and}$$

$$\underbrace{2\cdots2}_{\alpha_1=m_1}\underbrace{23\cdots23}_{\beta_2}\underbrace{13\cdots13}_{\beta_1=m_2-\beta_2}\underbrace{3\cdots3}_{\gamma_2}\underbrace{2\cdots2}_{\gamma_1}\underbrace{1\cdots1}_{m_3-\gamma_1-\gamma_2},$$

where $\alpha_1, \beta_2 \geq 0$ and $\beta_1+\beta_2 \leq m_2$ (see *(iii)* and *(iv)* in the statement of the Lemma). In particular, we can see that if β_2 is non-zero, then $\alpha_1 = m_1$. This completes the proof. $\square$

We are now ready to prove the theorem.

Proof of Theorem 4.3. Theorem 4.2 implies *(a)*. To prove *(b)* and *(c)* we will first use the properties of the algorithm that produces the standard monomial basis to obtain a polytope $\tilde{\Delta}$ that must contain Δ. We will then show that the vertices of $\tilde{\Delta}$ are contained in Δ, which implies $\tilde{\Delta} = \Delta$.

For $I = I(3) = (1,2,1)$, recall that we can find a basis for the space of sections $H^0(Z_{I(3)}, O(m))$ by looking at standard monomials which can be realized as products of minors of the coordinates

$$\mu_{I(3)}(\Phi(t_1,t_2,t_3)) \left[\begin{pmatrix} 1 \\ t_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t_1 & 1 \\ 0 & t_2 \end{pmatrix}, \begin{pmatrix} 1 \\ t_1 + t_3 \\ t_2 t_3 \end{pmatrix} \right] \in Gr(1) \times Gr(2) \times Gr(1)$$

where $\Theta_1 x^{(1)} = 1$, $\Theta_2 x^{(1)} = t_1$, $\Theta_{12} x^{(2)} = 1$, $\Theta_{13} x^{(2)} = t_2$, $\Theta_{23} x^{(2)} = t_1 t_2$, $\Theta_1 x^{(3)} = 1$, $\Theta_2 x^{(3)} = t_1 + t_3$, and $\Theta_3 x^{(3)} = t_2 t_3$ (see (2.5), (2.7), and Example 2.23). By Definition 2.21, a standard monomial Θ_τ in $\Theta(\mathcal{T}((1,2,1),(m_1,m_2,m_3)))$ is a product of minors of the coordinates listed above, where the minors are determined by the columns of the standard monomial τ. In Lemma 4.4, we saw exactly what form each standard tableau $\tau \in \mathcal{T}((1,2,1),(m_1,m_2,m_3))$ must have. In particular, columns in the 1-sector must be 2 or 1; columns in the 2-sector must be 23, 13, or 12; and columns in the 3-sector must be 3, 2, or 1. Therefore, a standard monomial Θ_τ must be a product of the minors listed above. In other words, for τ having one of the four forms listed in Lemma 4.4, a standard monomial Θ_τ must be a polynomial of the form $t_1^{\alpha_1} t_2^{\beta_1} (t_1 t_2)^{\beta_2} (t_1 + t_3)^{\gamma_1} (t_2 t_3)^{\gamma_2}$, where $\alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2 \geq 0$, $\alpha_1 \leq m_1$, $0 \leq \beta_1 + \beta_2 \leq m_2$, and $0 \leq \gamma_1 + \gamma_2 \leq m_3$. The image of this polynomial under the lowest term valuation ν is $(\alpha_1 + \beta_2, \beta_1 + \beta_2 + \gamma_2, \gamma_1 + \gamma_2)$ by Lemma 2.7.

Let $x = \alpha_1 + \beta_2$, $y = \beta_1 + \beta_2 + \gamma_2$, $z = \gamma_1 + \gamma_2$. The image of the standard monomial basis in $\mathbb{R}^3$ will be (x,y,z) subject to a set of restrictions. These restrictions include the following. We must have that

$$0 \leq z \leq m_3, \tag{4.1}$$

since $0 \leq \gamma_1 + \gamma_2 \leq m_3$, and

$$0 \leq x \leq m_1 + m_2, \tag{4.2}$$

since $0 \leq \alpha_1 \leq m_1$ and $0 \leq \beta_2 \leq m_2$. Moreover, $0 \leq \beta_1 + \beta_2 \leq m_2$ implies that $0 \leq \beta_1 + \beta_2 + \gamma_2 \leq m_2 + \gamma_2 \leq m_2 + \gamma_2 + \gamma_1$. This implies that

$$0 \leq y \leq z + m_2. \tag{4.3}$$

Now, if $x - m_1 > 0$, then this implies that $\alpha_1 + \beta_2 - m_1 > 0$, which implies that $\beta_2 > 0$, since $\alpha_1 \leq m_1$. By Lemma 4.4, if β_2 is non-zero, then $\alpha_1 = m_1$. Therefore, $x = \alpha_1 + \beta_2$ implies that $\beta_2 = x - m_1$. This implies that

$y = \beta_1 + \beta_2 + \gamma_2 = x - m_1 + \beta_1 + \gamma_2$, which implies that

$$y \geq x - m_1. \tag{4.4}$$

These four restrictions, (4.1), (4.2), (4.3), (4.4), cut out a polytope, $\tilde{\Delta}$, given by the inequalities $0 \leq x \leq m_1 + m_2, 0 \leq z \leq m_3, x - m_1 \leq y \leq z + m_2, y \geq 0$ (notice that these are the same inequalities given in *(c)*). This polytope must contain the Newton-Okounkov body Δ, since Δ will be cut out by the four inequalities listed (and possibly more), i.e. $\Delta \subseteq \tilde{\Delta}$.

It is not hard to see that the polytope $\tilde{\Delta}$ has nine vertices

$$(0,0,0),\ (0,0,m_3),\ (0,m_2+m_3,m_3),\ (0,m_2,0),\ (m_1,0,0),\ (m_1,0,m_3),$$
$$(m_1+m_2,m_2,0),\ (m_1+m_2,m_2+m_3,m_3),\ (m_1+m_2,m_2,m_3).$$

By Lemma 4.4, the set of tableaux

$$\{\underbrace{1\ldots1}_{m_1}\underbrace{12\ldots12}_{m_2}\underbrace{1\ldots1}_{m_3},\ \underbrace{1\ldots1}_{m_1}\underbrace{13\ldots13}_{m_2}\underbrace{1\ldots1}_{m_3},\ \underbrace{1\ldots1}_{m_1}\underbrace{12\ldots12}_{m_2}\underbrace{2\ldots2}_{m_3},$$
$$\underbrace{1\ldots1}_{m_1}\underbrace{13\ldots13}_{m_2}\underbrace{3\ldots3}_{m_3},\ \underbrace{2\ldots2}_{m_1}\underbrace{12\ldots12}_{m_2}\underbrace{1\ldots1}_{m_3},\ \underbrace{2\ldots2}_{m_1}\underbrace{12\ldots12}_{m_2}\underbrace{2\ldots2}_{m_3},$$
$$\underbrace{2\ldots2}_{m_1}\underbrace{23\ldots23}_{m_2}\underbrace{1\ldots1}_{m_3},\ \underbrace{2\ldots2}_{m_1}\underbrace{23\ldots23}_{m_2}\underbrace{2\ldots2}_{m_3},\ \underbrace{2\ldots2}_{m_1}\underbrace{23\ldots23}_{m_2}\underbrace{3\ldots3}_{m_3}\}$$

are contained in the set of standard tableaux $\mathcal{T}((1,2,1),(m_1,m_2,m_3))$. Their respective standard monomials are

$$1,\ t_2^{m_2},\ (t_1+t_3)^{m_3},\ t_2^{m_2}(t_2t_3)^{m_3},\ t_1^{m_1},\ t_1^{m_1}(t_1+t_3)^{m_3},$$
$$t_1^{m_1}(t_1t_2)^{m_2},\ t_1^{m_1}(t_1t_2)^{m_2}(t_1+t_3)^{m_3},\ t_1^{m_1}(t_1t_2)^{m_2}(t_2t_3)^{m_3}.$$

The image of these nine polynomials under ν are exactly the vertices of $\tilde{\Delta}$ listed above. They are also contained in Δ, since $\Delta = \mathrm{conv}(\nu(\Theta(\mathcal{T}(1,2,1),(m_1,m_2,m_3))))$. Therefore, $\tilde{\Delta} \subseteq \Delta$, and hence $\Delta = \tilde{\Delta}$. This proves *(b)* and *(c)*.

Given *(c)* we can compute the volume given in *(d)*:

$$\begin{aligned}\mathrm{vol}\Delta &= \int_0^{m_1}\int_0^{m_3}\int_0^{z+m_2} \mathrm{d}y\mathrm{d}z\mathrm{d}x + \int_{m_1}^{m_1+m_2}\int_0^{m_3}\int_{x-m_1}^{z+m_2} \mathrm{d}y\mathrm{d}z\mathrm{d}x\\ &= \frac{1}{2}m_1m_3^2 + m_1m_2m_3 + \frac{1}{2}m_2m_3^2 + \frac{1}{2}m_2^2m_3.\end{aligned}$$

Note that in Figure 4.1, the purple face is the plane $y = z + m_2$, and the blue face is the plane $y = x - m_1$.

□

Remark 4.5. Since $\dim(Z_{(1,2,1)}) = 3$, we know from equation (1.2) in §1.2 that the degree of the projective embedding of $Z_{(1,2,1)}$ in $\mathbb{P}(H^0(Z_{(1,2,1)}, \mathcal{O}(m_1,m_2,m_3)))$ equals $3!\mathrm{vol}\Delta = 3m_1m_3^2 + 6m_1m_2m_3 + 3m_2m_3^2 + 3m_2^2m_3$.

In the next theorem we show that, when m has a certain form, these Newton-Okounkov bodies Δ are affinely equivalent to the well-known Gelfand-Zetlin polytopes from representation theory. For more details about Gelfand-Zetlin polytopes, see e.g. [KST12, §3.1].

Definition 4.6. *The 3-dimensional* ***Gelfand-Zetlin polytope*** $GZ_{(\lambda_1,\lambda_2,\lambda_3)}$*, corresponding to a set of strictly increasing integers* $(\lambda_1 < \lambda_2 < \lambda_3)$*, can be defined by the inequalities* $\lambda_1 \leq y \leq \lambda_2$*,* $\lambda_2 \leq x \leq \lambda_3$*,* $y \leq z \leq x$*.*

Theorem 4.7. *The Newton-Okounkov body* $\Delta := \Delta(Z_{(1,2,1)}, \mathcal{O}(0, m_2, m_3), \nu)$ *is affinely equivalent to the Gelfand-Zetlin polytope* $GZ_{(0,m_2,m_2+m_3)}$*.*

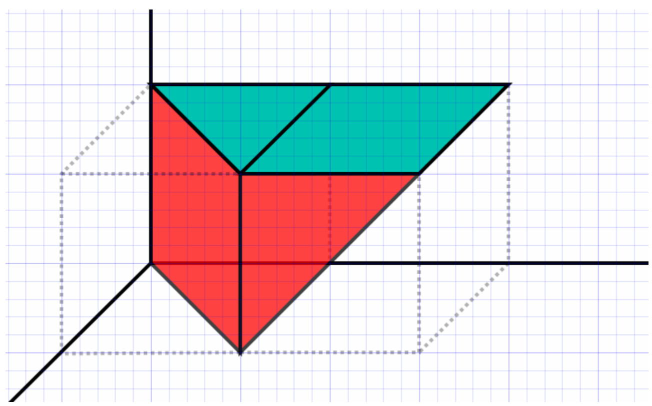


Figure 4.2: Newton-Okounkov body $\Delta(Z_{(1,2,1)}, O(0, m_2, m_3), \nu)$.

Proof. By Theorem 4.3, we know Δ is the polytope determined by the inequalities $0 \leq x \leq m_2, 0 \leq z \leq m_3, x \leq y \leq z + m_2$. Consider the affine transformation $\psi : \Delta \to GZ_{(0,m_2,m_2+m_3)}$, where

$$\psi(x, y, z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} m_2 \\ 0 \\ 0 \end{pmatrix}.$$

Since ψ is invertible, it suffices to show that $\psi(\Delta) \subseteq GZ_{(0,m_2,m_2+m_3)}$. If (x', y', z') are in the image $\mathrm{im}(\psi)$, then (x', y', z') has the form $(z + m_2, x, y)$

for some $(x, y, z) \in \Delta$. But then $0 \leq x \leq m_2$ implies that $0 \leq y' \leq m_2$. Similarly, $0 \leq z \leq m_3$ implies that $m_2 \leq z + m_2 \leq m_2 + m_3$, which implies that $m_2 \leq x' \leq m_2 + m_3$. Similarly, $x \leq y \leq z + m_2$ implies that $y' \leq z' \leq x'$. By Definition 4.6, $GZ_{(0,m_2,m_2+m_3)}$ is the polytope determined by the inequalities $m_2 \leq x' \leq m_2 + m_3$, $0 \leq y' \leq m_2$, and $y' \leq z' \leq x'$. Therefore $\psi(\Delta) \subseteq GZ_{(0,m_2,m_2+m_3)}$.

□

Part II

Peterson Varieties

Chapter 5

Background

The second part of this thesis gives a computation of Newton-Okounkov bodies of a Peterson variety, which is a special type of Hessenberg variety. The connections between Hessenberg varieties and combinatorics has been evident for some time. For instance, Harada and Tymoczko proved a Schubert-calculus-type Monk formula for the equivariant cohomology of Peterson varieties in purely combinatorial and manifestly-positive terms [HT11]. More recently, a study of the cohomology of Hessenberg varieties led to a proof of the Shareshian-Wachs conjecture, which is in turn related to the famous Stanley-Stembridge conjecture in combinatorics (cf. [BC15], see also [AHHM15]). Thus, it is of interest to compute combinatorial invariants associated to Hessenberg varieties and to relate them to known results.

In this thesis, we focus on Peterson varieties, which are a special case of Hessenberg varieties, and are in a certain sense the "simplest" of the regular nilpotent Hessenberg varieties, in a sense we make precise below. In fact, our results in this thesis apply to the single special case Pet_3, the Peterson variety in $Fl(\mathbb{C}^3)$. However, it is worth mentioning here that we have generalized some of the tools used in this thesis to the case of general n, notably the flat family constructed in Chapter 6, in an ongoing collaboration with H. Abe and M. Harada. In particular, at the time of this writing, we are optimistic that we can compute the Newton-Okounkov bodies of Peterson varieties Pet_n for general n using similar arguments to those in this thesis, at least under some technical hypotheses on the choice of line bundle. We intend to discuss this and related results in forthcoming work [ADH].

The above being said, because the motivation for the results in Part II of this thesis stems from the broader context and also because our later arguments (particularly the flat family mentioned above) require a discussion of more general Hessenberg varieties, we briefly recall below some history of this research area, give the basic definitions, and record some explicit computations and facts concerning the case of $Pet_3 \subseteq Fl(\mathbb{C}^3)$.

We thank Dave Anderson for teaching us the essence of the ideas and techniques used in Chapter 6.

5.1 Hessenberg Varieties

Hessenberg varieties are subvarieties of the full flag variety G/B. Historically, they have arisen in many contexts, including geometric representation theory, numerical analysis, mathematical physics, combinatorics, and algebraic geometry, among others [Fun03, DMPS92, Kos96, Rie03, Ful99, BC04]. Furthermore, there are many special cases of Hessenberg varieties, such as Springer varieties, which play a fundamental role in geometric representation theory. Another special case, namely Peterson varieties, arise in the study of the quantum cohomology of flag varieties. In Part II of this thesis, we will primarily be concerned with Peterson varieties.

Peterson varieties were introduced in the 1990's by D. Peterson. Since then, the geometry and combinatorics of Peterson varieties have been of particular interest and have been actively researched. For example, as briefly mentioned above, Kostant [Kos96] showed that Peterson varieties have a dense subvariety whose coordinate ring is isomorphic to the quantum cohomology of the flag variety, and Rietsch [Rie03] showed that the quantum parameters can be realized as principal minors of certain Toeplitz matrices. In another direction, Insko-Yong [IY12] have explicitly described the singular locus of Peterson varieties of type A. There is much that is still unknown about the Peterson variety. For additional background, see [HT11].

In what follows, let $G = GL_n\mathbb{C}$. These definitions can be generalized, but here we will always work in type A.

We begin with the definition of a Hessenberg variety.

Definition 5.1. *Let $X : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator and $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ a function satisfying $h(i+1) \geq h(i)$ for all $1 \leq i \leq n-1$ and $h(i) \geq i$ for all $1 \leq i \leq n$. Such a h is called a* ***Hessenberg function****. The* ***Hessenberg variety*** *associated to X and h is defined to be*

$$Hess(X, h) := \{V_\bullet \in Fl(\mathbb{C}^n) \mid XV_i \subseteq V_{h(i)} \forall i\}.$$

In particular, any Hessenberg variety $Hess(X, h)$ is, by definition, a subvariety of the flag variety $Fl(\mathbb{C}^n)$.

Remark 5.2. Since $Hess(X, h)$ and $Hess(gXg^{-1}, h)$ are isomorphic varieties $\forall\ g \in G$, we may always assume that X is in Jordan form with respect to the standard basis on $\mathbb{C}^n$.

Two important special cases of Hessenberg varieties are the *regular semisimple* Hessenberg varieties and the *regular nilpotent* Hessenberg varieties. We will encounter these varieties in Chapter 6.

Definition 5.3. *A Hessenberg variety $Hess(X, h)$ is called **regular semisimple** if X is a semisimple operator with distinct eigenvalues. Equivalently, there is a basis of $\mathbb{C}^n$ with respect to which X is diagonal with distinct entries along the diagonal.*

Definition 5.4. *A Hessenberg variety $Hess(X, h)$ is called **regular nilpotent** if X is a principal nilpotent operator. Equivalently, the Jordan canonical form of X has a single Jordan block with eigenvalue zero, so up to a change of basis X is of the form:*

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

5.2 Peterson Varieties

We will now give the precise definition of the *Peterson variety.* This variety will be our main object of study in the chapters that follow.

Definition 5.5. *If X is a principal nilpotent operator and h is the Hessenberg function defined by $h(i) = i+1$ for $1 \leq i \leq n-1$ and $h(n) = n$, then $Hess(X, h)$ is called a **Peterson Variety**. In this case, we denote $Hess(X, h)$ by Pet_n.*

In the example that follows, we explicitly describe the subvarieties Pet_2 and Pet_3 in $Fl(\mathbb{C}^2)$ and $Fl(\mathbb{C}^3)$ respectively. We also give the defining equations of $Pet_3 \subset Fl(\mathbb{C}^3)$.

Example 5.6. For $n = 2$, we have $Fl(\mathbb{C}^2) = \{0 \subset V_1 \subset V_2 = \mathbb{C}^2\}$, and let $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case, we always have $XV_1 \subseteq V_2 = \mathbb{C}^2$, so $Pet_2 = Fl(\mathbb{C}^2) \cong \mathbb{P}^1$.

For $n = 3$, $Fl(\mathbb{C}^3) = \{0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^3\}$, and let $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Suppose $V_1 = \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle$ and $V_2 = \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right\rangle$. We need $XV_1 = \left\langle \begin{pmatrix} b \\ c \\ 0 \end{pmatrix} \right\rangle \subset V_2$. Note that we may express V_1 as $V_1 = \left\langle \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix} \right\rangle$ if $c \neq 0$, $V_1 = \left\langle \begin{pmatrix} \gamma \\ 1 \\ 0 \end{pmatrix} \right\rangle$ if $c = 0$ and $b \neq 0$, and $V_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$ if $c = b = 0$. Then XV_1 equals $\left\langle \begin{pmatrix} \beta \\ 1 \\ 0 \end{pmatrix} \right\rangle$, $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$, and $\left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle$, in each case, respectively. In the first two cases, V_1 and XV_1 are linearly independent, so V_2 is spanned by V_1 and V_2. In the last case, $XV_1 = \{0\}$, so the requirement $XV_1 \subseteq V_2$ is vacuous. Therefore, Pet_3 consists of the following four types of flags:

$$\left\{ \left\langle \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix}, \begin{pmatrix} \beta \\ 1 \\ 0 \end{pmatrix} \right\rangle, \mathbb{C}^3 \right\}, \left\{ \left\langle \begin{pmatrix} \gamma \\ 1 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} \gamma \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \mathbb{C}^3 \right\},$$
$$\left\{ \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \delta \\ 1 \end{pmatrix} \right\rangle, \mathbb{C}^3 \right\}, \left\{ \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle, \mathbb{C}^3 \right\},$$

where α, β, γ, and δ are free parameters.

We will now find the defining equations of Pet_3 in $\mathbb{P}^2 \times \mathbb{P}^2$. We have $Pet_3 \subset Fl(\mathbb{C}^3)$, and can map it into $\mathbb{P}^2 \times \mathbb{P}^2$ using the Plücker mapping, p. Given an element x of the flag represented by a matrix

$$x = \left[\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \right]$$

let P_i denote the leftmost 1×1 minor in row i, and let P_{ij} denote the leftmost

2×2 minor in rows i and j. Then the Plücker mapping maps x to $p(x) = ([P_1 : P_2 : P_3], [P_{12} : P_{13} : P_{23}]) \in \mathbb{P}^2 \times \mathbb{P}^2$. Recall that $Fl(\mathbb{C}^3)$ is cut out in $\mathbb{P}^2 \times \mathbb{P}^2$ by the equation $P_1P_{23} - P_2P_{13} + P_3P_{12}$ (the Plücker relations of $Gr(1)$ and $Gr(2)$ in $\mathbb{P}^2$ are trivial, and the incidence relation $V_1 \subset V_2 \subset V_3$ gives this single defining equation). For Pet_3, $XV_1 \subset V_2 \iff$

$$\det \left[\begin{pmatrix} x_{11} & x_{12} & x_{21} \\ x_{21} & x_{22} & x_{31} \\ x_{31} & x_{32} & 0 \end{pmatrix} \right] = 0 \iff x_{21}P_{23} - x_{31}P_{13} = 0 \iff P_2P_{23} - P_3P_{13} = 0.$$

Thus we conclude $Pet_3 = Z(P_2P_{23} - P_3P_{13}, P_1P_{23} - P_2P_{13} + P_3P_{12}) \subset \mathbb{P}^2 \times \mathbb{P}^2$.

Finally, we record the following facts about Peterson varieties which we use in later chapters [IY12].

Theorem 5.7. *[IY12, Cor. 1.8, Lemma 4.2] Consider $Pet_n \subset GL_n\mathbb{C}/B$.*

(i) Pet_n is an irreducible variety of dimension $n-1$.

(ii) Pet_3 is normal and has a unique singular point at the identity

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right].$$

.

Chapter 6

The Degree of Pet_3

The goal of this chapter is to compute the degree of the Peterson variety Pet_3 with respect to a chosen Plücker mapping. By equation (1.2) in §1.2, which relates the volume of a Newton-Okounkov body with the degree of a projective embedding, this computation allows us to explicitly describe $\Delta = \Delta(Pet_3, R(V_\lambda), \nu)$ in the next chapter.

We compute the degree of Pet_3 in an indirect manner, as we now describe. It turns out that Pet_3 can be realized as the special fibre of a flat family of Hessenberg varieties, the generic fibre of which happens to be a smooth projective toric variety. The fact that the generic fibre is toric allows us to use standard techniques in equivariant topology, and in particular the Atiyah-Bott-Berline-Vergne localization formula, to compute its degree. Then the well-known result that the degree is constant along fibres of a flat family yields our result.

We briefly outline the contents of this chapter. In Section 6.1 we describe the scheme structure of our family X of Hessenberg varieties. In Section 6.2 we show that the natural morphism $p : X \to Y = \mathrm{Spec}\mathbb{C}[t] \cong \mathbb{C}$ is flat. In

Section 6.3 we show that the fibres of this family are reduced and hence can be naturally identified with the Hessenberg varieties (in the classical algebraic geometry sense) defined in the previous chapter. Finally, in Section 6.4 we use the Atiyah-Bott-Berline-Vergne localization formula to compute the degree of a generic fibre of the flat family, and hence obtain the degree of Pet_3.

Although some of the arguments in this chapter can readily be generalized to the case of general n, for simplicity we restrict the discussion in this chapter to the special case $n = 3$.

6.1 The Definition of the Family $p : X \to Y$

Let $\mathfrak{gl}_3(\mathbb{C})$ denote the Lie algebra of $\mathrm{GL}_3(\mathbb{C})$, i.e. 3×3 matrices with entries in $\mathbb{C}$. For distinct complex numbers $\gamma_1, \gamma_2, \gamma_3$ and $z \in \mathbb{C}$ let

$$M_z := \begin{pmatrix} z\gamma_1 & 1 & 0 \\ 0 & z\gamma_2 & 1 \\ 0 & 0 & z\gamma_3 \end{pmatrix}.$$

Consider the subset

$$Y := \{M_z \in \mathfrak{gl}_3(\mathbb{C}) \mid z \in \mathbb{C}\}$$

in the Lie algebra of $\mathfrak{gl}_3(\mathbb{C})$. Evidently Y is an affine line (not going through the origin) in the vector space $\mathfrak{gl}_3(\mathbb{C})$, and we can give Y the induced reduced closed subscheme structure. In particular we have

$$Y \cong \mathrm{Spec}\mathbb{C}[t]$$

as schemes. Next consider the subset

$$X := \{(V_\bullet, M_z) \in Fl(\mathbb{C}^3) \times Y \mid M_z V_i \subset V_{h(i)}, 1 \leq i \leq 3\} \subset Fl(\mathbb{C}^3) \times Y \quad (6.1)$$

where $h(1) = 2$, $h(2) = 3$, and $h(3) = 3$. Notice that there is a natural projection map

$$p : X \to Y$$
$$(V_\bullet, M_z) \mapsto M_z$$

and that each (set-theoretic) fibre $X_z := p^{-1}(M_z) \subseteq Fl(\mathbb{C}^3)$ is the Hessenberg variety $Hess(M_z, h)$ (see Definition 5.1). In particular, when $z = 0$ we get (set-theoretically) the Peterson variety Pet_3 and when $z \neq 0$ we get a regular semisimple Hessenberg variety (see Definitions 5.5 and 5.3, respectively). As mentioned above, the basic idea of this chapter is to view X as a family of Hessenberg varieties over the affine line $Y \cong \text{Spec}\mathbb{C}[t]$ and to use the flatness of this family to compute the degree of the fibre $X_0 \cong Pet_3$. As a first step in implementing this plan, we must describe the scheme structure on X and also prove that the projection $p : X \to Y$ is a flat family. We must also verify that the scheme-theoretic fibre of $z \in \text{Spec}\mathbb{C}[t] \cong \mathbb{C}$, which by slight abuse of notation we also denote by X_z, is reduced; this would allow us to identify the scheme-theoretic fibre X_z with the corresponding Hessenberg varieties (in the classical algebraic geometry sense) mentioned above. We do the first of these tasks in the remainder of this section. The second task, of proving flatness, occupies Section 6.2, and the proof that the scheme-theoretic fibres are reduced is given in Section 6.3.

First recall that $Fl(\mathbb{C}^3)$ has a standard scheme structure which can be described explicitly using an open cover by affines, indexed by the permutation group S_3 (see for example [IY12, Section 2]). We already specified, above, a scheme structure on Y. Let $Fl(\mathbb{C}^3) \times Y$ denote the product scheme. We give X a closed subscheme structure by locally giving generators for an ideal cutting out the subset X. Since we work locally when we describe these ideals, we also check that our scheme structures are compatible with respect to gluing affine charts.

We use the following open cover of $Fl(\mathbb{C}^3)$ (see for example [IY12, §2]). Let S_3 denote the symmetric group on 3 elements and let U_- denote the group of 3×3 lower triangular unipotent matrices, so an element $u \in U_-$ has the form

$$u = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}$$

for some $a, b, c \in \mathbb{C}$. For each $w \in S_3$ and $u \in U_-$ we write

$$U_w := \{[wu] \mid u \in U_-\} \subset Fl(\mathbb{C}^3)$$

where we use $[g]$ to denote the flag represented by the matrix $g \in GL_n(\mathbb{C})$ as in Section 1.3. Each $U_w \subset Fl(\mathbb{C}^3)$ is isomorphic to $U_- \cong \mathbb{C}^3$ and $Fl(\mathbb{C}^3)$ decomposes into these six open cells

$$Fl(\mathbb{C}^3) = \bigcup_{w \in S_3} U_w$$

where

$$\begin{aligned} U_{(123)} &= \left[\begin{pmatrix} 1 & 0 & 0 \\ u_1 & 1 & 0 \\ u_2 & u_3 & 1 \end{pmatrix}\right], U_{(213)} = \left[\begin{pmatrix} v_1 & 1 & 0 \\ 1 & 0 & 0 \\ v_2 & v_3 & 1 \end{pmatrix}\right], U_{(132)} = \left[\begin{pmatrix} 1 & 0 & 0 \\ w_2 & w_3 & 1 \\ w_1 & 1 & 0 \end{pmatrix}\right] \\ U_{(231)} &= \left[\begin{pmatrix} x_2 & x_3 & 1 \\ 1 & 0 & 0 \\ x_1 & 1 & 0 \end{pmatrix}\right], U_{(312)} = \left[\begin{pmatrix} y_1 & 1 & 0 \\ y_2 & y_3 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right], U_{(321)} = \left[\begin{pmatrix} z_2 & z_3 & 1 \\ z_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right] \end{aligned} \tag{6.2}$$

and all the parameters are complex parameters. Next we find the (set-theoretic) intersection of the subset X with each of these affine open cells. Consider, for instance, the affine cell $U_{(123)}$. Recalling that X is defined by the conditions

$M_z V_i \subset V_{h(i)}$ for all i, we find that on $U_{(123)}$ this is equivalent to the condition that

$$\det \begin{pmatrix} 1 & 0 & z\gamma_1 + u_1 \\ u_1 & 1 & z\gamma_2 u_1 + u_2 \\ u_2 & u_3 & z\gamma_3 u_2 \end{pmatrix} = 0$$

which in turn holds if and only if

$$(\gamma_{31} u_2 + \gamma_{12} u_1 u_3) z + (u_1^2 u_3 - u_1 u_2 - u_2 u_3) = 0,$$

where we denote

$$\gamma_{ij} := \gamma_i - \gamma_j.$$

Based on the above, we can equip the intersection $U_{(123)} \cap X$ with a scheme structure by defining

$$U_{(123)} \cap X := \operatorname{Spec} \mathbb{C}[u_1, u_2, u_3, t] / \left((\gamma_{31} u_2 + \gamma_{12} u_1 u_3) t + (u_1^2 u_3 - u_1 u_2 - u_2 u_3) \right). \tag{6.3}$$

For each of the other five open cells, by computing their corresponding determinantal equations, it is not hard to see that we can give the following scheme structure to their intersections with X:

$$\begin{aligned} U_{(213)} \cap X &:= \operatorname{Spec} \mathbb{C}[v_1, v_2, v_3, t] / \left((\gamma_{23} v_2 + \gamma_{12} v_1 v_3) t + (v_2^2 + v_3 - v_1 v_2 v_3) \right), \\ U_{(132)} \cap X &:= \operatorname{Spec} \mathbb{C}[w_1, w_2, w_3, t] / \left((\gamma_{12} w_2 + \gamma_{31} w_1 w_3) t + (w_2^2 - w_1 - w_1 w_2 w_3) \right), \\ U_{(231)} \cap X &:= \operatorname{Spec} \mathbb{C}[x_1, x_2, x_3, t] / \left((\gamma_{12} x_2 + \gamma_{23} x_1 x_3) t + (x_2^2 x_3 + 1 - x_1 x_2) \right), \\ U_{(312)} \cap X &:= \operatorname{Spec} \mathbb{C}[y_1, y_2, y_3, t] / \left((\gamma_{23} y_2 + \gamma_{31} y_1 y_3) t + (1 - y_2 y_3) \right), \\ U_{(321)} \cap X &:= \operatorname{Spec} \mathbb{C}[z_1, z_2, z_3, t] / \left((\gamma_{31} z_2 + \gamma_{23} z_1 z_3) t + (z_3 - z_1) \right). \end{aligned} \tag{6.4}$$

For notational simplicity, henceforth we denote by A_w the coordinate rings of the $U_w \cap X$ and by I_w the ideals appearing in the RHS of (6.4), so that

$$U_w \cap X := \operatorname{Spec}(A_w)$$

and I_w is the ideal of relations defining A_w, so for example

$$I_{(123)} = \left((\gamma_{31}u_2 + \gamma_{12}u_1u_3)t + (u_1^2u_3 - u_1u_2 - u_2u_3)\right).$$

It is well known that the open charts U_w of the flag $Fl(\mathbb{C}^3)$, each of which is isomorphic to $\mathbb{C}^3$ (where for example we regard $U_{(123)}$ as the scheme $\operatorname{Spec}\mathbb{C}[u_1, u_2, u_3]$), glue together to give a well-defined scheme structure on $Fl(\mathbb{C}^3)$. To show that the schemes (6.4) and (6.3) also glue, we must show that there exist isomorphisms (i.e. change-of-coordinate morphisms) $U_w \cap U_v \subset U_w \to U_w \cap U_v \subset U_v$ for all pairs w, v which satisfy the usual cocycle conditions.

We show how to do this concretely for a concrete pair of w and v. It is not hard to see that the intersection $(X \cap U_{(123)}) \cap (X \cap U_{(213)})$ can be described as $D(u_1) \subseteq U_{(123)}$ and $D(v_1) \subseteq U_{(213)}$. (Here we follow the notation of [Vak15], so in particular, for an element $f \in A$, the subset $D(f)$ in $\operatorname{Spec} A$ is defined as $\{\mathfrak{p} \mid f \notin \mathfrak{p}\}$.) Indeed, if $u_1 \neq 0$, then the flag

$$\left[\begin{pmatrix} 1 & 0 & 0 \\ u_1 & 1 & 0 \\ u_2 & u_3 & 1 \end{pmatrix}\right]$$

in $U_{(123)}$ is equal to the flag

$$\left[\begin{pmatrix} \frac{1}{u_1} & 1 & 0 \\ 1 & 0 & 0 \\ \frac{u_2}{u_1} & u_2 - u_1u_3 & 1 \end{pmatrix}\right]$$

in $U_{(213)}$. Similarly, if $v_1 \neq 0$, then

$$\left[\begin{pmatrix} v_1 & 1 & 0 \\ 1 & 0 & 0 \\ v_2 & v_3 & 1 \end{pmatrix}\right] \sim \left[\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{v_1} & 1 & 0 \\ \frac{v_2}{v_1} & v_2 - v_1v_3 & 1 \end{pmatrix}\right].$$

Recall that the coordinate ring of $D(f) \subseteq \operatorname{Spec} A$ is the localization A_f. With this in mind, we can define an isomorphism between the coordinate rings of $D(u_1)$ and $D(v_1)$ by

$$\left({}^{\mathbb{C}[v_1,v_2,v_3,t]}\Big/_{I_{(213)}} \right)_{v_1} \xrightarrow{\varphi} \left({}^{\mathbb{C}[u_1,u_2,u_3,t]}\Big/_{I_{(123)}} \right)_{u_1}$$

$$v_1 \mapsto \frac{1}{u_1}$$

$$v_2 \mapsto \frac{u_2}{u_1}$$

$$v_3 \mapsto u_2 - u_1u_3.$$

The inverse can be easily checked to be

$$\left({}^{\mathbb{C}[u_1,u_2,u_3,t]}\Big/_{I_{(123)}} \right)_{u_1} \xrightarrow{\psi} \left({}^{\mathbb{C}[v_1,v_2,v_3,t]}\Big/_{I_{(213)}} \right)_{v_1}$$

$$u_1 \mapsto \frac{1}{v_1}$$

$$u_2 \mapsto \frac{v_2}{v_1}$$

$$u_3 \mapsto v_2 - v_1v_3.$$

Similar calculations with respect to other pairs of intersections confirms that the six coordinate patches indeed glue together to give a well-defined global closed subscheme structure on X. It is also not difficult to check directly the cocycle conditions. Moreover, the description given above evidently realizes X as a closed subscheme of $Fl(\mathbb{C}^n) \times Y$.

6.2 Flatness of the Family $p : X \to Y$

As already mentioned, there is a natural projection map

$$p : X \to Y$$

$$(V_\bullet, M_z) \mapsto M_z$$

whose fibres are set-theoretically identifiable with Hessenberg varieties. In this section we will show that this is in fact a flat morphism of schemes with respect to the scheme structures on X and Y defined in the previous section. In other words, the fibres of this morphism form a flat family. For more details about flatness see for example [Har77, Ch.3, §9].

We first recall the definition of a flat morphism.

Definition 6.1. *We say a morphism of schemes $p : X \to Y$ is **flat at** $x \in X$ if the stalk $(\mathcal{O}_X)_x$ is a flat $(\mathcal{O}_Y)_{p(x)}$-module. The morphism is called **flat** if it is flat at every $x \in X$.*

Since X can be covered by six open affine schemes, $U_{w_i} \cap X = \mathrm{Spec}(A_{w_i})$, we can check for flatness by checking it on each open chart. The following proposition will help us do just that, since $Y = \mathrm{Spec}(\mathbb{C}[t])$ is an affine scheme.

Proposition 6.2. *[Har77, Proposition 9.2] Let $B \to A$ be a ring homomorphism. Then this is a flat ring homomorphism if and only if the corresponding morphism of affine schemes $\mathrm{Spec}A \to \mathrm{Spec}B$ is flat.*

Using this proposition, we can see that it suffices to show that for each $w \in S_3$, the morphism $\mathrm{Spec}\, A_w \to \mathrm{Spec}\, \mathbb{C}[t]$ corresponds to a flat ring homomorphism $\mathbb{C}[t] \to A_w$, i.e., A_w is flat with respect to the $\mathbb{C}[t]$-module structure specified by this ring homomorphism. In order to show that A_w is a flat $\mathbb{C}[t]$-module, we can take advantage of the fact that $\mathbb{C}[t]$ is a PID. We recall the following useful criterion for this special case.

Proposition 6.3. *[Har77, Example 9.1.3] Let B be a principal ideal domain and let $B \to A$ be a ring homomorphism. Then this morphism is flat (i.e. A is a flat B-module) if and only if A is a torsion-free B-module.*

Before proceeding, we must first concretely describe the ring homomorphism $\varphi_w : \mathbb{C}[t] \to A_w$ corresponding to $U_w \cap X = \mathrm{Spec}\, A_w \to Y = \mathrm{Spec}\, \mathbb{C}[t]$

for each $w \in S_3$. From the description of the intersections $U_w \cap X$ given in Section 6.1, it is not hard to see that for each $w \in S_3$ the ring homomorphism φ_w is given by the composition of the canonical inclusion of $\mathbb{C}[t]$ into a larger polynomial ring (where the variable t gets sent to the variable t) and the quotient map of the polynomial ring to its quotient by the ideal I_w. So for example $\varphi_{(123)} : \mathbb{C}[t] \to A_{(123)}$ is the composition of the usual inclusion $\mathbb{C}[t] \hookrightarrow \mathbb{C}[u_1, u_2, u_3, t]$ with the quotient homomorphism $\mathbb{C}[u_1, u_2, u_3, t] \to A_{(123)} := \mathbb{C}[u_1, u_2, u_3, t]/I_{(123)}$. This ring homomorphism then defines the $\mathbb{C}[t]$-module structure on A_w.

We are now ready to prove that our morphism is flat.

Proposition 6.4. *The morphism of schemes $p : X \to Y$ is flat.*

Proof. By our discussion above, it suffices to show that for each $w \in S_3$, the coordinate ring A_w contains no torsion element as a $\mathbb{C}[t]$-module. In other words, for each A_w we want to show that zero is the only element of A_w that can be annihilated by a non-zero element of $\mathbb{C}[t]$.

For concreteness, let us first consider

$$A_{(123)} = \mathbb{C}[u_1, u_2, u_3, t]/\left((\gamma_{31}u_2 + \gamma_{12}u_1u_3)t + (u_1^2u_3 - u_1u_2 - u_2u_3)\right).$$

Let $f \in \mathbb{C}[u_1, u_2, u_3, t]$, $g \neq 0 \in \mathbb{C}[t]$ and suppose $fg \equiv 0$ in $A_{(123)}$. We wish to show that $f \equiv 0$ in $A_{(123)}$.

Given $fg \equiv 0$, this implies that there exists a $h \in \mathbb{C}[u_1, u_2, u_3, t]$ such that

$$fg = h\left((\gamma_{31}u_2 + \gamma_{12}u_1u_3)t + (u_1^2u_3 - u_1u_2 - u_2u_3)\right) \in \mathbb{C}[u_1, u_2, u_3, t].$$

Since $g \in \mathbb{C}[t]$ does not contain any positive powers of the variables u_1, u_2, u_3, it cannot be a multiple of $(\gamma_{31}u_2 + \gamma_{12}u_1u_3)t + (u_1^2u_3 - u_1u_2 - u_2u_3)$ in $\mathbb{C}[u_1, u_2, u_3, t]$. Since $\mathbb{C}[u_1, u_2, u_3, t]$ is a UFD, it now suffices to show that $(\gamma_{31}u_2 + \gamma_{12}u_1u_3)t +$

$(u_1^2u_3 - u_1u_2 - u_2u_3)$ is irreducible in $\mathbb{C}[u_1, u_2, u_3, t]$, since that would imply that it divides f.

To see this, suppose $F, G \in \mathbb{C}[u_1, u_2, u_3, t]$ are such that

$$FG = (\gamma_{31}u_2 + \gamma_{12}u_1u_3)t + (u_1^2u_3 - u_1u_2 - u_2u_3). \tag{6.5}$$

Since the right-hand side has degree 1 with respect to the variable t, we can assume without loss of generality that

$$F = F_1t + F_0 \text{ for } F_1, F_0 \in \mathbb{C}[u_1, u_2, u_3], \quad G = G_0 \in \mathbb{C}[u_1, u_2, u_3].$$

Then (6.5) is equivalent to

$$F_1G_0 = \gamma_{31}u_2 + \gamma_{12}u_1u_3 \text{ and } F_0G_0 = u_1^2u_3 - u_1u_2 - u_2u_3. \tag{6.6}$$

Since the right-hand side of the first equation in (6.6) is a degree-1 polynomial with respect to the variable u_2 with a non-zero constant coefficient, it is not difficult to see that either F_1 or G_0 must be a constant polynomial.

Now suppose in order to obtain a contradiction that $F_1 = k$, where $k \in \mathbb{C}$ is non-zero. Then by the first equation in (6.6) we have

$$G_0 = \frac{1}{k}(\gamma_{31}u_2 + \gamma_{12}u_1u_3). \tag{6.7}$$

Substituting G_0 into the second equation in (6.6) we obtain

$$F_0G_0 = F_0\frac{1}{k}(\gamma_{31}u_2 + \gamma_{12}u_1u_3) = u_1^2u_3 - u_2(u_1 + u_3). \tag{6.8}$$

Note that the RHS of (6.8) is degree-1 in the variable u_2. Since the factor $\frac{1}{k}(\gamma_{31}u_2 + \gamma_{12}u_1u_3)$ is also degree-1 in u_2 (recall $\gamma_{31} \neq 0$ by assumption on the eigenvalues γ_i), we must have that F_0 is degree-0 with respect to the variable u_2. Now, comparing coefficients in front of the u_2 term, we see then that we must have

$$F_0 = \frac{-k}{\gamma_{31}}(u_1 + u_3).$$

Taking the G_0 found in (6.7), this means that

$$F_0G_0 = -u_2(u_1+u_3) - \frac{\gamma_{12}}{\gamma_{31}}u_1u_3(u_1+u_3)$$

which is a contradiction, since this does not equal the F_0G_0 found in (6.8). (Recall as before that $\gamma_{12} \neq 0$ and $\gamma_{31} \neq 0$.) Therefore, $G_0 = G$ must be a constant polynomial. Hence $(\gamma_{31}u_2 + \gamma_{12}u_1u_3)t + (u_1^2u_3 - u_1u_2 - u_2u_3)$ is irreducible, and thus $A_{(123)}$ is torsion-free.

Similar arguments show that A_w is $\mathbb{C}[t]$-torsion-free for the non-identity elements in S_3. Since flatness is a local property, we conclude $p : X \to Y$ is a flat morphism, as desired.

$\square$

6.3 Reducedness of the Fibres of the Flat Family

Having shown that the family $p : X \to Y$ is flat in Section 6.2, our next task – which we accomplish in this section – is to prove that the fibres of p are reduced schemes. This will allow us to compute the degree of Pet_3 in the next section.

We fix a choice of a (closed) point $z \in Y \cong \mathbb{C} \cong \operatorname{Spec}\mathbb{C}[t]$ for the duration of this discussion. We first recall (cf. for example [Har77, Ch. 2.3, pg. 89]) that the fibre over z of the morphism $p : X \to Y$ is defined to be the scheme-theoretic fibre product

$$X_z := X \times_Y \operatorname{Spec} k(z)$$

where $k(z)$ denotes the residue field of the local ring $\mathbb{C}[t]_{(t-z)}$:

$$k(z) := \mathbb{C}[t]_{(t-z)}/\mathfrak{m} \cong \mathbb{C} \quad ; \quad \frac{f(t)}{g(t)} + \mathfrak{m} \mapsto \frac{f(z)}{g(z)}.$$

In particular, the $\mathbb{C}[t]$-module structure on $k(z) \cong \mathbb{C}$ is specified by the ring homomorphism $\mathbb{C}[t] \to k(z) \cong \mathbb{C}$ which evaluates a polynomial at the value z, i.e., $f(t) \mapsto f(z)$.

For our purposes, it suffices to know that in the case of affine schemes, the fibre product $\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B$ of $\varphi : \operatorname{Spec} A \to \operatorname{Spec} C$ and $\psi : \operatorname{Spec} B \to \operatorname{Spec} C$ is given by $\operatorname{Spec}(A \otimes_C B)$, where the C-algebra structure of A and B are specified by the maps $C \to A$ and $C \to B$ corresponding to φ and ψ. Using the open cover $X = \cup_{w \in S_3} \operatorname{Spec} A_w$ of X constructed in Section 6.1, we may therefore describe a fibre X_z as follows:

$$\begin{aligned} X_z := & X \times_Y \operatorname{Spec} k(z) = \left(\bigcup_{w \in S_3} \operatorname{Spec} A_w \right) \times_Y \operatorname{Spec} k(z) \\ &= \bigcup_{w \in S_3} \left(\operatorname{Spec} A_w \times_{\operatorname{Spec} \mathbb{C}[t]} \operatorname{Spec} k(z) \right) \\ &\cong \bigcup_{w \in S_3} \operatorname{Spec} \left(A_w \otimes_{\mathbb{C}[t]} k(z) \right). \end{aligned} \tag{6.9}$$

Recall that each of the rings A_w above are of the form $\mathbb{C}[a, b, c, t]/I_w$ for some variables a, b, c and an ideal I_w, and is equipped with the natural $\mathbb{C}[t]$-module structure induced by the natural inclusion $\mathbb{C}[t] \hookrightarrow \mathbb{C}[a, b, c, t]$. The following is then immediate from the given module structure on $k(z) \cong \mathbb{C}$.

Lemma 6.5. *Let $k(z) \cong \mathbb{C}$ be as above. Let I be an ideal of $\mathbb{C}[a, b, c, t]$ and equip $\mathbb{C}[a, b, c, t]/I$ with the natural $\mathbb{C}[t]$-module structure as above. Then*

$$(\mathbb{C}[a, b, c, t]/I) \otimes_{\mathbb{C}[t]} k(z) \cong \mathbb{C}[a, b, c]/I_{t=z}$$

as $\mathbb{C}$-algebras, where $I_{t=z}$ denote the ideal of $\mathbb{C}[a, b, c]$ obtained from I by setting t equal to z, i.e.,

$$I_{t=z} := \{ f(a, b, c, z) \in \mathbb{C}[a, b, c] \mid f(a, b, c, t) \in I \}.$$

Henceforth we let B_w denote the ring $A_w \otimes_{\mathbb{C}[t]} k(z)$. From Lemma 6.5 it immediately follows that

$$\begin{aligned}
\mathrm{Spec}(B_{(123)}) &\cong \mathrm{Spec}\mathbb{C}[u_1, u_2, u_3]/\left((\gamma_{31}u_2 + \gamma_{12}u_1u_3)z + (u_1^2u_3 - u_1u_2 - u_2u_3)\right),\\
\mathrm{Spec}(B_{(213)}) &\cong \mathrm{Spec}\mathbb{C}[v_1, v_2, v_3]/\left((\gamma_{23}v_2 + \gamma_{12}v_1v_3)z + (v_2^2 + v_3 - v_1v_2v_3)\right),\\
\mathrm{Spec}(B_{(132)}) &\cong \mathrm{Spec}\mathbb{C}[w_1, w_2, w_3]/\left((\gamma_{12}w_2 + \gamma_{31}w_1w_3)z + (w_2^2 - w_1 - w_1w_2w_3)\right),\\
\mathrm{Spec}(B_{(231)}) &\cong \mathrm{Spec}\mathbb{C}[x_1, x_2, x_3]/\left((\gamma_{12}x_2 + \gamma_{23}x_1x_3)z + (x_2^2x_3 + 1 - x_1x_2)\right),\\
\mathrm{Spec}(B_{(312)}) &\cong \mathrm{Spec}\mathbb{C}[y_1, y_2, y_3]/\left((\gamma_{23}y_2 + \gamma_{31}y_1y_3)z + (1 - y_2y_3)\right),\\
\mathrm{Spec}(B_{(321)}) &\cong \mathrm{Spec}\mathbb{C}[z_1, z_2, z_3]/\left((\gamma_{31}z_2 + \gamma_{23}z_1z_3)z + (z_3 - z_1)\right)
\end{aligned} \tag{6.10}$$

and from (6.9) it follows that these six affine schemes form an open cover of the fibre X_z.

We now wish to show that each scheme-theoretic fibre X_z is in fact a *reduced* scheme, and hence corresponds to a Hessenberg *variety* in the classical algebraic-geometric sense. In particular, X_0 corresponds to the Peterson variety Pet_3 and X_z for $z \neq 0$ corresponds to the regular semisimple Hessenberg variety

$$Hess(M_z, h), \tag{6.11}$$

with Hessenberg function $h(1) = 2$, $h(2) = 3$, and $h(3) = 3$ and matrix

$$M_z = \begin{pmatrix} z\gamma_1 & 1 & 0 \\ 0 & z\gamma_2 & 1 \\ 0 & 0 & z\gamma_3 \end{pmatrix}.$$

For reference, we recall the definition of reducedness (see e.g. [Vak15, Definition 5.2.1]).

Definition 6.6. *A ring is said to be* ***reduced*** *if it has no non-zero nilpotents. A scheme X is* ***reduced*** *if $\mathcal{O}_X(U)$ is reduced for every open set U of X.*

The following is the main result of this section.

Proposition 6.7. *The fibres X_z of $p : X \to Y$ are reduced for all closed points $z \in Y$.*

Proof. It is clear from Definition 6.6 that reducedness is a local property, so it suffices to show that each $\operatorname{Spec} B_w$ is reduced. It is also straightforward [Vak15, Exercise 5.2.B] to see that an affine scheme $\operatorname{Spec} B$ is reduced if B is a reduced ring, so in our case it suffices to show that each B_w is reduced. By Lemma 6.5, each of the rings B_w is a polynomial ring modulo a principal ideal, generated by a polynomial which we denote here by f_w. To show that B_w is reduced, it now suffices to show that this ideal is radical, and since prime ideals are radical, we will show that the generator f_w is irreducible.

The argument showing that each f_w is irreducible is similar to the argument given in Proposition 6.4. One difference, however, is that in the argument below we must treat the cases $z = 0$ and $z \neq 0$ separately. Since the proof for all six possibilities of $w \in S_3$ are nearly identical, we only provide details for the case $w = (123)$ and leave the other cases to the reader.

Let $w = (123)$ and suppose $z \neq 0$. Then the generator f_w for the defining ideal of $B_{(123)}$ is

$$(\gamma_{31}u_2 + \gamma_{12}u_1u_3)z + (u_1^2u_3 - u_1u_2 - u_2u_3).$$

Suppose F and G are polynomials in $\mathbb{C}[u_1, u_2, u_3]$ such that

$$FG = (\gamma_{31}z - u_1 - u_3)u_2 + \gamma_{12}zu_1u_3 + u_1^2u_3. \tag{6.12}$$

The right-hand side of (6.12) is linear in the variable u_2. Therefore, without loss of generality, we may assume that

$$F = F_1u_2 + F_0 \text{ and } G = G_0 \text{ for } F_1, F_0, G_0 \in \mathbb{C}[u_1, u_3].$$

Moreover,

$$F_1 G_0 = \gamma_{31} z - u_1 - u_3 \text{ and } F_0 G_0 = \gamma_{12} z u_1 u_3 + u_1^2 u_3. \tag{6.13}$$

Since the right-hand side of the first equation in (6.13) is a degree-1 polynomial with respect to the variable u_1 with a non-zero constant coefficient, it is not difficult to see that either F_1 or G_0 must be a constant polynomial. Suppose in order to obtain a contradiction that $F_1 = k$ is a non-zero constant. Then the first equation of (6.13) implies that

$$G_0 = \frac{1}{k}(\gamma_{31} z - u_1 - u_3). \tag{6.14}$$

Substituting into the second equation of (6.13) tells us

$$F_0 G_0 = \frac{F_0}{k}(\gamma_{31} z - u_1 - u_3) = (\gamma_{12} z u_1 + u_1^2) u_3. \tag{6.15}$$

Note that both G_0 and the RHS of (6.15) are degree-1 polynomials in u_3. Hence F_0 must be degree-0 in u_3. Comparing coefficients in front of u_3 in (6.15), we conclude that that

$$F_0 = -k(\gamma_{12} z u_1 + u_1^2). \tag{6.16}$$

Multiplying the expressions for F_0 and G_0 from (6.16) and (6.14) respectively, we obtain

$$F_0 G_0 = (\gamma_{12} z u_1 + u_1^2) u_3 + (\gamma_{12} z u_1 + u_1^2)(u_1 - \gamma_{31} z)$$

which is not equal to the expression for $F_0 G_0$ in (6.15), yielding a contradiction. Therefore, $G = G_0$ must be a constant polynomial, and we conclude that $(\gamma_{31} u_2 + \gamma_{12} u_1 u_3) z + (u_1^2 u_3 - u_1 u_2 - u_2 u_3)$ is irreducible, as desired.

Next we consider the case $z = 0$. Then the generator for the defining ideal of $B_{(123)}$ is

$$u_1^2 u_3 - u_1 u_2 - u_2 u_3.$$

Suppose F and G are polynomials in $\mathbb{C}[u_1, u_2, u_3]$ such that

$$FG = (u_1^2 - u_2)u_3 - u_1u_2.$$

The right-hand side is degree-1 in u_3, so once again without loss of generality we may assume that

$$F = F_1u_3 + F_0 \text{ and } G = G_0 \text{ for } F_1, F_0, G_0 \in \mathbb{C}[u_1, u_2].$$

Moreover,

$$F_1G_0 = u_1^2 - u_2 \text{ and } F_0G_0 = -u_1u_2. \tag{6.17}$$

Since the first equation in (6.17) is linear in u_2 with a non-zero constant coefficient, again it is not hard to see that either F_1 or G_0 is a constant polynomial. If $F_1 = k$ is constant with $k \neq 0$, then $G_0 = \frac{1}{k}(u_1^2 - u_2)$. Substituting into the second equation of (6.17) gives

$$F_0G_0 = \frac{F_0}{k}(u_1^2 - u_2) = -u_1u_2.$$

By arguments similar to the above, F_0 must have degree 0 with respect to u_2. Comparing the coefficients in front of u_2, we must have that $F_0 = ku_1$, and so

$$F_0G_0 = u_1^3 - u_1u_2$$

which does not equal $-u_1u_2$, yielding a contradiction. Therefore, $G_0 = G$ must be a constant polynomial, and $u_1^2u_3 - u_1u_2 - u_2u_3$ is irreducible, as desired. $\square$

6.4 The Degree of Pet_3 via Equivariant Localization

The results in the previous sections allow us to compute the degree of the Peterson variety Pet_3, as we now explain. The key is the well-known fact that degrees are preserved under flat families.

Theorem 6.8. *[Har77, Corollary 9.10, pg. 263] Let T be a connected noetherian scheme, and let $X \subseteq \mathbb{P}^n \times T$ be a closed subscheme which is flat over T. For any $t \in T$, let X_t denote the fibre, considered as a closed subscheme of $\mathbb{P}^n \times \operatorname{Spec} k(t)$, where $k(t)$ is the residue field of t. Then the degree of X_t is independent of t.*

In our setting, recall that we can view the flag variety $Fl(\mathbb{C}^3)$ as a subvariety of some (large) projective space $\mathbb{P}^N$ via a Plücker mapping. Since the total space X of our flat family is by definition a closed subscheme of $Fl(\mathbb{C}^3) \times Y$, we may use the mapping $Fl(\mathbb{C}^3) \subseteq \mathbb{P}^N$ to view X as a closed subscheme of $\mathbb{P}^N \times Y$ as well. Also note that $\operatorname{Spec} \mathbb{C}[t] \cong \mathbb{C}$ is both connected (since it is irreducible) and noetherian (since $\mathbb{C}[t]$ is a PID). We have also shown in Section 6.2 that $p : X \to Y$ is flat. Hence Theorem 6.8 applies. Moreover, recall that we saw in Section 6.3 that the fibres X_z are reduced and therefore that X_0 can be identified with the Peterson variety Pet_3 of Definition 5.5 and X_z for $z \neq 0$ with a regular semisimple Hessenberg variety as in Definition 5.3. In particular, since each fibre is reduced, its degree (as a scheme) coincides with the degree of its embedding in $\mathbb{P}^N$ as a classical algebraic variety. Thus by Theorem 6.8 we may conclude

$$\deg(Pet_3) = \deg(Hess(M_z, h)) \text{ for any } z \neq 0,\ z \in \mathbb{C}. \tag{6.18}$$

In order to achieve our goal of computing $\deg(Pet_3)$ it therefore suffices to compute $\deg(Hess(M_z, h))$; this latter computation is simpler due to the fact that this regular semisimple Hessenberg variety is in fact a smooth projective toric variety.

Before proceeding we recall the cohomological formula for the degree of a projective algebraic variety. We fix for the duration of this discussion a choice of Plücker mapping (as described in Section 1.3) of $Fl(\mathbb{C}^3)$ into some projective space $\mathbb{P}^N$. Since $Hess(M_z, h)$ is a subvariety of $Fl(\mathbb{C}^3)$, by composing with the inclusion map $Hess(M_z, h) \hookrightarrow Fl(\mathbb{C}^3)$ we also obtain a mapping of the Hessenberg variety $Hess(M_z, h) \hookrightarrow \mathbb{P}^N$. Let $\mathcal{L}_{Hess(M_z,h)}$ denote the Plücker bundle restricted to $Hess(M_z, h)$ under this mapping, i.e., the pullback via this mapping of the anti-tautological line bundle $O(1)$ on $\mathbb{P}^N$. Then the degree of $Hess(M_z, h)$ with respect to this mapping is given by

$$\deg(Hess(M_z, h)) = \int_{Hess(M_z,h)} c_1(\mathcal{L}_{Hess(M_z,h)})^2 \tag{6.19}$$

where $c_1(\mathcal{L}_{Hess(M_z,h)})$ denotes the first Chern class of the line bundle $\mathcal{L}_{Hess(M_z,h)}$ [GH78, Section 1.3, pg. 171]. As already mentioned, for $z \neq 0$ and Hessenberg function h with $h(1) = 2, h(2) = h(3) = 3$, it is known that the Hessenberg variety $Hess(M_z, h)$ is a smooth projective toric variety [DMPS92]. This fact allows us to use the well-known Atiyah-Bott-Berline-Vergne formula to compute the integral on the RHS of (6.19). The Peterson variety Pet_3, in contrast, is *not* smooth (cf. Theorem 5.7); this explains why it is useful to first relate the degrees of Pet_3 and $Hess(M_z, h)$ and then to compute the degree of $Hess(M_z, h)$ instead. In fact, since the degree is constant along the fibres of a flat family, we may, without loss of generality, compute the degree of $Hess(M_1, h)$.

We now recall the famous theorem of Atiyah-Bott-Berline-Vergne.

Theorem 6.9. *[AB84, BV82] Let M be a compact complex manifold of real dimension $2m$ and let $T \cong (S^1)^n$ be a compact torus acting on M. Suppose the T-fixed point set M^T is finite. Then for $\alpha \in H_T^{2m}(M)$ we have*

$$\int_M \alpha = \sum_{p \in M^T} \frac{\alpha(p)}{e_T(p)},$$

where $e_T(p)$ denotes the T-equivariant Euler class of the normal bundle to the T-fixed point $p \in M^T$.

Before proceeding, some comments are in order regarding Theorem 6.9. The class α mentioned in the theorem and the notation $\int_M \alpha$ refers to an *equivariant* cohomology class and the *equivariant* integral $\int_M : H_T^*(M) \to H_T^{*-2m}(M)$ respectively, whereas the computation of the degree in (6.19) uses the ordinary integral $\int_M : H^*(M) \to H^{*-2m}(M)$ of an ordinary cohomology class $c_1(\mathcal{L}_{Hess(M_z,h)})^2$. These maps are related by the commutative diagram

$$\begin{array}{ccc} H_T^*(M) & \longrightarrow & H_T^{*-2m}(M) \\ \downarrow & & \downarrow \\ H^*(M) & \longrightarrow & H^{*-2m}(M) \end{array}$$

where the horizonal arrows are the integrals and the vertical arrows are forgetful maps. In particular, since $H_T^0(M) \cong H^0(M) \cong \mathbb{C}$ and the right vertical arrow is an isomorphism in degree 0, in order to compute the RHS of (6.19) it suffices to find an equivariant lift of the ordinary cohomology class $c_1(\mathcal{L}_{Hess(M_z,h)})^2$. The equivariant cohomology class $c_1^T(\mathcal{L}_{Hess(M_z,h)})^2$ coming from the equivariant first Chern class is a natural candidate. The discussion above therefore implies that it now suffices to compute

$$\int_{Hess(M_z,h)} c_1^T(\mathcal{L}_{Hess(M_z,h)})^2$$

using ABBV (Theorem 6.9).

In order to use the Atiyah-Bott-Berline-Vergne formula we will need to describe the variety $Hess(M_z, h)$ and its T-action more precisely. For the remainder of this discussion we fix $z = 1$. First observe that since M_1 has distinct eigenvalues, it is diagonalizable and hence there exists a $g \in GL_n(\mathbb{C})$ with

$$gM_1g^{-1} = A := \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}.$$

It is not difficult to see that for $g \in GL_n(\mathbb{C})$ as above and from the definition of the $GL_3(\mathbb{C})$-action on $Fl(\mathbb{C}^3)$ that we have

$$g \cdot Hess(M_1, h) = Hess(gM_1g^{-1}, h) = Hess(A, h).$$

Since the Plücker mapping is also $GL_3(\mathbb{C})$-equivariant, where the target $\mathbb{P}^N$ is realized as the projectivization of a $GL_3(\mathbb{C})$-representation in a well-known way [Bri05], it follows that the degree of $Hess(M_1, h)$ in $\mathbb{P}^N$ is equal to that of $Hess(A, h)$. So in what follows we compute the degree of $Hess(A, h)$ instead.

Next we recall the torus action on $Hess(A, h)$. Recall that the group $GL_3(\mathbb{C})$ acts on $Fl(\mathbb{C}^3)$ by left multiplication. This action restricts to an action of the usual maximal torus of $GL_3(\mathbb{C})$, namely, the subgroup of diagonal matrices

$$T = \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \mid t_i \in \mathbb{C} \right\} \cong \mathbb{C}^3.$$

Since A is itself a diagonal matrix, we have $tAt^{-1} = A$ for any $t \in T$. It is not hard to see that this implies that the action of T on $Fl(\mathbb{C}^3)$ preserves the subvariety $Hess(A, h)$, i.e., that T acts on $Hess(A, h)$. Moreover, since this action is defined simply by restricting the action on $Fl(\mathbb{C}^3)$, it is immediate that $Hess(A, h)^T \subseteq Fl(\mathbb{C}^3)^T$. It is well-known that the T-fixed point set of $Fl(\mathbb{C}^3)$ is given by the flags represented by the permutation matrices (which in turn we identify with S_3), so in particular, the fixed point set $Hess(A, h)^T$ is finite. In fact it is not difficult to see that any flag represented by a permutation matrix is contained in $Hess(A, h)$, so we have $Hess(A, h)^T \cong S_3$.

The above discussion shows that the ABBV formula of Theorem 6.9 applies to our situation. Identifying $Hess(A, h)^T$ with S_3 as above, we have

$$\int_{Hess(A,h)} c_1(\mathcal{L}_{Hess(A,h)})^2 = \sum_{w \in S_3} \frac{c_1^T(\mathcal{L}_{Hess(A,h)})^2(w)}{e_T(w)}. \tag{6.20}$$

To compute the RHS of (6.20), we therefore need to compute the restriction to the T-fixed points of the T-equivariant Chern classes $c_1^T(\mathcal{L}_{Hess(A,h)})$ and also the T-equivariant Euler classes of the normal bundle at the T-fixed points.

We begin with the equivariant Chern classes. Computations for the bundle $\mathcal{L}_{Hess(A,h)}$ will be more convenient using a different description of it, as follows. Following our conventions in Section 1.3, let $\lambda = (\lambda_1, \lambda_2, 0) \in \mathbb{Z}^3$ with $\lambda_1 \geq \lambda_2 \geq 0$ and let φ_λ denote the corresponding Plücker mapping of $Fl(\mathbb{C}^3)$. Let B denote the standard Borel subgroup in $GL_3(\mathbb{C})$ of invertible upper triangular matrices as in Section 1.3 and define a one-dimensional representation of B associated to λ as follows:

$$\text{for } b = \begin{pmatrix} t_1 & b_{12} & b_{13} \\ 0 & t_2 & b_{23} \\ 0 & 0 & t_3 \end{pmatrix} \in B \text{ and } z \in \mathbb{C} \text{ define } b \cdot z := t_1^{\lambda_1} t_2^{\lambda_2} z. \tag{6.21}$$

We can now define a line bundle L_λ over the flag variety $Fl(\mathbb{C}^3) \cong GL_3\mathbb{C}/B$ using the above action. More precisely, we define the total space of the bundle as a quotient

$$L_\lambda := GL_3\mathbb{C} \times_B \mathbb{C}_\lambda = GL_3(\mathbb{C}) \times \mathbb{C} \Big/ (gb, z) \sim (g, b \cdot z) \tag{6.22}$$

where the action of B on $\mathbb{C}$ is given by (6.21) and the right action of B on $GL_3(\mathbb{C})$ is given by right multiplication in the group $GL_3(\mathbb{C})$. The projection $\pi : L_\lambda \to Fl(\mathbb{C}^3)$ is given by projecting to the left factor, i.e. $\pi([g, b]) = gB \in Fl(\mathbb{C}^3) = GL_3(\mathbb{C})/B$. It is easy to see that left multiplication by T on the left factor of $GL_3(\mathbb{C}) \times \mathbb{C}$ induces a T-equivariant line bundle structure on L_λ.

It is well-known that this line bundle L_λ is T-equivariantly isomorphic to the Plücker bundle $\mathcal{L}^\lambda = \varphi_\lambda^*(O(1))$ over the flag variety [Ful97, §9.3, pg. 143]. Therefore the T-equivariant line bundle $\mathcal{L}_{Hess(A,h)}$ can also be equivalently described as the restriction to $Hess(A, h)$ of the line bundle $L_\lambda \to Fl(\mathbb{C}^3)$, and this will turn out to be more amenable to computation. By slight abuse of notation, below we will use the same notation L_λ to denote the restriction to $Hess(A,h)$ of the bundle L_λ above.

We now quickly recall some facts about equivariant cohomology.[1] We refer the reader to e.g. [Web12, GGK02] for more details on equivariant cohomology. For a T-equivariant line bundle L_λ the equivariant Chern class $c_1^T(L_\lambda)$ is an element in $H^2_T(Hess(A,h))$. Recall that for a T-fixed point $w \in Hess(A,h)^T$, the T-equivariant inclusion $w \hookrightarrow Hess(A,h)$ (where the T-action on the point w is trivial) induces a pullback

$$H^*_T(Hess(A,h)) \to H^*_T(w) \cong \mathbb{C}[u_1,u_2,u_3], \quad \alpha \mapsto \alpha(w) \tag{6.23}$$

where the isomorphism $H^*_T(w) \cong \mathbb{C}[u_1,u_2,u_3]$ can be derived from the definition of equivariant cohomology as $H^*_T(w) := H^*(ET \times_T \{w\}) \cong H^*(BT)$ where $ET \to BT$ is the universal bundle over the classifying space BT of T. Here the variables u_i each have cohomological degree 2. In the notation of the RHS of (6.20) we need to compute $c_1^T(L_\lambda)(w)$, i.e. the image of $c_1^T(L_\lambda) \in H^2_T(Hess(A,h))$ in $H^*_T(w) \cong \mathbb{C}[u_1,u_2,u_3]$ under the map (6.23). Since the polynomial variables have degree 2, the restriction must be a linear polynomial in the variables. Indeed, from the naturality of Chern classes it follows that $c_1^T(L_\lambda)(w)$ is the T-equivariant Chern class of the line bundle L_λ restricted to the T-fixed point $w \in Hess(A,h)^T \cong S_3$. Since w is a T-fixed point, the fibre over p is a (one-dimensional) T-representation, and it is well-known that the T-equivariant Chern class $c_1^T(L_\lambda|_w)$ is then exactly the T-weight of this representation.

The above discussion shows that we must compute the weights of the T-representation on the fibre of L_λ over each $w \in S_3$. To find these weights we

[1] We always work with cohomology with $\mathbb{C}$-coefficients.

will need the following. Let $w \in S_3$. For an element

$$t = (t_1, t_2, t_3) = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \in T$$

let t_w denote the "permuted" element

$$t_w := \begin{pmatrix} t_{w(1)} & 0 & 0 \\ 0 & t_{w(2)} & 0 \\ 0 & 0 & t_{w(3)} \end{pmatrix}.$$

For the remainder of this discussion we identify an element $w \in S_3$ with a permutation matrix in $GL_3(\mathbb{C})$ in the standard way, and multiplication of an element in S_3 with an element of T as multiplication of matrices in $GL_3(\mathbb{C})$. The following is then an easy matrix computation.

Lemma 6.10. *Given $w \in S_3$ and $t \in T$ we have $tw = wt_w$.*

We remind the reader that the dictionary between T-representations and linear polynomials in $H^2_T(w) \cong \mathbb{C}[u_1, u_2, u_3]$ is given as follows: if T acts on a 1-dimensional vector space $\mathbb{C}$ by

$$t \cdot z = (t_1, t_2, t_3) \cdot z = (t_1^{\beta_1} t_2^{\beta_2} t_3^{\beta_3})z$$

for some integers $\beta_i \in \mathbb{Z}$, then the corresponding linear polynomial (i.e. the "T-weight") is $\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3$. With respect to this dictionary, we compute the T-weights of the fibres $(L_\lambda)_w = L_\lambda|_w$ in the next lemma. Recall that our choice of λ is such that $\lambda = (\lambda_1, \lambda_2, 0)$.

Lemma 6.11. *Let $w \in S_3$. Let $\lambda = (\lambda_1, \lambda_2, 0) \in \mathbb{Z}^3$ with $\lambda_1 \geq \lambda_2 \geq 0$ and L_λ defined as above. Then the T-weight of the T-representation $(L_\lambda)_w$ is $\lambda_1 u_{w(1)} + \lambda_2 u_{w(2)}$.*

Proof. As above, we view an element $w \in S_3$ as a T-fixed point of $Fl(\mathbb{C}^3)$ by viewing it as a permutation matrix in $GL_3(\mathbb{C})$ (and its corresponding flag). A point in the fibre $(L_\lambda)_w$ can be represented by a pair (w, z) for some $z \in \mathbb{C}$. For $t \in T$ acting on the left component by left multiplication we have $t \cdot (w, z) = (tw, z)$. By Lemma 6.10 this equals (wt_w, z), which is equivalent to $(w, t_w \cdot z) = (w, t_{w(1)}^{\lambda_1} t_{w(2)}^{\lambda_2} t_{w(3)}^{0} z)$. Therefore, with respect to the association given above, the T-weight of this representation is $\lambda_1 u_{w(1)} + \lambda_2 u_{w(2)}$, as desired. □

From the discussion above, we conclude

$$c_1^T(L_\lambda)(w) = \lambda_1 u_{w(1)} + \lambda_2 u_{w(2)}. \tag{6.24}$$

The last computations required to evaluate the RHS of (6.20) are the T-equivariant Euler classes $e_T(w)$ of the normal bundle of $Hess(A, h)$ at w for each $w \in Hess(A, h)^T \cong S_3$. Since w is an isolated T-fixed point, the normal bundle is just the tangent space $T_w Hess(A, h)$ of $Hess(A, h)$ at w, and this is naturally a T-representation. It is well-known (see for example [Web12, §3, pg.11]) in this situation that the T-equivariant Euler class $e_T(w)$ is the product of the T-weights appearing in the decomposition of $T_w Hess(A, h)$ into T-weight spaces. Hence our main remaining task is to describe these weights. For this purpose, let α_1 and α_2 denote the positive simple roots of $GL_3\mathbb{C}$ with respect to the standard Borel subgroup B and let $\mathfrak{g}_{\alpha_i}$ denote the weight space generated by the simple root α_i. We recall the following.

Lemma 6.12. *[DMPS92, Lemma 7] Let $w \in S_3$ be a T-fixed point of $Hess(A, h)$. The tangent space of $Hess(A, h)$ at a w is isomorphic to $w \left(\oplus_{i=1}^{2} \mathfrak{g}_{\alpha_i}\right) w^{-1}$ as a T-representation, i.e.*

$$T_w Hess(A, h) \cong w \left(\oplus_{i=1}^{2} \mathfrak{g}_{\alpha_i}\right) w^{-1}$$

where the notation on the RHS indicates the S_3-action on the Lie algebra $\mathfrak{gl}_3(\mathbb{C})$ of $GL_3(\mathbb{C})$ by (matrix) conjugation.

From Lemma 6.12 it immediately follows that the T-equivariant Euler class is equal to the product of the weights of $w\mathfrak{g}_{\alpha_1}w^{-1}$ and $w\mathfrak{g}_{\alpha_2}w^{-1}$. We will now compute these weights more concretely, in terms of the variables u_i. Recall that E_{12} (the matrix with a 1 in the $(1,2)$-th entry and 0s elsewhere) and E_{23} spans $\mathfrak{g}_{\alpha_1}$ and $\mathfrak{g}_{\alpha_2}$, respectively, in $\mathfrak{gl}_3(\mathbb{C})$. Recall also, as mentioned in the statement of Lemma 6.12, that S_3 acts on $\mathfrak{gl}_3(\mathbb{C})$ by conjugation. Fix $w \in S_3$. Using Lemma 6.10 it is straightforward to compute

$$t \cdot wE_{12}w^{-1} = (t_{w(1)}t_{w(2)}^{-1})wE_{12}w^{-1} \text{ and } t \cdot wE_{23}w^{-1} = (t_{w(2)}t_{w(3)}^{-1})wE_{23}w^{-1}.$$

Hence, the weights of $w\mathfrak{g}_{\alpha_1}w^{-1}$ and $w\mathfrak{g}_{\alpha_2}w^{-1}$ are $t_{w(1)} - t_{w(2)}$ and $t_{w(2)} - t_{w(3)}$, respectively, and we obtain the following.

Lemma 6.13. *Let $w \in S_3$. The T-equivariant Euler class of the normal bundle to w in $Hess(A, h)$ is*

$$e_T(w) = (u_{w(1)} - u_{w(2)})(u_{w(2)} - u_{w(3)}). \tag{6.25}$$

With the above in place, we may now compute the degree of $Hess(A, h)$. Note that in the line bundles we consider we always have that $\lambda_1 \geq \lambda_2$ and $\lambda_3 = 0$ Therefore, for notational convenience in future sections, we denote $a_1 := \lambda_2$, $a_2 := \lambda_1 - \lambda_2$, and $a_3 := 0$ given a fixed $\lambda = (\lambda_1, \lambda_2, 0)$. In other words, we have $\lambda = (a_1 + a_2, a_1, 0)$.

Theorem 6.14. *The degree of $Hess(A, h)$ with respect to the mapping determined by the pullback of the Plücker line bundle $L_\lambda = \mathcal{L}_{Hess(A,h)}$ for $\lambda = (a_1 + a_2, a_1, 0)$ is $a_1^2 + 4a_1a_2 + a_2^2$, i.e.*

$$\deg(Hess(A, h)) = a_1^2 + 4a_1a_2 + a_2^2.$$

Proof. By Theorem 6.9, it suffices to compute the equation given in (6.20). By equations (6.24) and (6.25) we have that

$$c_1^T(L_\lambda)(w) = (a_1 + a_2)u_{w(1)} + a_1 u_{w(2)}$$

and

$$e_T(w) = (u_{w(1)} - u_{w(2)})(u_{w(2)} - u_{w(3)}).$$

Simplifying

$$\sum_{w \in S_3} \frac{((a_1 + a_2)u_{w(1)} + a_1 u_{w(2)})^2}{(u_{w(1)} - u_{w(2)})(u_{w(2)} - u_{w(3)})},$$

in Sage for example, gives $a_1^2 + 4a_1a_2 + a_2^2$. $\square$

By our discussion at the beginning of this section, we have the following corollary.

Corollary 6.15. *The degree of Pet_3 with respect to the mapping determined by the pullback of the Plücker line bundle $L_{(a_1+a_2,a_1,0)}|_{Pet_3} \cong \mathcal{L}^{(a_1+a_2,a_1,0)}|_{Pet_3}$ is $a_1^2 + 4a_1a_2 + a_2^2$, i.e.*

$$\deg(Pet_3) = a_1^2 + 4a_1a_2 + a_2^2.$$

Chapter 7

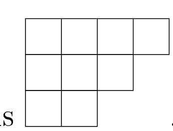
Computing a Newton-Okounkov Body of Pet_3

The results in the previous section allow us now to explicitly compute Newton-Okounkov bodies of Peterson varieties Pet_3, $\Delta(Pet_3, R(V_\lambda), \nu)$, where V_λ is the image of $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$ in $H^0(Pet_3, \mathcal{L}^\lambda \mid_{Pet_3})$ and $\mathcal{L}^\lambda$ is the Plücker bundle over $Fl(\mathbb{C}^3)$ corresponding to λ (see Definition 1.22).

7.1 Young Tableaux

In order to compute the Newton-Okounkov body of Pet_3, we need to consider line bundles over Pet_3 and their spaces of sections. To do this, we first recall some terminology (for more details see [Ful97]).

Definition 7.1. *A **Young diagram** of shape $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0) \in \mathbb{Z}^n$ is a collection of boxes arranged in left-justified rows of size λ_i.*

Example 7.2. The Young diagram of shape $\lambda = (4, 3, 2, 0)$ is  .

Note that a Young diagram of shape $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{n-1})$ is the same as one of shape $\lambda' = (\lambda_1 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0)$, that is, we may always add any number of "rows of length 0" to a Young diagram without changing the diagram. In this chapter it will be convenient for us to assume that the last entry in λ is 0, as we do in the definition below.

Definition 7.3. *The **conjugate** of $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0)$, denoted $\tilde{\lambda} = (\tilde{\lambda_1} \geq \ldots \geq \tilde{\lambda}_{\lambda_1} > 0)$, is the diagram obtained by flipping λ over its main diagonal. In other words, $\tilde{\lambda_i}$ is the length of column i in the Young diagram λ, counting from the left. There are $a_{n-i} := \lambda_i - \lambda_{i+1}$ columns of size i in the Young diagram of shape λ, where here we use the convention that $\lambda_n = 0$. Thus, $\tilde{\lambda} = (\underbrace{n-1, \ldots, n-1}_{a_1}, \underbrace{n-2, \ldots, n-2}_{a_2}, \ldots, \underbrace{1, \ldots, 1}_{a_{n-1}})$.*

Example 7.4. If $\lambda = (4, 3, 2, 0)$, as in Example 7.2, then $a_3 = 1$, $a_2 = 1$, and $a_1 = 2$, so $\tilde{\lambda} = (3, 3, 2, 1)$ and

$$\tilde{\lambda} = \ydiagram{3,3,2,1} \quad .$$

Definition 7.5. *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0) \in \mathbb{Z}^n$ be a Young diagram. A **semistandard Young tableau,** T**,** is a filling of a Young diagram of shape λ by positive integers (possibly with repetition) from the set $\{1, 2, \ldots, n\}$ such that the entries are weakly increasing across rows and strictly increasing down columns.*

Example 7.6. Let $\lambda = \ydiagram{2,1}$. Then there are 8 semistandard Young tableaux of shape λ:

$$\ytableaushort{11,2}\,,\ \ytableaushort{11,3}\,,\ \ytableaushort{12,2}\,,\ \ytableaushort{12,3}\,,\ \ytableaushort{13,2}\,,\ \ytableaushort{13,3}\,,\ \ytableaushort{22,3}\,,\ \ytableaushort{23,3}\,.$$

Similarly, $\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \cline{1-4} 4 & 4 \\ \cline{1-2}\end{array}$ is an example of a semistandard Young tableau of shape $\lambda = (5, 4, 2)$.

A semistandard Young tableau T induces a function which takes a matrix to a product of minors. For the following definition it is convenient to introduce some notation. Suppose C is a column in a semistandard Young tableau T. We temporarily denote by $|C|$ the length of this column, so C is of the form $\begin{array}{|c|}\hline r_1 \\ \hline r_2 \\ \hline \vdots \\ \hline r_{|C|} \\ \hline\end{array}$. Suppose the semistandard Young tableau T is of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0)$. For a square $n \times n$ matrix $A = (a_{ij})$ and a column C of T as above, let us denote by $\det(A(C))$ the determinant of the $|C| \times |C|$ submatrix of A obtained by taking the rows indexed by C of the $|C|$ leftmost columns of A, i.e.

$$\det(A(C)) := \det \begin{pmatrix} a_{r_1,1} & a_{r_1,2} & \cdots & a_{r_1,|C|} \\ a_{r_2,1} & a_{r_2,2} & \cdots & a_{r_2,|C|} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r_{|C|},1} & a_{r_{|C|},2} & \cdots & a_{r_{|C|},|C|} \end{pmatrix}.$$

Notice that a semistandard Young tableau T of shape $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0)$ has exactly λ_1 columns. Starting from the leftmost column we may label these columns as $C_1, \cdots, C_{\lambda_1}$. With this notation in place we may define the following.

Definition 7.7. *Let T be a a semistandard tableau T of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0)$. Following notation as above, we define $S(T)$, also denoted* $\det A(T)$*, as the product of the minors of A corresponding to the* *columns of T. More precisely,*

$$S(T) = \det A(T) = \prod_{i=1}^{\lambda_1} \det(A(C_i)).$$

We illustrate this definition in the following example:

Example 7.8. If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $\det A\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right)$ corresponds to the minor $P_{12}P_3 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} a_{31}$. Similarly, if $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{pmatrix}$, then

$$\begin{aligned} \det B\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 & 4 \\ \cline{1-4} 4 & 4 \\ \cline{1-2}\end{array}\right) &= P_{124}P_{234}P_{24}P_{34}P_3 \\ &= \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{41} & b_{42} & b_{43} \end{vmatrix} \begin{vmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{vmatrix} \begin{vmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \end{vmatrix} \begin{vmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{vmatrix} b_{31}. \end{aligned}$$

In this chapter, we will only be concerned with Young tableaux of shape $\lambda = (a_1 + a_2, a_1, 0)$ for $a_1, a_2 \in \mathbb{N}$. Note that these a_i are the same as those which appear in Definition 7.3. Since a semistandard Young tableau of shape λ must be a filling by integers $\{1, 2, 3\}$ and because columns of a semistandard Young tableau must be strictly increasing, the only possibilities for the columns of length 2 in such a T are $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$, $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}$ and $\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$. The only possibilities for the columns of length 1 are $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$. Moreover, because rows must be weakly increasing (from left to right), it is clear that a column $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$ must appear to the left of a $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}$ or a $\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$, and a $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}$ can only appear to the left of a $\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}$, etc. Thus it is not hard to see that we can uniquely represent a semistandard

Young tableau of shape $\lambda = (a_1 + a_2, a_1, 0)$ by recording the number of times each type of column appears. The following lemma formalizes this reasoning.

Lemma 7.9. *The set of semistandard Young tableaux of shape* $\lambda = (a_1 + a_2, a_1, 0)$ *is in one-to-one correspondence with the set*

$$\mathcal{S} = \{(k_1, \ldots, k_6) \in \mathbb{N}^6 \mid k_1 + k_2 + k_3 = a_1, k_4 + k_5 + k_6 = a_2, k_3 \neq 0 \Rightarrow k_4 = 0\}.$$

Proof. By definition, a Young tableau of shape $\lambda = (a_1 + a_2, a_1, 0)$, reading left to right, has a_1 columns of size 2 and a_2 columns of size 1. A semistandard Young tableau is a filling of λ from the set $\{1, 2, 3\}$ such that entries are weakly increasing across rows and strictly increasing down columns. Therefore, the columns of length two, reading from top to bottom, must be filled with 12's, 13's, or 23's. Moreover, since the rows are weakly increasing, the only way to arrange columns filled with 12, 13, and 23 is by placing all 12's to the left, followed by 13's, followed by 23's. Let k_1, k_2, k_3 denote the number of times 12, 13 and 23 appear in T respectively.

Next consider the columns of length one. These must be filled with 1, 2, or 3. Let k_4, k_5, k_6 denote the number of times 1, 2, and 3 appear in T respectively. Since the rows must be weakly increasing, all 1's must be placed in the leftmost columns of size 1, followed by the 2's, followed by the 3's. Again, since rows are weakly increasing, if $k_3 \neq 0$, then this implies that $k_4 = 0$, since we cannot have a row with a 1 to the right of a 2.

Therefore, any semistandard Young tableau of shape λ corresponds to exactly one element in $\mathcal{S}$. Conversely, any element in $\mathcal{S}$ corresponds to exactly one semistandard Young tableau T (the leftmost k_1 columns are filled with 12, the next k_2 columns are filled with 13, etc.). □

Based on the above lemma, henceforth we denote a semistandard Young tableau of shape $\lambda = (a_1 + a_2, a_1, 0)$ by

$$(12)^{k_1}(13)^{k_2}(23)^{k_3}(1)^{k_4}(2)^{k_5}(3)^{k_6}, \tag{7.1}$$

where $k_1+k_2+k_3 = a_1$, $k_4+k_5+k_6 = a_2$, and k_3 and k_4 cannot simultaneously be non-zero.

Example 7.10. If $\lambda = (5, 2, 0)$, then $a_2 = 3$, $a_1 = 2$, and the filling $\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 3 & 3 \\ \cline{1-2}\end{array}$ will be notated as $(13)^2(2)^2(3)$.

The tableau $\lambda = (a_1 + a_2, a_1, 0)$ also determines a mapping φ_λ of $Fl(\mathbb{C}^3)$ into projective space (see Definition 1.20). Let $\mathcal{L}^\lambda$ denote the Plücker bundle over $Fl(\mathbb{C}^3)$ corresponding to λ (see Definition 1.22).

We now describe a correspondence between semistandard Young tableaux T of shape λ and an element $S(T)$ in the space of sections $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$. It turns out that these $S(T)$ form a basis for $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$. The following is well-known.

Lemma 7.11. *Let T be a semistandard tableau of shape $\lambda = (a_1 + a_2, a_1, 0)$ and let A be a 3×3 matrix of indeterminates. Then the polynomial $S(T) =$ $\det A(T)$ can be interpreted as an element of $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$.*

Proof. The Plücker mapping φ_λ maps $Fl(\mathbb{C}^3)$ into a large projective space $\mathbb{P}^N$. Recall that elements of the space of sections $H^0(\mathbb{P}^N, O(1))$ can be interpreted as homogeneous polynomials of degree one in the homogeneous coordinates $x_0, \ldots, x_N$ of $\mathbb{P}^N$. If we restrict these sections to (the image of) $Fl(\mathbb{C}^3)$, then we will obtain sections in $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$. In other words, given an element of the flag $[A] = \left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right]$, homogeneous polynomials of degree one in the homogeneous coordinates of the image $\varphi_\lambda(A)$ will be sections of

$H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$. We now discuss how a semistandard Young tableau T reads off a homogeneous coordinate of $\varphi_\lambda(A)$, and hence determines a section of $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$.

Recall that the Plücker mapping φ_λ first maps $Fl(\mathbb{C}^3)$ into the product of Grassmannians $Gr(1) \times Gr(2) \cong \mathbb{P}^2 \times \mathbb{P}^2$. It then takes the a_2-th Veronese mapping (see Definition 1.17) of the first coordinate of $\mathbb{P}^2 \times \mathbb{P}^2$ and the a_1-th Veronese mapping of the second coordinate. Finally, it takes the Segre embedding (Definition 1.18) of the two factors. In terms of a 3×3 matrix A representing a flag $[A] \in Fl(\mathbb{C}^3)$, the a_2-th Veronese map on the first $\mathbb{P}^2 \cong Gr(1)$ factor corresponds to taking all monomials of degree a_2 in the entries (i.e. the 1×1 minors) of the leftmost column of A, and the a_1-th Veronese map on the second factor $\mathbb{P}^2 \cong Gr(2)$ corresponds to taking all possible products of a_1 many 2×2 minors of the leftmost columns of A. Finally, the last Segre map exactly multiplies the results of each of these Veronese maps, resulting in a product of the form $S(T) = \det A(T)$ as in Definition 7.7. The result follows. $\square$

It turns out that these sections of the form $S(T)$ form a basis of the space of sections. We have the following.

Theorem 7.12. *[Ful97, §9.3, Proposition 3 and §8.1, Theorem 1] The set of semistandard Young tableaux of shape* $\lambda = (a_1+a_2, a_1, 0)$ *are in one-to-one correspondence with a basis for the space of sections* $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$*. Moreover, the set of polynomials*

$$\{S(T) \mid T \textit{ is a semistandard Young tableau of shape } \lambda\},$$

when interpreted as an element of $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$ *as in Lemma 7.11, form a basis for* $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$ *.*

Example 7.13. Let $\lambda = \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$. Then $a_1 = a_2 = 1$. We consider the pullback bundle $\mathcal{L}^\lambda$, as in Example 1.23. A basis for the space of holomorphic sections $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$ is given by the 8 polynomials determined by the 8 semistandard Young tableaux of shape λ, as in Example 7.6. Applying Theorem 7.12, we obtain:

$$H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda) = \mathrm{span}_{\mathbb{C}}\{P_1P_{12}, P_1P_{13}, P_2P_{12}, P_2P_{13}, P_3P_{12}, P_3P_{13}, P_2P_{23}, P_3P_{23}\}.$$

7.2 Newton-Okounkov Body of Pet_3

In order to construct the Newton-Okounkov body of Pet_3, we use certain affine coordinates on an open dense subset of Pet_3:

$$\mathcal{U} := \left\{ \left[\begin{pmatrix} y & x & 1 \\ x & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] \mid x, y \in \mathbb{C} \right\}. \tag{7.2}$$

Lemma 7.14. *The set of flags $\mathcal{U}$ given in* (7.2) *is an open dense subset of Pet_3.*

Proof. It is well-known that the flag variety $Fl(\mathbb{C}^3)$ can be realized as the disjoint union of so-called *open Schubert cells* (for details see, for instance, [Bri05, §1.2]). In particular, the set of flags

$$Bw_0B := \left[\begin{pmatrix} y & z & 1 \\ x & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right]$$

is an open set in $Fl(\mathbb{C}^3)$ around the point corresponding to the permutation matrix $w_0 = 321$ (i.e. x=y=z=0). Since Pet_3 is a subvariety of $Fl(\mathbb{C}^3)$, the intersection $Pet_3 \cap Bw_0B$ is open in Pet_3. Recall that in order for a flag to lie in Pet_3, we need that $XV_1 \subset V_2$ (see Definition 5.5 and Example 5.6), where

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is not hard to see that

$$X(Bw_0B) = \left\langle \begin{pmatrix} x \\ 1 \\ 0 \end{pmatrix} \right\rangle \subset \left\langle \begin{pmatrix} y \\ x \\ 1 \end{pmatrix}, \begin{pmatrix} z \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

if and only if $x = z$. Therefore, the intersection $Pet_3 \cap Bw_0B$ is exactly the matrix given in (7.2). Finally, since the Peterson variety is irreducible by Theorem 5.7, an open subset must be dense. The result follows. $\square$

With respect to these coordinates, we consider the flag of subvarieties

$$Y_2 = \overline{\{x = y = 0\}} \subset Y_1 = \overline{\{x = 0\}} \subset Y_0 = Pet_3.$$

By Theorem 5.7, Pet_3 is irreducible, has dimension 2, and has a unique singular point at the identity. Therefore, since $\dim Y_i = 2 - i$ and each Y_i is nonsingular at the point Y_2, this flag is an admissible flag. We choose the system of parameters $y_1 := x, y_2 := y$ about this flag. Notice that $y_{k|Y_{k-1}}$ is a well-defined, not identically zero, rational function on Y_{k-1} and has a zero of first order on Y_k, so this is a valid system of parameters (see §1.2).

For a line bundle L, this admissible flag and system of parameters defines a geometric valuation $\nu : H^0(Pet_3, L) \to \mathbb{Z}^2$ as in §1.2. It is not hard to see that for the usual lexicographic order on $\mathbb{Z}^2$ with $x > y$, the valuation ν is the lowest term valuation (see Lemma 2.7 and Example 2.8).

We define V_λ to be the image of $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$ in $H^0(Pet_3, \mathcal{L}^\lambda \mid_{Pet_3})$, where $\lambda = (a_1 + a_2, a_1, 0)$. Below we compute the Newton-Okounkov body, Δ, for $R(V_\lambda)$ with respect to ν. Using the combinatorics of Young diagrams, we can explicitly write down elements in this subspace of sections restricted to the open affine coordinate chart $\mathcal{U}$.

Example 7.15. In Example 7.13, we found a basis for $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$, where $\lambda = (2,1,0)$. In particular, we found that

$$H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda) = \mathrm{span}_{\mathbb{C}}\{P_1P_{12}, P_1P_{13}, P_2P_{12}, P_2P_{13}, P_3P_{12}, P_3P_{13}, P_2P_{23}, P_3P_{23}\},$$

where $P_{c_1\cdots c_k}$ denotes the $k \times k$ subdeterminant of $[A] \in Fl(\mathbb{C}^3)$ formed by the rows $1 \leq c_1 < \cdots < c_k \leq n$ of $[A]$. In order to restrict these sections to $\mathcal{U}$, we instead take minors of an element $[B] \in \mathcal{U}$. With respect to the coordinates (7.2) on $\mathcal{U}$, we obtain $P_1 = y$, $P_{12} = y - x^2$, and $P_{13} = -x$. Therefore P_1P_{12} restricted to $\mathcal{U}$ is $y(y-x^2)$ and P_1P_{13} restricted to Pet_3 is $y(-x)$. Continuing in this manner, we can find a basis:

$$V_{(2,1,0)} = \mathrm{span}_{\mathbb{C}}\{y^2 - yx^2, -xy, x^3, -x^2, -x, y - x^2, -1\}.$$

Recall that in (7.1), we established the notation

$$(12)^{k_1}(13)^{k_2}(23)^{k_3}(1)^{k_4}(2)^{k_5}(3)^{k_6}$$

where $k_1+k_2+k_3 = a_1$, $k_4+k_5+k_6 = a_2$, and k_3 and k_4 cannot simultaneously be non-zero; this corresponds to a unique semistandard Young tableau. We also showed in Lemma 7.11 that each semistandard Young tableau corresponds to a section in $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$. As in Example 7.15, we can further restrict these sections to our coordinates on $\mathcal{U}$. In other words, each semistandard Young tableau $(12)^{k_1}(13)^{k_2}(23)^{k_3}(1)^{k_4}(2)^{k_5}(3)^{k_6}$ corresponds to a section in V_λ.

Lemma 7.16. *The semistandard Young tableau*

$$T := (12)^{k_1}(13)^{k_2}(23)^{k_3}(1)^{k_4}(2)^{k_5}(3)^{k_6},$$

(where $k_1+k_2+k_3 = a_1$, $k_4+k_5+k_6 = a_2$, and k_3 and k_4 cannot simultaneously be non-zero) corresponds to the section

$$(y-x^2)^{k_1}(-x)^{k_2}(1)^{k_3}y^{k_4}x^{k_5}1^{k_6} \in V_\lambda.$$

Proof. Let A denote the 3×3 matrix given in (7.2). By Definition 7.7, $S(T) = \det A(T)$ is equal to the product of minors

$$(P_{12})^{k_1}(P_{13})^{k_2}(P_{23})^{k_3}(P_1)^{k_4}(P_2)^{k_5}(P_3)^{k_6}.$$

Here,

$$P_{12} = \begin{vmatrix} y & x \\ x & 1 \end{vmatrix} = y - x^2, P_{13} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x,$$

$$P_{23} = \begin{vmatrix} x & 1 \\ 1 & 0 \end{vmatrix} = -1, P_1 = y, P_2 = x, P_3 = 1.$$

Therefore,

$$\det A(T) = (P_{12})^{k_1}(P_{13})^{k_2}(P_{23})^{k_3}(P_1)^{k_4}(P_2)^{k_5}(P_3)^{k_6} = (y-x^2)^{k_1}(-x)^{k_2}(1)^{k_3}y^{k_4}x^{k_5}1^{k_6}.$$

□

We need to establish the following lemma before computing Newton-Okounkov bodies of Pet_3.

Proposition 7.17. *If the convex hull, $\tilde{\Delta}$, of a set of points $\{\nu(s_1), \ldots, \nu(s_k) \mid s_i \in V_{(a_1+a_2,a_1,0)}\}$ has area equal to $\frac{1}{2}a_1^2 + 2a_1a_2 + \frac{1}{2}a_2^2$, then $\tilde{\Delta}$ is equal to the Newton-Okounkov body $\Delta(Pet_3, R(V_{(a_1+a_2,a_1,0)}), \nu)$.*

Proof. Recall that equation (1.2) in §1.2 relates the volume of a Newton-Okounkov body to a degree of a projective variety. In our case, the Peterson variety Pet_3 is embedded in a projective space by composing the natural inclusion $Pet_3 \hookrightarrow Fl(\mathbb{C}^3)$ with the Plücker mapping $Fl(\mathbb{C}^3) \hookrightarrow \mathbb{P}(V_\lambda)$.

It follows that the degree of $Pet_3 \subseteq \mathbb{P}(V_\lambda)$ is equal to 2! times the volume of $\Delta(Pet_3, R(V_{(a_1+a_2,a_1,0)}), \nu)$. In §6, we found that the degree of the Plücker mapping corresponding to $V_{(a_1+a_2,a_1,0)}$ is $a_1^2+4a_1a_2+a_2^2$. Therefore, we know that the area of the Newton-Okounkov body $\Delta(Pet_3, R(V_{(a_1+a_2,a_1,0)}), \nu)$ is $\frac{1}{2}a_1^2 + 2a_1a_2 + \frac{1}{2}a_2^2$.

By the definition of a Newton-Okounkov body (Definition 1.9), the image under ν of the sections $s_i \in V_{(a_1+a_2,a_1,0)}$ are contained in $\Delta(Pet_3, R(V_{(a_1+a_2,a_1,0)}), \nu)$. Therefore, $\tilde{\Delta} \subset \Delta(Pet_3, R(V_{(a_1+a_2,a_1,0)}), \nu)$. But by assumption, the area of $\tilde{\Delta}$ equals the area of $\Delta(Pet_3, R(V_{(a_1+a_2,a_1,0)}), \nu)$. Hence the two polytopes must in fact be equal, i.e.

$$\tilde{\Delta} = \Delta(Pet_3, R(V_{(a_1+a_2,a_1,0)}), \nu).$$

□

We will now compute the Newton-Okounkov body of Pet_3 with respect to ν and the space of sections $V_{(a_1+a_2,a_1,0)}$ defined above. We need to consider the cases $a_2 \geq a_1$ and $a_1 \geq a_2$ separately. For a discussion about how these results might be generalized for larger Peterson varieties, see §8.1.

Theorem 7.18. *The Newton-Okounkov body $\Delta(Pet_3, R(V_{(a_1+a_2,a_1,0)}), \nu)$, where $a_2 \geq a_1$, is the convex hull of the vertices $\{(0,0), (2a_1+a_2, 0), (0, a_1+a_2), (3a_1, a_2-a_1)\}$.*

Proof. First, notice that the area of the polytope described in the theorem is $3a_1(a_2-a_1) + \frac{1}{2}(3a_1)(2a_1) + \frac{1}{2}(a_2-a_1)^2 = \frac{1}{2}a_1^2 + 2a_1a_2 + \frac{1}{2}a_2^2$. Therefore, by Proposition 7.17, it suffices to show that the four vertices given in the statement of the theorem lie in $\nu(V_{(a_1+a_2,a_1,0)})$. We will deal with the four cases separately.

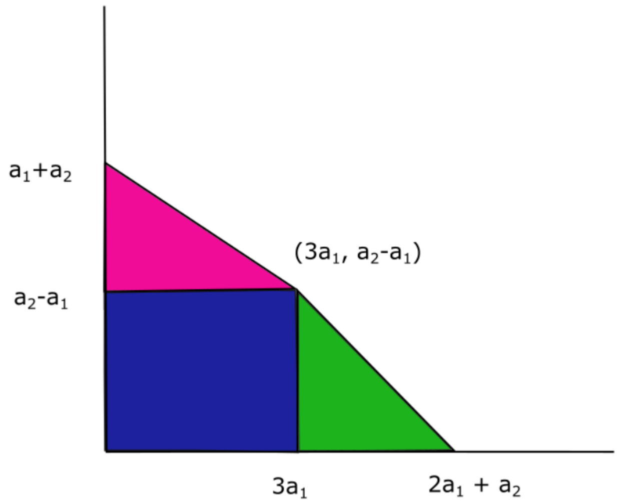


Figure 7.1: Newton-Okounkov body $\Delta(Pet_3, R(V_{(a_1+a_2,a_1,0)}), \nu)$ for $a_2 \geq a_1$.

Case $(\mathbf{0}, \mathbf{0})$: The filling $(23)^{a_1}(3)^{a_2}$ corresponds to the polynomial 1 (see Lemma 7.16), and $\nu(1) = (0,0)$ Hence, $(0,0)$ is in the image $\nu(V_{(a_1+a_2,a_1,0)})$.

Case $(\mathbf{0}, \mathbf{a_1} + \mathbf{a_2})$: The filling $(12)^{a_1}(1)^{a_2}$ corresponds to the polynomial $(y-x^2)^{a_1}y^{a_2}$, and $\nu((y-x^2)^{a_1}y^{a_2}) = (0, a_1 + a_2)$.

Case $(\mathbf{2a_1} + \mathbf{a_2}, \mathbf{0})$: Now consider $(12)^k(13)^{a_1-k}(1)^{a_1-k}(2)^{a_2-a_1+k}$. This semistandard Young tableau has corresponding polynomial

$$g_k := (y-x^2)^k x^{a_2} y^{a_1-k} = \left[\sum_{j=0}^{k}(-1)^j \binom{k}{j} y^{k-j}x^{2j}\right] x^{a_2}y^{a_1-k}$$
$$= \sum_{j=0}^{k}(-1)^j \binom{k}{j} x^{a_2+2j}y^{a_1-j}.$$

Concretely, $g_0 = x^{a_2}y^{a_1}$, $g_1 = x^{a_2}y^{a_1} - x^{a_2+2}y^{a_1-1}$, $g_2 = x^{a_2}y^{a_1} - 2x^{a_2+2}y^{a_1-1} + x^{a_2+4}y^{a_1-2}$, etc. Notice that each g_k consists of the same monomials as in g_{k-1} (with possibly different coefficients), plus one monomial not in g_{k-1}, namely, $x^{a_2+2k}y^{a_1-k}$. Therefore, by taking appropriate linear combinations of g_k's we can obtain any monomial of the form $x^{a_2+2j}y^{a_1-j}$. In particular, applying this to the case $j = a_1$, we see that there exists an element in $V_{(a_1+a_2,a_1,0)}$ whose expression with respect to our choice of coordinates is the monomial $x^{a_2+2a_1}$. Since $\nu(x^{2a_1+a_2}) = (2a_1 + a_2, 0)$, we conclude that $(2a_1 + a_2, 0)$ is in the image of $V_{(a_1+a_2,a_1,0)}$ under ν.

Note that another way to state the above argument is the following. The set of monomials $x^\alpha y^\beta$ that appear in the $a_1 + 1$ polynomials $\{g_0, \ldots, g_{a_1}\}$ is precisely:

$$\left\{x^{a_1}y^{a_1}, x^{a_2+2}y^{a_1-1}, x^{a_2+4}y^{a_1-2}, \ldots, x^{a_2+2a_1}\right\}. \tag{7.3}$$

In terms of this basis, the polynomial g_k has coordinates, up to a sign,

$$\left[\begin{pmatrix}1 & \binom{k}{1} & \binom{k}{2} & \cdots & \binom{k}{k-1} & 1 & 0 & \cdots & 0\end{pmatrix}\right]^T. \tag{7.4}$$

Therefore, to show that there exists a linear combination of g_k's which equal $x^{a_2+2a_1}$, it suffices to show that the $(a_1 + 1) \times (a_1 + 1)$ matrix consisting of the polynomials g_k, written in terms of the basis (7.3), is invertible. From (7.4), we can see that this matrix is upper triangular, and its diagonal entries are equal to ± 1. Therefore it is invertible.

Case $(\mathbf{3a_1}, \mathbf{a_2} - \mathbf{a_1})$: Now, consider $(12)^k(13)^{a_1-k}(1)^{a_2-k}(2)^k$. This semistandard Young tableau has corresponding polynomial

$$\begin{aligned} h_k := (y - x^2)^k x^{a_1} y^{a_2-k} &= \left[\sum_{j=0}^{k} (-1)^j \binom{k}{j} y^{k-j} x^{2j}\right] x^{a_1} y^{a_2-k} \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{a_1+2j} y^{a_2-j}. \end{aligned}$$

Concretely, $h_0 = x^{a_1}y^{a_2}$, $h_1 = x^{a_1}y^{a_2} - x^{2+a_1}y^{a_2-1}$, $h_2 = x^{a_1}y^{a_2} - 2x^{2+a_1}y^{a_2-1} + x^{4+a_1}y^{a_2-4}$, etc. As in the argument above, by taking appropriate linear combinations of h_k's we can obtain any monomial of the form $x^{a_1+2j}y^{a_2-j}$. In particular, when $j = a_1$ we can obtain the monomial $x^{3a_1}y^{a_2-a_1}$ which means $(3a_1, a_2 - a_1)$ is in the image.

Therefore, the four vertices $(0,0)$, $(0,a_1+a2)$, $(2a_2+a_1,0)$, and $(3a_2,a_1-a_2)$ are in the image of $V_{(a_1+a_2,a_1,0)}$ under ν. This completes the proof.

$\square$

In order to prove the $a_1 \geq a_2$ case, we will need the following terminology.

Definition 7.19. *An* ***upper triangular Pascal matrix*** *is an infinite matrix with the (i,j)-th entry equal to the binomial coefficient $\binom{j-1}{i-1}$, i.e.*

$$T := \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \cdots \\ 0 & \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots \\ 0 & 0 & \binom{2}{2} & \binom{3}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix},$$

where we take the convention that $\binom{j}{i} := 0$ if $i > j$ and $\binom{k}{0} = 1$.

Definition 7.20. *A* ***truncated Pascal matrix*** *is a matrix obtained from T by selecting some arbitrary finite subsets of the rows and columns of T of equal size, i.e.*

$$T(r,x) := \begin{pmatrix} \binom{x_0}{r_0} & \binom{x_1}{r_0} & \cdots & \binom{x_d}{r_0} \\ \binom{x_0}{r_1} & \binom{x_1}{r_1} & \cdots & \binom{x_d}{r_1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{x_0}{r_d} & \binom{x_1}{r_d} & \cdots & \binom{x_d}{r_d} \end{pmatrix},$$

for some sets $r = \{r_0 < r_1 < \cdots < r_d\}$ and $x = \{x_0 < x_1 < \cdots < x_d\}$, for $x_i, r_i \in \mathbb{N}$.

We will need the following result of Kersey.

Theorem 7.21 ([Ker13])**.** *Following the notation above, truncated Pascal matrices are invertible if and only if $r_i \leq x_i$, $\forall i$.*

We now compute the Newton-Okounkov body for the case $a_1 \geq a_2$.

Theorem 7.22. *The Newton-Okounkov body $\Delta(Pet_3, R(V_{(a_1+a_2,a_1,0)}), \nu)$, where $a_1 \geq a_2$, is the polytope with vertices $(0,0), (0, a_1 + a_2)$, $(2a_2 + a_1, 0)$, and $(3a_2, a_1 - a_2)$.*

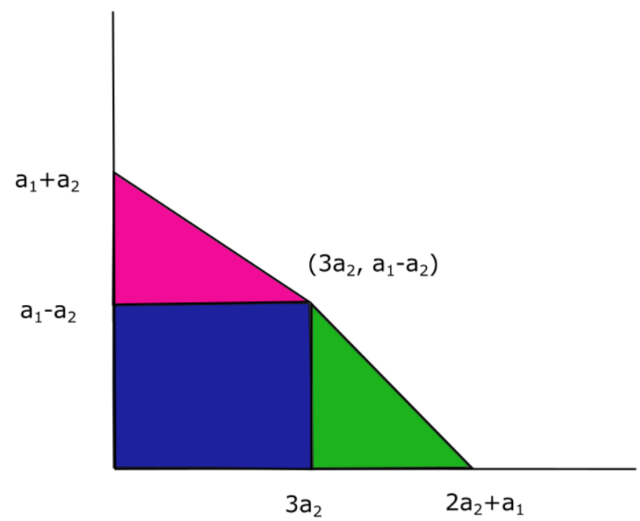


Figure 7.2: Newton-Okounkov body $\Delta(Pet_3, R(V_{(a_1+a_2,a_1,0)}), \nu)$ for $a_1 \geq a_2$.

Proof. As in Theorem 7.18, we notice that the area of the polytope described in the theorem is $3a_2(a_1 - a_2) + \frac{1}{2}(3a_2)(2a_2) + \frac{1}{2}(a_1 - a_2)^2 = \frac{1}{2}a_1^2 + 2a_1a_2 + \frac{1}{2}a_2^2$. Therefore, by Proposition 7.17, it suffices to show that the four vertices given in the statement of the theorem lie in $\nu(V_{(a_1+a_2,a_1,0)})$. We deal with the four cases separately.

Case $(\mathbf{0}, \mathbf{0})$: The filling $(23)^{a_1}(3)^{a_2}$ corresponds to the polynomial 1 (see Lemma 7.16), and $\nu(1) = (0, 0)$ Hence, $(0, 0)$ is in the image $\nu(V_{(a_1+a_2,a_1,0)})$.

Case $(\mathbf{0}, \mathbf{a_1} + \mathbf{a_2})$: The filling $(12)^{a_1}(1)^{a_2}$ corresponds to the polynomial $(y - x^2)^{a_1}y^{a_2}$, and $\nu((y - x^2)^{a_1}y^{a_2}) = (0, a_1 + a_2)$.

Case $(\mathbf{2a_2} + \mathbf{a_1}, \mathbf{0})$: Consider $(12)^k(13)^{a_1-k}(1)^{a_2-k}(2)^k$. This semistandard

Young tableau has corresponding polynomial

$$\begin{aligned} g_k := (y-x^2)^k x^{a_1} y^{a_2-k} &= \left[\sum_{j=0}^{k} (-1)^j \binom{k}{j} y^{k-j} x^{2j}\right] x^{a_1} y^{a_2-k} \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{a_1+2j} y^{a_2-j}. \end{aligned}$$

Concretely, $g_0 = x^{a_1}y^{a_2}$, $g_1 = x^{a_1}y^{a_2} - x^{a_1+2}y^{a_2-1}$, $g_2 = x^{a_1}y^{a_2} - 2x^{a_1+2}y^{a_2-1} + x^{a_1+4}y^{a_2-2}$, etc. Notice that each g_k consists of the same monomials as in g_{k-1} (with possibly different coefficients), plus one monomial not in g_{k-1}, $x^{a_1+2k}y^{a_2-k}$. Therefore, by taking appropriate linear combinations of g_k's we can obtain any monomial of the form $x^{a_1+2j}y^{a_2-j}$. In particular, when $j = a_2$ we can obtain the monomial $x^{a_1+2a_2}$, which means $(2a_2 + a_1, 0)$ is in the image of ν.

Case $(\mathbf{3a_2}, \mathbf{a_1} - \mathbf{a_2})$: To find the vertex $(3a_2, a_1 - a_2)$ we begin in a similar way. Consider $(12)^{a_1-a_2+k}(13)^{a_2-k}(1)^{a_2-k}(2)^k$ for $0 \le k \le a_2$. Define the polynomial h_k to be the polynomial associated to this semistandard Young tableau,

$$\begin{aligned} h_k &:= (y-x^2)^{a_1-a_2+k} x^{a_2} y^{a_2-k} \\ &= \left[\sum_{j=0}^{a_1-a_2+k} (-1)^j \binom{a_1-a_2+k}{j} y^{a_1-a_2+k-j} x^{2j}\right] x^{a_2} y^{a_2-k} \\ &= \sum_{j=0}^{a_1-a_2+k} (-1)^j \binom{a_1-a_2+k}{j} x^{a_2+2j} y^{a_1-j}, \text{ for } 0 \le k \le a_2. \end{aligned}$$

This defines a collection of $a_2 + 1$ polynomials. In particular,

$$h_{a_2} = \sum_{j=0}^{a_1} (-1)^j \binom{a_1}{j} x^{a_2+2j} y^{a_1-j}.$$

Notice that since $a_2 \le a_1$, the monomial $x^{3a_2}y^{a_1-a_2}$ corresponding to $j = a_2$ will have a non-zero coefficient in h_{a_2}, and the image of this term under ν is

$(3a_2, a_1 - a_2)$. We want to show that there exists a linear combination of h_k's such that the image of this linear combination under ν is $(3a_2, a_1 - a_2)$.

The set of monomials $x^\alpha y^\beta$ that appear in these $a_2 + 1$ polynomials $\{h_0, h_1, \ldots, h_{a_2}\}$ is precisely

$$\{x^{a_2}y^{a_1}, x^{a_2+2}y^{a_1-1}, x^{a_2+4}y^{a_1-2}, \ldots, x^{3a_2}y^{a_1-a_2}, \ldots, x^{2a_1+a_2-2}y, x^{a_2+2a_1}\}. \quad (7.5)$$

Note that these monomials are listed in increasing order with respect to our lexicographical order, i.e. $x^{a_2}y^{a_1}$ is the lowest term, $x^{a_2+2}y^{a_1-1}$ is the second lowest term, etc. For a fixed k, the set of monomials appearing in the polynomial h_k is exactly

$$\{x^{a_2}y^{a_1}, x^{a_2+2}y^{a_1-1}, \ldots, x^{a_2+2(a_1-a_2+k)}y^{a_2-k}\},$$

i.e. the first (leftmost) $a_1 - a_2 + k + 1$ elements in the ordered list (7.5). Recall we need to construct a linear combination of the h_k with the property that the lowest term of this linear combination is a constant multiple of the monomial $x^{3a_2}y^{a_1-a_2}$. In terms of the basis given in (7.5), the polynomial h_k has coordinates

$$\begin{pmatrix} \binom{a_1-a_2+k}{0} \\ -\binom{a_1-a_2+k}{1} \\ \vdots \end{pmatrix}.$$

A linear combination of the basis elements in (7.5) has lowest term $x^{3a_2}y^{a_1-a_2}$ if its coordinate vector is of the form $\begin{pmatrix} 0 & \cdots & 0 & r & * & \cdots & * \end{pmatrix}^T$, where r is a non-zero entry in the $(a_2 + 1)$-th row, and the entries below r are free. Therefore, we are looking for a solution to the matrix equation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & r & * & \cdots & * \end{pmatrix}^T,$$

where A is a $(a_2 + 1) \times (a_2 + 1)$ block matrix, X is a $(a_2 + 1) \times 1$ vector, and B, C, D, and Y are block matrices of the appropriate sizes. Choosing $Y = 0$, it suffices to show that

$$AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

has a solution. More precisely, we wish to show that the matrix equation

$$\underbrace{\begin{pmatrix} \binom{a_1-a_2}{0} & \binom{a_1-a_2+1}{0} & \cdots & \binom{a_1-1}{0} & \binom{a_1}{0} \\ -\binom{a_1-a_2}{1} & -\binom{a_1-a_2+1}{1} & \cdots & -\binom{a_1-1}{1} & -\binom{a_1}{1} \\ \binom{a_1-a_2}{2} & \binom{a_1-a_2+1}{2} & \cdots & \binom{a_1-1}{2} & \binom{a_1}{2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \end{pmatrix}}_{A} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{a_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

has a solution, where $c_j \in \mathbb{C}$. Here we take the convention that $\binom{\alpha}{\beta} = 0$ if $\beta > \alpha$. In particular, it would suffice to show that $\det(A) \neq 0$.

Consider the matrix $\tilde{A}$ obtained by multiplying the even rows of A by -1. Clearly $\det(\tilde{A}) = \pm\det(A)$, so $\det(A) \neq 0$ if and only if $\det(\tilde{A}) \neq 0$. Now observe that $\tilde{A}$ is the truncated Pascal matrix $T(\tilde{r}, \tilde{x})$, where $\tilde{r} = \{0, 1, \ldots, a_2\}$, and $\tilde{x} = \{a_1 - a_2, a_1 - a_2 + 1, \ldots, a_1\}$. Since $a_1 \geq a_2$, we have $r_i \leq x_i$ for all i. Therefore, by Theorem 7.21, $\tilde{A}$ is invertible, as desired. Hence there exists an element in $V_{(a_1+a_2,a_1,0)}$ whose image under ν is $(3a_2, a_1 - a_2)$. This completes the proof. $\square$

Chapter 8

Open Questions and Future Work

We now briefly assemble some open questions and directions for future work.

As was mentioned in Chapter 1, part of the motivation for the theory of Newton-Okounkov bodies is to associate combinatorial objects to algebraic varieties. Thus, given the explicit computation of Newton-Okounkov bodies given in this thesis, an obvious and natural question is to explore the connections between the combinatorics of these polytopes and the geometric/topological properties of the underlying varieties. For example, do these Newton-Okounkov bodies encode the cohomology ring of the original variety, Betti numbers, orbit types, and if so, how? This question is still wide open, and we hope to explore it further in future work. In what follows, we instead discuss some of our preliminary results concerning certain generalizations of the computations in this thesis.

More specifically, the results of this thesis concern very special cases of Bott-Samelson varieties and Hessenberg varieties, and also place restrictive conditions on the choices of the auxiliary data. Therefore, a natural direction for future work is to relax these conditions and also to work with more general classes of Peterson varieties, Hessenberg varieties, and Bott-Samelson varieties.

8.1 Peterson Varieties Pet_n for $n > 3$

In joint work with Hiraku Abe and Megumi Harada, I have begun work on generalizing the results of this thesis to the case of the Peterson variety Pet_n for general n (in particular, $n > 3$). Indeed, we already have preliminary results showing that the Peterson variety Pet_n for general n can be realized as a special fibre in a flat family of Hessenberg varieties, thus generalizing the results in Chapter 6. By the same reasoning as in Chapter 7, this should then allow us to compute the volume of the Newton-Okounkov body by computing the degree of a smooth projective toric variety.

We note that there is a natural generalization of our admissible flag of subvarieties in Pet_3 to the case of general n. More precisely, Insko and Yong [IY12, Theorem 1.4] proved that the point

$$w_0B := \left[\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}\right]$$

is nonsingular in Pet_n and by reasoning similar to that in Lemma 7.14, it is not hard to see that the following is an affine open neighbourhood around the point w_0B in Pet_n:

$$\mathcal{U} = \left\{ \begin{pmatrix} x_{n-1} & x_{n-2} & x_{n-3} & \cdots & x_1 & 1 \\ x_{n-2} & x_{n-3} & x_{n-4} & \cdots & 1 & 0 \\ x_{n-3} & x_{n-4} & \cdots & \iddots & \iddots & \vdots \\ \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\ x_1 & 1 & \iddots & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \right\}.$$

The following is then an admissible flag of subvarieties in Pet_n:

$$Y_{n-1} = \{w_0B\} \subset Y_{n-2} = \overline{\{x_1 = x_2 = \cdots = x_{n-2} = 0\}} \subset \cdots$$
$$\subset Y_2 = \overline{\{x_1 = x_2 = 0\}} \subset Y_1 = \overline{\{x_1 = 0\}} \subset Y_0 = \mathrm{Pet}_n$$

and using the system of parameters $x_1, \ldots, x_{n-1}$ about this flag, this geometric valuation corresponds to the lowest term valuation ν with respect to the lexicographical ordering $x_1 > x_2 > \cdots > x_{n-1}$. Thus the natural question which I intend to pursue together with Abe and Harada is to compute the Newton-Okounkov body of Pet_n with respect to this flag, for different choices of L_λ.

We also take a moment to observe that the problem outlined above appears to be non-trivial to solve. For instance, one challenge we face in the computations for higher values of n is that the corresponding degree (of Pet_n) gets large quite quickly. For instance, for Pet_4 and the line bundle corresponding to $a = (1, 1, 1, 0)$, the degree is 96 and for Pet_5 and $a = (1, 1, 1, 1, 0)$ the degree is 3000. Another obstacle is that even if we compute the sections corresponding to the set of semistandard Young tableaux and also obtain their images under ν, we would most likely need to take linear combinations of these sections to find the vertices of Pet_n (as we did in Theorems 7.18 and 7.22). Even using a computer, this could be quite challenging.

8.2 Regular Semisimple Hessenberg Varieties

We also have preliminary computations concerning Newton-Okounkov bodies of Hessenberg varieties different from the Peterson variety; more specifically, we have computed an example of a Newton-Okounkov body of a regular semisimple Hessenberg variety, i.e. the "generic fibre" in the flat family of Hessenberg varieties alluded to above.

Let h be the Hessenberg function $h(1) = 2, h(2) = h(3) = 3$ and let $Hess(A, h)$ be the regular semisimple Hessenberg variety discussed in Chapter 6. We use coordinates on an open set in $Hess(A, h)$ following the method described in §7.2. Recall that in order for a flag $V_\bullet$ to lie in $Hess(A, h)$, we need that $AV_1 \subset V_2$ where

$$A = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}$$

is the matrix corresponding to the Hessenberg variety $Hess(A, h)$. Using methods similar to those in Chapter 6 it is straightforward to compute that the intersection of $Hess(A, h)$ with the well-known open Bruhat cell

$$Bw_0B := \left\{ \left[\begin{pmatrix} y & z & 1 \\ x & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] \mid x, y, z \in \mathbb{C} \right\}$$

of $Fl(\mathbb{C}^3)$ is the subset

$$\left\{ \left[\begin{pmatrix} \frac{\gamma_2 - \gamma_3}{\gamma_1 - \gamma_3} xz & z & 1 \\ x & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] \mid x, z \in \mathbb{C} \right\}.$$

(Note that $\frac{\gamma_2 - \gamma_3}{\gamma_1 - \gamma_3} - 1 \neq 0$ since γ_1, γ_2, and γ_3 are distinct by assumption.) With respect to the coordinates x and z appearing above, we can consider the flag of subvarieties

$$Y_2 = \overline{\{x = z = 0\}} \subset Y_1 = \overline{\{x = 0\}} \subset Y_0 = Hess(A, h).$$

For any line bundle L, this flag gives rise to a geometric valuation $\nu : H^0(Hess(A, h), L) \to \mathbb{Z}^2$, which can be realized as the lowest term valuation with respect to the lexicographic ordering $x > z$.

By Theorem 6.14 we know that the degree of $Hess(A,h)$ with respect to the mapping determined by the pullback of the Plücker line bundle L_λ for $\lambda = (a_1+a_2, a_1, 0)$ is $a_1^2+4a_1a_2+a_2^2$, and hence the volume of its corresponding Newton-Okounkov body is $\frac{1}{2}a_1^2 + 2a_1a_2 + \frac{1}{2}a_2^2$.

Therefore, as in the Pet_3 case, it suffices to find points in the the Newton-Okounkov body of $Hess(A,h)$ whose convex hull is a polytope of area $\frac{1}{2}a_1^2 + 2a_1a_2 + \frac{1}{2}a_2^2$. As in Chapter 7 we define V_λ to be the image of $H^0(Fl(\mathbb{C}^3), \mathcal{L}^\lambda)$ in $H^0(Hess(A,h), \mathcal{L}^\lambda \mid_{Hess(A,h)} = L_\lambda)$, and consider the restrictions of sections over the flag corresponding to semistandard Young tableaux (see Lemma 7.16).

Consider the following semistandard Young tableaux, their corresponding restricted sections in $Hess(A,h)$, and their images under ν:

Young tableaux	Sections	Image under ν
$(23)^{a_1}(1)^{a_2}$	1	$(0,0)$
$(23)^{a_1}(2)^{a_2}$	x^{a_2}	$(a_2,0)$
$(13)^{a_1}(3)^{a_2}$	z^{a_1}	$(0,a_1)$
$(13)^{a_1}(1)^{a_2}$	$z^{a_1}(\frac{\gamma_2-\gamma_3}{\gamma_1-\gamma_3}xz)^{a_2}$	(a_2, a_1+a_2)
$(12)^{a_1}(2)^{a_2}$	$(\frac{\gamma_2-\gamma_3}{\gamma_1-\gamma_3}xz - xz)^{a_1}x^{a_2}$	(a_1+a_2, a_1)
$(12)^{a_1}(1)^{a_2}$	$(\frac{\gamma_2-\gamma_3}{\gamma_1-\gamma_3}xz - xz)^{a_1}(\frac{\gamma_2-\gamma_3}{\gamma_1-\gamma_3}xz)^{a_2}$	(a_1+a_2, a_1+a_2)

Let Δ denote the convex hull of these six points. It is not hard to see that the area of Δ is exactly $\frac{1}{2}a_1^2+2a_1a_2+\frac{1}{2}a_2^2$, and hence Δ is the Newton-Okounkov body $\Delta(Hess(A,h), R(V_{(a_1+a_2,a_1,0)}), \nu)$.

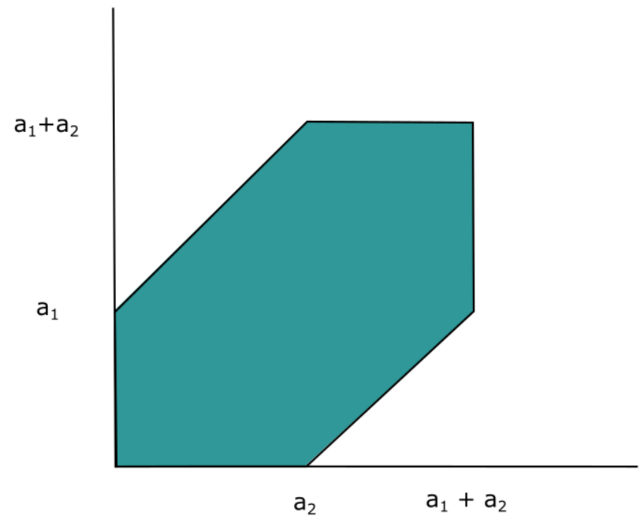


Figure 8.1: Newton-Okounkov body $\Delta(Hess(A,h), R(V_{(a_1+a_2,a_1,0)}), \nu)$.

Bibliography

[AB84] M.F. Atiyah and R. Bott. The Moment Map and Equivariant Cohomology. *Tolpology*, 23(1):1–28, 1984.

[ADH] H. Abe, L. DeDieu, and M. Harada. Flat Families of Hessenberg Varieties. In preparation.

[AHHM15] H. Abe, M. Harada, T. Horiguchi, and M. Masuda. The Cohomology Rings of Regular Nilpotent Hessenberg Varieties in Lie Type A. arXiv preprint arXiv:1512.09072, 2015.

[And13] D. Anderson. Okounkov Bodies and Toric Degenerations. *Mathematische Annalen*, 356(3):1183–1202, 2013.

[BC04] M. Brion and J.B. Carrell. The Equivariant Cohomology Ring of Regular Varieties. *Michigan Mathematics Journal*, 52(1):189–203, 2004.

[BC15] P. Brosnan and T. Chow. Unit Interval Orders and the Dot Action on the Cohomology of Regular Semisimple Hessenberg Varieties. arXiv preprint arXiv:1511.00773, 2015.

[BK07] M. Brion and S. Kumar. *Frobenius Splitting Methods in Geometry and Representation Theory*. Vol. 231. Springer Science & Business Media, 2007.

[Bri05] M. Brion. Lectures on the geomtry of flag varieties. In *Topics in Cohomological Studies of Algebraic Varieties*, pages 35–85. Birkhäuser Basel, 2005.

[BS58] R. Bott and H. Samelson. Applications of the Theory of Morse to Symmetric Spaces. *American Journal of Mathematics*, pages 964–1029, 1958.

[BV82] N. Berline and M. Vergne. Classes caractéristiques équivariantes. Formules de localisation en cohomologie équivariante. *C.R. Acad. Sci. Paris Sér. I Math.*, 295(9):539–541, 1982.

[CdS03] Ana Cannas da Silva. Symplectic toric manifolds. In *Symplectic Geometry of Integrable Hamiltonian Systems (Barcelona, 2001)*, Adv. Courses Math. CRM Barcelona, pages 85–173. Birkhäuser, Basel, 2003.

[Dem74] M Demazure. Désingularisation des variétés de Schubert généralisées. *Annales scientifiques de l'École Normale Supérieure*, 7(1):53–88, 1974.

[DMPS92] F. De Mari, C. Procesi, and M. A. Shayman. Hessenberg Varieties. *Transactions of the American Mathematical Society*, 332(2):529–534, 1992.

[Fuj15] N. Fujita. Newton-Okounkov Bodies for Bott-Samelson Varieties and String Polytopes for Generalized Demazure Modules. arXiv preprint arXiv:1503.08916, 2015.

[Ful93] W. Fulton. *Introduction to Toric Varieties*. Princeton University Press, 1993.

[Ful97] W. Fulton. *Young Tableaux: With Application to Representation Theory*. Cambridge University Press, 1997.

[Ful99] J. Fulman. Descent Identities, Hessenberg Varieties, and the Weil Conjectures. *Journal of Combinatorial Theory, Series A*, 87(2):390–397, 1999.

[Fun03] F.Y.C. Fung. On the Topology of Components of some Springer Fibers and their Relation to Kazhdan-Lusztig Theory. *Advances in Mathematics*, 178(2):244–276, 2003.

[GGK02] V. Ginzburg, V. Guillemin, and Y. Karshon. *Moment Maps, Cobordisms, and Hamiltonian Group Actions*. Number 98. American Mathematical Society, 2002.

[GH78] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley & Sons, 1978.

[Ham08] M.D. Hamilton. The Quantization of a Toric Manifold is Given by the Integer Lattice Points in the Moment Polytope. In *Toric topology*, volume 460 of *Contemp. Math.*, pages 131–140. American Mathematical Society, Providence, RI, 2008.

[Han73] H.C. Hansen. On Cycles in Flag Manifolds. *Mathematica Scandinavica*, 33:269–274, 1973.

[Har77] R. Hartshorne. *Algebraic Geometry (Vol. 52).* Springer Science and Business Media, 1977.

[HT11] M. Harada and J. Tymoczko. A Positive Monk Formula in the S^1-Equivariant Cohomology of Type A Peterson Varieties. *Proceedings of the London Mathematical Society*, 103(1):40–72, 2011.

[HY16] M. Harada and Jihyeon J. Yang. Newton-Okounkov Bodies of Bott-Samelson Varieties and Grossberg-Karshon Twisted Cubes. *Michigan Mathematical Journal*, 65(2):413–440, 2016.

[IY12] E. Insko and A. Yong. Patch Ideals and Peterson Varieties. *Transformation Groups*, 1(4):1011–1036, 2012.

[Jan87] J.C. Janzen. *Representations of Algebraic Groups.* Academic Press, London, 1987.

[Kav15] K. Kaveh. Crystal Bases and Newton-Okounkov Bodies. *Duke Mathematical Journal*, 164(13):2461–2506, 2015.

[Ker13] S. Kersey. Invertibility of Submatrices of Pascal's Matrix and Birkhoff Interpolation. arXiv preprint arXiv:1303.6159, 2013.

[Kir15] V. Kiritchenko. Newton-Okounkov Polytopes of Flag Varieties. arXiv preprint arXiv:1506.00362, 2015.

[KK12] K. Kaveh and A.G. Khovanskii. Newton-Okounkov Bodies, Semigroups of Integral Points, Graded Algebras and Intersection Theory. *Annals of Mathematics*, 176(2):925–978, 2012.

[Kos96] B. Kostant. Flag Manifold Quantum Cohomology, the Toda Lattice, and the Representation with Highest Weight ρ. *Selecta Mathematica*, 2(1):43–91, 1996.

[KST12] V. Kiritchenko, E. Smirnov, and V. Timorin. Schubert Calculus and Gelfand-Zetlin Polytopes. *Russian Mathematical Surveys*, 67(4):685–719, 2012.

[LLM02] V. Lakshmibai, P. Littelmann, and P. Magyar. Standard Monomial Theory for Bott-Samelson Varieties. *Compositio Mathematica*, 130:293–318, 2002.

[LM97] V. Lakshmibai and P. Magyar. Standard Monomial Theory for Bott-Samelson Varieties. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 324(11):1211–1215, 1997.

[LM98] V. Lakshmibai and P. Magyar. Standard Monomial Theory for Bott-Samelson Varieties of $GL(n)$. *Publications of RIMS Kyoto*, 34:229–248, 1998.

[LM09] R. Lazarsfeld and M. Mustata. Convex Bodies Associated to Linear Series. *Annales scientifiques de l'Ecole normale supérieure*, 42(5):783–835, 2009.

[LT04] N. Lauritzen and J.F. Thomsen. Line Bundles on Bott-Samelson Varieties. *Journal of Algebraic Geometry*, 13:461–473, 2004.

[Mag98] P. Magyar. Schubert Polynomials and Bott-Samelson Varieties. *Commentarii Mathematici Helvetici*, 73(4):603–636, 1998.

[Oko96] A. Okounkov. Brunn-Minkowski Inequality for Multiplicities. *Inventiones Mathematicae*, 125(3):405–411, 1996.

[Oko98] A. Okounkov. Multiplicities and Newton Polytopes. *American Mathematical Society Translations*, 181(2):231–244, 1998.

[Rie03] K. Rietsch. Totally Positive Toeplitz Matrices and Quantum Cohomology of Partial Flag Varieties. *Journal of the American Mathematical Society*, 16(2):363–392, 2003.

[SKKT00] K.E. Smith, L. Kahanpää, P. Kekäläinen, and W. Traves. *An Invitation to Algebraic Geometry*. Springer New York, 2000.

[SS14] D. Schmitz and H. Seppänen. Global Okounkov Bodies for Bott-Samelson Varieties. arXiv preprint arXiv:1409.1857, 2014.

[Vak15] R. Vakil. The Rising Sea: Foundations of Algebraic Geometry. http://math.stanford.edu/∼vakil/216blog/FOAGdec2915public.pdf, 2015.

[Web12] A. Weber. Equivariant Chern Classes and Localization Theorem. *Journal of Singularities*, 5:153–176, 2012.