THE DYNAMIC RESPONSE OF DOUBLE BILINEAR HYSTERETIC SYSTEMS

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BILINEAR HYSTERETIC SYSTEMS

BY

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SCOPE AND CONTENTS:

An investigation of the dynamic response of single and two degree of freedom systems with double bilinear hysteretic restoring force has been made. The stability of the two systems has also been examined.

Numerical integration and Digital-Analog simulation of the system equations of motion has been done using fourth order Runge-Kutta Method and MIMIC simulator to check the approximate analysis.

ABSTRACT

An investigation of the dynamic response of one and two degrees of freedom systems with double bilinear hysteretic restoring forces has been made using Krylov and Bogolieubov method of "variation of parameters" and "Ritz averaging method".

The single degree of freedom system response exhibits "jump phenomena" which has a tendency to disappear as the external excitation or the external damping is increased.

The two degree of freedom system exhibits an extra hump near the first natural frequency which tends to decrease as the slope (at which the first discontinuity occurs in the dispacement - restoring force characteristic) is increased or the external excitation is decreased. Stability of one and two degrees of freedom systems has also been examined.

Numerical integration of the equation of motion for single degree system has been made using Fourth-Order-Runge Kutta Method. Also a Digital Analog simulation has been done using MIMIC simulator language on IBM-7040. The approach of the system towards steady state has been found to be extremely slow in the time domain of the equations of motion.

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NOMENCLATURES

x, x_1, x_2 = Displacements of the masses at any instant

$\bullet \equiv \frac{d}{dt}$ = Total derivative with respect to time 't'

$\bullet\bullet \equiv \frac{d^2}{dt^2}$ = Second derivative with respect to time

$f(x, \dot{x})$ = Restoring force of the system as a function of x and $\frac{dx}{dt}$ at any instant

R = Amplitude of excitation

ω = Frequency of trigonometric excitation (external)

α = Slope of the normalised double bilinear hysteretic curve. The initial slope is unity and α is the slope at which the first discontinuity occurs.

A, A_1, A_2 = Maximum amplitudes of displacement

μ = Viscous damping coefficient

ϕ, ϕ_1, ϕ_2 = Phase angles

χ = Harmonic content

a_n, b_n = Fourier coefficients

X_1 = [Referring to Figure (9)] displacement of the first mass with respect to the moving base.

X_2 = Relative displacement between first and second masses

X_o = Static displacement

$E(\tilde{x}_1, \tilde{x}_2)$, $E(\tilde{x})$ = Equation deficiencies in the "Ritz" averaging method" [8]

ω_p = Frequency at which peak amplitude response occurs

A_p = Magnitude of the peak amplitude

$R_{critical}$ = Critical value of the amplitude of external excitation

(Note: All the above variables are dimensionless)

F_p = yield force (in lbs.)

y = displacement of the system mass (in inches)

K_1, K_2 = spring stiffness (in lbs/in.)

m = mass (in lbs. sec^2/in)

τ = time (in seconds)

CONTENTS

I. INTRODUCTION

Research in the field of structural damping was started about 200 years ago. In his "Memoir on Torsion", Coulomb hypothesised about the microstructural mechanisms of damping. He undertook experiments which proved that the damping of torsional oscillations is not caused by air friction but by internal losses in material. He also recognized that the mechanism operative at low stress may be different from those at high stresses. However, for more than a century these investigations were confined to the low stress levels and were mainly of academic interest. Helmholtz (1861), Sir William Thomson (1865), Gazz, Messer, Wiedmann (1874 to 1880), Tomlinson (1886-87) etc. have contributed much in this field. A short history of their work and an extensive bibliography can be found in reference [10]*

Experiments on hysteretic effects under cyclic loading was performed by Ewing (1889) and under cyclic bending by Voigt (1892). G.F.C. Searle (1908) also did extensive experimental work on the measurement of hysteretic damping [10].

All these early investigations were mainly concerned with the damping properties of specific materials under specific test conditions. The generalization of the behavioural pattern

* Square brackets indicate references given in the Biblography.

and their application to structures and loadings such as usually encountered in engineering practice has not been complete, but now a number of literatures are available on this topic. It is a significant property in the design of high speed aircraft, problems involving mechanical resonance, shaft whirl, instrument hysteresis, heating under cyclic stresses, viscoelastic coatings etc. Some typical examples of systems exhibiting hysteretic behaviour are - systems with coulomb damping, built up structures with rivetted, bolted or clamped construction (in which the combined effect of friction and elastic forces produce the hysteresis), elastoplastic elements like some steels, masonry in shear etc.

The hysteretic behaviour of many engineering systems is widely described by the bilinear hysteretic model in which the force-deflection relationship of the system is represented by a single parallelogram. This type of hysteretic model has been previously studied by L. S. Jacobsen, L. E. Goodman and J. H. Klumpp (on the dynamic properties of laminated beams with slip interface)[2], R. Tanabashi, G. V. Berg, T. Kobori and R. Minai (elasto-plastic system)[2] and W. T. Thomson [4](elastic analog simulation). A general analytical solution has been presented by T. K. Caughey [7],[9] and a thorough analysis given by W. D. Iwan [2].

In the study of such a model it was found that it gives rise to steady-state response curves, for trigonometric excitation, that are single valued and stable or marginally stable (in certain regions of the response curves). It has been found that in some cases the hysteresis loss specified by the bilinear model is higher than that which actually exists in the structures. In his recent paper [1] W. D. Iwan has remarked that "it may be due to marked rounding of the hysteresis loop or to a pinching together of the loop near the origin". Mostly for the latter reason a new model called the "Double bilinear hysteretic model" was introduced by him [1]. In this he also investigated the steady state response of a single degree of freedom system (with the restoring force characteristics of the double bilinear hysteretic type) using Krylov and Bogolieubov method. It was shown that such a system is stable or at least marginally stable and that it exhibits "jump phenomenon", a characteristic only encountered in non-linear systems. It is worth mentioning that the jump phenomenon is never obtained in the analysis of the bilinear hysteretic model [2]. Also this system has exactly half the hysteresis loss-given by bilinear model [1],[2] for the same amplitude of response.

In the present investigation a mathematical analysis for a single and two degrees of freedom systems with "Double bilinear hysteretic" restoring forces has been done using exact and approximate analytic techniques, numerical methods and digital analog simulation with hybrid elements. Dynamic stability for both the systems has been examined using Routh - Hurwitz stability criteria. Ritz averaging method [5],[6] has been introduced for the first time [8] in the analytical treatment of hysteretic systems in this text. It has been shown that Krylov - Bogolieubov method (of slowly varying parameters) and Ritz Averaging method both yield the same approximate solution.

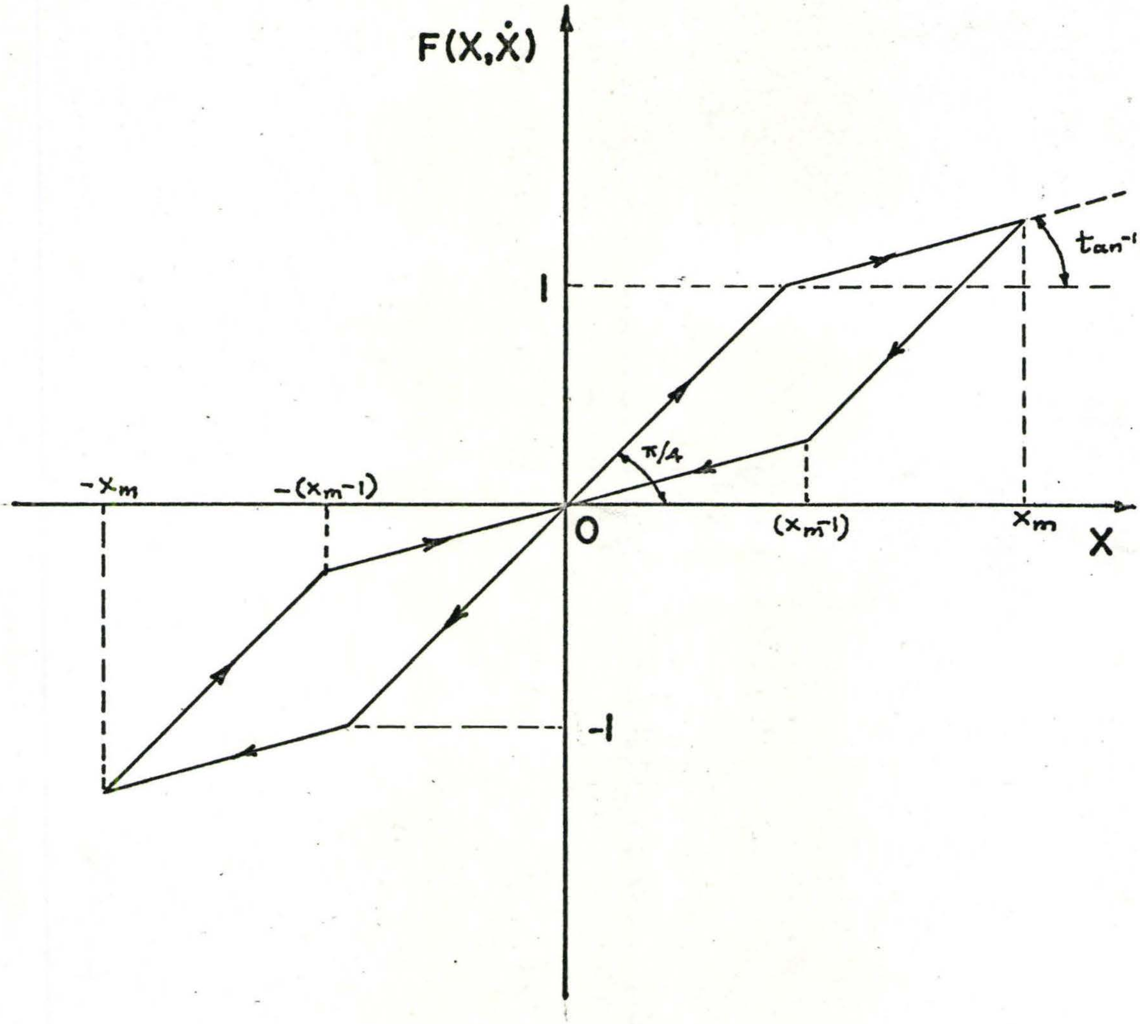
II. SINGLE DEGREE OF FREEDOM SYSTEM

IIA. The Equation of Motion:

Since the character of the restoring force is piece-wise linear, the equation of motion can be expressed in terms of a set of linear differential equations each valid within a certain region of the force-displacement curve.

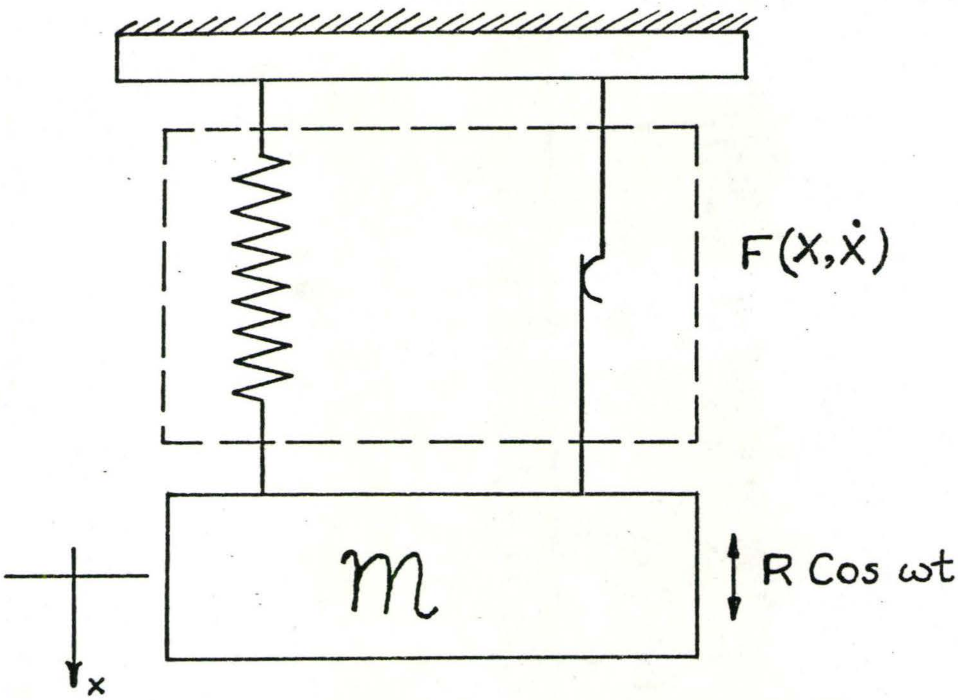
Let the force-displacement curve be represented as shown in Figure(1) and the restoring force in general be given by $y = f(x, \dot{x})$ where x is the displacement and $\dot{x}$ is the velocity of the mass m at any instant as shown in Fig. (1A). It is evident that y is an explicit function of x only but the function has different expressions depending on the sign of the velocity $\dot{x}$, hence it may be said to be an implicit function of sgn $(\dot{x})$.

For mathematical convenience, let us assume the system to be simplified - the mass and the spring forces adjusted to give dimensionless variables and coefficients, and also the hysteretic curve to have initial slope unity and the force at which a slope discontinuity first occurs is also unity as in Fig. (1). It has been shown in general



NORMALIZED DOUBLE BILINEAR HYSTERETIC RESTORING FORCE

FIGURE. 1



SINGLE DEGREE OF FREEDOM SYSTEM

FIG. 1A

[2] with particular reference to Bilinear hysteretic system that such simplifications do not impair the generality of the results, rather they facilitate the analytical treatment of the system to a great extent. The method adopted for normalization is shown in the Appendix.

On the above assumptions the equation of motion for a single degree of freedom system can be written as:

$$\ddot{x} + f(x, \dot{x}) = R \cos \omega t \tag{2.1}$$

where R is the amplitude and ω is the frequency of the external trigonometric excitation.

The restoring force $f(x,\dot{x})$ is defined by:

$$f(x,\dot{x}) \left.\begin{aligned}
&= x && 0 < x < 1,\ \dot{x} > 0 \\
&= \alpha x + (1-\alpha) && x > 1 > 0,\ \dot{x} > 0 \\
&= x - (1-\alpha)(x_m - 1) && x > (x_m - 1) > 0,\ \dot{x} < 0 \\
&= \alpha x && (x_m - 1) > x > 0,\ \dot{x} < 0 \\
&= x && |x| < 1,\ x < 0,\ \dot{x} < 0 \\
&= \alpha x - (1-\alpha) && |x| > 1,\ x < 0,\ \dot{x} < 0 \\
&= x + (1-\alpha)(x_m - 1) && |x| > (|x_m| - 1),\ x < 0,\ \dot{x} > 0 \\
&= \alpha x && |x| < (|x_m| - 1),\ x < 0,\ \dot{x} > 0
\end{aligned}\right\} \tag{2.2}$$

Thus the motion of the system is defined by eight different "differential equations" under different conditons of x and $\dot{x}$. On inspection it will be quite clear how the sign of $\dot{x}$ is implicitly involved in the expressions.

It should be noted that here the hysteresis curve under the steady state condition has been assumed to be symmetrical about the origin. The angle ψ is such that $\alpha = \tan\psi$ is always finite.

IIB. The Approximate Solution:

In finding the approximate solution of the system, the "Ritz Averaging Method" has been employed. In a series of papers by K. Klotter[5] and in the extensive work by F. R. Arnold[6], Ritz method has been well analysed and the case of application of this method to numerous nonlinear problems has also been eatablished. This method is sometimes also known as Ritz-Galerkin method [8] .

Let us introduce the identity, using equation (2.1):

$$E(x) \equiv \ddot{x} + f(x, \dot{x}) - R\,\mathrm{Cos}(\omega t) \tag{2.3}$$

The averaging method furnishes the following two conditions:

$$\int_0^{2\pi} E(\tilde{x})\,\mathrm{Cos}(\omega t)\,d(\omega t) = 0 \qquad \text{(a)}$$

(2.4)

$$\int_0^{2\pi} E(\tilde{x})\,\mathrm{Sin}(\omega t)\,d(\omega t) = 0 \qquad \text{(b)}$$

We assume the periodic solution $\tilde{x}(t)$ to be approximated by:

$$\tilde{x}(t) = A_1\,\mathrm{Cos}(\omega t) + B_1\,\mathrm{Sin}(\omega t), \tag{2.5}$$

clearly it is a two-term approximation of the complete series

$$\tilde{x}(t) = \sum_{k=1}^{n} a_k\,\xi_k(t), \tag{2.6}$$

where $\xi_k(t)$ is a given set of orthogonal functions and a_k represent the corresponding coefficients. On this assumption, equations (2.4) shall yield a set of equations which will be functions of the coefficients a_k. Equation (2.5) may also be written in the form

$$\tilde{x}(t) = A \cos(\omega t - \phi) \text{ , } (\omega t - \phi) = \theta \qquad (2.7)$$

where A is the amplitude of the displacement and ϕ is the phase angle relative to the excitation.

Substituting equation (2.7) in equation (2.3) the "Equation Deficiency" $E(\tilde{x})$ may be written as [8] :

$$E(\tilde{x}) \equiv -A\omega^2 \cos(\omega t - \phi) + f(\tilde{x}, \dot{\tilde{x}}) - R\cos(\theta + \phi) = 0 \qquad (2.8)$$

From equations (2.4) (assuming $\omega t = \beta$) we obtain:

$$-A\omega^2 \int_0^{2\pi} \cos\theta \cos\beta \, d\beta + \int_0^{2\pi} f(x, \dot{x}) \cos\beta \, d\beta - R \int_0^{2\pi} \cos^2\beta \, d\beta = 0$$

$$(2.9)$$

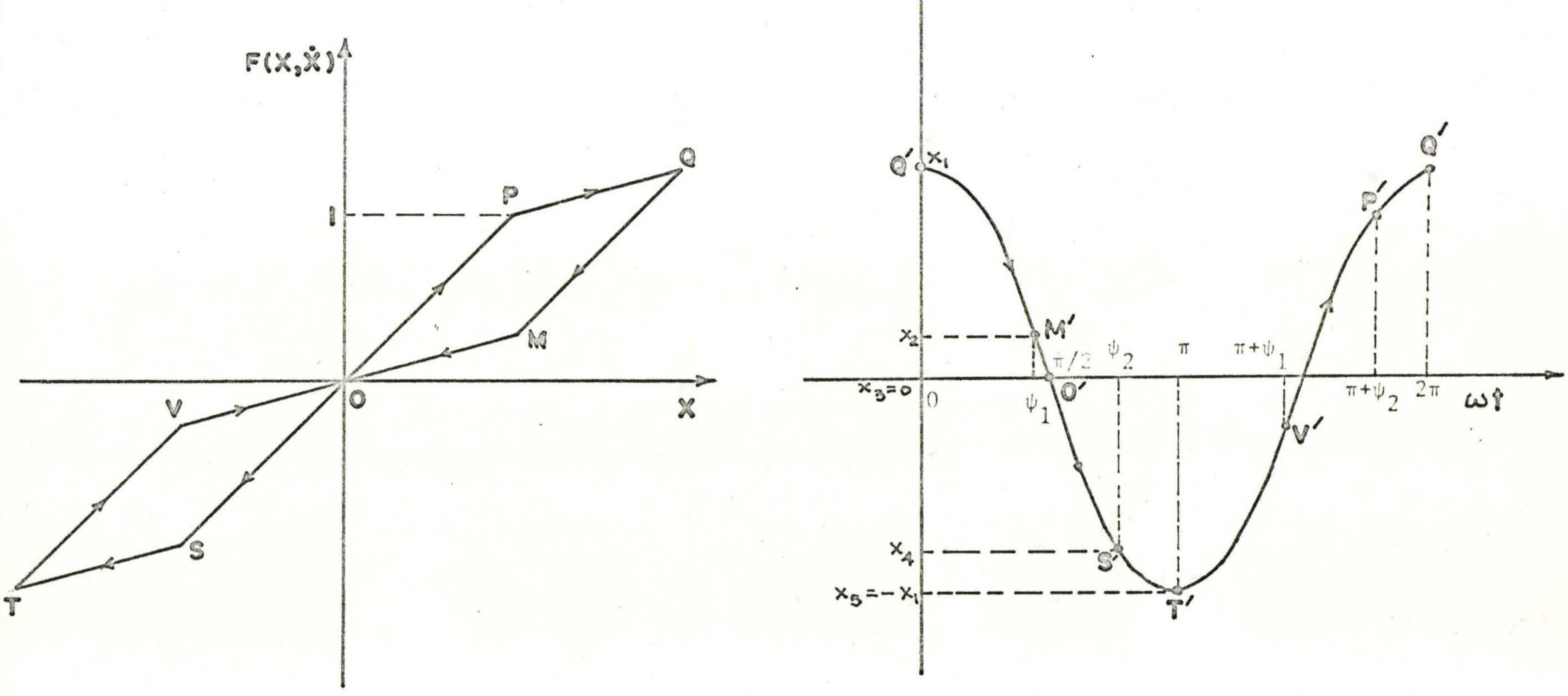
$$-A\omega^2 \int_0^{2\pi} \cos\theta \sin\beta \, d\beta + \int_0^{2\pi} f(x, \dot{x}) \sin\beta \, d\beta - R \int_0^{2\pi} \cos\beta \sin\beta \, d\beta = 0$$

Let

$$I_1 = \int_0^{2\pi} f(x, \dot{x}) \cos\beta \, d\beta = 2 \int_0^{\pi} f(x, \dot{x}) \cos\beta \, d\beta$$

$$\therefore \frac{1}{2} I_1 = \int_0^{\pi} f(x, \dot{x}) \cos\beta \, d\beta = \int_0^{\pi} f(x, \dot{x}) \cos(\theta + \phi) \, d\beta$$

which shows an averaging over half a cycle.



DIAGRAMMATIC INTERPRETATION

OF ψ_1 AND ψ_2

FIGURE. 2

Evaluating the integral over half a cycle Q'M'O'S'T' as shown in Figure (2) i.e. over the linear segments of the restoring force QM, MO, OS, ST during which $f(x, \dot{x})$ has the following expressions

$$
\begin{aligned}
f(x, \dot{x}) &= x - (1-\alpha)(A-1) && (A-1) < x < A \\
&= \alpha x && 0 < x < (A-1) \\
&= x && |x| < 1, \; x < 0 \\
&= \alpha x - (1-\alpha) && |A| > |x| > 1, \; x < 0
\end{aligned}
$$

Hence the integral I_1 has to be broken into four definite integrals integrated between the limits 0 to ψ_1, ψ_1 to $\pi/2$, $\pi/2$ to ψ_2 and ψ_2 to π respectively. Thus

$$
\begin{aligned}
\tfrac{1}{2} I_1 &= \int_0^{\psi_1} \left[A \cos\theta - (1-\alpha)(A-1)\right] \cos(\theta+\phi)\, d\theta \\
&+ \alpha \int_{\psi_1}^{\pi/2} A \cos\theta \cos(\theta+\phi)\, d\theta + \int_{\pi/2}^{\psi_2} A \cos\theta \cos(\theta+\phi)\, d\theta \\
&+ \int_{\psi_2}^{\pi} \left[\alpha A \cos\theta - (1-\alpha)\right] \cos(\theta+\phi)\, d\theta
\end{aligned}
$$

$$\left[\because \theta+\phi = \beta \quad \therefore d\theta = d\beta \;, \quad \phi = \text{Constant.}\right]$$

A diagrammatic interpretation of ψ_1 and ψ_2 is shown in Figure (2) where ψ_1 and ψ_2 have the algebraic values given by

$$\psi_1 = \cos^{-1}\left(\frac{A-1}{A}\right) \qquad \text{(a)}$$

$$\psi_2 = \cos^{-1}(-1/A) \qquad \text{(b)} \qquad (2.10)$$

Thus the integral I_1 is given by:

$$\frac{1}{2}I_1 = A\left[\cos\phi\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) - \frac{\sin\phi}{2}\left(\sin^2\theta\right)\right]_0^{\psi_1}$$

$$- (1-\alpha)(A-1)\left[\sin(\theta+\phi)\right]_0^{\psi_1} + \alpha A\left[\cos\phi\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right)\right.$$

$$\left. - \tfrac{1}{2}\sin\phi\left(\sin^2\theta\right)\right]_{\psi_1}^{\pi/2} + A\left[\cos\phi\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) - \tfrac{1}{2}\sin\phi\right.$$

$$\left.\left(\sin^2\theta\right)\right]_{\pi/2}^{\psi_2} + \alpha A\left[\cos\phi\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) - \tfrac{1}{2}\sin\phi\left(\sin^2\theta\right)\right]_{\psi_2}^{\pi}$$

$$- (1-\alpha)\left[\sin(\theta+\phi)\right]_{\psi_2}^{\pi}$$

which when simplified gives

$$I_1 = A\cos\phi(1-\alpha)\left[\psi_1 + \psi_2 + \tfrac{1}{2}\left(\sin 2\psi_1 + \sin 2\psi_2\right) + \frac{\pi A}{2}(3\alpha-1)\cos\phi\right.$$
$$+ A(1-\alpha)\sin\phi\left[3 - \left(\sin^2\psi_1 + \sin^2\psi_2\right)\right] + 2(1-\alpha)\left[\sin(\psi_2+\phi)\right.$$
$$\left.\left. - (A-1)\sin(\psi_1+\phi)\right]\right] \qquad (2.11)$$

Similarly the integral -

$$\int_0^{2\pi} f(x,\dot{x})\sin(\theta+\phi)\,d\theta = I_2 = 2\int_0^{\pi} f(x,\dot{x})\sin(\theta+\phi)\,d\theta$$

Therefore:

$$\frac{1}{2}I_2 = \int_0^{\psi_1}\left[A\cos\theta - (1-\alpha)(A-1)\right]\sin(\theta+\phi)\,d\theta$$

$$+\alpha\int_{\psi_1}^{\pi/2} A\cos\theta\sin(\theta+\phi)\,d\theta + A\int_{\pi/2}^{\psi_2}\cos\theta\sin(\theta+\phi)\,d\theta$$

$$+\int_{\psi_2}^{\pi}\left[\alpha A\cos\theta - (1-\alpha)\right]\sin(\theta+\phi)\,d\theta$$

$$= A\left[\sin\phi\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) + \tfrac{1}{2}\cos\phi\,\sin^2\theta\right]_0^{\psi_1} + (1-\alpha)(-1+A)\left[\cos(\theta+\phi)\right]_0^{\psi_1}$$
$$+ \alpha A\left[\sin\phi\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) + \tfrac{1}{2}\cos\phi\,\sin^2\theta\right]_{\psi_1}^{\pi/2} + A\left[\sin\phi\left(\tfrac{1}{2}\theta\right.\right.$$
$$\left.\left. + \tfrac{1}{4}\sin 2\theta\right) + \frac{\cos\phi}{2}\sin^2\theta\right]_{\pi/2}^{\psi_2} + (1-\alpha)\left[\cos(\theta+\phi)\right]_{\psi_2}^{\pi}$$
$$+ \alpha A\left[\sin\phi\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) + \tfrac{1}{2}\cos\phi\,\sin^2\theta\right]_{\psi_2}^{\pi} .$$

Simplifying the above expression we get:

$$I_2 = A(1-\alpha)\sin\phi\left[\psi_1 + \psi_2 + \tfrac{1}{2}(\sin 2\psi_1 + \sin 2\psi_2)\right] + \frac{\pi A}{2}\sin\phi(3\alpha-1)$$
$$+ A(1-\alpha)\cos\phi\left[\sin^2\psi_1 + \sin^2\psi_2 - 3\right] + 2(1-\alpha)$$
$$\left[(A-1)\cos(\psi_1+\phi) - \cos(\psi_2+\phi)\right] \qquad (2.12)$$

Also the integral $\int_0^{2\pi}\cos\theta\cos\beta\, d\beta = \pi\cos\phi$

$\int_0^{2\pi}\cos\theta\sin\beta\, d\beta = \pi\sin\phi$, $\int_0^{2\pi}\sin(\theta+\phi)\cos(\theta+\phi)\, d\theta = 0$,

$\int_0^{2\pi}\cos^2(\theta+\phi)\, d\theta = \pi$.

Substituting these relations in equations (2.9a) and (2.9b)

$$-A\omega^2\pi\cos\phi + I_1 - \pi R = 0 \qquad \text{(a)}$$
$$\qquad\qquad (2.13)$$
$$-A\omega^2\pi\sin\phi + I_2 = 0 \qquad \text{(b)}$$

Using the values of I_1 and I_2 from (2.11) and (2.12), equations (2.13a) and (2.13b) may be written as:

$$F \equiv -A\omega^2\pi\cos\phi + A(1-\alpha)\cos\phi\left[\psi_1+\psi_2+\tfrac{1}{2}(\sin 2\psi_1 + \sin 2\psi_2)\right]$$
$$+ \frac{\pi A}{2}(3\alpha-1)\cos\phi + A(1-\alpha)\sin\phi\left[3 - (\sin^2\psi_1 + \sin^2\psi_2)\right]$$
$$+ 2(1-\alpha)\left[\sin(\psi_2+\phi) - (A-1)\sin(\psi_1+\phi)\right] - \pi R = 0$$

$$G \equiv -A\omega^2\pi\sin\phi + A(1-\alpha)\sin\phi\left[\psi_1+\psi_2+\tfrac{1}{2}(\sin 2\psi_1 + \sin 2\psi_2)\right] +$$

$$\frac{\pi A}{2}(3\alpha-1)\sin\phi - A(1-\alpha)\cos\phi\left[3-\sin^2\psi_1-\sin^2\psi_2\right]$$

$$+2(1-\alpha)\left[(A-1)\cos(\psi_1+\phi)-\cos(\psi_2+\phi)\right] = 0$$

Multiplying F by $\cos\phi$ and G by $\sin\phi$ and adding, we get :

$$f \equiv (F\cos\phi + G\sin\phi) \equiv -A\omega^2\pi + A(1-\alpha)\left[\psi_1+\psi_2+\frac{1}{2}(\sin 2\psi_1+\sin 2\psi_2)\right]$$

$$+\frac{\pi A}{2}(3\alpha-1) + 2(1-\alpha)\left[\sin\psi_2-(A-1)\sin\psi_1\right] - \pi R\cos\phi = 0$$

$$g \equiv (F\sin\phi - G\cos\phi) \equiv A(1-\alpha)\left[3-(\sin^2\psi_1+\sin^2\psi_2)\right]$$

$$+2(1-\alpha)\left[\cos\psi_2-(A-1)\cos\psi_1\right] - \pi R\sin\phi = 0$$

Further simplification of f and g yields:

$$f \equiv -A\omega^2\pi + A\left\{(1-\alpha)\left[\psi_1+\psi_2-\frac{1}{2}(\sin 2\psi_1+\sin 2\psi_2)-\pi/2\right] + \alpha\pi\right\} - \pi R\cos\phi = 0$$

$$g \equiv 2(1-\alpha)(A-1)/A - \pi R\sin\phi = 0 \tag{2.14}$$

In these equations, on close examinations we can easily identify the coefficients C(A) and S(A) given by the following expressions in reference [1].

$$C(A) = \frac{A}{\pi}\left\{(1-\alpha)\left[\psi_1+\psi_2-\frac{1}{2}(\sin 2\psi_1+\sin 2\psi_2)-\frac{\pi}{2}\right] + \alpha\pi\right\} \tag{2.15}$$

$$S(A) = -\frac{2}{\pi}(1-\alpha)(A-1)/A$$

Hence equations (2.14) and (2.15) give

$$f \equiv -A\omega^2\pi + \pi C(A) - \pi R \cos\phi = 0 \qquad \text{(a)}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.16)$$

$$g \equiv -\pi S(A) - \pi R \sin\phi = 0 \qquad \text{(b)}$$

This gives the equations describing the steady-state response as

$$-A\omega^2 + C(A) = R\cos\phi \qquad \text{(a)}$$

$$\qquad\qquad\qquad\qquad\qquad (2.17)$$

$$S(A) = -R\sin\phi \qquad \text{(b)}$$

These can be immediately identified with the steady-state response equations obtained in reference [1] using the"method of slowly varying parameters"[1],[2] .

The response equation is given by

$$\omega^2 = \frac{C(A)}{A} \pm \left[\left(\frac{R}{A}\right)^2 - \left\{\frac{S(A)}{A}\right\}^2\right]^{1/2} \qquad (2.18)$$

where C(A) and S(A) have the expressions as in (2.15)

The steady state frequency response curves are shown in the Figures (3),(4) and (5). The equation for the locus of peak amplitude is obtained by setting

$$\left(\frac{R}{A}\right)^2 - \left[\frac{S(A)}{A}\right]^2 = 0 \qquad (2.19)$$

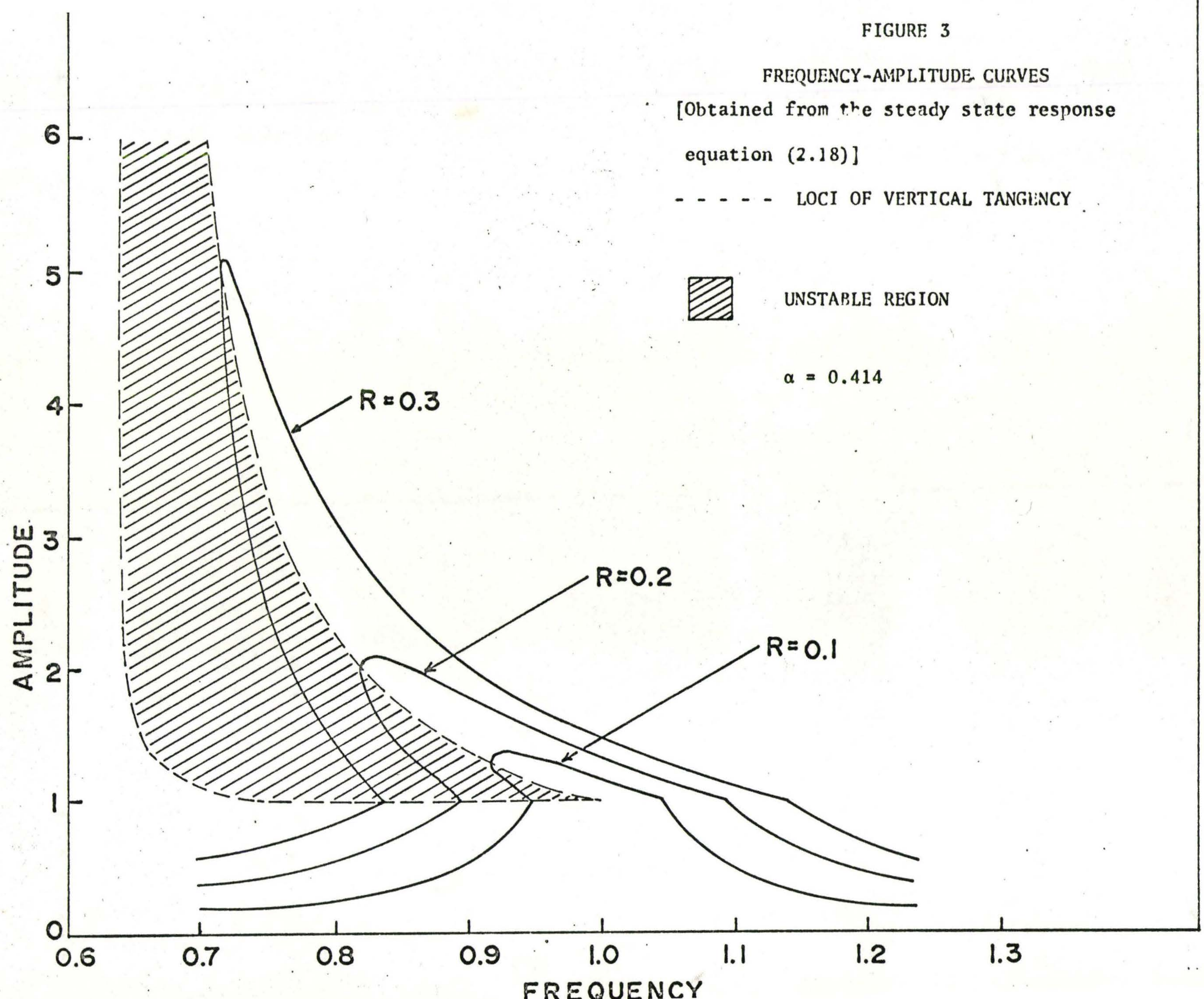


FIGURE 3

FREQUENCY-AMPLITUDE CURVES

[Obtained from the steady state response equation (2.18)]

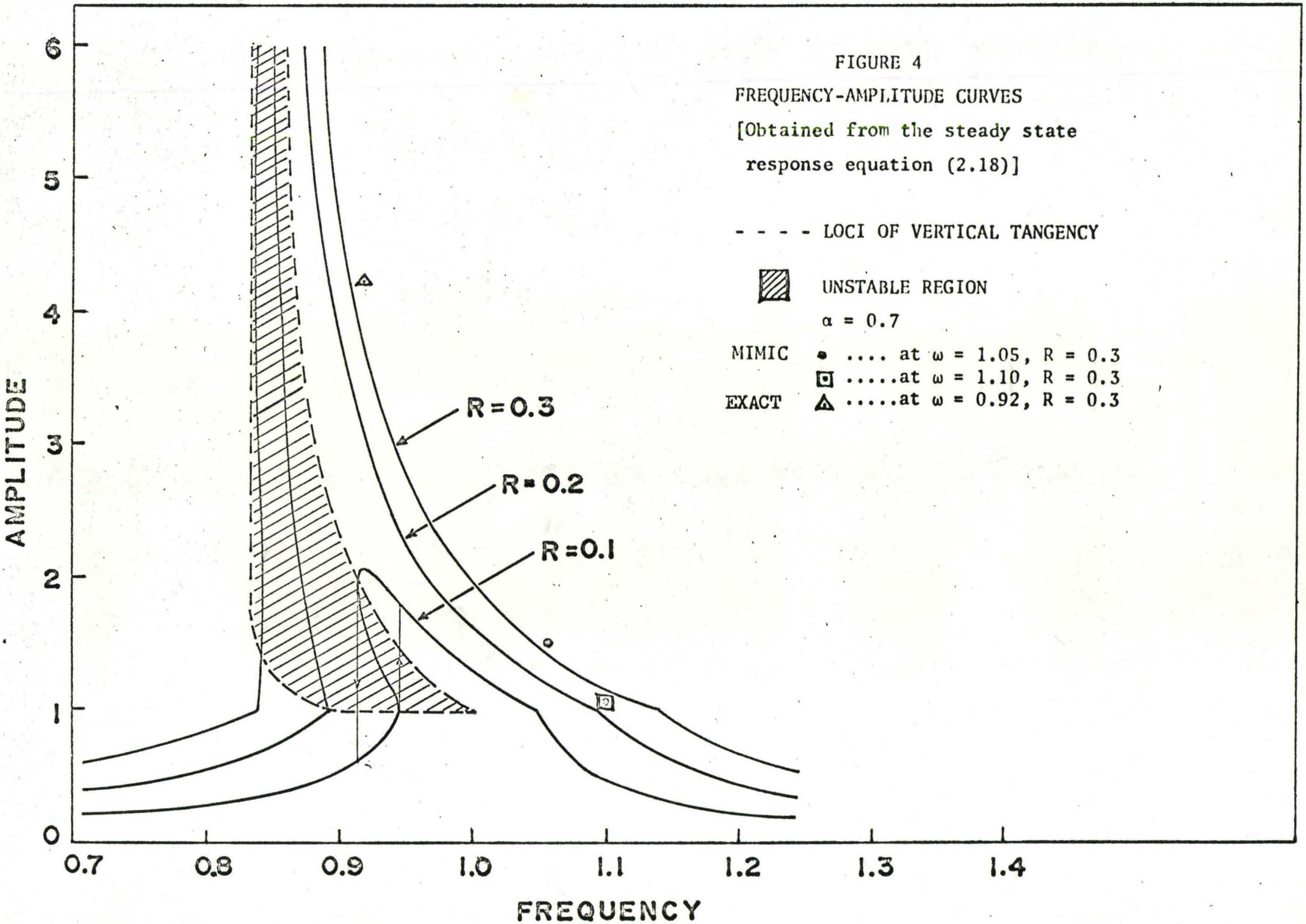


FIGURE 4
FREQUENCY-AMPLITUDE CURVES
[Obtained from the steady state response equation (2.18)]

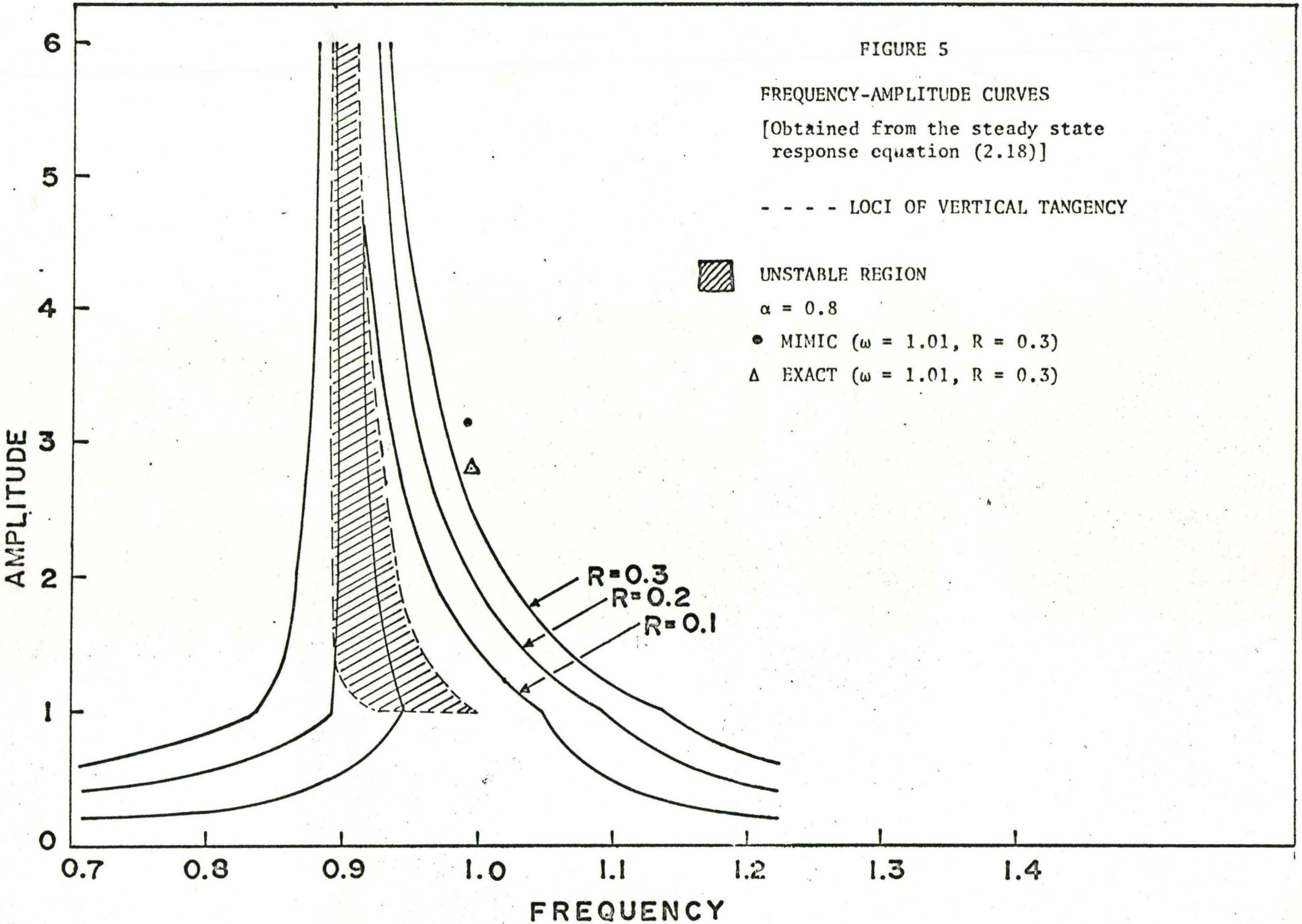


FIGURE 5

FREQUENCY-AMPLITUDE CURVES

[Obtained from the steady state response equation (2.18)]

which gives:

$$\omega_p^2 = \frac{C(A)}{A} \tag{2.20}$$

The magnitude of the peak amplitude response is obtained by putting the expression for S(A) in equation (2.19). Thus

$$A_p = \frac{2(1-\alpha)/\pi}{2(1-\alpha)/\pi - R} \tag{2.21}$$

This shows the existence of an unbounded resonance at -

$$R_{critical} = 2(1-\alpha)/\pi \tag{2.22}$$

The steady state response curves have the following features as shown in Figures (3),(4) and (5).

(i) All the curves have a tendency to lean towards the lower frequency which is characteristic of "soft spring systems". The same tendency is also seen in the steady state response curves for "bilinear hysteretic system".

(ii) For sufficiently low amplitudes of excitation (R), the "jump phenomena" is observed (Figures (3) and (5)) which is unlike the behaviour of a standard bilinear hysteretic system.

Similar results have been summarized in reference [1] and the equations for the loci of vertical tangency and stability have also been obtained. These details will be discussed in a more general way later in this section.

IIC. THE EXACT SOLUTION

Assuming the hysteretic curve to be symmetrical while the system is in steady state, an exact solution may be found by taking one-by-one the piecewise linear segments. The following notations will be used in this section with reference to Figure (2):

t_2, t_3, t_4 and t_5 are the time required for the displacements QM, MO, OS, and ST respectively of the mass m [Figure (1A)] starting from point Q.

$x_1, \dot{x}_1$ = displacement and velocity at Q.

(here $\dot{x}_1 = 0$ and $x_1 = A$ = the amplitude)

$x_2, \dot{x}_2$ — displacement and velocity at M ($x_2 = x_1 - 1$ and $\dot{x}_2$ is unknown)

$x_3, \dot{x}_3$ — displacement and velocity at O ($x_3 = 0$, $\dot{x}_3$ is unknown)

$x_4, \dot{x}_4$ — displacement and velocity at S ($x_4 = 1$, $\dot{x}_4$ is unknown)

$x_5, \dot{x}_5$ — displacement and velocity at T ($x_5 = -x_1$, $\dot{x}_5 = 0$)

As in Figure (2) following the segments, QM, MO, OS and ST starting from point Q at t=0, it can be seen that Q is the point where $\frac{dx}{dt} = 0$ and $x = x_1$ (the amplitude); at M we have no idea about the velocity of the system (except that it has negative sign) but the coordinate $x_2 = x_1 - 1$. Again at O, $x_3 = 0$ but the velocity $\dot{x}_3$ is unknown. At S, $x_4 = -1$, $\dot{x}_4$ is unknown and at T, $x_5 = -x_1$, $\dot{x}_5 = 0$, since here, x passes through a minima.

The motion along QM may be described by the differential equation

$$\ddot{x} + x - (x_1-1)(1-\alpha) = R\cos(\omega t+\phi) \qquad (2.23)$$

with the initial conditions -

$$\begin{aligned} x(0) &= x_1 \\ \dot{x}(0) &= 0 \end{aligned} \qquad (2.24)$$

Applying Laplace's transform to the equation (2.23)

$$S^2 x(s) - s\,x(0) - \dot{x}(0) + x(s) = (x_1-1)(1-\alpha)/s$$
$$+ R\left[\cos\phi\left(\frac{s}{s^2+\omega^2}\right) - \sin\phi\left(\frac{\omega}{s^2+\omega^2}\right)\right]$$

which, on putting the initial conditions (2.24) yields

$$x(s) = s x_1/(s^2+1) + (x_1-1)(1-\alpha)\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + R\left[-\frac{\sin\phi}{1-\omega^2}\left(\frac{\omega}{s^2+\omega^2} - \frac{\omega}{1+s^2}\right) + \frac{\cos\phi}{1-\omega^2}\left(\frac{s}{s^2+\omega^2} - \frac{s}{s^2+1}\right)\right]$$

The inverse Laplace transform gives -

$$\begin{aligned} x(t) &= x_1\cos t + (x_1-1)(1-\alpha)(1-\cos t) + R\left[\frac{\cos\phi}{1-\omega^2}(\cos\omega t - \cos t) - \sin\phi(\sin\omega t - \omega\sin t)/(1-\omega^2)\right] \\ &= \cos t\left[x_1 - (x_1-1)(1-\alpha) - \frac{R\cos\phi}{1-\omega^2}\right] + \frac{\omega R\sin\phi}{1-\omega^2}\sin t \\ &\quad + (x_1-1)(1-\alpha) + \frac{R}{1-\omega^2}\cos(\omega t+\phi) \end{aligned} \qquad (2.25)$$

If t_2 is the time taken from Q to M (starting from Q), then for $\omega \neq 1$,

$$x_2 = (x_1-1) = \left[x_1 - (x_1-1)(1-\alpha) - \frac{R\cos\phi}{1-\omega^2}\right]\cos t_2 + \frac{\omega R\sin\phi}{1-\omega^2}\sin t_2$$
$$+ (x_1-1)(1-\alpha) + \frac{R}{1-\omega^2}\cos(\omega t_2+\phi) \qquad (2.26)$$

Differentiating equation (2.25), we get:

$$\dot{x}_2 = -\left[x_1 - (x_1-1)(1-\alpha) - \frac{R\cos\phi}{1-\omega^2}\right]\sin t_2 + \frac{\omega R\sin\phi}{1-\omega^2}\cos t_2 - \frac{R\omega}{1-\omega^2}\sin(\omega t_2 + \phi) \qquad (2.27)$$

For motion from M to O:

$$\ddot{x} + \alpha x = R\cos[\omega(t+t_2)+\phi] \qquad (2.28)$$

with initial conditions

$$X(0) = X_1 - 1$$
$$\dot{X}(0) = \dot{x}_2 \qquad (2.29)$$

which gives the solution

$$x(t) = \frac{\dot{x}_2}{\sqrt{\alpha}}\sin\sqrt{\alpha}t + (x_1-1)\cos\sqrt{\alpha}t + R\left[\frac{\cos(\omega t_2+\phi)}{\alpha-\omega^2}(\cos\omega t - \cos\sqrt{\alpha}t) - \frac{\sin(\omega t_2+\phi)}{\alpha-\omega^2}\cdot\left(\sin\omega t - \frac{\omega}{\sqrt{\alpha}}\sin\sqrt{\alpha}t\right)\right]$$

or

$$X(t) = \cos\sqrt{\alpha}t\left[(x_1-1) - R\frac{\cos(\omega t_2+\phi)}{\alpha-\omega^2}\right] + \frac{\sin\sqrt{\alpha}t}{\sqrt{\alpha}}\left[\dot{x}_2 + \frac{R\omega\sin(\omega(t_2)+\phi)}{\alpha-\omega^2}\right] + \frac{R}{\alpha-\omega^2}\cos[\omega(t+t_2)+\phi] \qquad (2.30)$$

If t_3 is the time required from M to O,

$$x(t_3) = 0, \quad \dot{x}(t_3) = \dot{x}_3$$

From which we obtain

$$\cos\sqrt{\alpha}t_3\left[(x_1-1) - R\frac{\cos(\omega t_2+\phi)}{\alpha-\omega^2}\right] + \frac{\sin\sqrt{\alpha}t_3}{\sqrt{\alpha}}\left[\dot{x}_2 + \frac{R\omega}{\alpha-\omega^2}\sin(\omega t_2+\phi)\right] + \frac{R}{\alpha-\omega^2}\cos[\omega(t_2+t_3)+\phi] = 0 \qquad (2.31)$$

$$\dot{x}_3 = -\left[(x_1-1) - R\,\frac{\cos(\omega t_2+\phi)}{\alpha-\omega^2}\right]\sqrt{\alpha}\,\sin\sqrt{\alpha}\,t_3 + \cos\sqrt{\alpha}\,t_3\left[\dot{x}_2 + \frac{R\omega}{\alpha-\omega^2}\sin(\omega t_2+\phi)\right] - \frac{R\omega}{\alpha-\omega^2}\sin\left[\omega(t_2+t_3)+\phi\right]$$

(2.32)

Following the path OS:

$$x(0) = 0\,, \quad \dot{x}(0) = \dot{x}_3$$

$$\ddot{x} + x = R\cos\left[\omega(t+t_2+t_3)+\phi\right] \qquad (2.33)$$

which gives the solution -

$$x(t) = \sin t\left[\dot{x}_3 + \frac{\omega R}{1-\omega^2}\sin\{\omega(t_2+t_3)+\phi\}\right] - \frac{R}{1-\omega^2}\cos t \times \cos\left[\omega(t_2+t_3)+\phi\right] + \frac{R}{1-\omega^2}\cos\left[\omega(t+t_2+t_3)+\phi\right]$$

(2.34)

Let t_4 be the time from O to S,

$$x(t_4) = -1 = x_4\,, \quad \dot{x}(t_4) = \dot{x}_4$$

$$\left[\dot{x}_3 + \frac{\omega R}{1-\omega^2}\sin(\omega t_2+\omega t_3+\phi)\right]\sin t_4 - \frac{R}{1-\omega^2}\cos(\omega t_2+\omega t_3+\phi)\cos t_4 + \frac{R}{1-\omega^2}\cos\left[\omega(t_2+t_3+t_4)+\phi\right] + 1 = 0 \qquad (2.35)$$

$$\dot{x}_4 = \left[\dot{x}_3 + \frac{\omega R}{1-\omega^2}\sin(\omega t_2+\omega t_3+\phi)\right]\cos t_4 + \frac{R}{1-\omega^2}\sin t_4 \times \cos(\omega t_2+\omega t_3+\phi) - \frac{\omega R}{1-\omega^2}\sin\left[\omega(t_2+t_3+t_4)+\phi\right] \qquad (2.36)$$

Finally, considering the motion from S to T,

$$x(0) = x_4 = -1\,, \quad \dot{x}(0) = \dot{x}_4$$

$$\ddot{x} + \alpha x - (1-\alpha) = R\cos\left[\omega(t+t_2+t_3+t_4)+\phi\right] \qquad (2.37)$$

Equation (2.37) has the solution

$$X(t) = \sin\sqrt{\alpha}t\left[\frac{\dot{x}_4}{\sqrt{\alpha}} + \frac{R\omega}{\sqrt{\alpha}(\alpha-\omega^2)}\sin\{\omega(t_2+t_3+t_4)+\phi\}\right]$$

$$+\cos\sqrt{\alpha}t\left[\frac{\alpha-1}{\alpha} - 1 - \frac{R}{\alpha-\omega^2}\cos\{\omega(t_2+t_3+t_4)+\phi\right]$$

$$+\frac{1-\alpha}{\alpha} + \frac{R}{\alpha-\omega^2}\cos\{\omega(t+t_2+t_3+t_4)+\phi\} \quad (2.38)$$

Let t_5 be the time from S to T, then

$$x(t_5) = x_5 = -x_1, \quad \dot{x}(t_5) = \dot{x}_5 = 0$$

From (2.38) we obtain -

$$0 = x_1 + \sin\sqrt{\alpha}t_5\left[\frac{\dot{x}_4}{\sqrt{\alpha}} + \frac{R\omega}{\sqrt{\alpha}(\alpha-\omega^2)}\sin\{\omega(t_2+t_3+t_4)+\phi\}\right]$$

$$-\cos\sqrt{\alpha}t_5\left[\frac{1}{\alpha} + \frac{R}{\alpha-\omega^2}\cos\{\omega(t_2+t_3+t_4)+\phi\}\right]$$

$$+\frac{1-\alpha}{\alpha} + \frac{R}{\alpha-\omega^2}\cos\{\omega(t_2+t_3+t_4+t_5)+\phi\} \quad (2.39)$$

$$\dot{x}_5 = 0 = \sqrt{\alpha}\cos\sqrt{\alpha}t_5\left[\frac{\dot{x}_4}{\sqrt{\alpha}} + \frac{R\omega}{\sqrt{\alpha}(\alpha-\omega^2)}\sin\{\omega(t_2+t_3+t_4+\phi)\}\right]$$

$$+\sqrt{\alpha}\sin\sqrt{\alpha}t_5\left[\frac{1}{\alpha} + \frac{R}{\alpha-\omega^2}\cos\{\omega(t_2+t_3+t_4)+\phi\}\right]$$
$$-\frac{R\omega}{\alpha-\omega^2}\sin\left[\omega(t_2+t_3+t_4+t_5)+\phi\right] \quad (2.40)$$

and finally

$$t_2+t_3+t_4+t_5 = \pi/\omega \quad (2.41)$$

Thus we have a set of nine equations viz (2.25), (2.27), (2.31), (2.32), (2.35), (2.36), (2.39), (2.40) and (2.41) in nine unknowns viz. $x_1, \phi, t_2, t_3, t_4, t_5, \dot{x}_2, \dot{x}_3, \dot{x}_4$. Due to highly transcendental nature of these equations it is difficult to adopt any direct method, but the following iterative procedure was employed:

(i) Guided by the approximate solution, some initial values for x_1 and ϕ were assumed.

(ii) Equation (2.26) was iterated for t_2.

(iii) Knowing t_2, $\dot{x}_2$ was calculated from equation (2.27).

(iv) Equation (2.31) was now iterated for t_3.

(v) The values of t_2 and t_3 being known, $\dot{x}_3$ was calculated from equation (2.32).

(vi) Equation (2.35) was iterated for t_4.

(vii) $\dot{x}_4$ was calculated from equation (2.36).

(viii) t_5 was now obtained from equation (2.41).

(ix) Equation (2.39) was iterated for the new value of ϕ (say ϕ_1).

(x) Putting this value of ϕ_1 in (2.40) x_5 was obtained.

(xi) x_5 was compared with x_1, if they are equal in magnitude, the iteration was stopped, otherwise the process was repeated with the new values of ϕ_1 and $x_1=|x_5|$ till $|x_5|=x_1$ within certain limits of accuracy.

Due to very slow convergence, only a few points on the response curves obtained by the approximate theory were tried and were found to have a comparable agreement. These points have been shown as Δ's in Figures (4) and (5).

IID. ADDITION OF VISCOUS DAMPING

The system described in section IIA was without any external damping. We shall now consider the same system with viscous damping, i.e. proportional to the first power of the velocity, added to it. The system motion may be described by the equation:

$$\ddot{x} + 2\mu\dot{x} + f(x, \dot{x}) = R\cos\omega t \tag{2.42}$$

where μ = damping coefficient.

Here we have a choice of solving this equation either by the "Ritz averaging method" or by "the method of slowly varying parameters". For reasons which will appear obvious later in this section we select "the method of slowly varying parameters". However, it should be borne in mind that the "Ritz averaging method" will also yield exactly the same system response equation as the other method.

Let $\quad x(t) = A(t)\cos(\omega t - \phi(t)) = A(t)\cos\theta \quad$ (2.43)

where $A(t)$ and $\phi(t)$ are slowly varying functions of the time.

Differentiating (2.43) we get

$$\dot{x} = \dot{A}(t)\cos\theta - \omega A(t)\sin\theta + \dot{\phi}(t)A(t)\sin\theta \tag{2.44}$$

By analogy to Lagrange's method of variation of parameters:

$$\dot{A}(t)\cos\theta + \dot{\phi}(t)A(t)\sin\theta = 0 \tag{2.45}$$

and

$$\dot{x} = -\omega A(t)\sin\theta \tag{2.46}$$

Now:

$$\ddot{x} = -\omega\dot{A}(t)\sin\theta - \omega^2 A(t)\cos\theta + \omega A(t)\dot{\phi}(t)\cos\theta \quad (2.47)$$

Substituting (2.46) and (2.47) in the equation (2.42) -

$$-\omega\dot{A}(t)\sin\theta - \omega^2 A(t)\cos\theta + \omega A(t)\dot{\phi}(t)\cos\theta$$
$$-2\mu\omega A(t)\sin\theta + F(A,\theta) = R\cos\omega t \quad (2.48)$$

Multiplying (2.45) by $\omega\sin\theta$ and (2.48) by $\cos\theta$ and adding, we get:

$$\omega A(t)\dot{\phi}(t) - \omega^2 A(t)\cos^2\theta + F(A,\theta)\cos\theta$$
$$-2\mu\omega A(t)\sin\theta\cos\theta = R\cos(\theta+\phi(t))\cos\theta \quad (2.49)$$

Averaging (2.49) over one cycle of θ $(0 \rightarrow 2\pi)$

$$\omega\dot{\phi}A - \frac{\omega^2 A}{2} + \frac{1}{2\pi}\int_0^{2\pi} F(A,\theta)\cos\theta\, d\theta = \frac{R}{2}\cos\phi \quad (2.50)$$

where ($\dot{\phi}$) and (A) are average values over one cycle of θ .

Multiplying (2.45) by $\omega\cos\theta$ and (2.48) by $\sin\theta$, subtracting and averaging, we get:

$$-\omega\dot{A} - \mu\omega A + \frac{1}{2\pi}\int_0^{2\pi} F(A,\theta)\sin\theta\, d\theta = -\frac{R}{2}\sin\phi \quad (2.51)$$

Let

$$C(A) = \frac{1}{\pi}\int_0^{2\pi} F(A,\theta)\cos\theta\, d\theta \quad (a)$$

$$(2.52)$$

$$S(A) = \frac{1}{\pi}\int_0^{2\pi} F(A,\theta)\sin\theta\, d\theta \quad (b)$$

As shown in section IIB, equation (2.15) C(A) and S(A) have the following values:

$$C(A) = \frac{A}{\pi}\left[(1-\alpha)\left\{\psi_1 + \psi_2 - \sin 2\psi_1/2 - \sin 2\psi_2/2 - \pi/2\right\} + \alpha\pi\right]$$

when $A \geqslant 1$. (2.53)

$C(A) = A$ $\qquad A \leq 1$

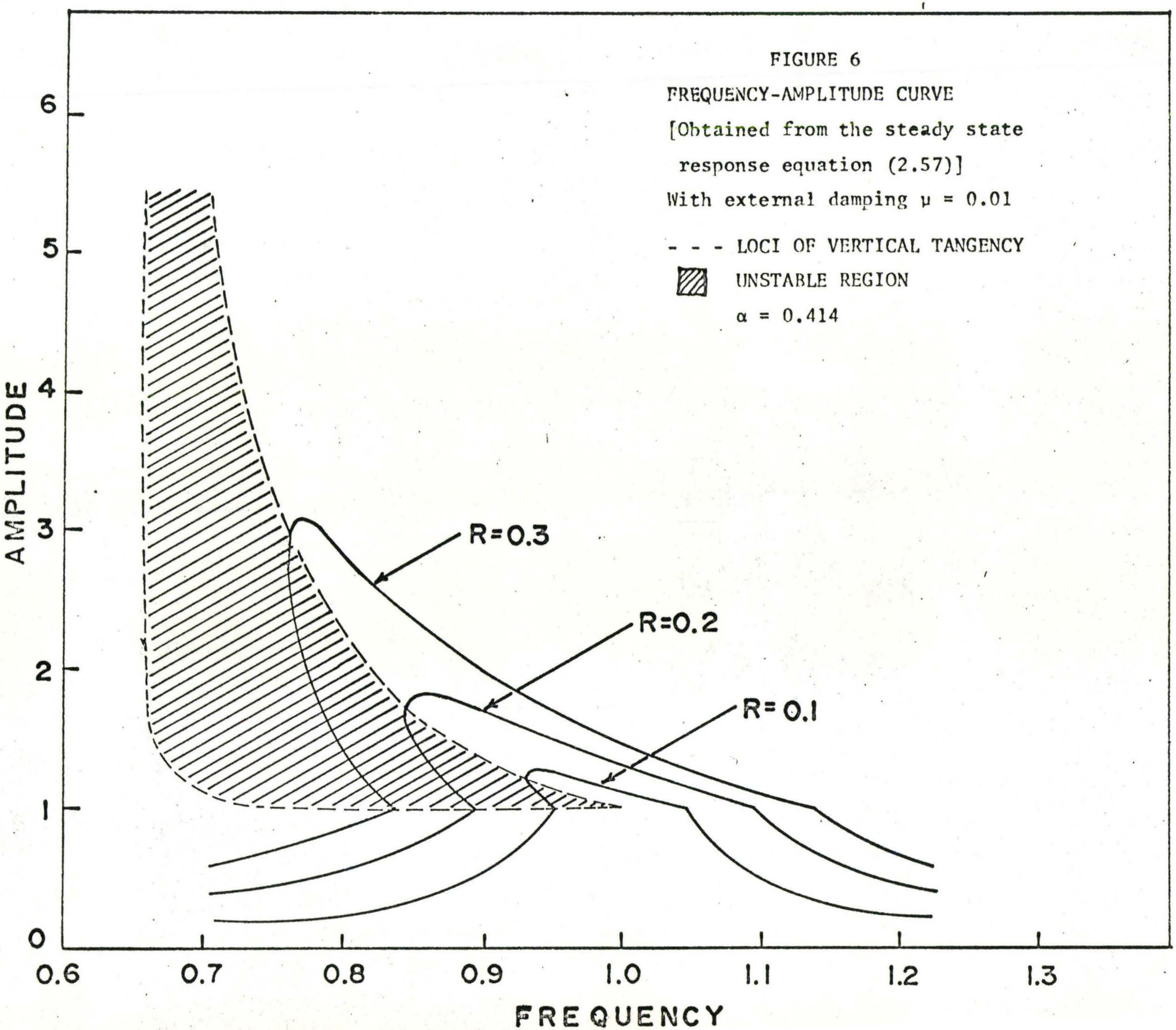


FIGURE 6

FREQUENCY-AMPLITUDE CURVE

[Obtained from the steady state response equation (2.57)]

With external damping $\mu = 0.01$

- - - LOCI OF VERTICAL TANGENCY

UNSTABLE REGION

$\alpha = 0.414$

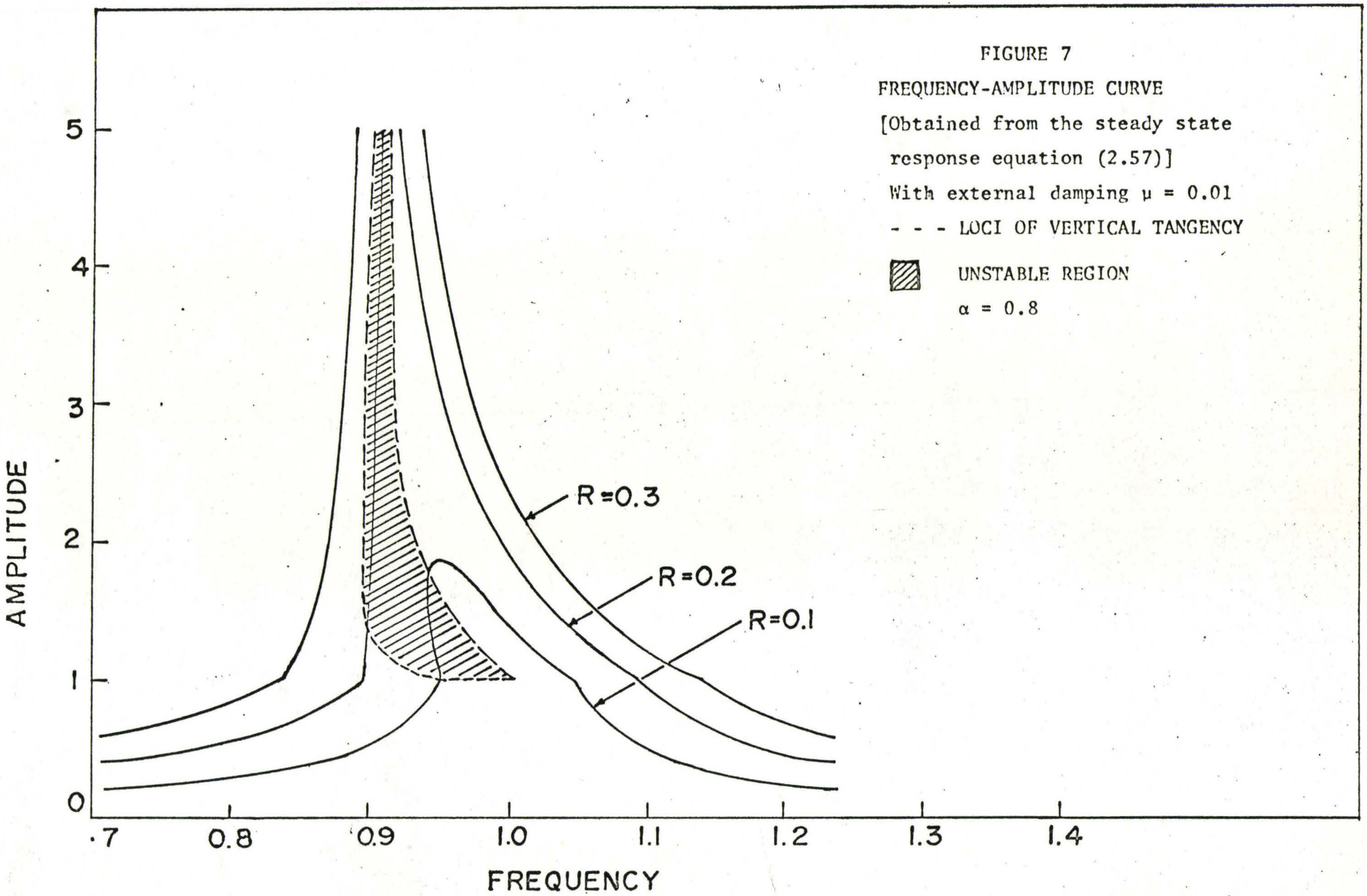


FIGURE 7

FREQUENCY-AMPLITUDE CURVE

[Obtained from the steady state response equation (2.57)]

With external damping $\mu = 0.01$

- - - LOCI OF VERTICAL TANGENCY

UNSTABLE REGION

$\alpha = 0.8$

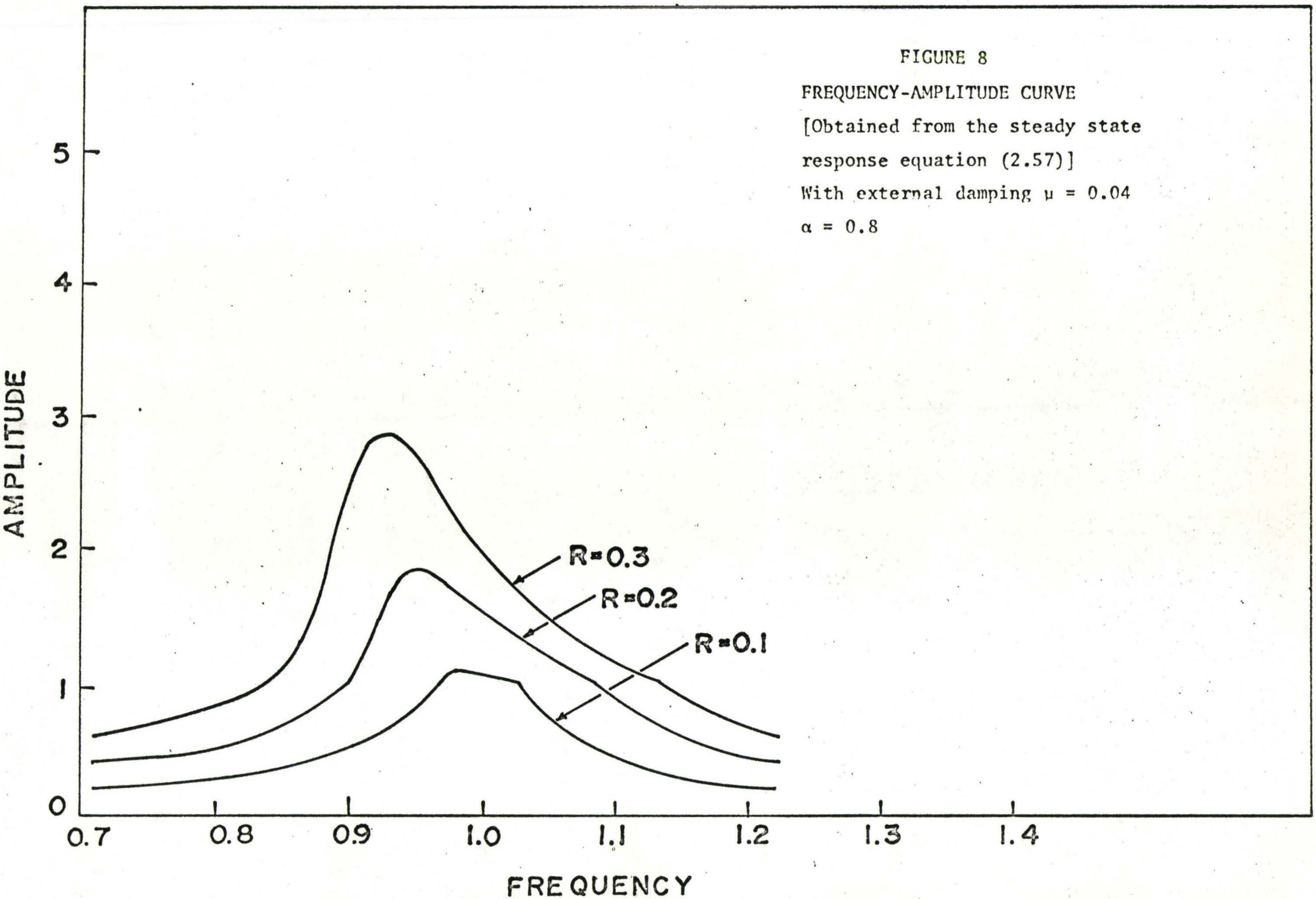


FIGURE 8

FREQUENCY-AMPLITUDE CURVE

[Obtained from the steady state response equation (2.57)]

With external damping $\mu = 0.04$

$\alpha = 0.8$

$$S(A) = -\frac{2}{\pi}(1-\alpha)(A-1)/A \qquad A \geqslant 1. \qquad (2.54)$$

$$S(A) = 0 \qquad A \leqslant 1.$$

where Ψ_1 and Ψ_2 have the same meaning as explained in section IIB with reference to Figure (2). Substituting (2.52) in (2.50) and (2.51):

$$2\omega\dot{\phi} A - \omega^2 A + C(A) = R \cos \phi \qquad \text{(a)}$$

$$(2.55)$$

$$-2\omega\dot{A} - 2\mu\omega A + S(A) = -R \sin \phi \qquad \text{(b)}$$

For steady state motion the average value of $\dot{\phi}$ and $\dot{A}$ over a cycle of θ will be zero. Thus

$$-\omega^2 A + C(A) = R \cos \phi \qquad \text{(a)} \qquad (2.56)$$

$$-2\mu\omega A + S(A) = -R \sin \phi \qquad \text{(b)}$$

Eliminating $\cos\phi$ and $\sin\phi$ from these equations we obtain

$$\omega^4 A^2 + 2\omega^2 A\left(2\mu^2 A - C(A)\right) - 4\omega\mu A\, S(A) + C(A)^2 + S(A)^2 - R^2 = 0 \qquad (2.57)$$

which is the required steady state response equation. This can be easily solved by employing Newton-Raphson's iteration method, each time once at low value of $\omega (<1)$ and other at high frequency $(\omega > 1)$ for various values of the parameters α, R and μ.

Putting $\mu = 0$, with slight manipulation equation (2.57) can be easily reduced to equation (2.18).

IIE. LOCI OF VERTICAL TANGENCY

The loci of the points where the slope of the steady-state response curve becomes vertical may be obtained by setting

$$\left(\frac{\partial \omega}{\partial A}\right)_{A=A_0} = 0 \tag{2.58}$$

where A_0 is the steady state amplitude. Differentiating equation (2.57) with respect to A, we obtain:

$$4\omega^3\left(\frac{\partial \omega}{\partial A}\right)A^2 + 2\omega^4 A + 8\omega\left(\frac{\partial \omega}{\partial A}\right)\mu^2 A^2 + 8\omega^2\mu^2 A$$
$$- 4\omega\left(\frac{\partial \omega}{\partial A}\right)C\cdot A - 2\omega^2\left(\frac{\partial C}{\partial A}\right)A - 2\omega^2 C - 4\mu\left(\frac{\partial \omega}{\partial A}\right)A\cdot S$$
$$- 4\omega\mu S - 4\omega\mu A\left(\frac{\partial S}{\partial A}\right) + 2\left(\frac{\partial C}{\partial A}\right)C + 2\left(\frac{\partial S}{\partial A}\right)S = 0$$

which when simplified, gives:

$$\left(\frac{\partial \omega}{\partial A}\right)\left[4\omega^3 A^2 + 8\omega\mu^2 A^2 - 4\omega C A - 4\mu A S\right] + 2\omega^4 A$$
$$+ 8\omega^2\mu^2 A - 2\omega^2\left(\frac{\partial C}{\partial A}\right)A - 2\omega^2 C - 4\omega\mu S$$
$$- 4\omega\mu A\left(\frac{\partial S}{\partial A}\right) + 2\left[C\frac{\partial C}{\partial A} + S\frac{\partial S}{\partial A}\right] = 0$$

Now using condition (2.58) for loci of vertical tangency:

$$\omega^4 + 4\omega^2\mu^2 - \omega^2\left(\frac{\partial C}{\partial A}\right) - \omega^2\left(\frac{C}{A}\right) - 2\omega\mu\left(\frac{S}{A}\right) - 2\omega\mu\left(\frac{\partial S}{\partial A}\right)$$
$$+ (C/A)\left(\frac{\partial C}{\partial A}\right) + \frac{S}{A}\left(\frac{\partial S}{\partial A}\right) = 0 \tag{2.59}$$

Now $\qquad \dfrac{\partial}{\partial A}(C/A) = \dfrac{1}{A}\dfrac{\partial C}{\partial A} - C/A^2$

or $\qquad \left(\dfrac{\partial C}{\partial A}\right) = A\dfrac{\partial}{\partial A}(C/A) + C/A \qquad (2.60)$

Similarly $\qquad \left(\dfrac{\partial S}{\partial A}\right) = A\dfrac{\partial}{\partial A}\left(\dfrac{S}{A}\right) + S/A \qquad (2.61)$

Using these relations and the expressions for C(A) and S(A) from (2.53) and (2.54), we get:

$$\frac{\partial}{\partial A}\left(\frac{C}{A}\right) = -\frac{2(1-\alpha)}{\pi A^2}\left(\sin\psi_1 + \sin\psi_2\right) \tag{2.62}$$

$$\left(\frac{\partial S}{\partial A}\right) = -\frac{2}{\pi}(1-\alpha)/A^2 \tag{2.63}$$

Substituting (2.62) and (2.63) into the equation (2.59) and simplifying we get:

$$\omega^4 + 2\omega^2\left[\mu^2 + \frac{1-\alpha}{\pi A}(\sin\psi_1 + \sin\psi_2) - C/A\right] + \left(\frac{C}{A}\right)^2 - \frac{2S}{\pi A^3}(1-\alpha)$$
$$-2\omega\mu\left[\frac{S}{A} + \frac{2(1-\alpha)}{\pi A^2}\right] - \frac{2C}{\pi A^2}(1-\alpha)(\sin\psi_1 + \sin\psi_2) \tag{2.64}$$

which gives the general equation for loci of vertical tangency for the system with added viscous damping.

For $\mu=0$, ie, without external damping,

$$\omega^4 + 2\omega^2\left[\frac{1-\alpha}{\pi A}(\sin\psi_1 + \sin\psi_2) - C/A\right] + \left(\frac{C}{A}\right)^2 - \frac{2S}{\pi A^3}(1-\alpha)$$
$$-\frac{2C}{\pi A^2}(1-\alpha)(\sin\psi_1 + \sin\psi_2)$$

Substituting $\omega_p^2 = \frac{C}{A}$ which gives the locus of peak amplitudes,

$$\omega^4 + 2\omega^2\left[\frac{1-\alpha}{\pi A}(\sin\psi_1 + \sin\psi_2) - \omega_p^2\right] + \omega_p^4 - \frac{2S}{\pi A^3}(1-\alpha)$$
$$-\frac{2}{\pi A}\omega_p^2(1-\alpha)(\sin\psi_1 + \sin\psi_2) = 0$$

Substituting for $(\frac{S}{A})$ and rearranging we get:

$$\omega^2 = \omega_p^2 - \frac{1-\alpha}{A\pi}\left[(\sin\psi_1 + \sin\psi_2) \pm \left\{2 - (\sin\psi_1 - \sin\psi_2)^2\right\}^{1/2}\right] \tag{2.64a}$$

which gives two loci of vertical tangency for $A>1$.

In Figures (3), (4), (5), (6) and (7), the loci of vertical tangency have been shown by dotted lines, In Figures (3), (4) and (5) (which correspond to system without external damping), it can be seen that the area bounded by the loci of vertical tangency, ie. the unstable region, increases by decreasing α . Also comparing Figures (3) and (5) with Figures

(6) and (7) respectively, it can be concluded that the unstable region bounded by the loci of vertical tangency decreases by addition of viscous damping to the system. No loci of vertical tangency exists for the system with high external damping, $\mu = 0.04$, (all other parameters remaining the same), as shown in Figure (8).

IIF. STABILITY

In this section we shall investigate the stability of the steady state response of the system. Let us assume infinitesimal perturbations ξ and ψ on the steady state values A_0 and ϕ_0.

Thus

$$A = A_0 + \xi \qquad \text{(a)}$$
$$\phi = \phi_0 + \psi \qquad \text{(b)} \qquad (2.65)$$

Using equations (2.55a) and (2.55b)

$$2(A_0+\xi)\omega\dot{\psi} - (A_0+\xi)\omega^2 + C(A_0+\xi) = R\cos(\phi_0+\psi)$$

Expanding $C(A_0+\xi)$ by Taylor's theorem and retaining only two terms we get:

$$2A_0\omega\dot{\psi} + 2\xi\omega\dot{\psi} - A_0\omega^2 - \xi\omega^2 + C(A_0) + \xi\left(\frac{\partial C}{\partial A}\right)_{A=A_0}$$
$$= R\cos\phi_0 - R\psi\sin\phi_0$$

Since ξ and $\dot{\psi}$ are infinitesimal quantities of the same order, $2\omega\xi\dot{\psi}$ is an infinitesimal quantity of second order compared to the rest of the terms. Hence it may be assumed to be negligible. Thus

$$2A_0\omega\dot{\psi} - A_0\omega^2 - \xi\omega^2 + C(A_0) + \xi\left(\frac{\partial C}{\partial A}\right)_{A=A_0} = R\cos\phi_0 - R\psi\sin\phi_0 \qquad (2.66)$$

Also

$$2\dot{\xi}\omega + 2\mu\omega A_o + 2\mu\omega\xi - S(A_o) + \xi\left(\frac{\partial S}{\partial A}\right)_{A=A_o} = R\sin\phi_o + R\psi\cos\phi_o \tag{2.67}$$

Now let

$$\xi = \xi_o e^{\lambda t} \qquad \text{(a)}$$

$$\psi = \psi_o e^{\lambda t} \qquad \text{(b)} \tag{2.68}$$

Using (2.56) along with (2.67) we get

$$2A_o\omega\dot{\psi} - \xi\omega^2 + \xi\left(\frac{\partial C}{\partial A}\right)_{A=A_o} = -R\psi\sin\phi_o \qquad \text{(a)}$$

$$2\dot{\xi}\omega + 2\mu\omega\xi - \xi\left(\frac{\partial S}{\partial A}\right)_{A=A_o} = R\psi\cos\phi_o \qquad \text{(b)} \tag{2.69}$$

Also eliminating $R\sin\phi_o$ and $R\cos\phi_o$ with equations (2.56)

$$2A_o\omega\dot{\psi} + \left[2\mu\omega A_o - S(A_o)\right]\psi + \left(\frac{\partial C}{\partial A} - \omega^2\right) = 0 \qquad \text{(a)}$$

$$\left[\omega^2 A_o - C(A_o)\right]\psi + 2\omega\dot{\xi} + \left[2\mu\omega - \left(\frac{\partial S}{\partial A}\right)_{A=A_o}\right] = 0 \qquad \text{(b)} \tag{2.70}$$

Substituting (2.68) in equations (2.70), yields

$$\left[2A_o\omega\lambda + 2A_o\omega\mu - S(A_o)\right]\psi_o + \left(\frac{\partial C}{\partial A} - \omega^2\right)\xi_o = 0 \qquad \text{(a)}$$

$$\left[\omega^2 A_o - C(A_o)\right]\psi_o + \left(2\omega\lambda + 2\omega\mu - \frac{\partial S}{\partial A}\right)\xi_o = 0 \qquad \text{(b)} \tag{2.71}$$

The frequency equation is obtained by setting the determinant of the coefficients of equations (2.71) equal to zero.

$$\begin{vmatrix} 2A_o\omega(\lambda+\mu) - S(A_o) & \left(\frac{\partial C}{\partial A}\right)_{A=A_o} - \omega^2 \\ \omega^2 A_o - C(A_o) & 2\omega(\lambda+\mu) - \left(\frac{\partial S}{\partial A}\right)_{A=A_o} \end{vmatrix} = 0 \tag{2.72}$$

Solving (2.72), we get:

$$(2\omega\lambda)^2 + 2\omega\lambda\left[4\mu\omega - \frac{S(A_o)}{A_o} - \left(\frac{\partial S}{\partial A}\right)_{A=A_o}\right] + \left[2\mu\omega - \frac{S(A_o)}{A_o}\right]$$
$$\times\left[2\mu\omega - \left(\frac{\partial S}{\partial A}\right)_{A=A_o}\right] + \left[\left(\frac{\partial C}{\partial A}\right)_{A=A_o} - \omega^2\right]\left(\frac{C(A_o)}{A_o} - \omega^2\right) = 0 \quad (2.73)$$

From (2.54), (2.62) and (2.63), $S(A_o)$, $\left(\frac{\partial S}{\partial A}\right)_{A=A_o}$, $C(A_o)$ and $\left(\frac{\partial C}{\partial A}\right)_{A=A_o}$ can be substituted in equation (2.73). Thus (2.73) is essentially a quadratic equation in λ which may be written as

$$\nu^2 + 2b\nu + e = 0 \quad (2.74)$$

where $\nu = 2\omega\lambda$

$$b = \frac{1}{2}\left[4\mu\omega - \frac{S(A_o)}{A_o} - \left(\frac{\partial S}{\partial A}\right)_{A=A_o}\right]$$
$$e = \left[2\mu\omega - \frac{S(A_o)}{A_o}\right]\left[2\mu\omega - \left(\frac{\partial S}{\partial A}\right)_{A=A_o}\right]$$
$$+ \left[\left(\frac{\partial C}{\partial A}\right)_{A=A_o} - \omega^2\right]\left(\frac{C(A_o)}{A_o} - \omega^2\right) \quad (2.75)$$

From (2.74):

$$\nu_{1,2} = -b \pm \sqrt{b^2 - e} \quad (2.76)$$

If the agebraic sign of the real part of any of the roots is positive, the system is asymptotically unstable, meaning thereby that any disturbance in the system will grow with time. If all the roots have negative real parts the system is asymptotically stable. If any root has a real part that is zero, neutral stability is indicated.

The determination of signs of the real parts of the roots may be carried out by either of the two methods:

(i) Nyquist numerical method

(ii) Routh-Hurwitz analytical method

Both these methods indicate the presence or absence of roots with positive real parts.

Now according to Routh-Hurwitz criteria of stability, if there is characteristic equation of nth degree in general;

$$a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots\cdots + a_{n-1}\lambda + a_n = 0 \quad (2.77)$$

then n determinants are formed as follows -

$$\Delta_1 = a_1$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} \quad (2.78)$$

$$\Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{2n-1} & \cdots & \cdots & \cdots & \cdots & a_n \end{vmatrix}$$

and for stability, all these determinants should separately be greater than zero. But since Δ_n is composed all of zeros, except for one term $a_n, \Delta_n = a_n \Delta_{n-1}$. Thus for stability it is required that both $a_n > 0$, $\Delta_{n-1} > 0$. In our case the frequency equation (2.73) can be written as:

$$a_o\lambda^2 + a_1\lambda + a_2 = 0 \qquad (2.79)$$

where

$$a_2 = \left[2\mu\omega - \frac{S(A_o)}{A_o}\right]\left[2\mu\omega - \left(\frac{\partial S}{\partial A}\right)_{A=A_o}\right] + \left[\left(\frac{\partial C}{\partial A}\right)_{A=A_o} - \omega^2\right]\left[\frac{C(A_o)}{A_o} - \omega^2\right] \qquad (2.80)$$

$$\Delta_{n-1}\Big|_{n=2} = \Delta_1 = a_1 = 2\omega\left[4\mu\omega - \frac{S(A_o)}{A_o} - \left(\frac{\partial S}{\partial A}\right)_{A=A_o}\right]$$

Substituting the values of $S(A_0)$, $C(A_0)$, etc. from equations (2.60), (2.61), (2.53) and (2.54), we obtain:

$$\begin{aligned} a_1 &= 8\omega^2\mu & \ldots A_o \leq 1 \\ a_1 &= 2\omega\left[4\mu\omega + \frac{2}{\pi A_o}(1-\alpha)\right] & \ldots A_o > 1 \end{aligned} \qquad (2.81)$$

Thus since ω, $(1-\alpha)$ and μ are always positive,

$$a_1 > 0 \qquad (2.82)$$

$$a_2 = \left[2\mu\omega + \frac{2}{\pi}(1-\alpha)(A_o-1)/A_o^2\right]\left(2\mu\omega + \frac{2}{\pi}(1-\alpha)/A_o^2\right) - \frac{2(1-\alpha)}{\pi A_o}(\sin\psi_1 + \sin\psi_2)\left[\frac{1-\alpha}{\pi}\left(\psi_1 + \psi_2 - \frac{\sin 2\psi_1}{2} - \frac{\sin 2\psi_2}{2} - \frac{\pi}{2}\right) + \alpha - \omega^2\right] + \left(\frac{C(A_o)}{A_o} - \omega^2\right)^2 \ldots A_o > 1$$

and $a_2 = 4\mu^2\omega^2 + (1-\omega^2)^2 \; \ldots\ldots A_o \leq 1 \qquad (2.83)$

For $A_o \leq 1$, $a_2 > 0$ hence with the equation (2.82) it can be seen that the system is always stable for $A_o \leq 1$.

For $A_o \geq 1$ the stability condition is

$$a_2 > 0 \tag{2.84}$$

Hence, from (2.83):

$$\left[2\mu\omega + \frac{2}{\pi}(1-\alpha)(A_o-1)\left(2\mu\omega + \frac{2}{\pi}(1-\alpha)/A_o^2\right)/A_o^2 + \left(\frac{C(A_o)}{A_o} - \omega^2\right)^2\right]$$

$$> \frac{2(1-\alpha)}{\pi A_o}\left[\frac{1-\alpha}{\pi}\left\{\psi_1 + \psi_2 - \frac{1}{2}\left(\sin 2\psi_1 + \sin 2\psi_2\right) - \frac{\pi}{2}\right\} + \alpha - \omega^2\right]\left(\sin\psi_1 + \sin\psi_2\right) \tag{2.85}$$

In our case $\alpha = 0.7$ to 0.9, and for $\mu \geqslant 0$ the stability criterion is given by (2.85)

For $\mu = 0$, i.e., no external damping, for stability we have $(A_o > 1)$

$$\left[\frac{4}{\pi^2}(1-\alpha)^2(A_o-1)/A_o^4 + \left(\omega_p^2 - \omega^2\right)^2\right] > \frac{2(1-\alpha)}{\pi A_o}\left[\frac{1-\alpha}{\pi}\left\{\psi_1 + \psi_2 - \frac{1}{2}\left(\sin 2\psi_1 + \sin 2\psi_2\right) - \frac{\pi}{2}\right\} + \alpha - \omega^2\right]\left(\sin\psi_1 + \sin\psi_2\right) \tag{2.85a}$$

or

$$\left[\frac{4}{\pi^2}(1-\alpha)^2(A_o-1)/A_o^4 + \left(\omega_p^2 - \omega^2\right)^2\right] + \left(\omega^2 - \omega_p^2\right)\left[\frac{2(1-\alpha)}{\pi A_o}\left(\sin\psi_1 + \sin\psi_2\right)\right] > 0 \tag{2.85b}$$

With the help of (2.64a) it can be shown that when ω lies outside the range of frequencies defined by the two loci of vertical tangency, only then this condition is satisfied. The unstable regions are shown in Figures (3), (4), (5), (6) and (7) as shaded area.

III. STEADY STATE RESPONSE OF A TWO-DEGREE-OF FREEDOM DOUBLE BI-LINEAR HYSTERETIC SYSTEM

IIIA. EQUATIONS OF MOTION

We start with the following assumptions -

(i) The system has the configuration and characteristic properties as shown in Figures (1) and (9).

(ii) Both masses are equal.

(iii) Identical normalized hysteresis-loops as in the case of single degree of freedom system, Figure (1).

Normalizing the system in the manner as shown in appendix for two degree of freedom system, we have the following equations of motion:

$$\ddot{x}_1 + f(x_1, \dot{x}_1) - f(x_2, \dot{x}_2) = R \cos \omega t \qquad (3.1)$$

$$\ddot{x}_2 + 2 f(x_2, \dot{x}_2) - f(x_1, \dot{x}_1) = 0 \qquad (3.2)$$

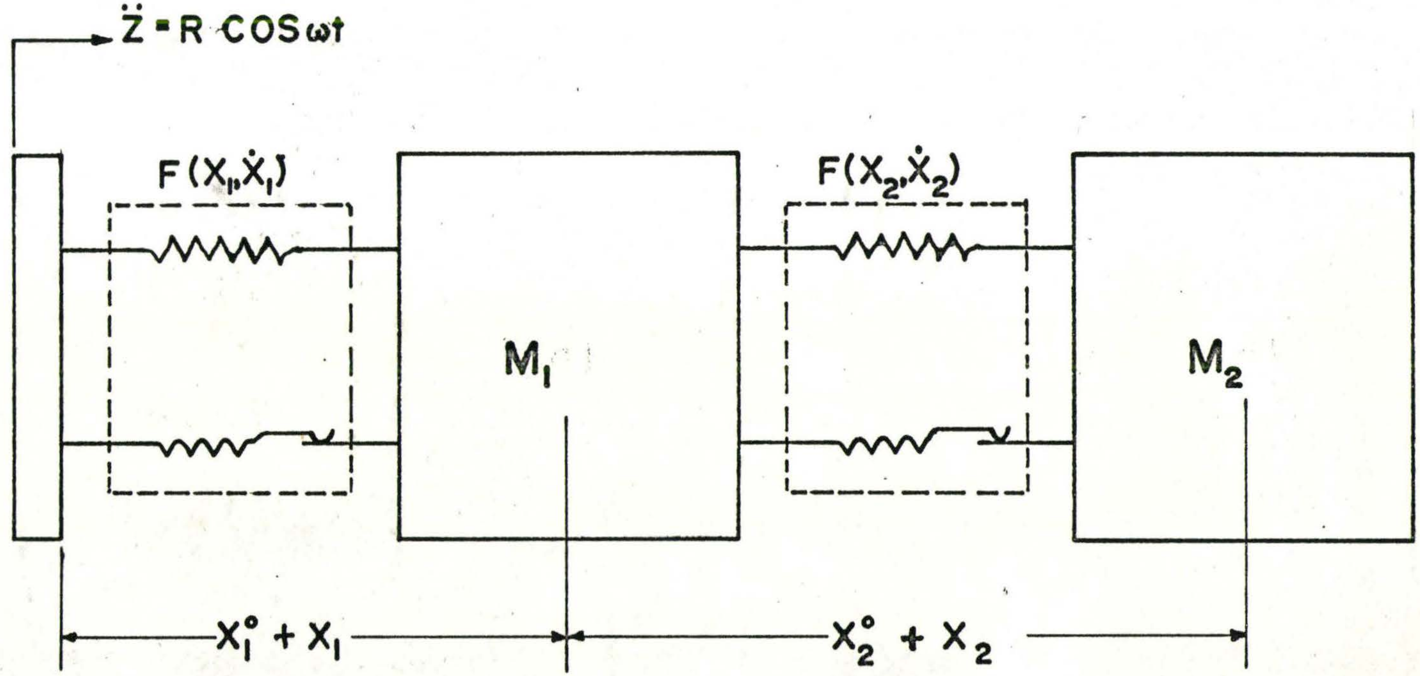
This describes the motion of the two masses with the hysteretic restoring force.

IIIB. KRYLOV AND BOGOLIEUBOV SOLUTION

Let us assume the solutions to be

$$x_1 = A_1 \cos \theta_1 \qquad (a)$$

$$x_2 = A_2 \cos \theta_2 \qquad (b) \qquad (3.3)$$



TWO DEGREES OF FREEDOM DOUBLE BILINEAR HYSTERETIC SYSTEM

X_1° & X_2° are static displacements

FIGURE 9

where

$$\theta_1 = \omega t - \phi_1 \quad \text{(a)}$$
$$(3.4)$$
$$\theta_2 = \theta_1 - \phi_2 \quad \text{(b)}$$

where A_1, A_2 are amplitudes and ϕ_1, ϕ_2 are phase angles of the mass M_1 and M_2 respectively with corresponding suffixes. A_1, A_2, ϕ_1 and ϕ_2 are assumed to be slowly varying functions of time t.

Differentiating (3.3a) and (3.3b) with respect to time we get:

$$\dot{x}_1 = \dot{A}_1 \cos\theta_1 - \omega A_1 \sin\theta_1 + A_1 \dot{\phi}_1 \sin\theta_1 \quad \text{(a)}$$
$$(3.5)$$
$$\dot{x}_2 = \dot{A}_2 \cos\theta_2 - \omega A_2 \sin\theta_2 + A_2 \dot{\phi}_2 \sin\theta_2 \quad \text{(b)}$$

But by Krylov and Bogolieubov Method[16]since A_1, A_2, ϕ_1 and ϕ_2 are all slowly varying functions of time from equation (3.5) we assume:

$$\dot{A}_1 \cos\theta_1 + A_1 \dot{\phi}_1 \sin\theta_1 = 0 \quad \text{(a)}$$
$$(3.6)$$
$$\dot{A}_2 \cos\theta_2 + A_2 \dot{\phi}_2 \sin\theta_2 + A_2 \dot{\phi}_1 \sin\theta_2 = 0 \quad \text{(b)}$$

Hence

$$\dot{x}_1 = -\omega A_1 \sin\theta_1 \quad (3.7a)$$

$$\dot{x}_2 = -\omega A_2 \sin\theta_2 \tag{3.7b}$$

Differentiating (3.7) once again, we have

$$\ddot{x}_1 = -\omega^2 A_1 \cos\theta_1 - \omega \dot{A}_1 \sin\theta_1 + \omega A_1 \dot{\phi}_1 \cos\theta_1 \qquad \text{(a)}$$

(3.8)

$$\ddot{x}_2 = -\omega^2 A_2 \cos\theta_2 - \omega \dot{A}_2 \sin\theta_2 + \omega A_2 \dot{\phi}_2 \cos\theta_2 \qquad \text{(b)}$$

Substituting this in equations (3.1) and (3.2)

$$-\omega^2 A_1 \cos\theta_1 - \omega \dot{A}_1 \sin\theta_1 + \omega A_1 \dot{\phi}_1 \cos\theta_1 + f(A_1, \theta_1) - f(A_2, \theta_2) = R\cos(\theta_1 + \phi_1) \qquad \text{(a)}$$

(3.9)

$$-\omega^2 A_2 \cos\theta_2 - \omega \dot{A}_2 \sin\theta_2 + \omega A_2 \dot{\phi}_2 \cos\theta_2 + \omega A_2 \dot{\phi}_1 \cos\theta_2 + 2f(A_2, \theta_2) - f(A_1, \theta_1) = 0 \qquad \text{(b)}$$

Multiplying equation (3.6a) by $\omega \cos\theta_1$ and equation (3.9a) by $\sin\theta_1$ and subtracting

$$-\omega \dot{A}_1 - \omega^2 A_1 \cos\theta_1 \sin\theta_1 + f(A_1, \theta_1)\sin\theta_1 - f(A_2, \theta_2)\sin\theta_1 = R\sin\theta_1 \cos(\theta_1 + \phi_1) \tag{3.10}$$

Integrating and averaging over a cycle of θ_1 as in the case of single degree of freedom system we obtain

$$-\omega \dot{A}_1 + \frac{1}{2\pi}\int_0^{2\pi} f(A_1, \theta_1)\sin\theta_1 \, d\theta_1 - \frac{\sin\phi_2}{2\pi}\int_0^{2\pi} f(A_2 \theta_2)\cos\theta_2 \, d\theta_2 - \cos\phi_2 / 2\pi \int_0^{2\pi} f(A_2, \theta_2)\sin\theta_2 \, d\theta_2 = -\frac{R}{2}\sin\phi_1 \tag{3.11}$$

Again, multiplying equation (3.6a) by $\omega \operatorname{Sin}\theta_1$ and equation (3.9a) by $\operatorname{Cos}\theta_1$, adding and averaging over one cycle of θ_1, gives

$$\omega A_1 \dot{\phi}_1 - \frac{\omega^2 A_1}{2} + \frac{1}{2\pi}\int_0^{2\pi} f(A_1,\theta_1)\operatorname{Cos}\theta_1\, d\theta_1 - \frac{\operatorname{Cos}\phi_2}{2\pi}\int_0^{2\pi} f(A_2,\theta_2)\operatorname{Cos}\theta_2\, d\theta_2 + \frac{\operatorname{Sin}\phi_2}{2\pi}\int_0^{2\pi} f(A_2,\theta_2)\operatorname{Sin}\theta_2\, d\theta_2 = R\operatorname{Cos}\phi_1 \quad (3.12)$$

Similarly, multiplying equation (3.6b) by $\omega \operatorname{Cos}\theta_2$ and (3.9b) by $\operatorname{Sin}\theta_2$, subtracting and averaging over a cycle of θ_2 gives

$$-\omega \dot{A}_2 + 2\,\frac{1}{2\pi}\int_0^{2\pi} f(A_2,\theta_2)\operatorname{Sin}\theta_2\, d\theta_2 - \frac{\operatorname{Cos}\phi_2}{2\pi}\int_0^{2\pi} f(A_1,\theta_1)\operatorname{Sin}\theta_1\, d\theta_1 - \frac{\operatorname{Sin}\phi_2}{2\pi}\int_0^{2\pi} f(A_1,\theta_1)\operatorname{Cos}\theta_1\, d\theta_1 = 0 \quad (3.13)$$

Finally, multiplying equation (3.6b) by $\omega \operatorname{Sin}\theta_2$ and equation (3.9b) by $\operatorname{Cos}\theta_2$, adding and averaging over a cycle of θ_2 gives

$$-\frac{\omega^2 A_2}{2} + \omega A_2\dot{\phi}_2 + \omega A_2\dot{\phi}_1 + 2\,\frac{1}{2\pi}\int_0^{2\pi} f(A_2,\theta_2)\operatorname{Sin}\theta_2\, d\theta_2 - \frac{\operatorname{Cos}\phi_2}{2\pi}\int_0^{2\pi} f(A_1,\theta_1)\operatorname{Cos}\theta_1\, d\theta_1 - \frac{\operatorname{Sin}\phi_2}{2\pi}\int_0^{2\pi} f(A_1,\theta_1)\operatorname{Sin}\theta_1\, d\theta_1 = 0 \quad (3.14)$$

Let

$$C_i(A_i) = \frac{1}{\pi}\int_0^{2\pi} f(A_i,\theta_i)\operatorname{Cos}\theta_i\, d\theta_i \quad (3.15)$$

$$i = 1, 2$$

$$S_i(A_i) = \frac{1}{\pi}\int_0^{2\pi} f(A_i,\theta_i)\operatorname{Sin}\theta_i\, d\theta_i \quad (3.16)$$

Now the equations (3.11) through (3.14) may be written as follows -

$$-2\omega \dot{A}_1 + S_1(A_1) - C_2(A_2)\sin\phi_2 - S_2(A_2)\cos\phi_2 = -R\sin\phi_1 \tag{3.17}$$

$$-\omega^2 A_1 + 2\omega A_1\dot{\phi}_1 + C_1(A_1) - C_2(A_2)\cos\phi_2 + S_2(A_2)\sin\phi_2 = R\cos\phi_1 \tag{3.18}$$

$$-2\omega \dot{A}_2 + 2S_2(A_2) - S_1(A_1)\cos\phi_2 + C_1(A_1)\sin\phi_2 = 0 \tag{3.19}$$

$$-\omega^2 A_2 + 2\omega A_2\dot{\phi}_2 + 2C_2(A_2) - S_1(A_1)\sin\phi_2 - C_1(A_1)\cos\phi_2 + 2\omega A_2\dot{\phi}_1 = 0 \tag{3.20}$$

$C_i(A_i)$ and $S_i(A_i)$ in (3.15) and (3.16) may be evaluated in a similar manner as in Chapter II and are given by -

$$\begin{aligned} C_i(A_i) &= \frac{A_i}{\pi}\left\{(1-\alpha)\left[\psi_1^{i} + \psi_2^{i} - \frac{\sin 2\psi_1^{i}}{2} - \frac{\sin 2\psi_2^{i}}{2} - \frac{\pi}{2}\right] + \alpha\pi\right\} \quad \cdots\; A_i \geq 1 \\ &= A_i \quad \cdots\; A_i \leq 1 \end{aligned} \tag{3.21}$$

$$\begin{aligned} S_i(A_i) &= -\frac{2}{\pi}(1-\alpha)(A_i-1)/A_i \quad \cdots\; A_i \geq 1 \\ &= 0 \quad \cdots\; A_i \leq 1 \end{aligned} \tag{3.22}$$

where $\cos\psi_1^{(i)} = (A_i - 1)/A_i \qquad i=1,2 \quad$ (a)

(3.23)

$$\cos\psi_2^{(i)} = \left(-\frac{1}{A_i}\right) \qquad i=1,2 \quad \text{(b)}$$

STEADY STATE RESPONSE

The steady state response can be obtained by setting $\dot{A}_1, \dot{A}_2, \dot{\phi}_1, \dot{\phi}_2$ all equal to zero in equations (3.17) to (3.20):

Thus:

$$S_1(A_1) - C_2(A_2)\operatorname{Sin}\phi_2 - S_2(A_2)\operatorname{Cos}\phi_2 = -R\operatorname{Sin}\phi_1 \quad (3.24)$$

$$-\omega^2 A_1 + C_1(A_1) - C_2(A_2)\operatorname{Cos}\phi_2 + S_2(A_2)\operatorname{Sin}\phi_2 = R\operatorname{Cos}\phi_1 \quad (3.25)$$

$$2S_2(A_2) - S_1(A_1)\operatorname{Cos}\phi_2 + C_1(A_1)\operatorname{Sin}\phi_2 = 0 \quad (3.26)$$

$$-\omega^2 A_2 + 2C_2(A_2) - C_1(A_1)\operatorname{Cos}\phi_2 - S_1(A_1)\operatorname{Sin}\phi_2 = 0 \quad (3.27)$$

Squaring and adding (3.24) and (3.25) to eliminate ϕ_1 we get

$$\left[-\omega^2 A_1 + C_1 - C_2\operatorname{Cos}\phi_2 + S_2\operatorname{Sin}\phi_2\right]^2 + \left[S_1 - C_2\operatorname{Sin}\phi_2 - S_2\operatorname{Cos}\phi_2\right]^2 = R^2 \quad (3.28)$$

where the functional sign $S_1(A_1)$, $C_1(A_1)$ etc. are dropped and simply S_1, C_1, S_2, C_2 have been written for them.

Also, from equations (3.26) and (3.27), we get

$$\operatorname{Sin}\phi_2 = \left[-2C_1S_2 + S_1(-\omega^2 A_2 + 2C_2)\right] / (C_1^2 + S_1^2) \quad (3.29)$$

$$\operatorname{Cos}\phi_2 = \left[C_1(-\omega^2 A_2 + 2C_2) + 2S_1S_2\right] / (C_1^2 + S_1^2) \quad (3.30)$$

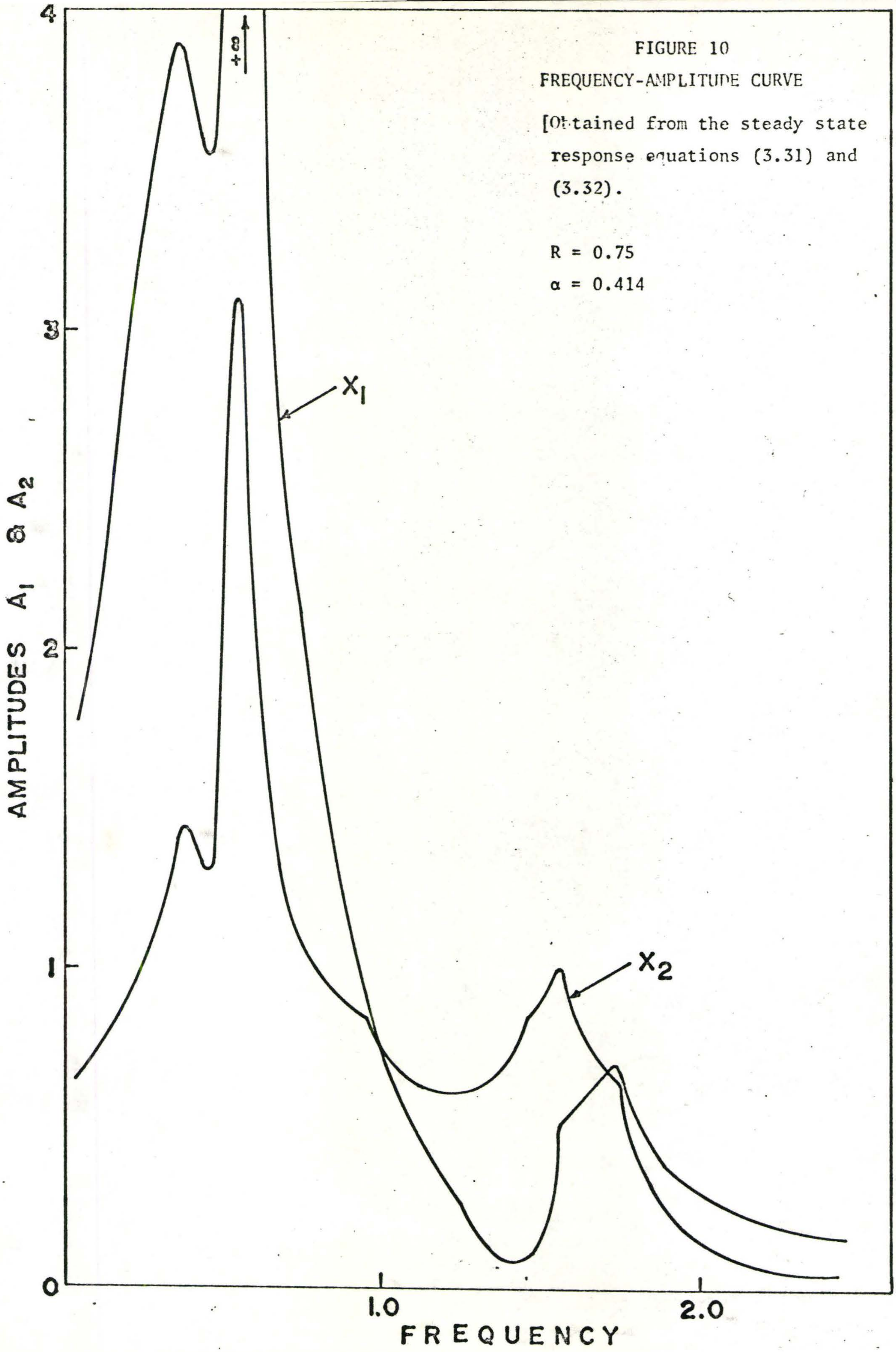


FIGURE 10

FREQUENCY-AMPLITUDE CURVE

[Obtained from the steady state response equations (3.31) and (3.32).

$R = 0.75$

$\alpha = 0.414$

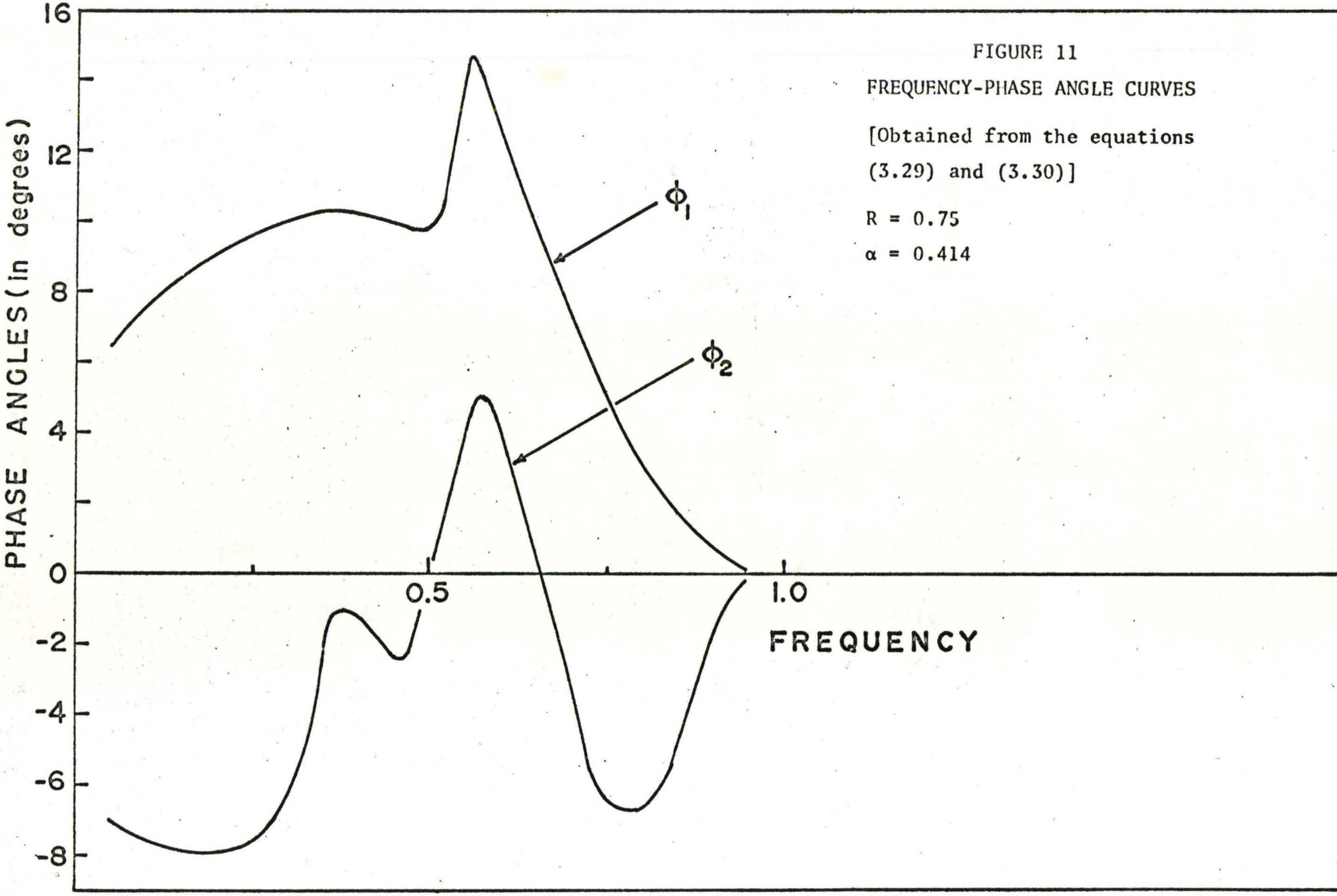


FIGURE 11

FREQUENCY-PHASE ANGLE CURVES

[Obtained from the equations (3.29) and (3.30)]

R = 0.75

α = 0.414

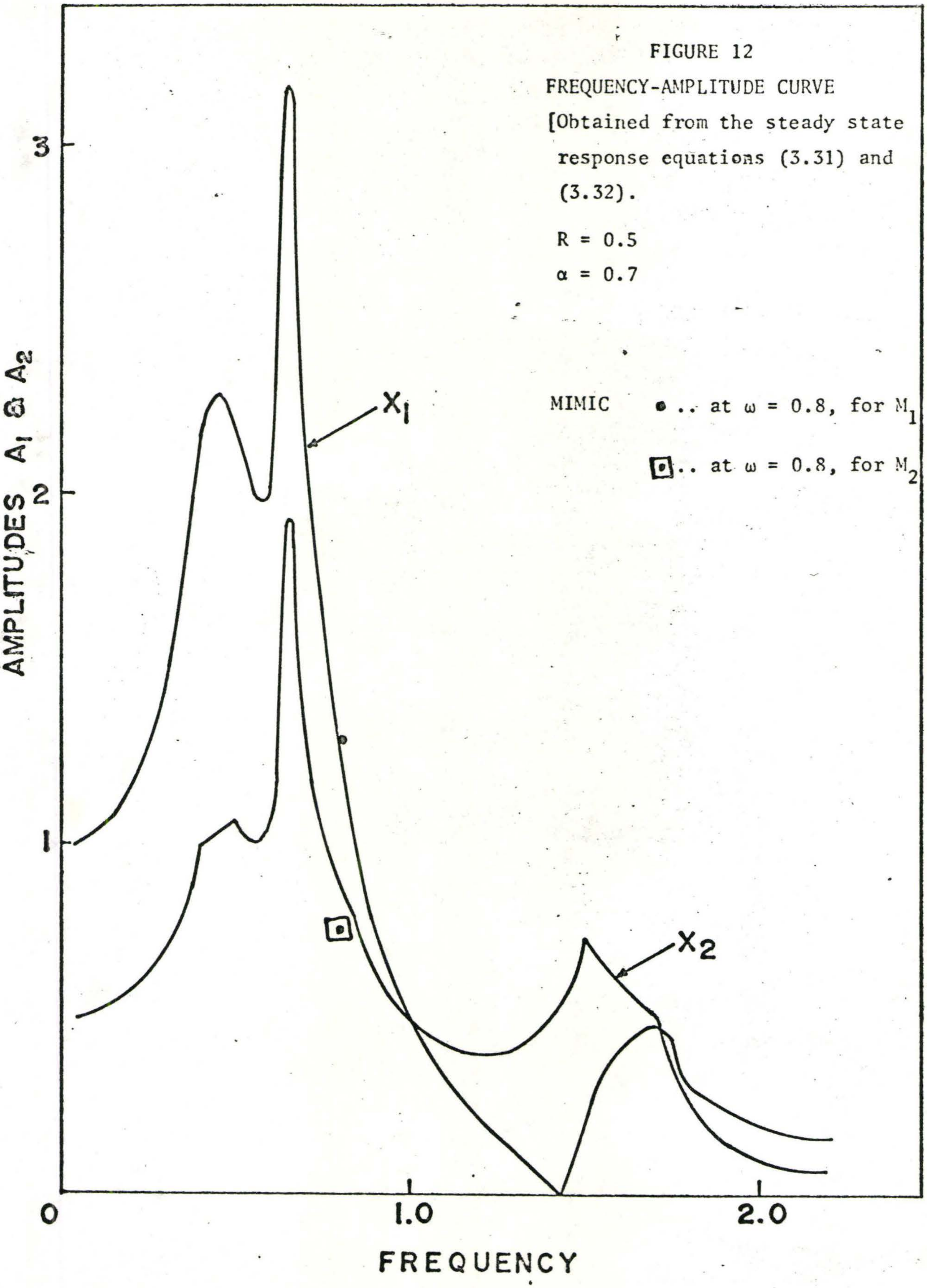


FIGURE 12

FREQUENCY-AMPLITUDE CURVE

[Obtained from the steady state response equations (3.31) and (3.32).

$R = 0.5$

$\alpha = 0.7$

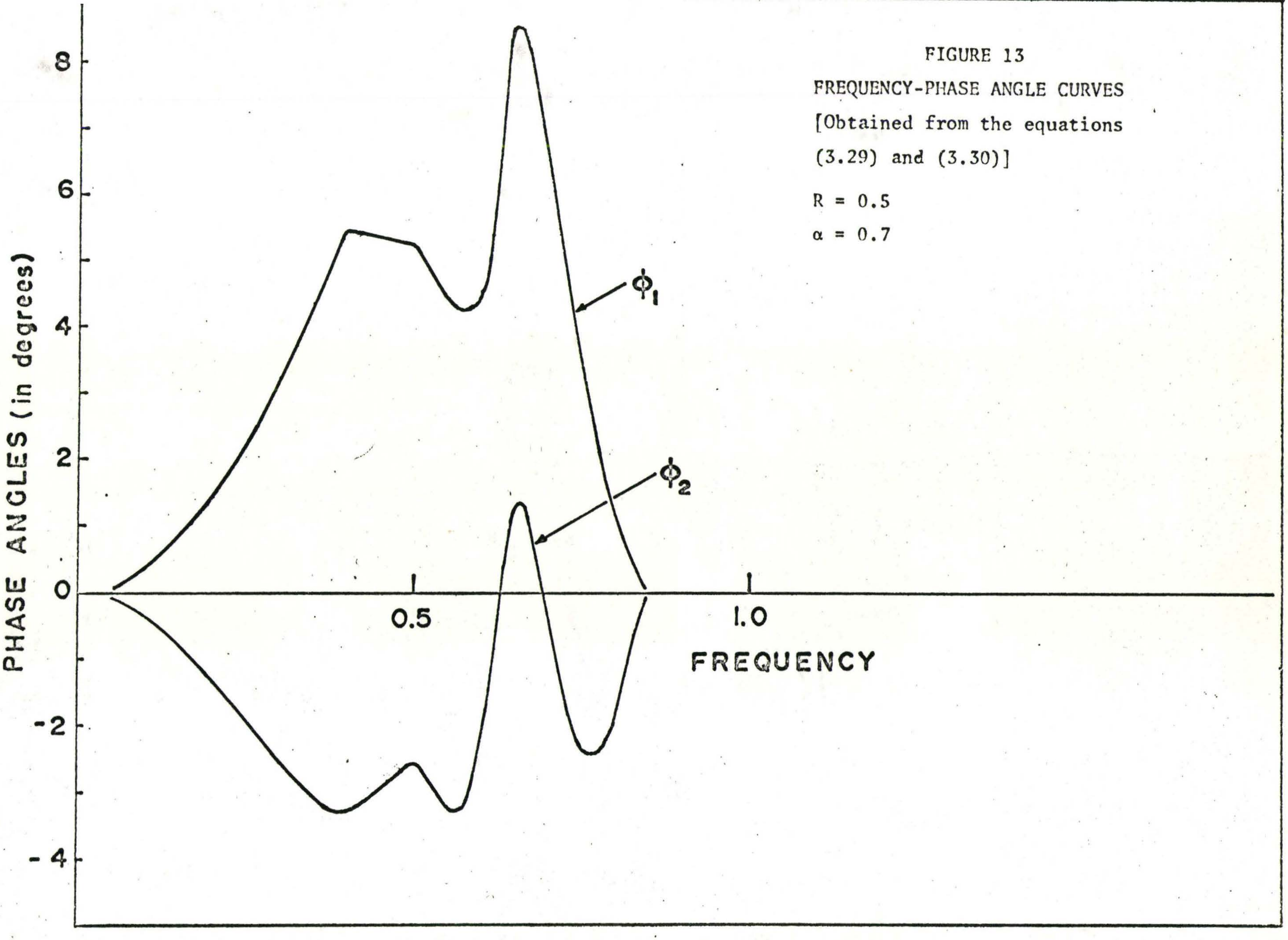


FIGURE 13

FREQUENCY-PHASE ANGLE CURVES

[Obtained from the equations (3.29) and (3.30)]

$R = 0.5$

$\alpha = 0.7$

Substituting for Sin ϕ_2 and Cos ϕ_2 in equation 3.28.

$$\left(-\omega^2 A_1 + C_1/2\right)^2 + \left(-\omega^2 A_2 + 2C_2\right)\left[\frac{\omega^4 A_1 A_2 C_1}{(C_1^2 + S_1^2)} - \frac{\omega^2 A_2}{2}\right] + \omega^4 A_1 A_2 \frac{2 S_1 S_2}{(C_1^2 + S_1^2)} + \frac{(\omega^2 A_2)^2}{4} + S_1^2/4 - R^2 = 0 \tag{3.31}$$

From (3.29) and (3.30)

$$\left(-\omega^2 A_2 + 2C_2\right)^2 - \left(C_1^2 + S_1^2 - 4S_2^2\right) = 0 \tag{3.32}$$

Theoretically, (3.31) and (3.32) constitute two equations in two unknowns A_1 and A_2 and it should be possible to find A_1 and A_2 from them. However, due to the highly involved nature of the equations ordinary iterative methods fail to give a fast result. We apply "gradient method" for this purpose. The left hand expressions in (3.31) and (3.32) are designated as Y(1) and Y(2) respectively. A new function $F = Y(1)^2 + Y(2)^2$ is defined. The gradient of F is now calculated by approximating as:

$$\frac{\partial F}{\partial A_i} = \frac{F(A_i + h) - F(A_i)}{2h} \quad \text{at an}$$

arbitrary A_i, where h = initial size of the step. The function is then evaluated at steps of h^2 in this direction until F begins to increase and the process is repeated. The procedure was conveniently accomplished by using the CALL GRAD subroutine based on the gradient method described above on the IBM-7040 computer, with quite fast results.

III C. SOLUTION BY RITZ AVERAGING METHOD

In this sub-section the equations (3.1), (3.2) will be solved by the Ritz averaging method described for single degree of freedom system in Chapter II. Designating the two equations as $E_1(x_{1,}x_2)$ and $E_2(x_{1,}x_2)$.

$$E_1(x_1,x_2) \equiv \ddot{x}_1 + f(x_1,\dot{x}_1) - f(x_2,\dot{x}_2) - R\cos\omega t = 0 \qquad (3.33)$$

$$E_2(x_1,x_2) \equiv \ddot{x}_2 + 2f(x_2,\dot{x}_2) - f(x_1,\dot{x}_1) = 0 \qquad (3.34)$$

Let an approximate solution be assumed as

$$\tilde{x}_1 = \sum_{n=1}^{n} a_n \phi_n(t) \qquad (a)$$

$$\tilde{x}_2 = \sum_{n=1}^{n} b_n \psi_n(t) \qquad (b) \qquad (3.35)$$

In this case, $E_1(\tilde{x}_1,\tilde{x}_2)$, $E_2(\tilde{x}_1,\tilde{x}_2)$ will be different from $E_1(x_1,x_2)$, $E_2(x_1,x_2)$ and therefore they will not necessarily be equal to zero. $E_1(\tilde{x}_1,\tilde{x}_2)$, $E_2(\tilde{x}_1,\tilde{x}_2)$ (called equation deficiency [8]) will vary from instant to instant, but over an arbitrary duration of time T, it will be possible to demand that each of n weighted averages of the deficiency must vanish. According to Ritz averaging criterion, the existence of the weight functions $\omega_1(t)$, $\omega_2(t)$.... $\omega_n(t)$ are postulated and $\omega_1(t)$ is placed equal to $\phi_1(t)$, $\omega_2(t)$ equal $\phi_2(t)$ .... etc. and the weight functions $v_1(t) = \psi_1(t)$, $v_2(t) = \psi_2(t)$ etc.

Thus:

$$\int_0^T E_1(\tilde{x}_1, \tilde{x}_2)\,\phi_1(t)\,dt = 0 \;, \quad \int_0^T E_2(\tilde{x}_1, \tilde{x}_2)\,\phi_1(t)\,dt = 0$$

$$\int_0^T E_1(\tilde{x}_1, \tilde{x}_2)\,\psi_1(t)\,dt = 0 \;, \quad \int_0^T E_2(\tilde{x}_1, \tilde{x}_2)\,\psi_1(t)\,dt = 0 \qquad (3.36)$$

etc.

thus yielding 2 n algebraic equations from which $(a_1, a_2 \ldots\ldots a_n)$, $(b_1, b_2, \ldots\ldots b_n)$ can be calculated . In the present case as a two term approximation we shall chose

$$\tilde{x}_1 = A_1 \cos\theta_1 \qquad (a)$$

$$\tilde{x}_2 = A_2 \cos\theta_2 \qquad (b) \qquad (3.37)$$

where

$$\theta_1 = (\omega t - \phi_1) \qquad (a)$$

$$\theta_2 = (\theta_1 - \phi_2) \qquad (b) \qquad (3.38)$$

where A_1, A_2 are amplitudes and ϕ_1, ϕ_2 are phase angles for the two masses. Substituting these expression in (3.36) and averaging over a period 2π

$$\int_0^{2\pi} E_1(\tilde{x}_1, \tilde{x}_2) \cos\omega t\,dt = 0 \qquad (a)$$

$$\int_0^{2\pi} E_1(\tilde{x}_1, \tilde{x}_2) \sin\omega t\,dt = 0 \qquad (b)$$

$$\int_0^{2\pi} E_2(\tilde{x}_1, \tilde{x}_2) \cos\omega t\,dt = 0 \qquad (c) \qquad (3.39)$$

$$\int_0^{2\pi} E_2(\tilde{x}_1, \tilde{x}_2) \sin\omega t\,dt = 0 \qquad (d)$$

These equations will lead to four algebraic equations in four unknowns A_1, A_2, ϕ_1 and ϕ_2.

Let $\omega t = \sigma$, then

$$-A_1\omega^2 \int_0^{2\pi} \cos\theta_1 \cos\sigma \, d\sigma + \int_0^{2\pi} f(\tilde{x}_1, \dot{\tilde{x}}_1)\cos\sigma \, d\sigma - \int_0^{2\pi} f(\tilde{x}_2, \dot{\tilde{x}}_2)\cos\sigma \, d\sigma$$
$$- R\int_0^{2\pi} \cos^2\sigma \, d\sigma = 0 \qquad (3.40a)$$

$$-A_1\omega^2 \int_0^{2\pi} \cos\theta_1 \sin\sigma \, d\sigma + \int_0^{2\pi} f(\tilde{x}_1, \dot{\tilde{x}}_1)\sin\sigma \, d\sigma$$
$$- \int_0^{2\pi} f(\tilde{x}_2, \dot{\tilde{x}}_2)\sin\sigma \, d\sigma - R\int_0^{2\pi} \cos\sigma \sin\sigma \, d\sigma = 0 \qquad (3.40b)$$

$$-A_2\omega^2 \int_0^{2\pi} \cos\theta_2 \cos\sigma \, d\sigma + 2\int_0^{2\pi} f(\tilde{x}_2, \dot{\tilde{x}}_2)\cos\sigma \, d\sigma$$
$$- \int_0^{2\pi} f(\tilde{x}_1, \dot{\tilde{x}}_1)\cos\sigma \, d\sigma = 0 \qquad (3.40c)$$

$$-A_2\omega^2 \int_0^{2\pi} \cos\theta_2 \sin\sigma \, d\sigma + 2\int_0^{2\pi} f(\tilde{x}_2, \dot{\tilde{x}}_2)\sin\sigma \, d\sigma$$
$$- \int_0^{2\pi} f(\tilde{x}_1, \dot{\tilde{x}}_1)\sin\sigma \, d\sigma = 0 \qquad (3.40d)$$

Assuming that the double bilinear hysteretic curves are completely symmetrical under steady state conditions, the integrals

$$\int_0^{2\pi} f(\tilde{x}_1, \dot{\tilde{x}}_1)\sin\sigma \, d\sigma \,, \quad \int_0^{2\pi} f(\tilde{x}_2, \dot{\tilde{x}}_2)\cos\sigma \, d\sigma$$

etc can be evaluated easily by considering only half cycle of the motion.

Let $\psi_{1,2}$ and $\xi_{1,2}$ denote the phase angles at which discontinuities occur in the displacement - restoring force curve [as explained in Figure (2)] for the mass M_1 and M_2, respectively.

The integrals in (3.40) can be evaluated as follows:

$$\frac{1}{2}I_1 = \int_0^{\pi} f(\tilde{x}_1, \dot{\tilde{x}}_1)\cos\theta\, d\theta = \int_0^{\psi_1}\left[A_1\cos\theta_1 - (1-\alpha)(A_1-1)\right]\cos(\theta_1+\phi_1)\,d\theta_1$$

$$+ \int_{\psi_1}^{\pi/2}\alpha A_1\cos\theta_1\cos(\theta_1+\phi_1)\,d\theta_1 + \int_{\pi/2}^{\psi_2} A_1\cos\theta_1\cos(\theta_1+\phi_1)\,d\theta_1$$

$$+ \int_{\psi_2}^{\pi}\left[\alpha A_1\cos\theta_1 - (1-\alpha)\right]\cos(\theta_1+\phi_1)\,d\theta_1$$

$$= A_1\left[\cos\phi_1\left(\tfrac{1}{2}\theta_1 + \tfrac{1}{4}\sin 2\theta_1\right) - \sin\phi_1\left(\tfrac{1}{2}\sin^2\theta_1\right)\right]_0^{\psi_1}$$

$$-(1-\alpha)(A_1-1)\left[\sin(\theta_1+\phi_1)\right]_0^{\psi_1} + \alpha A_1\left[\cos\phi_1\left(\tfrac{1}{2}\theta_1 + \tfrac{1}{4}\sin 2\theta_1\right)\right.$$

$$\left.-\sin\phi_1\left(\tfrac{1}{2}\sin^2\theta_1\right)\right]_{\psi_1}^{\pi/2} + A_1\left[\cos\phi_1\left(\tfrac{1}{2}\theta_1+\tfrac{1}{4}\sin 2\theta_1\right) - \sin\phi_1\left(\tfrac{1}{2}\sin^2\theta_1\right)\right]_{\pi/2}^{\psi_2}$$

$$+\alpha A_1\left[\cos\phi_1\left(\tfrac{1}{2}\theta_1 + \tfrac{1}{4}\sin 2\theta_1\right) - \sin\phi_1\left(\tfrac{1}{2}\sin^2\theta_1\right)\right]_{\psi_2}^{\pi}$$

$$-(1-\alpha)\left[\sin(\theta_1+\phi_1)\right]_{\psi_2}^{\pi}$$

Therefore:

$$I_1 = \int_0^{2\pi} f(\tilde{x}_1, \dot{\tilde{x}}_1)\cos\sigma\, d\sigma$$

$$= A_1\cos\phi_1(1-\alpha)\left[\psi_1 + \psi_2 + \tfrac{1}{2}\left(\sin 2\psi_1 + \sin 2\psi_2\right)\right]$$

$$+ \frac{\pi A_1}{2}(3\alpha-1)\cos\phi_1 + A_1(1-\alpha)\sin\phi_1\left[3 - \left(\sin^2\psi_1 + \sin^2\psi_2\right)\right]$$

$$+ 2(1-\alpha)\left[\sin(\psi_2+\phi_1) - (A_1-1)\sin(\psi_1+\phi_1)\right] \qquad (3.41)$$

Similarly:

$$I_2 = \int_0^{2\pi} f(\tilde{x}_2, \dot{\tilde{x}}_2)\cos\sigma\, d\sigma = \frac{\pi A_2}{2}\cos(\phi_1+\phi_2)(3\alpha-1)$$
$$+ A_2\cos(\phi_1+\phi_2)(1-\alpha)\left[\xi_1+\xi_2+\tfrac{1}{2}(\sin 2\xi_1+\sin 2\xi_2)\right]$$
$$+A_2(1-\alpha)\sin(\phi_1+\phi_2)\left[3-\sin^2\xi_1-\sin^2\xi_2\right]+2(1-\alpha)\left[\sin(\xi_2+\phi_1+\phi_2)\right. \quad (3.42)$$
$$\left.-(A_2-1)\sin(\xi_1+\phi_1+\phi_2)\right]$$

$$J_1 = \int_0^{2\pi} f(\tilde{x}_1, \dot{\tilde{x}}_1)\sin\sigma\, d\sigma = A_1(1-\alpha)\cos\phi_1(\sin^2\psi_1+\sin^2\psi_2-3)$$
$$+ A_1(1-\alpha)\sin\phi_1\left[\psi_1+\psi_2+\tfrac{1}{2}(\sin 2\psi_1+\sin 2\psi_2)\right]+\frac{\pi A_1}{2}(3\alpha-1)\sin\phi_1$$
$$+2(1-\alpha)\left[(A_1-1)\cos(\psi_1+\phi_1)-\cos(\psi_2+\phi_1)\right] \quad (3.43)$$

$$\frac{1}{2}J_2 = \int_0^{\pi} f(\tilde{x}_2, \dot{\tilde{x}}_2)\sin\sigma\, d\sigma = \frac{A_2}{2}(1-\alpha)\cos(\phi_1+\phi_2)(\sin^2\xi_1+\sin^2\xi_2-3)$$
$$+\frac{A_2(1-\alpha)}{2}\sin(\phi_1+\phi_2)\left[\xi_1+\xi_2+\tfrac{1}{2}(\sin 2\xi_1+\sin 2\xi_2)\right]$$
$$+\frac{\pi}{4}A_2(3\alpha-1)\sin(\phi_1+\phi_2)+(1-\alpha)\left[(A_2-1)\cos(\xi_1+\phi_1+\phi_2)\right.$$
$$\left.-\cos(\xi_2+\phi_1+\phi_2)\right]$$

Therefore:

$$J_2 = A_2(1-\alpha)\cos(\phi_1+\phi_2)\left[\sin^2\xi_1+\sin^2\xi_2-3\right]+A_2(1-\alpha)\sin(\phi_1+\phi_2)$$
$$\times\left[\xi_1+\xi_2+\tfrac{1}{2}(\sin 2\xi_1+\sin 2\xi_2)\right]+\frac{\pi A_2}{2}(3\alpha-1)\sin(\phi_1+\phi_2)$$
$$+2(1-\alpha)\left[(A_2-1)\cos(\xi_1+\phi_1+\phi_2)-\cos(\xi_2+\phi_1+\phi_2)\right]$$

(3.44)

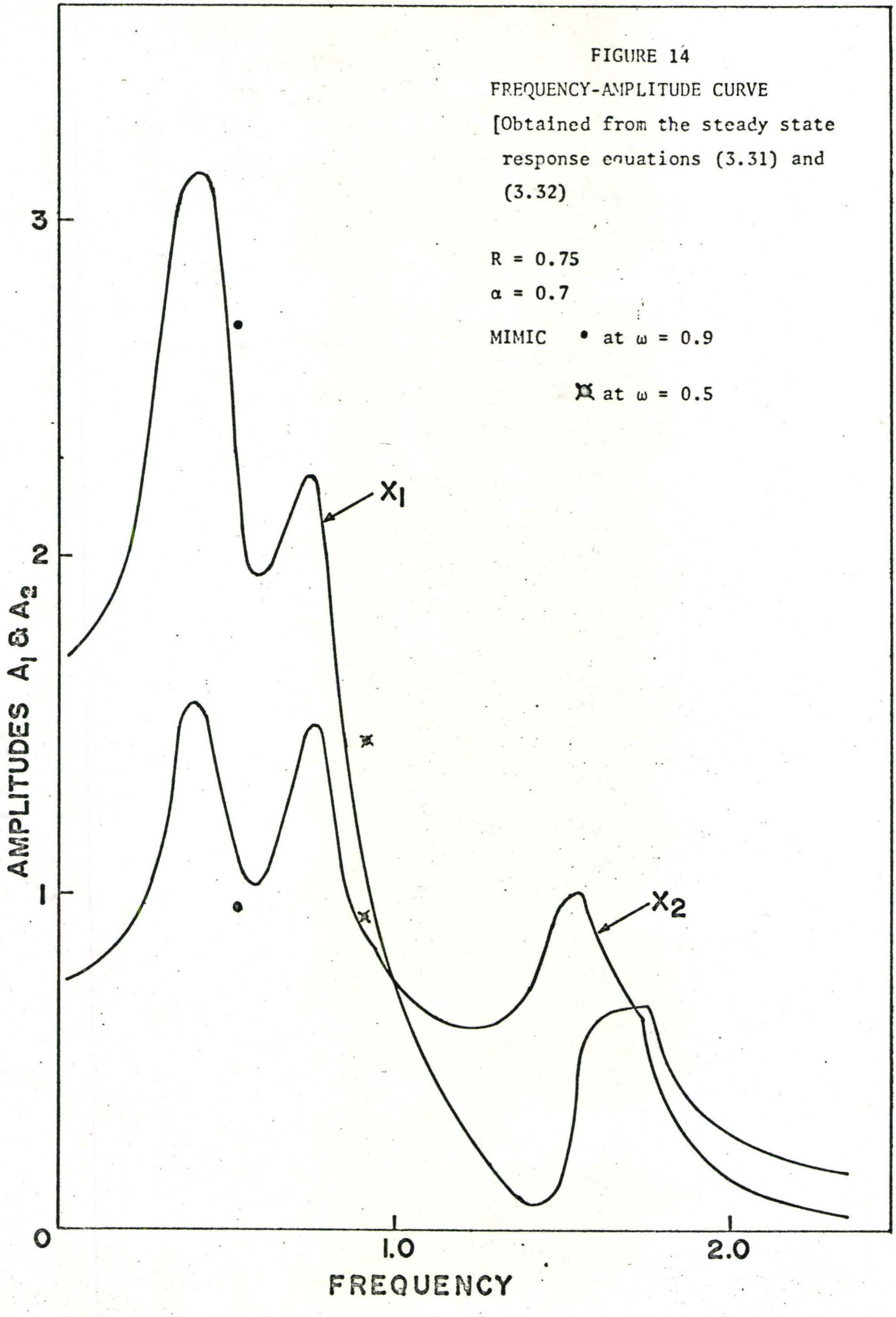


FIGURE 14
FREQUENCY-AMPLITUDE CURVE
[Obtained from the steady state response equations (3.31) and (3.32)
R = 0.75
α = 0.7
MIMIC • at ω = 0.9
¤ at ω = 0.5
X1
X2
AMPLITUDES A1 & A2
FREQUENCY
0
1
2
3
1.0
2.0

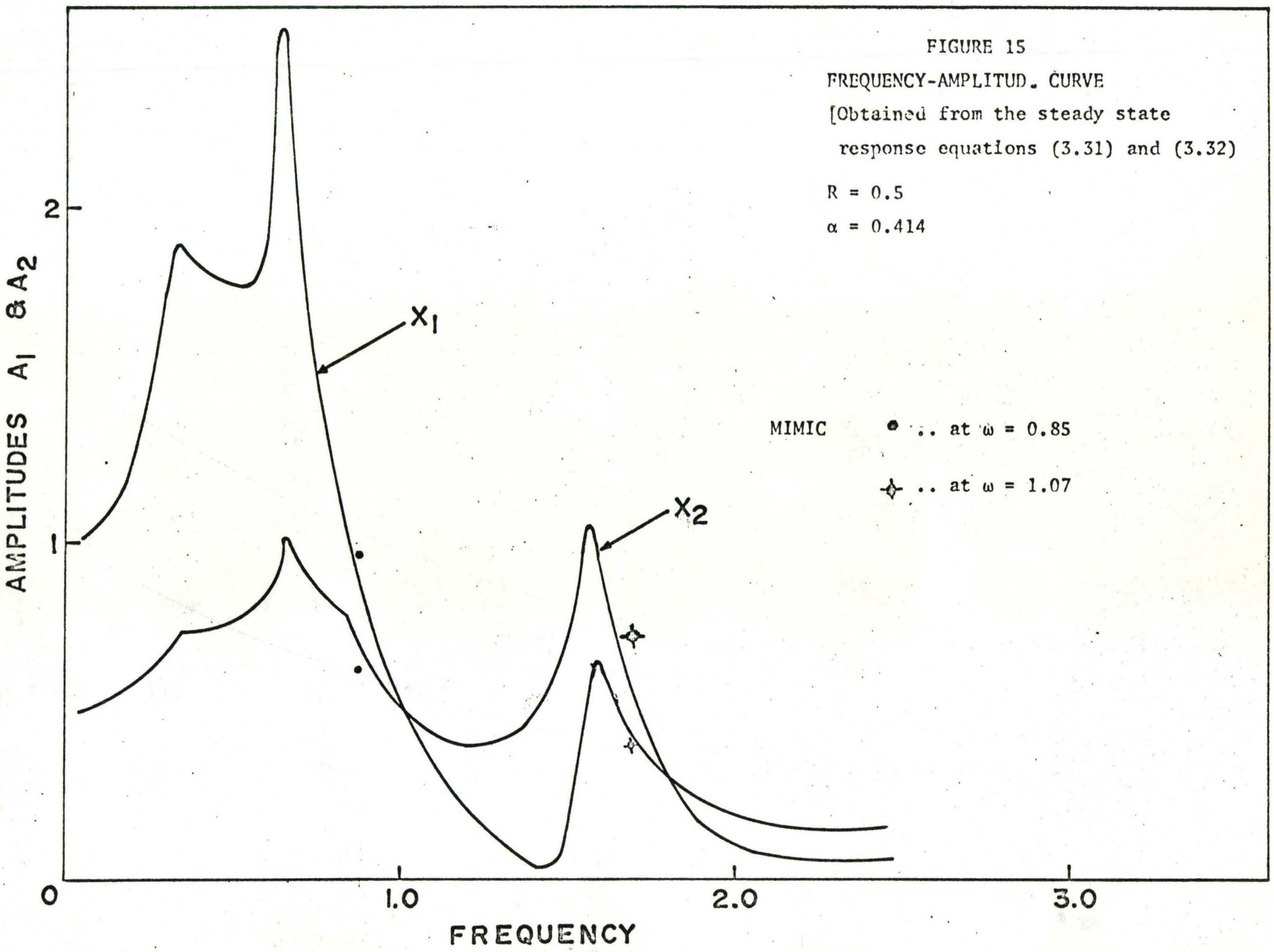


FIGURE 15

FREQUENCY-AMPLITUD. CURVE

[Obtained from the steady state response equations (3.31) and (3.32)

R = 0.5

α = 0.414

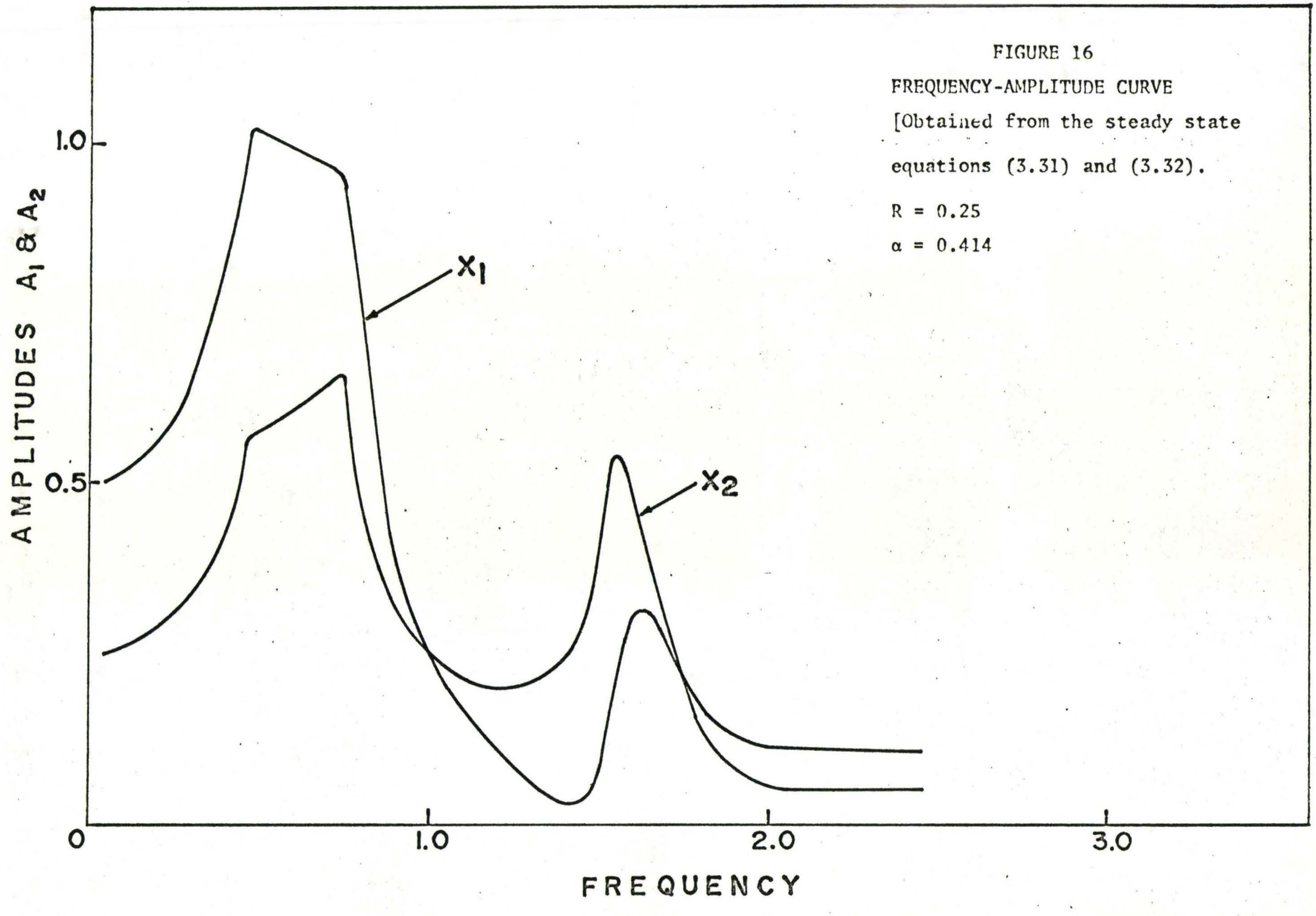


FIGURE 16

FREQUENCY-AMPLITUDE CURVE

[Obtained from the steady state equations (3.31) and (3.32).

R = 0.25

α = 0.414

Now the equations (3.40) with the help of (3.41), (3.42), (3.43) and (3.44), may be written as

$$\begin{aligned}
&-A_1\omega^2\pi\operatorname{Cos}\phi_1 + I_1 - I_2 - \pi R = 0 && \text{(a)}\\
&-A_1\omega^2\pi\operatorname{Sin}\phi_1 + J_1 - J_2 = 0 && \text{(b)}\\
&-A_2\omega^2\pi\operatorname{Cos}(\phi_1+\phi_2) + 2I - I_1 = 0 && \text{(c)}\\
&-A_2\omega^2\pi\operatorname{Sin}(\phi_1+\phi_2) + 2J_2 - J_1 = 0 && \text{(d)}
\end{aligned} \qquad (3.45)$$

By suitable algebraic manipulations;

$$\begin{aligned}
&F \equiv -2A_1\omega^2\pi\operatorname{Cos}\phi_1 - A_2\omega^2\pi\operatorname{Cos}(\phi_1+\phi_2) + I_1 - 2\pi R = 0 && \text{(a)}\\
&G \equiv -2A_1\omega^2\pi\operatorname{Sin}\phi_1 - A_2\omega^2\pi\operatorname{Sin}(\phi_1+\phi_2) + J_1 = 0 && \text{(b)}\\
&f \equiv -A_1\omega^2\pi\operatorname{Cos}\phi_1 - A_2\omega^2\pi\operatorname{Cos}(\phi_1+\phi_2) + I_2 - \pi R = 0 && \text{(c)}\\
&g \equiv -A_1\omega^2\pi\operatorname{Sin}\phi_1 - A_2\omega^2\pi\operatorname{Sin}(\phi_1+\phi_2) + J_2 = 0 && \text{(d)}
\end{aligned} \qquad (3.46)$$

Let $(\phi_1+\phi_2) = \mu_1$, then,

$$\begin{aligned}
&(F\operatorname{Cos}\phi_1 + G\operatorname{Sin}\phi_1) \equiv -2A_1\omega^2\pi - A_2\omega^2\pi\operatorname{Cos}\phi_2 + \pi C(A_1)\\
&\qquad - 2\pi R\operatorname{Cos}\phi_1 = 0 && \text{(a)}\\
&(F\operatorname{Sin}\phi_1 - G\operatorname{Cos}\phi_1) = A_2\omega^2\pi\operatorname{Sin}\phi_2 - \pi S(A_1)\\
&\qquad - 2\pi R\operatorname{Sin}\phi_1 = 0 && \text{(b)}\\
&(f\operatorname{Cos}\mu_1 + g\operatorname{Sin}\mu_1) = -A_1\omega^2\pi\operatorname{Cos}\phi_2 - A_2\omega^2\pi\\
&\qquad + \pi C(A_2) - \pi R\operatorname{Cos}\mu_1 = 0 && \text{(c)}\\
&(f\operatorname{Sin}\mu_1 - g\operatorname{Cos}\mu_1) = A_2\omega^2\pi\operatorname{Sin}\phi_2 - \pi S(A_2)\\
&\qquad - \pi R\operatorname{Sin}\mu_1 = 0 && \text{(d)}
\end{aligned} \qquad (3.47)$$

where $C(A_1)$, $C(A_2)$, $S(A_1)$, $S(A_2)$ have the same meaning and values as explained in subsection IIIB.

A few more steps and manipulations similar to these will now result in the same quations as obtained by the "method of slowly varying parameters", i.e. equations (3.31) and (3.32). The response curves have been shown in Figures (10), (12), (14), (15) and (16) with different values of the parameters α and R using "Gradient Method" of iteration [Sec. IIIB.] In all the cases which have been examined with the given parameters, no tendency for the "jump phenomena"have been observed. One extra hump near the first resonance has been found which tends to disappear subsequently as the amplitude of external excitation R is decreased. The frequency $\sim$ phase angle characteristics have been shown in Figures (11) and (13) corresponding to the response curves (10) and (12).

IIID. SOME APPROXIMATIONS

Approximate values of $C_i(A_i)$ and $S_i(A_i)$ for large A_i

Here we shall find approximate expressions for $C_i(A_i)$ and $S_i(A_i)$ for large values of A_i. Here $\psi_1^{(i)}$ and $\psi_2^{(i)}$ denote the phase angles for the motion of the two masses M_1 and M_2 (i = 1,2 corresponding to the subscript 1 and 2 for the masses M_1 and M_2) and ψ_1 and ψ_2 have the same meaning as explained in Figure (2), viz.

$$\psi_1^{(i)} = \cos^{-1}\left(\frac{A_i-1}{A_i}\right)$$

$$\psi_2^{(i)} = \cos^{-1}(-1/A_i)$$

Expanding $\psi^{(i)}$ in terms of $\sin\psi^{(i)}$ [17] we get

$$\psi^{(i)} = \sin\psi^{(i)} + \frac{1}{6}\sin^3\psi^{(i)} + \cdots \tag{3.48}$$

Thus

$$\sin^2\psi_1^{(i)} = 1 - \cos^2\psi_1^{(i)} = (2A_i - 1)/A_i^2 \tag{3.49}$$

substituting in (3.48) -

$$\psi_1^{(i)} = (2A_i-1)^{1/2}/A_i + \frac{1}{6}(2A_i-1)^{3/2}/A_i^3 + \cdots$$

which can be reduced to

$$\psi_1^{(i)} = \frac{\sqrt{2}}{A_i^{1/2}} + \frac{\sqrt{2}}{12}\frac{1}{A_i^{3/2}} + O\left(1/A_i^{5/2}\right) \tag{3.50}$$

For obtaining the expression for $\psi_2^{(i)}$ we use the Maclaurin's series -

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \cdots \tag{3.51}$$

From which we get -

$$\psi_2^{(i)} = \frac{\pi}{2} + \frac{1}{A_i} + O\left(\frac{1}{A_i^3}\right) \tag{3.52}$$

Now from equations (3.21) we can derive the relations

$$C_i(A_i) = \frac{1-\alpha}{\pi}\left[2 - \frac{7\sqrt{2}}{6A_i^{1/2}}\right] + \alpha A_i \tag{3.53}$$

$$S_i(A_i) = -\frac{2}{\pi}(1-\alpha)\left[1 - \frac{1}{A_i}\right] \tag{3.54}$$

Approximations of $C_i(A_i)$ and $S_i(A_i)$ for $A_i = 1 + \epsilon$ where $\epsilon \ll 1$

For the case when $A_i = 1 + \epsilon$ where $\epsilon \ll 1$.
ψ may be expanded in the form:

$$\psi = \pi - \left(\sin\psi + \frac{1}{6}\sin^3\psi + \cdots\right) \tag{3.55}$$

$$\psi_1^{(i)} = \cos^{-1}\left(\frac{A_i - 1}{A_i}\right) = \cos^{-1}\left(\frac{\epsilon}{1+\epsilon}\right) \tag{3.55a}$$

$$\sin\psi_1^{(i)} = \left[1 - \frac{\epsilon^2}{(1+\epsilon)^2}\right]^{1/2}$$

or

$$\sin\psi_1^{(i)} = \left(1 - \frac{\epsilon^2}{2}\right) + o(\epsilon^3) \tag{3.56}$$

$$\sin\psi_2^{(i)} = \left[1 - \frac{1}{(1+\epsilon)^2}\right]^{1/2}$$

from which

$$\sin\psi_2^{(i)} = \sqrt{2}\,\epsilon^{1/2} - \frac{3\sqrt{2}}{4}\epsilon^{3/2} + o\left(\epsilon^{5/2}\right) \tag{3.57}$$

Using the Mclaurin's Expression for ψ and equation (3.55a)

$$\psi_1^{(i)} = \frac{\pi}{2} + \left(\frac{\epsilon}{1+\epsilon}\right) + o\left(\frac{\epsilon}{1+\epsilon}\right)^3$$

$$\therefore \quad \psi_1^{(i)} = \frac{\pi}{2} + \epsilon - \epsilon^2 + o(\epsilon^3) \tag{3.58}$$

Using the expression (3.55)

$$\psi_2^{(i)} = \pi - \sqrt{2}\,\epsilon^{1/2} - \frac{11}{4}\sqrt{2}\,\epsilon^{3/2} + o\left(\epsilon^{5/2}\right) \tag{3.59}$$

$$\sin 2\psi_1^{(i)} = 2\sin\psi_1^{(i)}\cos\psi_1^{(i)}$$

$$= 2\epsilon(1-\epsilon) + o(\epsilon^3) \tag{3.60}$$

$$\sin 2\psi_2^{(i)} = 2\sin\psi_2^{(i)}\cos\psi_2^{(i)}$$

$$\sin 2\psi_2^{(i)} = -2\sqrt{2}\,\epsilon^{1/2}(1-\tfrac{7}{4}\epsilon) + o(\epsilon^{5/2}) \tag{3.61}$$

From the expression for $C_i(A_i)$

$$C_i(A_i) = (1+\epsilon)\left[1 - \frac{9\sqrt{2}}{2\pi}\epsilon^{1/2}(1-\alpha)\right] + o(\epsilon^3) \tag{3.62}$$

Similarly -

$$S_i(A_i) = -\frac{2}{\pi}(1-\alpha)\epsilon(1-\epsilon) + o(\epsilon^3) \tag{3.63}$$

IIIE. Large Amplitude Steady State Behaviour

Let the steady state amplitudes A_1 and A_2 both become very large but with the following restraint -

$$N = A_1/A_2 \tag{3.64}$$

Now from the equation (3.32) -

$$\omega^2 = 2C_2/A_2 \mp \left[C_1^2 + S_1^2 - 4S_2^2\right]^{1/2}/A_2 \tag{3.65}$$

Using equations (3.53), (3.54) and (2.22) and designating $R_{critical}$ as δ where

$$\delta = 2(1-\alpha)/\pi \tag{3.66}$$

$$\therefore\ \omega^2 = \delta\left[\frac{2}{A_2} - \frac{7\sqrt{2}}{6}/A_2^{3/2}\right] + \alpha \mp \left\{\delta\left(1 - \frac{7\sqrt{2}}{12}/A_1^{1/2}\right) + \alpha A_1\right\}$$

$$\left[1 + \frac{\delta^2\left(1-\frac{1}{A_1}\right)^2 - 4\delta^2\left(1-1/A_2\right)^2}{\left\{\delta\left(1-\frac{7\sqrt{2}}{12}/A_1^{1/2}\right) + \alpha A_1\right\}^2}\right]^{1/2}/A_2$$

Eliminating A_1 using (3.64), we get

$$\omega^2 = \alpha(1 \mp N) + \delta(2 \mp 1)\frac{1}{A_2} - \delta\frac{7\sqrt{2}}{6}\left(1 \mp \frac{1}{N^{1/2}}\right)\frac{1}{A_2^{3/2}}$$

$$\pm \frac{3\delta^2}{2\alpha N}(1-\delta)\frac{1}{A_2^2} + o\left(\frac{1}{A_2^{5/2}}\right) \qquad (3.67)$$

Also from (3.31), we have

$$R^2 = \left(-\omega^2 A_1 + C_1/2\right)^2 + \left(-\omega^2 A_2 + 2C_2\right)\left[\frac{\omega^4 A_1 A_2 C_1}{C_1^2 + S_1^2}\right.$$

$$\left. - \frac{\omega^2 A_2}{2}\right] + \omega^4 A_1 A_2 \frac{2S_1 S_2}{C_1^2 + S_1^2} + \frac{\omega^4 A_2^2}{4} + \frac{S_1^2}{4}$$

which yields the following approximate value

$$R^2 = A_2^2\left[N^2\left(-\omega^2 + \frac{\alpha}{2}\right)^2 + \frac{\omega^2}{2}\left(2\frac{\omega^2}{\alpha} - 1\right)\left(2\alpha - \omega^2\right) + \frac{\omega^4}{4}\right]$$

$$+ A_2\delta N\left(-\omega^2 + \frac{\alpha}{2}\right) + \left(1 + \frac{4\omega^4}{\alpha^2 N}\right)\frac{\delta^2}{2} + o\left(\frac{1}{A_2^{1/2}}\right) \qquad (3.68)$$

Now taking the same approximation $o\left(\frac{1}{A_2^{1/2}}\right)$ in equation (3.67)

$$\omega^2 \simeq \alpha(1 \mp N) \qquad (3.69)$$

Substituting this in (3.68)

$$R^2 = \frac{\alpha^2 A_2^2}{4}\left[4N^4 \mp 6N + 3\right] - \frac{A_2\delta N\alpha}{2}(1 \mp 2N)$$

$$+ \frac{\delta^2}{2}\left(4N + 1 \mp 8 + \frac{4}{N}\right) + o\left(\frac{1}{A_2^{1/2}}\right) \qquad (3.70)$$

For finite excitation R_{cr} there can be unbounded resonance if and only if the coefficients of A_2^2 and A_2 vanish identically, i.e.

$$4N^4 \mp 6N + 3 = 0 \qquad (3.71)$$

$$1 \mp 2N = 0 \qquad (3.72)$$

Equation (3.72) gives

$$N = \pm \frac{1}{2} \qquad (3.73)$$

Within the limits of the accuracy involved in the approximations, this value of N seems to satisfy the equation (3.71) approximately.

In that case:

$$R_{critical} = \frac{\delta}{\sqrt{2}}\left(4N + 1 \mp 8 + \frac{4}{N}\right)^{1/2} \qquad (3.74)$$

From (3.69) we obtain

$$\omega^2 = \frac{\alpha}{2}\left(2 \mp N\right) \qquad (3.75)$$

This analysis illustrates the possibility of two unbounded resonances which may occur for finite value of the excitation amplitude R given by (3.74).

IIIF. Stability of the Steady State Solution

Giving infinitesimal perturbations ξ_i, ψ_i to the stable amplitudes and phase differences, A_i^0 and ϕ_i^0 respectively,

$$\begin{aligned} A_i &= A_i^0 + \xi_i \\ \phi_i &= \phi_i^0 + \psi_i \end{aligned} \qquad i = 1,2 \qquad (3.76)$$

It may be shown [2] using equations (3.17) through (3.20) and (3.24) through (3.27) that:

$$-2\omega\dot{\xi}_1 + S_1'\xi_1 - \xi_2\left(C_2'\sin\phi_2 + S_2'\cos\phi_2\right) - \left(\omega^2 A_1 - C_1\right)\psi_1$$
$$- \left(\psi_1 + \psi_2\right)\left(C_2\cos\phi_2 - S_2\sin\phi_2\right) = 0$$

$$-\omega^2\xi_1 + 2\omega A_1\dot{\psi}_1 + C_1'\xi_1 - \xi_2\left(C_2'\cos\phi_2 - S_2'\sin\phi_2\right) - \psi_1 S_1$$
$$+ \left(\psi_1 + \psi_2\right)\left(C_2\sin\phi_2 + S_2\cos\phi_2\right) = 0$$

(3.77)

$$-2\omega\dot{\xi}_2 + 2S_2'\xi_2 + \xi_1\left(C_1'\sin\phi_2 - S_1'\cos\phi_2\right) + \psi_2\left(-\omega^2 A_2 + 2C_2\right) = 0$$

$$-\omega^2\xi_2 + 2\omega A_2\left(\dot{\psi}_1 + \dot{\psi}_2\right) + 2C_2'\xi_2 - \xi_1\left(C_1'\cos\phi_2 + S_1'\sin\phi_2\right)$$
$$-2\psi_2 S_2 = 0$$

where $C_i' = \left.\dfrac{\partial C_i}{\partial A_i}\right]_{A_i = A_i^\circ}$

$i = 1,2$ (3.78)

and $S_i' = \left.\dfrac{\partial S_i}{\partial A_i}\right]_{A_i = A_i^\circ}$

Assuming the infinitesimal perturbations to be time variants

$$\xi_i = \bar{\xi}_i\, e^{\lambda t}$$

(3.79)

$$\psi_i = \bar{\psi}_i\, e^{\lambda t}$$

With the above assumptions, it has been shown in reference [2] for a very general case that the frequency equation is given by

$$a_0 = \frac{1}{A_1A_2}\Big[\!\Big[-S_1'\left[4S_1'S_2 I - 4S_1S_2S_1' - S_1(\omega^2A_2 - 2C_2)(\omega^2 - 2C_2') + (\omega^2A_2 - 2C_2)(\omega^2 - 2C_2')I\right] - K\left[2S_2LI - 2S_1S_2L - S_1(\omega^2A_2 - 2C_2)N + (\omega^2A_2 - 2C_2)NI\right] - (C_1 - \omega^2A_1)\left[L\{2S_2J - (\omega^2 - 2C_2')I\} - 2S_2'\{2S_2(-\omega^2 + C_1') - NI\} + (-\omega^2A_2 + 2C_2)\{(-\omega^2 + C_1')(\omega^2 - 2C_2') - NJ\}\right] - M\left[L\{S_1(\omega^2 - 2C_2') - 2S_2J\} - 2S_2'\{S_1N - 2S_2(-\omega^2 + C_1')\} + (\omega^2A_2 - 2C_2)\{(-\omega^2 + C_1')(\omega^2 - 2C_2') - NJ\}\right]\Big]\!\Big] \tag{3.81}$$

$$a_1 = \frac{1}{A_1A_2}\Big[\!\Big[-S_1'\left[4A_1S_2S_2' + 2A_2S_1S_2' - 2S_2I + 2S_1S_2 + (\omega^2A_2 - 2C_2)\{A_2J + A_1(\omega^2 - 2C_2')\}\right] + 4S_2'S_2I - 4S_1S_2S_2' - S_1(\omega^2A_2 - 2C_2)(\omega^2 - 2C_2') + (\omega^2A_2 - 2C_2)(\omega^2 - 2C_2')I - K\left[L(2A_1S_2 + S_1A_2) + (\omega^2A_2 - 2C_2)\{A_1N + A_2(-\omega^2 + C_1')\}\right] - (C_1 - \omega^2A_1)\left[-A_2LJ + 2S_2'A_2(-\omega^2 + C_1') + 2S_2(-\omega^2 + C_1') - NI\right] - M\left[-A_1L(\omega^2 - 2C_2') - 2S_2(-\omega^2 + C_1') + (S_1 + 2A_1S_2')N\right]\Big]\!\Big] \tag{3.82}$$

$$a_2 = \frac{1}{A_1A_2}\Big[-S_1'(2A_1'S_2 - A_2S_1 - 2A_1A_2S_2') + 4A_1S_2S_2' + 2A_2S_1S_1'$$
$$-2S_2I + 2S_1S_2 + A_2(C_1 - \omega^2A_1)(-\omega^2 + C_1') + (\omega^2A_2 - 2C_2)$$
$$\{A_2J + A_1(\omega^2 - 2C_2')\} + A_1A_2KL + A_1MN\Big] \quad (3.83)$$

$$a_3 = \frac{1}{A_1A_2}\Big[-2A_1S_2 - 2A_1A_2S_2' - A_2S_1 - A_1A_2S_1'\Big] \quad (3.84)$$

where

$$\begin{aligned} I &= C_2 \sin\phi_2 + S_2 \cos\phi_2 \\ J &= S_2' \sin\phi_2 - C_2' \cos\phi_2 \\ K &= C_2' \sin\phi_2 + S_2' \cos\phi_2 \\ L &= C_1' \sin\phi_2 - S_1' \cos\phi_2 \\ M &= C_2 \cos\phi_2 - S_2 \sin\phi_2 \\ N &= C_1' \cos\phi_2 + S_1' \sin\phi_2 \end{aligned} \quad (3.84a)$$

(Note: Equations (3.80) through (3.84) have already been derived in Reference[2]in detail, and are very general in nature up to this point, hence without going into the derivations, these expressions will directly be used for the forthcoming analysis.)

According to Routh-Hurwitz stability criteria, the coefficients a_j ($j = 0,1,2,3$) of the frequency equation (3.80) must satisfy the conditions as specified in Chapter II. Using the conditions for stability;

$$a_j > 0, \quad j = 0,1,2,3 \qquad (3.85)$$

and

$$a_1 a_2 a_3 > a_1^2 + a_0 a_3^2 \qquad (3.86)$$

The criteria is simple, but the complicated expressions for a_0, a_1, a_2 and a_3 hinders any direct and general approach to the solution. Therefore, examination shall be made of only two limiting cases in which much simplifications are possible.

Stability of Small Amplitude Oscillations

Under the circumstances when $A_1, A_2 > 1$ the double-bilinear hysteretic system reduces to a simple undamped linear system where the oscillation is simple harmonic and the steady state solutions are marginally stable. But since this does not reveal the typical nature of the double - bilinear system behaviour, the following cases are now considered -

(i) $A_2 < 1$, $A_1 = 1 + \epsilon$, where $\epsilon \ll 1$

Then

$$C_2/A_2 = 1 \qquad (a)$$

$$C_2' = 1 \qquad (b) \quad (3.87)$$

$$S_2 = S_2' = 0 \qquad (c)$$

From equations (3.62), (3.63):

$$C_1 = (1+\epsilon)\left[1 - \frac{9\sqrt{2}}{2\pi}\epsilon^{1/2}(1-\alpha)\right]$$

$$\approx (1+\epsilon) - \frac{9\sqrt{2}}{2\pi}(1-\alpha)\epsilon^{1/2} + 0(\epsilon^{3/2})$$

$$C_1 = (1+\epsilon) - \frac{9}{4}\sqrt{2}\,\mu\,\epsilon^{1/2} \qquad \text{(d)}$$

$$S_1 = -\frac{2}{\pi}(1-\alpha)\epsilon + o(\epsilon^2) = -\mu\epsilon \qquad \text{(e)}$$

$$S_1' = -\frac{2}{\pi}(1-\alpha)\frac{1}{(1+\epsilon)^2} = -\mu(1-2\epsilon) + o(\epsilon^2) \qquad \text{(f)}$$

$$C_1' = -\mu\left(-\epsilon + 1 + \frac{13}{4}\sqrt{2}\,\epsilon^{1/2}\right) + 1 + o(\epsilon^{3/2}) \qquad \text{(g)}$$

Under these conditions

$$a_3 = -\frac{1}{A_1 A_2}(A_2 S_1 + A_1 A_2 S_1')$$

$$= -\frac{S_1}{A_1} - S_1' = \frac{\delta\epsilon}{1+\epsilon} + \delta(1-2\epsilon)$$

$$= \delta\epsilon + \delta(1-2\epsilon) = \delta(1-\epsilon) > 0 \qquad (3.88)$$

$$a_2 = \frac{1}{A_1 A_2}\Big[A_2 S_1 S_1' + A_2(C_1 - \omega^2 A_1)(C_1' - \omega^2) + (\omega^2 A_2 - 2C_2)\{A_2 J + A_1(\omega^2 - 2C_2')\} + A_1 A_2 KL + A_1 MN\Big]$$

where $J = -\cos\phi_2 = C_1 A_2(-\omega^2 + 2)/(C_1^2 + S_1^2)$

$$KL = \sin\phi_2\,(C_1'\sin\phi_2 - S_1'\cos\phi_2)$$

$$MN = A_2\cos\phi_2\,(C_1'\cos\phi_2 + S_1'\sin\phi_2)$$

$$\omega^2 = 2 \mp 1/A_2 = 2 \mp N/A_1 \approx (2 \mp N) \qquad (3.89)$$

Hence -

$$a_2 = (1-\omega^2)(1-\delta-\omega^2) + (\omega^2-2)\left[A_2^2(2-\omega^2) + \omega^2 - 2\right] + (1-\delta)$$

Substituting for ω^2 from (3.89)

$$a_2 = 2N^2 \pm N(2-\delta) + (1-2\delta) \tag{3.90}$$

when the positive sign is implied $a_2 > 0$. For the negative sign, $a_2 > 0$, for $\delta \leq 1$. But for $\delta \nleq 1$, a_2 may be less than zero for some values of δ and N.

$$a_1 = \frac{1}{A_1 A_2}\Big[-S_1'(\omega^2 A_2 - 2C_2)\{A_2 J + A_1(\omega^2 - 2C_2')\}$$
$$- S_1(\omega^2 A_2 - 2C_2)(\omega^2 - 2C_2') + (\omega^2 A_2 - 2C_2)(\omega^2 - 2C_2')I$$
$$- K\{L A_2 S_1 + (\omega^2 A_2 - 2C_2)(A_1 N + A_2(-\omega^2 + C_1'))\}$$
$$- (C_1 - \omega^2 A_1)\{-A_2 J L - N I\}$$
$$- M\{-A_1 L(\omega^2 - 2C_2') + S_1 N\}\Big]$$

which reduces to:

$$a_1 = \delta\left[N^2 \mp 2N \pm 1 + 1\right] + O(\epsilon) \tag{3.91}$$

which is always positive for the present case ($N > 1, \delta < 1$)

Similarly -

$$a_0 = N^4 \mp N^3(2 + \delta) + N^2(\delta \mp 1) + N(1 \pm 1 \mp \delta + \delta)$$
$$\pm (1 - \delta) + O(\epsilon) \tag{3.92}$$

which is also positive for all $N > 1$ and $\delta < 1$.

Thus the first criterion (3.85) is satisfied for all $N > 1$ and all $\delta < 1$. For $\omega^2 \approx (2-N)$, it has been found that the criterion (3.86) viz.

$$a_1 a_2 a_3 > a_1^2 + a_0 a_3^2$$

is satisfied, and hence the system is stable. But for $\omega^2 \approx (2+N)$ although the condition (3.85) is satisfied, but (3.86) is not, hence the system is either marginally stable or unstable.

(ii) <u>when $A_1 \leq 1$ and $A_2 = (1 + \epsilon)$ where $\epsilon \ll 1$</u>

Then $C_1 = A_1$, $C_1' = 1$

$$S_1 = S_1' = 0, \quad S_2 = -\delta\epsilon, \quad S_2' = -\delta(1-2\epsilon)$$

$$C_2 = 1 + \epsilon - \frac{9}{4}\sqrt{2}\,\delta\,\epsilon^{1/2} \tag{3.93}$$

$$C_2' = -\delta(1 - \epsilon + \frac{13}{4}\sqrt{2}\,\epsilon^{1/2}) + 1$$

Using the expression (3.84)

$$a_3 = \frac{1}{A_1 A_2}\left(-2A_1 S_2 - 2A_1 A_2 S_2'\right)$$

$$= 2\left(\frac{\delta\epsilon}{A_2} + \delta(1-2\epsilon)\right)$$

$$\approx 2\delta + O(\epsilon) \tag{3.94}$$

Again from expression (3.88)

$$a_2 = N^2(1\mp 1) + 2N(\delta \mp 1) + (2 \pm 1 \mp \delta) \tag{3.95}$$

Here $N < 1$, hence for $\delta \leq 1$, a_2 is always positive.

From equation (3.82) -

$$a_1 = 2\delta N^2 + \delta N(\mp 5 - 1) + (4\delta \pm 1) \tag{3.96}$$

which is again found to be positive for all $N < 1$, $\delta < 1$.

Also from (3.81) -

$$a_0 = N^4 \mp 2N^3(1+\delta) + N^2(5\delta \mp 1) + N(\mp 3\delta \pm 1 + 1) \pm (1-\delta) \tag{3.97}$$

This term can also be seen to be positive for all $N < 1$, $\delta \leq 1$.

For $\omega^2 = 2 - N$,

$$P_1 \equiv [a_1 a_2 a_3 - (a_1^2 + a_0 a_3^2)] = 8\delta^2 N^3(3-N) - 16\delta^3 N^2(3-N) - 4\delta^2 N(2+3N) + 4\delta^3(10N-1) - 2\delta(1-\delta) + 4\delta N(3-N) - 1. \tag{3.98}$$

It can be seen from this expression that $P_1 > 0$ for $\delta \approx 1$, $N \approx 1$, the system is stable. For $\delta \ll 1$, $N \approx 1$ or $\delta \approx 1$, $N \ll 1$ or $\delta \ll 1$, $N \ll 1$, in all the cases the system is either marginally stable or unstable.

For $\omega^2 = 2 + N$,

$$P_2 \equiv [a_1 a_2 a_3 - (a_1^2 + a_0 a_3^2)] = 8\delta^2 N^3(\delta + 4) - 4\delta N^4(1-\delta) + 8\delta^2 N^2(4-\delta) + 4\delta^3(3N+1) + 2\delta^2(26N-3) + 6\delta(1-2N) + (6\delta - 1) \tag{3.99}$$

which can be seen to be positive for all values of $N < 1$, $\delta < 1$.

Hence the system is completely stable for $\omega^2 = 2 + N$.

Thus the above analysis gives the stability criteria for the system for ($A_1 = 1 + \epsilon$ and $A_2 < 1$) and ($A_2 = 1 + \epsilon$ and $A_1 < 1$) where ϵ is infinitesimal.

IV NUMERICAL INTEGRATION AND DIGITAL-ANALOG SIMULATION OF THE SYSTEM EQUATIONS OF MOTION

IVA NUMERICAL INTEGRATION

Equations (2.1) for the system with hysteretic damping only, (2.42) for the system with added viscous damping and equations (3.1) and (3.2) for the two degrees of freedom system were programmed for numerical integration using fourth-order Runge-Kutta method [13] . The subroutine used was "CALL RKG" on IBM-7040. A sample programme is attached in the appendix which was made for the equation (2.1), other programmes being exactly on the same pattern. To minimize rounding off and other inherent errors, the interval was chosen to be very small in the time domain ($\Delta t = 0.01$). The starting point was taken to be

$$(x)_{t=0} = 0$$

$$\left(\frac{dx}{dt}\right)_{t=0} = 0$$

The approach towards a steady state value was very-very slow, hence only a few points were tried to test the approximate results. In all the cases the numerical results were found to be quite close to but slightly higher than the analytical results.

IVB. DIGITAL-ANALOG SIMULATION

There are several methods which have been developed recently for Analog Simulation using Digital Computer. Among these are MIDAS[14], PACTOLUS[14], MIMIC [15] etc. Essentially they all consist of functional blocks eg. integrators, summers,

BLOCK DIAGRAM FOR DIGITAL-ANALOGUE SIMULATION

two degrees of freedom system

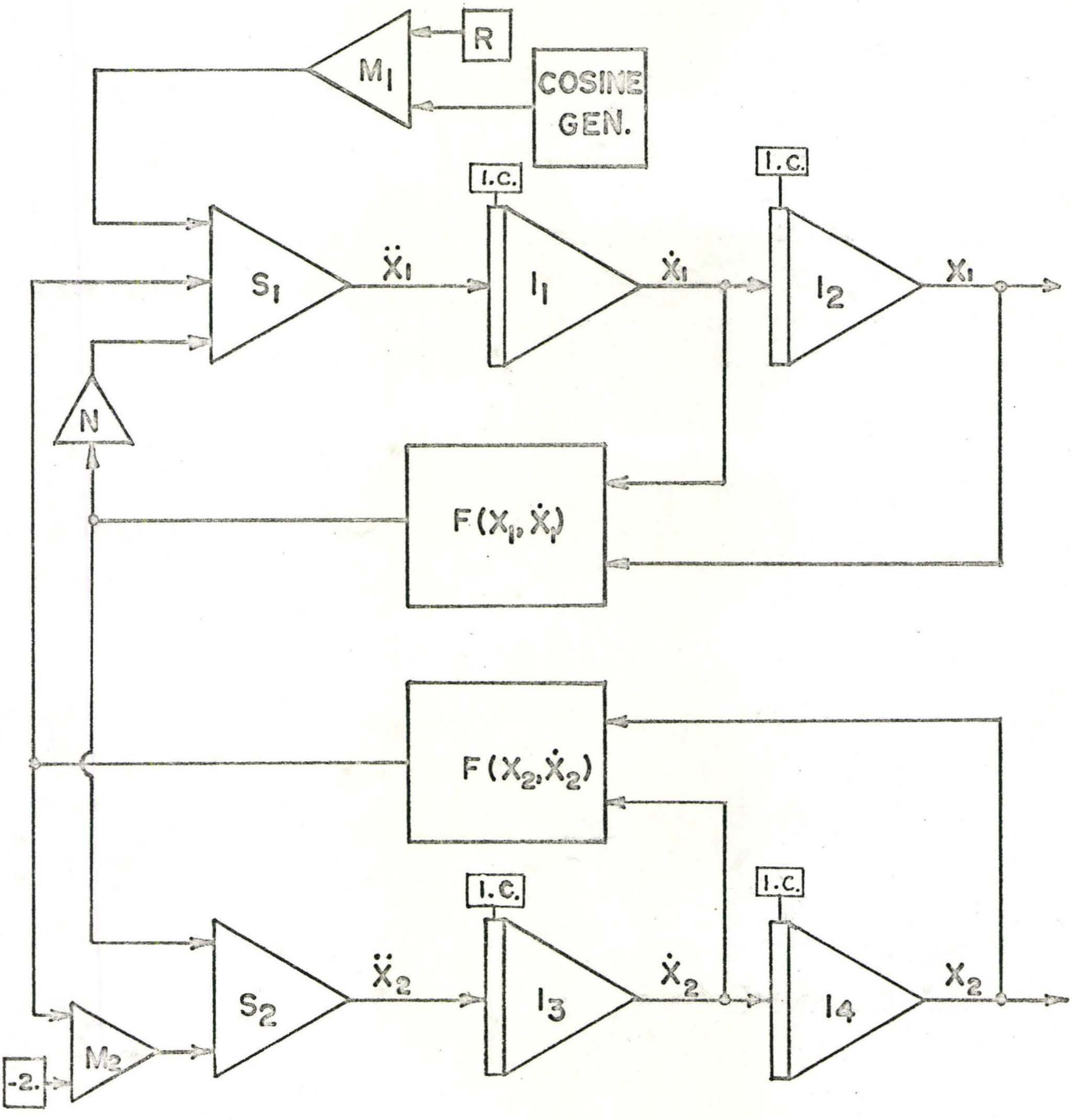
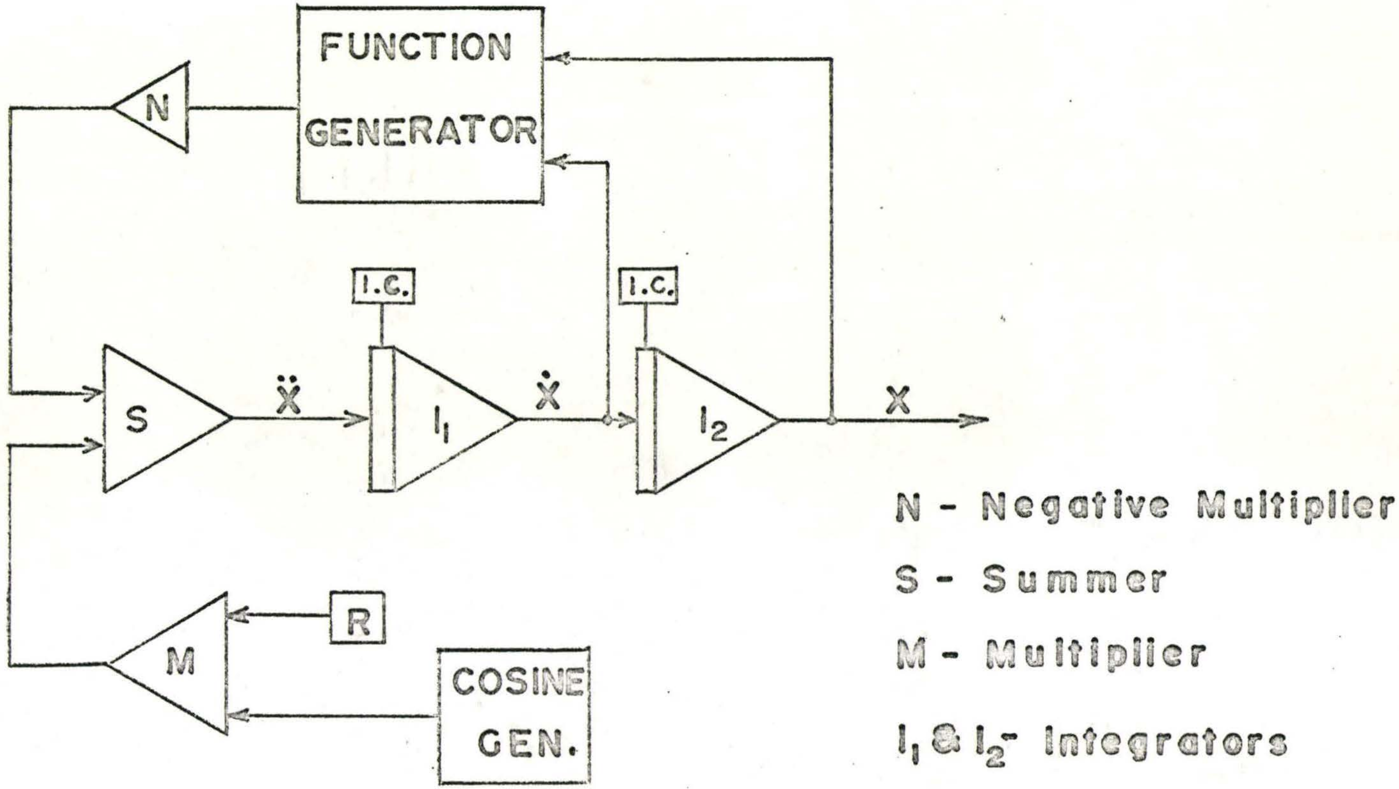


FIGURE. 17

BLOCK DIAGRAM FOR DIGITAL-ANALOGUE SIMULATION

single degree of freedom system

FIGURE.18



multipliers etc. and they all have translating and processing decks or tapes which must be put along with the programme. The programming is started by making block diagrams as in the case of Electronic Analog Computation, and then writing up in a language special to the system described above. As in the case of Analog Computer the functional units or blocks must be interconnected, or patched, however, with a digital-analog simulation patching is accomplished by a sequence of interconnection statements, which are punched on cards rather than actual wiring. It is convenient in the sense that once the cards are punched, the programme can be re-run with different initial conditions and parameters without the repatching and testing inconvenience usually encountered with the Analog Computers. Also the "scaling" problem is completely eliminated. The output is in terms of digital printouts which can be plotted to get the desired curve. MIMIC has got some hybrid elements like Monostable Multi-Vibrator (MMV), Track and store (TAS), Flip-Flop (FLF), Zero Order Hold (ZOH) etc. as special functional blocks. This makes the simulation more convenient. PACTOLUS is similar to MIDAS, but is almost obsolete now due to the improved features of MIMIC and MIDAS.

For the present problem, MIMIC was more useful for the reason that a hybrid element "Track and Store" (TAS) had to be used to generate the piecewise linear function as shown in Figure (1). An explanation of the logical variables and the various elements are given in the Appendix along with the sample programmes.

A comparison between the results obtained from the MIMIC and Numerical integration by fourth order Runge-Kutta method shows that they tally remarkably well up to third place of decimal. The slight difference is due to the greater accuracy of the MIMIC execution and output. Although for single degree of freedom system Runge-Kutta and MIMIC both can be programmed with the same ease, but for two and higher degrees of freedom system, MIMIC is much easier to programme.

IVC HARMONIC ANALYSIS

(1) For the single degree of freedom system, let us consider the Fourier solution of the form

$$x(t) = \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

near the steady state. It is now possible to calculate the coefficients a_n and b_n if we know the numerical values of x(t) at small equal intervals of time for one cycle near the steady state. From the solution obtained by the MIMIC at time intervals 0.01 the various values of x(t) for one cycle were put as data along with the programme for finding the Fourier coefficients. A sample programme is attached in the appendix which can determine up to twenty coefficients. More numbers of coefficients can be determined simply be increasing the upper limit of J in the DO Loop. The magnitude of the coefficients for particular values of α , R and frequency is given in Table I.

TABLE I

FOURIER COEFFICIENTS

$\alpha = 0.7$, $R = 0.3$, Frequency $\omega = 1.1$

SINGLE DEGREE OF FREEDOM SYSTEM

$a_0 = -0.0003968659$	
$a_1 = 1.361404000$	$b_1 = 0.01741116$
$a_2 = -0.008708884$	$b_2 = 0.01196286$
$a_3 = 0.005930529$	$b_3 = 0.004271753$
$a_4 = 0.001884598$	$b_4 = 0.0006426263$
$a_5 = -0.006963342$	$b_5 = 0.004897478$
$a_6 = 0.002329363$	$b_6 = 0.005619695$
$a_7 = 0.004357267$	$b_7 = -0.002224481$
$a_8 = -0.004185004$	$b_8 = 0.001494067$
$a_9 = -0.001351962$	$b_9 = 0.006989534$
$a_{10} = 0.004824445$	$b_{10} = -0.002122467$

$$X_{10} = 0.0165536$$

It can be inferred that only a_1 has significant contribution to the total amplitude. Other coefficients are too small as compared to a_1.

The harmonic content of x (steady state) is defined [2] as

$$X = \left[\frac{\sum_{n=2}^{\infty}(a_n^2 + b_n^2)}{(a_1^2 + b_1^2)} \right]^{1/2}$$

Restricting our analysis to the coefficients given in table I, we may define

$$X_{10} = \left[\frac{\sum_{n=2}^{10} (a_n^2 + b_n^2)}{(a_1^2 + b_1^2)} \right]^{1/2}$$

which gives the percentage harmonic content in the first ten harmonics with respect to the fundamental. This has been calculated and given in Table I.

(2) <u>Two Degrees of Freedom System</u>

MIMIC solutions have been obtained for two degrees of freedom system. A sample programme has been attached in the appendix. The following harmonic analysis has been done for a particular set of parameters indicated in Table II. Due to very long time taken by the computer in giving a reasonable steady state solution, only a few sets of data were tried, indicated by thick dots on the curves obtained by approximate solution.

In Table II, x_1 and x_2 are the displacements of the masses M_1 and M_2 as referred to Figure (9). The parameters at which these values have been obtained is also shown on the top of the table.

TABLE II

$\alpha = 0.8$, $R = 0.5$, $\omega = 0.7$

$$\chi_{10}(1) = 0.031036 \, , \, \chi_{10}(2) = 0.049427566$$

n	x_1 a_n	b_n	x_2 a_n	b_n
o	-0.07605551		-0.04681889	
1	1.1732217	0.02250781	1.081294	0.05960166
2	0.01903797	0.03802776	-0.05058914	-0.017025
3	-0.001984013	0.01783301	-0.00114383	-0.001816730
4	-0.002484549	0.01399160	0.0002336187	-0.0009194572
5	-0.001999055	0.01481208	0.0009830913	-0.002979263
6	-0.001243880	0.01122836	0.0005388436	-0.00002526127
7	0.0002854142	0.008862810	-0.0004799221	0.0001072911
8	0.0008288734	0.007734805	0.00006795314	-0.00001643026
9	0.0007092457	0.006301769	-0.0003185970	0.00004282973
10	0.0002236938	0.005281545	0.0001151447	-0.0001048617

where a_n and b_n are coefficients of cosine and sine terms in the Fourier series. The harmonic content $\chi(1)$ and $\chi(2)$ for the ten coefficients have been calculated separately for x_1 and x_2.

A comparison between the values of χ_{10}'s in the cases of single and two degree of freedom systems shows that the harmonic content of two degrees of freedom system is much higher than that of the single degree of freedom system for the same α, although the smoothest portion of the curve was chosen for the harmonic analysis. This high harmonic content in the ~~case of~~ two degrees of freedom case could be one of the reasons for the marked irregularities in the response curves, specially in the resonance region. On examination of Table II, it seems that second harmonics also have values of coefficients (a_2's and b_2's) much higher than those in Table I, hence they have also a marked influence on the response curves for two degrees of freedom system.

V CONCLUSION

From the previous analysis, conclusions can be drawn showing the main features of the double bilinear hysteretic systems of single and two degrees of freedom.

SINGLE DEGREE OF FREEDOM SYSTEM

For the slope $\alpha > 0.5$, the response curves have the characteristic features of a soft spring similar to the case of Bilinear hysteretic system. All the curves have a tendency of leaning towards lower frequencies with a steep slope. As the amplitude of excitation is increased the peak response occurs at a progressively lower frequency.

The system exhibits unbounded amplitude resonance at finite amplitude of excitation R greater than $2(1-\alpha)/\pi$. It should be noted that corresponding critical amplitude of excitation for the bilinear hysteretic system is $4(1-\alpha)/\pi$, which is exactly two times the value in the present case. It shows the existence of two loci of vertical tangencies for $A > 1$, but they do not exist at all for $A < 1$. This means that for $A > 1$, the curves become triple valued over certain ranges of frequency. When the response amplitude is triple valued, the values which lie within the region bounded by the two loci of vertical tangencies represents an unstable solution.

A comparison of the curves at different α's shows that the area of the unstable region increases with the decrease in α and it decreases rapidly as α tends to 1.

The system response shows "jump phenomena" for lower values of α. As the amplitude of excitation R is increased this "jump" gradually diminishes. Also for higher values of α, this phenomenon is less marked. For larger amount of viscous damping, "jump" gradually disappears, but for low values of α (≤ 0.7), μ and R, it can still be observed.

Digital-analog solution shows that the system comes to a steady-state very sluggishly if started from initial zero, zero (velocity and displacements) condition, and the harmonic content χ_{10} (up to ten harmonics) is within 2% in general for sinusoidal excitation.

A general exact solution gives nine equations in nine unknowns, all the equations being of highly transcendental nature.

TWO DEGREES OF FREEDOM SYSTEM

An approximate analysis gives response curves which are single valued within the range of investigation and are definitely asymmetric with usually a small extra peak near the first resonance. There are critical levels of excitation above which the system will exhibit unbounded amplitude resonance. For the normalised system, these critical excitation levels are $R_{cr.} = \frac{\delta}{\sqrt{2}}\left(4N-7+\frac{4}{N}\right)^{1/2}$ and $\frac{\delta}{\sqrt{2}}\left(4N+9+\frac{4}{N}\right)^{1/2}$ respectively for lower and higher frequency response peaks, where $\delta = \frac{2(1-\alpha)}{\pi}$ and $N = \frac{A_1}{A_2}$ where A_1 and A_2 may attain the values infinity but their ratio N remains finite.

Stability of the system has been examined only in certain limiting cases. For $A_1 = 1 + \epsilon$ and $A_2 < 1$, it has been found to give stable solution for $\omega^2 = (2-N)$ and $\delta << 1$, but for $\omega^2 = (2+N)$ the system is either marginally stable or unstable. Again for $A_1 < 1$, $A_2 = 1 + \epsilon$, the system is completely stable for $\omega^2 = 2+N$, but for $\omega^2 = (2-N)$ the system is stable only when $\delta \approx 1$, $N \approx 1$. Here $\epsilon << 1$ and $N = A_1/A_2$ in each of the cases.

Digital analog study shows that the system is still the more sluggish in attaining a steady state value and the time (in the domain of the equations of motion) taken is more for this case than that for the single degree of freedom system. The harmonic contents χ_{10} within 5% for the first ten harmonics.

A comparison can be made between the steady state characteristics of two degree of freedom Double Bilinear Hysteretic System and Bilinear hysteretic System [2] . On examining the response curves they show some similar features e.g. higher values of amplitude is attained at the first natural frequency and the next peak is always smaller. Also as it passes from the first resonant frequency to the other, x_1 and x_2 invariably intersect at $\omega = 1.0$ in all the cases. The distinguishing features are that with the same type of analysis in both the cases, the response curves of the double bilinear system gives one extra peak near the first resonant frequency which tends to disappear as the amplitude of excitation is decreased or as α tends to 1. No such extra peak is observed in the case of bilinear hysteretic system [2]. The extra peak in the response curves of double bilinear hysteretic systems may be attributed to the higher nonlinearity (as we have already seen in the case of the single degree of freedom system in Chapter II) and more harmonic content as compared to Bilinear hysteretic system.

Although the available data related to the dynamic response of real structures do not give any indication of jump phenomenon in the response curve, nor conform very closely to the restoring force characteristics of the double bilinear hysteretic type, the study of such a model is non-the-less interesting and significant. Interesting, because it exhibits phenomenon quite distinct from that of all other hysteretic models presented and studied so far, and significant from the view point of its lower hysteresis loss as compared to the corresponding Bilinear model for the same amplitude. However, the author feels that a general type of analysis as done by Pisarenko [12] using hysteretic curve of a very general nature might be much more useful in dealing with actual systems and structures. In Pisarenko's work [12] an analysis of oscillations of elastic systems taking into account the dissipation of energy in the material has been given, using the ideas of the theory of asymptotic expansion in non-linear mechanics. In the same work a series of important and practical problems e.g. vibration of turbine blades of constant and variable cross sections and vibrations of bars etc. has also been analyzed.

APPENDIX

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$JOB            003715 SAHAY              100   010   030
$IBJOB          NODECK
$IBFTC
C     SOLUTION OF SINGLE DEGREE OF FREEDOM DOUBLE BILINEAR HYSTERETIC SY
C     -STEM BY RITZ AVERAGING METHOD.B=ALFA,DELB=CHANGE IN ALFA,R=AMPL-
C     ITUDE OF THE APPLIED HARMONIC FORCE,DELR=CHANGE IN R,A=AMPLITUDE
C     OF THE NORMALISED SYSTEM,DELA=CHANGE IN A,X=THETA1,Y=THETA2
      READ(5,1)B,DELB,NB,R,DELR,NR,A,DELA,NA
    1 FORMAT(2F10.0,I2,2F10.0,I2,2F10.0,I2)
      DO 50 I=1,NB
      WRITE(6,2)B
    2 FORMAT(20X,6H ALFA=,F20.9/)
      RR=R
      DO 40 J=1,NR
      WRITE(6,3)RR
    3 FORMAT(20X,6H    R=,F20.9/)
      AA=A
      DO 30 K=1,NA
      IF(AA-1.)10,10,20
   10 C=AA
      S=0.
      GO TO 25
   20 X=ARCOS((AA-1.)/AA)
      Y=ARCOS(-1./AA)
```

```
      C=(AA/3.1416)*((1.-B)*(X+Y-SIN(2.*X)/2.-SIN(2.*Y)/2.-1.5708)+
     1B*3.1416)
      S=-(1.-B)*COS(X)/1.5708
   25 D=(RR/AA)**2-(S/AA)**2
      E=(ABS(D))**0.5
      FR1=(C/AA+E)**0.5
      F=C/AA-E
      FR2=(ABS(F))**0.5
      WRITE(6,4)AA,FR1,FR2,D,F
    4 FORMAT(1H-,2X,5E16.7)
   30 AA=AA+DELA
   40 RR=RR+DELR
   50 B=B+DELB
      STOP
      END
$ENTRY
0.414     0.100     3 0.100     0.100     3  0.200   0.200     30
$IBSYS
```

```
$JOB            003715 SAHAY            100   010   030
$IBJOB          NODECK
$IBFTC
C     RESPONSE CURVES FOR SINGLE DEGREE OF FREEDOM SYSTEM WITH EXTERNAL
C     DAMPING  GAMA =DAMPING COEFFICIENT
      GAMA=0.01
      RR=0.
      AB=0.
      B=0.414
      DO 8 II=1,3
      RR=RR+0.1
      AB=AB+0.2
      WRITE(6,4)RR
    4 FORMAT(50X,3H R=,F13.8/)
      W1=0.71
      W2=1.225
      AA=AB
    9 N=1
      M=1
      IF(AA-1.)10,10,20
   10 C=AA
      S=0.
      GO TO 30
   20 X=ARCOS((AA-1.)/AA)
```

```
      Y=ARCOS(-1./AA)
      C=(AA/3.1416)*((1.-B)*(X+Y-SIN(2.*X)/2.-SIN(2.*Y)/2.-1.5708)+
     1B*3.1416)
      S=-(1.-B)*COS(X)/1.5708
   30 E=2.*(2.*GAMA**2-C/AA)
      F=-4.*GAMA*(S/AA)
      G=(C/AA)**2+(S/AA)**2-(RR/AA)**2
   40 FW1=W1**4+E*W1*W1+F*W1+G
      DW1=4.*W1**3+2.*E*W1+F
      Z1=W1-FW1/DW1
      IF(ABS(FW1).LE.1.E-06)GO TO 50
      N=N+1
      IF(N.GE.60)GO TO 60
      W1=Z1
      GO TO 40
   50 WRITE(6,5)AA,Z1,FW1
    5 FORMAT(4X,2HA=,F13.8,4X,3H W=,F13.8,4X,4H FW=,F13.8/)
      W1=Z1
   60 FW2=W2**4+E*W2*W2+F*W2+G
      DW2=4.*W2**3+2.*E*W2+F
      Z2=W2-FW2/DW2
      IF(ABS(FW2).LE.1.E-06)GO TO 70
      M=M+1
      IF(M.GE.60)GO TO 8
      W2=Z2
      GO TO 60
```

```
   70 WRITE(6,6)Z2,FW2
    6 FORMAT(23X,3H W=,F13.8,4X,4H FW=,F13.8/)
   80 AA=AA+0.2
      IF(AA.GT.5.0)GO  TO 8
      GO TO 9
    8 CONTINUE
  100 STOP
      END
$ENTRY
$IBSYS
```

```
$JOB           003715 SAHAY              100   010   030
$IBJOB         NODECK
$IBFTC
C     DETERMINATION OF THE LOCI OF VERTICAL TANGENCY OF SINGLE DEGREE
C     OF FREEDOM DOUBLE BILINEAR HYSTERETIC SYSTEM.X1=PSI1,X2=PSI2,WPSQ
C     =SQUARE OF THE FREQUENCY AT THE PEAK AMPLITUDE RESPONSE,WVA,WVB
C     ARE THE FREQUENCIES AT WHICH CONSTANT AMPLITUDE LINES CUT THE
C     LOCI OF VERTICAL TANGENCY.
      ALFA=0.414
      A=1.2
   10 X1=ARCOS((A-1.)/A)
      X2=ARCOS(-1./A)
      C=A*((1.-ALFA)*(X1+X2-SIN(2.*X1)/2.-SIN(2.*X2)/2.-1.57)+ALFA*3.14)
     U/3.14
      WPSQ=C/A
      ZA=WPSQ-(1.-ALFA)*(SIN(X1)+SIN(X2))/(3.14*A)
      ZB=(1.-ALFA)* ((2.-(SIN(X1)-SIN(X2))**2)**0.5)/(3.14*A)
      WVA=(ZA-ZB)**0.5
      WVB=(ZA+ZB)**0.5
      WRITE(6,1)A,WVA,WVB
    1 FORMAT(3E20.9/)
      IF(A.GT.6.)GO TO 20
      A=A+0.2
      GO TO 10
```

```
   20 STOP
      END
$ENTRY
$IBSYS
```

```
$JOB            003715 SAHAY                100   010   030
$IBJOB               MAP,NODECK
$IBFTC    FITS
C      RUNGE-KUTTA SOLUTION TO THE DIFFERENTIAL EQUATION FOR SINGLE
C      DEGREE OF FREEDOM DOUBLE BILINEAR HYSTERETIC SYSTEM
C      F IS THE AUXILLIARY SUBROUTINE IN WHICH THE EQUATIONS ARE DEFINED.
C      YI IS AN ARRAY IN WHICH THE INITIAL VALUES ARE GIVEN.Y IS AN ARRAY
C      IN WHICH THE STORED VALUES ARE RETURNED.R=AMPLITUDE  OF EXTERNAL
C      EXCITATION.W=FREQUENCY OF EXCITATION.AL=ALFA.XMX1 IS THE MAXIMUM
C      AMPLITUDE IN THE POSITIVE QUADRANT OF THE FORCE-DISPLACEMENT CURVE
C      XMX2 IS THE MAXIMUM ABSOLUTE AMPLITUDE IN THE NEGATIVE QUADRANT.
C      H=STEP LENGTH.N3 IS THE NUMBER OF INTEGRATION STEPS BETWEEN EACH
C      EACH STORED VALUE.
       EXTERNAL F
       DIMENSION YI(3),Y(3,3000)
       COMMON R,W,AL,XMX1,XMX2
       R=0.3
       W=1.10
       AL=0.7
       XMX1=0.
       XMX2=0.0
       READ(5,1)YI
       H=0.0025
       N3=4
C      THREE FIRST ORDER EQUATIONS(REALLY ONE SECOND ORDER)
```

```
      CALL RKG(F,Y,3,3000,N3,YI,H)
      DO 10 I=1,3000
   10 WRITE(6,2)(Y(K,I),K=1,3)
      CALL EXIT
    1 FORMAT(3F10.4)
    2 FORMAT(3F20.9)
      END
$IBFTC FUNC1
      SUBROUTINE F(YY,YK)
C     YK IS THE ARRAY OF DERIVATIVES AND YY THE ARRAY OF  FUNCTIONS
C     DEFINED BY THE DIFFERENTIAL EQUATIONS.
      DIMENSION YY(3),YK(3)
      COMMON R,W,AL,XMX1,XMX2
C     THIS IS THE EQUATION FOR THE INDEPENDENT VARIABLE,INITIALLY 0.
      YK(1)=1.
      YK(2)=YY(3)
      IF(YY(2).LE.0.)XMX1=0.00
      IF(YY(2).GE.0.)XMX2=0.00
      IF(YY(2).GE.0..AND.YY(3).GE.0..AND.YY(2).LE.1.)YK(3)=-YY(2)+
     1R*COS(W*YY(1))
      IF(YY(2).GE.0..AND.YY(3).LE.0..AND.YY(2).LE.1..AND.XMX1.LE.0.)
     1YK(3)=-YY(2)+R*COS(W*YY(1))
      IF(YY(2).LE.0..AND.YY(3).LE.0..AND.ABS(YY(2)).LE.1.)
     1YK(3)=-YY(2)+R*COS(W*YY(1))
```

```
      IF(YY(2).LE.0..AND.YY(3).GE.0..AND.ABS(YY(2)).LE.1..AND.XMX2.LE.0.
     1)YK(3)=-YY(2)+R*COS(W*YY(1))
      IF(YY(2).GE.1..AND.YY(3).GE.0.)YK(3)=-AL*YY(2)-(1.-AL)+R*COS(W*
     1YY(1))
      IF(YY(2).GE.1..AND.YY(3).GE.0.)XMX1=YY(2)
      IF(YY(2).GE.0..AND.YY(3).LE.0..AND.XMX1.GT.1.)YK(3)=-YY(2)+
     1(1.-AL)*(XMX1-1.)+R*COS(W*YY(1))
      IF(YY(2).GE.0..AND.YY(3).LE.0..AND.YY(2).LE.(XMX1-1.) .AND.XMX1.
     1GT.1.)YK(3)=-AL*YY(2)+R*COS(W*YY(1))
      IF(ABS(YY(2)).GE.1..AND.YY(3).LE.0..AND.YY(2).LT.0.)
     1YK(3)=-AL*YY(2)+(1.-AL)+R*COS(W*YY(1))
      IF(ABS(YY(2)).GE.1..AND.YY(3).LE.0..AND.YY(2).LT.0.)XMX2=ABS(YY(2)
     1)
      IF(YY(2).LE.0..AND.YY(3).GE.0..AND.XMX2.GE.1.)
     1YK(3)=-YY(2)-(1.-AL)*(XMX2-1.)+R*COS(W*YY(1))
      IF(YY(2).LE.0..AND.YY(3).GE.0..AND.ABS(YY(2)).LE.(XMX2-1.).
     1AND.XMX2.GE.1.)YK(3)=-AL*YY(2)+R*COS(W*YY(1))
      RETURN
      END
$ENTRY
0.00      0.00      0.00
$IBSYS
```

```
$JOB          003715 SAHAY              100   010   030
$EXECUTE           MIMIC
C      DIGITAL ANALOGUE SOLUTION FOR ONE DEGREE OF FREEDOM DOUBLE
C      BILINEAR HYSTERETIC SYSTEM.XMX1,XMX2ARE MAXIMUM VALUES OF X
C      FOR MASS M  IN POSITIVE AND NEGATIVE QUADRANTS RESPECTIVELY
                      CON(W,R,DT,ALFA)
                      CON(Q,P,F,G)
          DTMIN       EQL(0.01)
          DTMAX       EQL(DTMIN)
          Y1          FSW(X,FALSE,TRUE,TRUE)
          Y2          FSW(DX1,FALSE,TRUE,TRUE)
          Y3          FSW((ABS(X)-1.),TRUE,FALSE,FALSE)
          A1          AND(Y1,Y2,Y3)
 A1       Y           EQL(X)
          PQ          FSW(XMX1,TRUE,TRUE,FALSE)
          A3          AND(Y1,COM(Y2),Y3,PQ)
 A3       Y           EQL(X)
          A2          AND(Y1,Y2,COM(Y3))
 A2       Y           EQL(ALFA*X+1.-ALFA)
          XMX1        TAS(X,A2,F)
          COY1        COM(Y1)
 COY1     XMX1        EQL(0.00)
          XM1         EQL(XMX1-1.)
          Y4          FSW((X-XM1),FALSE,TRUE,TRUE)
```

```
        A4      AND(Y1,COM(Y2),Y4,COM(PQ))
A4      Y       EQL(X-(1.-ALFA)*XM1)
        A5      AND(Y1,COM(Y2),COM(Y4))
A5      Y       EQL(ALFA*X)
        A6      AND(COM(Y1),COM(Y2),Y3)
A6      Y       EQL(X)
        A11     AND(COM(Y1),Y2,Y3,RS)
A11     Y       EQL(X)
        A7      AND(COM(Y1),COM(Y2),COM(Y3))
A7      Y       EQL(ALFA*X-1.+ALFA)
        XMX2    TAS(X,A7,G)
Y1      XMX2    EQL(0.00)
        XM2     EQL(ABS(XMX2)-1.)
        Y5      FSW((ABS(X)-XM2),FALSE,TRUE,TRUE)
        RS      FSW(ABS(XMX2),TRUE,TRUE,FALSE)
        A9      AND(COM(Y1),Y2,Y5,COM(RS))
A9      Y       EQL(X+(1.-ALFA)*XM2)
        A10     AND(COM(Y1),Y2,COM(Y5))
A10     Y       EQL(ALFA*X)
C     DX1,DX2 ARE THE FIRST AND SECOND DIFFERENTIALS WITH RESPECT TO
C     TIME OF THE DISPLACEMENT OF MASS M
        DX2     ADD(-Y,YZ,R*COS(W*T))
        DX1     INT(DX2,P)
        X       INT(DX1,Q)
                FIN(T,75.0)
```

```
          HDR(T,X,DX1,Y,XMX1,XMX2)
          HDR
          OUT(T,X,DX1,Y,XMX1,XMX2)
          END
1.10      0.30      0.01      0.70
0.00      0.00      0.00      0.00
$IBSYS
```

```
$JOB           003715 SAHAY              100   010   030
$IBJOB         NODECK
$IBFTC
C      GRADIENT METHOD APPLIED FOR THE ITERATION OF STEADY STATE EQUATION
C      OF TWO DEGREE OF FREEDOM DOUBLE BILINEAR HYSTERETIC SYSTEM,SOLUTIO
C      OBTAINED BY K.AND B. METHOD.
C      A(1),A(2) ARE THE AMPLITUDES OF THE MASSES M1 AND M2.W=FREQUENCY
C      X1=PSI1(1),Y1=PSI2(1),X2=PSI1(2),Y2=PSI2(2).ALFA=0.7.
C      Z1,Z2=PHASE ANGLES FOR THE MASSES M1,M2 RESPECTIVELY.
       DIMENSION A(2),Y(2)
       COMMON W,W2,W4
       COMMON C1,C2,S1,S2
       EXTERNAL  FUNC
       READ(5,1)A
       W=1.50
       GO TO 11
    10 W=W+0.05
       IF(W.GT.2.5)GO TO 5
    11 W2=W**2
       W4=W**4
       WRITE(6,2)A,W
       H=0.1
       I=1
       VL=100000.
    20 CALL GRAD(FUNC,2,A,H,10,V,Y)
```

```
      WRITE(6,3)I,A,V,Y
      IF(I.GT.30.OR.V.GT..99* VL.OR.V.LT.1.E-8) GO TO 12
      I=I+1
      VL=V
      GO TO 20
   12 SZ2=(-2.*C1*S2+S1*(-W2*A(2)+2.*C2))/(C1*C1+S1*S1)
      CZ2=( 2.*S1*S2+C1*(-W2*A(2)+2.*C2))/(C1*C1+S1*S1)
      R=0.5
      SZ1=(C2*SZ2+S2*CZ2-S1)/R
      Z1=ARSIN(SZ1)
      Z2=ARSIN(SZ2)
      WRITE(6,4)Z1,Z2
      GO TO 10
    1 FORMAT(2F10.4)
    2 FORMAT(1H0,29X,3F20.9/)
    3 FORMAT(20X,I10,2F20.9/5X,3E20.9//)
    4 FORMAT(2F20.9/)
    5 END
      FUNCTION  FUNC(A,Y)
      DIMENSION A(2),Y(2)
      COMMON W,W2,W4
      COMMON C1,C2,S1,S2
      IF(A(1)-1.)30,30,40
   30 C1=A(1)
      S1=0.
```

```
      GO TO 50
   40 X1=ARCOS((A(1)-1.)/A(1))
      Y1=ARCOS(-1./A(1))
      C1=(A(1)/3.14)*(0.300*(X1+Y1-SIN(2.*X1)/2.-SIN(2.*Y1)/2.-1.57)
     1+2.198)
      S1=-0.1910*COS(X1)
   50 IF(A(2)-1.)60,60,70
   60 C2=A(2)
      S2=0.
      GO TO 80
   70 X2=ARCOS((A(2)-1.)/A(2))
      Y2=ARCOS(-1./A(2))
      C2=(A(2)/3.14)*(.300*(X2+Y2-SIN(2.*X2)/2.-SIN(2.*Y2)/2.-1.57)+2.2)
      S2=-0.1910*COS(X2)
   80 Y(1)=(-W2*A(1)+C1/2.)**2+(-W2*A(2)+2.*C2)*(W4*A(1)*A(2)*C1/(C1*C1
     1+S1*S1)-W2*A(2)/2.)+2.*S1*S2*A(1)*A(2)*W4/(C1*C1+S1*S1)+W4*A(2)**2
     2/4.+S1*S1/4.-0.25
      Y(2)=(-W2*A(2)+2.*C2)**2-C1*C1-S1*S1 +4.*S2*S2
      FUNC=Y(1)**2+Y(2)**2
      RETURN
      END
$ENTRY
0.1810     0.7270
$IBSYS
```

```
$JOB            003715 SAHAY               100   010   030
$EXECUTE        MIMIC
C     DIGITAL ANALOGUE SOLUTION FOR TWO DEGREE OF FREEDOM  DOUBLE
C     BILINEAR  HYSTERETIC SYSTEM.XMX1,XMX2 ARE MAXIMUM VALUES OF X1
C     FOR MASS M1 IN POSITIVE AND NEGATIVE QUADRANTS.ZMX1,ZMX2 ARE
C     MAXIMUM VALUES OF X2 FOR MASS M2 IN IN POSITIVE AND
C     NEGATIVE QUADRANTS OF DISPLACEMENT RESTORING FORCE CHARACTERISTICS
                     CON(W,R,DT,ALFA)
                     CON(Q,P,F,G)
          DTMIN      EQL(0.01)
          DTMAX      EQL(DTMIN)
          Y1         FSW(X,FALSE,TRUE,TRUE)
          Y2         FSW(DX1,FALSE,TRUE,TRUE)
          Y3         FSW((ABS(X)-1.),TRUE,FALSE,FALSE)
          A1         AND(Y1,Y2,Y3)
   A1     Y          EQL(X)
          PQ         FSW(XMX1,TRUE,TRUE,FALSE)
          A3         AND(Y1,COM(Y2),Y3,PQ)
   A3     Y          EQL(X)
          A2         AND(Y1,Y2,COM(Y3))
   A2     Y          EQL(ALFA*X+1.-ALFA)
          XMX1       TAS(X,A2,F)
          COY1       COM(Y1)
   COY1   XMX1       EQL(0.00)
          XM1        EQL(XMX1-1.)
          Y4         FSW((X-XM1),FALSE,TRUE,TRUE)
```

```
          A4       AND(Y1,COM(Y2),Y4,COM(PQ))
A4        Y        EQL(X-(1.-ALFA)*XM1)
          A5       AND(Y1,COM(Y2),COM(Y4))
A5        Y        EQL(ALFA*X)
          A6       AND(COM(Y1),COM(Y2),Y3)
A6        Y        EQL(X)
          A11      AND(COM(Y1),Y2,Y3,RS)
A11       Y        EQL(X)
          A7       AND(COM(Y1),COM(Y2),COM(Y3))
A7        Y        EQL(ALFA*X-1.+ALFA)
          XMX2     TAS(X,A7,G)
Y1        XMX2     EQL(0.00)
          XM2      EQL(ABS(XMX2)-1.)
          Y5       FSW((ABS(X)-XM2),FALSE,TRUE,TRUE)
          RS       FSW(ABS(XMX2),TRUE,TRUE,FALSE)
          A9       AND(COM(Y1),Y2,Y5,COM(RS))
A9        Y        EQL(X+(1.-ALFA)*XM2)
          A10      AND(COM(Y1),Y2,COM(Y5))
A10       Y        EQL(ALFA*X)
          YA1      FSW(Z,FALSE,TRUE,TRUE)
          YA2      FSW(DZ1,FALSE,TRUE,TRUE)
          YA3      FSW((ABS(Z)-1.),TRUE,FALSE,FALSE)
          AZ1      AND(YA1,YA2,YA3)
```

```
AZ1     YZ      EQL(Z)
        PZQ     FSW(ZMX1,TRUE,TRUE,FALSE)
        AZ3     AND(YA1,COM(YA2),YA3,PZQ)
AZ3     YZ      EQL(Z)
        AZ2     AND(YA1,YA2,COM(YA3))
AZ2     YZ      EQL(ALFA*Z+1.-ALFA)
        ZMX1    TAS(Z,AZ2,F )
        CYA1    COM(YA1)
CYA1    ZMX1    EQL(0.00)
        ZM1     EQL(ZMX1-1.)
        YA4     FSW((Z-ZM1),FALSE,TRUE,TRUE)
        AZ4     AND(YA1,COM(YA2),YA4,COM(PZQ))
AZ4     YZ      EQL(Z-(1.-ALFA)*ZM1)
        AZ5     AND(YA1,COM(YA2),COM(YA4))
AZ5     YZ      EQL(ALFA*Z)
        AZ6     AND(COM(YA1),COM(YA2),YA3)
AZ6     YZ      EQL(Z)
        AZ11    AND(COM(YA1),YA2,YA3,RZS)
AZ11    YZ      EQL(Z)
        AZ7     AND(COM(YA1),COM(YA2),COM(YA3))
AZ7     YZ      EQL(ALFA*Z-1.+ALFA)
        ZMX2    TAS(Z,AZ7,G )
YA1     ZMX2    EQL(0.00)
        ZM2     EQL(ABS(ZMX2)-1.)
        YA5     FSW((ABS(Z)-ZM2),FALSE,TRUE,TRUE)
        RZS     FSW(ABS(ZMX2),TRUE,TRUE,FALSE)
```

```
        AZ9       AND(COM(YA1),YA2,YA5,COM(RZS))
 AZ9    YZ        EQL(Z+(1.-ALFA)*ZM2)
        AZ10      AND(COM(YA1),YA2,COM(YA5))
 AZ10   YZ        EQL(ALFA*Z)
C     X AND Z ARE FOR X1 AND X2(IN THE TEXT),DX1DZ1 AND DX2,DZ2 ARE
C     FIRST AND SECOND DIFFERENTIALS  W.R.T. TIME
        DX2       ADD(-Y,YZ,R*COS(W*T))
        DX1       INT(DX2,P)
        X         INT(DX1,Q)
        DZ2       ADD(Y,-2.*YZ)
        DZ1       INT(DZ2,P)
        Z         INT(DZ1,Q)
                  FIN(T,75.0)
                  HDR(T,X,DX1,Z,DZ1)
                  HDR
                  OUT(T,X,DX1,Z,DZ1)
                  END
0.800       0.500       0.0100      0.700
0.00        0.00        0.00        0.00
$IBSYS
```

```
$JOB           003715 SAHAY          100   010   030
$IBJOB         NODECK
$IBFTC
C      842 POINT FOURIER ANALYSIS   MASS M1,M2
C      ARRAY A CONTAINS THE VALUES OF THE DISPLACEMENTS OF THE SYSTEM
C      AT TIME INTERVALS 0.01 AS OBTAINED FROM THE MIMIC  SOLUTION
C      NEAR THE STEADY STATE
       DIMENSION A(842),B(842)
C      ARRAYS C AND S ARE FOURIER COEFFICIENTS(COSINE AND SINE TERMS)
       DIMENSION  C(10),S(10)
       N=842
       READ(5,1)A
     1 FORMAT(8F10.6)
       DO 100 I=1,N
   100 B(I)=A(I)
       ZN=N
       AAVG=0.0
       DO 10 I=1,N
    10 AAVG=AAVG+B(I)
       SAVG=AAVG/ZN
       WRITE(6,2)SAVG
     2 FORMAT(/9H C( 0) = ,E14.8)
       DO 20 J=1,10
       ZJ=J
       SSUM=0.0
```

```
      CSUM=0.0
      AL(J)=6.2831853*ZJ/ZN
      BT=2.0/ZN
      DO 30 I=1,N
      A1=I-1
      CSUM=CSUM+COS(AL(J)*A1)*BT*B(I)
      SSUM=SSUM+SIN(AL(J)*A1)*BT*B(I)
   30 CONTINUE
      C(J)=CSUM
      S(J)=SSUM
      V2=CSUM*CSUM+SSUM*SSUM
      V1=SQRT(V2)
      ANG=ATAN2(CSUM,SSUM)
      ANG=ANG*57.29578
      WRITE(6,5)J,C(J),J,S(J),J,V1,J,ANG
    5 FORMAT(3H C(,I2,4H) = ,E14.8,8H      S(,I2,4H) = ,E14.8,5X,2HV(,I2
     1,4H) = ,E14.8,5X,4HANG(,I2,4H) = ,F9.4,4HDEGS)
   20 CONTINUE
      STOP
      END
$ENTRY
```

NORMALIZATION OF DOUBLE-BILINEAR HYSTERETIC SYSTEM

Let

F_n = yield force (in lbs.)

K_1 = spring stiffness before the break off point (segment OA) as shown in Figure (19) (in lbs./in.)

K_2 = spring stiffness after the break off point (segment AB), (in lbs./in.)

m = mass of the system (in $\frac{\text{lbs. x sec}^2}{\text{in}}$)

Y_m = maximum amplitude under steady state condition (in inches).

Y = actual displacement of the system (in inches)

τ = time (in seconds)

$P(\tau)$ = external force of excitation (in lbs.)

With reference to Figure (19) we can see that there will be eight different equations of motion along the eight segments of the restoring force curve. Here we shall take two typical segments, and try to normalize the corresponding equations of motion.

The equation of motion along the segment BC may be written as:

$$m \frac{d^2y}{d\tau^2} + K_1 Y + (\text{sgn} \frac{dy}{d\tau})(K_1 |Y_m| - F_n)(K_1 - K_2)/K_1 = P(\tau) \quad \ldots \text{(A)}$$

The equation of motion along the segment AB may be written as:

$$m \frac{d^2y}{d\tau^2} + K_2 Y + (\text{sgn} \frac{dy}{d\tau}) F_n \frac{(K_1 - K_2)}{K_1} = p(\tau) \quad \ldots\ldots\ldots\ldots \text{(B)}$$

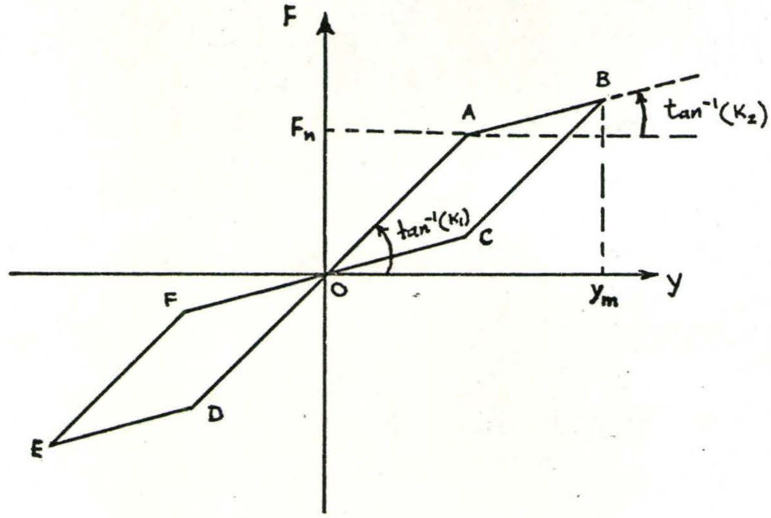

FIGURE 19

UN-NORMALIZED DOUBLE BILINEAR HYSTERETIC MODEL

(F = Restoring Force, y = displacement)

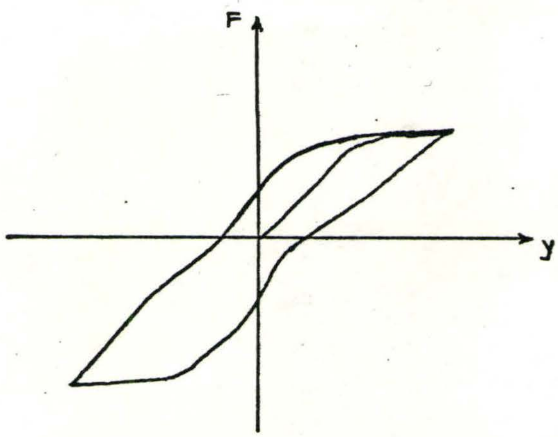

FIGURE 20

Representing the hysteretic curve in some real structures. Note the "pinching together near the origin". A similar figure was shown by W. D. Iwan [1] at the Winter Annual Meeting, Chicago, (Nov. 7-11, 1965) of the A.S.M.E.

If we substitute the following quantities for normalization:

$$x = \frac{K_1}{F_n} Y$$

$$t = \sqrt{\frac{K_1}{m}}\ \tau$$

$$p(t) = \frac{P(\tau)}{F_n} \qquad \ldots. \qquad (C)$$

$$\alpha = \frac{K_2}{K_1}$$

Then the equations (B) and (A) may be reduced to the form:

$$\frac{d^2x}{dt^2} + \alpha x + (\operatorname{sgn} \frac{dx}{dt})\ (1-\alpha) = p(t) \qquad \ldots. \qquad (D)$$

and

$$\frac{d^2x}{dt^2} + x + (\operatorname{sgn} \frac{dx}{dt})\ (1-\alpha)\ (|x_m|-1) = p(t) \qquad \ldots. \qquad (E)$$

Here p(t) is the normalized external trigonometric excitation, given by:

$$p(t) = R \operatorname{Cos} \omega t \qquad \ldots. \qquad (F)$$

Thus the equation of motion along any segment may be normalized by using the substitutions (C).

Equations (D) or (E) may also be written in general as:

$$\frac{d^2x}{dt^2} + f(x,\dot{x}) = p(t) \qquad \ldots. \qquad (G)$$

where $f(x,\dot{x})$ designates the segment on the normalized hysteretic curve.

NORMALIZATION OF TWO DEGREE OF FREEDOM SYSTEM

Let

AK_1, AK_2 = spring stiffness corresponding to mas M_1

BK_1, BK_2 = spring stiffness corresponding to mass M_2

y_1 = displacement of mass M_1

y_2 = displacement of mass M_2

[measured in a similar manner as x_1 and x_2 in Figure (9)]

y_{m1} = maximum displacement of mass M_1

y_{m2} = maximum displacement of mass M_2

[on the un-normalized curve Figure (19)]

Now, let us suppose that the restoring force characteristics for the masses M_1 and M_2 are similar [Figure (19)] but not the same. Thus let, at a particular moment, mass M_1 execute its motion along BC, while mass M_2 has its motion along AB.

Then the equations of motion are given by:

$$M_1 \frac{d^2 y_1}{d\tau^2} + AK_1 y_1 + \left(\operatorname{sgn} \frac{dy_1}{d\tau}\right)\left(AK_1 |y_{m1}| - F_{n1}\right)\left(AK_1 - AK_2\right)/AK_1$$
$$- BK_2 y_2 - \left(\operatorname{sgn} \frac{dy_2}{d\tau}\right) F_{n2} \left(BK_1 - BK_2\right)/BK_1 = p(\tau) \quad \ldots (H)$$

$$M_2 \frac{d^2 y_2}{d\tau^2} + 2BK_2 y_2 + 2\left(\text{sgn}\,\frac{dy_2}{d\tau}\right) F_{n2}\left(BK_1 - BK_2\right)/BK_1$$

$$- AK_1 y_1 - \left(\text{sgn}\,\frac{dy_1}{d\tau}\right)\left(AK_1 |y_{m1}| - F_{n1}\right)\left(AK_1 - AK_2\right)/AK_1 = 0 \cdot \text{(I)}$$

According to the postulate $M_1 = M_2$, $\frac{AK_2}{AK_1} = \frac{BK_2}{BK_1} = \alpha$ (since the hysteretic curves are supposed to be similar for the simplicity of the analysis).

Substituting the following quantities:

$$\begin{aligned} x_1 &= \frac{AK_1}{F_{n1}} \, y_1 \\ x_2 &= \frac{BK_1}{F_{n2}} \, y_2 \\ t &= \sqrt{\frac{AK_1}{M_1}}\,\tau = \sqrt{\frac{BK_1}{M_2}}\,\tau \\ p(t) &= \frac{p(\tau)}{F_{n1}} \end{aligned} \quad - - - \quad \text{(J)}$$

the equations (H) and (G) may be reduced to the form:

$$\ddot{x}_1 + x_1 + \left(\text{sgn}\,\frac{dx_1}{dt}\right)(1-\alpha)\left(|x_{m1}| - 1\right) - \alpha x_2$$

$$- \left(\text{sgn}\,\frac{dx_2}{dt}\right)(1-\alpha) = p(t) \quad - - - \quad \text{(K)}$$

$$\ddot{x}_2 + 2\alpha x_2 + 2\left(\operatorname{sgn}\frac{dx_2}{dt}\right)(1-\alpha)$$
$$- x_1 - \left(\operatorname{sgn}\frac{dx_1}{dt}\right)(1-\alpha)\left(|x_{m_1}|-1\right) = 0 \quad \cdots (L)$$

In a general manner, these last two equations may also be written as:

$$\ddot{x}_1 + f(x_1, \dot{x}_1) - f(x_2, \dot{x}_2) = p(t) \quad \cdots (M)$$

$$\ddot{x}_2 + 2f(x_2, \dot{x}_2) - f(x_1, \dot{x}_1) = 0 \quad \cdots (N)$$

BIBLIOGRAPHY

(1) IWAN, W. D.: The Steady State Response of the Double Bilinear Hysteretic Model, Paper No. 65-WA/APM-4, Journal of Applied Mechanics, 1965.

(2) IWAN, W. D.: The Dynamic Response of Bilinear Hysteretic Systems, Ph.D. Thesis, California Institute of Technology, 1961.

(3) RUZICKA, J. E.: Forced Vibration in Systems with Elastically Supported Dampers, M. Sc. Thesis, Massachusetts Institute of Technology, June, 1957.

(4) THOMSON, W. T.: Analog Computer for Nonlinear System with Hysteresis, Journal of Applied Mechanics, Vol. 24, June 1957.

(5) KLOTTER, K.: Nonlinear Vibration Problems Treated by the Averaging Method of W. Ritz, Proceedings of the First U. S. National Congress of Applied Mechanics, ASME, 1952.

(6) ARNOLD, F. R.: Steady-State Behaviour of Systems Provided With Nonlinear Dynamic Vibration Absorbers, Journal of Applied Mechanics, December 1955.

(7) CAUGHEY, T. K.: Sinusoidal Excitation of a System with Bilinear Hysteresis, Journal of Applied Mechanics, Vo. 27, No. 4 (1960).

(8) JACOBSEN, L. S. and R. S. AYRE: Engineering Vibrations, McGraw-Hill Book Company Inc., New York, N.Y. (1958)

(9) CAUGHEY, T. K.: Forced Oscillations of a Semi-Infinite Rod Exhibiting Weak Bilinear Hysteresis, Journal of Applied Mechanics, Vol. 27, No. 4 (1960).

(10) RUZICKA, J. E. (edited by), Structural Damping, ASME Annual Meeting, Atlantic City, N. J., December 1959.

(11) Shock and Vibration Handbook - edited by Harris and Crede, Volume II, McGraw-Hill Book Company, Inc.

(12) PISARENKO : Vibration of Elastic Systems Taking Account of Energy Dissipation in the Material - An English Translation of "Kolebaniia uprugukh sistem c uchetom rasseianiia energii v materiale", Kiev 1955, Academy of Sciences of the Ukrainian SSR, Report prepared by the University of Minnesota, Department of Aeronautics and Engineering Mechanics, February 1962.

(13) NIELSEN, Kay L.: Methods in Numerical Analysis, Second Edition, The MacMillan and Company.

(14) HARNETT, K. T., F. G. Sanson, L. M. WARSHAWAKY: MIDAS Programming Guide, Technical Documentary Report, No. SEG-TDR-64-1, Wright Patterson Air Force Base, Ohio, January 1964.

(15) CRESS, P.: MIMIC - A Digital-Analogue Simulator, Computing Centre, University of Waterloo, Waterloo, Ontario, February 1966.

(16) HAYASHI, C.: Nonlinear Oscillations in Physical Systems, McGraw-Hill Book Company, Inc., 1964.

(17) C.R.C. - Standard Mathematical Tables, Twelfth Edition 1963