TESTING FOR EXPONENTIALITY

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ABSTRACT

Several test statistics, which are known, can be used for testing for exponentiality. A new test statistic T_E is proposed. T_E is based on a censored sample and is similar to Tiku's T statistic for testing for normality. The distribution of T_E tends to normality with increasing sample size. Besides, T_E is easy to compute and is both origin and scale invariant. The power of T_E for non-exponential distributions is comparable with Shapiro & Wilk statistic W-exponential.

PREFACE

Occasions arise in statistical practice when it is necessary to test, whether a random variable x has the exponential distribution, on the basis of η independent observations. Examples are to be found in life-testing where x measures the length of life of, say, an electronic tube and also in the study of the distribution of intervals between events occurring in time, e.g., time between failures for a demonstration test of a system.

Several test statistics (Karl Pearson x^2, Kolmogorov - Smirnov D, Cramer - Von Mises W^2, Kuiper V, Anderson - Darling A, Watson U^2, Shapiro and Wilk W-Exponential), which are known, can be used for this problem.

A new statistic T_E for testing for exponentiality, similar to Tiku's T statistic for testing for normality, is proposed. T_E is origin and scale invariant and is easy to compute. The distribution of T_E tends to normality very rapidly with increasing sample size η (effectively $\eta > 20$). Against non-exponential distributions with skewness $\sqrt{\beta_1} > 2$ (the skewness of exponential), T_E is slightly more powerful than Shapiro and Wilk W-Exponential, although against non-exponential distributions with $\sqrt{\beta_1} < 2$, the power of T_E is slightly smaller than W-Exponential.

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CONTENTS

CHAPTER 1

TESTING FOR DISCRIMINATION BETWEEN TWO MODELS WITH UNKNOWN LOCATION AND SCALE PARAMETERS

When one wishes to select one of the two, location and scale parameter models, the values of the parameters being unknown, one must have a test which takes the alternative model into account. Let x_1, x_2, x_n be a random sample from a distribution with density function $f_o(x;\theta,\sigma)$ or $f_1(x;\theta,\sigma)$ where

$$f_j(x;\theta,\sigma) = \frac{1}{\sigma} \; g_j\left(\frac{x-\theta}{\sigma}\right); \qquad -\infty < \theta < \infty, \; \sigma > 0,$$

g_o and g_1 are functions of $\left(\frac{x-\theta}{\sigma}\right)$.

Suppose that the null hypothesis is

$$Ho: \;\; x \sim f_o(x;\theta,\sigma) = \frac{1}{\sigma} \; g_o\left(\frac{x-\theta}{\sigma}\right),$$

and the alternative hypothesis is

$$H_1: \;\; x \sim f_1(x;\theta,\sigma) = \frac{1}{\sigma} \; g_1\left(\frac{x-\theta}{\sigma}\right)$$

Let LR be defined by

$$LR = \frac{\underset{\theta,\sigma}{\text{Max}} \prod_{i=1}^{n} f_1(x_i;\theta,\sigma)}{\underset{\theta,\sigma}{\text{Max}} \prod_{i=1}^{n} f_o(x_i;\theta,\sigma)}$$

In general, if $x_1, x_2, \ldots, x_n$ represent n independent observations on a continuous variate x whose density function is of the form

$$f(x;\theta,\sigma) = \frac{1}{\sigma}\, g\left(\frac{x-\theta}{\sigma}\right) ;$$

$$(\theta,\sigma) \in \Omega,\ \Omega = \{(\theta,\sigma);\ -\infty < \theta < \infty,\ \sigma > o\},$$

then the likelihood function is given by

$$L(x;\theta,\sigma) = \frac{1}{\sigma^n} \prod_{i=1}^{n} g\left(\frac{x_i-\theta}{\sigma}\right) .$$

If θ_o and σ_o represent the true values of θ and σ for a given case, the density of the standardized variate $z=(x-\theta_o)/\sigma_o$ is given by $g(z)$, which is independent of both the parameters.

The maximum likelihood estimates of θ and σ are the values $\hat{\theta}$ and $\hat{\sigma}$ which satisfy

$$L(x;\hat{\theta},\hat{\sigma}) = \underset{\Omega}{\text{Max}}\ L(x;\theta,\sigma),$$

or

$$\frac{1}{(\hat{\sigma})^n} \prod_{i=1}^{n} g\left(\frac{x_i-\hat{\theta}}{\hat{\sigma}}\right) = \underset{\Omega}{\text{Max}}\ \frac{1}{(\sigma)^n} \prod_{i=1}^{n} g\left(\frac{x_i-\theta}{\sigma}\right),$$

or

$$\frac{1}{(\sigma_o)^n}\ \frac{1}{\left(\frac{\hat{\sigma}}{\sigma_o}\right)^n} \prod_{i=1}^{n} g\left\{\frac{\left(\frac{x_i-\theta_o}{\sigma_o}\right) - \left(\frac{\hat{\theta}-\theta_o}{\sigma_o}\right)}{\hat{\sigma}/\sigma_o}\right\}$$

$$= \underset{\Omega}{\text{Max}} \frac{1}{(\sigma_0)^n} \frac{1}{(\frac{\sigma}{\sigma_0})^n} \prod_{i=1}^{n} g\left[\left\{\left(\frac{x_i-\theta_0}{\sigma_0}\right) - \left(\frac{\theta-\theta_0}{\sigma_0}\right)\right\} / \frac{\sigma}{\sigma_0}\right],$$

or

$$\left(\frac{1}{\hat{\sigma}_s}\right)^n \prod_{i=1}^{n} g\left(\frac{z_i-\hat{\theta}_s}{\hat{\sigma}_s}\right) = \underset{\Omega^*}{\text{Max}}\left(\frac{1}{\sigma^*}\right)^n \prod_{i=1}^{n} g\left(\frac{z_i-\theta^*}{\sigma^*}\right),$$

where

$$\hat{\sigma}_s = \frac{\hat{\sigma}}{\sigma_0}, \quad \hat{\theta}_s = \frac{\hat{\theta}-\theta_0}{\sigma_0},$$

$$\sigma^* = \frac{\sigma}{\sigma_0}, \quad \theta^* = \frac{\theta-\theta_0}{\sigma_0}$$

and z_i's correspond to observations on a standardised variable. Since $\Omega = \Omega^*$, $\hat{\theta}_s$ and $\hat{\sigma}_s$ correspond exactly to the maximum likelihood estimators of θ and σ when the sampling is actually on a standardised variate z. Thus the joint density of $\hat{\theta}_s$ and $\hat{\sigma}_s$ does not depend on θ and σ. Therefore the density of $\hat{\theta}_s/\hat{\sigma}_s$ is also independent of parameters; see Antle and Bain [1].

From this it follows that the distribution of LR is independent of θ and σ. Hence, for any location and scale parameter model one can construct tables for critical values; see Dumonceaux et.al. [8].

Thus for testing normality against exponentiality

$$LR = \sqrt{n \sum_{i=1}^{n} (x_i-\bar{x})^2} / \sum_{i=1}^{n} (x_i - \min x_i).$$

One would reject H_0: that the observations are from a normal population when LR exceeds its critical value.

Similarly for testing exponentiality against normality

$$LR = \sum_{i=1}^{n} (x_i - \min x_i) \Big/ \sqrt{n \sum_{i=1}^{n} (x_i - \bar{x})^2} \ .$$

If LR $\geq$ critical value, we reject the hypothesis H_0: that the observations are from exponential population.

Note that the use of likelihood ratio test necessitates specification of the functional form of the alternative distribution.

CHAPTER 2

TESTING ANY HYPOTHESIZED DISTRIBUTION AGAINST UNSPECIED ALTERNATIVES

2.1 Introduction

Let $x_1, x_2, \ldots, x_n$ be a random sample of observations on a continuous random variable x with cumulative distribution function (c.d.f.) F(x) and probability density function (p.d.f.) f(x). The general problem of goodness of fit consists in testing the null hypothesis H_0: that $F(x) = F_0(x; \theta_1, \theta_2, \ldots, \theta_s)$ for every x, against unspecified alternatives. Here the θ_i's denote the parameters of the hypothesized c.d.f. F_0. In the sequel $f_0(x; \theta_1, \theta_2, \ldots, \theta_s)$ will denote the hypothesized p.d.f. Two cases of this problem are of interest:

case 1. H_0 simple

case 2. H_0 composite .

2.2 Simple Hypothesis

In this case the c.d.f. F_0 under H_0 is completely specified as to its functional form (such as normality, exponentiality, etc.) as well as to the values of the parameters θ_i involved. This case is easier to handle and has been worked extensively and has a long and well known history going back to Karl Pearson's Chi-square test.

In the standard application of the Chi-square test, the n observations in a random sample from a population are classified into k mutually exclusive classes. The null hypothesis gives the probability p_{oi} that an observation falls into the i^{th} class ($i = 1,2,.., k$). The quantities $m_i = np_{oi}$ are called expected frequencies and

$$\sum_{i=1}^{k} p_{oi} = 1, \sum_{i=1}^{k} m_i = n .$$

The joint density of the observed n_i's falling in the respective classes is a multinomial distribution

$$\frac{n!}{n_1!\, n_2! \ldots n_k!} \quad p_{o1}^{n_1}\; p_{o2}^{n_2} \ldots p_{ok}^{n_k} \; .$$

The likelihood function is

$$L(n_1, n_2,.., n_k/p_{o1}, p_{02},..,p_{ok}) \propto \prod_{i=1}^{k} (p_{oi})^{n_i} \; .$$

On the other hand, if the true c.d.f. is $F_1(x)$, where F_1 may be any distribution function, we may denote the probabilities in k classes by p_{1i}, $i = 1, 2,.., k$ and the likelihood function by

$$L(n_1, n_2,.., n_k/ p_{11}, p_{12},.., p_{1k}) \propto \prod_{i=1}^{k} (p_{1i})^{n_i}$$

which is maximized when we substitute the maximum likelihood estimators $\hat{p}_{1i} = \frac{n_i}{n}$ for p_{1i}.

Then the likelihood ratio statistic for testing H_o against any composite alternative hypothesis H_1: $F(x) = F_1(x)$ is therefore

$$L = \frac{L(n_1, n_2, \ldots, n_k \, / \, p_{01}, p_{02}, \ldots, p_{0k})}{L(n_1, n_2, \ldots, n_k \, / \, \hat{p}_{11}, \hat{p}_{12}, \ldots, \hat{p}_{1k})},$$

$$= (n)^n \prod_{i=1}^{k} \left(\frac{p_{oi}}{n_i}\right)^{n_i}.$$

H_o is rejected when L is small enough. The exact distribution of L is unknown. However, as $n \longrightarrow \infty$ and when H_o holds,

$$-2 \log (L) = 2 \sum_{i=1}^{k} n_i \log \left(\frac{n_i}{n \, p_{oi}}\right) \quad \ldots\ldots\ldots\ldots\ldots(2.2.1)$$

is asymptotically distributed as Chi-square with (k-1) degrees of freedom (d.f.).

Karl Pearson proposed the statistic

$$X^2 = \sum_{i=1}^{k} \frac{(n_i - m_i)^2}{m_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.2.2)$$

for testing H_o.

As $n \longrightarrow \infty$, under H_o X^2 follows Chi-square distribution with (k-1)d.f. if m_i's are known.

The two distinct statistics (2.2.1) and (2.2.2) thus have the same distribution asymptotically, given H_o. More than this, they are asymptotically equivalent statistics when H_o holds. This is true because if we write $\Delta_i = (n_i - np_{oi}) \, / \, np_{oi}$,

$$-2 \log L = 2 \sum_{i=1}^{k} n_i \log (1 + \Delta_i),$$

$$= 2 \sum_{i=1}^{k} \{(n_i - np_{oi}) + np_{oi}\} \{\Delta_i - \tfrac{1}{2} \Delta_i^2 + 0 (n^{-3/2})\},$$

$$= 2 \sum_{i=1}^{k} \{(n_i - np_{oi})\Delta_i + np_{oi}\Delta_i - \frac{np_{oi}}{2} \Delta_i^2 + 0(n^{-1/2})\},$$

$$= \sum_{i=1}^{k} \{np_{oi} \Delta_i^2 + 0(n^{-\frac{1}{2}})\},$$

$$= X^2 \{1 + 0(n^{-\frac{1}{2}})\}.$$

Tests Based on Distances

Let $y_1 \leq y_2 \leq \quad \leq y_n$ denote the ordered observations in a complete sample of size n and the probability integral transformation be denoted by

$$z_i = \int_{-\infty}^{y_i} f(x)\, dx$$

Let

$$D^+ = \max_{1 \leqslant i \leqslant n} \{\frac{i}{n} - z_i\} \quad \text{and}$$

$$D^- = \max_{1 \leqslant i \leqslant n} \{z_i - \frac{i-1}{n}\} .$$

The test statistics based on distances are as follows:

Cramer-Von Mises $\quad W^2 = \sum_{i=1}^{n} \{z_i - \frac{2i-1}{2n}\}^2 + \frac{1}{12n}$.

Kolmogorov-Smirnov $D = \max\{D^+, D^-\}$.

Kuiper $V = D^+ + D^-$.

Watson $U^2 = W^2 - n(\bar{z} - .5)^2$; $\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$.

Anderson-Darling $A = \frac{1}{n}\ [-\{\sum_{i=1}^{n} (2i - 1)(\log z_i + \log(1 - z_{n-i+1}))\}]-n$

If we consider a random variable z, connected with x by the relation $z = \int_{-\infty}^{x} f(x)dx$, then z is monotonic non-decreasing function of x and $o < z < 1$. Further if f(z) is the p.d.f. of z then

$$f(z) = f(x)\frac{dx}{dz} = 1 .$$

Hence in the interval [o,1] all values of z are equally likely, or z is uniformly distributed in the interval [o,1], no matter what the probability function of x. Therefore, if we have n independent random observations x_i's ($i = 1,2,\ldots,n$) following a known continuous probability distribution which is completely specified by H_o, the hypothesis to be tested, then by means of the transformation

$$z_i = \int_{-\infty}^{x_i} f(x_i/H_o)\ dx_i,$$

the x_i's can be transformed into n independent random observations z_i's which are uniformly distributed.

Thus, if we consider the Kolmogorov-Smirnov test D, the distri-tion of D is independent of F_o. Tables of percentage points of D for various values of n are given by Birnbaum [3], Miller [14] and others.

This test has at least two advantages over Chi-square test:

(i) It can be used with small sample sizes where the validity of Chi-square test would be questionable.

(ii) It is generally more powerful than Chi-square test.

2.3 Composite Hypothesis

This case arises when only the functional form of F_0 is given but one or more of the parameters θ_i's $(i = 1,2,.., s)$ are unspecified. This case is in fact of more relevance since situations are extremely rare in practice where the c.d.f. to be tested is completely specified. The following test statistics

$$\sqrt{b_1} = \sqrt{n} \sum_{i=1}^{n} (y_i - \bar{y})^3 / [\sum_{i=1}^{n} (y_i - \bar{y})^2]^{3/2} ,$$

$$b_2 = n \sum_{i=1}^{n} (y_i - \bar{y})^4 / [\sum_{i=1}^{n} (y_i - \bar{y})^2]^2,$$

and

$$u = \sqrt{(n - 1)}\, (y_n - y_1) / [\sum_{i=1}^{n} (y_i - \bar{y})^2]^{\frac{1}{2}}$$

are all scale and origin invariant and hence are appropriate for testing the composite hypotheses.

It is well-known that the classical Chi-square test of goodne of fit can be modified to fit this case by properly estimating th unspecified parameters.

In estimating the parameters for this test, one may use maximum likelihood (or equivalent) estimates based on

(i) the cell frequencies or,

(ii) the original observations.

If only the numbers n_i of observations falling into the i^{th} of the k cells are available, there is no difficulty. In this case let $\tilde{p}_{oi}$ be any best asymptotically normal (B.A.N.) estimate of p_{oi}, such as minimum Chi-square or maximum likelihood estimate. Then under certain suitable regularity conditions, the asymptotic distribution of

$$\tilde{R} = \sum_{i=1}^{k} \frac{(n_i - n\tilde{p}_{oi})^2}{n\tilde{p}_{oi}}$$

is that of a Chi-square with (k-s-1) d.f., where s is the number of parameters being estimated.

If the original observations $x_1, x_2, \ldots, x_n$ are available, one is tempted to use more efficient estimates, such as the maximum likelihood estimate $\hat{p}_{oi}$ based on all the data. The distribution of

$$\hat{R} = \sum_{i=1}^{k} \frac{(n_i - n\hat{p}_{oi})^2}{n\hat{p}_{oi}}$$

differs from that of $\tilde{R}$. If we let

$$R = \sum_{i=1}^{k} \frac{(n_i - np_{oi})^2}{np_{oi}}$$

which has a limiting Chi-square distribution with (k-1) d.f., then limiting distribution of $\hat{R}$ lies between those of $\tilde{R}$ and R. Chernoff & Lehman [5] have shown that the asymptotic distribution of $\tilde{R}$ is that of

$$\sum_{i=1}^{k-s-1} u_i^2 + \sum_{i=1}^{k-1} \lambda_i u_i^2 ,$$

where u_i's are independently normally distributed with zero mean, and unit variance and λ_i's are between o and 1 and may depend on the s parameters $\theta_1, \theta_2, \ldots, \theta_s$.

But there are several objections to this test, such as the lack of "proper" estimates in many situations and the arbitrariness in the choice of the class intervals. Also its validity is questionable when the sample size n is small.

Bearing in mind that the Kolmogorov-Smirnov statistic D has, in general, a higher power than Chi-square statistic when H_0 is simple, one might try to modify D to the case when H_0 is composite. If we replace unknown parameters by sample estimates, the z_i's obtained by the probability integral transformation will no longer be independent, neither will they be uniformly distributed.

David & Johnson [7] have shown that if F_0 depends only on a location θ, and a scale σ, and $\hat{\theta}$ and $\hat{\sigma}$ are "proper" estimates of θ and σ respectively, then the distribution of the random variable $z = F_0(x; \hat{\theta}, \hat{\sigma})$ under H_0 depends only on the functional form of F_0, but not on the parameters θ and σ. Thus if $\hat{D}$ is defined by

$$\hat{D} = \sup_{-\infty \leq x \leq \infty} \left| S_n(x) - F_0(x; \hat{\theta}, \hat{\sigma}) \right|$$

where $S_n(x)$ is Empirical Distribution Function, then the distribution of $\hat{D}$ is non-parametric, and therefore $\hat{D}$ can be used as a test statistic. Its percentiles are given by Lilliefors [12], [13] for the normal and exponential distributions using Monte Carlo calculations. He has also shown that in these two cases the test based on $\hat{D}$ is more powerful than the Chi-square for any sample size against certain alternatives.

Another modification of D based on the unique minimum variance unbiased estimator of F_o is given by Srinivasan [23] as follows:

Assume that $F_o(x; \theta_1, \theta_2,.., \theta_s)$ is such that $t_1, t_2,...t_s$ are joint complete sufficient statistics for the parameters $\theta_1, \theta_2,...,\theta_s$. For a fixed real number u define the random variable Z as

$$Z = \begin{cases} 1 \text{ if } x_1 \leqslant u \\ o \text{ otherwise} \end{cases} .$$

Then it is obvious that Z is an unbiased estimator of $F_o(u; \theta_1, \theta_2,...,\theta_s)$. under H_o, and by the Rao-Blackwell theorem,

$$\tilde{F}_o(u; \theta_1, \theta_2,...,\theta_s) = E\,[Z/t_1, t_2,...,t_s]$$

is the unique minimum variance unbiased estimator of $F_o(u; \theta_1, \theta_2,...,\theta_s)$. Define the statistic $\tilde{D}$ as

$$\tilde{D} = \sup_{-\infty \leqslant X \leqslant \infty} \left|S_n(x) - \tilde{F}_o(x; \theta_1, \theta_2,...,\theta_s)\right| .$$

If the distribution of $\tilde{D}$ is independent of $\theta_1, \theta_2, \ldots, \theta_s$ then it would serve as an appropriate statistic for testing H_0. Srinivasan [23] has derived the estimator $\tilde{F}_0$ for the case of normal, mean and variance unknown, and exponential, $\theta = o$ and σ unknown, and showed that $\tilde{D}$ is non-parametric in both cases.

The percentage points of $\tilde{D}$ for various values of n have been tabulated by Srinivasan [23]. If we choose the alternatives to be lognormal and Chi-square with 1 d.f., $\tilde{D}$ is slightly more powerful than the test given by Lilliefors [12], [13] (see also Schafer et. al. [17]). However, no general recommendations can be made regarding which test is to be preferred in a given practical situation since the choice would obviously depend on the alternatives that one has in mind.

Shapiro & Wilk [19] proposed a statistic W which is obtained by dividing the square of an appropriate linear combination of the order statistics by the usual symmetric estimate of variance. Let $\underset{\sim}{\alpha}' = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ denote the vector of expected values of standardized normal ordered observations and $\underset{\sim}{V} = (v_{ij})$ be the corresponding $n \times n$ variance - covariance matrix.

Let $\underset{\sim}{y}' = (y_1, y_2, \ldots, y_n)$ denote a sample of ordered observations. If the y_i's come from a random sample from a normal distribution with mean μ and variance σ^2, the best linear unbiased estimator of σ is

$$\hat{\sigma} = \frac{\underset{\sim}{\alpha}' \underset{\sim}{V}^{-1} \underset{\sim}{y}}{\underset{\sim}{\alpha}' \underset{\sim}{V}^{-1} \underset{\sim}{\alpha}} .$$

provided V^{-1} exists.

Let $S^2 = \sum_{i=1}^{n} (y_i-\bar{y})^2$, $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$, denote the usual symmetric unbiased estimate of $(n-1)\sigma^2$. The W test statistic for testing for normality is defined as

$$W = \frac{(\underset{\sim}{a}' \underset{\sim}{y})^2}{S^2},$$

$$= (\sum_{i=1}^{n} a_i y_i)^2 / \sum_{i=1}^{n} (y_i - \bar{y})^2,$$

where $\underset{\sim}{a}' (a_1, a_2, \ldots a_n) = \underset{\sim}{\alpha}' \underset{\sim}{V}^{-1} / (\underset{\sim}{\alpha}' \underset{\sim}{V}^{-1} \underset{\sim}{V}^{-1} \underset{\sim}{\alpha})^{\frac{1}{2}}$.

The coefficients $\{a_i\}$ are the normalized best linear unbiased coefficients tabulated in Sarhan & Greenberg [16].

W is scale and origin invariant and hence is appropriate for a test of the composite hypothesis of normality. It has not been possible, for general n, to obtain an explicit form of the distribution of W. Shapiro & Wilk [19] have supplied percentage points of the null distribution of W for samples of size 3 to 50. Subsequent investigation revealed that this test has good power properties; see Shapiro et. al. [21]. It is an omnibus test, that is, it is appropriate for detecting deviations from normality due either to skewness or kurtosis and is generally superior to "distance" tests. It also generally dominates standard tests $\sqrt{b_1}$, b_2 and u for testing for normality.

Shapiro and Wilk did not extend their test beyond samples of size 50. A number of reasons indicate that it is best not to make such an extension. D'Agostino [6] presented a test D of normality applicable for samples of size 50 or larger which possesses the desirable omnibus property.

The statistic D is defined as

$$D = \frac{T}{n^2 S_1},$$

where

$$T = \sum_{i=1}^{n} \{i - \tfrac{1}{2}(n + 1)\} y_i ,$$

and

$$S_1^{\,2} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 ,\ \bar{x} \text{ being the sample mean.}$$

If the sample is drawn from a normal distribution, the expected value of D and its asymptotic standard deviation are, respectively,

$$E(D) = \frac{(n-1)}{2\sqrt{(2\,n\Pi)}} \quad \frac{\Gamma(\frac{n}{2} - \frac{1}{2})}{\Gamma(\frac{n}{2})} ,$$

$$\doteq (2\sqrt{\Pi}\,)^{-1} ,$$

and

$$a.s.d.(D) = \frac{.02998598}{\sqrt{n}} .$$

An approximate standardized variable, possessing asymptotically mean zero and variance unity, is

$$y = \frac{D - (2\sqrt{\Pi})^{-1}}{a.s.d.(D)} .$$

Values of the percentage points of y, obtained using the Cornish-Fisher expansion, are available for a number of different sample sizes. The statistic D is most powerful when the type of deviation from normality is unknown. Simulation results of powers for various alternatives, when the sample size is 50, indicate that the test compares favourably with the Shapiro-Wilk W test, $\sqrt{b_1}$, b_2 and u,

A test statistic for the exponential distribution can be obtained using the same principle as employed in defining the W statistic for testing for normality; see Shapiro & Wilk [20]. The W-exponential statistic for testing the composite hypothesis of exponentiality is, therefore,

$$W = \frac{n(\bar{y} - y_1)^2}{(n - 1)S^2} ,$$

where

$$S^2 = \sum_{i=1}^{n} (y_i - \bar{y})^2 .$$

The null distribution of W was studied through empirical sampling. The empirical cumulative distribution of W was obtained for sample sizes $n = 2(1)100$; see Shapiro & Wilk [20].

2.4 Other Tests For Exponentiality

Given a random sample $x_1, x_2, \ldots, x_n$, we wish to test the hypothesis that this sample comes from an exponential distribution. Let $y_1 \leq y_2 \leq \ldots \leq y_n$ denote the above sample observations arranged in ascending order of magnitude.

Under the assumption that this sample comes from an exponential distribution with p.d.f. $f(x) = \frac{1}{\sigma} \exp(-\frac{x}{\sigma})$, $(x > 0, \sigma > 0)$, it is well known that

$$D_i = (n-i)(y_{i+1} - y_i) \quad (i = 1, 2, \ldots, n)$$

are independently distributed, each having the same exponential density f(x). The following test statistics have been proposed; see Bartholomew [2]:

$$M = -2\{\sum_{i=1}^{n} \log D_i - n \log \bar{D}\} ,$$

$$S = \frac{1}{n^2} \sum_{i=1}^{n} (\frac{D_i}{\bar{D}})^2 ,$$

$$W = \sum_{i=1}^{n} \left|D_i - \bar{D}\right| / 2n\bar{D} ,$$

where

$$\bar{D} = \frac{1}{n} \sum_{i=1}^{n} D_i .$$

Jackson [10] gave a test statistic which is based on a direct comparison between the ordered observations and the corresponding expected values of the order statistics. If $y_1, y_2, \ldots, y_n$ are the order statistics for a random sample from the distribution with p.d.f. $\frac{1}{\sigma} \exp(-\frac{x}{\sigma})$, it is known that

$$\frac{1}{\sigma} E(y_r) = \sum_{i=1}^{r} (n - i + 1)^{-1} = t_{rn}, \text{ say.}$$

Then the test statistic is

$$T_n = \sum_{r=1}^{n} t_{rn}\, y_r / \sum_{r=1}^{n} y_r.$$

The distribution of T_n tends asymptotically to normality.

Two techniques, based on the transformations, J and K, of the observations x_i for testing for exponentiality have been proposed by Seshadri, Csorgo and Stephens [18].

Transformation J:

$$\text{Let } Z_r = \sum_{i=1}^{r} x_i/s \qquad (1 \leqslant r \leqslant n-1),$$

where

$$s = \sum_{i=1}^{n} x_i.$$

Then z_r are $(n-1)$ ordered random variables from $U(o,1)$, the uniform distribution between o and 1, if and only if x_i have an exponential distribution with parameters θ and σ.

Transformation K:

$$\text{Let } d_i = (n - i + 1)\,(y_i - y_{i-1}) \qquad (1 \leqslant i \leqslant n)$$

where

$$(y_0 = \theta),$$

$$D_i = \frac{d_i}{S_A} \text{ for all } i, \text{ where } S_A = \sum_{i=1}^{n} d_i,$$

and

$$Z_r' = \sum_{i=1}^{r} D_i \qquad (1 \leqslant r \leqslant n-1).$$

Then z_r' are $(\eta-1)$ ordered U(o,1) random variables if and only if y_i are ordered observations from exponential distribution with parameters θ and σ.

In the practical applications, θ is not known, so that d_1 cannot be calculated. D_i must be found omitting d_1, ie.,

$$D_i = \frac{d_i}{S^*} \quad (2 \leqslant i \leqslant \eta-1),$$

where

$$S^* = \sum_{i=2}^{r} d_i \ ,$$

and

$$Z_r' = \sum_{i=2}^{r} D_i \qquad (2 \leq y \leq \eta-1) \ .$$

Therefore, under the null hypothesis,

(i) the J transformation yields a set of $(\eta-1)$ observations from U(o,1), and

(ii) the K transformation yields a set of $(\eta-2)$ observations from U(o,1).

These two transformations enable us to use the test statistics based on "distance" for testing for exponentiality, even though the parameters θ and σ are unknown. The statistics may be computed as

$$D^+ = \max_i \{\frac{i}{\eta} - z_i\},$$

$$D^- = \max_i \{z_i - \frac{(i-1)}{\eta}\},$$

$$D = \max\{D^+, D^-\},$$

$$V = D^+ + D^-,$$

$$W^2 = \sum_{i=1}^{n} \{z_i - (\frac{2i-1}{2n})\}^2 + \frac{1}{12n},$$

and

$$U^2 = W^2 - n(\bar{z} - .5)^2.$$

It may, however, be noted that transformations with the data inflate the power and therefore exaggerate the power properties of these tests; see Durbin [9]. Shapiro & Wilk [20 p.370] mention other difficulties with such transformations.

2.5 Tiku's T Test For Testing For Normality

Tiku [27] proposed a statistic T, based on a censored sample, as a test for normality. Since the end observations are more sensitive to non-normality, especially to long tailedness, we censor r_1 smallest and r_2 largest observations to obtain the censored sample

$$y_a, y_{a+1}, \ldots, y_b \quad (a = r_1 + 1,\ b = n - r_2).$$

Under the assumption of normality, an efficient estimator of the population standard deviation σ can be obtained as (see Tiku [25]):

$$\sigma_c = \{B + \surd(B^2 + 4AC)\}/2A,$$

where

$$q_1 = \frac{r_1}{n}, \; q_2 = \frac{r_2}{n}, \; A = 1-q_1-q_2,$$

$$B = q_2\alpha_2 y_b - q_1\alpha_1 y_a - (q_2\alpha_2 - q_1\alpha_1)K,$$

and

$$C = \frac{1}{n}\sum_{i=a}^{b} y_i^2 + q_2 b_2 y_b^2 - q_1 b_1 y_a^2 - (1-q_1-q_2+q_2b_2-q_1b_1)K^2$$

here

$$K = (\frac{1}{n}\sum_{a}^{b} y_i + q_2 b_2 y_b - q_1 b_1 y_a)/(1-q_1-q_2+q_2b_2-q_1b_1),$$

and

α_1 and b_1, and α_2 and b_2 are chosen to give good fits

$$f(z)/P(z) \simeq \alpha_1 + b_1 Z \text{ and } f(z)/Q(z) \simeq \alpha_2 + b_2 Z;$$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \; P(z) = 1-Q(z) = \int_{-\infty}^{Z} f(z)dz,$$

α and b are functions of η.

The approximate bias in σ_c is

$-\sigma/n(1-q_1-q_2)$; see Tiku [28]. For $q_1 = q_2 = o$

$$\sigma_c = S = \left\{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right\}^{1/2}$$

The statistic T is defined as

$$T = (1 - \frac{1}{n})\sigma_c/(1 - \frac{1}{nA})S \quad , \quad (A = 1 - q_1 - q_2) \ o \leq T \leq \infty.$$

T is both, orgin and scale invariant. The mean of T is approximately 1; see Kendall & Stuart [11 p.232]. For $q_1 = o$, $q_2=[\frac{1}{2}+.6n]/n$ and $q_1 = q_2 = [\frac{1}{2}+.3n]/n$, a close approximation to the variance of T is (see Tiku [27])

$$V(T) \simeq \frac{1}{n}\{(1-\frac{1}{n})^2/(1-\frac{1}{nA})^2\} [1/\{2(1-q_1-q_2) - (q_2\alpha_2t_2 - q_1\alpha_1t_1)\}$$

$$-.5- .0532(q_1 + q_2) + 1.1284(q_2\alpha_2 - q_1\alpha_1)],$$

where

$$P(t_1) = q_1 \text{ and } Q(t_2) = 1 - P(t_2) = q_2.$$

Small values of T lead to the rejection of H_o(H_o: the sample comes from a normal distribution). The distribution of T under H_o tends to normality with increasing sample size n (effectively $n > 30$).

The power of T against non-normal distributions is generally an increasing function of $q_1 + q_2$. For $q_1 + q_2 > 0.6$, the power of T is not much higher than for $q_1 + q_2 = 0.6$, and since for $q_1 + q_2 > 0.6$, the null distribution of T tends to normality more slowly than for $q_1 + q_2 = 0.6$, the following choice of $r_1 = nq_1$ and $r_2 = nq_2$ has been suggested:

(i) Choose $r_1 = o$ and $r_2 = [\frac{1}{2} + 0.6n]$ if the non-normal distribution is positively skewed, ie., has its longer tail on the right hand side. Note that if the distribution of x is negatively skewed, then the distribution of $y = -x$ is positively skewed.

(ii) Choose $r_1 = [\frac{1}{2} + 0.3n]$ and $r_2 = [\frac{1}{2} + 0.3n]$, if the non-normal distribution is symmetric.

Of course, the assumption is that one has "a priori" knowledge whether the alternative non-normal distribution is skew or symmetric. Against skew distributions,and symmetric distributions having large kurtosis (ie., having long tails on both sides), T is generally more powerful than W and other goodness of fit statistics. Against symmetric distributions having kurtosis less than 3, T has very low power.

The lower percentage points of T for the above choice of r_1 and r_2 were determined empirically for sample sizes $n = 10(1) 30$.

For $n > 30$, the normal approximation of the distribution of T may be used, with $E(T) = 1$ and variance of T as $V(T)$ given above; see Tiku [27].

It may be noted that T admits straightforward generalization to multi-sample situations. Most of the goodness-of-fit tests do not.

CHAPTER 3

A NEW STATISTIC SIMILAR TO TIKU'S T, FOR TESTING FOR EXPONENTIALITY

3.1 Introduction

Given a random sample $x_1, x_2, \ldots, x_n$ of size n, we want to test the null hypothesis

H_o: that these observations come from an exponential distribution with p.d.f.

$$f_o(x) = \frac{1}{\sigma} \exp\{-(\frac{x-\theta}{\sigma})\}, \; x > \theta, \; \sigma > o$$

$$(\theta, \sigma \text{ unknown}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.1.1),$$

against the alternative hypothesis

H_1: that these observations come from a non-exponential distribution. In practical situations, this alternative distribution will have a long tail on the right hand side.

Let $y_1, y_2, \ldots, y_n$ denote the above sample observations arranged in ascending order of magnitude. Since the end observations on the right hand side would be particularly sensitive to long tails, we censor r largest observations to obtain the censored sample

$$y_1, y_2, \ldots, y_{n-r} \; .$$

Under the assumption of exponentiality, Tiku [26] obtained the following estimator of σ from his modified maximum likelihood equation,

$$\sigma_c = [\frac{1}{n}\sum_{i=1}^{n-r} y_i + qy_{n-r} - y_1]/(1-q), \; q = \frac{r}{n}$$

The bias in σ_c is $-\sigma/(n-r)$ and variance of σ_c is given by

$$V(\sigma_c) = \sigma^2(n-r-1)/(n-r)^2; \text{ see Tiku [26].}$$

Note that when $r = o$,

$$\sigma_c = S = \frac{1}{n}\sum_{i=1}^{n} y_i - y_1 = \bar{y} - y_1$$

is the maximum likelihood estimator of σ obtained from a complete sample of size n,

where

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n} x_i \; .$$

The bias in S is $-\sigma/n$ and variance of S is given by

$$V(S) = \frac{(n-1)}{n^2}\sigma^2 .$$

Define the statistic T_E as

$$T_E = \frac{\sigma_c/(1-\frac{1}{n-r})}{S/(1-\frac{1}{n})},$$

$$= (1-\frac{1}{n})\sigma_c/(1-\frac{1}{n-r})S \; , \; o < T_E < \infty .$$

Note that T_E is origin and scale invariant.

3.2 Mean And Variance Of T_E

For large n, the mean of T_E is approximately 1; see also Tiku [27] and Kendall & Stuart [11 p.232]. For example, for q = 0.5, the empirical values of the mean of T_E are:

0.999 for $n = 10$,
1.000 for $n = 20$,
0.999 for $n = 30$,
0.999 for $n = 50$.

Variance of T_E is obtained as

$$V(T_E) = \frac{(1-\frac{1}{n})^2}{(1-\frac{1}{n-r})^2} \quad V(\frac{\sigma_c}{S})$$

where

$$V(\frac{\sigma_c}{S}) \doteq [\frac{E(\sigma_c)}{E(S)}]^2 \quad [\frac{V(\sigma_c)}{E^2(\sigma_c)} + \frac{V(S)}{E^2(S)} - \frac{2Cov(\sigma_c,S)}{E(\sigma_c)\,E(S)}] \ldots (3.2.1);$$

see Kendall & Stuart [11 p.232].

We notice that both, σ_c and S, are linear functions of order statistics $y_1, y_2, \ldots, y_n$, ie.,

$$\sigma_c = \underset{\sim}{L}'\underset{\sim}{y} \quad \text{and } s = \underset{\sim}{m}'\,\underset{\sim}{y},$$

where

$$\underset{\sim}{y}' = [y_1, y_2, \ldots y_{n-r-1}, y_{n-r}, y_{n-r+1}, \ldots y_n],$$

$$\underset{\sim}{L}' = \frac{1}{n-r}[-(n-1), 1, \ldots, 1, \quad r+1, o, \ldots, o],$$

$$\underset{\sim}{m}' = \frac{1}{n}[-(n-1), 1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots 1].$$

Therefore, $Cov(\sigma_c, S) = \underset{\sim}{L}' \underset{\sim}{V} \underset{\sim}{m}$, where $\underset{\sim}{V}$ is the $n \times n$ variance-covariance matrix of ordered observations. The elements of $\underset{\sim}{V}$ can easily be obtained by using the relation

$$V(y_j) = Cov(y_j, y_k) = \sigma^2 \sum_{i=1}^{j} \frac{1}{(n-i+1)^2}$$

$$j < k$$

(see Sarhan & Greenberg [16]).

Thus

$$Cov(\sigma_c, S) = \sigma^2 \underset{\sim}{L}' \begin{bmatrix} \frac{1}{n^2} & \frac{1}{n^2} & \cdots\cdots\cdots\cdots & \frac{1}{n^2} \\ \frac{1}{n^2} & \frac{1}{n^2} + \frac{1}{(n-1)^2} & \cdots & \frac{1}{n^2} + \frac{1}{(n-1)^2} \\ \frac{1}{n^2} & \frac{1}{n^2} + \frac{1}{(n-1)^2} & \cdots & \left\{ \frac{1}{n^2} + \frac{1}{(n-1)^2} + \ldots + 1 \right\} \end{bmatrix} \underset{\sim}{m}.$$

The above expression can easily be simplified, and reduces to

$$Cov(\sigma_c, s) = \sigma^2 \frac{(n-r-1)}{n(n-r)} .$$

We therefore, obtain from equation (3.2.1)

$$V\left(\frac{\sigma_c}{\cdot}\right) \doteq \frac{\left(1 - \frac{1}{n-r}\right)^2}{\left(1 - \frac{1}{n}\right)^2} \left[\frac{1}{n-r-1} - \frac{1}{n-1} \right].$$

Thus

$$V(T_E) \doteq \frac{r}{(n-1)(n-r-1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.2.2).$$

In spite of its approximating nature, equation (3.2.2) provides accurate values. For example, for $r = n/2$ and $n = 10, 20, 30$ and 100, equation (3.2.2) gives $V(T_E) = 0.136, 0.0525, 0.0369$ and 0.0103 respectively. The empirical values of the variance for these values of n are 0.126, 0.0557, 0.0362 and 0.0102 respectively.

3.3 Testing For Exponentiality

We propose T_E as a test statistic for testing for exponentiality. As an omnibus procedure, the test based on T_E is to be used as a two tailed test. However, if one knows "a priori" the class of alternative non-exponential distribution H_1, one can improve the sensitivity of the test by employing the left or right tail. This is true, because for distributions having $\sqrt{\beta_1} > 2$, the mean of T_E shifts to the left, and small values of T_E will constitute the critical tail. For distributions with $\sqrt{\beta_1} < 2$, the opposite is true.

Calculations show that the power of T_E against some non-exponential distributions H_1 is an increasing function of q and for others, the power is rather a decreasing function of q. The choice $q = [\frac{1}{2} + 0.5n]/n$ ([k] being integer value of k) was a very good compromise. Besides, for this value of q, the distribution of T_E tends to normality very rapidly. It may be noted that the asymptotic normality of T_E can be established from the work of Moore [15], Shorak [22] and Stigler [24], and also follows from the fact that σ_c is asmptotically identical with the maximum likelihood estimator and therefore $\sigma_c/\sigma \simeq \sigma_c/\hat{S}$ is asymptotically normally distributed.

3.4 Percentage Points of T_E

The null distribution of T_E was studied by empirical sampling. For sample size $n = 10(1)20$, 30, 50 and 100, the lower and upper 1%, 2.5%, 5% and 10% points of T_E for the above choice of q were determined from M samples, where M = 15000 for $n = 10(1)20$, M = 10000 for $n = 30$, M = 8000 for $n = 50$ and M = 4000 for $n = 100$. The percentage points are given in Table 1. The percentage points were not calculated for $n < 10$, because it is unlikely that one will be doing a goodness-of-fit test with so few observations.

3.5 Sensitivity Properties

The values of the power of T_E have been obtained empirically (based on 2000 random samples) against some non-exponential distributions H_1. For each sample size ($n = 10,20,30,50$) and alternative distribution, two entries are given in Table 2, namely, the proportion which fell below α% point, and the proportion which fell above $100\ (1-\alpha)$% points; $\alpha = 1.0$, 2.5, 5.0 and 10.0. The sum of the two entries in the table is, therefore, the power of a two tailed test of size 2α%.

It may be noted that alternative non-exponential distributions fall into two categories with respect to the behaviour of T_E statistic - those which lead to an excess of small values and those which lead to an excess of large values. The statistic T_E exhibits a shift to smaller values for alternatives with Pearson coefficient $\sqrt{\beta_1} > 2$ ($\sqrt{\beta_1} = 2$ for exponential distribution) and a shift to larger values for alternatives with $\sqrt{\beta_1} < 2$.

The values of the power of T_E for some significance levels are given in Table 3 and compared with the values of the power of Shapiro & Wilk [20] W-exponential statistic. It is clear that against alternative distributions with $\sqrt{\beta_1} \geqslant 2$, T_E is slightly more powerful than W-exponential and for alternatives with $\sqrt{\beta_1} < 2$, its power is slightly smaller. On the whole, the two statistics W and T_E are of comparable magnitudes so for as their sensitivity to non-exponential distributions is concerned. It may be noted that the values of the power of W-exponential for Chi-square ($v = 3$) as reported in Shapiro & Wilk [20] are in error.

3.6 Null Distribution of T_E

As indicated earlier, the asymptotic normality of T_E can be rigorously established. To study its null distribution for small samples, we did extensive Monte Carlo simulations. For $q = [\frac{1}{2}+0.5n]/n$, the empirical (β_1, β_2) values of T_E for $n = 10, 20, 30, 50$ and 100 are (0.013, 2.502), (0.002, 2.710), (0.000, 2.755), (0.000, 2.885) and (0.000, 2.894), respectively; $\sqrt{\beta_1}$ and β_2 are Pearson coefficients of "skewness" and "kurtosis". As is indicated by these values, the distribution of T_E tends to normality with increasing sample size n(effectively $n > 20$). To verify this more fully, the empirical percentage points of T_E were compared with the normal approximation; see Table 4. It is clear that for $n > 20$, the distribution of T_E can successfully be approximated by a normal distribution with mean 1 and variance given by (3.2.2).

It may be noted that the distribution of Shapiro & Wilk W-exponential becomes unmanageable with increasing sample size. The rapid convergence of the distribution of T_E to normality seems therefore a considerable gain.

3.7 The Statistic T_E For Known Location Parameter

If the location parameter θ in the exponential distribution (3.1.1) is known, the estimator σ_c is given by (see Tiku [26])

$$\sigma_{c(\theta)} = \{ \sum_{i=1}^{n-r} y_i + r y_{n-r} \} / (n-r) .$$

Note that

$$E(\sigma_{c(\theta)}) = \sigma$$

and

$$V(\sigma_{c(\theta)}) = \sigma^2/(n-r) .$$

For known θ , define

$$T_{E(\theta)} = \sigma_{c(\theta)}/\hat{\sigma}(\theta) , \quad \hat{\sigma}(\theta) = \bar{y} .$$

The distribution of $T_{E(\theta)}$ is approximately normal with mean 1 and variance $r/\{n(n-r)\}$. For $r = [0.5 + 0.5n]$ and $n > 10$, the normal approximation provides accurate values as is clear from the following values of α% points of $T_{E(\theta)}$:

α	Lower					Upper			
	1	2.5	5	10		10	5	2.5	1
					$n=10$				
Approximate	.265	.380	.480	.595		1.41	1.52	1.62	1.74
Empirical	.346	.418	.496	.595		1.40	1.50	1.58	1.66
					$n=15$				
Approximate	.359	.460	.547	.647		1.35	1.45	1.54	1.64
Empirical	.413	.488	.558	.649		1.35	1.44	1.53	1.61
					$n=30$				
Approximate	.576	.642	.700	.766		1.23	1.30	1.36	1.42
Empirical	.590	.646	.699	.763		1.23	1.30	1.35	1.41

It may be noted that the power of $T_{E(\theta)}$ is, on the whole, of the same magnitude as Kolmogorov-Smirnov type test-statistics, $\tilde{D}$ (Srinivasan [23]) and $\hat{D}$ (Lilliefors [13]); see Schafer et. al [17]. For example, we have the following values of the power based on a two-tailed test of size 5%:

		$n = 10$			$n = 20$		
Alternative		$\tilde{D}_n$	$\hat{D}_n$	$T_{E(\theta)}$	$\tilde{D}_n$	$\hat{D}_n$	$T_{E(\theta)}$
Lognormal							
	$\sigma=2.0$	.67	.61	.64	.90	.89	.91
	$\sigma=2.4$	.81	.77	.80	.97	.97	.97
Chi-squared	$(\nu=1)$	.30	.25	.31	.48	.44	.47

3.8 Generalization to Multisample Situation

If $y_{1,i}$, $y_{2,i}$, ..., $y_{n_i,i}$, i = 1,2,..,k, are k independent samples of ordered observations from k exponential populations

$$(1/\sigma)\exp\{-(x-\theta_i)/\sigma\}\ ,\ i = 1,2,..,k, \ldots (3.8.1),$$

with a common scale parameter σ, then the k-sample versions of the above statistics are

$$T_E = \sum_{i=1}^{k}\left(1-\frac{1}{n_i-r_i}\right)^{-1}\sigma_{ci}/\sum_{i=1}^{k}\left(1-\frac{1}{n_i}\right)^{-1} S_i \ldots (3.8.2),$$

and if θ_i's are known

$$T_{E(\theta)} = \sum_{i=1}^{k}\sigma_{c_i}(\theta)/\sum_{i=1}^{k}\hat{\sigma}_i(\theta) \ldots\ldots\ldots\ldots\ldots\ldots (3.8.3).$$

Here

$$\sigma_{c_i} = \left(\sum_{j=1}^{n_i-r_i} y_{j,i} + r_i\, y_{n_i-r_i,i} - n_i\, y_{1,i}\right)/(n_i-r_i),$$

$$\sigma_{c_i(\theta)} = \left(\sum_{j=1}^{n_i-r_i} y_{j,i} + r_i\, y_{n_i-r_i,i}\right)/(n_i-r_i)$$

and similarly for S_i and $\hat{\sigma}_i(\theta)$.

(r_i is the number of observations censored on the right hand side in the i^{th} sample). For large n_i, the distributions of T_E and $T_{E(\theta)}$ are approximately normal with mean 1 and

$$V(T_E) \simeq \frac{1}{k^2} \{ \sum_{i=1}^{k} (n_i-r_i-1)^{-1} - \sum_{i=1}^{k} (n_i-1)^{-1} \} ,$$

$$V(T_{E(\theta)}) \simeq \frac{1}{k^2} \{ \sum_{i=1}^{k} (n_i-r_i)^{-1} - \sum_{i=1}^{k} (n_i)^{-1} \} .$$

It needs further study of the distributions of the above generalized versions of T_E for small samples and their power properties.

TABLE 1

VALUES OF THE LOWER AND UPPER PERCENTAGE POINTS

OF T_E

n	Lower				Upper			
	1%	2.5%	5%	10%	10%	5%	2.5%	1%
10	.274	.346	.425	.536	1.475	1.599	1.699	1.799
11	.259	.338	.418	.523	1.499	1.636	1.745	1.875
12	.330	.413	.481	.583	1.423	1.537	1.635	1.731
13	.321	.394	.475	.576	1.434	1.559	1.657	1.765
14	.371	.445	.525	.620	1.382	1.489	1.578	1.662
15	.375	.446	.518	.608	1.393	1.503	1.594	1.700
16	.413	.491	.566	.645	1.353	1.446	1.527	1.606
17	.395	.475	.552	.643	1.368	1.468	1.549	1.641
18	.451	.519	.587	.672	1.327	1.418	1.497	1.579
19	.447	.515	.586	.666	1.338	1.434	1.506	1.592
20	.483	.552	.615	.693	1.307	1.391	1.465	1.542
30	.569	.629	.687	.752	1.247	1.311	1.364	1.429
50	.674	.723	.764	.814	1.181	1.234	1.279	1.329
100	.766	.798	.832	.871	1.128	1.166	1.194	1.233

TABLE 2

VALUES OF THE POWER OF T_E FOR LOWER AND UPPER 1,2.5, 5 and 10 PERCENTAGE POINTS.

n	%	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
10	1	.12	.00	.00	.97	.39	.00	.65	.00	.00	.22
		.00	.02	.03	.00	.00	.08	.00	.40	.04	.00
	2.5	.20	.01	.01	.98	.49	.00	.72	.00	.00	.30
		.01	.05	.07	.00	.00	.16	.00	.56	.08	.00
	5	.28	.02	.01	.98	.58	.00	.79	.00	.01	.36
		.02	.09	.12	.00	.00	.27	.00	.70	.14	.01
	10	.40	.04	.03	.99	.69	.01	.85	.00	.02	.46
		.03	.17	.23	.00	.01	.42	.00	.80	.26	.02
20	1	.28	.00	.00	1.00	.73	.00	.93	.00	.00	.47
		.00	.04	.07	.00	.00	.30	.00	.87	.07	.00
	2.5	.40	.00	.00	1.00	.80	.00	.96	.00	.00	.56
		.00	.08	.14	.00	.00	.46	.00	.93	.15	.00
	5	.48	.01	.00	1.00	.87	.00	.97	.00	.00	.64
		.00	.14	.23	.00	.00	.61	.00	.97	.25	.00
	10	.60	.02	.00	1.00	.91	.00	.98	.00	.01	.72
		.01	.24	.39	.00	.00	.76	.00	.99	.39	.01
30	1	.41	.00	.00	1.00	.88	.00	.99	.00	.00	.65
		.00	.07	.13	.00	.00	.58	.00	.98	.14	.00
	2.5	.53	.00	.00	1.00	.93	.00	1.00	.00	.00	.71
		.00	.12	.23	.00	.00	.72	.00	.99	.24	.00
	5	.63	.01	.00	1.00	.95	.00	1.00	.00	.00	.78
		.00	.19	.35	.00	.00	.82	.00	1.00	.36	.00
	10	.73	.01	.00	1.00	.98	.00	1.00	.00	.01	.84
		.00	.32	.52	.00	.00	.91	.00	1.00	.52	.00
50	1	.66	.00	.00	1.00	.99	.00	1.00	.00	.00	.86
		.00	.10	.30	.00	.00	.89	.00	1.00	.25	.00
	2.5	.76	.00	.00	1.00	1.00	.00	1.00	.00	.00	.90
		.00	.20	.46	.00	.00	.95	.00	1.00	.40	.00
	5	.83	.00	.00	1.00	1.00	.00	1.00	.00	.00	.93
		.00	.33	.62	.00	.00	.98	.00	1.00	.56	.00
	10	.90	.00	.00	1.00	1.00	.00	1.00	.00	.00	.96
		.00	.47	.75	.00	.00	.99	.00	1.00	.69	.00

(1) Chi-square $v = 1$, (2) Chi-square $v=3$, (3) Chi-square $v = 4$, (4) Weibull $k = .2$, (5) Weibull $k = .5$, (6) Weibull $k = 2$, (7) Lognormal $\sigma = 2.4$, (8) Beta $a = 2$, $b = 1$, (9) Halfnormal, (10) Half Cauchy.

TABLE 3

VALUES OF THE POWER OF T_E AND W-EXPONENTIAL
FOR 5, 10 AND 20 PER CENT SIGNIFICANCE LEVELS

Alternative H_1	%	n = 10		n = 20		n = 30		n = 50	
		T_E	W-Exp.	T_E	W-Exp.	T_E	W-Exp.	T_E	W-Exp.
Chi-Squared	5	.21	.17	.40	.28	.53	.42	.76	.58
v = 1	10	.30	.27	.48	.40	.63	.54	.83	.73
	20	.43	.38	.61	.55	.73	.69	.90	.83
v = 3	5	.06	.08	.08	.10	.12	.17	.20	.25
	10	.11	.13	.15	.19	.20	.26	.33	.36
	20	.21	.22	.26	.31	.33	.40	.47	.50
v = 4	5	.08	.12	.14	.22	.23	.35	.46	.57
	10	.13	.18	.23	.31	.35	.46	.62	.69
	20	.26	.29	.39	.46	.52	.60	.75	.81
Weibull k =.2	5	.98	.93	1.00	1.00	1.00	1.00		
	10	.98	.96	1.00	1.00	1.00	1.00		
	20	.99	.98	1.00	1.00	1.00	1.00		
k =.5	5	.49	.43	.80	.73	.93	.90	1.00	.99
	10	.58	.54	.87	.82	.95	.94	1.00	.99
	20	.70	.66	.91	.89	.98	.97	1.00	1.00

continued....

TABLE 3 (Continued)

Alternative H_1	%	n = 10		n = 20		n = 30		n = 50	
		T_E	W-Exp.	T_E	W-Exp.	T_E	W-Exp.	T_E	W-Exp.
k =2.0	5	.16	.26	.46	.63	.72	.88	.95	.99
	10	.27	.38	.61	.75	.82	.93	.98	1.00
	20	.43	.52	.76	.86	.91	.97	.99	1.00
Lognormal, σ = 2.4	5	.72	.67	.96	.93	1.00	.99	1.00	1.00
	10	.79	.77	.97	.96	1.00	.99	1.00	1.00
	20	.85	.84	.98	.98	1.00	1.00	1.00	1.00
Beta, a = 2 b = 1	5	.56	.72	.93	.98	.99	1.00	1.00	1.00
	10	.70	.82	.97	1.00	1.00	1.00	1.00	1.00
	20	.80	.90	.99	1.00	1.00	1.00	1.00	1.00
Halfnormal	5	.08	.11	.15	.21	.24	.34	.40	.55
	10	.15	.18	.25	.33	.36	.46	.56	.70
	20	.28	.30	.40	.48	.53	.63	.69	.84
Half Cauchy	5	.30	.40	.56	.68	.71	.83	.90	.95
	10	.37	.48	.64	.74	.78	.86	.93	.98
	20	.48	.58	.73	.81	.84	.90	.96	.99

TABLE 4

100P PER CENT POINTS OF T_E

P	η = 20		η = 30		η = 50		η = 100	
	Empirical	Approx.	Emp.	Approx.	Emp.	Approx.	Emp.	Approx.
Lower								
.01	.483	.438	.569	.553	.674	.661	.766	.764
.025	.552	.526	.629	.623	.723	.714	.798	.801
.05	.615	.602	.687	.684	.764	.760	.832	.833
.10	.693	.690	.752	.754	.814	.813	.871	.870
Upper								
.10	1.307	1.310	1.247	1.246	1.181	1.187	1.128	1.130
.05	1.391	1.398	1.311	1.316	1.234	1.234	1.166	1.167
.025	1.465	1.474	1.364	1.377	1.279	1.286	1.194	1.199
.01	1.542	1.562	1.429	1.447	1.329	1.339	1.233	1.236

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