CHANGE OF RINGS THEOREMS

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SCOPE AND CONTENTS: The intention of this thesis is to gather together the results of various papers concerning the three change of rings theorems, generalizing them where possible, and to determine if the various results, although under different hypotheses, are in fact, distinct.

PREFACE

Classically, there exist three theorems which relate the two homological dimensions of a module over two rings. We deal with the first and last of these theorems. J. R. Strooker and L. W. Small have significantly generalized the "Third Change of Rings Theorem" and we have simply reorganized their results as Chapter 2. J. M. Cohen and C. U. Jensen have generalized the "First Change of Rings Theorem", each with hypotheses seemingly distinct from the other. However, as Chapter 3 we show that by developing new proofs for their theorems we can, indeed, generalize their results and by so doing show that their hypotheses coincide. Some examples due to Small and Cohen make up Chapter 4 as a completion to the work.

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CHAPTER 0

INTRODUCTION

The theme of our work will be to relate the two possible dimensions of a module in the case that it is a module over two rings. Classically, three such theorems exist, in which the two rings, R and S, are related by the fact that $S = R/(x)$ where (x) is the principal ideal generated by a central non-zero divisor, x, of R. Two contrasting situations are considered: that in which the R-module A is annihilated by x, that is, $xA = 0$ and that in which x acts faithfully on A, that is, x is a non-zero divisor for A. Formalizing these two concepts we have the following theorems (cf. (6) and (5), chapter 4):

THEOREM A (First Change of Rings Theorem). Let R be a ring with unit and x a central element of R which is a non-zero divisor. Write $S = R/(x)$. Let A be a non-zero S-module with $\text{p.dim}_S A = n < \infty$. Then

$$\text{p.dim}_R A = n + 1.$$

(This may be written

$$\text{p. dim}_R A = \text{p.dim}_S (A/xA) + 1$$

since $xA = 0$.)

THEOREM A'. Let R be a ring with unit and x a central element of R which is a non-zero divisor. Write $S = R/(x)$. Let A be a non-zero S-module with $i.\dim_S A = n < \infty$. Then

$$i.\dim_R A = n + 1.$$

(This may be written

$$i.\dim_R A = i.\dim_S (A/xA) + 1.)$$

THEOREM B (Second Change of Rings Theorem). Let R be a ring with unit and x a central element in R. Write $S = R/(x)$. Let A be an R-module and suppose that x is a non-zero divisor both on R and on A. Then

$$p.\dim_S (A/xA) \leqq p.\dim_R A.$$

THEOREM B'. Let R be a ring with unit and x a central element in R. Write $S = R/(x)$. Let A be an R-module and suppose x is a non-zero divisor both on R and on A. Then

$$i.\dim_S (A/xA) \leqq i.\dim_R A - 1$$

except when A is R-injective (in which case $A = xA$).

THEOREM C (Third Change of Rings Theorem). Let R be a left-Noetherian ring, x a central element in the Jacobson radical of R. Write $S = R/(x)$. Let A be a finitely generated R-module. Assume x is a non-zero divisor on both R and A. Then

$$p.\dim_R A = p.\dim_S (A/xA).$$

<u>THEOREM C'</u>. Let R be a commutative Noetherian ring and x an element in the Jacobson radical of R. Write S = R/(x). Let A be a finitely generated non-zero R-module with x a non-zero divisor of A. Then

$$\mathrm{i.dim}_R A = \mathrm{i.dim}_R (A/xA).$$

If, further, x is a non-zero divisor of R, then

$$\mathrm{i.dim}_R A = 1 + \mathrm{i.dim}_S (A/xA).$$

The Second Change of Rings Theorem adds nothing to our discussion since it is incorporated into the statement and proof of Theorem C, and we shall not consider it here. In the proof of Theorem C, (6), it is seen that by a simple induction step one needs consider only the case in which A/xA being R/(x)-projective implies A is R-projective. Thus Strooker (9) considered cases in which R/I-projectives could be lifted to R-projectives where I is a suitable ideal of R. Strooker's main result is somewhat stronger than an auxiliary result of Small ((8),Lemma 1) who also introduces a theorem concerning global dimension. Prior to proving these results (Chapter 2) we shall prove some preliminary propositions all of which have some intrinsic interest. In Chapter 3 we shall prove the results of Jensen (4), and Cohen (2) concerning a rather wide generalization of Theorems A and A' along with a theorem concerning weak dimensions. We shall prove, as a conclusion to

Chapter 3 that, in fact, Cohen's result and Jensen's result are equivalent. Some consequences of these change of rings theorems will be discussed in Chapter 4 by exhibiting several applications and examples.

In the following we shall assume all rings have units and are not necessarily commutative, and that "module" denotes "left module" unless it is otherwise stated. Also, unless otherwise stated, all dimensions refer to left dimensions. We adopt the convention that the dimension of a zero module $\dim(0) = -\infty$. If not so stated, the dimension of a module is assumed to be finite. The numbering of 'propositions' follows the decimal convention in which the first digit indicates the section number. Main theorems are numbered consecutively regardless of the section they appear in and Corollaries, following the decimal code, have as their first digit the number of the theorem to which they are Corollaries. All theorems herein hold equally well if "module" were taken to mean "right module" throughout although proofs would have to be modified accordingly.

CHAPTER 1

PRELIMINARIES

The propositions in this section will be used a great deal later on but are placed here because they are interesting in their own right. The following proposition can be found with identical proof for projective dimension in (6), Theorem 1.2.

PROPOSITION 1.1. Let

$$0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0$$

be an exact sequence of R-modules.

(1) If two of the weak dimensions $w.\dim_R A$, $w.\dim_R B$, $w.\dim_R C$ are finite, so is the third.

(2) If $w.\dim_R A > w.\dim_R B$, then

$$w.\dim_R C = w.\dim_R A.$$

(3) If $w.\dim_R A < w.\dim_R B$, then

$$w.\dim_R C = w.\dim_R B + 1.$$

(4) If $w.\dim_R A = w.\dim_R B$, then

$$w.\dim_R C \leqq w.\dim_R A + 1.$$

PROOF. If any of A, B, C are zero, the results are trivial. Suppose, therefore, that A, B, C are non-zero and let A be flat. Then $w.\dim_R B \geqq w.\dim_R A$ and obviously $w.\dim_R C = w.\dim_R B + 1$ unless B is flat in which case

$w.\dim_R C = 1$ satisfying (4). Suppose C is flat. Then $w.\dim_R B = w.\dim_R A = 0$ which satisfies (4). We proceed, therefore, under the assumption that A, C are not flat. Write $A = P/D$ with P flat. Then B has the form E/D, where $D \subset E \subset P$, and $C \simeq P/E$. Thus $w.\dim_R E = w.\dim_R C - 1$, $w.\dim_R D = w.\dim_R A - 1$, $w.\dim_R (E/D) = w.\dim_R B$. We have the exact sequence

$$0 \to D \to E \to B \to 0.$$

By using it and induction on the sum of the two finite dimensions we get (1) of the theorem. Hence we assume all three dimensions are finite and we do induction on their sum. The induction assumption on D, E, B gives the following information when translated back to A, B, C:

(a) If $w.\dim_R C > w.\dim_R A$, then $w.\dim_R B = w.\dim_R C - 1$,

(b) If $w.\dim_R C < w.\dim_R A$, then $w.\dim_R B = w.\dim_R A$,

(c) If $w.\dim_R C = w.\dim_R A$, then $w.\dim_R B \leqq w.\dim_R A$.

But these are just a logical rearrangement of the statements of the theorem so we are finished.

PROPOSITION 1.2. Let

$$0 \to C \to A \to B \to 0$$

be an exact sequence of R-modules.

(1) If two of the dimensions $i.\dim_R A$,

$i.\dim_R B$, $i.\dim_R C$ are finite, so is the third.

(2) If $i.\dim_R A > i.\dim_R B$, then

$i.\dim_R C = i.\dim_R A$.

(3) If $i.\dim_R B > i.\dim_R A$, then

$i.\dim_R C = i.\dim_R B + 1$.

(4) If $i.\dim_R B = i.\dim_R A$, then

$i.\dim_R C \leqq i.\dim_R B + 1$.

The proof of Proposition 1.2 is exactly dual to that of Proposition 1.1 and will be omitted.

PROPOSITION 1.3.† Let R and S be two rings and let $\varepsilon : R \to S$ be a unitary homomorphism. Let B be any S-module. Then

$$p.\dim_R B \leqq p.\dim_S B + \dim_R S.$$

PROOF. If $B = 0$, the result is trivial since $p.\dim_R B = -\infty$. Suppose $B \neq 0$ and let $p.\dim_S B = h < \infty$, $p.\dim_R S = k < \infty$. Let C be any R-module and let

$$(*) \quad 0 \to C \to Q^0 \to Q^1 \to \cdots$$

be an R-injective resolution of C. Now, applying the functor $\mathrm{Hom}_R(S,\_)$ to (*) we have the sequence

$$0 \to \mathrm{Hom}_R(S,C) \to \mathrm{Hom}_R(S,Q^0) \to \cdots \to \mathrm{Hom}_R(S,Q^k) \xrightarrow{\phi}$$

$$\mathrm{Hom}_R(S,Q^{k+1}) \to \cdots .$$

The sequence $0 \to \mathrm{Im}\,\phi \to \mathrm{Hom}_R(S,Q^{k+1}) \to \cdots \quad (**)$ is exact since $p.\dim_R S = k$ yielding $\mathrm{Ext}^m_R(S,A) = 0$ for all R-modules A and all $m > k$. Also by (1), pg. 30,

†. The Propositions 1.3, 1.4 and 1.5 are due to Jensen (4).

Proposition 6.1, $Hom_R(S,Q^j)$ is S-injective for all j yielding (**) an S-injective resolution of Im ϕ = D. Apply the functor $Hom_S(B,\_)$ to (**) giving the sequence

$$0 \to Hom_S(B,D) \to Hom_S(B,Hom_R(S,Q^{k+1})) \to \dots \to$$

$$\to Hom_S(B,Hom_R(S,Q^{h+k+1})) \to \dots$$

which is exact after h steps because $p.dim_S B = h$ yielding $Ext^m_S(B,A) = 0$ for all S-modules A and all m > h. Thus the homology group at

$$Hom_S(B,Hom_R(S,Q^{h+k+1})) \simeq Hom_R(B,Q^{h+k+1}),$$

$Ext_R^{h+k+1}(B,C) = 0$ for any R-module C. Hence

$$p.dim_R B \leqq h + k.$$

<u>PROPOSITION 1.4</u>. Let R and S be two rings and let $\varepsilon : R \to S$ be a unitary homomorphism. Let B be any S-module. Then

$$i.dim_R B \leqq i.dim_S B + r.w.dim_R S.$$

<u>PROOF</u>. Let $i.dim_S B = h < \infty$, $w.dim_R S = k < \infty$. Let C be any R-module and let

$$\dots \to P_2 \to P_1 \to P_0 \to C \to 0 \quad (*)$$

be an R-projective resolution of C. Now, applying the functor $S \otimes_R \_$ to (*) we obtain the sequence

$$\dots \xrightarrow{\phi} S \otimes_R P_{k+1} \to S \otimes_R P_k \to \dots \to S \otimes_R P_0 \to S \otimes_R C \to 0.$$

The sequence

$$\dots \to S \otimes_R P_{k+1} \to \text{Coker } \phi \to 0 \quad (**)$$

is exact since $r.w.\dim_R S = k$ yielding $\mathrm{Tor}_m^R(S,A) = 0$ for all R-modules A and all $m > k$. Also, by (1) pg. 30 Proposition 6.1, $S \otimes_R P_j$ is S-projective for all j yielding (**) an R-projective resolution of Coker $\phi = D$. Apply, now, the functor $\mathrm{Hom}_S(\_,B)$ to the sequence (**) giving the sequence

$$0 \to \mathrm{Hom}_S(D,B) \to \mathrm{Hom}_S(S \otimes_R P_{k+1},B) \to \cdots$$

which becomes exact after h steps since $i.\dim_S B = h$ yielding $\mathrm{Ext}_S^m(A,B) = 0$ for all S-modules A and all $m > h$. Thus the homology group at

$$\mathrm{Hom}_S(S \otimes_R P_{k+h+1},B) \simeq \mathrm{Hom}_R(P_{k+h+1},B),$$

$\mathrm{Ext}_R^{k+h+1}(C,B) = 0$ for any R-module C. Hence

$$i.\dim_R B \leqq h + k.$$

PROPOSITION 1.5. Let R and S be two rings and let $\varepsilon : R \to S$ be a unitary homomorphism. Let B be any S-module. Then

$$w.\dim_R B \leqq w.\dim_S B + w.\dim_R S.$$

PROOF. We first show that if P is R-right-flat, then $P \otimes_R S$ is S-right-flat: let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of left S-modules. Apply the functor $(P \otimes_R S) \otimes_S \_$ to obtain the sequence

$$0 \to (P \otimes_R S) \otimes_S A \to (P \otimes_R S) \otimes_S B \to (P \otimes_R S) \otimes_S C \to 0 \quad (*)$$

which is isomorphic to the sequence

$$0 \to P \otimes_R A \to P \otimes_R B \to P \otimes_R C \to 0. \quad (**)$$

But P is R-right-flat implying that (**) is exact which means (*) is exact, proving that $P \otimes_R S$ is S-right-flat. Let C be an arbitrary right R-module and let

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to C \to 0$$

be an R-flat resolution of C. Applying the functor $\_ \otimes_R S$ we obtain the exact sequence

$$\cdots \to P_{k+2} \otimes_R S \xrightarrow{\phi} P_{k+1} \otimes_R S \to \text{Coker } \phi \to 0 \quad (***)$$

since $w.\dim_R S = k$. As we have seen $P_j \otimes_R S$ is S-right-flat so (***) is an S-flat resolution of Coker ϕ. Apply the functor $\_ \otimes_S B$ to obtain the sequence

$$\cdots \to P_{h+k+2} \otimes_R S \otimes_S B \xrightarrow{\phi'} P_{h+k+1} \otimes_R S \otimes_S B \to \text{coker } \phi' \to 0$$

which is exact since $w.\dim_S B = h$. But this means the homology group of

$$P_{h+k+1} \otimes_R S \otimes_S B \cong P_{h+k+1} \otimes_R B,$$

$\text{Tor}^R_{h+k+1}(C,B) = 0$. Since C was arbitrary we have

$$w.\dim_R B \leqq h + k.$$

The following theorem is due to Small and may be found as Lemma 1, (8).

<u>THEOREM 1</u>. Let R be a ring and I a two-sided ideal contained in the Jacobson radical of R. Write

$S = R/I$. Let A be a finitely generated R-module such that A possesses a free resolution of finitely generated modules and with $\mathrm{Tor}_p^R(S,A) = 0$ for all $p > 0$. Then

$$\mathrm{p.dim}_R A = \mathrm{p.dim}_S(A/IA).$$

PROOF. Suppose $A \neq 0$ since otherwise the result is trivial. Let F be a free R-module and let

$$0 \to L \to F \to A \to 0 \quad (*)$$

be exact. Since $\mathrm{Tor}_p^R(S,A) = 0$ for all $p > 0$ we have from the long exact sequence of (*) that $\mathrm{Tor}_p^R(S,L) = 0$ for all $p > 0$. We shall leave to the last the proof that A/IA is S-projective if and only if A is R-projective. We proceed by induction on $n = \mathrm{p.dim}_R A < \infty$. From (*) we see that $\mathrm{p.dim}_R L = n - 1$ and since $\mathrm{Tor}_1^R(S,A) = 0$ (and for any R-module C, $S \otimes_R C \simeq C/IC$) we have the exact sequence

$$0 \to L/IL \to F/IF \to A/IA \to 0.$$

But F/IF is S-free and $\mathrm{Tor}_p^R(S,L) = 0$ for all $p > 0$ so by the induction hypothesis $\mathrm{p.dim}_S(L/IL) = n - 1$ giving $\mathrm{p.dim}_S(A/IA) = n$. It remains to start off the induction at $n = 0$. Obviously if A is R-projective, then A/IA is S-projective. Suppose A/IA is S-free. Let $\{e_\alpha\}_{\alpha \in J}$ be elements of A whose images under the natural map $\eta : A \to A/IA$ are the basis for A/IA. Let E be the submodule of A generated by $\{e_\alpha\}_{\alpha \in J}$. Now $E + IA = A$

so we have $I(A/E) = A/E$ which means $A = E$ by Nakayama.†
Let F' be the free R-module with basis $\{x_\alpha\}_{\alpha \in J}$ and let $g : F' \to A$ be the map such that $g(x_\alpha) = e_\alpha$. Then we have an exact sequence

$$0 \to L' \to F' \to A \to 0.$$

Since $\mathrm{Tor}_1^R(S,A) = 0$ we have, from the long exact sequence, the sequence

$$0 \to S \otimes_R L' \to S \otimes_R F' \xrightarrow{1_S \otimes_R g} S \otimes_R A \to 0.$$

Now $1_S \otimes_R g$ is an isomorphism of the S-free modules F'/IF' and A/IA since generators are mapped onto generators, giving $L'/IL' = 0$ or $L' = IL'$ in which case by Nakayama† $L' = 0$. Hence A is R-free. Suppose now A/IA is S-projective. We have an exact sequence

$$0 \to L/IL \to F/IF \to A/IA \to 0$$

which splits yielding

$$F/IF \simeq L/IL \oplus A/IA \simeq (L \oplus A)/I(L \oplus A),$$

which is S-free so that $L \oplus A$ is R-free which implies that A is R-projective. Hence the theorem holds at $n = 0$ and it is proved.

<u>REMARKS</u>. (1) If R is left-Noetherian, then any finitely generated R-module possesses a free resolution of finitely generated modules.

(2) We use the fact that I is contained in the radical of R and that A is finitely generated

only at the points marked † in the proof to satisfy Nakayama's Lemma which yields that $IB = B$ implies $B = 0$ for all finitely generated R-modules B. We can drop the restriction on A by assuming that

(i) R is left-perfect in which case $IB = B$ implies $B = 0$ for <u>all</u> R-modules B, where I is contained in the radical of R, or

(ii) I is nilpotent in which case $IB = B$ implies $B = 0$ for <u>all</u> R-modules B.

CHAPTER 2

THE THIRD CHANGE OF RINGS THEOREM

In the proof of the classical third change of rings theorem, as in the proof of Theorem 1, the induction step is easy when $\mathrm{Tor}_1^R(S,A) = 0$ for A an R-module, S = R/I where I is a suitable ideal of the ring R. The difficulty arises at n = 0. That is, if A/IA is S-projective, is A R-projective? Strooker dealt, therefore, with the problem of when such S-projectives could be lifted to R-projectives. The main result of Strooker's paper (9) uses the following lemma (Lemma 0, (9)). First we need a definition. In any category, an epimorphism $f : A \to B$ is called a <u>cover</u> if any morphism $g : X \to A$ such that fg is an epimorphism, must needs be an epimorphism. Sloppily we say that A is a cover of B.

<u>LEMMA 2.1</u>. Let R be a ring and I a two-sided ideal of R. Write S = R/I. Let A be a finitely generated non-zero S-module. If E is an R-cover of A, then E is finitely generated. If A is S-projective then $E/IE \simeq A$.

<u>PROOF</u>. Let t_X denote the natural map $X \to X/IX$ and if $f : X \to Y$ let $\bar{f}$ denote the map $X/IX \to Y/IY$ for X, Y R-modules. Let $f' : L \to A$ be an epimorphism where L

is a finitely generated S-free module. Let B be a free R-module on the same number of generators as L and define $g : B \to L$ by mapping generators to generators; g is an epimorphism. Let $s : E \to A$ be an R-cover of A and let $f : B \to E$ be such that $sf = f'g$. Now $f'g$ is an epimorphism and by the cover property of s we have that f is an epimorphism. Thus E is finitely generated. Now suppose A is S-projective. Then since $\bar{s} : E/IE \to A$ is onto there exists a map $h : A \to E/IE$ with $\bar{s}h = 1_A$. Let $F = t_E^{-1}(h(A))$. Then $s(F) = \bar{s}t_E(F) = \bar{s}(h(A)) = A$. But s is a cover so $F = E$ and hence $A = E/IE$.

The main result of Strooker's paper is the following

PROPOSITION 2.2 (Strooker). Let R be a left-Noetherian ring and I a two-sided ideal contained in the Jacobson radical of R. Let A be a finitely generated projective non-zero $S = R/I$-module. Suppose the R-module E is an R-cover of A and that $\mathrm{Tor}_1^R(S,E) = 0$. Then E is uniquely determined up to isomorphism and is finitely generated projective. Moreover, $E/IE \simeq A$.

PROOF. By Lemma 2.1, we have that E is finitely generated and that $E/IE \simeq A$. Let f be an epimorphism of a finitely generated free module L onto E and let $g : D \to L$ be the kernel of f. Since $\mathrm{Tor}_1^R(S,E) = 0$ the bottom row in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \xrightarrow{g} & L & \xrightarrow{f} & E & \longrightarrow & 0 \\ & & \downarrow t_D & & \downarrow t_L & & \downarrow t_E & & \\ 0 & \longrightarrow & D/ID & \xrightarrow{\bar{g}} & L/IL & \xrightarrow{\bar{f}} & E/IE & \longrightarrow & 0 \end{array}$$

is also exact. Since E/IE is S-projective the bottom row splits and we have a map $\bar{h} : L/IL \to D/ID$ such that $\bar{h}\bar{g} = 1_{D/ID}$. Since L is projective we have a map $h : L \to D$ such that $t_D h = \bar{h} t_L$. If we can show hg to be an automorphism of D we will have split the top row and so E would be projective. The commutative diagram shows that $t_D hg = \bar{h} t_L g = \bar{h}\bar{g} t_D = t_D$. Since R is Noetherian, D is finitely generated, so t_D is a cover and hg is surjective. (The lemma of Nakayama states that $f : A \to B$ is a cover if A is finitely generated and $\ker f \subset J(R)A$.) Thus hg is an epimorphism of the Noetherian module D onto itself and so it is an automorphism (N. Bourbaki, Éléments de Mathématique, Algèbre, Chap. 8, Lemma 3, pg. 23). Hence E is a projective cover of E/IE and so is uniquely determined up to isomorphism.

REMARKS. (1) We remark at this point that Proposition 2.2 generalizes Remark (1) of Theorem 1 since $f : A \to A/IA$ is a cover by Nakayama since $I \subset J(R)$ and $\mathrm{Tor}_p^R(S,A) = 0$ for all $p > 0$ gives, in particular, $\mathrm{Tor}_1^R(S,A) = 0$. Proposition 2.2 gives that if A/IA is S-projective, A is R-projective and by the familiar induction argument we have $\mathrm{p.dim}_R A = \mathrm{p.dim}_S(A/IA)$ which is the result of Theorem 1.

(2) Proposition 2.2 yields the classical third change of rings theorem for x a central non-zero divisor in the radical of R which acts faithfully on A and $I = (x)$, that is A/xA being $S = R/(x)$-projective implies A is R-projective.

PROOF. By Nakayama the natural map $A \to A/xA$ is a cover since $x \in J(R)$. Also, from the short exact sequence

$$0 \to xR \to R \to S \to 0$$

we have the long exact sequence

$$0 \to \mathrm{Tor}_1^R(S,A) \to xR \otimes_R A \to R \otimes_R A \to S \otimes_R A \to 0$$

or, isomorphically,

$$0 \to \mathrm{Tor}_1^R(S,A) \to xR \otimes_R A \to A \to A/xA \to 0.$$

Since x acts faithfully on A, $xR \otimes_R A \simeq xA$ $(x \otimes_R e \rightsquigarrow xe)$ and so the sequence

$$0 \to xA \to A \to A/xA \to 0$$

is exact yielding $\mathrm{Tor}_1^R(S,A) = 0$. Now Proposition 2.2 applies and the result follows.

(3) In Proposition 2.2 we can drop the requirement that R be Noetherian if we assume that E is finitely presented.

PROOF. Using the latter part of the proof of Theorem 1 we have an exact sequence

$$0 \to L' \to F' \to E \to 0$$

where E replaces the A of Theorem 1. Since E is finitely presented we have that L' is finitely generated. Also, since $\mathrm{Tor}_1^R(S,E) = 0$ the proof moves to its conclusion. Since $E/IE \approx A$ is S-projective, we have that E is R-projective. Thus E is uniquely determined up to isomorphism and we have the result of Proposition 2.2.

Strooker showed that he could remove the requirement that R be left-Noetherian by requiring that E be finitely presented and that R be a direct limit of a directed set of left-Noetherian rings. We have shown that by Theorem 1, we need only the assumption that E is finitely presented and hence we may drop the Noetherian requirement entirely, assuming R to be an arbitrary ring.

We turn our attention now to a change of rings theorem concerning global dimensions which is due to Small ((8), Theorem 1).

THEOREM 2 (Small). Let R be a ring and I a two-sided ideal in the radical of R. Write $S = R/I$. Suppose (1) I is nilpotent, or

(2) R is left-Noetherian, or

(3) R is left-perfect.

Then

$$\text{l.gl.dim } R \leq \text{l.gl.dim } S + \text{r.w.dim}_R S.$$

PROOF. (1) Suppose I is nilpotent. Let A be an arbitrary non-zero R-module and suppose $\text{r.w.dim}_R S = n < \infty$. Consider an R-free resolution of A

$$0 \to X_n \to F_{n-1} \to \cdots \to F_0 \to A \to 0$$

where the F_i are R-free. By shifting along this sequence we have $\text{Tor}_p^R(S, X_n) = \text{Tor}_{n+p}^R(S, A)$ for all $p > 0$. Since $\text{r.w.dim}_R S = n$, $\text{Tor}_{n+p}^R(S, A) = 0$ for all $p > 0$. Hence X_n satisfies the hypothesis of Theorem 1 (Remark (2), (ii)), so that

$$\text{p.dim}_R X_n = \text{p.dim}_S (X_n / IX_n) \leq \text{l.gl.dim } S.$$

Now by "tacking" on a resolution of X_n onto

$$\cdots \to F_{n-1} \to \cdots \to F_0 \to A \to 0$$

we see that

$$\text{p.dim}_R A \leq \text{p.dim}_R X_n + \text{r.w.dim}_R S \leq \text{r.w.dim}_R S + \text{l.gl.dim } S.$$

Since A was arbitrary,

$$\text{l.gl.dim } R \leq \text{r.w.dim}_R S + \text{l.gl.dim } S.$$

(2) If R is left-Noetherian and A is finitely generated the X_n, F_{n-1}, ..., F_0 may all be taken to be finitely generated and Theorem 1, Remark (1) applies.

(3) If R is left-perfect then Theorem 1, Remark (2) (i) applies and the theorem is proved.

CHAPTER 3

THE FIRST CHANGE OF RINGS THEOREM

Propositions 1.3, 1.4 and 1.5 give inequalities regarding dimensions involving a change of rings. We wish now to look at conditions on the ideal I which will yield equality. Theorems A and A' give one particular case in which equality holds but both Jensen (4) and Cohen (2) developed rather wide generalizations of these. Before considering these results we need three lemmas:

LEMMA 3.1. Let R be a ring and I a two-sided ideal of R. Write $S = R/I$. If B is any S-module with $\text{p.dim}_R B = 1$ and $\text{p.dim}_S B \leqq 2$ then $I \otimes_R B$ is S-projective.

PROOF. From the exact sequence

$$0 \to I \to R \to S \to 0$$

we obtain the exact sequence

$$0 \to \text{Tor}_1^R(S,B) \to I \otimes_R B \xrightarrow{f} R \otimes_R B \to S \otimes_R B \to 0. \quad (*)$$

But $R \otimes_R B \simeq B$ so that for $i \otimes b \in I \otimes_R B$, $f(i \otimes b) = ib = 0$ since B is an S-module. Hence $f = 0$ and $\text{Tor}_1^R(S,B) \simeq I \otimes_R B$. ($I \otimes_R B$ is an S-module since $I(I \otimes_R B) = I \otimes_R IB = 0$ since $IB = 0$.) We now let F be an R-free module so that

$$0 \to K \to F \to B \to 0$$

is exact. But $p.\dim_R B = 1$ which means K is R-projective. Now applying the functor $S \otimes_R -$ we have the exact sequence

$$0 \to \mathrm{Tor}_1^R(S,B) \to S \otimes_R K \to S \otimes_R F \to S \otimes_R B \to 0. \quad (**)$$

These are all S-modules and, in fact, $S \otimes_R K$ and $S \otimes_R F$ are S-projective ((1)). Since (**) is an S-exact sequence of projectives, by dimension shifting we see that $\mathrm{Tor}_1^R(S,B)$ is also S-projective since $p.\dim_S B \leq 2$. ($\mathrm{Ext}_S^1(\mathrm{Tor}_1^R(S,B),C) \simeq \mathrm{Ext}_S^3(B,C)$ since $S \otimes_R B \simeq B$ by (*).) But $I \otimes_R B \simeq \mathrm{Tor}_1^R(S,B)$ and so $I \otimes_R B$ is S-projective.

LEMMA 3.2. Let R be a ring and I a two-sided ideal of R. Write $S = R/I$. If B is any S-module with $i.\dim_R B = 1$ and $i.\dim_S B \leq 2$ then $\mathrm{Hom}_R(I,B)$ is S-injective.

PROOF. From the exact sequence

$$0 \to I \to R \to S \to 0$$

we obtain the exact sequence

$$0 \to \mathrm{Hom}_R(S,B) \to \mathrm{Hom}_R(R,B) \xrightarrow{f} \mathrm{Hom}_R(I,B) \to \mathrm{Ext}_R^1(S,B) \to 0. \quad (*)$$

But $\mathrm{Hom}_R(R,B) \simeq B$ and f is just the restriction of $\sigma \in \mathrm{Hom}_R(R,B)$ to I. But $\sigma(i) = i\sigma(1) = ib = 0$ for $i \in I$ and $b = \sigma(1) \in B$. Thus $f = 0$ and $\mathrm{Hom}_R(S,B) \simeq B$ and $\mathrm{Hom}_R(I,B) \simeq \mathrm{Ext}_R^1(S,B)$. ($\mathrm{Hom}_R(I,B)$ is an S-module since for $\sigma \in \mathrm{Hom}_R(I,B)$, $i\sigma(i') = ib = o$ for i, $i' \in I$ and $b \in B$.) We let Q be R-injective so that

$$0 \to B \to Q \to K \to 0$$

is exact. But $i.\dim_R B = 1$ so K is R-injective. Now applying the functor $\mathrm{Hom}_R(S,\_)$ we obtain the S-exact sequence

$$0 \to \mathrm{Hom}_R(S,B) \to \mathrm{Hom}_R(S,Q) \to \mathrm{Hom}_R(S,K) \to \mathrm{Ext}_R(S,B) \to 0.$$

But $\mathrm{Hom}_R(S,Q)$ and $\mathrm{Hom}_R(S,K)$ are S-injective (1) so by a shifting argument $\mathrm{Ext}_R(S,B)$ is S-injective since $i.\dim_S B \leqq 2$ and so $\mathrm{Hom}_R(I,B)$ is S-injective.

<u>LEMMA 3.3</u>. Let R be a ring and I a two-sided ideal of R. Write $S = R/I$. If B is any S-module with $w.\dim_R B = 1$ and $w.\dim_S B \leqq 2$ then $I \otimes_R B$ is S-flat.

<u>PROOF</u>. From the exact sequence

$$0 \to I \to R \to S \to 0$$

we obtain the exact sequence

$$0 \to \mathrm{Tor}_1^R(S,B) \to I \otimes_R B \xrightarrow{f} R \otimes_R B \to S \otimes_R B \to 0$$

where $f(i \otimes b) = ib = 0$ so $\mathrm{Tor}_1^R(S,B) \simeq I \otimes_R B$ and $S \otimes_R B \simeq B$. Let P be R-flat so that

$$0 \to K \to P \to B \to 0$$

is an R-flat resolution of B. But $w.\dim_R B = 1$ so K is R-flat. Now applying the functor $S \otimes_R \_$ we obtain the S-exact sequence

$$0 \to \mathrm{Tor}_1^R(S,B) \to S \otimes_R K \to S \otimes_R P \to B \to 0$$

and since $S \otimes_R K$, $S \otimes_R P$ are S-flat we have $\mathrm{Tor}_1^R(S,B)$ is S-flat since $w.\dim_S B \leqq 2$. Hence $I \otimes_R B \simeq \mathrm{Tor}_1^R(S,B)$ is S-flat.

The following result was proved by Jensen ((4), Theorem 1) for R a commutative ring, but we shall prove it for an arbitrary ring.

THEOREM 3 (Jensen). Let R be a ring and I a two-sided ideal of R such that R is a bidirect summand of a bidirect sum of copies of ${}_RI_R$. Write $S = R/I$. If B is any S-module with finite S-dimension then:

(1) if ${}_RI$ is R-projective then $p.\dim_R B = p.\dim_S B + 1$;

(2) if I_R is R-flat then $i.\dim_R B = i.\dim_S B + 1$;

(3) if ${}_RI$ is R-flat then $w.\dim_R B = w.\dim_S B + 1.$

PROOF. Suppose $A \neq 0$ since otherwise the results are trivial. (1) From the short exact sequence

$$0 \to I \to R \to S \to 0$$

we have the exact sequence

$$0 \to \mathrm{Tor}_1^R(S,B) \to I \otimes_R B \xrightarrow{f} B \to S \otimes_R B \to 0.$$

If B is R-projective, then $\mathrm{Tor}_1^R(S,B) = 0$ yielding $I \otimes_R B = 0$ since $f = 0$ as B is an S-module. Thus $I^m \otimes_R B = 0$ for all $m > 0$ and since R is a bidirect summand of a direct sum of copies of I,

$$\oplus I \otimes_R B = (K \oplus R) \otimes_R B \simeq (K \otimes_R B) \oplus B$$

where K is some R-bimodule. Thus B = 0. Therefore, no non-zero S-module is R-projective and for $0 = n = p.\dim_S B$, $p.\dim_R B = 1$ by Proposition 1.3. We now do induction on n. Let F be S-free so that

$$0 \to K \to F \to B \to 0 \quad (*)$$

is exact. Then $p.\dim_S K = n - 1$ so by the induction hypothesis $p.\dim_R K = n$. By Proposition 1.1 for projective dimension we have for $n > 1$, $p.\dim_R B = n + 1$ where (*) is now considered as an R-exact sequence and $p.\dim_R F = 1$ from above. If $n = 1$ we obtain $p.\dim_R B \leq 2$. Suppose, therefore, that $p.\dim_R B = 1$ ($p.\dim_R B \neq 0$ since B is an S-module) and from $p.\dim_S B = 1$ we shall show the contradiction that $p.\dim_S B = 0$. By Lemma 3.1 we have $I \otimes_R B$ is S-projective. Hence for all $m > 0$, $I^m \otimes_R B$ is S-projective. But there exists an R-bimodule K with $\oplus I = K \oplus R$ (as above). Thus

$$\oplus I \otimes_R B = (K \oplus R) \otimes_R B = (K \otimes_R B) \oplus (R \otimes_R B) \simeq (K \otimes_R B) \oplus B.$$

Thus B is a direct summand of an S-projective module $\oplus I \otimes_R B$ and so is S-projective. Therefore we have $p.\dim_R B = 2$ when $p.\dim_S B = 1$ and the proof is finished.

(2) From the short exact sequence

$$0 \to I \to R \to S \to 0$$

we have the exact sequence

$$0 \to \mathrm{Hom}_R(S,B) \to \mathrm{Hom}_R(R,B) \xrightarrow{f} \mathrm{Hom}_R(I,B) \to \mathrm{Ext}_R(S,B) \to 0.$$

If B is R-injective, then $\mathrm{Ext}_R^1(S,B) = 0$ and since $f = 0$ for B an S-module, we have $\mathrm{Hom}_R(I,B) = 0$. Hence $\mathrm{Hom}_R(\oplus I,B) = 0$. Since $\mathrm{Hom}_R(\oplus I,B) = \mathrm{Hom}_R(K \oplus R,B)$ we have

$$\mathrm{Hom}_R(K,B) \oplus \mathrm{Hom}_R(R,B) \approx \mathrm{Hom}_R(K,B) \oplus B = 0$$

which implies $B = 0$. Hence no non-zero S-module is R-injective and by Proposition 1.4 we have $i.\dim_R B = 1$, that is $i.\dim_R B = i.\dim_S B + 1$, where $0 = n = i.\dim_S B$. By induction and Proposition 1.2 it is sufficient to consider the case in which $i.\dim_R B = 1$ and $i.\dim_S B = 1$. By Lemma 3.2 we have $\mathrm{Hom}_R(I,B)$ is S-injective. Since there exists an R-bimodule K with $\oplus I = K \oplus R$ we see that since $\mathrm{Hom}_R(\oplus I,B) \approx \Pi\, \mathrm{Hom}_R(I,B)$ is S-injective, B is S-injective which is a contradiction. Therefore, we have $i.\dim_R B = 2$ when $i.\dim_S B = 1$ and the proof is finished.

(3) From the exact sequence

$$0 \to I \to R \to S \to 0$$

we obtain the exact sequence

$$0 \to \mathrm{Tor}_1^R(S,B) \to I \otimes_R B \xrightarrow{f} R \otimes_R B \to S \otimes_R B \to 0.$$

If B is R-flat then by an argument similar to that in part (1) B must be zero. Hence no non-zero flat R-module is an S-module and by Propostion 1.5 we have $w.\dim_R B = 1 = w.\dim_S B + 1$ where $0 = n = w.\dim_S B$. By induction and Proposition 1.1 it is sufficient to consider the case in which $w.\dim_R B = 1$ and $w.\dim_S B = 1$. By Lemma 3.3 we have that $I \otimes_R B$ is S-flat and so $\oplus I \otimes_R B$ is S-flat. Since there exists an R-bimodule K with $\oplus I = K \oplus R$, we have $\oplus I \otimes_R B = (K \oplus R) \otimes_R B \approx (K \otimes_R R) \oplus B$ in which case B is a direct summand of an S-flat module and so is S-flat which is a contradiction. This finishes the proof.

COROLLARY 3.1 (Jensen). Let I be generated by an R-sequence of two-sided ideals in R ($I_1 \subset I_2 \subset \ldots \subset I_m = I$) of length m such that R/I_{i-1} is a bidirect summand of a bidirect sum of copies of I_i/I_{i-1} for all $1 \leq i \leq m$. Then for any S-module B

(1) if I_i/I_{i-1} is projective as a left R/I_i-module then $p.\dim_R B = p.\dim_S B + m$;

(2) if I_i/I_{i-1} is flat as a right R/I_i-module then $i.\dim_R B = i.\dim_S B + m$;

(3) if I_i/I_{i-1} is flat as a left R/I_i-module then $w.\dim_R B = w.\dim_S B + m$.

PROOF. I_1 has the properties of I in Theorem 3, so by the theorem $p.\dim_R B = p.\dim_{R/I_1} B + 1$. Now

$$p.\dim_{R/I_1} B = p.\dim_{R/I_2} B + 1$$

since $R/I_2 \approx R/I_1/I_2/I_1$. By induction it is easily seen that $p.\dim_R B = p.\dim_S B + m$. Similarly parts (2) and (3) are proved.

We now look at a rather interesting result due to Jensen ((4), Theorem 3).

PROPOSITION 3.4. Let R be a ring and I a two-sided ideal of R such that R is a bidirect summand of a bidirect sum of a finite number of copies of ${}_RI_R$. If $i.\dim_R R \leq 1$, then $S = R/I$ is self-injective.

PROOF. From the short exact sequence

$$0 \to I \to R \to S \to 0$$

we obtain the exact sequence

$$0 \to \mathrm{Hom}_R(S,R) \to \mathrm{Hom}_R(R,R) \xrightarrow{f} \mathrm{Hom}_R(I,R) \to \mathrm{Ext}^1_R(S,R) \to 0.$$

Thus $\mathrm{Ext}^1_R(S,R) \simeq \mathrm{Coker}\ f$. For some $n > 0$ there exists an R-bimodule K with $I^m = K \oplus R$ and so there exists R-R-homomorphisms i, p such that

$$I^n \underset{i}{\overset{p}{\rightleftarrows}} R$$

with $pi = 1_R$. We then have

$$\mathrm{Hom}_R(R^n,R) \xrightarrow{f^n} \mathrm{Hom}_R(I^n,R) \xrightarrow{i^*} \mathrm{Hom}_R(R,R) \simeq R$$

where i^* is the dual of i. Since f^n is a restriction map and since elements of $\mathrm{Hom}_R(R^n,R)$ are determined by how they act on the unit vectors (0, ..., 1, 0, ..., 0) we see that $\mathrm{Im}\ (i^*f^n) \subseteq I$. We therefore have the map $\bar{i}$: Coker $f^n \to S$. Now $p \in \mathrm{Hom}_R(I^n,R)$, represents an element

$\bar{p} \in \operatorname{Coker} f^n$ and $i^*(p) = p' = 1_R$. Thus $\widetilde{i^*(\bar{p})} = 1_S$ which means $\widetilde{i^*}$ is an S-epimorphism. But S is S-projective and so S is a direct summand of Coker f^n. We now show Coker f^n is S-injective. Consider the exact sequence

$$0 \to R \to Q \to K \to 0$$

where Q is R-injective. K is R-injective since $i.\dim_R R \leq 1$. This gives rise to the sequence

$$0 \to \operatorname{Hom}_R(S,R) \to \operatorname{Hom}_R(S,Q) \to \operatorname{Hom}_R(S,K) \to \operatorname{Ext}^1_R(S,R) \to 0.$$

Now $\operatorname{Hom}_R(S,R) = 0$ since otherwise for $0 \neq s \in \operatorname{Hom}_R(S,R)$, $j(s(\bar{1})) = s(j) = 0$ but if $s(\bar{1}) = r \in R$, $jr \neq 0$ for $\bar{1} = 1 + I$ and $j \in I$. We therefore have the exact sequence

$$0 \to \operatorname{Hom}_R(S,Q) \to \operatorname{Hom}_R(S,K) \to \operatorname{Ext}^1_R(S,R) \to 0$$

and since $\operatorname{Hom}_R(S,Q)$ is S-injective the sequence splits yielding $\operatorname{Ext}^1_R(S,R)$ a direct summand of $\operatorname{Hom}_R(S,K)$ which is S-injective. Thus $\operatorname{Ext}^1_R(S,R) \simeq \operatorname{Coker} f$ is S-injective and since S is a direct summand of Coker f^n, S is self-injective.

<u>THEOREM 4</u> (Cohen). Let R be a ring and let I be a two-sided ideal of R such that there exists an R-bimodule I' which is bidirect in a bidirect sum of copies of R and such that R is a bidirect summand of $I' \otimes_R I$. Write S = R/I. If A is any S-module with finite S-dimension, then

(1) if ${}_R I$ is R-projective then $p.\dim_R A = p.\dim_S A + 1$;

(2) if I_R is R-flat then

$i.\dim_R A = i.\dim_S A + 1$;

(3) if ${}_R I$ is R-flat then

$w.\dim_R A = w.\dim_S A + 1$.

PROOF. If $A = 0$ the results are trivial. Hence suppose $A \neq 0$. (1) By part (1) of Theorem 3 we need only consider the case in which $p.\dim_R A = 1$ and $p.\dim_S A = 1$. By Lemma 3.1 we have that $I \otimes_R A$ is S-projective. Let $K \simeq \oplus R$ with I' a bidirect summand. Since R is a bidirect summand of $I' \otimes_R I$ we have $A \simeq R \otimes_R A$ is a bidirct summand of $(I' \otimes_R I) \otimes_R A \simeq I' \otimes_R (I \otimes_R A)$ which is a bidirect summand of

$$K \otimes_R (I \otimes_R A) \simeq \oplus R \otimes_R (I \otimes_R A) \simeq \oplus (I \otimes_R A)$$

which is S-projective. Hence A is S-projective and we are finished. Parts (2) and (3) are proved similarly.

COROLLARY 4.1. Let I be generated by an R-sequence of two-sided ideals of R ($I_1 \subset I_2 \subset \ldots \subset I_m = I$) of length m such that for each i there exists an R/I_{i-1}-bimodule I_i' which is bidirect in a bidirect sum of copies of R/I_{i-1} and such that R/I_{i-1} is a bidirect summand of $I_i' \otimes_R (I_i/I_{i-1})$. Then

(1) if I_i/I_{i-1} is left-projective as an R/I_{i-1}-module then $p.\dim_R B = p.\dim_S B + m$;

(2) if I_i/I_{i-1} is right-flat as an R/I_{i-1}-module then $i.\dim_R B = i.\dim_S B + m$;

(3) if I_i/I_{i-1} is left-flat as an R/I_{i-1}-module then $w.\dim_R B = w.\dim_S B + 1$.

PROOF. The proof is identical to that of Corollary 3.1.

COROLLARY 4.2 (Cohen). If

(1) $I = (x)$, $x \in R$ a central non-zero divisor or

(2) I is finitely generated rank 1 projective (for every maximal ideal $M \subset R$, $I_M \simeq R_M$), R commutative,

then for every S-module B, $p.\dim_R B = p.\dim_S B + 1$ (also for injective and weak dimensions).

PROOF. (1) Let $I' = R$ in the theorem. Then

$$I' \otimes_R I \simeq I \simeq R$$

where $x : R \to I = xR$ is the isomorphism. The theorem applies.

(2) Let I be finitely generated. Then

$$\mathrm{Hom}_R(I,R)_M \simeq \mathrm{Hom}_{R_M}(I_M,R_M) \simeq R_M$$

since $I_M \simeq R_M$, I is of rank 1. Therefore, the evaluation map $I \otimes_R \mathrm{Hom}_R(I,R) \longrightarrow R$ is locally an isomorphism at each maximal ideal $M \subset R$, hence is an isomorphism. Thus we let $I' \simeq \mathrm{Hom}_R(I,R)$ and the theorem applies.

(The distinction between left and right dimensions dissappears under the assumption that R is commutative.)

REMARK. Let R be commutative and let I be invertible. Then we let $I' = I^{-1}$ in which case $I' \otimes_R I \simeq I^{-1}I \simeq R$ and the theorem applies. Note also that I invertible means I is finitely generated rank 1 projective in which case (2) applies. But Bourbaki in Algèbre Commutative chapters 1-2, pg. 179, Exercise 12 gives an example of a finitely generated rank 1 projective which is not invertible and so (2) is strictly stronger.

PROPOSITION 3.5. If R is any ring, then I is a two-sided ideal of R such that R is a bidirect summand of a bidirect sum of copies of ${}_R I_R$ if and only if there exits an R-bimodule I' such that I' is a bidirect summand of a bidirect sum of copies of R and so that R is a bidirect summand of $I' \otimes_R I$.

PROOF. Since R is bidirect summand of $\oplus\, I$ and $I \simeq R \otimes_R I$ we have $\oplus\, I \simeq \oplus\, (R \otimes_R I) = \oplus\, R \otimes_R I$. Let $I' = \oplus\, R$. Conversely, we have $K \oplus I' = \oplus\, R$ for some R-bimodule K. Hence $(K \otimes_R I) \oplus (I' \otimes_R I) = \oplus\, R \otimes_R I \simeq \oplus\, I$. Since R is a bidirect summand of $I' \otimes_R I$ we have R is a bidirect summand of $\oplus\, I$.

Although Theorems 3 and 4 are attributed to Jensen and Cohen, respectively, we have significantly changed the hypotheses required to prove them. In

Jensen's original result he required that R be commutative and I be an ideal of R which is a faithfully projective R-module (Pis a faithfully projective R-module if and only if $\mathrm{Hom}_R(P,\_)$ is a faithfully exact functor). By ((3), Proposition 2.4) we note that if I is faithfully projective then there exists a positive integer n such that $I^n = I \oplus I \oplus \ldots \oplus I$ has a direct summand isomorphic to R, thus satisfying the hypotheses of Theorem 3. Therefore, Theorem 3 is a significant generalization of Jensen's result.

Cohen, on the other hand, assumes R to be an arbitrary ring, not necessarily commutative, but he requires that the ideal I be right flat and left projective such that there exists an R-bimodule I' which is a bidirect summand of $I' \otimes_R I$. Cohen considers only part (1) of Theorem 4 and as can be seen our proof eliminates the necessity of assuming I to be right flat. Since Theorems 3 and 4 include the results of Jensen and Cohen, respectively, Proposition 3.5 shows that, indeed, Jensen's result and Cohen's result coincide.

CHAPTER 4

APPLICATIONS AND EXAMPLES

M. Auslander and D. Buchsbaum in their note "Homological Dimension in Noetherian Rings II", Trans. Amer. Math. Soc., 88(1958). pp. 194-206 showed that if R is a commutative Noetherian ring of global dimension n then $R[[X]]$ has global dimension n + 1. Small shows that the assumption of commutivity may be dropped.

PROPOSITION 4.1 (Small). If R is a right Noetherian ring and gl.dim R = n, then gl. dim $R[[X]] = n + 1$.

PROOF. Since X is a central non-zero divisor in $R[[X]]$, we have $(X) \simeq R[[X]]$ and so is projective. Thus

$$\text{r.w.dim}_{R[[X]]} R[[X]]/(X) \leqq 1.$$

But $X \in J(R)$ so by Theorem 2 we have

$$\text{l.gl.dim } R[[X]] \leqq 1 + \text{l.gl.dim } R[[X]]/(X) = n + 1.$$

However, by Kaplansky, (6) Theorem 1.3

$$\text{l.gl.dim } R[[X]]/(X) \geqq n + 1$$

and so the result.

EXAMPLE 4.2 (Small). Let R be the integers localized at a prime and Q be the rationals. Consider the ring S consisting of all 2 X 2 matrices

$$\begin{pmatrix} r & q_1 \\ 0 & q_2 \end{pmatrix}$$

with $r \in R$, q_1, $q_2 \in Q$. S is right-Noetherian but <u>not</u> left-Noetherian, and

$$J(S) = \left\{ \begin{pmatrix} a & q \\ 0 & 0 \end{pmatrix} \middle| \, a \in J(R),\ q \in Q \right\}.$$

Now $\text{l.w.dim}_S J(S) = 0$ but $\text{l.p.dim}_S J(S) = 1$ since ${}_R Q$ is flat but not projective. Since $S/J(S) \simeq R/J(R) \oplus Q$, a direct sum of fields, $\text{r.gl.dim } S/J(S) = 0$. By Theorem 2, therefore, $\text{r.gl.dim } S = 1$. However, $\text{l.gl.dim } S = 2$, for, $\text{l.p.dim}_S J(S) = 1$ so $\text{l.gl.dim } S \geqq 2$. Now $R \oplus Q$ is Noetherian on both sides so we have

$$\text{l.gl.dim } (R \oplus Q) = \text{r.gl.dim } (R \oplus Q) \leqq 1.$$

Using the inequality of Theorem 2 and $\text{r.w.dim}_R (R \oplus Q) \leqq 1$ we have $\text{l.gl.dim } S \leqq 1 + 1 = 2$. ($R \oplus Q \simeq S/N(S)$ where $N(S)$ is the maximal nilpotent ideal of S.) However, if we tried to apply the inequality of Theorem 2 to S with $I = J(S)$ we would obtain $\text{l.gl.dim } S = 1$ which is false. Thus Theorem 2 cannot be generalized in this direction. Suppose we now apply the inequality of Theorem 2 with S and $I = N(S)$. Since $S/N(S) \simeq R \oplus Q$ and $\text{l.w.dim}_S S/N(S) = \text{r.w.dim}_S S/N(S) = 1$ we obtain

$$\text{r.gl.dim } S \leqq 2 \text{ and } \text{l.gl.dim } S \leqq 2.$$

Thus, in the former case the inequality is strict while in the latter it is not. Therefore, the inequalities are the best possible (for small dimensions).

EXAMPLE 4.3. Let the ideal I be contained in the radical of a ring R where I satisfies the hypotheses of Theorem 3, (2) and either let I be nilpotent or let R be left-Noetherian or left-perfect. Suppose l.gl.dim S is finite. Then by Theorem 2 we have that

$$\text{l.gl.dim } R \leq \text{l.gl.dim } S + 1$$

since I is right-flat. But by Theorem 3 we have for any S-module B,

$$\text{i.dim}_R B = \text{i.dim}_S B + 1$$

and so

$$\text{l.gl.dim } R = \text{l.gl.dim } S + 1.$$

EXAMPLE 4.4 (Cohen). We wish to show that for the hypotheses of Theorem 3,(1), in the case where R is commutative, that I faithful and projective is not sufficient.

Let $S = \{f : I \to R^1 \text{ continuous}\}$ where I is the unit interval $[0,1]$ and R^1 is the real line. Let

$$J = \{f \in S \mid \text{there exists an } \varepsilon > 0 \text{ and } f[0,\varepsilon] = 0\}.$$

Then J is projective : let

$$f_n(t) = \begin{cases} 0 & 0 \leq t \leq 1/(n+1) \\ n(n+1)(t - 1/(n+1)) & 1/(n+1) \leq t \leq 1/n \\ 1 & 1/n \leq t \leq 1. \end{cases}$$

Let $g_n = (f_{n+1} - f_n)^{\frac{1}{2}}$, $g_0 = (f_1)^{\frac{1}{2}}$. Then

$$\sum_{i=0}^{n} g_i^2 = f_{n+1},\ g_i \in J.$$

Define $\theta : \sum_{i=0}^{n} S \longrightarrow J$ by $\theta(a_0, a_1, \ldots) = \sum_{i=0}^{n} a_i g_i$. Given $h \in J$ observe that for some N, $hf_n = h$ for all $n \geqq N$, hence $hg_n = 0$ for almost all n. So we can define $\sigma : J \longrightarrow \sum_{i=0}^{n} S$ by $\sigma(h) = (hg_0, hg_1, \ldots)$. Then

$$\theta\sigma(h) = \sum_{i=0}^{\infty} hg_i = \sum_{i=0}^{n} hg_i = h\sum_{i=0}^{n} g_i = hf_{n+1} = h,$$

for n sufficiently large. Thus J is projective. J is clearly faithful. Let $S^* = S/J$, the ring of evaluations at 0 (that is, constant terms). Let $\alpha \in S$ be given by $\alpha(t) = t$, $t \in I$. Let $A = (\alpha)$. Let $M = S/A$. $A^2 \neq A$ so $\text{p.dim}_S M = 1$. (If M is left-flat, then

$$S/A \otimes_S A \simeq S/A \cdot A = 0.$$

But $S/A \otimes_S A = 0$ implies $A = A^2$.) Observe that $J \subset A$ and α is a non-zero divisor of S^*. Thus

$$M = S/A \simeq S^*/(A/J) \simeq S^*/\alpha S^*$$

so $\text{p.dim}_{S^*} M = 1$. Thus $\text{p.dim}_S M = \text{p.dim}_{S^*} M = 1$. Thus faithful and projective is not sufficient.

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