THE MEASURE ALGEBRA OF A LOCALLY COMPACT GROUP

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SCOPE AND CONTENTS: Let G be a locally compact group (= locally compact Hausdorff topological group). By the measure algebra of G we mean the Banach *-algebra M(G) of bounded regular Borel measures on G. The major results of this work are a structure theorem for norm decreasing isomorphisms of measure algebras, and a characterization of those Banach algebras which are isometric and isomorphic to the measure algebra of some locally compact group. We also obtain some results on subalgebras of M(G) and on representations of G.

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In memory of my father

Philip Rigelhof

INTRODUCTION

Let G be a locally compact group (= locally compact Hausdorff topological group). By the measure algebra of G we mean the Banach *-algebra M(G) of bounded regular Borel measures on G. The major results of this work are a structure theorem for norm decreasing isomorphisms of measure algebras, and a characterization of those Banach algebras which are isometric and isomorphic to the measure algebra of some locally compact group. We also obtain some results on subalgebras of M(G) and on representations of G.

The first chapter of this work is composed of those definitions and results from the theory of topological vector spaces, integration, topological algebras etc., which are needed in future chapters. In this introduction frequent use of the contents of Chapter I is made without explicit reference.

In addition to the norm topology on M(G) there are other topologies on M(G) which have to some extent been investigated. In sections 1 and 2 of Chapter II of this work we study two of these, namely the $\sigma(M(G),C_o(G))$-topology and the so-topology (see Chapter II for the definitions). Using these topologies on M(G) a number of results on M(G) and $L^1(G)$ are obtained.

The first of these concerns a problem raised by A. B. Simon in (21). Let S be a Borel subset of a locally compact group G and let L(S) be the subspace of $L^1(G)$ consisting of functions which are zero almost everywhere outside S. If S is a semigroup it is not difficult to show that L(S) is a subalgebra. Simon asked the following question: If L(S) is a subalgebra of $L^1(G)$, is there a semigroup T such that L(S)= L(T)? We have been able to give an affirmative answer in a number of special cases (theorems 3, 4 and 5 of Chapter II). These results are then used to generalize another result due to Simon (theorem 7 of chapter II).

In the third chapter we prove a structure theorem for norm decreasing isomorphisms of measure algebras. Let F and G be locally compact groups, α an isomorphism and homeomorphism of F onto G and γ a continuous character on F. For μ in M(F) and f in $C_o(G)$, let $T\mu(f) = \mu(\gamma(f\circ\alpha))$. Then the mapping $\mu \to T\mu$ is an isometric *-isomorphism of M(F) onto M(G) (Chapter III, lemma 2). The main result (theorem 2) of this chapter is that every norm decreasing isomorphism of M(F) onto M(G) is of the above form and therefore is an isometric *-isomorphism. A number of other results follow from this, including a theorem due to Wendel on isomorphisms of $L^1(F)$ onto $L^1(G)$ (theorem 3 of Chapter III).

In view of the isomorphism theorem of Chapter III, one would expect in principle to be able to characterize those Banach algebras which are isometric and isomorphic to the measure algebra of some locally compact group. In Chapter IV we obtain such a characterization. This characterization is largely in terms of properties of the extreme points of the unit ball. This is to be expected since the extreme points of the unit ball of M(G) are scalar multiples (of absolute value 1) of the Dirac measures, and the Dirac measures play a special role in the proof of the isomorphism theorem.

The problem of characterizing $L^1(G)$ and $M(G)$ has been the subject of recent papers by Greenleaf (10) and Rieffel (18). Rieffel had identified those abelian Banach algebras which are isometrically isomorphic to $M(G)$ for some locally compact abelian group G. He also obtained a characterization of $L^1(G)$ for G locally compact abelian. His work is largely based on properties of multiplicative linear functionals on the algebra. Greenleaf characterizes those Banach algebras which are isometric and isomorphic to $L^1(G)$ for G compact. His approach to the problem is similar to the one we have used. In fact it was his characterization of $M(G)$ for G finite which motivated our work (theorem 1.5.1 of (10)).

The final chapter of this work is concerned with direct integral decompositions of the left regular representation of certain locally compact groups.

There is a general theorem due to Mautner on direct integral decompositions of representations of a locally compact group having a countable basis for the open sets on a separable Hilbert space. Each decomposition of a given representation is in a sense determined by an abelian W*-algebra of the commutant of the representation (for the precise statement see theorems 10 and 11 of Chapter I). If this subalgebra is maximal abelian then Mautner's theorem states that the decomposition is into irreducibles. The existence of a maximal abelian W*-subalgebra of the commutant of a given representation is a consequence of the axiom of choice. Thus the uniqueness of the decomposition can be questioned. Examples have been given by Mackey (16) and Yoshizawa

(25) that show that these decompositions may not be unique.

If a discrete group G has an abelian subgroup S, then the left regular representation of G can be expressed as a direct integral of representations induced by the characters on S. This has been shown by Godement in (8). (Also see Mackey (16)). Godement also found necessary and sufficient conditions on the group G and subgroup S for these representations to be irreducible.

The relationship of this construction to the decomposition obtained via Mautner's theorem is the subject of Chapter IV. We show that if S is an open abelian subgroup of a separable locally compact unimodular group then one can shoose an abelian W*-algebra $\mathfrak{R}(S)$ (depending on S) of the commutant of the left regular representation such that the corresponding direct integral decomposition gives representations which are (equivalent to) the representations induced by characters on S. We also find necessary and sufficient conditions on G and S so that $\mathfrak{R}(S)$ is a maximal abelian W*-subalgebra.

CHAPTER I

PRELIMINARIES

1. Topological vector spaces

1.1 Convex sets. By a real (resp. complex) TVS we shall mean a Hausdorff topological vector space over the real (resp. complex) field. A subset C of a real or complex vector space E is said to be convex if $x, y \in C$ implies $ax + (1 - a)y \in C$, $0 \leq a \leq 1$. An element x of a convex set C is said to be an extreme point of C if $x = ay + (1 - a)z$, $y, z \in C$, $0 < a < 1$ implies $x = y = z$. The convex hull of a subset C of E is the intersection of all convex subsets of E which contain C. The convex hull is a convex set and coincides with $\{ax + (1 - a)y : x,y \in C, 0 \leq a \leq 1\}$. A real (resp. complex) locally convex space E is a real (resp. complex) TVS, which has a fundamental system of convex neighborhoods of the origin of E. In the following whenever we speak of a vector space without specifying the field we shall mean that the field may be either the real or complex field.

Theorem 1. (Krein - Milman) Let E be a locally convex space, and let K be a compact convex subset of E. Then K is the closure of the convex hull of its extreme points.

Proof: A proof may be found in (2)(Chapitre II, §4, théorème 1) or in (6)(Chapter V, §8.4).

The next lemma is in a sense a converse to the theorem of Krein-Milman.

Lemma 1. Let K be a compact subset of a locally convex space E whose closed convex hull is compact. Then the only extreme points of the closure of the convex hull of K are the points in K.

A proof may be found in (2)(Chapitre II, §4, proposition 4) or in (6)(Chapter V, § 8.4).

1.2 Uniform spaces and completeness. Let E be a topological vector space and let $\{U\}$ be a fundamental system of neighborhoods of 0 in E. For each $U \in \{U\}$, let $L(U) = \{(x,y) \in E \times E : x - y \in U\}$. Then $\{L(U) : U \in \{U\}\}$ is a base for a uniformity on E, and E becomes a uniform space. Thus E has a unique completion $\widehat{E}$ and E is said to be complete if $E = \widehat{E}$. A Banach space is a normed vector space which is complete. Each linear continuous mapping of a topological vector space into a topological vector space is uniformly continuous. In a complete space the closure of the convex hull of a compact set is compact, ((2), Chapitre II, §4).

1.3 Subsets of a TVS. A subset C of a vector space E is said to be circled if $aC \subseteq C$ for all $|a| \leq 1$. C is absorbing if for each $x \in E$, there is an $a > 0$ such that $bx \in C$ for all $0 < |b| \leq a$. Any

TVS E has a fundamental system of closed, circled, absorbing neighborhoods of the origin. Let E be a locally convex space, a closed, convex circled and absorbing subset of E is called a barrel. If each barrel of E is a neighborhood of $0 \in E$, then E is said to be barrelled.

A subset C of a vector space E is bounded if and only if for any neighborhood U of $0 \in E$, there is a $a > 0$ such that for all $0 \leq |b| \leq a$, $bC \subseteq U$. A TVS is said to be quasi-complete if every closed and bounded subset is complete.

1.4 Function spaces and equicontinuous sets. Let E, F be topological vector spaces and let C(E,F) be the space of all continuous linear mappings of E into F given the topology of simple convergence i.e. the coarsest topology such that for each $x \in E$ the mapping $f \to f(x)$ is continuous. If E is a barrelled space and F a quasi-complete locally convex space, then C(E,F) is quasi-complete ((2) Chapitre III, §3 No. 7).

A subset $H \subseteq C(E,F)$ is said to be equicontinuous if for each neighborhood U of $0 \in F$, $\bigcap_{f \in H} f^{-1}(U)$ is a neighborhood of 0 in E.

Let H be an equicontinuous subset of C(E,F) where E and F are locally convex spaces. Then : (a) $\overline{H}$ is equicontinuous, (b) the convex circled hull of H is equicontinuous (c) H is relatively compact if and only if for each $x \in E$, $H(x) = \{f(x): f \in H\}$ is relatively compact in F, (d) if in addition E is barrelled then every bounded subset of C(E,F) is equicontinuous, ((2) Chapitre III, §3, No. 5 and No. 6).

Theorem 2. Let E be a barrelled space and let F be a complete locally convex space. If $H \subseteq C(E,F)$ is compact, then the closure of the convex hull of H is compact.

Proof: Since H is compact, H is bounded and therefore equicontinuous by (d). By (c), H(x) is relatively compact in F so that the convex hull of H(x) is relatively compact since F is complete. Again by (c), the convex hull of H is relatively compact and this proves the theorem.

1.5 The dual of a TVS. Let E be a TVS, then the dual of E written E' is the set of all continuous linear functionals on E. The coarsest topology on E' such that for each $x \in E$ the mapping $x' \to x'(x)$ is continuous, is called the weak topology or the $\sigma(E',E)$-topology.

If E is a normed vector space, then we define a norm on E' by $||x'|| = \sup \{ |x'(x)| : ||x|| \leq 1 \}$ and with respect to this norm E' becomes a Banach space. The unit sphere (i.e. $\{x' : ||x'|| \leq 1\}$ of the dual of a Banach space is compact in the $\sigma(E',E)$-topology. It follows from this that the norm is lower semicontinuous in the weak topology.

Let A be a subset of a locally convex space E. Let $A^o = \{ x' \in E' : |x'(x)| \leq 1 \text{ for all } x \in A \}$, and for each subset $B \subseteq E'$ let $B^o = \{ x \in E : |x'(x)| \leq 1 \text{ for all } x' \in B \}$. If B is a subset of E' which contains $0 \in E'$, then B^{oo} is the $\sigma(E',E)$-closure of the convex hull of B.

Let $\mathcal{V}$ be the family of all bounded subsets of E, then there is a Hausdorff locally convex topology on E' having $\{ A^o : A \in \mathcal{V} \}$ as a subbase. This topology is called the strong topology on E'. If

E is normed, then the topology induced by the norm on E' coincides with the strong topology. If E is barrelled then each weakly bounded subset of E' is strongly bounded, and $(E')_\sigma$ is quasi-complete ((2) Chapitre IV, §2 No. 2).

Let N be a subspace of E. Then N^o is a $\sigma(E',E)$-closed subspace of E'. Each $x' \in E'$ when restricted to N defines an element of N', and if $x' - y' \in N^o$ then $< n,x' > = < n,y' >$ for each $n \in N$. Thus we may define a mapping $T : E'/N^o \rightarrow N'$ by putting $< n,Tx' > = < n,x' >$ where $x' \in E'/N^o$.

Theorem 3. The mapping T defined above is a one-one linear mapping of E'/N^o onto N'. If E is a normed space then T is an isometry, and if N is closed then the $\sigma(E'/N^o,N)$-topology equals the quotient weak topology on E'/N^o.

Proof: See ((2) Chapitre IV, §5, No. 4 proposition 10, and §1, No. 5 proposition 7).

1.6 The adjoint of a linear mapping. Let E, F be locally convex spaces and let T be a continuous linear mapping of E into F. For each y' in F' we define a linear functional T'y' on E by $T'y'(x) = y'(Tx)$. Then $T'y' \in E'$ and the mapping $T' : y' \rightarrow T'y'$ is a linear mapping which is continuous for the weak and strong topologies. For T(E) to be dense in F it is necessary and sufficient that T' be a one-one mapping of F' into E'. ((2) Chapitre IV, §4 No. 1). If E and F are normed spaces, then $||T'|| = ||T||$. ((2) Chapitre IV, §5 No. 3).

2. Semitopological groups and topological groups.

2.1 Definitions. A semitopological group G is a group given a topology such that for each $y \in G$, the mappings $x \to yx$ and $x \to xy$ are continuous. A topological group G is a group given a topology such that the mapping $(x,y) \to xy^{-1}$ is a continuous mapping of $G \times G$ onto G.

If G is a locally compact Hausdorff semitopological group, then G is a topological group ((7) theorem 2 or (13) Exercise B2, p. 41).

2.2 Neighborhood systems of a topological group. A subset U of a topological group G is called symmetric if $U^{-1} = U$ where $U^{-1} = \{x : x^{-1} \in U\}$. Each topological group G has a fundamental system $\{U\}$ of closed neighborhoods of the identity e such that:

(i) each U is symmetric

(ii) for each U in $\{U\}$ there is a V in $\{U\}$ such that $V^2 \subseteq U$.

(iii) for each U in $\{U\}$ and x in G there is a V in $\{U\}$ such that $V \subseteq x^{-1}Ux$

((13) § 20, theorem 3 or (12) 4.5, 4.6 and 4.7).

For any subset $A \subseteq G$ and any neighborhood U of the identity $\bar{A} \subseteq AU$, in fact $\bar{A} = \bigcap AU = \bigcap UA$ where the intersection is over the family of all neighborhoods of the identity. Consequently any open subgroup of a topological group is closed. For any compact set K and any open set U such that $K \subset U$ there is a neighborhood V of the identity such that $KV \subseteq U$. ((13) § 20 proposition 4, or (12) 4.10).

2.3 Uniform structures. Let G be a topological group with $\{U\}$ as the system of all neighborhoods of e. For U in $\{U\}$ define

$$L(U) = \{(x,y) : x^{-1}y \in U\}$$

$$R(U) = \{(x,y) : xy^{-1} \in U\}$$

The family $\{L(U) : U \in \{U\}\}$ (resp. $\{R(U) : U \in \{U\}\}$ forms a base for a uniformity called the left (resp. right) uniform structure on G. With respect to either the right or left uniform structure, G is a uniform space.

Let G be a locally compact group (= locally compact Hausdorff topological group). Then G is complete in either the right or left uniform structures ((13) §26 theorem 3). It follows from this that a subgroup of a locally compact group is locally compact if and only if it is closed.

If f is a complex valued function on a locally compact group G , which "vanishes at infinity", then f is right and left uniformly continuous ((12) Chapter IV, 15.4).

2.4 The character group. Let G be a locally compact group. A character t on G is a homomorphism of G into the group Z of complex numbers of absolute value 1.

Let G be a locally compact abelian group and let $\widehat{G}$ be the set of all continuous characters on G. We define a group operation in $\widehat{G}$ by $t_1t_2(x) = t_1(x)t_2(x)$. It follows that $t(e) = 1$ and $t(x^{-1}) = \overline{t(x)}$ for $t \in \widehat{G}$ and $x \in G$. For every compact set $F \subseteq G$ and every $\varepsilon > 0$ let $T(F,\varepsilon) = \{t \in G : |t(x) - 1| < \varepsilon$ for all $x \in F\}$. Then $\{T(F,\varepsilon) : F$ is compact and $\varepsilon > 0\}$ is a basis at e for a topology on $\widehat{G}$, and $\widehat{G}$ given this topology is a locally compact abelian group called the character group of G ((13) Chapter 8, (12) Chapter VI, §23)

Throughout the remainder of this work all topological spaces are to be taken as Hausdorff.

3. Measure Theory

3.1 Definitions. Let X be a set and $\mathfrak{M}$ a σ-algebra of subsets of X. A positive measure μ on $\mathfrak{M}$ is a function on $\mathfrak{M}$ into the extended reals such that

(1) $\mu(\emptyset) = 0$

(2) $\mu(A) \geq 0$ for any $A \in \mathfrak{M}$

(3) $\mu(\bigcup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty} \mu(A_i)$ if $A_i \in \mathfrak{M}$ for all i, and $A_i \cap A_j = \emptyset$ for $i \neq j$.

A complex measure μ on $\mathfrak{M}$ is a complex-valued function on $\mathfrak{M}$ satisfying (1) and (3) above. We shall frequently use the word measure to mean either a positive measure or a complex measure.

The total variation of a measure μ is the measure $|\mu|$ defined for each $A \in \mathfrak{M}$ by $|\mu|(A) = \sup \Sigma |\mu(A_i)|$ the supremum being over all finite disjoint unions $A = \bigcup A_i$, $A_i \in \mathfrak{M}$. The total variation of a measure μ is a positive measure. If μ is itself a positive measure then $\mu = |\mu|$. The total mass of a measure μ is $|\mu|(X)$.

Let f be any nonnegative function on X. Then the integral $\int_X f(x)d\mu(x)$ (or $\int_X f d\mu$ or $\int f d\mu$) is defined as

$\sup \{ \Sigma_{i=1}^{n} [\inf(f(x) : x \in A_i)] \mu(A_i) : X = \bigcup_{i=1}^{n} A_i , A_i \cap A_j = \emptyset$ whenever $i \neq j$, and $A_i \in \mathcal{M} \}$. If f is any extended real-valued function, let $f^+ = \max(f,0)$ $f^- = -\min(f,0)$, and if $\int f^+ d\mu$ or $\int f^- d\mu$ is finite we put $\int f d\mu = \int f^+ d\mu - \int f^- d\mu$. If f is complex-valued, then there are real-valued functions f_1, f_2 with $f = f_1 + if_2$. If $\int f_1 d\mu$, $\int f_2 d\mu$ are defined and finite, we put $\int f d\mu = \int f_1 d\mu + i \int f_2 d\mu$.

Let X be a set and $\mathcal{P}(X)$ the set of all subsets of X. A Carathéodory outer measure μ is a function on $\mathcal{P}(X)$ into the extended reals such that (1) $\mu(\emptyset) = 0$, (2) $\mu(A) \geq 0$ for all $A \in \mathcal{P}(X)$

(3) $\mu(\bigcup_{n=1}^{\infty} A_i) \leq \Sigma_{i=1}^{\infty} \mu(A_i)$ $A_1, A_2, \ldots\ldots\ldots \in \mathcal{P}(X)$

A set $A \in \mathcal{P}(X)$ is said to be μ-measurable if for every $S \in \mathcal{P}(X)$ we have $\mu(S) \geq \mu(S \cap A) + \mu(S \cap (X \setminus A))$

The set of all μ-measurable subsets of X is a σ-algebra $\mathcal{M}$ and μ is a positive measure on $\mathcal{M}$. ((11) Chapter II, §11).

Let X be a locally compact space. The Borel subsets of X are the elements of the σ-algebra generated by the closed sets of X.[1] A measure μ is called a Borel measure if its domain is the Borel subsets of X, and if $|\mu(K)| < \infty$ for each compact set K.

A measure μ on a σ-algebra $\mathcal{M}$ whose domain includes the Borel sets of X is said to be outer regular if for every $A \in \mathcal{M}$

$$\mu(A) = \inf \{\mu(U) : U \text{ is open and } A \subseteq U\}.$$

μ is said to be inner regular if for every $A \in \mathcal{M}$

$$\mu(A) = \sup \{\mu(K) : K \text{ is compact and } K \subseteq A\}.$$

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1. This definition and the definition of a regular measure differ from that in Halmos (11), but is the definition used in Hewitt and Ross (12), and a number of other authors.

A measure μ is said to be regular if it is outer regular and if for each open set V,

$$\mu(V) = \sup \{ \mu(K) : K \text{ is compact and } K \subseteq V \}.$$

If μ is a regular measure and $A \in \mathcal{M}$ with $|\mu(A)| < \infty$, then

$\mu(A) = \sup \{ \mu(K) : K \text{ is compact and } K \subseteq A \}$ ((12) Chapter III, 11.34).

3.2 $C_o(X)$ and $M(X)$. Let X be a locally compact space and let C(X) be the collection of all bounded continuous complex-valued functions on X. C(X) is a Banach space with norm given by $||f|| = \sup \{ |f(x)| : x \in X \}$. K(X) is the subspace of C(X) consisting of functions whose support is compact, and $C_o(X)$ is the closure of K(X) in C(X). $C_o(X)$ consists of all functions f such that for given $\varepsilon > 0$ there is a compact set K such that $|f(x)| < \varepsilon$ for $x \in X \setminus K$. $C^+(X)$ is the set of positive functions in C(X). $C_o^+(X)$, $K^+(X)$ are defined similarly.

Let $y \in X$ and let V be an open neighborhood of y whose closure is compact. By the complete regularity of X, there is a continuous function f, $0 \leq f(x) \leq 1$ such that $f(y) = 1$ and $f(X \setminus V) = 0$. Thus $f \in K^+(X)$. Consequently K(X), $C_o(X)$ and C(X) separate the points of X.

Let $C_o(X)'$ be the dual of $C_o(X)$. Then since $C_o(X)$ is a Banach space, $C_o(X)'$ is also a Banach space, and the norm of an element $I \in C_o(X)'$ is given by $||I|| = \sup \{ |I(f)| : ||f|| \leq 1 \}$.

Let M(X) be the set of all regular Borel measures on X having finite total mass. There is a natural isomorphism between $C_o(X)'$ and M(X). This isomorphism is a consequence of the fact that each positive linear functional I on K(X) can be extended to a linear functional on a much larger class of functions. We now outline this extension, and once this is done we shall outline the proof of the above mentioned isomorphism between $C_o(X)'$ and M(X). For the details we refer the reader to (12)(Chapter III, § 11 and §13).

3.3 Extension of a positive linear functional. Let I be a positive linear functional on K(X), i.e. a linear functional such that $f \geq 0$ implies $I(f) \geq 0$. We define a functional $\bar{I}$ on a class M^+ of all positive lower semicontinuous functions f on X by

$$\bar{I}(f) = \sup\{ I(g) : g \in K^+(X) \text{ and } g \leq f \} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Then:

(1) $\bar{I}(f) = I(f)$ if $f \in K^+(X)$

(2) $\bar{I}(f + g) = \bar{I}(f) + \bar{I}(g)$ $f, g \in M^+$

(3) $\bar{I}(af) = a\bar{I}(f)$ $a > 0$, $f \in M^+$

(4) $f \leq g \Rightarrow \bar{I}(f) \leq \bar{I}(g)$ $f, g \in M^+$

(5) if D is a subset of M^+, directed by $\leq$, then $\bar{I}(\sup\{ f : f \in D \}) = \sup\{\bar{I}(f) : f \in D \}$.

$\bar{I}$ is then extended to the set F^+ of all positive functions f on X by

$$\bar{\bar{I}}(f) = \inf\{\bar{I}(g) : g \in M^+ \text{ and } g \geq f \}.$$

Then:

(1) $\bar{\bar{I}}(g) = \bar{I}(g)$ if $g \in M^+$

(2) $\bar{\bar{I}}(f + g) \leq \bar{\bar{I}}(f) + \bar{\bar{I}}(g)$ $f, g \in F^+$

(3) $\bar{\bar{I}}(af) = a\bar{\bar{I}}(f)$ $a > 0$ $f \in F^+$

(4) If $\{f_n\}_{n=1}^{\infty} \subseteq F^+$ and $f_1 \leq f_2 \leq \ldots\ldots\ldots\ldots\ldots \leq f_n \leq \ldots\ldots$, then $\bar{\bar{I}}(\lim_{n\to\infty} f_n) = \lim_{n\to\infty} \bar{\bar{I}}(f_n)$.

For any subset A of X, let χ_A be its characteristic function. Define a set function μ by $\mu(A) = \bar{\bar{I}}(\chi_A)$. Then μ is a Carathéodary outer measure.

A subset A of X is said to be μ-null if $\mu(A) = 0$. If $A \cap K$ is μ-null for each compact set K then A is said to be locally μ-null. It can be shown that a set A is locally μ-null if and only if each x in X has a neighborhood V such that $\mu(A \cap V) = 0$.

Let $\mathfrak{M}_\mu$ be the σ-algebra of μ-measurable subsets of X. Then $\mathfrak{M}_\mu$ contains every Borel set and every locally null set. The following properties of μ can be shown.

(i) μ is a positive regular measure on $\mathfrak{M}_\mu$.

(ii) $\mu(K)$ is finite for each compact set K.

(iii) $\mu(X) = \sup\{I(f) : f \in K^+(X),\ f \leq 1\}$.

(iv) the functional I is bounded if and only if $\mu(X) < \infty$ and then $||I|| = \mu(X)$

(v) for each nonnegative function f on X, $\bar{\bar{I}}(f) = \int f d\mu$.

(vi) if λ is a regular measure on X; $\mathfrak{M}_\lambda$ the σ-algebra of λ-measurable subsets and $\lambda(K)$ is finite for each compact set K, then $\int f d\lambda = \int f d\mu$ for all $f \in K^+(X)$ implies $\lambda(A) = \mu(A)$ for all $A \in \mathfrak{M}_\lambda \cap \mathfrak{M}_\mu$.

For $f \in C_o^+(X)$ let $|I|(f) = \sup\{|I(g)| : g \in C_o(X),\ |g| \leq f$ and extend $|I|$ by linearity to $C_o(X)$. Then $|I|$ is a positive linear functional on $C_o(X)$; $|I| = I$ if and only if I is positive; and

$||(|I|)|| = ||I||$ ((12) Chapter III, 14.5).

3.4 The dual of $C_o(X)$. If I is in $C_o(X)'$ then there are positive linear functionals I_1, I_2, I_3, $I_4 \in C_o(X)'$ such that $I = I_1 - I_2 + I_3 - I_4$. We may apply the construction of the preceeding section to each I_i and obtain a positive regular measure μ_i (i = 1, 2, 3, 4) such that $I_i(f) = \int f d\mu_i$ for each $f \in C_o(X)$. The restriction of μ_i to the Borel subsets of X is a Borel measure, which we again denote by μ_i. Let $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$. Thus we have for each $I \in C_o(X)'$ a Borel measure μ such that $I(f) = \int f d\mu$. Moreover by (v) this measure is uniquely determined.

Theorem 4. The mapping $I \to \mu$ given by the above is a one-one linear mapping of $C_o(X)'$ onto M(X) such that

(i) $I(f) = \int_X f d\mu$ for all $f \in C_o(X)$

(ii) $||I|| = |\mu|(X)$

(iii) $|I|(f) = \int f d|\mu|$ for all $f \in C_o(X)$

For a proof see (12)(Chapter III, 14.10 and 14.14).

In view of the above theorem we now drop the distinction between elements of $C_o(X)'$ and elements of M(X). In particular for $\mu \in M(X)$, $f \in C_o(X)$ the symbols $\mu(f)$, $\int f d\mu$, $\int_X f(x)d\mu(x)$ etc. all have the same meaning, i.e. each is equal to I(f), where μ corresponds to I uniquely.

3.5 The support of a measure. The support of a measure μ, written Supp(μ) is the smallest closed set whose complement is μ-null.

Equivalently, Supp(μ) is the set of all x such that for any neighborhood V of x, there is an $f \in K(X)$, with $f(X \setminus V) = 0$ and $\mu(f) \neq 0$. If $|\mu|$ is the total variation of μ, then $Supp(\mu) = Supp(|\mu|)$ ((3) Chapitre III, §2 No. 2).

Let $x \in X$, then the linear functional ε_x on $C_o(X)$ defined by $\varepsilon_x(f) = f(x)$, for f in $C_o(X)$, is called the Dirac measure at the point x. Clearly $Supp(\varepsilon_x) = \{x\}$. Conversely if μ is a measure and $Supp(\mu) = \{x\}$ then there is a scalar a such that $\mu = a\varepsilon_x$.

A measure μ is said to be discrete or purely discontinuous if its support is a countable subset of X. A measure μ is said to be continuous if $\mu(\{x\}) = 0$ for all $x \in X$. Each measure $\mu \in M(X)$ has a unique decomposition, $\mu = \mu_d + \mu_c$ where μ_d is purely discontinuous and μ_c is continuous. Each purely discontinuous measure $\mu \in M(X)$ has the form $\mu = \Sigma_{x \in X}\, a_x \varepsilon_x$ where ε_x is the Dirac measure at the point x, and a_x is a complex number, (the number of x such that $a_x \neq 0$ is countable) and $||\mu|| = \Sigma_{x \in X} |a_x|$ ((12) Chapter V, §19).

<u>3.6 Dirac measures and extreme points of the unit ball</u>. The following theorems give the relationship between Dirac measureson a locally compact space X and extreme points of the unit ball of M(X). Theorem 5 below is taken from Dunford and Schwartz (6) p.441. Theorem 6 for the special case of X compact also appears in Dunford and Schwartz. The proof given here is an easy adaptation of theirs.

<u>Theorem 5</u>. Let X be a compact space and let N be a closed subspace of C(X). Then every extreme point of the unit ball of N' is of

the form $a\varepsilon_x$ where $|a| = 1$ and $x \in X$.

Proof: Let S (resp. S') be the unit ball of N (resp. N') and let A be the subset of N' of all elements $a\varepsilon_x$ where $|a| = 1$ and $x \in X$, together with the zero measure. Then

$$A^o = \{f \in N : |a\varepsilon_x(f)| \leq 1 \text{ for all } x \in X\}$$
$$= \{f \in N : |f(x)| \leq 1 \text{ for all } x \in X\}$$
$$= S$$

Therefore $A^{oo} = S'$ and by §1.5 the $\sigma(N',N)$-closure of the convex hull of A is S' which is a $\sigma(N',N)$-compact set by §1.5. Since each $f \in N$ is continuous, the mapping $x \to \varepsilon_x$ is a continuous mapping of X into $(N')_\sigma$, therefore $\{\varepsilon_x : x \in X\}$ is $\sigma(N',N)$-compact. Consequently $A = \{a\varepsilon_x : x \in X, |a| = 1\} \cup \{0\}$ is $\sigma(N',N)$-compact since it is the product of a compact set of complex numbers and a $\sigma(N',N)$-compact subset of N'. Thus lemma 1 of §1.1 applies and we have that every extreme point of S' is an element of A. Clearly 0 is not an extreme point of S', so that each extreme point is of the form $a\varepsilon_x$, $x \in X$ and $|a| = 1$.

If X is a locally compact non-compact space, let X^∞ be its one-point compactification. Then $C_o(X)$ may be identified with a subspace of $C(X^\infty)$ and $C_o(X)$ is closed in $C(X^\infty)$ since $C_o(X)$ is complete. Hence the above theorem applies and we have that every extreme point of the unit ball of $M(X) = C_o(X)'$ has the form $a\varepsilon_x$, $|a| = 1$, $x \in X^\infty$.

Theorem 6. Let X be a locally compact space. The extreme points of the unit ball of M(X) are the measures $a\varepsilon_x$, $|a| = 1$, $x \in X$.

Proof: By the above remarks if $\mu \in M(X)$ is such an extreme point then there is an $x \in X^\infty$ and an a, $|a| = 1$ such that $\mu = a\varepsilon_x$. If $x = \infty$ (i.e. $x \in X^\infty \setminus X$) then ε_x is 0 on $C_o(X)$ and therefore is the zero measure. Since $0 = 1/2\,\lambda + 1/2\,(-\lambda)$ for any $\lambda \in M(X)$, 0 cannot be the extreme point of the unit ball. To prove the theorem it remains to show that for given $x \in X$, $|a| = 1$, $a\varepsilon_x$ is an extreme point of the unit ball. For this first we show that ε_x is an extreme point. Let $\varepsilon_x = a\mu + (1 - a)\lambda$ where $0 < a < 1$ and $||\mu|| \leq 1$, $||\lambda|| \leq 1$. Then we have to show that $\mu = \lambda = \varepsilon_x$. Let $f \in C_o(X)$, $||f|| \leq 1$, and $f(x) = 0$. For each integer $n > 0$ let $W_n = \{y : |f(y)| < 1/n\}$ and let V_n be a compact neighborhood of x such that $V_n \subseteq W_n$. There is a $g_n \in K(X)$ such that $0 \leq g_n(y) \leq 1$ for all $y \in X$ and $g_n(y) = 1$, $y \in V_n$, $g_n(y) = 0$ for $y \notin W_n$. Let $f_n = f - g_n f$, then $f_n \to f$ in the norm of $C_o(X)$ and $f_n(\mathring{V}_n) = 0$. If $y \notin W_n$ then $|f_n(y) + g_n(y)| = |f(y)| \leq 1$ and if $y \in W_n$ then

$$\begin{aligned} |f_n(y) + g_n(y)| &= |f(y) - g_n(y)f(y) + g_n(y)| \\ &\leq |f(y)|(1 - g_n(y)) + g_n(y) \\ &\leq 1. \end{aligned}$$

Consequently $||f_n + g_n|| \leq 1$.

Now $\varepsilon_x(g_n) = 1$ and $\varepsilon_x(f_n) = 0$ so that

$$a\mu(f_n + g_n) + (1 - a)\lambda(f_n + g_n) = 1.$$

Since $||f_n + g_n|| \leq 1$ we have $|\mu(f_n + g_n)| \leq 1$ and $|\lambda(f_n + g_n)| \leq 1$ so that $\mu(f_n + g_n) = \lambda(f_n + g_n) = 1.$. Similarly since $||g_n|| \leq 1$ we obtain $\mu(g_n) = \lambda(g_n) = 1$. Therefore $\mu(f_n) = \lambda(f_n) = 0$. Since $f_n \to f$ we have $\mu(f) = \lambda(f) = 0$. Therefore $\ker \varepsilon_x \subseteq \ker \mu$ and $\ker \varepsilon_x \subseteq \ker \lambda$. Consequently there are scalars a, β such that $\mu = \alpha\varepsilon_x, \lambda = \beta\varepsilon_x$, and $|\alpha| \leq 1$, $|\beta| \leq 1$. Thus $\alpha = \beta = 1$, since

$a\alpha + (1 - a)\beta = 1$.

Let a be a scalar with $|a| = 1$ and let $x \in X$. If $a\varepsilon_x = b\mu + (1 - b)\lambda$ with $0 < b < 1$ and $||\mu|| \leq 1$, $||\lambda|| \leq 1$, then multiplying by $\bar{a}$ we have, $\varepsilon_x = b\bar{a}\mu + (1 - b)\bar{a}\lambda$ and since $||\bar{a}\mu|| = ||\mu|| \leq 1$, $||\bar{a}\lambda|| = ||\lambda|| \leq 1$ the above applies and we have $\varepsilon_x = \bar{a}\mu = \bar{a}\lambda$ which gives $a\varepsilon_x = \mu = \lambda$. Thus $a\varepsilon_x$ is an extreme point of the unit ball.

4. The $L^p(X)$ spaces.

Throughout this section X is a locally compact space; I a positive linear functional on K(X) and μ the measure constructed from I as in §3.3.

4.1 Definitions and elementary facts. A function f is said to be μ-null (resp. locally μ-null) if there is a μ-null (resp. locally μ-null) set N such that $f(x) = 0$, $x \in X \setminus N$. When no confusion will arise we shall drop the μ. We shall also say that $f(x) = 0$ almost everywhere in place of saying that f is null.

If f is locally null but not null then for each p, $1 \leq p < \infty$, $\int |f|^p d\mu = \infty$ ((12) Chapter III, 12.2). In particular if N is a locally μ-null set then $\mu(N) = \infty$ or $\mu(N) = 0$.

For each positive real number p, let $\mathcal{L}^p(X,\mu)$ be the set of all complex valued measurable functions on X such that $\int |f|^p d\mu < \infty$. Let $\mathcal{N}$ be the set of all μ-null functions on X, and put $L^p(X,\mu) = \mathcal{L}^p(X,\mu)/\mathcal{N}$. When no confusion can arise we write $L^p(X)$ for $L^p(X,\mu)$. We shall also allow ourselves the luxury of being imprecise and calling elements of

$L^p(X,\mu)$ functions. For $1 \leq p < \infty$, $L^p(X)$ is a Banach space with norm given by $||f||_p = [\int |f|^p d\mu]^{1/p}$. $L^2(X)$ is a Hilbert space with the inner product of $f, g \in L^2(X)$ given by $(f, g) = \int f\bar{g}d\mu$.

Let $\mathfrak{B}$ be the set of all measurable bounded functions on X and let $\mathcal{O}$ be the subset of $\mathfrak{B}$ consisting of all locally null functions on X. $\mathfrak{B}$ is a Banach space with norm given by $||f|| = \sup\{|f(x)| : x \in X\}$ and $\mathcal{O}$ is a closed subspace of $\mathfrak{B}$. Thus $\mathfrak{B}/\mathcal{O}$ is a Banach space which we denote by $L^\infty(X,\mu)$ or simply $L^\infty(X)$. We feel obligated to point out that this definition of $L^\infty(X)$ differs from that used by many writers. However this definition is the one used by Hewitt and Ross (12) and N. Bourbaki (3). It is clear that $K(X) \subseteq L^p(X)$ for all $p \geq 1$.

For given p, $1 < p < \infty$, let $q = p/(1 - p)$, for $p = 1$ let $q = \infty$ and for $p = \infty$ let $q = 1$. We shall need the following facts about $L^p(X)$ and $L^q(X)$:

(i) for $f \in L^p(X)$, $||f||_p = \sup\{|\int fgd\mu| : g \in K(X), ||g||_q \leq 1\}$.

(ii) (Hölder's inequality) if $f \in L^p(X)$ and $g \in L^q(X)$ then $fg \in L^1(X)$ and $\int |fg|d\mu \leq ||f||_p ||g||_q$.

(iii) $K(X)$ is norm dense in $L^p(X)$, $1 \leq p < \infty$.

4.2 Absolute continuity. Let J be a positive linear functional on $K(X)$ and let λ be the measure constructed from J as in §3.3. λ is said to be absolutely continuous with respect to μ if each locally μ-null set is locally λ-null. If λ is absolutely continuous with respect to μ then there is a positive μ-measurable function g such that $\int fd\lambda = \int fgd\mu$ for all $f \in K(X)$ (Lebesgue- Radon- Nikodym Theorem (12) Chapter III, 12.17).

The measures λ and μ are said to be equivalent if they are absolutely continuous with respect to each other.

Let λ be any regular measure on X. Then λ is said to be absolutely continuous with respect to μ if $|\lambda|(F) = 0$ for every compact set F such that $\mu(F) = 0$. If $\lambda \in M(X)$ and λ is absolutely continuous with respect to μ, then there is an $f \in L^1(X,\mu)$ such that $\lambda(A) = \int_A f d\mu$ for each Borel set A and $||f||_1 = ||\lambda||$((12) Chapter III, 14.17 and 14.19). The function f is called the Radon- Nikodym derivative of λ with respect to μ, and will sometimes be denoted by $\frac{d\lambda}{d\mu}$.

4.3 The product of a function and a measure. Let $f \in L^1(X,\mu)$ then we define $f\mu$ to be the measure defined for each Borel set A by $f\mu(A) = \int_A f d\mu$. Then $f\mu \in M(X)$, $f\mu$ is absolutely continuous with respect to μ and $||f\mu|| = ||f||_1$.

5. M(G) and $L^1(G)$.

5.1 The measure algebra. Let G be a locally compact group. For any function f on G and each $y \in G$, ${}_yf$ is the function defined by ${}_yf(x) = f(yx)$. Since for given y the mapping $x \rightarrow yx$ is a homeomorphism of G onto itself it follows that $f \in C_o(G)$ implies ${}_yf \in C_o(G)$. For $\mu \in M(G)$, and $f \in C_o(G)$ we may thus define a function $\bar{\mu}(f)$ by $\bar{\mu}(f)(x) = \mu({}_xf)$. Then $\bar{\mu}(f) \in C_o(G)$ ((12) Chapter V, 19.5). For $\mu, \lambda \in M(G)$ we define the convolution $\mu * \lambda$ by $\mu*\lambda(f) = \mu(\bar{\lambda}(f))$.

Proposition 1. M(G) is a Banach algebra[1] with convolution as

1. See §6.3 for the definitions of Banach algebra and Banach *-algebra.

multiplication.

Proof: M(G) is a Banach space since it is the dual of the Banach space $C_o(G)$ (§3.4). We now show that for $\mu, \lambda \in M(G)$ $||\mu*\lambda|| \leq ||\mu|| \, ||\lambda||$. Let $f \in C_o(G)$. Then

$$|\mu*\lambda(f)| = |\mu(\bar{\lambda}(f))| \leq ||\mu|| \, ||\bar{\lambda}(f)|| = ||\mu|| \sup_{x \in G}|\lambda({}_x f)|$$
$$\leq ||\mu|| \, ||\lambda|| \sup||{}_x f|| \leq ||\mu|| \, ||\lambda|| \, ||f||.$$

Consequently $$||\mu*\lambda|| \leq ||\mu|| \, ||\lambda||.$$

To prove associativity first note that for $x, y \in G$ we have

$$\bar{\lambda}({}_x f)(y) = \lambda({}_y({}_x f)) = \lambda({}_{xy} f) = \bar{\lambda}(f)(xy) = {}_x(\bar{\lambda} f)(y)$$

so that $$\bar{\lambda}({}_x f) = {}_x(\bar{\lambda} f).$$

Now $$(\overline{\mu*\lambda})(f)(y) = \mu*\lambda({}_y f) = \mu(\bar{\lambda}({}_y f)) = \mu({}_y(\bar{\lambda} f))$$
$$= \mu(\bar{\lambda}(f))(y).$$

Let $\nu \in M(G)$ then

$$(\nu*(\mu*\lambda))(f) = \nu(\overline{\mu*\lambda}(f)) = \nu(\mu(\bar{\lambda}(f))) = (\nu*\mu)(\bar{\lambda}(f))$$
$$= ((\nu*\mu)*\lambda)(f).$$

Thus convolution is associative. It is easily seen that for any complex number c we have $\mu*(c\lambda) = c(\mu*\lambda) = (c\mu)*\lambda$ and that the distributive laws hold.

If f is any function on G we define a function $f^\sim$ by $f^\sim(x) = f(x^{-1})$. Since the mapping $x \to x^{-1}$ is a homeomorphism of G, $f \in C_o(G)$ implies $f^\sim \in C_o(G)$. We now define a mapping $\mu \to \mu^*$ of M(G) onto itself by $\mu^*(f) = \overline{\mu(\bar{f}^\sim)}$ where $^-$ is the complex conjugate.

Theorem 7. M(G) is a Banach *-algebra.

Proof: It is straightforward to verify that (1) $(\mu + \lambda)^* =$

$\mu^* + \lambda^*$, (2) $(\alpha\mu)^* = \bar{\alpha}\mu^*$, (3) $\mu^{**} = \mu$ and that $||\mu^*|| = ||\mu||$. For the proof that $(\mu*\lambda)^* = \lambda^* * \mu^*$ see (12)(Chapter V, 20.22).

The algebra M(G) is called the measure algebra of the locally compact group G.

For each $x \in G$, let ε_x be the Dirac measure at x, and let G^ε be the collection of all Dirac measures.

Proposition 2. G^ε is a group and the mapping $x \to \varepsilon_x$ is a homomorphism.

Proof: For $x, y \in G$, $f \in C_o(G)$, $\bar{\varepsilon}_x(f)(y) = \varepsilon_x({}_yf) = {}_yf(x) = f(yx) = f_x(y)$. Thus $\varepsilon_x * \varepsilon_y(f) = \varepsilon_x(f_y) = f(xy) = \varepsilon_{xy}(f)$. Thus $\varepsilon_x * \varepsilon_y = \varepsilon_{xy}$. From this it follows that $\varepsilon_{x^{-1}} * \varepsilon_x = \varepsilon_x * \varepsilon_{x^{-1}} = \varepsilon_e$ where e is the identity of G. Thus to show ε_x^{-1} exists and $\varepsilon_x^{-1} = \varepsilon_{x^{-1}}$ it suffices to show that $\varepsilon_e * \mu = \mu * \varepsilon_e = \mu$ for all $\mu \in M(G)$. Now

$$(\varepsilon_e * \mu)(f) = \varepsilon_e(\bar{\mu}(f)) = \bar{\mu}(f)(e) = \mu(f) \text{ and}$$
$$(\mu * \varepsilon_e)(f) = \mu(\bar{\varepsilon}_e(f)) = \mu(f).$$

Thus ε_e is the identity of M(G).

5.2 Haar measure. Let G be a locally compact group. There exists a positive regular measure m on G finite for each compact set, which is not identically zero and is left translation invariant, i.e. for each $x \in G$ and each Borel set $A \subseteq G$, $m(xA) = m(A)$. The measure m is called the Haar measure on G and is unique to within a positive constant factor.

The idea of the proof is to construct a positive translation invariant linear functional I on K(G) (by translation invariant we mean $I({}_x f) = I(f)$ for each $x \in G$) and then apply the extension procedure outlined in §3.3. For the construction of such a linear functional the reader is referred to Hewitt and Ross ((12) Chapter IV, 15), Husain ((13) Chapter VI) or Loomis ((15) Chapter VI).

For $x \in G$, consider the measure m_x defined for each Borel set A by $m_x(A) = m(Ax)$. Then m_x is a left invariant measure so there is a real number $\Delta(x) > 0$ such that $m(Ax) = \Delta(x)m(A)$. The function $\Delta : x \to \Delta(x)$ is a continuous homomorphism of G into the positive reals ((12) Chapter IV, 15.11) and is called the modular function. G is said to be unimodular if $\Delta(x) = 1$ for all $x \in G$. If G is compact then $\Delta(G)$ is a compact subgroup of the positive reals and consequently $\Delta(G) = \{1\}$ so that every compact group is unimodular. Clearly every abelian locally compact group is also unimodular.

<u>5.3 $M_a(G)$ and $L^1(G)$</u>. Let $M_a(G)$ be the subspace of M(G) consisting of all measures μ which are absolutely continuous with respect to the Haar measure m on G. Let $h \in L^1(G) = L^1(G,m)$ then the measure hm is in $M_a(G)$ and the mapping $h \to hm$ is a linear one-one mapping of $L^1(G)$ into $M_a(G)$ which preserves norms (§4.3). If $\mu \in M_a(G)$ then by the Radon- Nikodym theorem there is an $h \in L^1(G)$ such that $hm = \mu$. Consequently the above mapping is an isometry. Thus we may identify $L^1(G)$ with $M_a(G)$. It turns out that $M_a(G)$ is a closed two sided ideal in M(G) ((12) Chapter V, 19.18). Thus if $\mu \in M(G)$ and $h \in L^1(G)$, $\mu * hm$ and $hm * \mu$ are absolutely continuous with respect to m. We define

$\mu * h$ (resp. $h * \mu$) to be the Radon- Nikodym derivative of $\mu * hm$ (resp. $hm * \mu$) with respect to m. Then the following formulas hold for g, $h \in L^1(G)$, and $\mu \in M(G)$.

(i) $\mu * h(x) = \int h(y^{-1}x) d\mu(y)$

(ii) $h * \mu(x) = \int \Delta(y^{-1}) h(xy^{-1}) d\mu(y)$

(iii)
$$\begin{aligned} h * g(x) &= \int h(xy) g(y^{-1}) dm(y) \\ &= \int h(y) g(y^{-1}x) dm(y) \\ &= \int \Delta(y^{-1}) h(y^{-1}) g(yx) dm(y) \\ &= \int \Delta(y^{-1}) h(xy^{-1}) g(y) dm(y) \end{aligned}$$

((12) Chapter V, 20.9 and 20.10).

For each $\mu \in M(G)$ we define a mapping T_μ of $L^1(G)$ onto itself by $T_\mu h = \mu * h$, $h \in L^1(G)$. The mapping $\mu \rightarrow T_\mu$ is a one-one mapping of $M(G)$ into $C(L^1(G), L^1(G))$ (see §1.3 for the definition).

Theorem 8. (Wendel). The image of $M(G)$ in $C(L^1(G), L^1(G))$ by the mapping $\mu \rightarrow T_\mu$ is closed.

For the proof see (24) theorem 2.

The so-topology on $M(G)$ is the coarsest topology such that the above embedding $M(G) \rightarrow C(L^1(G), L^1(G))$ is continuous. It follows from theorem 8 and §1 that $M(G)_{so}$ is a locally convex space which is quasi-complete.

5.4 The $L^p(G)$ space. Let $f \in L^p(G)$ $(1 \leq p \leq \infty)$ and let μ be a measure in $M(G)$. Then $\int f(y^{-1}x) d\mu(y)$ exists and is finite for all

$x \notin N$ where $N \subseteq G$ is m-null if $1 \leq p < \infty$. If we define a function $\mu * f$ by $\mu * f(x) = \int f(y^{-1}x)d\mu(y)$, $x \notin N$ and $\mu * f(x) = 0$, $x \in N$ then we have $||\mu * f||_p \leq ||\mu||\ ||f||_p$ ((12) Chapter V 20.12).

Let $1 \leq p \leq \infty$, and let $q = p/(p - 1)$ if $p \neq 1$ or ∞, $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$. Let μ be in $M(G)$ and suppose that $\int \Delta(y)^{-1/q}\, d|\mu|(y)$ is finite. Let f be in $L^p(G)$. Then the integral

$$\int \Delta(y^{-1})f(xy^{-1})d\mu(y) = f*\mu(x) \qquad (1)$$

exists and is finite for all $x \in G \setminus N$ where N is m-null if $1 \leq p < \infty$ and locally m-null if $p = \infty$. Equation (1) defines a function in $L^p(G)$ for which

$$||f*\mu||_p \leq ||f||_p \int_G \Delta(y)^{-1/q}\, d|\mu|(y)$$

((12) Chapter V, 20.13). In particular if G is unimodular then the convolution $f*\mu$ exists for each $f \in L^p(G)$, $\mu \in M(G)$ and $||f*\mu||_p \leq ||f||_p||\mu||$.

If $1 < p < \infty$ and $q = p/(p - 1)$ then for $f \in L^p(G)$, $g^{\sim} \in L^q(G)$, the integral $\int_G f(xy)g(y^{-1})dy = f*g(x)$ exists for all $x \in G$ and defines a function in $C_o(G)$. ((12) Chapter V, 20.16). From this it follows that if A is a set of positive finite measure then AA^{-1} is a neighborhood of $e \in G$. Consequently if S is a subgroup of G which contains a set of positive measure then S is open and hence closed.

In a later section we shall need the following lemma which is given in Hewitt and Ross (12)(Chapter V, 20.15).

<u>Lemma 2</u>. Let f be in $L^p(G)$ ($1 \leq p < \infty$) and ε a positive number. There is a neighborhood U of the identity e in G such that

(i) $||\mu * f - f||_p < \varepsilon$

for all positive measures μ in $M(G)$ such that $\mu(G) = 1$ and $\mu(G \setminus U) = 0$. There is also a neighborhood V of e such that

(ii). $||f*\mu - f||_p < \varepsilon$

for all positive measures μ in $M(G)$ such that $\mu(G) = 1$ and $\mu(G \setminus V) = 0$.

6. Algebras, *-algebras and topological algebras.

6.1 Definitions and elementary facts. By an algebra we mean a linear associative algebra over the complex field. An algebra A is called a *-algebra if there is a mapping $x \to x^*$ of A onto itself such that for $x, y \in A$ and any complex number c

(i) $x^{**} = x$

(ii) $(cx)^* = \bar{c}x^*$ where $\bar{c}$ is the complex conjugate of c

(iii) $(x + y)^* = x^* + y^*$

(iv) $(xy)^* = y^*x^*$

A topological algebra is an algebra A given a topology u such that A_u is a TVS and such that for each $y \in A$ the mappings $x \to xy$ and $x \to yx$ are continuous.[1]

A topological *-algebra A is a topological algebra which is a *-algebra and such that the mapping $x \to x^*$ is continuous.

We now state a number of elementary facts concerning topological algebras and *-algebras. Let A be a topological algebra (resp. topological *-algebra) then

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1. It should be noted that some authors require multiplication to be jointly continuous.

(a) the closure of a subalgebra (resp. *-subalgebra) of A is a subalgebra (resp. *-subalgebra) of A.

(b) the closure of an abelian subalgebra (resp. abelian *-subalgebra) of A is an abelian subalgebra (resp. abelian *-subalgebra) of A.

(c) if N is a closed two sided ideal of A then A/N is a topological algebra.

6.2 The Jacobson radical and semi-simplicity. Let A be an algebra and let $x \in A$. An element y in A is called a left (resp. right) quasi-inverse for x if $x + y - yx = 0$ (resp. $x + y - xy = 0$). If y is both a left and a right quasi-inverse for x then y is called a quasi-inverse for x.

The Jacobson radical of A is the set of all $y \in A$ such that for each x in A, and each scalar a, $ay + xy$ has a left quasi-inverse. A is said to be semi-simple if the Jacobson radical of A consists of only the zero element. There are various characterizations of the Jacobson radical, - we refer the reader to (17)(Chapter II, §7) for these.

6.3 Banach algebras and W*-algebras. A Banach algebra A is an algebra whose underlying vector space is a Banach space and whose norm satisfies the inequality $||xy|| \leq ||x|| \, ||y||$ for each $x, y \in A$. If in addition A has a unit u then we shall require $||u|| = 1$. A Banach *-algebra is a *-algebra which is a Banach algebra such that $||x^*|| = ||x||$ for each x in A. A C*-algebra is a Banach *-algebra in which $||x^*x|| = ||x||^2$ is satisfied for all x in A.

If p is a multiplicative linear functional on a Banach algebra A, then $||p|| \leq 1$. If A has a unit u then $||p|| = 1$ ((12) C.21, note that commutivity is not used in the proof).

Theorem 9. (Kakutani). Let A be a Banach algebra having a unit u. Then u is an extreme point of the unit ball of A.

Proof: Let A' be the dual of A and let S' be the unit ball of A'. Now suppose $u = ax + (1 - a)y$, $0 < a < 1$, $||x|| \leq 1$, $||y|| \leq 1$. We shall now show that $x = y = u$. Let x' be an extreme point of S'. Let x_1' be defined by $x_1'(z) = x'(xz)$ and $x_2'(z) = x'(yz)$ for any $z \in A$. Then $ax_1'(z) + (1 - a)x_2'(z) = x'(ax + (1 - a)y)z) = x'(z)$ and since x' is an extreme point and $||x_1'|| \leq 1$, $||x_2'|| \leq 1$ we have $x_1' = x_2' = x'$. Therefore $x'(z) = x'(xz) = x'(yz)$ for any $z \in A$ and any extreme point x' of S'. For a given z in A, the mapping $x' \to x'(z - xz)$ of A' into the complex numbers is a $\sigma(A',A)$-continuous linear functional on A'. By the above, this mapping is zero on the extreme points of S'. Consequently it is zero on the $\sigma(A',A)$-closure of the convex span of the extreme points of S'. By the Krein- Milman theorem (§1.1), S' is the $\sigma(A',A)$-closure of the extreme points of S'. Therefore $x'(z - xz) = 0$ for all $x' \in S'$, and thus $z = xz$ for all $z \in A$. Taking $z = u$ we have $x = u$, and then $y = u$. This completes the proof.

Let H be a Hilbert space and let B(H) be the algebra of all bounded operators on H. Then B(H) is a C*-algebra where the norm of an element T is given by $||T|| = \sup \{||Tx|| : ||x|| \leq 1\}$. T* is the conjugate of T defined by $(Tx,y) = (x,T^*y)$ for $x, y \in H$.

In addition to the norm topology on B(H) a number of other topologies are commonly used. We will use only one of these, the weak operator or w-topology. For $x, y \in H$ define a semi-norm $p_{x,y}(A) = |(Ax,y)|$, where (x,y) is the scalar product of $x, y \in H$. Then the w-topology is the topology given by the family $\{p_{x,y} : x,y \in H\}$ of semi-norms. A W*-algebra (also called a vonNeumann algebra) is a *-subalgebra of B(H) which is w-closed.

Let $\mathfrak{A}$ be a subset of B(H), then the commutant of $\mathfrak{A}$, denoted by $\mathfrak{A}^c$ is the subset of B(H) consisting of all operators S such that $ST = TS$ for all T in $\mathfrak{A}$. The W*-algebra generated by $\mathfrak{A}$ is the smallest W*-algebra containing $\mathfrak{A}$. If $T \in \mathfrak{A} \Rightarrow T^* \in \mathfrak{A}$ for all $T \in \mathfrak{A}$, then the W*-algebra generated by $\mathfrak{A}$ equals $\mathfrak{A}^{cc} (= (\mathfrak{A}^c)^c)$ ((17) Chapter VII, 34). In particular if $\mathfrak{A}$ is a W*-algebra then $\mathfrak{A} = \mathfrak{A}^{cc}$.

7. Representations of locally compact groups.

7.1 Definitions. Let G be a locally compact group. A representation of G is a pair (L,H) where H is a Hilbert space and L is a homomorphism $x \to L_x$ of G into the group of unitary operators on H, such that for each $f \in H$, $x \to L_x f$ is a continuous function from G into H.

A subspace $V \subseteq H$ is said to be invariant under L if $L_x V \subseteq V$ for all $x \in G$. A representation is said to be irreducible if the only closed invariant subspaces of H are $\{0\}$ and H.

7.2 The regular representations. Let p be a real number $1 \leq p < \infty$, then for $f \in L^p(G)$ the mapping $x \to {}_xf$ is right uniformly continuous, ((12) Chapter V, 20.4) and the mapping $x \to f_x$ is continuous. Let L_x be the operator on $L^2(G)$ defined by $L_xf = {}_{x^{-1}}f$ and let L be the mapping $x \to L_x$. Then $(L, L^2(G))$ is a representation of G which we call the left regular representation. Let R_x be the operator on $L^2(G)$ defined by $R_xf = f_x$ and let R be the mapping $x \to R_x$. Then $(R, L^2(G))$ is a representation of G which we call the right regular representation of G.

Let G be a locally compact unimodular group. For each $\mu \in M(G)$, let R_μ be the operator on $L^2(G)$ defined by $R_\mu f = f*\mu$ for $f \in L^2(G)$. Then each R_μ is in the commutant of $\{L_x : x \in G\}$. Let $\mathfrak{R}$ be the W*-algebra generated by $\{R_\mu : \mu \in M(G)\}$. Note that $R_{\varepsilon_x}f = f*\varepsilon_x = f_{x^{-1}} = R_{x^{-1}}f$. It follows that the W*-algebra generated by $\{R_x : x \in G\}$ is $\mathfrak{R}$. It is known that $\mathfrak{R}$ is the commutant of $\{L_x : x \in G\}$ ((20) theorem).

7.3 Induced representations. Let G be a locally compact unimodular group and let S be an open abelian subgroup of G. Then S is also closed and locally compact. Let t be a continuous character of S. We define a representation (L^t, H) of G as follows. Let K be the vector space of all functions f on G such that $f(xs) = t(s)f(x)$ for all x in G and s in S. Choose an element from each left coset of S and let G/S be the set obtained in this manner. For $f \in K$ we define $||f||^2 = \Sigma_{x \in G/S} |f(x)|^2$. Let H be the set of all $f \in K$ such that $||f|| < \infty$. Then H is a Hilbert space with the inner product given by $(f,g) = \Sigma_{x \in G/S} f(x)\overline{g(x)}$. For each $x \in G$ we define an operator L_x^t on H by $L_x^tf(y) = f(x^{-1}y)$, and let L^t be the mapping $x \to L^t_x$. Then (L^t, H) is a representation of G which is called the representation induced by the character t. A more general treatment of the above construction can be found in (16) (Chapter III).

8. Direct Integrals of Hilbert Spaces and Representations.

In this section we shall outline the theory of direct integrals of Hilbert spaces and of representations. Unless another reference is given, the proof of any assertion we make can be found in (4)(Chapitre II).

Let X be a locally compact space, and μ a positive regular measure on X. Let $\{H(t) : t \in X\}$ be a family of Hilbert spaces and let $\mathcal{F} = \prod_{t \in X} H(t)$. An element $f \in \mathcal{F}$ is called a vector field on X.

<u>Definition:</u> A family $\{H(t) : t \in X\}$ of Hilbert spaces is said to be μ-measurable if there is a subspace $\mathcal{G} \subseteq \mathcal{F}$ such that

(i) for each $f \in \mathcal{G}$, the function $t \to ||f(t)||$ of X into the positive reals is measurable.

(ii) if $g \in \mathcal{F}$ is such that for each $f \in \mathcal{G}$, the function $t \to (g(t), f(t))$ is μ-measurable then $g \in \mathcal{G}$, where $(g(t), f(t))$ is the inner product in $H(t)$ for each t.

(iii) there exists a sequence of elements $(f_1, f_2, \ldots\ldots)$ of $\mathcal{G}$ such that $(f_1(t), f_2(t) \ldots\ldots)$ is total in $H(t)$, for each $t \in X$.

The elements of $\mathcal{G}$ are called μ-measurable vector fields.

If there exists in $\mathcal{F}$ a sequence $(f_1, f_2, \ldots)$ of vector fields such that (1) the functions $t \to (f_i(t), f_j(t))$ are measurable for $i, j = 1, 2\ldots$ and (2) $(f_1(t), f_2(t)\ldots)$ is total in $H(t)$ for each $t \in X$, then there exists a unique $\mathcal{G} \subseteq \mathcal{F}$ satisfying (i), (ii) and (iii) above. A necessary and sufficient condition for a vector field g to be μ-measurable is that $t \to (g(t), f_i(t))$ be μ-measurable for each $i = 1, 2\ldots$.

Let $\mathfrak{H}$ be the subset of $\mathcal{G}$ consisting of all μ-measurable vector fields f such that $\int ||f(t)||^2 d\mu(t) < \infty$, and let $\mathfrak{H}_0$ be the subset of $\mathfrak{H}$ of all vector fields f such that $\int ||f(t)||^2 d\mu(t) = 0$. Define an inner product in $\mathfrak{H}/\mathfrak{H}_0$ by $(f,g) = \int (f(t), g(t)) d\mu(t)$, and then $\mathfrak{H}/\mathfrak{H}_0$ becomes a Hilbert space which we denote by $\int_X H(t) d\mu(t)$. We shall write H in place of $\int_X H(t) d\mu(t)$. The space H is called the direct integral of the family $\{H(t) : t \in X\}$.

Suppose that for each $t \in X$, T^t is a bounded linear operator on $H(t)$. If $t \to (T^t f(t), g(t))$ is μ-measurable for each $f, g \in \mathcal{G}$ then $t \to T^t$ is said to be a μ-measurable operator field. A necessary and sufficient condition for $t \to T^t$ to be μ-measurable is that $t \to (T^t f_i(t), f_j(t))$ be μ-measurable for each i and j.

If $t \to T^t$ is a μ-measurable operator field and if $\sup_t ||T^t|| < \infty$ then there is a unique bounded linear operator T on H such that for each f in H $(Tf)(t) = T^t f(t)$ almost everywhere. It can then be shown that $||T|| = \sup_t ||T^t||$. In particular if $h \in L^\infty(X,\mu)$ then there is a unique operator T_h on H such that $(T_h f)(t) = h(t) f(t)$ almost everywhere for each $f \in H$. Let $\mathcal{Z} = \{T_h : h \in L^\infty(X,\mu)\}$ then $\mathcal{Z}$ is a W*-algebra called the algebra of diagonalizable operators. If T is a bounded linear operator on H then T is said to be decomposable if there exists a μ-measurable operator field $t \to T^t$ such that for each $f \in H$, $(Tf)(t) = T^t f(t)$ almost everywhere. A bounded operator T on H is decomposable if and only if $TT_h = T_h T$ for all $h \in L^\infty(X,\mu)$.

If we are given a Hilbert space H, and an abelian W*-algebra $\mathfrak{Z}$, then there exists a locally compact space X, a positive measure μ on X, a μ-measurable family $\{H(t) : t \in X\}$ of Hilbert spaces and an isometry V of $H' = \int H(t)d\mu(t)$ onto H such that the mapping $T \to VTV^{-1}$ maps the algebra of diagonalizable operators on H' onto $\mathfrak{Z}$.

We now define the direct integral of representations. Let X be a locally compact space, μ a positive regular measure on X. A family $\{(L^t,H(t)) : t \in X\}$ of representations of a group G is said to be μ-measurable if the family $\{H(t) : t \in X\}$ of Hilbert spaces is μ-measurable and if for each $x \in G$, the operator field $t \to L_x^t$ is μ-measurable. Since $||L_x^t|| = 1$ for all t, there exists a linear operator L_x on $H = \int H(t)d\mu(t)$ such that $(L_x f)(t) = L_x^t f(t)$ almost everywhere. The mapping $x \to L_x$ is denoted by $\int L^t d\mu(t)$, and the pair $(\int L^t d\mu(t), \int H(t)d\mu(t))$ is a representation of G called the direct integral of the representations $\{(L^t,H(t)) : t \in X\}$.

<u>Theorem 10</u>. (Mautner). Let G be a locally compact group having a countable basis for the open sets, and let (L,H) be a representation of G on the separable Hilbert space H. Let $\mathfrak{A}$ be the commutant of $\{L_x : x \in G\}$ and let $\mathfrak{Z}$ be an abelian W*-subalgebra of

$\mathfrak{A}$. Then there is a compact subset I of the real line, a positive regular measure μ on I, and a μ-measurable family of representations $\{(L^t, H(t)) : t \in I\}$ of G, and an isometry V of $\int H(t)d\mu(t)$ onto H such that

(i) $V^{-1}L_x V = \int L_x^t d\mu(t)$ for each $x \in G$.

(ii) $A \to VAV^{-1}$ maps the algebra of diagonalizable operators on $\int H(t)d\mu(t)$ onto $\mathfrak{Z}$.

If $\mathfrak{Z}$ is a maximal abelian W*-subalgebra of $\mathfrak{A}$, then there is a set $N \subseteq I$ of μ-measure zero such that for $t \in I \setminus N$, L^t is irreducible.

Proof: See (17) §41, theorem 3.

We shall close this section with a theorem concerning the uniqueness of such decompositions.

Theorem 11. Let G be a locally compact group having a countable basis for the open sets, X a locally compact space having a countable basis for the open sets, μ a positive measure on X, $\{(L^t, H(t)) : t \in X\}$ a μ-measurable family of representations of G,

$$H = \int H(t)d\mu(t) \quad \text{and } L = \int L^t d\mu(t)$$

and let $\mathfrak{Z}$ be the algebra of diagonalizable operators on H. Define in an analagous fashion X_1, μ_1, $\{L_1^{t_1}, H_1(t_1)) : t_1 \in X_1\}$, H_1, L_1 and $\mathfrak{Z}_1$. If there exists an isometry U of H onto H_1 such that the mapping $A \to UAU^{-1}$ maps $\mathfrak{Z}$ onto $\mathfrak{Z}_1$ and L_x to L_{1x} for each $x \in G$, then there exists:

(i) a set $N \subseteq X$ of μ-measure zero, a set $N_1 \subseteq X_1$ of μ_1-measure zero,

(ii) a Borel isomorphism θ of $X \setminus N$ onto $X_1 \setminus N_1$ which maps μ to a measure $\tilde{\mu}_1$ equivalent to μ_1:

(iii) an isometry $V(t)$ for each $t \in X \setminus N$ of $H(t)$ onto $H_1(\theta(t))$ such that $V(t)L_x^t = L_{1x}^{\theta(t)}V(t)$ for each $x \in G$ and $t \in X \setminus N$.

Proof: Note that a locally compact space having a countable basis for the open sets is a separable complete metric space and therefore in the terminology of (5), the measure spaces (X,μ) and (X_1, μ_1) are standard. The proof of the above theorem is almost identical to the proof of 8.2.4 of (5). (8.2.4 is proven for representations of C*-algebras, the proof for representations of locally compact groups is an easy modification of the proof given).

CHAPTER II

THE so-TOPOLOGY AND ITS APPLICATIONS

1. The weak topology

Throughout this chapter G is a locally compact Hausdorff topological group. The weak topology on M(G) is the $\sigma(M(G),C_o(G))$-topology, i.e. the coarsest topology such that for each f in $C_o(G)$ the mapping $\mu \to \mu(f)$ of M(G) into the complex numbers is continuous. We shall frequently write σ in place of $\sigma(M(G),C_o(G))$.

Proposition 1. G is isomorphic and homeomorphic to G^{ε}_{σ}.

Proof: By proposition 2, §5, Chapter I, the mapping $x \to \varepsilon_x$ is a homomorphism of G onto G^{ε}, and since $C_o(G)$ separates points, this mapping is an isomorphism. Since each $f \in C_o(G)$ is continuous, $x \to \varepsilon_x$ is a continuous mapping of G onto G^{ε}_{σ}. Let V be a compact neighborhood of the identity e in G. Then there is an $f \in C_o^+(G)$, $0 \leq f(x) \leq 1$, such that $f(e) = 1$ and $f(x) = 0$ for $x \notin V$. Let $W = \{x : |f(x) - f(e)| < 1\}$ then $W \subseteq V$ and W is the image of the neighborhood $\{\varepsilon_x : |\varepsilon_x(f) - \varepsilon_e(f)| < 1\}$ of the identity ε_e of G^{ε} under the mapping $\varepsilon_x \to x$.

Proposition 2. The mappings $\mu \to \mu*\lambda$, $\mu \to \lambda*\mu$ and $\mu \to \mu^*$, of M(G) into itself are weakly continuous.

Proof: $(\mu*\lambda)(f) = \mu(\bar{\lambda}(f))$ so that $\mu \to \mu*\lambda$ is weakly continuous. Since $\mu^*(f) = \overline{\mu(\overline{f^\sim})}$, $\mu \to \mu^*$ is weakly continuous. The mapping $\mu \to \lambda*\mu$ may be written as $\mu \to \mu^* * \lambda^* \to (\mu^* * \lambda^*)^* = \lambda*\mu$, so that it is continuous.

Corollary 1. $M(G)_\sigma$ is a topological *-algebra.

Proof: $M(G)_\sigma$ is a locally convex space since it is the dual of a Banach space and thus in view of the above proposition $M(G)_\sigma$ is a topological *-algebra.

Corollary 2. The weak closure of a subalgebra (resp. *subalgebra) of $M(G)$ is a subalgebra (resp. *-subalgebra) of $M(G)$.

Proof: This is true for any topological *-algebra. (Chapter 1, §6).

Definition: Let S be a Borel subset of G and let $M(S) = \{\mu \in M(G) : |\mu|(G \sim S) = 0\}$.

Lemma 1. Let S be a closed subset of G, and let $\mu \in M(G)$. Then $\mu \in M(S)$ if and only if $\mu(f) = 0$ for each $f \in K(G)$ such that $\mathrm{Supp}(f) \cap S = \emptyset$.

Proof: Since S is closed, $G \sim S$ is open so that $|\mu|(G \sim S) = 0$ $\Rightarrow$ $\mathrm{Supp}\,(|\mu|) \subseteq S$. Since $\mathrm{Supp}\,(|\mu|) = \mathrm{Supp}(\mu)$ we have $\mathrm{Supp}\,(\mu) \subseteq S$. Consequently if $\mathrm{Supp}(f) \cap S = \emptyset$ then $\mu(f) = 0$.

For the other part note that S closed implies that $\chi_{G \sim S}$ (the characteristic function of $G \sim S$) is lower semicontinuous. Thus by Chapter 1, §3.3

$$|\mu|(G\sim S) = \sup\{|\mu|(f) : f \in K^+(G) \text{ and } f \leq \chi_{G\sim S}\}.$$

Recall (Chapter I, §3.3) that for $f \in C_o^+(G)$

$$|\mu|(f) = \sup\{|\mu(g)| : |g| \leq f,\ g \in C_o(G)\}.$$

Therefore if $\mu(f) = 0$ for all f such that $f \in K(G)$ and $\mathrm{Supp}(f)\cap S = \emptyset$, then $|\mu|(f) = 0$ for all $f \in K^+(G)$ with $\mathrm{Supp}(f)\cap S = \emptyset$. Then $|\mu|(f) = 0$ for all $f \in K^+(G)$ with $f \leq \chi_{G\sim S}$ so that $|\mu|(G\sim S) = 0$. Hence $\mu \in M(S)$.

Proposition 3. (i) M(S) is a norm closed subspace of M(G). (ii). M(S) is weakly closed if and only if S is closed.

Proof: (i) $\mu,\lambda \in M(S) \Rightarrow |\mu(A) + \lambda(A)| \leq |\mu|(A) + |\lambda|(A) = 0$ for all Borel sets $A \subseteq G\sim S \Rightarrow \mu + \lambda \in M(S)$. For any $\mu \in M(G)$ and any complex number c, $|c\mu| = |c||\mu|$ so that $\mu \in M(S)$ implies $c\mu \in M(S)$. Hence M(S) is a subspace of M(G). Now to show that M(S) is norm closed suppose (μ_n) is a sequence in M(S) and $||\mu_n - \mu|| \to 0$. Then for any Borel set $A \subseteq G\sim S$

$$|\mu(A)| = |\mu_n(A) - \mu(A)| \leq |\mu_n - \mu|(A) \leq ||\mu_n - \mu||$$

hence $|\mu|(G\sim S) = 0$ so that $\mu \in M(S)$.

(ii) Suppose S is closed. Let $f \in K(G)$ be such that $\mathrm{Supp}(f)\cap S = \emptyset$. Then if μ is in the weak closure of M(S) we must have $\mu(f) = 0$, thus by lemma 1, $\mu \in M(S)$.

Now suppose M(S) is weakly closed. Let $a \in \overline{S}$ then since the mapping $x \to \varepsilon_x$ is a continuous mapping of G into $M(G)_\sigma$, $\varepsilon_a \in M(S)$. Therefore $\varepsilon_a(G\sim S) = |\varepsilon_a|(G\sim S) = 0$ so that $a \in S$, and S is closed.

Corollary. Let K be a closed subset of G. Then $T = \{\mu \in M(G) : \mathrm{Supp}(\mu) \subseteq K\}$ is a weakly closed subspace of M(G).

Proof: It is clear that $T \subseteq M(K)$. If $\mu \in M(K)$, then $|\mu|(G \sim K) = 0$ and $G \sim K$ open implies $\mathrm{Supp}(|\mu|) \subseteq K$. This means $\mu \in T$ since $\mathrm{Supp}(\mu) = \mathrm{Supp}(|\mu|)$. Therefore $T = M(K)$ which is weakly closed by the above proposition since K is closed.

Remark. It is worth noting that if K is not closed then $|\mu|(G \sim K) = 0$ does not mean that $\mathrm{Supp}(|\mu|) \subseteq K$. To see this let μ be a nonzero continuous measure, and let $x \in \mathrm{Supp}(\mu)$. Put $K = G \sim \{x\}$, then $\mathrm{Supp}(|\mu|) \nsubseteq K$, but $|\mu|(G \sim K) = 0$ since $G \sim K = \{x\}$ and μ and therefore $|\mu|$ is continuous.

Proposition 4. Let V be the linear span of the set of Dirac measures. Then for any $\mu \in M(G)$, $\mu \in \mathrm{Cl}_\sigma\{\lambda \in V : ||\lambda|| \leq ||\mu||$ and $\mathrm{Supp}(\lambda) \subseteq \mathrm{Supp}(\mu)\}$.

Proof: Put $A = \{\lambda \in V : ||\lambda|| \leq ||\mu||$ and $\mathrm{Supp}(\lambda) \subseteq \mathrm{Supp}(\mu)\}$. Since A is convex and contains 0, it suffices to show that $\mu \in A^{oo}$ (Chapter I, §1, 5) where A^o is the polar of A in $C_o(G)$ and A^{oo} is the polar of A^o in M(G). If $\mu = 0$, then the proposition is clear. Assume $\mu \neq 0$, let $f \in A^o$ then in particular $|\varepsilon_x(f)| \leq 1/||\mu||$ for all $x \in \mathrm{Supp}(\mu)$. i.e. $|f(x)| \leq 1/||\mu||$ for $x \in \mathrm{Supp}(\mu)$. Thus $|\mu(f)| \leq \int |f(x)| d|\mu|(x) \leq (1/||\mu||) \int d|\mu|(x) \leq 1$. Thus $\mu \in A^{oo}$.

Corollary 1. G^ε is total in $M(G)_\sigma$.

Corollary 2. Given any $\mu \in M(G)$ there is a net $(\mu_j : j \in J)$ in $A = \{\lambda \in V : \mathrm{Supp}(\lambda) \subseteq \mathrm{Supp}(\mu), ||\lambda|| \leq ||\mu||\}$ such that $\mu_j \xrightarrow{\sigma} \mu$ and $||\mu|| = \lim ||\mu_j||$.

Proof: By proposition 4 there is a net $(\mu_j : j \in J) \subseteq A$ such that $\mu_j \xrightarrow{\sigma} \mu$. Since $\mu \to ||\mu||$ is lower semicontinuous in the weak topology, we have that $\lim\inf ||\mu_j|| \geq ||\mu||$. Since $\mu_j \in A$ for each j, we have $||\mu_j|| \leq ||\mu||$. Therefore $\lim\sup ||\mu_j|| \leq ||\mu||$. Thus $\lim ||\mu_j||$ exists and $\lim ||\mu_j|| = ||\mu||$.

2. The so-topology.

The so-topology was defined in Chapter I, §5.3, it is the coarsest topology such that for each f in $L^1(G)$, the mapping $\mu \to \mu * f$ of $M(G)$ into $L^1(G)$ is continuous. As in chapter I, §5, $M_a(G)$ is the subset of $M(G)$ of measures that are absolutely continuous with respect to the Haar measure m on G, and we know that $M_a(G)$ is isometric and isomorphic to $L^1(G)$ via the mapping $\mu \to \frac{d\mu}{dm}$. In this section we show that propositions 1, 3 and 4 remain true if one replaces the $\sigma(M(G), C_o(G))$-topology by the so-topology, and that a weakened version of proposition 2 holds.

Proposition 5. The mapping $x \to \varepsilon_x$ is a continuous mapping of G into $M(G)_{so}$.

Proof: Let f be in $L^1(G)$, we shall show the mapping $x \to \varepsilon_x * f$ is a continuous mapping of G into $L^1(G)$. First note that $\varepsilon_x * f(y) = \int f(z^{-1}y)d\varepsilon_x(z) = f(x^{-1}y) = {}_{x^{-1}}f(y)$, hence $\varepsilon_x * f = {}_{x^{-1}}f$.

The mapping $x \to {}_x f$ is continuous from G into $L^1(G)$ (Chapter I, §7.2), and since G is a topological group $x \to x^{-1}$ is a continuous map. Therefore $x \to {}_{x^{-1}}f = \varepsilon_x * f$ is continuous since it is the composite of continuous maps.

Proposition 6. $M(G)_{so}$ is a topological algebra.

Proof: By chapter I, §5.3 $M(G)_{so}$ is a locally convex space, so it remains to show that multiplication is continuous in each variable separately. For this let $(\mu_j : j \in J)$ be a net in $M(G)$ and suppose that $\mu_j \xrightarrow{so} \mu$. Let λ be in $M(G)$ and f be in $L^1(G)$. Then since $\lambda * f$ is in $L^1(G)$, (Chapter I, §5.3), we have that $\|\mu_j * \lambda * f - \mu * \lambda * f\| \to 0$ so that $\mu_j * \lambda \xrightarrow{so} \mu * \lambda$. Moreover

$$\|\lambda * \mu_j * f - \lambda * \mu * f\| \leq \|\lambda\| \; \|\mu_j * f - \mu * f\| \to 0$$

so that $\lambda * \mu_j \xrightarrow{so} \lambda * \mu$. This proves the proposition.

Lemma 2. Let $f \in C_o(G)$ and $\varepsilon > 0$ be given. Then there is a positive measure λ in $M_a(G)$, $\|\lambda\| = 1$, such that $|\int f(xy)d\lambda(y) - f(x)| \leq \varepsilon$ for all x in G. If in addition V is a given neighborhood of the identity e of G, then we may choose λ such that $\text{Supp}(\lambda) \subseteq V$.

Proof: Since f is in $C_o(G)$, f is left uniformly continuous (Chapter I, §7.2), thus there is a neighborhood U of e in G such that $|f(xy) - f(x)| \leq \varepsilon$ for y in U and any x in G. Let λ be any positive measure in $M_a(G)$ such that $\text{Supp}(\lambda) \subseteq U$ and $\|\lambda\| = 1$. Then

$$\begin{aligned} \left| \int f(xy)d\lambda(y) - f(x) \right| &= \left| \int f(xy)d\lambda(y) - \int f(x)d\lambda(y) \right| \\ &\leq \int_U |f(xy) - f(x)| d\lambda(y) \\ &\leq \varepsilon \|\lambda\| = \varepsilon. \end{aligned}$$

If we are given a neighborhood V of e in G, then we choose λ such that $\mathrm{Supp}(\lambda) \subseteq U \cap V$, $||\lambda|| = 1$, and the second assertion follows in the same manner as above.

Proposition 7. Let S be a Borel subset of G. S is closed if and only if M(S) is so-closed.

Proof: First suppose that M(S) is so-closed and that $a \in \overline{S}$. The mapping $x \rightarrow \varepsilon_x$ is so-continuous by proposition 5, so that $\varepsilon_a \in \mathrm{Cl}_{so}M(S) = M(S)$. By the definition of M(S), we have $\varepsilon_a(G \setminus S) = 0$. Therefore a is in S, and S is closed.

Now suppose S is closed and let $\mu \in \mathrm{Cl}_{so}M(S)$. Then there is a net $(\mu_j : j \in J)$ in M(S) such that $\mu_j \xrightarrow{so} \mu$ and $\mathrm{Supp}(\mu_j) \subseteq S$ since S is closed. Let $f \neq 0$ be in K(G) and suppose that $\mathrm{Supp}(f) \cap S = \emptyset$. Then since $\mathrm{Supp}(f) \subseteq G \setminus S$ which is open, there is a symmetric neighborhood V of e in G such that $(\mathrm{Supp}(f))V \cap S = \emptyset$ (Chapter I, §2.2). Let $\varepsilon > 0$ be given. By lemma 2 there is a λ in $M_a(G)$ with $\mathrm{Supp}(\lambda) \subseteq V$ and $||\lambda|| = 1$, such that

$$\left|\int f(xy)d\lambda(y) - f(x)\right| \leq \varepsilon/2\ ||\mu|| \text{ for all } x \text{ in } G.$$

If y is in V and $f(xy) \neq 0$ then $xy \in \mathrm{Supp}(f)$ so that $x \in \mathrm{Supp}(f)V^{-1} = \mathrm{Supp}(f)V$. Therefore $\int_V f(xy)d\lambda(y) = 0$ whenever x is in S. Then, since $\mathrm{Supp}(\mu_j) \subseteq S$, we have

$$(\mu_j * \lambda)(f) = \iint f(xy)d\lambda(y)d\mu_j(x) = 0 \quad \text{and}$$

$$|\mu*\lambda(f) - \mu(f)| \leq \int\left|\int f(xy)d\lambda(y) - f(x)\right|d|\mu|(x)$$
$$\leq \varepsilon/2.$$

Since $\mu_j \xrightarrow{so} \mu$ and $\lambda \in M_a(G)$ there is a j_o such that $j \geq j_o$ implies

$$||\mu_j*\lambda - \mu*\lambda|| \leq \varepsilon/2||f||.$$

Thus for $j \geq j_o$

$$|\mu(f)| \leq |\mu(f) - \mu*\lambda(f)| + |\mu_j*\lambda(f) - \mu*\lambda(f)| + |\mu_j*\lambda(f)|$$
$$\leq \varepsilon/2 + ||\mu_j*\lambda - \mu*\lambda||\,||f|| + 0$$
$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

We must therefore have $\mu(f) = 0$ for any $f \in K(G)$ with $\mathrm{Supp}(f) \cap S = \emptyset$. Thus lemma 1 applies and we have $\mu \in M(S)$.

The next proposition is due to Greenleaf (9).

<u>Proposition 8</u>. On norm bounded subsets of $M(G)$, $\sigma(M(G), C_o(G)) \subset so$.

<u>Proof</u>: Let $(\mu_j : j \in J)$ be a norm bounded net and suppose $\mu_j \xrightarrow{so} \mu$. Put $M = \max(\sup||\mu_j||, ||\mu||)$, then M is finite. Given $\varepsilon > 0$ and $f \in C_o(G)(f \neq 0)$, then by lemma 2 there is a λ in $M_a(G)$, $||\lambda|| = 1$ and such that

$$|\int f(xy)d\lambda(y) - f(x)| \leq \varepsilon/3M \quad \text{for all } x \text{ in } G.$$

Since $\mu_j*\lambda(f) = \int\int f(xy)d\lambda(y)d\mu_j(x)$ we have

$$|\mu_j(f) - \mu_j*\lambda(f)| = |\int f(x)d\mu_j(x) - \int\int f(xy)d\lambda(y)d\mu_j(x)|$$
$$\leq \int |f(x) - \int f(xy)d\lambda(y)|d|\mu_j|(x)$$
$$\leq ||\mu_j||\ \varepsilon/3M \leq \varepsilon/3$$

Similarly $|\mu(f) - \mu*\lambda(f)| \leq \varepsilon/3$.

Thus
$$|\mu_j(f) - \mu(f)| \leq |\mu_j(f) - \mu_j*\lambda(f)| + |\mu_j*\lambda(f) - \mu*\lambda(f)| + |\mu*\lambda(f) - \mu(f)|$$
$$\leq \varepsilon/3 + ||\mu_j*\lambda - \mu*\lambda||\,||f|| + \varepsilon/3$$

Since $\mu_j \xrightarrow{so} \mu$ there is a j_o such that $j \geq j_o$ implies $||\mu_j*\lambda - \mu*\lambda|| \leq \varepsilon/3||f||$. Then for $j \geq j_o$,

$$|\mu_j(f) - \mu(f)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

This shows that $\mu_j \xrightarrow{\sigma} \mu$.

Corollary 1. The mapping $\mu \to ||\mu||$ of $M(G)$ into the reals is lower semi-continuous in the so-topology.

Proof: Let $a > 0$ then $\{\mu : ||\mu|| \leq a\}$ is clearly norm bounded and $\sigma(M(G),C_o(G))$-closed, hence by proposition 8, so-closed.

Corollary 2. On G^{ε}, so $= \sigma(M(G),C_o(G))$.

Proof: If $x \in G$, then $||\varepsilon_x|| = 1$ and hence G^{ε} is a norm bounded subset of $M(G)$, so that proposition 8 applies and we have $\sigma(M(G),C_o(G)) \subset$ so on G^{ε}. To show that so $\subset \sigma(M(G),C_o(G))$ on G^{ε}, note that the identity mapping $G^{\varepsilon}_{\sigma} \to G^{\varepsilon}_{so}$ is the composite of the mappings $\varepsilon_x \to x$ and $x \to \varepsilon_x$. The first of these is continuous by proposition 1, and the second by proposition 5.

Corollary 3. G is homeomorphic and isomorphic to G^{ε}_{so}.

Proposition 9. Let V be the linear span of G^{ε}. For any $\mu \in M(G)$ we have $\mu \in Cl_{so}\{\lambda \in V : Supp(\lambda) \subseteq Supp(\mu)$ and $||\lambda|| \leq ||\mu||\}$.

Proof: It is clear that for $\mu = 0$ the result is easy. First suppose $\mu \neq 0$ and that $Supp(\mu)$ is compact. Let Z be the set of complex numbers of absolute value less than or equal to 1. Put $A = \{||\mu|| Z\varepsilon_x : x \in Supp(\mu)\}$ and $B = \{\lambda \in V : Supp(\lambda) \subseteq Supp(\mu)$, and $||\lambda|| \leq ||\mu||\}$.

Let convA be the convex hull of A, we first show that convA = B. Let $\lambda \in$ convA, then $\lambda = \sum_1^n \alpha_i ||\mu|| z_i \varepsilon_{x_i}$, $\alpha_i > 0$,

$\Sigma\, \alpha_i = 1$, so that $||\lambda|| \leq ||\mu||\sum_1^n \alpha_i = ||\mu||$. Clearly $\lambda \in V$ and $\mathrm{Supp}(\lambda) \subseteq \mathrm{Supp}(\mu)$, so that $\lambda \in B$. If $\lambda \in B$ and $\lambda = 0$ then $\lambda \in A$. Suppose $\lambda \neq 0$ and $\lambda = \Sigma\, \beta_i \varepsilon_{x_i}$, $x_i \in \mathrm{Supp}(\mu)$, $\beta_i \neq 0$, then $||\lambda|| = \sum_1^n |\beta_i|$ (Chapter 1. §3.5). Put $\alpha_i = |\beta_i|/||\lambda||$, $z_i = ||\lambda||\beta_i/(|\beta_i|\ ||\mu||)$ then for all i, we have $0 < \alpha_i \leq 1$; $\sum_1^n \alpha_i = 1$; $|z_i| \leq 1$, and $\lambda = \sum_1^n \alpha_i ||\mu|| z_i \varepsilon_{x_i}$ so that $\lambda \in \mathrm{conv}A$. Thus we conclude $B = \mathrm{conv}A$.

Observe that $A = CD$, where $D = \{\, \varepsilon_x : x \in \mathrm{Supp}(\mu)\}$ which is $\sigma(M(G),C_o(G))$-compact by proposition 1, and is therefore so-compact by corollary 2 to proposition 8, and $C = \{\, ||\mu||z : z \text{ complex } |z| \leq 1\}$ which is a compact set of complex numbers. Thus A is so-compact. By theorems 2 and 8 of Chapter I, Cl_{so} convA is so-compact and since on norm bounded sets $\sigma \subset so$, Cl_{so} convA is $\sigma(M(G),C_o(G))$-compact and therefore $\sigma(M(G),C_o(G))$-closed. Thus $\mathrm{Cl}_{so}\,\mathrm{conv}A = \mathrm{Cl}_{\sigma}\,\mathrm{conv}A$ so that $\mathrm{Cl}_{so}B = \mathrm{Cl}_{\sigma}B$ and now the result follows from proposition 4 if $\mathrm{Supp}(\mu)$ is compact.

If $\mathrm{Supp}(\mu)$ is not compact then let $0 \neq f \in L^1(G)$ and $\varepsilon > 0$ be given. Since each $\mu \in M(G)$ is inner regular, there is a compact set K such that $|\mu|(G\setminus K) \leq \varepsilon/2||f||$. Let μ_1 be the measure such that $\mu = \mu_1$ on K and $|\mu_1|(G\setminus K) = 0$. Then $||\mu - \mu_1|| \leq \varepsilon/2||f||$. By the above there is a $\mu_2 \in \{\lambda \in V : \mathrm{Supp}(\lambda) \subseteq \mathrm{Supp}(\mu_1),\ ||\lambda|| \leq ||\mu_1||\}$ $\subseteq \{\lambda \in V : \mathrm{Supp}(\lambda) \subseteq \mathrm{Supp}(\mu),\ ||\lambda|| \leq ||\mu||\}$ such that $||\mu_1 * f - \mu_2 * f|| \leq \varepsilon/2$. Thus

$$||\mu * f - \mu_2 * f|| \leq ||\mu * f - \mu_1 * f|| + ||\mu_1 * f - \mu_2 * f|| \leq \varepsilon$$

and this proves the proposition.

<u>Corollary 1</u>. G^{ε} is total in $M(G)_{so}$.

<u>Corollary 2.</u> Let S be a closed subset of G. Then $\{\lambda \in V : \mathrm{Supp}(\lambda) \subseteq S\}$ is dense in $M(S)_{so}$.

<u>Proof:</u> Since S is closed we have $M(S) = \{\lambda : \mathrm{Supp}(\lambda) \subseteq S\}$. Let $\mu \in M(S)$, then by proposition 9, $\mu \in Cl_{so}\{\lambda \in V : \mathrm{Supp}(\lambda) \subseteq \mathrm{Supp}(\mu), ||\lambda|| \leq ||\mu||\} \subseteq Cl_{so}\{\lambda \in V : \mathrm{Supp}(\lambda) \subseteq S\}$.

<u>Definition.</u> An approximate unit in a topological algebra A is a net $(x_j : j \in J)$ such that for each $y \in A$, $y = \lim yx_j = \lim x_j y$. It is known that $M_a(G)$ has an approximate unit ((12) Chapter V, 20.27).

<u>Proposition 10.</u> Let $\mu \in M(G)$, then $\mu \in Cl_{so}\{\lambda \in M_a(G) : ||\lambda|| \leq ||\mu||\}$.

<u>Proof:</u> Let $(e_j : j \in J)$ be an approximate unit of norm 1 in $M_a(G)$, then for any $\mu \in M(G)$, $\mu * e_j \xrightarrow{so} \mu$; $\mu * e_j \in M_a(G)$ and $||\mu * e_j|| \leq ||\mu||$.

<u>Proposition 11.</u> Let F and G be locally compact groups and let T be a norm continuous isomorphism of M(F) onto M(G). Then T is continuous on norm bounded sets as a mapping of $M(F)_{so}$ onto $M(G)_\sigma$, where $\sigma = \sigma(M(G), C_o(G))$.

<u>Proof:</u> Let $(\mu_j : j \in J)$ be a norm bounded net in M(F) and suppose $\mu_j \xrightarrow{so} u$. Let $M = \sup\{||T\mu_j||, ||T\mu||\}$, then M is finite since T is bounded. Since K(G) is norm dense in $C_o(G)$ it suffices to show that $T\mu_j(f) \rightarrow T\mu(f)$ for nonzero $f \in K(G)$, (for $f = 0$ it is obvious).

For given $f \in K(G)$ and $\varepsilon > 0$ by the argument used in Lemma 2 and proposition 8, there is a $g \in L^1(G) \cap L^2(G)$, $g \neq 0$ such that

$$\left.\begin{array}{l} |(T\mu_j * g)(f) - T\mu_j(f)| \leq \varepsilon/3 \text{ and} \\ |(T\mu * g)(f) - T\mu(f)| \leq \varepsilon/3 \end{array}\right\} \cdots\cdots\cdots\cdots (1)$$

Since $g \in L^2(G)$, we have that $T\mu_j * g \in L^2(G)$ for each j (Chapter 1, §5.4) and that $T\mu * g \in L^2(G)$. The Schwarz inequality yields that

$$|(T\mu_j*g)(f) - (T\mu*g)(f)| \leq ||T\mu_j*g - T\mu*g||_2 ||f||_2$$

We claim that there is a $\lambda \in M_a(F)$ such that

$$||g - T\lambda*g||_2 \leq \varepsilon/9M||f||_2 \ldots\ldots\ldots\ldots(2)$$

To prove this let E be the closure of the subspace $\{T\lambda*g : \lambda \in M_a(F)\}$ of $L^2(G)$ and let $E^\perp$ be the orthogonal complement of E. Then $g = g_1 + g_2$ where $g_1 \in E$ and $g_2 \in E^\perp$. Then we have

$$0 = (g_2, T\lambda*g) = (g_2, T\lambda*g_1) + (g_2, T\lambda*g_2) \text{ for all } \lambda \in M_a(F)$$

(where (,) is the inner product in $L^2(G)$). Since $g_1 \in E$, we have $T\lambda*g_1 \in E$ for all $\lambda \in M_a(F)$ and therefore $(g_2, T\lambda*g_2) = 0$ for all $\lambda \in M_a(F)$. Since $M_a(F)$ is an ideal in $M(F)$ we have $(g_2, T(T^{-1}(T\lambda)^{*}*\lambda)*g_2) = 0$ for all $\lambda \in M_a(F)$. Since $(\nu*g_2, \nu*g_2) = (g_2, \nu^{*}*\nu*g_2)$ for any $\nu \in M(G)$ ((12) Chapter V, 20.20), we have

$$||T\lambda*g_2||_2^2 = (T\lambda*g_2, T\lambda*g_2) = (g_2, (T\lambda)^{*}*T\lambda*g_2) = (g_2, T(T^{-1}(T\lambda)^{*}*\lambda)*g_2) = 0$$

for all $\lambda \in M_a(F)$. Thus $T\lambda*g_2 = 0$ for all $\lambda \in M_a(F)$. To show that $g_2 = 0$, let $\varepsilon' > 0$ be given. Since $K(G)$ is dense in $L^2(G)$ there is an $h \in K(G)$ such that $||g_2 - h||_2 \leq \varepsilon'$. Then $||T\lambda*h|| = ||T\lambda*h - T\lambda*g_2|| \leq ||T\lambda||\varepsilon'$. Therefore $T\lambda*h = 0$ for all $\lambda \in M_a(F)$. Let m be the Haar measure on G, then $hm \in M_a(G)$, and $T\lambda*hm = 0$ for all $\lambda \in M_a(F)$ (Chapter I, §5.3). Therefore $\lambda*T^{-1}hm = 0$ for all $\lambda \in M_a(G)$, this means that $h = 0$ and therefore $g_2 = 0$. Thus $g = g_1 \in E$, so that (2) holds.

Since $\mu_j \xrightarrow{so} \mu$ there is a j_o such that $j \geq j_o$ implies

$$||\mu_j*\lambda - \mu*\lambda|| \leq \varepsilon/9||T|| \; ||g||_2||f||_2.$$

Thus for $j \geq j_o$, we have

$$||T\mu_j*g - T\mu*g||_2 \leq ||T\mu_j*g - T\mu_j*T\lambda*g||_2 + ||T\mu_j*T\lambda*g - T\mu*T\lambda*g||_2 + ||T\mu*T\lambda*g - T\mu*g||_2$$

$$\leq ||T\mu_j|| \; ||g - T\lambda * g||_2 + ||T\mu_j * T\lambda - T\mu * T\lambda|| \; ||g||_2$$
$$+ ||T\mu|| \; ||T\lambda * g - g||_2$$
$$\leq \varepsilon/3 ||f||_2 \quad \ldots\ldots (3)$$

We now have using (1) and (3), for $j \geq j_o$

$$|T\mu_j(f) - T\mu(f)| \leq |T\mu_j(f) - (T\mu_j * g)(f)| + |T\mu_j * g(f) - T\mu * g(f)|$$
$$+ |T\mu * g(f) - T\mu(f)|$$
$$\leq \varepsilon/3 + ||T\mu_j * g - T\mu * g||_2 ||f||_2 + \varepsilon/3$$
$$\leq \varepsilon.$$

<u>Remark</u>. The following example shows that the requirement that T be an isomorphism is essential. Let F be any nondiscrete locally compact group and let G be the group having only one element. For $\mu \in M(F)$, let $\mu = \mu_d + \mu_c$ where μ_d is a purely discontinuous measure and μ_c is a continuous measure (see Chapter 1, § 3.5). Then for any $x \in F$, $\mu(\{x\}) = \mu_d(\{x\})$ so that

$$\Sigma_{x \in F} |\mu(\{x\})| = \Sigma_{x \in F} |\mu_d(\{x\})| = ||\mu_d|| < \infty \quad \text{(Chapter 1, §3.5)}$$

Thus we may define a mapping $T : M(F) \to M(G)$ by

$$T\mu = [\Sigma_{x \in F} \mu(\{x\})] \varepsilon_e \qquad \mu \in M(F), \; e \in G$$

and we have $||T\mu|| = ||\mu_d|| \leq ||\mu||$. We now show that T is a homomorphism. Let $\mu, \lambda \in M(F)$ and let a, b be scalars. Then

$$T(a\lambda + b\mu) = [\Sigma_{x \in F} (a\lambda + b\mu)(\{x\})] \varepsilon_e$$
$$= \Sigma_{x \in F} a\lambda(\{x\}) \varepsilon_e + \Sigma_{x \in F} b\mu(\{x\}) \varepsilon_e$$
$$= aT\lambda + bT\mu$$

Now let $\lambda = \lambda_d + \lambda_c$ where λ_c is a continuous measure and λ_d is purely discontinuous. Then $\mu * \lambda_c$ is a continuous measure since

$$\mu*\lambda_c(\{x\}) = \int \lambda_c(\{y^{-1}x\})d\mu(y) = 0, \text{ for each } x \in F.$$

Thus

$$\begin{aligned}\mu*\lambda(\{x\}) &= (\mu*\lambda_d)(\{x\}) = \int\mu(\{xy^{-1}\})d\lambda_d(y)\\ &= \Sigma_{y\in F}\mu(\{xy^{-1}\})\lambda_d(\{y\})\\ &= \Sigma_{y\in F}\mu(\{xy^{-1}\})\lambda(\{y\})\end{aligned}$$

Thus

$$\begin{aligned}T(\mu*\lambda) &= (\Sigma_{x\in F}\ \Sigma_{y\in F}\mu(\{xy^{-1}\})\lambda(\{y\}))\varepsilon_e\\ &= (\Sigma_{y\in F}\ (\Sigma_{x\in F}\mu(\{xy^{-1}\})\lambda(\{y\})))\varepsilon_e\\ &= \Sigma_{y\in F}\ (\Sigma_{x\in F}\mu(\{x\}))\lambda(\{y\})\varepsilon_e\\ &= \Sigma_{x\in F}\mu(\{x\})(\Sigma_{y\in F}\lambda(\{y\}))\varepsilon_e\\ &= (\Sigma_{x\in F}\mu(\{x\}))\varepsilon_e*(\Sigma_{y\in F}\lambda(\{y\}))\varepsilon_e\\ &= T\mu*T\lambda\end{aligned}$$

Thus we have shown that $T : M(F) \to M(G)$ in a continuous homomorphism.

From the definition of T, and the requirement that F is not a discrete group we have $T\mu = 0$ if $\mu \in M_a(F)$. Now let $\mu \in M(F)$ be such that $\Sigma_{x\in F}\mu(\{x\}) \neq 0$ and let $(e_j : j \in J)$ be an approximate unit in $M_a(F)$. Then $\mu*e_j \xrightarrow{so} \mu$. $||\mu*e_j|| \leq ||\mu||$, and $T(\mu*e_j) = 0$ for all $j \in J$ since $\mu*e_j \in M_a(F)$, but $T\mu \neq 0$.

3. Subalgebras of $M_a(G)$

Definitions. Let S be a Borel subset of G. Then we define $L(S) = M_a(G)\cap M(S)$. If m is the Haar measure on G, then D(S) is defined to be the set of all x in G such that each m-measurable neighborhood of x meets S in a set of positive Haar measure.

Observe that L(S) is a norm closed subspace of $M_a(G)$.

Remark. The operator D defined above was first used by Simon (1). The next proposition is a summary of the properties of D(S), some of which were first demonstrated by Simon in (20).

Proposition 12. Let S be a Borel subset of a locally compact group G and let m be the Haar measure in G. Then

(i) $D(S)$ is closed and $D(S) \subseteq \overline{S}$.

(ii) $D(S) = \emptyset \Leftrightarrow S$ is locally m-null $\Leftrightarrow S \cap D(S) = \emptyset$

(iii) $S \subseteq T \Rightarrow D(S) \subseteq D(T)$

(iv) $S \setminus D(S)$ is locally null.

(v) $L(S) \subseteq L(D(S))$

Proof: Let $x \in \overline{D(S)}$, and let U be an open neighborhood of x, then there is a $y \in D(S) \cap U$ and U is an open neighborhood of y, so that $m(S \cap U) > 0$. Thus $x \in D(S)$. It is clear that $D(S) \subseteq \overline{S}$.

(ii) $D(S) = \emptyset \Leftrightarrow$ each $x \in G$ has a measurable neighborhood U such that $m(S \cap U) = 0 \Leftrightarrow$ S is locally null. (Chapter I, §3.3). Since $D(S) = \emptyset \Rightarrow S \cap D(S) = \emptyset$, to complete the proof of (ii) it suffices to show that $S \cap D(S) = \emptyset \Rightarrow$ S is locally null. Let K be a compact subset of G, and for each x in S let U_x be a neighborhood of x such that $m(U_x \cap S) = 0$. Then $\{ U_x : x \in S \}$ is an open covering of K so that there are $x_i \in S$ $(1 \leq i \leq n)$ such that $K \cap S \subseteq \bigcup_{i=1}^{n} U_{x_i} \cap S$. Thus $m(K \cap S) = 0$ so that S is locally null.

The proof of (iii) is very simple.

(iv) $S \setminus D(S) \subseteq S$ so that $D(S \setminus D(S)) \subseteq D(S)$ by (iii). Thus it is clear that $(S \setminus D(S)) \cap D(S \quad D(S)) = \emptyset$ so by (ii) $S \setminus D(S)$ is locally null.

(v) Note that $G \setminus D(S) \subseteq (G \setminus S) \cup (S \setminus D(S))$ thus for $\lambda \in M(G)$

we have $|\lambda|(G \setminus D(S)) \leq |\lambda|(G \setminus S) + |\lambda|(S \setminus D(S))$. If $\lambda \notin L(S)$ then $|\lambda|(G \setminus S) = 0$. By (iv) $S \setminus D(S)$ is locally m-null, so that $\lambda \notin L(S) \Rightarrow |\lambda|(G \setminus D(S)) = 0$ and then $\lambda \notin L(D(S))$.

<u>Corollary 1</u>. Let K be any compact subset, then $m(K \cap S) = m(K \cap S \cap D(S))$.

<u>Proof</u>: Clearly $m(K \cap S \cap D(S)) \leq m(K \cap S)$. By (iv) of the above proposition $S \setminus D(S)$ is locally null, thus

$m(K \cap S \setminus K \cap S \cap D(S)) = m(K \cap (S \setminus D(S))) = 0$ so that $m(K \cap S) = m(K \cap S \cap D(S))$.

<u>Corollary 2</u>. $D(D(S)) = D(S)$

<u>Proof</u>: Since D(S) is closed, $D(D(S)) \subseteq D(S)$ by (i) of the proposition. Now suppose $x \notin D(D(S))$ then there is a compact neighborhood V of x such that $m(V \cap D(S)) = 0$. By Corollary 1, $m(V \cap S) = m(V \cap S \cap D(S)) \leq m(V \cap D(S)) = 0$, so that $x \notin D(S)$.

<u>Proposition 13</u>. Let S and T be Borel subsets of G and suppose that $L(S) = L(T)$. Then $S \setminus T$ and $T \setminus S$ are locally m-null.

<u>Proof</u>: Let K be a compact subset of G, put $F = (S \setminus T) \cap K$ and let χ_F be the characteristic function of F. We shall show that $m(F) = 0$. If $m(F) > 0$, then $\chi_F m$ is a nonzero measure which is in L(S) but not in L(T) and this is a contradiction. Thus for any compact subset K of G, $m(K \cap (S \setminus T)) = 0$, which means that $S \setminus T$ is locally m-null. Similarly $T \setminus S$ is locally m-null.

Corollary 1. Let S be a Borel subset of G. $L(S) = \{0\}$ if and only if S is locally null.

Proof: Suppose $L(S) = \{0\}$. Then $L(\emptyset) = \{0\} = L(S)$. By proposition 13, S must be locally null. Now suppose that S is locally null, then $|\lambda|(S) = 0$ for any $\lambda \in M_a(G)$, so that $L(S) = \{0\}$.

Corollary 2. Let S be a Borel subset of G. $L(S) = \{0\}$ if and only if $D(S) = \emptyset$.

Proof: By proposition 12(ii) S is locally null if and only if $D(S) = \emptyset$. Thus corollary 2 follows from corollary 1.

Theorem 1. $L(S)$ and $L(D(S))$ are dense in $M(D(S))_\sigma$.

Proof: Since $D(S)$ is closed, $M(D(S))$ is weakly closed by proposition 3. Moreover $L(S) \subseteq L(D(S))$ by proposition 12(v) so that it suffices to show that $M(D(S)) \subseteq Cl_\sigma L(S)$. If $D(S) = \emptyset$, then $L(S)$ and $M(D(S))$ both consist only of the zero measure so that we may assume $D(S) \neq \emptyset$. To prove the theorem it suffices in virtue of proposition 4 to show that for $x \in D(S)$, $f \in C_o(G)$ and $\varepsilon > 0$ there is a $\lambda \in L(S)$ such that $|\lambda(f) - \varepsilon_x(f)| \leq \varepsilon$. Let $U = \{y : |f(y) - f(x)| \leq \varepsilon\}$, then U is a closed neighborhood of x so that $m(U \cap S) > 0$ where m is the Haar measure in G. Let $\chi_{U \cap S}$ be the characteristic function of $U \cap S$, and define λ by

$$\lambda(g) = (1/m(U \cap S)) \int_G g \chi_{U \cap S} dm, \quad \text{for } g \in C_o(G).$$

Then by putting $g = \chi_{G \setminus S}$ in the definition of λ, we see that λ is in $L(S)$. Now

$$|\lambda(f) - \varepsilon_x(f)| = (1/m(U \cap S)) \left| \int_G f \chi_{U \cap S} dm - f(x) m(U \cap S) \right|$$

$$\leq 1/m(U \cap S) \int_{U \cap S} |f(y) - f(x)| dm(y) \leq \varepsilon.$$

Corollary 1. $M_a(G)$ is dense in $M(G)_\sigma$.

Corollary 2. (Simon) Let S be a Borel set in G. If L(S) is an algebra then D(S)is a semigroup (possibly empty if S is locally null). If $D(S) \neq \emptyset$ and L(S) is a *-algebra then D(S) is a subgroup.

Proof: If L(S) is an algebra then M(D(S)) is an algebra by theorem I and corollary 2 to proposition 2. Now since the mapping $\varepsilon_x \to x$ of G^ε onto G is an isomorphism by proposition 1 and since $\{\varepsilon_x : x \in D(S)\}$ is a semigroup in M(D(S)) under convolution, D(S) is a semigroup. If L(S) is a *-algebra, and if $D(S) \neq \emptyset$, then M(D(S)) is a *-algebra (Corollary 2 to proposition 2 and theorem 1) and hence D(S) is a subgroup since $\varepsilon_x^* = \varepsilon_{x^{-1}}$.

Remark. In (20) Simon showed that if there is a semigroup T such that L(S) = L(T) then L(S) is an algebra. As Simon has noted the above result yields a partial converse of this statement: If for a closed subset S of G, L(S) is an algebra then there is a semigroup T such that L(S) = L(T). (Proof: Take T = D(S) which is a semigroup by Corollary 2. By proposition 12(v), $L(S) \subseteq L(D(S))$. Since S is closed, $D(S) \subseteq S$ (proposition 12(i), consequently L(D(S)) = L(S).).

Simon raised the following question: If S is a Borel subset of G, such that L(S) is an algebra, then is there a semigroup T such that L(S) = L(T)? We shall return to this question later and provide an (affirmative) answer in a number of special cases.

Lemma 3. Let S be a Borel subset of G, and let $x \in D(S)$.

For any $f \in L^p(G)$ $(1 \leq p < \infty)$ and any $\varepsilon > 0$ there is a neighborhood U of x such that

$$||\mu*f - \varepsilon_x*f||_p \leq \varepsilon$$

for all positive measures μ in M(G) such that $||\mu|| = 1$ and $\mu(G \setminus (U \cap S)) = 0$. There is also a neighborhood V of x such that

$$||f*\mu - f*\varepsilon_x||_p \leq \varepsilon$$

for all positive measures μ in M(G) such that $||\mu|| = 1$ and $\mu(G \setminus (V \cap S)) = 0$.

<u>Proof</u>: By Lemma 2, Chapter I, §5.4, there is a neighborhood U_1 of e in G such that $||\mu*f - f|| \leq \varepsilon$ for every positive measure μ in M(G) such that $||\mu|| = 1$ and $\mu(G \setminus U_1) = 0$. Let $U = xU_1$, then U is a neighborhood of x. Let $\mu \in M(G)$ be such that $||\mu|| = 1$ and $\mu(G \setminus (S \cap U)) = 0$. Then

$$\begin{aligned} \varepsilon_{x^{-1}}*\mu(G \setminus U_1) &= \mu(x(G \setminus U_1)) \\ &= \mu(G \setminus U) \\ &\leq \mu(G \setminus (S \cap U)) = 0. \end{aligned}$$

Moreover $||\varepsilon_{x^{-1}}*\mu|| = ||\mu|| = 1$. Therefore $||\varepsilon_{x^{-1}}*\mu*f - f|| \leq \varepsilon$. Consequently $||\mu*f - \varepsilon_x*f|| \leq \varepsilon$.

The second assertion follows similarly from the second assertion of Lemma 2, Chapter I, §5.4.

<u>Corollary 1</u>. Let S be a Borel subset of G and let $x \in D(S)$. Given $\lambda \in M_a(G)$ and $\varepsilon > 0$ there is a $\mu \in L(S)$ such that $||\mu*\lambda - \varepsilon_x*\lambda|| \leq \varepsilon$. There is also a ν in L(S) such that $||\lambda*\nu - \lambda*\varepsilon_x|| \leq \varepsilon$.

<u>Proof</u>: Let f be the Radon-Nikodym derivative of λ with respect to the Haar measure m on G. Then f is in $L^1(G)$, and by the lemma there

is a neighborhood U of x such that

$$||\nu*f - \varepsilon_x*f||_1 \leq \varepsilon$$

for all positive measures ν in M(G) such that $||\nu|| = 1$ and $\nu(G \setminus (U \cap S)) = 0$. Let K be a compact neighborhood of x such that $K \subseteq U$, then $\mu = (1/m(K \cap S))\chi_{K \cap S}m$ is in L(S) and satisfies the requirements of the lemma. Therefore $||\mu*f - \varepsilon_x*f||_1 \leq \varepsilon$. Now since $L^1(G)$ is isometric and isomorphic to $M_a(G)$ we have $||\mu*\lambda - \varepsilon_x*\lambda|| \leq \varepsilon$.

The second assertion follows similarly.

Corollary 2. Let S be a Borel subset of G and let $x \in D(S)$. Let $x \in D(S)$, $f \in L^2(G)$ and $\varepsilon > 0$ be given. Then there is an $h \in L^2(G)$ with $hm \in L(S)$ such that $||f*h - f*\varepsilon_x||_2 \leq \varepsilon$.

Proof: The proof is similar to the proof of Corollary 1.

Theorem 2. L(S) and L(D(S)) are dense in $M(D(S))_{so}$.

Proof: Since D(S) is closed, M(D(S)) is so-closed by proposition 7. By proposition 12(v), $L(S) \subseteq L(D(S))$ so that it suffices to show that $M(D(S)) \subseteq Cl_{so}L(S)$. If $D(S) = \emptyset$ then M(D(S)) and L(S) both consist of only the zero measure (proposition 13, corollary 2) so that we may assume $D(S) \neq \emptyset$. For any $x \in D(S)$, $\lambda \in M_a(G)$ and $\varepsilon > 0$ there is by corollary 1 to lemma 3, a μ in L(S) such that $||\varepsilon_x*\lambda - \mu*\lambda|| \leq \varepsilon$. The theorem now follows from corollary 2 to proposition 9.

Corollary. Let S be a Borel subset of G. If L(S) is a subalgebra of $M_a(G)$, then for any $\mu \in M(D(S))$ and any $\lambda \in L(S)$ we have

$\mu*\lambda \in L(S)$.

Proof: By the above theorem there is a net $(\mu_j : j \in J)$ in $L(S)$ such that $\mu_j \xrightarrow{so} \mu$, thus $||\mu_j*\lambda - \mu*\lambda|| \to 0$. Since $L(S)$ is an algebra, $\mu_j*\lambda \in L(S)$ and since $L(S)$ is norm closed, we have $\mu*\lambda \in L(S)$.

Remark. If $L(S)$ is a subalgebra of $M_a(G)$, then it is a subalgebra of $M(G)$. Moreover whenever $L(S)$ is a subalgebra of $M(G)$, $M(D(S))$ is also a subalgebra of $M(G)$ (see the proof of corollary 2 of theorem 1). The corollary says that if $L(S)$ is a subalgebra of $M(G)$ then it is a left ideal in $M(D(S))$.

Lemma 4. Let S be a Borel subset of G. If $L(S)$ is an algebra and if $e \in D(S)$, then $L(D(S)) = L(S)$.

Proof: By proposition 12(v) we have $L(S) \subseteq L(D(S))$. To show the reverse inclusion, let $\lambda \in L(D(S))$ and $\varepsilon > 0$ be given. By corollary 1 to lemma 3, there is a $\mu \in L(S)$ such that $||\lambda*\mu - \lambda|| \leq \varepsilon$. By corollary 1 to theorem 2, $\lambda*\mu \in L(S)$. Since $L(S)$ is norm closed, $\lambda \in L(S)$. Thus $L(S) = L(D(S))$.

We now proceed to generalize a number of results of Simon (21). In the remainder of this section we identify $M_a(G)$ with $L^1(G)$. Thus for a Borel set S, we identify $L(S)$ with a subspace of $L^1(G)$. From the definition of $L(S)$ it is clear that a given $f \in L^1(G)$ is in $L(S)$ if and only if $\int_{G\setminus S}|f|dm = 0$.

Theorem 3. Let S be a nonempty Borel subset of G. If $L(S)$ is a $*$-algebra then there is a closed subgroup T such that $L(S) = L(T)$. If $D(S) \neq \emptyset$ then we may take $T = D(S)$.

Proof: If $D(S) = \emptyset$, then $L(S) = \{0\}$ (proposition 13, corollary 2), and since S is nonempty G cannot be discrete, so if we take $T = \{e\}$, then $L(T) = \{0\} = L(S)$. If $D(S) \neq \emptyset$, and if $L(S)$ is a $*$-algebra then $D(S)$ is a subgroup by corollary 2 to theorem 1. Thus $e \in D(S)$ and by lemma 4, $L(S) = L(D(S))$. Taking $T = D(S)$ (which is a subgroup) we have that T is closed (proposition 12(i)) and $L(S) = L(T)$.

Lemma 5. Let $S \subseteq G$ be a Borel semigroup and $T \subseteq G$ a Borel subgroup. If $L(S) = L(T) \neq \{0\}$ then $S = T$.

Proof: First suppose that $T \not\subseteq S$. Then $T \setminus S \cap T \neq \emptyset$. Let $y \in T \setminus S \cap T$ and let $x \in S \cap T$. Since T is a subgroup and S is a semigroup, $yx^{-1} \in T$, $yx^{-1} \notin S \cap T$ so that $yx^{-1} \in T \setminus S \cap T$. Therefore $y(S \cap T)^{-1} \subseteq T \setminus S \cap T$. Since $L(S) \neq \{0\}$, there is a compact set K in S with $m(K) > 0$ and since $L(S) = L(T)$, the measure $\chi_K m$ is in $L(T)$. Thus $\chi_K m(G \setminus T) = 0$ and since $\chi_K m \neq 0$ we must have $m(K \cap T) \neq 0$. Consequently $S \cap T$ contains a set of nonzero Haar measure so that $(S \cap T)^{-1}$ contains a set of nonzero measure. By the left invariance of the Haar measure, $T \setminus S \cap T$ contains a set of nonzero measure, and this contradicts the fact that $T \setminus S$ is locally null (proposition 13) because $T \setminus S = T \setminus S \cap T$. Therefore $T \subseteq S$. To show the reverse inclusion suppose that $S \setminus T \neq \emptyset$. Let $y \in S \setminus T$ and $x \in T \subseteq S$. Then $yx^{-1} \in S \setminus T$ so that $yT \subseteq S \setminus T$. Thus $S \setminus T$ is not locally null which is a contradiction (proposition 13). Therefore $S \subseteq T$ and we have shown $S = T$.

Definitions. Let S be a Borel subset of G, if $L(S)$ is a subalgebra of $L^1(G)$ we shall call $L(S)$ a vanishing algebra. A vanishing algebra $L(S)$ is called a maximal vanishing algebra if it is proper and if for every vanishing algebra $L(T)$ where T is a Borel set, $L(S) \subseteq L(T) \Rightarrow L(S) = L(T)$ or $L(T) = L^1(G)$.

Theorem 4. Let S be a Borel subset of G. If $L(S)$ is a non-zero maximal vanishing algebra then there is a maximal proper closed semi-group $T \subset G$ such that $L(S) = L(T)$.

Proof: Since $L(S) \subseteq L(D(S))$ (proposition 12(v)), by the maximality of $L(S)$ we must have that $L(D(S)) = L^1(G)$ or $L(D(S)) = L(S)$. By corollary 2 to theorem 1, $D(S)$ is a semigroup, thus if $L(D(S)) = L^1(G)$ then $D(S) = G$ by lemma 5. Hence by lemma 4 $L(S) = L(D(S)) = L^1(G)$ which contradicts the maximality of $L(S)$. Therefore we have $L(S) = L(D(S))$. Let T be a maximal proper closed semigroup containing $D(S)$, then T is a Borel set and $L(D(S)) \subseteq L(T)$ by proposition 12(iii), so that $L(S) \subseteq L(T)$. Since $L(T)$ is a subalgebra (because S is a semigroup), either $L(T) = L(S)$ or $L(T) = L^1(G)$ by the maximality of $L(S)$. If $L(T) = L^1(G)$ then $T = G$ by lemma 5 so that we must have $L(S) = L(T)$.

Theorem 5. Let G be a compact group, and let S be a Borel subset of G. If $L(S)$ is a nonzero subalgebra of $L^1(G)$ then $D(S)$ is an open and closed subgroup and $L(S) = L(D(S))$.

Proof: If $L(S)$ is a nonzero subalgebra then $D(S) \neq \emptyset$ and

D(S) is therefore a closed and hence compact semigroup (corollary 2 to theorem 1). Since $D(S) \subseteq G$, D(S) is a compact semigroup satisfying the left and right cancellation laws, and therefore is a group ((12)Chapter II, 9.16). Lemma 4 applies and we have $L(S) = L(D(S))$. Since $L(S) \neq \{0\}$, D(S) must contain a set of positive measure and is therefore open (Chapter 1, §5.4).

<u>Corollary 1</u>. If G is compact then every vanishing algebra is a *-algebra.

<u>Corollary 2</u>. If G is compact then G is connected if and only if there are no proper nonzero vanishing algebras in $L^1(G)$.

<u>Theorem 6.</u> (Beck, Corson and Simon (1)) Let S be a Borel subsemigroup of a compact group G. If S contains a set of nonzero finite measure then S is an open and closed subgroup of G.

<u>Proof</u>: Since S is a semigroup, L(S) is an algebra and if S contains a set of nonzero measure then $L(S) \neq \{0\}$. Thus by theorem 5, D(S) is an open and closed subgroup of G and $L(S) = L(D(S))$. By lemma 5, $D(S) = S$.

<u>Remarks</u>. Simon (21) proved theorem 3 for S a closed subset of G. He obtained a weaker version of theorem 4 in the same paper. (cf. 3.19, and 3120 of (21)). Theorem 5 and its corollaries were also obtained by him with the restriction that G be abelian as well as compact. Theorem 4 can be used to strengthen another result

of Simon (22).

Theorem 7. (Simon). Let S be a measurable semigroup of a locally compact abelian group G. If L(S) is a maximal proper closed subalgebra of $L^1(G)$, then G is (isomorphic and homeomorphic to) either a discrete subgroup of the reals or the real line itself.

For a proof see (19) theorems 9.2.5 and 8.1.6.

Theorem 8. Let S be a Borel subset of a locally compact abelian group G. If L(S) is a maximal proper closed subalgebra of $L^1(G)$, then G is (isomorphic and homeomorphic to) either a discrete subgroup of the reals or the real line itself.

Proof: Since $L^1(G)$ always contains a nonzero proper subalgebra, the maximality of L(S) implies that $L(S) \neq 0$. By theorem 4, there is a closed (and therefore measurable) semigroup T such that $L(S) = L(T)$. Theorem 8 now follows from theorem 7 applied to T.

CHAPTER III

NORM DECREASING ISOMORPHISMS OF MEASURE ALGEBRAS

The main result (theorem 2) of this chapter concerns isomorphisms of measure algebras. The proof of this theorem requires a characterization of the Dirac measures in M(G). This characterization is given in Theorem 1. It should be noted that theorem 1 could be easily derived from a theorem of Wendel (24). It is only for the sake of completeness that we have included an independent proof.

Lemma 1. Let X be a locally compact space and let μ be a measure in M(X). Then μ is a Dirac measure if and only if $||\mu|| = 1$ and $|\mu(f)| = \mu(|f|)$ for all f in $C_o(X)$.

Proof: Clearly any Dirac measure satisfies the stated properties. Now suppose μ is such that $||\mu|| = 1$ and $|\mu(f)| = \mu(|f|)$. Assume there are $x_o, y_o \in \mathrm{Supp}(\mu)$, $x_o \neq y_o$. Since X is locally compact there is a real valued $f \in C_o(X)$ such that $f(x_o) < 0$, $f(y_o) > 0$. Clearly $|f| + f \geq 0$, $|f| - f \geq 0$; and $|f| + f \neq 0$, $|f| - f \neq 0$ on $\mathrm{Supp}(\mu)$. Therefore μ positive implies $\mu(|f|) > - \mu(f)$, and $\mu(|f|) > \mu(f)$ so that $\mu(|f|) > |\mu(f)|$ which is a contradiction. Thus $\mathrm{Supp}(\mu)$ consists of a single point x. Since μ is positive and since $||\mu|| = 1$ we have $\mu = \varepsilon_x$.

Proposition 1. Let X be a locally compact space and let μ be in $M(X)$. Then $\mu = \gamma\varepsilon_x$ for some x in X and some complex number γ, $|\gamma| = 1$ if and only if $||\mu|| = 1$ and $|\mu(f)| = |\mu|(|f|)$ for all f in $C_o(X)$.

Proof: "If part": Let $f \in C_o(X)$ be such that $f \geq 0$, and $\mu(f) \neq 0$. Put $\gamma_f = |\mu|(f)/\mu(f)$, clearly $|\gamma_f| = 1$. Let $g \in C_o(X)$, $g \geq 0$ be such that $|\mu|(g) = |\mu|(f)$, then

$$\gamma_{f+g}\mu(f + g) = 2|\mu|(f) = 2\gamma_f\mu(f),$$

and $\gamma_f\mu(f) = \gamma_g\mu(g)$, so we have that

$$\gamma_{f+g}(1 + \gamma_f/\gamma_g)\mu(f) = 2\gamma_f\mu(f)$$

and hence $|1 + \gamma_f/\gamma_g| = 2$ which means $\gamma_f = \gamma_g$. For any $a > 0$, $\gamma_{ag} = \gamma_g$ so that $\gamma_f = \gamma_g$ for any pair $f,g \in C_o^+(X)$ with $\mu(f) \neq 0$ and $\mu(g) \neq 0$. Thus putting $\gamma = \overline{\gamma_f}$ for some $f \in C_o^+(X)$ with $\mu(f) \neq 0$ we have $\gamma|\mu|(f) = \mu(f)$ for any $f \in C_o^+(X)$. Then by the linearity of $|\mu|$ and μ we have $\gamma|\mu|(f) = \mu(f)$ for any $f \in C_o(X)$. Thus we have shown that there is a γ with $|\gamma| = 1$ such that $\gamma|\mu| = \mu$. Clearly $|\mu|$ satisfies the conditions of lemma 1. Hence $|\mu|$ is a Dirac measure.

To prove the "only if" part suppose that $\mu \in M(X)$ and there is a γ, $|\gamma| = 1$ and an x in X such that $\mu = \gamma\varepsilon_x$. Then $||\mu|| = |\gamma|\,||\varepsilon_x|| = 1$ and for any $f \in C_o(X)$, $|\mu(f)| = |\gamma|\,|\varepsilon_x(f)| = |f(x)| = \varepsilon_x(|f|) = |\varepsilon_x|(|f|) = |\gamma\varepsilon_x|(|f|) = |\mu|(|f|)$.

Theorem 1. Let G be a locally compact group and let $\mu \in M(G)$. Then $\mu = \gamma\varepsilon_x$ for some x in G and some complex number γ, $|\gamma| = 1$ if and only if $||\mu*\lambda|| = ||\lambda||$ for all λ in $M(G)$.

Proof: $(\Rightarrow)$ $(\gamma\varepsilon_x * \lambda)(f) = \gamma\int\int f(uv)d\varepsilon_x(u)d\lambda(v) = \gamma\int f(xv)d\lambda(v)$ $\gamma\lambda({}_xf)$. Thus

$$||\gamma\varepsilon_x*\lambda|| = \sup_{||f||\leq 1}|\gamma\varepsilon_x*\lambda(f)| = \sup_{||f||\leq 1}|\lambda({}_xf)| = \sup_{||f||\leq 1}|\lambda(f)| = ||\lambda||.$$

$(\Leftarrow)$. First observe that for any $\mu \in M(G)$, $||\mu*\lambda|| = ||\lambda||$ for all λ in $M(G)$ implies that $||\mu|| = 1$, because $||\mu|| = ||\mu*\varepsilon_e|| = ||\varepsilon_e|| = 1$. Now let $f \in K(G)$ and let m be the Haar measure on G. Then since $fm \in M_a(G)$ we have by hypothesis $||\mu*fm|| = ||fm||$. Moreover $\mu*fm \in M_a(G)$ and $M_a(G)$ is isometric to $L^1(G)$ (Chapter I, §5.3) so that $||\mu*fm|| = ||fm||$ implies $||\mu*f||_1 = ||f||_1$. Clearly

$$||\mu*f||_1 = \int |\mu*f(x)|dm(x)$$
$$= \int \left| \int f(y^{-1}x)d\mu(y)\right|dm(x) \qquad \text{(Chapter I, §5.3),}$$

we have

$$\int \left| \int f(y^{-1}x)d\mu(y)\right|dm(x) = \int | f(x)|dm(x)$$
$$= \int|f(y^{-1}x)|dm(x) \quad \text{(by the left invariance of Haar measure)}$$
$$= \int|f(y^{-1}x)|dm(x)\int d|\mu|(y) \quad \text{(since } ||\mu|| = \int d|\mu|(y) = 1)$$

Now using Fubini's theorem we obtain

$$\int|f(y^{-1}x)|dm(x)\int d|\mu|(y) = \int\int|f(y^{-1}x)|d|\mu|(y)dm(x)$$

so that

$$\int\left|\int f(y^{-1}x)d\mu(y)\right|dm(x) = \int\int|f(y^{-1}x)d|\mu|(y)dm(x) \quad \text{..........(1)}$$

Since for any f in $K(G)$, $|\int fd\mu| \leq \int|f|d|\mu|$ always, for any x in G we have $|\mu({}_{x^{-1}}f)| \leq |\mu|(|{}_{x^{-1}}f|)$. Now suppose $|\mu({}_{x^{-1}}f)| < |\mu|(|{}_{x^{-1}}f|)$ on some set of positive measure. Then

$$\int\left|\int f(x^{-1}y)d\mu(y)\right|dm(x) < \int\int|f(x^{-1}y)|d|\mu|(y)dm(x).$$

Replacing f by the function $f^{\sim}$ where $f^{\sim}(x) = f(x^{-1})$ we see that (1) is violated. Thus $|\mu({}_{x^{-1}}f)| = |\mu|(|{}_{x^{-1}}f|)$ almost everywhere. But since both sides of this equation are continuous, (Chapter I, § 5.1), equality holds everywhere. Putting $x = e$ we have $|\mu(f)| = |\mu|(|f|)$. The theorem now follows from proposition 1.

Lemma 2. Let F and G be locally compact groups, α an isomorphism and homeomorphism of F onto G and γ a continuous character on F. Let T be the mapping of M(F) into M(G) defined by

$$T\mu(f) = \mu(\gamma(f\circ\alpha)) \qquad \lambda \in M(F), \quad f \in C_0(G),$$

then T is an isometric *-isomorphism of M(F) onto M(G), and T is a bicontinuous mapping of M(F) with the $\sigma(M(F),C_0(F))$-topology onto M(G) with the $\sigma(M(G),C_0(G))$-topology.

Proof: Let S be the mapping of $C_0(G)$ into $C_0(F)$, defined by $Sf = \gamma(f\circ\alpha)$. Clearly S is well defined and linear. Since $|\gamma| = 1$ we have

$$||Sf|| = ||\gamma(f\circ\alpha)|| = ||f\circ\alpha|| = \sup_{x\in G}|f(\alpha(x))| = \sup_{\alpha(x)\in F}|f(\alpha(x))|$$

$$= ||f||, \text{ since } \alpha \text{ is a homeomorphism.}$$

For any $g \in C_0(F)$, $g\circ\alpha^{-1} \in C_0(G)$ and $\bar{\gamma}\circ\alpha^{-1}$ is continuous so that $(\bar{\gamma}\circ\alpha^{-1})(g\circ\alpha^{-1}) \in C_0(G)$. Furthermore, since

$$S(\bar{\gamma}\circ\alpha^{-1})(g\circ\alpha^{-1}) = \gamma(((\bar{\gamma}\circ\alpha^{-1})(g\circ\alpha^{-1}))\circ\alpha) = \gamma(\bar{\gamma}g) = |\gamma|g = g$$

we have that S is onto. Let S' be the adjoint of S. Then S' is an isometry of M(F) onto M(G), and S' is a bicontinuous mapping of $M(F)_\sigma$ onto $M(G)_\sigma$ (Chapter 1, §1.6). For any $\mu \in M(F)$ and $f \in C_0(G)$ we have

$$S'\mu(f) = \mu(Sf) = \mu(\gamma(f\circ\alpha)) = T\mu(f)$$

hence $S' = T$ so we have that T is an isometry of M(F) onto M(G). To show $T(\lambda*\mu) = T\lambda*T\mu$ for $\mu,\lambda \in M(F)$, let $f \in C_o(G)$, then we have

$$(T(\lambda*\mu))(f) = (\lambda*\mu)(\gamma(f\circ\alpha))$$

$$= \lambda(\bar{\mu}(\gamma(f\circ\alpha)), \text{ where } \bar{\mu}(f)(x) = \mu({}_xf).$$

On the other hand

$$(T\lambda*T\mu)(f) = T\lambda(\overline{T\mu}(f))$$

$$= \lambda(\gamma(\overline{T\mu}(f)\circ\alpha)).$$

Thus to show that $T(\lambda*\mu) = T\lambda*T\mu$ it suffices to show that $\bar{\mu}(\gamma(f\circ\alpha)) = \gamma(\overline{T\mu}(f)\circ\alpha)$. For $x \in F$ we have

$$\bar{\mu}(\gamma(f\circ\alpha))(x) = \mu({}_x\gamma_x(f\circ\alpha))$$

$$= \gamma(x)\mu(\gamma({}_x(f\circ\alpha))) \quad \text{since } {}_x\gamma(y) = \gamma(x)\gamma(y)$$

$$= \gamma(x)\mu(\gamma({}_{\alpha(x)}f)\circ\alpha)) \quad \text{since } \alpha \text{ is an isomorphism}$$

$$= \gamma(x)T\mu({}_{\alpha(x)}f) \quad \text{by the definition of } T$$

$$= \gamma(x)\overline{T\mu}(f)(\alpha(x)) \text{ where } \overline{T\mu}(f)(y) = T\mu({}_yf) \text{ for } y \text{ in } G$$

$$= \gamma(x)(\overline{T\mu}(f)\circ\alpha)(x)$$

and thus $\bar{\mu}(\gamma(f\circ\alpha)) = \gamma(\overline{T\mu}(f)\circ\alpha)$.

To complete the proof it remains to show that $T\mu^* = (T\mu)^*$. Let $\mu \in M(F)$, $f \in C_o(G)$, then

$$T\mu^*(f) = \mu^*(\gamma(f\circ\alpha)) = \overline{\mu(\overline{(\gamma(f\circ\alpha))^{\sim}})}.$$

But $(\gamma(f\circ\alpha))^{\sim}(x) = \gamma(x^{-1})f(\alpha(x^{-1})) = \overline{\gamma(x)}f((\alpha(x))^{-1}) = \overline{\gamma(x)}(f^{\sim}\circ\alpha)(x)$.

Thus $(T\mu^*)(f) = \overline{\mu(\gamma(\overline{f^{\sim}}\circ\alpha))} = \overline{T\mu(\overline{f^{\sim}})} = (T\mu)^*(f)$, i.e. $T\mu^* = (T\mu)^*$.

Theorem 2 below is a converse to this theorem; that every norm decreasing isomorphism of M(F) onto M(G) has the form of T.

<u>Lemma 3</u>. Let F, G be locally compact groups and let T be a norm decreasing isomorphism of M(F) onto M(G). Then for each x in F

$||T\varepsilon_x * \mu|| = ||\mu||$ for all μ in $M(G)$.

Proof: Let L_x be the operator on $M(G)$ defined by $L_x\mu = T\varepsilon_x * \mu$. To show that $L_x^{-1} = L_{x^{-1}}$ we note that T is a homomorphism of $M(F)$ onto $M(G)$ and therefore it maps the unit of $M(F)$ to the unit of $M(G)$. Moreover

$$L_{x^{-1}} L_x \mu = (T\varepsilon_{x^{-1}} * T\varepsilon_x) * \mu = T(\varepsilon_{x^{-1}} * \varepsilon_x) * \mu = T\varepsilon_e * \mu = \mu.$$

Hence $L_{x^{-1}} L_x = I$. Similarly $L_x L_{x^{-1}} = I$. Now since T is norm decreasing,

$$||L_x\mu|| = ||T\varepsilon_x * \mu|| \le ||T\varepsilon_x||\ ||\mu|| \le ||\mu||$$

and $$||L_x^{-1}\mu|| = ||T\varepsilon_{x^{-1}} * \mu|| \le ||T\varepsilon_{x^{-1}}||\ ||\mu|| \le ||\mu||.$$

Hence L_x and $L_{x^{-1}}$ are norm decreasing operators on $M(G)$. If μ is such that $||L_x\mu|| < ||\mu||$, then

$$||\mu|| = ||L_{x^{-1}} L_x \mu|| \le ||L_x\mu|| < ||\mu||, \text{ a contradiction.}$$

Then we must have

$$||L_x\mu|| = ||\mu|| \quad \text{i.e.} \quad ||T\varepsilon_x * \mu|| = ||\mu|| \text{ for each } \mu \in M(G).$$

We now give the main result of this chapter.

Theorem 2. Let F and G be locally compact groups and let T be a norm decreasing isomorphism of $M(F)$ onto $M(G)$. Then there is an isomorphism and homeomorphism α of F onto G, and a continuous character γ on F such that

$$(T\mu)(f) = \mu(\gamma(f \circ \alpha)) \qquad \mu \in M(F), \qquad f \in C_0(G).$$

Proof: For $x \in F$, we have by lemma 3, $||T\varepsilon_x * \mu|| = ||\mu||$ for all $\mu \in M(G)$. Thus by theorem 1, there is a complex number $\gamma(x)$ depending upon x with $|\gamma(x)| = 1$, and an element $\alpha(x)$ of G

such that $T\varepsilon_x = \gamma(x)\varepsilon_{\alpha(x)}$. Consider the mappings

$$\gamma : x \rightarrow \gamma(x) \quad \text{and } \alpha : x \rightarrow \alpha(x).$$

We first show that α is a homomorphism of F into G, and γ is a homomorphism of F into the complex numbers of absolute value 1.

Clearly $T\varepsilon_{xy} = \gamma(xy)\varepsilon_{\alpha(xy)}$,

and $\quad T\varepsilon_x * T\varepsilon_y = \gamma(x)\gamma(y)\varepsilon_{\alpha(x)} * \varepsilon_{\alpha(y)} = \gamma(x)\gamma(y)\varepsilon_{\alpha(x)\alpha(y)}$.

Since $\quad T\varepsilon_x * T\varepsilon_y = T(\varepsilon_x * \varepsilon_y) = T\varepsilon_{xy}$, we have

$$\gamma(xy)\varepsilon_{\alpha(xy)} = \gamma(x)\gamma(y)\varepsilon_{\alpha(x)\alpha(y)}:$$

and since the Dirac measure are pairwise linearly independent, we have

$$\gamma(xy) = \gamma(x)\gamma(y)$$

and $\quad \alpha(xy) = \alpha(x)\alpha(y)$.

Since T is an isomorphism, T maps the unit of M(F) onto the unit of M(G), so that

$$\gamma(e) = 1$$

and $\quad \alpha(e) = e'$

where e (resp. e') is the unit of F (resp. G). Thus α is a homorphism of F into G, and γ is a homomorphism of F into the complex numbers.

We now show that γ is continuous. Since $T\varepsilon_x = \gamma(x)\varepsilon_{\alpha(x)}$ we have for $f \in C_o^+(X)$, $|T\varepsilon_x(f)| = |\gamma(x)||\varepsilon_{\alpha(x)}(f)| = \varepsilon_{\alpha(x)}(f)$. Consequently $\gamma(x)\varepsilon_{\alpha(x)}(f) = \gamma(x)|T\varepsilon_x(f)| = T\varepsilon_x(f)$. Thus it suffices to show that the mapping $x \rightarrow T\varepsilon_x(f)$ is continuous at e in F where f is such that $T\varepsilon_e(f) \neq 0$. By the definition of the weak topology the mapping $T\varepsilon_x \longrightarrow T\varepsilon_x(f)$ is continuous from $M(G)_\sigma$ to C (the complex numbers). The mapping $x \rightarrow \varepsilon_x$ is continuous from F to F_σ^ε by proposition 1, of Chapter II and the mapping $\varepsilon_x \rightarrow T\varepsilon_x$ is continuous from F_{so}^ε into $M(G)_\sigma$ by proposition 11 of Chapter II. Since by corollary 2 to proposition 8 of Chapter II, $F^{\varepsilon}_{so} = F^{\varepsilon}_{\sigma}$, we have that γ is continuous. The continuity of α follows by considering the mappings.

$$x \to \varepsilon_x \to T\varepsilon_x = \gamma(x)\varepsilon_{\alpha(x)} \to \varepsilon_{\alpha(x)} \to \alpha(x) .$$

The only mapping we have to check is the mapping $\gamma(x)\varepsilon_{\alpha(x)} \to \varepsilon_{\alpha(x)}$, of a subset of $M(G)_{\sigma}$ into $M(G)_{\sigma}$. But since this mapping is multiplication by the continuous character $x \to \overline{\gamma(x)}$, it is continuous. Thus α is continuous since it may be written as a composite of continuous mappings. Moreover since each of the above mappings is one to one we have that α is a continuous one to one homomorphism. Now consider T^{-1}. By the open mapping theorem ((14) Chapter 3, §2) T^{-1} is a bounded isomorphism of $M(G)$ onto $M(F)$. Thus proposition 11 of Chapter II applies and we have that $\varepsilon_{\alpha(x)} \to T^{-1}\varepsilon_{\alpha(x)} = \overline{\gamma(x)}\varepsilon_x$ is continuous from G^{ε}_{so} into $M(F)_{\sigma}$. The continuity of α^{-1} restricted to $\alpha(F)$ now follows by considering the mappings

$$\alpha(x) \to \varepsilon_{\alpha(x)} \to T^{-1}\varepsilon_{\alpha(x)} = \overline{\gamma(x)}\varepsilon_x \to \varepsilon_x \to x.$$

Thus F is homeomorphic to $\alpha(F)$ and since a locally compact group is complete (Chapter I, §2.3) $\alpha(F)$ is complete and therefore closed.

Now suppose α is not onto, then there is a y in $G \setminus \alpha(F)$ and a compact neighborhood V of y such that $V \cap \alpha(F) = \emptyset$ because $\alpha(F)$ is closed. Since $T^{-1}\varepsilon_y$ is in $M(F)$, by proposition 9 of Chapter II, there is a net $(\mu_j : j \in J)$ such that

$$\mu_j = \Sigma_{i=1}^{n_j} b_{i,j}\varepsilon_{x_{i,j}} \quad , \quad x_{i,j} \in F \quad , \quad b_{i,j} \text{ complex,}$$

$$||\mu_j|| \leq ||T^{-1}\varepsilon_y||$$

and $\mu_j \xrightarrow{so} T^{-1}\varepsilon_y$. Thus by proposition 11 of Chapter II, $T\mu_j \xrightarrow{\sigma} \varepsilon_y$. Note that $T\varepsilon_x = \gamma(x)\varepsilon_{\alpha(x)}$ (shown above) implies

$$T\mu_j = \Sigma_{i=1}^{n_j} b_{i,j}\gamma(x_{i,j})\varepsilon_{\alpha(x_{i,j})}$$

Since G is locally compact there is a function f in $C_o(G)$ such that $f(y) = 1$ and $f(G\setminus V) = 0$ and $0 \leq f(x) \leq 1$ for all $x \in G$.

Since $T\mu_j \xrightarrow{\sigma} \varepsilon_y$ we have

$$\Sigma_{i=1}^{n_j} b_{i,j}\gamma(x_{i,j})\varepsilon_{\alpha(x_{i,j})}(f) \rightarrow f(y).$$

But since $\alpha(x_{i,j}) \in \alpha(F)\cap(G\setminus V)$, $\varepsilon_{\alpha(x_{i,j})}(f) = f(\alpha(x_{i,j})) = 0$, so that $f(y) = 0$, a contradiction. Thus α is onto.

All that remains now is to establish the formula

$(T\mu)(f) = \mu(\gamma(f\circ\alpha))$.

Let T_1 be the mapping defined by

$(T_1\mu)(f) = \mu(\gamma(f\circ\alpha))$, $\mu \in M(F)$, $f \in C_o(G)$.

By lemma 2 we have that T_1 is an isomorphism and isometry from M(F) onto M(G). Hence in view of proposition 11 of Chapter II, T_1 is continuous on norm bounded sets from $M(F)_{so}$ onto $M(G)_\sigma$. Now observe that

$$T_1\varepsilon_x(f) = \varepsilon_x(\gamma(f\circ\alpha)) = \gamma(x)f(\alpha(x)) = \gamma(x)\varepsilon_{\alpha(x)}(f)$$
$$= T\varepsilon_x(f)$$

Thus T and T_1 coincide on F^ε, and by proposition 9 of Chapter II each $\mu \in M(F)$ is a so-adherence point of a norm bounded set of linear combinations of Dirac measures so we have $T = T_1$. This completes the proof.

<u>Corollary 1</u>. Every norm decreasing isomorphism of M(F) onto M(G) is an isometric *-isomorphism.

Proof: This follows from lemma 2 and the above theorem.

Corollary 2. Let T be a norm decreasing isomorphism of M(F) onto M(G), then T is a bicontinuous mapping of M(F) with the $\sigma(M(F), C_o(F))$-topology onto M(G) with the $\sigma(M(G), C_o(G))$-topology.

Proof: This follows from lemma 2 and the above theorem.

Corollary 3. Each norm decreasing isomorphism of M(F) onto M(G) maps $M_a(F)$ onto $M_a(G)$.

Proof: Let m (resp. m_1) be the left invariant Haar measure on F (resp. G) and let $\alpha(m)$ be the measure defined by $\alpha(m)(f) = m(f \circ \alpha)$, $f \in K(G)$ where α is the homomorphism of F onto G given in theorem 2. Then $\alpha(m)$ is a left invariant Haar measure consequently $\alpha(m) = cm_1$ for some $c > 0$. Now let $\lambda \in M_a(F)$ and put $h = d\lambda/dm$, then $h \in L^1(F)$ and $h \circ \alpha^{-1} \in L^1(G)$. For $f \in C_o(G)$

$$T\lambda(f) = Thm(f) = hm(\gamma(f \circ \alpha)) = m(\gamma h(f \circ \alpha))$$

$$= cm_1((\gamma \circ \alpha^{-1})(h \circ \alpha^{-1})f) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

So that $T\lambda \in M_a(G)$. The reverse inclusion follows similarly by considering T^{-1}, since it is an isometric isomorphism of M(G) onto M(F), (Corollary 1.).

Corollary 4. $M_a(G)$ is invariant under norm decreasing automorphisms of M(G).

The following example shows that a *-isomorphism of M(F) onto M(G) need not be norm decreasing. Let F and G be finite abelian groups of order n, and suppose that F and G are not isomorphic. Let $\gamma_1, \gamma_2, \ldots\ldots\ldots\ldots \gamma_n$ be the n characters of F and define functions f_j, $j = 1, 2, \ldots\ldots\ldots n$ on F by

$$f_j(x) = (1/n)\gamma_j(x) \quad , \quad x \in F.$$

<u>Proposition 2</u>. $f_j * f_k = 0$ for $j \neq k$; $f_j * f_j = f_j$, for all j, and $f_j^* = f_j$.

<u>Proof</u>: $(f_j * f_k)(x) = \Sigma_{y \in G} f_j(y) f_k(y^{-1}x)$

$$= (1/n^2)\Sigma_{y \in G}\gamma_j(y)\gamma_k(y^{-1}x)$$

$$= (1/n)\gamma_k(x)((1/n)\Sigma_{y \in G}\gamma_j(y)\overline{\gamma_k(y)})$$

So that the first two assertions follow from the well known orthogonality relations for characters on abelian groups.

Since $f_j^*(x) = \overline{f_j(x^{-1})} = \overline{(1/n)\gamma_j(x^{-1})} = (1/n)\overline{\overline{\gamma_j(x)}} = f_j(x)$ the last assertion follows.

Let $\gamma_1', \gamma_2', \ldots\ldots \gamma_n'$ be the n characters on G, and define functions g_j, $j = 1, 2, \ldots\ldots n$ by

$$g_j(x) = (1/n)\gamma_j'(x) \qquad x \in G$$

It follows as above that $g_j * g_k = 0$ for $j \neq k$, $g_j * g_j = g_j$, and $g_j^* = g_j$.

Since M(F) (resp. M(G)) has dimension n, it follows from the above that $(f_j : j = 1, 2, \ldots\ldots n)$ (resp. $(g_j : j = 1, 2, \ldots\ldots n)$) algebraically generate M(F) (resp.M(G)). Thus the mapping T defined by $Tf_j = g_j$ can be extended to an isomorphism of M(F) onto M(G). We now show that T is a *-isomorphism. Let $f \in M(F)$, then

$f = \Sigma_1^n c_j f_j$, and $Tf = \Sigma_1^n c_j g_j$. Thus

$$Tf^* = T(\Sigma c_j f_j)^* = T(\Sigma \bar{c}_j f_j) = \Sigma \bar{c}_j g_j = \Sigma \bar{c}_j g_j^* = (\Sigma c_j g_j)^* = (Tf)^*$$

By theorem 2, if T were norm decreasing then F and G would be isomorphic.

This example was first considered by Wendel (23) in a slightly different context.

One can however show the following:

<u>Proposition 3</u>. If T is a *-isomorphism of M(F) onto M(G), then T is bounded.

<u>Proof</u>: Let $(L, L^2(G))$ be the left regular representation of M(G) (Chapter I, §7.2). Since $L \circ T$ is a *-representation of M(F) by operators on a Hilbert space, $L \circ T$ is bounded ((12) Chapter V, 21.22) so that if $\mu_n \xrightarrow{n} \mu$ and $T\mu_n \xrightarrow{n} \lambda$, then

$$L(T\mu) = \lim_n (L \circ T)(\mu_n) = L(\lim_n T\mu_n) = L(\lambda).$$

Since L is 1 : 1, $T\mu = \lambda$ i.e. the graph of T is closed so that T is continuous by the closed graph theorem ((14) Chapter 3, §2).

As a further consequence we shall derive a theorem due to Wendel (24) on isomorphisms of $L^1(F)$. First we need the following lemma.

<u>Lemma 4</u>. Let T be a bounded isomorphism of $M_a(F)$ onto $M_a(G)$. Then there is a unique bounded isomorphism $\bar{T}$ of M(F) onto M(G) which extends T. Moreover $||\bar{T}|| = ||T||$.

<u>Proof</u>: Clearly T is continuous as a mapping of $M_a(F)_{so}$ onto

$M_a(G)_{so}$. Since $M(G)_{so}$ is quasi-complete (Chapter I, §5.3) and since each μ in $M(F)$ is a so-adherence point of a bounded set in $M_a(F)$ (Chapter II proposition 10), T has a unique extension $\bar{T}$ to a continuous linear mapping of $M(F)_{so}$ onto $M(G)_{so}$ ((2) Chapitre III, §2, No. 5).

To show that $\bar{T}(\mu*\lambda) = T\mu*T\lambda$ for μ, λ in $M(F)$, let $(\mu_j : j \in J)$ and $(\lambda_k : k \in K)$ be nets in $M_a(F)$ such that $\mu_j \xrightarrow{so} \mu$ and $\lambda_j \xrightarrow{so} \lambda$. Then since multiplication is separately continuous in $M(F)_{so}$ (Chapter II proposition 6) we have

$$\mu*\lambda = \lim_j(\lim_k \mu_j*\lambda_k),$$

Therefore the so-continuity of $\bar{T}$ implies that

$$\bar{T}(\mu*\lambda) = \lim_j(\lim_k \bar{T}(\mu_j*\lambda_k)).$$

Since $\bar{T} = T$ on $M_a(F)$ we have $\bar{T}(\mu_j*\lambda_k) = T\mu_j*T\lambda_k$. Now using the fact that multiplication is separately continuous in $M(G)_{so}$ we have

$$\lim_j(\lim_k Tu_j*T\lambda_k) = \bar{T}\mu*\bar{T}\lambda.$$

Combining the above we have $\bar{T}(\mu*\lambda) = \bar{T}\mu*\bar{T}\lambda$.

To show that $\bar{T}$ is one-one, let λ, $\mu \in M(F)$ and suppose $\bar{T}\mu = \bar{T}\lambda$. If $\lambda \neq \mu$ then there is a ν in $M_a(F)$ such that $\lambda*\nu \neq \mu*\nu$. Then $T(\lambda*\nu) = \bar{T}\lambda*T\nu = \bar{T}\mu*T\nu = T(\mu*\nu)$ which contradicts the assertion that T is an isomorphism because $\lambda*\nu$ and $\mu*\nu$ are in $M_a(F)$. Therefore $\bar{T}$ is one-one.

We now show that T is onto. Let $\mu' \in M(G)$, by proposition 10 of chapter II there is a net $(\mu'_j : j \in J)$ in $M_a(G)$ such that $\mu'_j \xrightarrow{so} \mu'$ and $||\mu'_j|| \leq ||\mu'||$. By the open mapping theorem ((14) Chapter 3, § 2) T^{-1} is bounded, so that $(T^{-1}\mu'_j : j \in J)$ is a bounded Cauchy net in $M_a(F)_{so}$. Since $M(F)_{so}$ is quasi-complete there is a μ in $M(F)$ such that $T^{-1}\mu'_j \xrightarrow{so} \mu$. Then $T(T^{-1}\mu'_j) \xrightarrow{so} T\mu$ so that

$\bar{T}\mu = \mu'$ and $\bar{T}$ is onto.

We next show $||\bar{T}|| = ||T||$. Clearly $||T|| \leq ||\bar{T}||$. To show the reverse inequality let $\mu \in M(F)$ be given. By proposition 10 of Chapter II there is a net $(\mu_j : j \in J)$ in $M_a(F)$ such that $\mu_j \xrightarrow{so} \mu$ and $||\mu_j|| \leq ||\mu||$ so that $T\mu_j \xrightarrow{so} T\mu$. Since the mapping $\lambda \rightarrow ||\lambda||$ is lower semicontinuous in the so-topology (corollary 1 to proposition 8), we have $||\bar{T}\mu|| \leq \lim\inf ||T\mu_j|| \leq \lim\inf ||T||\,||\mu_j|| \leq ||T||\,||\mu||$. Therefore $||\bar{T}|| \leq ||T||$ and hence $||\bar{T}|| = ||T||$.

Finally we show that $\bar{T}$ is unique as a norm bounded isomorphism which extends T. Let S be any norm bounded isomorphism of M(F) onto M(G) such that S = T on $M_a(F)$. By proposition 11 of Chapter II, S and T are continuous on norm bounded sets as a mapping of $M(F)_{so}$ onto $M(G)_{\sigma}$. Since each μ in M(F) is an so-adherence point of a norm bounded net in $M_a(F)$ (proposition 10 of Chapter II) we have $S\mu = \bar{T}\mu$, ie. $S = \bar{T}$.

<u>Theorem 3</u>. (Wendel) Let T be a norm decreasing isomorphism of $L^1(F)$ onto $L^1(G)$. Then there is a homeomorphism and isomorphism α of F onto G, a continuous character γ on G and a constant c such that

$$Tg = c\gamma g \circ \alpha^{-1} \qquad g \in L^1(F)$$

<u>Proof</u>: Let m (resp. m_1) be the Haar measure on F (resp. G). For λ in $M_a(F)$ let $\frac{d\lambda}{dm}$ be the Radon-Nikodym derivative of λ with respect to m. Then the mapping $\lambda \rightarrow \frac{d\lambda}{dm}$ is an isometric *-isomorphism of $M_a(F)$ onto $L^1(F)$ (Chapter I, §5.3). If $h \in L^1(G)$ then hm_1 is in $M_a(G)$ and the mapping $h \rightarrow hm_1$ is an isometric *-isomorphism of $L^1(G)$

onto $M_a(G)$. Define a mapping S of $M_a(F)$ into $M_a(G)$ by $S\lambda = (T\frac{d\lambda}{dm})m_1$, then S is a norm decreasing isomorphism of $M_a(F)$ onto $M_a(G)$ since it is the composite of $\lambda \to \frac{d\lambda}{dm} \to T\frac{d\lambda}{dm} \to (T\frac{d\lambda}{dm})m_1$; the first and last of these are isometries and the middle mapping is norm decreasing. By lemma 4, S has a unique extension $\bar{S}$ to a norm decreasing isomorphism of M(F) onto M(G). Let $g \in L^1(F)$, and consider the measure gm. By theorem 2, $Sgm(f) = gm(\gamma'(f\circ\alpha))$ for all $f \in C_o(G)$, where γ' is a continuous character on F and α is an isomorphism and homeomorphism of F onto G given by lemma 2. Put $\gamma = \gamma'\circ\alpha$, define c as in corollary 3 of theorem 2, then (1) of corollary 3 becomes

$$Sgm = c\gamma(g\circ\alpha^{-1})m_1 = (Tg)m_1$$

Therefore $Tg = c\gamma g\circ\alpha^{-1}$, and this proves the theorem.

<u>Remark</u>. Recently Greenleaf (9) has shown that if T is a norm decreasing homomorphism (not necessarily an isomorphism) of $M_a(F)$ onto $M_a(G)$ then there is a closed normal subgroup $F_o \subseteq F$, an isometric isomorphism T_1 of $M_a(F/F_o)$ onto $M_a(G)$ and a norm decreasing homorphism T_2 of $M_a(F)$ onto $M_a(F/F_o)$ such that $T = T_1\circ T_2$. T_2 is the mapping defined by $T_2\mu(f) = \mu(f\circ\pi)$, $\mu \in M_a(F)$, $f \in C_o(F/F_o)$ and π is the canonical map $F \to F/F_o$.

To show that the analogous result does not hold for norm decreasing homomorphisms of M(F) onto M(G), consider the example following proposition 11 of Chapter II. Here G is the group having only 1 element, F is any nondiscrete locally compact group and T is the mapping defined by

$$T\mu = \Sigma_{x\in F}\mu(\{x\})\varepsilon_e \qquad \mu \in M(F)$$

We have shown that T is a homomorphism and that $||T|| \leq 1$ (see the remark after proposition 11 of Chapter II). If there were a closed normal subgroup $F_o \subseteq F$ and an isometric isomorphism T_1 of $M(F/F_o)$ onto $M(G)$, then F/F_o must be isomorphic to G by theorem 2, consequently F/F_o consists of a single element so that $F_o = F$. If T_2 is the mapping of $M(G)$ given $T_2\mu(1) = \mu(1\circ\pi)$ where π is the canonical map: $F \rightarrow F/F_o$, then T_2 has the form $T_2\mu = \mu(F)\varepsilon_e$ for $\mu \in M(F)$. If μ is a continuous measure such that $\mu(F) \neq 0$, then $T\mu = 0$, $T_2\mu \neq 0$ and since T_1 is an isometry $T_1\circ T_2\mu \neq 0$. Thus $T \neq T_1\circ T_2$.

CHAPTER IV

CHARACTERIZATIONS OF M(G)

In this chapter we show that certain of the properties of M(G) which we developed in the preceeding parts of this work characterizes those Banach algebras which are isometrically isomorphic to M(G) for some locally compact group G.

Definitions. An element x of a Banach algebra A is called a left (resp. right) inner translator of A if the mapping $y \to xy$ (resp. $y \to yx$) is an isometry of A onto itself. If x is both a left and right inner translator of A, then x is said to be an inner translator.

An element z of A is called a left (resp. right) annihilator of A if $zx = 0$ (resp. $xz = 0$) for all x in A.

Remarks. It is clear that the product of two left (resp. right) inner translators is a left (resp. right) inner translator. The zero element is never a left or right inner translator.

If a Banach algebra has a unit u, then we shall always require $||u|| = 1$. Thus if x is left (resp. right) inner translator $||x|| = ||xu|| = ||u|| = 1$.

Proposition 1. If a Banach algebra A has a left inner translator x, and if 0 is the only left annihilator of A, then A has a unit u, x^{-1} exists, and x, x^{-1} are inner translators.

Proof: Let T_x be the mapping $y \to xy$. By hypothesis T_x is an isometry of A onto itself, hence T_x^{-1} exists. Let $u = T_x^{-1}x$, then for any $y \in A$, $uy = y$ so that u is a left unit. For y, $z \in A$ we have

$(zu - z)y = zT_x^{-1}\, T_x y - zy = 0$, so that $zu - z$ is a left annihilator. Thus by hypothesis we have $zu = z$ for all $z \in A$, so that u is also a right unit and hence a unit. Put $y = T_x^{-1}u$, then

$$yx = T_x^{-1}ux = u$$

and $$xy = xT_x^{-1}u = T_xT_x^{-1}u = u.$$ Hence $y = x^{-1}$.

Finally to show that x and x^{-1} are inner translators we observe the following. Since T_x is an isometry of A onto itself, T_x^{-1} is also an isometry of A onto itself and $T_x^{-1} = T_{x^{-1}}$, so that x^{-1} is a left inner translator. Since $||T_x|| = 1$, we have $||x|| = ||T_x u|| = 1$ so that for each y in A, $||yx|| \leq ||y||$. Similarly $||x^{-1}|| = 1$ and $||yx^{-1}|| \leq ||y||$. Now suppose x is not a right inner translator, that means for some y, $||yx|| < ||y||$ since $y \to yx$ is onto. But then $||y|| = ||yxx^{-1}|| \leq ||yx|| < ||y||$ a contradiction. Hence x is an inner translator. Similarly for x^{-1}.

Corollary 1. If a Banach algebra A has an inner translator x, then A has a unit, x^{-1} exists, and x^{-1} is an inner translator of A.

Proof: If A has an inner translator, then 0 is the only left annihilator. Hence the corollary follows from proposition 1.

Corollary 2. If A is a semi-simple Banach algebra, and if A has a left inner translator x, then A has a unit u, x^{-1} exists, and x and x^{-1} are inner translators.

Proof: If y is a left annihilator, then for any scalar α and any x in A, $\alpha y + xy$ has $-(\alpha y + xy)$ as a quasi-inverse, thus by Chapter I, §6.2, y is in the Jacobson radical and hence $y = 0$ by hypothesis. Therefore 0 is the only left annihilator so that the corollary follows from proposition 1.

Proposition 2. Let A be a Banach algebra. A necessary and sufficient condition for any element x of A to be an inner translator of A is that A has a unit u, x^{-1} exists and $||x|| \leq 1$, $||x^{-1}|| \leq 1$.

Proof: Necessity follows from corollary 1 above and the remark preceeding proposition 1. To prove sufficiency, note that since x^{-1} exists, the mappings $y \to yx$ and $y \to xy$ are one-one and onto. If the first mapping, say, is not an isometry, there is a y in A such that $||xy|| < ||y||$. Hence $||y|| = ||x^{-1}xy|| \leq ||xy|| < ||y||$ which is a contradiction. Thus $||xy|| = ||y||$ and similarly $||yx|| = ||y||$. Therefore x is an inner translator.

Corollary. Let A and B be Banach algebras and suppose T is a norm decreasing homomorphism of A onto B. If x is an inner translator of A, then Tx is an inner translator of B.

Proof: By proposition 2, A has a unit u, x^{-1} exists and $||x|| \leq 1$, $||x^{-1}|| \leq 1$. Since T is a homomorphism onto, Tu is the unit in B and $(Tx)^{-1} = Tx^{-1}$. Since $||T|| \leq 1$ we have $||Tx|| \leq 1$, and $||(Tx)^{-1}|| = ||Tx^{-1}|| \leq ||x^{-1}|| \leq 1$. Thus by proposition 2, Tx is an inner translator.

Proposition 3. Let A be a Banach algebra, having 0 as its only left annihilator and let x be a left inner translator of A. Then x is an extreme point of the unit ball of A.

Proof: By proposition 1 above, A has a unit u, x^{-1} exists, and $||x^{-1}|| \leq 1$ by proposition 2. Now suppose

$$x = ay + (1 - a)z\ , \quad 0 < a < 1 \quad ||y|| \leq 1\ , \quad ||z|| \leq 1$$

then $\quad u = ax^{-1}y + (1 - a)x^{-1}z$

and $\quad ||x^{-1}y|| \leq 1,\ ||x^{-1}z|| \leq 1.$

Now by Theorem 9, Chapter 1, u is an extreme point of the unit ball so that we must have x = y = z, i.e. x is an extreme point.

Remark. In an arbitrary Banach algebra an extreme point of the unit ball need not be an inner translator. We now give an example of a Banach algebra having a unit and an extreme point of the unit ball which is not an inner translator.

Let H be an infinite dimensional separable Hilbert space, and let $(e_n : n = 1, 2, \ldots\ldots)$ be an orthonormal basis in H. We define a linear operator T on H by means of the equations

$$Te_n = e_{n+1} \qquad n = 1, 2, 3, \ldots\ldots$$

Let (,) denote the inner product in H. For $x \in H$ we have ((2) Chapitre V, §2 No. 3)

$$||x||^2 = \Sigma_1^\infty |(x, e_n)|^2.$$

Therefore

$$||Tx||^2 = \Sigma_1^\infty |(Tx, e_n)|^2$$
$$= \Sigma_1^\infty |(x, T^* e_n)|^2.$$

Note that T^* satisfies the equations

$$T^* e_n = e_{n-1} \qquad \text{if } n = 2, 3, \ldots\ldots$$
$$T^* e_1 = 0.$$

Thus

$$||Tx||^2 = \Sigma_2^\infty |(x, e_{n-1})|^2$$
$$= \Sigma_1^\infty |(x, e_n)|^2$$
$$= ||x||^2$$

Consequently $||T|| = 1$.

Let B(H) be the Banach *-algebra of all bounded linear operators on H. By the above we have that T is an element of the unit ball of B(H). We now show that T is an extreme point of the unit ball of B(H). For this, suppose that

$$T = aR + (1 - a)S \quad , \quad 0 < a < 1 \; , \quad ||R|| \leq 1 \quad , \quad ||S|| \leq 1.$$

Note that $T^*T = I$ (the unit of B(H)). Therefore

$$I = aT^*R + (1 - a)T^*S \, ,$$

Now $||T^*R|| \leq ||T^*|| \, ||R|| = ||T|| \, ||R|| \leq 1$ and $||T^*S|| \leq 1$, we have $I = T^*R = T^*S$ because I is an extreme point of the unit ball (Chapter I, theorem 9). We also have

$$T^* = aR^* + (1 - a)S^*$$

which gives

$$T^*R = aR^*R + (1 - a)S^*R$$

Since $I = T^*R$ and I is an extreme point of the unit ball, we have

$$I = R^*R = S^*R.$$

Consequently since we also have $T^*R = I = R^*T$

$$R^*R - R^*TT^*R = 0$$

Thus $R^*(I - TT^*)R = 0$.

Observe that

$$(I - TT^*)^*(I - TT^*) = I - TT^*$$

So that

$$((I - TT^*)R)^*(I - TT^*)R = R^*(I - TT^*)R = 0.$$

In virtue of the equality $||V^*V|| = ||V||^2$ for any V in $B(H)$ we must have

$$(I - TT^*)R = 0.$$

Therefore $R = TT^*R = T$. So that we have $R = T = S$ which shows that T is an extreme point. To show that T is not an inner translator let P be the operator defined by the equations

$$Pe_1 = e_1 \quad \text{and} \quad Pe_n = 0 \qquad n = 2, 3, \ldots..$$

Then P is in $B(H)$ and $PT = 0$. Therefore T is not a right inner translator and hence not an inner translator. Thus the Banach *-algebra $B(H)$ is the desired example.

<u>Lemma 1</u>. Let A be a Banach algebra and suppose that A satisfies the conditions,

(1) 0 is the only left annihilator of A

(2) each extreme point of the unit ball S of A is a left inner translator.

Then the extreme points S^ε of S form a group.

Proof: Conditions (1), (2) and proposition 3 imply that an element of A is a left inner translator if and only if it is an extreme point of the unit ball. The product of left inner translators is a left inner translator and this together with proposition 1 implies that the set of all left inner translators is closed under multiplication and inverses. Clearly u is a left inner translator and thus S^ε is a group.

Theorem 1. Let A be a Banach algebra, S its unit ball and S^ε the set of extreme points of S. Suppose that

(1) there is a Banach space E such that A is the dual of E,

(2) multiplication is $\sigma(A,E)$-continuous in each variable separately,

(3) 0 is the only left annihilator of A,

(4) each x in S^ε is a left inner translator of A,

(5) $S^\varepsilon \cup \{0\}$ is $\sigma(A,E)$-closed,

(6) there is a nonzero multiplicative linear functional p on A,

(7) let $G = \{x \in S^\varepsilon : p(x) = 1\}$ where p is the nonzero multiplicative linear functional given by (6). Then

(i) for each f in E, there is a g in E such that $\overline{x(f)} = x(g)$ for each x in G, and

(ii) for f and g in E there is an h in E such that $x(h) = x(f)x(g)$.

Then G is a locally compact group and A is isometric and isomorphic to M(G). If S^{ε} is closed then G is compact. G is unique to within isomorphism and homeomorphism. Conversely if G is a locally compact group then M(G) satisfies (1) to (7).

<u>Proof</u>: Conditions (3), (4) and lemma 1 imply that S^{ε} is a group and A has a unit u (Proposition 1). Since p is multiplicative, it follows that G is a subgroup of S^{ε}. We divide the proof into a number of assertions.

I. For any $x \in S^{\varepsilon}$, $\overline{p(x)} = p(x^{-1})$ and $|p(x)| = 1$.

Since p is a nonzero multiplicative linear functional on a Banach algebra with a unit u we have $p(u) = 1$ and $||p|| = 1$ (Chapter I, §6). Let $x \in S^{\varepsilon}$, then $|p(x)| \leq 1$ and $|p(x^{-1})| \leq 1$. If for some $x \in S^{\varepsilon}$ we have $|p(x)| < 1$

$$1 = p(u) = |p(x^{-1}x)| = |p(x^{-1})p(x)| = |p(x^{-1})||p(x)| < 1,$$

which is an absurdity, thus we must have $|p(x)| = 1$. Then $p(x)\overline{p(x)} = 1 = p(u) = p(x^{-1}x) = p(x^{-1})p(x)$ we also have $\overline{p(x)} = p(x^{-1})$.

Let C be the complex numbers and Z the complex numbers of absolute value 1. Let g be the mapping of $C \times A$ onto A given by $g(a,x) = ax$, and let g_1 be its restriction to $Z \times G$.

II. g_1 is a homeomorphism of $Z \times G$ onto S^{ε}.

If x is an element of S^{ε}, and $a \in Z$ then ax is an extreme point so that g_1 maps $Z \times G$ into S^{ε}. Let $x \in S^{\varepsilon}$ then $p(x) \neq 0$, $x/p(x) \in G$, and $p(x) \in Z$ by I, so that g_1 is onto. To show that g_1 is one-one, let $x,y \in G$, $a,b \in Z$ and suppose $ax = by$, then

$$a = ap(x) = p(ax) = p(by) = bp(y) = b.$$

Hence $x = y$. Since g is continuous and open so is g_1.

III. G is a locally compact group, and if S^ε is $\sigma(A,E)$-closed, G is compact.

By (5), $S^\varepsilon \cup \{0\}$ is a $\sigma(A,E)$-closed subset of the unit ball S, hence $S^\varepsilon \cup \{0\}$ is $\sigma(A,E)$-compact since S is. Therefore S^ε is locally compact. By II $Z \times G$ is homeomorphic to S^ε and this means that G is locally compact. If S^ε is $\sigma(A,E)$-closed then $Z \times G$ is homeomorphic to a compact space and consequently G is compact. By (2) multiplication is weakly continuous in each variable separately so that G is a locally compact Hausdorff semitopological group. It is known (cf. Chapter I, § 2) that a locally compact semitopological group is a topological group. Thus G is a locally compact group.

For $f \in E$, let $\hat{f}$ be the function on G defined by $\hat{f}(x) = x(f)$, where $x \in G$.

IV. $f \to \hat{f}$ is a norm decreasing linear mapping of E into $C_o(G)$.

It is clear that this mapping is linear and since $||x|| \leq 1$, we have $|\hat{f}(x)| \leq ||f||$ so that $\hat{f}$ is bounded, and $||\hat{f}|| = \sup \{ |\hat{f}(x)| : x \in G\} \leq ||f||$, consequently $f \to \hat{f}$ is norm decreasing. By the definition of the weak topology we have that $\hat{f}$ is continuous. To show that $\hat{f}$ is in $C_o(G)$, first note that if S^ε is compact, G is compact and hence $C_o(G) = C(G)$ so that $\hat{f} \in C_o(G)$. If G is not compact, then S^ε is not compact so that 0 is a $\sigma(A,E)$-adherence point of S^ε since

$S^{\varepsilon} \cup \{0\}$ is $\sigma(A,E)$-compact. Now let $\varepsilon > 0$ be given and suppose $f \neq 0$. Clearly $U = \{ x \in A : |x(f)| < \varepsilon \} \cap (S^{\varepsilon} \cup \{0\})$ is an open $\sigma(A,E)$-neighborhood of 0 in $S^{\varepsilon} \cup \{0\}$ so that $W = S^{\varepsilon} \setminus U$ is compact in S^{ε}. Since $Z \times G$ is homeomorphic to S^{ε}, $g_1^{-1}(W)$ is compact in $Z \times G$. Let K be the image of $g_1^{-1}(W)$ by the projection mapping $Z \times G \rightarrow G$, then K is compact in G. Thus for $f \in E$, we have found a compact set K such that $|\hat{f}(x)| < \varepsilon$ for $x \notin K$ because $G \setminus K \subseteq U$.

V. Let $\hat{E}$ be the image of E in $C_o(G)$ under the mapping $f \rightarrow \hat{f}$. $\hat{E}$ is dense in $C_o(G)$.

If for $x,y \in G$, $\hat{f}(x) = \hat{f}(y)$ for all $\hat{f} \in \hat{E}$ then $x(f) = y(f)$ for all $f \in E$, hence $x = y$, so that E separates the points of G. If $x \in G$, then $x \neq 0$, so there is an $f \in E$ such that $x(f) \neq 0$. i.e. $\hat{f}(x) \neq 0$. Thus given $x \in G$, we can find an $\hat{f} \in \hat{E}$ such that $\hat{f}(x) \neq 0$. Further, for any $f \in E$, by (7)(i), there is a $g \in E$ such that $x(g) = \overline{x(f)}$. i.e. $\hat{g}(x) = \overline{\hat{f}(x)}$ for all $x \in G$. By (7)(ii) $\hat{E}$ is a subalgebra of $C_o(G)$. Thus $\hat{E}$ is a subalgebra of $C_o(G)$ which separates the points of G, does not vanish at any point of G, and is closed under complex conjugation, hence the Stone-Weierstrass theorem applies, and we may conclude that $\hat{E}$ is dense in $C_o(G)$.

Let T be the adjoint of the mapping of $f \rightarrow \hat{f}$. i.e. $T\mu(f) = \mu(\hat{f})$ for $\mu \in C_o(G)' = M(G)$ and $f \in E$. Note that $T\varepsilon_x = x$, so that by the linearity of T we have

$$T(\Sigma_1^n a_i \varepsilon_{x_i}) = \Sigma_1^n a_i x_i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (*)$$

VI. T is a norm decreasing one-one linear mapping of M(G) into A, and T is continuous as a mapping of $M(G)_\sigma$ into A_σ.

This follows from IV, V, and the general properties of adjoint mappings (cf. Chatper I, §1.6).

Let S^M be the unit ball of M(G).

VII. $(S^M)_\sigma$ is homeomorphic to S_σ, and T is an isometry of M(G) onto A.

Since $(S^M)_\sigma$ is compact, and T is one-one and continuous, to prove the first assertion, it suffices to show that $T(S^M) = S$. By VI, $T(S^M) \subseteq S$ so it suffices to show that $T(S^M) \supseteq S$. For this let $x \in S$, then since S is convex and $\sigma(A,E)$-compact, the Krein-Milman theorem (Chapter 1, §1.1) applies and there is a net $(x_j : j \in J)$ such that $x_j \xrightarrow{\sigma} x$ and $x_j = \sum_{i=1}^{n_j} a_{i,j} x_{i,j}$ where

$\sum_{i=1}^{n_j} a_{i,j} = 1$, $a_{i,j} > 0$ and the $x_{i,j}$ are extreme points (hence are in S^ε) of the unit ball in A. Putting $y_{i,j} = x_{i,j} / p(x_{i,j})$, we have $y_{i,j} \in G$ for any i, j. Considering

$$\mu_j = \sum_{i=1}^{n_j} a_{i,j} p(x_{i,j}) \varepsilon_{y_{i,j}}$$ we see that μ_j is an element of

the unit ball S^M of M(G) for each j and $T\mu_j = x_j$ by (*). Since S^M is weakly compact, there is a $\mu \in S^M$ and there is a subnet $(\mu_{j(i)}) \subseteq (\mu_j)$ such that $\mu_{j(i)} \xrightarrow{\sigma} \mu$. Since T is continuous as a map of $M(G)_\sigma$ into A_σ we have that $T\mu_{j(i)} \xrightarrow{\sigma} T\mu$. $(x_{j(i)}) \subseteq (x_j)$, and $T\mu_{j(i)} = x_{j(i)} \xrightarrow{\sigma} x$, it follows that $T\mu = x$. This shows T maps S^M onto S and hence is a homeomorphism because S^M and S are compact. Hence T maps M(G) onto A.

To show that T is an isometry suppose there is a $\mu \in M(G)$ such that $||T\mu|| < ||\mu||$. Since S^M is mapped onto S and T is one-to-one, $||T^{-1}|| \leq 1$. Thus $||\mu|| = ||T^{-1}T\mu|| \leq ||T\mu|| < ||\mu||$, a contradiction.

Finally in order to show that A is isometric and isomorphic to M(G), we have to show the following:

VIII. For $\mu, \lambda \in M(G)$, $T(\mu*\lambda) = T\mu T\lambda$

First let $\mu, \lambda \in V$ (the linear span of the Dirac measures). Then

$$\mu = \Sigma_{i=1}^{n} a_i \varepsilon_{x_i} \quad \text{and} \quad \lambda = \Sigma_{i=1}^{m} b_i \varepsilon_{y_i}$$ where a_i, b_i are complex numbers. We have

$$\mu*\lambda = \Sigma_{i=1}^{n} \Sigma_{j=1}^{m} a_i b_j \varepsilon_{x_i} * \varepsilon_{y_j} = \Sigma_{i=1}^{n} \Sigma_{j=1}^{m} a_i b_j \varepsilon_{x_i y_j}$$ (by proposition 2 of Chapter I). Thus using (*)

$$T(\mu*\lambda) = \Sigma_{i=1}^{n} \Sigma_{j=1}^{m} a_i b_j x_i y_j = (\Sigma a_i x_i)(\Sigma b_i y_i) = (T\mu)(T\lambda) \ldots\ldots\ldots\ldots (\ddagger)$$

Now let $\mu, \lambda \in M(G)$ then by corollary 1 to proposition 4 there are nets $(\mu_j : j \in J)$, $(\lambda_k : k \in K)$ in V such that $\mu_j \xrightarrow{\sigma} \mu$ and $\lambda_k \xrightarrow{\sigma} \lambda$. Then since multiplication in M(G) is separately weakly continuous, (proposition 2, Chapter II) we have

$$\mu*\lambda = \lim_j (\lim_k \mu_j*\lambda_k).$$

Since T is continuous $M(G)_\sigma \to A_\sigma$ we have

$$T(\mu*\lambda) = \lim_j T(\lim_k \mu_j*\lambda_k) = \lim_j (\lim_k T(\mu_j*\lambda_k))$$

$$= \lim_j (\lim_k T\mu_j T\lambda_k)$$ by the $(\ddagger)$ above. By hypothesis

(2) of the theorem multiplication in A is weakly continuous in each variable separately, thus

$$\lim_{j}(\lim_{k} T\mu_j T\lambda_k) = \lim_{j} T\mu_j T\lambda = T\mu T\lambda$$ which proves the assertion.

Thus A is isometric and isomrophic to M(G). By theorem 2 of Chapter III, G is unique to within isomorphism and homeomorphism. This completes the proof of the first part of the theorem.

For the converse part of the theorem let G be a locally compact group. Then we have in our previous notation $A = M(G)$, $E = C_o(G)$. Clearly M(G) satisfies (1) and (3). By proposition 2 of Chapter II, M(G) satisfies (2). Let Z be the complex numbers of absolute value 1, then ZG^{ε} is the set of extreme points of the unit sphere of M(G) (Chapter 1, §3, theorem 6). Thus theorem 1 of Chapter III shows that M(G) satisfies (4). If G is compact then G^{ε} is $\sigma(M(G),C_o(G))$-compact so that $ZG^{\varepsilon} = S^{\varepsilon}$ is $\sigma(M(G),C_o(G))$-compact, since it is the continuous image of the compact set $Z \times G^{\varepsilon}_{\sigma}$ and therefore (5) is satisfied, because $\sigma(A,E)$ is Hausdorff. If G is not compact, let G^{∞} be the one point compactification of G. If $(x_j : j \in J)$ is a net in G which converges to ∞, then $f(x_j) \to 0$ for each $f \in C_o(G)$ so that $\varepsilon_{x_j} \xrightarrow{\sigma} 0$. Thus the mapping $x \to \varepsilon_x$ has a continuous extension to G^{∞} and this extension is one to one (because $0 \notin G^{\varepsilon}$) and therefore a homeomorphism of G^{∞} onto $(G^{\varepsilon} \cup \{0\})_{\sigma}$. Thus $(G^{\varepsilon} \cup \{0\})_{\sigma}$ is compact, so that $Z(G^{\varepsilon} \cup \{0\})_{\sigma} = S^{\varepsilon} \cup \{0\}$ is compact, and therefore $\sigma(M(G),C_o(G))$-closed because $\sigma(M(G),C_o(G))$ is Hausdorff, and (5) is satisfied.

We now show that (6) & (7) are satisfied. For this define a linear functional p by $p(\mu) = \mu(G) = \int d\mu$. Now

$$(\mu*\lambda)(G) = \int\int d\mu d\lambda \quad \text{(by Chapter I, §5.1)}$$
$$= \int d\mu \int d\lambda \quad \text{by Fubini's theorem}$$
$$= \mu(G)\lambda(G)$$

So that $p(\mu*\lambda) = p(\mu)p(\lambda)$. Note that $G^{\varepsilon} = \{\mu \in S^{\varepsilon} : p(\mu) = 1\}$, which corresponds to G in 7(i). Since $f \in C_o(G) \Rightarrow \bar{f}$ (the complex conjugate of f) is in $C_o(G)$, it follows that for any $f \in C_o(G)$, $\mu \in G^{\varepsilon}$, $\mu(\bar{f}) = \overline{\mu(f)}$ because μ is real. Thus choosing $\bar{f}$ for g, 7(i) is satisfied. 7(ii) is satisfied since $C_o(G)$ is an algebra.

This completes the proof of the theorem.

The next proposition is due to Greenleaf, and has appeared in (10) in a less general form.

Proposition 4. Let E be a Banach space and let N be a $\sigma(E',E)$-closed subspace of the dual E' of E. Let π be the canonical mapping $E' \rightarrow E'/N$ then π maps the unit ball of E' onto the unit ball of E'/N.

Proof: Since N is $\sigma(E',E)$-closed, N is norm closed so that E'/N is a Banach space with the norm of an element $\pi(x)$ given by

$$||\pi(x)|| = \inf\{||x + n|| : n \in N\} \leq ||x||. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Hence $||x|| \leq 1$ implies $||\pi(x)|| \leq 1$, i.e. π maps the unit ball of E' into the unit ball of E'/N. To show that π is onto. let $\pi(x) \in E'/N$ with $||\pi(x)|| \leq 1$. By (1) there are $x_j \in x + N$ such that

$$||x_j|| \leq ||\pi(x)|| + 1/j||x|| \qquad j = 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots$$

the sequence $(x_j : j = 1, 2\ldots\ldots)$ is then norm bounded and therefore contained in a $\sigma(E',E)$ compact subset of E'. Thus there is a $y \in E'$ and a subsequence $(x_{j(i)}) \subseteq (x_j)$ such that $x_{j(i)} \xrightarrow{\sigma} y$. Since N and hence $x + N$ are $\sigma(E',E)$-closed $y \in x + N$ and therefore $\pi(y) = \pi(x)$. Since the norm is $\sigma(E',E)$ lower semicontinuous, we have $||y|| \leq \liminf ||x_{j(i)}|| \leq ||\pi(x)|| \leq 1$. Thus for each element $\pi(x)$ of the unit ball of E'/N there is an element y of the unit ball of E' such that $\pi(y) = \pi(x)$.

The next proposition is also due to Greenleaf (10) in the case that G is a compact group. It should be noted that our proof is new, and somewhat simpler than his.

Proposition 5. Let G be a locally compact group; N a weakly closed two sided ideal in M(G); S^{ε} (resp. S^{π}) the set of extreme points of the unit sphere of M(G)(resp. M(G)/N), and π the canonical mapping $M(G) \to M(G)/N$. Then $\pi(S^{\varepsilon}) = S^{\pi}$.

Proof: We first show that $S^{\pi} \subseteq \pi(S^{\varepsilon})$. Recall that M(G)/N can be identified with the dual of N^{o} the polar of N in $C_o(G)$ (Chapter I, §1.5). If G is not compact, let G^{∞} be the one point compactification of G, and if G is compact put $G^{\infty} = G$. Consider $N^{o} \subseteq C(G^{\infty})$; N^{o} is $\sigma(C_o(G),M(G))$-closed and hence norm closed. Let $\mu \in S^{\pi}$, then by theorem 5 of chapter I, there is a complex number c, $|c| = 1$ and an x in G such that $\mu(f) = c\varepsilon_x(f)$ for f in N^{o}, and this means that $c\varepsilon_x \in \pi^{-1}(\mu)$ by theorem 6 of Chapter I, $c\varepsilon_x \in S^{\varepsilon}$, thus $S^{\pi} \leq \pi(S^{\varepsilon})$.

If $\mu \in S^{\varepsilon}$, then μ is an inner translator (theorem I) so that by corollary 1 to proposition 2, $\pi(\mu)$ is an inner translator because π is norm decreasing. By proposition 3, $\pi(\mu)$ is in S^{π}. Thus $\pi(S^{\varepsilon}) = S^{\pi}$.

Theorem 2. Let A be a Banach algebra which satisfies conditions

(1) to (5) of theorem 1. Then there is a locally compact group G, and a weakly closed two-sided ideal N in M(G) such that A is isometric and isomorphic to M(G)/N. Conversely if N is a weakly closed two-sided ideal in M(G), then M(G)/N satisfies (1) to (5).

Proof: Let S and S^{ε} be as in theorem 1, and take G to be S^{ε}, then G is a locally compact group (see the proof of theorem 1). For f in E, let $\hat{f}$ be thefunction on G given by $\hat{f}(x) = x(f)$. Then $f \to \hat{f}$ is a norm decreasing linear mapping of E into $C_o(G)$ (see the proof of IV in theorem 1). Let T be the adjoint of $f \to \hat{f}$, then we have that T is a norm decreasing and continuous linear mapping of $M(G)_{\sigma}$ onto A_{σ}. The arguments to show that $T(\mu * \lambda) = T\mu T\lambda$ and $T(S^M) = S$ are similar (and easier) than those used in the proofs of VII and VIII of theorem 1. Now let $N = \ker T$, then N is a weakly closed two-sided ideal in M(G). Let π be the canonical mapping $M(G) \to M(G)/N$ and let T_1 be the mapping $M(G)/N \to A$ such that $T = T_1 \circ \pi$. Clearly T_1 is one-one and onto. We now show that T_1 is an isometry. By proposition 4, $\pi(S^M)$ is the unit sphere in M(G)/N and since $T(S^M) = S$ we have $T_1(\pi(S^M)) = S$. i.e. T_1 maps the unit sphere of M(G)/N onto the unit sphere of A. Thus $||T_1|| \leq 1$ and $||T_1^{-1}|| \leq 1$ and this means that T_1 is an isometry (see the calculation used in the proof of proposition 1). This completes the proof of the first assertion.

Now let N be a weakly closed two sided ideal in M(G). We shall show that M(G)/N satisfies (1) to (5) of theorem 1. M(G)/N may be identified with the dual of N^o. Since N^o is $\sigma(C_o(G), M(G))$-closed in $C_o(G)$, N^o is norm closed and therefore a Banach space. Thus (1) is satisfied. To show that (2) is satisfied note that since N^o

is $\sigma(C_o(G),M(G))$-closed and since $N^{oo} = N$, the $\sigma(M(G)/N,N^o)$-topology equals the quotient weak topology on $M(G)/N$ (Chapter 1, §1.5). Thus it suffices to show that for $\dot{\lambda} \in M(G)/N$, the mapping $\dot{\mu} \to \dot{\lambda}\dot{\mu}$ is continuous in the quotient weak topology and this is true because the quotient of a topological algebra is a topological algebra. Let S^{ε} (resp. S^{π}) be the set of extreme points of the unit ball of $M(G)$ (resp. $M(G)/N$). Let $\dot{\mu} \in S^{\pi}$, then by proposition 5 there is a $\mu \in S^{\varepsilon}$ such that $\pi(\mu) = \dot{\mu}$. Since μ is an inner translator, we have by the corollary to proposition 2 that $\pi(\mu)$ is an inner translator hence (3) and (4) are satisfied. To show that $S^{\pi} \cup \{0\}$ is $\sigma(M(G)/N,N^o)$-closed, note that π is weakly continuous, hence since $S^{\varepsilon} \cup \{0\}$ is weakly compact, and since $\pi(S^{\varepsilon}) = S^{\pi}$ (proposition 5), $\pi(S^{\varepsilon} \cup \{0\}) = S^{\pi} \cup \{0\}$ is $\sigma(M(G)/N,N^o)$-compact and hence $\sigma(M(G)/N,N^o)$-closed, so that (5) is satisfied.

CHAPTER V

ABELIAN *-SUBALGEBRAS OF $L^1(G)$ AND REPRESENTATION THEORY

1. Maximal abelian *-subalgebras of $L^1(G)$.

Definitions: Let A be a *-algebra and let B be an abelian *-subalgebra of A. B is said to be a maximal abelian *-subalgebra of A, if for each abelian *-subalgebra B_1 such that $B \subseteq B_1 \subseteq A$ we have $B_1 = B$.

An element x of a *-algebra A is called normal if $xx^* = x^*x$.

Remarks. Note that we do not require a maximal abelian *-subalgebra to be a proper subalgebra. Thus if A is an abelian *-algebra then the only maximal abelian *-subalgebra of A is A itself.

Let A be a *-algebra, then for any $x \in A$, the set of all finite linear combinations of elements of the form x^*x, $(x^*x)^2$, $(x^*x)^3$,.......... is a nonzero abelian *-subalgebra of A, consequently a maximal abelian *-subalgebra is always nonzero. Moreover it is a consequence of Zorn's lemma that any abelian *-subalgebra is contained in a maximal abelian *-subalgebra.

Proposition 1. Let A be a *-algebra and let B be an abelian *-subalgebra of A. Then B is a maximal abelian *-subalgebra of A if and only if for each normal element $y \in A$; $xy = yx$ for all $x \in B$ implies $y \in B$.

Proof: First note that $xy = yx$ for all $x \in B$ implies $yx^* = x^*y$ so that by applying * to each side we have $xy^* = y^*x$. Thus if y commutes with B, so does y^*. If y is normal, then the *-algebra generated by $\{y\} \cup B$ is an abelian *-subalgebra which contains B. Thus if y is not in B, B is not maximal. Conversely if B is not maximal then there is an abelian *-subalgebra $B_1 \subseteq A$ such that $B \subset B_1$. Let $y \in B_1 \setminus B$, then y is normal and y commutes with every element of B. The violates the condition of the proposition.

Let S be a Borel subset of a locally compact group G. Let L(S) be the subset of $L^1(G)$ which consists of functions which vanish almost everywhere outside of S, and D(S) the set of all $x \in G$ such that every measurable neighborhood of x meets S in a set of positive Haar measure (see Chapter II, §3).

If S is an open and closed subset of G, then S is locally compact and each function $f \in K(S)$ can be extended to a function on G by simply putting $f(x) = 0$ for $x \notin S$. In this way we identify K(S) with a subspace of K(G). If μ is a regular measure on G, then its restriction to S is a regular measure on S. Moreover since S is closed, every regular measure on S is obtained in this way ((3) Chapitre III, §2, No. 1 and Chapitre V, §7, No. 2). If S is an open subgroup of G, then S is also closed, consequently we may identify K(S) with a subspace of K(G) and M(S) with a subspace of M(G). In addition the restriction of the Haar measure on G to S is the Haar measure on S.

Thus we may identify $L(S)$ with $L^1(S)$.

Lemma 1. Let G be a unimodular locally compact group and let S be an open abelian subgroup of G. Let $f \in L^p(G)$ $(1 \leq p \leq \infty)$, then a necessary and sufficient condition for f to commute with $L(S)$ is that there exist a subset N of G, having Haar measure zero and such that

$$f(x) = f(y^{-1}xy) \text{ for all } y \in S \setminus N.$$

Proof: Since $K(S)$ is dense in $L(S) = L^1(S)$ we have that f commutes with $L(S)$ if and only if $f*g = g*f$ for all $g \in K(S)$. Since $f*g$ is continuous (Chapter I, §5.4) a necessary and sufficient condition for $f*g = g*f$ is that $f*g(x) = g*f(x)$ for all $x \in G$.

Now $f*g(x) = \int f(xy)g(y^{-1})dm(y)$ and since G is unimodular,

$$\int f(xy)g(y^{-1})dm(y) = \int f(xy^{-1})g(y)dm(y).$$

Thus $f*g(x) = g*f(x)$ gives

$$\int f(xy^{-1})g(y)dm(y) = \int g(y)f(y^{-1}x)dm(y).$$

Then

$$\int f(xy^{-1}) - f(y^{-1}x)g(y)dm(y) = 0 \text{ for all } x \in G, \text{ and any}$$

$g \in K(S)$ if and only if there is a set N of measure zero such that

$$f(xy^{-1}) - f(y^{-1}x) = 0 \text{ for } y \in S \setminus N.$$

Putting $z = xy^{-1}$ we have

$$f(z) = f(y^{-1}zy) \text{ for } y \in S \setminus N.$$

Definitions: If S is a subset of G, put $F(A) = \bigcup\{x^{-1}Ax: x \in D(S)\}$ if $D(S) \neq \emptyset$ and $F(A) = \emptyset$ if $D(S) = \emptyset$, for any subset $A \subseteq G$

Proposition 2. If S and A are Borel subsets, then F(A) is a Haar measurable set.

Proof: D(S) is closed and therefore a Borel set, thus A x D(S) is a Borel set. Let g be the mapping of G x G into itself defined by $g((x,y)) = (y^{-1}xy, y)$, then g is a homeomorphism of G x G onto itself. Let p_1 be the projection $(x,y) \to x$. Observe that $F(A) = p_1 \circ g(A \times D(S))$. Since a homeomorphism takes Borel sets to Borel sets and since p_1 takes Borel sets to Haar measurable sets, we have that F(A) is measurable.

Theorem 1. Let G be a locally compact unimodular group, S a Borel subset of G, and suppose that L(S) is a subalgebra. A necessary and sufficient condition for L(S) to be a maximal abelian *-subalgebra of $L^1(G)$ is that D(S) be an open abelian subgroup with the property that $m(F(A)) = \infty$ for every Borel subset of A satisfying

(i) $m(A \cap S) = 0$ where m is the Haar measure on G.

(ii) $0 < m(A) < \infty$.

Proof: We first show that the condition is necessary. If L(S) is a maximal abelian *-subalgebra then $L(S) \neq \{0\}$. Thus by theorem 3 of Chapter II, D(S) is a closed subgroup of G and L(D(S)) = L(S). Since $L(D(S)) \neq \{0\}$, D(S) contains a set of positive measure and is therefore open (Chapter I, §5.4). Since we can identify L(D(S)) with $L^1(D(S))$ and since $L^1(G)$ is abelian if and only if G is abelian it follows that D(S) is abelian. Now suppose there is a Borel set A satisfying (i) and (ii) but such that $m(F(A)) < \infty$. Define a function f on G by $f(x) = 1$ for $x \in F(A) \cup F(A)^{-1}$ and $f(x) = 0$ otherwise. Since G is unimodular $m(F(A)^{-1}) = m(F(A)) < \infty$, and since F(A) and $F(A)^{-1}$ are measurable subsets, $f \in L^1(G)$. If $x \in F(A) \cup F(A)^{-1}$ then for any $y \in D(S)$, $y^{-1}xy \in F(A) \cup F(A)^{-1}$. If for any $x \in G$, $y^{-1}xy \in F(A) \cup F(A)^{-1}$ for some $y \in D(S)$, then $x = yy^{-1}xyy^{-1} \in F(A) \cup F(A)^{-1}$. This means that $f(x) = f(y^{-1}xy)$ for all $y \in D(S)$ and any $x \in G$. Thus lemma 1 applies to D(S) and f commutes with $L(D(S)) = L(S)$. By condition (ii) above f is not the zero element of $L^1(G)$ because $A \subseteq F(A)$, and by condition (i), $f \notin L(S)$. Moreover $f(x) = f(x^{-1}) = f^*(x)$ so that f is normal. Thus proposition 1 applies and L(S) cannot be maximal. Therefore we have shown the necessity of the condition.

We now show sufficiency. If D(S) is an open abelian subgroup, then $e \in D(S)$ so that lemma 4 of Chapter II applies and we have $L(S) = L(D(S))$. Thus L(S) is an abelian *-subalgebra of $L^1(G)$ since L(D(S)) is an abelian *-subalgebra of $L^1(G)$. To show that L(S) is maximal let $f \in L^1(G)$ and suppose that f commutes with $L(S) = L(D(S))$. We will show that $f \in L(S)$ i.e. $f = 0$ almost everywhere outside S. Let $c = \sup\{|f(x)| : x \in G \setminus S\}$. If $c = 0$ there is nothing to prove. Suppose that $c \neq 0$. Let a be chosen such that $0 < a < c$ and let $B = \{x \in G \setminus S : |f(x)| \geq a\}$, then B is not locally null and therefore contains a compact set A such that $0 < m(A) < \infty$. Clearly $m(A \cap S) = 0$ so that A satisfies (i) and (ii). Since f commutes with L(D(S)), by lemma 1 we have that there is a subset N of S such that $f(x) = f(y^{-1}xy)$ for all $y \in S \setminus N$. Consequently $|f(x)| \geq a$ almost everywhere for $x \in F(A)$. Then $\int_G |f| dm \geq \int_{F(A)} |f| dm \geq a\, m(F(A))$.
By hypothesis $m(F(A)) = \infty$ and we have a contradiction. Therefore $c = 0$. Then $f(x) = 0$ almost everywhere for $x \notin S$. i.e. $f \in L(S)$.

By proposition 1, L(S) is maximal.

Corollary. Let G,S be as in theorem 1 and suppose that L(S) is a maximal abelian *-subalgebra of $L^1(G)$. If $f \in L^p(G)(1 \leq p < \infty)$ and $f(x) = f(y^{-1}xy)$ almost everywhere for $y \in S$ and $x \in G$, then $f(x) = 0$ almost everywhere for $x \notin S$.

Proof: Let $c = \sup\{|f(x)|^p : x \in G \setminus S\}$ and suppose $c \neq 0$. Let $B = \{x \in G \setminus S : |f(x)|^p \geq a\}$ where $0 < a < c$. Then there is a compact set $A \subseteq B$ satisfying conditions (i) and (ii) of the theorem. Thus since L(S) is a maximal abelian *-subalgebra, $m(F(A)) = \infty$. Consequently $\int |f|^p dm \geq \int_{F(A)} |f|^p dm \geq a\, m(F(A)) = \infty$. Thus we must have $c = 0$.

Examples. We now give examples of groups G having an abelian subgroup S such that L(S) is a maximal abelian *-subalgebra of $L^1(G)$. For this first note that if G is discrete, and S is an abelian subgroup of G then by theorem 1, a necessary and sufficient condition for L(S) to be a maximal abelian *-subalgebra of $L^1(G)$ is that for each $y \notin S$, the set $\{x^{-1}yx : x \in S\}$ be infinite.

Let $G = \{(a,b) : a,b \text{ real}, a > 0\}$ and give G the discrete topology. Define $(a,b)(a',b') = (aa', ab' + b)$, then $(a,b)^{-1} = (1/a, -b/a)$ and G is a group. Moreover since G is discrete, G is unimodular. Let S be the subset of G of all elements of the form (1,c), then S is an abelian subgroup. For $(a,b) \in G$, $a \neq 1$ we have

$$(1,c)^{-1}(a,b)(1,c) = (1, -c)(a, ac + b)$$

$= (a, (a - 1)c + b)$

Consequently $\{(1,c)^{-1}(a,b)(1,c) : (1,c) \in S\}$ is infinite and L(S) is a maximal abelian *-subalgebra of $L^1(G)$.

A second example is the free group on two generators. If we take S to be the subgroup generated by one generator then it is easily seen that $\{xyx^{-1}: x \in S\}$ is infinite for any $y \notin S$. By the above L(S) is maximal in $L^1(G)$.

2. Maximal abelian *-subalgebras of $\mathfrak{R}$.

Throughout this section G denotes a locally compact unimodular group. Let $(L, L^2(G))$ be the left regular representation of G, i.e. for each $x \in G$, L_x is the operator on $L^2(G)$ defined by $L_x f(y) = f(x^{-1}y)$ for each f in $L^2(G)$. For each $x \in G$, let R_x be the operator on $L^2(G)$ defined by $R_x f = f_x$ for $f \in L^2(G)$, and for $\mu \in M(G)$ let R_μ be the operator defined by $R_\mu f = f * \mu$. If $h \in L^1(G)$, then $hm \in M(G)$ and put $R_h = R_{hm}$. Let $\mathfrak{R}$ be the W*-algebra generated by $\{R_\mu : \mu \in M(G)\}$, then it is known that $\mathfrak{R}$ is the commutant of $\{L_x : x \in G\}$, and that $\mathfrak{R}$ is generated by $\{R_x : x \in G\}$, (cf. Chapter I, §7.2).

If H is a Hilbert space and B(H) is the algebra of all bounded operators on H, recall that the w-topology on B(H) is the weak operator topology; i.e. the topology defined by the semi-norms $A \to |(Af,g)|$; $f,g \in L^2(G)$, where (f,g) is the scalar product of $f,g \in L^2(G)$.

<u>Proposition 3</u>. The mapping $\mu \to R_\mu$ is a linear continuous

mapping of $M(G)_\sigma$ into $\mathcal{R}_w$.

Proof: Clearly $\mu \to R_\mu$ is linear. To show continuity let $f,g \in L^2(G)$ and $\mu \in M(G)$. Then since G is unimodular the convolution $f^{*}*g$ exists and is an element of $C_o(G)$ (Chapter I, §5), to prove the proposition it clearly suffices to show that $(R_\mu f,g) = \mu(\overline{f^{*}*g})$. Using the unimodularity of G we have

$$\begin{aligned}(R_\mu f,g) &= \int(\int f(xy^{-1})d\mu(y))\overline{g(x)}dm(x)\\ &= \int\int f(xy^{-1})\overline{g(x)}dm(x)d\mu(y) \qquad \text{by Fubini's theorem}\\ &= \int\int \overline{f^{*}(yx^{-1})g(x)}dm(x)d\mu(y)\\ &= \int \overline{(f^{*}*g)(y)}d\mu(y)\\ &= \mu(\overline{f^{*}*g}).\end{aligned}$$

Proposition 4. Let S be an open subgroup of G, and $\mathcal{R}(S)$ be the W*-algebra generated by $\{R_x : x \in S\}$. Then the w-closure of $\{R_f : f \in L(S)\cap L^2(G)\}$ equals $\mathcal{R}(S)$.

Proof: If $f \in L(S)\cap L^2(G)$ then $\text{Supp}(fm) \subseteq S$ since S is an open subgroup and is therefore closed. Thus by proposition 4 of Chapter II, we have $\{fm : f \in L(S)\cap L^2(G)\} \subseteq Cl_\sigma\{\lambda \in V : \text{Supp}(\lambda) \subseteq S\}$ (where V is the linear span of the Dirac measures). By proposition 3, the mapping $\mu \to R_\mu$ is a continuous map of $M(G)_\sigma$ into $\mathcal{R}_w$, so that

$$\{R_\mu : \mu \in Cl_\sigma\{\lambda \in V : \text{Supp}(\lambda) \subseteq S\}\} \subseteq \mathcal{R}(S)$$

Thus $\{R_f : f \in L(S)\cap L^2(G)\} \subseteq \mathcal{R}(S)$ and since $\mathcal{R}(S)$ is w-closed, we have $Cl_w\{R_f : f \in L(S)\cap L^2(G)\} \subseteq \mathcal{R}(S)$. For the reverse inclusion note that $\{f : f \in L(S)\cap L^2(G)\}$ is norm dense in $L(S)$ (Chapter I, §4.1) so that $Cl_\sigma\{fm : f \in L(S)\cap L^2(G)\} = Cl_\sigma\{fm : f \in L(S)\}$. By theorem 1 of Chapter II, $\{\varepsilon_x : x \in S\} \subseteq Cl_\sigma\{fm : f \in L(S)\} = Cl_\sigma\{fm : f \in L(S)\cap L^2(G)\}$.

($S = D(S)$ because S is open). Thus the continuity of $\mu \to R_\mu$ as a map of $M(G)_\sigma$ into $\mathcal{R}_w$ implies that $\{R_x : x \in S\} \subseteq Cl_w \{R_f : f \in L(S) \cap L^2(G)\}$. Thus $Cl_w \{R_f : f \in L(S) \cap L^2(G)\}$ contains a set of generators for $\mathcal{R}(S)$ and consequently $\mathcal{R}(S) = Cl_w \{R_f : f \in L(S) \cap L^2(G)\}$ since this latter set is a W*-algebra, because $L(S)$ is an algebra.

<u>Proposition 5</u>. Let S be an open subgroup of G and let $L^2(S)$ be the subset of $L^2(G)$ consisting of functions which vanish almost everywhere outside S. For any $f \in L^1(G)$, $f \in L(S)$ if and only if $R_f(L^2(S)) \subseteq L^2(S)$.

<u>Proof</u>: If $f \in L(S)$ and $g \in L(S) \cap L^2(G)$ then $R_f g \in L(S) \cap L^2(G) \subseteq L^2(S)$ since $L(S)$ is an algebra. $L(S) \cap L^2(G)$ is dense in $L^2(S)$, thus by the continuity of R_f we have $R_f(L^2(S)) \subseteq L^2(S)$ since $L^2(S)$ is closed.

Now suppose $L^2(S)*f \subseteq L^2(S)$. Let $\varepsilon > 0$ be given, then by corollary 2 to lemma 3 of Chapter II there is an $h \in L(S) \cap L^2(G)$ such that $||h*f - f|| \leq \varepsilon$ since $e \in D(S) = S$. Thus $f \in L(S)$ since $L(S)$ is a norm closed algebra.

<u>Theorem 2</u>. Let S be an open abelian subgroup of G. Then $\mathcal{R}(S)$ is a maximal abelian *-subalgebra of $\mathcal{R}$ if and only if $L(S)$ is a maximal abelian *-subalgebra of $L^1(G)$.

<u>Proof</u>: Suppose that $\mathcal{R}(S)$ is a maximal abelian *-subalgebra of $\mathcal{R}$. Let $f \in L^1(G)$ and suppose that f is normal and commutes with $L(S)$. Since the w-closure of $\{R_g : g \in L(S)\}$ is $\mathcal{R}(S)$ (proposition 4), we have that R_f commutes with $\mathcal{R}(S)$. Since $\mathcal{R}(S)$ is maximal, and since $R_f^* = R_{f^*}$, R_f is normal, and therefore $R_f \in \mathcal{R}(S)$ (proposition 1). Since

$L^2(S)$ is invariant under $\mathfrak{R}(S)$ because S is a subgroup, we have by proposition 5, $f \in L(S)$. By proposition 1, $L(S)$ is maximal.

Suppose that $L(S)$ is maximal. To show that $\mathfrak{R}(S)$ is maximal it is sufficient in view of proposition 1 to show that $\mathfrak{R} \cap \mathfrak{R}(S)' \subseteq \mathfrak{R}(S)$ (where $\mathfrak{R}(S)'$ is the commutant of $\mathfrak{R}(S)$). Let $T \in \mathfrak{R} \cap \mathfrak{R}(S)'$, then for any $f \in L(S) \cap L^2(G)$ and any $g \in K(G) \subseteq L^2(G)$ we have

$$TR_f g = T(g*f) = TL_g f = L_g Tf$$
$$= g*Tf$$

Let R_{Tf} be the bounded linear extension to $L^2(G)$ of the operator defined on $K(G)$ by $R_{Tf} g = g*Tf$. The above shows that $R_{Tf} = TR_f$. Thus $R_{Tf} \in \mathfrak{R} \cap \mathfrak{R}(S)'$ since T and R_f are elements of $\mathfrak{R} \cap \mathfrak{R}(S)'$ and $\mathfrak{R} \cap \mathfrak{R}(S)'$ is an algebra. Let $g \in K(G)$ and $h \in L(S)$ then

$$R_{Tf} R_h g = R_h R_{Tf} g$$

which gives $g*(h*Tf) = g*(Tf*h)$.

i.e. $R_{h*Tf - Tf*h} g = 0$ for $g \in K(G)$

therefore $h*Tf = Tf*h$.

By lemma 1 we have that there is a set N of measure zero such that $Tf(x) = Tf(y^{-1}xy)$ for all $y \in S \setminus N$ and any $x \in G$. By corollary to theorem 1 we therefore have $Tf(x) = 0$ almost everywhere for $x \notin S$. It follows that $TR_f = R_{Tf} \in \mathfrak{R}(S)$ whenever $R_f \in \mathfrak{R}(S)$. Let $g, f \in L^2(G)$ and $\varepsilon > 0$ be given. By corollary 2 to lemma 3 of Chapter II, there is an $h \in L(S) \cap L^2(G)$ such that

$$||g*h - g||_2 \leq \varepsilon / ||T|| \; ||f||$$

By the above $TR_h \in \mathfrak{R}(S)$ since $R_h \in \mathfrak{R}(S)$,

and $\quad |((T - TR_h)g, f)| \leq ||T|| \; ||g - g*h||_2 \; ||f|| \leq \varepsilon.$

i.e. $T \in \mathfrak{R}(S)$, since $\mathfrak{R}(S)$ is w-closed. Thus $\mathfrak{R} \cap \mathfrak{R}(S)' \subseteq \mathfrak{R}(S)$ so that $\mathcal{R}(S)$ is maximal.

3. Decomposition of the Left Regular Representation.

Throughout this section G is a locally compact unimodular group with a countable basis for the open sets and S is an open (and therefore closed) abelian subgroup of G. Let $(L, L^2(G))$, $\mathcal{R}(S)$ and $\mathcal{R}$ be as in §2. The commutant of $\{L_x : x \in G\}$ is $\mathcal{R}$ (Chapter I, § 7.2) thus by applying Mautner's theorem (Chapter I, § 8 theorem 10) taking $\mathcal{R}(S)$ as $\mathfrak{Z}$ we obtain a compact subset I of the real line, a positive regular measure λ on I, a λ-measurable family $\{(L^t, H(t)) : t \in I\}$ of representations of G, and an isometry V of $\int H(t)d\lambda(t)$ onto $L^2(G)$ such that

1° $V^{-1}L_xV = \int L_x^t d\lambda(t)$

2° $A \to VAV^{-1}$ maps the algebra of diagonalizable operators onto $\mathcal{R}(S)$

3° if $\mathcal{R}(S)$ is a maximal abelian *-subalgebra of $\mathcal{R}$, then there is a set $N \subseteq I$ of λ-measure zero such that for $t \in I \setminus N$, L^t is an irreducible representation.

The purpose of this section is to identify the representations L^t. Note that there are groups G having open abelian subgroups such that the decomposition of the left regular representation G is not unique (cf. (16) Chapter III, and (25)).

Let $\hat{S}$ be the character group of S. For each $t \in \hat{S}$ we define a linear functional p_t on $L^1(G)$ by

$$p_t(f) = \int_S f(x)\overline{t(x)}dm(x) \quad \text{.................................}(1)$$

<u>Lemma 2</u>. Let μ be the Haar measure on $\hat{S}$. Then

(i) for each $f \in L^1(G)$, $t \to p_t(f)$ is a continuous function on $\hat{S}$.

(ii) for $f, g \in L^1(G) \cap L^2(G)$ and any $x \in G$, we have

$(f,g) = \int_{\hat{S}} p_t(g^* * f) d\mu(t)$

<u>Proof</u>: (i) follows by noting $t \to p_t(f)$ is the Fourier transform of f restricted to S.

To prove (ii) first note that if $f \in L^1(G) \cap L^2(G)$ then $f^* * f$ is a positive definite function which is in $L^1(G)$, consequently the Fourier inversion formula ((13) §47 theorem 5) applies and we have for $x \in S$

$$f^* * f(x) = \int_{\hat{S}} p_t(f^* * f) t(x) d\mu(t)$$

In particular putting $x = e$

$$f^* * f(e) = \int_{\hat{S}} p_t(f^* * f) d\mu(t).$$

But
$$f^* * f(e) = \int_G f^*(y^{-1}) f(y) dm(y)$$
$$= \int_G \overline{f(y)} f(y) dm(y)$$
$$= ||f||_2^2.$$

Consequently $||f||_2^2 = \int_{\hat{S}} p_t(f^* * f) d\mu(t)$..........................(2)

Now given $f, g \in L^1(G) \cap L^2(G)$

$$4(f,g) = ||f+g||_2^2 - ||f-g||_2^2 + i||f+ig||_2^2 - i||f-ig||_2^2$$

Applying (2) we have that

$$(f,g) = \int_{\hat{S}} p_t(g^* * f) d\mu(t).$$

Lemma 3. Let $h \in L(S)$, then $(R_h f, g) = \int_S p_t(g^* * f)\,\hat{h}(t)d\mu(t)$, where $\hat{h}$ is the Fourier transform of h.

Proof: By lemma 2 we have

$(f*h, g) = \int_{\hat{S}} p_t(g^* * f * h)d\mu(t)$ thus it is sufficient to show that $p_t(g^* * f * h) = p_t(g^* * f * h)\hat{h}(t)$. Now

$$p_t(g^* * f)\hat{h}(t) = \int_S (g^* * f)(x)\overline{t(x)}dm(x) \int_S h(y)\overline{t(y)}dm(y)$$

$$= \int_S (g^* * f)(xy^{-1})\overline{t(xy^{-1})}dm(x) \int_S h(y)\overline{t(y)}dm(y)$$

$$= \int_S \int_S (g^* * f)(xy^{-1})h(y)\overline{t(x)}dm(x)dm(y) \quad \text{by Fubini's theorem}$$

$$= \int_S(\int_G (g^* * f)(xy^{-1})h(y)dm(y))\overline{t(x)}dm(x)$$

$$= \int_S (g^* * f * h)(x)\overline{t(x)}dm(x)$$

$$= p_t(g^* * f * h).$$

Choose an element from each left coset of S, and let G/S be the collection of elements so obtained. Then for any $f \in L^1(G)$ we have by (14) §33

$$\int_G f dm = \Sigma_{x \in G/S} \int_S f(xy)dm(y) \text{..........................(1)}$$

Lemma 4. For $f \in L^1(G)$, we have $p_t(f^* * f) = \Sigma_{x \in G/S} |p_t({}_x f)|^2$

Proof: $f^* * f(x) = \int_G f^*(y)f(y^{-1}x)dm(y)$

$$= \int \overline{f(y^{-1})}f(y^{-1}x)dm(y)$$

$$= \int \overline{f(y)}f(yx)dm(y)$$

$$= \int \overline{f(y)} f_x(y)dm(y)$$

$$= \Sigma_{z \in G/S} \int_S \overline{f(zy)} f_x(zy)dm(y) \qquad \text{by (3).}$$

Hence $p_t(f^* * f) = \int_S \Sigma_{z \in G/S} \int_S \overline{f(zy)} f_x(zy)dm(y)\overline{t(x)}dm(x)$

$$= \Sigma_{z \in G/S} \int_S \int_S \overline{f(zy)} f_x(zy)\overline{t(x)}dm(y)dm(x)$$

$$= \Sigma_{z \in G/S} \int_S \int_S \overline{f(zy)} f_x(zy)\overline{t(yx)}t(y)dm(y)dm(x)$$

$$= \Sigma_{z \in G/S} \int_S {}_z\overline{f(y)}t(y)dm(y) \int_S {}_zf(yx)\overline{t(yx)}dm(yx)$$

$$= \Sigma_{z \in G/S} p_t({}_zf)\overline{p_t({}_zf)}$$

$$= \Sigma_{z \in G/S} |p_t({}_zf)|^2$$

Corollary. p_t is a positive linear functional on $L^1(G)$ and $|p_t(f)| \leq ||f||$, for $f \in L^1(G)$.

Proof: By lemma 4, $p_t(f^* * f) \geq 0$. The inequality $|p_t(f)| \leq ||f||$, is immediate from formula (1).

Let $(f_1, f_2, \ldots\ldots)$ be a countable subset of $L^1(G) \cap L^2(G)$ which is an orthonormal basis of $L^2(G)$. Such a set always exists since G has a countable basis for the open sets and K(G) is dense in $L^2(G)$. Since S is open and closed in G, S also has a countable basis, so there is a countable subset $(g_1, g_2, \ldots\ldots\ldots)$ in $L^1(S) \cap L^2(S)$ which is dense in $L^2(S)$. Let $M = \{R_{g_i} f_j : i,j = 1,2.......\}$. We retain these notations throughout.

Lemma 5. M is total in $L^2(G)$.

Proof: Since $(f_1, f_2, \ldots\ldots)$ is dense in $L^2(G)$ to prove the theorem it suffices to show that for given f_k and $\varepsilon > 0$, there is a $g_j \in L^2(S)$ such that $||R_{g_j} f_k - f_k||_2 \leq \varepsilon$. By corollary 2 to lemma 3 of Chapter II, there is a $\lambda \in L(S) \cap L^2(S)$ such that

$$||R_\lambda f_k - f_k|| \leq \varepsilon/2$$

Since $(g_1, g_2, \ldots\ldots)$ is dense in $L^2(S)$ there is a g_j such that $||g_j - \lambda||_2 \leq \varepsilon/2\ ||f_k||_1$. Then

$$\begin{aligned} ||f_k - R_{g_j} f_k||_2 &\leq ||R_{g_j} f_k - R_\lambda f_k||_2 + ||R_\lambda f_k - f_k||_2 \\ &\leq ||f_k * (g_j - \lambda)||_2 + \varepsilon/2 \\ &\leq ||f_k||_1 ||g_j - \lambda||_2 + \varepsilon/2 \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

This proves the lemma.

For each $t \in \hat{S}$ let $N^t = \{f \in L^1(G) \cap L^2(G) : p_t(f^* * f) = 0\}$, and let $H'(t)$ be the quotient space $L^1(G) \cap L^2(G)/N^t$. For $f(t), g(t) \in H'(t)$ we put

$$(f(t), g(t)) = p_t(g^* * f) \qquad g \in g(t) \qquad f \in f(t) \ldots\ldots\ldots\ldots (4)$$

It follows from the corollary to lemma 4 and 21.17 of (12) that (4) defines an inner product in $H'(t)$. It follows from the invariance of the Haar measure on G, that $L_x N^t \subseteq N^t$ for each $x \in G$. Thus we may define an operator L'^t_x on $H'(t)$ by $L'^t_x f(t) = (L_x f)(t)$. If we now complete $H'(t)$ relative to the inner product we obtain a Hilbert space $\hat{H}(t)$ and a representation $\hat{L}^t$ of G on $\hat{H}(t)$. Note that for each $t \in \hat{S}$,

$\{f_i(t) : i = 1, 2, \ldots\}$ is a total subset of $\hat{H}(t)$.

Proposition 7. Let μ be the Haar measure on $\hat{S}$. Then $\{(\hat{L}^t, \hat{H}(t)) \; ; \; t \in \hat{S}\}$ is a μ-measurable family of representations.

Proof: Let $\{f_i' \; ; \; i = 1, 2, \ldots\}$ be the subset of $\prod \hat{H}(t)$ such that for each i, $f_i'(t) = f_i(t)$. Then $\{f_i'(t) \; ; \; i = 1, 2, \ldots\}$ is total in $\hat{H}(t)$ for each t and the function $t \to (\hat{L}_x^t f_i'(t), f_j'(t)) = p_t(f_j^* * L_x f_i)$ is μ-measurable since it is continuous by lemma 2.

In view of proposition 7 we may construct the direct integral of the representations $\{(\hat{L}^t, \hat{H}(t)) : t \in \hat{S}\}$ using the measure μ (see Chapter I, §8). Let $\hat{H} = \int_{\hat{S}} \hat{H}(t) d\mu(t)$.

For any Hilbert space H, B(H) denotes the algebra of all bounded operators on H.

Theorem 3. There is an isometry W of $\hat{H}$ onto $L^2(G)$ such that

(i) for each $x \in G$, $W^{-1} L_x W = \int_S \hat{L}_x^t d\mu(t)$

(ii) the mapping $A \to WAW^{-1}$ of $B(\hat{H})$ onto $B(L^2(G))$ maps the algebra of diagonalizable operators onto $\mathcal{R}(S)$.

Proof: Let $T_{\hat{g}_k}$ be the operator on $\hat{H}$ defined by

$$(T_{\hat{g}_k} f')(t) = \hat{g}_k(t) f'(t) \quad \text{for any } f' \in \hat{H}.$$

Let K' be the subset of H consisting of elements of the form $T_{\hat{g}_k} f_i'$, $i, k = 1, 2, \ldots$ where f_i' is as in proposition 7. We first show that the linear span K of K' is dense in H, for this it suffices to show that the orthogonal complement $K^\perp$ of K is zero. Let $h' \in K^\perp$

then

$$(T_{\hat{g}_k} f_i', h') = 0 \quad \text{for all } i,k = 1, 2, \ldots\ldots .$$

Thus for any fixed i,

$$\int \hat{g}_k(t)(f_i'(t), h'(t))d\mu(t) = 0, \qquad k = 1, 2, \ldots\ldots .$$

Since $(g_1, g_2, \ldots\ldots)$ is dense in $L^2(S)$, we have by the Plancherel theorem ((13) §47, theorem 6) that $(\hat{g}_1, \hat{g}_2, \ldots\ldots)$ is dense in $L^2(\hat{S})$. This implies that there is a set N_i of μ-measure zero such that

$$(f_i'(t), h'(t)) = 0 \quad \text{for } t \notin N_i.$$

Putting $N = \bigcup_{i=1}^{\infty} N_i$ we have

$$(f_i'(t), h'(t)) = 0, \quad i = 1, 2, \ldots\ldots \quad \text{and } t \notin N.$$

Since $\{f_i'(t) : i = 1, 2, \ldots.\}$ is dense in $\hat{H}(t)$ for each t, we have $h'(t) = 0$ almost everywhere. Thus $h' = 0$. This proves that K is dense in $\hat{H}$.

We now define a mapping W' of K into $L^2(G)$. For $h' \in K$ we have

$$h' = \Sigma_{i=1}^{p} c_i T_{\hat{g}_{n_i}} f_i' \ , \ c_i \text{ complex numbers, and we put}$$

$$W'h' = \Sigma_{i=1}^{p} c_i R_{g_{n_i}} f_i \ .$$

Then

$$||\Sigma_{i=1}^{p} c_i R_{g_{n_i}} f_i||^2 = \Sigma_{i,j=1}^{p} c_i \bar{c}_j (R^*_{g_{n_i}} R_{g_{n_i}} f_i, f_j)$$

$$= \Sigma_{i,j} c_i \bar{c}_j (R_{g^*_{n_i} * g_{n_i}} f_i, f_j) \ , \text{ since } g \to R_g \text{ is a representation of } L(S)$$

$$= \Sigma_{i,j} c_i \bar{c}_j \int \widehat{(g^*_{n_j} * g_{n_i})}(t)(f_i(t), f_j(t)d\mu(t) \quad \text{by lemma 3}$$

$$= \Sigma_{i,j} c_i \bar{c}_j \int \overline{\hat{g}_{n_j}(t)}\, \hat{g}_{n_i}(t)\, (f_i(t), f_j(t))d\mu(t) \ , \text{ since the Fourier transform is multiplicative}$$

$$= \Sigma_{i,j} c_i \bar{c}_j \int \ (\hat{g}_{n_i}(t) f_i'(t), \hat{g}_{n_j}(t) f_j(t))d\mu(t)$$

$$= \int (\Sigma_i \, c_i T_{\hat{g}_{n_i}} f_i'(t), \Sigma_j \, c_j T_{\hat{g}_{n_j}} f_j'(t)) d\mu(t)$$

$$= \int || \Sigma_{i=1}^{p} \, c_i T_{\hat{g}_{n_i}} f_i'(t) ||^2 d\mu(t)$$

$$= || \Sigma_{i=1}^{p} \, c_i T_{\hat{g}_{n_i}} f_i' ||^2$$

$$= ||h'||^2.$$

Consequently, $||W'h'||^2 = ||h'||^2$ which shows that W' is well defined and an isometry of K into $L^2(G)$. Thus W' has an extension W which is an isometry of $\hat{H}$ into $L^2(G)$. Since the range of W contains the set M which is total in $L^2(G)$ by lemma 5, and since $W(\hat{H})$ is complete and hence closed in $L^2(G)$, we have that W is onto.

We now show that (i) of the theorem holds. Since $(f_1, f_2, \ldots.)$ is total in $L^2(G)$, $(f_1', f_2', \ldots\ldots)$ is total in H, since $W^{-1}f_i = f_i'$ and W^{-1} is an isometry. Thus to show (i) it suffices to show that $(W^{-1}L_x Wf_i', f_j') = (\int \hat{L}_x^t d\mu(t) f_i', f_j')$ because $W^{-1}L_x W$ and $\int \hat{L}_x^t d\mu(t)$ are bounded operators and $\{f_i\}$, $\{f_j'\}$ are respectively total in the spaces. Now

$$(\int \hat{L}_x^t d\mu(t) f_i', f_j') = \int (\hat{L}_x^t f_i'(t), f_j'(t)) d\mu(t) \quad \text{by the definition of } \int L_x^t d\mu(t)$$

$$= \int ((L_x f_i)(t), f_j(t)) d\mu(t) \quad \text{by the definition of } \hat{L}_x^t \text{ and } f_i'$$

$$= \int p_t(f_j^* * L_x f_i) d\mu(t)$$

$$= (L_x f_i, f_j) \quad \text{by lemma 2 (ii)}$$

$$= (W^{-1}L_x f_i, W^{-1}f_j) \text{ , since } W^{-1} \text{ is an isometry}$$

$$= (W^{-1}L_x Wf_i', f_j') \text{ , since } Wf_j' = f_j$$

We now show (ii). Let $g \in L(S)$. Then $T_{\hat{g}}$ is a diagonalizable operator and

$$(T_{\hat{g}}f_i', f_j') = \int (\hat{g}(t)f_i'(t), f_j'(t))d\mu(t)$$

$$= \int \hat{g}(t)(f_i(t), f_j(t)d\mu(t)$$

$$= (R_g f_i, f_j) \qquad \text{by lemma 3}$$

$$= (W^{-1}R_g\, Wf_i', f_j')$$

so that $T_{\hat{g}} = W^{-1}R_gW$. The mapping $A \rightarrow WAW^{-1}$ is a continuous mapping of $B(H)_{\bar{w}}$ into $B(L^2(G))_w$. Therefore $Cl_w\{T_g : g \in L(S)\}$ is mapped onto $Cl_w\{R_g : \in L(S)\}$. By proposition 4, $Cl_w\{R_g : g \in L(S)\} = \mathcal{R}(S)$. Therefore to complete the proof is suffices to show that the algebra of diagonalizable operators is the w-closure of $\{T_{\hat{g}} : g \in L(S)\}$. Since the algebra of diagonalizable operators is $\{T_h : h \in L^\infty(\hat{S})\}$ it suffices to show the following:

1° the mapping $h \rightarrow T_h$ of $L^\infty(\hat{S})$ into $B(\hat{H})$ is continuous when one gives $L^\infty(\hat{S})$ the $\sigma(L^\infty(\hat{S}), L^1(\hat{S}))$ topology and $B(\hat{H})$ the w-topology.

2° $\{\hat{g} : g \in L(S)\}$ is $\sigma(L^\infty(\hat{S}), L^1(\hat{S}))$ dense in $L^\infty(\hat{S})$.

1° follows from the formula $(T_hf, g) = \int h(t)(f(t), g(t))d\mu(t)$ by observing that the function $t \rightarrow (f(t), g(t))$ is in $L^1(\hat{S})$. 2° follows by noting that $\{\hat{g} : g \in L(S)\}$ is norm dense in $C_o(\hat{S})$ ((19) Chapter I, 1.2.4) and that $C_o(\hat{S})$ is $\sigma(L^\infty(\hat{S}), L^1(\hat{S}))$ dense in $L^\infty(\hat{S})$. This latter fact follows by noting that the polar of $C_o(\hat{S})$ in $L^1(\hat{S})$ is the zero element. This completes the proof.

<u>Theorem 4</u>. Let I, λ, L^t, N, V and H(t) be as in the beginning

of this section. There is a set $N_1 \subset I$ of λ-measure zero, a set $N_2 \subset \hat{S}$ of μ-measure zero, a one-one mapping θ of $I \setminus N_1$ onto $\hat{S} \setminus N_2$ and for each $t \in I \setminus N_1$ there is an isometry $T(t)$ of $H(t)$ onto $H(\theta(t))$ such that $\hat{L}_x^{\theta(t)} T(t) = T(t) L_x^t$ for all $x \in G$.

Proof: Let $U = W^{-1}V$, then in view of theorem 3 and (2°) of paragraph I, § 3, U is an isometry of $\int_I H(t)d\lambda(t)$ onto $\hat{H} = \int_S \hat{H}(t)d\mu(t)$, and the mapping $A \to UAU^{-1}$ maps the algebra of diagonalizable operators of $\int_I H(t)d\lambda(t)$ onto the algebra of diagonalizable operators of $\hat{H}$, and $\int L_x^t d\lambda(t)$ onto $\int \hat{L}_x^t d\mu(t)$ for each $x \in G$. Thus by theorem 11 of Chapter I, § 8 the theorem follows.

To complete the identification of the representations L^t, we now show that the representations $\hat{L}^t$ are "equivalent" to representations induced by characters on S. (The definition of induced representations is given in Chapter I, § 7.3).

Proposition 7. Let $(U^t, H(t))$ be the representation of G induced by a character t of S. Then there is an isometry V^t of $\hat{H}(t)$ onto $H(t)$ such that $V^t \hat{L}_x^t = U_x^t V^t$ for all $x \in G$.

Proof: We define a mapping V^t of $H(t)$ into $H(t)$ by

$$(V^t f(t))(x) = p_t({}_x f) \quad x \in G \quad , f(t) \in L^1(G) \cap L^2(G)/N^t$$

To show that $V^t f(t) \in H(t)$ we must show

1° $(V^t f(t))(xy) = t(y) V^t f(t)(x) \quad x \in G, \ y \in S$

2° $||V^t f(t)|| < \infty$, where $||V^t f(t)||^2 = \Sigma_{x \in G/S} |V^t f(t)(x)|^2$.

To show 1°, let $x \in G$, $y \in S$, then

$$V^t f(t)(xy) = p_t({}_{xy} f) = \int_S f(xyz) t(z) dm(z)$$

$$= \int_S f(xyz)t(y)\overline{t(yz)}dm(z)$$

$$= t(y) \int_S f(xyz)\overline{t(yz)}dm(z)$$

$$= t(y)p_t({}_xf)$$

$$= t(y)(V^t f(t))(x)$$

To show 2° we shall show $||V^t f(t)|| = ||f(t)||$. This follows immediately from lemma 4, since $||f(t)||^2 = p_t(f^* * f)$ and $||V^t f(t)||^2 = \Sigma_{x \in G/S} |p_t({}_xf)|^2$. Clearly V^t is linear, so that $||V^t f(t)|| = ||f(t)||$ implies that V^t is well defined.

Let $K = L^1(G) \cap L^2(G)/N^t$, we now show that $V^t(K)$ is dense in $H(t)$. For this it suffices to show $(V^t(K))^\perp = 0$. Let $g \in (V^t(K))^\perp$, then $\Sigma_{x \in G/S} V^t f(t)(x)g(x) = 0$ for all $f(t) \in K$. To show that $g = 0$ it suffices to show that for given $x_o \in G/S$ there is an $f \in L^1(G) \cap L^2(G)$ such that $y \neq x_o$. Let V be a compact neighborhood of $e \in G$ such that $V \subseteq S$ (recall that S is open). Let h be the characteristic function of x_oV. Let t' be the function on G given by $t'(y) = t(y)$ for $y \in S$, and $t'(y) = 0$ otherwise. Put $f(y) = t'(x_o^{-1}y)h(y)$. Then

$$p_t(x_of) = \int_S t(y)h(x_oy)\overline{t(y)}dm(y) = \int_S h(x_oy)dm(y) \neq 0.$$

If $y \in G/S$ and $y \neq x_o$, then $yS \cap x_oS = \emptyset$ so that $yS \cap x_oV = \emptyset$ since $h(yS) = 0$,

$$p_t({}_yf) = \int_S t'(yx_o^{-1}z)h(yz)\overline{t(z)}dm(z) = 0.$$

Thus $g = 0$ and $V^t(K)$ is dense in $H(t)$. Since $K$is dense in $\hat{H}(t)$ we have that V^t can be extended to an isometry (which we again denote by V^t) of $\hat{H}(t)$ onto $H(t)$. Now to complete the proof of the theorem we show that for $f(t) \in K$ we have $V^t\hat{L}_x^t f(t) = U_x^t V^t f(t)$. For $y \in G$, we have $V^t\hat{L}_x^t f(t)(y) = V^t(L_xf)(t)(y) = p_t({}_y(L_xf)) = p_t({}_{x^{-1}y}f)$

$$= V^t \hat{L}_x^t f(t)(x^{-1}y) = U_x^t V^t f(t)(y).$$

and this proves the proposition.

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