A STUDY OF REPULSIVE CORE POTENTIALS
IN THE
NEUTRON-PROTON SYSTEM AT LOW ENERGIES

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SCOPE AND CONTENTS: The variational parameters, r_o,P,Q, in the expansion $k \cot\delta = -\frac{1}{a} + r_o k^2 - P r_o^3 k^4 + Q r_o^5 k^6$, link theory and experiment. Using "experimental" values of a and r_o, P is calculated in both the ^1S and ^3S neutron-proton states, for the square, gaussian, exponential, and Yukawa wells with repulsive cores. Comparison of the theoretical values of P with values derived from experiment indicates that the potential shape is at least as singular as the Yukawa $(\frac{1}{r})$, and that if a core exists, it is small, of the order of .1 or $.2 \times 10^{-13}$cm. Q is calculated in the ^1S state for the square and Yukawa wells with repulsive cores, and found to be so small as to be undefined by the present experimental data.

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I INTRODUCTION

Insight into the nature of the nuclear interaction potential of the neutron-proton system may be derived by analysis of neutron-proton scattering data and the properties of the deuteron, in particular the deutron binding energy, photodisintegration cross sections, and neutron capture cross sections.

The present investigation is based on the expansion of a quantity closely allied to the phase shift in a power series in the kinetic energy of the system. The phase shift, usually denoted by δ , is the difference in phase between the asymptotic form of the radial wave function in the absence of a scattering potential and the actual wave function of the state describing the interacting neutron and proton. In principle since a power series has infinitely many terms, definition of the expansion requires the specification of infinitely many parameters, the coefficients of every power of the energy. In fact, however, the present experimental data is not sufficiently accurate to determine more than the first three coefficients of the expansion. Having determined, by at least squares procedure, the best experimental values of these parameters, the goal is to find by trial theoretical nuclear potentials which yield these values.

Specifically we shall subject to trial "repulsive core" potentials, characterized by a short range repulsive force interior to an attractive well, defined by

$$V = \infty \quad \text{for } r < r \text{ core} \qquad (1.1)$$

$$V = -V_0 f\left(\frac{r}{r_b}\right) \quad \text{for } r > r \text{ core}$$

It will be realized that for V negative the force between the neutron and proton is attractive, and that for V positive the force between the nucleous is repulsive. It should be mentioned that the repulsive core potential was selected as our subject of study because it has been shown (J2) that such a potential adequately accounts for the qualitative features of the neutron-proton scattering data at intermediate and high energies, and the question arises as to whether the repulsive core gives an acceptable fit to the low energy data. Although calculations based on the neutron-proton scattering cross section for neutrons of energy 4.75 Mev have recently been carried out which support the validity of the repulsive core hypothesis (H2), this question has not yet been resolved; the pursuit of a partial answer is our present concern.

By restricting our discussion to low energy neutron-proton systems we reduce the complexity of the problem in that S states, only, enter the considerations and tensor forces may be neglected. The singlet S state and the deuteron ground state, a combination of the triplet S and triplet D, will, then, be the two basic states dealt with. It is to be expected that the potentials describing the interaction in these two states will have different values of the parameters.

The phase shift expansion for the neutron-proton system in the S state may be written, (B10),

$$k\cot\delta = -\frac{1}{a} + \frac{1}{2} r_o k^2 - P r_o^3 k^4 + Q r_o^5 k^6 \qquad (1.2)$$

where k is the wave number in the centre of gravity system.

Since the quantities a, r_o, P, Q, in (1.2) are fundamental in this method of analysis their significance needs elaboration. The first term, a, is Fermi's scattering length at zero energy; geometrically it is the radial intercept of the asymptotic wave function. It follows that negative values of, a, characterize potentials which have no bound states, and positive values potentials which have bound states.

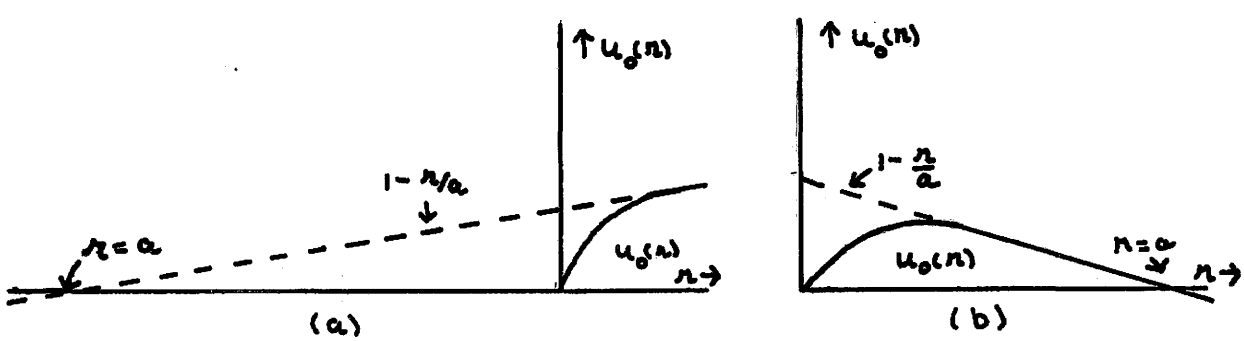

Fig. 1. Typical zero energy wave functions $u_o(r)$ for wells with zero core. In (a) the scattering length a is negative, characteristic of an unbound state; in (b) the scattering length is positive, characteristic of a bound state.

The parameter r_o is called the "effective range" since it has the dimensions of length and is of the same magnitude as the width of the potential. The pioneers of this work (B10) designated the potentials by two parameters, fixing the width and depth of the well. Obviously it is possible to choose two parameters for any reasonable shape which yield the desired values of the scattering length and effective range. For this reason the approximation for $k \cot \delta$ given by the first two terms of (1.2) is called the shape independent approximation. Now the most resent data is good enough to define the third term within meaningful limits. Therefore it is to be hoped that information about the detailed shape of the potential can be gained from the parameter P. The value of Q turns out to be so small that the fourth term is negligible.

Expansion (1.2) has been rigorously established using a variational method (B10); just as rigorous but a less complicated derivation was given by Bethe (B5). He considers the neutron-proton system with kinetic energy, E_1 in the laboratory system and potential V(r), and writes the Schrödinger equation in the form,

$$\frac{d^2u_1}{dr^2} + k_1^2u_1 - \frac{2\mu}{\hbar^2}V(r)u_1 = 0 \qquad (1.3)$$

where ru_1 is the radial wave function, μ the reduced mass, $\frac{M_p M_n}{M_p + M_n}$, and k the wave number of the system equal to

$$\left(\frac{2}{\hbar^2}\mu \frac{M_p}{M_n + M_p} E_1\right)^{\frac{1}{2}}$$

For a second energy E_2 it follows that

$$\frac{d^2u_2}{dr^2} + k_2^2 u_2 - \frac{2\mu}{\hbar^2} V(r) u_2 = 0 \tag{1.4}$$

Manipulation of (1.3) and (1.4) gives

$$u_2u_1' - u_1u_2'\Big]_o^R = (k_2^2 - k_1^2)\int_o^R u_1u_2 \, dr \tag{1.5}$$

R is an arbitrary integration limit. Subscript 1 refers to the first energy, subscript 2 to the second. The asymptotic representations of u_1 and u_2 normalized to unity at the origin are,

$$\psi_1 = \frac{\sin(k_1r + \delta_1)}{\sin\delta_1} \qquad \psi_2 = \frac{\sin(k_2r + \delta_2)}{\sin\delta_2} \tag{1.6}$$

Analogous to (1.5) we may write

$$\psi_2\psi_1' - \psi_1\psi_2'\Big]_o^R = (k_2^2 - k_1^2)\int_o^R \psi_1\psi_2 \, dr \tag{1.7}$$

Since each u equals its asymptotic form ψ for r greater than the arbitrary value R, and since $u(o) = 0$, by substracting (1.5) from (1.7) we obtain

$$\psi_1\psi_2' - \psi_2\psi_1'\Big]_{r=o} = (k_2^2 - k_1^2)\int_o^\infty (\psi_1\psi_2 - u_1u_2) \, dr \tag{1.8}$$

From (1.6) and (1.8) we deduce that

$$k_2 \cot\delta_2 - k_1 \cot\delta_1 = (k_2^2 - k_1^2)\int_o^\infty (\psi_1\psi_2 - u_1u_2) \, dr \tag{1.9}$$

Equation (1.9) enables us to express explicitly the parameters of (1.2) in terms of quantities intimately related to the wave function. For $k_1 = 0$ we obtain immediately from (1.2) the Fermi scattering length

$$k_1 \cot\delta_1 = -\frac{1}{a} = -\alpha \tag{1.10}$$

For any energy E, (1.9) may be rewritten

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2} k^2 \rho \,(0E) \qquad (1.11)$$

where

$$\frac{1}{2} \rho\,(0,E) = \int_0^{\infty} (\psi_0 \psi - u_0 u)\, dr \qquad (1.12)$$

and u_0 is the wave function at zero energy with asymptotic form ψ_0. For two different energies E_1 and E_2 it is useful to define

$$\frac{1}{2} \rho\,(E_1 E_2) = \int_0^{\infty} (\psi_1 \psi_2 - u_1 u_2)\, dr \qquad (1.13)$$

It is to be noticed that in the range of energies considered the quantity (1.13) is not sensitive to changes in energy. This is because the potential is much larger than the kinetic energy inside the range of the nuclear force, the only region where ψ and u are appreciably different. The effective range ,r_o, is defined to be

$$r_o = 2\int_0^{\infty} (\psi_o^2 - u_o^2)\, dr = \rho\,(o,o) \qquad (1.14)$$

We have observed that r_o, $\rho\,(0,E)$, $\rho(E_1 E_2)$ are all very nearly equal.

In order to evaluate P we expand the wave function u and its asymptotic form ψ in terms of the energy

$$u = u_o + k^2 v_1 + k^4 v_2 + \cdots \qquad (1.15)$$

$$\psi = \psi_o + k^2 \; X_1 + k^4 \; X_2 + \cdots \qquad (1.16)$$

The functions v_1, v_2, etc. satisfy differential equations obtained by expanding the Schrödinger equation in a power series in k^2. For example v_1 satisfies

$$\frac{d^2v_1}{dr^2} - \frac{2\mu}{\hbar^2} V(r)v_1 = -u_o \qquad (1.17)$$

The functions ψ_o, X_1, X_2, etc. are the asymptotic forms of u_o, v_1 v_2, etc. and satisfy the boundary conditions $\psi_o = 1$ at $r = o$ $X_1 = X_2 = X_3 = o$ at $r = r_c$. Hence

$$\psi_o = 1 - \frac{r}{a} = 1 - \alpha r \qquad (1.18)$$

$$X_1 = \frac{1}{2} r(r_o - r) + \frac{1}{6}\frac{r^3}{a} \qquad (1.19)$$

From the basic expansion (1.2) the difference, $k_2 \cot \delta_2 - k_1 \cot \delta_1$ to terms of order k^4 is given by $\frac{1}{2} r_o(k_2^2 - k_1^2) - Pr_o^3 (k_2^4 - k_1^4)$.

Therefore, using (1.9) we may write

$$\rho(E_1E_2) = r_o - 2Pr_o^3(k_2^2 + k_1^2) \qquad (1.20)$$

We obtain the desired integral for P by writing

$$\frac{1}{2}\rho(EO) - \frac{1}{2}\rho(o,o) = \int_o^\infty [\psi_o(\psi - \psi_o) - u_o(u - u_o)]\, dr = k^2 \int_o^\infty (\psi_o X_1 - u_o v_1) dr \qquad (1.21)$$

From (1.21) and (1.20) with $E_2 = o$, we obtain

$$r_o^3\, P = \int_o^\infty (u_o v_1 - \psi_o X_1)\, dr \qquad (1.22)$$

By similar reasoning the coefficients of the higher terms in (1.2) may be derived. In particular the fourth coefficient Qr_o^5 is given by

$$Qr_o^5 = \int_o^\infty (\psi_o X_2 - u_o v_2) dr = \int_o^\infty (X_1^2 - v_1^2) dr, \quad (J.1) \qquad (1.23)$$

The expansion (1.2) is applicable to the negative energy state. The gound state wave function is given by

$$\psi_g = e^{-\gamma r} \qquad (1.24)$$

Where γ^{-1}, the "deuteron radius" is related to the binding energy of the deuteron ϵ by $\epsilon = \frac{\hbar^2}{2}\gamma^2$. Taking state 2 in (1.8) to be the ground state we obtain

$$\alpha = \gamma - \tfrac{1}{2}\gamma^2 \rho(0,-\epsilon) \tag{1.25}$$

Equation (1.25) provides a good value for $\rho(o, -\epsilon)$ since α and γ are well known. Hereafter the subscripts s and t will be used to distinguish the parameters of the unbound singlet state, and the bound triplet state respectively.

In our notation the trial potentials are specified by three parameters V_o, r_b, and r_c. V_o and r_b determine the depth and compactness of the well. The relation of the parameters V_o and r_b to the well depth parameter s and intrinsic range b used by Blatt and Jackson (B10) should be clarified. In (B10) b and s are defined for potentials V(r) with zero cores. Usually the scattering length is finite. However the well depth may be adjusted to give an infinite scattering length, that is to give a wave function having an asymptotic representation parallel to the r axis. The adjusted potential is denoted by $V^R(r)$ and the corresponding wave function by U_o^R. The following two relations then define s and b

$$V(r) = sV^R(r) \tag{1.26}$$

$$b = 2\int_0^\infty (1 - u_o^R(r)^2)\, dr \tag{1.27}$$

From (1.27) it is obvious that in the case of the uncored potential b equals the effective range r_o when s equals one. We prove in section IV that the wave function of the cored potential at r is equal to the wave function of the uncored potential at $(r-r_c)$ that is $u_c(r) = u(r-r_c)$. Hence $b = 2\int_0^{\infty}[1-u_c^R(r+r_c)^2]\,dr$ which is not the effective range. In the notation of section IV, $b = r_o'$ for $s = 1$. For the four conventional shapes without cores the relations between b, r_b , s and V_o are as follows,

Square $$r_b = b$$ $$V_o = 102.276\,\frac{s}{b^2}$$

Gaussian $$r_b^2 = \frac{b^2}{2.0604}$$ $$V_o = 229.208\,\frac{s}{b^2} \qquad (1.28)$$

Exponential $$r_b = \frac{b}{3.5412}$$ $$V_o = 751.541\,\frac{s}{b^2}$$

Yukawa $$r_b = \frac{b}{2.1196}$$ $$V_o = 313.404\,\frac{s}{b^2}$$

The application of equation (1.2) in utilizing the neutron-proton scattering data to gain knowledge of the potential shape is straight forward. The variational parameters a, r_o, P, provide a link between the experimental cross sections and the theoretical potentials in that $k\cot\delta$ can be evaluated at various energies from the experimental cross sections, providing a set of experimental values which can be compared to values of the variational parameters calculated for arbitrarily selected potentials. The cross section is

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta \quad (S1) \qquad (1.29)$$

In the case of the photoelectric effect it is shown by Bethe and Longmaire (B6) that the dependence of the cross section on the variational parameters is contained in the factor $\frac{1}{1-\rho_t(-\epsilon, -\epsilon)}$ which arises from the normalization of the ground state wave function and also that the photomagnetic cross section involves the effective range quantities $\rho_t(-\epsilon, -\epsilon)$ and $\rho_s(E,E)$. It has already been remarked that the available experimental data is adequate to determine only the first three coefficients of expansion (1.2). Potentials which lead to the same values for these coefficients are equivalent fits although the coefficients of the higher terms may be different.

It is well to realize the limitations of the theoretical possibilities of finding the interaction potential from the phase shift by the method described. Although the potential defines a unique phase shift, it has not been satisfactorily established that the converse is true. In fact Bargmann (B1a) showed, for the case of central forces, that it was possible to choose two different potentials and obtain the same phase shift. In fact he has derived families of phase equivalent potentials (B1b). Nevertheless Levinson, (L1), (L2), in a mathematically rigorous treatment, proved that the phase shift defines the potential uniquely in the S state provided V(r) is of a constant sign. If $V(r) \geq 0$ and continuous, we must also have $\int_0^\infty rV(r)\,dr < \infty$ and if $V(r) \leq 0$ and continuous, $\int_0^\infty r\,|V(r)|\,dr + \int_0^\infty r^2\,|V(r)|\,dr < \infty$. This does not therefore prove that repulsive core potentials are uniquely defined.

In sections II to IV, P is evaluated for the four conventional shapes at representative values of the possible values of r_c for both the singlet and triplet states, with parameters having the values in units of 10^{-13}cm. $a_s = -23.68$ $r_{os} = 2.6$; $a_t = 5.28$ $r_{ot} = 1.56$. In section V a plot of Q against r_c is found for the square and Yukawa shapes in the singlet state with parameters $a_s = -23.68$ and $r_{os} = 2.6$. Section VI constitutes a determination of the values of the variational parameters r_o,P, by a least squares analysis of the data and contains conclusions resulting from comparison of these with the shape dependent values.

II SINGLET SQUARE WELLS CONSISTENT WITH $a_s = -23.68 \times 10^{-13}$ cm., $r_{os} = 2.6 \times 10^{-13}$ cm. AND THE SHAPE DEPENDENT PARAMETER FOR A SELECTED WELL

The square well potentials V(r), defined by V_o, r_b, and r_c are of the form,

$$
\begin{aligned}
V &= \infty \qquad && r < r_c \\
V &= -V_o \qquad && r_c < r < r_b \\
V &= o \qquad && r > r_b
\end{aligned}
\tag{2.1}
$$

We proceed to derive sets of parameters V_o, r_b, r_c consistent with the chosen values of a_s and r_{os}.

To find r_{os} it is necessary to know u_o. From equation (1.3) we see that u_o satisfies

$$
\begin{aligned}
u_o'' + D^2 u_o &= o \qquad && r_c < r < r_b \\
u_o'' &= o \qquad && r > r_b
\end{aligned}
\tag{2.2}
$$

Where $D^2 = \frac{2\mu}{\hbar^2} V_o$

Consequently

$$
\begin{aligned}
u_o &= o \qquad && r < r_c \\
u_o &= A \sin (Dr + \epsilon\ ) \qquad && r_c < r < r_b \\
u_o &= \psi_o = 1 - \frac{r}{a_s} \qquad && r > r_b
\end{aligned}
\tag{2.3}
$$

Since u is everywhere continuous and has a continuous derivative except at $r = r_c$, we have

$$1 - \frac{r_b}{a_s} = A \sin (Dr_b + \epsilon\ ) \tag{2.4a}$$

$$-\frac{1}{a_s} = AD \cos (Dr_b + \epsilon\ ) \tag{2.4b}$$

$$0 = A \sin (Dr_c + \epsilon\ ) \tag{2.4c}$$

These equations yield

$$A^2 = \left(1 - \frac{r_b}{a_s}\right)^2 + \frac{a_s}{D^2}^2 \tag{2.4d}$$

$$Dr_c + \epsilon \ = n\pi$$

Now r_{os} can be written

$$r_{os} = 2\int_0^{r_b} \left(1 - \frac{r}{a_s}\right)^2 dr - 2\int_{r_c}^{r_b} A^2 \sin{}^2 (Dr + \epsilon\ )\, dr$$

$$= \frac{2}{3} a_s (1 - y^3) - \left(y^2 + \frac{1}{(a_s D)^2}\right)\left(r_b - r_c - \frac{\sin 2\theta}{2D}\right) \tag{2.5}$$

Where

$$Dr_b + \epsilon \ = \theta + n\pi \tag{2.6}$$

$$1 - \frac{r_b}{a_s} = y \tag{2.7}$$

From (2.4a), (2.4d), and (2.7) we deduce that

$$y^2 = \left(y^2 + \frac{1}{(a_s D)^2}\right) \sin{}^2\theta \quad \text{or} \quad y = \frac{|\tan\theta|}{|a_s|\, D} \tag{2.8}$$

Also

$$r_b - r_c = \frac{\theta}{D} \tag{2.9}$$

Substitution of (2.8) and (2.9) in (2.5) gives

$$a_s^2 D^3 = \left(\frac{2}{3} |\tan{}^3\theta| - \theta \sec^2\theta + \tan\theta\right) \div \left(r_o - \frac{2}{3} a_s\right) = \frac{F(\theta)}{r_o - \frac{2}{3} a_s} \tag{2.10}$$

The value of the angle θ in (2.10) is restricted. Because the wave function does not vanish between r_c and r_b, and [illegible] Dr_c [illegible], we [illegible] that Dr_b [illegible] .

because $Dr_c + \epsilon = n\pi$, we see that $Dr_b + \epsilon = (n+1)\pi$. Of course, $r_b > r_c$, therefore $0 < \theta < \pi$. More explicitly it can be seen by using (2.7) and (2.8) that the inequality $r_b \geq \frac{\theta}{D}$ requires that

$$D \leq |a_s|^{-1}(|\tan\theta| - \theta) = L \tag{2.11}$$

Equations (2.10), (2.8), (2.7), (2.9), were applied to calculate sets of parameters D, r_b, r_c, listed in table 1, corresponding to different values of θ satisfying the inequality (2.11) with $r_{os} = 2.6 \times 10^{-13}$ cm. and $a_s = 23.68 \times 10^{-13}$ cm.

TABLE I VALUES OBTAINED FOR THE SQUARE WELL PARAMETERS D, r_b, and r_c, COMPATIBLE WITH $r_{os} = 2.6 \times 10^{-13}$ cm. and $a_s = -23.68 \times 10^{-13}$ cm.

θ	$r_c(10^{-13}$cm.)	$r_b(10^{-13}$cm)	$D(10^{13}$cm$^{-1})$	$L\ (10^{13}$cm$^{-1})$
1.51	.04516	2.4466	.62879	.62998
1.53	.43691	2.0433	.95242	.96997
1.54	.63195	1.8428	1.2719	1.3057
1.55	.82743	1.6439	1.8985	1.9649
1.57	1.2200	1.2512	50.366	52.956

The variational parameter P is defined in (1.22) by $r_o^3 P = \int_0^\infty (u_o v_1 - \psi_o X_1)dr$. We recall that $X_1 = \frac{1}{2}r(r_o - r) + \frac{1}{6}\frac{r^3}{a_s}$

For the square well the equation (1.17) for v_1 is

$$v_1'' + D^2 v_1 = -u_o \tag{2.12}$$

Therefore

$$v_1 = 0 \qquad r_c > r$$
$$v_1 = B\sin(Dr + \delta) + \frac{A}{2D} r\cos(Dr + \epsilon) \qquad r_c < r < r_b \tag{2.13}$$
$$v_1 = X_1 \qquad r > r_b$$

The continuity conditions on v_1 and v_1' yield the following equations for δ and B

$$\tan(Dr_b + \delta) = \frac{\left(r_b(r_o - r_b) + \frac{r_b^3}{3a_s} - \frac{A}{D} r_b\cos\theta\right)D}{r_o - 2r_b + \frac{r_b^2}{a_s} + Ar_b\sin\theta - \frac{A}{D}\cos\theta} \tag{2.14}$$

$$B = -\frac{A}{2D} r_c \cos(\delta - \epsilon) \tag{2.15}$$

Substituting the functions u_o, v, ψ_o, X_1 into equation (1.22) we obtain for P_s

$$r_{os}^3 P_s = \int_{rc}^{rb}\left[A\sin(Dr + \epsilon)\left(B\sin(Dr + \delta) + \frac{A}{2D} r\cos(Dr + \epsilon)\right)\right]dr$$
$$- \int_0^{rb}\left(1 - \frac{r}{a_s}\right)\left[\frac{1}{2}r(r_o - r) + \frac{1}{6}\frac{r^3}{a_s}\right] dr = I_1 + I_2 - I_3 \tag{2.16}$$

Where I_1, I_2, I_3 are defined by

$$\frac{4D}{AB} I_1 = 4D\int_{r_c}^{r_b}\sin(Dr + \epsilon)\sin(Dr + \delta)dr = 2\theta\cos(\epsilon + \delta) - \sin(2\theta - \delta - \epsilon)$$
$$+ \sin(\delta - \epsilon) \tag{2.17}$$

$$\frac{16D^3}{A^2} I_2 = 8D\int_{r_c}^{r_b} r \sin(Dr+\epsilon)\cos(Dr+\epsilon)\,dr = 4Dr_b\sin^2\theta - 2\theta \sin 2\theta \quad (2.18)$$

$$I_3 = \int_0^{r_b}\psi_o X_1\,dr = \frac{r_o r_b}{4}^2 - \frac{1}{6}\left(1+\frac{r_o}{a_s}\right)r_b^3 + \frac{1}{6}\frac{r_b^4}{a_s} - \frac{r_b^5}{30a_s}^2 \quad (2.19)$$

The constants A, B, ϵ, δ, and the integrals I_1, I_2, I_3, essential in the calculation of P for $\theta = 1.55$, are listed together with the value obtained in table II.

TABLE II P_s AND QUANTITIES ON WHICH ITS VALUE DEPENDS FOR A SELECTED θ OF 1.55

A	B	ϵ	δ	I_1	I_2	I_3	P
1.0696	.78585	-1.5708	1.8721	.25766	.09842	1.0452	-.039210

III CALCULATIONS OF THE VARIATIONAL PARAMETERS r_{os} AND P_s FOR AN EXPONENTIAL WELL WITH A CORE OF $.4575 \times 10^{-13}$ cm.

The exponential potential is of the form

$$V = \infty \qquad r \leq r_c$$

$$V = -V_o e^{\frac{-r-r_c}{b}} \qquad r > r_c \tag{3.1}$$

The calculation of r_{os} involves solving the differential equation

$$\frac{d^2u_o}{dr^2} - \frac{2\mu}{\hbar^2} V(r)u_o = 0 \tag{3.2}$$

with the boundary conditions $u_o(r_c) = 0$ and $u_o(r) = \psi_o$ for $r > r_a = \lambda r_o$ where λ is an arbitrarily selected number. The solution requires the application of a numerical method; Milne's method (M3) has been chosen. By means of the substitutions

$$x^2 = e^{\frac{-r}{r_b}} \tag{3.3}$$

$$v = \frac{2\mu V_o}{\hbar^2} r_b^2 \, e^{\frac{r_c}{r_b}} \tag{3.4}$$

$$U = \sqrt{x}\, u_o \tag{3.5}$$

equation (3.2) can be written in terms of the dimensionless quantity x as

$$\frac{d^2 U}{dx^2} + \left(\frac{1}{4x^2} + 4v\right)U = 0 \tag{3.6}$$

The boundary conditions become

$$U(x_c)=0 \tag{3.7}$$

$$U(x)= \left(1-\frac{2r_b}{a_s}\log x\right)\sqrt{x} \text{ for } x< e^{-\frac{\lambda}{2}\frac{r_o}{r_b}} \tag{3.8}$$

We fit a_s by using as starting values in the integration of (3.6) the asymptotic values given by (3.8) which depend only on the scattering length. According to the substitution (3.3) integrating in with respect to r is equivalent to integrating out with respect to x, that is from smaller to larger values of x. The substitution is advantageous because $\frac{dr}{dx}= \frac{-2r_b}{x}$ which means that for a fixed increment, h, in x the corresponding change in r decreases as x increases, so that the substitution provides an automatic refining of the interval of integration in r inside the effective range, the region in which the wave function changes most rapidly. Five approximate starting values were obtained from (3.8) and were used in (3.16) to calculate approximate values of $\frac{d^2U}{dx^2}$. These values of the second derivative were used in Milne's formulae, (appendix I) to derive values of the first derivative. Milne's formulae were used again with the known values of $\frac{dU}{dx}$ to derive better starting values of U at the five points. The process was repeated until the values of U at some stage were the same as those of the previous stage. Having obtained the starting values the value of the general U is predicted by

$$U_{n+1} = 2U_{n-1} - U_{n-3} + 4h^2\left(U''_{n-1} + \frac{1}{3}\delta^2 U''_{n-1}\right) + \frac{16}{241}h^6 U^{(6)} \quad (3.9)$$

This predicted value is corrected by

$$U_{n+1} = 2U_n - U_{n-1} + h^2\left(U''_n + \frac{\delta^2 U''_n}{12}\right) - \frac{h^6 U^{(6)}}{240} \quad (3.10)$$

where $\delta^2 U''_n$ is defined by

$$\delta^2 U''_n = U''_{n+1} - 2U''_n + U''_{n-1} . \quad (3.11)$$

The error of 3.10 is about $\frac{1}{17}$ the difference between the predicted and the corrected value; therefore, this difference provides a check on the accuracy.

With $V_o = 295.2$ Me.v., $r_c = .4575 \times 10^{-13}$ cm., $r_s = .4400 \times 10^{-13}$ cm, chosen because this set of parameters was known to give reasonable values of a_s and r_o (B7), equations (3.6) and (3.8) become

$$\frac{d^2U}{dx^2} + \left(\frac{1}{4x^2} + 15.583\right)U = 0 \quad (3.12)$$

$$U(x) = (1 - .037162 \ln x)\sqrt{x} \quad (3.13)$$

The starting values are recorded in table III.

TABLE III STARTING VALUES USED IN THE INTEGRATION OF $\frac{d^2U}{dx^2} + \left(\frac{1}{4x^2} + 15.583\right) = 0$ with $U(x) = (1 - .037162 \ln x)\sqrt{x}$ for $x < .02$

x	.02	.03	.04	.05	.06
U (x)	.16198	.19563	.2233	.24705	.26789

$U(x)$ was evaluated at intervals, h, equal to .01 from $x = .07$ to $x = .16$ and at intervals, h, equal to .04 from $x = .16$ to .60. $U(x)$ was found to vanish at $x = x_c = .5962$ or at $r = r_c = .4553 \times 10^{-13}$ cm. For $r = r_a = 3.443 \times 10^{-13}$ cm., u_o equals its asymptotic form. The asymptotic integral $\int_o^{r_a} \psi_o^2(r)\,dr$ equals 3.9673×10^{-13}. The integral of $u_o{}^2$ from the origin to the asymptotic value of r equals $-2r_b \int_{x_c}^{x_a} \frac{U^2(x)}{x^2}\,dx$

We obtain for this integral, by numerical integration using Gregory's formula and Simpson's formula (appendix I), 2.6090 and 2.691 respectively. Finally we obtain for r_{os}, $r_{os} = 2.553 \times 10^{-13}$ cm.

To form the integrand $u_o v_1 - \psi_o X_1$ required in the calculation of P_s equation (1.17) must be solved numerically. Making the substitution

$$V = \frac{\sqrt{x}\, v_1}{r_b{}^2} \tag{3.14}$$

and using (3.3) and (3.4) equation (1.17) reduces to

$$\frac{d^2V}{dx^2} + \left(\frac{1}{4x^2} + 4v\right) V = \frac{-4\ U}{x^2} \tag{3.15}$$

$$V = \frac{\sqrt{x}}{r_b^2}\ X_1 = \sqrt{x}\left(2\ln x \ln x_{os} - 2\ln^2 x - \frac{4}{3}\frac{r_b}{a_s}\ln^3 x\right) \quad r > r_a \tag{3.16}$$

Equation (3.15) was solved by the method used in obtaining the solution of (3.6). V was found to vanish at $x = x_c = .5972$ giving $r_c = .4536 \times 10^{-13}$ cm. We observe that the two values of r_c obtained from (3.6) and (3.15) agree to within 0.5%. Recalling the definition we evaluate P_s by writing

$$r_{os}^3 P_s = \int_0^R (u_o v_1 - \psi_o X_1)\,dr + \int_R^{r_a} (u_o v_1 - \psi_o X_1)\,dr \qquad (3.17)$$

The integral for P_s has been separated into two parts o to R, and R to r_a, to gain accuracy since both products $u_o v$, and $\psi_o X_1$, in the integrand, are large for $r > R$, although the integral of their difference can be accurately evaluated using numerical integration formulas. The calculation of P_s now requires the calculation of three integrals,

$$I_1 = \int_{x_R}^{x_c} \frac{UV}{x^2}\,dx, \quad I_2 = \int_0^R \psi_o X_1\,dr, \quad I_3 = \int_{x_a}^{x_R} \left(\frac{UV}{x^2} - \frac{\psi_o X_1}{r_b^2 x} \right) dx.$$

The second can be integrated explicitly, the other two numerically using Weddle's and Simpson's formulae; their values are tabulated with the final result in table IV.

TABLE IV $$r_{os}^3 P_s = 2r_b^3 \int_{x_R}^{x_c} \frac{UV}{x^2}\,dx - \int_0^R \psi_o X_1\,dr + 2r_b^3 \int_{x_a}^{x_R} \left(\frac{UV}{x^2} - \frac{\psi_o X_1}{r_b^2 x} \right) dx$$

$$r_{os}^3 P_s = 2r_b^3 (I_1 + I_3) - I_2$$

r_{os}	r_b	R	r_c	I_1	I_2	I_3	P_s
2.553	.4400	2.223	.4550	4.3634	1.3456	.03974	-.02571

The unit of length is 10^{-13} cm.

IV THE SHAPE DEPENDENT PARAMETER, P, FOR SINGLET AND TRIPLET STATES WITH $r_{os}=2.6$, $a_s=-23.68$, $r_{ot}=1.56$, $a_t=5.28$ IN UNITS OF 10^{-13} cm., AS A FUNCTION OF THE CORE RADIUS FOR THE SQUARE, GAUSSIAN, EXPONENTIAL, YUKAWA, WELL SHAPES.

If P is known for a given shape of potential without a core it is possible to find P for a potential of the same shape with a core. Let V(r) be any potential without a core. Define the "cored" potential by

$$V_c(r) = \infty \qquad r < r_c$$
$$V_c(r) = V(r-r_c) \qquad r > r_c \tag{4.1}$$

Let $\psi(r)=\sin(kr+\delta)$ be the asymptotic form of the solution of

$$\frac{d^2u(r)}{dr^2} + (k^2 - V(r))u_{(r)} = 0 \tag{4.2}$$

with $u(o)= 0$

Make a change of variable $r' = r+r_c$. Then, of course, $u(r) = u(r'-r_c)$. In the variable r' equation (4.2) are

$$\frac{d^2u(r')}{dr'^2} + (k^2-V'(r'))u(r') = 0 \tag{4.3}$$

$$u(r_c) = 0$$

where $V'(r') = V(r'-r_c)$, that is V' is the potential V moved out a distance r_c. Also, since $u(r_c) = 0$, V' can be assumed infinite for $r' < r_c$. Now the wave function of the cored potential satisfies

$$\frac{d^2u_c}{dr^2} + (k^2 - V_c(r))u_c = 0$$

$$u_c(r_c) = 0 \tag{4.4}$$

But $V_c(r) = V'(r)$. Hence equation (4.4) is identical with (4.3) except that the variable is called r rather than r'. Consequently

$$u_c(r) = u(r-r_c) \tag{4.5}$$

and

$$\psi_c(r) = \sin(kr + \delta_c) = \psi(r-r_c) = \sin(kr-kr_c+\delta) \tag{4.6}$$

Therefore
$$\delta_c = \delta - kr_c \tag{4.7}$$

In the following analysis we shall use unprimed letters $\delta, \alpha, r_o, P, Q$, to refer to quantities determined by the cored potential and primed letters $\delta', \alpha', r_o', P', Q'$, to refer to the potential without a core.

Using relation (4.7) and expansion (1.2) expressions for α', r_o' can be found in terms of α, r_o and r_c; also P can be expressed as a function of α', r_o', and P'. Blatt and Jackson (B10) give expansions for P' in terms of ($\alpha' r_o'$) for the four shapes. Therefore we are able to find P at various values of r_c for each of the four shapes.

We have, to terms of order k^4,

$$k \cot\delta' = -\alpha' + \tfrac{1}{2}k^2 r_o' - P'k^4 r_o'^3$$

$$= k\cot(\delta + r_c) = \frac{k\cot\delta \; k\cot kr_c - k^2}{k\cot\delta + k\cot kr_c} \tag{4.8}$$

Also

$$k\cot\delta = -\alpha + \tfrac{1}{2}k^2 r_o - Pk^4 r_o^3$$

$$k\cot kr_c = \frac{1}{r_c}\left(1 - \frac{k^2 r_c^2}{3} - \frac{k^4 r_c^4}{45}\right) \tag{4.9}$$

Substitution of (4.9) into (4.8) and equating the coefficients of successive powers of k^2 gives

$$\alpha' = \frac{\alpha}{1-\alpha r_c} \tag{4.10a}$$

$$(1-x)^2 r_o' = r_o - \frac{2}{3}(x^2-3x+3)r_c \quad \text{where } x = \alpha r_c \tag{4.10b}$$

$$(1-x)^3 P' r_o'^3 = P r_o^3 (1-x) - \frac{r_c^3}{45}(x^3-6x^2+15x-15) - \frac{r_c r_o}{6}(3-x) + \frac{r_c r_o^2}{4} \tag{4.10c}$$

In an analogous way we obtain from the equation

$$k \cot\delta = k \cot(\delta' - kr_c) \tag{4.11}$$

the relations

$$\alpha = \frac{\alpha'}{1+\alpha' r_c} \tag{4.12a}$$

$$(1+x')^2 r_o = r_o' + \frac{2}{3}(x'^2+3x+3)r_c \quad \text{where } x' = \alpha' r_c \tag{4.12b}$$

$$(1+x')^3 P r_o^3 = P' r_o'^3 (1+x') - \frac{r_c^3}{45}(x'^3+6x'^2+15x'+15) - \frac{r_o' r_c^2}{6}(3+x') - \frac{r_c r_o'^2}{4} \tag{4.12c}$$

The values of $(\alpha'_s\ r_o'_s)$, P_s', and P_s are given in table V and in Fig. 2 and Fig. 3, and the values of $(\alpha_t' r_{ot}')$, P_t', P_t in table VI and Fig. 4 and Fig. 5, for various core radii, for square, gaussian, exponential, and Yukawa shapes and in every case for the "experimental" values, in units of 10^{-13} cm., $a_s = -23.68$, $r_{os} = 2.6$, $a_t = 5.28$, $r_{ot} = 1.56$

TABLE V $\alpha_s' r_{os}'$, P_s' and P_s at VARIOUS VALUES OF r_c FOR FOUR SHAPES

$a_s = -23.68\ (10^{-13}\ \text{cm.})$ $r_{os} = 2.6\ (10^{-13}\ \text{cm.})$

r_c	$\alpha' r_o'$	P_s'				P_s			
		Square	Gaussian	Exponential	Yukawa	Square	Gaussian	Exponential	Yukawa
0000	-.10980	-.030679	-.01764	.009813	.05445	-.03068	-.01764	.009813	.05445
.2000	-.09043	-.03103	-.01776	.01018	.05628	-.03438	-.02664	-.01035	.01654
.43691	-.06836	-.00801	-.01789	.01060	.05836	-.03721	-.03362	-.02609	-.01274
.82743	-.03377	-.03208	-.01810	.01126	.06162	-.03923	-.03876	-.03745	-.03601
1.0000	-.01919	-.03234	-.01818	.01154	.06299	-.03950	-.03940	-.03921	-.03887
1.2345	.0000	-.0327	-.0183	.0119	.0648	-.03951	-.03951	-.03951	-.03951

TABLE VI $\alpha_t' r_{ot}'$, P_t' AND P_t AT VARIOUS VALUES OF r_c FOR FOUR SHAPES

$a_t = 5.28(10^{-13}$ cm.) $r_{ot} = 1.56(10^{-13}$ cm.)

r_c	$\alpha' r_o'$	P_t'				P_t			
		Square	Gaussian	Exponential	Yukawa	Square	Gaussian	Exponential	Yukawa
0	.2954	-.03813	-.02007	.01751	.09262	-.03813	-.02007	.01751	.09262
.1	.2734	-.03773	-.01994	.01709	.09055	-.04155	-.02872	-.00202	.05454
.2	.2498	-.03729	-.01979	.01664	.08833	-.04421	-.03547	-.01759	.01848
.3	.2245	-.03683	-.01964	.01616	.08594	-.04625	-.04062	-.02889	-.00603
.4	.1972	-.03632	-.01948	.01564	.08337	-.04781	-.04443	-.03739	-.02380
.6	.1364	-.03520	-.01911	.01449	.07764	-.05007	-.04920	-.04740	-.04401
.7	.1026	-.0345	-.0189	.0138	.0744	-.05107	-.05073	-.05007	-.04883
.8	.0666	-.0339	-.0186	.0131	.0710	-.05205	-.05197	-.05184	-.05154
.9	.0028	-.0327	-.0184	.0119	.0650	-.04958	-.04958	-.04958	-.04958

TABLE VII A COMPARISON OF THE VALUES OF P_s FOR A SQUARE WELL WITH

$r_c = .8274(10^{-13}$cm.) AND AN EXPONENTIAL WELL WITH $r_c = .4550(10^{-13}$cm)

Shape	$r_c(10^{-13}$cm.)	P_s	
Square	.8274	From Section II -.03921	From Section IV -.03923
Exponential	.4550	From Section III -.02571	From Section IV -.02570

In table VII a comparison of the values of Ps for the square well and exponential well with core radii of .8274 (10^{-13}cm.) and .4550 (10^{-13}cm.) respectively obtained from equation (4.12c) is made with the corresponding values from section II and section III. The good agreement supports the validity of the independent determinations.

Included in Fig. 5, for the sake of comparison, is a plot, from (H2), of P_t versus r_c for a Hulthén potential, a potential similar to the Yukawa, namely of the form

$$V(r) = \infty \qquad r < r_c$$

$$= \frac{-V_o e^{\frac{-r-r_c}{r_b}}}{1 - e^{\frac{-r-r_c}{r_b}}} \qquad r > r_c \tag{4.13}$$

However in obtaining this plot r_{ot} was not held constant; the potential was adjusted at each value of r_c to give the experimental values of a_t and $\rho_t(0,-\epsilon)$. Then $\rho_t(-\epsilon, -\epsilon)$ and r_{ot} were calculated from their definitions and the equation

$$\rho_t(-\epsilon, -\epsilon) = \rho_t(0,-\epsilon) + 2P_t\, r_{ot}^3 \gamma^2 \tag{4.14}$$

was solved for P_t. Because of the difference in the forms of the Hulthén and Yukawa potentials and the fact that in the case of the Hulthén the constant value 2.6(10^{-13}cm.) was not taken for r_{ot}, although in section VI we show that $\frac{dP_t}{dr_o}$ is small, the small differences in the corresponding values of P_t for the two potentials are justifiable.

As an immediate consequence of equation (4.12c) we see that as the core radius approaches its limiting value, P approaches a limiting value independent of the potential shape. Since r_o' must be positive it follows from (4.10b) that the maximum value of the core radius, $r_{c\ max}$, is

$$r_{c\ max} = \frac{3}{2}\,\frac{r_o}{x^2-3x+3} \tag{4.15}$$

For $r_c = r_{c\ max}$, that is for $r_o'=0$, we obtain from (4.12c) the limiting, shape independent, value of P,

$$P = -\left(\frac{r_c}{r_o}\right)^3 \frac{15+15x'+6x'^2+x'^3}{45\ (1+x')^3} \tag{4.16}$$

We remark that the effect of the core is to decrease the dependence of P on the shape, so that, for example, all shapes considered have P negative in the triplet state for $r_c > .30$ (10^{-13}cm.)

It should also be pointed out that the shapes considered here are not "cut off" at r_c, but are displaced from the origin. Thus in considering the Yukawa potential we are considering an attractive potential which has an infinity just at r_c where the infinite repulsive potential starts. Since potentials deduced from meson theories would be expected to cut off with a finite (though possibly large) value at the core, the potentials considered here may be said to cover the range of reasonable shapes.

V INVESTIGATION OF THE MAGNITUDE OF THE VARIATIONAL PARAMETER Q

In order to justify the use of the first three terms in expansion (1.2) as an approximation for $k \cot \delta$ it is necessary to have a knowledge of the magnitude of the neglected terms. The ratio of the fourth to the third term is

$$\frac{Q}{P}(kr_o)^2 \tag{5.1}$$

At the low energies considered $kr_o < 1$. To test the validity of the approximation we must know the magnitude of Q . Extending the procedure of section IV we equate the coefficients of k^6 in equation (4.11) to obtain

$$\begin{aligned} Q = {} & \frac{x'y^5}{(1+x')^2}\left[-\frac{.01481}{1+x'} - \frac{.03481}{(1+x')^2} + \frac{(x'+3)(x'+6)}{135x'(1+x')} + .002116\,(1+x') + .00528\right] \\ & + \frac{1}{(1+x')^4}\left[(3x'^2+18x'+30)\frac{y^4z}{90} + \frac{x'+4}{12}y^3z^2 + \frac{y^2z^3}{8}\right] \\ & - \frac{1}{(1+x')^3}\left(yz^4 + \frac{3+x'}{3}y^2z^3\right)P' + \frac{1}{(1+x')^2}z^5 Q' \end{aligned} \tag{5.2}$$

Where $x' = \alpha' r_c$, $y = \dfrac{r_c}{r_o}$, $z = \dfrac{r_o'}{r_o}$

It can be seen from (5.2) that the limiting values of Q for maximum core radius are, like those of P, independent of shape and given by

$$Q_{rc\,max} = \frac{x'}{(1+x')^2}y^5\left[-\frac{.01481}{1+x'} - \frac{.03481}{(1+x')^2} + \frac{(x'+3)(x'+6)}{135x'(1+x')} + .0021164(1+x') + .00528\right] \tag{5.3}$$

The limiting values of Q for the singlet and triplet states are compared in table VIII with the corresponding limiting values of P ; we see that in the limit of maximum r_c the ratio of Q to P is about 1 to 10.

Using (5.2) we have determined the behaviour of Q as a function of r_c for the singlet state and for two shapes, the square and Yukawa. In using (5.2) to evaluate Q at any value of r_c it is, of course, essential to know Q' as a function of r_c; all the other quantities in (5.2) are known functions of r_c. In appendix II we show that Q' as well as P', is a function only of ($\alpha' r_o'$); a plot of $\alpha' r_o'$ against r_c is given in Fig. 2. Since ($\alpha' r_o'$) is small we may write approximately

$$Q' = a + b(\alpha' r_o') \tag{5.4}$$

and the problem of finding the functional relation between Q' and r_c becomes the problem of finding the constants a and b. Different procedures were used to find a and b for the two shapes dealt with.

TABLE VIII RATIO Q/P FOR MAXIMUM CORE RADIUS

State	P	Q	Q/P
Singlet $a_s = -23.68(10^{-13}\text{cm.})$ $r_{os} = 2.6(10^{-13}\text{cm.})$	-.03951	.00375	-.094
Triplet a_t 5.28(10^{-13}cm.) r_{ot} 1.56(10^{-13}cm.)	-.04958	.00618	-.124

The constant, a, is the value of Q for α and r_c both equal to zero. For the square well, with $\alpha = r_c = 0$, equations (2.3), (2.13) reduce to

$$u_o = \sin \frac{\pi}{2}\frac{r}{r_o}, \quad \psi_o = 1 \tag{5.5a}$$

$$v_1 = \frac{r_o}{\pi} r \cos\left(\frac{\pi}{2}\frac{r}{r_o}\right), \quad X_1 = \tfrac{1}{2} r(r_o - r) \tag{5.5b}$$

Exact integration of (1.23), using the functions (5.5b) gives

$$Q = .001713 = a \tag{5.6}$$

Also, as we shall show immediately, it is possible by exact integration of (1.23) with finite values of α and r_c, to obtain Q for a square well. Consequently using the values of Q obtained from integration, Q' can be calculated at different values of r_c, that is at different values of $(\alpha' r_o')$, from (5.2). Then the slope b can be found from

$$Q' = b + (\alpha' r_o') \tag{5.7}$$

Substituting (1.19), (2.13), in (1.23) and integrating we obtain

$$Q\, r_o^5 = I_1 - (I_2 + I_3 + I_4) \tag{5.8}$$

Where

$$I_1 = \left[\frac{r_o^2}{12} - \frac{r_o}{8} r_b + \left(\frac{1}{20} + \frac{r_o}{30 a_s}\right) r_b^2 - \frac{r_b^3}{36 a_s} + \frac{1}{252}\frac{r_b^4}{a_s^2}\right] r_b^3 \tag{5.9a}$$

$$I_2 = \frac{B^2}{2}\left[r_b - r_c + \frac{1}{2D}\left(\sin 2(\delta - \epsilon) - \sin 2\varphi\right)\right] \tag{5.9b}$$

$$\frac{2D}{AB} I_3 = \sin(\delta - \epsilon)\frac{r_b^2 - r_c^2}{2} + \frac{1}{2D}\left[r_c \cos(\delta - \epsilon) - r_b \cos(\theta + \varphi)\right] + \frac{1}{4D^2}\left\{\sin(\theta + \varphi) - \sin(\delta - \epsilon)\right\} \tag{5.9c}$$

$$\frac{8D^2}{A^2} I_4 = \frac{1}{3}(r_b^3 - r_c^3) + \frac{1}{2D} r_b^2 \sin 2\theta + \frac{1}{2D^2} (r_b \cos 2\theta - r_c) - \frac{\sin 2\theta}{4D^3} \quad (5.9d)$$

$$(\phi = \delta + Dr_b)$$

Q was obtained by integration at five different values of r_c; Q' was evaluated at two values of r_c and b was calculated. The expansion (5.4) for Q' was then used together with (5.2) to obtain Q at various values of r_c. The results are given in table IX and Fig. 6. The best value for b was taken as .0020, so that

$$Q' = .001713 + .0020(\alpha' r_o') \quad (5.10)$$

The agreement between the values of Q at the core radii .8274(10^{-13}cm.) and 1.220(10^{-13}cm.) obtained from direct integration and the values obtained using (5.4) and (5.2) checks the accuracy of the calculation.

TABLE IX VALUES OF Q_s and Q_s' USED TO OBTAIN b AND VALUES OF Q_s OBTAINED FROM (5.2) WITH $Q_s' = .001713 + .0020(\alpha_s' r_{os}')$

r_c	$\alpha_s' r_{os}'$	Q_s from integration of (1.23)	Q_s'	$b = \frac{Q'-a}{\alpha_s' r_{os}'}$	Q from (5.2)
0	-.10979				
.04516	-.10538	.001764	.00164		.00149
.1	-.09973				.00165
.2	-.09042				.00196
.3	-.08236				.00243
.4369	-.06835	.003152	.00154	.00248	.00276
.5	-.06262				.00311
.6	-.05364				.00323
.63195	-.05081	.003483	.00161	.00197	.00354
.8274	-.03377	.003644			.00364
1	-.01918				.00368
1.2200		.003743			
1.2345	.0				.00375

Because the analytic form of the function v_1 is unknown the derivation of expansion (5.4) for the Yukawa potential is more laborious and in fact requires the numerical solution of four differential equations. Equation (5.4) can be written

$$Q' = Q'\Big]_{\alpha=0} + \frac{\partial Q'}{\partial(\alpha' r_o')}\Big]_{\alpha=0} (\alpha' r_o') \tag{5.11}$$

From differentiation of (5.2) we have at $\alpha = 0$,

$$\begin{aligned} r_o' z^5 \frac{dQ'}{d\alpha' r_o'} &= (.37555\, y^5 + \tfrac{5}{4} y^3 z^2 + \tfrac{1}{2} y^2 z^3 - 3(yz^4 + \tfrac{8}{3} y^2 z^3)P' \\ &\quad + 2z^5 Q')r_c + (\tfrac{51}{45} y^5 + .0942(yz^4 + y^2 z^3))r_o' \\ &\quad + (\tfrac{2}{3} y^5 + \tfrac{4}{3} y^4 z + y^3 z^2 + y^2 z^3 - (yz^4 + 2y^2 z^3)P') \frac{1}{r_o} \frac{\partial r_o}{\partial \alpha} \\ &\quad - (\tfrac{y^4}{3} + \tfrac{2}{3} y^3 z + \tfrac{3}{8} y^2 z^2 - (4yz^3 + 3y^2 z^2) P' + 5z^4 Q')\frac{\partial z}{\partial \alpha} + \frac{\partial Q}{\partial \alpha} \end{aligned} \tag{5.12}$$

To obtain (5.11) it is therefore necessary to calculate Q', $\frac{1}{r_o}\frac{\partial r_o}{\partial \alpha}$, and $\frac{\partial Q}{\partial \alpha}$ at $\alpha = 0$

Consider, first, $Q'_{\alpha=0}$. From (5.2) we obtain with $\alpha = 0$,

$$z^5 Q' = Q - y^2(.1\dot{3}y^3 + .\dot{3}yz^2 + .\dot{3}yz^2 + .125z^3) + (yz^4 + y^2 z^3)P' \tag{5.13}$$

Hence we require $Q_{,\alpha=0}$. We recall that

$$r_o^5 Q = \int_0^\infty X_1^2\,dr - \int_{r_c}^\infty v_1^2\,dr \tag{5.14}$$

Numerical integration formulas can only be applied with accuracy to obtain the integrals of slowly changing functions.

Consequently although X_1^2 can be integrated exactly, since the increments in v_1^2 for a fixed increment in r at asymptotic values of r, are relatively large Q cannot be obtained with sufficient accuracy from the difference of the two integrals in (5.14). Hence rather than solving a differential equation for v_1 we write Q in the form

$$r_o{}^5 Q = \int_0^\infty (X_1 - v_1)(X_1 + v_1)\,dr \tag{5.15}$$

We make the substitutions

$$x = \frac{r}{r_b} \qquad y = \ln x \tag{5.16a}$$

$$u_o = \sqrt{x}\, U \qquad \psi_o = \sqrt{x}\, \Psi \tag{5.16b}$$

$$v_1 = \sqrt{x}\, r_b{}^2\, V \qquad X_1 = \sqrt{x}\, r_b{}^2 X \tag{5.16c}$$

In terms of X, V, and y defined in (5.16), from (5.15) Q may be written in the form

$$r_o{}^5 Q = \int_{-\infty}^{\infty} x^2 (X - V)(X + V)\,dy \tag{5.17}$$

Also in terms of the quantities defined in (5.16) the differential equations for v_1 and X_1 take the form

$$\left(\frac{d^2V}{dy^2} - \tfrac{1}{4}V\right) + \frac{Ke^{-x}x^2}{x - x_c}\, V = -x^2 U \tag{5.18}$$

$$\frac{d^2X}{dy^2} - \tfrac{1}{4}X = -x^2\Psi \quad \text{where } K = 2r_b{}^2\mu\, \frac{V_o^R}{\hbar^2}\, e^{\frac{r_c}{r_b}} \tag{5.19}$$

Equations (5.18) can be combined to give an equation for (X - V) namely,

$$\frac{d^2}{dy^2}(X - V) + \left(\frac{Kx^2e^{-x}}{x - x_c} - \tfrac{1}{4}\right)(X - V) = \frac{Kx^2e^{-x}}{x - x_c}\, X - x^2(\Psi - U) \tag{5.20}$$

In a like manner the differential equations for ψ_o and u_o provide an equation for $\Psi - U$, namely

$$\frac{d^2}{dy^2}(\Psi - U) + \left(\frac{Kx^2e^{-x}}{x-x_c} - \frac{1}{4}\right)(\Psi - U) = \frac{Kx^2e^{-x}}{x-x_c}\Psi \tag{5.21}$$

In both (5.20) and (5.21) we have the boundary condition $X - V = \Psi - U = 0$ for asymptotic values of r. To find $Q_{\alpha=0}$ then, we solve (5.21) for Ψ-U, form the R.H.S. of (5.20), and solve for (X- V). From (5.16c) and the definition of X_1 it follows that

$$X = \frac{\sqrt{x}}{2}(x_o - x) \tag{5.22}$$

We can, therefore, obtain (X + V), form the integrand of (5.17) and integrate to obtain Q. Specifically we choose the set of numbers, $r_o = 2.7(10^{-13}$cm.) $r_c = .21432\ (10^{-13}$cm.) $r_b = 1.0716(10^{-13}$cm.). With this set of values $x_c = .2$, and $r_o' = 2.2714(10^{-13}$cm.). Consequently from the definition of V_o in (1.28) and the definition of K in (5.19) we have K = 2.05171. The differential equations (5.20) and (5.21) were solved by Milne's method; the equations were integrated in from an asymptotic value of r about five times r_o, insuring accurate starting values. U and V were found to vanish at values of r equal to $.2141(10^{-13}$cm.) and $.2142(10^{-13}$cm.) respectively; the significance is that, since the selected value of the core radius is $.2143(10^{-13}$cm.), the obtained functions Ψ -U and X - V, are reliable to within about .5%. From integration of (5.17) we obtain for Q,

$$Q_{\alpha=0} = .009526 \tag{5.22a}$$

and from (5.13)

$$Q'_{\alpha=0} = .0113 \tag{5.23}$$

To evaluate, $\frac{1}{r_o}\frac{\partial r_o}{\partial \alpha, \alpha=0}$, we write

$$\frac{\partial r_o}{\partial \alpha} = -4\int_0^\infty (r\psi_o + u_o \frac{\partial}{\partial \alpha} u_o)dr \tag{5.24}$$

We obtain $\frac{\partial}{\partial \alpha} u_o$ from the differential equation

$$\frac{d^2}{dr^2}\frac{\partial u_o}{\partial \alpha} - V(r)\frac{\partial u_o}{\partial \alpha} = u_o \frac{\partial V(r)}{\partial \alpha} \tag{5.25}$$

Realizing that the dependence of V(r) on α is contained in the quantity s, defined in (1.26), we may write

$$\frac{\partial V(r)}{\partial \alpha} = \frac{\partial V}{\partial s}\frac{\partial s}{\partial \alpha} \tag{5.26}$$

For the Yukawa, Blatt and Jackson (B10), give

$$s = 1.0 + .6361\ (\alpha' b) \tag{5.27}$$

Therefore,

$$\frac{\partial V}{\partial \alpha} = .6361\ bV \tag{5.28}$$

To carry out the integration (5.24) we again use the substitutions (5.16a) and (5.16c) and write the integral in the form

$$\frac{\partial r_o}{\partial \alpha} = -4\int_0^{r_c} r\ dr - 4r_b \int_{y_c}^{y_a} (r_b x^2 + x^2 U \frac{\partial U}{\partial \alpha})\ dy \tag{5.29}$$

The differential equation (5.25) takes the form

$$\frac{d^2}{dy^2}\frac{\partial U}{\partial \alpha} + \left[\frac{Kx^2e^{-x}}{x-x_c} - \frac{1}{4}\right]\frac{\partial U}{\partial \alpha} = -1.445\ \frac{Kx^2e^{-x}}{x-x_c}\ U \tag{5.30}$$

Milnes method is used to solve (5.30); starting values are given by

$$\frac{\partial \Psi}{\partial \alpha} = -r_b \sqrt{x} \tag{5.31}$$

Again the accuracy of the calculation is supported by the fact that $\frac{\partial U}{\partial \alpha}$ vanishes at $r = .2180(10^{-13})$; that is at a value which differs from r_c by about 2%. We obtain by integration

$$\frac{1}{r_o} \frac{\partial r_o}{\partial \alpha}_{,\alpha=0} = -2.0410 \ (10^{-13}\text{cm.}) \tag{5.32}$$

Lastly to evaluate $\frac{\partial Q}{\partial \alpha}_{,\alpha=0}$, differentiation of (5.17) yields,

$$r_o^5 \frac{\partial Q}{\partial \alpha} + 5r_o^4 \frac{\partial r_o}{\partial \alpha} Q = \int_0^{r_c} 2X_1 \frac{\partial X_1}{\partial \alpha} dr + r_b^5 \int_{y_c}^{y_a} \left[x^2(X-V) \frac{\partial}{\partial \alpha}(X+V) + x^2(X+V) \frac{\partial}{\partial \alpha}(X-V)\right] dy \tag{5.33}$$

Analogous to equation (5.30) we obtain an equation for $\frac{\partial}{\partial \alpha}(X-V)$ by differentiating equation (5.20), which we solve by Milne's method. Of course

$$\frac{\partial}{\partial \alpha}(X+V) = 2\frac{\partial}{\partial \alpha}X - \frac{\partial}{\partial \alpha}(X-V) \tag{5.34}$$

Hence, we can form the integrand in (5.33) and, knowing $\frac{\partial r_o}{\partial \alpha}$ and Q, we can obtain $\frac{\partial Q}{\partial \alpha}$. The result is

$$\frac{\partial Q}{\partial \alpha}_{,\alpha=0} = .08546 \tag{5.35}$$

In this case it is found by interpolation that $\frac{\partial V}{\partial \alpha}$ vanishes at $r = .240(10^{-13}\text{cm.})$ rather than at $r_c = .2143(10^{-13}\text{cm.})$. This discrepancy, which can be attributed to accumulated round off error, has however no appreciable significance in the final result since the integral in (5.33) contributes only about 10% of the value of $\frac{\partial Q}{\partial \alpha}$, the main contribution, .09721, coming from the term $\frac{5}{r_o}\frac{\partial r_o}{\partial \alpha}Q$. Inserted in equation (5.12) the values obtained for Q', $\frac{1}{r_o}\frac{\partial r_o}{\partial \alpha}$, and $\frac{\partial Q}{\partial \alpha}$ give

$$\frac{\partial Q'}{\partial \alpha' r_o', \alpha=0} = .1134 \qquad (5.36)$$

Finally the linear expansion for Q' is

$$Q' = .0113 + .113(\alpha' r_o') \qquad (5.37)$$

Values of Q' from (5.37) and of Q from (5.2) are listed in table X and a plot of Q against r_c is given in Fig. 6.

TABLE X Q_s and Q_s' FOR A YUKAWA SHAPE

$a_s = -23.68(10^{-13}\text{cm.})$, $r_{os} = 2.6(10^{-13}\text{cm.})$

r_c	Q'	Q
0	-.0012	-.0012
.1	.0000	-.0014
.2	.0010	-.0014
.3	.0020	-.0009
.4369	.0035	.0001
.5	.0042	.0006
.6	.0052	.0016
.8274	.0075	.0029
1	.0091	.0035
1.235	.0113	.0037

In section III the function v_1 was calculated for the particular exponential, $V_o = 295.2$ Mev, $r_c = .4575(10^{-13}$cm.) $r_b = .4400(10^{-13}$cm.). Therefore, Q can be calculated for this exponential directly from its definition (1.23). For accuracy the range of integration, from the origin to the asymptotic value of r, has been divided into two parts O to R and R to r_a, as in Section III. The integral for Q, in the notation of Section III, is written in the form

$$r_o^5 Q = \int_o^R X_1^2\, dr - 2r_b^5 \int_{x_R}^{x_c} \frac{V^2}{x^2}\, dx + 2r_b^5 \int_{x_a}^{x_R} \left(\frac{X_1^2}{xr_b^4} - \frac{V^2}{x^2}\right) dx \qquad (5.38)$$

The calculated value .00187 is plotted on Fig. 6; as one would expect, because of the shape of the exponential, the Q for the exponential is seen to be between the square and Yukawa values at this core radius.

From Fig 3 and Fig. 6 we see that the ratio $\frac{Q}{P}$ for the square well decreases monotomically from its shape independent value, .09, at maximum r_c to .04 at $r_c = 0$. For the Yukawa at $r_c = 0$, $\frac{Q}{P} = .02$. Of course, for the Yukawa $P = 0$ at $r_c = .325$; but for $P = 0$ a measure of the accuracy of (5.2) is given by the ratio of the fourth term to the second, that is by $\frac{2Qr_o^5k^6}{r_ok^2}$, not $\frac{Q}{P}$. For an energy of 5 M.ev. the ratio

$2Q(r_ok)^4 = 2.3 \times 10^{-4}$. Therefore, from our investigation of Q for the square and Yukawa wells, which are representative of the reasonable well shapes, we conclude that its magnitude is small enough to make the first three terms in the expansion for k cotδ a valid approximation.

Jackson and Blatt (J1), using the wave function for proton-proton scattering have calculated Q_s' for the four conventional shapes. The values quoted are: square .00179, gaussian -.00073, exponential .00089, Yukawa .019. It will be observed that their square well value agrees with ours, while their Yukawa Q_s' is an order of magnitude larger than ours and their other three.

VI EXPERIMENTAL VALUES OF THE VARIATIONAL PARAMETERS a_s, a_t, r_{os}, r_{ot}, P_s and P_t.

The experiments which determine the unknown r_{os}, r_{ot}, P_s and P_t, do not measure the unknowns directly but rather functions of them. Four exact equations are sufficient, of course, to determine four unknowns. The experiments, however, yield a set of N equations,

$$F_n(r_{os}, r_{ot}, P_s, P_t) = A_n \tag{6.1}$$

The number of equations is greater than the number of unknowns and the right hand sides of the equations, being experimental numbers are not exact. Hence the quest for the best set of values does not involve simply the solution of four exact, simultaneous, equations, but rather some unbiased analytic technique. To find the best values we apply the method of least squares in a manner analogous to that used by J. DuMond and E. Cohen (D2). In order to understand the method of application, let us consider the general case of q unknows, X_1, X_2, X_q, overdetermined by a set of N equations,

$$F_n(X_1 \cdots X_q) = A_n \qquad n = 1 \text{ to } N \tag{6.2}$$

The equations (6.2) are linearized by choosing a set of origin values X_{10}, X_{20}, X_{qo}, sufficiently close to the anticipated solution X_1, X_2, X_q so that when the equations (6.2) are written in terms of the small dimensionless quantities $x_1 = \dfrac{X_1 - X_{10}}{X_{10}}$, $xq = \dfrac{Xq - X_{qo}}{X_{qo}}$, second order

terms in these quantities may be neglected. The observational equations (6.2), then take the form

$$a_{1n}x1 + a_{2n}x_2 + \cdots + a_{qn}x_n = a_n \qquad (6.3)$$

Let us denote the true values of the unknowns by $\bar{x}_1, \bar{x}_2, \cdots \bar{x}_q$. Substitution of these values in (6.3) would not give the numeric a_n, since a_n is a measurement subject to error. However we may write $a_{1n}\bar{x}_1 + a_{2n}\bar{x}_2 + \cdots + a_{qn}\bar{x}_q = a_n(1-r_n)$ (6.4) where r_n is the actual relative error in the measured quantity. We assume that the probability distribution of the error, r_n, is Gaussian. Hence σ_n, being the standard deviation, the probability that the error r_n lies between r and $r + dr$ is

$$P_n(r < r_n < r + dr) = \frac{1}{2\pi^{\frac{1}{2}}\sigma_n} \exp\left(\frac{-r^2}{2\sigma_n^2}\right)dr \qquad (6.5)$$

Introduction of the factors $(1-r_n)$ in (6.4) increases the number of unknowns from q to $q + N$ so that the system is no longer overdetermined but in fact can be satisfied by any one of infinitely many combinations of the unknowns $x_1 \cdots x_q$, $r_1 \cdots r_n$. The Axiom of Maximum Likelihood provides us with the condition which provides a means of finding the best solution. The axiom states that "of all the possible choices for the set of residuals r_n, the best choice is that whose probability of occurence is maximum". The probability of obtaining a set of residuals $r_1, \cdot\cdot\, r_N$ is

$$P(r_1, r_2, r_3, \cdot\cdot\,) = 2\pi^{-\frac{1}{2}N}(\sigma_1\sigma_2 \cdot\cdot \cdot\sigma_N)^{-1} \exp\left(-\tfrac{1}{2}\left(\frac{r_1^{\,2}}{\sigma_1^{\,2}} + \frac{r_2^{\,2}}{\sigma_2^{\,2}} + \cdot\cdot\right)\right) \tag{6.6}$$

The probability, P , is a maximum when the exponent in (6.6) is a minimum. The function to be minimized is, therefore, $Q = \sum \frac{(r_n^{\,2})}{n^2}$ and the minimum condition is obtained by equating to zero the partial derivatives of Q with respect to each variable $x_1 \cdot \cdot\, x_q$. The equations which express the minimum conditions are called the "normal equations". The normal equations are

$$\begin{aligned} b_{11}x_1 + b_{12}x_2 + \cdot\cdot\, + b_{1q}x_q &= c_1 \\ b_{21}x_1 + b_{22}x_2 + \cdot\cdot + b_{2q}x_q &= c_2 \\ & \\ b_{q1}x_1 + b_{q2}x_2 + \cdot\cdot\, + b_{qq}x_q &= c_q \end{aligned} \tag{6.7}$$

The quantities b_{ij} are given by

$$b_{ii} = \sum_n P_n\, a_{in}^{\,2}$$

$$b_{ij} = \sum_n P_n\, a_{in} a_{jn} \tag{6.8}$$

Where P_n is the weight of the n^{th} equation and is related to the standard deviation of a_n by an arbitrary constant C. The quantities b_{ij} are symmetric, that is $b_{ij} = b_{ji}$.

The well known solution of (6.7) is

$$x_1 = d_{11} c_1 + d_{12} c_2 + \cdot\cdot + d_{1q} c_q$$

$$x_q = d_{q1} c_1 + d_{q2} c_2 + \cdot\cdot + d_{qq} c_q \qquad (6.9)$$

where the element d_{ij} is the minor b_{ji} in the determinant of the bs divided by the determinant itself. A measure of the consistency of the set of values is given by the minimum value of Q, which is usually denoted by

$$\chi^2 = \sum \frac{R_n^2}{n^2} \qquad (6.10)$$

R_n is the residual r_n for the solution (6.9).
The mean square error in x_i is given by C_{dii}. "A priori" the value of C is taken to be

$$C_I = P_n \sigma_n^2 \qquad (6.11)$$

If $\frac{\chi^2}{N-q}$ is different from the expected value one, "a posteriori"

$$C_E = C_I \frac{\chi^2}{N-q} \qquad (6.12)$$

and the mean square error in x_i must be adjusted accordingly. If $\frac{\chi^2}{N-q}$ is very much larger than one, the consistency of the equations is to be suspected. In this case χ^2 could be evaluated for different sets of equations selected from the inconsistent set in order to ascertain which of the equations of the set contains the concealed error.

Before proceeding further with the application of the method to our problem, it is advantageous to discuss some quantities which will be needed in the observational equations. We require D, defined by

$$D = \tfrac{1}{4}\left(\rho_t(-\epsilon\,,\,-\epsilon\,) + \rho_s(E,\ E)\right) \tag{6.13}$$

D is introduced in evaluating the matrix element of the magnetic dipole transition to ^1S state of energy E. The matrix element is proportional to

$$M_m = N_g N_s \int_0^\infty u_g u_s \, dr \tag{6.14}$$

where N_g and N_s are the normalizing factors for the ground and singlet states. From elementary integration of the wave functions

$$\int_0^\infty \Psi_g \Psi_s \, dr = \frac{\gamma + k \cot \delta_s}{\gamma^2 + k^2} \tag{6.15}$$

We write

$$2D = 2\int_0^\infty (\Psi_g \Psi_s - u_g u_s)\,dr = \int_0^\infty (\Psi_g{}^2 - u_g{}^2)\,dr + \int_0^\infty (\Psi_s{}^2 - u_s{}^2)\,dr$$
$$- \int_0^\infty \left[(\Psi_g - \Psi_s)^2 - (u_g - u_s)^2\right] dr \tag{6.16}$$

The third integral in (6.16) proves to be negligible (B6), (S2), so that the matrix element can be written in terms of D in the form (6.13).

We require the difference between the magnetic moments of the proton, and neutron, $\mu_p - \mu_n$. From Sachs', Nuclear Theory, (S1)

$$\mu_p = 2.7955 \ \text{erg}/\text{gauss}$$
$$\mu_n = -1.91280 \ \text{erg}/\text{gauss} \tag{6.17}$$

We require the scattering lengths a_s and a_t; the values are obtained from measurements of the epithermal cross section, σ_o, and the coherent scattering amplitude f.

$$\sigma_o = \tfrac{3}{4}(4\pi a_t^2) + \tfrac{1}{4}(4\pi a_s^2) \qquad (6.18)$$

$$f = 2(\tfrac{3}{4}a_t + \tfrac{1}{4}a_s)$$

From the weighted averages of the most recent determinations of σ_o and f, listed in table XI, we find

$$a_s = -23.71(1 \pm .00059) \quad x10^{-13}\text{cm.} \qquad (6.19)$$

$$a_t = 5.383(1 \pm .0015) \times 10^{-13}\text{cm.}$$

k^2 is defined by

$$k^2 = \frac{2\mu}{\hbar^2} \text{Ecg.} \qquad (6.20)$$

Where Ec.g. is the energy in the centre of mass system. In the scattering experiments

$$\text{Ec.g.} = \frac{M_p}{M_n + M_p} E,$$ where E is the energy measured

in the laboratory system. In the photodisintegration the centre of mass is at rest, hence Ec.g. = E. In this case E is the total kinetic energy after disintegration and equals the difference between the gamma ray energy and the

TABLE XI RECENT VALUES OF σ_o, THE EPITHERMAL CROSS SECTION AND f , THE SCATTERING AMPLITUDE.

σ_o	Reference	f	Reference
20.36 ± .10	(M2)	-3.78 ± .02	(H5)
20.41 ± .14	(S6)	-3.80 ± .05	(S6)

binding energy of the deuteron. The conversion factors from E to k^2 in the scattering and photodisintegration are therefore, respectively

$$\frac{2\mu}{\hbar^2}\frac{M_p}{M_n+M_p} = .012052\text{x}10^{26}\,\text{cm}^{-2}\text{Mev}^{-1}$$
$$\frac{2\mu}{\hbar^2} = .02412\text{x}10^{26}\,\text{cm}^{-2}\text{Mev}^{-1} \tag{6.21}$$

We denote the binding energy of the deuteron by ϵ , and take according to (S2), $\epsilon = 2.226$ M.ev.
Hence, $\gamma^2 = \frac{2\mu\epsilon}{\hbar^2}$ is $.05369 \times 10^{26}\,\text{cm}^{-2}$. Also we use the quantity $\beta^1 = -\frac{1}{a_s}$, in the observational equations.

In our problem we have a total of thirteen observational equations which fall into four groups, namely:

(a) a single equation in which the measured quantity A_n is the binding energy of the deuteron.

(b) four equations in which A_n are the total neutron-proton scattering cross section at four energies from 1 Mev to 14 Mev.

(c) five equations in which A_n are the total photodisintegration cross sections for five different gamma rays.

(d) three equations in which A_n are the ratios of the photomagnetic to the photoelectric cross section for three different gamma rays.

These equations can be written in the form,

(a) $r_{ot} + 2P_t r_{ot}^3\gamma^2 = \rho(0, -\epsilon)$ (6.22)

(b) $3\pi\left[k^2+(a_t^{-1}-\frac{1}{2}\rho_t(0,E)k^2)^2\right]^{-1}+\pi\left[k^2+(a_s^{-1}-\frac{1}{2}\rho_s(0,E)\right.$

$$\left.\times k^2)^2\right]^{-1}=\sigma(E) \quad (S2), \quad (6.23)$$

(c) $\sigma_e(E)+\sigma_m(E)=\sigma(E)$ where

$$\sigma_e=\frac{8\pi}{3}\frac{e^2}{\hbar c}\frac{\hbar^2}{M}\frac{\epsilon^{\frac{1}{2}}E^{\frac{3}{2}}}{(E+\epsilon)^3}\frac{1}{1-\rho_t^1\gamma}$$

$$\sigma_m=\frac{2\pi}{3}\frac{e^2}{\hbar c}\left(\frac{\hbar}{Mc}\right)^2(\mu_p-\mu_n)^2\frac{\gamma k}{\gamma^2+k^2}\frac{1}{1-\rho_t^1}$$

$$\times\frac{\left[\gamma+\beta^1-\gamma^2 D+\frac{k^2}{2}\rho(0E)-k^2D\right]^2}{k^2+(\beta^1+\frac{k^2}{2}\rho(0,E))^2} \quad (B6),\ (S2),\ (6.24)$$

$$\rho_t^1=\rho_t(-\epsilon,-\epsilon)$$

(d) $\frac{\sigma_m}{\sigma_e}=\frac{\text{photomagnetic cross section}}{\text{photoelectric cross section}}$ (6.25)

We define the origin values r_{oso} and r_{oto} by

$$r_{os}=r_{oso}(1+s) \qquad r_{ot}=r_{oto}(1+t) \quad (6.26)$$

The observational equations (6.22) to (6.25) can now be expressed in terms of the four small dimensionless quantities s, t, P_s, P_t in the linearized form,

(a) $r_{oto}t+2P_t\, r_{oto}^3\gamma^2=\rho(0,-\epsilon)-r_{oto}$ (6.27)

$$\text{(b)}\ \frac{3\pi\, r_{oto}(a_t^{-1}-\frac{1}{2}r_{oto}k^2)k^2}{\left[k^2+(a_t^{-1}-\frac{1}{2}r_{oto}k^2)\right]^2}t-\frac{6\pi\, r_{oto}^3k^4(a_t^{-1}-\frac{1}{2}r_{oto}k^2)}{\left[k^2+(a_t^{-1}-\frac{1}{2}r_{oto}k^2)^2\right]^2}P_t$$

$$+\frac{\pi\, r_{oso}(a_s^{-1}-\frac{1}{2}r_{oso}k^2)k^2}{\left[k^2+(a_s^{-1}-\frac{1}{2}r_{oso}k^2)^2\right]^2}s-\frac{2\pi\, r_{oso}^3k^4(a_s^{-1}-\frac{1}{2}r_{oso}k^2)}{\left[k^2+(a_s^{-1}-\frac{1}{2}r_{oso}k^2)^2\right]^2}P_s$$

$$=\sigma(E)-\frac{3\pi}{k^2+(a_t^{-1}-\frac{1}{2}r_{oto}k^2)^2}-\frac{\pi}{k^2+(a_s^{-1}-\frac{1}{2}r_{oso}k^2)^2} \quad (6.28)$$

(c) $(MA+EF)t + MBs + (MC+EG)P_t + MDP_s = \sigma(E) - (E+M)$ (6.29)

$$\sigma_m = M(1 + At + Bs + CP_t + DP_s)$$

$$\sigma_e = E(1 + Ft + GP_t)$$

$$M = \frac{2\pi}{3} \frac{e^2}{\hbar c} \left(\frac{\hbar}{Mc}\right)^2 (\mu_p - \mu_n)^2 \frac{\gamma}{1-\gamma r_{oto}} \frac{k}{\gamma^2 + k^2}$$

$$\times \frac{\left[\gamma + \beta\,1 - \frac{\gamma^2}{4}(r_{oto} + r_{oso}) + \frac{k^2}{4}(r_{oso} - r_{oto})\right]^2}{k^2 + (\beta\,1 + \frac{1}{2} r_{oso} k^2)^2}$$

$$A = \frac{\gamma r_{oto}}{1-\gamma r_{oto}} - \frac{1}{2} \frac{(\gamma^2 + k^2) r_{oto}}{\gamma + \beta\,1 - \frac{\gamma^2}{4}(r_{oto} + r_{oso}) + \frac{k^2}{4}(r_{oso} - r_{oto})}$$

$$B = \frac{1}{2} \frac{r_{oso}(k^2 - \gamma^2)}{\gamma + \beta\,1 - \frac{\gamma^2}{4}(r_{oto} + r_{oso}) + \frac{k^2}{4}(r_{oso} - r_{oto})}$$

$$- \frac{r_{oso} k^2 (\frac{1}{2} r_{oso} k^2 + \beta\,1)}{k^2 + (\beta\,1 + \frac{r_{oso}}{2} k^2)^2}$$

$$C = \frac{4\gamma^3 r_{oto}^3}{1 - \gamma r_{oto}} - \frac{2\gamma^2(\gamma^2 + k^2)\, r_{oto}^3}{\gamma + \beta\,1 - \frac{\gamma^2}{4}(r_{oto} + r_{oso}) + \frac{k^2}{4}(r_{oso} - r_{oto})}$$

$$D = \frac{2\gamma^2 k^2 r_{oso}^3}{\gamma + \beta\,1 - \frac{\gamma^2}{4}(r_{oto} + r_{oso}) + \frac{k^2}{4}(r_{oso} - r_{oto})}$$

$$+ \frac{r_{oso}^3 k^4 (2\beta\,1 + r_{oso} k^2)}{k^2 + (\frac{r_{oso}}{2} k^2 + \beta\,1)^2}$$

$$E = \frac{8\pi}{3} \frac{e^2}{\hbar c} \frac{\gamma k^3}{(\gamma^2 + k^2)^3} \frac{1}{1 - \gamma r_{oto}}$$

$$F = \frac{r_{oto}\gamma}{1 - \gamma r_{oto}}$$

$$G = \frac{4r_{oto}^3\gamma^3}{1-\gamma r_{oto}}$$

$$\text{(d)} \quad \left[(A-F)t + Bs + (C-G)P_t + DP_s\right] \frac{M}{E} = \frac{\sigma m}{\sigma e} - \frac{M}{E} \qquad (6.30)$$

The experimental numbers which we use in (6.27) to (6.30) to obtain the thirteen observational equations are tabulated in table XII. The resulting thirteen equations are given in table XIII. In accordance with the previous notation the coefficients of t, s, P_t, and P_s, are denoted by a_{1n}, a_{2n}, a_{3n}, a_{4n}. The probable error of the R.H.S. of each equation has been adjusted to take into account the probable error in the energies. The adjustment is small, and in equations 6 to 13 is ignorable. The weight taken for each equation was $\frac{1}{(\text{Probable error})^2}$, so that "a priori" $C_I = \frac{1}{(.675)^2}$ and the probable error of x_i is $d_{ii}^{\frac{1}{2}}$. The normal equations (6.7) were formed; the "best" values were calculated from (6.9) and were used to calculate χ^2. The probable errors were adjusted according to (6.12). The results are in table XIV. The large value of χ^2 leads us to suspect inconsistency in the set of equations. The inconsistency is found to lie in the photodisintegration equations. Equations 11, 12, and 13, may be written in the form

TABLE XII EXPERIMENTAL NEUTRON-PROTON SCATTERING CROSS SECTIONS, DEUTERON PHOTODISINTEGRATION CROSS SECTIONS, AND THE RATIO OF PHOTOMAGNETIC TO PHOTOELECTRIC CROSS SECTIONS

Energy (Mev)	Scattering σ (Barns)	Reference	Photodisintegration $\sigma_e + \sigma_m$ (10^{-26} cm.^{2})	Reference	Ratio $\frac{\sigma_m}{\sigma_e}$	Reference
1.311	3.675 ± .020	(F2)				
2.532 ± .006	2.525 ± .009	(F1)				
4.749 ± .009	1.690 ± .0066	(H2)				
14.12 ± .04	.686 ± .007	(B4)				
14.10 ± .05	.689 ± .005	(P1)				
14.12 ± .008	.688 ± .0095	Average				
2.508 ± .003			.119 ± .008	(B8)	.600 ± .02	(B8)
			.107 ± .01	(S4)		
			.115 ± .0074	Average		
2.6143 ± .005			.1380 ± .0038	(H1)	.3600 ± .008	(B8)
2.754 ± .005			.145 ± .015	(S4)	.247 ± .007	(B8)
			.159 ± .006	(B8)	.317 ± .012	(M1)
			.157 ± .0056	Average	.295 ± .036	(G1)
4.45 ± .04			.243 ± .017	(B3)	.264 ± .0059	Average
6.14 ± .01			.219 ± .01	(B3)		
			.215 ± .012	(B2)		
			.218 ± .008	Average		

TABLE XIII THE LINEARIZED OBSERVATIONAL EQUATIONS AND THE DETERMINANT OF THE NORMAL EQUATIONS

Equation	a_{1n}	a_{2n}	a_{3n}	a_{4n}	a_n	Probable error in a_n
1	1.70	0	.5276	0	.0102	.014
2	21.07	-21.64	-1.925	4.992	2.1	2.93
3	24.86	-15.37	-4.389	6.846	.6	1.57
4	21.73	-11.34	-7.197	9.471	1.2	1.19
5	3.786	-6.651	-3.728	16.35	1.1	1.00
6	.05481	-.01546	.03402	.003006	.0163	.007
7	.06876	-.01318	.04268	.003625	.0193	.0038
8	.08625	-.011004	.05353	.004246	.013	.0056
9	.15512	-.002448	.09628	.006765	-.002	.017
10	.13618	-.000673	.08452	.007348	.004	.0078
11	-.1578	-.2606	-.09793	.05068	-.0644	.03
12	-.09911	-.1556	-.06151	.04279	-.0415	.008
13	-.06516	-.0955	-.04044	.03684	.012	.006

$$B = \begin{vmatrix} 16.6049 & 5.2031 & -.060365 & .19901 \\ 5.2031 & 1.9860 & .32998 & -.17828 \\ -.060365 & .32998 & 1.0141 & -.46275 \\ .19901 & -.17828 & -.46275 & .42610 \end{vmatrix} \times 10^{12}$$

TABLE XIV "BEST" VALUES WITH THEIR PROBABLE ERRORS, FROM THE NORMAL EQUATIONS

	t	s	P_t	P_s	r_{ot}	r_{os}
Value:	-.04415	.06649	.1964	.21697	1.625	2.880
Probable Error :-	.0394	.0767	.117	.123	.066	.207

$$\frac{\chi^2}{N-q} = 2.788$$

$$t + .6206\,P_t + 1.6514s - .3212P_s = -.3612 \pm .19$$
$$t + .6206\,P_t + 1.5700s - .4317P_s = -.3531 \pm .08 \qquad (6.31)$$
$$t + .6206\,P_t + 1.466s - .5654P_s = .2609 \pm .092$$

The L.H.S's are very nearly identical, so eliminating t, P_t will give two almost identical L.H.S's but quite different R.H.S's. Hence if these equations are taken seriously, large values of s and P_s arise. However we suspect one of the experimental residuals on the R.H.S.'s. Moreover, apart from this inconsistency, the values obtained could not be taken as the "best" values because P_t and P_s are so large that neglecting second order terms involving them in linearizing the observational equations would introduce an appreciable error.

Consequently a second least squares analysis was executed using the first five equations. The values obtained are

$$t = .00129 \pm .0077 \qquad r_{ot} = 1.702 \pm .013$$
$$s = -.03633 \pm .0722 \qquad r_{os} = 2.60 \pm .19$$
$$P_t = .00932 \pm .078 \qquad \chi^2 = .223$$
$$P_s = .0347 \pm .032$$

The small value of χ^2 substantiates the consistency of the equations and the unknowns are sufficiently small to make the approximation introduced by the linearization a valid one. However upon substitution into equation 6 to 13 it is found that the residuals of equations 6, 7, 8, 11, and 12, exceed the probable error of their R.H.S's by 1, 4, 1, 2, and 7, probable errors respectively, after the residuals had been made as small as possible by taking into account the error in the L.H.S's due to the errors in s, t, P_s, and P_t.

By inspection of the equations and comparison of these values with those in table XIV, another set was selected which proved to be more consistent.

With the set

$$t = -.008 \pm .05 \qquad r_{ot} = 1.686 \pm .085$$

$$s = -.05 \pm .05$$

$$P_t = .05 \pm .01 \qquad r_{os} = 2.56 \pm .13$$

$$P_s = .03 \pm .05$$

only the residuals of equations 7, and 12, exceed their probable error by more than one probable error and in these two cases the probable errors given by the experimenters, in comparison with those of the other equations, are small.

The capture cross section for thermal neutrons is given by

$$\sigma_H v = 2\pi \frac{e^2}{Mc} \frac{\hbar}{Mc} \left(\frac{\epsilon}{Mc^2}\right)^{\frac{3}{2}} (\mu_p - \mu_n)^2 \frac{[\gamma + \beta^{1} - \gamma^2 D]^2}{(1-\rho_t{}^1\gamma)\beta^{12}} \qquad (6.32)$$

With the chosen set of values we obtain

$$\sigma_H v = 6.83 \pm .55 \times 10^{-20} \text{cm.}^3/\text{sec.}$$

A recent value for the cross section for neutrons with velocity 2200 metres/second is .321 $\pm$ .005 barns (D1) giving

$$\sigma_H v = 7.06 \pm .11 \times 10^{-20} \text{cm.}^3/\text{sec.}$$

The set of values is in agreement with the experimental capture cross section.

We are able, then, to draw conclusions in connection with the shape of the potential, since the experiments indicate that P_t and P_s are positive. However, the values of r_{ot} and r_{os} used in the calculations of section IV differ slightly from those dictated by experiment. Therefore in order to compare the calculated values with the experimental results it is necessary to have an estimate of the derivative of P with respect to r_o. From (4.12c), since α^1 is small, we have

$$Pr_o^3 \doteq P^1 r_o^{1\,3} - \frac{r_c^3}{2} - \frac{r_o^1 r_c^2}{2} - r_c r_o^{1\,2} \tag{6.33}$$

It follows that

$$r_o^3 \frac{\partial P}{\partial r_o} \doteq 3P^1 r_o^1 - \frac{r_c^2}{2} - \tfrac{1}{2} r_c r_o^1 - 3Pr_o^2 + r_o^{1\,3} \frac{dP^1}{d\alpha^1 r_o^1} \frac{\partial}{\partial r_o}(\alpha^1 r_o^1) \tag{6.34}$$

For the triplet set of values $r_{ot} = 1.56\ (10^{-13}\text{cm})$ $a_t = 5.28(10^{-13}\text{cm.})$ and $r_c = .4(10^{-13}\text{cm.})$, for the Yukawa $\frac{\partial P_t}{\partial r_{ot}}$ is approximately .034. For $\Delta r_{ot} = .13$, the difference between the experimental value and the value used, ΔP_t for the Yukawa is .004. For the other shapes $\frac{\partial P_t}{\partial r_{ot}}$ is smaller and particularly for the square well equals approximately .018. Hence, we can justifiably compare the calculated values of P_t and P_s of section IV with the experimental values.

From Fig. 5 we see that $P_t < .05$ for all values of r_c except in the case of the Yukawa potential where $P_t > .05$ provided the core $< .110 \times 10^{-13}$cm. Also from Fig. 3 we see that the Yukawa is the only shape of the four investigated which gives $P_s > .03$ and for this $r_c < .115 \times 10^{-13}$cm. Now, the "best" values for P_t and P_s, although not very well fixed, are respectively .05 and .03. Consequently the conclusion is that a static repulsive core potential exists which gives an acceptable fit to the experimentally observed properties of the neutron-proton system at low energies but that unless this potential is more singular than the Yukawa, $(\frac{1}{r})$, the core must be small, of the order 0.1 or 0.2×10^{-13}cm. In an investigation of the charge independence of repulsive core potentials, J. Shapiro (S3) found, in agreement with our conclusion, that it is possible to obtain a potential which explains both the neutron-proton, and proton-proton scattering data, provided the potential posses a strong singularity at the origin $(\frac{1}{r^2})$ and that the singularity is cut off by a core of very small radius.

APPENDIX I

1. Five point formulas used to obtain starting values in the numerical solution of differential equations (M3)

$$y_1-y_0 = \frac{h}{720}\left(251y_0^1 + 646y_1^1 - 264y_2^1 + 106y_3^1 - 19y_4^1\right) + \frac{27h^6 y^{(6)}(s)}{1440}$$

$$y_2-y_0 = \frac{h}{90}\left(29y_0^1 + 124y_1^1 + 24y_2^1 + 4y_3^1 - y_4^1\right) + \frac{16h^6 y^{(6)}(s)}{1440}$$

$$y_3-y_0 = \frac{h}{80}\left(27y_0^1 + 102y_1^1 + 72y_2^1 + 42y_3^1 - 3y_4^1\right) + \frac{27h^6 y^{(6)}(s)}{1440}$$

$$y_4-y_0 = \frac{4h}{90}\left(7y_0^1 + 32y_1^1 + 12y_2^1 + 32y_3^1 + 7y_4^1\right) - \frac{8h^7 y^{(7)}(s)}{945}$$

2. Formulas used to obtain the numerical integrals: Weddle's Formula (M3)

$$y_6-y_0 = \frac{3h}{10}\left(y_0^1 + 5y_1^1 + y_2^1 + 6y_3^1 + y_4^1 + 5y_5^1 + y_6^1\right) - \frac{h^7 y^{(7)}}{140} - \frac{9h^6 y^{(9)}}{1400}$$

Simpson's Formula (M3)

$$y_2-y_0 = \frac{h}{3}\left(y_0^1 + 4y_1^1 + y_2^1\right) - \frac{h^5 y^{(5)}(s)}{90}$$

Gregory's Formula (J3)

$$\frac{1}{h}\int_{x_0}^{x_0+nh} f(x)dx = \tfrac{1}{2} f(x_0) + f(x_0+h) + \cdots + f(x_0+\overline{n-1}h) + \tfrac{1}{2} f(x_0+nh)$$

$$-\frac{1}{12}\left(\nabla f(x_0+nh) - \Delta f(x_0)\right) - \frac{1}{24}\left(\nabla^2 f(x_0+nh) + \Delta^2 f(x_0)\right)$$

$$-\frac{19}{720}\left(\nabla^3 f(x_0+nh) - \Delta^3 f(x_0)\right) - \frac{3}{160}\left(\nabla^4 f(x_0+nh) + \Delta^4 f(x_0)\right).$$

APPENDIX II

1. To verify that P' is a function only of $\alpha' r_o'$ write

$$P' = \frac{1}{r_o'^3}\int_0^\infty (\psi_o X_1 - u_o v_1)\,dr = \frac{1}{(\alpha' r_o')^3}\int_0^\infty [\psi_o(\alpha'^2 X_1) - u_o(\alpha'^2 v_1)]\,d(\alpha' r)$$

$$= \frac{1}{(\alpha' r_o')^3}\int_0^\infty [(1-\beta)(\alpha'^2 X_1) - u_o(\alpha'^2 v_1)]\,d\beta. \qquad (1)$$

Hence it need only be shown that $\alpha'^2 X_1$, u_o, $\alpha'^2 v_1$, are functions of $\alpha' r_o'$.

We have immediately

$$\alpha'^2 X_1 = \tfrac{1}{2}\beta(\alpha' r_o' - \beta) + \tfrac{1}{6}\beta^3 \qquad (2)$$

Consider $u_o(r)$. From

$$\frac{d^2 u_o}{dr^2} + V(r)u_o = 0 \qquad u_o(o) = 0 \qquad \psi_o = 1 - \alpha' r \qquad (3)$$

it follows that

$$\frac{d^2 u_o(\beta)}{d\beta^2} + \frac{V}{\alpha'^2}u_o(\beta) = 0 \qquad u_o(o) = 0 \qquad \psi_o = 1-\beta \qquad (4)$$

We may write $\frac{V}{\alpha'^2} = f(\beta, C_1, C_2)$ where C_1 and C_2 are the two constants specifying the potential. Then from (4), $u_o = u_o(C_1, C_2, A, B)$. But A, B, and C_1 are fixed by the two equations, $u_o(o) = 0$ and $\psi_o = 1-\beta$. Hence

$u_o = u_o(\beta, C_2)$

The definition of r_o' serves to fix C_2. We have

$$\alpha' r_0' = \tfrac{1}{2} \int_0^\infty \left[(1-\beta)^2 - u_o^2(\beta,\ C_2) \right] d\beta \qquad (5)$$

From which $C_2 \equiv g(\alpha' r_0')$

Hence both $\dfrac{V}{\alpha'^2}$ and u_o are functions only of $(\alpha' r_0')$.

Consider $v_1(r)$. The equation for $v_1(r)$ can be written in the form

$$\frac{d^2(\alpha'^2 v_1)}{d\beta^2} + \frac{V}{\alpha'^2}(\alpha'^2 v_1) = -u_o \qquad (6)$$

Since u_o and $\alpha'^2 X_1$ are functions only of $\alpha' r_0'$, it follows that $\alpha'^2 v_1$ is a function only of $\alpha' r_0'$.

2. It follows immediately that Q' is a function only of $\alpha' r_0'$, because

$$Q' = \frac{1}{r_0'^5} \int_0^\infty (X_1^2 - v_1^2)\,dr = \frac{1}{(\alpha' r_0')^5} \int_0^\infty \left[(\alpha'^2 X_1)^2 - (\alpha'^2 v_1)^2 \right] d\beta. \qquad (7)$$

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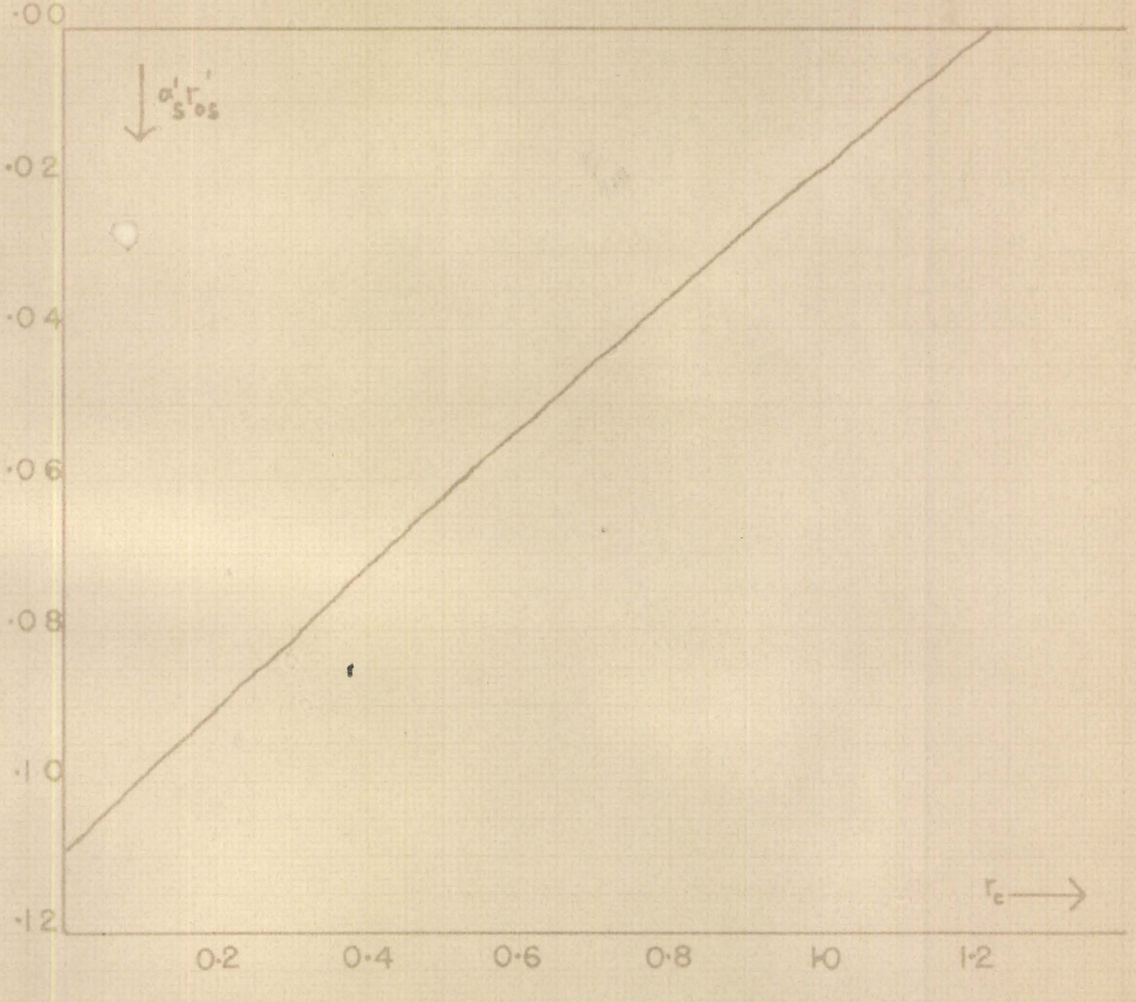

Fig. 2. $\alpha' r'_{os}$ vs. r_c for $r_{os} = 2{\cdot}6$, $a_s = -23{\cdot}68$

Length Unit $= 10^{-13}$ cm.

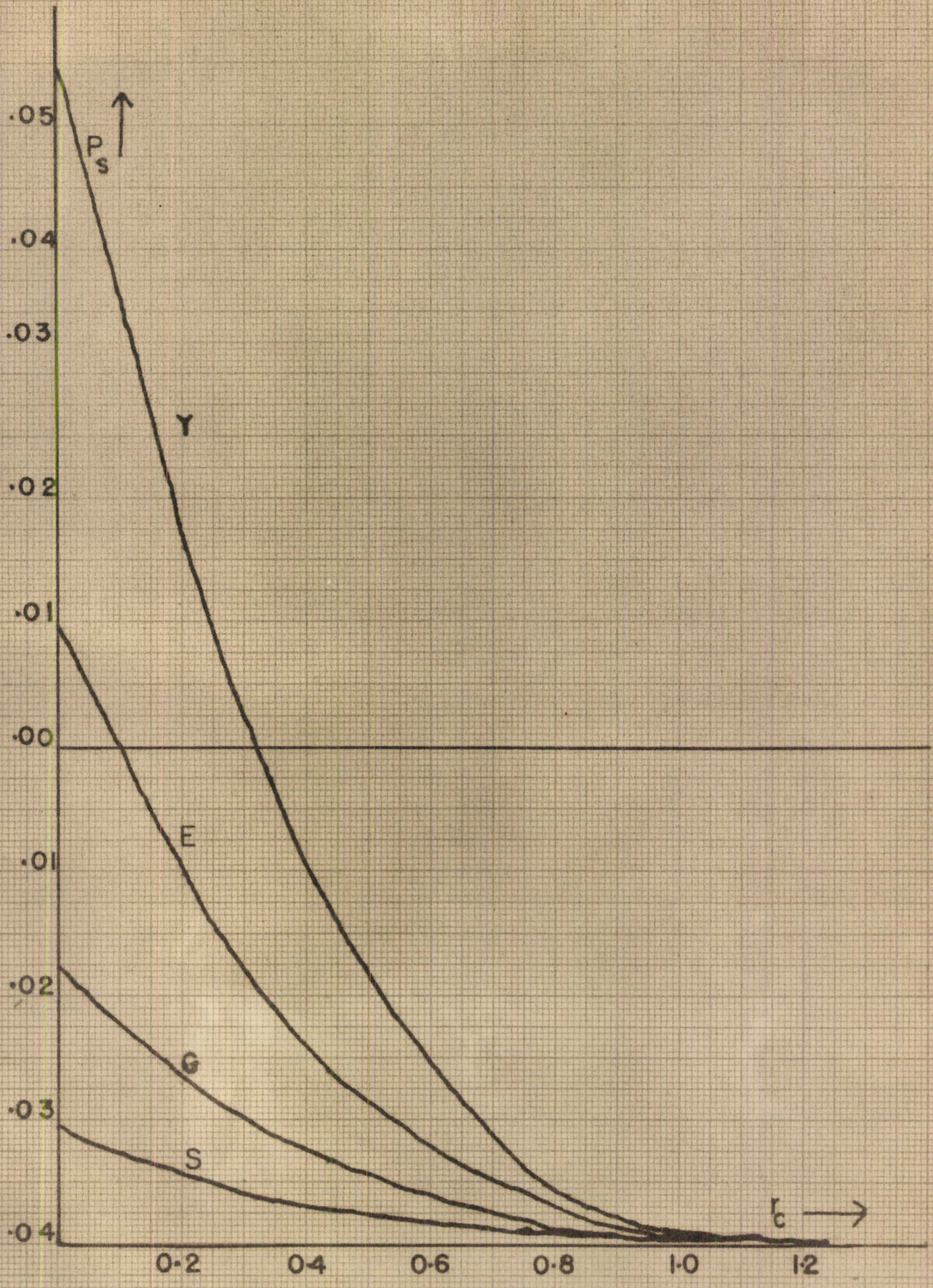

Fig.3. P_s vs. r_c for the four well shapes, r_{os}=2·6 a_s=-23·68

Length Unit=10^{-13} cm.

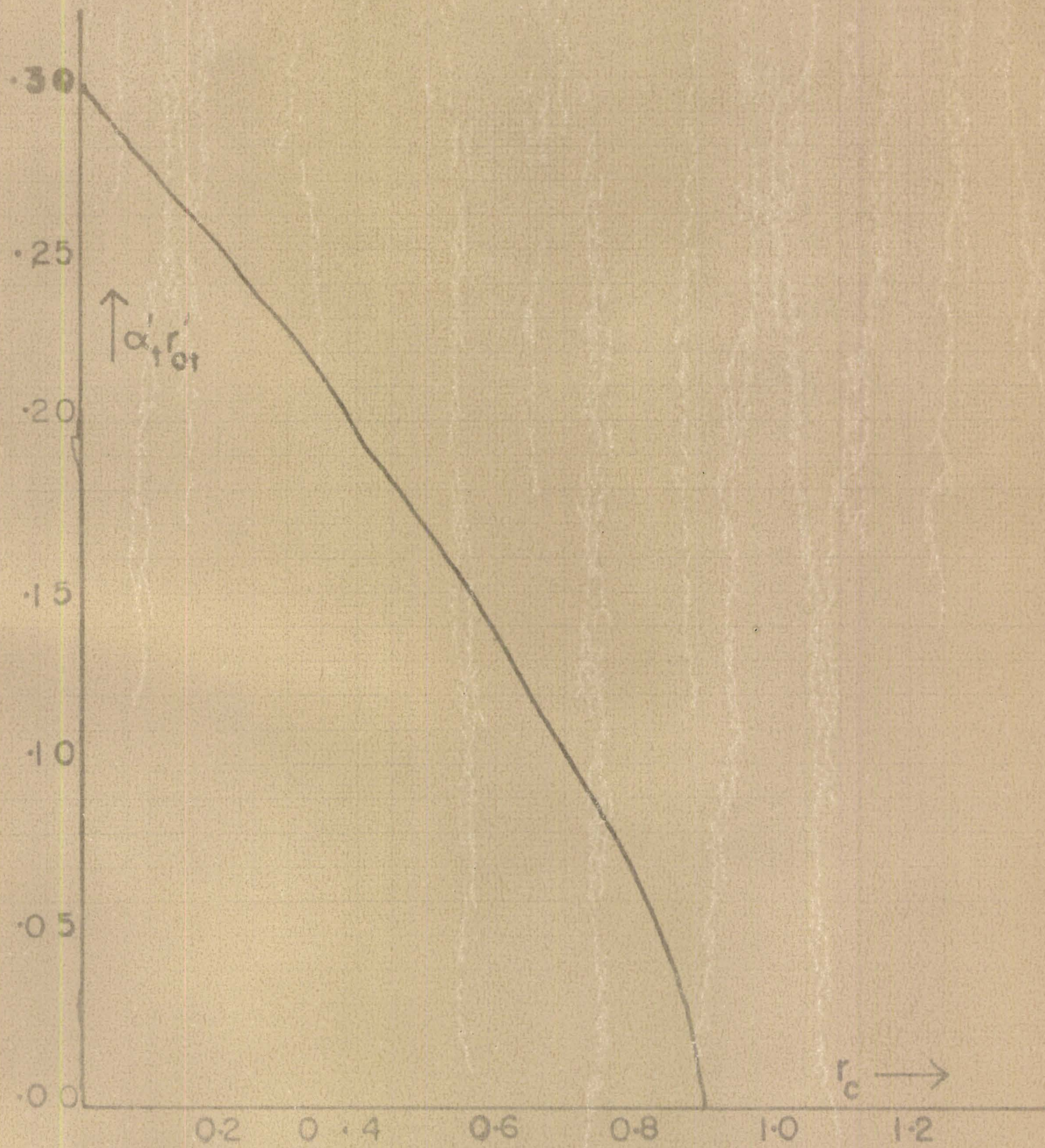

Fig. 4. $\alpha' r'_{ot}$ vs. r_c for $r_{ot} = 1{\cdot}56$, $a_t = 5{\cdot}28$

Length Unit $= 10^{-13}$ cm.

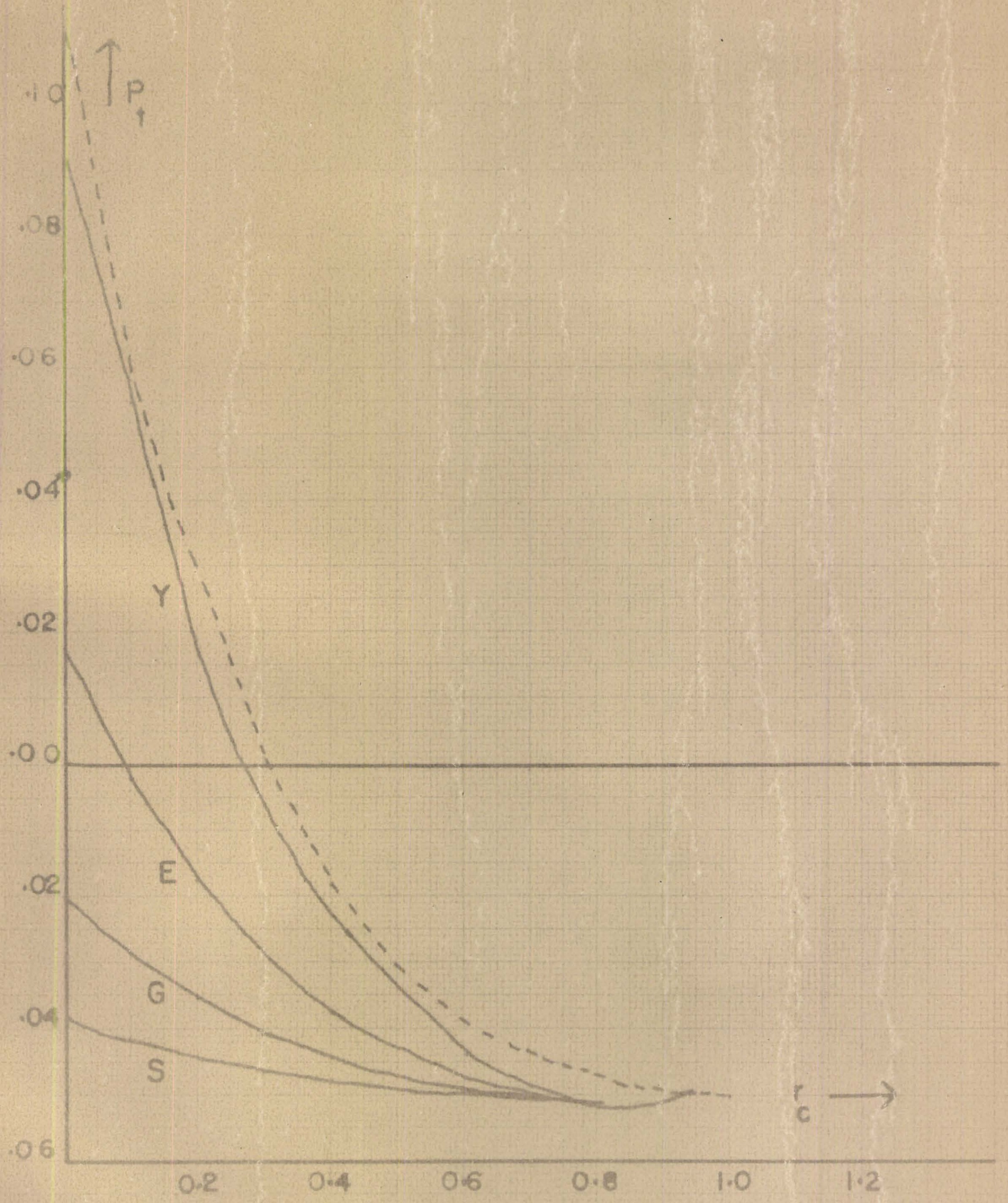

Fig. 5. P_t vs. r_c for the four well shapes, r_{ot}=1·56=a_t=5·28.

- - - - for a Hulthén shape (H2).

Length Unit = 10^{-13} cm.

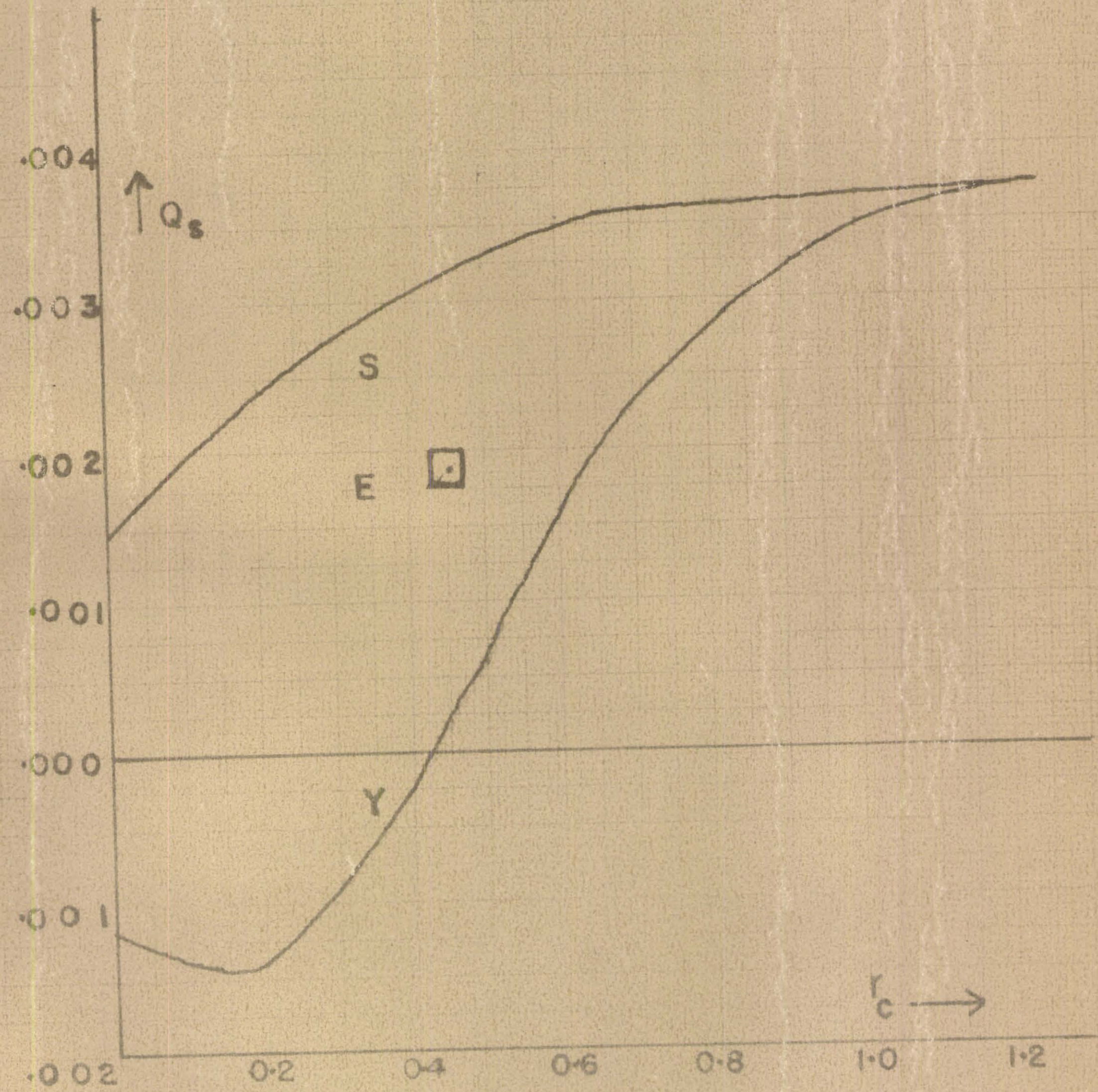

Fig.6. Q_s vs. r_c for Square and Yukawa shapes, $r_{os} = 26$, $a_s = 23{\cdot}68$

⊡ Exponential shape, $r_{os} = 25{\cdot}5$, $a_s = 23{\cdot}68$

Length Unit $= 10^{13}$ cm

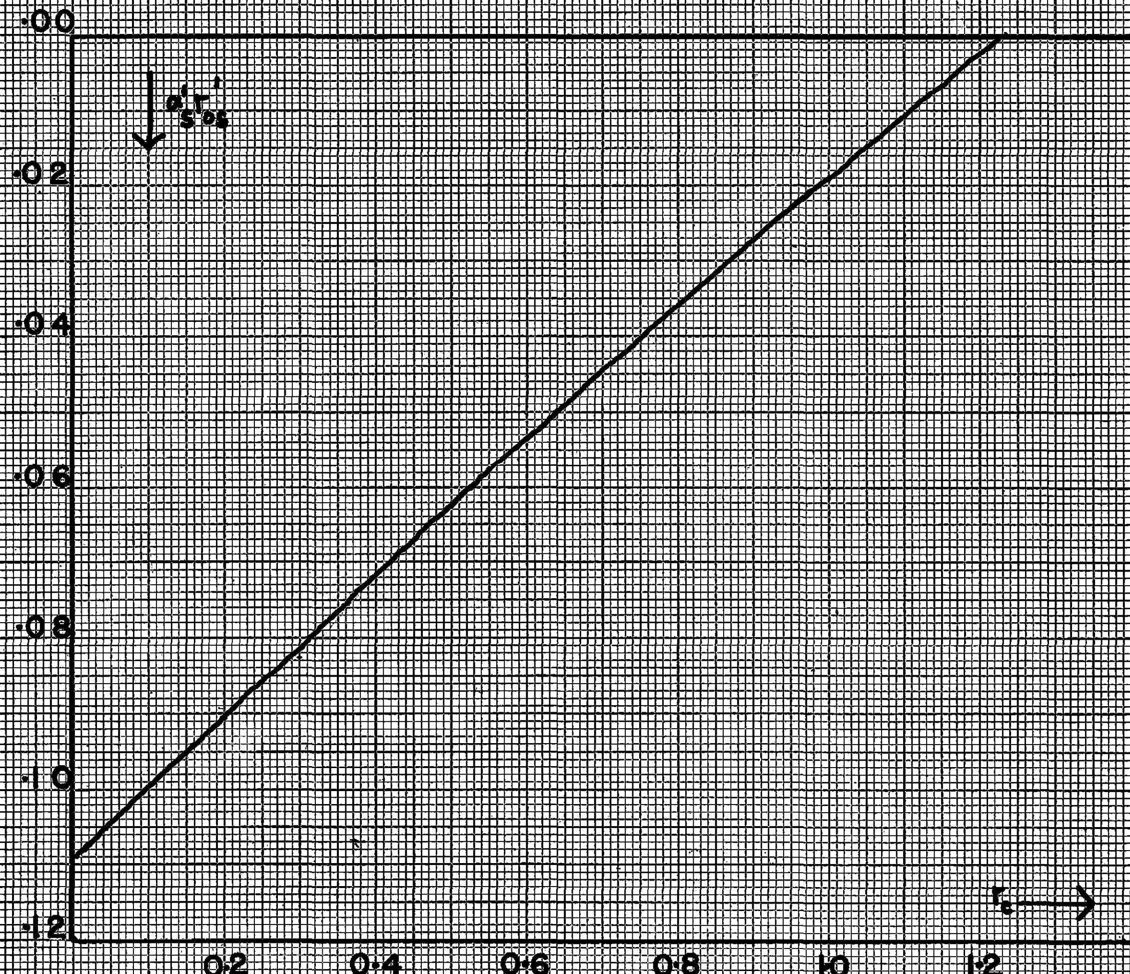

Fig.2. $\alpha r_{os}'$ vs. r_c for $r_{os} = 2{\cdot}6$, $a_s = {\cdot}2368$

Length Unit $= 10^{-13}$ cm.

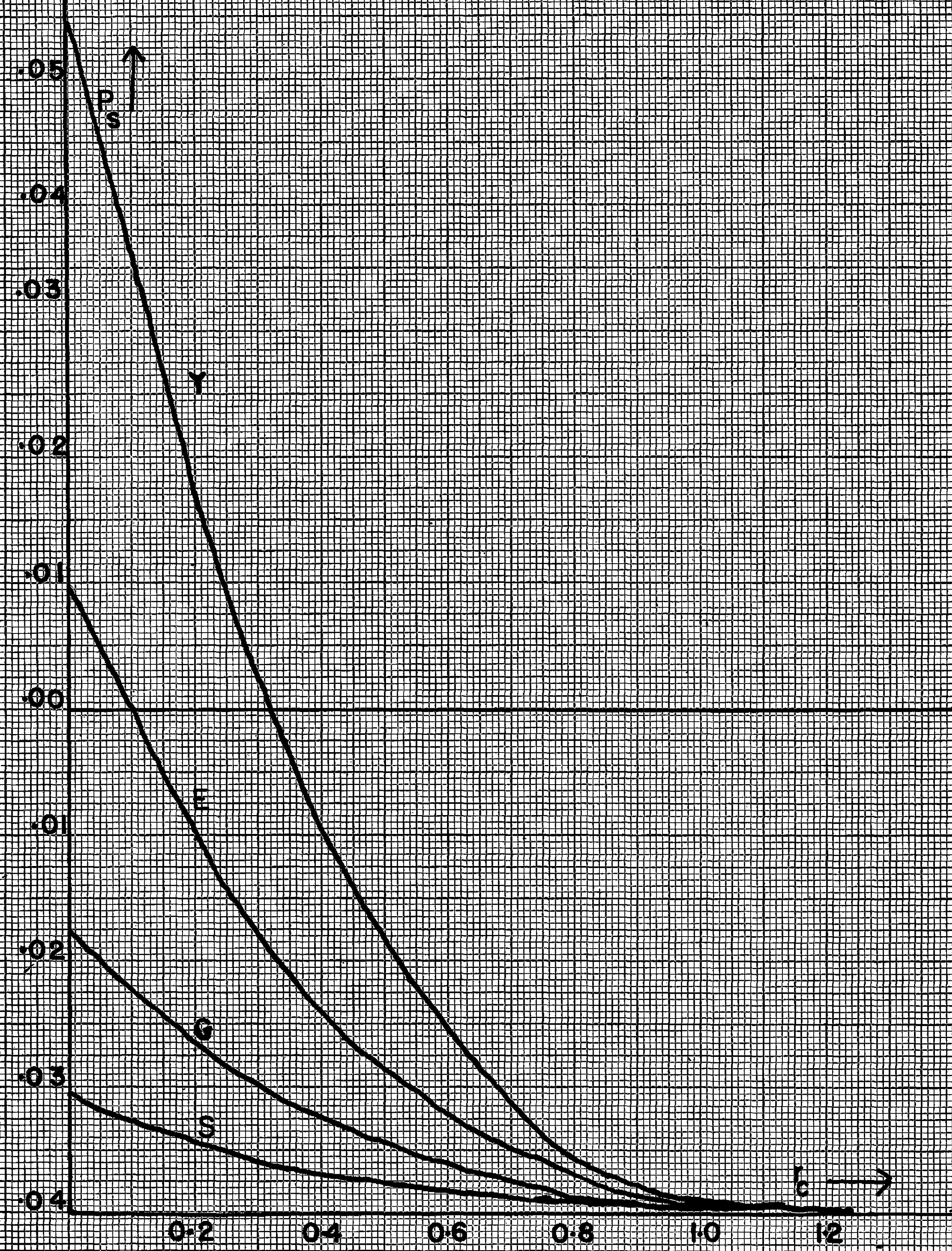

Fig.3. P_s vs. r_c for the four well shapes, $r_{os}=2\cdot6$ $a_s=-23\cdot68$
Length Unit $=10^{-13}$ cm.

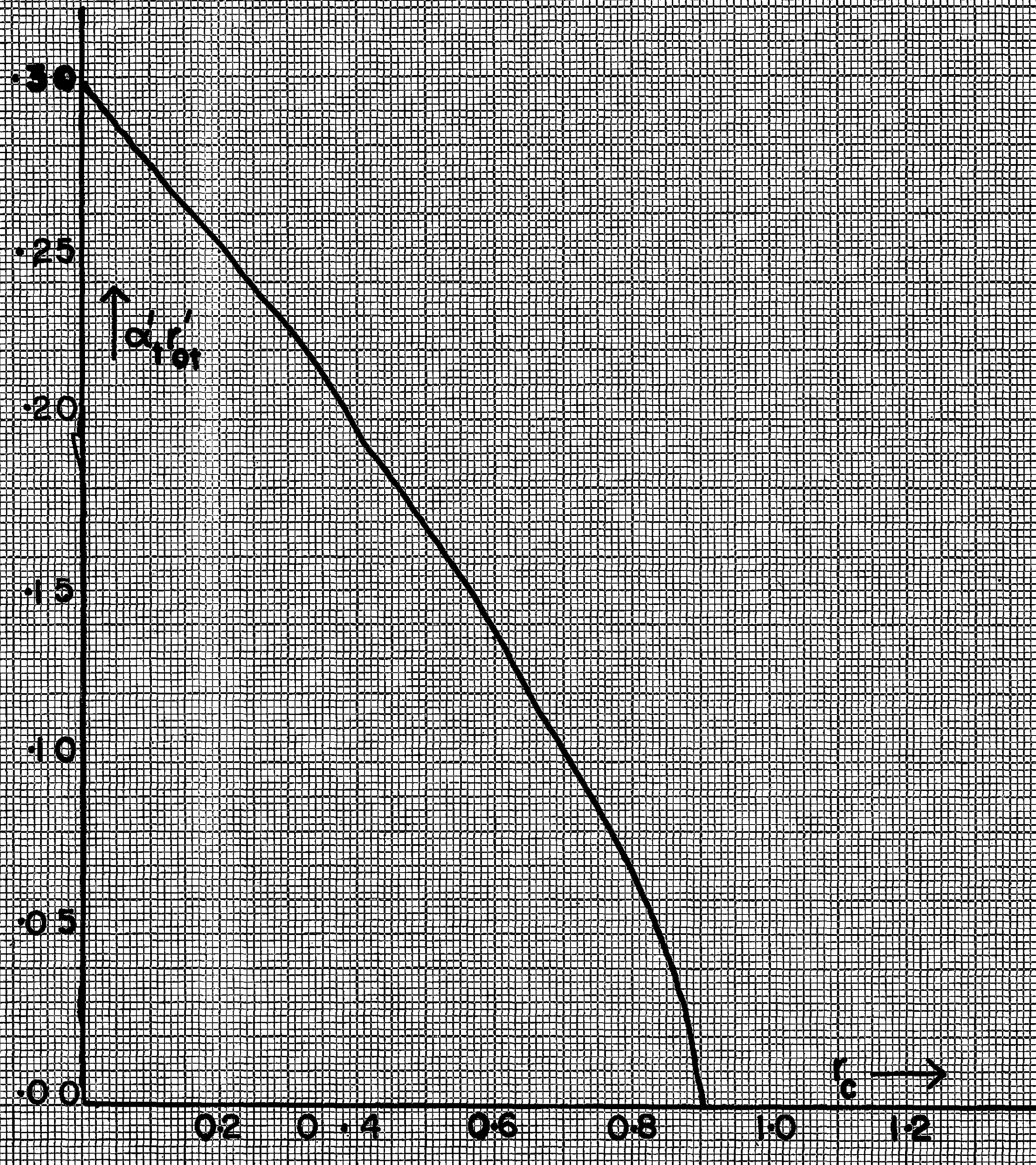

Fig. 4. $\alpha'_t r'_{ot}$ vs. r_c for $r_{ot} = 1{\cdot}56$, $a_t = 5{\cdot}28$

Length Unit $= 10^{-13}$ cm.

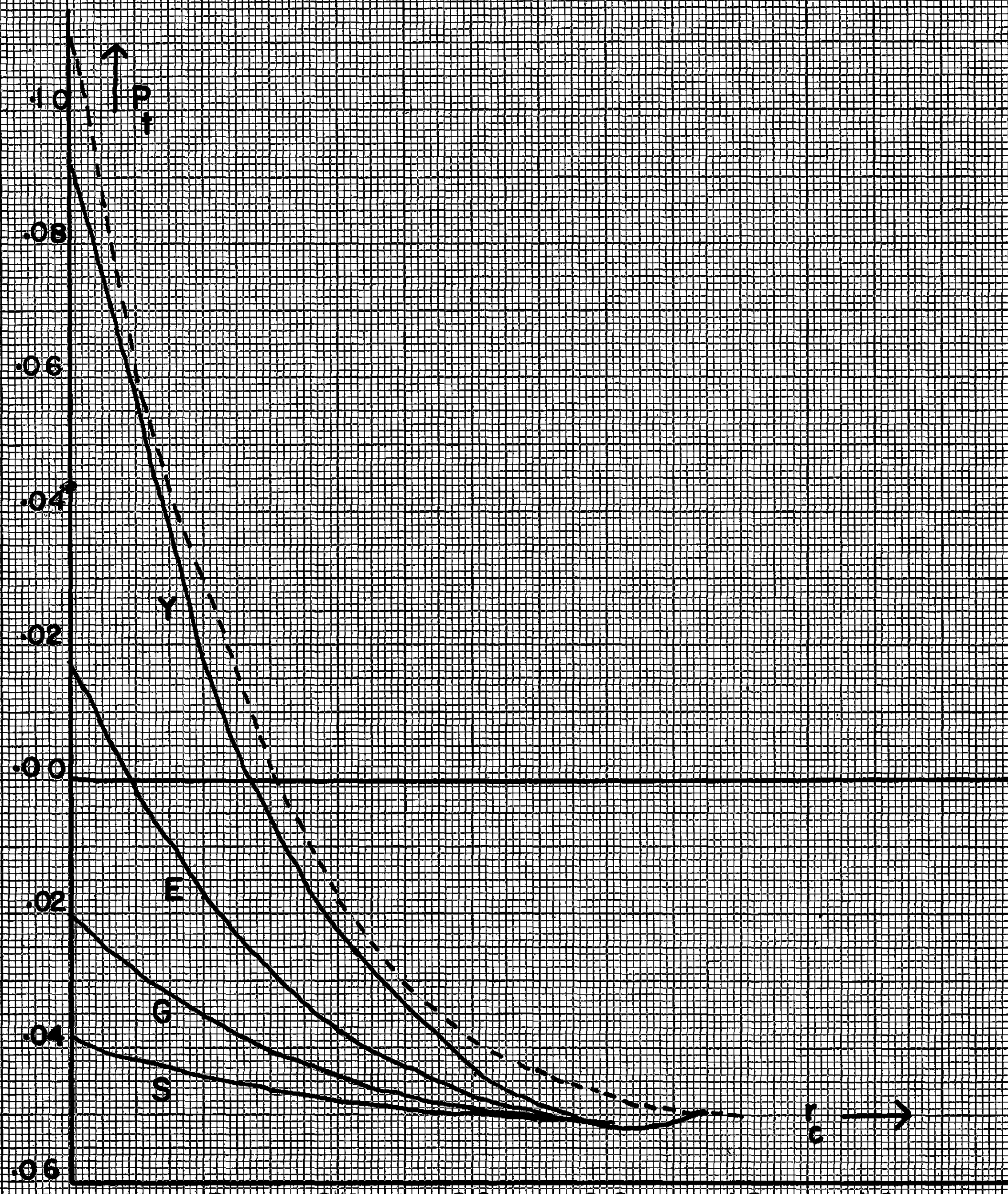

Fig. 5. P_t vs. r_c for the four well shapes, $r_{0t}=1\cdot56=a_t=5\cdot28$.

— — — for a Hulthén shape (H2).

Length Unit = 10^{-13} cm.

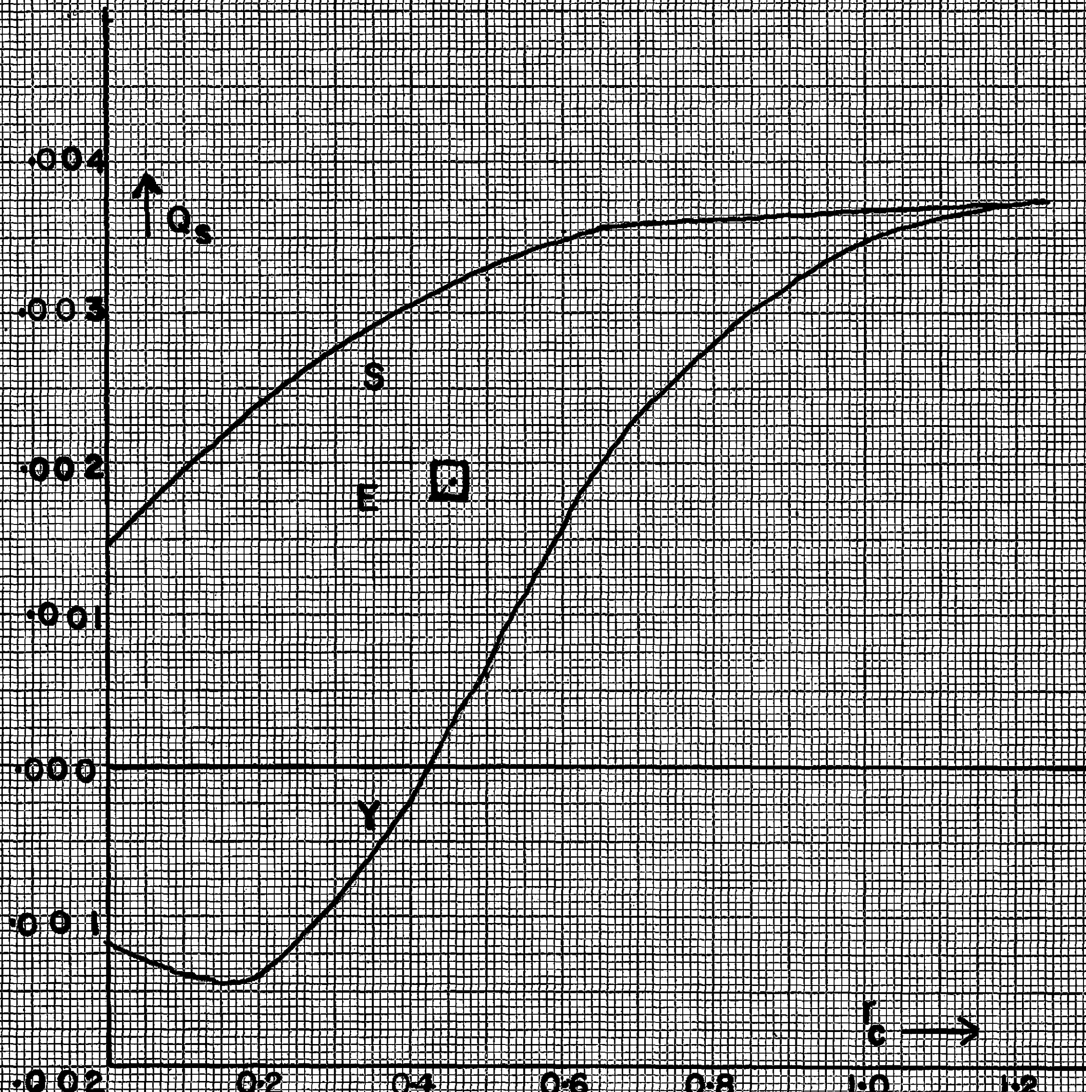

Fig.6. Q_s vs. r_c for Square and Yukawa shapes,

$r_{os} = 2.6$, $a_s = 23{\cdot}68$

⊡ Exponential shape, $r_{os} = 25{\cdot}5$, $a_s = 23{\cdot}68$

Length Unit $= 10^{13}$ cm

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DEPARTMENT OF PHYSICS

Comments on Ralph Pristo's Thesis

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