TWO-PION EXCHANGE THREE-BODY FORCE IN NUCLEAR MATTER

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by

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SCOPE AND CONTENTS:

The two-pion exchange three-body nuclear force is discussed, and an effective two-body interaction derived which is capable of reproducing approximately its effect in nuclear matter. Treating the three-body interaction as a perturbation, the first order contribution to the binding energy of nuclear matter is derived using the actual three-body interaction, the effective interaction, and a recently suggested method in which the three-body effects are taken approximately into account by modifying the pion mass in the one-pion exchange potential. Although the latter method leads to a simple prescription for calculating the three-body effects when no nucleon-nucleon cut-off is applied, the calculation is shown to be considerably more difficult for the realistic case when a cut-off is introduced, and the modified pion mass is momentum dependent.

On the other hand, the effective interaction is found to reproduce quite well the actual three-body effects in first order. This fact is used as a basis for calculating the second order contribution using the effective interaction, and so an estimate of the three-body effects in nuclear matter is obtained. For a reasonable value of nucleon-nucleon cut-off, the three-body forces are shown to contribute approximately 6 MeV additional binding to nuclear matter.

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CHAPTER 1

INTRODUCTION

The study of the two nucleon interaction has been a fundamental problem in nuclear physics, and although the processes involved are not yet completely understood, enough experimental data has been accumulated so that accurate phenomenological[(1)] or semi-theoretical[(2)] potentials can be constructed. These two-body potentials have then been applied in calculations involving many-body systems, on the assumption that many-body interactions are negligible. However, from a meson-theoretical point of view, many-body forces may well be important, and in particular the three-body force may play a significant role in determerming the binding energy of a many-nucleon system.

The possibility of three-body forces in many-nucleon systems has been discussed for some time[(3,4)], and the effects due to both two-pion and three-pion exchange processes in some many-body systems have been considered. Bhaduri, Loiseau and Nogami[(5,6)] (to be referred to as BLN) have pointed out that for the case of hypernuclei the three-body two-pion exchange process may be expected to play an important role and have calculated the effects of such a force to first order in nuclear matter. More recently, Brown and Green[(7)] (to be referred to as BG) have examined the NNN interaction in nuclear matter, and have shown that, under certain assumptions, the three-body interaction due to the two-pion exchange (TPE) process may be considered as an effective two-body interaction, similar in form to the one-pion-exchange potential (OPEP), but with a modified pion mass. By using this method, BG do not need to find the effective interaction in coordinate space, and so the perturbation calculation is simple, at least for the case in which no nucleon-nucleon cut-off is introduced. As will be shown later, the prescription is not as simple when an NN cut-off is applied and the modified pion mass is momentum dependent.

In this work, the effective interaction due to the three-body force will be derived in coordinate space, and the contribution of this interaction to the binding energy of nuclear matter will be calculated to second order. Considerable attention will be devoted to the first order calculation, where it will be shown how this interaction is related to the more conventional three-body interaction of BLN, and to the effective mass approach of BG. Although the primary process to be considered is the TPE, some higher order effects will be taken into account through the introduction of pionic form factors[8,9]. It will be seen that these form factors tend to suppress the high momentum components of the potential, so that even though the effective interaction derived here and the potential of BLN differ in some singular contact terms characteristic of high momentum components, for realistic form factors the two give very similar results when calculated to first order.

As will be shown in Chapter II, the perturbation series will require a knowledge of the two-body as well as the three-body force in order to obtain the total second order contribution. In principle, one of the two-body interactions mentioned earlier which fit all the available two-nucleon scattering data could be used. Instead, the tail of the OPEP will be used, in the spirit of the Moszkowski-Scott separation method[10]. Two reasons motivate this choice. Firstly, it is well known that for large distances (r>2F) the OPEP approximates quite well the actual nucleon-nucleon interaction. Secondly, by using the OPEP with a cut-off at small distances, analytic evaluation of most of the required results will be possible, a situation which would not normally prevail with a more complicated NN interaction. For comparison purposes, the OPEP contribution to nuclear matter will also be calculated. This will permit on accurate estimation of the relative importance of the three-body interaction, both in first and second order.

In Chapter II the Rayleigh-Schrodinger perturbation series to second order will be discussed briefly, and the single exchange terms separated out for special consideration. In Chapter III the three-body interaction for the TPE process will be discussed, and from it the effective two-body interaction, which is to be used in the remaining calculations, will be derived. The first order effects of the BLN potential, the BG potential, and the effective interaction expressed in coordinate space will be discussed in detail, and differences and similarities pointed out. The second order calculation for the effective interaction will be derived in Chapter V, and numerical results given in Chapter VI. Finally, the importance of the three-body interaction will be discussed, and some of the remaining difficulties and ambiguities mentioned.

CHAPTER II

PERTURBATION SERIES TO SECOND ORDER

Nuclear matter, although a theoretical construct, has been a useful many-body system for testing the effects of an assumed nuclear force. It consists of an infinite array of equal numbers of protons and neutrons, uniformly distributed in space, but with the Coulomb interaction between the protons inoperative. This means that the only interaction between the nucleons is due to the nuclear force, and because the system is infinite in extent, surface effects do not complicate the analysis. The main interest here will not be in constructing a combined two and three-body interaction which can reproduce the expected binding energy per particle at the equilibrium density, but rather in determining the relative importance of the three-body as compared to the two-body force.

The approximate effect of the TPE three-body force to nuclear matter will be estimated using the Rayleigh-Schrodinger perturbation series to second order. The system Hamiltonian will be written as

$$H_t = H_0 + H \ , \qquad (1)$$

where H_0 is the unperturbed Hamiltonian, consisting of any effective one-body potential generated by the system of nucleons, and H is the perturbation which in this case will consist of the sum of a two and three-body interaction. Because of the translational invariance of nuclear matter, any effective one-body interaction must be independent of the position variable, and so the unperturbed system wave functions, $|\phi_n\rangle (n=0,1,2\ \ldots)$, must be Slater determinants of plane waves. These single particle wave functions will be assumed to satisfy periodic boundary conditions in a large but finite volume Ω, which will eventually be allowed to approach infinity.

The interaction term H may be written as

$$H = V + W \tag{2}$$

where V and W are given by

$$V = \frac{1}{2} \sum_{i,j} v(\underset{\sim}{r}_i, \underset{\sim}{r}_j) \text{ and } W = \frac{1}{2} \sum_{i,j,k} w(\underset{\sim}{r}_i, \underset{\sim}{r}_j; \underset{\sim}{r}_k) \ , \tag{3}$$

with v and w being respectively the OPEP, and the three-body potential to be derived in the following chapter. The variable $\underset{\sim}{r}_\nu$ is the position vector of the particle in the ν^{th} state, and the sums are over all occupied single particle states. To second order, the energy shift ΔE due to the perturbation H is given by

$$\Delta E = \langle\phi_0|V+W|\phi_0\rangle + \langle\phi_0|(V+W)\ 1/b\,(V+W)|\phi_0\rangle \ , \tag{4}$$

where

$$\frac{1}{b} = \frac{1 - |\phi_0\rangle\langle\phi_0|}{E_0 - H_0} ,$$

with E_0 being the unperturbed ground state energy. The terms containing only the two-body interaction are

$$\Delta E^{(1)}(\mathrm{OPEP}) = \langle\phi_0|V|\phi_0\rangle , \tag{5}$$

and

$$\Delta E^{(2)}(\mathrm{OPEP}) = \langle\phi_0|V\ 1/b\ V|\phi_0\rangle . \tag{6}$$

The standard reduction of these expressions in terms of plane wave states gives

$$\Delta E^{(1)}(\mathrm{OPEP}) = \frac{1}{2} \sum_{i,j} \langle ij|v|ij-ji\rangle , \tag{7}$$

and

$$\Delta E^{(2)}(\mathrm{OPEP}) = \frac{1}{2} \sum_{i,j} \sum_{i',j'} \frac{\langle ij|v|i'j'\rangle\langle i'j'|v|ij-ji\rangle}{\varepsilon_i+\varepsilon_j-\varepsilon_i'-\varepsilon_j'} , \tag{8}$$

where ε_ν is the energy of the particle in the state ν. The state labels i and j contain all the quantum numbers necessary to specify the single particle states, and the sums over i and j are to be taken over all occupied levels, while the corresponding sums over i' and j' are over all unoccupied levels. The diagramatic representation of these two-body contributions is given in figure 1, where the wavy line denotes the interaction v. (For simplicity, all the terms of the perturbation series will be represented by open-ended, rather than closed-loop, diagrams).

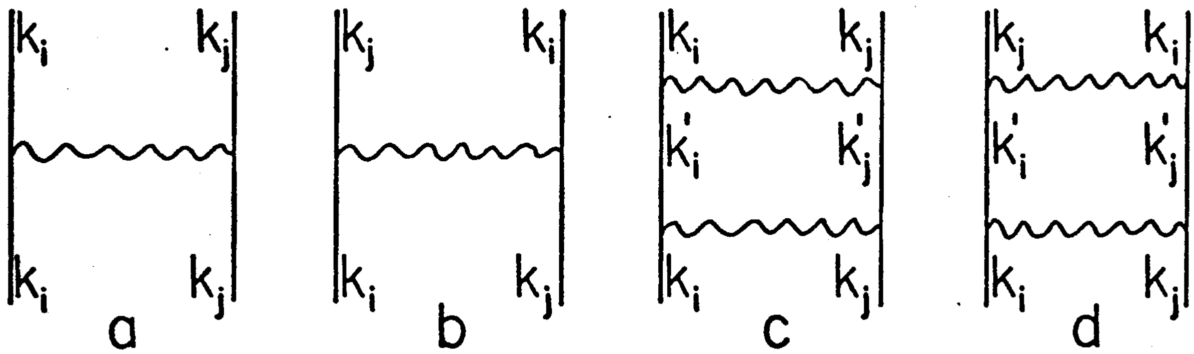

Figure 1. Schematic of the first and second order two-body contributions to nuclear matter. a) and b) are the first order, c) and d) the second order direct and exchange contributions. The wavy line denotes the two-body interaction.

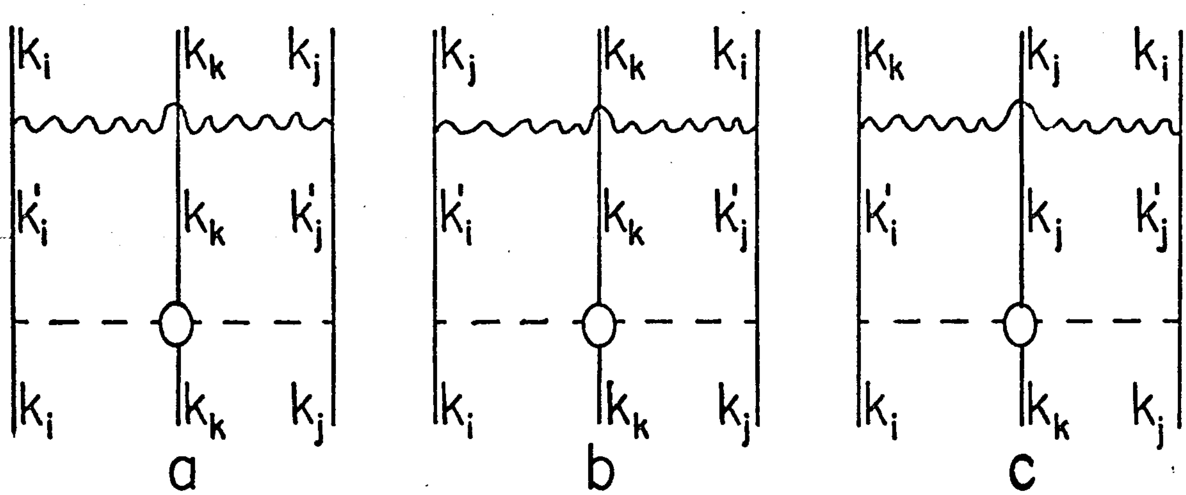

Figure 2. Schematic of the three-body cross term contributions, <V 1/b W>, to nuclear matter. a) and b) represents the direct and single exchange contributions, while c) is one of two double exchange contributions.

Consider next the cross term involving both V and W, given by

$$\Delta E^{(2)}(\mathrm{C.T.}) = \langle\phi_0|V\ 1/b\ W|\phi_0\rangle\ , \tag{9}$$

where the abbreviation C.T. stands for "cross term". Because of the presence of the two-body operator, any intermediate states can differ in at most two single particle excitations from the ground state. Since single particle excitations are not permitted because of the requirement of the conservation of linear momentum, only two particle excitations are possible, and these give rise to several diagrams, some of which are shown in figure 2. The assumption will now be made that the major contribution of the three-body force comes from the single exchange terms, so that diagrams such as figure 2c may be neglected. The validity of this assumption will be discussed later, but accepting it for the time being permits $\Delta E^{(2)}$(C.T.) to be written as

$$\Delta E^{(2)}(\mathrm{C.T.}) = \frac{1}{2} \sum_{i,j,k} \sum_{i',j'} \frac{\langle ijk|v|i'j'k\rangle\langle i'j'k|w|ijk-jik\rangle}{\varepsilon_i+\varepsilon_j-\varepsilon_i'-\varepsilon_j'} . \tag{10}$$

Because of the Hermitian nature of V and W,

$$\langle\phi_0|V\ 1/b\ W|\phi_0\rangle = \langle\phi_0|W\ 1/b\ V|\phi_0\rangle\ , \tag{11}$$

and so the total contribution to the second order energy due to the cross terms is 2 $\Delta E^{(2)}$(C.T.).

Finally, there are the terms involving only three-body interactions, given by

$$\Delta E^{(1)}(\mathrm{T.B.}) = \langle\phi_0|W|\phi_0\rangle \ , \tag{12}$$

$$\Delta E^{(2)}(\mathrm{T.B.}) = \langle\phi_0|W\ 1/b\ W|\phi_0\rangle \ , \tag{13}$$

where the abbreviation T.B. stands for "three body". Again assuming that the major contribution to $\Delta E^{(1)}$(T.B.) and $\Delta E^{(2)}$(T.B.) is given by single exchange terms, these expressions simplify to

$$\Delta E^{(1)}\mathrm{T.B.}) = \frac{1}{2} \sum_{i,j,k} \langle ijk|w|ijk-jik\rangle \ , \tag{14}$$

and

$$\Delta E^{(2)}(\mathrm{T.B.}) = \frac{1}{2} \sum_{i,j,k} \sum_{i',j',\ell} \frac{\langle ijk|w|i'j'k\rangle\langle i'j'\ell|w|ijk-ji\ell\rangle}{\varepsilon_i+\varepsilon_j-\varepsilon_i'-\varepsilon_j'} \ . \tag{15}$$

The diagramatic representation of these contributions is given in figures 3 and 4.

As can be seen from figures 2 through 4, if only single exchange terms are considered, the nucleon in the k^{th} (or ℓ^{th}) state is always a spectator. This fact permits the derivation of an effective interaction, for on inspecting equations (10), (14), and (15), it will be seen that a term of the form $\langle k|w|k\rangle$ (or $\langle\ell|w|\ell\rangle$) can be immediately evaluated, and the sum over k (or ℓ) performed to give N, the number of nucleons present. This integration over $\underset{\sim}{r}_k$ gives $u(\underset{\sim}{r}_i,\underset{\sim}{r}_j)$, the effective interaction due to the three-body force. Then

$$U = \frac{1}{2} \sum_{i,j} u(\underset{\sim}{r}_i,\underset{\sim}{r}_j) \ ,$$

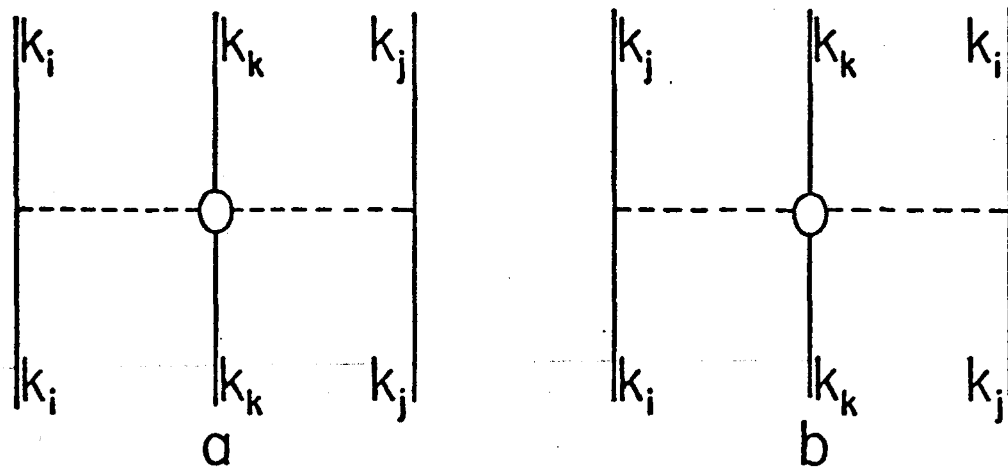

Figure 3. Schematic of the first order contribution of the three-body force to nuclear matter. a) is the direct contribution, b) the single exchange contribution.

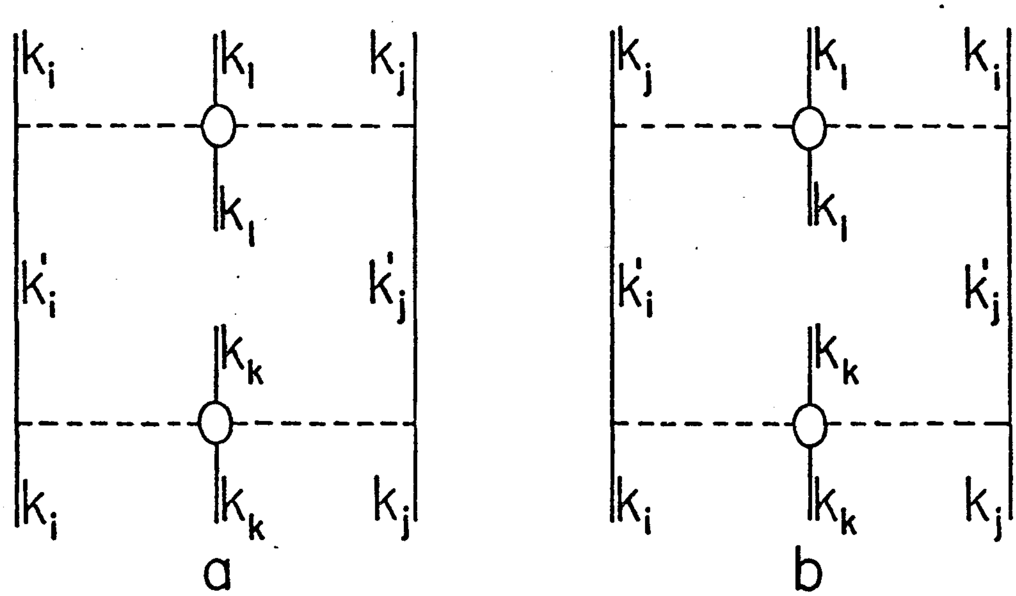

Figure 4. Schematic of the second order three-body contribution, <U 1/b U>, to nuclear matter. a) is the direct contribution, b) the single exchange contribution.

with

$$u(\underset{\sim}{r}_i,\underset{\sim}{r}_j) = \frac{N}{\Omega}\int d\underset{\sim}{r}_k \; w(\underset{\sim}{r}_i,\underset{\sim}{r}_j;\underset{\sim}{r}_k) \; , \tag{16}$$

and equations (10), (14), and (15) become

$$\Delta E^{(2)}(\mathrm{C.T.}) = \frac{1}{2}\sum_{i,j}\sum_{i',j'} \frac{\langle ij|v|i'j'\rangle\langle i'j'|u|ij-ji\rangle}{\varepsilon_i+\varepsilon_j-\varepsilon_i'-\varepsilon_j'} \; , \tag{17}$$

$$\Delta E^{(1)}(\mathrm{T.B.}) = \frac{1}{2}\sum_{i,j} \langle ij|u|ij-ji\rangle \; , \tag{18}$$

and

$$\Delta E^{(2)}(\mathrm{T.B.}) = \frac{1}{2}\sum_{i,j}\sum_{i',j'} \frac{\langle ij|u|i'j'\rangle\langle i'j'|u|ij-ji\rangle}{\varepsilon_i+\varepsilon_j-\varepsilon_i'-\varepsilon_j'} \; . \tag{19}$$

In the following chapter the form of u, based on the definition given in equation (16) will be found, and in Chapters IV and V explicit expressions for the first and second order contributions of v and u will be derived. It is worth noting that similar expressions hold for a Λ-particle embedded in nuclear matter, the only difference being in the form of u. In this case, the perturbation expansion to second order given above is exact, with the k^{th} nucleon state being replaced by that of the Λ-particle, for the Λ-particle is of necessity a spectator, and so only single exchange terms can contribute.

CHAPTER III

THE THREE-BODY POTENTIAL

Quantum field theory attributes the nuclear force to the exchange of virtual mesons between the interacting nucleons. Besides being an intutively satisfactory description of the interaction mechanism, this method of analysis permits the nuclear force to be described in terms of a series of processes, the importance of which depends mainly on the nucleon separation. Hence at large distances, the one-pion exchange process is believed to be the main contributor to the two-nucleon force, while at smaller distances, higher order processes, such as the exchange of more than one meson, or the exchange of heavier mesons, become important. In a similar way, the three-body force can be described in terms of meson exchange, and the lowest order process which contributes is the two-pion exchange depicted in figure 5a.

Although the TPE is the fundamental process to be considered, it is nevertheless possible to include in an approximate way higher order effects, such as those depicted in figure 5b, through the use of pionic form factors[7]. Their effect is similar to that obtained by introducing an NN cut-off when considering the two-nucleon interaction, in that they suppress the effects of the short range part of the

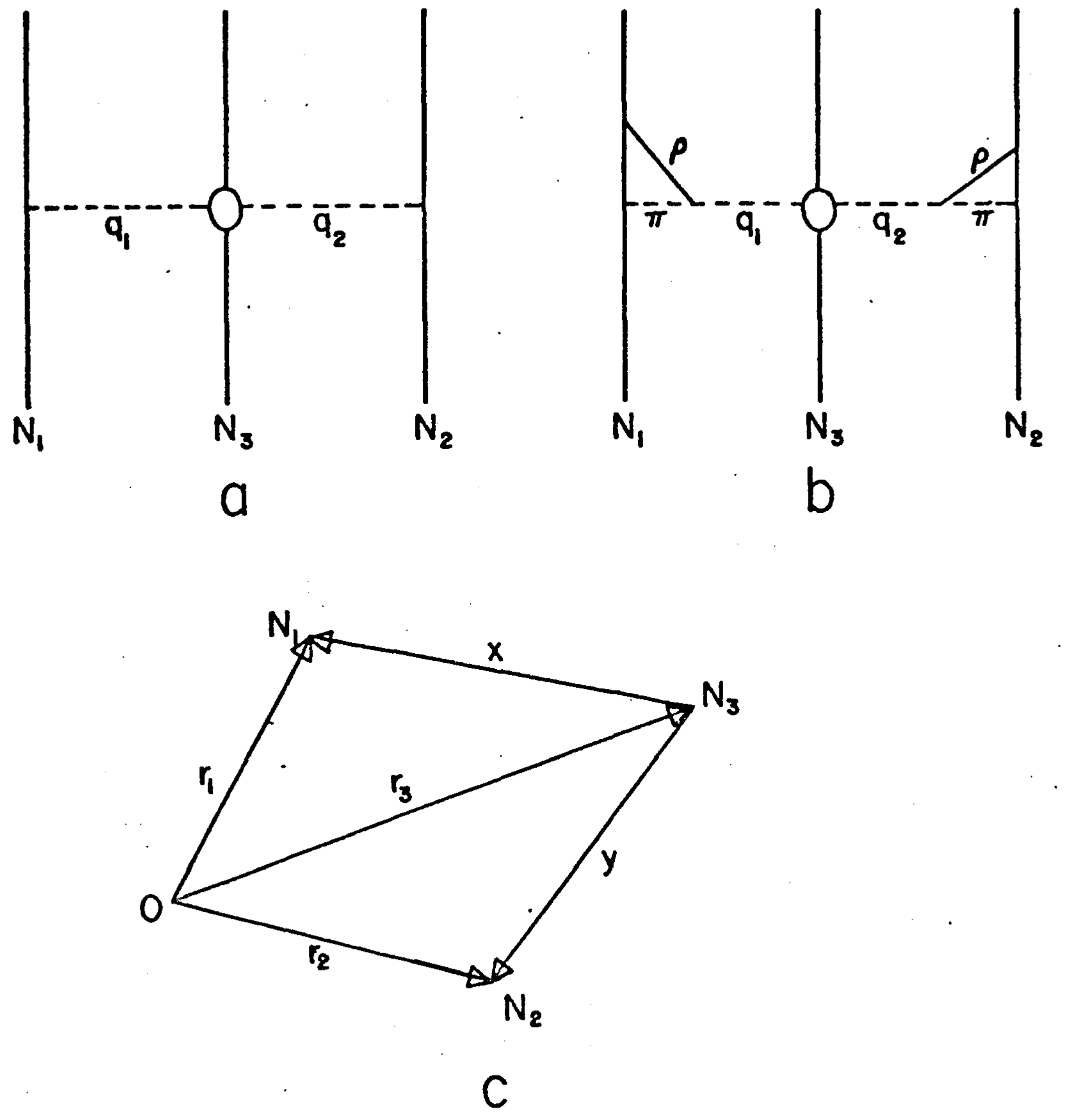

Figure 5. The process which gives rise to the TPE three-body force is shown in a), where the blob on the N_3 line represents the N* resonance. In c) the geometry of the situation is sketched, while b) is a schematic of some higher order processes which are included in the form factors.

potential, although the suppression is less severe than with a cut-off. The functional dependence of the form factors has considerable theoretical and experimental basis[8,9], and the following form will be assumed:

$$\frac{K^2(q^2)\ K'(q^2)}{q^2 + \mu^2} = \int_0^\infty \frac{\alpha(m^2)\,dm^2}{q^2 + m^2} , \tag{20}$$

where K and K' are the vertex and propagator form factors respectively, μ is the pion mass, and the spectral function, $\alpha(m^2)$, will be assumed to have the form

$$\alpha(m^2) = \delta(m^2-\mu^2) - (1-\xi)\ \delta(m^2-\eta^2) , \tag{21}$$

where ξ and η are constants which can be partially fixed by experiment. Different values of ξ and η give rise to different form factors, and calculations will be performed with two sets of reasonable values.

The three-body interaction due to figure 5a will be derived in the static approximation in which the assumption is made that the three nucleons are at rest, and that the energy of the exchanged pions approaches zero. For the single exchange terms, the nucleon labelled N_3 in figure 5a always appears as a spectator particle, and so the S-matrix for this process is identical to that for the ΛNN case given in reference 5, where N_3 takes the place of the Λ particle. Then in units of $\hbar = c = 1$,

$$S = \frac{4\pi f_N^2}{(2\pi)^6 \mu^2}\ \underset{\sim}{\tau}_1 \cdot \underset{\sim}{\tau}_2 \int d\underset{\sim}{q}_1\ d\underset{\sim}{q}_2\ \frac{(\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{q}_1)(\underset{\sim}{\sigma}_2 \cdot \underset{\sim}{q}_2) \langle \underset{\sim}{q}_1 | S_{\pi N}^{(3)} | \underset{\sim}{q}_2 \rangle}{(q_1^2+\mu^2)(q_2^2+\mu^2)}$$

$$\times \; K(q_1^2)\; K'(q_1^2)\; K(q_2^2)\; K'(q_2^2)\, e^{-i(\underset{\sim}{q}_1 \cdot \underset{\sim}{r}_1 - \underset{\sim}{q}_2 \cdot \underset{\sim}{r}_2)} \;, \tag{22}$$

where f_N^2 is the pseudovector πN coupling constant, $\underset{\sim}{q}_1$ and $\underset{\sim}{q}_2$ are the momenta of the exchanged pions, and $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ are the coordinates of the nucleons N_1 and N_2. $\underset{\sim}{\sigma}_1$ and $\underset{\sim}{\sigma}_2$ are the Pauli spin matrices for the two nucleons, while $\underset{\sim}{\tau}_1$ and $\underset{\sim}{\tau}_2$ are the corresponding isospin matrices. The scattering matrix, $\langle \underset{\sim}{q}_1 | S_{\pi N}^{(3)} | \underset{\sim}{q}_2 \rangle$, for a zero-energy pion is given by[3]

$$\langle \underset{\sim}{q}_1 | S_{\pi N}^{(3)} | \underset{\sim}{q}_2 \rangle = 2\pi i \;\delta(0)\; K(q_1^2)\; K(q_2^2)\{(A+B)[(\underset{\sim}{\sigma}_3 \cdot \underset{\sim}{q}_2)(\underset{\sim}{\sigma}_3 \cdot \underset{\sim}{q}_1) + (\underset{\sim}{\sigma}_3 \cdot \underset{\sim}{q}_1)(\underset{\sim}{\sigma}_3 \cdot \underset{\sim}{q}_2)] + 2D\}\; e^{i(\underset{\sim}{q}_1 - \underset{\sim}{q}_2)\cdot \underset{\sim}{r}_3} \;, \tag{23}$$

where A, B, and D are constants in the nonrelativistic approximation, with (A+B) being related to the p-wave πN scattering, while D is related to the s-wave scattering. Since experimentally the πN s-wave scattering is known to be very small[11], D will be taken to be zero. When equation (23) is substituted in equation (22), the S matrix takes the form

$$S = -2\pi i \;\delta(0) w \;, \tag{24}$$

where

$$w = -\frac{4\pi F_N^2}{(2\pi)^6 \mu^2}(A+B)\,\underset{\sim}{\tau}_1 \cdot \underset{\sim}{\tau}_2 \int d\underset{\sim}{q}_1 d\underset{\sim}{q}_2\, (\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{q}_1)\,(\underset{\sim}{\sigma}_2 \cdot \underset{\sim}{q}_2)\, \frac{K^2(q_1^2)K'(q_1^2)K^2(q_2^2)K'(q_2^2)}{(q_1^2+\mu^2)(q_2^2+\mu^2)}$$

$$\times [(\underset{\sim}{\sigma}_3 \cdot \underset{\sim}{q}_2)(\underset{\sim}{\sigma}_3 \cdot \underset{\sim}{q}_1) + (\underset{\sim}{\sigma}_3 \cdot \underset{\sim}{q}_1)(\underset{\sim}{\sigma}_3 \cdot \underset{\sim}{q}_2)]\; e^{-i(\underset{\sim}{q}_1 \cdot \underset{\sim}{r}_1 - \underset{\sim}{q}_2 \cdot \underset{\sim}{r}_2)}\, e^{i(\underset{\sim}{q}_1 - \underset{\sim}{q}_2)\cdot \underset{\sim}{r}_3} \;. \tag{25}$$

w is a function of the three nucleon coordinates, $\underset{\sim}{r}_1$, $\underset{\sim}{r}_2$ and

$\underset{\sim}{r}_3$, and is interpreted as the three-body interaction. From this expression it will be shown how the potential of BLN, the modified pion mass approach of BG, and the effective interaction in coordinate space may be derived.

In order to derive the three-body potential of BLN, consider initially the case in which

$$H(q^2) \equiv K^2(q^2)\ K'(q^2) = 1\ . \tag{26}$$

Because the integrals over $\underset{\sim}{q}_1$ and $\underset{\sim}{q}_2$ in equation (25) are divergent, it is necessary to remove the singular terms appearing in $w(\underset{\sim}{r}_1,\underset{\sim}{r}_2;\underset{\sim}{r}_3)$ due to these divergences, and the standard procedure is to replace

$$\underset{\sim}{\sigma}.\underset{\sim}{q} \quad \text{by} - \underset{\sim}{\sigma}.\underset{\sim}{\nabla}_r\ . \tag{27}$$

where the coordinate variable $\underset{\sim}{r}$ corresponds to the momentum variable $\underset{\sim}{q}$, and remove it from the integral. The integrations over $\underset{\sim}{q}_1$ and $\underset{\sim}{q}_2$ may now be done using equation A2, and after applying the gradient operators using equation A7, the following expression for w is obtained:

$$w(x,y) = -\frac{C_p}{9}\ \underset{\sim}{\tau}_1\cdot\underset{\sim}{\tau}_2\{[\underset{\sim}{\sigma}_2\cdot\underset{\sim}{\sigma}_3+S_{23}(\hat{y})\ \ T_\mu(y)]\quad Y_\mu(y),$$

$$[\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_3+S_{13}(\hat{x})\ \ T_\mu(x)]\quad Y_\mu(x)\}_+\quad , \tag{28}$$

where $\underset{\sim}{x} = \underset{\sim}{r}_1-\underset{\sim}{r}_3$, $\underset{\sim}{y} = \underset{\sim}{r}_2-\underset{\sim}{r}_3$, and $\{A,B\}_+ = AB + BA$. The functions appearing in equation (28) are defined as follows:

$$S_{\ell m}(\hat{r}) = 3(\underset{\sim}{\sigma}_\ell.\hat{r})(\underset{\sim}{\sigma}_m.\hat{r}) - \underset{\sim}{\sigma}_\ell.\underset{\sim}{\sigma}_m\ ,$$

$$T_\mu(r) = 1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2} ,$$

$$Y_\mu(r) = e^{-\mu r}/\mu r ,$$

with C_p given by[12]

$$C_p = \frac{\mu^4 f_N^2 (A+B)}{4\pi} = \frac{\mu^4 f_N^2}{9\pi^2} \int_0^\infty \frac{\sigma_{33}(p)\,dp}{p^2+\mu^2} , \tag{29}$$

where $\sigma_{33}(p)$ is the cross section in the $\pi + N \to N^*$ reaction. The anticommutator in equation (28) may be expanded to give more explicitly

$$\begin{aligned} w(\underset{\sim}{x},\underset{\sim}{y}) = &- \frac{C_p}{9}(\underset{\sim}{\tau}_1 \cdot \underset{\sim}{\tau}_2)\{2\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2 + 2S_{12}(\hat{y})\ T_\mu(y) + 2S_{12}(\hat{x})\ T_\mu(x) \\ &+ [18(\underset{\sim}{\sigma}_1 \cdot \hat{y})(\underset{\sim}{\sigma}_2 \cdot \hat{x})(\hat{x} \cdot \hat{y}) - 2S_{12}(\hat{x}) - 2S_{12}(\hat{y}) \\ &- 2\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2\ T_\mu(x)\ T_\mu(y)]\}\ Y_\mu(x)\ Y_\mu(y) . \end{aligned} \tag{30}$$

Consider now the case when form factors are included. The only difference is that

$$\int d\underset{\sim}{q}_1 \frac{e^{-i\underset{\sim}{q}_1 \cdot \underset{\sim}{x}}}{q_1^2 + \mu^2} \to \int d\underset{\sim}{q}_1 \int_0^\infty \frac{\alpha(m^2)\,dm^2}{q_1^2 + m^2} e^{-i\underset{\sim}{q} \cdot \underset{\sim}{x}} = 2\pi^2 \int_0^\infty m\alpha(m^2)\ Y_m(x)\,dm^2 , \tag{31}$$

where the explicit form for the form factors has been substituted from equation (20). If the gradient operators are now applied to both sides of the correspondence given in equation (31), the following relations may easily be obtained:

$$\mu^3 Y_\mu(x) \to \int_0^\infty m^3 \alpha(m^2)\ Y_m(x)\,dm^2 . \tag{32}$$

and

$$\mu^3 T_\mu(x)\ Y_\mu(x) \to \int_0^\infty m^3 \alpha(m^2)\ T_m(x)\ Y_m(x)\, dm^2 \ . \tag{33}$$

In the next chapter, the first order effects of $w(\underset{\sim}{x},\underset{\sim}{y})$ in nuclear matter will be derived, but the important point to note now is that in deriving equation (30), the singular contact interaction terms with respect to the variables $\underset{\sim}{x}$ and $\underset{\sim}{y}$ have been explicitly removed from $w(\underset{\sim}{x},\underset{\sim}{y})$.

The modified pion mass approach of BG is based on the definition of the effective interaction given in equation (16). An examination of equation (25) shows that if w is substituted in equation (16) the $\underset{\sim}{r}_3$ integration may immediately be done to give $(2\pi)^3\ \delta(\underset{\sim}{q}_1-\underset{\sim}{q}_2)$. The $\underset{\sim}{q}_1$ integration may then be done, so that

$$u(\underset{\sim}{r}) = \frac{f_N^2}{2\mu^2\pi^2}\,\underset{\sim}{\tau}_1\cdot\underset{\sim}{\tau}_2 \int d\underset{\sim}{q}\,(\underset{\sim}{\sigma}_1\cdot\underset{\sim}{q})\,(\underset{\sim}{\sigma}_2\cdot\underset{\sim}{q})\ \frac{\delta\mu^2}{(q^2+\mu^2)^2}\ e^{-i\underset{\sim}{q}\cdot\underset{\sim}{r}} \ , \tag{34}$$

where

$$\delta\mu^2 = -2\rho(A+B)\ q^2 H^2(q^2) \ , \tag{35}$$

and $\rho = N/\Omega$, the density of nuclear matter. Now if $|\delta\mu^2| << \mu^2+q^2$,

$$\frac{\delta\mu^2}{(q^2+\mu^2)^2} \simeq -\left[\frac{1}{q^2+\bar{\mu}^2} - \frac{1}{q^2+\mu^2}\right] \ , \tag{36}$$

where $\bar{\mu}^2 = \mu^2 + \delta\mu^2$. If equation (36) is substituted in equation (34), $u(\underset{\sim}{r})$ becomes

$$u(\underset{\sim}{r}) = -\frac{f_N^2}{2\mu^2\pi^2}\,\underset{\sim}{\tau}_1\cdot\underset{\sim}{\tau}_2 \int d\underset{\sim}{q}\,(\underset{\sim}{\sigma}_1\cdot\underset{\sim}{q})\,(\underset{\sim}{\sigma}_2\cdot\underset{\sim}{q})\,[\frac{1}{q^2+\bar{\mu}^2} - \frac{1}{q^2+\mu^2}]\; e^{-i\underset{\sim}{q}\cdot\underset{\sim}{r}} . \tag{37}$$

However, the OPEP (without form factors) is given by[11]

$$v(\underset{\sim}{r}) = -\frac{f_N^2}{2\mu^2\pi^2}\,\underset{\sim}{\tau}_1\cdot\underset{\sim}{\tau}_2 \int d\underset{\sim}{q}\,(\underset{\sim}{\sigma}_1\cdot\underset{\sim}{q})\,(\underset{\sim}{\sigma}_2\cdot\underset{\sim}{q})\,\frac{e^{-i\underset{\sim}{q}\cdot\underset{\sim}{r}}}{q^2+\mu^2} . \tag{38}$$

Hence the effective interaction $u(\underset{\sim}{r})$ may be viewed as containing two parts, one the usual OPEP with pion mass μ, the other an OPEP with a modified pion mass $\bar{\mu}$. This is the prescription used by BG to calculate the three-body effects, and its first order contribution will be discussed in the following chapter. Note however, that because of the integration over $\underset{\sim}{r}_3$, $u(\underset{\sim}{r})$ contains contact terms with respect to the variables $\underset{\sim}{r}_1-\underset{\sim}{r}_3$ and $\underset{\sim}{r}_2-\underset{\sim}{r}_3$, and so for this reason it is expected that the effects of $u(\underset{\sim}{r})$ will be somewhat different from those of $w(\underset{\sim}{x},\underset{\sim}{y})$.

The alternative method to introducing the modified pion mass, and as will be seen in the next chapter the more desirable one, is to integrate equation (34) with respect to $\underset{\sim}{q}$ and so find $u(\underset{\sim}{r})$ in coordinate space. Removing the singular contact terms according to the prescription given in equation (27), and substituting for the form factors from equation (20) gives

$$u(\underset{\sim}{r}) = \frac{4C_p\rho}{\pi\mu^6}(\underset{\sim}{\tau}_1\cdot\underset{\sim}{\tau}_2)\,(\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\nabla})\,(\underset{\sim}{\sigma}_2\cdot\underset{\sim}{\nabla})\int d\underset{\sim}{q}\; q^2\{\frac{1}{q^2+\mu^2} - \frac{(1-\xi)}{q^2+\eta^2}\}^2\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{r}} . \tag{39}$$

The integrals are evaluated in Appendix A and the operator $(\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\nabla})(\underset{\sim}{\sigma}_2\cdot\underset{\sim}{\nabla})$ applied. The final result is

$$u(\underset{\sim}{r}) = C(\underset{\sim}{\tau}_1\cdot\underset{\sim}{\tau}_2)[\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2\; g_c(r) + S_{12}(\hat{r})\; g_t(r)] \;, \tag{40}$$

where

$$C = \frac{4\pi\rho C_p}{3\mu^3} \;,$$

$$g_c(r) = c_1\; Y_\mu(r) + c_2 E_\mu(r) + c_3\; Y_\eta(r) + c_4 E_\eta(r) \;,$$

$$g_t(r) = c_5 T_\mu(r)\; Y_\mu(r) + c_2(E_\mu(r)+Y_\mu(r)) + c_6 T_\eta(r) Y_\eta(r) + c_4(E_\eta(r)+Y_\eta(r)) \;,$$

$$E_m(r) = e^{-mr} \;,$$

and the constants c_1 through c_6 are given by equation (A.10). For comparison purposes and later use note that the OPEP is given by

$$v(\underset{\sim}{r}) = B\; \underset{\sim}{\tau}_1\cdot\underset{\sim}{\tau}_2[\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2\; f_c(r) + S_{12}(\hat{r})\; f_t(r)] \;, \tag{41}$$

where $B = f_N^2\mu/3$, $f_c(r) = Y_\mu(r)$, and $f_t(r) = T_\mu(r)\; Y_\mu(r)$.

Using the form factors given in Table I and the constants given in Table II, the central and tensor parts, $Cg_c(r)$ and $Cg_t(r)$, of $u(\underset{\sim}{r})$ are plotted in figures 6 and 7, and compared with the central and tensor parts, $Bf_c(r)$ and $Bf_t(r)$, of OPEP. The introduction of form factors modifies the short range part of the interaction, and in the case of form factor III introduces an extreme suppression in the tensor part of $u(\underset{\sim}{r})$. Since the major contribution to the binding energy comes from the tensor part of the interaction,

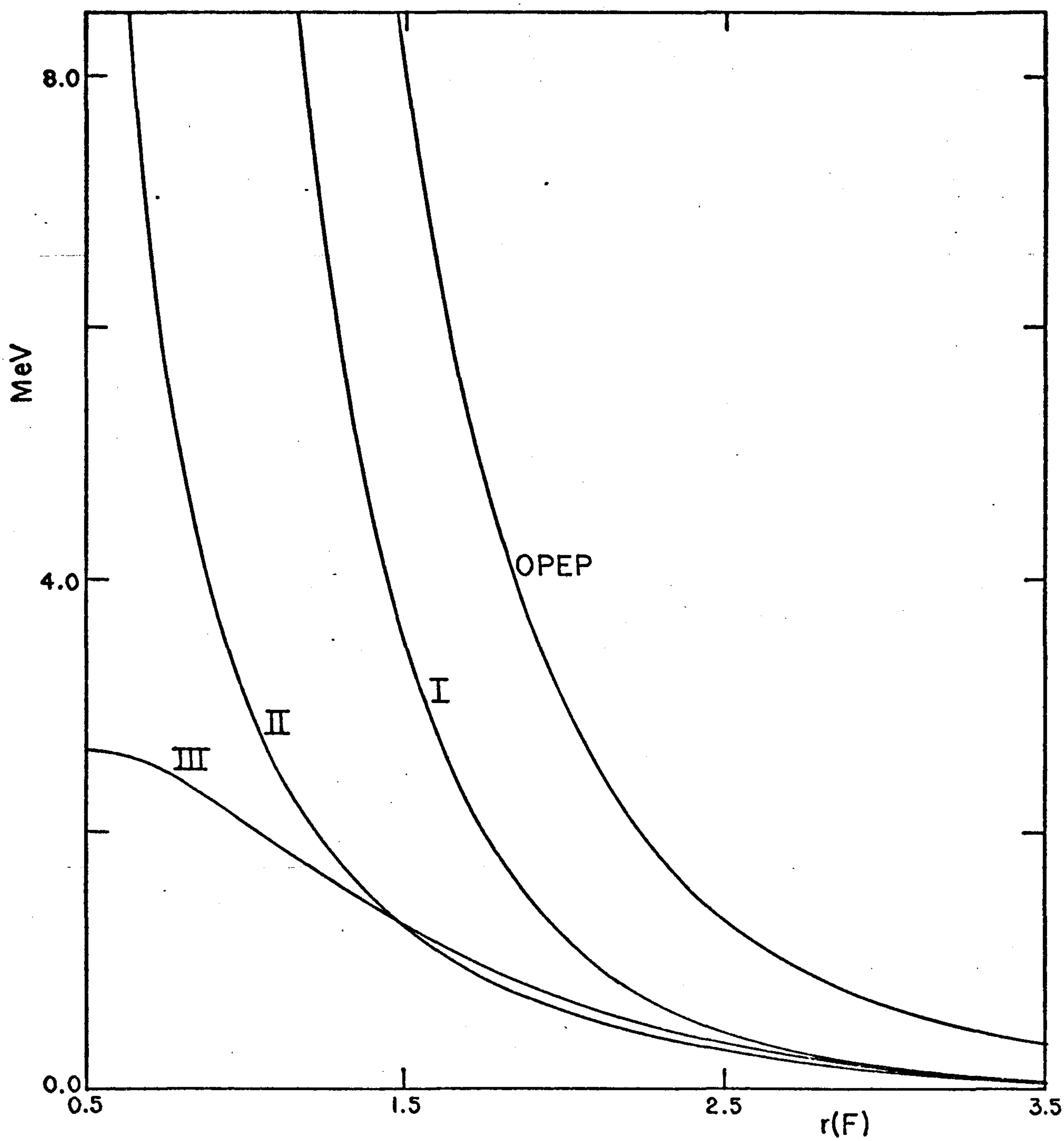

Figure 6. The central part of the OPEP, $B.f_c(r)$, and the central part of the effective interaction, $C.g_c(r)$, for the form factors I, II, and III, plotted against r.

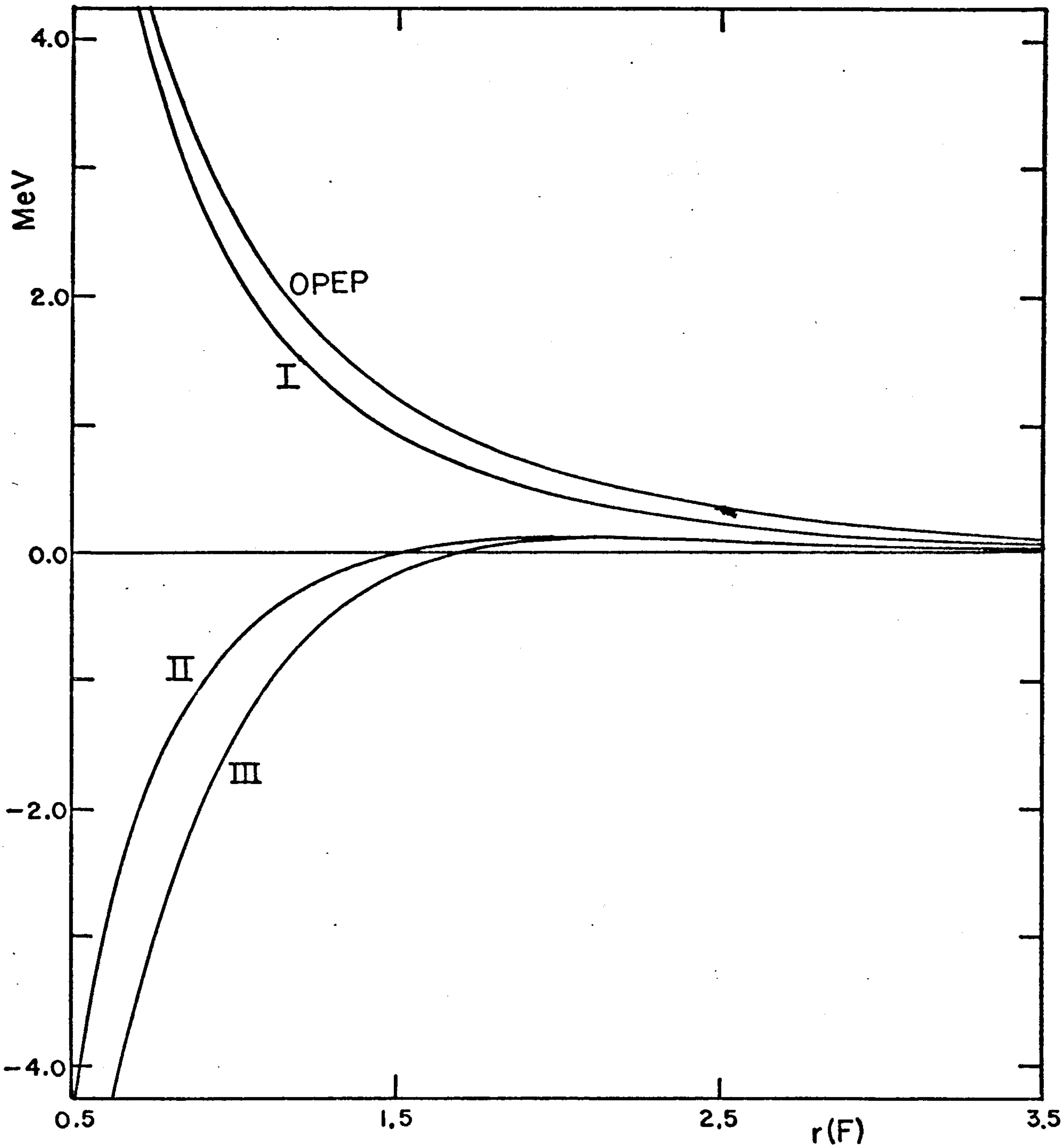

Figure 7. The tensor part of the OPEP, $B.f_t(r)$, and the tensor part of the effective interaction $C.g_t(r)$, for the form factors I, II, and III, plotted against r.

it is this suppression at short distances which characterizes the main effect of the form factors.

CHAPTER IV

THE FIRST ORDER CONTRIBUTIONS OF W AND U

Three different ways of describing the NNN interaction have been introduced in the previous chapter, and it is now desirable to understand what the important difference between these prescriptions are when each is applied to the calculation of the three-body effect in nuclear matter. The fact that the effective interaction $u(\underset{\sim}{r})$ contains some singular contact terms which are absent in $w(\underset{\sim}{x},\underset{\sim}{y})$ has already been pointed out, and these are expected to lead to differences in the contributions of $u(\underset{\sim}{r})$ and $w(\underset{\sim}{x},\underset{\sim}{y})$. In this chapter, the first order contribution of the NNN interaction will be examined in detail, and the differences obtained when it is calculated according to each of the above prescriptions discussed. In order to obtain the effects of the contact terms in $u(\underset{\sim}{r})$, and to understand the modifications introduced by the form factors, the first order contribution of both w and u will first be calculated with d, the NN cut-off, taken to be zero. The results will then be generalized to the more realistic case in which d is non-zero, and the contributions of w and u will be compared using the form factors given in Table I. The difficulty with the modified pion mass approach of BG when an NN cut-off is introduced will

ξ	η	Comments
1.0	(arbitrary)	Corresponds to no form factor
0.28	1.676	ref. 8
0.0	2.214	ref. 9. Eq. 56 is satisfied

Table I. Parameters for use in the spectral function given by equation (21). Each row corresponds to a different pionic form factor.

Parameter	Value	Comments
$\hbar c$	197.32 MeV-F	
k_F	1.36 F^{-1}	ref. 20
ρ	0.170 F^{-3}	ref. 20
$\hbar^2/M$	41.5 MeV-F^{-2}	ref. 20
f_N^2	0.0800	ref. 21
μ	0.700 F^{-1}	Average of μ^o, μ^+, and μ^-
C_p	0.460 MeV	ref. 21
B	3.68 MeV	ref. 22
C	0.955 MeV	

Table II. Numerical values for constants used in calculating the first and second order contributions.

be discussed, and finally, the first order contribution of u, given by equation (40), will be obtained.

The prescription for introducing form factors into $w(\underset{\sim}{x},\underset{\sim}{y})$ has already been obtained (equations (32) and (33)), and so $\langle W\rangle/N$ may first be calculated without form factors, and their effect introduced when the calculation is complete. BLN[6] have calculated the first order effect for Λ-particle in nuclear matter, and their result can be carried over unaltered to the NNN case, with one exception. If the total isospin I, and its 3-component, I_3, are denoted by (I, I_3), then for the πN system the i-spin is $(1/2, I_3)$ or $(3/2, I_3)$, while for N^* it is $(3/2, I_3)$. Hence for the reaction $\pi+N\rightarrow N^*$, only four of the six initial states are coupled to the final state. On the other hand, for the $\pi\Lambda$ system, I=1 only, and so all initial states are coupled to the final state. This leads to the relation

$$\langle W\rangle_{NN}/N = (4/6).(C_p(NN)/C_p(\Lambda N)).\langle W\rangle_{\Lambda N} . \tag{42}$$

Using equations (4) and (5) of reference (6),

$$\frac{\langle W\rangle}{N} = \frac{\langle W\rangle_{NN}}{N} = \frac{C_p\rho^2}{4}\int D^2(k_F|\underset{\sim}{x}-\underset{\sim}{y}|)\{1+ (3\cos^2\theta_{xy}-1)\ T_\mu(x)$$

$$\times\ T_\mu(y)\}\ Y_\mu(x)\ Y_\mu(y)d\underset{\sim}{x}\ d\underset{\sim}{y}\ , \tag{43}$$

where $D(k_Fr) = 3j_1(k_Fr)/k_Fr$, $\cos\theta_{xy} = \underset{\sim}{x}.\underset{\sim}{y}/xy$ (the geometry is shown in figure 5c), and j_1 is a spherical Bessel function of order one. Now define

$$F(q) = \int d\underset{\sim}{r}\, D^2(k_F r) e^{-i\underset{\sim}{q}\cdot\underset{\sim}{r}} , \tag{44}$$

so that

$$D^2(k_F r) = \frac{1}{(2\pi)^3} \int F(q) e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}} d\underset{\sim}{q} . \tag{45}$$

Writing

$$I_c = \mu^6 \int D^2(k_F|\underset{\sim}{x}-\underset{\sim}{y}|)\; Y_\mu(x)\; Y_\mu(y) d\underset{\sim}{x}\; d\underset{\sim}{y} ,$$

it is shown in Appendix B that by using equation (45), I_c may be reduced to

$$I_c = 8\mu^4 \int_0^\infty dq\; q^2 \frac{F(q)}{(q^2+\mu^2)^2} . \tag{46}$$

Similarly, if

$$I_t = \mu^6 \int D^2(k_F|\underset{\sim}{x}-\underset{\sim}{y}|)(3\cos^2\theta_{xy}-1) Y_\mu(x) T_\mu(x) Y_\mu(y) T_\mu(y) d\underset{\sim}{x}\; d\underset{\sim}{y} ,$$

I_t may be written in q-space as

$$I_t = 16 \int_0^\infty dq\; q^6 \frac{F(q)}{(q^2+\mu^2)^2} . \tag{47}$$

On collecting terms,

$$\frac{\langle W \rangle}{N} = \frac{2C_p\rho^2}{\mu^6}[\mu^4 \int_0^\infty dq\; q^2 \frac{F(q)}{(q^2+\mu^2)^2} + 2 \int_0^\infty dq\; q^6 \frac{F(q)}{(q^2+\mu^2)^2}] . \tag{48}$$

Consider now the prescription given in equations (32) and (33) for introducing form factors. Multiplying equation (32) by $x^2 j_o(qx)$, integrating over all x, and making use of equation (D1) gives

$$\frac{\mu^2}{(q^2+\mu^2)} \rightarrow \int_o^\infty dm^2 \; \frac{m^2\alpha(m^2)}{q^2 + m^2} \; . \tag{49}$$

Similarly, on multiplying equation (33) by $x^2 j_2(qx)$, integrating over all x, and using equation (D2), the following correspondence is obtained:

$$\frac{1}{q^2+\mu^2} \rightarrow \int_o^\infty dm^2 \; \frac{\alpha(m^2)}{q^2 + m^2} \; . \tag{50}$$

Then when the effects of form factors are included, $\langle W\rangle/N$ becomes

$$\frac{\langle W\rangle}{N} = \frac{2C_p\rho^2}{\mu^6}\int_o^\infty dq \; q^2 \; F(q) \left[\left(\int_o^\infty dm^2 \; \frac{m^2\alpha(m^2)}{q^2 + m^2}\right)^2 + \; 2q^4 \left(\int_o^\infty \frac{dm^2\alpha(m^2)}{q^2 + m^2}\right)^2 \right] .$$

This expression may be rearranged slightly to give

$$\frac{\langle W\rangle}{N} = \frac{2C_p\rho^2}{\mu^6}\int_o^\infty dq \; q^2 \; F(q) \left[\left(\int_0^\infty dm^2\alpha(m^2) - q^2\int_o^\infty \frac{dm^2\alpha(m^2)}{q^2 + m^2}\right)^2 \right.$$

$$\left. + \; 2q^4 \left(\int_o^\infty \frac{\alpha(m^2)dm^2}{q^2 + m^2}\right)^2 \right] \; . \tag{51}$$

The first order contribution of $u(\underset{\sim}{r})$ is less complicated to derive. From equation (C1) only the exchange term of equation (18) contributes, and upon doing the i-spin and spin sums using equations (C2) and (C4) <U>/N becomes

$$\frac{\langle U\rangle}{N} = -\frac{3f_N^2}{N\mu^2\pi^2}\sum_{k_i,k_j}\langle k_i\ k_j|u'(\underset{\sim}{r})|k_j\ k_i\rangle , \tag{52}$$

where

$$u'(\underset{\sim}{r}) = \int d\underset{\sim}{q}\,\frac{\delta\mu^2 q^2 e^{-i\underset{\sim}{q}\cdot\underset{\sim}{r}}}{(q^2+\mu^2)^2} .$$

Replacing the sums over k_i and k_j by integrals, and using the relation

$$\int d\,\underset{\sim}{k}\,e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} = \frac{4\pi}{3}k_F^3\,D(k_F r) ,$$

equation 52 becomes

$$\frac{\langle U\rangle}{N} = -\frac{3f_N^2\rho}{16\pi^2\mu^2}\int u'(\underset{\sim}{r})\ D^2(k_F r)\,d\underset{\sim}{r} \tag{53}$$

In terms of F(q) and $u'(q) = (2\pi)^3\ \delta\mu^2 q^2/(q^2+\mu^2)^2$, the Fourier of $u'(\underset{\sim}{r})$, equation (53) gives

$$\frac{\langle U\rangle}{N} = -\frac{3f_N^2\rho}{16\pi^2\mu^2}\int d\underset{\sim}{q}\,\frac{\delta\mu^2 q^2 F(q)}{(q^2+\mu^2)^2} . \tag{54}$$

Substituting for $\delta\mu^2$ from equation (35), this expression may be rewritten as

$$\frac{\langle U\rangle}{N} = \frac{6C_p\rho^2}{\mu^6}\int_0^\infty dq\ q^6\left(\int_0^\infty dm^2\ \frac{\alpha(m^2)}{q^2+m^2}\right)^2 F(q) . \tag{55}$$

Comparing $\langle U\rangle/N$ with $\langle W\rangle/N$ given by equation (51), the two are seen to agree only if

$$\int_0^\infty dm^2\ \alpha(m^2) = 0\ . \tag{56}$$

Thus, provided that the spectral function, $\alpha(m^2)$, is chosen in such a way that equation (56) is satisfied, the contact terms appearing in $\langle U\rangle/N$ will be of no importance. If the spectral function given in equation (21) is substituted in equation (56), the condition $\xi=0$ is obtained. For one of the form factors used in later calculations, this condition will be satisfied, while for the other $\xi=0.28$.

In order to ascertain how sensitive the difference between $\langle W\rangle/N$ and $\langle U\rangle/N$ is to the value of ξ, both quantities were calculated for the more realistic case of non-zero d. When a cut-off is introduced, the expression corresponding to equation (51) may be found by substituting for $D^2(k_F r)$ the quantity $D^2(k_F r)\ \theta(r-d)$, so that $F(q)$ is replaced by $\tilde{F}(q)$, where

$$\tilde{F}(q) = F(q) - 4\pi\int_0^d D^2(k_F r)\ j_o(qr)r^2 dr\ . \tag{57}$$

Details of the evaluation of $\langle W\rangle/N$ are given in Appendix F, and the results are compared with the contribution of $\langle U\rangle/N$ in Table III. For form factor III the results are identical, as equation (56) would predict. Perhaps more important is the fact that the results are very nearly the same for form

Form Factor	I			II			III		
d(F)	0.8	1.0	1.2	0.8	1.0	1.2	0.8	1.0	1.2
<W>/N	-1.77	-1.30	-0.906	0.316	0.088	-0.049	1.05	0.491	0.117
<U>/N	-3.17	-2.40	-1.72	0.378	0.088	-0.082	1.05	0.491	0.117
<U-W>/N	-1.40	-1.10	-0.81	0.062	0.000	-0.033	0.00	0.000	0.000

Table III. Comparison of the first order contributions of the three-body interaction W, and the effective interaction U. All energies are in MeV.

factor II, where $\xi=0.28$. This result will lend support to the assumption that even for form factor II the delta-functions in u give a negligible contribution, and so u may be used to calculate the second order three-body contribution to nuclear matter.

BG calculate the first order contribution of u by making use of equation (37), according to which

$$\frac{\langle U\rangle}{N} = \frac{\langle V(\mu=\bar{\mu})\rangle}{N} - \frac{\langle V(\mu=\ \mu\rangle}{N} \tag{58}$$

However, on comparing equations (34) and (38), the OPEP is seen to be related to $u(\mathbf{r})$ through the correspondence

$$\frac{\delta\mu^2}{q^2+\mu^2} \to -\frac{1}{q^2+\mu^2} \, ,$$

which when substituted in equation (54) gives

$$\frac{\langle V\rangle}{N} = \frac{3f_N^2\rho}{16\pi^2\mu^2}\int d\mathbf{q}\,\frac{q^2F(q)}{q^2+\mu^2} \, . \tag{59}$$

Then if no NN cut-off is introduced, the three-body effects can be obtained by modifying the pion mass in the OPEP contribution as equation (58) indicates, and this is the prescription given by BG. However, when an NN cut-off is introduced, the result is more complicated because of the q dependence of $\bar{\mu}$. BG make the error of replacing $1/(q^2+\bar{\mu}^2)$ by the Fourier transform of the cut-off Yukawa, and so in effect have neglected the momentum dependence of $\bar{\mu}$. Instead, what should be done is to find the potential corresponding to $1/(q^2+\bar{\mu}^2)$ in coordinate space, cut it off at r=d, and then find the Fourier transform of this cutoff potential. However, because of mathematical difficulties in carrying out this procedure, and because the simple result given by equation (58) will no longer hold, it does not seem useful to pursue this method any further.

Consider now the first order contribution of the effective interaction, $u(\underset{\sim}{r})$, given by equation (40). Because of equation (C1) only the central part contributes, and using techniques exactly similar to those used in obtaining equation (54), <U>/N becomes

$$\frac{\langle U\rangle}{N} = -\frac{9}{4}\,\frac{C\rho}{\pi}\int g_c(q)\;F(q)q^2dq\;, \tag{60}$$

where

$$g_c(q) = \int_d^{\infty} j_o(qr)\;g_c(r)r^2dr\;.$$

There is however, one important difference between equations (54) and (60) which should be noted at this point. In deriving the effective interaction in coordinate space, the contact terms with respect to the variable $\underset{\sim}{r}$ have been explicitly removed, while in equation (54) the contribution of such contact terms still remain. Of course, if an NN cut-off is introduced, equations (54) and (60) will give indentical results, and for convenience equation (60) will be used throughout in the first order calculation of u. The contribution of v is given by a similar expression which can be found by setting C to B and $g_c(r)$ to $f_c(r)$ in equation (60), and when equation (F1) for F(q) is subsituted in equation (60), the integral may be done numerically to obtain both $\langle U \rangle/N$ and $\langle V \rangle/N$.

The examination in this chapter of the first order contribution of the three-body force to nuclear matter has indicated that the effective interaction, u(r), describes quite adequately the first order three-body effects involving the single exchange terms, and on this basis u will be used in the following chapter to calculate the second order three-body effects.

CHAPTER V

THE SECOND ORDER CONTRIBUTION OF U

In the previous chapter, the first order contributions of w and u were compared, in order to see how the form factors modify the short range part of the interaction. The conclusion was reached that provided the form factors at least approximately satisfied equation (56), the effect of the spruious contact terms in u would be negligible. The same conclusion will be assumed to hold for the second order calculation, and so u rather than w may be used to calculate the second order contribution of the three-body interaction. Because of the similarity in form between u and v, the equations obtained for the calculation of u may easily be modified to permit a second order calculation of the OPEP contribution, and so the two and three-body results may easily be compared. To show how the calculation proceeds, the contribution of the cross term, $\langle U\ 1/b\ V\rangle$, will be calculated, and the method of obtaining the OPEP contribution, $\langle V\ 1/b\ V\rangle$, and the pure three-body term, $\langle U\ 1/b\ U\rangle$, from this result will be given.

From equation (17) the direct contribution of $\langle U\ 1/b\ V\rangle$ is

$$\frac{\Delta E_d^{(2)}}{N}(\mathrm{C.T.}) = \frac{1}{2N} \sum_{i,j} \sum_{i',j'} \frac{\langle ij|v|i'j'\rangle\langle i'j'|u|ij\rangle}{\varepsilon_i+\varepsilon_j-\varepsilon_i'-\varepsilon_j'} . \qquad (61)$$

The single particle energies, ε_γ, will be taken to be the free particle kinetic energy, $\hbar^2 k_\gamma^2/2M$, where M is the nucleon mass. Then

$$\varepsilon_i+\varepsilon_j-\varepsilon_i'-\varepsilon_j' = \frac{\hbar^2}{2M}(k_i^2+k_j^2-k_i'^2-k_j'^2) \quad . \tag{62}$$

Momentum conservation in nuclear matter requires that

$$\underset{\sim}{q} \equiv \underset{\sim}{k}_i' - \underset{\sim}{k}_i = \underset{\sim}{k}_j - \underset{\sim}{k}_j' \, , \tag{63}$$

so that equation (62) simplifies to

$$\varepsilon_i+\varepsilon_j-\varepsilon_i'-\varepsilon_j' = -\frac{h^2}{M}\underset{\sim}{q}\cdot(\underset{\sim}{q}+\underset{\sim}{k}_i-\underset{\sim}{k}_j) \quad . \tag{64}$$

Because of the condition given by equation (63), the four sums over momenta appearing in equation (61) reduces to three sums over the variables $\underset{\sim}{k}_i$, $\underset{\sim}{k}_j$ and $\underset{\sim}{q}$, where the sums are restricted so that

$$\begin{aligned} &|\underset{\sim}{k}_i|<k_F \, , \quad |\underset{\sim}{k}_j|<k_F \, , \\ &|\underset{\sim}{q}+\underset{\sim}{k}_i|>k_F \, , \quad |\underset{\sim}{k}_j-\underset{\sim}{q}|>k_F \, . \end{aligned} \tag{65}$$

According to equations (C13) and (C14) any terms linear in $S_{12}(\hat{r})$ cannot contribute to $\Delta E_d^{(2)}$(C.T.)/N, and so equation (61) reduces to two terms, one involving only the central parts of v and u, the other involving only the tensor parts. According to the results given in Appendix C, the i-spin and spin sums for the central contribution will give a factor of 12x12, while the i-spin and spin sums of the tensor contribution will give a factor of 12x24. With these substitutions equation (61) may be written as

$$\frac{\Delta E_d^{(2)}}{N}(\text{C.T.}) = -\frac{B.C}{2N}\cdot\frac{M}{\hbar^2}\left(\frac{4\pi}{\Omega}\right)^2 \sum_{k_i,k_j,q} \frac{[144f_c(q)g_c(q)+288f_t(q)g_t(q)]}{\underset{\sim}{q}\cdot(\underset{\sim}{q}+\underset{\sim}{k}_i-\underset{\sim}{k}_j)} \tag{66}$$

where the j_0 and j_2 transforms have been defined and evaluated in Appendix D. Using the relations

$$\sum_k \rightarrow \frac{\Omega}{(2\pi)^3}\int d\underset{\sim}{k}\ , \text{ as } \Omega \rightarrow \infty\ , \tag{67}$$

and

$$\int \frac{d\underset{\sim}{k}_i\ d\underset{\sim}{k}_j}{\underset{\sim}{q}\cdot(\underset{\sim}{q}+\underset{\sim}{k}_i-\underset{\sim}{k}_j)} = \frac{4\pi^2k_F^5}{15q}\,P(q/2k_F)\ , \tag{68}$$

discussed in Appendix E, equation (66) simplifies to

$$\frac{\Delta E_d^{(2)}}{N}(\text{C.T.}) = -\frac{18}{5\pi^2}\frac{M}{\hbar^2}B.C.k_F^2\int_0^\infty dqq\ P(q/2k_F)[f_c(q)g_c(q)+2f_t(q)g_t(q)]\ , \tag{69}$$

where the constants B and C have been defined by equations (40) and (41). Since the function P is known analytically (equation (E2)) this integral may be evaluated numerically, and a value for $\Delta E_d^{(2)}(\text{C.T.})/N$ may be obtained.

The direct OPEP contribution, $\Delta E_d^{(2)}(\text{OPEP})/N$, may now easily be found by setting C to B, g_c to f_c, and g_t to f_t in equation (69). Hence

$$\frac{\Delta E_d^{(2)}}{N}(\text{OPEP}) = -\frac{18}{5\pi^2}\frac{M}{\hbar^2}B^2k_F^2\int_0^\infty dq\ qP(q/2k_F)[f_c^2(q) + 2f_t^2(q)]\ . \tag{70}$$

Similarly, the pure three-body contribution, $\Delta E_d^{(2)}(\mathrm{T.B.})/N$, may be found by setting B to C, f_c to g_c, and f_t to g_t to obtain

$$\frac{\Delta E_d^{(2)}}{N}(\mathrm{T.B.}) = -\frac{18}{5\pi^2}\frac{M}{\hbar^2}C^2k_F^2\int_0^\infty dq\; q\; P(q/2k_F)[g_c^2(q) + 2g_t^2(q)]. \tag{71}$$

The exchange contribution of <U 1/b V> is

$$\frac{\Delta E_{ex}^{(2)}}{N}(\mathrm{C.T.}) = -\frac{1}{2N}\sum_{i,j}\sum_{i',j'}\frac{\langle ij|v|i'j'\rangle\langle i'j'|u|ji\rangle}{\varepsilon_i+\varepsilon_j-\varepsilon_i'-\varepsilon_j'}. \tag{72}$$

Making use of the spin sums given in Appendix C, evaluation proceeds as for the direct term, so that

$$\frac{\Delta E_{ex}^{(2)}}{N}(\mathrm{C.T.}) = \frac{B.C}{2N}\frac{M}{\hbar^2}\left(\frac{4\pi}{\Omega}\right)^2\sum_{k_i,k_j,q}\frac{[36f_c(q)g_c(s)-144P_2(\cos\theta_{sq})f_t(q)g_t(s)]}{\underset{\sim}{q}\cdot(\underset{\sim}{q}+\underset{\sim}{k}_i-\underset{\sim}{k}_j)}, \tag{73}$$

where $\underset{\sim}{s} = \underset{\sim}{q}+\underset{\sim}{k}_i-\underset{\sim}{k}_j$. Replacing the sums by integrals using equation (67), and using the generalized Euler functions G_0 and G_2, discussed in Appendix E and tabulated in reference (13), equation (73) becomes

$$\frac{\Delta E_{ex}^{(2)}}{N}(\mathrm{C.T.}) = \frac{27}{2\pi^3}B.C\frac{M}{\hbar^2}\int qdq\int sds[G_0(q/k_F,s/k_F)\; f_c(q)\; g_c(s)$$

$$- 4G_2(q/k_F,s/k_F)\; f_t(q)\; g_t(s)], \tag{74}$$

where q runs from 0 to ∞, and v runs from $\max(0,q-2k_F)$ to $q + 2k_F$.

The same correspondence as was used with the direct contribution may be used to find $\Delta E_{ex}^{(2)}$(OPEP)/N and $\Delta E_{ex}^{(2)}$(T.B.)/N, with the results

$$\frac{\Delta E_{ex}^{(2)}}{N}(\text{OPEP}) = \frac{27}{2\pi^3} B^2 \frac{M}{\hbar^2} \int qdq \int sds\,[G_0 f_c(q) f_c(s) - 4G_2 f_t(q) f_t(s)] \,, \tag{75}$$

and

$$\frac{\Delta E_{ex}^{(2)}}{N}(\text{T.B.}) = \frac{27}{2\pi^3} C^2 \frac{M}{\hbar^2} \int qdq \int sds\,[G_0 g_c(q) g_c(s) - 4G_2 g_t(q) g_t(s)] \,, \tag{76}$$

where it is understood that G_0 and G_2 have the same arguments as in equation (74).

All the required first and second order results have now been obtained. In Appendix F some of the details concerned with the numerical evaluation of these contributions has been discussed, and in the following chapter the various results given in Tables IV and V will be discussed, with a view to determining the importance of the three-body force in nuclear matter.

CHAPTER VI

DISCUSSION OF RESULTS

In Chapter IV the first order contributions of w and u were considered in detail, and the effects of the form factors on the short-range part of the effective interaction, u, discussed. Assuming that the three-body interaction could accurately be described by $u(\underset{\sim}{r})$, the second order contributions were derived in Chapter V. Using the parameters given in Table II and the integration methods discussed in Appendix F, the first and second order contributions of the OPEP, v, and the effective interaction, u, have been calculated and tabulated in Tables IV and V. The contributions for various values of the cut-off have been given, although the most realistic value is probably for d=1F, and in the following discussion all results taken from Table IV and V will refer to the column in which d=1F.

Although the two-body contributions given in Table IV are not of primary importance here, a brief discussion of these results will point out several facts which also hold for the three-body contributions. The first order contribution, $\Delta E^{(1)}(\mathrm{OPEP})/N$, is seen to be small -- about 3 MeV attraction, while the second order contribution is almost 20 MeV attraction. This would seem to indicate that the perturbation

Cut-off (F)	0.8	1.0	1.2
$\Delta E^{(1)}$ (OPEP)/N	-4.04	-3.16	-2.35
$\Delta E_d^{(2)}$ (OPEP)/N	-28.0	-15.7	-8.94
$\Delta E_{ex}^{(2)}$ (OPEP)/N	-7.9(4)	-3.7(8)	-1.7(7)
$\Delta E^{(2)}$ (OPEP)/N	-35.9	-19.5	-10.7
$\Delta E_{ex}^{(2)}/\Delta E_d^{(2)}$ (%)	28.4	24.1	19.8
$\Delta E^{(1)}/\Delta E^{(2)}$ (%)	11.3	16.2	22.0
$\Delta E(v)/N$	-39.9	-22.7	-13.1

Table IV. First and second order OPEP contributions. The direct and exchange contributions are shown separately, and the total contribution, $\Delta E(v)/N$ is given. All energies are in MeV.

series is not converging, and that higher order terms should be considered. However, as was pointed out in Chapter IV only the central part of the OPEP contributes is first order, while the tensor part gives a large contribution in second order, and so it is possible that by the third order the contributions will have become small. Third order calculations have been performed using the OPEP and this possibility has been confirmed.[16]

The total OPEP contribution, $\Delta E(v)/N$, is seen to be about 23 MeV attraction, which when combined with the average kinetic energy per particle of 23 MeV gives a net binding energy per particle of approximately zero. However, at d=0.8F, the potential energy contribution is now sufficient to give a binding energy of approximately 17 MeV per particle, and so the contribution is very sensitive to the value of the NN cut-off. The result will also be seen to hold for the contribution of the effective interaction, and BLN, in their consideration of the first order contribution of w, have reached a similar conclusion.

Turning now to the effective interaction contributions given in Table V it is seen that for a realistic form factor (either II or III) the first order contribution of u ranges from 0.1 MeV to 0.5 MeV repulsion. BG in using their modified pion mass approach concluded that the first order contribution of the effective interaction is approximately 1.2 MeV repulsion. The reason for this disagreement can be

Form Factor	I			II			III		
cut-off (F)	0.8	1.0	1.2	0.8	1.0	1.2	0.8	1.0	1.2
$\Delta E^{(1)}$ (T.B.)/N	-3.17	-2.40	-1.72	0.379	0.088	0.082	1.05	0.491	0.117
$\Delta E_d^{(2)}$ (C.T.)/N	-25.6	-13.8	-7.50	-7.32	-4.74	-3.04	-6.44	-4.66	-3.22
$\Delta E_{ex}^{(2)}$ (C.T.)/N	-7.2(8)	-3.3(4)	-1.4(9)	-1.9(9)	-1.0(9)	-0.57(6)	-1.6(8)	-1.0(4)	-0.59(4)
$\Delta E_d^{(2)}$ (T.B.)/N	-5.89	-3.06	-1.60	-0.521	-0.378	-0.267	-0.483	-0.398	-0.313
$\Delta E_{ex}^{(2)}$ (T.B.)/N	-1.6(9)	-0.74(8)	-0.32(3)	-0.12(6)	-0.08(0)	-0.04(7)	-0.09(2)	-0.07(3)	-0.05(0)
$\Delta E^{(2)}$/N(Total)	-40.5	-20.9	-10.9	-9.96	-6.28	-3.93	-8.70	-6.17	-4.18
$\Delta E^{(1)}/\Delta E^{(2)}$ (%)	7.82	11.5	15.8	3.80	1.40	2.10	12.1	7.95	2.79
$\Delta E(u)/N$	-43.7	-23.3	-12.6	-9.58	-6.19	-3.84	-7.65	-5.68	-4.06
$\Delta E(u)/\Delta E(v)$ (%)	110.	103.	96.2	24.0	27.3	29.3	19.2	25.0	31.0

Table V. First and second order contributions of the effective interaction u. The total contribution, $\Delta E(u)/N$, is given, as well as the ratio, $\Delta E(u)/\Delta E(v)$, of the three-body and two-body contributions. All energies are in MeV.

traced to the error in BG's method of imposing on NN cut-off, already discussed in Chapter IV. In second order, the total contribution of u is approximately 6 MeV additional attraction, so that altogether the total three-body contribution, $\Delta E(u)/N$, would appear to be approximately 5.5 to 6 MeV of additional binding to nuclear matter. When compared with the OPEP value of 23 MeV, the three-body force is seen to contribute 20% to 25% of the two-body contribution to the potential energy per particle.

From equation (42), a partial estimation of the contribution of the three-body ΛNN interaction to the binding energy of a Λ-particle in nuclear matter can be obtained, using the results given in Table V. The first order contribution, $\Delta E^{(1)}(\Lambda NN)$, can be unambiguously determined, and is given in the first row of Table VI. In second order, the term $\langle U_{\Lambda NN}\ 1/b\ U_{\Lambda NN}\rangle$, where the subscripts indicate explicitly the origin of the effective interaction, will give no contribution, since for a Λ-particle it will proportional to 1/N. However, the term $\langle V_{NN}\ 1/b\ U_{\Lambda NN}\rangle$ will contribute, and its contribution is given in Table VI as $\Delta E^{(2)}(\Lambda NN)$. The total contribution, $\Delta E(\Lambda NN)$, is seen to be unrealistically large, for the binding energy of a Λ-particle in nuclear matter is believed to be approximately 30 MeV[6] (note that for a Λ-particle in nuclear matter the kinetic energy is zero), and so this result would indicate that the three-body force gives rise to almost all of the binding energy. However, it is

Form Factor	I			II			III		
cut-off (F)	0.8	1.0	1.2	0.8	1.0	1.2	0.8	1.0	1.2
$\Delta E^{(1)}(\Lambda NN)$	-15.1	-11.4	-8.19	1.80	0.424	0.390	5.00	2.34	0.557
$\Delta E^{(2)}(\Lambda NN)$	-193.	-99.5	-51.9	-47.4	-29.9	-18.7	-41.4	-29.4	-19.9
$\Delta E(\Lambda NN)$	-208.	-111.	-60.1	-45.6	-29.5	-18.3	-36.4	-27.1	-19.3
$\Delta E^{(1)}/\Delta E^{(2)}$ (%)	7.82	11.5	15.8	3.80	1.40	2.10	12.1	7.95	2.79

Table VI. First and second order contributions of the three-body ΛNN force to Λ-particle binding in nuclear matter. All energies are in MeV.

well known that the two-body ΛN forces alone tend to give an overestimate of the Λ-particle binding, and in fact the three-body forces were originally introduced as a possible mechanism to suppress the two-body effects.

The reasons for the unrealistically large values of $\Delta E(\Lambda NN)$ are not clear. One possibility is that the second order term given by $<V_{\Lambda N}\ 1/b\ W_{\Lambda NN}>$ may give a sizeable contribution and so perhaps cancel some or all of the contribution due to $<V_{NN}\ 1/b\ W_{\Lambda NN}>$. This term cannot be calculated as readily as the other second order terms, since the Λ-particle will no longer be a spectator, and so an effective interaction cannot be determined. Another alternative is that the two-pion exchange process may not be the only one which gives a sizeable contribution to the three-body force, and for example the double exchange of the σ-meson has been suggested as a possible contributor[23].

There is also some question as to whether the use of form factors as opposed to the introduction of explicit cut-offs in the variables $\underset{\sim}{x}$ and $\underset{\sim}{y}$ gives a realistic method for calculating the contributions of the three-body force. When the first order contributions given in Table VI are compared with those given by BLN in reference 6, the agreement is only qualitative in nature. Both methods of calculation predict that in first order the three-body force gives rise to a slight repulsion which is very sensitive to the NN cut-off, but the quantitative agreement is very poor. This indicates that the method of calculation used here is quite ambiguous, and that if explicit cut-offs had been introduced in the variables $\underset{\sim}{x}$ and $\underset{\sim}{y}$, a very different second order contribution might have been obtained.

No further examination of these questions will be undertaken here. However, the several ambiguities in the calculation of the three-body contributions both to nuclear matter and to Λ-particle binding should be noted, and all numerical values should be regarded with suspicion until more detailed calculations can be performed.

In conclusion, a brief discussion of the main assumptions which have gone into the calculations of the results summarized in Tables IV through VI will be given.

The derivation of the three-body force has been carried out in the static limit, and although the exchanged pions cannot strictly be of zero energy, the main contribution to the first order energy comes from a region near $q=1F^{-1}$, while in second order the corresponding value is $q=2F^{-1}$. Because the main contribution comes from such small q values, the assumption of the static limit would appear justified. However, unless a more precise three-body interaction is derived, and the calculation repeated with it, no definite conclusion can be reached.

A second assumption concerned with the three-body force is that the TPE process is the main contributor to the three-body interaction. Higher order processes, such as the exchange of more than two, or of heavier mesons can be expected to be of some importance, and some such processes have been considered by other workers[19]. However, in the same way that the OPEP tail can reproduce approximately the effects of the two-nucleon force, it is perhaps reasonable to suppose that the tail of the effective interaction due to the TPE process can reproduce reasonably well the three-body effects.

In connection with the perturbation expansion, two basic assumptions have been made. In the first place, the series has been truncated after second order, on the assumption that higher order terms are not important. Because the tensor contribution in second order has led to a value very much larger than the first order contribution, this assumption would seem to be quite unjustified. However, as has already been mentioned, the effects of the third order terms have been calculated for the OPEP, and the conclusion was reached that third and higher order terms are indeed quite small. Because the effective interaction is weaker than the OPEP (figures 6 and 7) a similar convergence can be expected for this case as well.

The second assumption concerned with the perturbation series, and one which is less easily justified, is that only single exchange terms contribute. The double exchange terms have been calculated in first order for the three-body interaction w[19], and found to be approximately 12% of the single exchange term. In the results presented in Table V, the assumption has been made that the single exchange terms also dominate the second order contribution, although it would appear possible that the great number of double exchange terms which contribute in second order may combine to give a contribution comparable in size to that of the single exchange terms. It is worth noting, however, that for the ΛNN interaction, the first order term and any second order terms involving V_{NN} can only have contributions from the single-exchange terms, because in these cases the Λ-particle is always a spectator.

The results of the calculations presented here indicate that the three-body force can contribute approximately 6 MeV additional attraction in nuclear matter. This number is not meant to be a precise estimation of the three-body effects, for many ambiguities still remain in the calculation. In particular the results are sensitive to the form of the two-body interaction, and clearly a more realistic form of the two-nucleon force could be used. Also, the NN cut-off is a very important, but quite arbitrary parameter, and a calculation which avoided the introduction of a cut-off would be desirable. However, this would require a detailed knowledge of not only the two-body, but also the three-body force for arbitrarly small distances. Such a detailed description of the three-body force is not available at this time, either from a theoretical or experimental point of view.

APPENDIX A

SOME RESULTS REQUIRED IN THE EVALUATION OF W AND U

The following integrals will be of use in calculating the effective interaction in coordinate space:

$$\int \frac{d\mathbf{q}\, q^2\, e^{-i\mathbf{q}\cdot\mathbf{r}}}{(q^2+\mu^2)(q^2+\eta^2)} = \frac{4\pi}{r} \int_0^\infty \frac{q^3 \sin qr\, dq}{(q^2+\mu^2)(q^2+\eta^2)} = 2\pi^2 \frac{[\mu^3 Y_\mu(r) - n^3 Y_\eta(r)]}{\mu^2 - \eta^2}, \tag{A1}$$

where use has been made of no.5 p.63 and no. 25 p.66 of the Bateman tables(14). On setting $\eta=0$ the Fourier transform of the Yukawa function is obtained:

$$\int \frac{d\mathbf{q}\, e^{-i\mathbf{q}\cdot\mathbf{r}}}{q^2+\mu^2} = 2\pi^2\, \mu Y_\mu(r) \ . \tag{A2}$$

Using no. 39 p.68 of reference (14) the following result may easily be obtained:

$$\int d\mathbf{q}\, \frac{q^2\, e^{-i\mathbf{q}\cdot\mathbf{r}}}{(q^2+\mu^2)^2} = \frac{4\pi}{r} \int_0^\infty \frac{q^3 \sin qr\, dq}{(q^2+\mu^2)^2} = \pi^2 \mu [2Y_\mu(r) - E_\mu(r)] \ . \tag{A3}$$

Finally, the following two integrals will be required in evaluating the first order contribution of W:

$$\int \frac{d\mathbf{q}\, e^{-i\mathbf{q}\cdot\mathbf{r}}}{(q^2+\mu^2)^2} = \frac{\pi^2}{\mu} E_\mu(r) \ , \tag{A4}$$

using no. 39 p.68 of the Bateman tables, and

$$\int \frac{d\mathbf{q}\; e^{-i\mathbf{q}\cdot\mathbf{r}}}{(q^2+\mu^2)(q^2+\eta^2)} = 2\pi^2 \; \frac{[\eta Y_\eta(r) - \mu Y_\mu(r)]}{\mu^2 - \eta^2} \;, \tag{A5}$$

using no. 25 p.66.

The effect of the operator $(\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla})$ on an arbitrary function of r, say f(r), will be required. Now

$$(\sigma_1\cdot\nabla)(\sigma_2\cdot\nabla)f(r) = \boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}[(\boldsymbol{\sigma}_2\cdot\mathbf{r})\;\frac{1}{r}\,\frac{df(r)}{dr}] \;,$$

and making use of the relations $\boldsymbol{\nabla}(\boldsymbol{\sigma}_2\cdot\mathbf{r}) = \boldsymbol{\sigma}_2$ and

$$\boldsymbol{\nabla}(\frac{1}{r}\,\frac{df}{dr}) = \mathbf{r}[\frac{1}{r^2}\,\frac{d^2f}{dr^2} - \frac{1}{r^3}\,\frac{df}{dr}] \;,$$

this expression gives

$$(\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla})f(r) = (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(\frac{1}{r}\,\frac{df}{dr}) + (\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})(\frac{1}{r^2}\,\frac{d^2f}{dr^2} - \frac{1}{r^3}\,\frac{df}{dr}) \;. \tag{A6}$$

If $f(r) = Y_\mu(r)$, equation (A6) gives

$$(\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla})\; Y_\mu(r) = \frac{\mu^2}{3}\;[\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+S_{12}(\hat{r})\;\;T_\mu(r)]\;\;Y_\mu(r) \;, \tag{A7}$$

while for $f(r) = E_\mu(r)$,

$$(\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla})E_\mu(r) = \frac{\mu^2}{3}[\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+S_{12}(\hat{r})]E_\mu(r)+\frac{\mu^2}{3}[-2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+S_{12}(\hat{r})]Y_\mu(r) \,. \tag{A8}$$

From equation (39) the effective interaction is given by

$$u(\mathbf{r}) = \frac{4C_p\rho}{\pi\mu^6}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}) \int d\mathbf{q}\; q^2[\frac{1}{(q^2+\mu^2)^2}$$

$$- \frac{2(1-\xi)}{(q^2+\mu^2)(q^2+\eta^2)} + \frac{(1-\xi)^2}{(q^2+\eta^2)^2}] \; e^{-i\underset{\sim}{q}\cdot\underset{\sim}{r}} \; . \qquad \text{(A9)}$$

Using equations (A1) and (A3) the integrals may be done to give

$$\pi^2 [(2\mu - 4\frac{(1-\xi)\mu^3}{\mu^2 - \eta^2}) Y_\mu(r) - \mu E_\mu(r) + (2(1-\xi)^2\eta + 4\frac{(1-\xi)\eta^3}{\mu^2 - \eta^2}) \; Y_\eta(r)$$

$$-(1-\xi)^2 \eta E_\eta(r)] \; .$$

Applying equations (A7) and (A8) $u(\underset{\sim}{r})$ may be written as in equation (40), with c_1 through c_6 given by

$$c_1 = 4 - 4(1-\xi) \; \mu^2/(\mu^2-\eta^2) \; ,$$

$$c_2 = -1 \; ,$$

$$c_3 = 4(1-\xi)^2(\eta/\mu)^3 + 4(1-\xi)\eta^2(\eta/\mu)^3/(\mu^2-\eta^2) \; ,$$

$$c_4 = -(1-\xi)^2(\eta/\mu)^3 \; ,$$

$$c_5 = 2 - 4(1-\xi) \; \mu^2/(\mu^2-\eta^2) \; ,$$

$$c_6 = 2(1-\xi)^2(\eta/\mu)^3 + 4(1-\xi) \; \eta^2(\eta/\mu)^3/(\mu^2-\eta^2) \; . \qquad \text{(A10)}$$

APPENDIX B

SOME RESULTS REQUIRED IN OBTAINING <W>/N

On substituting for $D^2(k_F|\underset{\sim}{x}-\underset{\sim}{y}|)$ in terms of $F(q)$ in I_c one obtains

$$I_c = \frac{\mu^6}{(2\pi)^3} \int d\underset{\sim}{q}\ F(q) \left(\int Y_\mu(x) e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}}\ d\underset{\sim}{x} \right)^2 .$$

But

$$\int Y_\mu(x) e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}}\ d\underset{\sim}{x} = 4\pi\ f_c(q) \quad ,$$

where $f_c(q)$ is given by equation (D1), and so

$$I_c = 8\mu^4 \int_0^\infty dq\ \frac{q^2 F(q)}{(q^2+\mu^2)^2} . \tag{B1}$$

Similarly, when D^2 is replaced by its Fourier transform in I_t one obtains

$$I_t = \frac{\mu^6}{(2\pi)^3} \int d\underset{\sim}{q}\ F(q) \int d\underset{\sim}{x} d\underset{\sim}{y}\ e^{i\underset{\sim}{q}\cdot(\underset{\sim}{x}-\underset{\sim}{y})} (3\cos^2\theta_{xy}-1) Y_\mu(x) T_\mu(x) Y_\mu(y) T_\mu(y) . \tag{B2}$$

Now

$$3\cos^2\theta_{xy} - 1 = \frac{8\pi}{5} \sum_m Y^*_{2m}(\hat{x})\ Y_{2m}(\hat{y}) \quad , \tag{B3}$$

and

$$e^{i\underset{\sim}{q}\cdot\underset{\sim}{x}} = 4\pi \sum_{\ell,\eta} i^\ell\ Y^*_{\ell\eta}(\hat{q})\ Y_{\ell\eta}(\hat{x})\ j_\ell(qx) \quad , \tag{B4}$$

with a similar expression for $e^{-i\underset{\sim}{q}\cdot\underset{\sim}{y}}$. When equations (B3) and (B4) are substituted in equation (B2), the angular integrations over $\underset{\sim}{x}$, $\underset{\sim}{y}$ and $\underset{\sim}{q}$ may be done, so that

$$I_t = 16\mu^2 \int dq\, q^2\, F(q) \left(\int_0^\infty j_2(qx)\, T_\mu(x)\, Y_\mu(x) x^2 dx \right)^2 . \qquad \text{(B5)}$$

Substituting for the integral over x from equation (D2), I_t becomes

$$I_t = 16 \int_0^\infty dq \frac{q^6 F(q)}{(q^2+\mu^2)^2} . \qquad \text{(B6)}$$

APPENDIX C

EVALUATION OF THE FIRST AND SECOND ORDER SPIN SUMS

I. First Order

The matrix elements for the central contribution are of the form $\langle ij|(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\ k(r)|ij\rangle$ and $\langle ij|(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$ $k(r)|ji\rangle$, while for the tensor contribution they are $\langle ij|(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\ S_{12}(\hat{r})\ k(r)|ij\rangle$ and $\langle ij|(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\ S_{12}(\hat{r})\ k(r)|ji\rangle$, where $k(r)$ stands for any of the radial functions appearing in the OPEP or the effective interaction. The state labels i and j include all the quantum numbers necessary to specify the single particle states, so that, for example, $|i\rangle = |\tau_i m_i k_i\rangle$ where τ_i is the 3-component of the i-spin, m_i the z-component of the spin, and k_i the linear momentum, all with reference to the i^{th} state. The first order contribution requires a sum over all occupied states, and because of the nature of the matrix elements, the i-spin and spin sums may be separated from the sum over momentum, and done individually.

The direct i-spin sum is

$$\sum_{\tau_i,\tau_j} \langle\tau_i\tau_j|\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2|\tau_i\tau_j\rangle = \sum_{\tau_i,\tau_j} \langle\tau_i\tau_j|2P^{\tau}_{12}-1|\tau_i\tau_j\rangle\ ,$$

where P^{τ} is the i-spin exchange operator[15]. The sum is now easily evaluated to give

$$2\sum_{\tau_i} - \sum_{\tau_i,\tau_j} = 0\ . \qquad \text{(C1)}$$

A similar result holds for the direct spin sum. The exchange i-spin sum is

$$\sum_{\tau_i,\tau_j} \langle \tau_i \tau_j | 2P_{12}^{\tau} - 1 | \tau_j \tau_i \rangle = 6 \ , \qquad \text{(C2)}$$

with a similar result for the exchange spin sum.

For the tensor contribution, the spin sum may be separated from the radial part by taking $\hat{r}$ as the quantization axis to give

$$\sum_{m_i,m_j} \langle m_i m_j | 3\sigma_z^{(1)} \sigma_z^{(2)} - \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2 | m_i m_j \rangle = 0 \ . \qquad \text{(C3)}$$

Similarly the exchange spin sum gives zero, so that to first order only the central part of the potential contributes.

Finally, the evaluation of the first order contribution of u as given by equation (52) requires the following relation:

$$\sum_{m_i,m_j} \langle m_i m_j | (\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{q}) (\underset{\sim}{\sigma}_2 \cdot \underset{\sim}{q}) | m_j m_i \rangle = 2q^2 \ . \qquad \text{(C4)}$$

This may be proven by taking $\hat{q}$ as the quantization axis to give

$$\sum_{m_i m_j} \langle m_i m_j | \sigma_z^{(1)} \sigma_z^{(2)} | m_j m_i \rangle \ .$$

The sums are now easily performed to give equation (C4).

II. Second Order

The i-spin sum for the central direct contribution is

$$\sum_{\tau_i,\tau_j} \sum_{\tau_i',\tau_j'} \langle \tau_i\tau_j | (\underset{\sim}{\tau}_1 \cdot \underset{\sim}{\tau}_2) | \tau_i'\tau_j' \rangle \langle \tau_i'\tau_j' | (\underset{\sim}{\tau}_1 \cdot \underset{\sim}{\tau}_2) | \tau_i \tau_j \rangle .$$

Closure may be used to reduce this sum to

$$\sum_{\tau_i\tau_j} \langle \tau_i\tau_j | (2P_{12}^{\tau} - 1)^2 | \tau_i\tau_j \rangle = 12 . \tag{C5}$$

The central exchange i-spin sum is

$$\sum_{\tau_i,\tau_j} \langle \tau_i\tau_j | (2P_{12}^{\tau} - 1)^2 | \tau_j\tau_i \rangle = -6 . \tag{C6}$$

Similar results hold for the direct and exchange spin sums.

The spin sums involving the operator $S_{12}(\hat{r})$ may best be evaluated by first expressing S_{12} in q-space and then performing the summations. A typical matrix element appearing in the second order contribution can be expressed as follows:

$$\langle k_i k_j | S_{12}(\hat{r}) \; k(r) | k_i' k_j' \rangle = \frac{1}{\Omega} \int e^{i\underset{\sim}{q} \cdot \underset{\sim}{r}} S_{12}(\hat{r}) \; k(r) d\underset{\sim}{r} , \tag{C7}$$

where $\underset{\sim}{q}$ is given by equation (63). Using equation (B4) and the relation[13]

$$S_{12}(\hat{r}) = \sum_{\mu} T^2_{-\mu}(\underset{\sim}{\sigma}_1 \underset{\sim}{\sigma}_2) \; Y^*_{2-\mu}(\hat{r}) , \tag{C8}$$

where $T^2_{-\mu}$ is a second rank tensor involving only the spin variables, equation (C7) becomes, on performing the angular integration over $\underset{\sim}{r}$,

$$<k_i k_j|S_{12}(\hat{r})\ k(r)|k_i' k_j'> = -\frac{4\pi}{\Omega} S_{12}(\hat{q})\ k_2(q)\ , \tag{C9}$$

where

$$k_2(q) = \int_d^\infty j_2(qr)\ k(r) r^2 dr\ . \tag{C10}$$

A similar expression holds for the exchange matrix element, with $\underset{\sim}{q}$ replaced everywhere by $\underset{\sim}{s} = \underset{\sim}{q}+\underset{\sim}{k}_i-\underset{\sim}{k}_j$.

Consider now the direct spin sum given by

$$\sum_{m_i, m_j} <m_i m_j|S_{12}^2(\hat{q})|m_i m_j>\ ,$$

where closure has been used. On writing $S_{12}(\hat{q})$ as in equation (C3) the sums may readily be performed to give

$$\sum_{m_i, m_j} <m_i m_j|S_{12}^2(\hat{q})|m_i m_j> = 24\ . \tag{C11}$$

The exchange sum is given by

$$\sum_{m_i, m_j} <m_i m_j|S_{12}(\hat{q})\ S_{12}(\hat{s})|m_j m_i>\ .$$

This expression is slightly more difficult to evaluate because of the appearance of both $\hat{s}$ and $\hat{q}$ in the matrix elements. However, by again taking $\underset{\sim}{q}$ as the quantization axisk and expressing $(\underset{\sim}{\sigma}_1 \cdot \hat{s})(\underset{\sim}{\sigma}_2 \cdot \hat{s})$ in terms of it, the sum may be evaluated in a straightforward but tedious manner to give

$$\sum_{m_i, m_j} \langle m_i m_j | S_{12}(\hat{q})\ S_{12}(\hat{x}) | m_j m_i \rangle = 24\ P_2(\cos\theta_{sq}) \ . \tag{C12}$$

A final result which will prove useful is

$$\sum_{m_i, m_j} \langle m_i m_j | S_{12}(\hat{q}) (\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2) | m_i m_j \rangle = 0 \ . \tag{C13}$$

This may be proven by taking $\underset{\sim}{q}$ as the quantization axis, and replacing $\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2$ by $2P_{12}^{\sigma}-1$ to obtain

$$3 \sum_{m_i, m_j} \{2\langle m_i m_j | \sigma_z^{(1)} \sigma_z^{(2)} | m_j m_i \rangle - \langle m_i m_j | \sigma_z^{(1)} \sigma_z^{(2)} | m_i m_j \} - 12 \ .$$

The sums are easily performed to give the required result. Similarly it may be proven that for the exchange term,

$$\sum_{m_i, m_j} \langle m_i m_j | S_{12}(\hat{q}) (\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2) | m_j m_i \rangle = 0 \ . \tag{C14}$$

APPENDIX D

EVALUATION OF SOME INTEGRAL TRANSFORMS

The j_0 transform of $f_c(r)$ is given by

$$f_c(q) = \int_d^\infty j_0(qr) \frac{e^{-\mu r}}{\mu r} r^2 dr .$$

This integral is easily evaluated to give

$$f_c(q) = \frac{e^{-\mu d}}{\mu q} \left[\frac{\mu \sin qd + q \cos qd}{\mu^2 + q^2}\right] . \qquad \text{(D1)}$$

The j_2 transform of $f_t(r)$ is

$$f_t(q) = \int_d^\infty j_2(qr) \; h_2(i\mu r) r^2 dr ,$$

where $h_2(i\mu r) = T_\mu(r) \; Y_\mu(r)$ is a spherical Honkel function of second order. This integral has been done in reference (16), and the result is

$$f_t(q) = \frac{e^{-\mu d}}{\mu q} \left[3 j_1(qd) \left(\frac{1+\mu d}{\mu^2 d}\right) - \frac{\mu \sin qd + q \cos qd}{\mu^2 + q^2}\right] . \quad \text{(D2)}$$

For the effective interaction, the required j_0 transform is

$$g_c(q) = \int_d^\infty j_0(qr) \; g(r) r^2 dr ,$$

$$= \int_d^\infty j_0(qr)[c_1Y_\mu(r) + c_2E_\mu(r) + c_3Y_\eta(r) + c_4E_\eta(r)]r^2dr \ .$$

Now

$$\int_d^\infty j_0(qr)\ E_\mu(r)r^2dr = \mu d\ f_c(q)\ -\frac{e^{-\mu d}}{q}\ \frac{\sin qd}{\mu^2 + q^2} + \frac{2\mu^2\ f_c(q)}{\mu^2 + q^2} \ ,$$

and introducing the notation $f_c^{(\mu)}(q)$ and $f_c^{(\eta)}(q)$, g_c becomes

$$\begin{aligned} g_c(q) &= c_1f_c^{(\mu)}(q) + c_3f_c^{(\eta)}(q) + c_2[\mu df_c^{(\mu)}(q) \\ &\quad - \frac{e^{-\mu d}}{q}\ \frac{\sin qd}{\mu^2 + q^2} + \frac{2\mu^2 f_c^{(\mu)}(q)}{\mu^2 + q^2}] \qquad \text{(D3)} \\ &\quad + c_4[\eta df_c^{(\eta)}(q)\ -\frac{e^{-\eta d}}{q}\ \frac{\sin qd}{\eta^2 + q^2} + \frac{2\eta^2 f_c^{(\eta)}(q)}{\eta^2 + q^2}] \ . \end{aligned}$$

The j_2 transform of $g_t(r)$ is given by

$$\begin{aligned} g_t(q) &= c_5 \int_d^\infty j_2(qr)\ T_\mu(r)\ Y_\mu(r)r^2dr \\ &\quad + c_2 \int_d^\infty j_2(qr)[E_\mu(r) + Y_\mu(r)]r^2dr \\ &\quad + c_6 \int_d^\infty j_2(qr)\ T_\eta(r)\ Y_\eta(r)r^2dr \\ &\quad + c_4 \int_d^\infty j_2(qr)[E_\eta(r) + Y_\eta(r)]r^2dr \ . \end{aligned}$$

The first and third integrals are just $f_t^{(\mu)}(q)$ and $f_t^{(\eta)}(q)$, so it is only necessary to evaluate $\int_d^\infty j_2(qr)[E_\mu(r) + Y_\mu(r)]r^2dr$. This integral may be evaluated by writing $E_\mu(r) + Y_\mu(r) = -i\mu r\, h_1(i\mu r)$, and intragrating by parts. The final result is

$$j_2\{E_\mu(r) + Y_\mu(r)\} \equiv \frac{e^{-\mu d}}{\mu} d^2 j_2(qd) + \frac{e^{-\mu d}}{\mu q}[\frac{\mu \sin qd}{\mu^2 + q^2}$$

$$- qd(\frac{\mu \cos qd - q \sin qd}{\mu^2 + q^2})] + \frac{2q^2}{\mu^2 + q^2} f_c^{(\mu)}(q). \tag{D4}$$

Then

$$g_t(q) = c_5 f_t^{(\mu)}(q) + c_2 j_2\{E_\mu(r) + Y_\mu(r)\}$$

$$+ c_6 f_t^{(\eta)}(q) + c_4 j_2\{E_\eta(r) + Y_\eta(r)\}, \tag{D5}$$

where the values for the constants c_1 through c_6 are the same as given in equation (A10).

APPENDIX E

THE EULER FUNCTIONS P, G_0, AND G_2

The direct contribution to the second order energy involves an integral over the variables $\underset{\sim}{k}_i$ and $\underset{\sim}{k}_j$ which is independent of the form of the two-body interaction. This integral was first evaluated analytically by Euler[17] and the result is[18]

$$\int \frac{d\underset{\sim}{k}_i \, d\underset{\sim}{k}_j}{\underset{\sim}{q} \cdot (\underset{\sim}{q}+\underset{\sim}{k}_i-\underset{\sim}{k}_j)} = \frac{4\pi^2}{15} k_F^5 \frac{P(q/2k_F)}{q} , \tag{E1}$$

where

$$P(u) = (4+\tfrac{15}{2}u-5u^3+\tfrac{3}{2}u^5)\ln(1+u) + 29u^2 - 3u^4 + (4-\tfrac{15}{2}u$$
$$+ 5u^3-\tfrac{3}{2}u^5)\ln(1-u) - 40u^2 \ln 2 \text{ for } u<1 ,$$
$$= (4-20u^2-20u^3+4u^5)\ln(1+u) + 4u^3 + 22u + (-4+20u^2-20u^3$$
$$+ 4u^5)\ln(u-1) + (40u^3-8u^5)\ln u \text{ for } u>1 . \tag{E2}$$

It may easily be shown that

$$\lim_{u\to 1} P(u) = 26-32 \ln 2 .$$

The use of this function greatly simplifies the second order calculation, for it reduces the direct contribution to a single integration over q.

Because of the appearance of the variable $\mathbf{s}$ in the exchange contribution (equation (73)), the integrations can no longer be expressed in terms of P. However, the integrations appearing in the exchange contribution which are independent of the potential have been done numerically and tabulated by Sprung[13]. The correspondence between his generalized Euler functions, G_0 amd G_2, and the integrals appearing in the exchange contribution may be seen as follows:

According to equation (73) the central exchance contribution is proportional to the following integral:

$$\frac{\Delta E_{ex}^{(2)}}{N} \text{ (central) } \alpha \int \frac{d\mathbf{k}_i \, d\mathbf{k}_j \, d\mathbf{q}}{\mathbf{q}\cdot\mathbf{s}} \, y(q,s) \ , \tag{E3}$$

where y(q,s) is a product of j_0 transforms. Introduce the following change of variables:

$$\begin{aligned} \mathbf{k}_j &= \mathbf{k}_i' - \mathbf{s} \ , \\ \mathbf{k}_i &= \mathbf{k}_i' - \mathbf{z} \ , \\ \mathbf{q} &= \mathbf{z} \ . \end{aligned} \tag{E4}$$

The Jacobian for this transformation may be shown to be unity, so that equation (E3) becomes

$$\frac{\Delta E_{ex}^{(2)}}{N} \text{ (central) } \alpha \int \frac{d\mathbf{k}_i' \, d\mathbf{s} \, d\mathbf{q}}{\mathbf{q}\cdot\mathbf{s}} \, y(q,s) \ , \tag{E5}$$

where the proportionality constant is the same as in equation

(E3). In equation (E5), the portion of the integral which is independent of q or s is

$$\int \frac{d\Omega_q \, d\Omega_s \, dk_i'}{\cos \theta_{sq}} \equiv k_F^3 \, 16\pi^2 \, G_0(q/k_F, s/k_F) \, , \tag{E6}$$

where G_0 has been tabulated by Sprung in steps of $q/k_F = s/k_F = 0.1$. In on exactly similar manner, the tensor exchange contribution may be expressed in terms of G_2, and so the exchange contribution is reduced to a two-dimensional integral over the variable q and s.

APPENDIX F

NOTE ON THE NUMERICAL EVALUATION OF THE FIRST AND SECOND ORDER CONTRIBUTIONS

The exact values to be taken for the parameters appearing in the potentials v and u are in some cases rather arbitrary, either because of a lack of a unique definition (as for μ) or because the quantity is not well known experimentally (as for f_N^2). In Table II a consistent set of values for the required parameters are given, all assumed to be accurate to three significant figures.

The most difficult first order integration to perform is that for determining <W>/N in the case of non-zero d. If **F**(q) is replaced by $\tilde{F}(q)$ in equation (51), and the q integration attempted directly, the integrand will be found to oscillate, and fall off very slowly. The alternate method is to subsitute for $\tilde{F}(q)$, and then reverse the order of the r and the q integration. The q integration can be done exactly, and there remains an integral over the finite interval, o to d, involving r. The final result is

$$\frac{<W>}{N}(d\neq 0) = \frac{<W>}{N}(d=0) - \frac{2c_p\rho^2}{\mu^6}\left\{\int_o^d D^2(k_F r)(a_1 Y_\mu(r) + a_2 Y_\eta(r)\right.$$

$$\left. a_3 E_\mu(r) + a_4 E_\eta(r))r^2 dr - 4\pi^2\xi^2\right\},$$

where

$$a_1 = 4\pi^2\mu^3(-2+2(1-\xi)(\mu^2+\eta^2)/(\mu^2-\eta^2)-(1-\xi)\eta^2/(\mu^2-\eta^2)) ,$$

$$a_2 = 4\pi^2\eta^3(1-\xi)\{-2(1-\xi)-2(\mu^2+\eta^2)/(\mu^2-\eta^2)+\mu^2/(\mu^2-\eta^2)\} ,$$

$$a_3 = 3\pi^2\mu^3 ,$$

$$a_4 = 3\pi^2(1-\xi)^2\eta^3 .$$

All the first order integrations were done by dividing the interval of integration into an arbitrary number of subdivisions (five was found to be sufficient) and applying a ten-point Gauss quadrature routine in each subdivision. The first order contribution of $\langle U\rangle/N$ was obtained using equation (60) rather than equation (55), the two results being indentical provided that d is non-zero.

The evaluation of the first order contributions requires an explicit form for F(q). This may be obtained in a straightforward but tedious manner from equation (44), and the final result is

$$F(q) = \frac{3\pi^2}{k_F^3}[2 - 3(\frac{q}{2k_F}) + (\frac{q}{2k_F})^3]\ \theta(2k_F-q) , \qquad \text{(F1)}$$

where $\theta(x)$ is zero for $x<0$, and is unity otherwise.

Since the second order contribution involves integrals over infinite domains, a check is necessary to determine at what point the integration may be truncated. For the direct contributions, the same method of integration as was used in the first order calculation was applied, and it was found that carrying the integration out to $q=30F^{-1}$ was sufficient to guarantee three figure accuracy, with the major contribution coming from a region near $q=2F^{-1}$.

Because of the manner in which the functions G_0 and G_2 were tabulated by Sprung, a Simpson's rule integration routine was used in the calculation of the exchange contributions. The generalized Euler functions have been tabulated only as far as $q/k_F = 5.9$, with asymptotic expansions given for larger q values. The calculation of the direct contribution would indicate that q values to at least $q/k_F = 10$ are important, and so the asymptotic forms were used to extend the tables. However, since these expansions have an accuracy of approximately 1%, the third figure of the exchange contributions may be inaccurate. For this reason the last digit in the exchange contributions is enclosed in brackets in Tables IV and V.

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