EXTENSIONS OF THE KRULL-SCHMIDT-AZUMAYA THEOREM

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AZUMAYA THEOREM

By

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PREFACE

The question which prompted this dissertation is the following: "How unique is a direct sum decomposition of a given R-module?" The classical result in this direction is the so-called Krull-Schmidt-Azumaya Theorem, proved by Gorô Azumaya in [1]. It gives an answer to the question in the case that the given R-module is a direct sum of submodules with local endomorphism ring. It is generalizations and extensions of this theorem that this paper is concerned with. The results of this thesis are stated and proved in a more general categorical setting than mod-R. Moreover, we do not resort to the embedding theorem, with the idea in mind that further generalizations in those categories we are considering and similar results in other sorts of categories may be suggested.

Chapter I lays some necessary categorical groundwork. In Chapter 2 we combine results of S. B. Conlon [2] and S. Elliger [4] within our categorical setting to obtain a generalization of the Krull-Schmidt-Azumaya Theorem. We consider representations of an object as an essential extension of a direct sum of summands (rather than simply direct sum decompositions), and we allow certain summands other than those with local endomorphism ring. Chapter 3, following [10], (which was in turn applying the results of [3]) extends the concept of "local endomorphism ring" to the concept of "the exchange property" and produces certain coproduct uniqueness theorems. Finally, in Chapter 4, we consider decomposition of injectives and we see that certain problems involving coproduct decompositions can be eliminated in the case where the objects concerned are injective. We present a uniqueness theorem due to R. B. Warfield [10] and draw conclusions from this with the aid of the "spectral category" (which will be defined and examined).

ACKNOWLEDGEMENTS

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NOTATION

We will, in general, represent the objects of a category by capital latin letters (A, B, C, ...) and the morphisms by small Greek letters (α, β, γ, ...). For certain categorical notions where ambiguity arises as to whether an object or a morphism is referred to, (for instance the image of a morphism) we adopt the following convention: If an object is being referred to we capitalize the initial letter of label (for this notion). Thus Im α and Ker α are objects while im α and ker α are the corresponding morphisms.

$A \leq B$ p. 1

$\bigoplus_I A_i$ = the coproduct of objects A_i ($i \in I$) in an abelian category.

$A \lhd B$, $B \rhd A$ p. 12

$|I|$ = the cardinality of the set I.

endo (A) = the ring of endomorphisms of the object A.

E_A (ι_i: I, I') p. 24

E(A) = the injective hull of the object A.

$\underline{G}$<A, B> = the set of morphisms in the category $\underline{G}$ whose domain is A and whose co-domain is B.

$\underline{S(G)}$ p. 73

Chapter 1

Introduction

§1. Some Basic Notions.

In this chapter we will develop some elementary properties of coproducts in Grothendieck categories, those properties which we will find useful in describing uniqueness properties of certain coproducts. First however, some comments on notation seem necessary.

Given objects A and B in an arbitrary category, we will write $A \leq B$ if there is a monomorphism with domain A and codomain B. When we wish to distinguish one such monomorphism, unless stated otherwise, this distinguished monomorphisms will be labelled σ_{AB}, or simply σ_A if the codomain is evident from the context. Thus, we will write $C = A \oplus B$ (in an abelian category) to mean $A \leq C$, $B \leq C$ and

$$\begin{array}{ccc} A & & \\ & \searrow^{\sigma_A} & \\ & & C \\ & \nearrow_{\sigma_B} & \\ B & & \end{array}$$

is a coproduct diagram. Similarly, if $A \leq C$ and $B \leq C$, we write $D = A \cap B$ to mean

$$\begin{array}{ccccc} & & A & & \\ & \overset{\sigma_{DA}}{\nearrow} & & \overset{\sigma_{AC}}{\searrow} & \\ D & & & & C \\ & \underset{\sigma_{DB}}{\searrow} & & \underset{\sigma_{BC}}{\nearrow} & \\ & & B & & \end{array}$$

is a pullback diagram.

If $A \leq B$ and $B \leq C$, then obviously $A \leq C$, and if we have distinguished monomorphisms σ_{AB}, σ_{BC} and σ_{AC}, then, unless we state otherwise, they will be chosen so that $\sigma_{AC} = \sigma_{BC}\sigma_{AB}$.

Definition 1.1: A Grothendieck category is an abelian category which is cocomplete, well powered, and which satisfies the Grothendieck condition, which we give in the following form: If $A_i \leq C$ is an upwards directed family of objects $(i \in I)$, (i.e. if the distinguished monomorphisms $\sigma_{A_i}: A_i \to C$ form an upwards directed family), and if $B \leq C$, then $(\bigcup_I A_i) \cap B = \bigcup_I (A_i \cap B)$.

Note that we do not require the existence of a generator in a Grothendieck category. For a treatment of such categories, and of category theory in general, the reader is referred to Pareigis [8]. We will assume for the remainder of this dissertation that all objects and morphisms are the objects and morphisms of a Grothendieck category, unless otherwise stated.

It should be noted that an example of a Grothendieck category is the category of (left) modules over a ring with identity. The principal application of the theory developed in the following chapters is to precisely such categories.

Finite products and arbitrary coproducts exist in a Grothendieck category and finite products and coproducts can be identified; that is, there are finite biproducts. We will write $A = \bigoplus_I A_i$ if A is the coproduct of the A_i $(i \in I)$, and we will say that A is the direct sum of A_i $(i \in I)$. It is well known that an object C in a Grothendieck (or even any additive) category is the coproduct of objects A and B if and only if there exist "injections" σ_A: $A \to C$ and σ_B: $B \to C$ (which are monomorphisms) and "projections" π_A: $C \to A$ and π_B: $C \to B$ (which are epimorphisms) such that $\pi_A \sigma_A = 1_A$, $\pi_B \sigma_B = 1_B$, $\pi_B \sigma_A = 0$, $\pi_A \sigma_B = 0$ and $\sigma_A \pi_A + \sigma_B \pi_B = 1_C$. These results are to be found in [8] (pages 167-168). It is evident that all these conditions are not needed. For the purposes of this dissertation we will often find the following characterization of the coproduct useful:

Proposition 1.2: Let $A \leq C$ and $B \leq C$. Then $C = A \oplus B$ if and only if there are morphisms $\pi_A: C \to A$ and $\pi_B: C \to B$ such that $\pi_B \sigma_A = 0$, $\pi_A \sigma_B = 0$ and $\sigma_A \pi_A + \sigma_B \pi_B = 1_C$.

Proof: That these conditions are necessary is obvious by the preceeding remark.

Their sufficiency follows since $1_C = \sigma_A \pi_A + \sigma_B \pi_B$ implies that $\sigma_A = (\sigma_A \pi_A + \sigma_B \pi_B)\sigma_A = \sigma_A \pi_A \sigma_A$ so $1_A = \pi_A \sigma_A$, and similarly $1_B = \pi_B \sigma_B$. (This further implies that π_A and π_B are epimorphic.) Thus $C = A \oplus B$ by the preceeding remark. ∎

We should also note that if $C = A \oplus B$, then $A \cap B = 0$.

Lemma 1.3. Let $A \leq B \leq C$ be objects such that A is a direct summand of C. Then A is a direct summand of B.

Proof: Suppose $C = A \oplus X$ with projections π_A and π_X. We use the convention that σ_A, σ_B, σ_X (and so on) are the distinguished monomorphisms with codomain C. We will show that $B = A \oplus (B \cap X)$.

Define $\xi: B \to B$ by $\xi = 1_B - \sigma_{AB} \pi_A \sigma_B$. Then

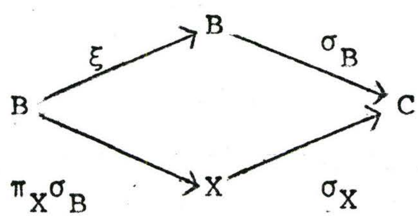
$$\sigma_B \xi = \sigma_B - \sigma_B \sigma_{AB} \pi_A \sigma_B$$

$$= (1_C - \sigma_B \sigma_{AB} \pi_A) \sigma_B$$

$$= (1_C - \sigma_A \pi_A) \sigma_B$$

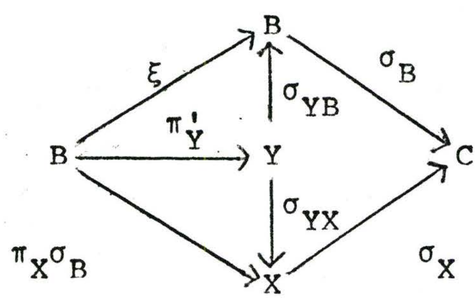
$$= \sigma_X \pi_X \sigma_B \quad \text{since} \quad 1_C = \sigma_A \pi_A + \sigma_X \pi_X.$$

Hence the diagram



commutes.

Let $Y = B \cap X$. Then there is a unique $\pi'_Y \colon B \to Y$ such that



commutes.

Now, define $\pi'_A \colon B \to A$ by $\pi'_A = \pi_A \sigma_B$. Then

$\pi'_Y \sigma_{AB} = 0$, since

$$\sigma_{YX} \pi'_Y \sigma_{AB}$$

$$= \pi_X \sigma_B \sigma_{AB}$$

$$= \pi_X \sigma_A$$

$$= 0,$$

and

$$\pi_A'\sigma_{YB} = \pi_A\sigma_B\sigma_{YB}$$
$$= \pi_A\sigma_X\sigma_{YX}$$
$$= 0.$$

Also

$$\sigma_B = (\sigma_A\pi_A + \sigma_X\pi_X)\sigma_B$$
$$= \sigma_A\pi_A\sigma_B + \sigma_X\sigma_{YX}\pi_Y'$$
$$= \sigma_B\sigma_{AB}\pi_A\sigma_B + \sigma_X\sigma_{YX}\pi_Y'$$
$$= \sigma_B\sigma_{AB}\pi_A' + \sigma_X\sigma_{YX}\pi_Y'$$
$$= \sigma_B\sigma_{AB}\pi_A' + \sigma_B\sigma_{YB}\pi_Y'$$
$$= \sigma_B(\sigma_{AB}\pi_A' + \sigma_{YB}\pi_Y')$$

so $1_B = \sigma_{AB}\pi_A' + \sigma_{YB}\pi_Y'$.

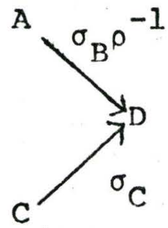
Hence, by Proposition 1.2, $B = A \oplus Y$. ▮

We remark that it is also true that if $A \leq B \leq C$ where A is a direct summand of B and B is a direct summand of C, then A is a direct summand of C. The proof of this is trivial.

Lemma 1.4: Let $D = A \oplus C = B \oplus X$ with projections π_A, π_B and π_C to A, B and C respectively.

Then $D = B \oplus C$ if and only if $\pi_A \sigma_B$ is an isomorphism.

<u>Proof</u>: (a) Assume $\pi_A \sigma_B$ is an isomorphism. Let $\rho = \pi_A \sigma_B$. Thus we obtain



and hence there is a unique $\tau : D \to D$ such that $\tau \sigma_C = \sigma_C$ and $\tau \sigma_A = \sigma_B \rho^{-1}$ (by the universality of the coproduct). It follows that $\tau \sigma_C \pi_C = \sigma_C \pi_C$ and $\tau \sigma_A \pi_A = \sigma_B \rho^{-1} \pi_A$ and hence

$$\tau = \tau(\sigma_A \pi_A + \sigma_C \pi_C)$$

$$= \sigma_B \rho^{-1} \pi_A + \sigma_C \pi_C .$$

Now τ is an isomorphism:

(i) τ is an epimorphism: If we suppose that $\phi\tau = 0$ for some ϕ, then $\phi\sigma_C = \phi\tau\sigma_C = 0$ so $\phi\sigma_C\pi_C = 0$. But then $0 = \phi\tau = \phi(\sigma_B \rho^{-1} \pi_A + \sigma_C \pi_C)$ so $\phi\sigma_B \rho^{-1} \pi_A = 0$. Hence

$$\phi\sigma_B = \phi\sigma_B \rho^{-1} \rho$$

$$= \phi\sigma_B \rho^{-1} \pi_A \sigma_B$$

$$= 0.$$

Therefore $\phi\sigma_A\pi_A = 0$ and $\phi\sigma_B\pi_B = 0$ so $\phi = 0$. Hence τ is epimorphic.

(ii) τ is a monomorphism: If we suppose that $\tau\psi = 0$ for some ψ, then

$$\begin{aligned}\pi_A\psi &= \rho\rho^{-1}\pi_A\psi \\ &= \pi_A(\sigma_C\pi_C + \sigma_B\rho^{-1}\pi_A)\psi \\ &= 0.\end{aligned}$$

Also $\sigma_C\pi_C\psi = 0$ so

$$\begin{aligned}\psi &= (\sigma_A\pi_A + \sigma_C\pi_C)\psi \\ &= 0.\end{aligned}$$

Hence τ is monomorphic.

Thus τ is an isomorphism and τ^{-1} exists. We define $\pi_B' = \rho^{-1}\pi_A\tau^{-1}$ and $\pi_C' = \pi_C\tau^{-1}$. Then

$$\begin{aligned}\pi_B'\sigma_C\pi_C &= \rho^{-1}\pi_A\tau^{-1}\sigma_C\pi_C \\ &= \rho^{-1}\pi_A\sigma_C\pi_C \quad \text{since } \tau\sigma_C = \sigma_C \\ &= 0.\end{aligned}$$

Hence $\pi_B'\sigma_C = 0$.

Also

$$\pi_C'\sigma_B\rho^{-1}\pi_A = \pi_C\tau^{-1}\sigma_B\rho^{-1}\pi_A$$

$$= \pi_C\sigma_A\rho\rho^{-1}\pi_A$$

$$= 0$$

so $\pi_C'\sigma_B = 0$. Further,

$$\sigma_B\pi_B' + \sigma_C\pi_C' = \sigma_B\rho^{-1}\pi_A\tau^{-1} + \sigma_C\pi_C\tau^{-1}$$

$$= (\sigma_B\rho^{-1}\pi_A + \sigma_C\pi_C)\tau^{-1}$$

$$= \tau\tau^{-1}$$

$$= 1_D.$$

Thus, by Proposition 1.2, $D = B \oplus C$.

(b) Assume $B \oplus C = D = A \oplus C$ and let π_B', π_C' be the projections onto B and C respectively, resulting from the direct sum $D = B \oplus C$. Then we know

$$\sigma_A\pi_A + \sigma_C\pi_C = 1_D = \sigma_B\pi_B' + \sigma_C\pi_C'.$$

Hence

$$\pi_A = \pi_A(\sigma_B\pi_B' + \sigma_C\pi_C')$$

$$= \pi_A\sigma_B\pi_B' \quad \text{which is therefore epimorphic.}$$

Thus $\pi_A\sigma_B$ is epimorphic.

Also, if for some ϕ, $\pi_A\sigma_B\phi = 0$ then

$$\begin{aligned}\phi &= \pi_B\sigma_B\phi \\ &= \pi_B(\sigma_A\pi_A + \sigma_C\pi_C)\sigma_B\phi \\ &= \pi_B\sigma_A\pi_A\sigma_B\phi + \pi_B\sigma_C\pi_C\sigma_B\phi \\ &= 0.\end{aligned}$$

Hence $\pi_A\sigma_B$ is monomorphic. Therefore $\pi_A\sigma_B$ is an isomorphism. ▮

<u>Lemma 1.5</u>: Suppose $\phi\colon A\to B$ and $\psi\colon B\to A$ are morphisms such that $\psi\phi$ is an automorphism of A. Then $B = \operatorname{Im}\phi \oplus \operatorname{Ker}\psi$.

<u>Proof</u>: Let $\pi_\phi = (\psi\phi)^{-1}\psi$. Also, let $\sigma_\psi = \ker\psi$. Since $\psi(1_B - \phi(\psi\phi)^{-1}\psi) = 0$, there is a unique morphism, say $\pi_\psi\colon B\to\operatorname{Ker}\psi$ such that

$$\begin{aligned}\sigma_\psi\pi_\psi &= 1_B - \phi(\psi\phi)^{-1}\psi \\ &= 1_B - \phi\pi_\phi.\end{aligned}$$

That is $1_B = \sigma_\psi\pi_\psi + \phi\pi_\phi$.

Further, $\pi_\psi\phi = 0$ since

$$\begin{aligned}\sigma_\psi\pi_\psi\phi &= \phi - \phi(\psi\phi)^{-1}\psi\phi \\ &= 0\end{aligned}$$

and $\pi_\phi \sigma_\psi$ is a monomorphism; and $\pi_\phi \sigma_\psi = (\psi\phi)^{-1}\psi\sigma_\psi = 0$ since $\sigma_\psi = \ker \psi$. Thus, by Proposition 1.2, $B = \operatorname{Im} \phi \oplus \operatorname{Ker} \psi$ (i.e. $B = A \oplus \operatorname{Ker} \psi$ where $\phi: A \to B$ is the injection). ▮

We note that the projection onto A in the direct sum $B = A \oplus \operatorname{Ker} \psi$ in the above lemma is given by $(\psi\phi)^{-1}\psi$.

<u>Lemma 1.6</u>: If $B \leq C$, $\bigoplus_I A_i \leq C$ and $B \cap \bigoplus_I A_i \neq 0$ then there is a finite subset $J \subseteq I$ such that $B \cap \bigoplus_J A_i \neq 0$.

<u>Proof</u>: Let L be the collection of finite subsets of I and let $A_J = \bigoplus_J A_i$ for each $J \in L$. Then $\{A_J : J \in L\}$ is a directed set in $\bigoplus_I A_i$ and $\bigcup_L A_j = \bigoplus_I A_i$.

Now

$$0 \neq B \cap \bigoplus_I A_i = B \cap (\bigcup_L A_J)$$

$$= \bigcup_L (B \cap A_J)$$

by the Grothendieck property. Hence, there is some $J \in L$ such that $B \cap A_J \neq 0$. That is, there is a finite $J \subseteq I$ such that $B \cap \bigoplus_J A_i \neq 0$. ▮

§2. Essential Monomorphisms and Injectives

Definition 1.7: A monomorphism σ is called essential if whenever $\phi\sigma$ is a monomorphism then ϕ is a monomorphism. If $\sigma: A \to B$ is an essential monomorphism, then we say A is essential in B and B is an essential extension of A, and we write $A \lhd B$ and $B \rhd A$.

Proposition 1.8: $\sigma: A \to B$ is an essential monomorphism if and only if, whenever $X \leq B$, $X \neq 0$, then $A \cap X \neq 0$.

Proof: (a) Assume σ is essential and $X \leq B$ with $X \cap A = 0$. Then $A \overset{\sigma}{\to} B \overset{\nu}{\to} B/X$ (where $\nu = \text{cok}\ \sigma_X$) is a monomorphism, since $A \cap X = 0$. This implies that ν is monomorphic and thus $X = \text{Ker}\ \nu = 0$.

(b) Assume, that for any $X \leq B$, $X \neq 0$ it follows that $A \cap X \neq 0$, and that we are given $\phi\sigma$ monomorphic for some $\phi: B \to C$. Then $A \cap \text{Ker}\ \phi = 0$ since $A \cap \text{Ker}\ \phi \to A \xrightarrow{\sigma} B \xrightarrow{\phi} C$ is the 0 morphism. Hence $\text{Ker}\ \phi = 0$ so ϕ is monomorphic. ∎

We conclude this chapter with two lemmas of R. B. Warfield ([10], p. 265-266). The first shows that in a Grothendieck category, a subobject of an injective object has an injective hull. It is well known ([8], p. 199-201) that every object will have an injective hull if we equip our category with a generator.

Lemma 1.9. ([10], Lemma 3): Let D be an injective object and let $A \leq D$. Then there is an injective $E \leq D$ such that E is an essential extension of A.

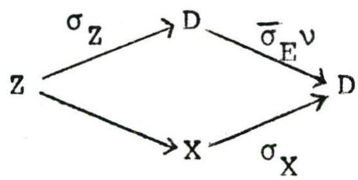
Proof: By Zorn's Lemma (on the partially ordered set of subobjects of D) we can find $E \leq D$ such that E is a maximal essential extension of A in D (up to isomorphism). We can apply Zorn's Lemma since the union of any chain of essential extensions of A is also an essential extension. Also by Zorn's Lemma, we can find $X \leq D$ maximal (up to isomorphism) with respect to $A \cap X = 0$.

Let $\nu = \text{cok}\, \sigma_X$. Then $\nu\sigma_A \colon A \to D/X$ is a monomorphism: suppose $\nu\sigma_A\phi = 0$. Then $\sigma_A\phi$ factors uniquely over $\text{Ker}\, \nu = X$, i.e. $\sigma_A\phi = \sigma_X\psi$ for some ψ. But then ϕ and ψ factor over $A \cap X = 0$. Thus $\phi = 0$ so $\nu\sigma_A$ is monomorphic.

But $\nu\sigma_A = \nu\sigma_E\sigma_{AE}$ and $\sigma_{AE} \colon A \to E$ is essential so $\nu\sigma_E$ is monomorphic. Hence, since D is injective, there is a $\overline{\sigma}_E \colon D/X \to D$ which extends σ_E to D/X. That is

$\overline{\sigma}_E \nu \sigma_E = \sigma_E$.

Thus $E \leq \operatorname{Im} \overline{\sigma}_E$. Also, let Z be the inverse image of X under $\overline{\sigma}_E \nu$. That is, the diagram



is a pullback. Then $Z \cap A = Z \cap E = 0$ and $X \leq Z$ so by the maximality of X we can assume $Z = X$. But then $\operatorname{Im} \overline{\sigma}_E \cap X = 0$ since $\operatorname{Im} \overline{\sigma}_E \nu = \operatorname{Im} \overline{\sigma}_E$. This implies E is essential in $\operatorname{Im} \overline{\sigma}_E$: If there is $Y \leq \operatorname{Im} \overline{\sigma}_E$ such that $Y \cap E = 0$ then $X \cap Y = 0$ so $(X \oplus Y) \leq D$ and $(X \oplus Y) \cap A = 0$ contradicting the maximality of X, unless $Y = 0$.

Therefore $E \leq \operatorname{Im} \overline{\sigma}_E$ and A is essential in $\operatorname{Im} \overline{\sigma}_E$, and hence, by the maximality of E, $E = \operatorname{Im} \overline{\sigma}_E$. (That is σ_E and $\operatorname{im} \overline{\sigma}_E$ represent the same subobject). Now $E \xrightarrow{\sigma_E} D \xrightarrow{\nu} D/X \xrightarrow{\operatorname{coim} \overline{\sigma}_E} \operatorname{Im} \overline{\sigma}_E = E$ is an automorphism, so E is a summand of D by Lemma 1.5. Hence E is injective.

We have therefore found E, an injective which is a maximal essential extension of A in D. That is, E is the injective hull of A in D. ▮

We note at this point that any two injective hulls of an object are isomorphic.

<u>Lemma 1.10</u>. ([10], Lemma 1): Let $A = \bigoplus_I A_i$, and $X \leq A$. Then X is essential in A if and only if $X \cap A_i$ is essential in A_i for all $i \in I$.

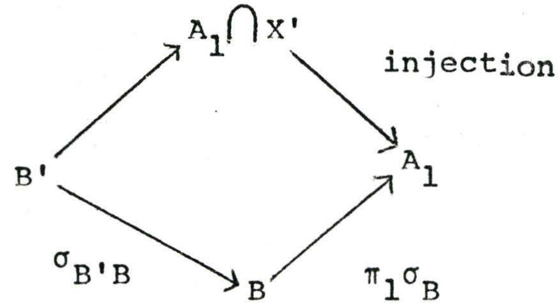
<u>Proof</u>: That the condition is necessary is trivial. To prove its sufficiency we assume that $X \cap A_i$ is essential in A_i for all $i \in I$. Let $B \leq A$, $B \neq 0$. Then, by Lemma 1.6 there is a finite $J \subseteq I$ such that $B \cap \bigoplus_J A_i \neq 0$. Hence, we need only show that if $B \leq \bigoplus_J A_i$, J finite, then $A \cap X \neq 0$. But then we need only consider the case $|J| = 2$, for the others will follow inductively.

Thus we may state what we must prove as follows: Let $A' = A_1 \oplus A_2$, $B \leq A'$ $(B \neq 0)$ with $X' \leq A'$, $X' \cap A_i$ essential in A_i for $i = 1, 2$. Then we want to show that $B \cap X' \neq 0$.

Let $\sigma_1: A_1 \to A'$ and $\sigma_2: A_2 \to A'$ be the injections and let π_1, π_2 be the corresponding projections. If $\pi_1\sigma_B = 0$ then $B \leq A_2$ so $B \cap X' \neq 0$. Hence assume $\pi_1\sigma_B \neq 0$. Then $\text{Im}\ \pi_1\sigma_B \cap X' \neq 0$.

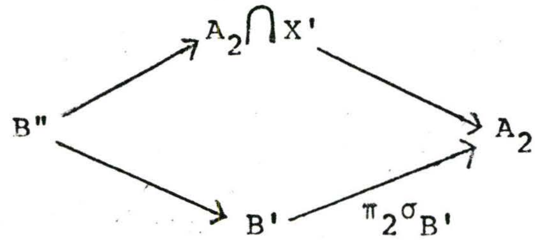
Let B' be the inverse image of $A_1 \cap X'$ under

$\pi_1\sigma_B$. That is



is a pullback. $B' \neq 0$ since $\pi_1\sigma_B \neq 0$.

If $\pi_2\sigma_{B'} = 0$, then $B' \leq A_1$ so $0 \neq B' \cap X' \leq B \cap X'$. Hence we assume $\pi_2\sigma_{B'} \neq 0$ and we let B'' be the inverse image of $A_2 \cap X'$ under $\pi_2\sigma_{B'}$. That is



is a pullback. Then $\pi_2\sigma_{B''} \neq 0$.

Hence $\sigma_{B''} \neq 0$ and $\sigma_{B''}$ factors over $\sigma_{X'}$, since $\sigma_{B''}$ factors over $A_1 \cap X'$ and $A_2 \cap X'$. Therefore $B \cap X' \neq 0$. ▮

Chapter 2

A Generalization of the Krull-Schmidt-Azumaya Theorem

§1. Local Endomorphism Rings and Idempotents

In this chapter we will prove a generalization of the classical Krull-Schmidt-Azumaya Theorem for an arbitrary Grothendieck category. We will examine representations of an object as essential extensions of direct sums of its summands, and determine their uniqueness properties. Specifically we show that if $C \rhd \bigoplus_{I \cup I'} \operatorname{Im} \iota_i$ and $C \rhd \bigoplus_{J \cup J'} \operatorname{Im} \kappa_j$ where $\{\iota_i \in \text{endo}(C): i \in I \cup I'\}$ and $\{\kappa_j \in \text{endo}(C): j \in J \cup J'\}$ are sets of orthogonal idempotents, $\operatorname{Im} \iota_i$ and $\operatorname{Im} \kappa_j$ have local endomorphism rings for $i \in I$, $j \in J$ and $\operatorname{Im} \iota_i$, $\operatorname{Im} \kappa_j$ have no summands with local endomorphism ring for $i \in I'$, $j \in J'$, then $\bigoplus_I \operatorname{Im} \iota_i = \bigoplus_J \operatorname{Im} \kappa_j$ and the summands of these two direct sums are pairwise isomorphic.

This easily seen to be a generalization of the usual Krull-Schmidt-Azumaya Theorem for Grothendieck categories as in [8] (p. 193-195) where only direct sum decompositions (and not essential extensions of direct sums) are considered, and where all summands have local endomorphism rings. Further, this result contains results of Conlon [2], and S. Elliger ([4],Satz 6.1);the former allowing summands other than those with local endomorphism ring, the latter considering decompositions as essential extensions of direct sums of summands (arising from sets of orthogonal idempotents in the endomorphism ring). In both these papers,the theorems are stated for decomposition of R-modules, however their extension to more general Grothendieck categories is quite elementary. Both Elliger and R. B. Warfield ([10], p. 264-272) have also considered the case where all the objects involved are injective and have derived some even stronger uniqueness properties in this case. These we will examine in Chapter 4.

We first note that the endomorphisms of an object in a Grothendieck category form a unitary ring, where the addition is defined by the additive structure of the morphisms, and the multiplication is given by the composition of morphisms.

<u>Definition 2.1</u>: A unitary ring is called <u>local</u> if the set of non-units is additively closed. It is easy to check that this is equivalent to saying that the ring has a unique maximal left ideal (which is also a unique maximal right ideal).

We see then that if endo (A) is a local ring, an automorphism of A can never be the sum of non-automorphisms.

Proposition 2.2: If A is an object with endo (A) local, then A is direct sum indecomposible.

Proof: Assume endo (A) is local and $A = A_1 \oplus A_2$ with $A_1, A_2 \neq 0$. Let $\sigma_1: A_1 \to A$, $\sigma_2: A_2 \to A$ be the injections and $\pi_1: A \to A_1$ and $\pi_2: A \to A_2$ be the projections. Then $\pi_2 \sigma_1 \pi_1 = 0$ and $\pi_1 \sigma_2 \pi_2 = 0$ so $\sigma_1 \pi_1$ and $\sigma_2 \pi_2$ are non-units in endo (A). But $1_A = \sigma_1 \pi_1 + \sigma_2 \pi_2$, a unit. This is a contradiction, since endo (A) is local, so either $A_1 = 0$ or $A_2 = 0$. ∎

Lemma 2.3. ([2], Lemma 2): Let $C = A \oplus B$ where endo (A) is a local ring. Let $\pi_A: C \to A$ be the projection and, as usual, σ_A the injection. Then, for any $\phi: C \to C$, either $\phi\sigma_A$ or $(1_C - \phi)\sigma_A$ is a monomorphism, and, if we call this monomorphism σ, $C = \text{Im}\,\sigma \oplus B$. Further, $\pi_A\sigma$ is an automorphism of A.

Proof: We know $\pi_A\sigma_A = 1_A$ and so

$$1_A = \pi_A\left[\phi + (1_C - \phi)\right]\sigma_A$$

$$= \pi_A\phi\sigma_A + \pi_A(1_C - \phi)\sigma_A ,$$

a unit in endo (A). As endo (A) is local, we must have either $\pi_A \phi \sigma_A$ or $\pi_A(1_C - \phi)\sigma_A$ is an automorphism of A. Hence, either $\phi\sigma_A$ or $(1_C - \phi)\sigma_A$ is a monomorphism. Call this monomorphism σ. Then $\pi_A \sigma$ is an automorphism of A and by Lemma 1.5

$$C = \operatorname{Im} \sigma \oplus \operatorname{Ker} \pi_A$$
$$= \operatorname{Im} \sigma \oplus B. \quad ▮$$

Proposition 2.4: If $\iota \in \text{endo}(A)$ is an idempotent then $\operatorname{Im} \iota$ is a direct summand of A with projection $\pi = \text{coim}\ \iota$. Conversely, if B is a direct summand of A, then there is an idempotent $\iota \in \text{endo}(A)$ such that $B = \operatorname{Im} \iota$ and $\pi = \text{coim}\ \iota$ is the projection onto B.

Proof: If $\iota \in \text{endo}(A)$ is an idempotent, take $\pi = \text{coim}\ \iota$, $\sigma = \text{im}\ \iota$ so $\iota = \sigma\pi$. Then $\pi\sigma = 1$ since $\sigma\pi\sigma\pi = \sigma\pi$. It follows from Lemma 1.5 that π is a projection onto $\operatorname{Im} \iota_i$.

Conversely, if π is a projection, with corresponding injection σ, then $\sigma\pi$ is an idempotent and $\operatorname{Im}(\sigma\pi) = \operatorname{Im} \pi$. ▮

Proposition 2.5: If $\{\iota_i \in \text{endo}(A) : i \in I\}$ is a set of non-trivial orthogonal idempotents, then $\bigoplus_I \operatorname{Im} \iota_i \leq A$.

Proof: We can easily see that $\bigoplus_{I'} \operatorname{Im} \iota_i = \operatorname{Im}(\Sigma_{I'} \iota_i)$

for each finite $I' \subseteq I$. (Thus $\bigoplus_{I'} \operatorname{Im} \iota_i \leq A$ for each finite $I' \subseteq I$.) Let L be the collection of all finite subsets of I and $A_J = \bigoplus_J \operatorname{Im} \iota_i = \operatorname{Im}(\sum_J \iota_i)$ for each $J \in L$. Then $\{A_J : J \in L\}$ is an upwards directed set. It follows from the Grothendieck property that $\bigoplus_I \operatorname{Im} \iota_i = \bigcup_L A_J \leq A$. ▮

<u>Lemma 2.6</u>. ([2], Lemma 4): Let A be an object, $\{\iota_i \in \operatorname{endo}(A) : i \in I \cup I'\}$ a set of orthogonal idempotents such that $\operatorname{endo}(\operatorname{Im} \iota_i)$ is local for $i \in I$ and $\operatorname{Im} \iota_i$ has no direct summands with local endomorphism ring for $i \in I'$. Assume further that $\bigoplus_{I \cup I'} \operatorname{Im} \iota_i$ is essential in A. Let $\kappa \in \operatorname{endo}(A)$ be any idempotent such that $\operatorname{endo}(\operatorname{Im} \kappa)$ is local. Then there is at least one and at most finitely many $i \in I$ such that $\rho \iota_i \sigma$ is an automorphism of $\operatorname{Im} \kappa$, where $\sigma = \operatorname{im} \kappa$ and $\rho = \operatorname{coim} \kappa$. For any such ι_i, $\operatorname{Im} \iota_i \kappa = \operatorname{Im} \iota_i$.

<u>Proof</u>: Since $\bigoplus_{I \cup I'} \operatorname{Im} \iota_i$ is essential in A, $\operatorname{Im} \kappa \cap (\bigoplus_{I \cup I'} \operatorname{Im} \iota_i) \neq 0$. Hence, by Lemma 1.6 there is a finite $J \subseteq I$ and as finite $J' \subseteq I'$ such that $\operatorname{Im} \kappa \cap (\bigoplus_{J \cup J'} \operatorname{Im} \iota_i) \neq 0$.

If $\rho \iota_{i_0}$ is to be an automorphism, then necessarily $i_0 \in J \cup J'$. If i_0 were not in $J \cup J'$, letting $\sigma' : \operatorname{Im} \kappa \cap (\bigoplus_{J \cup J'} \operatorname{Im} \iota_i) \to A$ be the injection, then $\iota_{i_0} \sigma' = 0$.

But $\sigma' \neq 0$. Also σ' factors over σ. Hence $\rho\iota_{i_0}\sigma$ is not an automorphism of $\mathrm{Im}\,\kappa$. It follows that if $\rho\iota_i\sigma$ is an automorphism then $i \in J \cup J'$ and hence there are at most finitely many $i \in I \cup I'$ such that $\rho\iota_i\sigma$ is an automorphism of $\mathrm{Im}\,\kappa$.

Let $\iota = \sum_{J \cup J'} \iota_i$. Since $\mathrm{Im}\,\kappa$ is a summand of A, and endo $(\mathrm{Im}\,\kappa)$ is local, by Lemma 2.3 either $\iota\sigma$ or $(1_A - \iota)\sigma$ is a monomorphism. Now, again with $\sigma' : \mathrm{Im}\,\kappa \cap \bigoplus_{J \cup J'} \mathrm{Im}\,\iota_i \longrightarrow A$ the injection, $\kappa\sigma' = \sigma'$ and $\iota\sigma' = \sigma'$. If $(1_A - \iota)\sigma$ were monomorphic, then $(1_A - \iota)\sigma'$ would also be monomorphic; but

$$\begin{aligned}(1_A - \iota)\sigma' &= \sigma' - \iota\sigma' \\ &= 0.\end{aligned}$$

Therefore $\iota\sigma$ is monomorphic.

It now follows by Lemma 2.3 that $\rho\iota\sigma$ is an automorphism of $\mathrm{Im}\,\kappa$. But then $\rho\iota\sigma = \sum_{J \cup J'} (\rho\iota_i\sigma)$ a unit, and since endo $(\mathrm{Im}\,\kappa)$ is local, there is some $i_0 \in J \cup J'$ such that $\rho\iota_{i_0}\sigma$ is an automorphism of $\mathrm{Im}\,\kappa$. Hence, by Lemma 1.3 $\mathrm{Im}\,\iota_{i_0}\sigma = \mathrm{Im}\,\iota_{i_0}\kappa$ is a direct summand of $\mathrm{Im}\,\iota_{i_0}$.

Now, because $\rho\iota_{i_0}\sigma$ is an automorphism, $\iota_{i_0}\sigma : \mathrm{Im}\,\kappa \to A$ is a monomorphism, so $\mathrm{Im}\,\iota_{i_0}\sigma \cong \mathrm{Im}\,\kappa$. Thus endo $(\mathrm{Im}\,\iota_{i_0}\sigma)$

is local. But we have just shown that $\operatorname{Im} \iota_{i_0}\sigma$ is a direct summand of $\operatorname{Im} \iota_{i_0}$. Hence we reject the case that $i_0 \in J'$ and obtain $\operatorname{Im} \iota_{i_0}\kappa = \operatorname{Im} \iota_{i_0}$ for some $i_0 \in I$. ∎

It should be reiterated that the above lemma tells us that any summand of A with local endomorphism ring is isomorphic to one of the summands in the given representation.

§2. A Krull-Schmidt-Azumaya Theorem

Definition 2.7: Let $A \triangleright \bigoplus_{I \cup I'} \operatorname{Im} \iota_i$ where $\{\iota_i \in \text{endo}(A) : i \in I \cup I'\}$ is a set of orthogonal idempotents such that endo $(\operatorname{Im} \iota_i)$ is local for $i \in I$ and $\operatorname{Im} \iota_i$ has no summands with local endomorphism ring for $i \in I'$. Then we call $\bigoplus_{I \cup I'} \operatorname{Im} \iota_i$ a Krull-Schmidt decomposition of A, and write $A = E_A(\iota_i : I, I')$.

This definition is somewhat more general than the definition of a Krull-Schmidt decomposition in [2]. This is also a generalization of what is called in [4] the inter-direct sum of direct summands. Since the decomposition is defined relative to A it seems necessary to include A in the notation. Further, it is important to distinguish I and I'. For these reasons, the notation $E_A(\iota_i : I, I')$, though cumbersome, carries the required information.

The following Lemma and Theorem are generalizations of a Lemma and Theorem of Conlon ([2], p. 111-112), and we follow generally his methods of proof.

Lemma 2.8: Let

$$A = E_A(\iota_i : I, I')$$
$$= E_A(\kappa_j : J, J')$$

and let $J_0 \subseteq J$ be finite. Then there exists $I_0 \subseteq I$, $|I_0| = |J_0|$, such that $\bigoplus_{I_0} \operatorname{Im} \iota_i \oplus \operatorname{Im} (1_A - \sum_{J_0} \kappa_j) = A$.

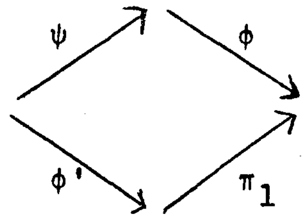
Proof: Let π_i, ρ, be the projections arising from ι_i and κ_j respectively $(i \in I \cup I', j \in J \cup J')$ and let σ_i, σ_j' be the corresponding injections. Assume $J_0 = \{1, 2, \ldots, n\}$ for notational convenience.

Applying Lemma 2.6 for $\operatorname{Im} \kappa_1$, we can find $i_1 \in I$ (say $\iota_{i_1} = \iota_1$) such that $\rho_1 \iota_1 \sigma_1'$ is an automorphism of $\operatorname{Im} \kappa_1$, and such that $\operatorname{Im} \iota_1 \kappa_1 = \operatorname{Im} \iota_1$.

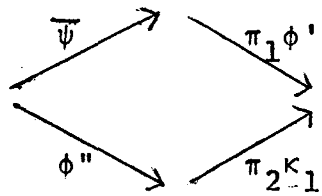
It follows that $\rho_1 \sigma_1 : \operatorname{Im} \iota_1 \to \operatorname{Im} \kappa_1$ is an isomorphism: It is clearly epimorphic. To prove it is a monomorphism, assume that for some ϕ, $\rho_1 \sigma_1 \phi = 0$. Then

$$0 = \rho_1 \sigma_1 \phi \psi$$
$$= \rho_1 \sigma_1 \pi_1 \phi'$$
$$= \rho_1 \iota_1 \phi'$$

where



is a pullback. But $\operatorname{Im}\pi_1 = \operatorname{Im}\iota_1 = \operatorname{Im}\iota_1\kappa_1$ so, taking ϕ'', $\overline{\psi}$ such that



is a pullback, we obtain $\overline{\psi}$ is epimorphic (since $\pi_1\kappa_1$ is). Thus we have $\pi_1\phi'\overline{\psi} = \pi_1\kappa_1\phi''$ and hence

$$0 = \rho_1\sigma_1\pi_1\phi'\overline{\psi}$$

$$= \rho_1\sigma_1\pi_1\kappa_1\phi''$$

$$= \rho_1\iota_1\sigma_1'\rho_1\phi''.$$

But $\rho_1\iota_1\sigma_1'$ is an automorphism so $\rho_1\phi'' = 0$. Therefore $\pi_1\phi'\overline{\psi} = 0$, which implies $\pi_1\phi' = 0$ since $\overline{\psi}$ is epimorphic. Hence $\phi = \pi\phi' = 0$. Thus $\rho_1\sigma_1$ is monomorphic and hence isomorphic.

It now follows by Lemma 1.4 that

$$\operatorname{Im}\iota_1 \oplus \left(\bigoplus_{j=2}^{n} \operatorname{Im}\kappa_j\right) \oplus \operatorname{Im}\left(1_A - \sum_{j=1}^{n}\kappa_j\right) = A.$$

We proceed inductively. Suppose

$\bigoplus_{i=1}^{k-1} \text{Im } \iota_i \oplus (\bigoplus_{j=k}^{n} \text{Im } \kappa_j) \oplus (\text{Im } 1_A - \sum_{j=1}^{n} \kappa_j) = A$. Let

ρ_k' be the projection onto $\text{Im } \kappa_k$ in this decomposition. Then, by Lemma 2.6 there is a $i_k \in I$ (say $\iota_{i_k} = \iota_k$) such that $\rho_k' \iota_k \sigma_k'$ is an automorphism of $\text{Im } \kappa_k$ and such that $\text{Im } \iota_k \kappa_k = \text{Im } \iota_k$.

$\rho_k' \iota_i = 0$ for $i = 1, 2, \ldots, k-1$, so $k \notin I_{0,k-1}$. Also $\rho_k' \sigma_k$: $\text{Im } \iota_k \to \text{Im } \kappa_k$ is an isomorphism, so, again by Lemma 1.4 we obtain $\bigoplus_{i=1}^{k} \text{Im } \iota_i \oplus (\bigoplus_{j=k+1}^{n} \text{Im } \kappa_j) \oplus \text{Im}(1_A - \sum_{j=1}^{n} \kappa_j) = A$

The result follows by induction. ∎

<u>Theorem 2.9</u>: Let

$$A = E_A(\iota_i : I, I')$$

$$= E_A(\kappa_j : J, J'),$$

let B be any object such that endo (B) is local and let $I_B = \{i \in I : \text{Im } \iota_i = B\}$, $J_B = \{j \in J : \text{Im } \kappa_j = B\}$. Then $|I_B| = |J_B|$.

<u>Proof</u>: If J_B is finite, then by Lemma 2.8 $|J_B| \leq |I_B|$. Assume J_B is infinite. Then, by Lemma 2.8 I_B is necessarily infinite.

Factor ι_i and κ_j into injections σ_i, σ_j' and projections π_i, ρ_j for $i \in I \bigcup I'$, $j \in J \bigcup J'$. Given $j_0 \in J_B$ we obtain $i_0 \in I$ such that $\rho_{j_0} \iota_{i_0} \sigma'_{j_0}$ is an automorphism of $\mathrm{Im}\, \kappa_{j_0}$ and such that $\mathrm{Im}\, \iota_{i_0} \kappa_{j_0} = \mathrm{Im}\, \iota_{i_0}$. Also, as in the proof of the previous lemma, it follows that $\rho_{j_0} \sigma_{i_0} : \mathrm{Im}\, \iota_{i_0} \to \mathrm{Im}\, \rho_{j_0}$ is an isomorphism. Thus $i_0 \in I_B$. Hence, given an element of J_B we have established a process to find a corresponding element of I_B.

But $i_0 \in I_B$ can be generated from at most finitely many $j_0 \in J_B$ in the above fashion, since if j_0 yields i_0 then $\rho_{j_0} \iota_{i_0} \sigma'_{j_0}$ is an automorphism, $\rho_{j_0} \sigma_{i_0}$ an isomorphism so $\rho_{j_0} \iota_{i_0} \sigma'_{j_0} \rho_{j_0} \sigma_{i_0}$ is an isomorphism. Hence $\rho_{j_0} \sigma_{i_0} \pi_{i_0} \kappa_{j_0} \sigma_{i_0}$ is an isomorphism (since $\iota_{i_0} = \sigma_{i_0} \pi_{i_0}$ and $\kappa_{j_0} = \sigma'_{j_0} \rho_{j_0}$). Thus $\pi_{i_0} \kappa_{j_0} \sigma_{i_0}$ is an isomorphism. But, by Lemma 2.6, for this i_0 there can be only finitely many such j_0 which make $\pi_{i_0} \kappa_{j_0} \sigma_{i_0}$ an isomorphism. This means that each $i_0 \in I$ can be produced by means of our given procedure, from only finitely many $j_0 \in J$.

We have therefore shown that for each $j \in J_B$ we can find $i \in I_B$, and that each $i \in I_B$ can be produced by only finitely many $j \in J_B$ in this way. Since J_B was assumed infinite, this implies $|J_B| \leq |I_B|$. We have already seen that this inequality holds in the finite case. By symmetry, it follows that $|I_B| = |J_B|$. ∎

Theorem 2.10: If

$$A = E_A(\iota_i : I, I')$$

$$= E_A(\kappa_j : J, J')$$

then there is a set bijection $f: I \to J$ such that $\mathrm{Im}\ \iota_i \cong \mathrm{Im}\ \kappa_{f(i)}$ and hence $\bigoplus_I \mathrm{Im}\ \iota_i \cong \bigoplus_J \mathrm{Im}\ \kappa_j$.

Proof: By Theorem 2.9 we obtain a set bijection of the indices of each isotype of summands. These bijections combine to give a bijection $f: I \to J$ which yields the required result. ∎

We notice that if $A = \bigoplus_I A_i = \bigoplus_J B_j$ where endo (A_i) and endo (B_j) are local for $i \in I$, $j \in J$ and if ι_i $(i \in I)$ and κ_j $(j \in J)$ are the corresponding idempotents, then

$$A = E_A(\iota_i : I, 0) = E_A(\kappa_j : J, 0)$$

and Theorem 2.10 for these decompositions yields the classical Krull-Schmidt-Azumaya Theorem.

Proposition 2.11: Let

$$A = E_A(\iota_i : I, I')$$

$$= E_A(\kappa_j : J, J')$$

such that I is finite (and hence J is finite). Then there are idempotents $\iota, \kappa \in$ endo (A) such that

$$A = \bigoplus_I \operatorname{Im} \iota_i \oplus \operatorname{Im} \iota$$

$$= \bigoplus_J \operatorname{Im} \kappa_j \oplus \operatorname{Im} \kappa$$

$\operatorname{Im} \iota \rhd \bigoplus_{I'} \operatorname{Im} \iota_i$, $\operatorname{Im} \kappa \rhd \bigoplus_{J'} \operatorname{Im} \kappa_j$, $\operatorname{Im} \iota \cong \operatorname{Im} \kappa$ and $\operatorname{Im} \iota$, $\operatorname{Im} \kappa$ have no direct summands with local endomorphism ring.

<u>Proof</u>: Put $\iota = 1_A - \sum_I \iota_i$, $\kappa = 1_A - \sum_J \kappa_j$

so $A = \bigoplus_I \operatorname{Im} \iota_i \oplus \operatorname{Im} \iota = \bigoplus_J \operatorname{Im} \kappa_j \oplus \operatorname{Im} \kappa$. $\operatorname{Im} \iota \rhd \bigoplus_{I'} \operatorname{Im} \iota_i$ since, if $X \leq \operatorname{Im} \iota$, with $X \cap \bigoplus_{I'} \operatorname{Im} \iota_i = 0$ then $X \cap \bigoplus_{I \cup I'} \operatorname{Im} \iota_i = 0$ so $X = 0$. Similarly $\operatorname{Im} \kappa \rhd \bigoplus_{J'} \operatorname{Im} \kappa_j$.

Further, $\operatorname{Im} \iota$ (and $\operatorname{Im} \kappa$) can have no direct summands with local endomorphism ring: Assume there is an idempotent ι' in endo (A) such that $\operatorname{Im} \iota'$ is a direct summand of $\operatorname{Im} \iota$, and endo ($\operatorname{Im} \iota'$) is local. Then, by Lemma 2.6 there is an $i \in I$ such that $\operatorname{Im} \iota_i \iota' = \operatorname{Im} \iota_i$ contradicting that $\operatorname{Im} \iota' \subseteq \operatorname{Im} \iota$.

Thus

$$A = \bigoplus_I \operatorname{Im} \iota_i \oplus \operatorname{Im} \iota$$

$$= \bigoplus_J \operatorname{Im} \kappa_j \oplus \operatorname{Im} \kappa$$

are Krull-Schmidt decompositions of A, and we can apply Lemma 2.8 to obtain $A = \bigoplus_J \operatorname{Im} \kappa_j \oplus \operatorname{Im} \iota$. Hence $\operatorname{Im} \iota \cong \operatorname{Cok} \sigma \cong \operatorname{Im} \kappa$ where $\sigma: \bigoplus_J \operatorname{Im} \kappa_j \to A$ is the injection. ▮

<u>Proposition 2.12.</u> ([2], Proposition 12): Let

$$A = E_A(\iota_i : I, I')$$

$$= E_A(\kappa_j : J, J').$$

Then $\bigoplus_I \operatorname{Im} \iota_i \bigcap \bigoplus_{J'} \operatorname{Im} \kappa_j = \bigoplus_J \operatorname{Im} \kappa_j \bigcap \bigoplus_{I'} \operatorname{Im} \iota_i$

$$= 0.$$

<u>Proof</u>: As usual we let σ_i, σ_j' be the injections and π_i, ρ_j the projections arising from ι_i and κ_j respectively for $i \in I \bigcup I'$, $j \in J \bigcup J'$. Consider any $i \in I$, let $J_0' = \{j_1, \ldots, j_n\} \subseteq J'$ and let $\kappa' = \sum_{J_0} \kappa_j$. Then by Lemma 2.3, either $\kappa' \sigma_i$ or $(1_A - \kappa')\sigma_i$ is a monomorphism. But $\kappa' \sigma_i$ cannot be a monomorphism, for if it were then $\operatorname{Im} \kappa' \sigma_i = \operatorname{Im} \kappa' \iota_i$ would be a summand of $\operatorname{Im} \kappa'$ by Lemma 1.4, a contradiction since $\operatorname{Im} \kappa' \sigma_i \cong \operatorname{Im} \iota_i$ has local endomorphism ring.

Hence $(1_A - \sum_{J_0'} \kappa_j)\sigma_i$ is a monomorphism for all $i \in I$ and all finite $J_0' \subseteq J'$. Also by Lemma 2.3 we obtain that we can substitute $\mathrm{Im}(1_A - \kappa')\sigma_{i_0}$ for $\mathrm{Im}\ \iota_{i_0}$ in $\bigoplus_I \mathrm{Im}\ \iota_i$.

Assume $\bigoplus_I \mathrm{Im}\ \iota_i \cap \bigoplus_{J'} \mathrm{Im}\ \kappa_j \neq 0$. Then there is a finite $J_0' \subseteq J'$ and a finite $I_0 \subseteq I$ such that

$$B = \bigoplus_{I_0} \mathrm{Im}\ \iota_i \cap \bigoplus_{J_0'} \mathrm{Im}\ \kappa_j$$

$$\neq 0.$$

Then for $\kappa' = 1_A - \sum_{J_0'} \kappa_j$, and for $\sigma_B: B \to A$ and $\sigma_B': B \to \bigoplus_{I_0} \mathrm{Im}\ \iota_i$ monomorphisms we have $(1_A - \kappa')\sigma_B = 0$. But $(1_A - \kappa')\sigma_i$ is monomorphic for $i \in I$, so $(1_A - \kappa')\sigma$ is monomorphic where $\sigma: \bigoplus_{I_0} \mathrm{Im}\ \iota_i \to A$ is the injection. Then

$$0 = (1_A - \kappa')\sigma_B$$

$$= (1_A - \kappa')\sigma\sigma_B'$$

which implies $\sigma_B' = 0$. Hence $\bigoplus_I \mathrm{Im}\ \iota_i \cap \bigoplus_{J'} \mathrm{Im}\ \kappa_j = 0$. $\bigoplus_J \mathrm{Im}\ \kappa_j \cap \bigoplus_{I'} \mathrm{Im}\ \iota_i = 0$ by symmetry. ▮

This proposition, together with Theorem 2.10, yields the Krull-Schmidt-Azumaya Theorem as it is found in [1].

That is, if $A = \bigoplus_{I} A_i$, endo(A_i) local $(i \in I)$ and $A = \bigoplus_{J} B_j$, B_j indecomposible $(j \in J)$, then these two decompositions of A are isomorphic.

Chapter 3

The Exchange Property

§1. Some Examples and Basic Notions

In this chapter, we will follow the results of R. B. Warfield ([10], § 3, p. 272-276), adding a slight generalization. Warfield was, in turn, applying certain proofs of P. Crawley and B. Jónsson [3] to the kind of categories that we are dealing with. We define a class of objects in a Grothendieck category which have a certain property, the exchange property; we show that this class is sufficiently large to be of interest (containing for example injective objects and objects with local endomorphism ring); and we prove some theorems concerning the uniqueness of certain direct sums with such objects as summands.

Definition 3.1: An object A in a Grothendieck category has the exchange property if, given any

$$B = A \oplus A'$$

$$= \bigoplus_{I} B_i$$

there exist $B_i' \leq B_i$ for $i \in I$ such that $B = A \oplus (\bigoplus_I B_i')$.

We say that A has the <u>finite exchange property</u> if this holds whenever the index set I is finite.

We note that the exchange property is preserved by isomorphism. Also, if A has the exchange property and

$$B = A \oplus A'$$

$$= \bigoplus_I B_i$$

then there exist $B_i'' \leq B_i$ as well as $B_i' \leq B_i$ $(i \in I)$ such that $B = A \oplus (\bigoplus_I B_i')$, $B_i = B_i' \oplus B_i''$ $(i \in I)$ and $A \cong \bigoplus_I B_i''$. This follows from Lemma 1.3.

<u>Lemma 3.2.</u> ([3], Lemma 5.1): If an indecomposable object A has the finite exchange property, then it has the exchange property.

<u>Proof</u>: Suppose $A \neq 0$ has the finite exchange property, and suppose

$$B = A \oplus A'$$

$$= \bigoplus_I B_i.$$

By Lemma 1.6, there is a finite $I_0 \subseteq I$ such that $A \cap \bigoplus_{I_0} B_i \neq 0$. Let $X = \bigoplus_{I-I_0} B_i$, so $B = \bigoplus_{I_0} B_i \oplus X$.

We can now apply the finite exchange property for A to obtain $B_i', B_i'' \leq B_i$ for $i \in I_0$ and $X', X'' \leq X$ such that $B = A \oplus (\bigoplus_{I_0} B_i') \oplus X'$, $B_i = B_i' \oplus B_i''$ for $i \in I_0$, $X = X' \oplus X''$, and $A \cong \bigoplus_{I_0} B_i'' \oplus X''$.

But A is indecomposable, so only one of these summands can be non-zero. If we assume $B_i'' = 0$ for all $i \in I_0$ we obtain a contradiction, for in this case $B_i' = B_i$ for $i \in I_0$ and hence $A \bigcap \bigoplus_{I_0} B_i = A \bigcap \bigoplus_{I_0} B_i' = 0$, contrary to our definition of I_0. Hence $X'' = 0$, $X' = X$ and

$$\begin{aligned} B &= A \oplus (\bigoplus_{I_0} B_i') \oplus X \\ &= A \oplus (\bigoplus_{I_0} B_i') \oplus (\bigoplus_{I-I_0} B_i). \end{aligned}$$

Thus A has the exchange property. ∎

This lemma enables us to show that any object with local endomorphism ring has the exchange property.

Proposition 3.3: ([9], Proposition 1): An indecomposable object has the exchange property if and only if it has local ring of endomorphisms.

Proof: (a) Assume endo(A) is local. Then A is indecomposable by Proposition 2.2. We need only show that A has the finite exchange property, and that it has the exchange property will follow by Lemma 3.2.

Suppose

$$B = A \oplus A'$$

$$= \bigoplus_I B_i$$

where I is finite. Let $\pi_i: B \to B_i$ $(i \in I)$ and $\pi_A: B \to A$ be the projections and $\sigma_i: B_i \to B$ $(i \in I)$ and $\sigma_A: A \to B$ be the injections.

Now $1_A = \sum_I \pi_A \sigma_i \pi_i \sigma_A$. But endo(A) is local, so there is an $i_0 \in I$ such that $\pi_A \sigma_{i_0} \pi_{i_0} \sigma_A$ is an automorphism of A. Then, by Lemma 1.5, $B_{i_0} = A \oplus \text{Ker } \pi_A \sigma_{i_0}$ with injection $\pi_{i_0} \sigma_A: A \to B_{i_0}$ and projection $(\pi_A \sigma_{i_0} \pi_{i_0} \sigma_A)^{-1} \pi_A \sigma_{i_0}: B_{i_0} \to A$. Therefore $B = A \oplus \text{Ker } \pi_A \sigma_{i_0} \oplus (\bigoplus_{I-\{i_0\}} B_i)$ with injection $\hat{\sigma}_A = \sigma_{i_0} \pi_{i_0} \sigma_A: A \to B$ and projection $\hat{\pi}_A = (\pi_A \sigma_{i_0} \pi_{i_0} \sigma_A)^{-1} \pi_A \sigma_{i_0} \pi_{i_0}: B \to A$.

But then $\hat{\pi}_A \sigma_A = 1_A$, so by Lemma 1.4, $B = A \oplus \text{Ker } \pi_A \sigma_{i_0} \oplus (\bigoplus_{I-\{i_0\}} B_i)$ with injection $\sigma_A: A \to B$. Trivially $\text{Ker } \pi_A \sigma_{i_0} \leq B$. Thus A has the finite exchange property, and by Lemma 3.2 A has the exchange property.

(b) Suppose A is indecomposable and endo(A) is not local. Then we will show that A does not have the exchange property.

As endo(A) is not local, there exist non-units $\alpha, \beta \in$ endo(A) such that $1_A = \alpha - \beta$. Let $B = A \oplus A$ with injections σ_1 and σ_2 and projections π_1 and π_2. We embed A in B in two ways: by $\delta = (1_A, 1_A) = \sigma_1 + \sigma_2$ and by $\eta = (\alpha, \beta) = \sigma_1\alpha + \sigma_2\beta$. It is easy to see that δ is monomorphic, and η is monomorphic since, if $(\sigma_1\alpha + \sigma_2\beta)\phi = 0$ then $\alpha\phi = 0 = \beta\phi$ so $\phi = (\alpha - \beta)\phi = 0$.

We will show $B = \text{Im}\,\eta \oplus \text{Im}\,\delta$ (i.e. $B = A \oplus A$ with injections η and δ). Let $\pi_\delta = \pi_1 - \alpha(\pi_1 - \pi_2)$. (We note that since $\pi_1 - \pi_2 = (\alpha - \beta)(\pi_1 - \pi_2)$, therefore

$$\pi_1 - \alpha(\pi_1 - \pi_2)$$

$$= \pi_2 - \beta(\pi_1 - \pi_2).)$$

Also let $\pi_\eta = \pi_1 - \pi_2$.

$$\text{Then } \delta\pi_\delta + \eta\pi_\eta = \left[(\sigma_1 + \sigma_2)(\pi_1 - \alpha(\pi_1 - \pi_2))\right] + \left[(\sigma_1\alpha + \sigma_2\beta)(\pi_1 - \pi_2)\right]$$

$$= \left[\sigma_1\pi_1 + \sigma_2\pi_1 - \sigma_1\alpha\pi_1 - \sigma_2\alpha\pi_1 + \sigma_1\alpha\pi_2 + \sigma_2\alpha\pi_2\right]$$

$$+ \left[\sigma_1\alpha\pi_1 + \sigma_2\beta\pi_1 - \sigma_1\alpha\pi_2 - \sigma_2\beta\pi_2\right]$$

$$= \sigma_1\pi_1 + \sigma_2\alpha\pi_2 - \sigma_2\beta\pi_2 - \sigma_2\alpha\pi_1 + \sigma_2\beta\pi_1$$

$$= \sigma_1\pi_1 + \sigma_2(\alpha - \beta)\pi_2 - \sigma_2(\alpha-\beta)\pi_1$$

$$= \sigma_1\pi_1 + \sigma_2\pi_2 - \alpha_2\pi_1$$

$$= 1_B.$$

Also

$$\pi_\delta\eta = (\pi_1 - \alpha(\pi_1 - \pi_2))(\sigma_1\alpha + \sigma_2\beta)$$

$$= \pi_1\sigma_1\alpha - \alpha\pi_1\sigma_1\alpha + \alpha\pi_2\sigma_1\alpha + \pi_1\sigma_2\beta - \alpha\pi_1\sigma_2\beta + \alpha\pi_2\sigma_2\beta$$

$$= \alpha - \alpha^2 + \alpha\beta$$

$$= \alpha(1_A - (\alpha - \beta))$$

$$= 0$$

and

$$\pi_\eta\delta = (\pi_1 - \pi_2)(\sigma_1 + \sigma_2)$$

$$= \pi_1\sigma_1 - \pi_2\sigma_1 + \pi_1\sigma_2 - \pi_2\sigma_2$$

$$= 0$$

Therefore, by Proposition 1.2, $B = \text{Im}\,\delta \oplus \text{Im}\,\eta$. ($B = A \oplus A$ with injections δ and η).

We will now assume that A has the exchange property and obtain a contradiction. If A has the exchange property, then there exist $A_1 \leq \text{Im}\,\pi_1$ and $A_2 \leq \text{Im}\,\pi_2$ such that $B = \text{Im}\,\eta \oplus A_1 \oplus A_2$. That is $B = A \oplus A_1 \oplus A_2$ with injections $\eta: A \to B$, $\sigma_{A_1}: A_1 \to B$ and $\sigma_{A_2}: A_2 \to B$ where $\sigma_{A_1} = \sigma_1 \sigma_{A_1 A}$ and $\sigma_{A_2} = \sigma_2 \sigma_{A_2 A}$. Hence we can find

$A_1', A_2'' \leq A$ where $A = A_1 \oplus A_1' = A_2 \oplus A_2'$ and $A \cong A_1' \oplus A_2'$. But A is indecomposable, so either $A_1' = 0$ or $A_2' = 0$. Thus $B = \text{Im}\,\eta \oplus A_1$ or $B = \text{Im}\,\eta \oplus A_2$. In the first case, by Lemma 1.4, $\pi_2 \eta$ is an isomorphism so β is an automorphism, a contradiction. Similarly we obtain a contradiction in the second case.

Thus A cannot have the exchange property and our result is proved. ∎

We can now show that $A = \bigoplus_I A_i$ has no summands with local endomorphism ring if and only if A_i has none for all $i \in I$. The necessity of this condition is obvious. To show the sufficiency we note that if A has a summand B, endo(B) local, then by the exchange property $B \cong \bigoplus_I A_i'$ for some $A_i' \leq A_i$ $(i \in I)$. But then $B \cong A_{i_0}'$ for some $i_0 \in I$ since B is indecomposable. Therefore $B = 0$.

Proposition 3.3 tells us that having the exchange property is a generalization of having local endomorphism ring. We will now show that another large class of objects, namely injectives, have the exchange property. From this and Proposition 3.3 we see that any indecomposable injective has a local ring of endomorphisms.

<u>Proposition 3.4.</u> ([10], Lemma 2): An injective object has the exchange property.

<u>Proof</u>: Let D be an injective and suppose

$$A = D \oplus X$$
$$= \bigoplus_I A_i.$$

We must construct $A_i' \leq A_i$ $(i \in I)$ such that $A = D \oplus (\bigoplus_I A_i')$.

Consider $B \leq A$ maximal with respect to $B = \bigoplus_I (B \cap A_i)$ and $B \cap D = 0$. We can find such a B, up to isomorphism, by Zorn's Lemma (applied to the partially ordered set of subobjects.)

Then we have $A/B = \bigoplus_I (A_i/(B \cap A_i))$. Let $\nu = \text{cok}\ \sigma_{BA}$. Since $B \cap D = 0$, it follows that $\nu\sigma_{DA}$ is a monomorphism: If $\nu\sigma_{DA}\phi = 0$ then $\sigma_{DA}\phi$ factors uniquely over $\text{Ker}\ \nu = B$. That is $\sigma_{DA}\phi = \sigma_{BA}\psi$ for some ψ.

But then $\phi = 0$ since $D \bigcap B = 0$.

Hence Im $\nu\sigma_{DA}$ is injective. Also $\nu\sigma_{DA}$ is an essential monomorphism. To show this, by Lemma 1.10 we need only show that Im $\nu\sigma_{DA} \bigcap (A_i/(B \bigcap A_i))$ is essential in $A_i/(B \bigcap A_i)$ for all $i \in I$. Say $\overline{A}_i = A_i/(B \bigcap A_i)$ for $i \in I$ and assume there is an $\overline{A}'_{i_0} \leq \overline{A}_{i_0}$ $(\overline{A}'_{i_0} \neq 0)$ for some $i_0 \in I$ such that $\overline{A}'_{i_0} \bigcap$ Im $\nu\sigma_{DA} = 0$. Let A'_{i_0} be such that

$$\begin{array}{ccccc} & & \overline{A}'_{i_0} & & \\ & \nearrow & & \searrow & \\ A'_{i_0} & & & & \overline{A}_{i_0} \\ & \searrow & & \nearrow & \\ & & A_{i_0} & & \end{array}$$

is a pullback where the morphism $A_{i_0} \to \overline{A}_{i_0}$ is coim $\nu\sigma_{i_0}$. Then $A'_{i_0} \neq 0$ since $\overline{A}'_{i_0} \neq 0$ and the canonical morphism $A_{i_0} \to \overline{A}_{i_0}$ is epimorphic. Also $A'_{i_0} \leq A_{i_0}$ and $B \bigcap A_{i_0} \leq A'_{i_0}$ where the associated monomorphism is not an isomorphism since $\nu\sigma_{A'_{i_0}} \neq 0$. Then $B' = \bigoplus_{I-\{i_0\}} (B \bigcap A_i) \oplus A'_{i_0}$ contradicts the maximality of B.

Thus $\nu\sigma_{DA}$ is an essential monomorphism and Im $\nu\sigma_{DA}$ is injective. Hence Im $\nu\sigma_{DA} = A/B$. That is $\nu\sigma_{DA}$ is an isomorphism, so $\nu\sigma_{DA}(\nu\sigma_{DA})^{-1}$ is an isomorphism and hence

$$A = \operatorname{Im} \sigma_{DA}(\nu\sigma_{DA})^{-1} \oplus \operatorname{Ker} \nu$$

$$= D \oplus B$$

$$= D \oplus (\bigoplus_I (B \cap A_i)).$$

Therefore D has the exchange property. ∎

We now prove some elementary properties of objects with the exchange property.

Lemma 3.5: Assume

$$A = A_0 \oplus A_1 \oplus X$$

$$= \bigoplus_I B_i \oplus X$$

where A_0 has the exchange property. Then there exist $B_i' \leq B_i$ for $i \in I$ such that $A = A_0 \oplus (\bigoplus_I B_i') \oplus X.$

Proof: Consider

$$A/X = \operatorname{Im} \nu\sigma_{A_0} \oplus \operatorname{Im} \nu\sigma_{A_1}$$

$$= \bigoplus_I \operatorname{Im} \nu\sigma_i$$

where $\sigma_i \colon B_i \to A$ is the injection. (That is

$$A/X = A_0 \oplus A_1$$

$$= \bigoplus_I B_i$$

with injections $\nu\sigma_{A_0}$, $\nu\sigma_{A_1}$ and $\nu\sigma_i$, for $i \in I$). For the sake of notation let us say $\overline{A}_0 = \text{Im}\ \nu\sigma_{A_0}$, $\overline{A}_1 = \text{Im}\ \nu\sigma_{A_1}$ and $\overline{B}_i = \text{Im}\ \nu\sigma_i$ for $i \in I$.

Now $\overline{A}_0$ has the exchange property, so there exist $\overline{B}_i' \leq \overline{B}_i$ $(i \in I)$ such that $A/X = \overline{A}_0 \oplus \bigoplus_I \overline{B}_i'$. Let B_i' be defined, for $i \in I$ such that

$$\begin{array}{ccccc} & & \overline{B}_i' & & \\ & \nearrow & & \searrow & \\ B_i' & & & & \overline{B}_i \\ & \searrow & & \nearrow & \\ & & B_i & & \end{array}$$

is a pullback where the morphism $B_i \to \overline{B}_i$ is the canonical morphism and the morphism $\overline{B}_i' \to \overline{B}_i$ the embedding. We will show that $A = A_0 \oplus (\bigoplus_I B_i') \oplus X$.

Let $B' = A_0 \oplus (\bigoplus_I B_i')$ with injection $\sigma_{B'}: B' \to A$. Then $A/X = \text{Im}\ \nu\sigma_{B'}$. Also $B' \cap X = 0$ so

$$\begin{aligned} A &= B' \oplus X \\ &= A_0 \oplus (\bigoplus_I B_i') \oplus X. \quad \blacksquare \end{aligned}$$

<u>Lemma 3.6</u>. ([3], Theorem 3.10): Let $A = A_0 \oplus A_1$. Then A has the exchange property if and only if A_0 and A_1 have the exchange property.

<u>Proof</u>: (a) Assume A_0 and A_1 have the exchange property and suppose

$$B = A \oplus X$$

$$= \bigoplus_I B_i .$$

By the exchange property for A_0 we can find $B_i' \leq B_i$ for $i \in I$ such that $B = A_0 \oplus (\bigoplus_I B_i')$. But we know $B = A_0 \oplus A_1 \oplus X$ so by the exchange property for A_1 and Lemma 3.5 there exist $B_i'' \leq B_i'$ such that

$$B = A_0 \oplus A_1 \oplus (\bigoplus_I B_i'')$$

$$= A \oplus (\bigoplus_I B_i'') .$$

Hence A has the exchange property.

(b) Assume A has the exchange property. Let $B = A_0 \oplus X = \bigoplus_I B_i$ and consider

$$C = A_1 \oplus B$$

$$= A \oplus X$$

$$= A_1 \oplus (\bigoplus_I B_i) .$$

We apply the exchange property for A to obtain $A_1' \leq A_1$, $B_i' \leq B_i$ $(i \in I)$ such that $C = A \oplus A_1' \oplus (\bigoplus_I B_i')$. But, noting that our monomorphism of A into C remains unchanged we see $A \cap A_1' = A_1'$ so necessarily $A_1' = 0$. Thus $C = A \oplus (\bigoplus_I B_i')$.

But also $C = A \oplus X$. If $\pi'_X: C \to X$ is the projection and $\sigma': \bigoplus_I B'_i \to C$ the injection, then by Lemma 1.4 $\pi'_X \sigma'$ is an isomorphism. But $\sigma' = \sigma_{BC}\sigma$ where $\sigma: \bigoplus_I B'_i \to B$ and $\sigma_{BC}: B \to C$ are injections; and $\pi'_X = \pi_X \pi_B$ where $\pi_B: C \to B$ and $\pi_X: B \to X$ are the projections. Therefore $\pi_X \pi_B \sigma_{BC} \sigma = \pi_X \sigma$ is an isomorphism and by Lemma 1.4 $B = A_0 \oplus (\bigoplus_I B'_i)$. Thus A_0 has the exchange property. ∎

<u>Corollary 3.7</u>. If endo(A) is a semi-perfect ring then A has the exchange property.

<u>Proof</u>: One characterization of a semi-perfect ring is that it contains a finite set of local orthogonal idempotents $\{\iota_i : i = 1, 2, \ldots, k\}$ such that $1 = \sum_{i=1}^{k} \iota_i$. But this means that $A = \bigoplus_{i=1}^{k} \operatorname{Im} \iota_i$ and $\operatorname{endo}(\operatorname{Im} \iota_i)$ is local for $i \in I$. Hence $\operatorname{Im} \iota_i$ has the exchange property and thus so does A, by Lemma 3.6. ∎

We have now shown that objects with the exchange property form a large class of objects in a Grothendieck category, including all objects with local endomorphism ring, all injectives and all finite direct sums of these.

§2. Uniqueness of Certain Direct Sum Decompositions

We will now prove a theorem, essentially due to Crawley and Jónsson [3] which illustrates the value of the exchange property with respect to uniqueness of direct sum decompositions. The theorem of Crawley and Jónsson is strengthened slightly here to allow summands in the decompositions other than those with the exchange property. Specifically, in addition to summands in our decomposition with the exchange property we allow a summand which has itself no summands with the exchange property. That we need only allow a single summand with this property, instead of many, in our statement of the following theorem stems from the following fact: $A = \bigoplus_I A_i$ has no summands with the exchange property if and only if A_i has none for all $i \in I$. (If A had a summand with the exchange property, say A', then A' is isomorphic to a direct sum of summands of the A_i, $i \in I$. But then each of these summands has the exchange property so some A_i has a summand with the exchange property. The converse is trivial.)

<u>Theorem 3.8</u>. ([3], Theorem 4.2): Let

$$A = \bigoplus_I A_i \oplus X$$
$$= \bigoplus_J B_j \oplus Y,$$

where I and J are countable, A_i and B_j have the exchange property for $i \in I$ and $j \in J$ and X and Y have no direct summands with the exchange property. Then $\bigoplus_I A_i \cong \bigoplus_J B_j$ and these direct sums have isomorphic refinements.

<u>Proof</u>: Since I and J are countable we can consider them to be the set of natural numbers. That is

$$A = \bigoplus_{i=0}^{\infty} A_i \oplus X$$
$$= \bigoplus_{j=0}^{\infty} B_j \oplus Y.$$

A_0 has the exchange property, so we can find $B_{0,j}$, $B'_{0,j} \leq B_j$, $(j = 0, 1, 2, \ldots)$, and Y', $Y'' \leq Y$ where $B_j = B_{0,j} \oplus B'_{0,j}$ $(j = 0, 1, \ldots)$, $Y = Y' \oplus Y''$,

$A = A_0 \oplus (\bigoplus_{i=0}^{\infty} B'_{0,j}) \oplus Y'$ and $A_0 \cong \bigoplus_{i=0}^{\infty} B_{0,j} \oplus Y''$.

But then Y'' is isomorphic to a summand of A_0, so has the exchange property. Hence $Y'' = 0$, $Y' = Y$ and $A = A_0 \oplus (\bigoplus_{i=0}^{\infty} B'_{0,j}) \oplus Y$ and $A_0 \cong \bigoplus_{i=0}^{\infty} B_{0,j}$.

Now $B'_{0,0}$, being a summand of B_0, has the exchange property so we can find, by Lemma 3.5,

$A_{i,0}, A'_{i,0} \leq A_i$, $(i = 1, 2, \ldots)$, such that $A_{i,0} \oplus A'_{i,0} = A_i$,
$A = A_0 \oplus B'_{0,0} \oplus (\bigoplus_{i=1}^{\infty} A'_{i,0}) \oplus X$ and $B'_{0,0} \cong \bigoplus_{i=1}^{\infty} A_{i,0}$.

(X will remain unchanged under application of the exchange property for $B'_{0,0}$ by the same argument we used for Y.)

Again, $A'_{1,0}$ has the exchange property so, considering the decomposition $A = A_0 \oplus (\bigoplus_{j=0}^{\infty} B'_{0,j}) \oplus Y$, we can find $B_{1,j}, B'_{1,j} \leq B'_{0,j}$ $(j = 1, 2, \ldots)$ with $B_{1,j} \oplus B'_{1,j} = B'_{0,j}$ (so $B_j = B_{0,j} \oplus B_{1,j} \oplus B'_{1,j}$) such that $A = A_0 \oplus B'_{0,0} \oplus A'_{1,0} \oplus (\bigoplus_{j=1}^{\infty} B'_{1,j}) \oplus Y$ and $A'_{1,0} \cong \bigoplus_{j=1}^{\infty} B_{1,j}$.

We continue inductively to obtain $A_{i,j}, A'_{i,j} \leq A_i$ for $0 \leq j < i$ and $B_{i,j}, B'_{i,j} \leq B_j$ for $0 \leq i \leq j$ satisfying:

(1) $A_i = \bigoplus_{j=0}^{i-1} A_{i,j} \oplus A'_{i,i-1}$

(2) $B_j = \bigoplus_{i=0}^{j} B_{i,j} \oplus B'_{j,j}$

(3) $A'_{i,i-1} \cong \bigoplus_{j=i}^{\infty} B_{i,j}$

and (4) $B'_{j,j} \cong \bigoplus_{i=j+1}^{\infty} A_{i,j}$.

We know $A_0 \cong \bigoplus_{j=0}^{\infty} B_{0,j}$ so we can find summands $A_{0,j} \leq A_0$ with $A_{0,j} \cong B_{0,j}$ $(j = 0, 1, \ldots)$ such that $A_0 = \bigoplus_{j=0}^{\infty} A_{o,j}$. Similarly, from (3) we can find summands $A_{i,j}$ of $A'_{i,i-1}$ for a given i and for $j \geq 0$ such that $A_{i,j} \cong B_{i,j}$ and $A'_{i,i-1} = \bigoplus_{j=i}^{\infty} A_{i,j}$. If we substitute this in (1) we obtain $A_i = \bigoplus_{j=0}^{\infty} A_{i,j}$. Similarly we find $B_{i,j} \cong A_{i,j}$ for $0 \leq j < i$ with $B_j = \bigoplus_{i=0}^{\infty} B_{i,j}$. Hence we have found $B_{i,j}$ and $A_{i,j}$ for $i, j = 0, 1, 2, \ldots$ such that $A_{i,j} \cong B_{i,j}$, $A_i = \bigoplus_{j=0}^{\infty} A_{i,j}$ and $B_j = \bigoplus_{j=0}^{\infty} B_{i,j}$. Thus $\bigoplus_{i=0}^{\infty} A_i \cong \bigoplus_{j=0}^{\infty} B_j$ and we have constructed the required isomorphic refinements. ∎

<u>Proposition 3.9</u>: Let

$$A = \bigoplus_{I} A_i \oplus X$$
$$= \bigoplus_{J} B_j \oplus Y$$

where I is finite, A_i, B_j have the exchange property for $i \in I$, $j \in J$, and X and Y have no direct summands with the exchange property. Then $X \cong Y$.

Proof: Since I is finite, $\bigoplus_I A_i$ has the exchange property. Hence there exist $B_j' \leq B_j$ for $j \in J$ such that $A = \bigoplus_I A_i \oplus (\bigoplus_J B_j') \oplus Y$. But then $X \cong \bigoplus_J B_j' \oplus Y$ so $\bigoplus_J B_j' = 0$ and $X \cong Y$. ∎

A definite weakness of Theorem 3.8 is the countability restriction on the index set of the summands. This restriction is difficult, if not impossible, to eliminate. However, it is possible to replace it by a countability restriction on each of the summands. The obvious choice is to require the summands to be countably generated, and to allow arbitrarily large indexing sets. This will work, but we can do somewhat better.

Definition 3.10: An object A is small if, for any object $B = \bigoplus_I B_i$ with projections $\pi_i \colon B \to B_i$, $(i \in I)$, and for any morphism $\phi \colon A \to B$, then $\pi_i \phi = 0$ for all but finitely many $i \in I$. A is countably small if $\pi_i \phi = 0$ for all but countably many $i \in I$.

Definition 3.11. A is σ-small if there exist $S_i \leq A$ $(i = 0, 1, 2 \ldots)$, $S_0 \leq S_1 \leq \ldots$, S_i small for all $i = 0, 1, 2 \ldots$ and $A = \bigcup_{i=0}^{\infty} S_i$.

Proposition 3.12: If A is σ-small then A is countably small.

Proof: If A is σ-small then there exist small $S_i \leq A$ $(i = 0, 1, \ldots)$ where $S_i \leq S_{i+1}$ and $A = \bigcup_{i=0}^{\infty} S_i$. Assume $B = \bigoplus_J B_j$ and $\phi: A \to C$, and let $\pi_j: B \to B_j$ $(j \in J)$ be the projections. Then $\mathrm{Im}(\pi_j \phi) = \bigcup_{i=0}^{\infty} \mathrm{Im}(\pi_j \phi \sigma_i)$ where $\sigma_i: S_i \to A$ is the injection. Now $\pi_j \phi \sigma_i \neq 0$ for only finitely many $j \in J$ so $\pi_j \phi \neq 0$ for only countably many $j \in J$. Thus A is countably small. ∎

Obviously if A is finitely generated then A is small and if A is countably generated than A is σ-small. Tom Head ([5], p. 235-237) gives an example of an R-module which is small but is not finitely (or even countably) generated. This is then also an example of an R-module which is σ-small but not countably generated. Thus our concepts of small and σ-small (countably small) are proper generalizations of the concepts of finitely generated and countably generated.

Lemma 3.13. ([6], Theorem 1): Let $A = \bigoplus_I A_i$ where A_i is countably small $(i \in I)$ and suppose $A = B \oplus X$. Then B is a direct sum of countably small objects.

Proof: We will construct an ascending chain $C_j \leq A$ ($j < M$ for some ordinal M) such that $C_j \leq C_{j+1}$, $C_0 = 0$, $A = \bigcup_{j<M} C_j$ and such that the following properties are satisfied:

(1) If $k < M$ is a limit ordinal then $C_k = \bigcup_{j<k} C_j$.

(2) C_{j+1}/C_j is countably small for $0 \leq j < M$.

(3) $C_j = \bigoplus_{I_j} A_i$ where $I_j \subseteq I$ for $0 \leq j \leq M$.

and (4) $C_j = B_j \oplus X_j$ where $B_j = C_j \cap B$ and $X_j = C_j \cap X$ for $0 \leq j < M$.

If we have constructed such C_j ($j < M$) then we can show inductively that B is a direct sum of countably small subobjects. $B_0 = 0$ is countably small; assume B_j is a direct sum of countably small objects for $j < k$. If k is a limit ordinal then the result is clear by property (1). Hence assume k is not a limit ordinal. We know B_j is a summand of B_{j+1} since it is a summand of A. Similarly X_j is a summand of X_{j+1}. Therefore there exist $B_j' \leq B_{j+1}$ and $X_j' \leq X_{j+1}$ such that

$$C_{j+1} = B_j \oplus B_j' \oplus X_j \oplus X_j'$$

$$= C_j \oplus B_j' \oplus X_j'.$$

Hence $B'_j \oplus X'_j \cong \bigoplus_{I_{j+1}-I_j} A_i \cong C_{j+1}/C_j$. Thus B'_j is countably small and $B_{j+1} = B_j \oplus B'_j$ is a direct sum of countably small objects for all $j < k$. In particular B_k is a direct sum of countably small objects. Therefore, by induction, B_j is a direct sum of countably small objects for all $j < M$. Hence $B = \bigcup_{j<M} B_j$ is a direct sum of countably small objects.

We must therefore construct C_j satisfying (1) - (4) and the required result will follow. Now $C_0 = 0$ and we apply the following recursive proceedure, assuming we have already constructed C_j for $j < k$ $(k < M)$:

If k is a limit ordinal, let $C_k = \bigcup_{j<k} C_j$.

If k is not a limit ordinal, choose $i_0 \notin I_{k-1}$. Let $\pi_1: A \to A_i$ $(i \in I)$, $\pi_B: A \to B$ and $\pi_X: A \to X$ be the projections and $\sigma_B: B \to A$ and $\sigma_X: X \to A$ the injections. Since A_{i_0} is countably small, $\pi_i \sigma_B \pi_B \sigma_{i_0} = 0$ and $\pi_i \sigma_X \pi_X \sigma_{i_0} = 0$ for all but countably many $i \in I$. Let $I_{B,1} = \{i \in I: \pi_i \sigma_B \pi_B \sigma_{i_0} \neq 0\}$ and $I_{X,1} = \{i \in I: \pi_i \sigma_X \pi_X \sigma_{i_0} \neq 0\}$ and let ${}_kI_1 = I_{B,1} \bigcup I_{X,1}$. Recursively we let $I_{B,n} = \{i \in I: \pi_i \sigma_B \pi_B \sigma_h \neq 0,\ h \in {}_kI_{n-1}\}$ and $I_{X,n} = \{i \in I: \pi_i \sigma_X \pi_X \sigma_h \neq 0,\ h \in {}_kI_{n-1}\}$ for each natural number n, and let ${}_kI_n = I_{B,n} \bigcup I_{X,n}$. Obviously ${}_kI_n$ is is countable for all n. Let $I_k' = \bigcup_{n=0}^{\infty} {}_kI_n$ and $I_k = I_k' \bigcup I_{k-1}$. It is easy to see that $\bigoplus_{I_k'} A_i$ is countably small, since I_k' is countable.

We now let $C_k = \bigoplus_{I_k} A_i$ and this C_k satisfies properties (1)-(4). (1)-(3) are trivial and (4) holds because $C_k = (C_k \bigcap B) \oplus (C_k \bigcap X)$ since $\{i \in I: \pi_i \sigma_{B_k} \neq 0\} \subseteq I_k$ and $\{i \in I: \pi_i \sigma_{X_k} \neq 0\} \subseteq I_k$. Thus we have the required result. ∎

<u>Lemma 3.14</u>. ([10], Lemma 5): Let

$$A = \bigoplus_{I} A_i$$
$$= \bigoplus_{J} B_j$$

where A_i and B_j are countably small for $i \in I$ and $j \in J$. Then there exist partitions of I and of J into countable disjoint subsets I_m and J_m for $m < M$ (M an ordinal) such that $\bigoplus_{I_m} A_i \cong \bigoplus_{J_m} B_j$ for all $m < M$.

<u>Proof</u>: The result is obtained through transfinite induction. We first show that there exist countable

$I_0 \subseteq I$, $J_0 \subseteq J$ such that $\bigoplus_{I_0} A_i = \bigoplus_{J_0} B_j$.

We take an arbitrary $i_0 \in I$, and for the sake of notation we will say $A_0 = A_{i_0}$. Let $\pi_i: A \to A_i$ $(i \in I)$ and $\rho_j: A \to B_j$ $(j \in J)$ be the projections and $\sigma_i: A_i \to A$ $(i \in I)$ and $\sigma_j': B_j \to A$ $(j \in J)$ the injections. Since A_0 is countably small, $\rho_j \sigma_0 = 0$ for all but countably many $j \in J$. That is $J_{0,0} = \{j \in J: \rho_j \sigma_0 \neq 0\}$ is countable. Now $\bigoplus_{J_{0,0}} B_j$ is countably small since $J_{0,0}$ is countable, and so (if $\sigma_{0,0}': \bigoplus_{J_{0,0}} B_j \to A$ is the injection),

$I_{0,1} = \{i \in I: \pi_i \sigma_{0,0}' \neq 0\}$ is countable. Continuing inductively, if we have $J_{0,n}$ countable with $\sigma_{0,n}': \bigoplus_{J_{0,n}} B_j \to A$ the injection, then we let $I_{0,n+1} = \{i \in I: \pi_i \sigma_{0,n}' \neq 0\}$; and if we have $I_{0,n}$ with injection $\sigma_{0,n}: \bigoplus_{I_{0,n}} A_i \to A$, we let $J_{0,n} = \{j \in J: \rho_j \sigma_{0,n} \neq 0\}$. Obviously $I_{0,n+1}$ and $J_{0,n}$ are countable for $n = 0,1,\ldots$. Now let $I_0 = \bigcup_{n=1}^{\infty} I_{0,n}$ and $J_0 = \bigcup_{n=0}^{\infty} J_{0,n}$. I_0 and J_0 are countable, and $\bigoplus_{I_0} A_i = \bigoplus_{J_0} B_j$ since $\{j \in J: \rho_j \sigma_i \neq 0,\ i \in I_0\} = J_0$ and $\{i \in I: \pi_i \sigma_j' \neq 0, j \in J_0\} = I_0$.

We now make the following induction hypothesis:

Assume, for an ordinal $k < M$ that for all $h < k$ and for all $m < h$ there exist countable $I_m \subseteq I$ and $J_m \subseteq J$ where the $I_m (m < h)$ and $J_m (m < h)$ are disjoint, such that $\bigoplus_{m<h} (\bigoplus_{I_m} A_i) = \bigoplus_{m<h} (\bigoplus_{J_m} B_j)$.

We want to show that there exist such I_m and J_m for all $m < k$. If k is a limit ordinal then the result is trivial. Thus assume k is not a limit ordinal. Then $k = h + 1$ for some h, and we are to construct I_h and J_h countable, such that $\bigoplus_{m<k} (\bigoplus_{I_m} A_i) = \bigoplus_{m<k} (\bigoplus_{J_m} B_j)$. Let $I_h' = I - \bigcup_{m<h} I_m$ and $J_h' = J - \bigcup_{m<h} J_m$ and choose $i_h \in I_h'$. By the same construction we used for i_0 we obtain countable $I_h'' \subseteq I$ and $J_h'' \subseteq J$ such that $\bigoplus_{I_h''} A_i = \bigoplus_{J_h''} B_j$. Let $I_h = I_h' \cap I_h''$ and $J_h = J_h' \cap J_h''$. Then $\bigoplus_{m<k} (\bigoplus_{I_m} A_i) = \bigoplus_{m<k} (\bigoplus_{J_m} B_j)$ since $\{j \in J: \rho_j \sigma_i \neq 0,\ i \in \bigcup_{m<k} I_m\} = \bigcup_{m<k} J_m$ and $\{i \in I: \pi_i \sigma_j' \neq 0,\ j \in \bigcup_{m<k} J_m\} = \bigcup_{m<k} I_m$.

If the ordinal M is chosen sufficiently large, the above induction will produce a partition of I into countable disjoint subsets I_m $(m<M)$, and hence a partition of J into countable disjoint subsets J_m $(m < M)$.

Now for a given $k < M$, the partitions have been constructed so that $\bigoplus_{m<k+1} (\bigoplus_{I_m} A_i) = \bigoplus_{m<k+1} (\bigoplus_{J_m} B_j)$ and $\bigoplus_{m<k} (\bigoplus_{I_m} A_i) = \bigoplus_{m<k} (\bigoplus_{J_m} B_j)$. Hence $\bigoplus_{I_k} A_i \cong \bigoplus_{J_k} B_j$ for all $k < M$. ∎

The preceeding lemma allows us to apply Theorem 3.8 to the case where the summands of our decomposition are countably small and the index set is arbitrarily large, for we can consider countable isomorphic subsums.

<u>Theorem 3.15</u>. ([10], Theorem 6): Suppose $A = \bigoplus_I A_i$ where A_i is countably small, and also that

$$A = \bigoplus_J B_j \oplus X$$
$$= \bigoplus_K C_k \oplus Y$$

where B_j, C_k have the exchange property $(j \in J,\ k \in K)$ and X and Y have no direct summands with the exchange property. Then $\bigoplus_J B_j \cong \bigoplus_K C_k$ and these sums have isomorphic refinements.

<u>Proof</u>: By Lemma 3.13 any direct sum decomposition of A refines to one with countably small summands, and by Lemma 3.6 any direct summand of an object with the

exchange property has the exchange property. Thus

$$A = \bigoplus_{J'} B_j' \oplus (\bigoplus_{J''} X_j)$$

$$= \bigoplus_{K'} C_k' \oplus (\bigoplus_{K''} Y_k)$$

where $\bigoplus_{J'} B_j'$ is a refinement of $\bigoplus_{J} B_j$, $\bigoplus_{K'} C_k'$ is a refinement of $\bigoplus_{K} C_k'$, $X = \bigoplus_{J''} X_j$ and $Y = \bigoplus_{K''} Y_k$, and such that each of the summands B_j', C_k', X_j and Y_k is countably small, and B_j' and C_k' have the exchange property for $j \in J'$ and $k \in K'$.

By Lemma 3.14 there exist partitions of J', K', J'' and K'' into countable disjoint subsets J_m', K_m', J_m'' and K_m'' respectively for $m < M$ (for some ordinal M), such that $\bigoplus_{J_m'} B_j' \oplus (\bigoplus_{J_m''} X_j) \cong \bigoplus_{K_m'} C_k' \oplus (\bigoplus_{K_m''} Y_k)$. We can now apply Theorem 3.8 for each index $m < M$ to obtain $\bigoplus_{J_m'} B_j' \cong \bigoplus_{K_m'} C_k'$ and these sums have isomorphic refinements. Thus $\bigoplus_{J} B_j \cong \bigoplus_{K} C_k$ and there exist isomorphic refinements. ∎

<u>Lemma 3.16</u>: If $A = \bigoplus_{I} A_i$, I countable and A_i σ-small ($i \in I$), and if $A = B \oplus X$ Then B is σ-small.

Proof: Say $I = \{0, 1, 2, \ldots\}$. A_i is σ-small for $i = 0, 1, 2 \ldots$ so there exist small $S_{i,j} \leq A_i$, $j = 0, 1, 2 \ldots$, with $S_{i,j} \leq S_{i,j+1}$ and $A_i = \bigcup_{j=0}^{\infty} S_{i,j}$ for each $i \in I$. Let $S_k = \bigoplus_{j=0}^{k} \bigoplus_{i=0}^{k} S_{i,j}$, $(k = 0, 1, 2 \ldots)$. Then S_k is small, and if $\pi_B: A \to B$ is the projection and $\sigma_k': C_k \to A$ are the injections for $k = 0, 1, 2, \ldots$, then $\operatorname{Im} \pi_B \sigma_k'$ is small and $\operatorname{Im} \pi_B \sigma_k' \leq \operatorname{Im} \pi_B \sigma_{k+1}'$. Also $B = \bigcup_{k=0}^{\infty} \operatorname{Im} \pi_B \sigma_k'$. Hence B is σ-small. ∎

Theorem 3.17. (Crawley, Jónsson, [10], Theorem 7): Let $A = \bigoplus_I A_i$ where A_i is σ-small and has the exchange property for $i \in I$. Then any two direct sum decompositions of A have isomorphic refinements.

Proof: A is σ-small so A is countably small and by Theorem 3.15 it suffices to prove that any direct summand of A is a direct sum of objects with the exchange property, (for then any decomposition can be refined to one whose summands have the exchange property.)

Assume then that $A = B \oplus X$ and then we are to find $C_k (k \in K)$ with the exchange property such that $B = \bigoplus_K C_k$. By Lemma 3.13 there exist B_j $(j \in J)$ and $X_j (j \in J')$ countably small such that $B = \bigoplus_J B_j$ and $X = \bigoplus_{J'} X_j$. Thus we have $A = \bigoplus_I A_i$

$$= \bigoplus_J B_j \oplus \left(\bigoplus_{J'} X_j\right)$$

where each of the summands is countably small. Applying Lemma 3.14 we can partition I, J and J' into countable disjoint subsets I_m, J_m and J'_m respectively for $m < M$ (for some ordinal M), such that

$\bigoplus_{I_m} A_i \cong \bigoplus_{J_m} B_j \oplus (\bigoplus_{J'_m} X_j)$ for all $m < M$. Thus

we need only show that if B' is a summand of $\bigoplus_{I'} A_i = A'$ where I' is countable, then $B' = \bigoplus_K C_k$ where C_k $(k \in K)$ has the exchange property.

But in this case A' is σ-small, so by Lemma 3.16 B' is σ-small. Hence there exist $S_i \leq B'$ $(i = 0, 1, 2, \ldots)$, S_i small, $S_i \leq S_{i+1}$ and $B' = \bigcup_{i=0}^{\infty} S_i$.

Assume, without loss of generality, that $S_0 = 0$. We will recursively construct C_k, direct summands of A' with the exchange property such that $S_n \leq \bigoplus_{k=0}^{n} C_k \leq B'$ for each natural number n. If we have such C_k $(k = 0, 1\ldots)$ then $B' = \bigoplus_{k=0}^{\infty} C_k$ and we have the required result.

Let $C_0 = 0$ and assume there exist C_k with the exchange property for $k \leq n$ (n a natural number), such that $\bigoplus_{k=0}^{n} C_k$ is a summand of A' and such that

$S_n \leq \bigoplus_{k=0}^{n} C_k \leq B'$. Now $\bigoplus_{k=0}^{n} C_k$ has the exchange property since each C_k does, so there exist $A_i' \leq A_i$ $(i \in I')$ such that $A' = \bigoplus_{k=0}^{n} C_k \oplus (\bigoplus_{I'} A_i')$. Since S_{n+1} is small, there is a finite $I'' \subseteq I'$ such that $S_{n+1} \leq \bigoplus_{k=0}^{n} C_k \oplus (\bigoplus_{I''} A_i')$. Let $C = \bigoplus_{k=1}^{n} C_k \oplus (\bigoplus_{I''} A_i')$ and suppose $A' = B' \oplus X'$. Then C has the exchange property so we can find $B_n', B_n'' \leq B'$, $X_n', X_n'' \leq X'$ with $B_n' \oplus B_n'' = B'$, $X_n' \oplus X_n'' = X'$ and $A' = C \oplus B_n' \oplus X_n'$. Let $B_n = B' \cap (C \oplus X_n')$.

We see that $S_{n+1} \leq B'$ and $S_{n+1} \leq S$ so $S_{n+1} \leq B_n$. Also $\bigoplus_{k=1}^{n} C_k \leq B_n$ so $B_n = \bigoplus_{k=1}^{n} C_k \oplus C_{n+1}$ for some $C_{n+1} \leq B_n$. Now $B' = B_n \oplus B_n'$ so

$$\begin{aligned} A' &= B' \oplus X' \\ &= B_n \oplus B_n' \oplus X' \\ &= C_{n+1} \oplus (\bigoplus_{k=0}^{n} C_k) \oplus B_n' \\ &\quad \oplus X_n' \oplus X_n''. \end{aligned}$$

But also
$$\begin{aligned} A' &= C \oplus B_n' \oplus X_n' \\ &= \bigoplus_{k=0}^{n} C_k \oplus (\bigoplus_{I''} A_i') \oplus B_n' \oplus X_n'. \end{aligned}$$

Therefore $C_{n+1} \oplus X_n'' \cong \bigoplus_{I''} A_i$. Hence C_{n+1} has the exchange property. That is, we have constructed C_{n+1} with the required properties.

By induction, therefore, B' is the direct sum of objects with the exchange property. Thus any direct sum decomposition of A refines to one with σ-small summands which have the exchange property. The result then follows from Theorem 3.15. ∎

<u>Corollary 3.18</u>: If $A = \bigoplus_I A_i$ where $\mathrm{endo}(A_i)$ is local and A_i is countably generated, then any other direct sum decomposition of A refines to one isomorphic to the given decomposition.

<u>Proof</u>: This is a direct consequence of Theorem 3.17. ∎

Chapter 4

Decompositions of Injectives

§1. A Uniqueness Theorem

In this chapter we will examine direct sum decompositions of injectives in a Grothendieck category. As we have already seen, injectives have the exchange property and indecomposable injectives have local endomorphism ring. Hence, noting that any direct summand of an injective is injective, we see that the results of Chapters 2 and 3 lend themselves naturally to the study of decompositions of injectives. By Theorem 2.10, any two direct sum decompositions of an injective into indecomposables are isomorphic, and by Theorem 3.8, any two countable direct sum decompositions of an injective have isomorphic refinements. Also, by Theorem 3.17, if an injective has a decomposition into countably generated summands, then any two direct sum decompositions have isomorphic refinements. However, when we are dealing with injectives these countability hypotheses can be removed. The following results are due to R. B. Warfield [10].

<u>Lemma 4.1</u>: Let D be injective, $D = A \oplus C = B \oplus C'$ with $A \cap B$ essential in A and in B. Then $D = B \oplus C$.

<u>Proof</u>: As usual we let π_A: $D \to A$ be the projection and σ_A: $A \to D$ and σ_B: $B \to D$ the injections. Let $X = A \cap B$ with injections σ_{XA}: $X \to A$, σ_{XB}: $X \to B$ and σ_X: $X \to D$ where $\sigma_B \sigma_{XB} = \sigma_X = \sigma_A \sigma_{XA}$. Then

$$\pi_A \sigma_B \sigma_{XB} = \pi_A \sigma_A \sigma_{XA}$$

$$= \sigma_{XA}.$$

Now σ_{XB} is essential and $\pi_A \sigma_B \sigma_{XB} = \sigma_{XA}$, a monomorphism, so $\pi_A \sigma_B$ is monomorphic.

$A \cong B$ since both are injective hulls of X; say ϕ: $B \to A$ is an isomorphism. Then, since $\pi_A \sigma_B$ is monomorphic and A is injective, ϕ can be extended to ϕ': $A \to A$ so that $\phi' \pi_A \sigma_B = \phi$. But then

$$\phi \sigma_{XB} = \phi' \pi_A \sigma_B \sigma_{XB}$$

$$= \phi' \pi_A \sigma_A \sigma_{XB}$$

$$= \phi' \sigma_{XA}.$$

Now σ_{XA} is essential and $\phi' \sigma_{XA} = \phi \sigma_{XB}$ is monomorphic so ϕ' is monomorphic. Also ϕ' is epimorphic since ϕ

is. Hence ϕ' is an isomorphism. Therefore $\pi_A\sigma_B = (\phi')^{-1}\phi$ is an isomorphism so by Lemma 1.4, $D = B \oplus C$. ∎

<u>Theorem 4.2</u>. ([10], Theorem 1): Let D be an injective object. Then any two direct sum decompositions of D have isomorphic refinements.

<u>Proof</u>: Consider

$$D = \bigoplus_{i<N} A_i$$
$$= \bigoplus_{j<M} B_j$$

where M and N are ordinal numbers. (We well order the summands in the direct sums to enable us to use transfinite induction). We will construct $C_{ij} \leq D$ for $i < N$, $j < M$ such that $D = \bigoplus_{i<N} \bigoplus_{j<M} C_{ij}$, $A_i \cong \bigoplus_{j<M} C_{ij}$ and $B_j \cong \bigoplus_{i<N} C_{ij}$. These will be constructed recursively with respect to the following conditions on $n \leq N$ and $m \leq N$: For all $i < n$ there exist $A_{im} \leq A_i$ and for $i < n$, $j < m$ there exist C_{ij} such that:

(1) $\bigoplus_{i<n} A_i = (\bigoplus_{i<n} A_{im}) \oplus (\bigoplus_{i<n} \bigoplus_{j<m} C_{ij})$ and

(2) $\bigoplus_{j<m} B_j \bigcap \bigoplus_{i<n} \bigoplus_{j<m} C_{ij}$ is essential in both

$(\bigoplus_{j<m} B_j) \bigcap (\bigoplus_{i<n} A_i)$ and in $\bigoplus_{i<n} \bigoplus_{j<m} C_{ij}$.

If we have these conditions satisfied for all $n \leq N$, $m \leq M$ then by (1) $D = \bigoplus_{i<N} A_{iM} \oplus (\bigoplus_{i<N} \bigoplus_{j<M} C_{ij})$ and by (2) $\bigoplus_{i<N} \bigoplus_{j<M} C_{ij}$ is essential in D, so $A_{iM} = 0$ for all $i < N$ and $D = \bigoplus_{i<N} \bigoplus_{j<M} C_{ij}$.

Further, by (1) for any $n < N$,

$$\bigoplus_{i<n} A_i = \bigoplus_{i<n} A_{iM} \oplus (\bigoplus_{i<n} \bigoplus_{j<M} C_j)$$

$$= \bigoplus_{i<n} \bigoplus_{j<M} C_{ij}$$

and $\bigoplus_{i<n+1} A_i = \bigoplus_{i<n+1} \bigoplus_{j<M} C_{ij}$ so $A_n \cong \bigoplus_{j<M} C_{nj}$ for all $n < N$. Also, for all $m < M$, $(\bigoplus_{j<m} B_j) \bigcap (\bigoplus_{i<N} \bigoplus_{j<m} C_{ij})$ is essential in both $(\bigoplus_{j<m} B_j) \bigcap (\bigoplus_{i<N} A_i) = \bigoplus_{j<m} B_j$ and in $\bigoplus_{i<N} \bigoplus_{j<m} C_{ij}$. Hence, by Lemma 4.1, $D = \bigoplus_{i<N} \bigoplus_{j<m} C_{ij} \oplus (\bigoplus_{j \geq m} B_j)$. Similarly $D = \bigoplus_{i<N} \bigoplus_{j<m+1} C_{ij} \oplus (\bigoplus_{j \geq m+1} B_j)$. Therefore $B_m \cong \bigoplus_{i<N} C_{im}$ for all $m < M$. Thus if C_{ij} $(i < N, j < M)$ and A_{ij} $(i < N, j \leq M)$ are constructed such that (1) and (2) are satisfied for all $n \leq N$ and $m \leq M$ we will have the required isomorphic refinements.

We proceed inductively. The conditions hold trivially for $n = 0$.

(a) For $n = 1$, $m = 1$ we take $C_{0,0}$ to be the injective hull of $A_0 \cap B_0$ (which exists by Lemma 1.9). Then $C_{0,0}$ is a direct summand of A_0 so we define $A_{0,1} \leq A_0$ such that $A_0 = A_{0,1} \oplus C_{0,0}$.

Assume we have $C_{i,0}$, $A_{i,1}$ for $i < n$ where $n < h < N$ for some h. If h is a limit ordinal then clearly $\bigoplus_{i<h} A_i = \bigoplus_{i<h} A_{i,1} \oplus (\bigoplus_{i<h} C_{i,0})$. Since ascending unions preserve essentiality by the Grothendieck property, (2) holds as well. If h is not a limit ordinal, then $h = k + 1$. Choose $C_{k,0} \leq \bigoplus_{i<h} A_i$ maximal with respect to:

(i) $C_{k,0} \cap (\bigoplus_{i<k} A_i) = 0$ and

(ii) $\bigoplus_{i\leq k} C_{i,0} \cap B_0$ is essential in $\bigoplus_{i\leq k} C_{i,0}$.

$C_{k,0}$ exists (up to isomorphism), by Zorn's Lemma.

$\bigoplus_{i\leq k} C_{i,0} \cap B_0$ is essential in $\bigoplus_{i\leq k} A_i \cap B_0$ since, for any $X \leq \bigoplus_{i\leq k} A_i \cap B_0$ with $X \cap \bigoplus_{i\leq k} C_{i,0} \cap B_0 = 0$, then $X \cap (\bigoplus_{i\leq k} C_{i,0}) = 0$ by (ii). This means that $C'_{k,0} = X \oplus C_{k,0}$ has properties (i) and (ii), contradicting the maximality of $C_{k,0}$, so $X = 0$. ($C'_{k,0}$ has property (i) by the essentiality of $\bigoplus_{i<k} C_{i,0} \cap B_0$ in $\bigoplus_{i<k} A_i \cap B$ and property

(ii) since $(\bigoplus_{i\leq k} C_{i,0} \oplus X) \cap B_0$ is essential in $\bigoplus_{i\leq k} C_{i,0} \oplus X$

by Lemma 1.10). Also $C_{k,0}$ is injective, for otherwise its injective hull satisfies (i) and (ii) and contradicts the maximality of $C_{k,0}$. Thus $C_{k,0}$ is a direct summand of D and hence has the exchange property.

Now $\bigoplus_{i<k} A_{i,1} \oplus (\bigoplus_{i\leq k} C_{i,0})$ is injective so

$\bigoplus_{i\leq k} A_i = \bigoplus_{i<k} A_{i,1} \oplus (\bigoplus_{i\leq k} C_{i,0}) \oplus C'$, for some $C' \leq D$.

But $\bigoplus_{i\leq k} A_i = \bigoplus_{i<k} A_{i,1} \oplus (\bigoplus_{i<k} C_{i,0}) \oplus A_k$. By the exchange property for $C_{k,0}$, we can find $A_{n,1} \leq A_n$ such that

$\bigoplus_{i\leq k} A_i = \bigoplus_{i\leq k} A_{i,1} \oplus (\bigoplus_{i\leq k} C_{i,0})$. Thus, we construct by transfinite recursion $C_{i,0}, A_{i,1}$ for $i < N$ such that (1) and (2) hold for $m = 1, n \leq N$.

(b) Assume we have constructed $A_{i,j+1}, C_{i,j}$ for all $i < N, j < m$ (where $m < M$).

If m is not a limit ordinal, then we choose, by Zorn's Lemma, $C_{0,m} \leq A_0$ maximal with respect to:

(i) $C_{0,m} \cap \bigoplus_{j<m} C_{0,j} = 0$ and

(ii) $\bigoplus_{j<m+1} C_{0,j} \cap \bigoplus_{j<m} B_j$ is essential in $\bigoplus_{j<m+1} C_{0,j}$.

Then, by the same argument we used in part (a) we obtain

$A_{0,m+1}$ such that $C_{0,m}$ and $A_{0,m+1}$ have the required properties.

Now assume we have $C_{i,m}$, $A_{i,m+1}$ $(i < n)$ for all $n < h$ where $h \leq N$. If h is a limit ordinal then, as in (a), we are done.

If h is not a limit ordinal, then $h = k + 1$ for some k and we choose $C_{k,m} \leq \bigoplus_{i \leq k} A_i$ maximal with respect to:

$$\text{(i)}\quad C_{k,m} \bigcap \left[\bigoplus_{i<k} A_{i,m} \oplus \left(\bigoplus_{j<m} C_{n,j}\right)\right] = 0$$

and (ii) $\left(\bigoplus_{i \leq k} \bigoplus_{j<m+1} C_{ij}\right) \bigcap \bigoplus_{j<m+1} B_j$ is essential in

$$\bigoplus_{i \leq k} \bigoplus_{j<m+1} C_{ij}.$$

Then, as before we see that $\left(\bigoplus_{i \leq k} \bigoplus_{j<m+1} C_{ij}\right) \bigcap \left(\bigoplus_{j<m+1} B_j\right)$ is essential in $\left(\bigoplus_{i \leq k} A_i\right) \bigcap \left(\bigoplus_{j<m+1} B_j\right)$ and C_{km} is injective, so by the exchange property we obtain $A_{n,m+1}$ and the required result.

Assume m is a limit ordinal. We know $\bigoplus_{i<n} A_i = \bigoplus_{i<n} A_{ik} \oplus \left(\bigoplus_{i<m} \bigoplus_{j<k} C_{ij}\right)$ and $\bigoplus_{j<k} B_j \bigcap \bigoplus_{i<n} \bigoplus_{j<k} C_{ij}$ is essential in both $\bigoplus_{i<n} \bigoplus_{j<k} C_{ij}$ and $\bigoplus_{i<n} A_i \bigcap \bigoplus_{j<k} B_j$ for any $k < m$ and for all $n \leq N$. In particular

$(\bigoplus_{j<k} B_j) \cap (\bigoplus_{i<N} \bigoplus_{j<k} C_{ij})$ is essential in both $\bigoplus_{i<N} \bigoplus_{j<k} C_{ij}$ and in $\bigoplus_{j<k} B_j$. By (1), $\bigoplus_{i<N} \bigoplus_{j<k} C_{ij}$ is a direct summand of D and so by Lemma 4.1, since $D = \bigoplus_{j<k} B_j \oplus (\bigoplus_{j\geq k} B_j)$ we obtain $D = \bigoplus_{i<N} \bigoplus_{j<k} C_{ij} \oplus (\bigoplus_{j\geq k} B_j)$. Similarly $D = \bigoplus_{i<N} \bigoplus_{j\leq k} C_{ij} \oplus (\bigoplus_{j>k} B_j)$. Therefore $B_k \cong \bigoplus_{i<N} C_{ik}$ for all $k < m$. Thus, for all $i < N$ and $j < m$, C_{ij} is isomorphic to a direct summand of B_j, and hence $\bigoplus_{j<m} C_{ij}$ is isomorphic to a direct summand of D and so is injective and has the exchange property.

By (1) for $n < N$ and $k < m$ we have $\bigoplus_{i<n} A_i = \bigoplus_{i<n} A_{ik} \oplus (\bigoplus_{i<n} \bigoplus_{j<k} C_{ij})$ and similarly $\bigoplus_{i\leq n} A_i = \bigoplus_{i\leq n} A_{ik} \oplus (\bigoplus_{i\leq n} \bigoplus_{j<k} C_{ij})$. Then $\bigoplus_{j<k} C_{nj}$ is isomorphic to a summand of A_n for all $n < N$ and for all $k < m$. Therefore $\bigoplus_{j<m} C_{nj}$ is isomorphic to a summand of A_n (since $\bigoplus_{j<m} C_{ij} \leq A_n$ and is injective).

Hence $\bigoplus_{i<n} \bigoplus_{j<m} C_{ij}$ is isomorphic to a summand of D, and hence has the exchange property. Therefore we can find $A_{im} \leq A_i$ for $i < n \leq N$ such that

$D = \bigoplus_{i<n} A_{im} \oplus (\bigoplus_{i<n} \bigoplus_{j<m} C_{ij})$. This is property (1).

Property (2) holds because $\bigoplus_{i<n} \bigoplus_{j<m} C_{ij} \bigcap \bigoplus_{j<m} B_j$ is essential in $\bigoplus_{i<n} \bigoplus_{j<m} C_{ij}$ by Lemma 1.10, and in $\bigoplus_{i<n} A_i \bigcap \bigoplus_{j<m} B_j$ since if $X \bigcap \bigoplus_{i<n} \bigoplus_{j<m} C_{ij} = 0$, $(X \leq \bigoplus_{i<n} A_i \bigcap \bigoplus_{j<m} B_j)$ then $X \bigcap \bigoplus_{i<n} \bigoplus_{j<k} C_{ij} = 0$ for all $k < m$ which implies $X = 0$ (by the Grothendieck property). ∎

§2. The Spectral Category

We shall now see how Theorem 4.2 yields a nice result concerning decompositions of injectives as the injective hull of a direct sum of direct summands. More specifically, we will obtain a generalization of Theorem 2.10 for injectives.

Definition 4.3: Let $\underline{G}$ be any complete Grothendieck category. Then the spectral category of $\underline{G}$, written $\underline{S(G)}$, is defined by:

ob $\underline{S(G)}$ is the class of injective objects of G

$\underline{S(G)}\langle A, B\rangle = \underline{G}\langle A, B\rangle/K$ where K is the group of morphisms whose kernel is essential in A.

This definition is a generalization of the definition of spectral category in [10] (p. 269-270), where only the specific category mod-R is considered.

If $\underline{G}$ is a complete Grothendieck category, and A is an injective object in $\underline{G}$, then we will write the corresponding object in $\underline{S(G)}$ As $\overline{A}$. Similarly if $\alpha: A \to B$ is a morphism in $\underline{G}$, where A and B are injective, then we will write the corresponding morphism in

S(G) as $\bar{\alpha}: \bar{A} \to \bar{B}$.

Obviously S(G) is a category and if α is a monomorphism (epimorphism) between injectives in G, then $\bar{\alpha}$ is a monomorphism (epimorphism) in S(G).

As we have seen in Chapter 1, if an object can be embedded in an injective, then it has an injective hull, and this injective hull is unique up to isomorphism. (We will write the injective hull of an object A as E(A)).

Now any direct sum of injectives in a complete Grothendieck category has an injective hull, since it can be embedded in the corresponding product, which is injective. (That a coproduct can be embedded in the corresponding product is proved in [7], (p. 83, Corollary 1.3).) For this reason we have limited ourselves to complete Grothendieck categories in this section. We note that a Grothendieck category with a generator is complete (B. Mitchell [7], p. 142). Thus our results will be valid for Grothendieck categories with generators.

Also, it should be noted that an intersection of injectives in G has an injective hull, as does a union of an upward directed set of subobjects of an injective.

Proposition 4.4. ([10], Theorem 4): For any complete Grothendieck category $\underline{G}$:

(1) $\underline{S(G)}$ is a Grothendieck category and $\bigoplus_I \bar{A}_i = \overline{E(\bigoplus_I A_i)}$ for any injectives A_i in $\underline{G}$ $(i \in I)$.

(2) Every object of $\underline{S(G)}$ is injective.

(3) $A \cong B$ in $\underline{G}$ if and only if $\bar{A} \cong \bar{B}$ in $\underline{S(G)}$, for injectives A and B in $\underline{G}$.

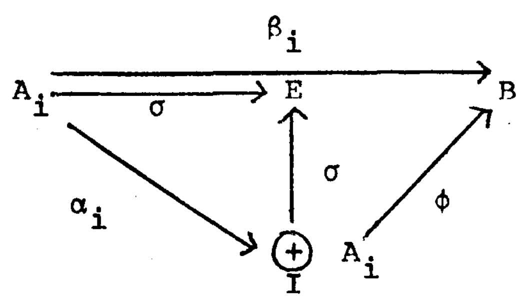
Proof: $\underline{S(G)}$ has a zero object and $\overline{\bigoplus_{i=1}^{n} A_i} = \bigoplus_{i=1}^{n} \bar{A}_i$ for injectives A_i in $\underline{G}$, $(i = 1, 2, \ldots, n)$. Also, $\underline{S(G)}\langle A, B\rangle$ is an additive abelian group.

Now, if $\phi \in \underline{G}\langle A, B\rangle$ where A and B are injective, then $\text{Ker}(\bar{\phi}) = \overline{E(\text{Ker}\ \phi)}$ so $\underline{S(G)}$ has kernels. Let us say E is the injective hull of $\text{Ker}\ \phi$. Then $A = E \oplus F$, for some F, since E is injective. If $\pi_F: A \to F$ is the projection and $\sigma_F: F \to A$ the injection, we obtain $\phi' = \phi\sigma_F: F \to B$. Then $\bar{\phi} = \bar{\phi}'\bar{\pi}_F$ (since ϕ and $\phi'\pi_F$ agree on F and on $\text{Ker}\ \phi$ which is essential in E.)

Also, ϕ' is monomorphic, so $\text{Im}\ \phi'$ is injective, and hence $B/\text{Im}\ \phi'$ is injective and $\overline{B/\text{Im}\ \phi'}$ is a cokernel for $\bar{\phi}$. This is so, since if $\nu: B \to B/\text{Im}\ \phi'$ is the canonical morphism, then $\overline{\nu\phi} = \overline{\nu\phi'\pi_F} = \bar{0}$. If also $\overline{\eta\phi} = \bar{0}$ then $\eta\phi = 0$ so $\eta\phi' = 0$ on an object essential in F. But ϕ' is monomorphic so $\bar{\eta}$ restricted to $\bar{F}$ is $\bar{0}$. Hence $\bar{\eta}$ factors over $\bar{\nu}$. Thus S(G) has kernels and cokernels.

If $\bar{\sigma}$ is a monomorphism and $\bar{\pi}$ an epimorphism then we see that $\mathrm{cok}(\ker \bar{\pi}) = \bar{\pi}$ and $\ker(\mathrm{cok}\ \bar{\sigma}) = \bar{\sigma}$. Hence S(G) is an abelian category.

Trivially S(G) is well powered. Also we obtain arbitrary coproducts as follows (so S(G) is cocomplete): Assume A_i, $i \in I$, is a set of injectives in G. Let E be the injective hull of their direct sum in G and let $\sigma_i\colon A_i \to E$ and $\alpha_i\colon A_i \to \bigoplus_I A_i$ be the injections for $i \in I$. Suppose $\beta_i\colon A_i \to B$ for $i \in I$. Then there exists $\phi\colon \bigoplus_I A_i \to B$ in G such that $\phi\alpha_i = \beta_i$. Now if $\sigma\colon \bigoplus_I A_i \to E$ is the injection then



commutes in G. By injectivity, ϕ extends to $\phi'\colon E \to B$ such that $\phi'\sigma_i = \beta_i$. If there is some $\psi\colon E \to B$ with $\psi\sigma_i = \beta_i$ then $\phi'\sigma\alpha_i = \psi\sigma\alpha_i$ so $\bigoplus_I A_i \leq \mathrm{Ker}(\phi' - \psi)$;

thus $\overline{\phi'} = \overline{\psi}$. Hence $\overline{E} = \bigoplus_I \overline{A}_i$ in S(G), that is

$\overline{E(\bigoplus_I A_i)} = \bigoplus_I \overline{A}_i$.

The Grothendieck property holds since $\bigcup_I \overline{A}_i = \overline{E(\bigcup_I A_i)}$ and $\overline{A} \cap \overline{B} = \overline{E(A \cap B)}$. Then, if $\{A_i : i \in I\}$ is an upward directed family of injectives in G, $A_i \leq D$ $(i \in I)$ and if $B \leq D$ we have

$$
\begin{aligned}
(\bigcup_I \overline{A}_i) \cap \overline{B} &= \overline{E(\bigcup_I A_i)} \cap \overline{B} \\
&= \overline{E(E(\bigcup_I A_i) \cap B)} \\
&= \overline{E((\bigcup_I A_i) \cap B)} \\
&= \overline{E(\bigcup_I (A_i \cap B))} \\
&= \bigcup_I \overline{E(A_i \cap B)} \\
&= \bigcup_I (\overline{A}_i \cap \overline{B})
\end{aligned}
$$

which is the Grothendieck property in S(G).

(2) Trivially every object in S(G) is injective.

(3) If $\overline{A} \cong \overline{B}$ in S(G) then there exist $\overline{\phi}: \overline{A} \to \overline{B}$ and $\overline{\psi}: \overline{B} \to \overline{A}$ such that $\overline{\phi}\overline{\psi} = 1_{\overline{B}}$ and $\overline{\psi}\overline{\phi} = 1_{\overline{A}}$. Hence $\psi\phi$ is an essential monomorphism. By injectivity there is a $\xi: A \to A$ such that $\xi\psi\phi = 1_A$ and by the

essentiality of $\psi\phi$, ξ is monomorphic as well as epimorphic. That is $\psi\phi$ is an automorphism of A. Similarly $\phi\psi$ is an automorphism of B. Hence ϕ is an isomorphism and $A \cong B$ in $\underline{G}$.

If A and B are injectives in $\underline{G}$, $A \cong B$, then trivially $\bar{A} \cong \bar{B}$. ∎

<u>Corollary 4.5</u>: In a complete Grothendieck category $\underline{G}$, assume D is an injective object such that

$$D = E(\bigoplus_I A_i)$$
$$= E(\bigoplus_J B_j)$$

where A_i and B_j are injective $(i \in I,\ j \in J)$. Then there exist injectives $A_{ij} \leq A_i$ and $B_{ij} \leq B_j$ for $i \in I$, $j \in J$ such that $A_i = E(\bigoplus_J A_{ij})$ $(i \in I)$, $B_j = E(\bigoplus_I B_{ij})$ $(j \in J)$, and $A_{ij} \cong B_{ij}$ $(i \in I,\ j \in J)$.

<u>Proof</u>: In <u>S(G)</u>,

$$\bar{D} = \bigoplus_I \bar{A}_i$$
$$= \bigoplus_J \bar{B}_j$$

so, by Theorem 4.2 there exist $\bar{A}_{ij} \leq \bar{A}_i$, $\bar{B}_{ij} \leq \bar{B}_j$ $(i \in I,\ j \in J)$ such that $\bar{A}_{ij} \cong \bar{B}_{ij}$, $\bar{A}_i = \bigoplus_J \bar{A}_{ij}$ and $\bar{B}_j = \bigoplus_I \bar{B}_{ij}$ for $i \in I$, $j \in J$. That is, in $\underline{G}$, by

Proposition 4.4, $A_i = E(\bigoplus_J A_{ij})$ for $i \in I$, $B_j = E(\bigoplus_I B_{ij})$ for $j \in J$, and $A_{ij} \cong B_{ij}$ for $i \in I$ and $j \in J$. ∎

Proposition 4.6: Let D be an injective object in a complete Grothendieck category $\underline{G}$. Then there exists a representation $D = E(\bigoplus_I A_i \oplus A')$ where A_i is injective and indecomposable $(i \in I)$ and A' is injective and has no indecomposable summands. If $D = E(\bigoplus_J B_j \oplus B')$ is any other such representation then $\bigoplus_I A_i \cong \bigoplus_J B_j$, the A_i and B_j are pairwise isomorphic and $A' \cong B'$.

Proof: Consider the sets of the form $\{A_i \leq D: i \in I\}$ where A_i is an indecomposable injective $(i \in I)$ and $\bigoplus_I A_i \leq D$. The set of all such sets is inductive, so has a maximal element by Zorn's Lemma. Say $\{A_i: i \in I\}$ is the maximal such set. Then $D = E(\bigoplus_I A_i) \oplus A'$ for some $A' \leq D$. That is $D = E(\bigoplus_I A_i \oplus A')$, and A' has no indecomposable summands by the maximality of $\{A_i: i \in I\}$.

If $D = E(\bigoplus_J B_j \oplus B')$ is any other such representation, then in $\underline{S(G)}$, $\bigoplus_I \overline{A}_i \oplus \overline{A}' = \overline{D} = \bigoplus_J \overline{B}_j \oplus \overline{B}'$.

By Theorem 4.2, these two direct sums have isomorphic refinements. But $\overline{A}_i$ and $\overline{B}_j$ are indecomposable ($i \in I$, $j \in J$) and $\overline{A}'$ and $\overline{B}'$ have no indecomposable summands, so $\bigoplus_I \overline{A}_i \cong \bigoplus_J \overline{B}_j$ and the A_i and B_j are pairwise isomorphic (and thus $\bigoplus_I A_i \cong \bigoplus_J B_j$). Therefore also $\overline{A}' \cong \overline{B}'$; hence $A' \cong B'$. ∎

This is then a generalization of Theorem 2.10 in the case where all the objects involved are injective.

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