THE ELECTRIC QUADRUPOLE INTERACTION IN BETA-DECAY

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By

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SCOPE AND CONTENTS: Although it has been customary in beta-decay theory to assume that the field in which the decay electron moves is spherically symmetrical, there is often a strong quadrupole interaction, comparable in magnitude at the nuclear surface to the central interaction. This will couple together different angular momentum states of both the electron and the daughter nucleus. In this work the wave functions of these coupled states have been obtained by an essentially exact solution of the Dirac equation for an electron in a non-central potential. It was found that the coupling between different daughter nuclear states was 1% at the very most, which is insufficient to account for some observed anomalies in branched decays of strongly deformed nuclei.

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CONTENTS

CHAPTER I - INTRODUCTION

PART I - GENERAL THEORY OF BETA-DECAY

β-decay is that form of radioactivity in which certain atomic nuclei emit either a positive or negative electron, transforming at the same time into the nucleus of adjacent atomic number, Z, appropriate to charge conservation. Closely allied to this process is that of orbital electron capture, in which, as an alternative to positon emission, the nucleus decreases its atomic number by unit amount through capture of one of the extra-nuclear atomic electrons.

In the early days one of the most puzzling features of β-decay was the fact that the decay electron was emitted with a continuous range of energies up to a certain maximum, despite there being a definite energy difference between the parent and daughter nuclei. In order to save energy conservation Pauli in 1933 introduced the hypothesis of the neutrino: the total decay energy is shared between the electron and a very light neutral particle, the neutrino, which must be emitted simultaneously with the electron. At the same time angular momentum will be conserved in the process if an odd half-integral spin is attributed to the neutrino. To anticipate later developments, it may be said that all the evidence is consistent with a neutrino of zero mass and spin $\frac{1}{2}$.

By the time of Pauli's suggestion the neutron-proton structure

of the nucleus had become accepted and it was realized that there were compelling reasons against the existence of electrons within a nucleus. Henceforth, β -decay had to be conceived as a process in which one type of nucleon (i.e. neutron or proton, as the case may be) transformed into the other with the simultaneous emission of an electron and a neutrino.

It was in the light of this situation that Fermi (1934) developed his theory of β -decay. Perceiving an analogy with the emission of photons during atomic transitions, which process is described in terms of the interaction of the electron and electromagnetic fields, Fermi postulated that β -decay could be similarly attributed to an interaction between the fields of the four particles directly involved: the neutron (n), the proton (p), the electron (e) and the neutrino (ν).

The subsequent history of the theory has been somewhat spectacular, there having occurred both drastic changes of empirical content and many refinements of formalism. Furthermore, there have been completely different theories proposed but despite all this the basic approach of Fermi has yet to be shown to be inconsistent with experiment and is the one generally accepted today.

The following two sections are intended to provide a background to the problem with which this thesis is concerned. They make no pretence to constituting a complete description of the theory of β -decay. For this one should refer to the various review articles in the literature, the most recent ones of which are to be found in the book of Siegbahn (1955). However, for a comprehensive account of the extensive developments that have occurred since early 1957 one must await the review article by Konopinski (1959).

1. THE FERMI THEORY.

In this section we outline the essential features of the Fermi theory as it stands in its present form. The Fermi theory can be correctly comprehended only in the language of second quantization, which we now discuss.

A quantum mechanical description of a single particle is given in terms of wave functions. Since these are functions in the configuration space of the particle a description of this kind will be formally incapable of handling situations in which particles are created or destroyed. A way out of this physical inadequacy of a single particle quantum theory is indicated by the fact that, as single particle theories, they are formally inconsistent. That is, both the Klein-Gordon equation and the Dirac equation require, in their different ways, a many-particle interpretation (see e.g. Dyson (1951)). The procedure is to regard the single particle wave functions, obtained as a first attempt at a quantum theory of particles, as classical fields which have to be quantized in some way or other. The entire content of such a theory will be determined by the choice of the field Lagrangian (or Hamiltonian) and the quantization rules. Since particles interact their fields will interact, so in considering the properties of any particular field it is necessary to include all the other fields that interact with it. The interaction between the fields is represented by additional terms in the Lagrangian (or Hamiltonian) of the entire system of fields. This is the program that one should follow. It is, of course, too ambitious and realistic approximations have to be made. Let us first, however, consider some features of the second quantization process (see Chapter XIII of Schiff

(1955) for some of the details).

The equation of motion of the field quantity $\hat{\psi}^{\alpha}$ is obtained from the Lagrangian (or Hamiltonian) in the usual way. $\hat{\psi}^{\alpha}$ has to be formally identified with the wave function ψ^{α} of a single particle α. So in quantum field theory it is really the field Lagrangian that determines the single particle equation. Nevertheless, single particle quantum theory has been quite successful, especially in the case of the Dirac theory of the electron. It does not work too well for the other Dirac (spin $\frac{1}{2}$) particles but it may be conjectured that failure is always due to interactions with other particles: in the case of nucleons with π mesons. Thus for free Dirac fields the Lagrangian is always chosen to give the Dirac single particle equation for the field equation.

Let the field $\hat{\psi}^{\alpha}(\underset{\sim}{r}, t)$ be expanded in terms of a complete set of orthonormal functions, $\psi_i^{\alpha}(W, \underset{\sim}{r})$ thus

$$\hat{\psi}^{\alpha}(\underset{\sim}{r}, t) = \sum_i \int dW \, \hat{a}_i^{\alpha}(W, t) \psi_i^{\alpha}(W, \underset{\sim}{r}) e^{-iWt} \qquad (1)$$

where

$$\int d\underset{\sim}{r} \, \psi_i^{\alpha}(W', \underset{\sim}{r})^{\dagger} \psi_j^{\alpha}(W, \underset{\sim}{r}) = \delta_{ij} \, \delta(W' - W) \qquad (2)$$

(the dagger here means Hermitian adjoint). The field quantities, $\hat{\psi}^{\alpha}$ are to be regarded as operators and in the expansion (1) the operator properties are carried by the $\hat{a}$'s rather than by the functions $\psi_i^{\alpha}(W, \underset{\sim}{r})$, which are essentially single particle wave functions of energy W.

The quantum conditions that $\hat{\psi}^{\alpha}(\underset{\sim}{r}, t)$ has to satisfy take the form of commutation or anti-commutation relations according as to whether we are dealing with Bose particles or Dirac particles. In both cases it is easy to show that the operators

$$\underset{\sim}{N}{}_i^{\alpha}(W,t) = \underset{\sim}{a}{}_i^{\alpha}(W,t)^{\dagger}\ \underset{\sim}{a}{}_i^{\alpha}(W,t) \qquad (3)$$

all commute with one another and also with the free field Hamiltonian. Now the wave function ψ^{α} can be expanded exactly as in (1), ψ^{α} and $\underset{\sim}{\psi}{}^{\alpha}$ being formally identical, whence $|a_i^{\alpha}(W,t)|^2$ is the probability of finding the particle in the state of wave function $\psi_i^{\alpha}(W,\underset{\sim}{x})$. Thus $\underset{\sim}{N}{}_i^{\alpha}(W,t)$ is interpreted as a field operator whose eigenvalues $n_i^{\alpha}(W)$ are the number of particles in this state. The complete set of normalized eigenvectors of $\underset{\sim}{N}{}_i^{\alpha}(W,t)$ form the basis of what may be termed the occupation number representation:

$$\underset{\sim}{N}{}_i^{\alpha}(W,t)\,|\cdots n_i^{\alpha}(W)\cdots n_j^{\alpha}(W')\cdots\rangle = n_i^{\alpha}(W)\,|\cdots n_i^{\alpha}(W)\cdots n_j^{\alpha}(W')\cdots\rangle \qquad (4)$$

For Dirac particles the n's can take only the values 0 and 1, because of the exclusion principle. It is then easy to show that

$$\underset{\sim}{a}{}_i^{\alpha}(W,t)\,|\cdots n_i^{\alpha}(W)\cdots n_j^{\alpha}(W')\cdots\rangle = (-)^{\nu_i(W)}\, n_i^{\alpha}(W)\,|\cdots -n_i^{\alpha}(W)\cdots n_j^{\alpha}(W')\cdots\rangle$$

$$\underset{\sim}{a}{}_i^{\alpha}(W,t)^{\dagger}\,|\cdots n_i^{\alpha}(W)\cdots n_j^{\alpha}(W')\cdots\rangle = (-)^{\nu_i(W)}\{1-n_i^{\alpha}(W)\}\,|\cdots 1-n_i^{\alpha}(W)\cdots n_j^{\alpha}(W')\cdots\rangle \qquad (5)$$

where the $\gamma_\lambda(W)$ is a quantity that depends on the ordering of states but whose actual value need not concern us. This immediately leads to the interpretation of the $\underset{\sim}{a}$'s and $\underset{\sim}{a}^\dagger$'s as destruction and creation operators, respectively, for particles in the appropriate states. Thus for particles α to be created there will have to be an interaction with another field β, the interaction Hamiltonian containing a term with $\underset{\sim}{\psi}^{\alpha\dagger}$ but not $\underset{\sim}{\psi}^{\alpha}$. This same term, representing the coupling between the α and β fields, must also have $\underset{\sim}{\psi}^{\beta\dagger}$ or $\underset{\sim}{\psi}^{\beta}$ appearing in it, depending on whether particles of type β are created or destroyed simultaneously with the creation of a particle α. Then in order for the interaction Hamiltonian to be Hermitian the complex conjugate of the first term must also be included. This will describe the inverse process of destruction of a particle α, etc.

We can now consider the field interaction Hamiltonian appropriate to β-decay. All four particles involved are treated as Dirac particles i.e. their free field equations are

$$\sum_{\lambda=1}^{4} \gamma_\lambda \frac{\partial \underset{\sim}{\psi}}{\partial x_\lambda} + m \underset{\sim}{\psi} = 0 \qquad (6)$$

Here $\underset{\sim}{\psi}$ is a four component spinor and γ_λ a particular 4 x 4 matrix (see, e.g. Dyson (1951)). m is the particle mass in units of electron mass. $\hbar$ and c have also been put equal to unity (we adhere to these units throughout the thesis). It is convenient to regard negaton decay as being associated with the destruction of a neutrino in a negative energy state rather than with the emission of a positive energy neutrino. Hence the interaction Hamiltonian describing negaton decay must contain

$\underset{\sim}{\psi}^{n}$, $\underset{\sim}{\psi}^{r}$, $\underset{\sim}{\psi}^{p\dagger}$, $\underset{\sim}{\psi}^{e\dagger}$. Since spinors change sign under a 2π rotation the Hamiltonian must be linear in the four field quantities or in their derivatives. Secondly, the Hamiltonian must be invariant under proper Lorentz transformations. Originally, it was assumed that the interaction would be invariant under space reflections as well, but this additional restriction was dropped at the suggestion of Lee and Yang (1956). The Fermi Ansatz for the interaction is now

$$H_K = \int d\underset{\sim}{r} \left(\underset{\sim}{\psi}^{p\dagger} O_K \underset{\sim}{\psi}^{n}\right)\left(\underset{\sim}{\psi}^{e\dagger} O_K \left[C_K + C_K' \gamma_5\right] \underset{\sim}{\psi}^{r}\right) + \text{c.c.} \qquad (7)$$

Here the O_K are various combinations of the matrices $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, such that the quantities $\left(\underset{\sim}{\psi}^{\alpha\dagger} O_K \underset{\sim}{\psi}^{\beta}\right)$ are Lorentz covariant. The scalar product of the two such covariant quantities is then invariant under all Lorentz transformations. However, the extra terms containing $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ will change sign under space reflections. The complex conjugate term makes the Hamiltonian Hermitian and describes positon decay. The C_K, C_K' measure the strength of the associated interaction term and are referred to as the coupling constants.

We now consider the form of the Lorentz covariant quantities $\left(\underset{\sim}{\psi}^{\alpha\dagger} O_K \underset{\sim}{\psi}^{\beta}\right)$. Since $\underset{\sim}{\psi}^{\alpha}$ and $\underset{\sim}{\psi}^{\beta}$ each have four components there are sixteen linearly independent bilinear terms of this kind. They can be grouped together into five different sets, each of which transforms as a tensor of different rank (see, e.g. Pauli (1958)). The five covariant forms are referred to as: (i) the scalar (S), (ii) the vector (V), (iii) the tensor (T) - actually an antisymmetric tensor of second rank - (iv) the axial vector (A) - an antisymmetric tensor of third rank - and (v) the pseudo-scalar (P). We shall then have just five

linearly independent β -decay interaction forms. Replacing the γ matrices by the Dirac matrices $\beta, \underset{\sim}{\sigma}$ and $\underset{\sim}{\alpha}$ we can now write the usual form of the operators O_K:

$$O_S = \beta \quad ; \qquad O_V = 1, \underset{\sim}{\alpha} \quad ;$$

$$O_T = \beta\underset{\sim}{\sigma}, \beta\underset{\sim}{\alpha} \quad ; \qquad O_A = \underset{\sim}{\sigma}, -\gamma_5 \quad ; \qquad O_P = \beta\gamma_5$$

There are no compelling a priori criteria determining either the absolute or the relative values of the different coupling constants; the matter has to be settled phenomenologically. It is sufficient to say that there is a large amount of evidence in favor of the currently accepted view that the interaction consists solely of a mixture of V and A forms with $C_A \simeq -1.2\, C_V$, $C_V' = C_V$, $C_A' = C_A$. Also, the coupling constants are real, corresponding to invariance of the interaction Hamiltonian under time reversal (see, e.g. Burgy et al. (1958)).

In writing down (7) there is an ambiguity in the position of the γ_5 matrix in the terms that change sign under space reflections. This is quite arbitrary and implies no physical restriction since a form of H_K in which γ_5 appears elsewhere can be written as linear combinations of the H_K defined in (7). Similar remarks apply to an interchange of $\underset{\sim}{\psi}^n$ and $\underset{\sim}{\psi}^\nu$. (7) is merely the conventional form of the Fermi theory.

It should be noted that the Fermi theory is not the only possible theory in which β -decay can be described in terms of a direct interaction between the n, p, e and ν fields. Konopinski and Uhlenbeck (1935) replaced $\underset{\sim}{\psi}^\nu$ by its derivatives. There are no

a priori reasons against this, but a conflict with experiment was later discovered and this so-called K-U modification was abandoned. More drastic alternatives to the Fermi theory replace the direct interaction between the four fields by an indirect interaction taking place through some meson. None of these modifications have become compellingly necessary and we shall ignore them.

Suppose we have a neutron in the state $\psi_h^n(W_n)$ and a neutrino in the state $\psi_j^\nu(W_\nu)$. The problem is to calculate the probability of transition to a proton in the state $\psi_\ell^p(W_p)$ and an electron in the state $\psi_m^e(W_e)$. The appropriate initial and final occupation number states are then

$$|0, \cdots 0, n_h^n(W_n) = 1, n_j^\nu(W_\nu) = 1, 0 \cdots\rangle = |i\rangle$$

and

$$|0, \cdots 0, n_\ell^p(W_p) = 1, n_m^e(W_e) = 1, 0 \cdots\rangle = |f\rangle$$

Now the β-decay interaction is known to be extremely weak so that it can well be treated by the standard time dependent perturbation theory (see, e.g. pp. 195-9 of Schiff (1955)). The relevant matrix element $\langle f|H_\beta|i\rangle$ will be that of the field interaction Hamiltonian (7) between the above occupation number states, these being orthonormal. Expanding (7) according to (1) and using (5) gives

$$\langle f|H_\beta|i\rangle = \sum_K \int d\underset{\sim}{r} \left(\psi_\ell^p(W_p)^\dagger O_K \psi_h^n(W_n)\right) + \left(\psi_m^e(W_e)^\dagger O_K [C_K + C_K' \gamma_5] \psi_j^\nu(W_j)\right) \tag{8}$$

This result is formally equivalent to the statement that

$$H_\beta = \sum_K O_K^N \cdot O_K^L \left[C_K + C_K' \gamma_5 \right] \left(\Gamma_N \Gamma_L + \text{c.c.} \right) \delta\left(\underset{\sim}{r}_N - \underset{\sim}{r}_L \right) \qquad (9)$$

where the labels N and L refer to nucleons and leptons, respectively. Γ_N is an operator transforming a neutron into a proton and Γ_L one transforming a neutrino into an electron. Matrix elements then have to be taken between the initial state $\left| \psi_h^n(W_n) \psi_j^\nu(W_\nu) \right\rangle$ and the final state $\left| \psi_\ell^p(W_p) \psi_m^e(W_e) \right\rangle$. While this formalism obscures the essential field theoretic content of the β-decay process it is nevertheless convenient for calculating transition probabilities and we shall in fact use it.

Since at least the lepton states form a continuum there will be a well defined transition probability per unit time. If we write the initial energy as $W^i = W_n + W_\nu$ and the final energy as $W^f = W_p + W_e$ there will be a transition rate from the initial states to unit energy interval of the final states given by

$$\omega(W_f) = 2\pi \rho(W^f) \left| \langle f | H_\beta | i \rangle \right|^2_{W^i = W^f} \qquad (10)$$

This is the usual form of the so-called Golden Rule ((29.12) of Schiff (1955)). The quantity $\rho(W_f)$ which appears here is referred to as the energy density of the final states; its use requires some careful consideration. Its occurrence is due to the fact that for convenience the continuum spectrum of states is often broken up into a discrete spectrum by confinement to a large but finite enclosure, within which the states are normalized. A definite number of states within a finite energy interval is then obtained by applying periodic boundary conditions

on the surface of the enclosure. However, it is easy to see that if we normalize our continuum functions over the whole of space, according to (2), then the state density just does not enter the problem and we have

$$w\,(W_F) = 2\pi \left| \langle F | H_\beta | i \rangle \right|^2_{W_i = W_F} \qquad (11)$$

Thus the state density is merely a weighting factor that has to be introduced to take care of the different normalizations. Clearly, for continuum states it has no direct physical meaning. Since it is not at all obvious that the imposition of periodic boundary conditions on the surface of the finite enclosure will always give the correct weighting factor we prefer to use the delta function normalization (2), whence the Golden Rule is (11).

There remains the question of the single particle wave functions appearing in (8). As already pointed out these should really be obtained from the field equations. In the case of the neutrino the only known interaction which it experiences is the β -decay one, so its wave function will be well described by the single particle Dirac equation

$$\sum_{\lambda=1}^{4} \gamma_\lambda \frac{\partial \psi}{\partial x_\lambda} = 0 \qquad (12)$$

the mass being assumed to be zero.

For the electron it is necessary, of course, to take account of the interaction with the electromagnetic field. The correct treatment of this, which constitutes the subject of quantum electrodynamics, involves the quantization of the electromagnetic field as well. However, what is known as the external field approximation is often quite successful. This treats the electromagnetic field as a classical Maxwell field

and assumes that its only sources are extraneous to the electron system e.g. a proton. It therefore ignores all interactions between electrons, both real and virtual. It will therefore break down for systems of more than one real electron and also for strong electric fields, since these will have an appreciable effect on the state of the vacuum (the so-called vacuum polarization process becomes important). However, since the condition for this to be serious is that $\alpha Z > 1$, Z being the charge of the source (see, e.g. Feshbach and Villars (1958)) it is customary to adopt this external field approximation in β -decay; that is, we ignore the radiative corrections. Then the single particle wave equation is simply the Dirac equation for external fields:

$$\sum_{\lambda=1}^{4} \gamma_\lambda \left(\frac{\partial}{\partial x_\lambda} + ie A_\lambda \right) \psi + \psi = 0 \qquad (13)$$

A_λ is the electromagnetic four-vector potential: $A_{1,2,3} = \underset{\sim}{A}$ and $A_4 = iV$, where V is the electrostatic potential.

It is the nucleon wave functions that present the major problem. For nucleons bound in a nucleus one has to fall back on the various nuclear models, which give wave functions of dubious reliability. This represents the greatest source of ambiguity in the understanding of β -decay transitions.

2. CALCULATION OF THE TRANSITION PROBABILITY.

In this section we outline the method of calculating β -decay transition probabilities, using the Golden Rule (11) and the matrix elements (8). What we are interested in is the probability of a nucleus with angular momentum $\underset{\sim}{I}$ and parity $\underset{\sim}{\Pi}$ decaying to one with

angular momentum I_f and parity Π_f, the electron (we shall confine ourselves to a negaton in order to be specific) being emitted with energy W_e.

The matrix elements that appear in (11) have the form (8). Since more than one nucleon in the nucleus can decay there should strictly be a summation over all nucleons. We shall represent this in the usual way by writing in place of the single nucleon functions the complete wave functions of the parent and daughter nuclei, $\underline{\Psi}^{M_i}_{I_i}$ and $\underline{\Psi}^{M_f}_{I_f}$ respectively.

Let the energy of the emitted neutrino be q. Then

$$W_e + q = W_o$$

where W_o is the observed energy end-point of the β -spectrum. (q is also the momentum of the neutrino, since it is assumed to be massless). The wave function of the emitted neutrino can be written as $\psi^{\nu}_{h}(q)$ where h denotes all the quantum numbers other than q. But in (8) it is the wave function of the destroyed negative energy neutrino that appears. The required wave function is then $A\,\psi^{\nu *}_{h}(q)$ where

$$A = i\,\alpha_2\,\beta \qquad (14)$$

(see, e.g. Dyson (1951)).

The wave equation (13) of the electron can be written in Hamiltonian form as

$$H_e\,\psi = \left(-i\,\underset{\sim}{\alpha}\cdot\underset{\sim}{\nabla} - \beta + V\right)\psi \qquad (15)$$

Here V is the electrostatic potential due to the daughter nucleus and $\underset{\sim}{A}$ has been put equal to zero since magnetic interactions are negligible. It is customary to assume that the electrostatic potential of the nucleus is spherically symmetrical, whence W_e must be a constant of

motion and we have

$$\left\{ i \underset{\sim}{\alpha} \cdot \underset{\sim}{\nabla} + \beta + (W_e - V_c) \right\} \psi = 0 \qquad (16)$$

In addition to W_e we have as constants of motion the total angular momentum, j, its z-component, μ , and the Dirac operator

$$K = \beta \left(\underset{\sim}{\sigma} \cdot \underset{\sim}{L} + 1 \right) \qquad (17)$$

where $\underset{\sim}{L}$ is the orbital angular momentum operator. K has the eigenvalues

$$K = \pm \left(j + \tfrac{1}{2} \right) \qquad (18)$$

It is convenient to introduce the quantity

$$\ell_K = |K| + \tfrac{1}{2} (S_K - 1) \qquad (19)$$

where S_K is the sign of K i.e. $K/|K| \cdot \ell_K$ may be regarded as the value of the orbital angular momentum in the non-relativistic limit..

An exact separation of the radial and angular parts of (16) is now possible. Simultaneous eigenstates of H_e, j , μ and K then take the form

$$\psi_K^{\mu}(W_e) = \begin{bmatrix} -i f_K(W_e) \chi_{-K}^{\mu} \\ g_K(W_e) \chi_K^{\mu} \end{bmatrix} \qquad (20)$$

where the $\chi_{\pm K}^{\mu_e}$ are just the two-component Pauli spinors

$$\chi_K^{\mu} = \sum_{\tau = \pm \frac{1}{2}} C\left(j \; \ell_K \; \tfrac{1}{2} \; ; \; \mu \; \mu - \tau \; \tau \right) Y_{\ell_K}^{\mu - \tau}(\theta, \phi) \begin{bmatrix} \frac{1}{2} + \tau \\ \frac{1}{2} - \tau \end{bmatrix} \qquad (21)$$

(the C quantities here are Clebsch-Gordan coefficients - see Appendix A). The f and g are real radial functions satisfying the pair of coupled first order equations

$$(W_e - V_c + 1) f - \frac{dg}{dr} - (K+1)\frac{g}{r} = 0$$
$$(W_e - V_c - 1) g + \frac{df}{dr} - (K-1)\frac{f}{r} = 0 \qquad (22)$$

There are two linearly independent solutions to this system, one regular at the origin and the other irregular. The solutions will depend on the potential V_c, but for the Coulomb case, $V_c = Ze^2/r$ i.e. a point nucleus, the calculation is straightforward. Rose (1937) gives the regular solutions to these relativistic Coulomb functions, normalized thus

$$\int r^2 dr \left\{ f_K(W_e') f_K(W_e) + g_K(W_e') g_K(W_e) \right\} = \delta(W_e' - W_e) \qquad (23)$$

so that $\psi_K^{\mu_e}(W_e)$ is normalized according to (2). Asymptotically the radial solutions take the form

$$f_K(W_e) \sim -\frac{1}{r}\sqrt{\frac{W_e - 1}{\pi p}} \sin(pr + \delta^c)$$
$$g_K(W_e) \sim \frac{1}{r}\sqrt{\frac{W_e + 1}{\pi p}} \cos(pr + \delta^c) \qquad (24)$$

where p is the asymptotic momentum corresponding to W_e.

$$p = \sqrt{W_e^2 - 1}$$

δ^c is the Coulomb phase shift

$$\delta^c = \frac{\alpha Z W_e}{p} \log 2pr - \arg\Gamma\left(\gamma + i\frac{\alpha Z W_e}{p}\right) + \eta - \frac{\pi\gamma}{2}$$

where

$$\gamma = (K^2 - \alpha^2 Z^2)$$

and $$\exp(2i\eta) = -\frac{\kappa - i\frac{\alpha Z}{p}}{\gamma + i\frac{\alpha ZW}{p}}$$

An irregular solution is formed by introducing any constant phase into the asymptotic form (24). Inside a nucleus of finite extent the solutions will not be Coulomb functions, but outside they will be linear combinations of regular and irregular Coulomb functions, although the normalization will have changed.

The Z = 0 solution for the regular electron functions is of some interest. It can easity be obtained from the general solution and is

$$f_{\kappa_e}(W_e) = S_{\kappa_e}\sqrt{\frac{W_e - 1}{\pi p}}\, p\, j_{\ell - \kappa_e}(pr)$$
$$g_{\kappa_e}(W_e) = \sqrt{\frac{W_e + 1}{\pi p}}\, p\, j_{\ell \kappa_e}(pr) \qquad (25)$$

where the j functions are spherical Bessel functions.

The neutrino wave equation can be treated in the same way. The solution may be obtained directly from (25) by replacing both $W_e \pm 1$ and p by q, whence

$$f_{\kappa_\nu}(q) = S_{\kappa_\nu}\sqrt{\frac{1}{\pi}}\, q\, j_{\ell - \kappa_\nu}(qr)$$
$$g_{\kappa_\nu}(q) = \sqrt{\frac{1}{\pi}}\, q\, j_{\ell \kappa_\nu}(qr) \qquad (26)$$

Since we are not going to concern ourselves with the direction of emission of either the neutrino or the electron this angular momentum representation is the natural one to use. Using (8) and (11) and summing over the lepton quantum numbers $\kappa_e, \kappa_\nu, \mu_e$ and μ_ν we have for the probability of emission of an electron in a unit energy interval

$$N(W_e) = 2\pi \sum_{\substack{\kappa_e \kappa_\nu \\ \mu_e \mu_\nu}} \left| \sum_K \int d\hat{r} \left(\Psi_{I_f}^{M_f \dagger} O_K \Psi_{I_i}^{M_i} \right) \times \left(\psi_{\kappa_e}^{\mu_e}(W_e)^\dagger O_K \left[C_K + C'_K \gamma_5 \right] A \psi_{\kappa_\nu}^{\mu_\nu}(q) \right)^* \right|^2 \tag{27}$$

This apparently differs by a factor of $\pi^2 \frac{p}{W_e}$ from the usual angular momentum representation expression for the transition probability (see (9.19) of Rose (1957)). This is simply because the latter normalizes within a finite sphere of unit radius, the appropriate solutions having been given by Rose (1937). It will be noticed that the implicit weighting factors, $\frac{1}{\pi}$ for the neutrinos and $\frac{1}{\pi} \frac{W_e}{p}$ for the electrons are just the state densities that would be obtained by imposition of periodic boundary conditions at the surface of the enclosure, the functions having assumed their asymptotic form (24). However, one cannot be sure how general such an identity is.

Now we consider the γ_5 term occurring in (27). It is easy to show that

$$\gamma_5 A \psi_{\kappa_\nu}^{\mu_\nu}(q)^* = -i S_{\kappa_\nu} A \psi_{-\kappa_\nu}^{\mu_\nu}(q)^* \tag{28}$$

Since there is an incoherent summation over all K_ν it follows that these terms can be ignored if we replace $C_K C_\ell$ by

$$\vartheta_K \vartheta_\ell = C_K C_\ell + C'_K C'_\ell \tag{29}$$

Henceforth we shall write in place of (27)

$$N(W_e) = 2\pi \sum_{\substack{K_e K_\nu \\ \mu_e \mu_\nu}} \left| \left\langle \Psi_{I_f}^{M_f} \gamma_{K_e}^{\mu_e}(W_e) \middle| H_\beta \middle| A \gamma_{K_\nu}^{\mu_\nu}(q)^* \Psi_{I_i}^{M_i} \right\rangle \right|^2 \tag{30}$$

where

$$H_\beta = \sum_K \vartheta_K O_K^N \cdot O_K^L \Gamma_N \Gamma_L \delta(\underset{\sim}{r}_N - \underset{\sim}{r}_L) \tag{31}$$

Now let us consider the quantity

$$O_K^N \cdot \gamma_{K_e}^{\mu_e}(W_e)^\dagger O_K^L A \gamma_{K_\nu}^{\mu_\nu}(q)^* \tag{32}$$

Then the matrix elements in (30) are essentially matrix elements of this quantity between the nuclear states. Recalling the explicit form (20) of the lepton functions, the lepton part of (32) can be expanded into a series of spherical harmonics with the aid of (A.23), it being noted that both lepton functions are to be evaluated at the same point in nucleon space.

If the O_K operator is a scalar (i.e. β, γ_5, $\beta\gamma_5$) then the matrix elements appearing in (30) can be expressed as

$$\left\langle \Psi_{I_f}^{M_f} \gamma_{K_e}^{\mu_e}(W_e) \middle| H_\beta \middle| A \gamma_{K_\nu}^{\mu_\nu}(q)^* \Psi_{I_i}^{M_i} \right\rangle =$$

$$\sum_\lambda \left\langle \Psi_{I_f}^{M_f} \middle| \Phi(K_e, K_\nu, \lambda, p, q, r) O_K Y_\lambda^M(\hat{r}) \middle| \Psi_{I_i}^{M_i} \right\rangle \tag{33}$$

in which the functions $\underline{\Phi}$ depend primarily on the lepton radial functions, and $\hat{\underline{r}}$ is the unit position vector. If O_K is a vector (i.e. $\underline{\sigma}$, $\underline{\alpha}$, $\beta\underline{\sigma}$ $\beta\underline{\alpha}$) it can be written in spherical form according to (A.27) and (A.28), whence from the inverse of (A.31) the scalar product between the two vector quantities of (32) can be expanded as a sum of spherical tensors. Then (33) is replaced by

$$\left\langle \underline{\Psi}^{M_f}_{I_f} \psi^{\mu_e}_{\kappa_e}(W_e) \left| H_\beta \right| A\, \psi^{\mu_\nu *}_{\kappa_\nu}(q)\, \underline{\Psi}^{M_i}_{I_i} \right\rangle =$$

$$\sum_{L,\lambda} \left\langle \underline{\Psi}^{M_f}_{I_f} \left| \underline{\Phi}(\kappa_e, \kappa_\nu, L, \lambda, \rho, q, r)\, T^{M}_{L\lambda}(\hat{\underline{r}}, O_K) \right| \underline{\Psi}^{M_i}_{I_i} \right\rangle \tag{34}$$

where $L = \lambda, \lambda \pm 1$, and is the tensor rank.

In both (33) and (34) what we have done essentially is to perform a multipole expansion on the nuclear operator (32), so that by reference to Appendix A it is possible to write down immediately the condition for non-vanishing of the matrix elements. For (33) we have the angular momentum selection rule

$$|\underline{I}_i - \underline{I}_f| \leq \lambda \leq \underline{I}_i + \underline{I}_f \tag{35}$$

and the parity selection rule

$$\pi(O_K)\,(-)^{\lambda} = \pi_i \pi_f \tag{36}$$

In (34) the tensor rank is L and not λ so that (35) has to be replaced by

$$|\underline{I}_i - \underline{I}_f| \leq L \leq \underline{I}_i + \underline{I}_f \tag{37}$$

We now note that for the energies encountered in β -decay $\underline{\Phi}(\cdots \lambda \cdots) \gg \underline{\Phi}(\cdots \lambda+1 \cdots)$ for r less than the nuclear radius. From (36) it is seen that non-vanishing matrix elements will

only occur for λ values differing by even amounts. This means that only the smallest value of λ that gives a non-vanishing contribution is of any significance. The error involved in ignoring the higher order terms is seldom greater than 2%.

In performing the integration over nuclear space in (33) and (34) $\underline{\Phi} r^{-\lambda}$ is presumed to be slowly varying over the nucleus (at the origin $\underline{\Phi}$ varies exactly as r^{λ}) and is therefore taken outside of the integral and evaluated at ρ , a quantity commensurate with the nuclear radius. Our matrix elements then have the form

$$\underline{\Phi}\rho^{-\lambda}\langle \underline{\Psi}^{M_f}_{\underline{I}_f} | O_K Y^m_\lambda(\hat{r}) | \underline{\Psi}^{M_i}_{\underline{I}_i} \rangle, \quad \underline{\Phi}\rho^{-\lambda}\langle \underline{\Psi}^{M_f}_{\underline{I}_f} | T^m_{L\lambda}(r, O_K) | \underline{\Psi}^{M_i}_{\underline{I}_i} \rangle$$

The lepton dependence of the transition probability resides wholly in the $\underline{\Phi}\rho^{-\lambda}$ factor, while the dependence on the nuclear wave functions is confined to the new matrix elements that appear here. It is these latter that are known in the terminology as the nuclear matrix elements; they represent the greatest source of ambiguity in β -decay theory. Nevertheless, it is possible to give an estimate of their order of magnitude. We have already seen that consecutive values of λ are associated with a marked decrease in the transition probability. At the same time we must note the effect of the operator

O_K on the nuclear wave functions. If this has even parity (β, $\underset{\sim}{\sigma}$, $\beta\underset{\sim}{\sigma}$) it couples together the large components of the nucleon four-spinors. On the other hand the operators $\underset{\sim}{\alpha}$, $\beta\underset{\sim}{\alpha}$, γ_5 and $\beta\gamma_5$ couple together the large and small components. Since the nucleons are moving slowly the matrix elements will depend considerably on whether the O_K is odd or even. In fact, changing O_K from an even to an odd parity operator has roughly the same effect as increasing λ by unity. This gives rise to the well known degree of forbiddenness classification of transition probabilities. A transition will be n'th forbidden if

$$n = \lambda \qquad \text{for } O_K \text{ even}$$
$$n = \lambda + 1 \qquad \text{for } O_K \text{ odd}$$

where, for a given O_K, λ will be the smallest value compatible with (36) and (35) or (37). Secondly, in any transition the only O_K of significance is the one associated with the lowest degree of forbiddenness: the contributions from the higher degrees are negligible.

It is instructive to be explicit and consider a particular transition: $\Delta I = 1$, $\pi_i = -\pi_f$. The vector interaction gives rise to the nuclear matrix elements (i) $\langle \mathcal{Y}^m_\lambda(\underset{\sim}{r}) \rangle$ and (ii) $\langle T^m_{L\lambda}(\underset{\sim}{r}, \underset{\sim}{\alpha}) \rangle$ while the axial vector interaction has associated with it (iii) $\langle \gamma_5 \mathcal{Y}^m_\lambda(\underset{\sim}{r}) \rangle$ and (iv) $\langle T^m_{L\lambda}(\underset{\sim}{r}, \underset{\sim}{\sigma}) \rangle$. For matrix elements (i) and (iii) we have from (35) that

$$| \leq \lambda \leq I_i + I_f$$

Now the parity condition (36) requires that λ be odd for (i) but even for (iii). Thus the former can contribute in the first degree of forbiddenness with $\lambda = 1$ while the latter cannot contribute below

the third degree and is therefore ignored. For (ii) and (iv) the angular momentum selection rule (37) becomes

$$| \leqslant L \leqslant I_i + I_f$$

where $L = \lambda, \lambda \pm 1$. For (ii) λ must be even and so it too will contribute in the first degree with $\lambda = 0, L = 1$. Finally, λ has to be odd for (iv) whence we shall have $\lambda = 1, L = 1$ contributing and if $I_i + I_f \neq 1$ there will also be a contribution from $\lambda = 1, L = 2$. Thus the lowest degree of forbiddenness in which the transition can occur is the first and we shall have the following matrix elements to consider:

$$\langle \mathcal{Y}^m_1(\underset{\sim}{r}) \rangle \qquad \langle T^m_{10}(\underset{\sim}{r}, \underset{\sim}{\alpha}) \rangle$$

$$\langle T^m_{11}(\underset{\sim}{r}, \underset{\sim}{\sigma}) \rangle \quad , \quad \langle T^m_{21}(\underset{\sim}{r}, \underset{\sim}{\sigma}) \rangle$$

The last of these, which corresponds to an angular momentum of 2 being carried off by the leptons, will vanish if $I_i + I_f = 1$ i.e. if either I_i or I_f is zero. Even when it does not vanish it turns out that its contribution is considerably smaller than those of the other matrix elements and it is often ignored altogether (it is related to the B_{ij} of Konopinski and Uhlenbeck (1941)). On the other hand it is easy to see that for a transition $\Delta I = 2$, with a parity change, this matrix element will still be non-vanishing although all the others will be identically zero. Thus this transition will be first forbidden, but significantly slower than the other first forbidden transitions. Since the transition probability depends on only one matrix element in this case the dependence on the lepton functions will be unique and the spectrum shape will be the same for all such transitions. They are referred to as unique transitions.

This example is sufficient to illustrate the general principle that the degree of forbiddenness, n, is odd if there is a parity change and even if there is not. The precise value of n is then given by

$$n = \Delta I \quad \text{or} \quad n = \Delta I - 1$$

the latter being the unique transitions, which occur in all degrees of forbiddenness. (Special attention has to be given to the case of $0 \rightarrow 0$ transitions, with a parity change.)

In the foregoing we have merely indicated the general approach to the problem of calculating transition probabilities. We have not discussed at all the details of the calculation of the lepton dependent functions, $\underline{\Phi}$, which multiply each matrix element and on which the spectrum shape and the total transition probability will depend. This calculation is straightforward but very tedious. The results for the various cases have been given by Konopinski and Uhlenbeck (1941), Greuling (1942) and Pursey (1951).

The angular momentum representation for the lepton states is clearly the most convenient one to use in establishing the selection rules for the different degrees of forbiddenness. On the other hand, the most elementary accounts of β -decay theory ignore the nuclear charge and represent the lepton states by plane waves. The allowed spectrum shape is then determined solely by the density of constant momentum states $= \rho W_e q^2$. It is of interest to note that if we take $Z = 0$ approximation (25) for the electron functions then our expression (30) reduces for allowed transitions to the plane wave result.

PART II - THE ELECTRIC QUADRUPOLE INTERACTION IN BETA-DECAY

In this part we consider the problem with which this thesis is concerned.

3. STATEMENT OF THE PROBLEM.

As we have already pointed out in the previous section it is customary in calculating the wave function of the decay electron to assume that the electrostatic field of the daughter nucleus in which the electron moves is spherically symmetrical. Now possession of quadrupole moments by nuclei shows that in general this cannot exactly be the case. The Hamiltonian of the interaction of a negaton with the quadrupole moment is

$$H_Q = -\frac{e^2}{2} Q_0 f(r) P_2(\cos\theta') \qquad (38)$$

where θ' is the polar angle of the electron position vector with respect to the nuclear symmetry axis and Q_0 is the intrinsic quadrupole moment of the nucleus (see, e.g. the article by Bohr and Mottelson in Siegbahn (1955)). $f(r)$ is a radial function given by $f(r) = 1/r^3$ outside of the nucleus. Inside the nucleus it depends very much on the charge distribution but for a uniform charge distribution of average radius ρ $f(r) = r^2/\rho^5$ (see, e.g. Wheeler (1953)).

Because the electrostatic interaction between the decay electron

and the daughter nucleus is non-central, their angular momenta, j_e and I_f, respectively, will no longer be constants of motion. Only asymptotically, when the electron is remote from the nucleus, will they become good quantum numbers.

One would expect the quadrupole coupling effect to be largest in the regions remote from closed shells i.e. $150 < A < 190$ and $A > 225$, for such nuclei are strongly deformed and have large quadrupole moments. Furthermore, their low lying states are very close together, the first excited state having an energy of less than 100 kev. Since adjacent states differ in angular momentum by two units and have the same parity an appreciable quadrupole coupling between the two states may be anticipated. The situation, in fact, is not unlike that obtaining in Coulomb excitation (see, e.g. Alder et al. (1956)).

In view of the considerable ambiguity that arises in the interpretation of β-decay transitions, due mainly to the uncertainty in the nuclear wave functions, one may well ask how quadrupole effects of the kind just mentioned could be unequivocally identified as such, even if they were appreciable. In order to see that this is possible we must first examine more closely the structure of the strongly deformed nuclei.

These nuclei are characterized by a comparative stability of the nuclear deformation so that its changes have only adiabatic influence on the motion of the individual nucleons. That is to say, there is a decoupling of the individual particle motion from the collective modes of motion that the nuclear shape can execute. These collective modes of motion are of two types: a vibrational one

involving changes of shape, generally with preservation of axial symmetry and the other a rotational one in which the nuclear shape merely changes its orientation with respect to a space fixed system of axes. The latter modes correspond simply to the motion of a symmetrical top, which has been well studied. It has energy eigenvalues of $\frac{1}{2\mathcal{J}} I(I+1)$, $\mathcal{J}$ being the effective moment of inertia about the symmetry axis and I the total angular momentum. The associated normalized eigenfunctions are $\sqrt{\frac{2I+1}{8\pi^2}}\, D^{I}_{MK}(\Theta_i)$, where the $D^{I}_{MK}(\Theta_i)$ is a function, defined in Section A.3, of the Euler angles of the nucleus in a space fixed system of coordinates. M is the component of the angular momentum I on the z-axis of the space fixed system and K that on the nuclear symmetry axis.

Strongly deformed nuclei may now be described in terms of their rotational and intrinsic states. Only the latter depends on the particle configuration: it will involve both the individual particle and the vibrational modes. Let us write the intrinsic wave function as χ_Ω. This is a function of all the particle coordinates in the nuclear fixed system; Ω is the component of the total particle angular momentum along the nuclear symmetry axis. Then the nuclear wave function in the space fixed system may be written as

$$\Psi^{M}_{I} = \sqrt{\frac{2I+1}{16\pi^2}} \left\{ \chi_\Omega D^{I}_{MK}(\Theta_i) + (-)^{I-j} \chi_{-\Omega} D^{I}_{M-K}(\Theta_i) \right\} \tag{39}$$

The second term here is introduced to preserve the necessary symmetry; the factor $(-)^j$ refers to the expansion of χ_Ω into eigenfunctions

of particle angular momentum j (see Moszkowski (1957)). In the event of K = 0 (39) reduces to

$$\Psi^{M}_{I} = \sqrt{\frac{2I+1}{8\pi^2}}\ \chi_{\Omega}\ D^{I}_{M\,0}\left(\Theta_i\right) \qquad (40)$$

and I is always even.

The first few excited states of a strongly deformed nucleus consist of pure rotational excitations, so that they all have the same intrinsic structure. Different rotational states based on the same intrinsic structure are said to form a rotational band. Their wave functions differ only in that they have different D-functions, which are, however, known. This immediately suggests that it should be possible to calculate the relative probability of transitions from a given parent state to the ground state and to the first excited state, i.e. the branching ratio of the two transitions.

This general problem has been considered by Alaga et al. (1955), who calculate the branching ratios of transitions (β or γ) of definite multipolarity to different members of the same rotational band. Referring to Appendix D, it will be seen that, since each matrix element has the same lepton coefficient in both branches, the ratio is

$$\frac{\omega\left(I'_f\right)}{\omega\left(I^2_f\right)} = \frac{\sum_{M_f}\left|\left\langle \Psi^{M_f}_{I'_f}\right| T^{M}_{L}(\omega)\left|\Psi^{M_i}_{I_i}\right\rangle\right|^2}{\sum_{M_f}\left|\left\langle \Psi^{M_f}_{I^2_f}\right| T^{M}_{L}(\omega)\left|\Psi^{M_i}_{I_i}\right\rangle\right|^2}$$

$$\left\{ \frac{C\begin{pmatrix} I_f' & I_i & L \\ K_f' & K_i & K_f'-K_i \end{pmatrix}}{C\begin{pmatrix} I_f^2 & I_i & L \\ K_f^2 & K_i & K_f^2-K_i \end{pmatrix}} \right\}^2 \qquad (28)$$

$$= \qquad (41)$$

In obtaining this last result use has been made of various properties of the Clebsch-Gordan coefficients listed in Appendix A. Thus, as was to be expected, the branching ratio is independent of the details of the nuclear structure and depends on what is essentially a geometrical factor.

It will be seen that the multipolarity has to be the same for both transitions; in the case of β-decay this means that they must both have the same degree of forbiddenness. Thus the applicability of (41) is restricted to the case of the parent nucleus being in the states 1^+ or 1^- so that the transitions to the 0^+ and 2^+ daughter states will both be allowed (unique) in the first case or first forbidden in the second. In both cases $L = 1$, $K_1 = 1$ and $K_f = 0$ whence (40) becomes

$$\frac{\omega\,(I_f = 0)}{\omega\,(I_f = 2)} = 2 \qquad (42)$$

This just gives the branching ratio for a particular electron energy. If the total energies of the two transitions differ appreciably then allowance must be made for this before the branching ratio we have defined can be compared with the ratio of the total decay rates. This may be difficult if the spectrum shape is not that of an allowed transition. It should also be noted that we have tacitly ignored the

contribution of the second rank tensor, B_{ij}, to the transition to the 2^+ state. Its contribution is known to be small, generally, but its possible significance must not be overlooked.

Now let us return to the case of quadrupole coupling between the 0^+ and 2^+ state of the daughter nucleus. Then a nucleus which is in a definite state, say 0^+, when the decay electron is remote from the nucleus, will actually have been in a mixture of the two states when the electron was close to it. Hence the relation (41) no longer holds and both the numerator and denominator there must be replaced by linear combinations of the form

$$a \langle I_f' | T_L^M(\omega) | I_i \rangle + b \langle I_f^2 | T_L^M(\omega) | I_i \rangle$$

where the a and b coefficients depend on the amount of coupling.

Thus it is in the departure of the branching ratios from Alaga's value (42) that quadrupole coupling effects of the kind considered will unequivocally reveal themselves, if they exist.

4. THE CASES.

The data presented by Alaga et al. (1955) indicates that the relation (41) holds well for electromagnetic transitions and also for several branched β -decays. However, they present two cases of a breakdown of the relation: Lu^{176} and Ta^{180}, which both have branching ratios $\simeq 1$. A possible, but unlikely, explanation of this is that the B_{ij} matrix element is enhancing the 2^+ transition.

We now consider the branched decay of Np^{236} (Gray (1956)). The transitions to both the 0^+ and 2^+ states have rates consistent

with first forbidden transitions; furthermore the spectrum is reported as having a strong distortion from the allowed shape. Hence we may assume that we are dealing with a branched decay of the kind that we are interested in. But the branching ratio of the two transitions is

$$\frac{w\,(I_f = 0)}{w\,(I_f = 2)} \simeq 4$$

This value has not been corrected for the difference in energies of the two transitions (the correction would be difficult to make in this case because the spectrum shape is not allowed) but this can be of little consequence since the energy difference is only about 8%.

Since here it is the ground state transition that is excessive the B_{ij} matrix element cannot be invoked. Now because rotational structure is particularly well developed in this mass number region (see, e.g. Hyde and Seaborg (1957)) there is here a prima facie case for a quadrupole coupling effect occurring in the β -decay of strongly deformed nuclei.

5. THE SCOPE OF THIS THESIS.

In this thesis we consider the coupling together of different angular momentum states of the decay electron and the daughter nucleus in the belief that this may be able to account for the above mentioned anomalies in the branching ratios.

A general expression for transition probabilities to states which have only asymptotic conservation of angular momenta is obtained in Chapter II. One of the major problems encountered there is that these states are non-orthogonal. In the next chapter we specialize

this expression to give the required branching ratio.

With $Q_o \simeq 10$ barns it is readily seen that at the nuclear surface the quadrupole interaction is nearly as large as the central interaction. One is therefore reluctant to use perturbation calculation of the electron functions and prefers an exact treatment. This is done in Chapter IV, where we set up systems of coupled differential equations which the electron radial functions satisfy exactly, the angular parts of the wave functions having been separated out. Some formal properties and the method of solution of these equations are considered in Chapter V.

The numerical work is described in Chapter VI, this being concerned mainly with the solution of the differential equations, which was performed on an electronic computer. We considered just the single case of Np^{236} and repeated the calculation for both a positive and a negative value of the quadrupole moment, since the sign does not appear to have been conclusively established for the transuranic elements (the fact that the anomalies in this region are in the opposite direction to those in the rare earth region suggests that Q_o may possibly be negative for the former). Because of the extreme lengthiness of the computation only one energy value was taken. However, we are concerned not so much with reproducing the measured branching ratio as with seeing whether or not the coupling of the kind we have described is at all significant. Our conclusions on this point are stated in the last chapter.

The angular momentum theory of which extensive use is made in this thesis is summarized in Appendix A.

CHAPTER II

THE TRANSITION PROBABILITY TO STATES OF COUPLED ANGULAR MOMENTA

1. THE BASIC STATES.

In the initial state the total energy, W^i , of the absorbed neutrino-parent nucleus system, the angular momenta of the parent nucleus, I_i , and of the neutrino, j_ν , and the neutrino energy, $-q$, are all conserved. We express the initial state in terms of a representation in which the z-component of these two angular momenta, M_i and μ_ν respectively, are diagonal and so write for the wave function of the basic state, $|I_i M_i, K_\nu \mu_\nu, q W^i\rangle$,

$$\Upsilon(K_\nu \mu_\nu, q) = \Psi^{M_i}_{I_i} A \psi^{\mu_\nu *}_{K_\nu}(q) \qquad (1)$$

where the parameters I_i , M_i and W^i have been omitted from the left hand side because we shall only be considering one state of the parent nucleus, $\Psi^{M_i}_{I_i}$. $A \psi^{\mu_\nu *}_{K_\nu}(q)$ is the wave function of the absorbed, negative energy neutrino (see Section 1. 2).

In the final state the angular momenta of the nucleus, I_f , and of the electron, j_e , and the energy of the electron, W_e , are none of them constants of motion when a quadrupole coupling is present. However, the complete final state system, electron plus daughter nucleus,

has its total angular momentum, J, the z-component thereof, M, and the total energy, W, conserved. We can therefore work in a representation in which J, M and W are diagonal and write

$$H|JMW\rangle = (H_{NUC} + H_e + H_Q)|JMW\rangle = W|JMW\rangle \qquad (2a)$$

in which H_e is the Dirac Hamiltonian of the electron in a central field and H_Q is the Hamiltonian of the quadrupole interaction between the nucleus and the electron.

The description of the final states is not yet complete for, as will be seen later, there remains a degeneracy which must correspond to the existence of other operators that commute with J, M and H. However, as we wish to remove this degeneracy by requiring that the electron wave function obeys certain asymptotic boundary conditions we shall not choose our representation to be one in which these extra operators, whatever they may be, are diagonal. Accordingly, we write our final state as $|JMW,\ell\rangle$ where ℓ is simply a label distinguishing between the degenerate states belonging to J, M and W that satisfy different asymptotic boundary conditions. (2a) then becomes

$$H|JMW,\ell\rangle = W|JMW,\ell\rangle \qquad (2b)$$

We expand these complete final states in terms of the discrete basic set consisting of the nuclear states that are coupled together, $\Psi_{I_f}^{M_f}$, and the angular momentum eigenfunctions of the electron. The angular part of these latter must be identical to the central field case whence from (1.26) they take the form

(34)

$$\psi^{\mu_e}_{K_e} = \begin{bmatrix} -i F(r)\, \chi^{\mu_e}_{-K_e} \\ G(r)\, \chi^{\mu_e}_{K_e} \end{bmatrix} \qquad (3)$$

in a system of space fixed axes.

The final states can then have their wave functions written as

$$\Upsilon(JMW, \ell) = \sum_{K_e \mu_e I_f} C\begin{pmatrix} J & j_e & I_f \\ M & \mu_e & M_f \end{pmatrix} \Psi^{M_f}_{I_f} \begin{bmatrix} -i F_{K_e}(JWI_f, \ell)\, \chi^{\mu_e}_{-K_e} \\ G_{K_e}(JWI_f, \ell)\, \chi^{\mu_e}_{K_e} \end{bmatrix} \qquad (4)$$

This represents a mixture of the nuclear states, each one of which is associated with all possible angular momentum states of the electron, (3). The angular dependence of these electron states is determined solely by angular momentum considerations but the complete states (4) still have to satisfy (2a). It will be seen in Chapter IV how the angular part can be removed from (2a), which then reduces to a system of first order coupled differential equations in the radial functions, $F_{K_e}(JWI_f, \ell)$, $G_{K_e}(JWI_f, \ell)$. The role of the parameters in these functions is self-evident; as this notation is somewhat cumbersome it will be condensed when no confusion can arise, but such changes will be announced. It should be noted that we have tacitly assumed that there is no radial dependence on M; this will be shown to be the case in Chapter IV.

In the limit of vanishing quadrupole these functions will pass over into the familiar relativistic Coulomb functions, f_{κ_e}, g_{κ_e}, (Section 1. 2). However, in general the electron states (3) will not be states of constant electron energy, because of the quadrupole coupling. An expansion in terms of central field solutions is possible but it would involve an integration over the continuum of electron energy. The present method is the more suitable.

It will be seen in Chapter V that the degeneracy in the states is equal to the number of central field states, labelled by (κ_e, I_f) , that are coupled together by the quadrupole interaction. Furthermore, it is noted that because the quadrupole potential falls off as fast as $1/r^3$ then these (κ_e, I_f) states uncouple asymptotically and it is, in fact, possible to remove the degeneracy by requiring that each $|JMW, \ell\rangle$ state belonging to a given (JMW) set of values asymptotically assumes the form of a different central field state (κ_e, I_f) . That is, we impose boundary conditions such that in any final state all but one of the pairs of F and G functions fall off faster than $1/r$ and this particular pair is different in each of the different states labelled by ℓ. We may say that the degeneracy is removed by requiring that the states have "asymptotic eigenvalues", κ_e^{ℓ}, I_f^{ℓ} . However, because these are not true eigenvalues the states so chosen will not necessarily be orthogonal, although they are certainly linearly independent.

The central field electron functions which our F and G functions approach in the region of vanishing quadrupole interaction are, of course, just particular linear combinations of regular and

irregular $f_{\kappa_e}(W_e), g_{\kappa_e}(W_e)$. We recall, then, from (1.24) that they have the asymptotic form:

$$f_{\kappa_e}(W_e) \sim -\frac{A}{r}\sqrt{\frac{W_e-1}{\pi\rho}}\sin\left(\rho r+\delta_e^c+\epsilon\right)$$

$$g_{\kappa_e}(W_e) \sim \frac{A}{r}\sqrt{\frac{W_e+1}{\pi\rho}}\cos\left(\rho r+\delta_e^c+\epsilon\right) \qquad (5)$$

where ϵ is a constant phase angle depending on the particular combination of regular and irregular solutions, and A is a constant factor that is unity for normalized central field functions.

Then, if we choose our states $|\bar{J}MW,\ell\rangle$ of the complete daughter system in the manner described, (4) will asymptotically become

$$\Upsilon(JMW,\ell) \sim \sum_{\mu_e}\bar{A}_{J,\ell}(W_\ell)\,C\begin{pmatrix} J & j_e^\ell & I_f^\ell \\ M & \mu_e & M_f\end{pmatrix}\Psi_{I_f^\ell}^{M_f}$$

$$\times\ \frac{1}{r}\left[\begin{array}{l} i\left(\dfrac{W_\ell-1}{\pi\rho_\ell}\right)^{\frac{1}{2}}\sin\left(\rho_\ell r+\delta_\ell^c+\epsilon_\ell\right)\chi_{-\kappa_e^\ell}^{\mu_e} \\ \left(\dfrac{W_\ell+1}{\pi\rho_\ell}\right)^{\frac{1}{2}}\cos\left(\rho_\ell r+\delta_\ell^c+\epsilon_\ell\right)\chi_{\kappa_e^\ell}^{\mu_e}\end{array}\right] \qquad (6)$$

Here $W_\ell = W - E_{I_f^\ell}$, $E_{I_f^\ell}$ being the excitation energy of the nuclear state of angular momentum I_f^ℓ, and ρ_ℓ and δ_ℓ^c are the values appropriate to the particular W_ℓ and κ_e^ℓ. $A_{J,\ell}(W_\ell)$ is just a numerical constant.

2. TIME DEPENDENT PERTURBATION THEORY.

The application of time dependent perturbation theory to this problem is complicated by the fact that not all the final states are orthogonal to each other. However, the necessary modifications to the usual form of the theory are worked out in Appendix B, from which it is seen that the only essential changes are in the form of the perturbation matrix element. It will be noticed there that we have allowed the states to fill the whole of space. If we had confined them to a finite enclosure then it will be recalled from Section 1.1 that a weighting factor, the state density, would have to be introduced. It is difficult to show that the appropriate density will be the one determined by periodic boundary conditions on the surface of the enclosure. We prefer to avoid the problem by treating our states as the true continuum states that they really are.

Because the β -decay Hamiltonian does not commute with the Hamiltonian of the free nucleon-lepton system there will as usual be a violation of energy conservation in the decay process. However, the interaction will not destroy any of the angular momentum constants of motion so we can write for the packet of final states obtained from the initial state $|I_i M_i, K_\nu \mu_\nu, q W^i\rangle$ after the perturbation has been switched on for time t

$$\Phi(JMW^i, l)\, e^{-iW^i t} = \int dW\, a(JMW, l)\, \Upsilon(JMW, l)\, e^{-iWt} \qquad (7)$$

Here the left hand side has been written in a way that anticipates the time independence of the $\bar{\Phi}$ factor (aside from a step-function - see (14)) and

$$a(JMW,\ell) = -\langle JMW,\ell \,|\, \mathcal{H} \,|\, I_i M_i, K_\nu \mu_\nu, q W^i \rangle \frac{e^{i(W-W^i)t} - 1}{W - W^i} \tag{8}$$

(8) has exactly the same time dependence as in the usual form of the theory for orthogonal states but the matrix element is no longer the straightforward matrix element of the perturbation, H_β, between the initial and final states. Rather is it a linear combination of these latter, thus:

$$\langle JMW,\ell \,|\, \mathcal{H} \,|\, I_i M_i, K_\nu \mu_\nu, q W^i \rangle = \sum_K \left\{ U^{-1} \right\}_{\ell K} \langle JMW, K \,|\, H_\beta \,|\, I_i M_i, K_\nu \mu_\nu, q W^i \rangle \tag{9}$$

where the matrix $\{U\}$ is given by

$$\{U\}_{mn}\, \delta(W'-W) = \langle JMW', m \,|\, JMW, n \rangle \tag{10}$$

Use has been made here of the fact that all final states are orthogonal to initial states and final states belonging to different values of J or M are also orthogonal to each other.

Now, because the final states $|JMW,\ell\rangle$ are not all orthogonal to each other, $|a(JMW,\ell)|^2$ is no longer the probability of finding the system in such a state. Hence we cannot follow the usual procedure of using the Golden Rule (see Section 1.1) for obtaining transition probabilities. Instead, we adopt the method of Greuling and Meeks

(1951) and Lee-Whiting (1958) and determine first the asymptotic form of the wave packet, $\Phi(JMW,\ell)$. Since the asymptotic form of the final states are similar to those of the central field case there will be no essential difference here and we shall find that the wave packet consists only of outgoing waves, the incoming waves removing themselves by destructive interference.

Then substituting into (7) the asymptotic form of $\Upsilon(JMW,\ell)$, (6), we have

$$\Phi(JMW,\ell) \sim -\frac{1}{2r}\int dW_\ell \; A_{J,\ell}(W_\ell)\,\frac{1-e^{i(W_\ell^\circ - q - W_\ell)}}{W_\ell + q - W_o^\ell}$$

$$\langle JMW,\ell \mid \mathcal{H} \mid I_i M_i, K_f \mu_f, q, W^i\rangle \sum_{\mu_\ell} C\begin{pmatrix} J & j_e & I_f^\ell \\ M & \mu_e & M_f \end{pmatrix} \underset{I_f^\ell}{\Psi}{}^{M_f}_{-} +$$

$$\left[\left(\frac{W_\ell - 1}{\pi p_\ell}\right)^{\frac{1}{2}}\left\{e^{i(p_\ell r + \delta_\ell^c + \epsilon_\ell)} - e^{-i(p_\ell r + \delta_\ell^c + \epsilon_\ell)}\right\}\chi^{\mu_e}_{-K_e^e}\right.$$

$$\left.\left(\frac{W_\ell + 1}{\pi p_\ell}\right)^{\frac{1}{2}}\left\{e^{i(p_\ell r + \delta_\ell^c + \epsilon_\ell)} + e^{-i(p_\ell r + \delta_\ell^c + \epsilon_\ell)}\right\}\chi^{\mu_e}_{K_e^\ell}\right] \qquad (11)$$

where we have made use of the relations

$$W^i = E_{I_i} - q$$
$$W = E_{I_f^l} + W_l$$

and have written for the total energy available to the leptons in a decay to the state I_f^l :

$$W_o^l = E_{I_i} - E_{I_f^l}$$

The integrand in (11) has a sharp maximum in the vicinity of $W_l = W_o^l - q$ i.e. $W = W^i$ and therefore we take outside of the integral all the factors that are slowly varying in W_l . We are then left with the following integrals, which are evaluated in Appendix C:

$$\int dW_l \frac{1 - e^{i(W_o^l - q - W_l)t}}{W_l - W_o^l + q} e^{i p_l r} = \begin{cases} 2\pi i e^{i \bar{p}_l r} & \text{if } 1 \ll r < \frac{\bar{p}_l}{\bar{W}_l} t \\ 0 & \text{if } 1 \ll r > \frac{\bar{p}_l}{\bar{W}_l} t \end{cases}$$

$$\int dW_l \frac{1 - e^{i(W_o^l - q - W_l)t}}{W_l - W_o^l + q} e^{-i p_l r} = 0 \qquad (12)$$

in which $\bar{W}_l = W_o^l - q$ and $\bar{p}_l$ is the corresponding momentum. Thus we see that the wave packet behaves asymptotically as a train of outgoing spherical waves of energy corresponding to energy conservation and moving with a velocity equal to that of the electron. In subsequent work the bars on $\bar{p}_l$ and $\bar{W}_l$ will be dropped, it being understood that

these are the values appropriate to energy conservation.

The complete asymptotic expression for the packet of final states becomes

$$\underline{\Phi}\left(JMW,\ell\right) = A_{J,\ell}\left(W_\ell\right)\left\langle JMW,\ell \,\middle|\, \mathcal{H} \,\middle|\, I_i M_i K_\nu \mu_\nu q W^i \right\rangle$$

$$\times \frac{\pi}{r}\, e^{i\left(\rho_\ell r + \delta_\ell^c + \epsilon_\ell - \frac{\pi}{2}\right)} \sum_{\mu_\ell} C\begin{pmatrix} J & j_e^\ell & I_f^\ell \\ M & \mu_e & M_f \end{pmatrix} \Psi_{I_f^\ell}^{M_f} \left[\begin{array}{c} \left(\frac{W_\ell - 1}{\pi \rho_\ell}\right)^{\frac{1}{2}} \chi_{-K_e}^{\mu_e} \\ \left(\frac{W_\ell + 1}{\pi \rho_\ell}\right)^{\frac{1}{2}} \chi_{K_e}^{\mu_e} \end{array}\right]$$

$$\text{for} \quad 1 \ll r < \frac{\rho_\ell}{W_\ell} t$$

$$= 0$$

$$\text{for} \quad 1 \ll r > \frac{\rho_\ell}{W_\ell} t \qquad (13)$$

This result is similar to the expression obtained by Greuling and Meeks (1951). However, as pointed out by Lee-Whiting (1958), an incorrect sign of the exponent occurring in equation (9) of the former leads to an apparent cancellation of the outgoing rather than the incoming wave.

3. FORMULATION OF THE TRANSITION PROBABILITY.

Let us take the system at a certain instant to be in the initial

state $\left| I_i M_i, K_\nu \mu_\nu, q W^i \right\rangle$. Suppose that at time t later β-decay to one of the states belonging to the packet $\Phi(JMW^i, \ell)$ has occurred, i.e. to one of the states characterized by quantum numbers J, M and by asymptotic quantum numbers K_e^ℓ, I_f^ℓ. Then the probability of finding the electron in a volume element $d\tau = r'^2 \sin\theta' d\theta' d\phi' dr'$ at the point $\underset{\sim}{r}'$ is

$$\left| \Phi(JMW^i, \ell) \right|^2_{\underset{\sim}{r} = \underset{\sim}{r}'} d\tau$$

so that the probability of the electron being at a distance between r' and $r' + dr$ is

$$P(JMW^i, \ell; r', t)\, dr = \int \sin\theta\, d\theta\, d\phi \left| \Phi(JMW^i, \ell) \right|^2_{r = r'} r'^2 dr \qquad (14)$$

If we require that r' be large then by substitution from (13) this reduces to

$$P(JMW^i, \ell; r', t) =$$

$$2\pi \frac{W_\ell}{\rho_\ell} \left| A_{I,\ell}(W_\ell) \left\langle JMW, \ell \right| \mathcal{H} \left| I_i M_i, K_\nu \mu_\nu q W^i \right\rangle \right|^2$$

$$\text{if } r < \frac{\rho_\ell}{W_\ell} t$$

$$= \quad 0$$

$$\text{if} \quad r \; > \; \frac{P_\ell}{W_\ell} \, t$$

(15)

But in determining transition rates we are interested in the probability of finding an electron at <u>any</u> point in space, not at one particular distance. Now (15) shows that $P(JMW^i, \ell; r, t)$ is independent of r, as long as this is large but smaller than $\frac{P_\ell}{W_\ell} t$. Outside of the sphere of radius $\frac{P_\ell}{W_\ell} t$ there are no electrons. Thus as time elapses there will be an increasing volume in which electrons are to be found, i.e. transitions will continue to occur. It is now readily seen that the transition rate from the particular initial state to the states belonging to the packet $\Phi(JMW^i, \ell)$ is

$$\omega(JMW^i, \ell) = \frac{P_\ell}{W_\ell} \times P(JMW^i, \ell; r, t)$$

$$= 2\pi \left| A_{J,\ell}(W_\ell) \langle JMW, \ell \,|\, \mathcal{H} \,|\, I_i M_i, K_\nu \mu_\nu, q W^i \rangle \right|^2$$

(16)

This result is equivalent to the statement that the asymptotic current is equal to the product of the particle density and the particle velocity. However, the use of such a standard result required special justification here because of the non-orthogonality of the states.

The summation over the initial states proceeds exactly as in the standard theory (see Section 1. 2). We also have to sum over the various uninteresting final state quantum numbers, J, M and K_e^{ℓ}, whence the rate of transitions that leave the daughter nucleus in a state I_f when the decay electron is far away is

$$\omega(I_f) = 2\pi \int^{W_o(I_f)} dW_\ell \sum_{\substack{J,M,\ell \\ K_\nu,\mu_\nu}} \left| \delta_{I_f^{\ell} I_f} A_{J,\ell}(W_e) \times \right.$$

$$\left. \langle J M W, \ell \,|\, \mathcal{H} \,|\, I_i M_i, K_\nu \mu_\nu, q W^i \rangle \right|^2 \qquad (17)$$

Here we have written W_e for W_ℓ and $W_o(I_f)$ for W_o^{ℓ} since we are only summing over states for which $I_f^{\ell} = I_f$

We now show that in the limit of vanishing quadrupole moment (17) reduces to the standard central field expression. In the central field case the final states $|J M W_i^{\nu}, \ell\rangle$ are orthogonal and the matrix U given by (10) will be diagonal:

$$U_{\ell K} = \delta_{\ell K} A^2_{J,\ell}(W_\ell)$$

Furthermore, the electron functions will no longer depend on J so the perturbation matrix (9) becomes

$$\left\langle JMW^i,\ell \,\middle|\, \mathcal{H} \,\middle|\, I_i M_i, K_\nu \mu_\nu, q W^i \right\rangle =$$

$$\frac{1}{2} \frac{1}{A_{J,\ell}(W_e)} \sum_{\mu_\ell} C\begin{pmatrix} J & j_e^\ell & I_f^\ell \\ M & \mu_e & M_f \end{pmatrix} \left\langle \mathcal{T}_{I_f^\ell}^{M_f} \psi_{K_e^\ell}^{\mu_e}(W_e) \,\middle|\, H_\beta \,\middle|\, \mathcal{T}_{I_i}^{M_i} A \psi_{K_\nu}^{\mu_\nu^*}(q) \right\rangle$$

and (17) reduces to

$$\omega(I_f) = 2\pi \int_1^{W_0(I_f)} dW_\ell \sum_{\substack{J, M, K_e \\ K_\nu, \mu_\nu}} \left| \sum_{\mu_e} \frac{1}{A_{J,K_e}(W_e)} \right.$$

$$\left. \times C\begin{pmatrix} J & j_e & I_f \\ M & \mu_e & M_f \end{pmatrix} \left\langle \mathcal{T}_{I_f}^{M_f} \psi_{K_e}^{\mu_e}(W_e) \,\middle|\, H_\beta \,\middle|\, \mathcal{T}_{I_i}^{M_i} A \psi_{K_\nu}^{\mu_\nu^*}(q) \right\rangle \right|^2 \tag{18}$$

Now we recall that for normalized electron functions $A_{J,K_e}(W_e)$ is unity so that in (18) we can cancel it with a similar factor in $\psi_{K_e}^{\mu_e}(W_e)$ which must thereafter be regarded as normalized. Finally, using orthogonality conditions (see Appendix A) for the Clebsch-Gordan coefficients we have the familiar expression

$$\omega(I_f) = 2\pi \int_1^{W_o(I_f)} dW_e \sum_{\substack{K_e K_\nu \\ \mu_e \mu_\nu M_f}} \left| \left\langle \Psi_{I_f}^{M_f} \psi_{K_e}^{\mu_e}(W_e) \middle| H_\beta \middle| \Psi_{I_i}^{M_i} A \psi_{K_\nu}^{\mu_\nu}(q)^* \right\rangle \right|^2$$

which is just (1.30) integrated over the electron energy, W_e. Thus not only is (17) checked for this special case but it can also be seen how the matter of normalization is handled automatically in the general case by the U matrix and the $A_{J,\ell}(W_e)$ factor.

CHAPTER III

BRANCHING RATIOS IN ROTATIONAL BANDS

In the last chapter we obtained the general expression (2.17) for β-decay transition probabilities to mixtures of the daughter nuclear states, the different states being coupled together by their quadrupole interaction with the decay electron and only separating when the decay electron is far away. In this chapter we specialize this result to the case of particular interest: that of first forbidden transitions of strongly deformed nuclei in which the coupled daughter states belong to the same rotational band.

It is convenient to rewrite the generalized matrix element (2.9) as

$$\langle J M W, \ell \,|\, \mathcal{H} \,|\, I_i M_i; k_\nu \mu_\nu, q W^i \rangle =$$

$$\sum_{k_e \mu_e I_f} C\begin{pmatrix} J & j_e & I_f \\ M & \mu_e & M_f \end{pmatrix} \left\langle \Psi_{I_f}^{M_f} \psi_{k_e}^{\mu_e}(JWI_f, \ell) \,\middle|\, H_\beta \,\middle|\, \Psi_{I_i}^{M_i} A \psi_{k_\nu}^{\mu_\nu}(q)^* \right\rangle \tag{1}$$

where we have used (2.4) and written

$$\sum_K \left\{U^{-1}\right\}_{\ell K} \begin{bmatrix} -i F_{K_e}(JWI_f, K)\, \chi^{\mu_e}_{-K_e} \\ G_{K_e}(JWI_f, K)\, \chi^{\mu_e}_{K_e} \end{bmatrix} =$$

$$\begin{bmatrix} -i f_{K_e}(JWI_f, \ell)\, \chi^{\mu_e}_{-K_e} \\ g_{K_e}(JWI_f, \ell)\, \chi^{\mu_e}_{K_e} \end{bmatrix} = \psi^{\mu_e}_{K_e}(JWI_f, \ell) \qquad (2)$$

The f, g and ψ functions appearing here are distinguished from the similar functions appearing in the central field case by means of the extra parameters; however, in the central field limit they become identical to the latter.

(2.17) now becomes

$$\omega(I_f) = 2\pi \int_1^{W_0(I_f)} dW_e \sum_{\substack{J,M,\ell \\ K_\nu,\mu_\nu}} \Big| S_{I_f \ell I_f}\, A_{J,\ell}(W_\ell) \times$$

$$\sum_{K_e \mu_e I_f} C\begin{pmatrix} J & j_e & I_f \\ M & \mu_e & M_f \end{pmatrix} \left\langle \Psi^{I_f}_{M_f} \psi^{\mu_e}_{K_e}(JWI_f, \ell) \Big| H_\beta \Big| \Psi^{M_i}_{I_i} A \psi^{\mu_\nu}_{K_\nu}(8)^* \right\rangle \Big|^2 \qquad (3)$$

The β -decay matrix element in (3) has exactly the same form as that appearing in (1.30) for the standard form of the theory, the only difference being in the radial part of the electron functions. Thus a multipole expansion in terms of matrix elements of the various spherical tensor operators between the initial and final nuclear states is possible, as before. Furthermore, it is now asserted that the degree of forbiddenness approximation described in Section 1.2 holds good here. That is, although lepton states that in the central field limit would give a dominant contribution to the transition probability are here coupled in with electron states corresponding to higher degrees of forbiddenness, these latter states are not sufficiently different from their central field value to give a contribution of any significance. Because of the great difference in magnitude of the contributions associated with the different orders of forbiddenness the necessary distortion of the electron states would require an extremely large quadrupole coupling. However, the final justification for our assertion must come from a knowledge of the numerical value of the electron functions (see Chapter VI.)

Thus in the present case we shall have the same β -decay matrix elements of the form

$$\left\langle \Psi_{I_f}^{M_f} \psi_{\kappa_e}^{\mu_e}(JWI_f, l) \middle| H_\beta \middle| \Psi_{I_i}^{M_i} A \psi_{\kappa_\nu}^{\mu_\nu *}(q) \right\rangle \tag{4}$$

as appear in the central field case. Secondly, the multipole expansion of each of these matrix elements will also be essentially the same as there, with the difference that the central field functions $f_{k_e}(w_e)$, $g_{k_e}(w_e)$ are here replaced by the $f_{k_e}(JWI_f, l)$, $g_{k_e}(JWI_f, l)$ of (2).

Let us now consider the cases of special interest: the first forbidden $1^- \rightarrow 0^+$ and $1^- \rightarrow 2^+$ transitions, which have already been considered to some extent for the central field limit in Section 1.2. It will be recalled from Section 1.4 that it was decided to ignore the contribution of the second rank tensor, $T^M_{21}(\underset{\sim}{r}, \underset{\sim}{\sigma})$ that occurs in the second transition. Thus the only terms in the multipole expansion of matrix elements (4) that we take are those containing the nuclear matrix elements of $Y^M_1(\underset{\sim}{r})$, $T^M_{11}(\underset{\sim}{r}, \underset{\sim}{\sigma})$, $T^M_{10}(\underset{\sim}{r}, \underset{\sim}{\alpha})$. The lepton states that give rise to these operators are

$$\begin{aligned} &k_e = 1 \quad k_\nu = 1, && k_e = -1 \quad k_\nu = -1, \\ &k_e = 2 \quad k_\nu = -1, && k_e = -2 \quad k_\nu = 1, \\ &k_e = 1 \quad k_\nu = -2, && k_e = -1 \quad k_\nu = 2. \end{aligned}$$

Then the coefficient of each of the three nuclear matrix elements

$$\langle Y^M_1(\underset{\sim}{r}) \rangle, \quad \langle T^M_{11}(\underset{\sim}{r}, \underset{\sim}{\sigma}) \rangle, \quad \langle T^M_{10}(\underset{\sim}{r}, \underset{\sim}{\alpha}) \rangle$$

occurring in the multipole expansion of every matrix element (4) corresponding to each of the above lepton states with all possible magnetic quantum numbers μ_e, μ_ν, M_f has to be calculated in the manner outlined in Section 1.2. Because there are thirty-two such matrix element expansions to be performed we do not list the results obtained. However, they were checked by showing that they yielded the correct

central field first forbidden transition probability (Konopinski and Uhlenbeck (1941), Pursey (1951)).

We now take cognizance of the fact that our nuclear states have their collective rotational motion and intrinsic particle structure completely separated. Thus the nuclear matrix elements, which we write in general as

$$\left\langle \Psi^{M_f}_{I_f} \middle| T^{M}_{L}(\omega) \middle| \Psi^{M_i}_{I_i} \right\rangle$$

can be factored into a product of two matrix elements, one involving only the rotational motion and the other only the intrinsic structure. Since the former can be evaluated we are left essentially with intrinsic matrix elements, $M_L(\omega)$ of $T^{M}_{L}(\omega)$. In Appendix D it is shown that

$$\left\langle \Psi^{M_f}_{I_f} \middle| T^{M}_{L}(\omega) \middle| \Psi^{M_i}_{I_i} \right\rangle =$$

$$\sqrt{\frac{1}{2I_f+1}}\, C\begin{pmatrix} I_f & I_i & L \\ M_f & M_i & m \end{pmatrix} C\begin{pmatrix} I_f & I_i & L \\ 0 & K_i & -K_i \end{pmatrix} M_L(\omega) \qquad (5)$$

In our problem the two daughter states both belong to the same rotational band so the intrinsic matrix elements $M_L(\omega)$ will be the same for both transitions.

Let us now write (3) as

$$\omega(I_f) = 2\pi \int_1^{W_0(I_f)} dW_e \sum_{J,\ell} \delta_{I_f^{\ell} I_f}\, A^{2}_{J,\ell}\, Q_{J,\ell} \qquad (6)$$

where $Q_{J\,l}$ can be expressed in terms of the intrinsic matrix elements $M_1(\underset{\sim}{r})$, $M_{11}(\underset{\sim}{r},\underset{\sim}{\sigma})$, $M_{10}(\underset{\sim}{r},\underset{\sim}{\alpha})$. Now because angular momentum is conserved during β-decay

$$\underset{\sim}{I_i} + \underset{\sim}{j_\nu} = \underset{\sim}{J}$$

and since $I_i = 1$ and the significant values of j_ν are $\frac{1}{2}$, $\frac{3}{2}$ it follows that the J values involved will be

$$J = \frac{1}{2}, \quad \frac{3}{2}, \quad \frac{5}{2}$$

If we write the electron functions as $f_{\kappa_e}(J I_f l)$, $g_{\kappa_e}(J I_f l)$, the W being superfluous, we have

$$Q_{\frac{1}{2},\,l} =$$

$$\frac{4}{243}\left\{ f_1^2\left(\tfrac{1}{2}\,0\,l\right) + g_{-1}^2\left(\tfrac{1}{2}\,0\,l\right)\right\} q^4 \left| \sqrt{\frac{1}{4\pi}}\, M_1(\underset{\sim}{r}) - \sqrt{\frac{1}{6}}\, M_{11}(\underset{\sim}{r},\underset{\sim}{\sigma}) \right|^2$$

$$+$$

$$\left| \frac{1}{3}\left[-i\sqrt{\frac{1}{6\pi}} \left\{ \frac{1}{3} q^2 f_1\left(\tfrac{1}{2}\,0\,l\right) + \frac{1}{\rho} q\, g_1\left(\tfrac{1}{2}\,0\,l\right)\right\} M_1(\underset{\sim}{r}) \right.\right.$$

$$+ \sqrt{\frac{2}{3}}\, q\, f_1\left(\tfrac{1}{2}\,0\,l\right) M_{10}(\underset{\sim}{r},\underset{\sim}{\alpha}) - i\,\frac{2}{3}\left\{ \frac{1}{3} q^2 f_1\left(\tfrac{1}{2}\,0\,l\right) - \frac{1}{\rho} q\, g_1\left(\tfrac{1}{2}\,0\,l\right)\right\} M_{11}(\underset{\sim}{r},\underset{\sim}{\sigma}) \Big]$$

$$\left. + \frac{iq}{\rho}\,\frac{1}{3\sqrt{3}}\, g_{-2}\left(\tfrac{1}{2}\,2\,l\right)\left[\sqrt{\frac{1}{4\pi}}\, M_1(\underset{\sim}{r}) + \sqrt{\frac{1}{6}}\, M_{11}(\underset{\sim}{r},\underset{\sim}{\sigma})\right] \right|^2$$

$$+ \left| \frac{1}{3}\left[-i\sqrt{\frac{1}{6\pi}} \left\{ \frac{1}{3} q^2 g_{-1}\left(\tfrac{1}{2}\,0\,l\right) - \frac{1}{\rho} q\, f_{-1}\left(\tfrac{1}{2}\,0\,l\right)\right\} M_1(\underset{\sim}{r}) \right.\right.$$

$$+\sqrt{\frac{2}{3}}\, q\, g_{-1}\left(\tfrac{1}{2}0\ell\right) M_{10}\left(\hat{r},\hat{\alpha}\right) - i\frac{2}{3}\left\{\frac{1}{3}q^2 g_{-1}\left(\tfrac{1}{2}0\ell\right) + \frac{1}{\rho} q f_{-1}\left(\tfrac{1}{2}0\ell\right)\right\} M_{11}\left(\hat{r},\hat{\sigma}\right)\Bigg]$$

$$+\frac{iq}{\rho}\frac{1}{3\sqrt{3}} f_2\left(\tfrac{1}{2}2\ell\right)\left[\sqrt{\frac{1}{4\pi}}\, M_1\left(\hat{r}\right) + \sqrt{\frac{1}{6}}\, M_{11}\left(\hat{\sigma},\hat{r}\right)\right]\Bigg|^2$$

$$Q_{\frac{3}{2},\ell} =$$

$$\frac{1}{1215}\left\{f_1^2\left(\tfrac{3}{2}2\ell\right) + g_{-1}^2\left(\tfrac{3}{2}2\ell\right)\right\} q^4 \left|\sqrt{\frac{1}{4\pi}}\, M_1\left(\hat{r}\right) - \sqrt{\frac{1}{6}}\, M_{11}\left(\hat{r},\hat{\sigma}\right)\right|^2$$

$$+\Bigg|\frac{1}{3\sqrt{2}}\Bigg[-i\sqrt{\frac{1}{6\pi}}\left\{\frac{1}{3}q^2 f_1\left(\tfrac{3}{2}2\ell\right) + \frac{1}{\rho} q g_1\left(\tfrac{3}{2}2\ell\right)\right\} M_1\left(\hat{r}\right)$$

$$+\sqrt{\frac{2}{3}}\, q f_1\left(\tfrac{3}{2}2\ell\right) M_{10}\left(\hat{r},\hat{\alpha}\right) - i\frac{2}{3}\left\{\frac{1}{3}q^2 f_1\left(\tfrac{3}{2}2\ell\right) - \frac{1}{\rho} q g_1\left(\tfrac{3}{2}2\ell\right)\right\} M_{11}\left(\hat{r},\hat{\sigma}\right)\Bigg]$$

$$-\frac{iq}{\rho}\left\{\frac{2}{3\sqrt{3}} g_{-2}\left(\tfrac{3}{2}0\ell\right) + \frac{1}{3\sqrt{3}} g_{-2}\left(\tfrac{3}{2}2\ell\right)\right\}\left[\sqrt{\frac{1}{4\pi}}\, M_1\left(\hat{r}\right) + \sqrt{\frac{1}{6}}\, M_{11}\left(\hat{r},\hat{\sigma}\right)\right]\Bigg|^2$$

$$+\Bigg|\frac{1}{3\sqrt{2}}\Bigg[-i\sqrt{\frac{1}{6\pi}}\left\{\frac{1}{3}q^2 g_{-1}\left(\tfrac{3}{2}2\ell\right) - \frac{1}{\rho} q f_{-1}\left(\tfrac{3}{2}2\ell\right)\right\} M_1\left(\hat{r}\right)$$

$$+\sqrt{\frac{2}{3}}\, q g_{-1}\left(\tfrac{3}{2}2\ell\right) M_{10}\left(\hat{r},\hat{\alpha}\right) - i\frac{2}{3}\left\{\frac{1}{3}q^2 g_{-1}\left(\tfrac{3}{2}2\ell\right) + \frac{1}{\rho} q f_{-1}\left(\tfrac{3}{2}2\ell\right)\right\} M_{11}\left(\hat{r},\hat{\sigma}\right)\Bigg]$$

$$-\frac{iq}{\rho}\left\{\frac{2}{3\sqrt{3}} f_2\left(\tfrac{3}{2}0\ell\right) + \frac{1}{3\sqrt{3}} f_2\left(\tfrac{3}{2}2\ell\right)\right\}\left[\sqrt{\frac{1}{4\pi}}\, M_1\left(\hat{r}\right) + \sqrt{\frac{1}{6}}\, M_{11}\left(\hat{r},\hat{\sigma}\right)\right]\Bigg|^2$$

$$Q_{\frac{5}{2},\ell} =$$

$$\frac{1}{135}\left\{f_1^2\left(\tfrac{5}{2}2\ell\right) + g_{-1}^2\left(\tfrac{5}{2}2\ell\right)\right\} q^4 \left|\sqrt{\frac{1}{4\pi}}\, M_1\left(\hat{r}\right) - \sqrt{\frac{1}{6}}\, M_{11}\left(\hat{r},\hat{\sigma}\right)\right|^2$$

The electron functions are to be evaluated at the nuclear radius ρ . To expand the squares that appear here would be extremely tedious and the resultant expression very lengthy and no more illuminating. (7) is quite adequate for any numerical calculation of the transition probability.

In the central field limit

$$f_{\kappa_e}(JWI_f, \ell) = \delta_{I_f \ell I_f}\, \delta_{\kappa_e \kappa_e^{\ell}}\, f_{\kappa_e}(W_e)$$

and likewise for the g functions. With $A_{J,\ell}(W_e)$ being unity it is easy to check that (7) substituted into (6) yields

$$\frac{w(I_f = 0)}{w(I_f = 2)} = 2$$

which is the result of Alaga et al. (1955), quoted in Section 1. 3 .

It will be seen that it is only in this limit that the branching ratio will be independent of the intrinsic matrix elements.

CHAPTER IV

THE COUPLED EQUATIONS OF THE ELECTRON WAVE FUNCTIONS

In this chapter we reduce the energy eigenvalue equation, (2.2a), of the electron-daughter nuclear states to a system of coupled differential equations in the electron radial functions.

Firstly, we recognize the separation of the collective rotational motion and the intrinsic structure of the nuclear states, according to (1.40). As we are dealing with just one rotational band only one intrinsic state, χ , is involved, so (2.2a) can be rewritten as

$$\left(H_{NUC} + H_e + H_Q\right) \left| \Phi(JMW)\,\chi \right\rangle = W \left| \Phi(JMW)\,\chi \right\rangle \tag{1}$$

where, from (2.4) and (1.40)

$$\Phi(JMW) = \sum_{\chi_e \mu_e I_f} \sqrt{\frac{2I_f+1}{8\pi^2}}\, C\begin{pmatrix} J & j_e & I_f \\ M & \mu_e & M_f \end{pmatrix} \begin{bmatrix} -i\,F_{\chi_e}(JWI_f)\,\chi^{\mu_e}_{-\chi_e} \\ G_{\chi_e}(JWI_f)\,\chi^{\mu_e}_{\chi_e} \end{bmatrix} D^{I_f}_{M_f 0}(\Theta_i) \tag{2}$$

It will be seen that we have dropped the label ℓ , since this refers simply to the degeneracy. However, a possible dependence of the electron radial function on M is allowed for.

Separating the nuclear Hamiltonian, thus:

$$H_{NUC} = H_{ROT} + H_{INT}$$

gives

$$H_{NUC}\left|\Phi(JMW)\chi\right\rangle = H_{ROT}\left|\Phi(JMW)\chi\right\rangle + E_0\left|\Phi(JMW)\chi\right\rangle$$

since the rotational band in question is the lowest i.e. the one based on $I_f = 0$. Then if we write

$$W - E_0 = W^* \tag{3}$$

and premultiply both sides of (1) with the conjugate of the intrinsic state vector, $\langle\chi|$, we have

$$\left(H_{ROT} + H_e + H_Q\right)\left|\Phi(JMW)\right\rangle = W^*\left|\Phi(JMW)\right\rangle \tag{4}$$

This is the equation that we now have to concern ourselves with. In what follows just two daughter states are considered, $I_f = 0$ and 2. We shall have to draw heavily on standard results of the quantum theory of angular momentum, a summary of which is included in Appendix A. However, we shall only refer specifically to this when ambiguity might otherwise occur.

Since the angular part of $\Phi(JMW)$ is known, being determined by angular momentum considerations, we attempt to remove it from (1). Let us introduce the quantities

$$\pm\Omega^{I_f' K_e'}_{J'M'} = \sum_{\mu_e'} \sqrt{\frac{2I_f'+1}{8\pi^2}}\, C\begin{pmatrix} J' & j_e' & I_f' \\ M' & \mu_e' & M_f' \end{pmatrix}\begin{bmatrix} \chi^{\mu_e'}_{-K_e'} \\ \chi^{\mu_e'}_{K_e} \end{bmatrix} D^{I_f'}_{M_f' 0}(\Theta_i) \tag{5}$$

and form from (1)

$$\left\langle \pm\Omega^{I_f' K_e'}_{J'M'}\right| H_{ROT} + H_e + H_Q \left|\Phi(JMW)\right\rangle = W^*\left\langle \pm\Omega^{I_f' K_e'}_{J'M'}\right|\left.\Phi(JMW)\right\rangle \tag{6}$$

Using the orthogonality conditions for the Clebsch-Gordan coefficients and the orthonormality of the Pauli spinors, $\chi^{\mu_e}_{\pm\kappa_e}$, and of the rotational wave functions,

$$\sqrt{\frac{2I_f+1}{8\pi^2}}\; D^{I_f}_{M_f 0}(\Theta_i)$$

(6) becomes

$$\left\langle \pm\Omega^{I_f'\kappa_e'}_{J'M'} \middle| H_{ROT} + H_e + H_Q \middle| \Phi(JMW) \right\rangle = \delta_{MM'}\delta_{JJ'} W^* \mathcal{R}^{\pm}_{\kappa_e'}(JM'WI_f') \quad (7)$$

where

$$\mathcal{R}^{\pm}_{\kappa_e}(JMWI_f) = -i F_{\kappa_e}(JMWI_f) \pm G_{\kappa_e}(JMWI_f) \quad (8)$$

We now consider each of the three terms on the left hand side of (7) in turn.

(i) H_{ROT}.

$$H_{ROT}\left|\Phi(JMW)\right\rangle = E_2^* \sum_{\kappa_e\mu_e} \sqrt{\frac{5}{8\pi^2}}\, C\begin{pmatrix} J & j_e & 2 \\ M & \mu_e & M_f \end{pmatrix} \psi^{\mu_e}_{\kappa_e}(JMW2)\, D^{2}_{M_f 0}(\Theta_i)$$

where $$E_2^* = E_2 - E_0$$

The various orthogonality and normalization conditions then give

$$\left\langle \pm\Omega^{I_f'\kappa_e'}_{J'M'} \middle| H_{ROT} \middle| \Phi(JMW) \right\rangle = \delta_{MM'}\delta_{JJ'}\delta_{I_f'2} E_2^* \mathcal{R}^{\pm}_{\kappa_e'}(J'M'W2) \quad (9)$$

(ii) H_e.

As H_e is the Dirac Hamiltonian in a central field, V_c, and the $\psi^{\mu_e}_{\kappa_e}(JMWI_f)$ are eigenstates of angular momentum, $H_e\psi^{\mu_e}_{\kappa_e}(JMWI_f)$ may be evaluated exactly as in the central field problem (see, e.g., pp.

334-5 of Schiff (1955)).

$$H_e \psi^{\mu_e}_{\kappa_e}(JMWI_f) =$$

$$\left[-i \left\{ (V_c - 1) F_{\kappa_e}(JMWI_f) + \frac{dG_{\kappa_e}(JMWI_f)}{dr} + (\kappa_e + 1) \frac{G_{\kappa_e}(JMWI_f)}{r} \right\} \chi^{\mu_e}_{-\kappa_e} \right.$$

$$\left. \left\{ (V_c + 1) G_{\kappa_e}(JMWI_f) - \frac{dF_{\kappa_e}(JMWI_f)}{dr} + (\kappa_e - 1) \frac{F_{\kappa_e}(JMWI_f)}{r} \right\} \chi^{\mu_e}_{\kappa_e} \right]$$

Then

$$\left\langle \pm \Omega^{I_f' \kappa_e'}_{J'M'} \middle| H_e \middle| \Phi(JMW) \right\rangle = \delta_{MM'} \delta_{JJ'} \int^{\pm}_{\kappa_e'} (J'M'WI_f') \tag{10}$$

where

$$\int^{\pm}_{\kappa_e}(JMWI_f) = \pm \left\{ (V_c + 1) G - \frac{dF}{dr} + (\kappa_e - 1) \frac{F}{r} \right\} - i \left\{ (V_c - 1) F + \frac{dG}{dr} + (\kappa_e + 1) \frac{G}{r} \right\} \tag{11}$$

(iii) H_Q

The Hamiltonian of the quadrupole interaction with a negaton is given by (1.38) as

$$H_Q = - \frac{e^2}{2} Q_0 f(r) P_2(\cos \theta')$$

where θ' is the polar angle of the electron position vector with respect to the symmetry axis of the nucleus. Let (θ, ϕ) and (θ'', ϕ'') be the angular coordinates of the electron position vector and the symmetry axis of the nucleus, respectively, in a space fixed system. Then with the spherical harmonics addition theorem (A.18) we have

$$H_Q = -\frac{2\pi}{5} e^2 Q_0 f(r) \sum_m Y_2^{m*}(\theta'', \phi'') Y_2^m(\theta, \phi) \qquad (12)$$

It can easily be shown with the aid of (A.18) and (A.19) that

$$D^{\ell}_{m0}(\Theta_i) = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m(\theta'', \phi'') \qquad (13)$$

Then we can rewrite (2) and (5) to give

$$\Phi(JMW) = \sum_{K_e \mu_e} \left\{ \sqrt{\frac{1}{8\pi^2}}\, \delta_{J j_e} \delta_{M \mu_e} \psi_{K_e}^{\mu_e}(JMW0) \right.$$

$$\left. + \sqrt{\frac{1}{2\pi}}\, C\begin{pmatrix} J & j_e & 2 \\ M & \mu_e & m \end{pmatrix} Y_2^m(\theta'', \phi'')\, \psi_{K_e}^{\mu_e}(JMW2) \right\}$$

$$\pm \Omega^{0K_e'}_{J'M'} = \sqrt{\frac{1}{8\pi^2}}\, \delta_{J' j_e'} \begin{bmatrix} \chi^{M'}_{-K_e'} \\ \pm \chi^{M'}_{K_e'} \end{bmatrix}$$

$$\pm \Omega^{2K_e'}_{J'M'} = \sqrt{\frac{1}{2\pi}} \sum_{\mu_e'} C\begin{pmatrix} J' & j_e' & 2 \\ M' & \mu_e' & m' \end{pmatrix} Y_2^{m'}(\theta'', \phi'') \begin{bmatrix} \chi^{\mu_e'}_{-K_e'} \\ \pm \chi^{\mu_e'}_{K_e'} \end{bmatrix}$$

The following integrals over the space of the Euler angles of the nucleus (the volume of this space being $8\pi^2$) then appear.

$$\int Y_2^m(\theta'', \phi'')\, d\Gamma = 0$$

$$\int Y_2^{m'*}(\theta'', \phi'') Y_2^m(\theta'', \phi'')\, d\Gamma = 2\pi \delta_{mm'}$$

$$\int Y_2^{m'*}(\theta'', \phi'') Y_2^{m*}(\theta'' \phi'')\, d\Gamma = 2\pi (-)^m \delta_{m', -m} \qquad (14)$$

$$\int Y_2^{m'*}(\theta'', \phi'') Y_2^{m''*}(\theta'', \phi'') Y_2^m(\theta'', \phi'')\, d\Gamma =$$

$$- \sqrt{\frac{10\pi}{7}}\, C \begin{pmatrix} 2 & 2 & 2 \\ m & m' & m'' \end{pmatrix}$$

These may all be evaluated by means of various relations in Section A.3.

The electron part of the matrix element of H_Q reduces essentially to

$$\left\langle \begin{bmatrix} \chi^{\mu_e'}_{-\kappa_e'} \\ \pm \chi^{\mu_e'}_{\kappa_e'} \end{bmatrix} \middle| Y_2^m(\theta'', \phi'') \middle| \begin{bmatrix} -iF\, \chi^{\mu_e}_{-\kappa_e} \\ G\, \chi^{\mu_e}_{\kappa_e} \end{bmatrix} \right\rangle =$$

$$C \begin{pmatrix} j_e' & j_e & 2 \\ \mu_e' & \mu_e & m \end{pmatrix} \left\{ -iF \left\langle \chi_{-\kappa_e'} \middle\| Y_2 \middle\| \chi_{-\kappa_e} \right\rangle \pm G \left\langle \chi_{\kappa_e'} \middle\| Y_2 \middle\| \chi_{\kappa_e} \right\rangle \right\}$$

$$= (-)^{j_e' + \ell_{\kappa_e'} - \frac{1}{2}} \left\{ \frac{5(2j_e+1)}{4\pi} \right\}^{\frac{1}{2}} C\begin{pmatrix} j_e' & j_e & 2 \\ \mu_e' & \mu_e & m \end{pmatrix} \times$$

$$\left\{ i F \ (2\ell_{-\kappa_e}+1)^{\frac{1}{2}} C\begin{pmatrix} \ell_{-\kappa_e'} & \ell_{-\kappa_e} & 2 \\ 0 & 0 & 0 \end{pmatrix} W\left(\ell_{-\kappa_e'} j_e' \ell_{-\kappa_e} j_e, \tfrac{1}{2} 2\right) \right.$$

$$\left. \pm \ G (2\ell_{\kappa_e}+1)^{\frac{1}{2}} C\begin{pmatrix} \ell_{\kappa_e'} & \ell_{\kappa_e} & 2 \\ 0 & 0 & 0 \end{pmatrix} W\left(\ell_{\kappa_e'} j_e' \ell_{\kappa_e} j_e, \tfrac{1}{2} 2\right) \right\}$$

(15)

this last step following from (A.37).

After some manipulation, involving extensive use of the results of Sections A.1 and A.2, we have

$$\left\langle \pm \Omega^{0\kappa_e'}_{J'M'} \middle| H_Q \middle| \Phi^{\mp}(JMW) \right\rangle =$$

$$(-)^{j_e' + \ell_{\kappa_e'} + \frac{1}{2}} \delta_{JJ'} \delta_{MM'} \sqrt{\frac{1}{20}} \ e^2 Q_0 f(\tau) \times$$

$$\sum_{\kappa_e} (2j_e+1)^{\frac{1}{2}} \left\{ \begin{array}{l} i F_{\kappa_e}(JMW2)(2\ell_{-\kappa_e}+1)^{\frac{1}{2}} C\begin{pmatrix} \ell_{-\kappa_e'} & \ell_{-\kappa_e} & 2 \\ 0 & 0 & 0 \end{pmatrix} W\left(\ell_{-\kappa_e'} j_e' \ell_{-\kappa_e} j_e, \tfrac{1}{2} 2\right) \\ \pm \\ G_{\kappa_e}(JMW2)(2\ell_{\kappa_e}+1)^{\frac{1}{2}} C\begin{pmatrix} \ell_{\kappa_e'} & \ell_{\kappa_e} & 2 \\ 0 & 0 & 0 \end{pmatrix} W\left(\ell_{\kappa_e'} j_e' \ell_{\kappa_e} j_e, \tfrac{1}{2} 2\right) \end{array} \right\}$$

(16)

and

$$\left\langle \pm \Omega^{2\kappa_e'}_{J'M'} \middle| H_Q \middle| \Phi(JMW) \right\rangle =$$

$$(-)^{\ell_{\kappa_e'}+1} \delta_{JJ'} \delta_{MM'} \sqrt{\frac{2j_e'+1}{20}} \, e^2 Q_0 f(r) \times$$

$$\sum_{\kappa_e} \Bigg[i (2\ell_{-\kappa_e}+1)^{\frac{1}{2}} C\begin{pmatrix} \ell_{-\kappa_e'} & \ell_{-\kappa_e} & 2 \\ 0 & 0 & 0 \end{pmatrix} W\left(\ell_{-\kappa_e}, j_e' \ell_{-\kappa_e} j_e, \tfrac{1}{2} 2\right) \times$$

$$\left\{ (-)^{J-\frac{1}{2}} \delta_{J j_e} F_{\kappa_e}(JMW0) + (-)^{j_e'+\frac{1}{2}} \left[\frac{50}{7}(2j_e+1)\right]^{\frac{1}{2}} W\left(j_e 2 J' 2, j_e' 2\right) F_{\kappa_e}(JMW2) \right\}$$

$$\pm (2\ell_{\kappa_e}+1)^{\frac{1}{2}} C\begin{pmatrix} \ell_{\kappa_e'} & \ell_{\kappa_e} & 2 \\ 0 & 0 & 0 \end{pmatrix} W\left(\ell_{\kappa_e'} j_e' \ell_{\kappa_e} j_e, \tfrac{1}{2} 2\right) \times$$

$$\left\{ (-)^{J-\frac{1}{2}} \delta_{J j_e} G_{\kappa_e}(JMW0) + (-)^{j_e'+\frac{1}{2}} \left[\frac{50}{7}(2j_e+1)\right]^{\frac{1}{2}} W\left(j_e 2 J' 2, j_e' 2\right) G_{\kappa_e}(JMW2) \right\} \Bigg] \tag{17}$$

We are now in a position to form the coupled equations for the radial functions. It will be seen that the functions are independent of M, as was tacitly assumed in Chapter II. They will depend on J and W but different values of these are not coupled together, since they

are constants of motion, of course. Hence the labels J and W will be dropped in what follows, since no confusion can arise. It is convenient to make the substitution $rF = \mathcal{F}$, $rG = \mathcal{G}$ whence the functions we shall have are $\mathcal{F}_{\kappa_e}(I_F)$, $\mathcal{G}_{\kappa_e}(I_f)$, I_f having the values 0 and 2. Taking $\pm \Omega^{0\kappa_e}_{JM}$, $\pm \Omega^{2\kappa_e}_{JM}$, (7) will now give us the following four equations for a given κ_e

$$\left(W^* - V_c + 1\right) \mathcal{F}_{\kappa_e}(0) + e^2 Q_0 f(r) \sum_{\kappa_e'} a^{\kappa_e}_{\kappa_e'} \mathcal{F}_{\kappa_e'} (2)$$

$$- \frac{d\mathcal{G}_{\kappa_e}(0)}{dr} - \kappa_e \frac{\mathcal{G}_{\kappa_e}(0)}{r} = 0$$

$$\left(W^* - V_c - 1\right) \mathcal{G}_{\kappa_e}(0) + e^2 Q_0 f(r) \sum_{\kappa_e'} b^{\kappa_e}_{\kappa_e'} \mathcal{G}_{\kappa_e'} (2)$$

$$+ \frac{d\mathcal{F}_{\kappa_e}(0)}{dr} - \kappa_e \frac{\mathcal{F}_{\kappa_e}(0)}{r} = 0$$

$$\left(W^* - E_2^* - V_c + 1\right) \mathcal{F}_{\kappa_e}(2) + e^2 Q_0 f(r) \sum_{\kappa_e'} \left\{ c^{\kappa_e}_{\kappa_e'} \mathcal{F}_{\kappa_e'}(0) + p^{\kappa_e}_{\kappa_e'} \mathcal{F}_{\kappa_e'}(2) \right\}$$

$$- \frac{d g_{\kappa_e}(2)}{dr} - \kappa_e \frac{g_{\kappa_e}(2)}{r} = 0$$

$$\left(W^* - E_2^* - V_c - 1\right) g_{\kappa_e}(2) + e^2 Q_0 f(r) \sum_{\kappa_e'} \left\{ d^{\kappa_e}_{\kappa_e'} g_{\kappa_e'}(0) + q^{\kappa_e}_{\kappa_e'} g_{\kappa_e'}(2) \right\}$$

$$+ \frac{d \mathcal{F}_{\kappa_e}(2)}{dr} - \kappa_e \frac{\mathcal{F}_{\kappa_e}(2)}{r} = 0 \qquad (18)$$

where

$$a^{\kappa_e}_{\kappa_e'} = (-)^{j_e + \ell_{\kappa_e} + \frac{1}{2}} \left\{ \frac{1}{20} (2j_e' + 1)(2\ell_{-\kappa_e'} + 1) \right\}^{\frac{1}{2}} \times$$

$$C\begin{pmatrix} \ell_{-\kappa_e} & \ell_{-\kappa_e'} & 2 \\ 0 & 0 & 0 \end{pmatrix} W\left(\ell_{-\kappa_e} j_e \ell_{-\kappa_e'} j_e', \tfrac{1}{2} 2\right)$$

$$b^{\kappa_e}_{\kappa_e'} = (-)^{j_e + \ell_{\kappa_e} - \frac{1}{2}} \left\{ \frac{1}{20} (2j_e' + 1)(2\ell_{\kappa_e'} + 1) \right\}^{\frac{1}{2}} \times$$

$$C\begin{pmatrix} \ell_{\kappa_e} & \ell_{\kappa_e'} & 2 \\ 0 & 0 & 0 \end{pmatrix} W\left(\ell_{\kappa_e} j_e \ell_{\kappa_e'} j_e', \tfrac{1}{2} 2\right)$$

$$c^{\kappa_e}_{\kappa_e'} = (-)^{\ell_{-\kappa_e}+J-\frac{1}{2}} \delta_{J j_e'} \left\{ \frac{1}{20} (2j_e+1)(2\ell_{-\kappa_e'}+1) \right\}^{\frac{1}{2}} \times$$

$$C\begin{pmatrix} \ell_{-\kappa_e} & \ell_{-\kappa_e'} & 2 \\ 0 & 0 & 0 \end{pmatrix} W(\ell_{-\kappa_e} j_e \ell_{-\kappa_e'} j_e', \tfrac{1}{2} 2)$$

$$d^{\kappa_e}_{\kappa_e'} = (-)^{\ell_{\kappa_e}+J-\frac{1}{2}} \delta_{J j_e'} \left\{ \frac{1}{20} (2j_e+1)(2\ell_{\kappa_e'}+1) \right\}^{\frac{1}{2}} \times$$

$$C\begin{pmatrix} \ell_{\kappa_e} & \ell_{\kappa_e'} & 2 \\ 0 & 0 & 0 \end{pmatrix} W(\ell_{\kappa_e} j_e \ell_{\kappa_e'} j_e', \tfrac{1}{2} 2)$$

$$p^{\kappa_e}_{\kappa_e'} = (-)^{\ell_{-\kappa_e}+j_e+\frac{1}{2}} \left\{ \frac{5}{14} (2j_e+1)(2j_e'+1)(2\ell_{-\kappa_e'}+1) \right\}^{\frac{1}{2}} \times$$

$$C\begin{pmatrix} \ell_{-\kappa_e} & \ell_{-\kappa_e'} & 2 \\ 0 & 0 & 0 \end{pmatrix} W(\ell_{-\kappa_e} j_e \ell_{-\kappa_e'} j_e', \tfrac{1}{2} 2)\, W(j_e' 2 J 2, j_e 2)$$

$$q^{\kappa_e}_{\kappa_e'} = (-)^{\ell_{\kappa_e}+j_e+\frac{1}{2}} \left\{ \frac{5}{14} (2j_e+1)(2j_e'+1)(2\ell_{\kappa_e'}+1) \right\}^{\frac{1}{2}} \times$$

$$C\begin{pmatrix} \ell_{\kappa_e} & \ell_{\kappa_e'} & 2 \\ 0 & 0 & 0 \end{pmatrix} W(\ell_{\kappa_e} j_e \ell_{\kappa_e'} j_e', \tfrac{1}{2} 2)\, W(j_e' 2 J 2, j_e 2)$$

The number of different k_e values coupled together depends on the value of J. We recall from Chapter III that the J values of interest to us are $J = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. We then have the following coupling scheme:

$$J = 1/2 \quad \begin{cases} I_f = 0 & j_e = 1/2 \\ I_f = 2 & j_e = 3/2,\ 5/2 \end{cases}$$

$$J = 3/2 \quad \begin{cases} I_f = 0 & j_e = 3/2 \\ I_f = 2 & j_e = 1/2,\ 3/2,\ 5/2,\ 7/2 \end{cases}$$

$$J = 5/2 \quad \begin{cases} I_f = 0 & j_e = 5/2 \\ I_f = 2 & j_e = 1/2,\ 3/2,\ 5/2,\ 7/2,\ 9/2 \end{cases}$$

The calculation of the coefficients in the coupled equations (18) is rather laborious; however, a simplification is introduced by noting that

$$a^{k_e}_{k_e'} = b^{k_e}_{k_e'} = a^{-k_e}_{-k_e'} = c^{k_e'}_{k_e} = d^{k_e'}_{k_e} \quad , \text{ etc.}$$

and

$$p^{k_e}_{k_e'} = p^{-k_e}_{-k_e'} = p^{k_e'}_{k_e} = q^{k_e}_{k_e'} \quad , \text{ etc.}$$

The coupled equations that we then obtain from (18) for the above J values are given in Appendix E. It will there be noticed that the equations for each J value separate into two sets that correspond to the fact that for non-vanishing values of the quadrupole coupling matrix element we must have

$$\ell_{K_e} + \ell_{K_e'} \quad = \text{even integer}$$

$$\ell_{-K_e} + \ell_{-K_e'} \quad = \text{even integer}$$

The one implies the other, of course.

CHAPTER V

THE SOLUTION OF THE COUPLED EQUATIONS

In this chapter we consider more fully the assertion made in Chapter II that the degeneracy in the nuclear-electron states belonging to a given set of eigenvalues, J, M, W, is equal to the number of central field states, which are labelled by the set (K_e, I_p), that are coupled together by the quadrupole interaction and that this degeneracy can be completely removed by requiring that each such state asymptotically assumes a different set (K_e, I_p) as constants of motion. Secondly, we outline the method by which we obtain solutions to the coupled electron wave functions that satisfy such boundary conditions, but leave the computational details to the next chapter.

1. GENERAL PROPERTIES OF THE COUPLED EQUATIONS.

Referring to the coupled equations (4.18) we abbreviate the notation for the electron functions still further: $\mathcal{F}_{K_e}(I_p)$ and $\mathcal{G}_{K_e}(I_p)$ are written as $\mathcal{F}_i$ and $\mathcal{G}_i$ respectively, both K_e and I_p values being implicit in the subscript i. When we wish to distinguish between different linearly independent solutions of a particular function we do so by means of a superscript, thus: $\mathcal{F}_i^j$, $\mathcal{F}_i^k$, etc. We recall that it is F_i and G_i that are the electron radial wave functions and not $\mathcal{F}_i$ and $\mathcal{G}_i$, which are respectively equal to rF_i and rG_i.

In equations (4.18), then, we have the 2n functions $\mathcal{F}_i, \mathcal{G}_i$, i = 1 to n, coupled together in 2n first order differential equations; n is clearly equal to the number of (K_e, I_f) sets that are coupled together. Let us write a particular set of solutions to the 2n functions as a 2n component vector:

$$\underset{\sim}{\gamma}^{j} = \begin{bmatrix} \mathcal{F}_1^j \\ \mathcal{G}_1^j \\ \vdots \\ \mathcal{F}_i^j \\ \mathcal{G}_i^j \\ \vdots \\ \mathcal{F}_n^j \\ \mathcal{G}_n^j \end{bmatrix} \qquad (1)$$

which will be known as a solution vector. There will be, in all, 2n linearly independent vectors of this form that are solutions to the coupled equations. They will form a 2n x 2n matrix of solutions, $\{\gamma\}$, known as a fundamental matrix. A necessary and sufficient condition that a solution matrix is a fundamental matrix i.e. that the columns constitute linearly independent solution vectors, is that

$$\det \{\gamma\} \neq 0 \qquad (2)$$

at any point r (see e.g. p. 69 of Coddington and Levinson (1955)).

Any solution vector may be expressed in terms of the fundamental matrix, thus:

$$\underset{\sim}{y} = \sum_j A_j \underset{\sim}{y}^j \qquad (3)$$

2. THE DEGREE OF DEGENERACY OF THE COUPLED STATES.

We now prove that half of the columns of a fundamental matrix to the equations (4.18) consist entirely of solutions that are regular at the origin and the remaining n solution vectors are all irregular at the origin. That is, any function, $\mathcal{F}_i, g_i$, has n linearly independent regular solutions, and n linearly independent irregular solutions; further, regular solutions of one function are coupled only to regular solutions of other functions, and similarly for the irregular solutions.

First, we write the equations (4.18) in the general form

$$\frac{d\mathcal{F}_i}{dr} - K_e^i \frac{\mathcal{F}_i}{r} = -f(r)\sum_{j=1}^{n} \beta_j^i g_j - \left(W_i - h(r) - 1\right) g_i$$

$$\frac{dg_i}{dr} + K_e^i \frac{g_i}{r} = f(r)\sum_{j=1}^{n} \beta_j^i \mathcal{F}_j + \left(W_i - h(r) + 1\right) \mathcal{F}_i \qquad (4)$$

Here use has been made of the fact that in (4.18)

$$a_{K_e'}^{K_e} = b_{K_e'}^{K_e}, \quad c_{K_e'}^{K_e} = d_{K_e'}^{K_e}, \quad p_{K_e'}^{K_e} = q_{K_e'}^{K_e}$$

$f(r)$ and $h(r)$ involve the radial dependence of the quadrupole and central electrostatic interactions respectively and so must both be non-singular everywhere, since we are considering an extended nucleus. Also

$$W_i = W - E_{I_f}^i \tag{5}$$

If we write

$$f_i^j = r^{\sigma_j} \sum_{t=0}^{\infty} p_t^i \, r^t$$

$$g_i^j = r^{\sigma_j} \sum_{t=0}^{\infty} q_t^i \, r^t \tag{6}$$

where the p and q are <u>not</u> the ones of (4.18), the indicial equation for σ_j is obtained by equating to zero the lowest power of r, which is $r^{\sigma_j - 1}$:

$$\begin{aligned} (\sigma_j + K_e^i)\, p_o^i &= 0 \\ (\sigma_j - K_e^i)\, q_o^i &= 0 \end{aligned} \tag{7}$$

This is a set of 2n homogeneous equations in the 2n variables p_o^i, q_o^i, i = 1 to n, so that the determinant, Δ, of their coefficients has to vanish:

$$\Delta = \prod_{i=1}^{n} \left(\sigma_j^2 - K_e^{i\,2} \right) = 0$$

whence

$$\sigma_j = \pm K_e^i \qquad (8)$$

Now because these roots of the indicial equation all differ from one another by an integer the solutions will not in general have the form (6): logarithmic terms will have to be added. Nevertheless, we know (see, e.g. Ince (1956)) that each positive root of σ_j defines regular solutions and each negative root irregular solutions. Also to each root $\sigma_j = \pm K_e^i$ there will belong independent solutions corresponding to each value of I_f^i. Thus to each set of values (K_e^i, I_f^i) there will correspond one regular solution and one irregular solution; furthermore, since a particular solution $\mathcal{F}_i^j, \mathcal{G}_i^j$, involves the same σ_j for all $\mathcal{F}_i, \mathcal{G}_i$ it is clear that only regular solutions are coupled to regular solutions. As the irregular solutions are not physically admissable it follows that the n-fold degeneracy has been established.

It will be seen that this result is very general and the form of the proof given above requires only that the potential functions, f(r) and h(r), are non-singular everywhere, which is the case for an extensive nucleus. Actually, the proof may be extended to include the case of these functions possessing regular singularities at the origin. Since singularities in the potentials must be regular if physically admissable solutions are to exist it may be said that the foregoing result is as general as possible, i.e. it holds as long as there exist any physically admissable solutions at all.

That the degeneracy must be n-fold may be made physically plausible by noting that for an extended charge distribution the

quadrupole potential must always vanish at the origin. Thus at the origin the 2n coupled equations will partially uncouple into n central field pairs of coupled equations and we may expect that at the origin the behavior of the functions will correspond to the solution of n separate central field equation pairs, each one of which is known to have just one regular solution. Out of the single physically admissable solution that exists for each of the n pairs, $\mathcal{F}_i$, $\mathcal{G}_i$, n linearly independent solution vectors for the 2n functions can be formed.

3. ASYMPTOTIC FORM OF THE SOLUTIONS.

Outside of the nucleus the quadrupole interaction falls off as $\frac{1}{r^3}$, whereas all other terms in the coupled equations (4.18) fall off as $\frac{1}{r}$ at the most. Thus the 2n coupled equations rapidly uncouple into n central field pairs of coupled equations and the solutions will assume the form of central field functions long before these latter tend to their asymptotic form. We can therefore write

$$\begin{aligned} \mathcal{F}_i &= r F_i \longrightarrow u_i = r f_i \\ \mathcal{G}_i &= r G_i \longrightarrow v_i = r g_i \end{aligned} \tag{9}$$

and recall from Section 1. 2 that f_i and g_i satisfy the equations

$$\begin{aligned} (W_i - V_c + 1) f_i - \frac{dg_i}{dr} - (K_e^i + 1)\frac{g_i}{r} &= 0 \\ (W_i - V_c - 1) g_i + \frac{df_i}{dr} - (K_e^i - 1)\frac{f_i}{r} &= 0 \end{aligned} \tag{10}$$

where W_i is given by (5). Each u_i, v_i pair will have two linearly independent solutions, one regular at the origin and the other irregular.

In the region of vanishing quadrupole interaction, then, a 2n solution vector of the coupled equations will have the form

$$\underset{\sim}{y}^{j} = \begin{bmatrix} C_1^j u_1 \\ v_1 \\ \\ C_i^j u_i \\ v_i \\ \\ C_n^j u_n \\ v_n \end{bmatrix} \tag{11}$$

where u_i and v_i are particular linear combinations of regular and irregular solutions.

Let us now consider a solution matrix that in the central field region has the form

$$\{y\} = \begin{bmatrix} u_1^1 & 0 & 0 & u_1^2 & 0 & 0 \\ v_1^1 & 0 & 0 & v_1^2 & 0 & 0 \\ 0 & \vdots & \vdots & 0 & & \vdots \\ 0 & & & 0 & & \\ \vdots & 0 & & \vdots & 0 & \\ & 0 & & & 0 & \\ & u_i^1 & & & u_i^2 & \\ & v_i^1 & & & v_i^2 & \\ & 0 & & & 0 & \\ & 0 & \vdots & & 0 & \\ & \vdots & 0 & & \vdots & \\ 0 & 0 & u_n^1 & 0 & 0 & u_n^2 \\ 0 & 0 & v_n^1 & 0 & 0 & v_n^2 \end{bmatrix} \qquad (12)$$

where $u_i^1, v_i^1; u_i^2, v_i^2$ are particular linearly independent solutions. That this matrix is a solution for some particular set $(\cdots u_i^1, v_i^1; u_i^2, v_i^2 \cdots\cdots)$ is self-evident. For it to be a fundamental matrix requires that at any value of r $\det\{y\} \neq 0$, by the theorem quoted in Section 1 of this chapter. By inspection

$$\det\{y\} = \pm \prod_{i=1}^{n} \left(u_i^1 v_i^2 - u_i^2 v_i^1 \right). \qquad (13)$$

the sign depending on n in a rather complicated way. Now if u_i^1, v_i^1 and u_i^2, v_i^2 are linearly independent for all i, as stipulated, then

by the same theorem each factor in the above product must be non-zero. Therefore there exists a fundamental matrix of the form of (12).

This fundamental matrix is such that each basic solution asymptotically assumes good quantum numbers $\left(K_e^i, I_f^i\right)$, there being two solutions belonging to each of the n such sets. Since any regular solution of (4.18) can only be formed as a linear combination of regular solutions it follows that just half of the columns of (12) correspond to regular solutions. The question arises as to which are these n columns of the matrix. It was proposed in Chapter II that a complete set of states of the electron-nuclear system could be obtained by requiring that each state asymptotically assumed a different $\left(K_e, I_f\right)$ set as good quantum numbers. This would indeed be the case if the first n columns of (12) (or the second n; it clearly does not matter) did correspond to regular solutions. But suppose this is not so. Then some of the $\left(K_e, I_f\right)$ sets coupled together by the quadrupole interaction would be completely absent from all possible states in the central field region. While we have not been able to prove the mathematical impossibility of this asymptotic inhibition of some of the coupled central field states it is seen to be physically absurd by noting that such inhibition must be independent of quadrupole strength and hence must persist even with vanishing quadrupole moment. But in the absence of coupling the inhibition can clearly not occur and hence we conclude that it never occurs.

In this and the preceding section we have succeeded in showing that the method of removal of the degeneracy adopted in Chapter II is indeed correct.

4. THE TREATMENT OF THE TWO-POINT BOUNDARY CONDITIONS.

Having shown that such a basic set of states exists we now have to consider the method of obtaining it. Since the coupled equations have to be solved numerically, i.e. by a step-by-step integration from some particular set of starting values, we are faced essentially with a two-point boundary condition problem, the solutions having to be regular at the origin and in the central field region have the form

$$\begin{matrix} \mathcal{F}_i^{\ j} \\ \mathcal{G}_i^{\ j} \end{matrix} \longrightarrow \begin{matrix} \delta_{ij} & u_i \\ & v_i \end{matrix} \qquad (14a)$$

where u_i and v_i are particular, but undetermined, linear combinations of regular and irregular central field solutions.

Suppose, however, that we compute <u>any</u> complete set of regular solutions. This is a relatively simple matter, since the integration is started at the origin (actually, a small distance away because of the singularity; this point is discussed in the next chapter) with suitable values for the functions, but no attention need be paid to the asymptotic boundary conditions. Now since we know that there exists <u>a</u> basic set that satisfies (14a) it follows that <u>any</u> solution whatsoever must in the central field region be of the form

$$\begin{matrix} \mathcal{F}_i^{\ j} \\ \mathcal{G}_i^{\ j} \end{matrix} \longrightarrow \begin{matrix} C_i^{\ j} & u_i \\ & v_i \end{matrix} \qquad (15a)$$

The essential content of this statement is that in the central field region the ratio f_i^j / g_i^j should be the same for all solutions j. If our computed functions do not satisfy this at some particular value of r then this must be due simply to the fact that the quadrupole interaction is still not negligible (to within the limits of computational accuracy) and we are not yet in what we have chosen to call the central field region. Values for u_i and v_i, containing an arbitrary multiplicative constant, may be obtained by putting $C_i^j = 1$ for any single convenient j, whereupon all other C values may be obtained from the computed functions.

Let us for convenience write the pair of functions f_i^j, g_i^j that satisfy the required boundary conditions (14a) as ϕ_{ij} and let the computed functions f_i^j, g_i^j be written as ξ_{ij}. Then as the central field region is approached we have

$$\{\phi\} \longrightarrow \begin{bmatrix} u_1 & \cdots & 0 & & 0 \\ v_1 & & 0 & & 0 \\ 0 & & \vdots & & \vdots \\ 0 & & 0 & & \\ \vdots & & u_i & & \\ & & v_i & & \\ & & 0 & & \\ & & 0 & & \\ & & \vdots & & \vdots \\ 0 & & 0 & & u_n \\ 0 & & 0 & & v_n \end{bmatrix} \qquad (14b)$$

and

$$\{\xi\} \longrightarrow \begin{bmatrix} C_1^1 u_1 & \cdots & C_1^j u_1 & \cdots & C_1^n u_1 \\ C_1^1 v_1 & & C_1^j v_1 & & C_1^n v_1 \\ \vdots & & \vdots & & \vdots \\ C_i^1 u_i & & C_i^j u_i & & C_i^n u_i \\ C_i^1 v_i & & C_i^j v_i & & C_i^n v_i \\ \vdots & & \vdots & & \vdots \\ C_n^1 u_n & & C_n^j u_n & & C_n^n u_n \\ C_n^1 v_n & & C_n^j v_n & & C_n^n v_n \end{bmatrix} \qquad (15b)$$

These are essentially (14a) and 15a) rewritten.

Then by virtue of (3) it follows that we can obtain our required basic set from the computed basic set by means of the linear transformation

$$\{\phi\} = \{\xi\} C^{-1} \qquad (16)$$

where the matrix C is such that $C_{ij} = C_i^j$. We can now write down the expression for the radial functions $F_{\kappa e}(\bar{J}WI_f, \ell)$, $G_{\kappa e}(\bar{J}WI_f, \ell)$ satisfying the required boundary conditions in terms of the computed functions $\mathcal{F}_i^j$, g_i^j :

$$F_{\kappa e}^i(\bar{J}WI_f^i, \ell) = \frac{1}{\tau} \sum_j \{C^{-1}\}_{ij} \mathcal{F}_i^j \qquad (17)$$

and similarly for the G function.

CHAPTER VI

THE COMPUTATION AND RESULTS

It was seen in Section 1.4 that the most likely case of a β-decay in which a quadrupole coupling between the different daughter nuclear states could be significant is that of Np^{236}. Because of the great amount of numerical work involved we decided to restrict ourselves to this single case. For the same reason we took only a single value of the energy, $W^* = 1.3$ (in units of electron rest energy). This means that we shall not be able to obtain accurate values of the branching ratio, since it is only in the central field limit that this is independent of the lepton functions. However, we recall from the end of Chapter III that it is only in this limit that the branching ratio is independent of the intrinsic nuclear matrix elements. Hence, if there is a quadrupole coupling it will be impossible to calculate the branching ratio exactly, anyway, no matter how many energy values are taken. All that we can really hope for is an order of magnitude estimate of the branching ratio, but this will at least permit us to say whether or not the quadrupole coupling of the kind considered will supply a plausible explanation of the observed anomalies.

The intrinsic quadrupole moment of Np^{236} has not been measured. The highest measured value in this mass number region is $|Q_o| = 14$ barns,

for U^{233} (see, e.g. Alder et al. (1956)). We decided to take $|Q_o| =$ 15 barns for Np^{236}, realizing that this value may be rather high. For reasons stated at the end of Section 1.5 both signs of the intrinsic quadrupole moment were considered.

The major computational problem is that of the solution of the coupled equations, listed in Appendix E, in the region close to the nucleus where the quadrupole interaction is appreciable. The general method followed has been outlined in the previous section: we first have to obtain any complete set of solutions that are regular at the origin. We obtained such a set by noting that at the origin the quadrupole coupling vanishes, so that the wave equations uncouple into central field equations, as at large distances from the nucleus. Then there will exist a complete set of solutions such that each solution assumes a different (κ_e, I_f) set as constants of motion at the origin. That is, there is a complete set which behaves at the origin in exactly the same way as the required set does asymptotically. Our method of computing any such solution, then, was to put all but one of the pairs F_i, g_i equal to zero at the origin, the remaining pair taking just its central field value. With these starting values the solutions to the equations were then computed by a step-by-step integration outwards from the origin. Different solutions were obtained by taking different pairs of functions, F_i, g_i, to be non-vanishing at the origin. In the event of there being no quadrupole coupling this would be the only pair of functions that did not vanish everywhere. We refer to this pair as the principal function pair for the particular solution.

The solutions of the coupled equations depend on the radial

variation of both the central and quadrupole potentials. Outside the nucleus, radius ρ, the central potential is

$$V_c = -\alpha Z/r \qquad r > \rho$$

Inside the nucleus the central potential depends on the charge distribution, but for a constant density of charge it is

$$V_c = \frac{\alpha Z}{2\rho^3}\left(r^2 - 3\rho^2\right) \qquad r < \rho$$

The radial dependence of the quadrupole potential has already been discussed in Section 1.3. We have

$$f(r) = 1/r^3 \qquad r > \rho$$

$$f(r) = r^2/\rho^5 \qquad r < \rho$$

Since we are dealing with deformed nuclei ρ must be regarded as an averaged value; we took it to be given by

$$\rho = 1.2\, A^{1/3} \times 10^{-13} \text{ cm.}$$

Although our solutions are chosen to be regular at the origin, this is still a singular point and hence cannot be contained within the domain of computation. Thus the integration must begin at a finite distance from the origin, whereupon there arises the question of starting values. The problem may be avoided by a slight modification of the charge distribution within the nucleus. We imagine there to be a small spherical hollow around the nuclear center, within which the quadrupole interaction vanishes and the central potential is constant, chosen to be equal to the value it would have at the center for a true uniform charge distribution:

$$V_c(0) = -\frac{3}{2}\,\frac{\alpha Z}{\rho}$$

Within this hollow it is possible to write down the solutions. From

(1.25) we have for the principal functions

$$\mathcal{F}_{K_e}(I_f) = S_{K_e}\sqrt{\frac{W_e'-1}{\pi\rho'}}\, r\, j_{\ell_{-K_e}}(\rho' r)$$

$$g_{K_e}(I_f) = \sqrt{\frac{W_e'+1}{\pi\rho'}}\, r\, j_{\ell_{K_e}}(\rho' r) \qquad (1)$$

where

$$W_e' = W - \delta_{I_f 2} E_2 - V_c(0)$$

The other functions vanish throughout the hollow, of course. The numerical integration of the equations is then begun on the surface of the hollow with the known value of these functions. Provided that the hollow is not too big (in our case its radius was one-fifth that of the nucleus) there can be little departure from the solutions for a true uniform charge distribution, which is an idealization, anyway.

The numerical solution of the equations was performed on the Bendix G15D computer at McMaster University, using interpretive programming. The method of solution followed was the variant of the fourth order Runge-Kutta method due to Gill (1951). We ran through trial computations with progressively smaller values of the integration interval, h, until further reduction made no significant difference to the computed solutions. The value of h finally chosen was $h = \rho/5$.

The integration of the equations has to be continued outwards until the quadrupole interaction is negligible. At $r = 5\rho$ the quadrupole interaction has fallen to 1% of the central interaction,

in our case, so that it may be expected that at this distance the functions have reached their central field form and the matching of the computed solutions to solutions having the required asymptotic form can be effected. Accordingly, the integration was halted at $r = 5\rho$, where it was found that the computed functions could be expressed in the form

$$\begin{matrix} f_i^j \\ g_i^j \end{matrix} = C_{ij} \begin{matrix} \mu_i \\ \nu_i \end{matrix} \qquad (2)$$

to within 1% usually (see (5.15a)). Hence we can be sure that we have indeed reached the central field region (to within the necessary degree of computational accuracy.)

We now have to consider the matrices of the computed C_{ij} coefficients for the different sets of equations. There are two sets of equations for each of the three J values and since we considered both signs of Q_o there are in all twelve sets of equations to be solved. It is clear from (2) that the rows of these matrices will refer to the different solutions of a single pair of functions, f_i, g_i, characterized by angular momentum quantum numbers (κ_e^i, I_f^i), while the columns refer to a single solution characterized by the quantum numbers (κ_e^j, I_f^j) being pure at the origin. Thus the diagonal elements correspond to the principal functions. We display the C matrices in Table I below, the elements being correct to within 1%. The values (κ_e, I_f) are shown alongside each row and at the head of each column. It will be clear from the foregoing what they refer to in the two cases.

TABLE I

THE C MATRICES

	$K_e = 3$ $I_f = 0$	$K_e = 1$ $I_f = 2$	$K_e = -2$ $I_f = 2$	$K_e = 3$ $I_f = 2$	$K_e = -4$ $I_f = 2$	$K_e = 5$ $I_f = 2$
$K_e = 3$ $I_f = 0$	1	-.09	0	0	0	0
$K_e = 1$ $I_f = 2$	0	1	0	0	0	0
$K_e = -2$ $I_f = 2$	0	0	1	0	0	0
$K_e = 3$ $I_f = 2$	0	.10	0	1	0	0
$K_e = -4$ $I_f = 2$	0	0	.13	0	1	0
$K_e = 5$ $I_f = 2$	-.35	0	0	0	0	1

(1) $J = 5/2$ SET I

$Q_o = +15$ barns

	$K_e = 3$ $I_f = 0$	$K_e = 1$ $I_f = 2$	$K_e = -2$ $I_f = 2$	$K_e = 3$ $I_f = 2$	$K_e = -4$ $I_f = 2$	$K_e = 5$ $I_f = 2$
$K_e = 3$ $I_f = 0$	1	.09	0	0	0	0
$K_e = 1$ $I_f = 2$	0	1	0	0	0	0
$K_e = -2$ $I_f = 2$	0	0	1	0	0	0
$K_e = 3$ $I_f = 2$	0	-.10	0	1	0	0
$K_e = -4$ $I_f = 2$	0	0	-.13	0	1	0
$K_e = 5$ $I_f = 2$	.34	0	0	0	0	1

(2) J = 5/2 SET I

Q_o = - 15 barns

	$K_e = -3$ $I_f = 0$	$K_e = -1$ $I_f = 2$	$K_e = 2$ $I_f = 2$	$K_e = -3$ $I_f = 2$	$K_e = 4$ $I_f = 2$	$K_e = -5$ $I_f = 2$
$K_e = -3$ $I_f = 0$	1	-.09	0	0	0	0
$K_e = -1$ $I_f = 2$	0	1	0	0	0	0
$K_e = 2$ $I_f = 2$	0	0	1	0	0	0
$K_e = -3$ $I_f = 2$	0	.10	0	1	0	0
$K_e = 4$ $I_f = 2$	0	0	.14	0	1	0
$K_e = -5$ $I_f = 2$	-.35	0	0	0	0	1

(3) J = 5/2 SET II

Q_o = + 15 barns

	$K_e = -3$ $I_f = 0$	$K_e = -1$ $I_f = 2$	$K_e = 2$ $I_f = 2$	$K_e = -3$ $I_f = 2$	$K_e = 4$ $I_f = 2$	$K_e = -5$ $I_f = 2$
$K_e = -3$ $I_f = 0$	1	.09	0	0	0	0
$K_e = -1$ $I_f = 2$	0	1	0	0	0	0
$K_e = 2$ $I_f = 2$	0	0	1	0	0	0
$K_e = -3$ $I_f = 2$	0	-.10	.0	1	0	0
$K_e = 4$ $I_f = 2$	0	0	-.14	0	1	0
$K_e = -5$ $I_f = 2$	.34	0	.0	0	0	1

(4) $J = 5/2$ SET II

$Q_o = -15$ barns

	$K_e = 2$ $I_f = 0$	$K_e = -1$ $I_f = 2$	$K_e = 2$ $I_f = 2$	$K_e = -3$ $I_f = 2$	$K_e = 4$ $I_f = 2$
$K_e = 2$ $I_f = 0$	1	0	0	0	0
$K_e = -1$ $I_f = 2$	0	1	0	0	0
$K_e = 2$ $I_f = 2$	0	0	1	0	0
$K_e = -3$ $I_f = 2$	0	.19	0	1	0
$K_e = 4$ $I_f = 2$	-.25	0	0	0	1

(5) J = 3/2 SET I

Q_o = + 15 barns

	$K_e = 2$ $I_f = 0$	$K_e = -1$ $I_f = 2$	$K_e = 2$ $I_f = 2$	$K_e = -3$ $I_f = 2$	$K_e = 4$ $I_f = 2$
$K_e = 2$ $I_f = 0$	1	0	0	0	0
$K_e = -1$ $I_f = 2$	0	1	0	0	0
$K_e = 2$ $I_f = 2$	0	0	1	0	0
$K_e = -3$ $I_f = 2$	0	-.19	0	1	0
$K_e = 4$ $I_f = 2$	.24	0	0	0	1

(6) J = 3/2 SET I

Q_o = - 15 barns

	$K_e = -2$ $I_f = 0$	$K_e = 1$ $I_f = 2$	$K_e = -2$ $I_f = 2$	$K_e = 3$ $I_f = 2$	$K_e = -4$ $I_f = 2$
$K_e = -2$ $I_f = 0$	1	0	0	0	0
$K_e = 1$ $I_f = 2$	0	1	0	0	0
$K_e = -2$ $I_f = 2$	0	0	1	0	0
$K_e = 3$ $I_f = 2$	0	.19	0	1	0
$K_e = -4$ $I_f = 2$	-.24	0	0	0	1

(7) J = 3/2 SET II

Q_o = + 15 barns

	$K_e = -2$ $I_f = 0$	$K_e = 1$ $I_f = 2$	$K_e = -2$ $I_f = 2$	$K_e = 3$ $I_f = 2$	$K_e = -4$ $I_f = 2$
$K_e = -2$ $I_f = 0$	1	0	0	0	0
$K_e = 1$ $I_f = 2$	0	1	0	0	0
$K_e = -2$ $I_f = 2$	0	0	1	0	0
$K_e = 3$ $I_f = 2$	0	-.19	0	1	0
$K_e = -4$ $I_f = 2$	.24	0	0	0	1

(8) J = 3/2 SET II

Q_o = - 15 barns

	$K_e = 1$ $I_f = 0$	$K_e = -2$ $I_f = 2$	$K_e = 3$ $I_f = 2$
$K_e = 1$ $I_f = 0$	1	0	0
$K_e = -2$ $I_f = 2$	0	1	0
$K_e = 3$ $I_f = 2$	-.16	0	1

(9) J = 1/2 SET I

Q_o = + 15 barns

	$K_e = 1$ $I_f = 0$	$K_e = -2$ $I_f = 2$	$K_e = 3$ $I_f = 2$
$K_e = 1$ $I_f = 0$	1	0	0
$K_e = -2$ $I_f = 2$	0	1	0
$K_e = 3$ $I_f = 2$	.16	0	1

(10) J = 1/2 SET I

Q_o = - 15 barns

	$K_e = -1$ $I_f = 0$	$K_e = 2$ $I_f = 2$	$K_e = -3$ $I_f = 2$
$K_e = -1$ $I_f = 0$	1	0	0
$K_e = 2$ $I_f = 2$	0	1	0
$K_e = -3$ $I_f = 2$	-.16	0	1

(11) J = 1/2 SET II

Q_o = + 15 barns

	$K_e = -1$ $I_f = 0$	$K_e = 2$ $I_f = 2$	$K_e = -3$ $I_f = 2$
$K_e = -1$ $I_f = 0$	1	0	0
$K_e = 2$ $I_f = 2$	0	1	0
$K_e = -3$ $I_f = 2$	.16	0	1

(12) J = 1/2 SET II

Q_o = - 15 barns

Since the off-diagonal elements are a measure of the quadrupole coupling it will be seen from the form of the above matrices that there is not a very large amount of coupling. However, the functions represented in the C matrices have (K_e, I_f) becoming constants of motion as the origin is approached, whereas we want (K_e, I_f) to be asymptotic constants of motion. The necessary transformation (5.17) requires the inverses of the C matrices. They were obtained on the computer but we do not display them because there turned out to be a very simple numerical relation between the C matrices and their inverses. The latter may be represented to within a high degree of accuracy by a reversal of sign of the off-diagonal elements, all the diagonal elements remaining unity. We can now immediately write down the expression for the radial functions having the required boundary conditions in terms of the computed functions

$$F_{K_e^{\hat{i}}}\left(JWI_{f,}^{\hat{i}}\ell\right) = \frac{1}{r}\left\{\mathcal{F}_{\hat{i}}^{\ell} + \sum_{j}\left(\delta_{\ell j} - 1\right) C_{j\ell}\, \mathcal{F}_{\hat{i}}^{j}\right\} \qquad (3)$$

and likewise for the G functions.

We now recall how the F and G functions are involved in the expression for the transition probability, (3.7) and (3.8). There the Q's involve the quantities $f_{K_e^{\hat{i}}}\left(JWI_f^{\hat{i}}, \ell\right)$, $g_{K_e^{\hat{i}}}\left(JWI_f^{\hat{i}}, \ell\right)$ defined by (3.2) as

$$f_{\hat{i}}^{\ell} = \sum_{K}\left\{U^{-1}\right\}_{\ell K} F_{K_e^{\hat{i}}}\left(JWI_f^{\hat{i}}, K\right) \qquad (4)$$

etc., where we have made an obvious abbreviation of notation.

The matrix U is given by (2.10) and is

$$U_{mn}\,\delta(W'-W) = \sum_{k_e I_f} \int \Big\{ F_{k_e}(JW'I_f,m)\,F_{k_e}(JWI_f,n) + G_{k_e}(JW'I_f,m)\,G_{k_e}(JWI_f,n) \Big\} r^2 dr \qquad (5)$$

The delta-function that is involved in this definition poses a problem in the calculation of U_{mn} since the integrand of (5) is only known numerically. However, the delta-function may be removed from (5) by integrating both sides of (5) over a small interval of W' around W. Then the double integration of the right hand side of (5) can be performed numerically now, if we integrate first over the finite interval of W', since the subsequent integral over r must be convergent.

Having seen how it is possible in principle to remove the delta-function from the definition of U_{mn} we can now say all we want to about the latter. If the functions were orthogonal then $\{U\}$ would, of course, be diagonal. Now our functions have been chosen to approach orthogonal functions asymptotically and we have seen, in fact, that it is only for $r < 5\rho$ at the most that there is any significant departure of our functions from the orthogonal functions. This departure from orthogonality arises from the coupling in of angular momentum values that disappear as the central field region is reached, at or before $r = 5\rho$. From the form of (5) it is clear that the functions associated with these angular momentum values give rise to the off-diagonal elements, which accordingly must be very small compared to the diagonal elements. Then to a good approximation we can write

$$f_i^{\ell} \simeq \frac{1}{A_{J,\ell}^2} F_{k_e^i}\left(\overline{J}W I_F^i, \ell\right)$$

$$\simeq \frac{1}{A_{J,\ell}^2 r}\left\{\mathcal{F}_i^{\ell} + \sum_j (\delta_{\ell j} - 1) C_{j\ell}\, \mathcal{F}_i^{j}\right\} \quad (6)$$

and likewise for g_i^{ℓ}. $A_{J,\ell}$ is the constant introduced in Section 2.1.

Now the condition that coupling is appreciable is that there be functions f_i^{ℓ}, g_i^{ℓ}, with different values of I_F^i but belonging to the same asymptotic value I_F^{ℓ}, that have comparable orders of magnitude at the nuclear radius, ρ. That is, we require that

$$\frac{F_{i\neq\ell}^{\ell}(\rho)}{F_{\ell}^{\ell}(\rho)} \quad \text{and} \quad \frac{g_{i\neq\ell}^{\ell}(\rho)}{g_{\ell}^{\ell}(\rho)}$$

be significantly different from zero for some i and ℓ values.

It happens that in general the solution of any particular function, having the largest value at the nuclear radius, is that solution for which the function is the principal function. That is, for given i

$$\left|\mathcal{F}_i^{i}(\rho)\right| \gg \left|\mathcal{F}_i^{\ell\neq i}(\rho)\right|$$

and likewise for g_i^i . Furthermore, in any particular solution, by far the largest functions at the nuclear surface are the principal functions for that solution i.e. for given l

$$\left| f_l^l(\rho) \right| >> \left| f_{i \neq l}^{\ l}(\rho) \right| , \text{ etc.}$$

It should be noted of course, that these remarks do not necessarily apply to the transformed functions satisfying the required boundary conditions.

(6) now gives –

$$F_l^l(\rho) = \frac{1}{A_{J,l}^2 \rho} f_l^l(\rho) \tag{7}$$

Also if $F_{i \neq l}^{\ l}(\rho)$ is to be comparable to $F_l^l(\rho)$ then the approximation

$$F_{i \neq l}^{\ l}(\rho) = - \frac{1}{A_{J,l}^2 \rho} C_{il} f_i^i(\rho) \tag{8}$$

is good. Thus we at last see the manner in which the off-diagonal elements of the C matrices describe the effective coupling. With the C matrices given above in Table I it is now possible to write down the possible couplings. These are displayed in Table II below.

TABLE II

THE COUPLED STATES

J Value	Asymptotic Quantum Numbers		Angular Momentum States that are Coupled in Close to the Nucleus	
	K_e	I_f	K_e	I_f
5/2	1	2	3	0
			3	2
	-1	2	-3	0
			-3	2
	3	0	5	2
	-3	0	-5	2
	2	2	4	2
	-2	2	-4	2
3/2	2	0	4	2
	-2	0	-4	2
	1	2	3	2
	-1	2	-3	2
1/2	1	0	3	2
	-1	0	-3	2

Now in every case the only states that are coupled in have $\kappa_e > 2$, which correspond to higher degrees of forbiddenness. From the computed values of these functions at the nuclear radius it turns out that, as in the central field case, higher degrees of forbiddenness can be ignored. The largest amount of coupling is about 1%: this is for the admixture of ($\kappa_e = \pm 3$, $I_f = 0$) states into states that asymptotically have $\kappa_e = 1$ $I_f = 2$.

In this chapter we have merely presented the essentials of the computational procedures and the numerical results. For more detail reference should be made to the author, c/o McMaster University, who is in possession of the complete computed electron wave functions.

CHAPTER VII

CONCLUSION

It has been seen in the previous chapter that the only coupling between different nuclear states is that which occurs through electron states belonging to higher degrees of forbiddenness. At the nuclear surface these are considerably smaller than the electron states corresponding to the first degree of forbiddenness, as in the central field case, so that the actual amount of coupling is very small: 1% is the largest amount of admixture in any state characterized by asymptotic constancy of electron and nuclear angular momenta.

Thus, although there is a certain amount of mixing of electron states associated with different nuclear states, the states that are mixed are such as to be of little consequence for β -decay. This is rather surprising, since a rather naive perturbation theory approach would indicate the largest coupling to occur between states of low and comparable electron angular momenta e.g. between the states ($\kappa_e = 2$, $I_f = 0$) and ($\kappa_e = 2$, $I_f = 2$). Instead these states are only very weakly coupled.

It will be noted that in this thesis we have adopted the model of β -decay taking place in the nuclear surface, since we have evaluated the electron functions at the radius of the charge distribution . Now as the effective radial distance of a decaying nucleon

from the center of the nucleus must be less than the surface radius it follows that the relative contribution of the electron states of higher degrees of forbiddenness must be even smaller than we have estimated. Hence a correct treatment of the effective nuclear radius would reduce the coupling still further.

It is seen from (3.7) that the effect of the coupling on the branching ratio will depend on the intrinsic nuclear matrix elements. However, a large anomaly in the branching ratio could only arise from a small coupling if there were a large cancellation between the different terms of the Q's in (3.7). But such a cancellation would mean an abnormally low transition probability, whereas Gray (1956) reports a quite normal value for Np^{236}.

We therefore conclude that the quadrupole interaction of the kind we have considered in this thesis is not of importance in β-decay, since its effect on the transition probability is unlikely to be more than 1%, at the very most.

There are now two possible ways of accounting for the observed anomaly in Np^{236} (in the case of Lu^{176} and Ta^{180} we can never be sure that the B_{ij} matrix element is not responsible, however unlikely this is).

(i) The anomaly may be spurious. That is to say, either the parent or daughter states may not possess pure intrinsic states. This would mean that Alaga's formula (our 1.41) is no longer applicable. Neither Np^{236} nor its daughter, Pu^{236}, appear to have had their level structure carefully examined. On the other hand it is well known that the transuranic nuclei generally have well developed rotational structure. In particular the spectra of Np^{237} and Pu^{238} (see e.g.

Strominger et al. (1958)) indicate that mixing of intrinsic states must be very small. Thus it seems possible that

(ii) the anomaly is real, in which case another explanation has to be sought. Without developing the idea at all it is now suggested that there is a possibility of the Dirac single electron theory breaking down and radiative corrections becoming necessary. That this is possible can be seen by remembering that at the nuclear surface the quadrupole interaction is comparable to the central potential. Hence on some regions of the nuclear surface the electric potential will correspond to an atomic number of the order of 180, so that contributions from the negative energy states may become important. In that case there would certainly be a departure of the electron wave functions from the Dirac form. Now there appears to be some evidence for this happening in Np^{236}: since $\alpha Z/\rho \gg (W_0 - 1)$ the spectrum shape should be statistical (see the article of Konopinski in the book of Siegbahn (1955)) whereas in actual point of fact the spectrum has a forbidden shape, according to Gray (1956). On the other hand it is rather difficult to see how a differential effect on the transitions to the two nuclear states could arise.

Nevertheless, the general problem of the polarization of the vacuum by quadrupole distributions of charge is a matter worthy of further investigation, since it may be of general importance for both β-decay and internal conversion in strongly deformed nuclei of high Z.

APPENDIX A

MISCELLANEOUS NOTES ON THE QUANTUM THEORY OF ANGULAR MOMENTUM

The following is an explanation of notation and statement of results occurring in the quantum theory of angular momentum, relevant to this thesis.

1. CLEBSCH-GORDAN COEFFICIENTS.

Consider the combination of two angular momenta, denoted by the usual quantum numbers (j_1, m_1) and (j_2, m_2), respectively, to form the angular momentum (J, M):

$$\underset{\sim}{J} = \underset{\sim}{j_1} + \underset{\sim}{j_2}$$

so

$$|j_1 - j_2| \leq J \leq j_1 + j_2 \tag{1}$$

and

$$m_1 + m_2 = M \tag{2}$$

Then there exists a unitary transformation connecting a representation in which (j_1, m_1) and (j_2, m_2) are diagonal with one in which (J, M) and j_1, and j_2 are diagonal, thus

$$\psi_J^M = \sum_{m_1} C\begin{pmatrix} J & j_1 & j_2 \\ M & m_1 & m_2 \end{pmatrix} \psi_{j_1}^{m_1} \psi_{j_2}^{m_2} \tag{3}$$

where the $C\begin{pmatrix} J & j_1 & j_2 \\ M & m_1 & m_2 \end{pmatrix}$ factors are the Clebsch-Gordan coefficients, denoted $\left(j_1\, j_2\, m_1, m_2 \,|\, j_1\, j_2\, J M\right)$ by Condon and Shortley (1935). They are zero unless (1) and (2) are satisfied.

There exists an inverse to (3)

$$\psi_{j_1}^{m_1} \psi_{j_2}^{m_2} = \sum_J C\begin{pmatrix} J & j_1 & j_2 \\ M & m_1 & m_2 \end{pmatrix} \psi_J^M \tag{4}$$

Tables of values of Clebsch-Gordan coefficients, sufficient for our purpose, are given on pp. 76-77 of Condon and Shortley (1935). In this connection it is convenient to note that

$$C\begin{pmatrix} J & j_1 & j_2 \\ 0 & 0 & 0 \end{pmatrix} = 0 \tag{5}$$

unless $J_1 + j_1 + j_2 = 2\ell$

where ℓ is an integer.

The following orthogonality conditions hold

$$\sum_{m_1} C\begin{pmatrix} J & j_1 & j_2 \\ M & m_1 & m_2 \end{pmatrix} C\begin{pmatrix} J' & j_1 & j_2 \\ M & m_1 & m_2 \end{pmatrix} = \delta_{JJ'} \tag{6}$$

$$\sum_J C\begin{pmatrix} J & j_1 & j_2 \\ M & m_1 & m_2 \end{pmatrix} C\begin{pmatrix} J & j_1 & j_2 \\ M & m_1' & m_2' \end{pmatrix} = \delta_{m_1 m_1'} \tag{7}$$

There are the following permutation relations:

$$C\begin{pmatrix} J & j_1 & j_2 \\ M & m_1 & m_2 \end{pmatrix} = (-)^{j_1+j_2-J} C\begin{pmatrix} J & j_2 & j_1 \\ M & m_2 & m_1 \end{pmatrix} = (-)^{j_1+j_2-J} C\begin{pmatrix} J & j_1 & j_2 \\ -M & -m_1 & -m_2 \end{pmatrix}$$

$$= (-)^{j_2+m_2} \left(\frac{2J+1}{2j_1+1}\right)^{\frac{1}{2}} C\begin{pmatrix} j_1 & J & j_2 \\ -m_1 & -M & m_2 \end{pmatrix} = (-)^{j_1-m_1} \left(\frac{2J+1}{2j_2+1}\right)^{\frac{1}{2}} C\begin{pmatrix} j_2 & j_1 & J \\ -m_2 & m_1 & -M \end{pmatrix} \tag{8}$$

(This is not a complete list but all other permutation relations can be generated from these.)

2. RACAH COEFFICIENTS.

These arise in the combination of three angular momenta. They may be defined by

$$\left\{(2e+1)(2f+1)\right\}^{\frac{1}{2}} W(a\,b\,c\,d\,;\,e\,f) =$$

$$\sum_{\alpha,\beta} C\begin{pmatrix} e & a & b \\ \alpha+\beta & \alpha & \beta \end{pmatrix} C\begin{pmatrix} c & e & d \\ \alpha+\beta+\delta & \alpha+\beta & \delta \end{pmatrix} C\begin{pmatrix} f & b & d \\ \beta+\delta & \beta & \delta \end{pmatrix} C\begin{pmatrix} c & a & f \\ \alpha+\beta+\delta & \alpha & \beta+\delta \end{pmatrix} \tag{9}$$

Reviews of their properties are given by Biedenharn, Blatt and Rose (1952) and by Rose (1957); however, the former contains more extensive tables.

Two relations of importance are

$$\sum_f \left\{(2e+1)(2f+1)\right\}^{\frac{1}{2}} W(abcd;ef)\, C\begin{pmatrix} f & b & d \\ \beta+\delta & \beta & \delta \end{pmatrix} C\begin{pmatrix} c & a & f \\ \alpha+\beta+\delta & \alpha & \beta+\delta \end{pmatrix}$$

$$= C\begin{pmatrix} e & a & b \\ \alpha+\beta & \alpha & \beta \end{pmatrix} C\begin{pmatrix} c & e & d \\ \alpha+\beta+\delta & \alpha+\beta & \delta \end{pmatrix} \qquad (10)$$

$$\left\{(2e+1)(2f+1)\right\}^{\frac{1}{2}} W(abcd;ef)\, C\begin{pmatrix} c & a & f \\ \alpha+\beta+\delta & \alpha & \beta+\delta \end{pmatrix} =$$

$$\sum_\beta C\begin{pmatrix} e & a & b \\ \alpha+\beta & \alpha & \beta \end{pmatrix} C\begin{pmatrix} c & e & d \\ \alpha+\beta+\delta & a+\beta & \delta \end{pmatrix} C\begin{pmatrix} f & b & d \\ \beta+\delta & \beta & \delta \end{pmatrix} \qquad (11)$$

W(abcd;ef) vanishes unless the following triangle conditions hold:

$$\Delta(abe)\,, \quad \Delta(edc)\,, \quad \Delta(baf)\,, \quad \Delta(afc)$$

where $\Delta(abe)$ means

$$|a-b| \leq e \leq a+b$$

There are twenty-four possible permutations of the six arguments which preserve the triangle relations. The result of any permutation may be generated from the following basic relations:

$$W(abcd;ef) = W(badc;ef) = W(cdab;ef)$$

$$= W(acbd;fe) = (-)^{e+f-a-d} W(ebcf;ad) = (-)^{e+f-b-c} W(aefd;bc) \qquad (12)$$

3. SPHERICAL HARMONICS AND THE ROTATION MATRICES.

The spherical harmonic functions, $Y_\ell^m(\theta,\phi)$, may be defined to satisfy the following relations

$$\underset{\sim}{L}^2 Y_m^\ell(\theta,\phi) = \ell(\ell+1) Y_\ell^m(\theta,\phi)$$

$$L_z Y_\ell^m(\theta,\phi) = m Y_\ell^m(\theta,\phi) \tag{13}$$

$$(L_x \pm i L_y) Y_\ell^m(\theta,\phi) = \left\{(\ell \mp m)(\ell \pm m + 1)\right\}^{\frac{1}{2}} Y_\ell^{m\pm 1}(\theta,\phi)$$

If the normalization condition

$$\int_{\Omega = 4\pi} Y_\ell^{m*}(\theta,\phi) Y_\ell^m(\theta,\phi)\, d\Omega = 1 \tag{14}$$

is imposed, we obtain the following unique expression:

$$Y_\ell^m(\theta,\phi) = \left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{\frac{1}{2}} (-)^m P_\ell^m(\theta) e^{im\phi} \tag{15}$$

where

$$P_\ell^m(\theta) = \frac{1}{2^\ell \ell!} \sin^m\theta \left[\frac{d}{d(\cos\theta)}\right]^{\ell+m} (\cos^2\theta - 1)^\ell \tag{16}$$

This is the definition customarily adopted in quantum mechanics; it is the one used by Condon and Shortley (1935). Characteristic of this definition is the relation

$$Y_\ell^{m*}(\theta,\phi) = (-)^m Y_\ell^{-m}(\theta,\phi) \tag{17}$$

which comes from the identity

$$P_\ell^{-m}(\theta) = (-)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\theta)$$

Suppose there are two unit vectors, (θ,ϕ), (θ'',ϕ''), such that the angle between them is θ'. Then

$$P_\ell(\theta') = \frac{4\pi}{2\ell+1} \sum_m Y_\ell^{m\,*}(\theta,\phi)\, Y_\ell^m(\theta'',\phi'') \qquad (18)$$

where P_ℓ is simply the P_ℓ^o of (16). This is the well-known spherical harmonics addition theorem.

Let us now consider the transformation properties of the spherical harmonics under rotations of the coordinate system. Such a rotation is specified by the three Euler angles, which we shall denote by Θ_i; it is not necessary to offer any definition of them here. Then if (θ,ϕ) are the coordinates of a point on the unit sphere in the original frame and (θ',ϕ') those of the same point in the new frame we can write

$$Y_\ell^m(\theta,\phi) = \sum_{m'} D^\ell_{mm'}(\Theta_i)\, Y_\ell^{m'}(\theta',\phi') \qquad (19)$$

The transformation matrix, $D^\ell_{mm'}(\Theta_i)$ is known as a rotation matrix, and, in fact, it constitutes a (2l + 1) dimensional irreducible representation of the rotation group (see, e.g. Wigner (1931)).

A great deal of confusion arises from the use of different definitions of the D-matrix. (19) is essentially the definition of Edmonds (1957) (note, however, that (θ,ϕ) and (θ',ϕ') are used there in a

different sense) but differs from that of Rose (1957), who apparently takes the complex congugate of the Edmonds definition. The latter has the distinct advantage of leading to the conventional form of the eigenfunctions of the symmetric top (see, e.g. Bohr (1952)), whereas Rose's definition does not. Thus it is important to realize that when the rotation matrices are used in this context the definition (19) is implied. However, the relations listed below are valid for both definitions.

Since the spherical harmonics are orthonormal the rotation matrices must be unitary:

$$\sum_{m} D^{l\,*}_{m'm}(\Theta_i)\, D^{l}_{m''m}(\Theta_i) = \delta_{m'm''} \qquad (20)$$

The orthogonality relations for the irreducible representations of a continuous group (see 10.11 of Wigner (1931)) lead to the following integral on the unit sphere, taken over all three Euler angles:

$$\int_{\Omega = 8\pi^2} d\Omega\; D^{l_1\,*}_{m_1 m_1'}(\Theta_i)\, D^{l_2}_{m_2 m_2'}(\Theta_i) = \frac{8\pi^2}{2l+1}\,\delta_{l_1 l_2}\,\delta_{m_1 m_2}\,\delta_{m_1' m_2'} \qquad (21)$$

By considering the transformation induced in both sides of (4) by a rotation we arrive at the so-called Clebsch-Gordan series:

$$D^{\ell_1}_{m_1 m_1'}(\Theta_i)\, D^{\ell_2}_{m_2 m_2'}(\Theta_i) = \sum_L C\begin{pmatrix} L & \ell_1 & \ell_2 \\ M & m_1 & m_2 \end{pmatrix} C\begin{pmatrix} L & \ell_1 & \ell_2 \\ M' & m_1' & m_2' \end{pmatrix} D^{L}_{MM'}(\Theta_i) \tag{22}$$

From (18) and (22) can be deduced

$$Y^{m_1}_{\ell_1}(\theta,\phi)\, Y^{m_2}_{\ell_2}(\theta,\phi) = \sum_L \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2L+1)}}\, C\begin{pmatrix} L & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix} C\begin{pmatrix} L & \ell_1 & \ell_2 \\ M & m_1 & m_2 \end{pmatrix} Y^{M}_{L}(\theta,\phi) \tag{23}$$

4. SPHERICAL TENSOR OPERATORS AND THEIR REDUCED MATRIX ELEMENTS.

A spherical tensor of rank L, T^M_L, is a set of (2L + 1) functions that transform under rotations like spherical harmonics of order L i.e.

$$T^M_L(\omega) = \sum_{M'} D^{L}_{MM'}(\Theta_i)\, T^{M'}_{L}(\omega) \tag{24}$$

where ω represents all the arguments of the function. If we have two spherical tensors, $T^{M_1}_{L_1}$, $T^{M_2}_{L_2}$, the quantity

$$T^M_L = \sum_{M_1} C\begin{pmatrix} L & L_1 & L_2 \\ M & M_1 & M_2 \end{pmatrix} T^{M_1}_{L_1} T^{M_2}_{L_2} \tag{25}$$

is a spherical tensor also, since it can be easily shown with the aid

of (6) and (22) that it transforms according to (24). Thus we have a prescription for forming tensors of arbitrary rank from two given tensors.

A special instance of this procedure can be obtained by noting that the spherical harmonics themselves must be spherical tensors. Now (23) can be inverted by using (7) to yield

$$\sum_{m_1} C\begin{pmatrix} L & \ell_1 & \ell_2 \\ M & m_1 & m_2 \end{pmatrix} Y_{\ell_1}^{m_1}(\theta,\phi)\, Y_{\ell_2}^{m_2}(\theta,\phi) = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi\,(2L+1)}}\; C\begin{pmatrix} L & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix} Y_L^M(\theta,\phi). \tag{26}$$

which is just (25), aside from a constant numerical factor.

But thus far all we have obtained are spherical tensors on the unit sphere. We now note that any vector $\underset{\sim}{A}$ must be a spherical tensor of rank 1. This may be shown most explicitly by writing the components as

$$A_{\pm 1} = \mp\sqrt{\tfrac{1}{2}}\,(A_x \pm i A_y); \qquad A_0 = A_z \tag{27}$$

Then

$$A_m = \sqrt{\tfrac{4\pi}{3}}\; A\; Y_1^m(\theta,\phi) \tag{28}$$

whence $\underset{\sim}{A}$ transforms as $Y_1^m(\theta,\phi)$. We can then form spherical tensors from the vector $\underset{\sim}{A}$ by means of (26). It is convenient to introduce the so-called solid harmonic:

$$\mathcal{Y}_L^M\left(\underset{\sim}{A}\right) = A^L Y_L^M(\theta,\phi) \tag{29}$$

whence from (26)

$$\sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}}\, C\begin{pmatrix} L & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix} A^{l_1+l_2-L}\, \mathcal{Y}_L^M(\underset{\sim}{A}) =$$

$$\sum_{m_1} C\begin{pmatrix} L & l_1 & l_2 \\ M & m_1 & m_2 \end{pmatrix} \mathcal{Y}_{l_1}^{m_1}(\underset{\sim}{A})\, \mathcal{Y}_{l_2}^{m_2}(\underset{\sim}{A}) \qquad (30)$$

This again is just (25) with extra numerical factors.

The matter of greatest interest is the possibility of forming spherical tensors of more than one vector argument: there is nothing to require that the two tensors on the right hand side of (25) be formed from the same vector $\underset{\sim}{A}$. In the theory of β-decay there is just one particular kind of tensor of this form that occurs:

$$T_{L\lambda}^M(\underset{\sim}{r},\underset{\sim}{A}) = \sum_{m_1} C\begin{pmatrix} L & \lambda & 1 \\ M & m_1 & m_2 \end{pmatrix} \mathcal{Y}_\lambda^{m_1}(\underset{\sim}{r})\, \mathcal{Y}_1^{m_2}(\underset{\sim}{A}) \qquad (31)$$

Here $\underset{\sim}{A}$ may be any vector other than $\underset{\sim}{r}$. It is always involved in (31) as a vector; the rank of the other solid harmonic is denoted by the subscript λ on the $T_{L\lambda}^M$.

In forming (31) attention must be paid to a possible non-commutation of $\underset{\sim}{A}$ and $\underset{\sim}{r}$. The parity of $T_{L\lambda}^M$ is clearly

$$\Pi = (-)^\lambda \Pi(\underset{\sim}{A}) \qquad (32)$$

We usually encounter spherical tensors as operators and we require their matrix element between two angular momentum eigenstates, (j_1,m_1) and (j_2,m_2). By remembering that the matrix element must be

invariant under rotations it is easy to show that (see pp. 263-4 of Wigner (1931)).

$$\langle j_1 m_1 | T^M_L | j_2 m_2 \rangle = C\begin{pmatrix} j_1 & L & j_2 \\ m_1 & M & m_2 \end{pmatrix} \langle j_1 \| T_L \| j_2 \rangle \tag{33}$$

where $\langle j_1 \| T_L \| j_2 \rangle$ is known as the reduced matrix element. The essential content of this result, the Wigner-Eckart theorem, is that the dependence of the matrix element on the magnetic quantum numbers resides wholly in the Clebsch-Gordan coefficient.

We now evaluate the reduced matrix element

$$\langle \chi_{K'} \| Y_2 \| \chi_K \rangle$$

where χ^μ_K is the Pauli spinor

$$\chi^\mu_K = \sum_{\Gamma = \pm\frac{1}{2}} C\begin{pmatrix} j & \ell_K & \frac{1}{2} \\ \mu & \mu-\Gamma & \Gamma \end{pmatrix} Y^{\mu-\Gamma}_{\ell_K}(\theta,\phi) \begin{pmatrix} \frac{1}{2}+\Gamma \\ \frac{1}{2}-\Gamma \end{pmatrix} \tag{34}$$

By (33) and (6)

$$\langle \chi_{K'} \| Y_2 \| \chi_K \rangle = \sum_m C\begin{pmatrix} j' & j & 2 \\ \mu' & \mu & m \end{pmatrix} \langle \chi^{\mu'}_{K'} | Y^m_2 | \chi^\mu_K \rangle \tag{35}$$

Now from (23) and (14)

$$\langle Y^M_L, Y^{m_1}_{\ell_1} Y^{m_2}_{\ell_2} \rangle = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2L+1)}}\, C\begin{pmatrix} L & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix} C\begin{pmatrix} L & \ell_1 & \ell_2 \\ M & m_1 & m_2 \end{pmatrix} \tag{36}$$

where the integration goes over the solid angle 4π. Then

$$\langle x^{\mu'}_{\kappa'} | Y^m_2 | x^{\mu}_{\kappa} \rangle = \sqrt{\frac{5(2\ell_\kappa+1)}{4\pi(2\ell_{\kappa'}+1)}}\, C\begin{pmatrix} \ell_{\kappa'} & \ell_\kappa & 2 \\ 0 & 0 & 0 \end{pmatrix} \times$$

$$\sum_r C\begin{pmatrix} j' & \ell_{\kappa'} & \frac{1}{2} \\ \mu' & \mu'-r & r \end{pmatrix} C\begin{pmatrix} j & \ell_\kappa & \frac{1}{2} \\ \mu & \mu-r & r \end{pmatrix} C\begin{pmatrix} \ell_{\kappa'} & \ell_\kappa & 2 \\ \mu'-r & \mu-r & m \end{pmatrix}.$$

Then with (35) and various of the listed relations for the Clebsch-Gordan and Racah coefficients we obtain

$$\langle x_{\kappa'} \| Y_2 \| x_\kappa \rangle = (-)^{\ell_{\kappa'}+j'-\frac{1}{2}} \left\{ \frac{5(2\ell_\kappa+1)(2j+1)}{4\pi} \right\}^{\frac{1}{2}} \times$$

$$C\begin{pmatrix} \ell_{\kappa'} & \ell_\kappa & 2 \\ 0 & 0 & 0 \end{pmatrix} W\left(\ell_{\kappa'} j' \ell_\kappa j, \tfrac{1}{2} 2\right) \qquad (37)$$

APPENDIX B

FIRST ORDER TIME-DEPENDENT PERTURBATION THEORY FOR NON-ORTHOGONAL STATES

We refer here to pp. 195-7 of Schiff (1955). In our case $|\mu_n\rangle$ are to be a set of linearly independent but non-orthogonal eigenstates of H_o.

Putting $\hbar = 1$ we have, with Schiff,

$$i\sum_\ell \int dW\, \dot{a}_\ell(W) \left|\mu_\ell(W)\right\rangle e^{-iWt} = \sum_\ell \int dW a_\ell(W) H' \left|\mu_\ell(W)\right\rangle e^{-iWt} \qquad (1)$$

Then

$$i\sum_\ell \int dW \langle \mu_K(W') \left|\mu_\ell(W)\right\rangle \dot{a}_\ell(W) e^{-iWt} =$$

$$\sum_\ell \int dW a_\ell(W) \langle \mu_K(W') \left|H'\right| \mu_\ell(W)\rangle e^{-iWt} \qquad (2)$$

where we can no longer make use of orthonormality on the left hand side.

However, we can still follow Schiff's first order procedure of assuming that at $t = 0$ there is only one state occupied i.e.

$$a_\ell(W)\Big|_{t=0} = \delta_{\ell i}\, \delta(W'-W'')$$

and that changes in all a_ℓ have a negligible effect on the right hand side of (2). We then have the essentially first order result

$$i \sum_\ell \int dW \langle \mu_K(W') | \mu_\ell(W) \rangle \dot{a}_\ell(W) e^{-i(W-W'')t} =$$

$$\langle \mu_K(W') | H' | \mu_i(W'') \rangle \qquad (3)$$

We now note that because the states $|\mu_\ell(W)\rangle$ belong to the continuum their scalar product will have a delta-function dependence on energy. This can be seen most easily by expressing the non-orthogonal states $|\mu_\ell(W)\rangle$ as linear combinations of orthonormal states, $|\nu_m(W)\rangle$, thus

$$|\mu_\ell(W)\rangle = \sum_m c_{\ell m}(W) |\nu_m(W)\rangle \qquad (4)$$

where

$$\langle \nu_m(W') | \nu_n(W) \rangle = \delta_{mn} \delta(W'-W) \qquad (5)$$

Then

$$\langle \mu_K(W') | \mu_\ell(W) \rangle = \delta(W'-W) U_{K\ell} \qquad (6)$$

where

$$U_{K\ell} = \sum_i c^*_{Ki}(W) c_{\ell i}(W) \qquad (7)$$

Then, with a re-arrangement of primes (3) becomes

$$i \sum_{\ell} U_{k\ell} \dot{a}_{\ell}(W)\, e^{-i(W-W')t} = \langle \mu_k(W) | H' | \mu_i(W') \rangle . \tag{8}$$

We then have

$$\dot{a}_{\ell}(W)\, e^{-i(W-W')t} = -i \sum_{k} \left\{ U^{-1} \right\}_{\ell k} \langle \mu_m(W) | H' | \mu_i(W') \rangle , \tag{9}$$

This has essentially the same form as Schiff's equation (29.5), the only difference being that the simple perturbation matrix $\langle \mu_{\ell}(W) | H' | \mu_i(W_i') \rangle$ is replaced by the more complicated one

$$\langle \mu_{\ell}(W) | \mathcal{H} | \mu_i(W') \rangle = \sum_{m} \left\{ U^{-1} \right\}_{\ell m} \langle \mu_m(W) | H' | \mu_i(W') \rangle \tag{10}$$

For the case of particular interest here, that of the constant perturbation switched on for a finite time interval, we simply have to rewrite (29.9) of Schiff, thus

$$a_{\ell}(W) = - \langle \mu_{\ell}(W) | \mathcal{H} | \mu_i(W') \rangle \frac{e^{i(W-W')t} - 1}{W - W'} \tag{11}$$

We now recall from Section 1.1 that in the correct formulation of β -decay the perturbation Hamiltonian is the field interaction Hamiltonian (1.6) and that the states between which transitions must

be presumed to be occurring are occupation number states. It is now shown that, as for orthogonal states, we can formally identify the H' above with the H_β of (1.8) and the states $|\mu_n\rangle$ with the $|JMW, n\rangle$ states.

The non-orthogonal states $|JMW, n\rangle$ can always be expressed as linear combinations of orthonormal states as in (4). The corresponding occupation number states are then

$$|N_n(W)\rangle = \sum_m C_{nm} \left| 0 \cdots \; 0, n_m(W) = 1, \; 0 \cdots \right\rangle \tag{12}$$

We shall then have

$$\langle N_m(W') | N_n(W) \rangle = \langle JMW', m | JMW, n \rangle \tag{13}$$

since both the set of states $|v_m(W)\rangle$ and that of the states $|0 \cdots 0, n_m(W) = 1, 0 \cdots\rangle$ are orthonormal. It now follows in the same way as for the orthogonal states in Section 1.1 that we can make the required formal identification.

APPENDIX C

THE EVALUATION OF THE INTEGRALS (2.13)

We offer here a derivation of the integrals (2.13), which are implicit in the papers of Greuling (1951) and Lee-Whiting (1958).

Let us write

$$W_l - W_o^l + q = W_l - \bar{W}_l = \omega$$

whence

$$p_l = \left\{ \bar{p}_l^{\,2} + 2\bar{W}_l \omega + \omega^2 \right\}^{\frac{1}{2}}$$

If we now put

$$\omega r = z \qquad \omega t = a z$$

the integrals become

$$I = \int_{-\infty}^{\infty} dz \, \frac{1 - e^{-iaz}}{z} \, e^{\pm i \bar{p}_l \left(1 + \frac{2\bar{W}_l}{\bar{p}_l^{\,2}} \frac{z}{r} + \frac{z^2}{\bar{p}_l^{\,2} r^2}\right)^{\frac{1}{2}} r} \tag{1}$$

Next we note that the integrand has a fairly strong maximum at $z = 0$, whence an approximate value of the integrals may be obtained by confining attention to the neighborhood of the origin. Thus we expand the surd occurring in the second exponential term around $z = 0$ to obtain

$$\left(1 + \frac{2\bar{W}_l}{\bar{p}_l^{\,2}} \frac{z}{r} + \frac{z^2}{\bar{p}_l^{\,2} r^2}\right)^{\frac{1}{2}} = 1 + \frac{\bar{W}_l}{\bar{p}_l^{\,2}} \frac{z}{r} + O\left(\frac{z^2}{r^2}\right) \tag{2}$$

Discarding all but the first power of z, (1) becomes

$$e^{\pm i\bar{p}_e r}\int_{-\infty}^{\infty} dz\,\frac{1-e^{-iaz}}{z}\,e^{\pm i\frac{\bar{w}_e}{\bar{p}_e}z}$$

which immediately leads to the required result by means of the identity

$$\int_{-\infty}^{\infty} dx\,\frac{e^{iAx}-e^{iBx}}{x} = \begin{cases} 2\pi i & \text{if } A \text{ +ve and } B \text{ -ve} \\ 0 & \text{if } A \text{ -ve and } B \text{ -ve} \end{cases} \qquad (3)$$

This method of evaluation can clearly be only approximate. Inspection of (2) shows, however, that the approximation will improve as r increases. In fact, (2.13) may be regarded as an asymptotic value of the integrals for large r.

This point may be illustrated by considering the case $r = 0$, which can be evaluated exactly:

$$\int_{-\infty}^{\infty} dz\,\frac{1-e^{-iaz}}{z} = \pi i \qquad \text{for a + ve}$$

which is quite different from (2.13).

APPENDIX D

MATRIX ELEMENTS IN ROTATIONAL STATES

We are concerned with the evaluation of matrix elements of the form

$$\langle \Psi_{I_f}^{M_f} \mid T_L^M(\omega) \mid \Psi_{I_i}^{M_i} \rangle$$

where the nuclear states have their rotational motion and intrinsic structure completely separated and the T_L^M are spherical tensor operators of rank L in the space of any single nucleon.

A transformation to the usual system of coordinates fixed in the nucleus can be effected by use of (A.24), thus:

$$T_L^m(\omega) = \sum_{m'} D_{mm'}^{L}(\Theta_i)\, T_L^{m'}(\omega') \qquad (1)$$

The nuclear states can be written

$$\Psi_{I_i}^{M_i} = \sqrt{\frac{2I_i+1}{16\pi^2}} \left\{ \chi_{\Omega}^{i} D_{M_i K_i}^{I_i}(\Theta_i) + (-)^{I_i - j} \chi_{-\Omega}^{i} D_{M_i -K_i}^{I_i}(\Theta_i) \right\}$$

$$\Psi_{I_f}^{M_f} = \sqrt{\frac{2I_f+1}{8\pi^2}}\, \chi^f D_{M_f 0}^{I_f}(\Theta_i) \qquad (2)$$

the daughter state having the simpler form since it is in the lowest rotational band of an even-even nucleus and hence $K_f = 0$ (see Section 1.3).

The matrix element then factors thus:

$$\frac{1}{8\pi^2}\sqrt{\frac{(2I_i+1)(2I_f+1)}{2}}\sum_{m'}\left\{\langle x^f|T_L^{m'}(\omega')|x^i_{\Omega}\rangle\langle D^{I_f}_{M_f 0}|D^{L}_{M_f-M_i,m'}|D^{I_i}_{M_i K_i}\rangle\right.$$

$$\left.+(-)^{I_i-j}\langle x^f|T_L^{m'}(\omega')|x^i_{-\Omega}\rangle\langle D^{I_f}_{M_f 0}|D^{L}_{M_f-M_i,m'}|D^{I_i}_{M_i-K_i}\rangle\right\} \quad (3)$$

In the first term here only $m' = -K_i$ contributes and in the second term only $m' = K_i$. By means of (A.21) and (A.22) we then have

$$\langle D^{I_f}_{M_f 0}|D^{L}_{M_f-M_i,-K_i}|D^{I_i}_{M_i K_i}\rangle =$$

$$\frac{8\pi^2}{2I_f+1}C\begin{pmatrix} I_f & I_i & L \\ M_f & M_i & M_f-M_i \end{pmatrix}C\begin{pmatrix} I_f & I_i & L \\ 0 & K_i & -K_i \end{pmatrix}$$

$$\langle D^{I_f}_{M_f 0}|D^{L}_{M_f-M_i,K_i}|D^{I_i}_{M_i,-K_i}\rangle =$$

$$\frac{8\pi^2}{2I_f+1}C\begin{pmatrix} I_f & I_i & L \\ M_f & M_i & M_f-M_i \end{pmatrix}C\begin{pmatrix} I_f & I_i & L \\ 0 & -K_i & K_i \end{pmatrix} \quad (4)$$

Noting that

$$C\begin{pmatrix} I_f & I_i & L \\ 0 & K_i & -K_i \end{pmatrix} = (-)^{I_i+L-I_f} C\begin{pmatrix} I_f & I_i & L \\ 0 & -K_i & K_i \end{pmatrix}$$

we then have

$$\left\langle \Psi^{M_f}_{I_f} \middle| T^{m}_{L}(\omega) \middle| \Psi^{M_i}_{I_i} \right\rangle =$$

$$\sqrt{\frac{1}{2I_f+1}}\, C\begin{pmatrix} I_f & I_i & L \\ M_f & M_i & M_f-M_i \end{pmatrix} C\begin{pmatrix} I_f & I_i & L \\ 0 & K_i & -K_i \end{pmatrix} M_L(\omega) \qquad (5)$$

where

$$M_L(\omega) = \sqrt{\frac{2I_i+1}{2}} \left\{ \left\langle x^f \middle| T^{-K_i}_{L}(\omega') \middle| x^{i}_{\Omega} \right\rangle \right.$$

$$\left. + (-)^{I_f+j-L} \left\langle x^f \middle| T^{K_i}_{L}(\omega') \middle| x^{i}_{-\Omega} \right\rangle \right\} \qquad (6)$$

APPENDIX E

THE COUPLED EQUATIONS

J = 1/2

SET I $\quad \mathcal{F}_1(0),\ g_1(0),\ \mathcal{F}_{-2}(2),\ g_{-2}(2),\ \mathcal{F}_3(2),\ g_3(2)$.

$$\left(W^* - V_c + 1\right)\mathcal{F}_1(0) - \frac{dg_1(0)}{dr} - \frac{g_1(0)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r)\left\{\sqrt{2}\,\mathcal{F}_{-2}(2) + \sqrt{3}\,\mathcal{F}_3(2)\right\} = 0$$

$$\left(W^* - V_c - 1\right)g_1(0) + \frac{d\mathcal{F}_1(0)}{dr} - \frac{\mathcal{F}_1(0)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r)\left\{\sqrt{2}\,g_{-2}(2) + \sqrt{3}\,g_3(2)\right\} = 0$$

$$\left(W^* - E_2 - V_c - 1\right)\mathcal{F}_{-2}(2) - \frac{dg_{-2}(2)}{dr} + 2\,\frac{g_{-2}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r)\left\{\sqrt{2}\,\mathcal{F}_1(0) + \mathcal{F}_{-2}(2) + \sqrt{\frac{6}{49}}\,\mathcal{F}_3(2)\right\} = 0$$

$$(W^* - E_2 - V_c + 1)\, f_3(2) - \frac{d g_3(2)}{dr} - 3\,\frac{g_3(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r)\left\{\sqrt{3}\, f_1(0) + \sqrt{\frac{6}{49}}\, f_{-2}(2) + \frac{4}{35} f_3(2)\right\} = 0$$

$$(W^* - E_2 - V_c - 1)\, g_{-2}(2) + \frac{d f_{-2}(2)}{dr} + 2\,\frac{f_{-2}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r)\left\{\sqrt{2}\, g_1(0) + g_{-2}(2) + \sqrt{\frac{6}{49}}\, g_3(2)\right\} = 0$$

$$(W^* - E_2 - V_c - 1)\, g_3(2) + \frac{d f_3(2)}{dr} - 3\,\frac{f_3(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r)\left\{\sqrt{3}\, g_1(0) + \sqrt{\frac{6}{49}}\, g_{-2}(2) + \frac{4}{35} g_3(2)\right\} = 0$$

SET II $\quad F_{-1}(0),\ g_{-1}(0),\ F_2(2),\ g_2(2),\ F_{-3}(2),\ g_{-3}(2)$

$$(W^* - V_c + 1) F_{-1}(0) - \frac{d g_{-1}(0)}{dr} + \frac{g_{-1}(0)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{2}\, F_2(2) + \sqrt{3}\, F_{-3}(2) \right\} = 0$$

$$(W^* - V_c - 1) g_{-1}(0) + \frac{d F_{-1}(0)}{dr} + \frac{F_{-1}(0)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{2}\, g_2(2) + \sqrt{3}\, g_{-3}(2) \right\} = 0$$

$$(W^* - E_2 - V_c + 1) F_2(2) - \frac{d g_2(2)}{dr} - 2 \frac{g_2(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{2}\, F_{-1}(0) + F_2(2) + \sqrt{\frac{6}{49}}\, F_{-3}(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c + 1\right) \mathcal{F}_{-3}(2) - \frac{d\,g_{-3}(2)}{dr} + 3\,\frac{g_{-3}(2)}{r}$$

$$+ \frac{1}{10}\, e^2 Q_0 f(r) \left\{ \sqrt{3}\, \mathcal{F}_{-1}(0) + \sqrt{\frac{6}{49}}\, \mathcal{F}_2(2) + \frac{4}{35}\, \mathcal{F}_{-3}(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c - 1\right) g_2(2) + \frac{d\,\mathcal{F}_2(2)}{dr} - 2\,\frac{\mathcal{F}_2(2)}{r}$$

$$+ \frac{1}{10}\, e^2 Q_0 f(r) \left\{ \sqrt{2}\, g_{-1}(0) + g_2(2) + \sqrt{\frac{6}{49}}\, g_{-3}(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c - 1\right) g_{-3}(2) + \frac{d\,\mathcal{F}_{-3}(2)}{dr} + 3\,\frac{\mathcal{F}_{-3}(2)}{r}$$

$$+ \frac{1}{10}\, e^2 Q_0 f(r) \left\{ \sqrt{3}\, g_{-1}(0) + \sqrt{\frac{6}{49}}\, g_2(2) + \frac{4}{35}\, g_{-3}(2) \right\} = 0$$

J = 3/2

SET I $\quad f_2(0),\ g_2(0),\ f_{-1}(2), g_{-1}(2),\ f_2(2), g_2(2)$

$f_{-3}(2),\ g_{-3}(2),\ f_4(2),\ g_4(2)$

$$(W^* - V_c + 1) f_2(0) - \frac{d g_2(0)}{dr} - 2\frac{g_2(0)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 F(r) \left\{ - f_{-1}(2) - f_2(2) + \sqrt{\frac{3}{7}} f_{-3}(2) \right.$$

$$\left. + \sqrt{\frac{18}{7}} f_4(2) \right\} = 0$$

$$(W^* - V_c - 1) g_2(0) + \frac{d f_2(0)}{dr} - 2\frac{f_2(0)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 F(r) \left\{ - g_{-1}(2) - g_2(2) + \sqrt{\frac{3}{7}} g_{-3}(2) \right.$$

$$\left. + \sqrt{\frac{18}{7}} g_4(2) \right\} = 0$$

$$(W^* - E_2 - V_c + 1) f_{-1}(2) - \frac{d g_{-1}(2)}{dr} + \frac{g_{-1}(2)}{r}$$

$$+ \frac{1}{10} e^2 \varphi_0 f(r) \left\{ - f_2(0) + f_2(2) - \sqrt{\frac{30}{7}} f_{-3}(2) \right\} = 0$$

$$(W^* - E_2 - V_c - 1) g_{-1}(2) + \frac{d f_{-1}(2)}{dr} + \frac{f_{-1}(2)}{r}$$

$$+ \frac{1}{10} e^2 \varphi_0 f(r) \left\{ - g_2(0) + g_2(2) - \sqrt{\frac{30}{7}} g_{-3}(2) \right\} = 0$$

$$(W^* - E_2 - V_c + 1) f_2(2) - \frac{d g_2(2)}{dr} - 2 \frac{g_2(2)}{r}$$

$$+ \frac{1}{10} e^2 \varphi_0 f(r) \left\{ - f_2(0) + f_{-1}(2) + \frac{10}{7} \sqrt{\frac{3}{28}} f_{-3}(2) \right.$$

$$\left. - \frac{6}{7} \sqrt{\frac{2}{7}} f_4(2) \right\} = 0$$

$$(W^* - E_2 - V_c - 1) g_2(2) + \frac{d f_2(2)}{dr} - 2 \frac{f_2(2)}{r}$$

$$+ \frac{1}{10} e^2 \varphi_0 f(r) \left\{ - g_2(0) + g_{-1}(2) + \frac{10}{7} \sqrt{\frac{3}{28}} g_{-3}(2) \right.$$

$$\left. - \frac{6}{7} \sqrt{\frac{2}{7}} g_4(2) \right\} = 0$$

$$(W^* - E_2 - V_c + 1)\, \mathcal{F}_{-3}(2) - \frac{d g_{-3}(2)}{dr} + 3\,\frac{g_{-3}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{\frac{3}{7}}\, \mathcal{F}_2(0) - \sqrt{\frac{30}{7}}\, \mathcal{F}_{-1}(2) + \frac{10}{7}\sqrt{\frac{3}{28}}\, \mathcal{F}_2(2) \right.$$

$$\left. - \frac{20}{49}\, \mathcal{F}_{-3}(2) + \frac{12}{49}\sqrt{\frac{3}{2}}\, \mathcal{F}_4(2) \right\} = 0$$

$$(W^* - E_2 - V_c - 1)\, g_{-3}(2) + \frac{d \mathcal{F}_{-3}(2)}{dr} + 3\,\frac{\mathcal{F}_{-3}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{\frac{3}{7}}\, g_2(0) - \sqrt{\frac{30}{7}}\, g_{-1}(2) + \frac{10}{7}\sqrt{\frac{3}{28}}\, g_2(2) \right.$$

$$\left. - \frac{20}{49}\, g_{-3}(2) + \frac{12}{49}\sqrt{\frac{3}{2}}\, g_4(2) \right\} = 0$$

$$(W^* - E_2 - V_c + 1)\, \mathcal{F}_4(2) - \frac{d g_4(2)}{dr} - 4\,\frac{g_4(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{\frac{18}{7}}\, \mathcal{F}_2(0) - \frac{6}{7}\sqrt{\frac{2}{7}}\, \mathcal{F}_2(2) + \frac{12}{49}\sqrt{\frac{3}{2}}\, \mathcal{F}_{-3}(2) \right.$$

$$\left. + \frac{100}{98}\, \mathcal{F}_4(2) \right\} = 0$$

$$(W^* - E_2 - V_c - 1)\, g_4(2) + \frac{d \mathcal{F}_4(2)}{dr} - 4\,\frac{\mathcal{F}_4(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{\frac{18}{7}}\, g_2(0) - \frac{6}{7}\sqrt{\frac{2}{7}}\, g_2(2) + \frac{12}{49}\sqrt{\frac{3}{2}}\, g_{-3}(2) \right.$$

$$\left. + \frac{100}{98}\, g_4(2) \right\} = 0$$

SET II $\mathcal{F}_{-2}(0),\ g_{-2}(0),\ \mathcal{F}_1(2),\ g_1(2),\ \mathcal{F}_2(2),\ g_{-2}(2)$

$\mathcal{F}_3(2),\ g_3(2),\ \mathcal{F}_{-4}(2),\ g_{-4}(2)$

$$(W^* - V_c + 1)\,\mathcal{F}_{-2}(0) - \frac{dg_{-2}(0)}{dr} + 2\,\frac{g_{-2}(0)}{r}$$

$$+ \frac{1}{10}\,e^2 Q_0 f(r)\left\{-\mathcal{F}_1(2) - \mathcal{F}_{-2}(2) + \sqrt{\tfrac{3}{7}}\,\mathcal{F}_3(2)\right.$$

$$\left. + \sqrt{\tfrac{18}{7}}\,\mathcal{F}_{-4}(2)\right\} = 0$$

$$(W^* - V_c - 1)\,g_{-2}(0) + \frac{d\mathcal{F}_{-2}(0)}{dr} + 2\,\frac{\mathcal{F}_{-2}(0)}{r}$$

$$+ \frac{1}{10}\,e^2 Q_0 f(r)\left\{-g_1(2) - g_{-2}(2) + \sqrt{\tfrac{3}{7}}\,g_3(2)\right.$$

$$\left. + \sqrt{\tfrac{18}{7}}\,g_{-4}(2)\right\} = 0$$

$$\left(W^* - E_2 - V_c + 1\right) f_1(2) - \frac{dg_1(2)}{dr} - \frac{g_1(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ - f_{-2}(0) + f_{-2}(2) - \sqrt{\frac{30}{7}} f_3(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c - 1\right) g_1(2) + \frac{df_1(2)}{dr} - \frac{f_1(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ - g_{-2}(0) + g_{-2}(2) - \sqrt{\frac{30}{7}} g_3(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c + 1\right) f_{-2}(2) - \frac{dg_{-2}(2)}{dr} + 2 \frac{g_{-2}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ - f_{-2}(0) + f_1(2) + \frac{10}{7}\sqrt{\frac{3}{28}} f_3(2) \right.$$

$$\left. - \frac{6}{7}\sqrt{\frac{2}{7}} f_{-4}(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c - 1\right) g_{-2}(2) + \frac{df_{-2}(2)}{dr} + 2 \frac{f_{-2}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ - g_{-2}(0) + g_1(2) + \frac{10}{7}\sqrt{\frac{3}{28}} g_3(2) \right.$$

$$\left. - \frac{6}{7}\sqrt{\frac{2}{7}} g_{-4}(2) \right\} = 0$$

$$(W^* - E_2 - V_c + 1) f_3(2) - \frac{d g_3(2)}{dr} - 3\frac{g_3(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{\tfrac{3}{7}} f_{-2}(0) - \sqrt{\tfrac{30}{7}} f_1(2) + \frac{10}{7}\sqrt{\frac{3}{28}} f_{-2}(2) \right.$$

$$\left. - \frac{20}{49} f_3(2) + \frac{12}{49}\sqrt{\frac{3}{2}} f_{-4}(2) \right\} = 0$$

$$(W^* - E_2 - V_c - 1) g_3(2) + \frac{d f_3(2)}{dr} - 3\frac{f_3(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{\tfrac{3}{7}} g_{-2}(0) - \sqrt{\tfrac{30}{7}} g_1(2) + \frac{10}{7}\sqrt{\frac{3}{28}} g_{-2}(2) \right.$$

$$\left. - \frac{20}{49} g_3(2) + \frac{12}{49}\sqrt{\frac{3}{2}} g_{-4}(2) \right\} = 0$$

$$(W^* - E_2 - V_c + 1) f_{-4}(2) - \frac{d g_{-4}(2)}{dr} + 4\frac{g_{-4}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{\tfrac{18}{7}} f_{-2}(0) - \frac{6}{7}\sqrt{\tfrac{2}{7}} f_{-2}(2) + \frac{12}{49}\sqrt{\tfrac{3}{2}} f_3(2) \right.$$

$$\left. + \frac{100}{98} f_{-4}(2) \right\} = 0$$

$$(W^* - E_2 - V_c - 1) g_{-4}(2) + \frac{d f_{-4}(2)}{dr} + 4\frac{f_{-4}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{\tfrac{18}{7}} g_{-2}(0) - \frac{6}{7}\sqrt{\tfrac{2}{7}} g_{-2}(2) + \frac{12}{49}\sqrt{\tfrac{3}{2}} g_3(2) \right.$$

$$\left. + \frac{100}{98} g_{-4}(2) \right\} = 0$$

J = 5/2

SET I $\quad f_3(0),\ g_3(0),\ f_1(2),\ g_1(2),\ f_{-2}(2),\ g_{-2}(2)$

$f_3(2),\ g_3(2),\ f_{-4}(2),\ g_{-4}(2),\ f_5(2),\ g_5(2)$

$$\left(W^* - V_c + 1\right) f_3(0) - \frac{d g_3(0)}{dr} - 3\,\frac{g_3(0)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ f_1(2) - \sqrt{\frac{2}{7}}\, f_{-2}(2) - \sqrt{\frac{8}{7}}\, f_3(2) + \sqrt{\frac{4}{21}}\, f_{-4}(2) + \sqrt{\frac{50}{21}}\, f_5(2) \right\} = 0$$

$$\left(W^* - V_c - 1\right) g_3(0) + \frac{d f_3(0)}{dr} - 3\,\frac{f_3(0)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ g_1(2) - \sqrt{\frac{2}{7}}\, g_{-2}(2) - \sqrt{\frac{8}{7}}\, g_3(2) + \sqrt{\frac{4}{21}}\, g_{-4}(2) + \sqrt{\frac{50}{21}}\, g_5(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c + 1\right) f_1(2) - \frac{dg_1(2)}{dr} - \frac{g_1(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ f_3(0) - \sqrt{\frac{2}{7}} f_{-2}(2) - \sqrt{\frac{8}{7}} f_3(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c - 1\right) g_1(2) + \frac{df_1(2)}{dr} - \frac{f_1(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ g_3(0) - \sqrt{\frac{2}{7}} g_{-2}(2) - \sqrt{\frac{8}{7}} g_3(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c + 1\right) f_{-2}(2) - \frac{dg_{-2}(2)}{dr} + 2\frac{g_{-2}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ -\sqrt{\frac{3}{7}} f_3(0) - \sqrt{\frac{2}{7}} f_1(2) - \frac{5}{7} f_{-2}(2) \right.$$

$$\left. + \frac{10}{49} f_3(2) - \frac{180}{245}\sqrt{\frac{3}{2}} f_{-4}(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c - 1\right) g_{-2} + \frac{df_{-2}(2)}{dr} + 2\frac{f_{-2}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ -\sqrt{\frac{2}{7}} g_3(0) - \sqrt{\frac{2}{7}} g_1(2) - \frac{5}{7} g_{-2}(2) \right.$$

$$\left. + \frac{10}{49} g_3(2) - \frac{180}{245}\sqrt{\frac{3}{2}} g_{-4}(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c + 1\right) f_3(2) - \frac{d g_3(2)}{dr} - 3\,\frac{g_3(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ -\sqrt{\frac{8}{7}} f_3(0) - \sqrt{\frac{8}{7}} f_1(2) + \frac{10}{49} f_{-2}(2) \right.$$

$$\left. - \frac{20}{49} f_3(2) + \frac{17\sqrt{6}}{147} f_{-4}(2) - \frac{50}{49}\sqrt{\frac{1}{3}} f_5(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c - 1\right) g_3(2) + \frac{d f_3(2)}{dr} - 3\,\frac{f_3(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ -\sqrt{\frac{8}{7}} g_3(0) - \sqrt{\frac{8}{7}} g_1(2) + \frac{10}{49} g_{-2}(2) \right.$$

$$\left. - \frac{20}{49} g_3(2) + \frac{17\sqrt{6}}{147} g_{-4}(2) - \frac{50}{49}\sqrt{\frac{1}{3}} g_5(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c + 1\right) f_{-4}(2) - \frac{d g_{-4}(2)}{dr} + \frac{4\, g_{-4}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{\frac{4}{21}} f_3(0) - \frac{180}{245}\sqrt{\frac{3}{2}} f_{-2}(2) + \frac{17\sqrt{6}}{147} f_3(2) \right.$$

$$\left. + \frac{50}{294} f_{-4}(2) + \frac{50}{147\sqrt{2}} f_5(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c - 1\right) g_{-4}(2) + \frac{d f_{-4}(2)}{dr} + 4 \frac{f_{-4}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{\frac{4}{21}}\, g_3(0) - \frac{180}{245} \sqrt{\frac{3}{2}}\, g_{-2}(2) + \frac{17\sqrt{6}}{147}\, g_3(2) \right.$$

$$\left. + \frac{50}{294}\, g_{-4}(2) + \frac{50}{147\sqrt{2}}\, g_5(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c + 1\right) f_5(2) - \frac{d g_5(2)}{dr} - 5 \frac{g_5(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \frac{10}{\sqrt{42}}\, f_3(0) - \frac{50}{49\sqrt{3}}\, f_3(2) + \frac{50}{147\sqrt{2}}\, f_{-4}(2) \right.$$

$$\left. + \frac{20}{21}\, f_5(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c - 1\right) g_5(2) + \frac{d f_5(2)}{dr} - 5 \frac{f_5(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \frac{10}{\sqrt{42}}\, g_3(0) - \frac{50}{49\sqrt{3}}\, g_3(2) + \frac{50}{147\sqrt{2}}\, g_{-4}(2) \right.$$

$$\left. + \frac{20}{21}\, g_5(2) \right\} = 0$$

SET II $\quad f_{-3}(0),\ g_{-3}(0),\ f_{-1}(2),\ g_{-1}(2),\ f_{2}(2),\ g_{2}(2),$

$f_{-3}(2),\ g_{-3}(2),\ f_{4}(2),\ g_{4}(2),\ f_{-5}(2),\ g_{-5}(2)$

$$(W^* - V_c + 1)\, f_{-3}(0) - \frac{d\, g_{-3}(0)}{dr} + 3\,\frac{g_{-3}(0)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ f_{-1}(2) - \sqrt{\tfrac{2}{7}}\, f_2(2) - \sqrt{\tfrac{8}{7}}\, f_{-3}(2) \right.$$

$$\left. + \sqrt{\tfrac{4}{21}}\, f_4(2) + \sqrt{\tfrac{50}{21}}\, f_{-5}(2) \right\} = 0$$

$$(W^* - V_c - 1)\, g_{-3}(0) + \frac{d f_{-3}(0)}{dr} + 3\,\frac{f_{-3}(0)}{r},$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ g_{-1}(2) - \sqrt{\tfrac{2}{7}}\, g_2(2) - \sqrt{\tfrac{8}{7}}\, g_{-3}(2) \right.$$

$$\left. + \sqrt{\tfrac{4}{21}}\, g_4(2) + \sqrt{\tfrac{50}{21}}\, g_{-5}(2) \right\} = 0$$

$$(W^* - E_2 - V_c + 1)\, f_{-1}(2) - \frac{d\, g_{-1}(2)}{dr} + \frac{g_{-1}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ f_{-3}(0) - \sqrt{\frac{2}{7}}\, f_2(2) - \sqrt{\frac{8}{7}}\, f_{-3}(2) \right\} = 0$$

$$(W^* - E_2 - V_c - 1)\, g_{-1}(2) + \frac{d\, f_{-1}(2)}{dr} + \frac{f_{-1}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ g_{-3}(0) - \sqrt{\frac{2}{7}}\, g_2(2) - \sqrt{\frac{8}{7}}\, g_{-3}(2) \right\} = 0$$

$$(W^* - E_2 - V_c + 1)\, f_2(2) - \frac{d\, g_2(2)}{dr} - 2\frac{g_2(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ -\sqrt{\frac{2}{7}}\, f_{-3}(0) - \sqrt{\frac{2}{7}}\, f_{-1}(2) - \frac{5}{7} f_2(2) \right.$$

$$\left. + \frac{10}{49} f_{-3}(2) - \frac{180}{245}\sqrt{\frac{3}{2}}\, f_4(2) \right\} = 0$$

$$(W^* - E_2 - V_c - 1)\, g_2(2) + \frac{d\, f_2(2)}{dr} - 2\frac{f_2(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ -\sqrt{\frac{2}{7}}\, g_{-3}(0) - \sqrt{\frac{2}{7}}\, g_{-1}(2) - \frac{5}{7} g_2(2) \right.$$

$$\left. + \frac{10}{49} g_{-3}(2) - \frac{180}{245}\sqrt{\frac{3}{2}}\, g_4(2) \right\} = 0$$

$$(W^* - E_2 - V_c + 1)\, f_{-3}(2) - \frac{d g_{-3}(2)}{dr} + 3\,\frac{g_{-3}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ -\sqrt{\frac{8}{7}}\, f_{-3}(0) - \sqrt{\frac{8}{7}}\, f_{-1}(2) + \frac{10}{49} f_2(2) - \frac{20}{49} f_{-3}(2) \right.$$

$$\left. + \frac{17}{147}\sqrt{6}\, f_4(2) - \frac{50}{49\sqrt{3}}\, f_{-5}(2) \right\} = 0$$

$$(W^* - E_2 - V_c - 1)\, g_{-3}(2) + \frac{d f_{-3}(2)}{dr} + 3\,\frac{f_{-3}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ -\sqrt{\frac{8}{7}}\, g_{-3}(0) - \sqrt{\frac{8}{7}}\, g_{-1}(2) + \frac{10}{49} g_2(2) - \frac{20}{49} g_{-3}(2) \right.$$

$$\left. + \frac{17}{147}\sqrt{6}\, g_4(2) - \frac{50}{49\sqrt{3}}\, g_{-5}(2) \right\} = 0$$

$$(W^* - E_2 - V_c + 1)\, f_4(2) - \frac{d g_4(2)}{dr} - 4\,\frac{g_4(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{\frac{4}{21}}\, f_{-3}(0) - \frac{180}{245}\sqrt{\frac{3}{2}}\, f_2(2) + \frac{17\sqrt{6}}{147} f_{-3}(2) \right.$$

$$\left. + \frac{50}{294}\, f_4(2) + \frac{50}{147\sqrt{2}}\, f_{-5}(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c - 1\right) g_4(2) + \frac{d f_4(2)}{dr} - 4\,\frac{f_4(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \sqrt{\frac{4}{21}}\, g_{-3}(0) - \frac{180}{245}\sqrt{\frac{3}{2}}\, g_2(2) + \frac{17\sqrt{6}}{147}\, g_{-3}(2) \right.$$

$$\left. + \frac{50}{294}\, g_4(2) + \frac{50}{147\sqrt{2}}\, g_{-5}(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c + 1\right) f_{-5}(2) - \frac{d g_{-5}(2)}{dr} + 5\,\frac{g_{-5}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \frac{10}{\sqrt{42}}\, f_{-3}(0) - \frac{50}{49\sqrt{3}}\, f_{-3}(2) + \frac{50}{147\sqrt{2}}\, f_4(2) \right.$$

$$\left. + \frac{20}{21}\, f_{-5}(2) \right\} = 0$$

$$\left(W^* - E_2 - V_c - 1\right) g_{-5}(2) + \frac{d f_{-5}(2)}{dr} + 5 \cdot \frac{f_{-5}(2)}{r}$$

$$+ \frac{1}{10} e^2 Q_0 f(r) \left\{ \frac{10}{\sqrt{42}}\, g_{-3}(0) - \frac{50}{49\sqrt{3}}\, g_{-3}(2) + \frac{50}{147\sqrt{2}}\, g_4(2) \right.$$

$$\left. + \frac{20}{21}\, g_{-5}(2) \right\} = 0$$

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