QUOTIENT-LIKE EXTENSIONS OF

RINGS OF FUNCTIONS

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SCOPE AND CONTENTS: In this thesis we study the quotient-like extensions of the rings of all real continuous functions on a topological space previously considered by Fine-Gillman-Lambek and certain generalizations of these. In particular we obtain a result on the absence of real maximal ideals in certain such rings; a connection between the maximal ideal spaces of such rings and projective covers of compact Hausdorff spaces; and a useful description of the relationship between the quotient-like extensions of rings of real functions and their complex counterpart.

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PREFACE

The concept of "ring of quotients" of a ring was introduced by R. Johnson [17], Y. Utumi [24] and others, and a generalized ring of quotients was studied by G. Findlay and J. Lambek [7]. N. Fine, L. Gillman and J. Lambek [8] considered this concept for the ring C(X) of all real-valued continuous functions on a completely regular Hausdorff space X, in which case the maximal ring of quotients is realized as the ring of all continuous functions on the dense open subsets in X, modulo the equivalence relation which identifies two functions if they agree on the intersection of their domains. In a recent paper [1] B. Banaschewski has generalized this result to arbitrary commutative semi-prime rings by describing the maximal ring of quotients as a ring of functions (with variable domain) modulo a suitable ideal.

In [8], many results actually concern extensions of the ring C(X) which are more general than their maximal rings of quotients, and it is a further study of these extensions that this thesis is devoted to. A brief synopsis of the material presented here is given below.

In Chapter 0, we collect together the basic definitions and theorems which we utilize in the ensuing chapters. In particular we list the definition of real compact space, ring of quotients of a ring, Archimedean F-ring, and some examples of dense ideals of C(X).

In Chapter I, we introduce the quotient-like extension $Q_{\mathfrak{D}}(X)$ associated with a filter base $\mathfrak{D}$ consisting of dense subsets of X in a manner suggested by [1] and observe that these extensions can also be described as direct limits of suitable direct systems which relates them to [8]. Further we give a necessary and sufficient condition for such a quotient-like extension to be a ring of quotient of C(X) and a self-contained proof of the particular case of Banaschewski's theorem (Theorem 6, Chapter I)(cf.[1]) as it applies to the rings C(X). Finally we describe certain relations between continuous mappings from a space X into a space Y and homomorphisms from a quotient-like extension of C(Y) into that of C(X).

In Chapter II, we consider the natural partial order on an extension ring $Q_{\mathfrak{D}}(X)$ of C(X), showing that the resulting partially ordered rings are Archimedean F-rings. Further we establish a necessary and sufficient condition for certain rings $Q_{\mathfrak{D}}(X)$ to be totally unreal, i.e. to have no unitary algebra homomorphisms to $\mathbb{R}$. Finally we introduce the m-topology on a commutative F-ring with unit, and apply this to the extensions $Q_{\mathfrak{D}}(X)$ of $C(X)$.

In Chapter III, we study a specific type of injective system $(B_\alpha, \varphi_{\alpha\beta})$ in which all B_α are normed algebras over the reals with unit and $\varphi_{\alpha\beta}$ are norm-preserving embeddings which generalizes the injective system $(C^*(D))(D \in \mathfrak{D})$ with respect to the restriction homomorphisms, where * denotes boundedness. The main object in this chapter is to obtain a connection with the projective cover of the Stone-Čech compactification of the underlying space (in the category of all compact Hausdorff spaces and continuous mappings); our result is that the maximal ideal space of $Q^*_{\mathfrak{D}}(X)$ is the projective cover of βX if $\mathfrak{D}$ contains all disconnected dense open subsets of X, where $Q^*_{\mathfrak{D}}(X)$ is the injective limit of the injective system $(C^*(D))(D \in \mathfrak{D})$, and βX denotes the Stone-Čech compactification of X. The analogous partial generalization for arbitrary commutative semi-simple rings with unit and Hausdorff maximal ideal space is proved independently.

In Chapter IV, we consider certain generalizations of the preceding work. In the first section, we replace the range space $\mathbb{R}$ (the field of real numbers) by $\mathbb{C}$ (the field of complex numbers), and show that analogous results to those obtained in Chapter I and II hold;

our main tool here is the complexification of a real algebra. We prove that the maximal ring of quotients of the ring of all complex continuous functions on X is isomorphic to the complexification of the maximal ring of quotients of its subring of real functions, and that the maximal ideal spaces of the two rings of quotients are homeomorphic. In the second section we discuss relevant extensions of the algebra $C^{\infty}(U)$ of all unrestrictedly differentiable functions on an open subset $U \subseteq R^n$, and show among other things that the maximal ring of quotients of $C^{\infty}(U)$ is totally unreal.

CHAPTER 0

Preliminaries

This chapter is a collection of all the basic definitions and results which will be needed in the ensuing chapters.

Section 1: Rings of Functions.

Let X be a topological space. Let $C(X)$ denote the ring of all continuous functions from X into the reals $\mathbb{R}$, under the functional operations; the subring of bounded functions in $C(X)$ is denoted by $C^*(X)$. It is obvious that $C(X)$ and $C^*(X)$ are commutative rings with unit 1. They are also lattices under the pointwise definition of order. It is clear that $C(X)$ is semi-prime, i.e. there is no nilpotent ideal except (o), and it is known as semi-simple.

In studying $C(X)$, we assume that the space X is a completely regular Hausdorff space. The standard reference for the theory of $C(X)$ is [12] and[23].

Definition: Let $f \in C(X)$. The set of points in X for which f vanishes is said to be the zero set of f and is denoted by $Z(f)$. The complement $X-Z(f)$ is said to be the cozero-set of f and is denoted by $Coz(f)$.

<u>Theorem 1</u>: Every completely regular space X has a compactification βX, with the following equivalent properties.

i) Every continuous mapping φ from X into any compact space Y has a continuous extensition $\bar{\varphi}$ from βX into Y.

ii) Every function f in $C^*(X)$ has an extension to a function f^β in $C(\beta X)$.

<u>Definition</u>: The compact space βX is said to be the Stone-Čech compactification of X.

<u>Theorem 2</u>: Let X be a completely regular space. X is open in βX if and only X is locally compact.

<u>Theorem 3</u>: Every continuous function on a dense open set V in X can be continued to an open set ($\supset V$) in βX. Every continuous function on a dense cozero-set in X can be continued to a cozero-set in βX.

<u>Definition</u>: A space X is said to be realcompact if every real maximal in C(X) is fixed.

<u>Theorem 4</u>: Every completely regular space X has a real-compactification υX, contained in βX, with the following equivalent properties.

i) Every continuous mapping φ from X into any real compact space Y has a continuous extension φ^{o} from νX into Y.

ii) Every function f in C(X) has an extension to a function f^{ν} in $C(\nu X)$.

Theorem 5: X is realcompact if and only if, to each homomorphism φ from C(X) onto $\mathbb{R}$ there corresponds a point x of X such that $\varphi(f) = f(x)$ for all $f \in C(X)$.

Theorem 6: Let φ be a unitary homomorphism from C(Y) into C(X). If Y is realcompact, then there exists a unique continuous mapping ψ of X into Y such that $\varphi(g) = g \circ \psi$ for all $g \in C(Y)$.

Section 2: Rings of Quotients.

According to a well-known theorem of algebra, an integral domain can be embedded in a field, called its field of quotients. The simplest form of this is the theorem concerning the integers and the rational numbers. Many generalizations have been given in which a "ring of quotients" is constructed for a given ring [e.g. 1, 7, 8, 17, 22, 24].

Theorem 7: Let A be a commutative ring with unit and S be a multiplicative submonoid of A consisting of non-zero divisors of A. Then there exists an extension ring

$E \supseteq A$ such that

i) All elements in S are invertible in E;

ii) E is generated by $A \cup S^{-1}$;

moreover, E is uniquely determined up to a unique isomorphism extending the identity on A.

Definition: The ring E obtained in Theorem 7 is called a classical ring of quotients of A with respect to S and is denoted by $A[S^{-1}]$. If S is the monoid of all non-zero divisors of A, then $A[S^{-1}]$ is called the full ring of quotients of A.

Definition: An ideal D in A is said to be dense if its only annihilator in A is 0, i.e. D is dense in A if for all $a \in A$, $aD = 0$ implies $a = 0$.

Note that a principal ideal (d) is dense precisely when d is a non-zero-divisor in A.

More generally, we can speak of denseness of any subring. If A is a subring of B, then we shall say that A is dense in B provided that A has no non-zero annihilator in B.

Definition: A ring A is called semi-prime if it has no nilpotent ideal except (0)-equivalently (for commutative A), if it has no nilpotent element except 0.

<u>Proposition 8</u>: For commutative ring A with unit 1, the following holds:

i) A is dense.

ii) If D is dense and $D \subset D'$ then D' is dense.

iii) If D and D' are dense, so are DD' and $D \cap D'$.

iv) If $A \neq 0$ then 0 is not dense.

Let B be a commutative ring containing A and having the same unit element e. For $b \in B$, we write

$$b^{-1}A = \{a \in A \mid ba \in A\}.$$

Obviously, $b^{-1}A$ is an ideal in A. For $b = 0$ or e, $b^{-1}A$ is dense; it is A itself.

<u>Definition</u>: An extension ring $B \supseteq A$ of a ring A is called a <u>ring of left quotients</u> (or <u>rational extension</u>) of A if for any a, b in B, $b \neq 0$, there exists a c in A and an integer k such that $ca + ka \in A$ and $cb + kb \neq 0$.

It is obvious that the reference to the integer k is redundant if A contains a unit, i.e. B is a ring of quotients of A if for every $b \in B$, $b^{-1}A$ is dense in B.

<u>Definition</u>: <u>A maximal ring of quotients</u> of a ring A is a ring $Q \supset A$ of quotients of A such that there exists no proper extension ring $E \supset Q$ which is also a ring of quotients of A, and denoted by $Q(A)$.

<u>Theorem 9</u>: Any ring A has a maximal ring of (left) quotients Q which is unique up to isomorphism over A.

Remark 1: Let $A < B < C$. Then C is a ring of quotients of A if and only if B is a ring of quotients of A and C is a ring of quotients of B.

Remark 2: If B is a ring of quotients of A and D is a dense ideal in A, then D is dense in B.

Theorem 10: Let $B > A$. If A is semi-prime, then B is a ring of quotients of A if and only if $b\cdot(b^{-1}A) \neq 0$ for all non-zero $b \in B$ -- that is, for $0 \neq b \in B$, there exists $a \in A$ such that $0 \neq ba \in A$.

Definition: A ring A is said to be von Neumann regular if for each element a, there exists an element x (in general, not unique) such that $axa = a$. In the commutative case, this may of course be written $a^2x = a$.

Theorem 11: If A is semi-prime, then the maximal ring of quotients of A is von Neumann regular, and the converse holds if A is commutative.

Some examples of dense ideal in $C(X)$.

1. Let $(X, \mathfrak{J})$ be a completely regular space. Then an ideal A in $C(X)$ is dense if the Zariski topology determined by A, i.e. the topology generated by the sets $\mathrm{Coz}(f)$, $f \in A$, coincides with $\mathfrak{J}$.

2. Every free ideal in $C(X)$ is dense.

Let $C_o(X)$ denote the family of all functions in $C(X)$ having compact support and $C_{\infty}(X)$ denote the family of all functions in $C(X)$ vanishing at infinity.

3. $C_o(X)$ (hence $C_{\infty}(X)$) is dense in $C(X)$ if X is locally compact.

4. An ideal A in $C(X)$(or $C^*(X)$) is dense if and only if $\bigcup \mathrm{Coz}(f)$ $(f \in A)$ is topologically dense in X.

5. A prime ideal P in $C(X)$ is dense if $Z[P]$ has a non-isolated cluster point.

6. Every maximal ideal in $C(X)$ is dense if X is perfect.

<u>Section 3</u>: <u>Lattice Ordered Rings</u>.

<u>Definition</u>: A partially ordered ring is a ring A together with a partial order relation $\geqslant$ such that for $a, b \in A$

i) $a \geqslant b$ implies $a + c \geqslant b + c$ for each $c \in A$, and

ii) $a \geqslant o$ and $b \geqslant o$ implies $ab \geqslant o$.

<u>Definition</u>: A partially ordered ring A is said to be Archimedean if for every pair a, b of elements of A, with $a \neq 0$, there is an integer n such that $na \nleq b$.

<u>Definition</u>: A ring homomorphism Θ of a lattice-ordered ring A into a lattice ordered ring A' is called

an ℓ-homomorphism if one of the following equivalent conditions is satisfied for $a, b \in A$:

i) $\theta(a \vee b) = \theta a \vee \theta b$ and $\theta(a \wedge b) = \theta a \wedge \theta b$;

ii) $\theta |a| = |\theta a|$;

iii) $a \wedge b = 0$ implies $\theta a \wedge \theta b = 0$.

<u>Definition</u>: A subset I of a lattice-ordered ring A is said to be an ℓ-ideal of A if:

i) I is a ring ideal of A, and

ii) $a \in I$, $b \in A$, and $|b| \leq |a|$ imply $b \in I$.

<u>Definition</u>: A lattice-ordered ring A is called an F-ring if the following holds:

$a \wedge b = 0$, $c \geqslant 0$, implies that $ca \wedge b = 0$ and $ac \wedge b = 0$.

<u>Note that</u> every totally ordered ring is F-ring, since, in a totally ordered ring, $a \wedge b = 0$ implies either $a = 0$ or $b = 0$.

If A is an F-ring, we will call a subring of the ring A a sub-F-ring if it is also a sublattice of the lattice A.

<u>Theorem 12</u>: If A is an F-ring, then:

i) Every sub-F-ring of A is F-ring;

ii) Every ℓ-homomorphic image of A is an F-ring.

<u>Definition</u>: An ideal I in a partially ordered ring A is said to be <u>convex</u> if whenever $0 \leqslant x \leqslant y$, and $y \in I$, then $x \in I$.

<u>Theorem 13</u>: Let I be an ideal in a partially ordered ring A. In order that A/I be a partially ordered ring, according to the definition:

$a + I \geqslant 0$ if there exists $x \in A$ such that $x \geqslant 0$ and $a \equiv x (\mathrm{mod}\ I)$, it is necessary and sufficient that I be convex.

<u>Theorem 14</u>: For a convex ideal I in a lattice ordered ring A, the following are equivalent.

i) I is ℓ-ideal.

ii) $a \in I$ implies $|a| \in I$.

iii) $a, b \in I$ implies $a \vee b \in I$.

iv) $a \vee b + I = (a + I) \vee (b + I)$--whence A/I is a lattice.

v) $a + I \geqslant 0$ iff $a \equiv |a| \ (\mathrm{mod}\ I)$.

<u>Section 4</u>: <u>Categories and Direct Limits</u>.

<u>Definition</u>: A category $\mathcal{C}$ consists of a class of <u>objects</u> and with each pair A, B of objects a set M(A,B) called the set of <u>morphisms</u> $f : A \longrightarrow B$ such that for any three objects A, B, C in $\mathcal{C}$ there is given a mapping

$M(A,B) \times M(B,C) \longrightarrow M(A,C)$ denoted by $(f,g) \rightsquigarrow g \cdot f$ which satisfies (1) $f : A \longrightarrow B$, $g : B \longrightarrow C$, $h : C \longrightarrow D$ implies $h \cdot (g \cdot f) = (h \cdot g) \cdot f$. (2) For each object A in $\mathcal{C}$ there exists a morphism e_A in $M(A,A)$ such that $e_A \cdot f = f$ for all f in $M(B,A)$ and $f \cdot e_A = f$ for all f in $M(A,B)$.

Definition: A set $(I, \geqslant)$ is called directed (up directed) if for any j, k in I there exists i in I such that $i \geqslant j, k$.

Definition: For a directed set I, a system $(D_i, f_{ij})_{i,\, j\, \in I}$ is called a direct system in $\mathcal{C}$ over I if $D_i \in \mathcal{C}$ for each $i \in I$ and for each pair (i, j) with $i \leqslant j$ the morphism $f_{ij} : D_i \longrightarrow D_j$ satisfies $f_{jk} \cdot f_{ij} = f_{ik}$ for $i \leqslant j \leqslant k$ and $f_{ii} = e_{D_i}$.

Definition: Let $(D_i, f_{ij})_{i,\, j\, \in I}$ be a direct system in $\mathcal{C}$. A family of morphisms $(D_i \longrightarrow X)_{i \in I}$ is called a compatible family with respect to the system if for any pair (i, j) with $i \leqslant j$, $D_i \longrightarrow D_j \longrightarrow X = D_i \longrightarrow X$. L is called the direct limit (or colimit) of the system(D_i, f_{ij}) in the category $\mathcal{C}$ if there exists a compatible $(D_i \longrightarrow L)_{i \in I}$ such that for any compatible family $(D_i \longrightarrow Y)_{i \in I}$ there exists a unique morphism $L \longrightarrow Y$ for which $D_i \longrightarrow L \longrightarrow Y = D_i \longrightarrow Y$ for each $i \in I$; $(D_i \longrightarrow L)_{i \in I}$ is called a limit family.

CHAPTER I

Algebraic properties of the ring $Q_{\mathcal{D}}(X)$.

Section 1: Direct limits.

Let X be a completely regular Hausdorff space and $\mathcal{D}$ be a filter base of dense subsets of X. For $D \in \mathcal{D}$, let $C_D(X)$ denote the ring of all real-valued functions f on X which have continuous restriction $f|D$ to D. Put $C_{\mathcal{D}}(X) = \bigcup_{D \in \mathcal{D}} C_D(X)$. Since $D \subseteq E$ implies $C_D(X) \supseteq C_E(X)$ and for $f, g \in C_{\mathcal{D}}(X)$ with $f \in C_D(X)$, $g \in C_E(X)$, $f \dotplus g \in C_{D'}(X)$ for some $D' \in \mathcal{D}$ with $D' \subseteq D \cap E$, $C_{\mathcal{D}}(X)$ is a ring of functions. Now let $Z_D(X) = \{ f \in C_D(X) \mid f|D = 0 \}$ and put $Z_{\mathcal{D}}(X) = \bigcup_{D \in \mathcal{D}} Z_D(X)$. Then $Z_D(X)$ is an ideal in $C_D(X)$, and $Z_{\mathcal{D}}(X)$ is an ideal in $C_{\mathcal{D}}(X)$ since for any $f \in C_{\mathcal{D}}(X)$ with $f \in C_D(X)$ and $h \in Z_{\mathcal{D}}(X)$ with $h \in Z_E(X)$, $fh \in Z_{D'}(X)$ for some $D' \in \mathcal{D}$ with $D' \subseteq D \cap E$, and $f \dotplus g \in Z_{\mathcal{D}}(X)$ whenever $f, g \in Z_{\mathcal{D}}(X)$. Finally we put $Q_{\mathcal{D}}(X) = C_{\mathcal{D}}(X)/Z_{\mathcal{D}}(X)$. Then $Q_{\mathcal{D}}(X)$ is a commutative semi-prime ring with unit.

Proposition 1:

i) $C_D(X) \cap Z_{\mathcal{D}}(X) = Z_D(X)$ for any $D \in \mathcal{D}$.

ii) The restriction $\rho_D : f \rightsquigarrow f|D$ induces an isomorphism $C_D(X)/Z_D(X) \longrightarrow C(D)$ for each $D \in \mathcal{D}$.

iii) The natural mapping $\nu : C_{\mathcal{S}}(X) \longrightarrow Q_{\mathcal{S}}(X)$ determines, for each $D \in \mathcal{S}$, an embedding $j_D : C(D) \longrightarrow Q_{\mathcal{S}}(X)$ such that $\nu(f) = j_D(f|D)$ for $f \in C_D(X)$.

iv) For any D, $E \in \mathcal{S}$ where $D \subseteq E$, $j_D(f|D) = j_E(f)$ for all $f \in C(E)$.

Proof:

i) Clearly $C_D(X) \cap Z\ (X) \supseteq Z_D(X)$. Let $f \in C_D(X) \cap Z_{\mathcal{S}}(X)$ and $f \in Z_E(X)$, $E \in \mathcal{S}$. Suppose $f \notin Z_D(X)$; then there exists a point $x_o \in D$ such that $f(x_o) \neq 0$, and $f|E \cap D = 0$. But D is completely regular, hence there exists an open neighborhood V of x_o such that $f|V \not\equiv 0$ where $E \cap D \cap V \neq \emptyset$. Hence $f|E \not\equiv 0$; this is a contradiction.

ii) We show ρ_D is an epimorphism. Let f in C(D). Define f^* on X by $f^*(x) = f(x)$ if $x \in D$ and $f^* = 0$ if $x \in X \setminus D$; then clearly $f^* \in C_D(X)$ and $f^*|D = f$. Hence there exists f^* in $C_D(X)$ such that $\rho_D(f^*) = f$; i.e. $_D$ is an epimorphism. $\mathrm{Ker}(\rho_D) = Z_D(X)$ is trivial; thus ρ_D induces an isomorphism:

$$C_D(X)/Z_D(X) \longrightarrow C(D).$$

iii) For $f \in C_D(X)$, $j_D(f|D) = \nu(f)$ is well defined, since for f_1^* and f_2^* in $C_D(X)$ such that $f_1^* | D = f|D = f_2^*|D$, then $f_1^* - f_2^* \in Z_D(X)$; i.e. $f_1^* - f_2^* \in Z_{\mathcal{S}}(X)$; thus $\nu(f_1^*) - \nu(f_2^*) = \nu(f_1^* - f_2^*) = 0$; i.e. $\nu(f_1^*) = \nu(f_2^*)$. Now for $f \in C(D)$, if $j_D(f) = 0$, then there exists f^* in $C_D(X)$ such that $f^*|D = f$; then $j_D(f^*|D) = \nu(f^*)$; this implies that $f^* \in Z_E(X)$ for some $E \in \mathcal{S}$. Hence $f^*|D \cap E = 0$. But $f^*|D \in C(D)$, and $E \cap D \subseteq D$ implies that $f^*|D = 0$; i.e. $f = 0$. Hence j_D is a monomorphism.

iv) Let $f \in C(E)$, and f^* be an element in $C_E(X) \subseteq C_D(X)$ such that $f^*|E = f$; then by (iii)

$$j_D(f^*|D) = j_D(f|D) = \nu(f^*);$$

on the other hand,

$$j_E(f^*|E) = j_E(f) = \nu(f^*). \qquad \text{Q.E.D.}$$

Now, the family of rings $(C(D))_{D \in \mathcal{S}}$ together with the restriction mappings $\rho_{DE} : C(E) \longrightarrow C(D)$ for each pair $D, E \in \mathcal{S}$ where $D \subseteq E$ form a direct system, and the proposition shows that the maps $j_D : C(D) \longrightarrow Q_{\mathcal{S}}(X)$ are compatible with respect to this system: i.e. if $D, E \in \mathcal{S}$, $D \subseteq E$, then the following diagram is commutative

$$\begin{tikzcd}
C(D) \arrow[r] & Q_{\mathcal{S}}(X) \\
C(E) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

<u>Theorem 2</u>: $Q_{\mathcal{S}}(X)$ is the direct limit of the direct system $(C(D), \rho_{DE})_{D,\, E \in \mathcal{S}}$, with $(j_D)_{D \in \mathcal{S}}$ as a family of limit homomorphisms, in the category of all rings with unit and unitary ring homomorphisms.

Proof: To prove this we have to establish that for any given unitary ring homomorphism $\varphi_D : C(D) \longrightarrow R$ (R is arbitrary ring with unit) such that for D, E in $\mathcal{D}$, with $D \subseteq E$: $\varphi_D(f|D) = \varphi_E(f)$ for $f \in C(E)$ (i.e. the family $(\varphi_D)_{D \in \mathcal{D}}$ is compatible), there exists a unique unitary ring homomorphism $\varphi : Q_{\mathcal{D}}(X) \longrightarrow R$ such that $\varphi \circ j_D = \varphi_D$ for each $D \in \mathcal{D}$ (cf. : Theory of Category by B. Mitchell Chapter II § 2.)

First we show the uniqueness: Let $\varphi' : Q_{\mathcal{D}}(X) \longrightarrow R$ be another ring homomorphism such that $\varphi' \circ j_D = \varphi_D$ for each $D \in \mathcal{D}$. If u be any element in $Q_{\mathcal{D}}(X)$, then there exists a member D in $\mathcal{D}$ such that for some $f \in C_D(X)$, $\nu(f) = u$, hence by iii) of Proposition 1

$$\varphi'(u) = \varphi'(\nu(f)) = \varphi'(j_D(f|D)) = \varphi_D(f|D)$$
$$= (\varphi \circ j_D)(f|D) = \varphi(\nu(f)) = \varphi(u).$$

Now we establish a ring homomorphism $\varphi : Q_{\mathcal{D}}(X) \longrightarrow R$ with $\varphi \circ j_D = \varphi_D$. Let $u \in Q_{\mathcal{D}}(X)$ an element; then there exists $f \in C_D(X)$ for some D such that $\nu(f) = u$. Define φ by $\varphi(u) = \varphi_D(f|D)$. We show that φ is well defined. For any $u \in Q_{\mathcal{D}}(X)$ if $u = \nu(f) = \nu(g)$ for some $f, g \in C_{\mathcal{D}}(X)$ with $f \in C_D(X)$, $g \in C_E(X)$, then by iii) $j_D(f|D) = j_E(g|E)$, and by iv) $j_{D\cap E}(f|D \cap E) = j_{D\cap E}(g|D \cap E)$ since $D \cap E \subseteq D, E$. But then by hypothesis $\varphi_D(f|D) = \varphi_{D\cap E}(f|D \cap E) = \varphi_{D\cap E}(g|D \cap E) = \varphi_E(g|E)$; this shows that φ depends only upon u, not upon the choice of representative f; thus φ is well-defined. Clearly φ is a ring homomorphism.

Finally for $f \in C(D)$, $(\varphi \circ j_D)(f) = \varphi(j_D(f^*|D)) = \varphi(\nu(f^*)) = \varphi_D(f^*|D) = \varphi_D(f)$ where $f^* \in C_D(X)$ such that $f^* | D = f$. Q.E.D.

<u>Remark</u>: The ring $Q_{\mathcal{S}}(X)$ can be made into a $\mathbb{R}$-algebra in the obvious way, namely $ru = \nu(r)u$ for $r \in \mathbb{R}$, $u \in Q_{\mathcal{S}}(X)$.

<u>Proposition 3</u>: Let $u_1, \ldots, u_k$ be elements in $Q_{\mathcal{S}}(X)$ and P be a polynomial in k indeterminates over $\mathbb{R}$ such that $P(u_1, \ldots, u_k) = 0$, then there exist $f_1, \ldots, f_k$ in $C_{\mathcal{S}}(X)$ with $P(f_1, \ldots, f_k) = 0$, and $u_i = \nu(f_i)$ for each $i = 1, \ldots, k$.

<u>Proof</u>: Let f_i in $C_{\mathcal{S}}(X)$ such that $u_i = \nu(f_i)(i = 1, \ldots, k)$, and let $P = \sum r_{n1, \ldots, nk} \cdot x_1^{n1} \ldots\ldots x_K^{nk}$ and $f_i \in C_{D_i}(X)$ ($i = 1, \ldots, k$); put $D = \bigcap_1^k D_i$. Then $P(u_1, \ldots, u_k) = \sum r_{n1, \ldots, nk}\, u_1^{n1} \ldots u_k^{nk} = \sum (r_{n1, \ldots, nk}) \cdot u_1^{n1} \ldots u_k^{nk} = \sum (r_{n1, \ldots, nk})\, \nu(f_1^{n1}) \ldots \nu(f_k^{nk}) = \sum j_D(r_{n1, \ldots, nk}) \cdot j_D(f_1^{n1}|D) \ldots j_D(f_k^{nk}|D) = j_D\left(\sum r_{n1,\ldots nk}(f_1^{n1}|D) \ldots (f_k^{nk}|D)\right) = 0$. Hence $P(f_1|D, \ldots, f_k|D) = 0$. Now define $f_i^*(i=1\ldots k)$ on X by $f_i^* = f_i$ D on D and $f_i^* = 0$, on $\sim D$;

then $f_i^* \in C_{\mathfrak{S}}(X)$ $(i = 1, \ldots, k)$. Also, $P(f_1^*, \ldots, f_k^*) = 0$. Moreover, $\nu(f_1^*) = j_D(f_1^*|D) = j_D(f_1|D) = \nu(f_1) = u_1$.
Q.E.D.

<u>Remark</u>: One also shows that the same result holds for several polynomials $P_1, \ldots, P_m$ such that $P_i(u_1, \ldots, u_k) = 0$ $(i = 1, \ldots, m)$. <u>We observe</u> that if 1_Q is the unit in $Q_{\mathfrak{S}}(X)$, then for any e in $C_{\mathfrak{S}}(X)$ such that $\nu(e) = 1_Q$ there exists a $D \in \mathfrak{S}$ such that $e = 1$ on D.

<u>Corollary 1</u>: u is invertible in $Q_{\mathfrak{S}}(X)$ iff there exists an element f in $C_{\mathfrak{S}}(X)$ such that $u = \nu(f)$ and f is invertible in $C_{\mathfrak{S}}(X)$.

<u>Proof</u>: There exists u' in $Q_{\mathfrak{S}}(X)$ such that $uu' = 1_Q$ i.e. $uu' - 1_Q = 0$. By the above proposition there exists f, f' and e in $C_{\mathfrak{S}}(X)$ such that $ff' - e = 0$ holds, and $1_Q = \nu(e)$, $u = \nu(f)$, $u' = \nu(f')$, and also $f(x) \cdot f'(x) = e(x) = 1$ for $x \in D$, for some $D \in \mathfrak{S}$.

Define f^*, f'^* by

$$f^*(x) = f(x) \text{ and } f'^*(x) = f'(x) \text{ on } D$$
$$f^*(x) = 1 \quad \text{ and } f'^*(x) = 1 \quad \text{ on } \sim D.$$

Then f^*, f'^* in $C_{\mathfrak{S}}(X)$. Thus f^* is invertible in $C_{\mathfrak{S}}(X)$ and $\nu(f^*) = j_D(f^*|D) = j_D(f|D) = \nu(f) = u$.

<u>Corollary 2</u>: $0 \neq u$ is a zero divisor in $Q_{\mathfrak{S}}(X)$ iff there exist f in $C_{\mathfrak{S}}(X)$ such that $u = \nu(f)$ and f is a zero divisor in $C_{\mathfrak{S}}(X)$.

Section 2: Ring of quotients with respect to an embedding.

Let $\mathcal{S}$ be a filter base of dense subsets of X. We note that $C(X) \subseteq C_{\mathcal{S}}(X)$ since $C(X) \subseteq C_D(X)$ for each $D \in \mathcal{S}$, but X need not itself belong to $\mathcal{S}$. Also we note that the natural mapping ν gives an embedding of C(X) into $Q_{\mathcal{S}}(X)$; it is clear because $C(X) \cap Z_{\mathcal{S}}(X) = 0$, and hence $f + Z_{\mathcal{S}}(X) = 0$ for $f \in C(X)$ implies $f = 0$.

To say a ring $B(\supset A)$ is a ring of quotients of A with respect to an embedding $\varphi : A \longrightarrow B$ means that the ring B is a ring of quotients of $\varphi(A) \subseteq B$. Also we note that $Q_{\mathcal{S}}(X)$ is a ring of quotients of C(X) with respect to the embedding ν if and only if each C(D), $D \in \mathcal{S}$ is a ring of quotients of C(X) with respect to the embedding given by the restriction $f \rightsquigarrow f|D$.

Proposition 4: Let $\mathcal{S}$ be a filter base of dense subsets of X. A necessary and sufficient condition for $Q_{\mathcal{S}}(X)$ to be a ring of quotients of C(X) with respect to the embedding ν is that: For each $D \in \mathcal{S}$, for any $f \in C(D)$ and open subset U of D, there exists an open subset V in X such that $V \cap D \subseteq U$ and $f|V \cap D$ has a continuous extension to V.

Proof: Sufficiency: Let f, $0 \neq g$ in C(D) and $U = \{x \in D \mid g(x) \neq 0\}$ an open subset of D. By hypothesis there is an open subset V in X such that $V \cap D \subseteq U$.

Let $a \in V \cap D$ and $\tilde{f}$ be the continuous extension of $f|V \cap D$ to V. Then there is a function h in $C(X)$ such that $h(a) \neq 0$ and $h|\complement V = 0$. Define a function u on X by

$$\begin{aligned} u(x) &= \tilde{f}(x) \cdot h(x) \quad \text{for } x \text{ in } V \\ &= 0 \qquad\qquad\quad \text{for } x \text{ in } \complement V. \end{aligned}$$

Then clearly $u \in C(X)$ and $f \cdot h|D = u|D$, and $g \cdot h|D \neq 0$ since $g(a) \cdot h(a) \neq 0$. This shows that $C(D)$ is a ring of quotients of $C(X)$ with respect to the embedding $u \rightsquigarrow u|D$.

<u>Necessity</u>: Let $f \in C(D)$ and U be any open subset of D. Then there is an open subset V' in X such that $U = V' \cap D$. Let $a \in V' \cap D$, then there is g in $C(D)$ with $g(a) \neq 0$ and $g(x) = 0$ for $x \in D \setminus U$. By assumption there is an h in $C(X)$ such that $f \cdot h|D = u|D$ for some $u \in C(X)$ and $g \cdot h|D \neq 0$ in $C(D)$. Let $c \in U$ such that $g(c) \cdot h(c) \neq 0$. Then there is a neighborhood V'' of c in X such that $h|V'' \neq 0$. Put $V = V' \cap V'' \neq \emptyset$, since $c \in V', V''$. Then $V \cap D \subseteq U$, and $f \cdot h|D = u|D$ implies $f = \dfrac{u|V \cap D}{h|V \cap D}$; then $\dfrac{u|V}{h|V}$ $(h|V \neq 0)$ is the desired continuous extension of f to the open set V in X. Q.E.D.

<u>Remark</u>: Evidently the condition in Proposition 4 holds for every dense open D.

<u>Lemma 5</u>: Let A be a countable dense subset of irrational numbers and let $\lambda: A \longrightarrow \{1,2, \ldots,\}$ be a one to one mapping. For each $x \in \mathbb{R}$ define a function f by:

$$f(x) = \sum_{\substack{a < x \\ a \in A}} 1/\lambda^2(a),$$

then f is continuous at each point of $\mathbb{Q}$ but $f(a^+) > f(a)$ at each point $a \in A$.

<u>Proof</u>: First we show that f is not continuous at each point of A. To see this take any $a_0 \in A$, then

$$\lim_{x \to a_0^-} f(x) = \sum_{\substack{a < x \\ x \to a_0^-}} 1/\lambda^2(a) = \sum_{a < a_0} 1/\lambda^2(a) = f(a_0);$$

on the other hand

$$\lim_{x \to a_0^+} f(x) = \sum_{\substack{a < x \\ a_0 \leftarrow x}} 1/\lambda^2(a) = \sum_{a < a_0} 1/\lambda^2(a) + \sum_{a_0 \leqslant a < x} 1/\lambda^2(a) > f(a_0).$$

Secondly we show that f is continuous at each point q of $\mathbb{Q}$: Clearly

$$\lim_{x \to q^-} f(x) = f(q);$$

on the other hand

$$\begin{aligned} \lim_{x \to q^+} f(x) &= \inf_{x > q} \Big(\sum_{a < x} 1/\lambda^2(a) \Big) \\ &= \inf_{x > q} \Big(\sum_{a < q} 1/\lambda^2(a) + \sum_{q < a < x} 1/\lambda^2(a) \Big) \\ &= f(q) + \inf_{x > q} \Big(\sum_{q < a < x} 1/\lambda^2(a) \Big). \end{aligned}$$

Let $n_o (\geq 1)$ be given natural number, and take a point $a_{n_o} \in A \cap [q, q + \frac{1}{n_o}]$ such that $\lambda(a_{n_o}) \leq \lambda(a)$ for all $a \in A \cap [q, q + \frac{1}{n_o}]$; next take n_1 (natural number $> n_o$) such that $a_{n_o} \notin A \cap [q, q + \frac{1}{n_1}]$, and pick a point $a_{n_1} \in A \cap [q, q + \frac{1}{n_1}]$ such that $\lambda(a_{n_1}) \leq \lambda(a)$ for all $a \in A \cap [q, a + \frac{1}{n_1}]$. Hence, inductively, take a natural number $n_k (> n_{k-1})$ such that $a_{n_{k-1}} \notin A \cap [q, q + \frac{1}{n_k}]$ and $a_{n_k} \in A \cap [q, q + \frac{1}{n_k}]$ such that $\lambda(a_{n_k}) \leq \lambda(a)$ for all $a \in A \cap [q, q + \frac{1}{n_k}]$. Let P be the set of all $n_k (k = 0, 1, 2, \ldots.)$ defined by above process. Then for any n_k, $n_{k'}$ with $n_k < n_{k'}$ in P, it is clear that $\lambda(a_{n_k}) < \lambda(a_{n_{k'}})$; now clearly

$$\inf_{n_k \in P} \left\{ \sum_{q < a_{n_k} < q + \frac{1}{n_k}} 1/\lambda^2(a_{n_k}) \right\} = 0.$$

Hence $\inf_{x > q} \left\{ \sum_{q < a < x} 1/\lambda^2(a) \right\} = 0.$ This concludes that f is continuous at the point q.

<u>Corollary</u>: $C(Q)$ is not a ring of quotients of $C(\mathbb{R})$ with respect to the embedding $f \rightsquigarrow f|Q$.

Proof: If f is the function defined in Lemma 5, then $g \equiv f|Q \in C(Q)$. Since every open subset V in $\mathbb{R}$ intersects with A, the function $g|V \cap Q$ can not be extended to V; by the previous proposition, $C(Q)$ is not a ring of quotients of $C(\mathbb{R})$.

Theorem 6: (Banaschewski [1].) $Q_{\mathcal{S}}(X)$ is a maximal ring of quotients of $C(X)$ with respect to the embedding $f \rightsquigarrow f + Z_{\mathcal{S}}(X)$ if $\mathcal{S}$ is the set of all dense open subsets of X.

Proof: From the Proposition 4, clearly $Q_{\mathcal{S}}(X)$ is a ring of quotients of $C(X)$. We shall show that for any A, a ring of quotients of $C(X)$, there exists a monomorphism of A into $Q_{\mathcal{S}}(X)$.

Take any $a \in A$ and consider $T = \{ f \in C \mid af \in C \}$, where C denotes $C(X)$; then T is a dense ideal in A; thus the set $V =_{df} \bigcup Coz(f)(f \in T)$ is a dense open subset of X; i.e. $V \in \mathcal{S}$.

For each $a \in A$ define a function $a^* : X \longrightarrow R$ by

$$\begin{aligned} a^*(x) &= (af)(x)/f(x) && \text{if } x \in Coz(f) \text{ for some } f \in T \\ &= 0 && \text{if } x \notin V; \end{aligned}$$

this is well defined because if $x \in Coz(f)$ and $x \in Coz(g)$ for some f and g in T. Then $((af)g)(x) = ((ag)f)(x)$ implies $(af)(x)/f(x) = (ag)(x)/g(x)$ for $x \in Coz(f) \cap Coz(g) = Coz(fg)$. Hence a^*, thus defined, belongs to $C_V(X)$, and clearly $a^* = a$ if $a \in C(X)$ since $a^*(x) = (af)(x)/f(x) = a(x)$

for some $\mathrm{Coz}(f)$ containing x.

Now we show that the mapping $a \rightsquigarrow a^* + Z\ (X)$ of A into $Q_{\mathcal{S}}(X)$ is a monomorphism. We show first that this mapping is a ring homomorphism. If $T_i = \{ f \in C \mid a_i f \in C \}$ for $a_i \in A (i=1,2)$. Then $T =_{df} T_1 \cap T_2$ is a dense ideal in A. For $V = \bigcup \mathrm{Coz}\,(f)(f \in T)$, then it is easy to see that $V = V_1 \cap V_2$ where $V_i = \bigcup \mathrm{Coz}(g)(g \in T_i)$.

Take an element $x \in V$. Then there is an $f \in T$ such that $x \in \mathrm{Coz}\,(f)$ where $f \in T_i(i=1,2)$; hence $a_1 f + a_2 f = (a_1 + a_2)f$ belongs to $C(X)$. Thus $(a_1 + a_2)^*(x) = (a_1 f)(x)/f(x) + (a_2 f)(x)/f(x) = a_1^*(x) + a_2^*)x)$ for $x \in V$: This shows that $(a_1 + a_2)^* - (a_1^* + a_2^*) \in Z_{\mathcal{S}}(X)$, and in the same manner $(a_1 \cdot a_2)^* - (a_1^* \cdot a_2^*) \in Z_{\mathcal{S}}(X)$. This concludes that the mapping $a \rightsquigarrow a^* + Z_{\mathcal{S}}(X)$ is a ring homomorphism.

Finally we show this mapping is one to one. For any a_1, a_2 in A, let $a_1^* - a_2^* \in Z_{\mathcal{S}}(X)$. If $f \in T = T_1 \cap T_2$, where T_1, T_2 are defined as above, then $f(a_1^* - a_2^*) = f \cdot a_1^* - f \cdot a_2^*$ in $Z_{\mathcal{S}}(X)$, since $Z_{\mathcal{S}}(X)$ is an ideal. Consider the equality:

$$f \cdot a_1 - f \cdot a_2 = (f \cdot a_1 - f \cdot a_1^*) + (f \cdot a_1^* - f \cdot a_2^*) + (f \cdot a_2^* - f \cdot a_2).$$

Note that $(f \cdot a_1 - f \cdot a_1^*)(x) = (f \cdot a_1)(x) - f(x) \cdot (fa_1)(x)/f(x) = 0$, for all $x \in \mathrm{Coz}(f) \subseteq V_1$; hence $(fa_1 - fa_1^*) | V = 0$, where $V = V_1 \cap V_2$; similarly $(fa_1^* - fa_2^*) \mid V = 0$.

Thus both $(fa_1 - fa_1^*)$ and $(fa_2^* - fa_2)$ belong to $Z_{\mathcal{S}}(X)$; consequently $f(a_1 - a_2) = 0$, on some dense open subsets of X, but $f(a_1 - a_2)$ is element of $C(X)$; hence

$f(a_1 - a_2) = 0$ for all $x \in X$. This implies that $T(a_1 - a_2) = 0$; but T is dense ideal in A; hence $a_1 = a_2$. This completes the proof.

Note: A maximal ring of quotients of C(X) is von Neumann regular.

Proposition 7: A necessary and sufficient condition for $Q_{\mathcal{S}}(X)$ to be a von Neumann regular ring is the following: For each $D \in \mathcal{S}$ and $f \in C(D)$, the subset $(E \cap \mathrm{Coz}(f)) \cup \mathbf{I}_E\mathbf{C}_E\,(E \cap \mathrm{Coz}(f))$ belongs to $\mathcal{S}$ for some $E \subseteq D$ in $\mathcal{S}$.

Proof. Sufficiency: Let $u \in Q_{\mathcal{S}}(X)$ with $u = \nu(f')$ for some $f' \in C_{\mathcal{S}}(X)$, and put $f' \mid D = f$ for some D in $\mathcal{S}$. Note that $E \cap \mathrm{Coz}(f)$ and $\mathbf{I}_E\mathbf{C}_E(\mathrm{Coz}(f) \cap E)$ are E-open subsets of E and disjoint. Put $E' = (E \cap \mathrm{Coz}(f)) \cup \mathbf{I}_E\mathbf{C}_E(E \cap \mathrm{Coz}(f))$. Define a function g on E' by

$$g(x) = 1/f(x) \quad \text{for } x \text{ in } E \cap \mathrm{Coz}(f)$$

$$= c \qquad \text{for } x \text{ in } \mathbf{I}_E\mathbf{C}_E\mathrm{Coz}(f).,$$

then clearly $g \in C(E')$; hence there is a g' in $C_{\mathcal{S}}(X)$ such that $G' \mid E' = g$ and $f'^2 g' = f'$ on E'; hence we have $u^2 v = u$ where $v = \nu(g')$.

<u>Necessity</u> : Let $D \in \mathcal{D}$ and $f \in C(D)$ be given, and let $u = \nu(f')$ where $f'|D = f$ for some $f' \in C_{\mathcal{D}}(X)$. By the hypothesis there is a v in $Q_{\mathcal{D}}(X)$ such that $u^2v = u$. Let $v = \nu(g')$ with $g' \in C_{D'}(X)$ and $g = g'|D'$. Then $u^2v = u$ implies that $f^2g = f$ on some $E \in \mathcal{D}$, $E \subseteq D \cap D'$. By a well-known theorem, we have

$$E = \mathbf{I}_E(E \cap Coz(f)) \cup \delta_E(E \cap Coz(f)) \cup \mathbf{I}_E\mathbf{C}_E(E \cap Coz(f)),$$

δ_E denotes the E-boundary, where $\mathbf{I}_E(E \cap Coz(f)) \cap \delta_E(E \cap Coz(f)) \cap \mathbf{I}_E\mathbf{C}_E(E \cap Coz(f)) = \emptyset$ and $\Gamma_E(E \cap Coz(f)) = \mathbf{I}_E(E \cap Coz(f)) \cup \delta_E(E \cap Coz(f))$.

Now we claim that $\delta_E(E \cap Coz(f)) = \emptyset$. If there is a point p in $\delta_E(E \cap Coz(f))$, then for any E-neighborhood U_p of p, $f^2g - f = 0$ implies

$$g(x) = 1/f(x)$$

for all x in $U_p \cap (E \cap Coz(f)) \neq \emptyset$, but then $g(x) = \infty$ as x tends to p, $p \in E$. This contradicts the fact that g is continuous on E.; thus we have $\delta_E(E \cap Coz(f)) = \emptyset$, and since $E \cap Coz(f)$ is E-open, we have

$$E = (E \cap Coz(f)) \cup \mathbf{I}_E\mathbf{C}_E(E \cap Coz(f));$$

this completes the proof.

<u>Remark 1</u>: The ring $Q_{\mathcal{D}}(X)$ can be a von Neumann regular ring and still be rationally in-complete. Let X be a η_1-set; that is, a totally ordered set with the property that for any nonempty countable subsets A and B

of X with $A < B$ there exist element u, v and w satisfying $u < A < v < B < w$. Then if the η_1-set X is endowed with the interval topology the space X is what is called a P-space without isolated points [12]. Hence $C(X)$ is not rationally complete. Take any proper dense open subset D of X and let $\mathfrak{D} = \{D\}$ then clearly $Q_{\mathfrak{D}}(X) = C(D)$, and hence $Q_{\mathfrak{D}}(X)$ is not rationally complete either. But D is again P-space, hence $C(D)$ is a von Neumann regular ring, and so is $Q_{\mathfrak{D}}(X)$. This is an example of a non-maximal ring of quotients of $C(X)$ which is a von Neumann regular ring.

Remark 2: The ring $Q_{\mathfrak{D}}(X)$ can be a von Neumann regular ring without being a ring of quotients of $C(X)$ (w.r.t. ν.). Take $X = \mathbb{R}$, and $\mathfrak{D}$ the set of all relatively open dense subsets of $\mathbb{Q}$, the rational numbers; then clearly $Q_{\mathfrak{D}}(\mathbb{Q})$ is the maximal ring of quotients of $C(\mathbb{Q})$ and a von Neumann regular ring. Moreover we have $C(X) \subset C(\mathbb{Q}) \subset Q_{\mathfrak{D}}(\mathbb{Q})$ with the embeddings:

$\rho_{\mathbb{Q}} : C(X) \longrightarrow C(\mathbb{Q})$ by $\rho_{\mathbb{Q}}(f) = f|\mathbb{Q}$ and $\nu' : C(\mathbb{Q}) \longrightarrow Q_{\mathfrak{D}}(\mathbb{Q})$ by $\nu'(g) = g + Z_{\mathfrak{D}}(\mathbb{Q})$; also the composition $\nu' \circ \rho_{\mathbb{Q}} : C(X) \longrightarrow Q_{\mathfrak{D}}(\mathbb{Q})$ is an embedding. As was shown previously, $C(\mathbb{Q})$ is not a ring of quotients of $C(X)$ w.r.t. the embedding $\rho_{\mathbb{Q}}$: Hence $Q_{\mathfrak{D}}(\mathbb{Q})$ is not a ring of quotients of $C(X)$. Now we show that $Q_{\mathfrak{D}}(X) \cong Q_{\mathfrak{D}}(\mathbb{Q})$. Define $\varphi : Q_{\mathfrak{D}}(X) \longrightarrow Q_{\mathfrak{D}}(\mathbb{Q})$ by $\varphi(u) = f|\mathbb{Q} + Z_{\mathfrak{D}}(\mathbb{Q})$ for $u \in Q_{\mathfrak{D}}(X)$ with $u = f + Z_{\mathfrak{D}}(X)$.

We show φ is onto: Take any $v \in Q_{\mathcal{B}}(Q)$ with $v = g + Z_{\mathcal{B}}(Q)$, define g^* on X by

$$g^*(x) = g(x) \text{ on } Q$$

$$= 0 \text{ on } \sim Q.,$$

then $g^* \in C_{\mathcal{B}}(X)$ and let $u = g^* + Z_{\mathcal{B}}(X)$; then $\varphi(u) = g^*|Q + Z_{\mathcal{B}}(Q) = g + Z_{\mathcal{B}}(Q) = v$; hence φ is onto. Since $f|Q + Z_{\mathcal{B}}(Q) = 0$ implies $f \in Z_{\mathcal{B}}(X)$, and hence φ is one to one. Clearly $\nu(f) = (\varphi^{-1} \circ \nu' \circ \rho_Q)(f)$; thus $Q_{\mathcal{B}}(X)$ is a von Neumann regular ring without being a ring of quotients of C(X) with respect to the embedding ν .

Section 3: Induced homomorphisms.

Let $\mathcal{D}$ and $\mathcal{F}$ be two distinct filter bases of dense subsets of X; we say $\mathcal{D}$ is finer than $\mathcal{F}$ if and only if for each $F \in \mathcal{F}$ there exists a D in $\mathcal{D}$ with $D \subseteq F$.

Proposition 8: Let $\mathcal{D}$ and $\mathcal{F}$ be two filter bases of dense subsets of X. If $\mathcal{D}$ is finer than $\mathcal{F}$, then there is a unique embedding $\varphi : Q_{\mathcal{F}}(X) \longrightarrow Q_{\mathcal{D}}(X)$ such that for any $F \in \mathcal{F}$ and $D \in \mathcal{D}$ with $D \subseteq F$ the following diagram commutes.

$$\begin{tikzcd}
Q_{\mathcal{F}}(X) \arrow[r, "\varphi"] & Q_{\mathcal{D}}(X) \\
C(F) \arrow[u, "j_F"] \arrow[r, "f \rightsquigarrow f|D"] & C(D) \arrow[u, "j_D"']
\end{tikzcd}$$

Proof: Define $\varphi : Q_{\mathcal{F}}(X) \longrightarrow Q_{\mathcal{D}}(X)$ by $\varphi(j_F(f)) = j_D(f|D)$ for $f \in C_F(X)$. Then Clearly φ is a monomorphism; and from the definition of φ the uniqueness is evident.

Remark 1: In particular if $\mathcal{D}$ is finer than $\mathcal{F}$ and $\mathcal{F}$ is finer than $\mathcal{D}$, then $\varphi : Q_{\mathcal{F}}(X) \longrightarrow Q_{\mathcal{D}}(X)$ is an isomorphism. To see this, define $\varphi' : Q_{\mathcal{D}}(X) \longrightarrow Q_{\mathcal{F}}(X)$ by $\varphi'(v) = j_{F'}(f|F')$ where $v = j_{D'}(f)$, $f \in C(D')$, $F' \subseteq D'$, $F' \in \mathcal{F}$, $D' \in \mathcal{D}$. Then $(\varphi \circ \varphi')(v) = \varphi(j_{F'}(f|F'))$ $= j_D(f|D)(D \subseteq F',\ D \in \mathcal{D}) = v$, and hence $\varphi \circ \varphi' = \text{id}$.

Similarly $\varphi' \circ \varphi = i_d$. Hence φ is an isomorphism.

Remark 2: Also if $\mathcal{F} \subseteq \mathcal{G}$, then clearly $\mathcal{G}$ is finer than $\mathcal{F}$, but the converse need not hold.

Remark 3: In particular, if $\mathcal{G}$ is finer than $\mathcal{F}$ and $Q_{\mathcal{F}}(X)$ is a maximal ring of quotients of $C(X)$, and $\mathcal{G}$ is such that $Q_{\mathcal{G}}(X)$ is a ring of quotients of $C(X)$ (always with respect to the embeddings), then φ is an isomorphism. Hence if $Q_{\mathcal{F}}(X)$ is a maximal ring of quotients of $C(X)$ and $\mathcal{G}$ (consisting of dense open subsets) is finer than $\mathcal{F}$, then φ is an isomorphism.

Example: Let $\mathcal{F}$ be the set of all dense open subsets of X and $\mathcal{G}$ be the set of all disconnected dense open subsets of X, then $\mathcal{G}$ is finer than $\mathcal{F}$; hence $Q_{\mathcal{F}}(X) \cong Q_{\mathcal{G}}(X)$.

The following discussion is another approach to obtain a maximal ring of quotients of $C(X)$.

Let $\bar{\Phi}$ be the family of all filter bases $\mathcal{F}$ of dense subsets of X such that $Q_{\mathcal{F}}(X)$ is a ring of quotients of $C(X)$. On $\bar{\Phi}$ we define an ordering in the following way; for two member $\mathcal{F}_\alpha$, $\mathcal{F}_\beta$ in $\bar{\Phi}$, $\mathcal{F}_\alpha \leq \mathcal{F}_\beta$ if and only if $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$; i.e. each member of $\mathcal{F}_\alpha$ is a member of $\mathcal{F}_\beta$. Then clearly the set $\bar{\Phi}$ with the ordering " $\leq$ " becomes a partially ordered set.

Now let $\mathcal{L}$ be a chain in $\bar{\Phi}$ and

$$\cup\mathcal{L} = \{ F_\alpha \mid F_\alpha \in \mathfrak{F}_\alpha, \text{ for some } \mathfrak{F}_\alpha \text{ in } \mathcal{L} \}.$$

Then $\cup\mathcal{L}$ is again a filter base of dense subsets of X; for let F_α, F_β in $\cup\mathcal{L}$ with $F_\alpha \in \mathfrak{F}_\alpha$ and $F_\beta \in \mathfrak{F}_\beta$, and $\mathfrak{F}_\alpha \preccurlyeq \mathfrak{F}_\beta$, then there exists F'_β in $\mathfrak{F}_\beta$ such that $F'_\beta \subseteq F_\alpha$. Since $\mathfrak{F}_\beta$ is a filter base, there is $F''_\beta \in \mathfrak{F}_\beta$ such that $F''_\beta \subseteq F'_\beta \cap F_\beta \subseteq F_\alpha \cap F_\beta$. And $Q_{\cup\mathcal{L}}(X)$ is a ring of quotients of C(X) since each C(F), $F \in \cup\mathcal{L}$ is a ring of quotients of C(X). Thus the partially ordered set $(\bar{\Phi}, \preccurlyeq)$ is inductive, hence there exists at least one maximal filter base $\mathcal{M}$ of dense subsets of X such that $Q_{\mathcal{M}}(X)$ is a ring of quotients of C(X).

<u>Remark</u>: Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two filter bases of dense subsets of X such that $D_1 \cap D_2$ in dense for all $D_1 \in \mathcal{D}_1$, $D_2 \in \mathcal{D}_2$. Denote

$$\mathcal{D}_1 \wedge \mathcal{D}_2 = \{ D_1 \cap D_2 \mid D_1 \in \mathcal{D}_1,\ D_2 \in \mathcal{D}_2 \},$$

then clearly $\mathcal{D}_1 \wedge \mathcal{D}_2$ is a filter base of dense subset, since for any two E' and E" in $\mathcal{D}_1 \wedge \mathcal{D}_2$

$$\begin{aligned} E' \cap E'' &= (D_1 \cap D_2) \cap (D'_1 \cap D'_2) \\ &= (D_1 \cap D'_1) \cap (D_2 \cap D'_2) \\ &\supseteq D \cap D' = E,\ D \in \mathcal{D}_1,\ D' \in \mathcal{D}_2. \end{aligned}$$

We also denote:

$$\mathcal{D}_1 \vee \mathcal{D}_2 = \{ D \mid D \in \mathcal{D}_1 \text{ or } D \in \mathcal{D}_2 \}.$$

$\mathcal{D}_1 \vee \mathcal{D}_2$ need not be a filter base of dense subsets, but we shall show that $(\mathcal{D}_1 \vee \mathcal{D}_2) \cup (\mathcal{D}_1 \wedge \mathcal{D}_2) \equiv \mathcal{D}$ is a filter base. For E', E" of $\mathcal{D}$, we have three cases:

i) If E', E" $\in \mathcal{D}_1 \vee \mathcal{D}_2$, then clearly E' $\cap$ E" contains some member E of $\mathcal{D}$.

ii) If E', E" $\in \mathcal{D}_1 \wedge \mathcal{D}_2$, then also clearly E' $\cap$ E" contains a member E of $\mathcal{D}$.

iii) If E' $\in \mathcal{D}_1 \vee \mathcal{D}_2$and E" $\in \mathcal{D}_1 \wedge \mathcal{D}_2$, then E' $\cap$ E" = E' $\cap$ $(D_1 \cap D_2)$, $D_1 \in \mathcal{D}_1$, $D_2 \in \mathcal{D}_2$; the either E' $\cap$ D_1 or E' $\cap$ D_2 contains some member of $\mathcal{D}_1 \wedge \mathcal{D}_2$; hence E' $\cap$ E" contains a member of $\mathcal{D}$. This shows that $\mathcal{D}$ is a filter base.

<u>Theorem 9</u>: For a maximal filter base $\mathcal{M}$ in $\mathfrak{F}$, the ring $Q_{\mathcal{M}}(X)$ is a maximal ring of quotients of C(X).

<u>Proof</u>: Let $\mathcal{O}$ be the set of all dense open subsets of X. We claim that $\mathcal{O} \preceq \mathcal{M}$, where $\mathcal{O}$ is of course a filter base of dense subsets. Note that $O \cap M$ is dense for all $O \in \mathcal{O}$, $M \in \mathcal{M}$.

Let $\mathcal{M}' = (\mathcal{O} \vee \mathcal{M}) \cup (\mathcal{O} \wedge \mathcal{M})$; then by above remark $\mathcal{M}'$ is a filter base of dense subsets containing $\mathcal{O}$ and $\mathcal{M}$; clearly $Q_{\mathcal{M}'}(X)$ is a ring of quotients of C(X); the maximality of $\mathcal{M}$ implies that $\mathcal{M} \approx \mathcal{M}'$ which means that $\mathcal{M} \preceq \mathcal{M}'$ and $\mathcal{M}' \preceq \mathcal{M}$. Hence $\mathcal{O} \preceq \mathcal{M}$; thus by the previous proposition the ring $Q_{\mathcal{O}}(X)$ can be embedded into the ring $Q_{\mathcal{M}}(X)$.

$Q_{\mathfrak{D}}(X)$ is known as the maximal ring of quotients of $C(X)$, hence $Q_{\mathfrak{M}}(X)$ is a maximal ring of quotients of $C(X)$. Q.E.D.

In what follows we shall describe the relation between a continuous mapping from a space X into a space Y and homomorphism from $Q_{\mathfrak{F}}(Y)$ into $Q_{\mathfrak{S}}(X)$ where $\mathfrak{S}$ and $\mathfrak{F}$ are filter bases of dense subsets of X and Y respectively.

Let $\varphi : X \longrightarrow Y$ be a continuous mapping with $\varphi^{-1}(F) \in \mathfrak{S}$ for each $F \in \mathfrak{F}$; then φ induces a homomorphism from $Q_{\mathfrak{F}}(Y)$ into $Q_{\mathfrak{S}}(X)$. Namely, a mapping $\varphi^* : Q_{\mathfrak{F}}(Y) \longrightarrow Q_{\mathfrak{S}}(X)$ defined by

$$\varphi^*(u) = f \circ \varphi + Z_{\mathfrak{S}}(X),$$

where $u = \nu(f)$, $f \in C_{\mathfrak{F}}(Y)$, is a homomorphism.

To verify this we define a mapping $\varphi\# : C_{\mathfrak{F}}(Y) \longrightarrow C_{\mathfrak{S}}(X)$ by

$$\varphi\#(f) = f \circ \varphi \qquad (f \in C_{\mathfrak{F}}(Y)).$$

Then evidently $\varphi\#$ is a homomorphims induced by φ and it carries the constant functions onto the constant functions; for any $x \in X$, $\varphi\#(r)(x) = (r \circ \varphi)(x) = r$.

Let $\nu : C_{\mathfrak{S}}(X) \longrightarrow Q_{\mathfrak{S}}(X)$ and $\mu : C_{\mathfrak{F}}(Y) \longrightarrow Q_{\mathfrak{F}}(Y)$ be the natural homomorphisms. Consider the following diagram:

$$\begin{tikzcd}
C_{\mathfrak{F}}(Y) \arrow[r, "\varphi\#"] \arrow[d, "\mu"'] & C_{\mathfrak{S}}(X) \arrow[r, "\nu"] & Q_{\mathfrak{S}}(X) \\
Q_{\mathfrak{F}}(Y) \arrow[urr, dashed, "\varphi^*"'] & &
\end{tikzcd}$$

We first show that the homomorphism $\nu \circ \varphi\#$ annuls the Ker $\mu = Z_{\mathcal{S}}(Y)$. Let $f \in$ Ker $\mu = Z_{\mathcal{S}}(Y)$, then $f|F \equiv 0$ for some $F \in \mathcal{S}$ and $(\nu \circ \varphi\#)(f) = \nu(\varphi\#(f)) = f \circ \varphi + Z_{\mathcal{D}}(X)$. Also $(f \circ \varphi)(\varphi^{-1}(F)) = 0$, $\varphi^{-1}(F) \in \mathcal{D}$; hence $(\nu \circ \varphi\#)(f) = 0$. Thus the homomorphism $\nu \circ \varphi\#$ induces a homomorphism $\varphi^* : Q_{\mathcal{S}}(Y) \longrightarrow Q_{\mathcal{D}}(X)$ and the diagram is commutative; i.e. $\varphi^*(u) = (\nu \circ \varphi\#)(f)$, where $u = \mu(f)$. Hence $\varphi^*(u) = \nu(f \circ \varphi) = f \circ \varphi + Z_{\mathcal{D}}(X)$.

<u>We call φ^* the homomorphism induced by φ</u>.

<u>Proposition 10</u>: Let $\varphi^* : Q_{\mathcal{S}}(Y) \longrightarrow Q_{\mathcal{D}}(X)$ be the homomorphism induced by a continuous mapping $\varphi : X \longrightarrow Y$. Then φ^* is a monomorphism if φ has dense image.

<u>Proof</u>: Since $\varphi^*(u) = f \circ \varphi + Z_{\mathcal{D}}(X)$, where $u = \mu(f)$, φ^* is monomorphism iff $f \circ \varphi \in Z_{\mathcal{D}}(X)$ implies $f \in Z_{\mathcal{S}}(Y)$. This is again equivalent to saying that there exists a $D \in \mathcal{D}$ such that $\varphi(D) \subseteq Z(f)$ implies there exists $F \in \mathcal{S}$ with $F \subseteq Z(f)$.

Now we show that $\varphi(D)$, for each $D \in \mathcal{D}$, is a dense subset of Y. Since $\varphi(X)$ is dense in Y, for any $0 \neq U$, $\varphi(X) \cap U \neq \emptyset$: Then this would mean that there is a point $x \in X$ such that $\varphi(x) \in U$; i.e. $\varphi^{-1}(U) \neq \emptyset$ and open in X; hence $D \cap \varphi^{-1}(U) \neq \emptyset$; thus $\varphi(D) \cap U \neq \emptyset$.

Finally let $u = \mu(f)$, $f \in C_F(Y)$ and $\varphi^*(u) = 0$, then there exists $D \in \mathcal{D}$ such that $\varphi(D) \subseteq Z(f)$; i.e. $f \mid \varphi(D) = 0$; thus $f \mid \varphi(D) \cap F = 0$, but $\varphi(D) \cap F$ is a dense subset of F since $\varphi^{-1}(F) \in \mathcal{D}$; hence $f \mid F = 0$; i.e. $F \subseteq Z(f)$. Q.E.D.

<u>Proposition 11</u>: Let $\varphi : X \longrightarrow Y$ be a homeomorphism with dense image such that every continuous function on a dense open set V in $\varphi(X)$ can be continued to an open set $(\supset V)$ in Y, then φ induces an isomorphism between a maximal ring of quotients of $C(Y)$ and that of $C(X)$.

<u>Proof</u>: Let $\mathcal{D}$ and $\mathcal{D}'$ be the set of all dense open subsets of X and Y respectively. First we show that $\varphi^{-1}(D') \in \mathcal{D}$ for each $D' \in \mathcal{D}'$. Let V be any non void open set in X. Suppose $V \cap \varphi^{-1}(D') = \emptyset$, then $\varphi(V) \cap D' = \emptyset$. But $\varphi(V)$ is an open set in $\varphi(X)$; hence there is an open set U in Y such that $\varphi(V) = \varphi(X) \cap U$; thus we have $\varphi(X) \cap U \cap D' = \emptyset$, which is a contradiction.

Hence φ induces a homomorphism φ^*, and φ^* is clearly one to one from the above proposition. Now we show φ^* is onto. To show this it is enough to show that $\varphi\# : C_{\mathcal{D}'}(Y) \longrightarrow C_{\mathcal{D}}(X)$ is onto.

Take any $h \in C_{\mathcal{D}}(X)$ with $h \in C_D(X)$ for some $D \in \mathcal{D}$. Clearly φ^{-1} is a continuous mapping from $\varphi(X)$ onto X. Hence $h \circ \varphi^{-1}$ is continuous on $\varphi(D)$ (open in $\varphi(X)$), and, by hypothesis there is a dense open $D' \supset \varphi(D)$ in $\mathcal{D}'$ and g in

$C_{D'}(Y)$ such that $g|\varphi(D) = h \circ \varphi^{-1}$. Then clearly $\varphi\#(g) = g \circ \varphi = h \circ \varphi^{-1} \circ \varphi = h$. Q.E.D.

Corollary 1: $Q_{\mathfrak{D}}(X) \cong Q_{\mathfrak{D}'}(\beta X)$, ($\mathfrak{D}$, $\mathfrak{D}'$ the set of all dense open subsets of X and βX respectively).

Corollary 2: If $\varphi : X \longrightarrow Y$ is a homeomorphism with open dense image, then φ induces an isomorphism.

Let $\varphi : X \longrightarrow Y$ be a continuous mapping with the following property:

(*) $\varphi^{-1}(F) \in \mathcal{D}$ for each $F \in \mathcal{F}$,

where $\mathcal{D}$ and $\mathcal{F}$ are filter bases of dense subsets of X and Y respectively. Then the homomorphism $\varphi^* : Q_{\mathcal{F}}(Y) \longrightarrow Q(X)$ induced by φ satisfies the condition:

(**) $\varphi^*(\mu(C(Y)) \subseteq \nu(C(X))$ where μ, ν are the embeddings, and for each $F \in \mathcal{F}$ there exists $D \in \mathcal{D}$ such that $\varphi^*(j_F(C(F))) \subseteq j_D(C(D))$.

Proposition 12: Let $\psi : Q_{\mathcal{F}}(Y) \longrightarrow Q_{\mathcal{D}}(X)$ be a unitary homomorphism and Y be a hereditary realcompact, where $\mathcal{D}$ and $\mathcal{F}$ are the sets of all dense open subsets of X and Y respectively. Then $\psi = \varphi^*$ for some continuous mapping $\varphi : X \longrightarrow Y$ satisfying (*) iff ψ satisfies (**).

<u>Proof</u>: We only need to show the "if" part. Without loss of generality we may put $j_F(C(F)) = C(F)$ and $j_D(C(D)) = C(D)$ for $F \in \mathcal{F}$ and $D \in \mathcal{D}$. Since F is realcompact, $\psi(C(F)) \subset C(D)$ implies that there exists a unique continuous mapping $\varphi' : D \longrightarrow F$ such that $\psi(f|F)(x) = ((f|F) \circ \varphi')(x)$ for each $x \in D$ and for all $f \in C(Y)$. While $\psi(C(Y)) \subset C(X)$ implies that there exists a unique continuous mapping $\varphi : X \longrightarrow Y$ such that $\psi(f)(x) = (f \circ \varphi)(x)$ for each $x \in X$ and for all $f \in C(Y)$. Since $f \equiv f|F$ in $Q_{\mathcal{F}}(Y)$, $\psi(f|F)(x) = \psi(f)(x)$ for each $x \in D$ and for all $f \in C(Y)$, and hence $(f|F)(\varphi'(x)) = f(\varphi'(x)) = f(\varphi(x))$ for each $x \in D$ and for all $f \in C(Y)$. Hence $\varphi'(x) = \varphi(x)$ for each $x \in D$. This means that $\varphi' = \varphi|D$. Then clearly $D \subseteq \varphi^{-1}(F)$. But D is dense and $\varphi^{-1}(F)$ is open, hence $\varphi^{-1}(F) \in \mathcal{D}$. Thus φ induces a homomorphism $\varphi^* : Q_{\mathcal{F}}(Y) \longrightarrow Q_{\mathcal{D}}(X)$. To show $\varphi^* = \psi$, let $u \in Q_{\mathcal{F}}(Y)$ with $u = \mu(f) = j_F(f|F)$ for some $F \in \mathcal{F}$. Put $D = \varphi^{-1}(F)$. Noting $j_D(g|D) = g|D$ and $j_F(f|F) = f|F$, $\varphi^*(u) = \nu(f \circ \varphi) = j_D((f \circ \varphi)|D) = (f \circ \varphi)|D = ((f|F) \circ \varphi)|D = \psi(f|F) = \psi(j_F(f|F)) = \psi(u)$. Q.E.D.

The following characterizes a prime ideal P in C(X) via the maximal ring of quotients of C(X)/P. We first show the main clue of this idea.

<u>Lemma 13</u>: Let C be a commutative ring with unit e and B be a C-module with the property that for a fixed prime ideal P in C there is no element b in B such that the order ideal $O(b) = \{ c \in C \mid cb = 0 \} = P$; then the same holds for any essential extension of B.

<u>Proof</u>: Let $Q \supset B$ be an essential extension of B, and suppose there were an element $q \in Q$ such that $O(q) = \{ c \in C \mid cq = 0 \} = P$; then $q \neq 0$ since P is a proper ideal. Thus there exists an element $c \in C$ such that $cq \in B$ and $cq \neq 0$ since $C_q \cap B \neq (o)$.

Now we show that $O(cq) = P$. Let $p \in P$, then $pcq = cpq = 0$ since $O(q) = P$; hence $p \in O(cq)$. Conversely let $a \in O(cq)$, then $acq = 0$; hence $ac \in O(q) = P$. But $c \notin P$ since $cq \neq 0$; thus $a \in P$ since P is prime. Hence $O(cq) = P$. But $cq \in B$; this is a contradiction to the hypothesis.

<u>Remark</u>: Let B be a rational extension ring of a ring C and $B' = \varphi(B)$ where $\varphi : B \longrightarrow B'$ is a ring isomorphism. Then B' can be made into a C-module in the following way: For any $b' \in B'$, define $cb' = \varphi(c)\cdot\varphi(b)$ where $b' = \varphi(b)$, $b \in B$. By straightforward checking this gives a C-module structure. Then the ring isomorphism becomes a C-module isomorphism. We observe that for any $b \in B$, $O(b) = O(\varphi(b))$, order ideal in C.

<u>Corollary 1</u>: Let X be without isolated points and B any rational extension of $C(X)$, then there is no b in B such that the order ideal $O(b)$ in $C(X)$ is prime.

<u>Proof</u>: It is enough to show that there is no f in $C(X)$ such that $O(f)$ in $C(X)$ is prime (by the above lemma). Suppose there were an f in $C(X)$ such that $O(f) = P$ for some prime ideal in $C(X)$; then $fg = 0$ for all $g \in P$, and $Coz(f) \subset Z(g)$. This implies that $Coz(f) \subset \bigcap Z(g) (g \in P)$. It suffices to assume P is a fixed ideal; then the associated prime Z-filter has a cluster point and hence $\bigcap Z(g)$ is a singleton. It follows that $Coz(f) = \emptyset$ since X has no isolated point. This is a contradiction. Q.E.D.

<u>Corollary 2</u>: Let P and P' be two prime ideals in $C(X)$ then $Q(C(X)/P) \cong Q(C(X)/P')$ over $C(X)$ if and only if $P = P'$. (Q denotes a maximal ring of quotients.)

<u>Proof</u>: The "if" part is trivial. For the "only if" part, assume $P \neq P'$. For any non-zero element $u \in C(X)/P$, clearly the order ideal $O(u) = P$ in $C(X)$; in other words for any non-zero element u in $C(X)/P$, $O(u) \neq P'$. This implies from the Lemma that there is no element u' in $Q(C(X)/P')$ such that $O(u') = P'$; in particular there is no element u' in $C(X)/P'$ such that $O(u') = P'$.

But this is a contradiction because for a non-zero element u' in $C(X)/P'$, $O(u') = P'$. Q.E.D.

<u>Corollary 3</u>: Let P be a prime and M be a maximal ideal in $C(X)$. If $C(X)/M$ is isomorphic to a ring of quotients of $C(X)/P$ over $C(X)$ then P is maximal.

Section 4: Classical ring of quotients.

Let S be a multiplicative submonoid of C(X) consisting of non-zero divisors of C(X) (abbreviation: m.s.) and $\mathfrak{F}(S)$ be the filter base generated by the cozero sets of S, then $C(X)[S^{-1}]$ is a classical ring of quotients of C(X) with respect to S. Now we have the following proposition.

Proposition 14: Let S be a m.s. of C(X) such that, for $f \in S$ and $g \in C(X)$, $Coz(f) = Coz(g)$ implies $g \in S$. Then $C(X)[S^{-1}] \cong Q_{\mathfrak{F}(S)}(X)$.

Proof: For an $f \in C(X)$ and a $g \in S$, define a mapping $fg^{-1} \rightsquigarrow (fg^{-1})^* + Z_{\mathfrak{F}(S)}(X)$ of $C(X)[S^{-1}]$ into $Q_{\mathfrak{F}(S)}(X)$ where $(fg^{-1})^*$ is defined by $(fg^{-1})^* \mid V = fg^{-1}$ and vanishing outside of V for some $V \in \mathfrak{F}(S)$. Then clearly this mapping is a homomorphism and one to one since if $(fg^{-1})^* \in Z_{\mathfrak{F}(S)}(X)$, then $fg^{-1} \mid V = 0$ for some $V \subseteq Coz(g)$: Thus $f \mid X = 0$; hence $fg^{-1} = 0$ in $C(X)[S^{-1}]$.

Now we show the mapping is onto. Let $u \in Q_{\mathfrak{F}(S)}(X)$ with $u = \nu(h)$, $h \in C_V(X)$, $V \in \mathfrak{F}(S)$. Then there exists a g in S such that $V = Coz(g)$. Let $g' = g/1+h^2$. Then $g' \in C(X)$ and $Coz(g) = Coz(g')$. Thus $g' \in S$. Define a function f on X by $f = hg'$. Then clearly $f \in C(X)$ and

$(fg'^{-1})^* + Z_{\mathcal{F}(S)}(X) = u$ since $(fg'^{-1})^* \mid V' = fg'^{-1} = h$ for some $V' \subseteq \mathrm{Coz}(g')$. Q.E.D.

Thus for a m.s. S of C(X) consisting of non-zero divisors of C(X) satisfying the condition in the proposition, the ring $Q_{\mathcal{F}(S)}(X)$ can be regarded as a classical ring of quotients of C(X) with respect to S, i.e. the image of the $f \in S$ under the underlying natural homomorphism are invertible and $Q_{\mathcal{F}(S)}(X)$ is generated by these inverses and the image of C(X).

The proof of the following statements are straightforward.

1. If S is the set of all non-zero divisors of C(X), then $Q_{\mathcal{F}(S)}(X)$ is the full ring of quotients of C(X).

2. For any m.s. S in C(X), the ring $Q_{\mathcal{F}(S)}(X)$ is a ring of quotients of C(X).

<u>Remark</u>: We have seen in (2) that for any m.s. S the ring $Q_{\mathcal{F}(S)}(X)$ is a ring of quotients of C(X) with respect to the obvious embedding. From the previous section if the filter basis $\mathcal{F}(S)$ is finer than the filter basis of all dense open subsets of X, then $Q_{\mathcal{F}(S)}(X)$ is a maximal ring of quotients of C(X).

Now we have a question: If $Q_{\mathcal{F}(S)}(X)$ is a maximal ring of quotients of $C(X)$, then does this imply that $\mathcal{F}(S)$ is finer than the filter basis of all dense open subsets of X? For the time being we shall leave this as a problem. However, we give here some examples that provide some partial answers to this question.

We note first that the classical ring of quotients of $C(X)$ is the maximal ring of quotients of it if and only if for any $f \in C(U)$, where U is a dense open, there exists a $g \in C(V)$, where V is a dense cozero set, such that $f|U \cap V = g U \cap V$ [8].

1. If X is separable space then every open dense subset contains a dense cozero set, hence $Q_{\mathcal{F}(S)}(X)$ is a maximal ring of quotients of $C(X)$ if S is the set of all non-zero divisors.

2. Every metric space need not be separable, but every open subset of a metric space is cozero set; hence every dense open set itself is a dense cozero set.

3. Let X be an η_1-set endowed with the interval topology; then X is P-space without isolated points; hence every zero set is open; more precisely, every continous function vanishing at a point p vanishes on a neighborhood of p. Thus no proper cozero set is dense. Clearly βX has no isolated points. Then βX is the space that is compact without isolated points such that the maximal and

classical (full) ring of quotients of $C(X)$ do not coincide.

4. Let X be a topological space in which every open subspace is paracompact such that every subset of X, all of whose points are isolated, is countable. Then as is well-known every open subspace of X is Lindelöf; thus every dense open subset itself is cozero dense subset. Hence the maximal and classical (full) ring of quotients of $C(X)$ coincide.

Let A be a finite set of non-zero divisors of $C(X)$ containing unit and S be the set of all finite products of elements of A, then S is a m.s. Since, for each $f \in C(X)$, $Coz(f^n) = Coz(f)$ for integer $n \geqslant 1$, we see that the filter basis $\mathfrak{F}(S)$ generated by the cozero sets of $f \in S$ is the set of all finite intersections of the cozero sets of $f \in A$. Let $T = \bigcap_{f \in A} Coz(f)$; then T is the smallest member of $\mathfrak{F}(S)$ contained in every member of $\mathfrak{F}(S)$. Hence we have the following.

<u>Remark</u>: If S is a m.s. generated by the finite set A, then $Q_{\mathfrak{F}(S)}(X)$ is a maximal ring of quotients of $C(X)$ if and only if $Coz(\prod_{f \in A} f)$ is discrete.

<u>Proof</u>: $Q_{\mathfrak{F}(S)}(X) \cong C(T)$, where $T = Coz(\prod f)(f \in A)$, and $C(T)$ has no proper rational extension iff T is discrete.

CHAPTER II

Order and Topological properties

of the ring $Q_{\mathcal{D}}(X)$

Section 1: Archimedean F-ring.

In the ring $C_{\mathcal{D}}(X)$ defined in Chapter I, one defines a partial order in the usual function way, that is, for any f and g in $C_{\mathcal{D}}(X)$, $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. Next for any f and g, the function k defined by the formula

$$k(x) = f(x) \vee g(x)$$

satisfies: $k \geqslant f$ and $k \geqslant g$; furthermore if $f \in C_D(X)$ and $g \in C_{D'}(X)$, $D, D' \in \mathcal{D}$, then $k \in C_{D \cap D'}(X)$, and for all h such that $h \geqslant f$ and $h \geqslant g$, we have $h \geqslant k$. Therefore $f \vee g$ exists in $C_{\mathcal{D}}(X)$: It is k, and $(f \vee g)(x) = f(x) \vee g(x)$. Dually $f \wedge g$ exists and $(f \wedge g)(x) = f(x) \wedge g(x)$, and $f \wedge g \in C_{D \cap D'}(X)$. This shows that the ring $C_{\mathcal{D}}(X)$ is closed under the meet and join and hence is a lattice ordered ring. Also the partial order is purely algebraically determined, i.e. $f \geqslant 0$ iff $f = g^2$ for some $g \in C_{\mathcal{D}}(X)$.

Now $Z_{\mathcal{D}}(X)$ is an ℓ-ideal, that is for $f \in Z_{\mathcal{D}}(X)$ if $|g| \leqslant |f|$, then $|f| \mid D = 0$ for some $D \in \mathcal{D}$, hence $|g| \mid D = 0$; thus $g|D \equiv 0$, i.e. $g \in Z_{\mathcal{D}}(X)$.

Hence the ring $Q_{\mathcal{D}}(X)$ inherits a partial order which is a lattice order. Thus one has $u \leqslant v$, $u = \nu(f)$, $v = \nu(g)$ if $f \leq g$, and the natural homomorphism ν is a ℓ-homomorphism, i.e. $\nu(f \vee g) = \nu(f) \vee \nu(g)$ and $\nu(f \wedge g) = \nu(f) \wedge \nu(g)$. Also one shows the following:

$\nu(f) \leqslant \nu(g)$ iff $f|D \leqslant g|D$ for some $D \in \mathcal{D}$.

Since $\nu(f) \leqslant \nu(g)$ implies $\nu(g - f) \geqslant 0$, it follows that there is an $h \geqslant 0$ in $C_{\mathcal{D}}(X)$ such that $(g - f) - h \in Z_{\mathcal{D}}(X)$, i.e. $g - f = h$ on some D in $\mathcal{D}$. The converse is trivial.

Now let $nu \leqslant v$ for all integers n, where $u = \nu(f)$, $v = \nu(g)$, then $nf \leqslant g$ holds on some D in $\mathcal{D}$; this implies that $f = 0$ on the D; hence $u = 0$. Also if $u \wedge v = 0$ and $w \geqslant 0$ where $u = \nu(f)$, $v = \nu(g)$ and $w = \nu(h)$, then $f \wedge g = 0$, $h \geqslant 0$ on some D in $\mathcal{D}$. This implies that for each $x \in D$, $f(x) = 0$, or $g(x) = 0$, or both are 0. Hence $(h(x) \cdot f(x)) \wedge g(x) = 0$, i.e. $hf \wedge g = 0$; thus $wu \wedge v = 0$. One has the following.

<u>Proposition 1</u>: For any filter base $\mathcal{D}$, the ring $Q_{\mathcal{D}}(X)$ is an archimedean F-ring.

For an F-ring one has the following identities ([9]):

i) if $w \geqslant 0$, then $(u \vee v)w = uw \vee vw$

$$(u \wedge v)w = uw \wedge vw$$

ii) $|uv| = |u| \cdot |v|$

iii) $u^2 \geqslant 0$ for each u

iv) $u \wedge v = 0$ implies $uv = 0$.

<u>Remark</u>: The partial order in the ring $Q_{\mathfrak{D}}(X)$ is also algebraically determined, i.e. $u \geqslant 0$ iff $u = v^2$, $v \in Q_{\mathfrak{D}}(X)$, and each positive element $u > 0$ has a unique positive square root $\sqrt{u}$, and for any u, $|u| = u \vee -u$ is also $\sqrt{u^2}$.

We recall that the mapping $j_D : C(D) \longrightarrow Q_{\mathfrak{D}}(X)$ defined by $j_D(f|D) = \nu(f)$, $f \in C_D(X)$, is an embedding. Since ν is a ℓ-homomorphism, so is j_D. Hence we have:

<u>Proposition 2</u>: $Q_{\mathfrak{D}}(X)$, with its partial order, is a direct limit of the direct system $(C(D), \rho_{DE})_{D,\, E \in \mathfrak{D}}$, with $(j_D)_{D \in \mathfrak{D}}$ as limit homomorphisms, in the category of all semi-prime commutative F-ring and lattice-order ring homomorphisms.

<u>Remark</u>: <u>For any two elements $u \geqslant 0$, $v \geqslant 0$ in a commutative semi-prime F-ring, if $u^2 = v^2$, then $u = v$.</u> To show this let $s = u(u - v)^2$ and $t = v(u - v)^2$, then both $s \geqslant 0$ and $t \geqslant 0$, and $s + t = (u + v)(u - v)^2 = 0$. Hence $s = t = 0$. Thus $(u - v)^3 = s - t = 0$. But the

ring is semi-prime, hence $u = v$.

Here one shows that the unique unitary ring homomorphism $\varphi: Q_{\mathfrak{D}}(X) \longrightarrow R$, coming up in the proof of the Theorem 2, Section 1, Chapter I, satisfies the two conditions i) $u \geqslant 0$ implies $\varphi(u) \geqslant 0$, and ii) $\varphi(|u|) = |\varphi(u)|$. Incidentally, the ring R here is an arbitrary commutative semi-prime F-ring in the category. The first condition is clear because there is an element v in $Q_{\mathfrak{D}}(X)$ such that $u = v^2$ and $\varphi(v)^2 \geqslant 0$ in the F-ring R. For the condition ii) let $u \in Q_{\mathfrak{D}}(X)$, then $[\varphi(|u|)]^2 = \varphi(|u|^2) = \varphi(u^2) = |\varphi(u)|^2$, and $\varphi(|u|) \geqslant 0$ and $|\varphi(u)| \geqslant 0$. It follows from the remark that $\varphi(|u|) = |\varphi(u)|$.

As usual for each $r \in \mathbb{R}$, $r \cdot u =_{df} \nu(r)u$ gives a vector lattice structure on the ring $Q_{\mathfrak{D}}(X)$, since for $r \geqslant 0$, $u = \nu(f) \geqslant 0$, we have $(rf) | D \geqslant 0$ for some $D \in \mathfrak{D}$. Now if the ring R is a vector lattice over, at least the field of rational numbers Q, then the conditions i) and ii) imply that the ring homomorphism preserves the meets and joins. Since $Q_{\mathfrak{D}}(X)$ is a vector lattice over $\mathbb{R}$, $u \vee v = 1/2 \cdot (u + v - |u - v|)$ and $\varphi(1/2 \cdot 1_Q) = 1/2 \cdot e_R$, φ being a unitary ring homomorphism; thus $\varphi(u \vee v) = 1/2 \cdot (\varphi(u) + \varphi(v) - |\varphi(u) - \varphi(v)|) = \varphi(u) \vee \varphi(v)$ since the $\mathbb{R}$ is a vector lattice; and the same hold for meets. We have the followings:

<u>Corollary 1</u>: If $\varphi : Q_{\mathcal{D}}(X) \longrightarrow \mathbb{R}$, the reals, is a unitary ring homomorphism, then it is an ℓ-homomorphism.

<u>Proof</u>: Define a homomorphism $\varphi_D : C(D) \longrightarrow \mathbb{R}$ for each $D \in \mathcal{D}$ by $\varphi_D = \varphi \circ j_D$. Then for any $D, E \in \mathcal{D}$ with $D \leq E$, we have $\varphi_D(f|D) = (\varphi \circ j_D)(f|D) = \varphi(j_D(f|D)) = \varphi(j_E(f)) = \varphi_E(f)$ for $f \in C(E)$. This means that the family $(\varphi_D)_{D \in \mathcal{D}}$ is compatible with respect to the direct system. Hence φ is the one which is uniquely determined. Thus is a ℓ-homomorphism.

<u>Corollary 2</u>: Let $\mathcal{D}$ be such that $\bigcap D \neq \emptyset$ $(D \in \mathcal{D})$; then there exists a unitary ring homomorphism, hence an ℓ-homomorphism, from $Q_{\mathcal{D}}(X)$ into the reals $\mathbb{R}$.

<u>Proof</u>: Let $p_o \in \bigcap D(D \in \mathcal{D})$ be a fixed point: for each member $D \in \mathcal{D}$, define a mapping $\varphi_D : C(D) \longrightarrow \mathbb{R}$ by $\varphi_D(f) = f(p_o)$, then clearly each φ_D is a ring homomorphism and moreover for each pair D, E in $\mathcal{D}$ with $D \subseteq E$ and each $f \in C(E)$, we have $\varphi_E(f) = f(p_o) = (f|D)(p_o) = \varphi_D(f|D)$; this means the family $(\varphi_D)_{D \in \mathcal{D}}$ is compatible with respect to the direct system. Hence there exists a unique unitary ring homomorphism $\varphi : Q_{\mathcal{D}}(X) \longrightarrow \mathbb{R}$.

Section 2: Real ideals.

Proposition 3: Every prime ideal P in $Q_{\mathcal{S}}(X)$ is an ℓ-ideal and the residue class ring $Q_{\mathcal{S}}(X)/P$ is totally ordered and the mapping $r \rightsquigarrow r + P$ is an order-preserving isomorphism of the real field into the residue class ring.

Proof: Let $|v| \leq |u|$ with $u \in P$ and $u = \nu(f)$, $v = \nu(g)$, then $|g| \leqslant |f|$ holds for some $D \in \mathcal{S}$. Define a function h on X by:

$$h(x) = g^2(x)/f(x) \text{ for } x \in D \setminus D \cap Z(f)$$
$$= 0 \quad \text{elsewhere.}$$

Since $|g| \leqslant |f|$, $g(x)/f(x)$ is bounded for each $x \in D - D \cap Z(f)$, and hence h is continuous on D, i.e. $h \in C_D(X)$. Thus $\nu(h)\cdot\nu(f) = \nu(hf) = \nu(g^2) = v^2$ is an element of the prime ideal P; hence $v \in P$, thus P is an ℓ-ideal.

Since $Q_{\mathcal{S}}(X)$ is an F-ring, for each u in $Q_{\mathcal{S}}(X)$ $|u|^2 = u^2$; hence $(u - |u|)(u + |u|) = 0$ holds; thus either $u - |u| \in P$ or $u + |u| \in P$; the former implies $u + P \geqslant 0$, the latter implies $u + P \leqslant 0$.

For the last part, clearly the mapping is an isomorphism and $r \geqslant 0$ implies $r + P \geqslant 0$.

<u>Definition</u>: A maximal ideal M in a ring A is called <u>real</u> iff its quotient field A/M is Archimedean.

<u>Remark</u>: From the Proposition 3, we have seen that for any maximal ideal $M \subset Q_{\mathcal{D}}(X)$, the quotient field $Q_{\mathcal{D}}(X)/M$ is a totally ordered field containing an isomorphic copy of the reals $\mathbb{R}$, and every Archimedean ordered field is order-isomorphic to a subfield of the field of reals $\mathbb{R}$, hence a maximal ideal M in $Q_{\mathcal{D}}(X)$ is real if and only if $Q_{\mathcal{D}}(X)/M$ is isomorphic with $\mathbb{R}$.

<u>Proposition 4</u>: If $\varphi : Q_{\mathcal{D}}(X) \longrightarrow \mathbb{R}$, the reals, is a nonzero homomorphism, then the kernel, $\ker\varphi \equiv M_{\varphi}$ is a real maximal ideal, and $\varphi \rightsquigarrow M_{\varphi}$ is a one-to-one correspondence between the homomorphisms from $Q_{\mathcal{D}}(X)$ into $\mathbb{R}$ and the real maximal ideals.

<u>Proof</u>: Observe the ring homomorphism $\varphi \circ \nu$ from $C(X)$ into $\mathbb{R}$ is an onto mapping; hence so is φ. Thus the $\ker\varphi$ is a real maximal ideal. Since distinct homomorphisms onto $\mathbb{R}$ have distinct kernels, the correspondence between the homomorphisms of $Q_{\mathcal{D}}(X)$ onto $\mathbb{R}$, and the real maximal ideals, is one to one.

<u>Remark</u>: Clearly for any $\mathcal{D}$, if $\bigcap D \neq \emptyset$ $(D \in \mathcal{D})$ then by the above proposition there are plenty of real maximal ideals, the number of such ideals is at least the

cardinal number $|\bigcap D|$. This implies that if the space X has isolated points, then the number of real maximal ideals is at least the number of isolated points, since every isolated point is contained in every member of $\mathfrak{D}$.

Definition: A ring (or algebra) is said to be *totally unreal* if it does not have any real ideal.

The necessary condition in the following proposition has been conjectured by Professor Banaschewski.

Proposition 5: Let each member D of $\mathfrak{D}$ be realcompact; then the ring $Q_{\mathfrak{D}}(X)$ is totally unreal if and only if $\bigcap D = \emptyset$ $(D \in \mathfrak{D})$.

Proof: *Sufficiency*: It is evident, since if $\bigcap D \neq \emptyset$ $(D \in \mathfrak{D})$, then by Corollary 2 to Proposition 2, there is a real maximal ideal.

Necessity: Suppose there were a real maximal ideal; then there is a unitary ring homomorphism φ from $Q_{\mathfrak{D}}(X)$ onto the reals $\mathbb{R}$. Let D be a member of $\mathfrak{D}$. Then the mapping $\varphi \circ j_D$ is again a homomorphism from C(D) onto $\mathbb{R}$ since the mapping $r \rightsquigarrow \varphi \circ j_D(r)$ is a nonzero homomorphism from R into R and hence is the identity mapping. Since each D is realcompact, to the homomorphism $\varphi \circ j_D$, there corresponds a point x_0 of D such that $\varphi \circ j_D(f) = f(x_0)$ for all

$f \in C(D)$. Now we claim that each member of $\mathcal{D}$ contains the point x_o. Let E be a member of $\mathcal{D}$; then there exist a member D' in $\mathcal{D}$ such that $D' \subseteq D \cap E$; hence $j_D(f) = j_{D'}(f|D')$ for all $f \in C(D)$. Similarly, the mapping $\varphi \circ j_{D'}$ is a homomorphism from $C(D')$ onto $\mathbb{R}$, and since D' is realcompact, there corresponds a point y_o of D' such that $(\varphi \circ j_{D'})(f') = f'(y_o)$ for all $f' \in C(D')$: In particular $(\varphi \circ j_{D'})(f|D') = f(y_o)$ for all $f \in C(D)$. Since $(\varphi \circ j_{D'})(f|D') = (\varphi \circ j_D)(f)$ for all $f \in C(D)$, it follows that $f(x_o) = f(y_o)$ for all $f \in C(D)$, and since D is a completely regular space we have $x_o = y_o$. Thus $x_o \in D'$ and hence $x_o \in E$. One concludes that $x_o \in \cap D$ $(D \in \mathcal{D})$ which is a contradiction to the hypothesis; this completes the proof.

Remark: If X is realcompact, and each point of X is a G_δ, then every subspace of X is realcompact[12].

Now we have the following:

Corollary 1: Let X be a separable realcompact space without isolated points such that every closed subset is a G_δ-set; then the maximal ring of quotients of C(X) is totally unreal.

Proof: Let A be a countable dense subset of X such that $A = \bigcup_{i \in I} \{a_i\}$, where the index set $I = \{1,2, \ldots\}$.

For each $i \in I$, let J_i be a countable index set; then $\{a_i\} = \bigcap_{j \in J_i} V_{i,j}$ where $V_{i,j}$ is an open set containing a_i for each j. Then $A = \bigcup_{i \in I} (\bigcap_{j \in J_i} V_{i,j}) = \bigcap_{\varphi \in \Phi} (\bigcup_{i \in I} V_{i,\varphi(i)})$ where Φ is the set of all functions φ with domain I such that $\varphi(i) \in J_i$ for each $i \in I$; hence A itself is an intersection of dense open sets. On the other hand, the set $X - \{a_1, a_2, \ldots, a_n\}$, $a_i \in A$ $(i = 1, \ldots, n)$ is a dense open subset of X. Hence $\bigcap_{i=1}^{\infty} (X - \{a_1, \ldots, a_i\}) = \mathbf{C}\left(\bigcup_{i=1}^{\infty} \{a_1, \ldots, a_i\}\right) = X - A$; thus $X - A$ is an intersection of dense open sets. Consequently, the intersection of all dense open subsets of X is empty, and also by above remark every subspace is realcompact. Hence the maximal ring of quotients of $C(X)$ is totally unreal.

<u>Corollary 2</u>: For a separable metric space X without isolated points, the maximal ring of quotients of $C(X)$ is totally unreal.

<u>Proof</u>: A separable metric space is realcompact, and every closed set is a G_δ-set, hence every dense open subset is realcompact; thus the proof is evident.

<u>Remark</u>: Every maximal ideal in $Q_{\mathcal{S}}(X)$ is real if and only if each member of $\mathcal{S}$ is pseudocompact.

Proof: For a member D of $\mathcal{D}$, an element $f \in C(D)$ and a maximal ideal M in $Q_{\mathcal{D}}(X)$ we have the following: For some natural number n; $|j_D(f) + M| = |j_D(f)| + M \leq n \cdot 1_{Q/M}$ iff $|j_D(f)| = j_D(|f|) \leqslant n \cdot 1_Q$ iff $|f| \leqslant n$ where $1_{Q/M}$ and 1_Q are units in $Q_{\mathcal{D}}(X)/M$ and $Q_{\mathcal{D}}(X)$ respectively. Hence the assertion holds.

Example: The following illustrates an example of a ring of quotients of C(X), in which every maximal ideal is real. Let ω_1 be the first uncountable ordinal. Let $Z = W(\omega_1) = \{\sigma | \sigma < \omega_1\}$, and $Y = Z \oplus \ldots \oplus Z$, i.e. the free join of finitely many Z; then clearly Y is locally compact and pseudocompact, and $|\beta Y - Y| > 1$. Let X be the one-point compactification of Y, and $\mathcal{D} = \{Y\}$. Then $Q_{\mathcal{D}}(X)$ is a ring of quotients of C(X) in which every maximal ideal is real.

Section 3: Residue class fields.

For a maximal ideal M in $Q_{\mathcal{S}}(X)$, let M' be the preimage of M under the natural homomorphism ν ; then M' is a maximal ideal in $C_{\mathcal{S}}(X)$ containing $Z_{\mathcal{S}}(X)$. Denote $Z(M') = \{Z(f) \mid f \in M'\}$. Then one checks that for any $f \in C_{\mathcal{S}}(X)$, $f \in M'$ if and only if $Z(f) \in Z(M')$.

Let $f \in C_{\mathcal{S}}(X)$ with $f \geqslant 0$, and let r be any positive real number. The function f^r, defined by

$$f^r(x) = (f(x))^r$$

is an element of $C_{\mathcal{S}}(X)$. Hence it is possible to define exponentiation in the ring $Q_{\mathcal{S}}(X)$ by: For any $r > 0$ and $u \geqslant 0$ in $Q_{\mathcal{S}}(X)$ define $u^r = \nu(f^r)$ where $u = \nu(f)$, f is non-negative; then u^r depends only upon u and r, not upon the particular representative f: For if $u = \nu(f) = \nu(g)$, then $f = g$ on some D in $\mathcal{S}$ if and only if $f^r = g^r$ on this D.

Furthermore for any $r > 0$, and $a \geqslant 0$ in $Q_{\mathcal{S}}(X)/M$, where M is a maximal ideal in $Q_{\mathcal{S}}(X)$, $a = M(u) \equiv u + M$, $u \in Q_{\mathcal{S}}(X)$, define a^r, by $a^r = M(u^r)$; then also a^r depends only upon a and r, not upon the particular representative u.

Since if $a = M(u) = M(v)$, then $u - v \in M$, then $f - g \in M'$ where f, g and M' are preimages of u, v and M respectively. Note that $Z(f - g) = Z(f^r - g^r)$, and since $Z(f - g) \in Z(M')$ hence $Z(f^r - g^r) \in Z(M')$; thus $f^r - g^r \in M'$. Consequently $\nu(f^r - g^r) = \nu(f^r) - \nu(g^r) = u^r - v^r \in M$, hence $a^r = M(u^r) = M(v^r)$.

Clearly the following are valid: For any $a > 0$ in $Q_{\mathcal{S}}(X)/M$, $a^r a^s = a^{r+s}$; $(a^r)^s = a^{rs}$; if $a < b$, then $a^r < b^r$; and if a is infinitely large, then so is a^r $(r > 0)$. Hence, in a completely analogous way to [12, §13.2] one obtains the corresponding results, that is:

Proposition 6: The transcendence degree over the reals $\mathbb{R}$ of a non Archimedean field $Q_{\mathcal{S}}(X)/M$ is at least $\mathfrak{c}$.

Definition: An ordered field K is said to be real-closed if every positive element is a square and every polynomial over K in one indeterminate of odd degree has a zero in K.

From the definition of exponentiation, for any maximal ideal M, every positive element of $Q_{\mathcal{S}}(X)/M$ is a square; for if $a > 0$ in $Q_{\mathcal{S}}(X)/M$, then $a = M(u)$, $u > 0$ and $u = v^2$, $v \in Q_{\mathcal{S}}(X)$; hence $a = M(u) = M(v)^2 = b^2$, $b \in Q_{\mathcal{S}}(X)/M$.

Now for any odd number n, let $S^n + u_1 S^{n-1} + \ldots + u_n$ be a polynomial over $Q\ (X)$ with one indeterminate S. Put $q(S) = S^n + f_1 S^{n-1} + \ldots + f_n$, where $f_i (i = 1, \ldots, n)$ are representatives of $u_i (i = 1, \ldots, n)$, and each f_i is continuous on some $D \in \mathcal{S}$. Define a mapping $f : X \longrightarrow \mathbb{R}^n$ by

$$f(x) = (f_1(x), \ldots, f_n(x)),$$

then clearly f is continuous on the D. Note that for each point $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, let $p_1(a), \ldots, p_n(a)$ denote the real parts of the (complex) roots of the polynomial

$P_a(S) = S^n + a_1S^{n-1} + \dots + a_n$, taken such that $p_1(a) \leq \leq p_2(a) \leq \dots \leq p_n(a)$; then the functions $p_i (i = 1, \dots, n)$: $\mathbb{R}^n \longrightarrow \mathbb{R}$ are continuous [12].

Define a function $g_i : X \longrightarrow \mathbb{R}$, by $g_i = p_i \circ f$, then $g_i (i = 1, \dots, n)$ is continuous on the same D, and clearly for each $x \in X$, $g_i(x) (i = 1, \dots, n)$ are the real parts of the roots of the polynomial $P_{f(x)}(S) = S^n + f_1(x)S^{n-1} + \dots + f_n(x)$.

On the other hand, since the field of reals is real-closed, the polynomial $P_{f(x)}(S)$ has at least one real zero.

Hence for each $x \in X$, there corresponds at least one index i $(1 \leq i \leq n)$ such that $g_i(x)$ is a zero of $P_{f(x)}(S)$; i.e. for each $x \in X$:

$$P_{f(x)}(g_i(x)) = (g_i^n + f_1 g_i^{n-1} + \dots + f_n)(x)$$
$$= q(g_i)(x) = 0.$$

This implies that $q(g_1) \dots q(g_n) = 0$ in $C_{\mathcal{D}}(X)$, hence $\nu(q(g_1)) \dots \nu(q(g_n)) = 0$ in $Q_{\mathcal{D}}(X)$. Since M is a prime ideal, hence there exists an index i $(1 \leq i \leq n)$ such that $\nu(q(g_i)) \in M$, i.e. $\nu(q(g_i)) = \nu(g_i)^n + \nu(f_1) \cdot \nu(g_i)^{n-1} + \dots + \nu(f_n) = v^n + u_1 v^{n-1} + \dots + u_n$, where $v = \nu(g_i)$, belongs to M. Thus one has the following.

<u>Proposition 7</u>: For any maximal ideal M in $Q_{\mathcal{D}}(X)$, the field $Q_{\mathcal{D}}(X)/M$ is real-closed.

Section 4: m-topology.

Let A be a commutative F-ring with unit e. For a positive invertible element u in A, define a set W_u by:

$$W_u = \{ s \mid |s| \leq u \}.$$

Let $\mathcal{L}$ denote the family of all W_u. We show that $\mathcal{L}$ is a filter base on A. Take W_u, W_v in $\mathcal{L}$, where u, v are positive invertible in A. Since A is F-ring, we have $(u \wedge v)\cdot(u^{-1} \vee v^{-1}) = u(u^{-1} \vee v^{-1}) \wedge v(u^{-1} \vee v^{-1}) = (e \vee uv^{-1}) \wedge (vu^{-1} \vee e) \geqslant e$; on the other hand $(u \wedge v)\cdot(u^{-1} \vee v^{-1}) = (u \wedge v)u^{-1} \vee (u \wedge v)v^{-1} = (e \wedge vu^{-1}) \vee (uv^{-1} \wedge e) \leqslant e$, i.e. $(u \wedge v)(u^{-1} \vee v^{-1}) = e$. Thus $u \wedge v$ is positive invertible, and clearly $W_{u \wedge v} \leqslant W_u \cap W_v$.

Now if A is divisible as an additive group, then one shows that (1): $o \in W$ for all $W \in \mathcal{L}$; (2): For any $W \in \mathcal{L}$, there exists $V \in \mathcal{L}$ such that $V + V \subseteq W$; (3): $W = -W$ for each $W \in \mathcal{L}$. (1) and (3) are trivial. For (2), let $W = W_v$. Since A is divisible additive group, there is a $u \in A$ such that $2u = v$, and u^{-1} exists, namely $u^{-1} = 2v^{-1}$. Then clearly $W_u + W_u \subseteq W_v$. Hence we have the following:

<u>If A is a commutative F-ring with unit e and divisible as an additive group, then there is a unique topology on A, compatible with the group structure of A, for which the family $\{ W_u + s \mid s \in A, W_u \in \mathcal{L} \}$ is a basis for the</u>

<u>topology. The resulting topology will be called the m-topology</u> .

Moreover, if A is convex, i.e. all $x(\geqslant e)$ are invertible, and the multiplicative group of all positive invertible elements of A is divisible, then one has that (4): For each $W \in \mathcal{L}$, there exists $V \in \mathcal{L}$ such that $VV \subseteq W$; (5): For each $a \in A$, $W \in \mathcal{L}$, there exists $V \in \mathcal{L}$ with aV, $Va \subseteq W$. For (4), let $W = W_v$; then there exists u, positive invertible, such that $u^2 = v$. Put $V = W_u$, then clearly $VV \subseteq W$. For (5), let $W = W_v$, $u = (e \vee |a|)^{-1}v$, and put $V = W_u$; then $aV \subseteq W$. Thus we have the following:

<u>Proposition 8</u>: If a commutative F-ring A with unit is divisible as an additive group and convex, and the multiplicative group of all positive invertible elements of A is divisible, then the m-topology on A is compatible with the ring structure.

<u>Corollary 1</u>: In addition to the proposition, if A is Archimedean, then the m-topology is Hausdorff.

<u>Proof</u>: For each positive integer i, there exists a positive invertible element u_i such that $iu_i = e$. Let $a \in \bigcap_i W_{u_i}$. Then $|a| \leqslant u_i$, i.e. $i|a| \leqslant iu_i = e$. This implies that $k|a| \leqslant e$ for all $k = \pm 1, \pm 2, \ldots,$. Thus $|a| = 0$, hence $a = 0$. Q.E.D.

Corollary 2: Under the same condition as in Corollary 1, the zero ideal of the ring A is m-closed.

Proof: Let $a \in \Gamma_m(o)$, where Γ_m denotes the closure operator with respect to the m-topology; then every neighborhood $W_u + a$ of a contains the o; in particular $W_{u_i} + a$ contains the o, where u_i is defined as in Corollary 1; i.e. $-a \in W_{u_i}$ for all positive integer i, hence $|-a| = |a| \leqslant u_i$ as Corollary 1, we have $a = 0$. Q.E.D.

It has been mentioned that the ring $Q_{\mathcal{S}}(X)$, for a filter base of dense subsets of X, is a commutative F-ring. Clearly it is convex and divisible as an additive group. Also the multiplicative group of all positive invertible elements of $Q_{\mathcal{S}}(X)$ is divisible. Thus we have the following proposition.

Proposition 9: The m-topology on $Q_{\mathcal{S}}(X)$, for any $\mathcal{S}$, is compatible with its ring structure.

Proposition 10: The ring $Q_{\mathcal{S}}(X)$ endowed with the m-topology is Q-ring, i.e. the set of all invertible elements is m-open.

Proof: Let s be an invertible element; then $|s| > 0$. If $t \in W_{|s|/2} + s$, then $|t - s| \leqslant |s|/2$.

Let $s = \nu(f)$, $t = \nu(h)$; then $|h - f| \leqslant |f|/2$; thus h is invertible in $C(D)$ for some $D \in \mathcal{D}$. Hence t is invertible. Q.E.D.

Corollary 1: Every maximal ideal in $Q_{\mathcal{D}}(X)$ is m-closed.

Corollary 2: Every ideal in the maximal ring of quotients of $C(X)$ is m-closed.

Proposition 11: If X has dense cozero sets $D \neq X$, then, for any $\mathcal{D}$ containing such D, the natural mapping $\nu : C(X) \longrightarrow Q_{\mathcal{D}}(X)$ is not m-continuous.

Proof: Take $h \in C(X)$ with $Coz(h) = D \in \mathcal{D}$ and $h(x) > 0$ for all $x \in D$. Let $p = \nu(h)$ and suppose $\nu^{-1}(W_p) = \{ f \in C(X) \mid |\nu(f)| \leqslant p\}$ is an m-neighborhood of 0 in $C(X)$, i.e. there exists a positive invertible $q \in C(X)$ such that $|g| \leqslant q$ implies $|\nu(g)| \leqslant p$. In particular, $\nu(q) \leq p$ and hence $q|E \leqslant h|E$ for some $E \in \mathcal{D}$, $E \subseteq D$. Thus, by continuity, $q \leqslant h$ and hence $q(x) = 0$ for all $x \notin D$ which contradicts the existence of q^{-1}. Hence ν is not m-continuous. Q.E.D.

Remark: As a simple example, consider an ordered field K. Evidently K satisfies all those conditions in Proposition 8; thus the m-topology defined on K is compatible with the ring structure of K.

CHAPTER III

Systems of Algebras and Their Limits.

Section 1: Injective and Projective Systems.

Let $(B_\alpha, \varphi_{\alpha\beta})_{\alpha,\beta\in I}$ be an injective system of $\mathbb{R}$-algebras with unit and unitary algebra homomorphisms. Then, the injective limit B, as a ring, can be made into an $\mathbb{R}$-algebra by defining $\lambda\varphi_\alpha(x) = \varphi_\alpha(\lambda x)$ $(x \in B_\alpha)$ where (φ_α) is the family of limit homomorphisms; this is then the injective limit in the category of all $\mathbb{R}$-algebras with unit and unitary algebra homomorphisms.

Moreover, if each B_α is a normed algebra with norm $\|\cdot\|_\alpha$, and the $\varphi_{\alpha\beta}$ are all norm-preserving embeddings, then one sees that the injective limit B can be made into a normed algebra, the norm $\|\cdot\|$ defined in the following way: If $u \in B$ with $u = \varphi_\alpha(f_\alpha)$ for some $\alpha \in I$, define $\|u\| = \|f_\alpha\|$. Then definition of $\|u\|$ is independent from the choice of the representation $\varphi_\alpha(f_\alpha)$. To see this, let $u = \varphi_\alpha(f_\alpha) = \varphi_\beta(f_\beta)$, $f_\alpha \in B_\alpha$, $f_\beta \in B_\beta$. Then there exists $\gamma \geq \alpha, \beta$ such that $\varphi_\gamma \circ \varphi_{\alpha\gamma}(f_\alpha) = \varphi_\gamma \circ \varphi_{\beta\gamma}(f_\beta)$. Since $\varphi_\alpha(\alpha \in I)$ is one to one, hence

$\| f_\alpha \|_\alpha = \| \varphi_{\alpha\gamma}(f_\alpha) \|_\gamma = \| \varphi_{\beta\gamma}(f_\beta) \| = \| f_\beta \|_\beta$. Clearly $\| u \| = 0$ iff $u = 0$; $\| uv \| = \| f_\gamma \cdot g_\gamma \| \leqslant \| f_\gamma \|_\gamma \cdot \| g_\gamma \| = \| u \| \cdot \| v \|$; $\| u + v \| \leqslant \| u \| + \| v \|$; and $\| \alpha u \| = |\alpha| \cdot \| u \|$. B as a normed algebra is then the injective limit of the injective system $(B_\alpha, \varphi_{\alpha\beta})$ in the category of all normed algebras (over the reals) with unit and norm-decreasing unitary algebra homomorphisms.

In what follows we shall discuss specific injective systems $(B_\alpha, \varphi_{\alpha\beta})$ in which all B_α are normed algebras over the reals with unit and the $\varphi_{\alpha\beta}$ are norm-preserving embeddings; B will be the injective limit.

<u>Definition</u>: An up-directed set I is called σ-directed, if for any countably many $\alpha_1, \alpha_2, \ldots,$ in I there exists $\alpha \in I$ such that $\alpha \geqslant \alpha_i$ for all i.

<u>Proposition 1</u>: If I is σ-directed and all B_α are complete, then B is complete.

<u>Proof</u>: Each B_α may be assumed to be a normed unitary subalgebra of B. Let (f_n) is a Cauchy sequence of B with $f_n \in B_{\alpha_n}$ for suitable α_n. Then there exists $\alpha \in I$ such that $\alpha \geqslant \alpha_i$ for all i, and hence (f_n) in B_α. Hence one has $f = \lim_{B_\alpha} f_n$ and since B_α is a normed subalgebra, $f = \lim_B f_n$.

<u>Proposition 2</u>: If $I' \geqslant I$ and the injective system on I is the restriction of one on I', then there exists a natural homomorphism from B into B' which is embedding.

<u>Proof</u>: Let $(B'_{\alpha'}, \varphi'_{\alpha'\beta'})$ and $(B_\alpha, \varphi_{\alpha\beta})$ be the injective systems over I' and I respectively. Since the family of restrictions $\varphi'_\alpha : B_\alpha \longrightarrow B'$ of limit homomorphism $\varphi'_{\alpha'}$ is compatible, there exists a unitary homomorphism $\varphi : B \longrightarrow B'$ such that $\varphi' = \varphi \circ \varphi_\alpha$ for each $\alpha \in I$, and clearly φ is one to one.

Now we discuss an injective system $(B_\alpha, \varphi_{\alpha\beta})$ with injective limit B and limit homomorphisms (φ_α) satisfying the following condition:

<u>(*)</u>: <u>Each B_α is a ring with the property that each prime ideal is contained in a unique maximal ideal and, for $\alpha \leqslant \beta$, if $P_\alpha \subset B_\alpha$ and $P_\beta \subset B_\beta$ are prime ideals such that $\varphi_\alpha(P_\alpha) \subseteq \varphi_\beta(P_\beta)$, then $\varphi_\alpha(M_\alpha) \subseteq \varphi_\beta(M_\beta)$ for the maximal ideals $M_\alpha \supseteq P_\alpha$ and $M_\beta \supseteq P_\beta$</u>.

We shall provide an example of an injective system which satisfies the condition (*) in Section 2.

<u>Lemma 3</u>: Let $(B_\alpha, \varphi_{\alpha\beta})$ be satisfying the condition (*) and M be a maximal ideal in B. For each α, let

M_α be the maximal ideal in B_α containing $\varphi_\alpha^{-1}(M)$; then $\bigcup_\alpha \varphi_\alpha(M_\alpha) = M$.

Proof: Since φ_α is monomorphism, we may put $M_\alpha \equiv \varphi_\alpha(M_\alpha)$. Let $M' = \bigcup_{\alpha\in I} M_\alpha$; then clearly $M' > M$ and is proper subset of B. From the condition (*), for any M_α, M_β there exists $\gamma \geqslant \alpha, \beta$ such that $M_\gamma \geqslant M_\alpha, M_\beta$. This implies that for any u, v in M' there exists γ such that $u, v \in M_\gamma$, and for any $u \in B$ and $v \in M'$ there exists γ such that $u \in B_\gamma$ and $v \in M_\gamma$. Hence M' is an ideal; i.e. $M' = M$.

Corollary: Let $(B_\alpha, \varphi_{\alpha\beta})$ be a system satisfying the condition (*) and such that all maximal ideals in B_α, for each α, are real; then the same holds for B.

Proof: Since $M = \bigcup_{\alpha\in I} M_\alpha$, $\mathbb{R}e_\alpha + M_\alpha = B_\alpha$ implies $\mathbb{R}e + M = B$. Q.E.D.

We have another approach to obtain this result as we shall see. In fact the following arguments will give something more.

Lemma 4: Let $(M_\alpha)_{\alpha\in I}$ be a system of maximal ideals such that each $M_\alpha \subset B_\alpha$ and, for $\alpha \leqslant \beta$, $\varphi_\alpha(M_\alpha) \subseteq \varphi_\beta(M_\beta)$. If, for each α, $M_\alpha = \ker \theta_\alpha$, where $\theta_\alpha : B_\alpha \longrightarrow \mathbb{R}$ is a unitary algebra homomorphism, then the family $(\theta_\alpha)_{\alpha\in I}$ is compatible with respect to the system $(B_\alpha, \varphi_{\alpha\beta})$.

<u>Proof</u>: $\varphi_\alpha(M_\alpha) \subseteq \varphi_\beta(M_\beta)$ implies $\varphi_\beta \circ \varphi_{\alpha\beta}(\ker \theta_\alpha) \subseteq \varphi_\beta(\ker \theta_\beta)$; whence $(\theta_\beta \circ \varphi_{\alpha\beta})(\ker \theta_\alpha) = 0$; i.e. $\ker \theta_\alpha = \ker(\theta_\beta \circ \varphi_{\alpha\beta})$: Let this be denoted by N. But there is at most one isomorphism from B_α/N onto $\mathbb{R}$, and therefore $\theta_\alpha = \theta_\beta \circ \varphi_{\alpha\beta}$. Q.E.D.

<u>Proposition 5</u>: Let $(B_\alpha, \varphi_{\alpha\beta})$ be a system satisfying condition (*). If all maximal ideals in B_α, for each α, are the kernels of unitary algebra homomorphism into $\mathbb{R}$ with norm ≤ 1, then the same holds for its injective limit B.

<u>Proof</u>: Let M be a maximal ideal in B, and let $M_\alpha \supset \varphi^{-1}(M)$ be the maximal ideal in B_α. From (*), we have $\varphi_\alpha(M_\alpha) \subseteq \varphi_\beta(M_\beta)$, for each $\alpha \leq \beta$. Let $M_\alpha = \ker \theta_\alpha$ where $\theta_\alpha : B_\alpha \longrightarrow \mathbb{R}$ is a unitary algebra homomorphism. By Lemma the family $(\theta_\alpha)_{\alpha \in I}$ is compatible with respect to the system $(B_\alpha, \varphi_{\alpha\beta})$. Hence there exists a unique unitary algebra homomorphism $\varphi : B \longrightarrow \mathbb{R}$ such that $\varphi \circ \varphi_\alpha = \theta_\alpha$ for each α. Now, for each α, $\varphi^{-1}(M) \subset \ker \theta_\alpha = \ker(\varphi \circ \varphi_\alpha)$. For any $u \in M$, and let $u = \varphi_\alpha(f_\alpha)$; then $f_\alpha \in \varphi_\alpha^{-1}(M)$; thus $\varphi(u) = \varphi(\varphi_\alpha(f_\alpha)) = 0$; i.e. $\varphi(M) = 0$. Hence $M \subseteq \ker\varphi$; i.e. $M = \ker\varphi$. Finally, for each $u \in B$, $|\varphi(u)| = |(\varphi \circ \varphi_\alpha)(f_\alpha)| = |\theta_\alpha(f_\alpha)| \leq \|f\|_\alpha = \|u\|$ if $\|\theta_\alpha\| \leq 1$. i.e. $\|\varphi\| \leq 1$.

<u>Notation</u>: $\Delta(A)$ denotes the space (endowed with the weak topology determined by the Gelfand representation of A) of all continuous unitary algebra homomorphisms of A into $\mathbb{R}$.

<u>Proposition 6</u>: Let all B_α be complete. Then, if for all B_α the norm is the spectral norm, the same holds for B.

<u>Proof</u>: It has to be shown that $\| u \| \leq \|\hat{u}\|_\infty$, where $u = \varphi_\alpha(f_\alpha)$. Since each B_α is complete normed subalgebra of B and symetric, any algebra homomorphism θ_α : $B_\alpha \longrightarrow \mathbb{R}$ can be extended to an algebra homomorphism $\bar{\theta}_\alpha$: $\bar{B} \longrightarrow \mathbb{R}$, where $\bar{B}$ is the norm completion of B. Let $\tilde{\theta}_\alpha = \bar{\theta}_\alpha | B$. Then $\| u \| = \| f_\alpha \|_\alpha = \sup_{\theta_\alpha \in \Delta(B_\alpha)} | \hat{f}_\alpha(\theta_\alpha)| =$

$$\sup_{\theta_\alpha \in \Delta(B_\alpha)} | \theta_\alpha(f_\alpha)| = \sup_{\theta_\alpha \in \Delta(B_\alpha)} | \tilde{\theta}_\alpha(u)| \leq \sup_{\theta \in \Delta(B)} | \theta(u)| =$$

$\| u \|_\infty$.

The dual system of $(B_\alpha, \varphi_{\alpha\beta})$ can be discussed as follows:

For any $\alpha \leq \beta$, define a mapping $\varphi^*_{\alpha\beta} : \Delta(B_\beta) \longrightarrow \Delta(B_\alpha)$ by $\varphi^*_{\alpha\beta}(\theta_\beta)(f_\alpha) = \theta_\beta(\varphi_{\alpha\beta}(f_\alpha))$ and $\varphi^*_\alpha : \Delta(B) \longrightarrow \Delta(B_\alpha)$ by $\varphi^*(\theta)(f_\alpha) = \theta(\varphi_{\alpha\beta}(f_\alpha))$ for each $f_\alpha \in B_\alpha$. Then it is not hard to show that $\varphi^*_{\alpha\beta}$ and φ^*_α, thus defined, are continuous. Now we have the following proposition.

<u>Proposition 7</u>: The correspondence $B_\alpha \rightsquigarrow \Delta(B_\alpha)$ $(B \rightsquigarrow \Delta(B))$, $\varphi_{\alpha\beta} \rightsquigarrow \varphi^*_{\alpha\beta}$ $(\varphi_\alpha \rightsquigarrow \varphi^*_\alpha)$ is a contravariant functor from the category of all normed rings with unit and continuous homomorphisms to the category of all completely regular Hausdorff spaces and continuous mappings.

As a result one has that the dual system $(\Delta(B_\alpha), \varphi^*_{\alpha\beta})$ is a projective system. Now we see that <u>$\Delta(B)$ is the projective limit, with respect to φ^*_α, of the dual system</u>; i.e. for any continuous mapping $F_\alpha : X \longrightarrow \Delta(B_\alpha)$, X a completely regular T_2-space, with $F_\alpha = \varphi^*_{\alpha\beta} \circ F_\beta$ for each $\alpha \leqslant \beta$, there exists a unique continuous mapping $F : X \longrightarrow \Delta(B)$ such that $\varphi^*_\alpha \circ F = F_\alpha$ for each α. The uniqueness of F, if it exists, is clear.

For each α, define $F^*_\alpha : B_\alpha \longrightarrow C^*(X)$ by $F^*_\alpha(f_\alpha) = \hat{f}_\alpha \circ F_\alpha$, then clearly F^*_α is an algebra homomorphism. Since $F_\alpha(x)(f_\alpha) = \hat{f}_\alpha(F_\alpha(x)) = (\hat{f}_\alpha \circ F_\alpha)(x) = \theta_x(\hat{f}_\alpha \circ F_\alpha)$ $= \theta_x(F^*_\alpha(f_\alpha)) = F^{**}_\alpha(\theta_x)(f_\alpha) = F^{**}_\alpha(x)(f_\alpha)$ where $\theta_x : C^*(X) \longrightarrow \mathbb{R}$ with $\theta_x(f) = f(x)$, $x \in X$ and $x \rightsquigarrow \theta_x$, $f \rightsquigarrow \hat{f}$, $f \in C^*(X)$, are $1-1$; hence $F^{**}_\alpha | X = F_\alpha$. Also $\| F^*_\alpha(f_\alpha) \|_X$ $= \| \hat{f}_\alpha \circ F_\alpha \|_X = \sup_{x \in X} | (\hat{f}_\alpha \circ F_\alpha)(x) | \leqslant \sup_{\theta \in \Delta(B_\alpha)} | \hat{f}_\alpha(\theta) |$

$\leqslant \| f_\alpha \|_\alpha$, $f_\alpha \in B_\alpha$, and hence F^*_α is norm-decreasing.

Now we show the family $(F^*_\alpha)_{\alpha \in I}$ is compatible with respect to the system $(B_\alpha, \varphi_{\alpha\beta})$. For $\alpha \leqslant \beta$, $F^*_\beta(\varphi_{\alpha\beta}(f_\alpha))(x)$ $= (\widehat{\varphi_{\alpha\beta}(f_\alpha)} \circ F_\beta)(x) = F_\beta(x)(\varphi_{\alpha\beta}(f_\alpha)) = (\varphi^*_{\alpha\beta} \circ F_\beta)(x)(f_\alpha)$

$= F_\alpha(x)(f_\alpha) = \hat{f}_\alpha(F_\alpha(x)) = F^*_\alpha(f_\alpha)(x)$; i.e. $F^*_\beta \circ \varphi_{\alpha\beta} = F^*_\alpha$. Hence there exists a unique algebra homomorphism $F^* : B \longrightarrow C^*(X)$ such that $F^*_\alpha = F^* \circ \varphi_\alpha$ for each α. Thus by the proposition there exists a continuous mapping $F : X \longrightarrow \Delta(B)$ such that $\varphi^*_\alpha \circ F = F_\alpha$ for each α, namely $F = F^{**}|X$, where $F^* : B \longrightarrow C^*(X)$ is the algebra homomorphism defined by $F^*(f) = \hat{f} \circ F$.

Proposition 8: If all B_α are complete, then the set O^{-1} of all invertible elements in $\varinjlim B_\alpha$ is open, and hence every maximal ideal in $\varinjlim B_\alpha$ is closed.

Proof: First we show that for every element $f \in \varinjlim B_\alpha$ for which $\| e - f \| < 1$ has an inverse element g. If $\| e - f \| < 1$, then there exists $\alpha \in I$ such that $\| e_\alpha - f_\alpha \|_\alpha < 1$ where $f = \varphi_\alpha(f_\alpha)$, $f_\alpha \in B_\alpha$. But B_α is complete, hence f has an inverse g_α in B_α; and $e = f \cdot g = \varphi_\alpha(f_\alpha) \cdot \varphi_\alpha(g_\alpha)$. Now let $U_0(e) = \{ g \in \varinjlim B_\alpha \mid \| e - g \| < 1 \}$, a neighborhood of unit. Take an element $f \in O^{-1}$, then $f \cdot f^{-1} = e$; and then by the continuity of multiplication, there exists a neighborhood $U(f)$ of f such that $U(f) \cdot f^{-1} \subseteq U_0(e)$. Hence, for arbitrary element $h \in U(f)$, $hf^{-1} \in U_0(e)$; i.e. $\| e - hf^{-1} \| < 1$; thus hf^{-1} has an inverse element $(hf^{-1})^{-1}$: $h \cdot f^{-1} \cdot (hf^{-1})^{-1} = e$; i.e. h is invertible, hence $h \in O^{-1}$. i.e. $U(f) \subseteq O^{-1}$.
Q.E.D.

Section 2: The ring $Q^*_{\mathcal{S}}(X)$.

For each member $D \in \mathcal{S}$, the subset $C^*_D(X)$ of $C_D(X)$, consisting of all bounded functions in $C_D(X)$, is also closed under the algebraic and order operations discussed in the previous chapters. Therefore $C^*_{\mathcal{S}}(X) = \bigcup C^*_D(X)(D \in \mathcal{S})$ is a subring and sublattice of $C_{\mathcal{S}}(X)$. Thus $Z^*_{\mathcal{S}}(X) = C^*_{\mathcal{S}}(X) \cap Z_{\mathcal{S}}(X)$ is an ideal in the ring $C^*_{\mathcal{S}}(X)$. We put $Q^*_{\mathcal{S}}(X) = C^*_{\mathcal{S}}(X)/Z^*_{\mathcal{S}}(X)$, and then one sees that the natural mapping $\nu^* : C^*_{\mathcal{S}}(X) \longrightarrow Q^*_{\mathcal{S}}(X)$ determines, for each $D \in \mathcal{S}$, an embedding $j^*_D : C^*(D) \longrightarrow Q^*_{\mathcal{S}}(X)$ such that $\nu^*(f) = j^*_D(f|D)$ for each $f \in C^*_D(X)$ and, for $D \subseteq E$, $j^*_D(f|D) = j^*_E(f)$ for all $f \in C^*(E)$. For $D \subseteq E$, we denote the restriction homomorphism $f \rightsquigarrow f|D : C^*(E) \longrightarrow C^*(D)$ by ρ^*_{ED}. Each $C^*(D)$, $D \in \mathcal{S}$ is a normed ring with the sup norm $\|f\| = \sup_{x \in D} |f(x)|$, $f \in C^*(D)$, and also $Q^*_{\mathcal{S}}(X)$ becomes a normed ring with norm defined in the manner described in Section 1. We have the following:

Proposition 9: The ring $Q^*_{\mathcal{S}}(X)$ is the injective limit of the injective system $(C^*(D), \rho^*_{ED})$ in the category of all normed rings with unit and norm-decreasing unitary homomorphisms.

The following shows the direct proof of semi-simplicity of the ring $Q^*_{\mathcal{S}}(X)$.

<u>Proposition 10</u>: The injective limit $Q^*_{\mathcal{D}}(X)$ is semi-simple for any filter base $\mathcal{D}$ of dense subsets of X.

<u>Proof</u>: Suppose the radical of $Q^*_{\mathcal{D}}(X)$ is not 0, then there is an element $u \neq 0$ in $Q^*_{\mathcal{D}}(X)$ such that $1 - ux$ is invertible in $Q^*_{\mathcal{D}}(X)$ for all $x \in Q^*_{\mathcal{D}}(X)$; i.e. $1 - u \cdot j^*_D(f)$ is invertible for all $f \in C^*(D)$ and all $D \in \mathcal{D}$. Let $u = j^*_E(g)$, $g \neq 0$ in $C^*(E)$. Then, in particular, $1 - j^*_E(g) \cdot j^*_E(f) = j^*_E(1 - gf)$ is invertible for all $f \in C^*(E)$. This would mean that $1 - gf|D'$ is invertible in $C^*(D')$ for some $D' \subseteq E$, $D' \in \mathcal{D}$ and for all $f \in C^*(E)$; i.e. for each f, $|1 - gf|\,|D' \geqslant r$ for some $r > 0$. But E is completely regular and D' is dense in E. Hence $1 - gf \geqslant r$ on E. Thus $1 - gf$ is invertible in $C^*(E)$ for all $f \in C^*(E)$. This is a contradiction.

<u>Proposition 11</u>: The injective system $(C^*(D))_{D \in \mathcal{D}}$ with the restriction homomorphisms satisfies the condition (*).

<u>Proof</u>: For $D \subseteq E$, we may assume that $C^*(E)$ is a subring of $C^*(D)$. Let $P \subset C^*(E)$ and $P' \subset C^*(D)$ be prime ideals such that $P \subset P'$, and $M \supset P$ and $M' \supset P'$ be the maximal ideals. Since $C^*(D)/M' = \mathbb{R}$; for each $f \in C^*(E)$ we have $f + M' = r$ for some $r \in \mathbb{R}$; i.e. $f - r \in M' \cap C^*(E)$. Thus $C^*(E)/C^*(E) \cap M' = \mathbb{R}$. Hence $C^*(E) \cap M'$ is

a maximal ideal in $C^*(E)$ containing P. Consequently $M \subset M'$. Q.E.D.

We now apply the discussion in the previous section to the normed algebra $Q^*_{\mathfrak{F}}(X)$ in order to show how various results in [8] are consequences of these general results concerning injective limits of normed algebras.

1. Proposition 1 implies that $Q^*_{\mathfrak{F}}(X)$ is a Banach algebra if $\mathfrak{F}$ is σ-filter base (i.e. closed under countable intersections) [8, Lemma 4.5].

2. Proposition 5 implies that every maximal ideal is real [8, Lemma 5.1] and closed.

3. Proposition 6 implies the norm is the spectral norm [8, Theorem 5.2].

4. Finally we have $\Omega(Q^*_{\mathfrak{F}}(X)) = \varprojlim_{D} \beta D$ [8, Theorem 6.8]. This follows from Proposition 7 and the fact that $\Omega(Q^*_{\mathfrak{F}}(X)) \cong \mathcal{M}(Q^*_{\mathfrak{F}}(X)) = \Delta(Q^*_{\mathfrak{F}}(X)) = \varprojlim_{D} (C^*(D))$, $\Delta(C^*(D)) \cong \mathcal{M}(C^*(D)) \cong \beta D$, where $\Omega(\cdot)$ and $\mathcal{M}(\cdot)$ denote the maximal ideal spaces with the Stone and the weak topology respectively.

<u>Definition</u>: An element z of a normed ring A is called topological zero-divisor if there exists a sequence $\{z_n\}$ in A such that $\inf_n \|z_n\| > 0$ and $\lim_{n\to\infty} \|zz_n\| = 0$.

Remark: As is well known, a zero-divisor is a topological zero-divisor and topological zero-divisors cannot be invertible.

Notation: For an element $f \in A$, we denote by $Sp_A(f)$ the set of all real numbers r such that $(f - re)^{-1}$ does not exist in A.

Lemma 12: If all B_α have the property that each non-invertible element is a topological zero divisor then the same holds for $\varinjlim B_\alpha$.

Lemma 13: Any non-invertible element in $C^*(D)$ is a topological zero divisor.

Proof: Let f be a non-invertible element in $C^*(D)$; then there exists a maximal ideal M_o in $C^*(D)$ containing f, and hence containing f^2. Thus the minimal element of $Sp_{C^*}(f^2)$ is 0. Let r_n be a sequence of negative real numbers converging to 0. Then $(f^2 - r_n e)$ is invertible. Put $z_n = (f^2 - r_n e)^{-1}/\|(f^2 - r_n e)^{-1}\|$. Clearly $\|z_n\| = 1$ and $\hat{f^2}(M_o) = 0$, hence

$$\|(f^2 - r_n e)^{-1}\| = \sup_{M \in \mathfrak{M}(C^*)} |\hat{f^2}(M) - r_n|^{-1} \geqslant |r_n|^{-1} \to \infty$$

as $n \to \infty$, where $\mathfrak{M}(C^*)$ is the maximal ideal space of C^*. Hence we have $(f - r_n e)z_n \longrightarrow fz_n \longrightarrow 0$

as $n \longrightarrow \infty$. Thus $fz_n \longrightarrow 0$ as $n \longrightarrow \infty$.
Q.E.D.

<u>Proposition 14</u>: For any $\mathcal{S}$, an element in $Q^*_{\mathcal{S}}(X)$ is a topological zero-divisor iff it is non-invertible.

Section 3: Maximal ideal spaces and projective covers.

In [3] and [13], the existence of projective covers in the category of compact Hausdorff spaces and continuous mappings has been established. The spaces were made up of filters in certain topologically defined lattices; in other words, for a compact Hausdorff space E, let R(E) be the collection of its regular open subsets, and $\Pi(E)$ be the space of maximal filters $\mathfrak{U} \subsetneq R(E)$ whose topology is generated by the sets $\Pi_V(E) = \{\mathfrak{U} \mid V \in \mathfrak{U}, \mathfrak{U} \in \Pi(E)\}$ for each $V \in R(E)$. Then $\Pi(E)$ is an extremally disconnected compact Hausdorff space. Denote by $\lim_E$ the mapping $\Pi(E) \longrightarrow E$ which assigns to each $\mathfrak{U} \in \Pi(E)$ its limit. Then $\lim_E$ is a projection, closed, continuous and essential mapping (i.e. $\lim_E$ maps no proper closed subset of $\Pi(E)$ onto E).

In the following, A^o denotes the set of all idempotents of A, and we prove:

Lemma 15: If A is a commutative regular ring with unit, then $\Omega(A) = \Omega(A^o)$.

Proof: Define a mapping $\Omega(A) \longrightarrow \Omega(A^o)$ by $M \rightsquigarrow M \cap A^o$. We show the mapping is one to one. Let $M \cap A^o = M' \cap A^o$ for any $M, M' \in \Omega(A)$. Take any $a \in M$. Suppose $a \notin M'$; then there exists $x \in A$ such that $a^2x = a$;

i.e. $a(1 - ax) \in M'$. Hence $1 - ax \in M'$. Since $1 - ax$ is an idempotent, $1 - ax \in M' \cap A^o$, and hence $1 - ax \in M \cap A^o$. This implies that $M \ni 1$, a contradiction. Hence $a \in M'$; i.e. $M = M'$. To show the mapping is onto, let $N \in \Omega(A^o)$. Put $M \equiv AN = \{ ae \mid a \in A, e \in N \}$. Then M is a proper subset, since if $1 \in M$, then $1 = ae$ for some $a \in A$ and $e \in N$, and hence $1 - e = 1\cdot(1 - e) = ae(1 - e) = 0$; i.e. $e = 1 \in N$, a contradiction. We show M is an ideal in A. Clearly $AM \subset M$. Now take any a_1e_1, a_2e_2 in M. Put $e = e_1 \boxplus e_2 \boxplus e_1e_2$ where $\boxplus$ is defined as $f \boxplus g = (f - g)^2$. Then clearly $e \in N$ and $e_1e = e_1$, $e_2e = e_2$. Thus $a_1e_1 \pm a_2e_2 = (a_1e_1 \pm a_2e_2)e \in M$. Hence M is a proper ideal in A. To show that M is a maximal ideal, take any $a \in A$ and let $a \notin M$. There exists $x \in A$ such that $a^2x = a$, hence $ax \notin M$, and hence $ax \notin N$. Since $ax \in A^o$ and N is a maximal ideal in the Boolean ring A^o, $1 - ax \in N$. Thus $1 - ax \in M$. This means that M is maximal. Since N is a maximal ideal in A^o and $M \supset N$, $M \cap A^o = N$. Thus the mapping is onto. For each $e \in A^o$, put $\Omega^o(e) = \{ N \in \Omega(A^o) \mid N \not\ni e \}$, then $\Omega^o(e)$ is a basic open set of $\Omega(A^o)$. Note that for any $a \in A$, there exists $x \in A$ such that $ax \in A^o$. By straightforward checking one shows that $\Omega(a) \mid A^o = \Omega^o(ax)$ for $a \in A$ and $\Omega(e) \mid A^o = \Omega^o(e)$ for $e \in A^o$. This shows that the mapping carries a basic open set onto a basic open set and the inverse image of a basic open set is again a basic open set. Hence the mapping $M \rightsquigarrow M \cap A^{o}$ is a homeomorphism. Q.E.D.

Note: For a subset U of a space E, let $U^{\perp} =_{df} C_E I_E U$. Then it is easy to see that a set V is regular open set of E iff $V = U^{\perp}$ for some open set U of E. Also it follows that V is regular open iff $V = V^{\perp\perp}$.

Let A be a semi-simple ring. For an ideal I in A, we write $I^* = \{ a \in A \mid aI = 0 \}$. I is called an annihilator ideal iff $I = I^{**}$. The family of all annihilator ideals of A will be denoted by $\mathfrak{J}(A)$. For an ideal I of A and a subset U of $\Omega(A)$, we define the set $\Omega(I)$ and $\Delta(U)$ as follows:

$$\Omega(I) = \{ M \in \Omega(A) \mid M \not\supset I \};$$
$$\Delta(U) = \bigcap M \ (M \in U).$$

Noting that $\Gamma_{\Omega(A)} U = \{ M \in \Omega(A) \mid M \supset \Delta U \}$ one can easily show the following identities:

(a) $U^{\perp} = (\Omega \circ \Delta)(U)$, where $(\Omega \circ \Delta)(U) =_{df} \Omega(\Delta(U))$;

(b) $I^* = (\Delta \circ \Omega)(I)$, where $(\Delta \circ \Omega)(I) =_{df} \Delta(\Omega(I))$.

The following lemma is due to [8].

Lemma 16: If A is semi-simple, then $\mathfrak{J}(A) \cong R(\Omega(A))$, and hence $Q(A)^{o} \cong R(\Omega(A))$.

Proof: Define mappings $\mathfrak{J}(A) \longrightarrow R(\Omega(A))$ by

$I \rightsquigarrow \Omega(I)$, and $R(\Omega(A)) \longrightarrow \mathcal{J}(A)$ by $V \rightsquigarrow \Delta(V)$. Since $I = I^{**}$ and by (a), (b), we have $\Omega(I) = \Omega(I^{**}) = (\Omega \circ \Delta \circ \Omega)(I^*) = (\Omega \circ \Delta)(\Omega(I^*)) = \Omega(I^*)^{\perp}$, and for $V = U^{\perp}$, $\Delta(V) = \Delta(U^{\perp}) = (\Delta \circ \Omega \circ \Delta)(U) = \Delta(U)^*$; hence the mappings $I \rightsquigarrow \Omega(I)$, $V \rightsquigarrow \Delta(V)$ are well defined. Since $\Omega(I \cap J) = \Omega(I) \cap \Omega(J) = \Omega(I^{**}) \cap \Omega(J^{**})$ and $\Omega(I^{**}) = \Omega(I^*)^{\perp}$, the mapping $I \rightsquigarrow \Omega(I)$ is a homomorphism. By a straightforward checking one shows that $\Omega \circ (\Delta \circ \Omega \circ \Delta) = i_d$ on $R(\Omega(A))$ and $(\Delta \circ \Omega \circ \Delta) \circ \Omega = i_d$ on $\mathcal{J}(A)$. For the last assertion, it is well known in [20 , pp 44] that $\mathcal{J}(A) \cong \mathcal{J}(Q(A) \cong Q(A)^{\circ}$. Q.E.D.

<u>Proposition 17</u>: If A is a commutative semi-simple ring with unit and $\Omega(A)$ is Hausdorff, then $\Omega(Q(A))$ is the projective cover of $\Omega(A)$.

<u>Proof</u>: Since $Q(A)$ is regular, we have $\Omega(Q(A)) \cong \Omega(Q(A)^{\circ})$, and since A is semi-simple, $Q(A)^{\circ} \cong R(\Omega(A))$. Let $\sqcap \equiv \sqcap(\Omega(A))$; i.e. the space of maximal filters $\mathfrak{M} \leq R(\Omega(A))$. Then clearly $\Omega(R(\Omega(A)) \cong \sqcap$ under the mapping $M \rightsquigarrow M' = \{ P \mid P' \in M,\ P' \text{ is the complement of } P\}$. Since $\Omega(A)$ is compact Hausdorff, $\lim_{\Omega(A)} : \sqcap \longrightarrow \Omega(A)$ is the projective cover. Q.E.D.

<u>Corollary 1</u>: Let $\mathcal{B}$ be such that $Q_{\mathcal{B}}(X)$ is the maximal ring of quotients of $C(X)$. Then the maximal ideal

space of $Q_{\mathcal{D}}(X)$ is the projective cover of βX.

Corollary 2: Under the same condition as Corollary 1, $\varprojlim_{D \in \mathcal{D}} (\beta D)$ is the projective cover of βX.

In the preceding paragraph we have obtained algebraically the result that the maximal ideal space of $Q_{\mathcal{D}}(X)$ is the projective cover of βX if $Q_{\mathcal{D}}(X)$ is the maximal ring of quotients of $C(X)$. Now we are interested in finding an analogous result in a more topological manner.

To proceed with this, let E be a compact Hausdorff space, $O(E)$ be its topology and $\Lambda(E)$ be the space of maximal filters $\mathfrak{U} \subseteq O(E)$ whose topology is generated by the sets $\Lambda_W(E) = \{\mathfrak{U} \mid W \in \mathfrak{U}, \mathfrak{U} \in \Lambda(E)\}$ for each $W \in O(E)$. Then $\Lambda(E)$ is an extremally disconnected compact Hausdorff space, and the mapping $\lim_E : \Lambda(E) \longrightarrow E$ is compact, closed, essential and continuous projection [3].

Now let X and Y be topological spaces such that X is dense in Y. If $\mathfrak{U} \subset O(Y)$ is a maximal filter in $O(Y)$, then $\mathfrak{U}|X \subset O(X)$ is a maximal filter in $O(X)$ as can be seen as follows: Clearly $\mathfrak{U}|X$ is a proper filter on X since X is dense in Y. Let $\mathfrak{M} \supset \mathfrak{U}|X$ be a filter, $\mathfrak{M} \subseteq O(X)$. Take any member U of $\mathfrak{M}$, then $U \cap V \neq \emptyset$ for all $V \in \mathfrak{U}|X$.

Let $U = U^1 \cap X$ and $V = V' \cap X$ where $U^1 \in O(Y)$ and $V' \in \mathcal{U}$. Then $U^1 \cap V' \neq \emptyset$ for all $V' \in \mathcal{U}$. Since $\mathcal{U}$ is maximal, $U^1 \in \mathcal{U}$. Hence $U \in \mathcal{U}|X$; i.e. $\mathcal{M} = \mathcal{U}|X$. Moreover if $\mathcal{U} \neq \mathcal{U}'$, $\mathcal{U}$, $\mathcal{U}' \leq O(Y)$ then $\mathcal{U}|X \neq \mathcal{U}'|X$; because if $\mathcal{U} \neq \mathcal{U}'$, then there exists $U^o \in \mathcal{U}$ and $V^o \in \mathcal{U}'$ such that $U^o \cap V^o = \emptyset$; thus $(U^o \cap X) \cap (V^o \cap X) = \emptyset$. This means that $\mathcal{U}|X \neq \mathcal{U}'|X$. Now we show that for any maximal filter $\mathcal{M} \subsetneq O(X)$ there exists a maximal filter $\mathcal{U} \subsetneq O(Y)$ such that $\mathcal{U}|X = \mathcal{M}$. To end this, we define $\mathcal{U}_{\mathcal{M}}$ by

$$\mathcal{U}_{\mathcal{M}} = \{ U' \in O(Y) \mid U' \cap X \in \mathcal{M} \}.$$

We show that $\mathcal{U}_{\mathcal{M}}$ is a maximal filter on Y: Clearly $\emptyset \notin \mathcal{U}_{\mathcal{M}}$ and $U' \cap V' \in \mathcal{U}_{\mathcal{M}}$ whenever $U', V' \in \mathcal{U}_{\mathcal{M}}$. Let $U' \in \mathcal{U}_{\mathcal{M}}$ and $U' \subsetneq W'$, $W' \in O(Y)$. Then $W' = U' \cup W'$, and hence $W' \cap X = (U' \cap X) \cup (W' \cap X)$; thus $U' \cap X \subsetneq W' \cap X$. But $W' \cap X \subsetneq O(X)$, and hence $W' \cap X \in \mathcal{M}$. Therefore $W' \in \mathcal{U}_{\mathcal{M}}$. To show the maximality of $\mathcal{U}_{\mathcal{M}}$, let $\mathcal{F} \supseteq \mathcal{U}_{\mathcal{M}}$ be a filter in O(Y). Take any member $U' \in \mathcal{F}$; then $U' \cap V' \neq \emptyset$ for all $V' \in \mathcal{U}_{\mathcal{M}}$. Hence $(U' \cap X) \cap (V' \cap X) \neq \emptyset$ for all $V' \in \mathcal{U}_{\mathcal{M}}$; in particular, $(U' \cap X) \cap V \neq \emptyset$ for all $V \in \mathcal{M}$; the maximality of $\mathcal{M}$ implies that $U' \cap X \in \mathcal{M}$; i.e. $U' \in \mathcal{U}_{\mathcal{M}}$. Consequently, $\mathcal{F} = \mathcal{U}_{\mathcal{M}}$ and $\mathcal{U}_{\mathcal{M}}|X = \mathcal{M}$. Finally we show that, for any $W \in O(Y)$, $\Lambda_W(Y)|X = \Lambda_{W \cap X}(X)$, where $\Lambda_W(Y)|X =_{df} \{ \mathcal{U}|X \mid \mathcal{U} \in \Lambda_W(Y) \}$. Clearly $\Lambda_W(\cancel{Y})|X \subsetneq \Lambda_{W \cap X}(X)$. Now take any $\mathcal{U} \in \Lambda_{W \cap X}(X)$, then $\mathcal{U}_{\mathcal{M}}$ is clearly contained in $\Lambda_W(Y)$, and $\mathcal{U}_{\mathcal{M}}|X = \mathcal{M}$.

Thus $\Lambda_W(Y)|X = \Lambda_{W \cap X}(X)$.

From this consideration we have the following consequence:

Lemma 18: Let X and Y be the topological spaces such that X is a dense subspace of Y; then $\Lambda(Y) \cong \Lambda(X)$, given by the mapping $\mathcal{U} \rightsquigarrow \mathcal{U}|X$, $\mathcal{U} \in \Lambda(Y)$.

Lemma 19: For any $\mathcal{U} \in \Lambda(X)$, if $U \in \mathcal{U}$, then $\mathcal{U} \in \Gamma_{\Lambda(X)} \lim_{\beta X}^{-1}(U)$.

Proof: Take an arbitrary member $W \in \mathcal{U}$, then $U \cap W \neq \emptyset$, and $\Lambda_W(X)$ is an open neighborhood of $\mathcal{U}$. Let $a \in U \cap W$. Take a member $\mathcal{N}$ in $\Lambda(X)$ which converges to the point a. Then every member of $\mathcal{N}$ intersects with W. Since $\mathcal{N}$ is a maximal filter, $\mathcal{N}$ contains W; i.e. $\mathcal{N}$ is a member of $\Lambda_W(X)$. On the other hand $\mathcal{N}$ converges to the point a of U, hence $\mathcal{N} \in \lim_{\beta X}^{-1}(U)$. Thus we have $\Lambda_W(X) \cap \lim_{\beta X}^{-1}(U) \neq \emptyset$. Q.E.D.

Lemma 20: Let $\varphi : K \longrightarrow \beta X$ be a projective cover in the category of compact Hausdorff spaces and continuous mappings. Then for each dense subset D of X, $\varphi^{-1}(D)$ is dense in K.

Proof: Note that D is also dense in βX. Since K is a compact space, $\Gamma_K \varphi^{-1}(D)$ is also a compact subset of K. Since φ is onto, $D = \varphi(\varphi^{-1}(D)) \subset \varphi(\Gamma_K \varphi^{-1}(D))$.

Hence $\varphi(\Gamma_K \varphi^{-1}(D))$ is dense in βX. But $\varphi(\Gamma_K \varphi^{-1}(D))$ is compact, hence is closed in βX; i.e. $\varphi(\Gamma_K \varphi^{-1}(D)) = \beta X$. Since K is projective cover, $\Gamma_K \varphi^{-1}(D)$ can not be a proper closed subset of K; i.e. $\Gamma_K \varphi^{-1}(D) = K$. Q.E.D.

Corollary: $K = \beta\varphi^{-1}(D)$ for each dense subset D of X.

Proof: It is well known [12, pp 96] that a compact space K is extremally disconnected if and only if $K = \beta S$ for every dense subspace S. Q.E.D.

Now take a member D of $\mathfrak{D}$, then a function $f \in C^*(D)$ defines a continuous function $f \circ \varphi$ on $\varphi^{-1}(D)$. Since $K = \beta\varphi^{-1}(D)$ the function $f \circ \varphi$ has a unique continuous extension $\tilde{f}$ to K. Let u_f be the element in $Q^*_{\mathfrak{D}}(X)$ with $u_f = \nu^*(f)$ and $f \in C^*(D)$ for some $D \in \mathfrak{D}$, where $\nu^* : C^*_{\mathfrak{D}}(X) \longrightarrow Q^*_{\mathfrak{D}}(X)$ is the natural mapping. Define a mapping $Q^*_{\mathfrak{D}}(X) \longrightarrow C(K)$ by $u_f \rightsquigarrow \tilde{f}$. Clearly this mapping is well defined and the mapping is a norm preserving monomorphism.

We have the following proposition for the projective cover K of βX.

Proposition 21: If $\mathfrak{D}$ contains all disconnected dense open subsets of X, then the maximal ideal space of $Q^*_{\mathfrak{D}}(X)$ is homeomorphic to K.

<u>Proof</u>: Since the mapping $u_f \rightsquigarrow \tilde{f}$ is a norm preserving monomorphism of $Q^*_{\mathfrak{D}}(X)$ into $C(K)$, it is enough to show that the family of all $\tilde{f}$ separates the point of K. Take any a, b in K with $a \neq b$.

Since $\Lambda(\beta X) \cong \Lambda(X)$ ($\cong K$), we may assume that a, b are members of $\Lambda(X)$, and hence $\varphi = \lim_{\beta X}$. Since $a \neq b$, there exists open sets U and V in βX such that $U \cap V = \emptyset$ and $U \cap X \in a$, $V \cap X \in b$. Then, by Lemma 19, $a \in \Gamma_K \varphi^{-1}(U \cap X)$ and $b \in \Gamma_K \varphi^{-1}(V \cap X)$. Let

$$D = (U \cap X) \cup (X \cap \mathrm{I}_{\beta X} \mathrm{C}_{\beta X} U);$$

then clearly $D \in \mathfrak{D}$; and define a function f on D by

$$f(x) = \begin{cases} 0 & \text{if } x \in U \cap X \\ 1 & \text{if } x \in X \cap \mathrm{I}_{\beta X} \mathrm{C}_{\beta X} U. \end{cases}$$

Then $f \in C^*(D)$. Thus $f \circ \varphi$ has an extension $\tilde{f}$ on K, and

$$\tilde{f}(a) = \lim_{\substack{z \to a \\ z \in \varphi^{-1}(U \cap X)}} (f \circ \varphi)(z) = 0,$$

$$\tilde{f}(b) = \lim_{\substack{z \to b \\ z \in \varphi^{-1}(V \cap X)}} (f \circ \varphi)(z) = 1.$$

Thus the family of $\tilde{f}$ separates the points of K. By the Stone-Weierstrass theorem the proposition holds. Q.E.D.

CHAPTER IV

Change of Range and Restricted Rings of Functions.

Section 1. Complex-valued functions.

We now replace the range space $\mathbb{R}$ by the complex number field $\mathbb{C}$. In this section $\mathbb{C}(X)$ will denote the ring of all complex-valued continuous functions defined on a space X, and C(X,R) will denote the ring of all real-valued continuous functions on the space X, We shall attempt to obtain the analogous results that we obtained in Chapter I & II.

Proposition 1: Let U be a dense subset of X. Then C(U) is a ring of quotients of $\mathbb{C}(X)$ iff C(U,R) is a ring of quotients of C(X,R).

Proof: Let $\mathbb{C}(U)$ be a ring of quotients of C(X). Take f, $g \neq 0$ in C(U,R), then there exists $h \in \mathbb{C}(X)|U$ such that $fh \in \mathbb{C}(X)|U$ and $gh \neq 0$. Let $h = u + iv$; then $fh = fu + ifv \in C(X)|U$, and hence fu, $fv \in C(X,R)|U$. Since $gh = gu + igv \neq 0$, hence $gu \neq 0$ or $gv \neq 0$. This shows that C(U,R) is a ring of quotients of C(X,R). Conversely, let $f, g \in \mathbb{C}(U)$ with $g \neq 0$ and $f = u + iv$, $g = s + it$. Without loss of generality we may assume $s \neq 0$. Then there exists $h \in C(X,R)|U$ such that $uh \in C(X,R)|U$ and $sh \neq 0$; and also there exists $h' \in C(X,R)|U$ with $vh' \in C(X,R)|U$ and $shh' \neq 0$. Then $hh' \in \mathbb{C}(X)|U$, $fhh' \in \mathbb{C}(X)|U$ and $ghh' \neq 0$. This completes the proof.

Let $\mathfrak{D}$ be a filter base of dense subsets of X. In the same way as described in the Chapter I, one obtains the ring $Q_{\mathfrak{D}}(X)$ associated with the ring $\mathbb{C}(X)$. The corresponding ring for the ring $C(X,R)$ will be denoted by $Q_{\mathfrak{D}}(X,R)$. It is easy to check that the ring $Q_{\mathfrak{D}}(X)$ is a ring of quotients of $\mathbb{C}(X)$ if and only if, for each $D \in \mathfrak{D}$, $\mathbb{C}(D)$ is a ring of quotients of $\mathbb{C}(X)$. Hence we have the following:

Corollary: $Q_{\mathfrak{D}}(X)$ is a ring of quotients of $\mathbb{C}(X)$ iff $Q_{\mathfrak{D}}(X,R)$ is a ring of quotients of $C(X,R)$.

Remark: In a completely analogous way to the Theorem 6, Chapter I, one shows that if $\mathfrak{D}$ is the set of all dense open subsets of X, then $Q_{\mathfrak{D}}(X)$ is the maximal ring of quotients of $\mathbb{C}(X)$. The necessary and sufficient condition for $Q_{\mathfrak{D}}(X)$ to be a von Neumann regular ring is the following: For each $D \in \mathfrak{D}$ and $f \in \mathbb{C}(D)$, the subset $(E \cap Coz(f)) \cup I_E C_E(E \cap Coz(f))$ belongs to $\mathfrak{D}$ for some $E \subseteq D$ in $\mathfrak{D}$ (cf. Proposition 7, Chapter I).

Let A be a commutative real algebra with unit, and let A_c denote the cartesian product $A \times A$ in which algebraic operations are so defined that (a,b), $a, b \in A$, behaves like $a + ib$. More precisely, define $(a,b) + (c,d) = (a + c, b + d)$, $(a,b)(c,d) = (ac - bd, ad + bc)$ and $(\alpha + i\beta)(a,b) = (\alpha a - \beta b, \alpha b + \beta a)$, $\alpha, \beta \in \mathbb{R}$. Then it is easy to verify that A_c is a complex algebra with unit $(1,0)$. <u>A_c is called the complexification of A</u>[23]. Also it is not hard to check that if I is an ideal in A then its complexification I_c is an ideal in A_c.

<u>Lemma 2</u>: Let $\mathcal{D}$ be a filter base of dense subsets of X. Then $Q_{\mathcal{D}}(X) \cong Q_{\mathcal{D}}(X,R)_c$ as $\mathbb{C}$-algebras.

<u>Proof</u>: Let $f \in Q_{\mathcal{D}}(X)$. We may assume that $f \in C(D)$ for some $D \in \mathcal{D}$. Then $f = Re(f) + iIm(f)$ and $Re(f)$, $Im(f) \in C(D,R)$; and hence $Re(f)$, $Im(f) \in Q_{\mathcal{D}}(X,R)$. Define a mapping: $Q_{\mathcal{D}}(X) \longrightarrow Q_{\mathcal{D}}(x,R)_c$ by $f \rightsquigarrow (Re(f),Im(f))$. Since $(Re(f + g), Im(f + g)) = (Re(f), Im(f)) + (Re(g), Im(g))$, $(R(fg), Im(fg)) = (Re(f)\cdot Re(g) - Im(f)\cdot Im(g), Re(f)\cdot Im(g) + Im(f)\cdot Re(g)) = (Re(f), Im(f))(Re(g), Im(g))$ and $(\alpha + i\beta)\cdot(Re(f) + iIm(f)) \rightsquigarrow (\alpha,\beta)(Re(f), Im(f)) = (\alpha + i\beta)(Re(f), Im(f))$, the mapping is clearly a $\mathbb{C}$-algebra homomorphism. Clearly the mapping is one to one. To show the ontoness, let $(u,v) \in Q_{\mathcal{D}}(X,R)_c$; then there exist f, g in $C(D,R)$ for some $D \in \mathcal{D}$ such that $u \equiv f$ and $v \equiv g$ in $Q_{\mathcal{D}}(X,R)$. Then

$f + ig \in Q_{\mathcal{S}}(X)$ and clearly (u,v) is the image of $f + ig$ under the mapping. Q.E.D.

<u>Proposition 3</u>: The maximal ring of quotients of $C(X)$ is isomorphic to the complexification of the maximal ring of quotients of $C(X,R)$.

<u>Proof</u>: Let $\mathcal{S}$ be the set of all dense open subsets of X. Then, by Lemma 2, $Q(\mathbb{C}(X)) \cong Q_{\mathcal{S}}(X) \cong Q_{\mathcal{S}}(X,R)_c \cong Q(C(X,R))_c$.

<u>Lemma 4</u>: For a commutative real algebra A with unit, and an ideal I in A, $A_c/I_c \cong (A/I)_c$ as $\mathbb{C}$-algebras under the mapping $(a,b) + I_c \rightsquigarrow (a + I,\ b + I)$.

<u>Observation</u>: For $\alpha, \beta \in \mathbb{R}$, $(\alpha + i\beta)((a,b) + I_c) = (\alpha, \beta)(a,b) + I_c$ gives a $\mathbb{C}$-module structure on A_c/I_c; and $\alpha(a + I) = \alpha a + I$ gives a $\mathbb{R}$-module structure on A/I and hence $\alpha + I = \alpha \cdot 1$.

<u>Proof</u>: Since $(a,b) + I_c + (c,d) + I_c = (a + c, b + d) + I_c \rightsquigarrow (a + c + I,\ b + d + I) = (a + I,\ b + I) + (c + I, d + I)$, $((a,b) + I_c)((c,d) + I_c) = (a,b)(c,d) + I_c = (ac - bd,\ ad + bc) + I_c \rightsquigarrow ((a + I)(c + I) - (b + I)(d + I),\ (a + I)(d + I) + (b + I)(c + I)) = (a + I, b + I)(c + I, d + I)$ and $(\alpha + i\beta)\cdot((a,b) + I_c) = (\alpha,\beta)(a,b) + I_c \rightsquigarrow (\alpha + I,\ \beta + I)\cdot(a + I, b + I) = (\alpha,\beta)(a + I,\ b + I) = (\alpha + i\beta)(a + I, b + I)$, hence the mapping is a $\mathbb{C}$-algebra homomorphism. Clearly the

mapping is onto. If a, b $\in$ I, then $(a,b) \in I_c$; hence the mapping is one to one. Q.E.D.

Definition: An ideal I in a complex algebra A is called complex if $A/I \cong \mathbb{C}$ (the field of complex numbers). A complex algebra is said to be totally uncomplex if it does not have any complex ideal. A complex ideal is automatically a maximal ideal.

Lemma 5: Let M' be a maximal ideal in $Q_{\mathcal{B}}(X,R)_c$. Then there exists a maximal ideal M in $Q_{\mathcal{B}}(X,R)$ such that $M' = M_c$, and moreover if M' is a complex ideal, then M is a real ideal.

Proof: Define $M_1 = \{ f \mid (f,h) \in M', f,h \in Q_{\mathcal{B}}(X,R)\}$ and $M_2 = \{ g \mid (h', g) \in M', h', g \in Q_{\mathcal{B}}(X,R)$. We show $M_1 = M_2$: Let $f \in M_1$; then $(f,h) \in M'$ for some h. Since M' is an ideal, $(0,1)(f,h) = (-h,f) \in M'$; hence $f \in M_2$; i.e. $M_1 \leq M_2$. Conversely, let $g \in M_2$; then $(h', g) \in M'$ for some h'. Similarly, $(0, -1)(h', g) = (g, -h') \in M'$; i.e. $g \in M_1$. Hence $M_1 = M_2$. Let $M \equiv M_1 = M_2$. Clearly, $1 \notin M$, since $(1,h) \notin M'$: Suppose $(1,h) \in M'$ for some h. Then $(1/1 + h^2, -h/1 + h^2) \cdot (1,h) = (1,0) \in M'$; a contradiction. Now we show M is an ideal in $Q_{\mathcal{B}}(X,R)$. Let $f,g \in M$. Then $(f,h), (g,h') \in M'$ for some h and h'. Clearly $f + g \in M$. Since M' is an ideal in $Q_{\mathcal{B}}(X,R)_c$, $(f,o)(g,h') = (fg,fh') \in M'$, hence $fg \in M$. Similarly, let $u \in Q_{\mathcal{B}}(X,R)$; then $(u,o) \in$

$Q_{\mathcal{S}}(X,R)_c$, and $(u,o)(f,h) = (uf, uh) \in M'$; thus $uf \in M$. Next, to show $M' = M_c$, it is enough to show that $M' = M \times M$. Clearly $M' \subseteq M_1 \times M_2 = M \times M$. Let $(f,g) \in M \times M$. Suppose $(f,g) \notin M'$; i.e. $(f,g) + M' \neq 0$ in $Q_{\mathcal{S}}(X,R)_c/M'$. Then there exists $(u,v) \in Q_{\mathcal{S}}(X,R)_c$ such that $((f,g) + M')\cdot((u,v) + M') = \mathbb{1} = (1,0) + M'$; i.e. $(1,0) - (f,g)(u,v) = (1 - (fu - gv), - (fv + gu)) \in M'$; hence $1 - (fu - gv) \in M$. This implies that $1 \in M$, a contradiction. Thus $M' = M_c$. Now we show that M is a maximal ideal. By Lemma 4, $(Q_{\mathcal{S}}(X,R)/M)_c$ is a field, denote by K. For any $f \in Q_{\mathcal{S}}(X,R)$, if $f \in M$, then $(f + M, o) \neq 0$ in K. Hence there exist g, h in $Q_{\mathcal{S}}(X,R)$ such that $(f + M, o)(g + M, h + M) = 1_K = (1 + M,o)$. i.e. $(1 - fg + M, fh + M) = 0$ in K. Hence $1 - fg \in M$. Thus M is a maximal ideal. Finally let M' be a complex ideal. By Lemma 4, $(Q_{\mathcal{S}}(X,R)/M)_c \cong \mathbb{C} = (\mathbb{R})_c$. Thus for any $f \in Q_{\mathcal{S}}(X,R)$ there exists $r \in \mathbb{R}$ such that $(f + M, o) = (r,o)$ and vice versa; i.e. $f + M = r$. Hence $Q_{\mathcal{S}}(X,R)/M = \mathbb{R}$; i.e. M is a real ideal.

<u>Remark</u>: The same holds for algebras A, A_c if all $1 + x^2$, $x \in A$ are invertible.

<u>Proposition 6</u>: If $Q_{\mathcal{S}}(X)$ is totally uncomplex, then $\cap\, \mathcal{S} = \emptyset$, and the converse holds, provided each member of $\mathcal{S}$ is real-compact.

<u>Proof</u>: Suppose $\cap \mathcal{S} \neq \emptyset$. Then there exists a real maximal ideal M in $Q_{\mathcal{S}}(X,R)$ such that $Q_{\mathcal{S}}(X,R)/M$ is the real field (Proposition 5, Chapter II). Hence M_c is a complex maximal ideal in $Q_{\mathcal{S}}(X,R)_c$. Therefore there exists a complex ideal in $Q_{\mathcal{S}}(X)$; a contradiction. For the converse, we note that $Q_{\mathcal{S}}(X,R)$ is totally unreal iff $\cap \mathcal{S} = \emptyset$; and by Lemma 6, the assertion holds. Q.E.D.

<u>Lemma 7</u>: Let K be a formally real field. Then its complexification K_c is a field.

<u>Proof</u>: Clearly K_c is a ring with unit (1, 0). Let $(a,b) \neq 0$ in K_c; then $a \neq 0$ or $b \neq 0$ in K. Since K is formally real $a^2 + b^2 \neq 0$. Clearly $(a/a^2 + b^2, -b/a^2 + b^2) \in K_c$, and hence $(a/a^2 + b^2, -b/a^2 + b^2)(a,b) = (1,0)$; i.e. each nonzero element of K_c is invertible.

<u>Corollary</u>: If M is a maximal ideal in $Q_{\mathcal{S}}(X,R)$, then M_c is a maximal ideal in $Q_{\mathcal{S}}(X,R)_c$.

<u>Proof</u>: $Q_{\mathcal{S}}(X,R)/M$ is a real-closed field, and hence a formally real field[25], thus $(Q_{\mathcal{S}}(X,R)/M)_c = Q_{\mathcal{S}}(X,R)_c/M_c$ is a field. Hence M_c is a maximal ideal in $Q_{\mathcal{S}}(X,R)_c$.

<u>Proposition 8</u>: For any filter base $\mathcal{S}$ of dense subsets of X, $\Omega(Q_{\mathcal{S}}(X,R)) \cong \Omega(Q_{\mathcal{S}}(X))$.

Proof: It suffices to show that $\Omega \equiv \Omega(Q_{\mathcal{S}}(X,R)) \cong \Omega_c \equiv \Omega(Q_{\mathcal{S}}(X,R)_c)$. By the previous Corollary one has a mapping $\theta : \Omega(Q_{\mathcal{S}}(X,R)) \longrightarrow \Omega(Q_{\mathcal{S}}(X,R)_c)$ by $\theta(M) = M_c$. For M, $N \in \Omega$, let $\theta(M) = \theta(N)$; i.e. $M \times M = N \times N$; then $M = N$, hence θ is one to one. By Lemma 5, θ is onto. To show θ is homeomorphism it is sufficient to prove that $\theta(\Omega(fg)) = \Omega_c((f,g))$ for any f, $g \in Q_{\mathcal{S}}(X,R)$. This follows from the fact that $fg \notin M$ iff $f \notin M$ and $g \notin M$ iff $(f,g) \notin M \times M = \theta(M)$. Q.E.D.

From the remark following Lemma 5 and the above proof, one obtains the more general.

Corollary: For any R-algebra A such that

(i) A modulo any maximal ideal is formally real, and

(ii) all $1 + a^2$, $a \in A$, are invertible, one has $\Omega(A) \cong \Omega(A_c)$.

Section 2: C^∞-functions.

The ring $C^\infty(U)$ of all real-valued unrestrictedly differentiable functions on an open subset U of $\mathbb{R}^n$ has the following properties: (a) for any $f \in C^\infty(U)$, if $f(x) \neq 0$ for all $x \in U$ then the inverse f^{-1} of f exists in $C^\infty(U)$, and if $f(x) > o$ for all $x \in U$ then there exists $g \in C^\infty(U)$ such that $f = g^2$; (b) for any $a \in U$ and real numbers r_1 and r_2 such that $0 < r_1 < r_2$, there exists $f \in C^\infty(U)$ such that

$$f(x) = \begin{cases} = 1 \\ \geq 0 \\ = 0 \end{cases} \text{for} \begin{cases} \|x - a\| \leqslant r_1 \\ \text{all } x \\ \|x - a\| \geqslant r_2 \end{cases}$$

(c) all algebra homomorphism $p: C^\infty(U) \longrightarrow \mathbb{R}$ are evaluation maps [2].

From these facts one has that the topology of U (always understood as a subspace of $\mathbb{R}^n$) is generated by the cozero sets of the functions of $C^\infty(U)$; namely, let V be an open subset of U and $a \in V$; then there exists r_1 and r_2 with $0 < r_1 < r_2$, and $\{ x \mid \|a - x\| < r_1\} \subset \{ x \mid \|a - x\| < r_2\} \subset V$, and a function $f_a \in C^\infty(U)$ such that $\{ x \mid \|a - x\| < r_1\} \subset \mathrm{Coz}(f_a) \subset V$. Thus $V = \bigcup \mathrm{Coz}(f_a) (a \in V)$. Hence for any algebraically dense ideal A in $C^\infty(U)$, the set $E = \bigcup \mathrm{Coz}(f) (f \in A)$ is topologically dense in U. Next, we check that for any dense open subset E of U, the ring $C^\infty(E)$ is a proper rational extension of $C^\infty(U)$ with respect to the mapping $f \rightsquigarrow f|E$. Clearly the ring $C^\infty(E)$ is a ring of quotients of $C^\infty(U)$, and the extension is proper as can be seen as follows: The function g, defined by $g(x) = \sum_{i=1}^{n} |x_1 - a_1|$, where $x = (x_1, \ldots, x_n)$, $a = (a_1, \ldots, a_n)$, $a \in U \setminus E$, is unrestrictedly differentiable on E but not on U. Since the ring $C^\infty(U)$ is locally inversion closed, one obtains that for the set $\mathcal{D}$ of all dense open subsets of U the ring of fractions $Q^\infty_{\mathcal{D}}(U) \equiv C^\infty_{\mathcal{D}}(U)/Z_{\mathcal{D}}(U)$ of $C^\infty(U)$ is the maximal ring of quotients of $C^\infty(U)$ (cf. [1] and Chapter I) where $C^\infty_{\mathcal{D}}(U)$ is the ring of all real-valued functions on U which are unrestrictedly differentiable on a member of $\mathcal{D}$ which evidently contains $Z_{\mathcal{D}}(U)$ as defined earlier (Chapter I) as an ideal. Also, for any filter base $\mathcal{D}$ of dense open subsets of U, the ring $Q^\infty_{\mathcal{D}}(U)$ is the limit of the directed system $(C^\infty(D))(D \in \mathcal{D})$ with mapping $\varphi_D : C^\infty(D) \longrightarrow Q^\infty_{\mathcal{D}}(U)$ as limit homomorphisms.

<u>Lemma 9</u>: For any filter base $\mathcal{D}$ of dense open subsets of U, there is a one-to-one correspondence from $\Delta \equiv \Delta(Q^\infty_{\mathcal{D}}(U))$ onto $\cap\mathcal{D}$.

<u>Proof</u>: Take $\varphi \in \Delta$, then $p_D \equiv \varphi \circ \varphi_D : C^\infty(D) \longrightarrow \mathbb{R}$ is an evaluation mapping; hence there exists $x_D \in D$ such that $p_D(f|D) = f(x_D)$ for all $f \in C^\infty(U)$. Now take any E then also there exists $x_E \in E$ such that $p_E(f|E) = f(x_E)$ for all $f \in C^\infty(U)$. But $p_D(f|D) = p_E(f|E)$, hence $f(x_D) =$

$f(x_E)$ for all $f \in C^{\infty}(U)$. Thus $x_D = x_E \equiv a$, i.e. $a = x_D$ for all $D \in \mathfrak{D}$. Hence $a \in \cap\mathfrak{D}$ and $\varphi(f) = g(a)$ for $f = \varphi_D(g)$, $g \in C^{\infty}(D)$ for some $D \in \mathfrak{D}$. Conversely, this defines a $\varphi \in \Delta$ for any $a \in \cap\mathfrak{D}$. Hence the corresponding $\Delta \longrightarrow \cap\mathfrak{D}$ by $\varphi \rightsquigarrow a$ is onto, and it is one-to-one since $C^{\infty}(U)$ separates the points of U.

<u>Proposition 10</u>: The maximal ring of quotients of $C^{\infty}(U)$ is totally unreal.

<u>Proof</u>: Let $\mathfrak{D}$ be the set of all dense open subsets of U; then $\cap\mathfrak{D} = \emptyset$, and hence $\Delta(Q^{\infty}_{\mathfrak{D}}(U))$ is void. Q.E.D.

<u>Definition</u>: A ring A is called real semi-simple iff the intersection of all real maximal ideal of A is zero.

<u>Proposition 11</u>: For any filter base $\mathfrak{D}$ of dense open subsets of U, the ring $Q^{\infty}_{\mathfrak{D}}(U)$ is real semi-simple iff $\cap\mathfrak{D}$ is dense in U.

<u>Proof</u>: Let $\cap\mathfrak{D}$ be dense in U. Suppose there were $0 \neq f \in \cap \operatorname{Ker}\varphi(\ \varphi \in \Delta)$. Then $f = \varphi_D(g)$, $g \in C^{\infty}(D)$ for some $D \in \mathfrak{D}$. By Lemma, if $g(x) \neq 0$ for some $x \in D$, then $x \notin \cap\mathfrak{D}$; since if $x \in \cap\mathfrak{D}$, then there exists $\varphi \in \Delta$ such that $\varphi(f) = g'(x) = 0$, $f = \varphi_{D'}(g')$ $g' \in C^{\infty}(D')$ for some $D' \in \mathfrak{D}$, and $g|E = g'|E$ for some $E \subseteq D \cap D'$, $E \in \mathfrak{D}$.

Hence $g(x) = 0$, a contradiction. Thus $g \mid \cap \mathcal{D} = 0$, and hence $g = 0$ on D. Hence $f = \varphi_D(g) = 0$, a contradiction; i.e. $f = 0$. Conversely, suppose $\cap \mathcal{D}$ is not dense in U, then there is an open subset $W \subseteq U$ such that $W \cap (\cap \mathcal{D}) = \emptyset$. Let $c \in W$; find $f \in C^\infty(U)$ such that $f(c) \neq 0$ and $f(x) = 0$ for $x \notin W$. Note that $g \equiv \varphi_D(f|D) = \varphi_E(f|E)$ for all $D, E \in \mathcal{D}$. Then, by Lemma, for each $\varphi \in \Delta$, there corresponds an element $a \in \cap \mathcal{D}$ such that $\varphi(g) = (f|D)(a) = f(a) = 0$, i.e. $g \in \cap \operatorname{Ker} \varphi$. But $g \neq 0$ because $f|D \neq 0$ for all $D \in \mathcal{D}$. i.e. $\cap \operatorname{Ker} \varphi \neq 0$, a contradiction. Thus $\cap \mathcal{D}$ is dense in U. Q.E.D.

Remark: Many conditions which are known to hold for any ring $C^\infty(U)$ carry over, in virtue of their form, to the ring $Q^\infty_{\mathcal{D}}(U)$ for any filter base $\mathcal{D}$ of dense open subsets of U, since $Q^\infty_{\mathcal{D}}(U)$ is the injective limit of rings of the type $C^\infty(D)$. For instance, the following which are related to the work in [2].

(1) For any $f \in Q^\infty_{\mathcal{D}}(U)$ and $\varphi \in \Delta(Q\ (U))$ with $\varphi(f) = 0$ there exists a $g \in Q^\infty_{\mathcal{D}}(U)$ such that $\varphi(g) = 0$ and $(1 - g)^2\ (1 - f) + \alpha \cdot 1$ is invertible in $Q^\infty_{\mathcal{D}}(U)$ for each non-zero $\alpha \in \mathbb{R}$.

(2) For any $f \in Q^\infty_{\mathcal{D}}(U)$, $1 + f^2$ is invertible.

(3) For the elements $u_1, \ldots, u_n \in Q^\infty_{\mathcal{D}}(U)$ corresponding to the n Cartesian coordinate functions on U and

any $\varphi \in \Delta(Q^{\infty}_{\mathcal{S}}(U))$, Ker($\varphi$) is the ideal generated by $u_1 - \varphi(u_1)\cdot 1, \ldots, u_n - \varphi(u_n)\cdot 1$.

(4) For any $f \in Q^{\infty}_{\mathcal{S}}(U)$ and $\varphi \in \Delta(Q^{\infty}_{\mathcal{S}}(U))$ such that $\varphi(f) \neq 0$ then there exist $\alpha \in \mathbb{R}$ and $g \in Q^{\infty}_{\mathcal{S}}(U)$ for which $\sum (u_i - \varphi(u_i)1)^2 + f^2 = \alpha^2 1 + g^2$, the $u_1, \ldots, u_n$ are as in (3).

PROBLEMS

For further investigation the following problems could be considered.

1. Study real semi-simple rings of quotients of $C(X)$.

2. Let A be a commutative semi-simple ring with unit whose maximal ideal space $\Omega(A)$ is Hausdorff. Find an explicit description of the mapping $\Omega(Q(A)) \longrightarrow \Omega(A)$ given by Proposition 17, Chapter III.

3. For any maximal ideal M in $Q^{\infty}_{\mathfrak{D}}(U)$, is the quotient field $Q^{\infty}_{\mathfrak{D}}(U)/M$ real closed.

4. Study the maximal ideal spaces of the rings $Q^{\infty}_{\mathfrak{D}}(U)$, in particular their relation to those of the rings $Q_{\mathfrak{D}}(U)$.

5. Let $\mathfrak{D}$ be a filter base of dense open subsets of U; study the derivations of $\mathbb{R}$-algebra $Q^{\infty}_{\mathfrak{D}}(U)$.

6. For any filter base $\mathfrak{D}$ of dense subsets of U, let $C^{\infty}(D)$ be the ring of all functions $f = g|D$ where $g \in C^{\infty}(V)$ for some open neighborhood V of D in U and define $Q^{\infty}_{\mathfrak{D}}(U)$ in the obvious manner, and study those extensions of $C^{\infty}(U)$.

7. Discuss the rings $Q^{(k)}_{\mathfrak{D}}(U)$ defined in terms of $C^{(k)}$-functions instead of C^{∞}-functions.

8. In place of the bounded functions with supremum norm, as used in the discussion of $Q^*_{\mathfrak{S}}(X)$, consider $C^{(k)}$-functions f whose derivatives up to order k are bounded with the usual norm $\| f \| = \sup_x \sum_{i=1}^{k} \left| f^{(i)}(x) \right| / i!$.

9. Generalize to suitable sheaves of rings in place of the rings of continuous (resp. C^∞-) functions.

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