THE LATTICE OF EQUATIONAL CLASSES OF COMMUTATIVE SEMIGROUPS

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OF

COMMUTATIVE SEMIGROUPS

by

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Doctor of Philosophy

McMaster University

May 1970

DOCTOR OF PHILOSOPHY (1970)
(Mathematics)

McMASTER UNIVERSITY
Hamilton, Ontario.

TITLE: The Lattice of Equational Classes of Commutative Semigroups

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M. Sc. (McMaster University)

SUPERVISOR: Professor G. Bruns

NUMBER OF PAGES: vii, 48

SCOPE AND CONTENTS: Commutative semigroup equations are described, and rules of inference for them are given. Then a skeleton sublattice of the lattice of equational classes of commutative semigroups is described, and a partial description is given of the way in which the rest of the lattice hangs on the skeleton.

ACKNOWLEDGEMENTS

I would like to express my sincerest appreciation to my supervisor, Dr. Gunter Bruns, for the invaluable guidance and good counsel he has given me throughout my mathematical career, and for his patience and advice during the preparation of this thesis.

Thanks are also due to the National Research Council of Canada and to McMaster University for financial support.

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INTRODUCTION

In 1935, Garrett Birkhoff introduced the notion of equational class, and proved that a class of algebras is equationally definable if and only if it is closed under the formation of homomorphic images, subalgebras and direct products. ([3]) In this same paper, he also proved that the collection of equational classes of algebras of a given type form a complete lattice.

The first results concerning equational classes of a particular kind of algebra were obtained in 1937 by B. H. Neumann [21] on equational classes of groups. An account of results obtained in this area up to 1967 and references for them may be found in Hanna Neumann's book [22].

In recent years there has been much interest in lattices of equational classes of algebras. Some of the results obtained concerning the lattice of equational classes of lattices may be found in Baker [2], Gratzer [9], [11], Jonsson [15], [16], McKenzie [20] and Wille [26], [27]. In 1964, Jacobs and Schwabauer [13] proved that the lattice of equational classes of algebras with one unary operation is isomorphic to NxN^{+} with a unit adjoined, where N is the lattice of natural numbers with their usual order and N^{+} the natural numbers with division. T. J. Head proved that the lattice of equational classes of commutative monoids is also isomorphic to $(NxN^{+}) \cup \{1\}$. Complete descriptions of the lattices of equational classes of idempotent semigroups and of distributive pseudo-complemented lattices have been given by J. A. Gerhard [8] and K. B. Lee [19] respectively.

Peter Perkins [23] has shown that every equational class of commutative semigroups can be defined by finitely many equations; it follows that the lattice of equational classes of commutative semigroups is at most countable. On the other hand, Trevor Evans, [6], using an example of A. K. Austin [1] has shown that there are uncountably many equational classes of semigroups. There are only countably many atoms in the lattice of equational classes of all semigroups (Kalicki and Scott, [18]); however, there are uncountably many groupoid atoms, ([17]). J. Jezek [14] has given necessary and sufficient conditions for the lattice of equational classes of all algebras of a given type to be countable.

The present paper is concerned with equational classes of commutative semigroups. In Chapter 1, we define what is meant by an equation in commutative semigroups, and give certain rules of inference for these equations. It is then shown that these rules are precisely what one would intuitively want; i.e., one equation e can be inferred from another equation f by these rules if and only if every commutative semigroup that satisfies f also satisfies e; this is the completeness theorem for commutative semigroups. In Chapter 2, a "skeleton sublattice" of the lattice of equational classes of commutative semigroups is defined; this sublattice is isomorphic to AxN^{+}, where A is the lattice of pairs (r,s) of non-negative integers with $r \leq s$ and $s \geq 1$, ordered component-wise, and N^+ is as defined above. Every other equational class "hangs between" two members of the skeleton sublattice; in Chapter 3 we investigate the intervals of the form $[\mathfrak{K}_1, \mathfrak{K}_2]$, where $\mathfrak{K}_1, \mathfrak{K}_2$ are members of the skeleton, and the relationships between these intervals. By restricting attention to a special type of equational class, one obtains a distributive sublattice of the whole lattice (Schwabauer [24]) which contains the skeleton; we show that this sublattice is actually a maximal modular sublattice.

CHAPTER 1

BASIC CONCEPTS

Section 1. Equations and Completeness

A semigroup is a pair (S,f) consisting of a set S and a binary operation f on S satisfying $f(f(a,b),c) = f(a,f(b,c))$ for all $a,b,c \in S$. (S,f) is called commutative if, for all $a,b \in S$, $f(a,b) = f(b,a)$. We will deal exclusively with commutative semigroups, and will write simply ab for $f(a,b)$ and S for (S,f).

The free commutative semigroup on countably many generators, $F(\omega)$, is the set of sequences $(u_n)_{n\in N}$ of non-negative integers, such that $u_n = 0$ for all but finitely many $n \in N$, and $\Sigma u_n \geq 1$, with component-wise addition. For convenience we write (u_n) for $(u_n)_{n\in N}$, and, if $u_n = 0$ for all $n > m$, we sometimes write $(u_1, u_2, \ldots u_m)$ for $(u_n)_{n\in N}$.

A commutative semigroup equation is a pair $((u_n), (v_n))$ of elements of $F(\omega)$. A commutative semigroup S is said to satisfy the equation $((u_n),(v_n))$ if, for every family $(a_n)_{n\in N}$ of elements of S,

$$\prod\{a_i^{u_i} \mid u_i \neq 0\} = \prod\{a_i^{v_i} \mid v_i \neq 0\}.$$

A class $\mathfrak{K}$ of commutative semigroups is said to satisfy an equation e (a set Σ of equations) if every semigroup in $\mathfrak{K}$ satisfies e (satisfies every equation in Σ).

For a set Σ of equations, we define a set $\Gamma\Sigma$ of equations as follows: $e \in \Gamma\Sigma$ if and only if there exists a finite sequence $e_1, e_2, \ldots e_m$ of equations such that $e_m = e$, and such that

(P) : for each $i \leq m$, one of the following holds:

(P1) $e_i \in \Sigma$ or $e_i = ((u_n),(u_n))$ for some $(u_n)_{n\in N} \in F(\omega)$.

(P2) There exists $j < i$ such that $e_j = ((u_n),(v_n))$ and $e_i = ((v_n),(u_n))$.

(P3) There exists $j < i$ and a permutation π of N such that $e_j = ((u_n),(v_n))$ and $e_i = ((u_{\pi(n)}), (v_{\pi(n)}))$.

(P4) There exists $j < i$ such that e_i is obtained from e_j by multiplication, i.e., $e_j = ((u_n),(v_n))$ and $e_i = ((u_n+w_n),(v_n+w_n))$ for some $(w_n)_{n\in N} \in F(\omega)$.

(P5) There exists $j < i$ such that e_i is obtained from e_j by substitution, i.e., $e_j = ((u_n),(v_n))$ and for some $p \in N$ and $(k_n)_{n\in N} \in F(\omega)$, $e_i = ((u_n + k_n u_p),(v_n + k_n v_p))$. (See note below).

(P6) There exists $j < i$ such that e_i is obtained from e_j by identification of variables, i.e., $e_j = ((u_n),(v_n))$ and there exist p,q with $1 \leq p < q$ such that $e_i =$

$$((u_1,\ldots u_{p-1},0,u_{p+1},\ldots u_{q-1},u_q + u_p, u_{q+1},\ldots),$$

$$(v_1,\ldots v_{p-1},0,v_{p+1},\ldots v_{q-1},v_q + v_p, v_{q+1},\ldots))$$

(P7) There exist $j,k < i$ such that $e_j = ((u_n),(v_n))$, $e_k = ((v_n),(w_n))$

and $e_i = ((u_n),(w_n))$.

<u>Note</u>: (P5) does not yield what intuitively is the result of substituting some term $(h_n)_{n\varepsilon N}$ for the pth variable in e_j to obtain

$$((u_1 + h_1u_p, \ldots u_{p-1} + h_{p-1}u_p,\ h_pu_p,\ u_{p+1} + h_{p+1}u_p, \ldots),$$

$$(v_1 + h_1v_p, \ldots v_{p-1} + h_{p-1}v_p,\ h_pv_p,\ v_{p+1} + h_{p+1}v_p, \ldots)) \text{ from}$$

$((u_n)_{n\varepsilon N},(v_n)_{n\varepsilon N})$. However, these two operations are equivalent modulo (P6). For example, to obtain the above equation from $((u_n),(v_n))$ using (P5) and (P6) : if $h_p \geq 1$ then apply (P5) with $k_i = h_i$ for $i \neq p$ and $k_p = h_p - 1$. If $h_p = 0$ then we may assume, in view of (P3), that $h_q \geq 1$ for some $q > p$, and then apply (P5) with $k_i = h_i$ for $i \neq q$, $k_q = h_q - 1$. The result will be

$$((u_1 + h_1u_p, \ldots u_{p-1} + h_{p-1}u_p,\ u_p,\ u_{p+1} + h_{p+1}u_p, \ldots u_q + h_qu_p - u_p, \ldots),$$

$$(v_1 + h_1u_p, \ldots v_{p-1} + h_{p-1}v_p,\ v_p,\ v_{p+1} + h_{p+1}v_p, \ldots v_q + h_qv_p - v_p, \ldots)).$$

If (P6) is then applied to identify the pth variable with the qth, one obtains the desired result.

A set Σ of equations is called closed if $\Sigma = \Gamma\Sigma$. We also write $\Sigma \rightarrow e$ for $e \in \Gamma\Sigma$, and, in the case Σ consists of exactly one equation, f, we write $f \rightarrow e$.

<u>Theorem 1.1</u>: (Completeness Theorem) $e \in \Gamma\Sigma$ if and only if every commutative semigroup that satisfies Σ also satisfies e.

<u>Proof</u>: It is enough to show that $\Gamma\Sigma$ is the smallest fully invariant congruence relation on $F(\omega)$ containing Σ. (See [4], Chapter 6, Section 10). But by (P1), $\Sigma \subseteq \Gamma\Sigma$. By (P1), (P2) and (P7), respectively, $\Gamma\Sigma$ is reflexive, symmetric and transitive. By (P4), $\Gamma\Sigma$ is compatible with the multiplication in $F(\omega)$. Thus $\Gamma\Sigma$ is a congruence relation containing Σ.

Now suppose $\phi: F(\omega) \longrightarrow F(\omega)$ is a homomorphism, and that $((u_n),(v_n)) \in \Gamma\Sigma$. We must show that $(\phi((u_n)_{n\in N}), \phi((v_n)_{n\in N})) \in \Gamma\Sigma$. For each $i \in N$, let α_i be the ith generator of $F(\omega)$, i.e., $\alpha_i = (\delta_{in})_{n\in N}$ where δ_{in} is 0 if $i \neq n$ and 1 if $i = n$. There exists $m \in N$ such that $u_n = v_n = 0$ for all $n > m$, and then $(u_n)_{n\in N} = \sum_{i=1}^{m} u_i\alpha_i$, $(v_n)_{n\in N} = \sum_{i=1}^{m} v_i\alpha_i$. Let $pr_i : F(\omega) \longrightarrow I^+$ (where I^+ is the non-negative integers) be the ith projection map, i.e., $pr_i((w_n)_{n\in N}) = w_i$, and define $\psi: F(\omega) \longrightarrow F(\omega)$ by :

$$pr_i \circ \psi = 0 \quad \text{for all} \quad i \leq m$$

$$pr_i \circ \psi = pr_{i-m} \circ \delta \quad \text{for} \quad i > m.$$

We will prove by induction that for each $n \leq m$,

$$(*) \quad \Big(\sum_{i=1}^{n} u_i\psi(\alpha_i) + (\underbrace{0,\ldots 0}_{n}, u_{n+1},\ldots u_m), \sum_{i=1}^{n} v_i\psi(\alpha_i) + (\underbrace{0,\ldots 0}_{n}, v_{n+1},\ldots v_m)\Big)$$

$\in \Gamma\Sigma$.

The case $n = 0$ is trivial, since we assumed that $((u_1,\ldots u_m),(v_1,\ldots v_m)) = ((u_n)_{n\in N}, (v_n)_{n\in N}) \in \Gamma\Sigma$.

Now assume (*) holds for some n with $0 \leq n < m$. There exists $k \geq 1$ such that $pr_k(\psi(\alpha_{n+1})) \geq 1$; in view of the definition of ψ, $k > m$. Define $(w_i)_{i\epsilon N} \epsilon F(\omega)$ as follows:

$$w_k = pr_k(\psi(\alpha_{n+1})) - 1$$

$$w_i = pr_i(\psi(\alpha_{n+1})) \quad \text{for} \quad i \neq k .$$

By the definition of ψ, $pr_j\ (\sum_{i=1}^{n} u_i\psi(\alpha_i)) = 0$ for $j \leq m$. Thus $pr_{n+1}\ (\sum_{i=1}^{n} u_i\psi(\alpha_i) + (0,\ldots 0,\ u_{n+1},\ldots u_m)) = u_{n+1}$, similarly $pr_{n+1}\ (\sum_{i=1}^{n} v_i\psi(\alpha_i) + (0,\ldots 0,\ v_{n+1},\ldots v_m)) = v_{n+1}$. Thus, by (P5), $(\sum_{i=1}^{n} u_i\psi(\alpha_i) + (0,\ldots 0, u_{n+1},\ldots u_m) + (u_{n+1}w_i)_{i\epsilon N}, \sum_{i=1}^{n} v_i\psi(\alpha_i) +$ $(0,\ldots 0,\ v_{n+1},\ldots v_m) + (v_{n+1}w_i)_{i\epsilon N}) \ \epsilon\ \Gamma\Sigma$. By (P6), by identifying the $(n+1)$st and the kth variables, we obtain

$$(\sum_{i=1}^{n+1} u_i\psi(\alpha_i) + (0,\ldots 0,\ u_{n+2},\ldots u_m), \sum_{i=1}^{n+1} v_i\psi(\alpha_i) + (0,\ldots 0, v_{n+2},\ldots v_m))$$

$\epsilon\ \Gamma\Sigma$. Thus by induction, (*) holds for all $n \leq m$. In particular, it holds for m, thus $(\sum_{i=1}^{m} u_i\psi(\alpha_i)\ , \sum_{i=1}^{m} v_i\psi(\alpha_i))\ \epsilon\ \Gamma\Sigma$. But, by (P3), in view of the definition of ψ, this implies that $(\sum_{i=1}^{m} u_i\phi(\alpha_i)\ , \sum_{i=1}^{m} v_i\phi(\alpha_i))\ \epsilon\ \Gamma\Sigma$. But $\sum_{i=1}^{m} u_i\phi(\alpha_i) = \phi(\sum_{i=1}^{m} u_i\alpha_i)$ and $\sum_{i=1}^{m} v_i\phi(\alpha_i) = \phi(\sum_{i=1}^{m} v_i\alpha_i)$, since ϕ is a homomorphism. This yields the desired result; thus $\Gamma\Sigma$ is a fully invariant congruence relation containing Σ. On the other hand, it is clear from (P1) to (P7) that if Θ is any fully invariant congruence relation containing Σ then $\Gamma\Sigma \subseteq \Theta$. This completes the proof.

For a class $\mathfrak{K}$ of commutative semigroups, let $\mathfrak{K}^*$ be the set of all equations satisfied by every member of $\mathfrak{K}$, then $\mathfrak{K}^*$ is closed. For a set Σ of equations, let Σ^* be the class of all commutative semigroups satisfying Σ; then Σ^* is equational. For equational classes $\mathfrak{K}$, $\mathfrak{K}'$, $\mathfrak{K} \subseteq \mathfrak{K}'$ if and only if $\mathfrak{K}'^* \subseteq \mathfrak{K}^*$, and for closed sets Σ, Σ' of equations, $\Sigma \subseteq \Sigma'$ if and only if $\Sigma'^* \subseteq \Sigma^*$.

Let $\mathcal{L}$ be the lattice of equational classes of commutative semigroups, and $\mathcal{L}'$ the lattice of closed sets of equations; then $\mathcal{L}$ is dually isomorphic to $\mathcal{L}'$ by the mapping $\mathfrak{K} \rightsquigarrow \mathfrak{K}^*$. For $\mathfrak{K}_1, \mathfrak{K}_2 \in \mathcal{L}$, $\mathfrak{K}_1 \wedge_{\mathcal{L}} \mathfrak{K}_2 = \mathfrak{K}_1 \cap \mathfrak{K}_2 = (\mathfrak{K}_1^* \vee_{\mathcal{L}'} \mathfrak{K}_2^*)^*$, and $\mathfrak{K}_1 \vee_{\mathcal{L}} \mathfrak{K}_2 = (\mathfrak{K}_1^* \wedge_{\mathcal{L}'} \mathfrak{K}_2^*)^* = (\mathfrak{K}_1^* \cap \mathfrak{K}_2^*)^*$.

Section 2. The Free Commutative Semigroups Satisfying $((r),(r+n))$.

For each pair (r,n) of natural numbers, define $f_{r,n} : I^+ \longrightarrow \{0,1,\dots r+n-1\}$ as follows:

$$f_{r,n}(k) = \begin{cases} k & \text{if} \quad k \leq r \\ r + [k-r]_n & \text{if} \quad k > r \end{cases}$$

where $[m]_n$ is the least non-negative residue of m modulo n, and I^+ is the set of non-negative integers.

For each $m \in N$, let $F_{r,n}(m) = \{(i_1,\dots i_m) \mid 0 \leq i_j \leq r+n-1, \ \Sigma i_j \geq 1\}$ and define a binary operation on $F_{r,n}(m)$ as follows: $(i_1,i_2,\dots i_m)(j_1,j_2,\dots j_m) = (f_{r,n}(i_1+j_1),\dots f_{r,n}(i_m+j_m))$. Then

$F_{r,n}(m)$ with this operation is a commutative semigroup. In fact, we will show that it is the free commutative semigroup on m generators satisfying $((r),(r+n))$. The free generators are the m m-tuples with one entry 1 and all other entries 0.

Let $(i_1,\ldots i_m) \in F_{r,n}(m)$. Then $(i_1,\ldots i_m)^r =$ $(f_{r,n}(ri_1),\ldots f_{r,n}(ri_m))$ and $(i_1,\ldots i_m)^{r+n} = (f_{r,n}((r+n)i_1),\ldots f_{r,n}((r+n)i_m))$. For each $j \leq m$, $ri_j \equiv (r+n)i_j$ (modulo n), and, if $ri_j \neq (r+n)i_j$ then $i_j \geq 1$, thus $ri_j \geq r$, $(r+n)i_j \geq r$, hence $f_{r,n}(ri_j) =$ $f_{r,n}((r+n)i_j)$. It follows that $(i_1,\ldots i_m)^r = (i_1,\ldots i_m)^{r+n}$. Thus $F_{r,n}(m)$ satisfies $((r),(r+n))$.

For each $j \leq m$, let β_j be the m-tuple with jth entry 1 and all other entries 0. Clearly $\{\beta_1,\beta_2,\ldots\beta_m\}$ generates $F_{r,n}(m)$. If S is any commutative semigroup satisfying $((r),(r+n))$ and $\phi : \{\beta_1,\ldots\beta_m\} \longrightarrow S$ is any mapping, then the mapping $\bar{\phi}: F_{r,n}(m) \longrightarrow S$ defined by $\bar{\phi}((i_1,\ldots i_m)) = \prod\{\phi(\beta_j)^{i_j} \mid 1 \leq j \leq m,\ i_j \neq 0\}$ is a homomorphic extension of ϕ.

It follows that $F_{r,n}(m)$ is the free commutative semigroup on m generators satisfying $((r),(r+n))$.

Section 3. Definition of the Invariants D,V,L,U.

An equation $((u_n),(v_n))$ is called non-trivial if $u_n \neq v_n$ for some $n \in N$. A set of equations is called non-trivial if it contains at least one non-trivial equation; an equational class $\mathfrak{K}$ is called non-trivial if $\mathfrak{K}^*$ is non-trivial.

For a non-trivial equation $e = ((u_n),(v_n))$, define

$D(e)$ = greatest common divisor of $\{|u_n - v_n| \mid n \in N, u_n \neq v_n\}$

$V(e)$ = minimum $\{u_n, v_n \mid n \in N, u_n \neq v_n\}$

$L(e)$ = minimum $\{\text{maximum } \{u_n \mid n \in N\}, \text{maximum } \{v_n \mid n \in N\}\}$

$$U(e) = \begin{cases} \text{minimum } \{\sum_{n\in N} u_n, \sum_{n\in N} v_n\} & \text{if } \sum_{n\in N} u_n \neq \sum_{n\in N} v_n \\ \sum_{n\in N} u_n + V(e) & \text{if } \sum_{n\in N} u_n = \sum_{n\in N} v_n \end{cases}$$

Note that $D(e)$, $L(e)$, and $U(e) \geq 1$, and that $V(e) \leq L(e) \leq U(e)$.

For example, if $e = ((0,1), (1,0))$ then $D(e) = 1$, $V(e) = 0$, $L(e) = 1$ and $U(e) = 1 + 0 = 1$. A semigroup S satisfies e if and only if S has at most one element.

If $e = ((1,0), (1,p))$ then $D(e) = p$, $V(e) = 0$, and $L(e) = U(e) = 1$. A commutative semigroup S satisfies e if and only if, for all $s,t \in S$, $s = st^p$, i.e., if and only if S is an abelian group satisfying $s^p = 1$ for all $s \in S$.

For a non-trivial set Σ of equations, we define

$D(\Sigma)$ = greatest common divisor of $\{D(e) \mid e \in \Sigma, e \text{ non-trivial}\}$

$V(\Sigma)$ = minimum $\{V(e) \mid e \in \Sigma, e \text{ non-trivial}\}$

$L(\Sigma)$ = minimum $\{L(e) \mid e \in \Sigma, e \text{ non-trivial}\}$

$U(\Sigma)$ = minimum $\{U(e) \mid e \in \Sigma, e \text{ non-trivial}\}$.

Section 4. Elementary Properties of D,V,L,U.

Lemma 1.1: $F_{r,n}(1)$ satisfies a non-trivial equation e if and only if $U(e) \geq r$ and $n|D(e)$.

<u>Proof</u>: Let $e = ((u_i)_{i\epsilon N}, (v_i)_{i\epsilon N})$ be a non-trivial equation with $r \leq U(e)$ and $n|D(e)$.

<u>Case 1</u>: $\Sigma u_i \neq \Sigma v_i$. Then $\Sigma u_i, \Sigma v_i \geq r$. The elements of $F_{r,n}(1)$ are the ~~1~~-tuples of positive integers $\leq r + n - 1$; we write simply k for (k). If $(k_i)_{i\epsilon N}$ is a family in $F_{r,n}(1)$ then

$$\Pi\{k_i^{u_i}|u_i \neq 0\} = \Pi\{1^{k_i u_i}|u_i \neq 0\} = f_{r,n}(\Sigma\{k_i u_i|u_i \neq 0\}), \text{ and}$$

$$\Pi\{k_i^{v_i}|v_i \neq 0\} = f_{r,n}(\Sigma\{k_i v_i|v_i \neq 0\}).$$ But $\Sigma\{k_i u_i|u_i \neq 0\} \geq \Sigma\{u_i|u_i \neq 0\} = \Sigma u_i \geq r$; similarly $\Sigma\{k_i v_i|v_i \neq 0\} \geq r$. Moreover, since $n|D(e)$, it follows that $n|u_i - v_i$ for all i, thus $\Sigma k_i u_i \equiv \Sigma k_i v_i$ (modulo n). This implies that $f_{r,n}(\Sigma k_i u_i) = f_{r,n}(\Sigma k_i v_i)$, thus $\Pi\{k_i^{u_i}|u_i \neq 0\} = \Pi\{k_i^{v_i}|v_i \neq 0\}$.

<u>Case 2</u>: $\Sigma u_i = \Sigma v_i$. Then $U(e) = \Sigma u_i + V(e) \geq r$. Let $(k_i)_{i\epsilon N}$ be a family in $F_{r,n}(1)$. If $k_i = 1$ for all i with $u_i \neq v_i$ then $\Sigma k_i u_i = \Sigma k_i v_i$. If $k_i > 1$ for some i with $u_i \neq v_i$ then, since $u_i \neq v_i$ implies that $u_i, v_i \geq V(e)$, it follows that $\Sigma k_i u_i \geq \Sigma u_i + V(e) \geq r$, $\Sigma k_i v_i \geq \Sigma v_i + V(e) \geq r$. Thus we again have $f_{r,n}(\Sigma k_i u_i) = f_{r,n}(\Sigma k_i v_i)$, hence

$$\Pi\{k_i^{u_i}|u_i \neq 0\} = \Pi\{k_i^{v_i}|v_i \neq 0\}.$$

It follows that $F_{r,n}(1)$ satisfies e.

For the converse, assume $e = ((u_i)_{i\varepsilon N}, (v_i)_{i\varepsilon N})$ is a non-trivial equation with $U(e) < r$. If $\Sigma u_i \neq \Sigma v_i$, then $r >$ minimum $\{\Sigma u_i, \Sigma v_i\}$; $(1)_{i\varepsilon N}$ is a family in $F_{r,n}(1)$, and $\prod \{1^{u_i} | u_i \neq 0\} = f_{r,n}(\Sigma u_i) \neq f_{r,n}(\Sigma v_i) = \prod \{1^{v_i} | v_i = 0\}$. Thus, in this case $F_{r,n}(1)$ does not satisfy e. If $\Sigma u_i = \Sigma v_i$, then we may assume without loss of generality that $V(e) = u_1$ (and then $v_1 > u_1$ and $\Sigma u_i + u_1 < r$). By (P5), $e \to ((2u_1, u_2, \ldots), (2v_1, v_2, \ldots))$ and by (P4), $e \to ((2u_1, u_2, \ldots), (v_1 + u_1, v_2, \ldots))$. Thus, by (P7), $e \to ((2v_1, v_2, \ldots), (v_1 + u_1, v_2, \ldots))$. $(1)_{i\varepsilon N}$ is a family in $F_{r,n}(1)$, and $\prod \{1^{w_i} | w_1 = 2v_1, w_i = v_i$ for $i \geq 2\} = f_{r,n}(\Sigma v_i + v_1) \neq f_{r,n}(\Sigma v_i + u_1) = \prod \{1^{x_i} | x_1 = u_1 + v_1, x_i = v_i$ for $i \geq 2\}$. Thus $F_{r,n}(1)$ does not satisfy $((2v_1, v_2, \ldots), (v_1 + u_1, v_2, \ldots))$ and hence does not satisfy e.

If $U(e) \geq r$ but $n \nmid D(e)$ then we may assume without loss of generality that $u_1 \neq v_1$ and $n \nmid |u_1 - v_1|$. As above, $e \to ((2v_1, v_2, \ldots), (v_1 + u_1, v_2, \ldots))$. But since $\Sigma v_i + v_1 \not\equiv \Sigma v_i + u_1$ (modulo n), it follows that $F_{r,n}(1)$ does not satisfy $((2v_1, v_2, \ldots), (v_1 + u_1, v_2, \ldots))$ and hence does not satisfy e.

This completes the proof.

<u>Lemma 1.2</u>: For each $m \geq 2$, $F_{r,n}(m)$ satisfies a non-trivial equation e if and only if $V(e) \geq r$ and $n | D(e)$.

<u>Proof</u>: $F_{r,n}(m)$ satisfies $((r), (r + n))$. $((r), (r + n)) \to ((r), (r + kn))$ for all $k \geq 1$. If $e = ((u_i)_{i\varepsilon N}, (v_i)_{i\varepsilon N})$ and

if $V(e) \geq r$, $n|D(e)$, then $u_k \neq v_k$ implies $u_k, v_k \geq r$ and $n | u_k - v_k$, thus $((r),(r+n)) \to ((u_k),(v_k))$. Thus $((r),(r+n)) \to ((u_i)_{i\epsilon N}, (v_i)_{i\epsilon N}) = e$. It follows that if $V(e) \geq r$ and $n|D(e)$ then $F_{r,n}(m)$ satisfies e.

Conversely, if $F_{r,n}(m)$ satisfies $e = ((u_i)_{i\epsilon N},(v_i)_{i\epsilon N})$, then, since $F_{r,n}(1)$ is isomorphic to a subsemigroup of $F_{r,n}(m)$, it follows that $F_{r,n}(1)$ satisfies e and thus $n|D(e)$. We may assume without loss of generality that $V(e) = u_1$ (and then $u_1 < v_1$). Let $a_1 = (1,0,\ldots 0) \epsilon F_{r,n}(m)$, and for $i \geq 2$, let $a_i = (0,1,0,\ldots 0) \epsilon F_{r,n}(m)$. Then $(a_i)_{i\epsilon N}$ is a family in $F_{r,n}(m)$ and $\prod\{a_i^{u_i} | u_i \neq 0\} = (f_{r,n}(u_1), f_{r,n}(\sum_{i\geq 2} u_i), 0,\ldots 0)$, and $\prod\{a_i^{v_i} | v_i \neq 0\} = (f_{r,n}(v_1), f_{r,n}(\sum_{i\geq 2} v_i), 0,\ldots 0)$. Since $F_{r,n}(m)$ satisfies e, it follows that $f_{r,n}(u_1) = f_{r,n}(v_1)$. But $u_1 \neq v_1$, thus $u_1, v_1 \geq r$. This means that $V(e) \geq r$.

<u>Theorem 1.2</u>: If $\Sigma \to e$ and e is non-trivial then $U(\Sigma) \leq U(e)$, $V(\Sigma) \leq V(e)$, $L(\Sigma) \leq L(e)$ and $D(\Sigma)|D(e)$.

<u>Proof:</u> Assume $\Sigma \to e$. Since $F_{U(\Sigma),D(\Sigma)}(1)$ satisfies Σ, it also satisfies e, thus $U(\Sigma) \leq U(e)$ and $D(\Sigma)|D(e)$. Moreover, $F_{V(\Sigma),D(\Sigma)}(2)$ satisfies Σ, and hence also e, and thus $V(\Sigma) \leq V(e)$.

To show that $L(\Sigma) \leq L(e)$, it is enough to show that if $e_1,\ldots e_m$ is a sequence of equations satisfying (P) and $L(e_i) \geq L(\Sigma)$ for all $i < m$ then $L(e_m) \geq L(\Sigma)$. Let $e_i = (\alpha_i,\beta_i)$ where $\alpha_i,\beta_i \epsilon F(\omega)$. Then $L(e_i) \geq L(\Sigma)$ means that there exists an entry

$\geq L(\Sigma)$ in each of α_i and β_i. But if this holds for all $i < m$, then it is clear that whichever of (P1) to (P7) e_m satisfies, there will be an entry $\geq L(\Sigma)$ in each of α_m and β_m, i. e., $L(e_m) \geq L(\Sigma)$.

Corollary 1: If $\Sigma \rightarrow \Sigma'$ then $U(\Sigma) \leq U(\Sigma')$, $V(\Sigma) \leq V(\Sigma')$, $L(\Sigma) \leq L(\Sigma')$ and $D(\Sigma)|D(\Sigma')$.

Corollary 2: D,V,L,U as operators on sets of equations are invariant under Γ, i.e., for any non-trivial set Σ of equations, $D(\Sigma) = D(\Gamma\Sigma)$, $V(\Sigma) = V(\Gamma\Sigma)$, $L(\Sigma) = L(\Gamma\Sigma)$ and $U(\Sigma) = U(\Gamma\Sigma)$.

For a non-trivial equational class $\mathfrak{K}$, define $D(\mathfrak{K}) = D(\mathfrak{K}^*)$, $V(\mathfrak{K}) = V(\mathfrak{K}^*)$, $L(\mathfrak{K}) = L(\mathfrak{K}^*)$ and $U(\mathfrak{K}) = U(\mathfrak{K}^*)$. Since for two equational classes $\mathfrak{K}_1, \mathfrak{K}_2$, $\mathfrak{K}_1 \subseteq \mathfrak{K}_2$ if and only if $\mathfrak{K}_1^* \rightarrow \mathfrak{K}_2^*$, it follows that if $\mathfrak{K}_1 \subseteq \mathfrak{K}_2$ then $U(\mathfrak{K}_1) \leq U(\mathfrak{K}_2)$, $V(\mathfrak{K}_1) \leq V(\mathfrak{K}_2)$, $L(\mathfrak{K}_1) \leq L(\mathfrak{K}_2)$ and $D(\mathfrak{K}_1)|D(\mathfrak{K}_2)$.

CHAPTER 2

THE SKELETON SUBLATTICE CONSISTING OF THE CLASSES $\Omega_{r,s,n}$

Section 1. Definition of the Skeleton

For non-negative integers r,s,n with $r \leq s$ and $n \geq 1$, let $\Omega_{r,s,n} = \{((r,s),(r+n,s)),\ ((s),\ (s+n))\}^*$. Then $U(\Omega_{r,s,n}) = s = L(\Omega_{r,s,n})$, $V(\Omega_{r,s,n}) = r$ and $D(\Omega_{r,s,n}) = n$.

Note that since $((0,s),(n,s)) \rightarrow ((s),\ (s+n))$ by (P6), $\Omega_{0,s,n} = \{((0,s),(n,s))\}^*$. Since $((r),(r+n)) \rightarrow ((r,r),(r+n,r))$ by (P4), $\Omega_{r,r,n} = \{((r),(r+n))\}^*$.

$\Omega_{0,1,p}$ is the class of all commutative groups G satisfying $x^p = 1$ for all $x \in G$. $\Omega_{0,1,1} = \{((0,1),\ (1,1))\}^*$ and since $((0,1),\ (1,1)) \rightarrow ((0,1),\ (1,0))$ it follows that $\Omega_{0,1,1}$ is the zero of the lattice $\mathcal{L}$.

Clearly, in view of (P4), if $r \leq t$ and $s \leq u$ then $\Omega_{r,s,n} \subseteq \Omega_{t,u,n}$. If in addition $n|m$ then a simple induction argument yields $\Omega_{r,s,n} \subseteq \Omega_{t,u,m}$. On the other hand, by the remark at the end of Chapter 1, if $\Omega_{r,s,n} \subseteq \Omega_{t,u,m}$ then $r \leq t$, $s \leq u$ and $n|m$. Thus $\Omega_{r,s,n} \subseteq \Omega_{t,u,m}$ if and only if $r \leq t$, $s \leq u$ and $n|m$.

Section 2. The Set $\Omega_{r,s,n}^*$ of Equations Holding in $\Omega_{r,s,n}$.

Theorem 2.1: For a non-trivial equation e, $e \in \Omega_{r,s,n}^*$ if and only if $r \leq V(e)$, $s \leq L(e)$ and $n|D(e)$.

Proof: The "only if" part is a direct consequence of the results of the last section of Chapter 1.

For the converse, let $e = ((u_i)_{i\varepsilon N}, (v_i)_{i\varepsilon N})$ and assume $r \leq V(e)$, $s \leq L(e)$ and $n|D(e)$. It follows directly from the definition of V,D, and L that there exist j,k with $u_j, v_k \geq s$, and that if $u_i \neq v_i$ then $n|u_i - v_i$ and $u_i, v_i \geq r$. We may assume without loss of generality that $u_1 \geq s$. But then $((r,s), (r+n,s)) \rightarrow ((u_1,u_2\ldots), (u_1,v_2,v_3,\ldots))$. If $v_1 \geq s$ then $((s), (s+n)) \rightarrow ((u_1),(v_1)) \rightarrow ((u_1,v_2,\ldots), (v_1,v_2,\ldots))$. If $v_1 < s$ then $v_j \geq s$ for some $j \geq 2$ and then $((r,s), (r+n,s)) \rightarrow ((u_1,v_j), (v_1,v_j)) \rightarrow ((u_1,v_2,\ldots), (v_1,v_2,\ldots))$.
Thus $\Omega_{r,s,n}{}^* \rightarrow e$, i.e., $e \in \Omega_{r,s,n}{}^*$.

Corollary 1: For an equational class $\mathfrak{K}$, $\Omega_{r,s,n} \subseteq \mathfrak{K}$ if and only if $r \leq V(\mathfrak{K})$, $s \leq L(\mathfrak{K})$ and $n|D(\mathfrak{K})$.

Corollary 2: $\Omega_{r,s,n} \vee \Omega_{t,u,m} = \Omega_{v,w,p}$, where $v = \max\{r,t\}$, $w = \max\{s,u\}$, p = least common multiple $\{n,m\}$.

Proof: Since $\Omega_{r,s,n} \subseteq \Omega_{v,w,p}$ and $\Omega_{t,u,m} \subseteq \Omega_{v,w,p}$, it follows that $\Omega_{r,s,n} \vee \Omega_{t,u,m} \subseteq \Omega_{v,w,p}$. Thus it is enough to show that $\Omega_{v,w,p} \subseteq \Omega_{r,s,n} \vee \Omega_{t,u,m}$, i.e., that $\Omega_{r,s,n}{}^* \cap \Omega_{t,u,m}{}^* \subseteq \Omega_{v,w,p}{}^*$. But e non-trivial and $e \in \Omega_{r,s,n}{}^* \cap \Omega_{t,u,m}{}^*$ implies by the theorem that $V(e) \geq r$, $L(e) \geq s$, $n|D(e)$ and $V(e) \geq t$, $L(e) \geq u$ and $m|D(e)$, thus $V(e) \geq v$, $L(e) \geq w$ and $p|D(e)$. It follows from the theorem that $e \in \Omega_{v,w,p}{}^*$, and this completes the proof.

Since every non-trivial equational class is contained in some $\Omega_{r,s,n}$ it follows from Corollary 2 that the class of all commutative semigroups is not the join of two smaller classes. This was also proved in Dean and Evans, [5].

Theorem 2.2: $\Omega_{r,s,n} \wedge \Omega_{t,u,m} = \Omega_{v,w,d}$, where $v = \min\{r,t\}$, $w = \min\{s,u\}$ and $d =$ greatest common divisor $\{n,m\}$.

Proof: Since $\Omega_{r,s,n} \supseteq \Omega_{v,w,d}$, and $\Omega_{t,u,m} \supseteq \Omega_{v,w,d}$, it follows that $\Omega_{r,s,n} \wedge \Omega_{t,u,m} \supseteq \Omega_{v,w,d}$. To show the reverse inclusion, it is enough to show that $\{((v,w),(v + d,w)),((w),(w + d))\} \subseteq (\Omega_{r,s,n} \wedge \Omega_{t,u,m})^* = \Omega_{r,s,n}^* \vee \Omega_{t,u,m}^*$.

Assume $s \leq u$. Then there exist natural numbers p,q such that $pn = qm + d$ and $pn \geq u$. By Theorem 2.1,

$$((s),(s + 2pn)) = ((s),\ (s + pn + qm + d)) \in \Omega_{r,s,n}^*$$

$$((s + pn + qm + d),\ (s + pn + d)) \in \Omega_{t,u,m}^*$$

$$((s + pn + d,\ s + d)) \in \Omega_{r,s,n}^* .$$

Thus $((s),\ (s + d)) \in \Omega_{r,s,n}^* \vee \Omega_{t,u,m}^*$. The case $u < s$ follows by symmetry, thus $(w),\ (w + d)) \in \Omega_{r,s,n}^* \vee \Omega_{t,u,m}^*$.

Now assume $r \leq t$. Then $v = r$. There exist natural numbers h,k such that $w + kd \geq s$, $r + hn \geq w$. Then:

$$((r,w),\ (r,w + kd)) \in \Omega_{r,s,n}^* \vee \Omega_{t,u,m}^*$$

$$((r,w + kd),\ (r + hn,\ w + kd)) \in \Omega_{r,s,n}^*$$

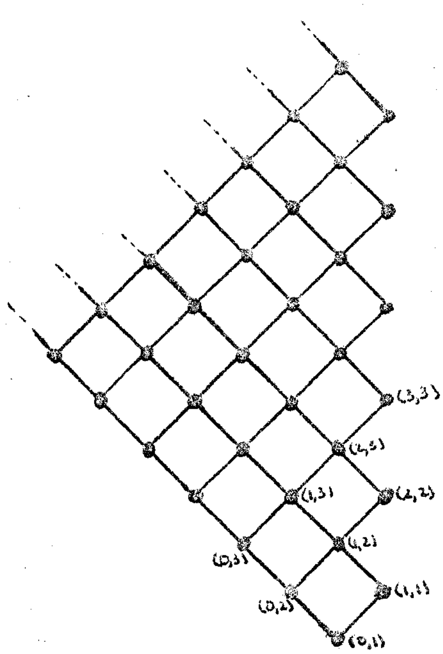
$$((r + hn,\ w + kd),\ (r + hn + d,\ w + kd)) \in \Omega_{r,s,n}^* \vee \Omega_{t,u,m}^*$$

$$((r + hn + d,\ w + kd),\ (r + d,\ w + kd)) \in \Omega_{r,s,n}^*$$

$$((r + d,\ w + kd),\ (r + d,\ w)) \in \Omega_{r,s,n}^* \vee \Omega_{t,u,m}^* .$$

Thus, $((v,w),\ (v + d,\ w)) \in (\Omega_{r,s,n}^* \vee \Omega_{t,u,m}^*) \vee \Omega_{t,u,m}^* =$ $\Omega_{r,s,n}^* \vee \Omega_{t,u,m}^*$. The case $t < r$ follows by symmetry. This completes the proof.

Let A be the lattice of pairs (r,s) of non-negative integers such that $r \leq s$ and $s \geq 1$, ordered component-wise, i.e., $(r,s) \leq (t,u)$ if and only if $r \leq t$ and $s \leq u$. Let N^+ be the lattice of natural numbers ordered by division. Then, by the above theorems, the map given by $(r,s,n) \rightsquigarrow \Omega_{r,s,n}$ is a lattice isomorphism of $A \times N^+$ onto a sublattice of $\mathcal{R}$.



A

Section 3. Equations Implying $\Omega_{r,s,n}^*$

Theorem 2.3: For a non-trivial equation e, $e \to \Omega_{r,s,n}^*$ if and only if $V(e) \le r$, $U(e) \le s$ and $D(e)|n$.

Proof: It follows from the results in the last section of Chapter 1 that if $e \to \Omega_{r,s,n}^*$ then $V(e) \le r = V(\Omega_{r,s,n})$, $U(e) \le s = U(\Omega_{r,s,n})$ and $D(e)|n = D(\Omega_{r,s,n})$.

For the converse, let $e = ((u_i)_{i\in N}, (v_i)_{i\in N})$ and assume $V(e) \le r$, $U(e) \le s$ and $D(e)|n$. For each $i \in N$,

$e \to ((u_1,\ldots u_{i-1}, 2u_i, u_{i+1},\ldots), (v_1,\ldots v_{i-1}, v_i + u_i, v_{i+1},\ldots)$ and

$e \to ((u_1,\ldots u_{i-1}, 2u_i, u_{i+1},\ldots), (v_1,\ldots v_{i-1}, 2v_i, v_{i+1},\ldots))$, thus $e \to$

$((v_1,\ldots v_{i-1}, v_i + u_i, v_{i+1},\ldots),(v_1,\ldots v_{i-1}, 2v_i, v_{i+1},\ldots))$. Let

$w_i = \sum_{j\in N} v_j + \min\{u_i,v_i\}$, and let $d_i = |u_i - v_i|$. Then for each $i \in N$, $e \to ((w_i),(w_i + d_i))$. Thus for each $i \in N$ with $u_i \ne v_i$, $e^* \subseteq \Omega_{w_i,w_i,d_i}$. By Theorem 2.2, $e^* \subseteq \Omega_{w,w,d}$ where $w = \min\{w_i | u_i \ne v_i\}$, and $d = \gcd\{d_i \mid d_i \ne 0\} = D(e)$.

If $\sum_{i\in N} u_i = \sum_{i\in N} v_i$ then $U(e) = w_j$ for some $j \in N$, and thus $e^* \subseteq \Omega_{U(e),U(e),D(e)}$. If $\Sigma u_i \ne \Sigma v_i$, then $e \to ((\sum_{i\in N} u_i), (\sum_{i\in N} v_i)) \to ((U(e)), (U(e) + h))$ where $h = |\Sigma u_i - \Sigma v_i|$ is divisible by $D(e)$. But then $e^* \subseteq \Omega_{U(e),U(e),D(e)}$.

Now assume without loss of generality that $V(e) = u_1$. Then

$$e \to ((u_1, \sum_{i\geq 2} u_i), (v_1, \sum_{i\geq 2} v_i)) \to ((u_1, U(e) + \sum_{i\geq 2} u_i), (v_1, U(e) + \sum_{i\geq 2} v_i)).$$

Since $D(e) \mid \sum_{i\geq 2} u_i - \sum_{i\geq 2} v_i$, and since $e \to ((U(e)), (U(e) + D(e)))$, it follows that $e \to ((u_1, U(e)), (v_1, U(e)))$. Thus $e^* \subseteq \Omega_{V(e),U(e),h}$, where $h = v_1 - u_1$ is divisible by $D(e)$. This, together with $e^* \subseteq \Omega_{U(e),\, U(e),\, D(e)}$ yields $e^* \subseteq \Omega_{V(e),\, U(e),\, D(e)}$. Since $V(e) \leq r$, $U(e) \leq s$ and $D(e) \mid n$ it follows that $e^* \subseteq \Omega_{r,s,n}$.

<u>Corollary</u>: For an equational class $\mathfrak{K}$, $\mathfrak{K} \subseteq \Omega_{r,s,n}$ if and only if $V(\mathfrak{K}) \leq r$, $U(\mathfrak{K}) \leq s$ and $D(\mathfrak{K}) \mid n$.

<u>Proof</u>: The "only if" part follows from the remark at the end of Chapter 1; the converse follows from the fact that if $\mathfrak{K}$ is a non-trivial equational class, then there exist equations $e_1, e_2, e_3 \in \mathfrak{K}^*$ such that $V(e_1) = V(\mathfrak{K})$, $U(e_2) = U(\mathfrak{K})$ and $D(e_3) = D(\mathfrak{K})$.

<u>Lemma 2.1</u>: For a non-trivial equation e, if $L(e) \leq t$ then there exists $k \in N$ such that $e \to ((\underbrace{t,t,\ldots t}_{k}), (t + D(e), \underbrace{t,t,\ldots t}_{k-1}))$.

<u>Proof</u>: Let e be a non-trivial equation with $L(e) \leq t$ and $D(e) = d$. We may assume without loss of generality that $e = ((u_1,\ldots u_n), (v_1,\ldots v_n))$ where $u_i \leq t$ for all $i \leq n$.
If $u_j < v_j$ for some $j \leq n$ then $e \to ((t,t,\ldots t), (v_1 + t-u_1,\ldots v_n+t-u_n))$ where $v_j + t - u_j > t$. If $u_i \geq v_i$ for all $i \leq n$, then $e \to ((u_1 + t - v_1,\ldots u_n + t - v_n), (t,\ldots t))$ where, since e is non-trivial, $u_i + t - v_i > t$ for some $i \leq n$. Thus, in either case,

there exist $w_2,\dots w_n$ and $s \geq 1$ such that $e \to ((\underbrace{t,\dots t}_{n}),(t+s,w_2,\dots w_n))$.

Choose h so that $t + hs \geq U(e)$ and let $k = h(n-1)+1$. For each m with $0 \leq m \leq h$, let $\alpha_m = (t + ms,\underbrace{w_2,\dots w_n,\dots w_2,\dots w_n}_{m(n-1)\text{ terms}},\underbrace{t,\dots t}_{k-m(n-1)-1})$.

By (P4), $e \to ((t + ms,\underbrace{t,\dots t}_{n-1}),(t + ms + s,w_2,\dots w_n))$ for each $m \geq 0$.

Thus, again by (P4), $e \to (\alpha_m, \alpha_{m+1})$ for each m with $0 \leq m < h$.

By (P7), it follows that $e \to (\alpha_o,\alpha_h) = ((\underbrace{t,\dots t}_{k}),(\underbrace{t+hs,w_2,\dots w_n,\dots w_n}_{k\text{ terms}}))$

and thus by (P4), $e \to (\underbrace{t + d,t,\dots t}_{k}),(\underbrace{t + hs + d,w_2,\dots w_n,\dots w_2,\dots w_n}_{k}))$.

But $t + hs \geq U(e)$, thus $e \to ((t + hs), (t + hs + d))$. It follows that $e \to ((\underbrace{t,\dots t}_{k}), (\underbrace{t + d, t,\dots t}_{k}))$.

Summarizing the results of this section and the preceding one, we have that for a non-trivial equational class $\mathfrak{K}$, $\Omega_{V(\mathfrak{K}),L(\mathfrak{K}),D(\mathfrak{K})} \subseteq \mathfrak{K} \subseteq \Omega_{V(\mathfrak{K}),U(\mathfrak{K}),D(\mathfrak{K})}$. Moreover these choices of the Ω's are the best possible in the following sense: if $\Omega_{r,s,n} \subseteq \mathfrak{K}$ then $\Omega_{r,s,n} \subseteq \Omega_{V(\mathfrak{K}),L(\mathfrak{K}),D(\mathfrak{K})}$ and if $\mathfrak{K} \subseteq \Omega_{r,s,n}$ then $\Omega_{V(\mathfrak{K}),U(\mathfrak{K}),D(\mathfrak{K})} \subseteq \Omega_{r,s,n}$. Thus if $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$ then $V(\mathfrak{K}) = r$, $L(\mathfrak{K}) \geq s$, $U(\mathfrak{K}) \leq t$ and $D(\mathfrak{K}) = n$.

It follows from this and from the results in Section 2 concerning meets and joins of different $\Omega_{r,s,n}$'s that for non-trivial equational classes $\mathfrak{K}_1,\mathfrak{K}_2$, $D(\mathfrak{K}_1 \wedge \mathfrak{K}_2) = \text{g.c.d.}\ \{D(\mathfrak{K}_1),D(\mathfrak{K}_2)\}$,

$D(\mathbb{K}_1 \vee \mathbb{K}_2) =$ l.c.m. $\{D(\mathbb{K}_1), D(\mathbb{K}_2)\}$, $V(\mathbb{K}_1 \wedge \mathbb{K}_2) =$ min $\{V(\mathbb{K}_1), V(\mathbb{K}_2)\}$, $V(\mathbb{K}_1 \vee \mathbb{K}_2) =$ max $\{V(\mathbb{K}_1), V(\mathbb{K}_2)\}$, $L(\mathbb{K}_1 \wedge \mathbb{K}_2) =$ min $\{L(\mathbb{K}_1), L(\mathbb{K}_2)\}$ and $U(\mathbb{K}_1 \vee \mathbb{K}_2) =$ max $\{U(\mathbb{K}_1), U(\mathbb{K}_2)\}$. Also, $U(\mathbb{K}_1 \wedge \mathbb{K}_2) = U(\mathbb{K}_1^* \cup \mathbb{K}_2^*) =$ min $\{U(\mathbb{K}_1), U(\mathbb{K}_2)\}$. Moreover, it follows from Lemma 2.1 that $L(\mathbb{K}_1 \vee \mathbb{K}_2) =$ max $\{L(\mathbb{K}_1), L(\mathbb{K}_2)\}$:

Let $L(\mathbb{K}_1) = t \geq L(\mathbb{K}_2)$. Then there exist $e_i \in \mathbb{K}_i^*$ with $L(e_i) \leq t$ for $i = 1,2$. By the lemma, there exists $k \in N$ such that $e_i \rightarrow ((\underbrace{t,t,\ldots t}_{k}), (t + d, \underbrace{t,\ldots t}_{k-1}))$ for $i = 1,2$ where $d =$ l.c.m. $\{D(\mathbb{K}_1), D(\mathbb{K}_2)\}$. Thus $((\underbrace{t,t,\ldots t}_{k}), (t + d, \underbrace{t,\ldots t}_{k-1})) \in \mathbb{K}_1^* \cap \mathbb{K}_2^* = (\mathbb{K}_1 \vee \mathbb{K}_2)^*$. It follows that $L(\mathbb{K}_1 \vee \mathbb{K}_2) \leq t$. But $L(\mathbb{K}_1 \vee \mathbb{K}_2) \geq L(\mathbb{K}_1) = t$, thus $L(\mathbb{K}_1 \vee \mathbb{K}_2) = t =$ max $\{L(\mathbb{K}_1), L(\mathbb{K}_2)\}$. Thus we have the following

<u>Theorem 2.4</u>: D is a lattice homomorphism from $\mathcal{L} - \{E\}$ to N^+, and V, L and U are lattice homomorphisms from $\mathcal{L} - \{E\}$ to the non-negative integers with their usual order, where E is the class of all commutative semigroups.

HANGING THE MEAT ON THE BONES

Section 1. The Intervals $[\Omega_{r,s,n}, \Omega_{t,u,m}]$

Since for each equational class $\mathfrak{K}$ there exist $r,s,t,n \in N$ with $\Omega_{r,s,n} \subseteq \mathfrak{K} \subseteq \Omega_{r,t,n}$, it follows that the interval $[\Omega_{r,s,n}, \Omega_{r,s,pn}]$ is a jump for p prime, and $[\Omega_{r,s,n}, \Omega_{r+1,s,n}]$ is a jump for all $r \geq 0$. Thus $[\Omega_{r,s,1}, \Omega_{r,s,m}]$ consists of exactly the classes $\Omega_{r,s,n}$ where $n|m$, and $[\Omega_{0,s,n}, \Omega_{s,s,n}]$ consists of exactly the classes $\Omega_{r,s,n}$ where $r \leq s$. Moreover, if $\mathfrak{K} \subset \Omega_{r,r,n}$ then either $\mathfrak{K} \subseteq \Omega_{r-1,r,n}$ or $\mathfrak{K} \subseteq \Omega_{r,r,m}$ for some $m < n$. Thus $\Omega_{1,1,1}$, the class of all semilattices, is an atom in $\mathcal{L}$, and, for p prime, $\Omega_{0,1,p}$, the class of all abelian groups G satisfying $x^p = 1$ for all $x \in G$, is an atom in $\mathcal{L}$. $\{((1,1,0), (0,0,2))\}^*$, the class of all semigroups with constant multiplication, is also an atom in $\mathcal{L}$. Moreover, it is an easy consequence of the above remarks that this exhausts the set of atoms in $\mathcal{L}$, a result proved by Kalicki and Scott in [18].

It remains only to investigate intervals of the form $[\Omega_{r,s,n}, \Omega_{r,t,n}]$ where $s < t$. It is easy to see that every such interval is infinite: for each $p \geq t$, let $\mathfrak{K}_p = \Omega_{r,s,n} \vee \{e_{t,p}\}^*$ where $e_{t,p} = ((\underbrace{1,1,\ldots,1}_{p},0), (\underbrace{0,\ldots,0}_{p},t))$. Then $\mathfrak{K}_p \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$ and if $p \leq q$ then $\mathfrak{K}_p \subseteq \mathfrak{K}_q$. Moreover, if $p > r + s$, then

$$f_p = ((r,s,\underbrace{1,1,\ldots 1}_{p-r-s}),(r+n,s,\underbrace{1,1,\ldots 1}_{p-r-s})) \varepsilon\ \mathfrak{K}_p{}^* \text{ but } f_p \notin \mathfrak{K}_{p+1}{}^*$$

since $e_{t,p} \nrightarrow f_p$. Thus $\{\mathfrak{K}_p | p \geq t,\ p > r + s\}$ is an infinite chain in $[\Omega_{r,s,n}, \Omega_{r,t,n}]$.

The following lemma will be useful in the rest of this chapter:

<u>Lemma 3.1</u>: If $\Omega_{r,t,n} \subseteq \mathfrak{K}$ for some t then $e \varepsilon (\mathfrak{K} \wedge \Omega_{r,s,n})^*$ if and only if there exist $\tau_1,\tau_2,\tau_3,\tau_4 \varepsilon F(\omega)$ such that $e = (\tau_1,\tau_4)$, and (τ_1,τ_2), $(\tau_3,\tau_4) \varepsilon \mathfrak{K}^*$, $(\tau_2,\tau_3) \varepsilon \Omega_{r,s,n}{}^*$.

<u>Proof</u>: The "if" part is trivial. On the other hand, if $e \varepsilon (\mathfrak{K} \wedge \Omega_{r,s,n})^* = \mathfrak{K}^* \vee \Omega_{r,s,n}{}^*$ then, since for arbitrary congruence relations Θ_1, Θ_2, $\Theta_1 \vee \Theta_2 = \bigcup\{\Theta_1 \circ \Theta_2 \circ \Theta_1 \ldots \circ \Theta_1 | n \geq 1,$ n odd$\}$ and since $\mathfrak{K}^*$ and $\Omega_{r,s,n}{}^*$ are congruence relations on $F(\omega)$, it follows that there exists a finite sequence $\tau_1,\tau_2,\ldots\tau_{2p} \varepsilon F(\omega)$ such that $e = (\tau_1,\tau_{2p})$ and $(\tau_i,\tau_{i+1}) \varepsilon \begin{cases} \mathfrak{K}^* & \text{for } i \text{ odd} \\ \Omega_{r,s,n}{}^* & \text{for } i \text{ even} \end{cases}$.

We may assume without loss of generality that (τ_i,τ_{i+1}) is non-trivial for all $i \neq 1$ or $2p - 1$, and that $p \geq 2$. But then, by Theorem 2.1, $L((\tau_i,\tau_{i+1})) \geq s$ for even i, i.e., for all i with $2 \leq i \leq 2p - 1$, τ_i has an entry $\geq s$. But then for all odd i with $3 \leq i \leq 2p - 3$, $V((\tau_i,\tau_{i+1})) \geq r$, $L((\tau_i,\tau_{i+1})) \geq s$, and

$n \mid D((\tau_i,\tau_{i+1}))$; thus $(\tau_i,\tau_{i+1}) \in \Omega_{r,s,n}{}^*$. It follows that $(\tau_2,\tau_{2p-1}) \in \Omega_{r,s,n}{}^*$. Thus we may take $\tau_1,\tau_2,\tau_{2p-1},\tau_{2p}$ for the four elements of $F(\omega)$ in the theorem statement.

Theorem 3.1: If $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$ and if $u \leq r$ then $\mathfrak{K} = (\mathfrak{K} \wedge \Omega_{u,t,n}) \vee \Omega_{r,s,n}$.

Proof: Since $\mathfrak{K} \wedge \Omega_{u,t,n} \subseteq \mathfrak{K}$ and $\Omega_{r,s,n} \subseteq \mathfrak{K}$ it follows that $(\mathfrak{K} \wedge \Omega_{u,t,n}) \vee \Omega_{r,s,n} \subseteq \mathfrak{K}$. Thus it is enough to show that if $e \in ((\mathfrak{K} \wedge \Omega_{u,t,n}) \vee \Omega_{r,s,n})^*$ then $e \in \mathfrak{K}^*$.

Assume $e \in ((\mathfrak{K} \wedge \Omega_{u,t,n}) \vee \Omega_{r,s,n})^* = (\mathfrak{K} \wedge \Omega_{u,t,n})^* \cap \Omega_{r,s,n}{}^*$. Then by Lemma 3.1, there exist $\tau_1,\tau_2,\tau_3,\tau_4 \in F(\omega)$ such that $(\tau_1,\tau_2), (\tau_3,\tau_4) \in \mathfrak{K}^*$, $(\tau_2,\tau_3) \in \Omega_{u,t,n}{}^*$ and $e = (\tau_1,\tau_4) \in \Omega_{r,s,n}{}^*$. But $\mathfrak{K}^* \subseteq \Omega_{r,s,n}{}^*$, thus $\{(\tau_1,\tau_4), (\tau_1,\tau_2), (\tau_3,\tau_4)\} \subseteq \Omega_{r,s,n}{}^*$. Since $\{(\tau_1,\tau_4), (\tau_1,\tau_2), (\tau_3,\tau_4)\} \to (\tau_2,\tau_3)$, it follows that $(\tau_2,\tau_3) \in \Omega_{r,s,n}{}^*$. Since $(\tau_2,\tau_3) \in \Omega_{u,t,n}{}^*$, we have that $(\tau_2,\tau_3) \in \Omega_{r,s,n}{}^* \cap \Omega_{u,t,n}{}^* = \Omega_{r,t,n}{}^*$. But $\Omega_{r,t,n}{}^* \subseteq \mathfrak{K}^*$, thus it follows that $e \in \mathfrak{K}^*$.

Theorem 3.2: If $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$ and if $m \mid n$ then $\mathfrak{K} = (\mathfrak{K} \wedge \Omega_{r,t,m}) \vee \Omega_{r,s,n}$.

Proof: By changing $\Omega_{u,t,n}$ to $\Omega_{r,t,m}$ in the proof of the last theorem and using the fact that $\Omega_{r,t,m}{}^* \cap \Omega_{r,s,n}{}^* = \Omega_{r,t,n}{}^*$, we obtain a proof of the theorem.

Corollary: If $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$ and $m|n$ then $\mathfrak{K} = (\mathfrak{K} \wedge \Omega_{r,t,m}) \vee \Omega_{0,1,n}$.

Proof: By the theorem, $\mathfrak{K} = (\mathfrak{K} \wedge \Omega_{r,t,m}) \vee \Omega_{r,s,n}$. But $\Omega_{r,s,m} \subseteq \mathfrak{K} \wedge \Omega_{r,t,m}$, and $\Omega_{r,s,m} \vee \Omega_{0,1,n} = \Omega_{r,s,n}$; this yields the desired result.

Section 2. The Sublattice $\mathcal{L}_n$ with Constant D

For $n \in N$, let $\mathcal{L}_n = \{\mathfrak{K} \in \mathcal{L} \mid D(\mathfrak{K}) = n\}$, and for each non-negative integer k, let $\mathcal{L}_{n,k} = \{\mathfrak{K} \in \mathcal{L}_n \mid V(\mathfrak{K}) = k\}$. Then the $\mathcal{L}_{n,k}$'s are pairwise disjoint, and $\mathcal{L}_n = \bigcup_{k \geq 0} \mathcal{L}_{n,k}$.

For $p \leq q$, define a mapping $\delta_{p,q,n} : \mathcal{L}_{n,q} \longrightarrow \mathcal{L}_{n,p}$ as follows: for $\mathfrak{K} \in \mathcal{L}_{n,q}$ with $U(\mathfrak{K}) = u$, $\delta_{p,q,n}(\mathfrak{K}) = \mathfrak{K} \wedge \Omega_{p,u,n}$. If $u \leq s$, then, since $\mathfrak{K} \subseteq \Omega_{q,u,n}$, $\mathfrak{K} \wedge \Omega_{p,s,n} = \mathfrak{K} \wedge \Omega_{q,u,n} \wedge \Omega_{p,s,n} = \mathfrak{K} \wedge \Omega_{p,u,n} = \delta_{p,q,n}(\mathfrak{K})$. It follows from this that $\delta_{p,q,n}$ is a meet homomorphism. Clearly if $p < q < r$ then $\delta_{p,r,n} = \delta_{p,q,n} \circ \delta_{q,r,n}$. By Theorem 3.1, if $\mathfrak{K} \in \mathcal{L}_{n,q}$ and $p < q$ then $\delta_{p,q,n}(\mathfrak{K}) \vee \Omega_{q,q,n} = \mathfrak{K}$, thus $\delta_{p,q,n}$ is one-to-one.

To see that $\delta_{p,q,n}$ is a lattice monomorphism it remains only to show that it preserves joins.

Let $\mathbb{R}_1, \mathbb{R}_2 \in \mathcal{L}_{n,q}$. Then $\Omega_{q,q,n} \subseteq \mathbb{R}_1, \mathbb{R}_2$, and there exists $u > q$ such that $\mathbb{R}_1, \mathbb{R}_2 \subseteq \Omega_{q,u,n}$. Then $\delta_{p,q,n}(\mathbb{R}_1 \vee \mathbb{R}_2) = (\mathbb{R}_1 \vee \mathbb{R}_2) \wedge \Omega_{p,u,n} \supseteq (\mathbb{R}_1 \wedge \Omega_{p,u,n}) \vee (\mathbb{R}_2 \wedge \Omega_{p,u,n})$ $= \delta_{p,q,n}(\mathbb{R}_1) \vee \delta_{p,q,n}(\mathbb{R}_2)$. On the other hand, if $e \in$ $(\delta_{p,q,n}(\mathbb{R}_1) \vee \delta_{p,q,n}(\mathbb{R}_2))^* = (\mathbb{R}_1{}^* \vee \Omega_{p,u,n}{}^*) \wedge (\mathbb{R}_2{}^* \vee \Omega_{p,u,n}{}^*)$, then, by Lemma 3.1, there exist $\tau_i \in F(\omega)$ for $1 \leq i \leq 6$ such that $e = (\tau_1, \tau_6)$ and $(\tau_1,\tau_2), (\tau_3,\tau_6) \in \mathbb{R}_1{}^*$, $(\tau_1,\tau_4),(\tau_5,\tau_6) \in \mathbb{R}_2{}^*$ and $(\tau_2,\tau_3), (\tau_4,\tau_5) \in \Omega_{p,u,n}{}^*$. If both (τ_2,τ_3) and (τ_4,τ_5) are non-trivial then τ_2,τ_3,τ_4 and τ_5 all contain an entry $\geq u$. But $(\tau_1,\tau_2) \in \mathbb{R}_1{}^* \subseteq \Omega_{q,q,n}{}^*$, $(\tau_1,\tau_4) \in \mathbb{R}_2{}^* \subseteq \Omega_{q,q,n}{}^*$ implies that $(\tau_2,\tau_4) \in \Omega_{q,q,n}{}^*$; it follows that $(\tau_2,\tau_4) \in \Omega_{q,u,n}{}^* \subseteq \mathbb{R}_1{}^*$. Thus $(\tau_1,\tau_4) \in \mathbb{R}_1{}^*$. Similarly $(\tau_5,\tau_6) \in \mathbb{R}_1{}^*$. Thus $e \in (\mathbb{R}_1{}^* \cap \mathbb{R}_2{}^*) \vee \Omega_{p,u,n}{}^*$. If (τ_2,τ_3) is trivial then $e = (\tau_1,\tau_6) \in \mathbb{R}_1{}^*$. Thus (τ_1,τ_4), (τ_5,τ_6) and $(\tau_1,\tau_6) \in \Omega_{q,q,n}{}^*$. Since $\{(\tau_1,\tau_4), (\tau_5,\tau_6), (\tau_1,\tau_6)\} \to (\tau_4,\tau_5)$, we have $(\tau_4,\tau_5) \in \Omega_{q,q,n}{}^*$. Thus $(\tau_4,\tau_5) \in \Omega_{q,q,n}{}^* \cap \Omega_{p,u,n}{}^* = \Omega_{q,u,n}{}^* \subseteq \mathbb{R}_2{}^*$. It follows that $e \in \mathbb{R}_1{}^* \cap \mathbb{R}_2{}^*$. Similarly if (τ_4,τ_5) is trivial then $e \in \mathbb{R}_1{}^* \cap \mathbb{R}_2{}^*$. Thus in any case, $e \in (\mathbb{R}_1{}^* \cap \mathbb{R}_2{}^*) \vee \Omega_{p,u,n}{}^* = \delta_{p,q,n}(\mathbb{R}_1 \vee \mathbb{R}_2)$. It follows that $\delta_{p,q,n}(\mathbb{R}_1) \vee \delta_{p,q,n}(\mathbb{R}_2) = \delta_{p,q,n}(\mathbb{R}_1 \vee \mathbb{R}_2)$.

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Thus, for $p < q$, $\delta_{p,q,n}$ is a lattice monomorphism of $\mathcal{L}_{n,q}$ into $\mathcal{L}_{n,p}$ with the property that $\delta_{p,q,n}(\mathbb{R}) \vee \Omega_{q,q,n} = \mathbb{R}$.

<u>Theorem 3.3</u>: For $\mathbb{R}_1, \mathbb{R}_2 \in \mathcal{L}_n$, the following are equivalent:

(1) $\mathbb{R}_1 \subseteq \mathbb{R}_2$

(2) $V(\mathbb{R}_1) \leq V(\mathbb{R}_2)$ and $\mathbb{R}_1 \subseteq \delta_{V(\mathbb{R}_1),V(\mathbb{R}_2),n}(\mathbb{R}_2)$

(3) $V(\mathbb{R}_1) \leq V(\mathbb{R}_2)$ and $\delta_{0,V(\mathbb{R}_1),n}(\mathbb{R}_1) \subseteq \delta_{0,V(\mathbb{R}_2),n}(\mathbb{R}_2)$

<u>Proof</u>: (1) $\Rightarrow$ (2) : Assume $\mathbb{R}_1 \subseteq \mathbb{R}_2$. Then $V(\mathbb{R}_1) \leq V(\mathbb{R}_2)$, and, if $u = \text{maximum } \{U(\mathbb{R}_1), U(\mathbb{R}_2)\}$ then

$$\mathbb{R}_1 = \mathbb{R}_1 \wedge \Omega_{V(\mathbb{R}_1),u,n} \subseteq \mathbb{R}_2 \wedge \Omega_{V(\mathbb{R}_1),u,n} = \delta_{V(\mathbb{R}_1),V(\mathbb{R}_2),n}(\mathbb{R}_2).$$

(2) $\Rightarrow$ (3) : If $\mathbb{R}_1 \subseteq \delta_{V(\mathbb{R}_1),V(\mathbb{R}_2),n}(\mathbb{R}_2)$ then

$$\delta_{0,V(\mathbb{R}_1),n}(\mathbb{R}_1) \subseteq \delta_{0,V(\mathbb{R}_1),n}(\delta_{V(\mathbb{R}_1),V(\mathbb{R}_2),n}(\mathbb{R}_2)) = \delta_{0,V(\mathbb{R}_2),n}(\mathbb{R}_2).$$

(3) $\Rightarrow$ (1) : Assume (3) holds. Then

$$\mathbb{R}_1 = \delta_{0,V(\mathbb{R}_1),n}(\mathbb{R}_1) \vee \Omega_{V(\mathbb{R}_1),V(\mathbb{R}_1),n} \subseteq \delta_{0,V(\mathbb{R}_2),n}(\mathbb{R}_2) \vee \Omega_{V(\mathbb{R}_1),V(\mathbb{R}_1),n}$$

$$\subseteq \delta_{0,V(\mathbb{R}_2),n}(\mathbb{R}_2) \vee \Omega_{V(\mathbb{R}_2),V(\mathbb{R}_2),n} = \mathbb{R}_2.$$

<u>Theorem 3.4</u>: The mapping $\mathbb{R} \rightsquigarrow (\delta_{0,V(\mathbb{R}),n}(\mathbb{R}), V(\mathbb{R}))$ is an embedding of $\mathcal{L}_n$ as a meet subsemilattice into $\mathcal{L}_{n,0} \times I^+$, where I^+ is the lattice of non-negative integers with their usual order.

Proof: Since $V(\mathbb{R}_1 \wedge \mathbb{R}_2) = \min \{V(\mathbb{R}_1), V(\mathbb{R}_2)\}$ and since the $\delta_{0,p,n}$'s are one-to-one, it is enough to show that if $\mathbb{R}_1, \mathbb{R}_2 \in \mathcal{L}_n$ then $\delta_{0,V(\mathbb{R}_1),n}(\mathbb{R}_1) \wedge \delta_{0,V(\mathbb{R}_2),n}(\mathbb{R}_2) = \delta_{0,V(\mathbb{R}_1 \wedge \mathbb{R}_2),n}(\mathbb{R}_1 \wedge \mathbb{R}_2)$. Assume $\mathbb{R}_1, \mathbb{R}_2 \in \mathcal{L}_n$ and let $u = \text{maximum } \{U(\mathbb{R}_1), U(\mathbb{R}_2)\}$. Then $\delta_{0,V(\mathbb{R}_1),n}(\mathbb{R}_1) \wedge \delta_{0,V(\mathbb{R}_2),n}(\mathbb{R}_2) = (\mathbb{R}_1 \wedge \Omega_{0,u,n}) \wedge (\mathbb{R}_2 \wedge \Omega_{0,u,n}) = (\mathbb{R}_1 \wedge \mathbb{R}_2) \wedge \Omega_{0,u,n} = \delta_{0,V(\mathbb{R}_1 \wedge \mathbb{R}_2),n}(\mathbb{R}_1 \wedge \mathbb{R}_2)$, and this completes the proof.

It will be shown in the next section that this embedding is not a lattice embedding, i.e., that it does not preserve joins.

Section 3. A Mapping Between Intervals of the Lattice

If r,s,t,u,n are non-negative integers such that $r < s \leq t < u$ and $n \geq 1$ then, since $\Omega_{s,t,n} \wedge \Omega_{r,u,n} = \Omega_{r,t,n}$ and $\Omega_{s,u,n} \wedge \Omega_{r,u,n} = \Omega_{r,u,n}$, it follows that the restriction of $\delta_{r,s,n}$ to $[\Omega_{s,t,n}, \Omega_{s,u,n}]$ is a lattice monomorphism mapping into $[\Omega_{r,t,n}, \Omega_{r,u,n}]$. Let

$\phi_{r,s,t,u,n} : [\Omega_{s,t,n}, \Omega_{s,u,n}] \longrightarrow [\Omega_{r,t,n}, \Omega_{r,u,n}]$ be the restriction of $\delta_{r,s,n}$. We will investigate which of the $\phi_{r,s,t,u,n}$'s are actually isomorphisms, i.e., for which values of r,s,t,u,n the image of $\phi_{r,s,t,u,n}$ is the whole interval $[\Omega_{r,t,n}, \Omega_{r,u,n}]$.

<u>Lemma 3.2</u>: $\phi_{0,1,t,u,n}$ maps onto $[\Omega_{0,t,n}, \Omega_{0,u,n}]$ for all $t,u,n \geq 1$.

<u>Proof</u>: Let $\mathfrak{K} \varepsilon [\Omega_{0,t,n}, \Omega_{0,u,n}]$. It is enough to show that for each non-trivial $e \varepsilon \mathfrak{K}^*$, there exists $\Sigma_e \subseteq \Omega_{1,t,n}{}^*$ such that $\{e\} \cup \Omega_{0,u,n}{}^* \leftrightarrow \Sigma_e \cup \Omega_{0,u,n}{}^*$, for then

$\mathfrak{K} = \phi_{0,1,t,u,n}((\bigcup_{e\varepsilon\mathfrak{K}^*} \Sigma_e)^* \wedge \Omega_{1,u,n})$.

Let $e \varepsilon \mathfrak{K}^*$ be non-trivial. Then $L(e) \geq t$ and $n|D(e)$. If $V(e) \geq 1$ then we can take $\Sigma_e = \{e\}$. If $V(e) = 0$, then we may assume without loss of generality that $e = ((u_i)_{i\varepsilon N}, (v_i)_{i\varepsilon N})$ where $u_1 = 0$, $v_1 > 0$ (and then $n|v_1$) and $u_2 > 0$. Then let

$\Sigma_e = \{((v_i)_{i\varepsilon N}, (2v_1, v_2, \ldots)), ((u_2, u_3, \ldots), (u_2 + v_1, u_3, \ldots))\}$.

Clearly $\Sigma_e \subseteq \Omega_{1,t,n}{}^*$ and $e \to \Sigma_e$. It remains only to show that $\Sigma_e \cup \Omega_{0,u,n}{}^* \to e$. But

$$((v_i)_{i\varepsilon N}, (uv_1, v_2, v_3, \ldots)) \varepsilon \Gamma\Sigma_e$$

$$((uv_1, v_2, v_3, \ldots), (0, u_2 + uv_1, u_3, \ldots)) \varepsilon \Omega_{0,u,n}{}^*$$

$$((0, u_2 + uv_1, u_3, \ldots), (0, u_2, u_3, \ldots)) \varepsilon \Gamma\Sigma_e.$$

Thus $\Sigma_e \cup \Omega_{0,u,n}{}^* \to e$. This completes the proof.

<u>Corollary</u>: $\delta_{0,1,n}$ is an isomorphism of $\mathcal{L}_{n,1}$ onto $\mathcal{L}_{n,0}$ for each $n \varepsilon N$.

Lemma 3.3: If $r > 0$ and $r + n < u$ then $\phi_{r,s,t,u,n}$ does not map onto $[\Omega_{r,t,n}, \Omega_{r,u,n}]$.

Proof: Let $e = ((r, r+n, t), (r+n, r, t))$, and let $\mathfrak{R} = e^* \wedge \Omega_{r,u,n}$. Then $\mathfrak{R} \in [\Omega_{r,t,n}, \Omega_{r,u,n}]$. If $\mathfrak{R} = \phi_{r,s,t,u,n}(\mathfrak{R}')$ for some $\mathfrak{R}' \in [\Omega_{s,t,n}, \Omega_{s,u,n}]$ then by Theorem 3.1, $\mathfrak{R} \vee \Omega_{s,t,n} = (\mathfrak{R}' \wedge \Omega_{r,u,n}) \vee \Omega_{s,t,n} = \mathfrak{R}'$, and this imples that $\mathfrak{R} = \phi_{r,s,t,u,n}(\mathfrak{R} \vee \Omega_{s,t,n}) = (\mathfrak{R} \vee \Omega_{s,t,n}) \wedge \Omega_{r,u,n}$. Thus to prove the lemma, it is enough to show that $\mathfrak{R} \neq (\mathfrak{R} \vee \Omega_{s,t,n}) \wedge \Omega_{r,u,n}$.

Since $e \in \mathfrak{R}^*$, it is enough to show that $e \notin ((\mathfrak{R} \vee \Omega_{s,t,n}) \wedge \Omega_{r,u,n})^* = (\mathfrak{R}^* \cap \Omega_{s,t,n}^*) \vee \Omega_{r,u,n}^*$. Suppose $e \in (\mathfrak{R}^* \cap \Omega_{s,t,n}^*) \vee \Omega_{r,u,n}^*$. Then there exist $\tau_1, \tau_2 \in F(\omega)$ such that $((r, r+n, t), \tau_1), (\tau_2, (r+n, r, t)) \in \mathfrak{R}^* \cap \Omega_{s,t,n}^*$ and $(\tau_1, \tau_2) \in \Omega_{r,u,n}^*$.

Now $((r, r+n, t), \tau_1) \in \mathfrak{R}^* = \Gamma e \vee \Omega_{r,u,n}^*$ implies that there exist $\tau_3, \tau_4 \in F(\omega)$ such that $((r, r+n, t), \tau_3) \in \Gamma e$, $(\tau_3, \tau_4) \in \Omega_{r,u,n}^*$ and $(\tau_4, \tau_1) \in \Gamma e$. But $((r, r+n, t), \tau_3) \in \Gamma e$ implies that $\tau_3 = (r, r+n, t)$ or $(r+n, r, t)$ in the case $r + n \neq t$, and that $\tau_3 = (r, r+n, r+n)$, $(r+n, r+n, r)$ or $(r+n, r, r+n)$ in the case $r + n = t$. In any case, since $r + n < u$ and $(\tau_3, \tau_4) \in \Omega_{r,u,n}^*$, it follows that $\tau_3 = \tau_4$. Thus $((r, r+n, t), \tau_1) \in \Gamma e$. A repetition of this argument yields $\tau_1 = \tau_2$. Thus $((r, r+n, t), (r+n, r, t)) \in \Omega_{s,t,n}^*$. But this is a contradiction since $r < s$. This completes the proof.

Section 4. Restriction of the mapping to Schwabauer classes

An equational class is called a Schwabauer class, or S-class, if it can be defined by equations of the form $((u_i)_{i\varepsilon N},(v_i)_{i\varepsilon N})$ where $u_i \leq v_i$ for all $i \varepsilon N$. Clearly all the $\Omega_{r,s,n}$'s are S-classes. The set of all S-classes forms a distributive sublattice $\mathcal{S}$ of the lattice of equational classes of commutative semigroups (Schwabauer, [25]) ; this will be proved in Section 9.

Lemma 3.4: If $r + n \geq t$ then $[\Omega_{r,s,n}, \Omega_{r,t,n}] \subseteq \mathcal{S}$.

Proof: Let $\mathfrak{K} \varepsilon [\Omega_{r,s,n}, \Omega_{r,t,n}]$ where $r + n \geq t$. To prove that $\mathfrak{K}$ is an S-class, it is enough to prove that every $e \varepsilon \mathfrak{K}^*$ with $e \notin \Omega_{r,t,n}{}^*$ is equivalent with an equation of the form $((u_i)_{i\varepsilon N}, (v_i)_{i\varepsilon N})$ where $u_i \leq v_i$ for all $i \varepsilon N$.

Assume $e \varepsilon \mathfrak{K}^*$ and $e \notin \Omega_{r,t,n}{}^*$. Since $\mathfrak{K}^* \subseteq \Omega_{r,s,n}{}^*$, $e = ((u_i)_{i\varepsilon N}, (v_i)_{i\varepsilon N})$ where $u_i \neq v_i$ implies $u_i, v_i \geq r$ and $n | u_i - v_i$. Since $e \notin \Omega_{r,t,n}{}^*$ we may assume without loss of generality that $u_i < t$ for all $i \varepsilon N$. But then, if $u_i > v_i$ for some i, it follows that $u_i = v_i + kn$ where $k \geq 1$ and $v_i \geq r$. But then $u_i \geq v_i + n \geq r + n \geq t$ and this is a contradiction. Thus $u_i \leq v_i$ for all $i \varepsilon N$.

For $t < u$, let $T_{t,u} = \{(u_i)_{i\varepsilon N} \varepsilon F(\omega) | u_i < u$ for all

$i,\ u_1 \geq t\}$. For $T \subseteq T_{t,u}$, and $n \geq 1$, let $T(n) = \{((u_i)_{i\varepsilon N},\ (u_1 + n,\ u_2,\ u_3,\ldots)) \mid (u_i)_{i\varepsilon N} \ \varepsilon\ T\}$. Then $T(n) \subseteq \Omega_{t,t,n}{}^*$, since if $(u_i)_{i\varepsilon N} \ \varepsilon\ T_{t,u}$ then $u_1 \geq t$, thus $((t),\ (t + n)) \rightarrow ((u_1),\ (u_1 + n)) \rightarrow ((u_i)_{i\varepsilon N},\ (u_1 + n,\ u_2,\ u_3,\ldots))$.

<u>Lemma 3.5</u>: $\mathfrak{K} \ \varepsilon\ [\Omega_{r,s,n},\ \Omega_{r,t,n}]$ is an S-class if and only if there exists $T \subseteq T_{s,t}$ such that $\mathfrak{K} = T(n)^* \wedge \Omega_{r,t,n}$.

<u>Proof</u>: Clearly if $\mathfrak{K} = T(n)^* \wedge \Omega_{r,t,n}$ for some $T \subseteq T_{s,t}$ then $\mathfrak{K}$ is an S-class.

On the other hand, if $\mathfrak{K}$ is an S-class there exists a set Σ of equations of the form $((u_i)_{i\varepsilon N},\ (v_i)_{i\varepsilon N})$ where $u_i \leq v_i$ for all $i \ \varepsilon\ N$ such that $\mathfrak{K} = \Sigma^*$. It is enough to show that for each $e \ \varepsilon\ \Sigma$ with $e \notin \Omega_{r,t,n}{}^*$ there exists $\bar{e} \ \varepsilon\ T_{s,t}(n)$ such that $\{e\} \cup \Omega_{r,t,n}{}^* \leftrightarrow \{\bar{e}\} \cup \Omega_{r,t,n}{}^*$, for then $\mathfrak{K} = \{\bar{e} \mid e \ \varepsilon\ \Sigma - \Omega_{r,t,n}{}^*\}^* \wedge \Omega_{r,t,n}$.

Let $e = ((u_i)_{i\varepsilon N},\ (v_i)_{i\varepsilon N}) \ \varepsilon\ \Sigma - \Omega_{r,t,n}{}^*$.

Since $\Sigma \subseteq \mathfrak{K}^* \subseteq \Omega_{r,s,n}{}^*$, there exists $j \ \varepsilon\ N$ with $u_j \geq s$, and if $u_i < v_i$ then $n \mid u_i - v_i$. Since $e \notin \Omega_{r,t,n}{}^*$, $u_i < t$ for all $i \ \varepsilon\ N$. Thus $e = ((u_i)_{i\varepsilon N},\ (u_i + k_i n)_{i\varepsilon N})$ where we may assume without loss of generality that $u_1 \geq s$. Let $\bar{e} = ((u_i)_{i\varepsilon N},\ (u_1 + n,\ u_2,\ u_3,\ldots))$. Then $\bar{e} \ \varepsilon\ T_{s,t}(n)$. Now $k_j \geq 1$ for some $j \ \varepsilon\ N$. Choose $k \ \varepsilon\ N$ so that $u_j + kk_j \geq t$. Then

$((u_i)_{i\epsilon N}, (u_i + kk_i n)_{i\epsilon N}) \ \epsilon\ \Gamma e$

$((u_i + kk_i n)_{i\epsilon N}, (u_1 + n + kk_1 n, u_2 + kk_2 n, u_3 + kk_3 n, \ldots)) \ \epsilon\ \Omega_{r,t,n}{}^*$

$((u_1 + n + kk_1 n, u_2 + kk_2 n, u_3 + kk_3 n, \ldots), (u_1 + n, u_2, u_3, \ldots)) \ \epsilon\ \Gamma e.$

Thus $\{e\} \cup \Omega_{r,t,n}{}^* \rightarrow \bar{e}$.

On the other hand, if $h \ \epsilon\ N$ is chosen so that $u_1 + hn \geq t$, then $((u_i)_{i\epsilon N}, (u_1 + hn, u_2, u_3, \ldots)) \ \epsilon\ \Gamma\bar{e}$

$((u_1 + hn, u_2, u_3, \ldots), (u_1 + hn + k_1 n, u_2 + k_2 n, u_3 + k_3 n, \ldots)) \ \epsilon\ \Omega_{r,t,n}{}^*$

$((u_1 + hn + k_1 n, u_2 + k_2 n, u_3 + k_3 n, \ldots), (u_i + k_i n)_{i\epsilon N}) \ \epsilon\ \Gamma\bar{e}$.

Thus $\{\bar{e}\} \cup \Omega_{r,t,n}{}^* \rightarrow e$. This completes the proof.

Corollary 1: $\phi_{r,s,t,u,n}$ restricted to $[\Omega_{s,t,n}, \Omega_{s,u,n}] \cap \gamma$ is an isomorphism of $[\Omega_{s,t,n}, \Omega_{s,u,n}] \cap \gamma$ onto $[\Omega_{r,t,n}, \Omega_{r,u,n}] \cap \gamma$.

Proof: It is an immediate consequence of the definition of $\phi_{r,s,t,u,n}$ that it maps S-classes to S-classes. We already know that $\phi_{r,s,t,u,n}$ is a lattice monomorphism, thus it is enough to show that for every S-class in $[\Omega_{r,t,n}, \Omega_{r,u,n}]$ there is an S-class in $[\Omega_{s,t,n}, \Omega_{s,u,n}]$ that maps onto it under $\phi_{r,s,t,u,n}$.

Let $\mathcal{R} \in [\Omega_{r,t,n}, \Omega_{r,u,n}] \cap \mathcal{Y}$. By the lemma, there exists $T \subseteq T_{t,u}$ such that $\mathcal{R} = T(n)^* \wedge \Omega_{r,u,n}$. But then $T(n)^* \wedge \Omega_{s,u,n} \in [\Omega_{s,t,n}, \Omega_{s,u,n}] \cap \mathcal{Y}$ and $\phi_{r,s,t,u,n}(T(n)^* \wedge \Omega_{s,u,n}) = T(n)^* \wedge \Omega_{s,u,n} \wedge \Omega_{r,u,n} = \mathcal{R}$. This completes the proof.

<u>Corollary 2</u>: If $r + n \geq u$ then $\phi_{r,s,t,u,n}$ is an isomorphism of $[\Omega_{s,t,n}, \Omega_{s,u,n}]$ onto $[\Omega_{r,t,n}, \Omega_{r,u,n}]$.

In view of Lemma 3.3 and the last corollary, $\phi_{r,s,t,u,n}$ maps onto $[\Omega_{r,t,n}, \Omega_{r,u,n}]$ for $r > 0$ if and only if $r + n \geq u$. For $1 < s \leq t < u$, since $\phi_{0,s,t,u,n} = \phi_{0,1,t,u,n} \circ \phi_{1,s,t,u,n}$, and since $\phi_{0,1,t,u,n}$ is an isomorphism by Lemma 3.2, it follows that $\phi_{0,s,t,u,n}$ maps onto $[\Omega_{0,t,n}, \Omega_{0,u,n}]$ if and only if $n + 1 \geq u$.

Thus $\phi_{r,s,t,u,n}$ maps onto $[\Omega_{r,t,n}, \Omega_{r,u,n}]$ if and only if $r = 0$ and $s = 1$, or $r = 0$ and $n + 1 \geq u$, or $r > 0$ and $r + n \geq u$.

From this we see that the embedding of $\mathcal{L}_n$ into $\mathcal{L}_{n,0} \times I^+$ in the last section does not preserve joins: Let $n \geq 1$. Let $p > n$. Then $\phi_{0,p,p,p+1,n}$ does not map onto $[\Omega_{0,p,n}, \Omega_{0,p+1,n}]$. Let

$\mathbb{R} \in [\Omega_{0,p,n}, \Omega_{0,p+1,n}]$ such that $\mathbb{R}$ is not in the image of $\phi_{0,p,p,p+1,n}$. Let $\mathbb{R}' = \Omega_{p,p,n}$. If the above-mentioned embedding preserved joins, then we would have

$$\delta_{0,V(\mathbb{R}),n}(\mathbb{R}) \vee \delta_{0,V(\mathbb{R}'),n}(\mathbb{R}') = \delta_{0,V(\mathbb{R} \vee \mathbb{R}'),n}(\mathbb{R} \vee \mathbb{R}').$$

But $\delta_{0,V(\mathbb{R}),n}(\mathbb{R}) = \delta_{0,0,n}(\mathbb{R}) = \mathbb{R}$, $\delta_{0,V(\mathbb{R}'),n}(\mathbb{R}') = \Omega_{p,p,n} \wedge \Omega_{0,p,n} = \Omega_{0,p,n}$, and $\Omega_{0,p,n} \vee \mathbb{R} = \mathbb{R}$. Thus we would have $\mathbb{R} = (\mathbb{R} \vee \mathbb{R}') \wedge \Omega_{0,p+1,n}$ and this would imply that $\mathbb{R}$ is in the image of $\phi_{0,p,p,p+1,n}$. Thus the embedding does not preserve joins.

Lemma 3.6: For all $n \geq 1$, both $[\Omega_{0,1,n}, \Omega_{0,2,n}]$ and $[\Omega_{1,1,n}, \Omega_{1,2,n}]$ are isomorphic to $\omega + 1$, i.e., to a countable ascending chain with unit adjoined.

Proof: The proof follows immediately from Lemmas 3.2, 3.4 and 3.5, and the fact that $T_{1,2} = \{(\underbrace{1,1,\ldots,1}_{m},0,0,\ldots) \mid m \geq 1\}$.

Section 5. The Relationship Between $\mathcal{L}_n$ and $\mathcal{L}_m$

For $n \neq m$, $\mathcal{L}_n$ and $\mathcal{L}_m$ are disjoint. $\mathcal{L} = \bigcup_{n \in N} \mathcal{L}_n \cup \{E\}$. If $n|m$, we define a mapping $\beta_{n,m} : \mathcal{L}_m \longrightarrow \mathcal{L}_n$ as follows: for $\mathbb{R} \in \mathcal{L}_m$ with $V(\mathbb{R}) = r$, $U(\mathbb{R}) = s$, $\beta_{n,m}(\mathbb{R}) = \mathbb{R} \wedge \Omega_{r,s,n}$. Then for any t,u with $\mathbb{R} \subseteq \Omega_{t,u,m}$ we have that $\mathbb{R} \wedge \Omega_{t,u,n} = (\mathbb{R} \wedge \Omega_{r,s,m}) \wedge \Omega_{t,u,n} = \mathbb{R} \wedge \Omega_{r,s,n} = \beta_{n,m}(\mathbb{R})$. It follows from this that $\beta_{n,m}$ is a meet-homomorphism. Moreover, by the corollary to Theorem 3.2, if $\mathcal{R} \in \mathcal{L}_m$ then $\mathcal{R} = \beta_{n,m}(\mathcal{R}) \vee \Omega_{0,1,m}$, thus $\beta_{n,m}$ is one-to-one. Thus, to show that $\beta_{n,m}$ is a lattice monomorphism of $\mathcal{L}_m$ into $\mathcal{L}_n$, it suffices to show that it preserves joins.

Let $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{L}_m$. We may assume without loss of generality that $r = V(\mathcal{R}_1) \leq V(\mathcal{R}_2) = s$. Let $u = \max\{U(\mathcal{R}_1), U(\mathcal{R}_2)\}$. Then $\beta_{n,m}(\mathcal{R}_1 \vee \mathcal{R}_2) = (\mathcal{R}_1 \vee \mathcal{R}_2) \wedge \Omega_{s,u,n} \supseteq (\mathcal{R}_1 \wedge \Omega_{r,u,n}) \vee (\mathcal{R}_2 \wedge \Omega_{s,u,n}) = \beta_{n,m}(\mathcal{R}_1) \vee \beta_{n,m}(\mathcal{R}_2)$. On the other hand, if $e \in (\beta_{n,m}(\mathcal{R}_1) \vee \beta_{n,m}(\mathcal{R}_2))^* = (\mathcal{R}_1^* \vee \Omega_{r,u,n}^*) \cap (\mathcal{R}_2^* \vee \Omega_{s,u,n}^*)$, then there exist $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6 \in F(\omega)$ such that $(\tau_1, \tau_2), (\tau_3, \tau_6) \in \mathcal{R}_1^*$, $(\tau_2, \tau_3) \in \Omega_{r,u,n}^*$, $(\tau_1, \tau_4), (\tau_5, \tau_6) \in \mathcal{R}_2^*$, $(\tau_4, \tau_5) \in \Omega_{s,u,n}^*$ and $e = (\tau_1, \tau_6)$. If both (τ_2, τ_3) and (τ_4, τ_5) are non-trivial, then each of $\tau_2, \tau_3, \tau_4, \tau_5$ contain an entry $\geq u$. $(\tau_1, \tau_2) \in \mathcal{R}_1^* \subseteq \Omega_{r,r,m}^*$, $(\tau_1, \tau_4) \in \mathcal{R}_2^* \subseteq \Omega_{r,r,m}^*$, thus $(\tau_2, \tau_4) \in \Omega_{r,r,m}^*$. Since both τ_2 and τ_4 contain an entry $\geq u$, it follows that $(\tau_2, \tau_4) \in \Omega_{r,u,m}^* \subseteq \mathcal{R}_1^*$. Thus $(\tau_1, \tau_4) \in \mathcal{R}_1^*$. Similarly $(\tau_5, \tau_6) \in \mathcal{R}_1^*$, hence $(\tau_1, \tau_6) \in (\mathcal{R}_1^* \cap \mathcal{R}_2^*) \vee \Omega_{s,u,n}^*$. If (τ_2, τ_3) is trivial, then $(\tau_1, \tau_6) \in \mathcal{R}_1^* \subseteq \Omega_{r,r,m}^*$. Since (τ_1, τ_4) and $(\tau_5, \tau_6) \in \mathcal{R}_2^* \subseteq \Omega_{r,r,m}^*$, it follows that $(\tau_4, \tau_5) \in \Omega_{r,r,m}^* \cap \Omega_{s,u,n}^* = \Omega_{s,u,m}^* \subseteq \mathcal{R}_2^*$. Thus $e \in \mathcal{R}_1^* \cap \mathcal{R}_2^*$. Similarly, if (τ_4, τ_5) is trivial then $e \in \mathcal{R}_1^* \cap \mathcal{R}_2^*$. Thus in any case, $e \in (\mathbb{K}_1^* \cap \mathbb{K}_2^*) \vee \Omega_{s,u,n}^* = ((\mathbb{K}_1 \vee \mathbb{K}_2) \wedge \Omega_{s,u,m})^* = (\beta_{n,m}(\mathbb{K}_1 \vee \mathbb{K}_2))^*$. It follows that $\beta_{n,m}(\mathbb{K}_1) \vee \beta_{n,m}(\mathbb{K}_2) = \beta_{n,m}(\mathbb{K}_1 \vee \mathbb{K}_2)$.

Thus $\beta_{n,m}$ is a lattice monomorphism of $\mathcal{L}_m$ into $\mathcal{L}_n$ with the property that for each $\mathbb{K} \in \mathcal{L}_m$, $\beta_{n,m}(\mathbb{K}) \vee \Omega_{0,1,m} = \mathbb{K}$. Moreover, $\beta_{n,m}$ retains the skeleton ; $\beta_{n,m}(\Omega_{r,s,m}) = \Omega_{r,s,n}$. Clearly if $n|m$ and $m|p$ then $\beta_{n,p} = \beta_{n,m} \circ \beta_{m,p}$.

Theorem 3.5: For non-trivial equational classes $\mathbb{K}_1, \mathbb{K}_2$, the following are equivalent:

(1) $\mathbb{K}_1 \subseteq \mathbb{K}_2$

(2) $D(\mathbb{K}_1)|D(\mathbb{K}_2)$ and $\mathbb{K}_1 \subseteq \beta_{D(\mathbb{K}_1),D(\mathbb{K}_2)}(\mathbb{K}_2)$

(3) $D(\mathbb{K}_1)|D(\mathbb{K}_2)$ and $\beta_{1,D(\mathbb{K}_1)}(\mathbb{K}_1) \subseteq \beta_{1,D(\mathbb{K}_2)}(\mathbb{K}_2)$.

Proof: (1) ⇒ (2): Assume $\mathbb{K}_1 \subseteq \mathbb{K}_2$. Then $D(\mathbb{K}_1)|D(\mathbb{K}_2)$, and there exist s,u with $\mathbb{K}_1 \subseteq \Omega_{s,u,D(\mathbb{K}_1)}$, $\mathbb{K}_2 \subseteq \Omega_{s,u,D(\mathbb{K}_2)}$. But then $\mathbb{K}_1 = \mathbb{K}_1 \wedge \Omega_{s,u,D(\mathbb{K}_1)} \subseteq \mathbb{K}_2 \wedge \Omega_{s,u,D(\mathbb{K}_1)} = \beta_{D(\mathbb{K}_1),D(\mathbb{K}_2)}(\mathbb{K}_2)$.

(2) ⇒ (3) : $\mathbb{K}_1 \subseteq \beta_{D(\mathbb{K}_1),D(\mathbb{K}_2)}(\mathbb{K}_2)$ implies that

$\beta_{1,D(\mathbb{K}_1)}(\mathbb{K}_1) \subseteq \beta_{1,D(\mathbb{K}_1)}(\beta_{D(\mathbb{K}_1),D(\mathbb{K}_2)}(\mathbb{K}_2)) = \beta_{1,D(\mathbb{K}_2)}(\mathbb{K}_2)$.

(3) $\Rightarrow$ (1) : If $D(\mathcal{K}_1) | D(\mathcal{K}_2)$ then $\beta_{1,D(\mathcal{K}_1)}(\mathcal{K}_1) \subseteq \beta_{1,D(\mathcal{K}_2)}(\mathcal{K}_2)$ implies that $\mathcal{K}_1 = \beta_{1,D(\mathcal{K}_1)}(\mathcal{K}_1) \vee \Omega_{0,1,D(\mathcal{K}_1)} \subseteq \beta_{1,D(\mathcal{K}_2)}(\mathcal{K}_2) \vee \Omega_{0,1,D(\mathcal{K}_1)} \subseteq \beta_{1,D(\mathcal{K}_2)}(\mathcal{K}_2) \vee \Omega_{0,1,D(\mathcal{K}_2)} = \mathcal{K}_2$.

Theorem 3.6: The mapping $\mathcal{K} \rightsquigarrow (\beta_{1,D(\mathcal{K})}(\mathcal{K}), D(\mathcal{K}))$ is an embedding of $\mathcal{L} - \{E\}$ as a meet subsemilattice into $\mathcal{L}_1 \times N^+$.

Proof: Since $\beta_{1,n}$ is one-to-one for each $n \in N$, the mapping in question is one-to-one. Since for non-trivial equational classes $\mathcal{K}_1, \mathcal{K}_2$, $D(\mathcal{K}_1 \wedge \mathcal{K}_2)$ is the greatest common divisor of $D(\mathcal{K}_1)$ and $D(\mathcal{K}_2)$, it is enough to show that $\beta_{1,D(\mathcal{K}_1 \wedge \mathcal{K}_2)}(\mathcal{K}_1 \wedge \mathcal{K}_2) = \beta_{1,D(\mathcal{K}_1)}(\mathcal{K}_1) \wedge \beta_{1,D(\mathcal{K}_2)}(\mathcal{K}_2)$. If $r,u \in N$ are chosen such that $\mathcal{K}_1 \subseteq \Omega_{r,u,D(\mathcal{K}_1)}$, $\mathcal{K}_2 \subseteq \Omega_{r,u,D(\mathcal{K}_2)}$, then

$\beta_{1,D(\mathcal{K}_1 \wedge \mathcal{K}_2)}(\mathcal{K}_1 \wedge \mathcal{K}_2) = \mathcal{K}_1 \wedge \mathcal{K}_2 \wedge \Omega_{r,u,1} = (\mathcal{K}_1 \wedge \Omega_{r,u,1}) \wedge (\mathcal{K}_2 \wedge \Omega_{r,u,1})$

$= \beta_{1,D(\mathcal{K}_1)}(\mathcal{K}_1) \wedge \beta_{1,D(\mathcal{K}_2)}(\mathcal{K}_2)$. This completes the proof.

It will be seen in the next section that this embedding does not preserve joins.

Combining the results of this section with those of Section 2, we see that $\mathcal{L}$ is isomorphic to a meet subsemilattice of $\mathcal{L}_{1,0} \times I^+ \times N^+$ with a unit adjoined.

Section 6. Another Mapping Between Intervals of the Lattice

For $r \leq s < t$ and $n|m$, $n \neq m$, let $\alpha_{r,s,t,n,m}$ be the restriction of $\beta_{n,m}$ to $[\Omega_{r,s,m}, \Omega_{r,t,m}]$. Then $\alpha_{r,s,t,n,m}$ is a lattice monomorphism of $[\Omega_{r,s,m}, \Omega_{r,t,m}]$ into $[\Omega_{r,s,n}, \Omega_{r,t,n}]$. We will investigate for which values of r,s,t,n,m $\alpha_{r,s,t,n,m}$ actually maps onto the whole interval $[\Omega_{r,s,n}, \Omega_{r,t,n}]$.

Lemma 3.7: If $r > 0$ and $r + n < t$ then $\alpha_{r,s,t,n,m}$ does not map onto $[\Omega_{r,s,n}, \Omega_{r,t,n}]$.

Proof: Let $e = ((r,r+n,s), (r+n,r,s))$ and let $\mathfrak{R} = e^* \wedge \Omega_{r,t,n}$. Then $\mathfrak{R} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$. If $\mathfrak{R}$ is in the image of $\alpha_{r,s,t,n,m}$ then $\mathfrak{R} = \alpha_{r,s,t,n,m}(\mathfrak{R}')$ for some $\mathfrak{R}' \in [\Omega_{r,s,m}, \Omega_{r,t,m}]$, and then by Theorem 3.2, $\mathfrak{R} \vee \Omega_{r,s,m} = \mathfrak{R}'$, thus $\mathfrak{R} = \alpha_{r,s,t,n,m}(\mathfrak{R} \vee \Omega_{r,s,m}) = (\mathfrak{R} \vee \Omega_{r,s,m}) \wedge \Omega_{r,t,n}$. Thus it is enough to show that $\mathfrak{R} \neq (\mathfrak{R} \vee \Omega_{r,s,m}) \wedge \Omega_{r,t,n}$. Clearly $e \in \mathfrak{R}^*$. We will show that $e \notin ((\mathfrak{R} \vee \Omega_{r,s,m}) \wedge \Omega_{r,t,n})^*$.

Assume $e \in ((\mathfrak{R} \vee \Omega_{r,s,m}) \wedge \Omega_{r,t,n})^* = (\mathfrak{R}^* \cap \Omega_{r,s,m}^*) \vee \Omega_{r,t,n}^*$. Then there exist $\tau_1, \tau_2 \in F(\omega)$ such that $((r,r+n,s),\tau_1)$, $(\tau_2, (r+n,r,s)) \in \mathfrak{R}^* \cap \Omega_{r,s,m}^*$ and $(\tau_1,\tau_2) \in \Omega_{r,t,n}^*$. But $((r,r+n,s),\tau_1) \in \mathfrak{R}^*$ implies that there exists $\tau_3,\tau_4 \in F(\omega)$ such that $((r,r+n,s),\tau_3)$, $(\tau_1,\tau_4) \in \Gamma e$, $(\tau_3,\tau_4) \in \Omega_{r,t,n}^*$. Since

$((r,r+n,s),\tau_3) \in \Gamma e$, it follows that, if $r+n \neq s$ and $r \neq s$ then $\tau_3 = (r,r+n,s)$ or $(r+n,r,s)$, if $r = s$ then $\tau_3 = (r,r+n,r)$, $(r+n,r,r)$ or $(r,r,r+n)$, and if $r+n = s$ then $\tau_3 = (r,r+n,r+n)$, $(r+n,r,r+n)$ or $(r+n,r+n,r)$. In any case, since $r+n < t$ and $s < t$ and $(\tau_3,\tau_4) \in \Omega_{r,t,n}{}^*$ it follows that (τ_3,τ_4) is trivial. Thus $((r,r+n,s),\tau_1) \in \Gamma e$. But then the same argument yields $\tau_1 = \tau_2$. But this implies that $((r,r+n,s),(r+n,r,s)) \in \mathfrak{K}^* \cap \Omega_{r,s,m}{}^* \subseteq \Omega_{r,s,m}{}^*$ and this is a contradiction. This completes the proof.

<u>Lemma 3.8</u>: $\alpha_{r,s,t,n,m}$ restricted to $[\Omega_{r,s,m}, \Omega_{r,t,m}] \cap \mathcal{Y}$ is an isomorphism of $[\Omega_{r,s,m}, \Omega_{r,t,m}] \cap \mathcal{Y}$ onto $[\Omega_{r,s,n}, \Omega_{r,t,n}] \cap \mathcal{Y}$.

<u>Proof</u>: It is clear from the definition of $\alpha_{r,s,t,n,m}$ that it maps S-classes to S-classes. Thus it is enough to show that for every $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}] \cap \mathcal{Y}$ there exists $\mathfrak{K}' \in [\Omega_{r,s,m}, \Omega_{r,t,m}] \cap \mathcal{Y}$ such that $\mathfrak{K} = \alpha_{r,s,t,n,m}(\mathfrak{K}')$.

Let $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}] \cap \mathcal{Y}$. By Lemma 3.5 there exists $T \subseteq T_{s,t}$ such that $\mathfrak{K} = T(n)^* \wedge \Omega_{r,t,n}$. Since $n|m$, $T(m)^* \wedge \Omega_{r,t,n} = T(n)^* \wedge \Omega_{r,t,n}$. Thus $\mathfrak{K} = T(m)^* \wedge \Omega_{r,t,m} \wedge \Omega_{r,t,n} = \alpha_{r,s,t,n,m}(T(m)^* \wedge \Omega_{r,t,m})$ and this completes the proof.

<u>Corollary</u>: If $r+n \geq t$ then $\alpha_{r,s,t,n,m}$ maps onto $[\Omega_{r,s,n}, \Omega_{r,t,n}]$.

<u>Proof</u>: Follows from the lemma, and Lemma 3.4.

Since $\alpha_{0,s,t,n,m} \circ \phi_{0,1,s,t,m} = \phi_{0,1,s,t,n} \circ \alpha_{1,s,t,n,m}$, and since $\phi_{0,1,s,t,n}$ and $\phi_{0,1,s,t,m}$ are isomorphisms, it follows that $\alpha_{0,s,t,n,m}$ maps onto $[\Omega_{0,s,n}, \Omega_{0,t,n}]$ if and only if $\alpha_{1,s,t,n,m}$ maps onto $[\Omega_{1,s,n}, \Omega_{1,t,n}]$. From the above results we have that $\alpha_{r,s,t,n,m}$ maps onto $[\Omega_{r,s,n}, \Omega_{r,t,n}]$ for $r > 0$ if and only if $r + n \geq t$. Thus $\alpha_{r,s,t,n,m}$ maps onto $[\Omega_{r,s,n}, \Omega_{r,t,n}]$ if and only if $r = 0$ and $n + 1 \geq t$ or $r > 0$ and $r + n \geq t$.

It follows from this that the embedding of $\mathcal{L}$ -{E} into $\mathcal{L}_1 \times N^+$ described in the last section does not preserve joins : let $\mathfrak{K}_1 \in \mathcal{L}_1$ such that $\mathfrak{K}_1 \notin$ image of $\beta_{1,n}$, and let $\mathfrak{K}_2 = \Omega_{0,1,n}$. Then $\beta_{1,1}(\mathfrak{K}_1) \vee \beta_{1,n}(\mathfrak{K}_2) = \mathfrak{K}_1 \vee (\Omega_{0,1,n} \wedge \Omega_{0,1,1}) = \mathfrak{K}_1 \vee \Omega_{0,1,1} = \mathfrak{K}_1$, but $\mathfrak{K}_1 \neq \beta_{1,n}(\mathfrak{K}_1 \vee \mathfrak{K}_2)$ since $\mathfrak{K}_1 \notin$ image of $\beta_{1,n}$.

<u>Section 7</u>. <u>The Sublattice of Schwabauer Classes</u>

It has already been mentioned that $\mathcal{S}$, the set of all S-classes, forms a distributive sublattice of $\mathcal{L}$. In this section, this, and the fact that $\mathcal{S}$ is a maximal modular sublattice, will be proved. We first give the following characterization of S-classes:

<u>Lemma 3.9</u>: $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$ is an S-class if and only if it satisfies : (1) for all u with $r < u \leq s$, $\mathfrak{K}$ is in the image of $\phi_{r,s,t,u,n}$

and (2) for all $m > n$ with $n|m$, $\mathfrak{R}$ is in the image of $\alpha_{r,s,t,n,m}$.

<u>Proof</u>: If $\mathfrak{R} \in [\Omega_{r,s,n}, \Omega_{r,t,n}] \cap \mathcal{S}$ then (1) and (2) follow from Lemma 3.5 Corollary 1 and Lemma 3.8 respectively.

On the other hand, if $\mathfrak{R}$ satisfies (1) and (2), then choose $m > n$ such that $r + m \geq t$ and $n|m$. Then by (2), $\mathfrak{R} = \mathfrak{R}' \wedge \Omega_{r,t,n}$ for some $\mathfrak{R}' \in [\Omega_{r,s,m}, \Omega_{r,t,m}]$. By Lemma 3.4, $\mathfrak{R}'$ is an S-class. Thus $\mathfrak{R}$ is an S-class and this completes the proof.

Let $\mathcal{S}_n = \{\mathfrak{R} \in \mathcal{S} \,|\, D(\mathfrak{R}) = n\} = \mathcal{L}_n \cap \mathcal{S}$. Then the $\mathcal{S}_n$ are pairwise disjoint and $\mathcal{S} = \bigcup_{n \in N} \mathcal{S}_n \cup \{E\}$. Moreover, from Lemma 3.8, $\beta_{n,m}$ restricted to $\mathcal{S}_m$ is an isomorphism of $\mathcal{S}_m$ onto $\mathcal{S}_n$. This implies that the mapping $\mathfrak{R} \rightsquigarrow (\beta_{1,D(\mathfrak{R})}(\mathfrak{R}), D(\mathfrak{R}))$ is a meet-monomorphism of $\mathcal{S} - \{E\}$ onto $\mathcal{S}_1 \times N^+$. But a mapping from one lattice to another that is one-to-one, onto, and meet preserving is also join preserving, i.e., it is a lattice isomorphism.

It follows that $\mathcal{S}$ is lattice isomorphic with $\mathcal{S}_1 \times N^+$ with a unit adjoined.

<u>Lemma 3.10</u>: $[\Omega_{0,1,1}, \Omega_{0,t,1}] \cap \mathcal{S}$ is distributive for all $t \geq 1$.

<u>Proof</u>: For $T \subseteq T_{1,t}$, define $\overline{T}$ to be the set of those sequences $(u_i)_{i \in N} \in T_{1,t}$ such that $((u_i)_{i \in N}, (u_1 + 1, u_2, u_3, \ldots)) \in \Gamma(T(1) \cup \Omega_{0,t,1}{}^*)$. Then $(u_i)_{i \in N} \in \overline{T}$ if and only if there exists

$(v_i)_{i\varepsilon N} \varepsilon T$ such that $\{(v_i)_{i\varepsilon N}, (v_1 + 1, v_2, v_3, \ldots)\} \cup \Omega_{0,t,1}{}^* \rightarrow ((u_i)_{i\varepsilon N}, (u_1 + 1, u_2, u_3, \ldots))$. Thus the set of all $T \subseteq T_{1,t}$ such that $T = \overline{T}$ is closed under unions and intersections. Moreover, if $T_1, T_2 \subseteq T_{1,t}$ and $T_1 = \overline{T}_1$, $T_2 = \overline{T}_2$ then $(T_1(1)^* \wedge \Omega_{0,t,1}) \wedge (T_2(1)^* \wedge \Omega_{0,t,1}) = (T_1 \cup T_2)(1)^* \wedge \Omega_{0,t,1}$ and $(T_1(1)^* \wedge \Omega_{0,t,1}) \vee (T_2(1)^* \wedge \Omega_{0,t,1}) = (T_1 \cap T_2)\ (1)^* \wedge \Omega_{0,t,1}$. Since for each $\mathfrak{R} \varepsilon [\Omega_{0,1,1}, \Omega_{0,t,1}] \cap \mathfrak{F}$ there exists $T \subseteq T_{1,t}$ such that $T = \overline{T}$ and $\mathfrak{R} = T(1)^* \wedge \Omega_{0,t,1}$, it follows that $[\Omega_{0,1,1}, \Omega_{0,t,1}] \cap \mathfrak{F}$ is isomorphic to a sublattice of the power set of $T_{1,t}$ and hence is distributive.

<u>Corollary</u>: $\mathfrak{F}_{1,0} = \mathfrak{L}_{1,0} \cap \mathfrak{F}$ is distributive.

<u>Proof</u>: This follows immediately from the lemma, and the fact that $\{[\Omega_{0,1,1}, \Omega_{0,t,1}] \cap \mathfrak{F} \mid t \geq 1\}$ forms an ascending chain and $\mathfrak{F}_{1,0} = \bigcup_{t \geq 1} [\Omega_{0,1,1}, \Omega_{0,t,1}] \cap \mathfrak{F}$.

Since for $p < q$, $\delta_{p,q,1}$ maps S-classes to S-classes, it follows that the mapping $\mathfrak{R} \rightsquigarrow (\delta_{0,V(\mathfrak{R}),1}(\mathfrak{R}), V(\mathfrak{R}))$ is a meet monomorphism of $\mathfrak{F}_1$ into $\mathfrak{F}_{1,0} \times I^+$. Moreover, this mapping preserves joins : let $\mathfrak{R}_1, \mathfrak{R}_2 \varepsilon \mathfrak{F}_1$, $V(\mathfrak{R}_1) = p$, $V(\mathfrak{R}_2) = q$. We may assume without loss of generality that $p \leq q$. Let $u = \max\{U(\mathfrak{R}_1), U(\mathfrak{R}_2)\}$. Then $\delta_{0,p,1}(\mathfrak{R}_1) \vee \delta_{0,q,1}(\mathfrak{R}_2) \vee \Omega_{q,q,1} = \delta_{0,p,1}(\mathfrak{R}_1) \vee \Omega_{p,p,1} \vee \delta_{0,q,1}(\mathfrak{R}_2) \vee \Omega_{q,q,1} = \mathfrak{R}_1 \vee \mathfrak{R}_2$. But

$\mathfrak{K}_2 \supseteq \Omega_{q,q,1}$, thus $\delta_{0,q,1}(\mathfrak{K}_2) \supseteq \Omega_{0,q,1}$, and thus $\delta_{0,p,1}(\mathfrak{K}_1) \vee \delta_{0,q,1}(\mathfrak{K}_2) \in [\Omega_{0,q,1}, \Omega_{0,u,1}] \cap \mathcal{V}$. It follows from Corollary 1 of Lemma 3.5 that there exists $\mathfrak{K} \in [\Omega_{q,q,1}, \Omega_{q,u,1}] \cap \mathcal{V}$ with $\delta_{0,q,1}(\mathfrak{K}) = \delta_{0,p,1}(\mathfrak{K}_1) \vee \delta_{0,q,1}(\mathfrak{K}_2)$. But then $\mathfrak{K}_1 \vee \mathfrak{K}_2 = \delta_{0,p,1}(\mathfrak{K}_1) \vee \delta_{0,q,1}(\mathfrak{K}_2) \vee \Omega_{q,q,1} = \delta_{0,q,1}(\mathfrak{K}) \vee \Omega_{q,q,1} = \mathfrak{K}$. Thus $\delta_{0,p,1}(\mathfrak{K}_1) \vee \delta_{0,q,1}(\mathfrak{K}_2) = \delta_{0,q,1}(\mathfrak{K}_1 \vee \mathfrak{K}_2)$.

It follows that $\mathcal{V}_1$ is lattice isomorphic to a sublattice of $\mathcal{V}_{1,0} \times I^+$. But $\mathcal{V}_{1,0}$ is distributive, thus $\mathcal{V}_1$ is also distributive.

Thus, since $\mathcal{V}$ is isomorphic to $\mathcal{V}_1 \times N^+$ with a unit adjoined, we can state the following :

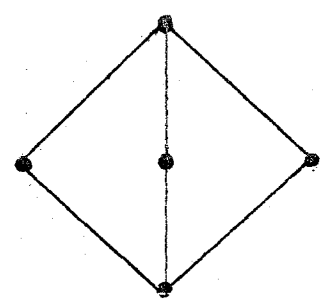
<u>Theorem 3.7</u>: $\mathcal{V}$ is distributive.

<u>Theorem 3.8</u>: $\mathcal{V}$ is maximal modular.

<u>Proof</u>: Let $\mathfrak{K}$ be any equational class not in $\mathcal{V}$, $\mathfrak{K} \in [\Omega_{r,s,n}, \Omega_{r,t,n}]$, say. Choose m such that $n|m$ and $r + m \geq t$. Then $\mathfrak{K}$ is not in the image of $\alpha_{r,s,t,n,m}$, and thus $\mathfrak{K} \neq (\mathfrak{K} \vee \Omega_{r,s,m}) \wedge \Omega_{r,t,n}$. But this implies that the sublattice of $\mathcal{L}$ generated by $\mathcal{V} \cup \{\mathfrak{K}\}$ is not modular. Thus $\mathcal{V}$ is a maximal modular sublattice.

Section 8. M_5 is a Sublattice

In the preceding section, we described a sublattice of $\mathcal{L}$ that is distributive, and has the property that any strictly larger sublattice is not even modular. One might well ask whether the set of maximal distributive sublattices of $\mathcal{L}$ coincides with the set of maximal modular sublattices of $\mathcal{L}$; this is the case if and only if every modular sublattice of $\mathcal{L}$ is distributive. This section is devoted to giving a description of a sublattice of $\mathcal{L}$ that is isomorphic to M_5, the five-element modular, non-distributive lattice.



M_5

Let $e = ((1,1,1,1,1,1,0), (0,0,0,0,0,0,6))$, and let $\Sigma = \{((u_i)_{i\varepsilon N}, (v_i)_{i\varepsilon N}) | \Sigma u_i, \Sigma v_i \geq 6\}$. Then it is clear from (P1) to (P7) that Σ is a closed set of equations and $\Sigma = \Gamma\{e\}$.

Let $e_1 = ((1,4),(2,3))$, $e_2 = ((1,4),(3,2))$ and $e_3 = ((3,2),(2,3))$. Then any two of e_1, e_2, e_3 imply the third by (P7). If $\mathfrak{K}_i = \{e_i, e\}^*$ for $i = 1,2,3$ and if $\mathfrak{K}_4 = \{e_1, e_2, e_3, e\}^*$ then $\mathfrak{K}_1 \wedge \mathfrak{K}_2 = \mathfrak{K}_1 \wedge \mathfrak{K}_3 = \mathfrak{K}_2 \wedge \mathfrak{K}_3 = \mathfrak{K}_4$.

Define sets $\Sigma_1, \Sigma_2, \Sigma_3$ of equations as follows:

$((u_i)_{i\in N}, (v_i)_{i\in N}) \in \Sigma_1$ if and only if there exist $j,k \in N$ such that $u_i = v_i = 0$ for all $i \neq j,k$, and either $u_j = 1, u_k = 4, v_j = 2, v_k = 3$, or $u_j = 2, u_k = 3, v_j = 1, v_k = 4$.

$((u_i)_{i\in N}, (v_i)_{i\in N}) \in \Sigma_2$ if and only if there exist $j,k \in N$ such that $u_i = v_i = 0$ for all $i \neq j,k$, and either $u_j = 1, u_k = 4, v_j = 3, v_k = 2$ or $u_j = 3, u_k = 2, v_j = 1, v_k = 4$.

$((u_i)_{i\in N}, (v_i)_{i\in N}) \in \Sigma_3$ if and only if there exist $j,k \in N$ such that $u_i = v_i = 0$ for all $i \neq j,k$, and $u_j = v_k = 3, u_k = v_j = 2$.

Then, for $i = 1,2,3$, $\Gamma\{e_i,e\} = \Sigma_i \cup \Sigma$, i.e., $\mathfrak{K}_i{}^* = \Sigma_i \cup \Sigma$. Since the Σ_i are pairwise disjoint and non-empty, it follows that $\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3$ are incomparable. Moreover, $\mathfrak{K}_1 \vee \mathfrak{K}_2 = (\Sigma_1 \cup \Sigma)^* \vee (\Sigma_2 \cup \Sigma)^* = ((\Sigma_1 \cup \Sigma) \cap (\Sigma_2 \cup \Sigma))^* = \Sigma^*$; similarly $\mathfrak{K}_1 \vee \mathfrak{K}_3 = \mathfrak{K}_2 \vee \mathfrak{K}_3 = \Sigma^*$.

It follows that $\{\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3, \mathfrak{K}_4, \Sigma^*\}$ forms a sublattice of $\mathcal{L}$ isomorphic to M_5.

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