# PERTURBATION THEORY WITH NON-LOCAL POTENTIALS

## IN NUCLEAR MATTER

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A Thesis

Submitted to the Faculty of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Science

McMaster University

September 1969

MASTER OF SCIENCE (1969)

McMASTER UNIVERSITY
Hamilton, Ontario.

TITLE:             Perturbation Theory with Non-local
                   Potentials in Nuclear Matter

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NUMBER OF PAGES:   v, 37

SCOPE AND CONTENTS:

In this thesis, the standard perturbation theory is
applied to nuclear matter.  The second order term is
simplified by the introduction of the $K(k,k',q)$ function
which takes care of the Pauli Exclusion Principle.  Various
potentials are used in the calculation of the first and
second order term of the perturbation expansion.  The results
are then discussed.

## ACKNOWLEDGEMENTS

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CHAPTER 1

INTRODUCTION

The main aim of nuclear physics is to understand the
structure of nuclei in terms of their elementary particles
and the interactions among them.  This at the present time
is a hard problem.  A simpler problem is that of nuclear
matter where a hypothetical system of equal numbers of
protons and neutrons fill all space at a uniform density.
The main calculation in this problem is to evaluate the
average binding energy per particle using the standard
perturbation theory. Euler [1] performed his first cal-
culations in 1937.  He had little idea of the nature of the
nuclear potential.  He used purely attractive gaussian
potentials which are central and local.  With the advent of
high energy accelerators in the period 1946-1954, it became
evident that the nuclear potential was much more compli-
cated than it had been thought.  As the study of scattering
cross sections was refined, it became also evident that a
so-called strong 'repulsive' core [2] existed at very short
range.  In this case, simple perturbation theory cannot
be applied.  The Brueckner-Goldstone theory was developed
for which the ground-state energy of a many-body system can
be calculated [3].  One of the virtues of this theory is

that it can take into account the strong short range re-
pulsion in the nucleon-nucleon potential.  However, the
theory is very complicated and the calculations are usually
hard and long to evaluate, such that there have been
periodic attempts in the 60's to come back to the simple
theory that perturbation theory is.  One of the ways to
resolve the dilemma was to introduce weaker potentials.
The first step was taken by Bressel and Kerman who re-
placed the hard core in the Hamada-Johnston potential by
a finite core[4].  As we shall see in this paper, the bind-
ing energy of nuclear matter using this potential is  too
large, at least up to second order.  Again Bressel, Kerman,
Rouben and Levy[5] replaced the square core of the Bressel-
Kerman potential with a separable potential so that the
complete form is

$$\langle \underset{\sim}{r} | V | \underset{\sim}{r}' \rangle = v(\underset{\sim}{r}) \; v(\underset{\sim}{r}') \qquad\qquad r < c$$

$$= v_{\ell}(\underset{\sim}{r}) \; \delta(\underset{\sim}{r} - \underset{\sim}{r}') \qquad r > c$$

but for  certain numerical factors.  The purely, separable
form had been earlier suggested and utilized by Tabakin[6];
adding the local tail brings the potential model into accord
with accepted theory, namely the one pion exchange tail.

In order to apply perturbation theory to this
mixed potential, or indeed to any potential whatever, Ko
and Sprung[7] developed a "kinematic function" $K(k'kq)$

which depends on the three "natural variables" of the problem. That is, if we think of the matrix elements $\langle \underset{\sim}{r}|V|\underset{\sim}{r}'\rangle$ and its Fourier transform

$$\langle \underset{\sim}{r}|V|\underset{\sim}{r}'\rangle \rightarrow \langle \underset{\sim}{k}|V|\underset{\sim}{k}'\rangle$$

then from rotational invariance, we can argue that $\langle \underset{\sim}{k}|V|\underset{\sim}{k}'\rangle$ depends, for any potential V, on the magnitude of the vectors $\underset{\sim}{k}$, $\underset{\sim}{k}'$ and the angle $\theta_{kk'}$ between these two vectors. In place of the angle $\theta_{kk'}$, we can introduce the length q of the vector $\underset{\sim}{q}$, which is the relative momentum and is given by

$$\underset{\sim}{q} = \underset{\sim}{k}'-\underset{\sim}{k} \ .$$

Hence the terminology of "natural variables" for $\underset{\sim}{k},\underset{\sim}{k}'$ and $\underset{\sim}{q}$.

The K-function contains all the effects of the Pauli exclusion principle. Ko[8] used this function to calculate some second order correction terms to the reference spectrum approximation in nuclear matter[9]. He also outlined roughly how his function would be used in standard perturbation theory calculations, but did not work out the details.

Here we have applied Ko's function to calculate second order terms in perturbation theory. In chapter 2 the formalism is worked out for a central but general non-local potential. The limits of local and separable forces

are considered too. In chapter 3 we have evaluated the first and second order terms in perturbation theory using different types of potentials often encountered in the current literature.

In the following chapters, we have used the following notation:

$k_\ell$ or, $\ell \equiv$ momentum of a particle

$|\ell\rangle$, $|m\rangle \equiv$ a single particle state inside the Fermi sea

$|a\rangle$, $|b\rangle \equiv$ a single particle state outside the Fermi sea.

When there is ambiguity we have used $k_\ell$ instead of $\ell$ for the momentum. Occurring as a subscript, $\ell$ usually means the orbital angular momentum value, but in chapter 3 it is used to distinguish the local force from the separable force.

# CHAPTER 2

## FORMALISM

In this chapter, we are going to discuss explicitly perturbation theory and how nuclear matter calculations can be carried out when we are dealing with non-local, but central potentials. We include the Pauli exclusion principle in the second order term by introducing a kinematic function of three variables $K(k,k',q)$ which has been evaluated by Ko from a geometrical point of view. We find this method is powerful and exact.

We want to develop a formalism that will be suitable for a non-local potential of the form

$$<R\underset{\sim}{r}|V|R'\underset{\sim}{r}'> = \delta(\underset{\sim}{R}-\underset{\sim}{R}') <\underset{\sim}{r}|V|\underset{\sim}{r}'> \qquad (2-1)$$

where

$$<\underset{\sim}{r}|V|\underset{\sim}{r}'> = \frac{\hbar^2}{m} \sum_\lambda \frac{(2\lambda+1)}{4\pi} \frac{\omega_\lambda(r,r')}{rr'} P_\lambda(\cos\theta_{rr'}) . \qquad (2-2)$$

Here, R and r denote the center of mass and relative coordinates respectively. It is interesting to look at the units of some of the factors in our expressions above which occur frequently . For example

$$[\hbar^2/m] \equiv MeV - Fermi^2$$

$$[\omega_\lambda(\underset{\sim}{r}r')] \equiv Fermi^{-3} .$$

For local forces

$$\langle \underset{\sim}{r}|V|\underset{\sim}{r}'\rangle = V(\underset{\sim}{r}) \; \delta(\underset{\sim}{r}-\underset{\sim}{r}') \qquad (2\text{-}3)$$

$$= \frac{\hbar^2}{m} \sum_{\ell} v_{\ell}(r) \; \frac{\delta(r-r')}{4\pi rr'} \; P_{\ell}(\cos \theta_{rr'}) \; . \qquad (2\text{-}3a)$$

We can incorporate both local and separable forces in (2-2) by making the identification

$$\omega_{\lambda}(r,r') \rightarrow v_{\lambda}(r) \; v_{\lambda}(r') \; \text{separable}$$

$$\rightarrow v(r) \; \delta(r-r') \; \text{local} \; .$$

## SCHROEDINGER EQUATION

There are many ways to incorporate separable potentials in nuclear matter calculations. Before we do that, we would like to see how a separable potential can be used in the Schroedinger equation and see how it is related to the familiar form for local potentials. The Schroedinger equation can be written in the form

$$(\nabla^2 + k^2) \; \psi(r) = \frac{m}{\hbar^2} \int \langle \underset{\sim}{r}|V|\underset{\sim}{r}'\rangle \; \psi(r') d^3 \underset{\sim}{r}' \; . \qquad (2\text{-}4)$$

Let

$$\psi(r) = \sum_{\mu} \frac{(2\mu+1)}{4\pi} \frac{U_{\mu}(r)}{r} P_{\mu}(\cos \theta_r) \; . \qquad (2\text{-}5)$$

Substituting this in the Schroedinger equation along with

equation (2-2) we have

$$\sum_{\mu} \frac{(2\mu+1)}{4} P_{\mu}(\cos \theta_r) \{\frac{1}{r} \frac{d^2}{dr^2} U_{\mu}(r)+k^2 \frac{U_{\mu}(r)}{r} - \frac{\mu(\mu+1)}{r^2} \frac{U_{\mu}(r)}{r}\}$$

$$= \frac{1}{(4\pi)^2} \sum_{\lambda} \frac{(2\lambda+1)}{r} \int \frac{\omega_{\lambda}(r,r')}{r'} P_{\lambda}(\cos \theta_{rr'}) \sum_{\mu} (2\mu+1) \frac{U_{\mu}(r')}{r'}$$

$$x\ P_{\mu}(\cos \theta_{r'})\ r'^2 dr' d\Omega' \ . \tag{2-6}$$

Using the property of the Legendre polynomials

$$\int_{-\pi}^{\pi} P_{\mu}(\cos \theta_{rr'}) P_{\lambda}(\cos \theta_{r'}) d\Omega' = \frac{4\pi}{2\mu+1} \delta_{\mu\lambda} P_{\mu}(\cos \theta_r)$$

and taking the summation over $\lambda$, the right-hand side of equation (2-6) becomes

$$= \sum_{\mu} \frac{(2\mu+1)}{4\pi} \int \omega_{\mu}(r,r')\ U_{\mu}(r')\ P_{\mu}(\cos \theta_r)\ dr' \ . \tag{2-7}$$

Equating term by term, we get the final result

$$(\frac{d^2}{dr^2} + k^2 - \mu(\mu+1))\ U_{\mu}(r)$$

$$= \int_0^{\infty} \omega_{\mu}(r,r')\ U_{\mu}(r')\ dr' \ . \tag{2-8}$$

For a separable force, the RHS becomes

$$v_{\mu}(r) \int_0^{\infty} v_{\mu}(r')\ U_{\mu}(r')\ dr' \ .$$

For a local potential, we get the much more familiar form

$$(\frac{d^2}{dr^2} + k^2 - \mu(\mu+1))\ U_{\mu}(r) = v_{\ell}(r)\ U_{\mu}(r) \ . \tag{2-9}$$

PERTURBATION THEORY

For the unperturbed system of non-interacting par-
ticles, the particles fill all the plane wave states up
to the fermi momentum, i.e. $|\underset{\sim}{k}| \leq k_f$. The plane waves are

$$\psi_k(r) = \frac{e^{i\underset{\sim}{k} \cdot \underset{\sim}{r}}}{\sqrt{\Omega}} . \tag{2-10}$$

We consider an arbitrary box of A particles of volume $\Omega$.
The density is a fixed quantity $\rho=A/\Omega$. For nuclear matter
the number of particles is related to the fermi momentum
and is equal to

$$A = \Omega \frac{4\pi/3 \ k_f^{\ 3}}{(2\pi)^3} \ 4 . \tag{2-11}$$

The factor 4 is to take into account the possible spin and
isospin states of the nucleons. Hence we have the well
known relationship of the number density $\rho$ to the related
fermi momentum $k_f$

$$\rho = \frac{2}{3\pi^2} k_f^{\ 3} . \tag{2-12}$$

In the calculation of the binding energy in nuclear matter
one must often evaluate the matrix elements $\langle ab|V|\ell m\rangle$. We
shall do this by introducing a complete set

$$\langle ab|V|\ell m\rangle = \int \langle ab|\underset{\sim}{r}_1\underset{\sim}{r}_2\rangle \ \langle \underset{\sim}{r}_1\underset{\sim}{r}_2|V|\underset{\sim}{r}_1'\underset{\sim}{r}_2'\rangle$$

$$x \ \langle \underset{\sim}{r}_1'\underset{\sim}{r}_2'|\ell m\rangle \ d^3\underset{\sim}{r}_1 \ d^3\underset{\sim}{r}_2 \ d^3\underset{\sim}{r}_1' \ d^3\underset{\sim}{r}_2' \tag{2-13}$$

$$= \frac{1}{\Omega^2} \int e^{-i\underset{\sim}{k}_a \cdot \underset{\sim}{r}_1} e^{-i\underset{\sim}{k}_b \cdot \underset{\sim}{r}_2} \langle \underset{\sim}{r}_1 \underset{\sim}{r}_2 | V | \underset{\sim}{r}_1' \underset{\sim}{r}_2' \rangle$$

$$\times \; e^{i\underset{\sim}{k}_\ell \cdot \underset{\sim}{r}_1'} e^{i\underset{\sim}{k}_m \cdot \underset{\sim}{r}_2'} \; d^3\underset{\sim}{r}_1 \; d^3\underset{\sim}{r}_2 \; d^3\underset{\sim}{r}_1' \; d^3\underset{\sim}{r}_2' \; . \qquad (2\text{-}13a)$$

If we perform the transformation to the center of mass and relative coordinates

$$\underset{\sim}{r}_1 = \underset{\sim}{R} + \underset{\sim}{r}/2 \qquad \underset{\sim}{r}_1' = \underset{\sim}{R}' + \underset{\sim}{r}'/2$$

$$\underset{\sim}{r}_2 = \underset{\sim}{R} - \underset{\sim}{r}/2 \qquad \underset{\sim}{r}_2' = \underset{\sim}{R}' - \underset{\sim}{r}'/2 \; , \qquad (2\text{-}14)$$

and also for the initial and intermediate momenta

$$\underset{\sim}{k}_a = \underset{\sim}{P} + \underset{\sim}{k}' \qquad \underset{\sim}{k}_\ell = \underset{\sim}{P} + \underset{\sim}{k}$$

$$\underset{\sim}{k}_b = \underset{\sim}{P} - \underset{\sim}{k}' \qquad \underset{\sim}{k}_m = \underset{\sim}{P} - \underset{\sim}{k} \; . \qquad (2\text{-}15)$$

The equation (2-13a) becomes, after integrating over R and R'

$$= \frac{\delta(\underset{\sim}{P} - \underset{\sim}{P}')}{\Omega} \int e^{-i\underset{\sim}{k}' \cdot \underset{\sim}{r}} \langle \underset{\sim}{r} | V | \underset{\sim}{r}' \rangle \; e^{i\underset{\sim}{k} \cdot \underset{\sim}{r}'} \; d^3\underset{\sim}{r} \; d^3\underset{\sim}{r}' \; . \qquad (2\text{-}16)$$

For local potential, we have

$$\langle ab | V | \ell m \rangle = \frac{\delta(\underset{\sim}{P} - \underset{\sim}{P}')}{\Omega} \int e^{-i\underset{\sim}{q} \cdot \underset{\sim}{r}} \; V(r) \; d^3\underset{\sim}{r} \qquad (2\text{-}17)$$

where

$$\underset{\sim}{q} = \underset{\sim}{k}' - \underset{\sim}{k} \; .$$

For non-local, separable potential

$$\langle ab|V|\ell m\rangle = \frac{\hbar^2}{m} \frac{\delta(\underline{P}'-\underline{P})}{\Omega} \int e^{-i\underline{k}'\cdot\underline{r}} \sum_{\ell m} \frac{(2\ell+1)}{4\pi} \frac{v_\ell(r)}{r} \frac{v_\ell(r')}{r'}$$

$$\times Y_\ell^m(\Omega_r) Y_\ell^{*m}(\Omega_{r'}) e^{i\underline{k}\cdot\underline{r}'} d^3\underline{r} \, d^3\underline{r}' \qquad (2\text{-}18)$$

$$= 4\pi \frac{\hbar^2}{m} \frac{\delta(\underline{P}'-\underline{P})}{\Omega} \sum_{\ell m} (2\ell+1) \int j_\ell(k'r) v_\ell(r) \, rdr$$

$$\times \int j_\ell(kr') v_\ell(r') r'dr' \, Y_\ell^m(\Omega_k) Y_\ell^{*m}(\Omega_{k'}) \qquad (2\text{-}19)$$

by using $e^{i\underline{k}\cdot\underline{r}} = 4\pi \sum_{\ell m} (i)^\ell j_\ell(kr) Y_\ell^{*m}(k) Y_\ell^m(r')$.

Define: $v_\ell(k) = \int j_\ell(kr') v_\ell(r') r'dr'$ . $\qquad (2\text{-}20)$

The final result is

$$\langle ab|V|\ell m\rangle = 4\pi \frac{\hbar^2}{m} \frac{\delta(\underline{P}'-\underline{P})}{\Omega} \sum_\ell (2\ell+1) v_\ell(k) v_\ell(k') P_\ell(\underline{k}\cdot\underline{k}')$$

$$. \quad (2\text{-}21)$$

ZEROTH ORDER PERTURBATION

For any state $\ell$ of momentum $\underline{\ell}$, the total energy is simply the kinetic energy for a free particle in that state

$$E_\ell = \frac{k^2\ell^2}{2m} . \qquad (2\text{-}22)$$

Summing over all states, we have for the unperturbed energy

of the system

$$E_0 = 4 \sum_{\ell} E_\ell \qquad (2\text{-}23)$$

$$= 4 \sum_{\ell} \frac{\hbar^2 \ell^2}{2m} . \qquad (2\text{-}23a)$$

Again the factor 4 accounts for the spins and isospins for nucleons in nuclear matter (i.e. degeneracy of momentum states). Using the well known relationship

$$\sum_{\ell} \rightarrow \frac{\Omega}{(2\pi)^3} \int d^3\underset{\sim}{\ell} \qquad (2\text{-}24)$$

$$E_0 = 4 \frac{\Omega}{(2\pi)^3} \int \frac{\hbar^2 \ell^2}{2m} d^3\underset{\sim}{\ell} \qquad (2\text{-}25)$$

$$= \frac{\hbar^2}{m} \frac{k_f^5 \Omega}{5\pi^2} . \qquad (2\text{-}25a)$$

The energy per particle is then from (2-12)

$$E_0/A = \frac{3}{5} \left(\frac{\hbar^2}{2m}\right) k_f^2 . \qquad (2\text{-}26)$$

FIRST ORDER PERTURBATION

We write the first order term of the perturbation expansion as

$$B^{(1)} = \frac{1}{2A} \sum_{\ell,m} \sum_{\sigma,\tau} [<\ell m|V|\ell m> - <m\ell|V|\ell m>] . \qquad (2\text{-}27)$$

DIRECT  EXCHANGE
TERM   TERM

For separable potential, the direct term only is

$$<\ell m|V|\ell m> = \frac{\hbar^2}{m} \frac{4\pi}{\Omega} \sum_\ell (2\ell+1) \, v_\ell(\underset{\sim}{k}) \, v_\ell(\underset{\sim}{k}') \qquad (2\text{-}28)$$

$$\underset{\sim}{k} = \underset{\sim}{k}' = \underset{\sim}{\ell} - \underset{\sim}{m} \qquad (2\text{-}29)$$

hence

$$= \frac{\hbar^2}{m} \frac{4\pi}{\Omega} \sum_\ell (2\ell+1) \, [v_\ell(\underset{\sim}{\ell}-\underset{\sim}{m})]^2 \qquad (2\text{-}30)$$

for the exchange term, we get the extra term

$$P_\ell(-\underset{\sim}{k}\cdot\underset{\sim}{k}) = (-)^\ell \, P_\ell(\underset{\sim}{k}\cdot\underset{\sim}{k}) \qquad \text{because } \underset{\sim}{k} = -\underset{\sim}{k}'$$

$$= (-)^\ell$$

hence

$$<m\ell|V|\ell m> = \frac{\hbar^2}{m} \frac{4\pi}{\Omega} \sum_\ell (2\ell+1) \, (-)^\ell \, [v_\ell(\underset{\sim}{\ell}-\underset{\sim}{m})]^2$$

$$B^{(1)} = \frac{1}{2A} \frac{\hbar^2}{m} \frac{4\pi}{\Omega} \sum_{\ell,m} \sum_\lambda \{v_\lambda^2(\underset{\sim}{\ell}-\underset{\sim}{m}) \, [\pm] \, (-1)^\lambda \, v_\lambda^2(\underset{\sim}{\ell}-\underset{\sim}{m})\} . \qquad (2\text{-}31)$$

The [±] sign in the square bracket arises from the fact that
we have dropped the summation over spins and isospins for
simplification. The + or − sign is $(-)^{S+T+1}$, and occurs
because of the symmetry or antisymmetry of the spin and iso-
spin parts of the wave function which have already been
eliminated.

Let us reduce the sum over $\ell,m$ into a simple sum
over $\underset{\sim}{q} = \underset{\sim}{\ell} - \underset{\sim}{m}$

$$m^2 = \ell^2 + q^2 - 2\ell q \cos\theta < 1 \qquad , \qquad k_f \equiv 1$$

or
$$\frac{\ell^2+q^2-1}{2\ell q} < \cos\theta < 1$$

but

$$\underset{\ell}{\Sigma}\ \underset{m}{\Sigma} \rightarrow \frac{\Omega}{(2\pi)^3}\ \frac{\Omega}{(2\pi)^3} \int_0^1 d^3\underset{\sim}{\ell}\ \int_0^1 d^3\underset{\sim}{m}\ .$$

Transforming

$$\int_0^1 d^3\underset{\sim}{\ell}\ \int_0^1 d^3\underset{\sim}{m} = \int_0^2 d^3\underset{\sim}{q}\ \int_0^1 d^3\underset{\sim}{\ell} \tag{2-32}$$

$$= 2\pi \int_0^2 d^3\underset{\sim}{q}\ \int_{q-1}^1 \ell^2 d\ell\ \int_{\frac{\ell^2+q^2-1}{2\ell q}}^1 d(\cos\theta_\ell)$$

$$= 8\pi^2 \int_0^2 q^2 dq\ \int_{q-1}^1 \ell^2 d\ell\ [1 - \frac{(\ell^2+q^2-1)}{2\ell q}]$$

$$= \frac{\pi^2}{3} \int_0^2 q^2(4+q)\ (2-q)^2\ dq$$

$$= \frac{\pi^2}{3}\ k_f^2\ 2^6 \int_0^1 \hat{q}^2(2+\hat{q})\ (1-\hat{q})^2\ d\hat{q} \tag{2-32a}$$

$$\therefore B^{(1)} = (2\ell+1)\ \frac{\hbar^2}{m}\ \frac{k_f^3}{\pi} \int_0^1 \hat{q}^2(2+\hat{q})\ (1-\hat{q})^2\ d\hat{q}$$

$$\times\ [\underset{\lambda}{\Sigma}\ v_\lambda^2(q)\ [\pm]\ (-)^\lambda\ v_\lambda^2(q)]\ . \tag{2-33}$$

SECOND ORDER PERTURBATION

The second order term is

$$B^{(2)} = \frac{-1}{2\rho\Omega} \sum_{\substack{\ell,m \\ a,b}} \frac{(<\ell m|V|ab> <ab|V|\ell m> - <\ell m|V|ab> <ab|V|m\ell \S)}{E_a + E_b - E_\ell - E_m}$$

DIRECT                           EXCHANGE

$$. \quad (2\text{-}34)$$

Due to the Pauli Exclusion Principle, we have the following restrictions over the summation

    i)   for unoccupied states $|\underset{\sim}{\ell}|, |\underset{\sim}{m}| < k_f$

    ii)  for unoccupied states $|\underset{\sim}{a}|, |\underset{\sim}{b}| > k_f$ ,

because of conservation of momentum

    iii)         $\underset{\sim}{\ell} + \underset{\sim}{m} = \underset{\sim}{a} + \underset{\sim}{b}$ .

The matrix element of equation (2-16) is simplified by writing it as

$$<\ell m|V|ab> = \frac{1}{\Omega} v(k,k') \qquad (2\text{-}35)$$

where

$$v(k,k') = \int e^{-i\underset{\sim}{k}' \cdot \underset{\sim}{r}} <\underset{\sim}{r}|V|\underset{\sim}{r}'> e^{-\underset{\sim}{k} \cdot \underset{\sim}{r}'} d^3\underset{\sim}{r} \, d^3\underset{\sim}{r}' \; .$$

The energy denominator for plane waves is

$$E_a + E_b - E_\ell - E_m = \frac{\hbar^2}{m} (\underset{\sim}{k}'^2 - \underset{\sim}{k}^2) \; . \qquad (2\text{-}36)$$

The direct term for separable, non-local potential over all

partial waves involves

$$\frac{16\pi^2}{\Omega^2} \left(\frac{\hbar^2}{m}\right)^2 \delta^2(\underset{\sim}{P}-\underset{\sim}{P}') \sum_\mu (2\mu+1) \, v_\mu(\underset{\sim}{k}) \, v_\mu(\underset{\sim}{k}')$$

$$P_\mu(\underset{\sim}{k}\cdot\underset{\sim}{k}') \sum_\lambda (2\lambda+1) \, v_\lambda(\underset{\sim}{k}) \, v_\lambda(\underset{\sim}{k}') \, P_\lambda(\underset{\sim}{k}\cdot\underset{\sim}{k}') \, . \tag{2-37}$$

Note well that the delta function has been built into the theory to satisfy the conservation of momentum restriction. For S-waves

$$B^{(2)}_{direct} = \frac{16\pi^2}{\Omega^2} \left(\frac{\hbar^2}{m}\right) \delta^2(\underset{\sim}{P}-\underset{\sim}{P}') \, [v^2(\underset{\sim}{k}) \, v^2(\underset{\sim}{k}')]_D \, . \tag{2-38}$$

Now we convert the summation into an integration (see equation (2-24))

$$\frac{1}{2\rho\Omega} \sum_{\substack{\ell,m \\ a,b}} \overset{\delta(\underset{\sim}{k}_a+\underset{\sim}{k}_b-\underset{\sim}{k}_\ell-\underset{\sim}{k}_m)}{\longrightarrow} \frac{1}{2\rho\Omega} \frac{\Omega}{(2\pi)^3} \frac{\Omega}{(2\pi)^3} \frac{\Omega}{(2\pi)^3} \int d^3\underset{\sim}{\ell} \int d^3\underset{\sim}{m} \int d^3\underset{\sim}{a}$$

$$\tag{2-39}$$

Using equations (2-39), (2-15), (2-36) into (2-38)

$$B^{(2)}_{direct} = \frac{1}{2\rho\Omega} \frac{\Omega^3}{2^4\pi^4} \frac{m}{\hbar^2} (16\pi^2) \left(\frac{\hbar^2}{m}\right)^2 \frac{8k_f^7}{\Omega^2} \int_0^\infty d^3\underset{\sim}{k}' \int_0^1 d^3\underset{\sim}{k}$$

$$\times \int_{\substack{|P\pm k|<1 \\ |P\pm k'|>1}} d^3\underset{\sim}{P} \frac{[v^2(\underset{\sim}{k}) \, v^2(\underset{\sim}{k}')]_D}{k'^2-k^2} \, .$$

The last integral contains all the kinematic effects of the Pauli Exclusion Principle in nuclear matter. Introducing the momentum transfer

$$\underset{\sim}{q} = \underset{\sim}{k}' - \underset{\sim}{k}$$

$$d(\cos \theta_{kk'}) = \frac{q}{k'k} dq$$

we then convert the integral over $\theta_{kk'}$ to an integral over q

$$B_{direct}^{(2)} = \frac{3}{2\pi^3} \left(\frac{\hbar^2}{m}\right) k_f^4 \int_0^\infty k'dk' \int_0^1 kdk \int_{|k'-k|}^{k'+k} qdq$$

$$\times \frac{[v^2(\underset{\sim}{k})\ v^2(\underset{\sim}{k}')]_D}{k'^2-k^2} \cdot \int_{\substack{|P\pm k|<1 \\ |P\pm k'|>1}} d^3\underset{\sim}{P} .$$

Following Ko, we define

$$K(k,k',q) = \int_{\substack{|P\pm k|<1 \\ |P\pm k'|>1}} d^3\underset{\sim}{P} .$$

The properties of this function have already been studied. Here we are only going to point out its main features for a given set of k, k', and q, satisfying the triangle equality

$$\underset{\sim}{q} = \underset{\sim}{k}' - \underset{\sim}{k} .$$

The value of $K(k,k',q)$ is the volume inside the inter-section of two spheres of unit radius excluding the portion outside two other spheres (see figure 3). Also it has been shown that for a local potential $v(k,k') \rightarrow v(q)$, from which we can deduce

$$P(q/2) \rightarrow \int_0^1 dk \int_{\max[|q-k|,k]}^{q+k} \frac{kk'\ K(k,k',q)}{k'^2-k^2}$$

here $P(q/2)$ is Euler's function.  The integration over k'
for the second order term runs from zero to infinity.  How-
ever as it turns out the potentials $v(k,k')$ are oscillatory,
decreasing functions which converge very rapidly so that
the k' integration need be carried only to several times
the Fermi momentum (about $8k_f$).  The total second order
term is

$$B^{(2)} = \frac{3k_f^4}{2\pi^3} \left(\frac{\hbar^2}{m}\right) \int_0^\infty k'dk' \int_0^1 kdk \int_{|k'-k|}^{k'+k} qdq \frac{K(k,k',q)}{k'^2-k^2}$$

$$x \sum_{\sigma,\tau} [v^2(\underset{\sim}{k}) \ v^2(\underset{\sim}{k'})_D \ [\pm] \ v^2(\underset{\sim}{k}) \ v^2(\underset{\sim}{k'})_E] \ .$$

# CHAPTER 3

## APPLICATIONS

Most phenomenological potentials used in nuclear matter calculations contain a strong short-range repulsion, whether a hard core as in the Hamada-Johnston potential or a "soft" core of the Bressel-Kerman type. In this chapter we will consider the various types of potentials using the method developed in the previous chapter.

We have in mind the potential of Kerman and Levy, which contains a non-local, separable core plus a long-range, local potential. This potential has so far been fitted only to the S-wave

i.e.
$$\langle \underset{\sim}{r}|V|\underset{\sim}{r}'\rangle = \frac{\hbar^2}{m} \quad \frac{1}{4\pi} \frac{v_s(\underset{\sim}{r})\, v_s(\underset{\sim}{r}')}{rr'} \qquad r<r_c$$

$$\qquad\qquad (3\text{-}1)$$

$$= \frac{\hbar^2}{m} \frac{v_\ell(r)}{4\pi} \frac{\delta(r-r')}{rr'} \quad r>r_c$$

For this kind of potential we replace (2-20) by

$$v_s(k) = \int_0^{r_c} j_0(kr')\, v_s(r')\, r'dr' \qquad\qquad (3\text{-}2)$$

and

$$v_\ell(q) = \int_{r_c}^{\infty} j_0(qr')\, v_\ell(r')\, r'dr' \;. \qquad\qquad (3\text{-}3)$$

Here we are considering only S-waves.

Hence

$$\langle \ell m|V|ab\rangle = \frac{1}{\Omega} [v_s(k) \, v_s(k') + v_\ell(q)].  \qquad (3-4)$$

If we denote by $\sigma$ and $\tau$ the total spin and isospin respectively, then summing (3-4) we have for direct and exchange term

$$\sum_{\sigma,\tau} \langle \ell m|v|\ell m\rangle = \frac{1}{\Omega} \left\{ \left[ \sum_{\sigma,\tau} v_s^2(k) \right]_D + \left[ \sum_{\sigma,\tau} v_\ell(0) \right]_D \right\}  \qquad (3-5)$$

$$\sum_{\sigma,\tau} \langle \ell m|v|m\ell\rangle = \frac{1}{\Omega} \left\{ \left[ \sum_{\sigma,\tau} v_s^2(k) \right]_E + \left[ \sum_{\sigma,\tau} v_\ell(2k) \right]_E \right\}  \qquad (3-6)$$

where

$$\left[ \sum_{\sigma,\tau} v_s^2(k) \right]_D = 9\{v_s^{33}(k)\}^2 + 3\{v_s^{31}(k)\}^2 + 3\{v_s^{13}(k)\}^2$$
$$+ \{v_s^{11}(k)\}^2 ,$$

$$\left[ \sum_{\sigma,\tau} v_s^2(k) \right]_E = -9\{v_s^{33}(k)\}^2 + 3\{v_s^{31}(k)\}^2 + 3\{v_s^{13}(k)\}^2$$
$$- \{v_s^{11}(k)\}^2 ,$$

$$\left[ \sum_{\sigma,\tau} v_\ell(0) \right]_D = 9\, v_\ell^{33}(0) + 3\, v_\ell^{31}(0) + 3_\ell v^{13}(0) + v_\ell^{11}(0) ,$$

and

$$\left[ \sum_{\sigma,\tau} v_\ell(2k) \right]_E = -9\, v_\ell^{33}(2k) + 3\, v_\ell^{31}(2k) + 3_\ell v^{13}(2k)$$
$$- v_\ell^{11}(2k) .  \qquad (3-7)$$

The first order term of the perturbation theory using (3-5) and (3-6) in (2-33) is

$$B^{(1)} = \frac{k_f^{\,3}}{12\pi} \left[ \sum_{\sigma,\tau} v_\ell(0) \right]_D + \frac{k_f^{\,3}}{\pi} \int_0^1 dk\, k^2\, (1-k)^2\, (2+k)$$

$$\times \left\{ \left[ \sum_{\sigma,\tau} v_s^{\,2}(k) \right]_D + \left[ v_s^{\,2}(k) \right]_E + \left[ \sum_{\sigma,\tau} v_\ell(2k) \right]_E \right\} \qquad (3\text{-}8)$$

The second order term is

$$B^{(2)} = -\frac{3k_f^{\,4}}{2\pi^3}\frac{\hbar^2}{m} \int_0^\infty k'dk' \int_0^1 kdk \int_{|k'-k|}^{k'+k} qdk\, \frac{K(k,k',q)}{k'^2-k^2}$$

$$\times \left\{ \left[ \sum_{\sigma,\tau} v_\ell^{\,2}(q) \right]_D + \left[ \sum_{\sigma,\tau} v_s^{\,2}(k)\, v_s^{\,2}(k') \right]_D \right.$$

$$+ \left[ \sum_{\sigma,\tau} v_s^{\,2}(k)\, v_s^{\,2}(k') \right]_E + 2\left[ \sum_{\sigma,\tau} v_s(k)\, v_s(k')\, v_\ell(q) \right]_D$$

$$+ \left[ \sum_{\sigma,\tau} v_s(k)\, v_s(k')\, v_\ell(q) \right]_E + \left[ \sum_{\sigma,\tau} v_s(k)\, v_s(k')\, v_\ell(x) \right]_E$$

$$\left. + \left[ \sum_{\sigma,\tau} v_\ell(q)\, v_\ell(x) \right]_E \right\} \qquad (3\text{-}9)$$

where $\underset{\sim}{x} = \underset{\sim}{k}' + \underset{\sim}{k}$, is the "crossed" momentum transfer, which gives $x = \{2(k'^2 + k^2) - q^2\}^{1/2}$. For an explicit derivation of equation (3-9), see reference (8). In evaluating these terms, we notice the following points

i) We can calculate for pure local, or pure separable potentials

i.e.

$$B^{(1)}_{\text{pure local}} = \frac{k_f^3}{12\pi} [ \sum_{\sigma,\tau} v_\ell(0) ]_D + \frac{k_f^3}{\pi} \int_0^1 dk k^2 (1-k)^2 (2+k)$$

$$\times [ \sum_{\sigma,\tau} v_\ell(2k) ]_E .$$

ii)   Each term separates into a sum of direct terms and exchange terms and there is no mixing of these.

iii)  There are mixing terms for separable and local terms in the second order.

## GAUSSIAN POTENTIAL

As an    application we consider a pure local gaussian potential

$$v = = V_0 \exp(-r^2/\beta^2) .$$

The first order direct and exchange terms are easy to calculate. For the second order term, the direct term, although much more complicated because of the Pauli principle which restricts the summation over intermediate states, has been worked out by several authors. Levinger et al. give[11]

$$B^{(2)}_{\text{direct}} = - (3/2^7 5\pi) x^2 (V_0^2/T_f) g(x) ,$$

$T_f$ being the kinetic energy of a particle at the Fermi surface:

$$T_f = (\frac{\hbar^2}{m}) k_F^2/2 .$$

$x = k_F \beta$ is a dimensionless parameter and $g(x)$ is an integral function which was originally calculated by H. Euler. Now

in evaluating the second order term, the exchange part is much more complicated than the direct. It was Swiatecki who first noted that the ratio of the exchange term to the direct term for first order is practically equal to the ratio of the exchange to direct for second order

$$\frac{\text{Exchange } (1^{st} \text{ order})}{\text{Direct } \quad (1^{st} \text{ order})} = \frac{\text{Exchange } (2^{nd} \text{ order})}{\text{Direct } \quad (2^{nd} \text{ order})} \qquad (3\text{-}10)$$

at least for $x = k_f \beta \to 0$ and $x = 2$. He conjectured that this would be true for the entire range $0 < x < 2$. Knowing the other three in equation (3-10), one can get an estimate of the second order exchange term. It can also be shown that the second order exchange term has a similar expression as the direct term

$$B^{(2)}_{\text{exchange}} = - (3/2^7 5\pi) \, x^2 \, V_0^2/T_f \cdot k(x) \ .$$

With the method developed in the previous section it is possible to calculate separately the direct and exchange terms exactly for first and second order. In Table 1 we have tabulated the values of the function $g(x)$ and $k(x)$ and the ratio of first order and second order exchange to direct. In figure 1 the ratio $r_1/r_2$ is plotted against x. The dotted line is Swiatecki's original conjecture, the dark line is what has been found. We see that $r_1/r_2$ function sags by about 15% at x=1.2 and then increases until it becomes slightly greater than unity at x=2.0.

In Table 2 the direct and exchange terms have been

computed and the results compared with those of Sprung. We find little disagreement. We must point out that the method developed in this present work is more precise than the one used by Sprung. For a potential, whose Fourier transforms is a constant the direct term is found to be exactly equal to the exchange term while Sprung's direct and exchange terms are slightly different. This is because the integrals are not converging, but the present method is selecting contributions from symmetrically placed regions.

## HARD AND FINITE CORE

The hard core potential have been historically used to fit the scattering data and phase shifts. Since the hard core does not lend itself easily to perturbation theory, finite cores have been proposed. Bressel used the well known Hamada-Johnston potentials and replaced the infinite hard cores by finite square potentials. The core radius is $x_c = .4852$ Compton wave lengths and for $x > x_c$, the Hamada-Johnston potential is defined as a function of $x = \mu r$, where $\frac{1}{\mu}$ is the meson Compton wave length. For the $^1S_0$ wave

$$V = 670 \text{ MeV} \qquad\qquad \text{for } r < r_c$$

$$= -3 \, G_c m_\pi c^2 \, \frac{e^{-x}}{x} \, (1 + a_c \, \frac{e^{-x}}{x} + b_c \, (\frac{e^{-x}}{x})^2) \quad r > r_c$$

$$G_c = .08/3.$$

$$\mu = 137.4/197.325 \, f^{-1}$$

$$a_c = 8.7075$$

$$b_c = 10.6 \quad . \tag{3-11}$$

In our calculation, we have taken $v^{31} = v^{13}$ and $v^{33} = v^{11} = 0$
in the summation over spin and isospin (see equations 3-7).
As expected the integration over $k'$ for second order is much
more slowly convergent and the integration had to be carried
out to about $14k_f$. Again we have confirmed as several
authors have that the second order term is much larger than
the first order. Comparing our results with those of Sprung
for several values of $k_f$ in Table 3, we notice that we are
in very close agreement. Sprung's programme integrates out
to $q=7k_f$; at these large arguments $|q| \simeq |k'|$. The fact that
we have had to integrate to $|k'| = 14k_f$ indicates that Sprung
should carry his integration to larger momenta.

SEPARABLE CORE POTENTIAL

The Kerman-Levy separable potential for $^1S_0$ two-
proton states is of the following form

$$V(r,r') = \frac{\hbar^2}{m} \frac{1}{r_c^3} \{ \frac{V_1(r) \, V_1(r')}{rr'} - \frac{V_2(r) \, V_2(r')}{rr'} \} r,r' < r_c$$

$$= \frac{\delta(r-r')}{4\pi r^2} V_0(r-r') \qquad\qquad r,r' > r_c$$

$V_0 \equiv$ Hamada-Johnston potential for $^1S_0$ wave (see

equation 3-11)

$r_c =$ radius of non-locality.

The functions $V_1$ and $V_2$ are polynomials in $r/r_c$. Kerman and Levy have supplied two different potential models of this type. In Table 4 we have numerical values for these two potentials, distinguished by $r_c = 1.104$ fm and $1.461$ fm. The first-order term is very sensitive to the value of $k_f$ almost tripling at $k_f = 1.50$ its value at $1.00$ while the second order term increases by about 50% over this range for $r_c = 1.104$. Although this potential was fitted to the $^1S_0$ state only, we have arbitrarily allowed it to act in the $^3S_1$ state as well. The odd state forces are taken to be zero. This is more or less reasonable, being a Serber mixture. Note, however that the separable parts of the potential act only in the S-state while the core part acts in D, G, .... waves. For the larger radius of non-locality $r_c = 1.461$ we find that the second order term varies more rapidly with $k_f$, increasing by more than a factor two as $k_f$ is raised from 1.0 to 1.5. In our calculations, we have used the rough estimate that $v^{31} = v^{13}$ and $v^{33} = v^{11} = 0$ in equation (3-7).

The separable potential gives for low values of $k_f$ not enough binding energy for large $k_f$, the desired value is close for example at $k_f = 1.50$ for $r_c = 1.104$, the total binding energy is $- 16.7860$. However as $k_f$ increases, the magnitude of the binding energy increases as some power of $k_f$ as in figure 2, such that there would be collapse of nuclear matter as $k_f$ becomes very large.

This difficulty might be overcome if a proper central

plus tensor force was used in the $^3S_1$ state. The tensor force contribution usually saturates easily as the density is increased.

## SOFT CORE POTENTIAL MODEL

Sprung and his co-workers have fitted a weak, local, static potential. This potential has a great advantage over the Hamada-Johnston and Bressel-Kerman potentials used earlier in this chapter in the sense that for perturbation calculations, the expansion seems to converge at least to second order. For the SSC-NP-2[10] at $k_f=1.40$ the binding energy per particle is approximately 21.84 MeV, which exceeds the experimental value. Mr. P. Banerjee* has used this potential in a reaction matrix calculation, which includes many higher order terms in perturbation theory. He finds that, roughly speaking, taking the first order term alone gives a result close to the reaction matrix calculation.

*Private Communication.

# CHAPTER 4

## CONCLUSIONS

In this paper we have seen how perturbation theory can be carried out for non-local potentials. We have also seen how the second order term can be easily calculated when one introduces the kinematic function already developed by Ko. We have also carried out the calculations for the Kerman-Bressel soft core potentials in which we have verified that the second order term is bigger than the first order.

We have also shown that Swiatecki's conjecture for local, gaussian potential is accurate to about 15%; not perhaps as good as he expected. We have performed also the calculations for the Kerman-Levy potential consisting of a non-local separable core with the Hamada-Johnston tail and have shown that the expected convergence of the perturbation expansion for non-local potentials is true, at least to second order. However we find that this potential gives too much binding energy. We have also performed the calculations for a recent developed local, static weak potential which also gives too much binding energy.

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## TABLE 1

NUMERICAL VALUES FOR EULER'S FUNCTIONS $g(x)$, $k(x)$ DEFINED
IN TEXT, FOR GAUSSIAN POTENTIALS. THE RATIO OF EXCHANGE
TO DIRECT TERMS IN FIRST $(r_1)$ AND SECOND $(r_2)$ ORDER PER-
TURBATION THEORY ARE SHOWN

| x | $g(x)$ | $k(x)$ | $r_2$ | $r_1$ |
|---|---|---|---|---|
| .2 | .0209 | .0209 | .9990 | .9881 |
| .4 | .1824 | .1806 | .9897 | .9536 |
| .6 | .4952 | .4794 | .9682 | .8999 |
| .8 | .8940 | .8322 | .9308 | .8317 |
| 1.0 | 1.3026 | 1.1365 | .8725 | .7546 |
| 1.2 | 1.6645 | 1.3179 | .7918 | .6740 |
| 1.4 | 1.9568 | 1.3555 | .6927 | .5944 |
| 1.6 | 2.1811 | 1.2795 | .5836 | .5191 |
| 1.8 | 2.3504 | 1.1156 | .4746 | .4503 |
| 2.0 | 2.478 | 0.9285 | .3747 | .3890 |
| 2.2 | 2.5753 | .7451 | .2893 | .3355 |
| 2.4 | 2.6508 | .5841 | .2204 | .2894 |
| 2.6 | 2.7103 | .4523 | .1669 | .2499 |

## TABLE 2

### NUMERICAL VALUES FOR GAUSSIAN POTENTIALS

### DIRECT AND EXCHANGE TERMS AND COMPARISONS

| x | | FIRST ORDER | | SECOND ORDER | |
|---|---|---|---|---|---|
| | | PRESENT WORK | SPRUNG'S WORK | PRESENT WORK | SPRUNG'S WORK |
| | Direct | - 7.0524 | - 7.0524 | - .9565 | - .9560 |
| | Exchange | - 5.3219 | - 5.3218 | - .8345 | - .8338 |
| 1 | Total | -12.3743 | -12.3742 | - 1.7910 | -1.7898 |
| | Ratio E/D | .7546 | .7546 | .8725 | .8722 |
| | Direct | -56.4224 | -56.4300 | - 7.2780 | -7.2760 |
| | Exchange | -21.9506 | -21.9520 | - 2.7270 | -2.7140 |
| 2 | Total | -78.3730 | -78.3820 | 10.0050 | -9.9900 |
| | Ratio E/D | .3890 | .3890 | .3747 | .3731 |

Units for all energy terms are MeV.

## TABLE 3

### BRESSEL-KERMAN CORE POTENTIALS

| $k_f$ | PRESENT WORK | | SPRUNG'S WORK | |
| | $B^{(1)}$ | $B^{(2)}$ | $B^{(1)}$ | $B^{(2)}$ |
| FERMI$^{-1}$ | MEV | MEV | MEV | MEV |
|---|---|---|---|---|
| 1.4 | -14.370 | -67.01 | -14.508 | -65.586 |
| 1.0 | - 7.6012 | 24.4990 | - 7.6110 | -23.712 |
| 1.2 | -10.9800 | -42.2831 | -10.998 | -41.25 |
| 1.36 | -13.7814 | -61.0043 | -13.806 | -60.246 |

# TABLE 4

## NUMERICAL VALUES OF KERMAN SEPARABLE CORE POTENTIALS

## FOR $^1S_0$ WAVE

| $k_f$ | $T_f$ | $B^{(1)}$ | $B^{(2)}$ | $E_T$ |
|---|---|---|---|---|
| | | $r_c$ = 1.104 Fermi | | |
| Fermi$^{-1}$ | MEV | MEV | MEV | MEV |
| 1.00 | 12.4410 | -15.0172 | -1.3572 | - 3.9334 |
| 1.20 | 17.9150 | -24.1291 | -1.5147 | - 7.7288 |
| 1.36 | 23.0075 | -33.3360 | -1.6656 | -11.9941 |
| 1.40 | 24.3843 | -35.9258 | -1.7119 | -13.2484 |
| 1.50 | 27.9922 | -42.9388 | -1.8394 | -16.7860 |
| | | $r_c$ = 1.461 | | |
| 1.00 | 12.4410 | -15.5370 | -1.7040 | - 4.8000 |
| 1.20 | 17.9150 | -24.6019 | -2.2782 | - 8.9651 |
| 1.36 | 23.0075 | -33.4312 | -3.0147 | -13.4384 |
| 1.40 | 24.3843 | -35.8530 | -3.2467 | -14.7154 |
| 1.50 | 27.9922 | -42.2746 | -3.9186 | -18.2010 |

## TABLE 5

## VALUES OF M. K. SRIVASTAVA ET AL. POTENTIAL

## AND COMPARISON WITH SPRUNG'S CALCULATION

| | $v^{31} = v^{31}$ | $v^{33} = v^{11} = 0$ | | |
|---|---|---|---|---|
| | PRESENT WORK | | SPRUNG'S CALCULATIONS | |
| $K_f$ | $B^{(1)}$ | $B^{(2)}$ | $B^{(1)}$ | $B^{(2)}$ |
| 1.00 | -16.147 | - 3.898 | -16.114 | - 3.856 |
| 1.20 | -25.673 | - 5.835 | -25.620 | - 5.804 |
| 1.36 | -35.053 | - 7.956 | -34.984 | - 7.922 |
| 1.40 | -37.649 | - 8.573 | -37.572 | - 8.540 |
| 1.50 | -44.589 | -10.257 | -44.506 | -10.2240 |

# FIGURE CAPTIONS

Figure 1.  Relationship of ratio $r_1/r_2$ with x.

Figure 2.  Total Energy as a function of $k_f$ for Different

separable core radii.

Figure 3.  Function of $K(k,k',q)$ as the volume of four

equal intersecting spheres.

FIGURE I

**FIGURE 2**

**FIGURE 3**