FINITENESS OF PARTIALLY ORDERED SEMIGROUPS

A FINITENESS CRITERION FOR PARTIALLY ORDERED SEMIGROUPS

AND

ITS APPLICATIONS TO UNIVERSAL ALGEBRA

by

EVELYN M. NELSON, B.Sc.

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AUTHOR: Evelyn M. Nelson

SUPERVISOR: Professor G. Bruns

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SCOPE AND CONTENTS: A finiteness criterion is given for finitely generated positively ordered semigroups and this is used to show that various semigroups of operators in universal algebra are finite.

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PREFACE

The motivation for this work came from the following question posed by Dr. George Grätzer: If one considers various operators on classes of universal algebras, (for example, the one that assigns to each class of algebras the class of homomorphic images of its member algebras), then what can be said about the finiteness of the partially ordered semigroups that various combinations of these operators generate?

In section one we prove a general theorem concerning the finiteness of finitely generated positively ordered semigroups. A corollary to this theorem is used in section two to give a partial answer to Dr. Grätzer's question.

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SECTION ONE

PARTIALLY ORDERED SEMIGROUPS

A partially ordered semigroup is a triple $(G, \leq, \varphi)$ consisting of a set G, a partial order $\leq$ on G and an associative binary operation φ on G such that, for all a, b, c, d $\in$ G, if $a \leq b$ and $c \leq d$ then $\varphi(a, c) \leq \varphi(b,d)$. As usual, we write ab instead of $\varphi(a,b)$, and G instead of $(G, \leq, \varphi)$.

A positively ordered semigroup is a partially ordered semigroup G such that for all a, b $\in$ G, $ab \geq a$ and $ab \geq b$.

Every maximal element b of a positively ordered semigroup G is a maximum, since for every c $\in$ G, one has $cb \geq b$ and hence cb=b. This together with $cb \geq c$ gives $b \geq c$. Consequently, b is a maximum.

Let G be a positvely ordered semigroup generated by a set S. The elements of G are products of finite sequences of elements in S. For any non-empty set $T \subseteq S$, let $[T]$ be the sub-semigroup of G generated by T. Define the T-length $l_T(a)$ of an element $a \in [T]$ to be the smallest natural number n for which there exists an n-element sequence in T, the product of which is a. If $[T]$ has a largest element, say u, define $n_T = l_T(u)$.

Assume now that S is finite and that for each $T \subseteq S$, $[T]$ has a largest element. We define numbers N_k, $1 \leq k \leq |S|$ as follows:

$$N_1 = \max. \{ n_T \mid T \subseteq S,\ |T| = 1 \} + 1$$

$$N_k = \max. \{ n_T \mid T \subseteq S,\ |T| = k \} \quad \text{for } k \geq 2.$$

Theorem 1: A positively ordered semigroup G generated by a finite set S is finite iff, for each $T \subseteq S$, $[T]$ has a largest element. In this case:

(*) If T is a k-element subset of S, then $l_T(a) < \prod_{i=1}^{k} N_i$ for every $a \in [T]$.

Proof: The necessity is clear since any finite positively ordered semigroup has a maximal element which, as remarked above, is a maximum.

Assume now that the condition of the theorem is satisfied, and that S contains n elements. It is sufficient to prove (*) since this implies that every element in G is a product of a finite sequence of elements from S of length strictly less than $\prod_{i=1}^{n} N_i$, and there are only finitely many such sequences. We prove (*) by induction on k.

k=1: Let $x \in S$. $[\{x\}]$ has a largest element y, say of $\{x\}$-length $n_{\{x\}}$. But then $y = x^{n_{\{x\}}}$. Since G is positively ordered, we have for any natural number $m \geq 1$: $x^{n_{\{x\}}+m} = x^{n_{\{x\}}} x^m \geq x^{n_{\{x\}}}$, hence, since $x^{n_{\{x\}}}$ is maximal, $x^{n_{\{x\}}+m} = x^{n_{\{x\}}}$. It follows that, for every element $a \in [\{x\}]$, $l_{\{x\}}(a) \leq n_{\{x\}} < N_1$.

Now assume that (*) is true for a given $k \geq 1$ and that $U \subseteq S$ contains k+1 elements. It will be sufficient to show that every element of $[U]$ that is a product of $\prod_{i=1}^{k+1} N_i$ elements of U is equal to a product of strictly less than $\prod_{i=1}^{k+1} N_i$ elements of U. Let $N = \prod_{i=1}^{k+1} N_i$ and $N^1 = \prod_{i=1}^{k} N_i$. Suppose $a \in [U]$ and $a = \prod_{i=1}^{N} y_i$, where $y_i \in U$ for each i, $1 \leq i \leq N$. Then $a = \prod_{i=1}^{N_{k+1}} \left(\prod_{j=1}^{N^1} a_{(i-1)N^1+j} \right)$. With $a_i = \prod_{j=1}^{N^1} a_{(i-1)N^1+j}$, we obtain $a = \prod_{i=1}^{N_{k+1}} a_i$. We distinguish two cases:

Case 1: For each $x \in U$ and each natural number i, $1 \leq i \leq N_{k+1}$, there exists a natural number j, $1 \leq j \leq N^1$, such that $y_{(i-1)N^1+j} = x$. $[U]$ has a largest element w, of U-length n_U, i.e., $w = \prod_{i=1}^{n_U} z_i$, $z_i \in U$ for all i, $1 \leq i \leq n_U$. By assumption, for each i, $1 \leq i \leq n_U$, there exists a j, $1 \leq j \leq N^1$, such that $z_i = y_{(i-1)N^1+j}$. Since G is positively ordered, and since $n_U \leq N_{k+1}$, $a = \prod_{i=1}^{N_{k+1}} a_i \geq \prod_{i=1}^{n_U} z_i = w$. But w is the largest element of $[U]$ and hence $a = w$. Thus $l_U(a) = l_U(w) = n_U \leq N_{k+1} < \prod_{i=1}^{k+1} N_i$.

Case 2: There exists an $x \in U$ and a natural number i_o, $1 \leq i_o \leq N_{k+1}$, such that $x \neq y_{(i_o-1)N^1+j}$ for all j, $1 \leq j \leq N^1$. This implies that $a_{i_o} \in [U - \{x\}]$ and, since $U - \{x\}$ has exactly k elements, by the induction hypothesis, it follows that a_{i_o} has $(U - \{x\})$-length strictly less than N^1, and hence has U-length strictly less than N^1. Now $a = (\prod_{i=1}^{i_o-1} a_i)\, a_{i_o} \prod_{i=i_o+1}^{k+1} a_i$, and since $l_U(a_i) \leq N^1$ for each i, $l_U(a) < N_{k+1}N^1 = N$. Hence, in either case, $l_U(a) < N$ for arbitrary $a \in [U]$.

Corollary: If G is a positively ordered semigroup generated by a finite set S of n idempotent elements, and if there exists a bijection $f: S \longrightarrow \{1, 2, .. n\}$ such that, for $x, y \in S$, $f(x) \leq f(y)$ implies $yx \leq xy$, then G is finite and all elements of G have S-length strictly less than $2 \cdot n!$

Proof: It is sufficient to prove that, if $T \subsetneq S$ contains k elements, then $[T]$ has a largest element and this element has T-length less than or equal to k, for then $N_k \leq k$ for $k \geq 2$ and $N_1 = 1+1 = 2$, and $\prod_{i=1}^{n} N_i \leq 2 \cdot n!$

To show this let $a = \prod_{i=1}^{k} y_i$ be an arbitrary product of a sequence in T. If $f(y_i) > f(y_{i+1})$ holds for some i, $1 \leq i \leq k$, then by interchanging y_i and y_{i+1} in this product, we obtain an element $a^1 \geq a$. This observation, together with an induction argument and the idempotence of the generators, shows that the product of all the elements of T in their natural order (via f) is the largest element of $[T]$, hence $n_T \leq k$.

It may be noted at this point that the result in the corollary can be improved, i.e., an upper bound less than 2n! can be found for the lengths of elements in the semigroup generated by n idempotents which satisfy the hypothesis of the corollary. For example, if the n idempotents are $x_1, \ldots x_n$ and if x_1 is the one that satisfies $x_i x_1 \leq x_1 x_i$ for all i, $1 \leq i \leq n$, then every element of the semigroup is a product of a finite sequence of the x_i's in which x_1 appears at most once. Considerations of this type will yield better results; however, for the purposes of section 2, it is sufficient to know that the semigroup is finite. In fact, even if we obtain the best possible result for the general case of the corollary, special properties of the operators considered in Section 2, such as CS=SC or $P_S S = SP = SP_S$, further reduce the size of the semigroups they generate. Hence the best results for the general case will still not yield the best results in the special cases we consider.

On the other hand the upper bound given in Theorem 1 for the lengths of the elements cannot be improved, at least in the case n=2, as the following example shows.

Let G_o be the free semigroup generated by two elements t, u, and let $\equiv_1$ be the smallest congruence relation on G_o such that $t^3 \equiv_1 t^2$ and $u^3 \equiv_1 u^2$. Let $G_1 = G_o/\equiv_1$. G_1 is a semigroup generated by v, w, the equivalence classes modulo $\equiv_1$ of t, u respectively, such that $v^3=v^2$, $w^3=w^2$. Let $\leq_1$ be the smallest quasi-order on G_1 satisfying

(i) $v \leq_1 w$,

(ii) If $a,b \in G_1$, then $a \leq_1 ab$ and $b \leq_1 ab$,

(iii) If $a,b,c,d \in G_1$, and $a \leq_1 b$, $c \leq_1 d$, then $ac \leq_1 bd$.

Define a congruence relation $\equiv_2$ on G_1 by $a \equiv_2 b$ iff $a \leq_1 b$ and $b \leq_1 a$. Let $G = G_1/\equiv_2$. In the usual way, $\leq_1$ induces a partial order $\leq$ on G. It is clear from the definition of $\leq_1$ that $(G, \leq)$ is a positively ordered semigroup, generated by x and y, the equivalence classes of v, w respectivley, and that $x < y$, $x^3 = x^2$ and $y^3 = y^2$. The largest element of $[\{x\}]$ is x^2, the largest element of $[\{y\}]$ is y^2 and hence $N_1 = 2+1 = 3$. The largest element of $G = [\{x,y\}]$ is y^2, thus $N_2=2$. Now x^2yx^2 has length $5 = N_1N_2-1$. This can be seen either by writing down explicitly the elements of G and a multiplication table for them, or by considering exactly which elements of G_o are identified with each other in G via the two congruence relations on G_o and G_1. The fact that there is an element of G with length 5 shows that, for the case n=2, the upper bound given in the theorem cannot be improved.

SECTION TWO

SEMIGROUPS OF OPERATORS ON UNIVERSAL ALGEBRAS

A universal algebra is a set together with a family of finitary operations defined on that set. Two universal algebras are of the same type if their families of operations have the same indexing set and, if the two families are $(f_\lambda)_{\lambda \in L}$ and $(g_\lambda)_{\lambda \in L}$ and if f_λ is an n-ary operation, then g_λ is also n-ary. All algebras under consideration will be of the same type. We assume that the reader is familiar with the notion of subalgebra, homomorphic image, isomorphic image, direct product, congruence relation, and quotient of universal algebras.[1] A universal algebra U is a subdirect product of the family $(A_i)_{i \in I}$ of universal algebras if it is a subalgebra of the direct product $\prod_{i \in I} A_i$ and if, for each i, the restriction to U of the natural projection $P_j: \prod A_i \longrightarrow A_j$ maps onto A_j.

If $(A_i)_{i \in I}$ is any family of universal algebras and if $\mathfrak{F}$ is a filter on I, then the relation $\equiv_{\mathfrak{F}}$ defined on $\prod_{i \in I} A_i$ by $f \equiv_{\mathfrak{F}} g$ iff $\{i \mid f(i) = g(i)\} \in \mathfrak{F}$, (where the elements of a direct product $\prod_{i \in I} A_i$ are denoted by choice functions on $(A_i)_{i \in I}$), is a congruence relation. The quotient algebra $\prod A_i / \equiv_{\mathfrak{F}}$ of $\prod A_i$ modulo this congruence relation is called a filter product of the family $(A_i)_{i \in I}$. If $\mathfrak{F}$ is an ultrafilter $\prod A_i / \equiv_{\mathfrak{F}}$ is called an ultraproduct.[2] Note that

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1. For a detailed discussion of these notions, see [4]. Numbers in square brackes refer to the bibliography at the end.

2. Ibid.

if $\mathfrak{F} = \{I\}$, the filter product reduces to an ordinary product.

A universal algebra U is called a cover of the family $(A_i)_{i \in I}$ of universal algebras if, for each i, A_i is a subalgebra of U, and if $U = \bigcup_{i \in I} A_i$.

It may be noted at this point that all these operators are invariant under isomorphism, i.e., if $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ are families of universal algebras such that A_i is isomorphic to B_i for each $i \in I$, and if one of the above operations is applied to both of these families of algebras, the two resulting algebras will be isomorphic. For example, if $\mathfrak{F}$ is a filter on I, then the filter products $\prod_{i \in I} A_i / \equiv_{\mathfrak{F}}$ and $\prod_{i \in I} B_i / \equiv_{\mathfrak{F}}$ are isomorphic.

A class of universal algebras will be called algebraic if it contains all isomorphic copies of the algebras contained in it.

For an arbitrary class K of universal algebras we define S(K) to be the smallest algebraic class containing all subalgebras of algebras in K. Then S is an operator on arbitrary classes of universal algebras. Similarly the operators H, P, P_S, P_F, P_U and C are defined, to correspond to homomorphic images, direct products, sub-direct products, filter products, ultraproducts and covers respectively.

For two operators X, Y, we define XY by $XY(K) = X(Y(K))$ for an arbitrary algebraic class K of universal algebras. This defines an associative binary operation and so we can consider the semigroup G generated by these operators. The relation $\leqslant$ is defined on G by: $X \leqslant Y$ iff $X(K) \subseteq Y(K)$ for all algebraic classes K of universal algebras. Then $(G, \leqslant)$ is a partially ordered semigroup. Moreover, since, if X is any one of S, H, P, P_S, P_F, P_U, or C, then $X(K) \supseteq K$ for all algebraic classes K of universal algebras, $(G, \leq)$ is a positively ordered semigroup.

Lemma 1: The operators S, H, P, P_S, P_F, P_U and C are idempotent.

Proof: The proof that $S^2=S$, $H^2=H$, $P^2=P$, ${P_S}^2 = P_S$, $C^2=C$ is trivial.

${P_F}^2=P_F$: Let K be an arbitrary algebraic class of universal algebras and let $A \in {P_F}^2(K)$. Since P_F is invariant under isomorphism, it is enough to consider $A=\prod_{i\in I}A_i/\equiv_{\mathfrak{F}}$ where $\mathfrak{F}$ is a filter on I, $A_i = \prod_{j\in J_i} B_{ij}/\equiv_{\mathfrak{F}_i}$ for each i, $\mathfrak{F}_i$ a filter on J_i and $B_{ij} \in K$. Let $S = \sum_{i\in I} J_i = \{(i,j) \mid i\in I,\ j\in J_i\}$. The filtered sum $\mathfrak{A}$ of the family $(\mathfrak{F}_i)_{i\in I}$ of filters is defined as follows:

$$\mathfrak{A} = \{M \subseteq S \mid \{i \mid M(i) \in \mathfrak{F}_i\} \in \mathfrak{F}\}, \text{ where } M(i) = \{j \mid j \in J_i \text{ and } (i,j)\in M\}.$$

It is easy to check that $\mathfrak{A}$ is a filter. For each $f \in \prod_{(i,j)\in S} B_{ij}$, and each $i\in I$, define $f(i,\cdot) \in A_i$ by $f(i,\cdot)(j) = f(i,j)$ for $j\in J_i$.

Define $\varphi : \prod_{(i,j)\in S} B_{ij} \longrightarrow A$ by

$$\varphi(f) = \left[\left(\left[f(i,\cdot)\right]_{\equiv_{\mathfrak{F}_i}}\right)_{i\in I}\right]_{\equiv_{\mathfrak{F}}} \quad \text{where}$$

square brackets denote congruence classes, and subscripts the congruence relations. For $f,g \in \prod_{(i,j)\in S} B_{ij}$,

$$\begin{aligned}
\varphi(f) = \varphi(g) \quad & \text{iff} \quad \{i \mid [f(i,\cdot)]_{\equiv_{\mathfrak{F}_i}} = [g(i,\cdot)]_{\equiv_{\mathfrak{F}_i}}\} \in \mathfrak{F} \\
& \text{iff} \quad \{i \mid \{j\in J_i \mid f(i,\cdot)(j) = g(i,\cdot)(j)\} \in \mathfrak{F}_i\} \in \mathfrak{F} \\
& \text{iff} \quad \{(i,j) \mid f(i,j) = g(i,j)\} \in \mathfrak{A} \\
& \text{iff} \quad f \equiv_{\mathfrak{A}} g.
\end{aligned}$$

This, together with the fact that φ is an epimorphism, yields that $\prod_{(i,j)\in S} B_{ij}/\equiv_{\mathfrak{A}}$ is isomorphic to A. Thus $A \in P_F(K)$. Hence $P_F^2 \leq P_F$ and thus $P_F^2 = P_F$.

If $\mathfrak{F}$, $\mathfrak{F}_i$ are all ultrafilters, then $\mathfrak{A}$ is also an ultrafilter: If $M \notin \mathfrak{A}$ then $\{i \mid M(i) \in \mathfrak{F}_i\} \notin \mathfrak{F}$. Since $\mathfrak{F}$ is an ultrafilter, this implies that $\{i \mid M(i) \notin \mathfrak{F}_i\} \in \mathfrak{F}$. But the $\mathfrak{F}_i$ are ultrafilters, hence $\{i \mid J_i - M(i) \in \mathfrak{F}_i\} \in \mathfrak{F}$. Now $J_i - M(i) = (S-M)(i)$. Hence $\{i \mid (S-M)(i) \in \mathfrak{F}_i\} \in \mathfrak{F}$. This implies that $S-M \in \mathfrak{A}$ and hence $\mathfrak{A}$ is an ultrafilter.

This, together with the above, yields that $P_U^2 \leq P_U$, and hence $P_U^2 = P_U$.

Lemma 2: $P_F S \leq SP_F$, $P_U S \leq SP_U$, $PS \leq SP$.[3]

Proof: Let K be an arbitrary class of universal algebras and let $A \in P_F S(K)$. It is enough to consider $A = \prod_{i\in I} A_i/\equiv_{\mathfrak{F}}$ where $\mathfrak{F}$ is a filter on I and, for each i, A_i is a subalgebra of B_i, and $B_i \in K$. Then the canonical homomorphism from $\prod_{i\in I} A_i/\equiv_{\mathfrak{F}}$ to $\prod_{i\in I} B_i/\equiv_{\mathfrak{F}}$ is a monomorphism, and hence $A \in SP_F(K)$. Thus $P_F S \leq SP_F$. The same argument shows that $P_U S \leq SP_U$, and, if $\mathfrak{F} = \{I\}$, that $PS \leq SP$.

Lemma 3: $P_F H \leq HP_F$, $P_U H \leq HP_U$, $PH \leq HP$.[4]

Proof: Let K be an arbitrary algebraic class of universal algebras and let $A \in P_F H(K)$. Since P_F is invariant under isomorphism it is enough to consider $A = \prod_{i\in I} A_i/\equiv_{\mathfrak{F}}$ where $\mathfrak{F}$ is a filter on I and, for each i, there is a $B_i \in K$ and an epimorphism $\varphi_i : B_i \rightarrow A_i$. Let $B = \prod_{i\in I} B_i/\equiv_{\mathfrak{F}}$, and define $\varphi: \prod_{i\in I} B_i \rightarrow \prod_{i\in I} A_i$ by $(\varphi(f))(i) = \varphi_i(f(i))$.

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3. PS ≤ SP is proved in [1] .

4. PH ≤ HP is proved in [1] .

It is then obvious that, for any $f, g \in \prod_{i \in I} B_i$, that $f \equiv_{\mathcal{F}} g$ (in $\prod B_i$) implies that $\varphi(f) \equiv_{\mathcal{F}} \varphi(g)$ (in $\prod A_i$). Hence φ induces an epimorphism $\bar{\varphi}$ of $\prod_{i \in I} B_i / \equiv_{\mathcal{F}}$ onto $\prod_{i \in I} A_i / \equiv_{\mathcal{F}}$. Hence $A \in HP_F(K)$, and thus $P_F H \leq HP_F$.

The same argument shows that $P_U H \leq HP_U$, and, if $\mathcal{F} = \{I\}$, that $PH \leq HP$.

Lemma 4: $P_F C \leq CP_F$, $P_U C \leq CP_U$, $PC \leq CP$.[4]

Proof: Let K be an arbitrary algebraic class of universal algebras and let $A \in P_F C(K)$. We may consider $A = \prod_{i \in I} A_i / \equiv_{\mathcal{F}}$, where $\mathcal{F}$ is a filter on I and, for each i, $A_i = \bigcup_{i \in I} B_{ij}$ where the B_{ij} are subalgebras of A_i and $B_{ij} \in K$. For each $r \in \prod_{i \in I} J_i$, let $A_r = \prod_{i \in I} B_{ir(i)} / \equiv_{\mathcal{F}}$. Then the canonical homomorphism $\varphi_r : A_r \longrightarrow A$ is a monomorphism. Moreover, $A = \bigcup_{r \in \prod_{i \in I} J_i} \varphi_r(A_r)$. Hence $A \in CP_F(K)$, and so $P_F C \leq CP_F$. The same argument shows that $P_U C \leq CP_U$, and, in the special case where $\mathcal{F} = \{I\}$, that $PC \leq CP$.

Theorem 2: The partially ordered semigroups generated by the following sets of operators are finite:

1. $\{H, S, P, P_S\}$
2. $\{H, S, P, P_F, C\}$
3. $\{H, S, P_F, P_U, C\}$.

Proof: These sets of operators generate positively ordered partially ordered semigroups, and, by Lemma 1, each of the operators in these sets is idempotent. We apply the corollary of Theorem 1:

1. Define $\varphi : \{H, S, P, P_S\} \longrightarrow \{1, 2, 3, 4\}$ by $\varphi(H)=1$,

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4. $PC \leq CP$ is proved in [2].

$\varphi(S) = 2$, $\varphi(P) = 3$, $\varphi(P_S) = 4$. It is sufficient to show that $SH \leq HS$, $PH \leq HP$, $PS \leq SP$, $P_SH \leq HP_S$, $P_SS \leq SP_S$, and $P_SP \leq PP_S$. The first, fourth and fifth of these inequalities are proved in [3] , page 44 ff. $PH \leq HP$ and $PS \leq SP$ are proved above. Since $P \leq P_S$, $P_SP = P_S \leq PP_S$. Hence the semigroup generated by H, S, P, P_S is finite.

2. Define $\varphi : \{H, S, P, P_F, C\} \longrightarrow \{1, 2, 3, 4, 5\}$ by $\varphi(C) = 1$, $\varphi(H) = 2$, $\varphi(S) = 3$, $\varphi(P) = 4$, $\varphi(P_F) = 5$. In addition to what has already been shown it is sufficient to show that $P_FH \leq HP_F$, $P_FS \leq SP_F$, $P_FC \leq CP_F$, $P_FP \leq PP_F$, $PC \leq CP$, $HC \leq CH$ and $SC \leq CS$. The first three and the fifth inequalities are proved in the lemmas above, $HC \leq CH$ and $SC \leq CS$ are proved in [2] and, since $P \leq P_F$, $P_FP = P_F \leq PP_F$. Hence the semigroup generated by H, S, P, P_F, C is finite.

3. Define $\varphi : \{H, S, P_F, P_U, C\} \longrightarrow \{1, 2, 3, 4, 5\}$ by $\varphi(C) = 1$, $\varphi(H) = 2$, $\varphi(S) = 3$, $\varphi(P_F) = 4$, $\varphi(P_U) = 5$. In addition to what is shown above, it is sufficient to show that $P_UH \leq HP_U$, $P_US \leq SP_U$, $P_UC \leq CP_U$ and $P_UP_F \leq P_FP_U$. The proofs of the first three inequalities are in the lemmas above, and since $P_U \leq P_F$, $P_UP_F = P_F \leq P_FP_U$. Hence the semigroup generated by H, S, P_F, P_U and C is finite.

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