DIRECT KINEMATIC SHELL THEORY

ON THE DIRECT KINEMATIC THEORY OF THIN ELASTIC SHELLS

by

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SCOPE AND CONTENTS:

This thesis consists of a rigorous development of the direct kinematic, small-displacement theory of thin elastic shells. The theory is developed, so as to facilitate a derivation of the equations of compatibility of middle-surface strains. These equations are developed by the kinematic approach and it is shown that this produces a more coherent relation of such equations to the general theory of shells, as no special techniques are required. The equations of compatibility are developed again by the formal Saint-Venant method; this development serves to substantiate the validity of the kinematic approach. At the same time, it provides many useful identities which are then employed as transformation relations, in order to compare the various forms of compatibility equations, as developed by other authors. A general comparison of kinematic shell theory with other nonkinematic methods is undertaken, and appended to the main discussion.

PREFACE

The reason for the encompassing character of this thesis on the Theory of Thin Elastic Shells, lies in the fact that no single work exists, which pursues a consistent and rigorous direct kinematic theory. In the opinion of the author, the direct kinematic exposition of the theory of thin shells offers a more intuitive conceptual grasp of the subject matter for physically-motivated professionals, such as engineers.

The lack of direct kinematic considerations in the available treatments of the 'conditions of compatibility of deformation of the elastic surface', causes this facet of the topic to be especially unsatisfactory for engineers. It is this direct kinematic treatment of the compatibility conditions which forms the core of the research in this thesis.

It soon became apparent, in the course of planning the material to be included herein, that one of two courses of action must be taken, either: to assume that any reader might be expected to be familiar with the basic kinematic concepts and to thus begin a discussion of compatibility *in media res*, or: to develop the entire theory from the very fundamentals of the direct method of vector analysis and thus include a large amount of material which is not original with this author. The latter approach having been selected as the better of the two, it is then essential that the following be noted.

The whole of Book I (Chapters 1, 2 and 3) does not originate with this author. These chapters represent, in fact, suitably-modified versions of the lectures as delivered by Professor G. AE. Oravas, during the course of the 1965-1966 session of lectures on the "Theory of Surface Structures".

In Book II, approximately half of Chapter 4 falls into the same classification as Book I, above; the remainder of Chapter 4, as well as the whole of Chapters 5 and 6, constitutes the original research of the author.

In this way, the direct kinematic analysis of the problem has been developed from the basic postulates, thereby requiring no *a priori* knowledge of this method on the part of the reader. Furthermore, the monographic form of this thesis has permitted an integrated and consistent development of the theory, without the necessity of introducing a multiplicity of interspersed explanatory footnotes (as would be otherwise required for the clarification of the various procedures and concepts employed).

The author takes this opportunity to express his sincere gratitude to his Research Supervisor, Professor G. AE. Oravas, not only for his omnipresent guidance through a multitude of difficulties, but also for his inspiration in the execution of this, and all endeavours. The author extends to Dr. W. K. Tso, of the Department of Civil Engineering and Engineering Mechanics, his sincere thanks for that gentleman's comments and suggestions, regarding specific points in the development of the compatibility equations. Sincere thanks are also due the National Research Council of Canada, whose award greatly facilitated the author's investigation. The author wishes also, to express his thanks to Miss Joan E. Armour, who typed the entire manuscript.

L. McLean

Hamilton, Ontario
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NOTATION

$\bar{r}$	position vector to an arbitrary point in the shell
s	arc length of curve
$\bar{t}$	tangent vector to space curves
$\bar{N}$	normal vector to space curves
$\bar{n}$	unit normal vector to space curves
κ	curvature (normal) of a space curve
ϕ	an angle/also an arbitrary scalar point-function
$\bar{e}_\phi$	unit vector specifying direction of rotation
$\bar{e}_n$	unit normal vector (in general)
R	Radius of curvature
$\bar{b}$	unit binormal vector to space curves
λ	a scalar factor/also an elastic constant
τ	torsion of a space curve
$\bar{D}$	the Darboux vector
α_1, α_2	parametric coordinates for the surface
$\bar{g}_1, \bar{g}_2$	base (surface tangent) vectors in direction α_1, α_2
$\bar{e}_1, \bar{e}_2$	unit (surface tangent) vectors in direction α_1, α_2
$\bar{e}_3$	unit normal to the surface
$d\bar{A}$	directed differential surface area
dA_n	absolute value of $d\bar{A}$, above
g_{11}, g_{12}, g_{22}	metric coefficients of first fundamental form
ds_1, ds_2	differential arc lengths along α_1, α_2
g_1, g_2	absolute value of $\bar{g}_1, \bar{g}_2$
A	an area of arbitrary nature

$\overline{N}^{(g)}$	geodesic curvature of a surface (directed)
$\overline{N}^{(n)}$	normal curvature of a surface (directed)
$\kappa^{(g)}$	geodesic curvature of a surface
$\kappa^{(n)}$	normal curvature of a surface
R_n	radius of normal curvature of a surface
b_{11}, b_{12}, b_{22}	coefficients of the second fundamental form
$\overline{\xi}$	an arbitrary vector point-function
I	symbolic representation of the first fundamental form
-II	symbolic representation of the second fundamental form
$\delta(\ )$	symbolic operator indicating the first variation
D	the determinant of a matrix
$\lambda_1, \lambda_2, \ldots$	the roots of an equation in λ
$C_1, C_2, \ldots$	constants
$\mathcal{I}$	a line of intersection of two planes
$\overline{\pi}$	an auxiliary position vector
$\overline{Q}$	a vector defined by other (previously-defined) vectors
$\overline{e}_t$	unit tangent vector to a curve in a surface
$\overline{e}_b$	unit binormal vector to a curve in a surface
$\overline{C}$	the CESÀRO-BURALI-FORTI vector
$\kappa^{(t)}$	geodesic torsion of a curve in a surface
$\kappa^{(3)}$	geodesic curvature of a curve in a surface
$\overline{\kappa}$	pure curvature vector (for a geodesic)
δ_{ij}	the KRONECKER Delta (for Cartesian systems)
$\overline{\overline{1}}$	the identity tensor, rank 2 (dyadic)
$\overline{\overline{\mathit{1}}}$	the planar identity tensor, rank 2 (dyadic)
III	symbolic representation of the third fundamental form

$(\ )_c$	the conjugate of a tensor
$\overline{e}_*^1$, $\overline{e}_*^2$	unit binormals to lines α_1 and α_2
$\kappa_{11} \equiv \kappa_1^{(n)}$	normal curvature, line 1 (α_1)
$\kappa_{12} \equiv \kappa_1^{(t)}$	geodesic torsion, line 1 (α_1)
$\kappa_{13} \equiv \kappa_1^{(3)}$	geodesic curvature, line (α_1)
$\kappa_{21} \equiv \kappa_2^{(t)}$	geodesic torsion, line 2 (α_2)
$\kappa_{22} \equiv \kappa_2^{(n)}$	normal curvature, line 2 (α_2)
$\kappa_{23} \equiv \kappa_2^{(3)}$	geodesic curvature, line 2 (α_2)
$\overline{C}_1$	the CESÀRO-BURALI-FORTI vector for line 1
$\overline{C}_2$	the CESÀRO-BURALI-FORTI vector for line 2
κ_g	the Gaussian curvature
ψ	an angle/also an arbitrary scalar point-function
F, F'	arbitrary scalar point-functions
$\overline{F}, \overline{F}'$	arbitrary vector point-functions
e	base of natural logarithms (ℓn)
γ_1, γ_2	differentials of the logarithms of g_2, g_1 w.r.t. s_1, s_2
$\overline{v}$	an arbitrary vector in the undeformed surface
$\overline{V}$	an arbitrary vector in the deformed surface
$\overline{e}$	an arbitrary unit vector, undeformed surface
$\overline{\Omega}_1, \overline{\Omega}_2$	$\overline{C}_1$ and $\overline{C}_2$, augmented by a geodesic curvature term
$\omega_{12} = -\omega_{21}$	angle between $\overline{e}_1$ and $\overline{e}_2$, if other than $\frac{\pi}{2}$
$D(\ )$	differential operator defined as $(\frac{\partial}{\partial s_1} + \gamma_1)(\ )$ or $(\frac{\partial}{\partial s_2} + \gamma_2)(\ )$
$\overline{r}^\circ$	position vector to the undeformed middle surface
$\overline{R}^\circ$	position vector to the deformed middle surface
dS	arc length in the deformed middle surface
$\overline{G}_1$, $\overline{G}_2$	base vectors for the deformed middle surface

G_{11}, G_{12}, G_{22}	metric coefficients for the deformed middle surface
$\overline{E}_1, \overline{E}_2$	unit tangent vectors for the deformed middle surface
$\overline{E}_3$	unit normal to the deformed surface
$\overline{u}^\circ$	displacement vector of middle surface
$u_1^\circ, u_2^\circ, u_3^\circ$	components of $\overline{u}^\circ$ in directions $\overline{e}_1, \overline{e}_2, \overline{e}_3$
$\varepsilon_{11}^\circ \equiv \phi_{11}$	longitudinal straining (direction $\overline{e}_1$) during deformation
ϕ_{12}	detrusion ($\overline{e}_1$ towards $\overline{e}_2$ about $\overline{e}_3$) during deformation
ϕ_{13}	rotation ($\overline{e}_1$ towards $\overline{e}_3$ about $-\overline{e}_2$) during deformation
ϕ_{21}	detrusion ($\overline{e}_2$ towards $\overline{e}_1$ about $-\overline{e}_3$) during deformation
$\varepsilon_{22}^\circ \equiv \phi_{22}$	longitudinal straining (direction $\overline{e}_2$) during deformation
ϕ_{23}	rotation ($\overline{e}_2$ towards $\overline{e}_3$ about $\overline{e}_2$) during deformation
m_1, m_2	incremental metric measures, accrued in deformation
$\overline{E}_*^1$	binormal to line 1, deformed configuration
$\overline{E}_*^2$	binormal to line 2, deformed configuration
$\overline{C}_1^\dagger$	$\overline{C}_1$ plus its first variation
$\overline{C}_2^\dagger$	$\overline{C}_2$ plus its first variation
K_{11}, K_{12}(etc)	same as κ_{11}, κ_{12}, but referring to the deformed case
α_3	parametric coordinate normal to the surface
$\overline{u}$	displacement vector of parallel surface
ds_1^*, ds_2^*	arc lengths in a parallel surface (undeformed)
dS_1^*, dS_2^*	arc lengths in a parallel surface (deformed).
h	shell thickness
a_1	ratio of ds_1 to ds_1^*
a_2	ratio of ds_2 to ds_2^*
$\overline{\overline{u}}$	the deformation tensor for a parallel surface
u_{11}, u_{12}(etc)	components of $\overline{\overline{u}}$ in directions $\overline{e}_1\overline{e}_1, \overline{e}_1\overline{e}_2$, (etc.)

$\bar{\bar{\varepsilon}}$	the strain tensor for a parallel surface
$\varepsilon_{11}, \varepsilon_{12}$(etc.)	components of $\bar{\bar{\varepsilon}}$ in directions $\bar{e}_1\bar{e}_1$, $\bar{e}_1\bar{e}_2$, (etc.)
$\bar{\bar{\varepsilon}}^{o}$	the strain tensor for the middle surface
Γ_1, Γ_2	same as γ_1, γ_2, but referring to the deformed case
$\bar{\Omega}_1^{+}$, $\bar{\Omega}_2^{+}$	$\bar{C}_1^{+}$ and $\bar{C}_2^{+}$ augmented by an incremental geodesic curvature
$\bar{\bar{P}}$	the tensor representing $\frac{\partial}{\partial \bar{r}} \times \bar{\bar{\varepsilon}}$ ("Curl $\bar{\bar{\varepsilon}}$")
$\bar{\bar{Q}}$	the tensor representing $\frac{\partial}{\partial \bar{r}} \times \bar{\bar{\varepsilon}} \times \frac{\partial}{\partial \bar{r}}$ ("Double Curl $\bar{\bar{\varepsilon}}$")
$\bar{f}$	body force intensity vector
$\bar{r}_c$	absolute acceleration of a mass centre
$\bar{\sigma}_n$	a stress vector on some normal face
ρ	mass density
$\bar{\bar{\sigma}}$	the stress tensor
σ_{11}, σ_{12}(etc)	components of $\bar{\bar{\sigma}}$ in the directions $\bar{e}_1\bar{e}_1$, $\bar{e}_1\bar{e}_2$(etc)
$\bar{\bar{F}}(\sigma)$	stress resultant tensor, defined through $\bar{\bar{\sigma}}$
$\bar{\bar{M}}(\sigma)$	stress couple tensor, defined through $\bar{\bar{\sigma}}$
P_1, P_2, P_3	boundary forces in directions $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$
$\mathcal{M}_1$, $\mathcal{M}_2$	moments caused by boundary stresses (directly)
μ, λ	CAUCHY-LAMÉ elastic constants
ν	POISSON's ratio
E	the modulus of elasticity
$\bar{\bar{T}}$, $\bar{\bar{D}}$	arbitrary tensors of rank 2 (dyadics)
$\bar{m}, \bar{n}, \bar{p}, \bar{q}$	arbitrary vectors
$E_{\alpha\beta\gamma}$	LEVI-CIVITA three-index density symbol
$\delta^{\alpha\beta\gamma}_{\mu\nu\pi}$	generalized KRONECKER Delta
ℓ^{β}_{α}	cosine of angle: defined by $\ell^{\beta}_{\alpha} = \bar{e}_{\beta} \cdot \bar{e}_{\alpha}$

BOOK I. DIFFERENTIAL GEOMETRY

CHAPTER 1

Differential Geometry of Space Curves

1.1. THE FUNDAMENTAL SYSTEM

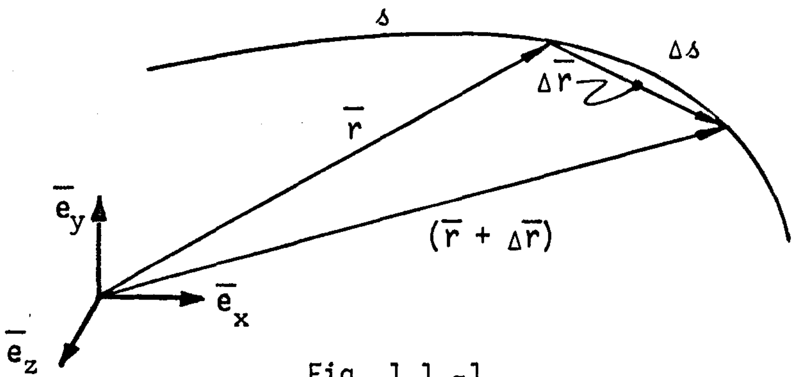

Fig. 1.1.-1.

A space curve may be specified by the *position vector (or radius vector)*, $\bar{r} \equiv \bar{r}(s)$, which can be considered to be a function of the arc length parameter, s, of the curve.

Two points on such a curve, separated by the (small) finite distance, Δs, along the curve are specified by $\bar{r}(s)$ and $\bar{r}(s + \Delta s)$. A Taylor Series expansion shows:

$$e^{\partial s \frac{\partial}{\partial s}}\, \bar{r}(s) = \bar{r}(s) + \Delta s \frac{\partial \bar{r}(s)}{\partial s} + \frac{1}{2!}\frac{\partial^2 \bar{r}(s)}{\partial s^2}\Delta s^2 + \ldots..$$

Neglecting terms of the second and higher orders as being negligible,

then,
$$\bar{r}(s + \Delta s) = \bar{r} + \Delta\bar{r}$$

where it is assumed, in all following discussion, that $\bar{r} \equiv \bar{r}(s)$ unless otherwise specified.

Thus, the relative positions of $\overline{r}(s)$ and $\overline{r}(s + \Delta s)$ may be given by

$$\Delta\overline{r} = \overline{r}(s + \Delta s) - \overline{r}(s)$$

1.2. THE TANGENT VECTOR

The tangent vector, $\overline{t}$, to a space curve will be defined as the limiting position of the secant, $\Delta\overline{r}$, as the arc length, Δs, approaches zero.

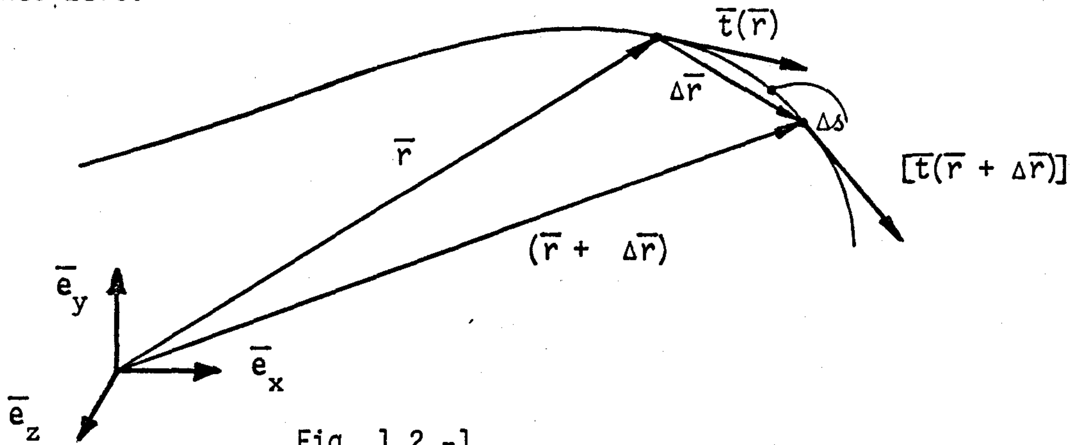

Fig. 1.2.-1.

From Fig. 1.2.-1.,

$$\left.\begin{aligned} \overline{r} &\equiv \overline{r}(s) \\ \overline{r}(s + \Delta s) &= \overline{r} + \Delta\overline{r} \end{aligned}\right\} \text{— as before}$$

Then

$$\lim_{\Delta s \to 0} \left[\frac{\Delta\overline{r}}{\Delta s}\right] = \frac{d\overline{r}}{ds} \equiv \overline{t}$$

Hence, $\overline{t}$ is referred to as the *tangent vector*.

Considering the magnitude of this vector:

$$|\overline{t}| = \lim_{\Delta s \to 0} \left|\frac{\Delta\overline{r}}{\Delta s}\right| = \lim_{\Delta s \to 0} \frac{|\Delta\overline{r}|}{|\Delta s|} = \lim_{\Delta s \to 0} \frac{|\Delta\overline{r}|}{\Delta s}$$

$$= \left|\frac{d\overline{r}}{ds}\right| = \frac{ds}{ds} = 1$$

this substantiates that $\overline{t}$ is a *unit tangent vector*.

Consequently, from $\frac{d\bar{r}}{ds} = \bar{t}$, $d\bar{r} = \bar{t}\,ds$

thus, $$d\bar{r} \cdot d\bar{r} = \bar{t}\,ds \cdot \bar{t}\,ds = (\bar{t} \cdot \bar{t})ds^2$$

or, $$d\bar{r} \cdot d\bar{r} = ds^2 \qquad \{\text{as } \bar{t} \cdot \bar{t} = 1\}$$

1.3. THE NORMAL VECTOR

The normal vector, $\bar{N}$, to a space curve will be defined as the change of the tangent vector, $\bar{t}$, per unit arc length; or, the rate of change of the tangent vector with respect to the arc length.

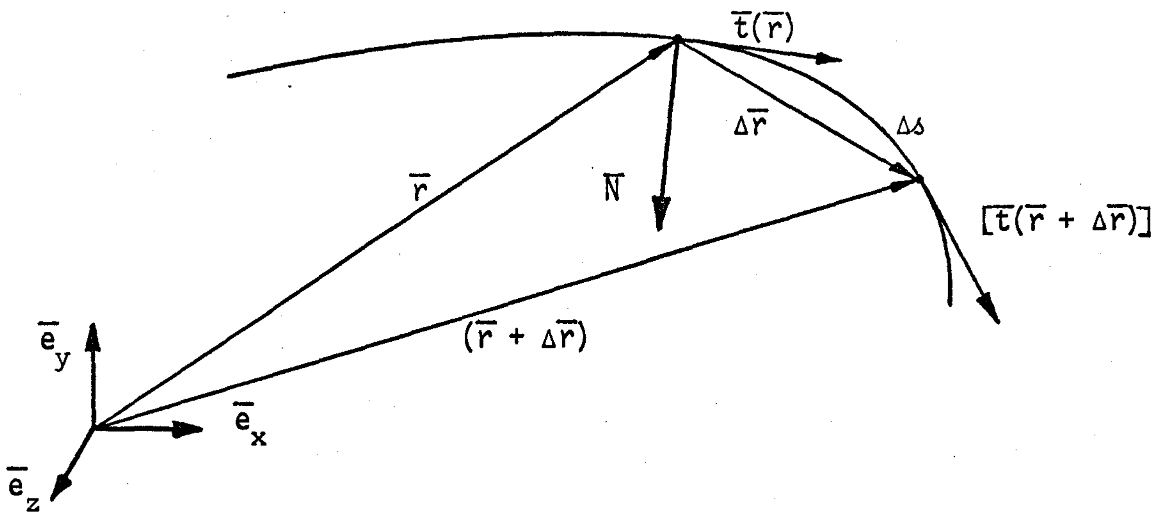

Fig. 1.3.-1.

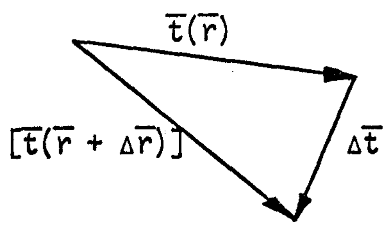

Fig. 1.3.-2.

From Fig. 1.3.-2., $\bar{t}(s) + \Delta\bar{t} = \bar{t}(s + \Delta s)$

or $\bar{t}(s + \Delta s) - \bar{t}(s) = \Delta\bar{t}$

Then, $\lim_{\Delta s \to 0} \left[\frac{\Delta \bar{t}}{\Delta s}\right] = \frac{d\bar{t}}{ds} \equiv \bar{N}$

Hence, $\bar{N}$ is referred to as the *normal vector.*

NOTE: $\bar{N} = \frac{d\bar{t}}{ds} = \frac{d}{ds}\left[\frac{d\bar{r}}{ds}\right] \equiv \frac{d^2\bar{r}}{ds^2}$

1.3.1. The Relation of Tangent and Normal Vectors

From the identity $\bar{t}\cdot\bar{t} = 1$

Then $\frac{d}{ds}(\bar{t}\cdot\bar{t}) = \frac{d}{ds}(1)$

or $\frac{d\bar{t}}{ds}\cdot\bar{t} + \bar{t}\cdot\frac{d\bar{t}}{ds} = 0$

$\bar{N}\cdot\bar{t} + \bar{t}\cdot\bar{N} = 0$ as $\frac{d\bar{t}}{ds} = \bar{N}$

so $2\,\bar{t}\cdot\bar{N} = 0$

Therefore, for nontrivial $\bar{t}$, $\bar{N}$, then $\bar{t}$ and $\bar{N}$ must be mutually perpendicular, in order for the dot product to vanish. Thus, $\bar{N}$ is at $\frac{\pi}{2}$ to $\bar{t}$.

1.3.2. Curvature

As $\bar{t}$ is a unit vector, then $\bar{N} = \frac{d\bar{t}}{ds}$ will not, in general, be of unit magnitude. If $\bar{n}$ is introduced as a unit vector in the direction of $\bar{N}$, then it may be said:

$\bar{N} = \kappa\bar{n}$ where κ is a constant

Obviously, as $\overline{N} = |\overline{N}|\ \overline{n} = \kappa\overline{n}$

then, $$\kappa = |\overline{N}| = \left|\frac{d\overline{t}}{ds}\right| = \left|\frac{d^2\overline{r}}{ds^2}\right|$$

Now, $$\left|\frac{d\overline{t}}{ds}\right| = \lim_{\Delta s \to 0} \left|\frac{\Delta\overline{t}}{\Delta s}\right| = \left|\lim_{\Delta s \to 0} \frac{\Delta\overline{t}}{\Delta s}\right|$$

By EUCLIDIAN geometry, Fig. 1.3.2.-1. is obtained:

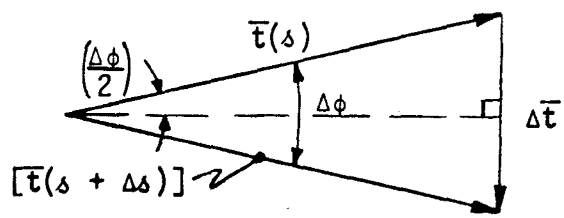

Fig. 1.3.2.-1.

From the above, $$|\overline{t}(s)| = |\overline{t}(s + \Delta s)| = 1$$

then, $$|\Delta\overline{t}| = 2|\overline{t}|\ \mathrm{Sin}\left(\frac{\Delta\phi}{2}\right)$$
$$= 2|\overline{t}|\ \left[\frac{\Delta\phi}{2} - \frac{1}{3!}\left(\frac{\Delta\phi}{2}\right)^3 + \cdots\cdots\right]$$

(Sine Series expansion)

so, $$|\Delta\overline{t}| \doteq 2|\overline{t}|\ \frac{\Delta\phi}{2} = \Delta\phi \qquad \ldots\ \{1.3.2.-1.\}$$

-to the first order of approximation,

as $|\overline{t}| = 1.$

Then it may be said:

$$\lim_{\Delta s \to 0} \left|\frac{\Delta\overline{t}}{\Delta s}\right| = \lim_{\Delta s \to 0} \frac{|\Delta\overline{t}|}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} = \frac{d\phi}{ds}$$

but $$\kappa = \left|\frac{d\overline{t}}{ds}\right| = \lim_{\Delta s \to 0} \left|\frac{\Delta\overline{t}}{\Delta s}\right|$$

so, $$\kappa = \frac{d\phi}{ds}$$

The quantity $\kappa = \frac{d\phi}{ds}$ is thus called the *curvature*, being defined as: the (angular) rate of change of tangent with respect to arc length.

NOTE: Prescribing the very small rotation, $\Delta\phi$, to be vectorial in character, as $\Delta\bar{\phi} = (\Delta\phi)\bar{e}_\phi$ where $\bar{e}_\phi$ is a unit vector in the direction of the axis of rotation (as is usual for the kinematic description), then:

for $\bar{t} \perp \Delta\bar{\phi}$, $\bar{t} \times \Delta\bar{\phi} = |\bar{t}||\Delta\bar{\phi}| \text{ Sin } \frac{\pi}{2} \; \bar{e}_n$

where $\bar{e}_n$ is normal to both $\bar{t}$ and $\bar{e}_\phi$

Thus, $$|\bar{t} \times \Delta\bar{\phi}| = |\bar{t}||\Delta\bar{\phi}| = |\bar{t}| \; \Delta\phi \qquad \text{..... \{1.3.2.-2.\}}$$

Then {1.3.2.-1.} and {1.3.2.-2.} are virtually identical. It is realized therefore, that if $\Delta\phi$ is small, a rotation may be validly expressed as a vector quantity. This holds as a first-order approximation, as shown above but is quite valid for angles $0° < \beta < \sim 6°$. For angles $> \sim 6°$, the approximation becomes poor (β no longer approximates Sin β) and the operations using the rotation as a vector fail to give commutative results: i.e., the order in which rotations are used will affect the result.

1.3.3. The Osculating Plane and Circle

The plane subtended by the vector double, $\{\bar{t}, \bar{n}\}$, is called the *osculating plane*. The *osculating circle*, defined as a circle passing through three consecutive points on the curve, has its centre at the terminus of the vector, $\bar{N} = \kappa\bar{n}$. The radius, R, of the osculating circle is thus seen to be the reciprocal of the curvature; i.e., $R = \frac{1}{\kappa}$.

It is to be noted that the equation of the space curve admits a unique determination of κ^2, but not of κ. In order to obtain the 'proper' (conceptually feasible) centre of the osculating circle, it is sometimes necessary to choose κ as the negative radical.

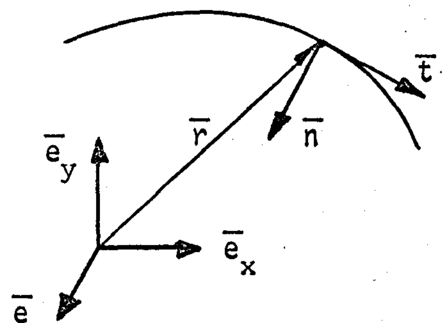

Fig. 1.3.3.-1.

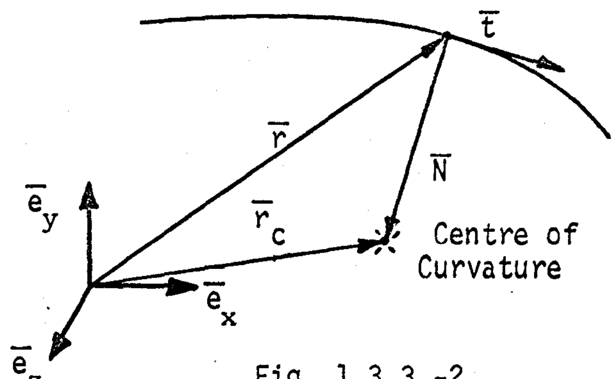

Fig. 1.3.3.-2.

This is interpreted geometrically, of course, as signifying that the 'proper' centre may be in a position, symmetric (about the point under consideration on the curve) to the one chosen.

1.4. THE BINORMAL VECTOR

Having defined the two orthogonal unit vectors, $\bar{t}$ and $\bar{n}$, the introduction of a third unit vector would then produce a unit vector triple, or *triad*. Prescribing the triad to be a *dextral* (right-handed), rectangular Cartesian triple, then it may be said:

$$\bar{b} = \bar{t} \times \bar{n} \qquad \ldots\ldots\ldots\ldots \qquad \{1.4.-1.\}$$

where $\bar{b}$ is referred to as the *binormal vector*, and is considered to be defined by {1.4.-1.} (above).

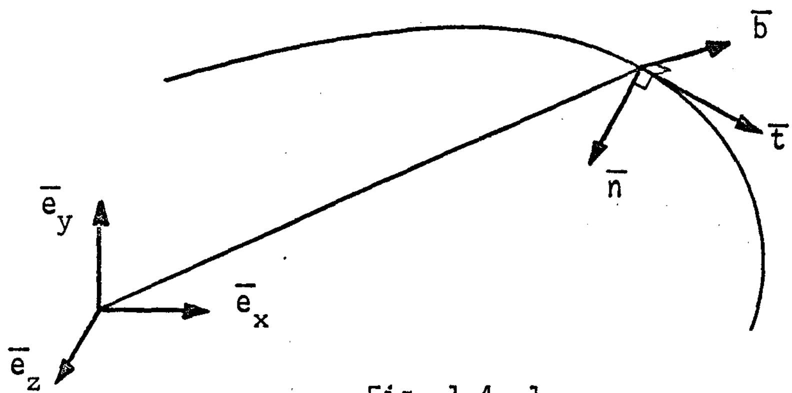

Fig. 1.4.-1.

The unit triple, $\{\bar{t}, \bar{n}, \bar{b}\}$, is thus constructed for a space curve, and is referred to as the FRENET Triad, after the French mathematician, Frederic FRENET (1816-1888), in 1847.*

1.5. TORSION

From the definition, $\bar{b} = \bar{t} \times \bar{n}$, then:

$$\frac{d\bar{b}}{ds} = \frac{d}{ds}(\bar{t} \times \bar{n}) = \frac{d\bar{t}}{ds} \times \bar{n} + \bar{t} \times \frac{d\bar{n}}{ds}$$

but, as

$$\frac{d\bar{t}}{ds} = \bar{N} = \kappa\bar{n}$$

then

$$\frac{d\bar{b}}{ds} = \kappa\bar{n} \times \bar{n} + \bar{t} \times \frac{d\bar{n}}{ds}$$

or

$$\frac{d\bar{b}}{ds} = \bar{t} \times \frac{d\bar{n}}{ds} \qquad \text{as } \bar{n} \times \bar{n} = 0$$

Thus,

$$\frac{d\bar{b}}{ds} \perp \bar{t} \quad (\text{and } \frac{d\bar{b}}{ds} \perp \frac{d\bar{n}}{ds})$$

so, $\frac{d\bar{b}}{ds}$ must lie in the plane $\{\bar{n}, \bar{b}\}$.

However, from $\bar{b} \cdot \bar{b} = 1$, then by a process exactly similar to that of § 1.3.1.,

$$\bar{b} \cdot \bar{b} = 1$$

so

$$\frac{d}{ds}(\bar{b} \cdot \bar{b}) = 2\,\bar{b} \cdot \frac{d\bar{b}}{ds} = 0$$

hence,

$$\frac{d\bar{b}}{ds} \perp \bar{b}$$

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* References are given chronologically, in the BIBLIOGRAPHY.

Therefore, if $\frac{d\bar{b}}{ds}$ is 1) in the plane $\{\bar{n}, \bar{b}\}$

and 2) perpendicular to $\bar{b}$

then $\frac{d\bar{b}}{ds}$ must be collinear with $\bar{n}$.

Hence,
$$\frac{d\bar{b}}{ds} = \lambda \, \bar{n}$$

where λ is a scalar factor.

This scalar multiplier, above, is usually given the symbolism $-\tau$, such that

$$\tau\bar{n} = -\frac{d\bar{b}}{ds} \qquad ....\{1.5.-1.\}$$

and τ is then referred to as the *torsion* of the curve. Scalar premultiplication of {1.5.-1.} by $\bar{n}$ shows

$$\bar{n} \cdot \tau\bar{n} = -\bar{n} \cdot \frac{d\bar{b}}{ds}$$

or,
$$\tau = -\bar{n} \cdot \frac{d\bar{b}}{ds}$$

1.5.1. The Relation of Torsion to Curvature

From
$$\tau = -\bar{n} \cdot \frac{d\bar{b}}{ds}, \text{ and } \bar{b} = \bar{t} \times \bar{n}$$

then
$$\tau = -\bar{n} \cdot [\frac{d}{ds}(\bar{t} \times \bar{n})]$$
$$= -\bar{n} \cdot [\frac{d\bar{t}}{ds} \times \bar{n} + \bar{t} \times \frac{d\bar{n}}{ds}]$$
$$= -\bar{n} \cdot [\kappa\bar{n} \times \bar{n} + \bar{t} \times \frac{d\bar{n}}{ds}]$$
$$= -\bar{n} \cdot \bar{t} \times \frac{d\bar{n}}{ds}$$

However, as $\bar{n} = \frac{1}{\kappa}\bar{N} = \frac{1}{\kappa}\frac{d\bar{t}}{ds}$

Then
$$\tau = -\frac{1}{\kappa}\frac{d\overline{t}}{ds}\cdot\overline{t} \times \frac{d}{ds}\left(\frac{1}{\kappa}\frac{d\overline{t}}{ds}\right)$$
$$= -\frac{1}{\kappa}\frac{d\overline{t}}{ds}\cdot\overline{t} \times \frac{1}{\kappa}\ \frac{d^2\overline{t}}{ds^2}$$
$$= -\frac{1}{\kappa^2}\frac{d\overline{t}}{ds}\cdot\overline{t} \times \frac{d^2\overline{t}}{dt^2}$$
$$= -\frac{1}{\kappa^2}\frac{d^2\overline{r}}{ds}\cdot\frac{d\overline{r}}{ds} \times \frac{d^3\overline{r}}{dr^3} \qquad \text{as } t = \frac{d\overline{r}}{ds}$$

and so, the torsion of the curve may be written as:
$$\tau = -\overline{n}\cdot\frac{d\overline{b}}{ds} = \left[\frac{1}{\kappa^2}\ \frac{d\overline{r}}{ds}\cdot\frac{d^2\overline{r}}{ds^2} \times \frac{d^3\overline{r}}{ds^3}\right]$$

This relationship connects the curvature, κ, and the torsion, τ, by means of the primitive quantities, $\frac{d^n\overline{r}}{ds^n}$, n = 1,2,3 - which are readily evaluated from the parametric representation of the curve.

1.6 THE FRENET-SERRET FORMULAS

From
$$\frac{d\overline{n}}{ds} = \frac{d}{ds}(\overline{b} \times \overline{t})$$
$$= \frac{d\overline{b}}{ds} \times \overline{t} + \overline{b} \times \frac{d\overline{t}}{ds}$$
$$= -\tau\overline{n} \times \overline{t} + \overline{b} \times \kappa\overline{n} \qquad \text{...... \{1.6.-1.\}}$$
$$= \tau\overline{b} - \kappa\overline{t}$$

and using the two previously-determined quantities, $\frac{d\overline{t}}{ds} = \kappa\overline{n}$ and $\frac{d\overline{b}}{ds} = -\tau\overline{n}$, then the FRENET-SERRET Formulas are revealed as:

$$\left.\begin{aligned} \frac{d\overline{t}}{ds} &= \kappa\overline{n} \\ \frac{d\overline{n}}{ds} &= -\kappa\overline{t} + \tau\overline{b} \\ \frac{d\overline{b}}{ds} &= -\tau\overline{n} \end{aligned}\right\} \qquad \text{...... \{1.6.-2.\}}$$

These relationships, existing between the unit vectors of the FRENET Triad and their arc length derivatives (in conjunction with the curvature and torsion), are named in honour of FRENET and the French applied mathematician, Joseph Alfred SERRET (1819-1885), in 1851.

1.7. THE DARBOUX VECTOR: A Kinematic Form of the FRENET-SERRET Formulas

From $\frac{d\bar{n}}{ds} = -\tau\bar{n} \times \bar{t} + \bar{b} \times \kappa\bar{n}$ {1.6.-2}; re-writing in altered form yields:

$$\frac{d\bar{n}}{ds} = \tau\bar{t} \times \bar{n} + \kappa\bar{b} \times \bar{n} = (\tau\bar{t} + \kappa\bar{b}) \times \bar{n}$$

or

$$\frac{d\bar{n}}{ds} = \bar{D} \times \bar{n}$$

where $\bar{D} = \tau\bar{t} + \kappa\bar{b}$ is called the DARBOUX Vector.

The FRENET-SERRET Formulas may be re-stated in terms of the DARBOUX Vector, as:

$$\left.\begin{aligned} \frac{d\bar{t}}{ds} &= \bar{D} \times \bar{t} \\ \frac{d\bar{n}}{ds} &= \bar{D} \times \bar{n} \\ \frac{d\bar{b}}{ds} &= \bar{D} \times \bar{b} \end{aligned}\right\} \text{...... \{1.7.-1.\}}$$

The advantages of such a representation are far more than the obvious ones of the succinct and symmetric form. The DARBOUX Vector admits kinematic interpretation as a *rotational* vector, existing in the *rectifying* $\{\bar{t}, \bar{b}\}$ *plane* and specifying the rates of rotation of the three unit vectors of the triad. This places the concepts of curvature and torsion on a firm conceptual footing: the curvature appears as the relative rotation of the space curve per unit arc length, about the binormal, $\bar{b}$; the torsion is interpreted as the relative rotation (or "twist") of the space curve per unit arc length, about the tangent, $\bar{t}$.

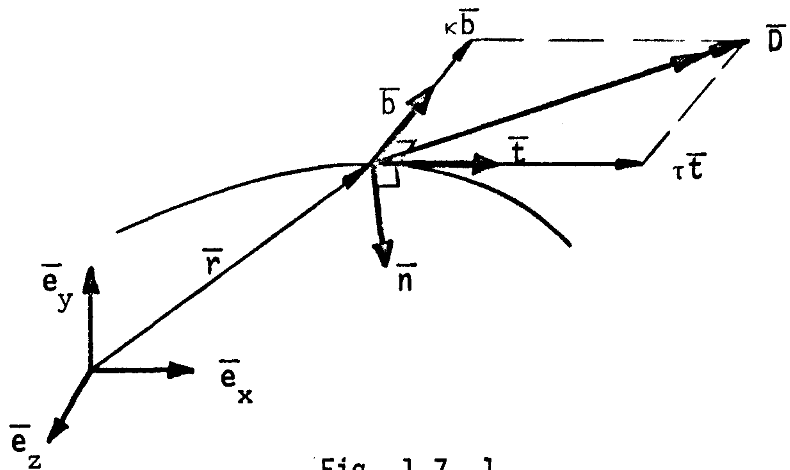


Fig. 1.7.-1.

The curvature, κ, thus exists in the capacity of the magnitude of a rotation vector, $\bar{\kappa} = \kappa\bar{b}$, and the torsion as the magnitude of a rotation vector, $\bar{\tau} = \tau\bar{t}$. Hence, the DARBOUX Vector represents the relative rotation of the FRENET Triad, as it moves a unit distance along the arc length of the curve.

The DARBOUX vector is so named, after the French applied mathematician, Jean-Gaston DARBOUX (1842-1917), who employed it in his lectures of 1887-1896.

CHAPTER 2

Differential Geometry of Surfaces

2.1. THE FUNDAMENTAL SYSTEM

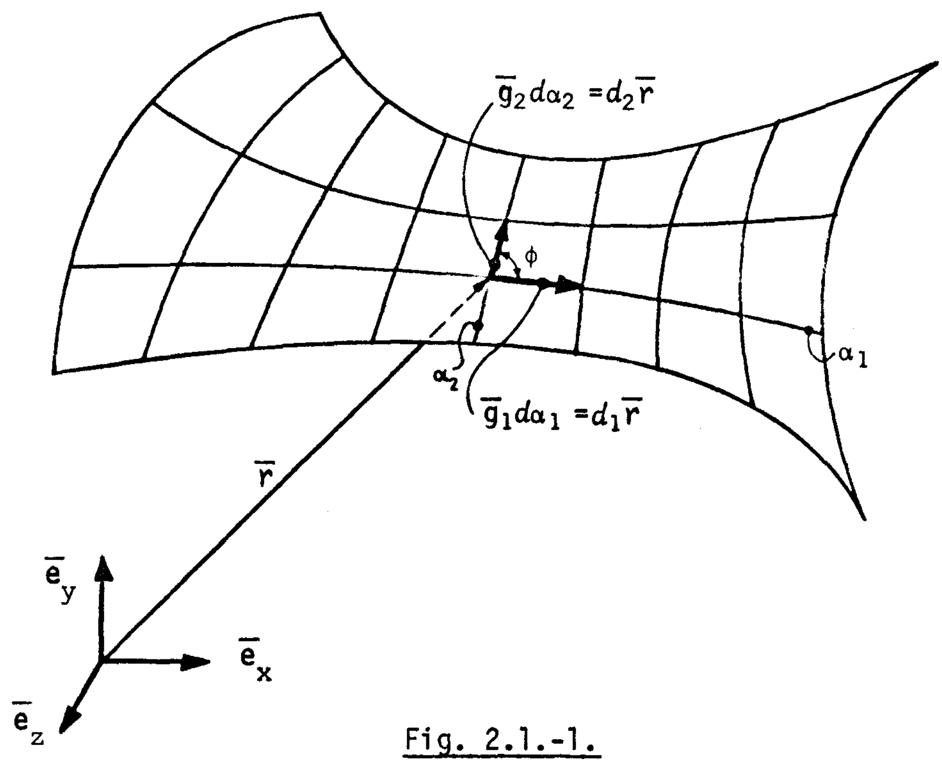


Fig. 2.1.-1.

The parametric coordinates, α_1 and α_2, trace out a coordinate 'net', in the surface. If one parameter is held constant while the other is varied (and vice-versa), the result is a set of space curves as shown above. The parametric coordinate α_1 is defined by the position vector

$$\bar{r}\ (\alpha_1, \alpha_2\ = \text{constant}) \qquad \ldots\ldots \quad \{2.1.-1.\}$$

and correspondingly, parametric coordinate α_2 is defined by the position vector

$$\bar{r}(\alpha_1 = \text{constant}, \alpha_2) \qquad \ldots\ldots \quad \{2.1.-2.\}$$

where in {2.1.-1.}, α_1 assumes arbitrary values, and in {2.1.-2.}, α_2 assumes arbitrary values.

2.2. THE TANGENT VECTOR

In a manner similar to that of § 1.2., the tangent vector to "parametric coordinate α_1" will be given by:

$$\bar{g}_1 \equiv \frac{\partial \bar{r}}{\partial \alpha_1}$$

The partial differentiation is employed, as α_2 = constant.

Similarly, a tangent vector to "parametric coordinate α_2" will be given by:

$$\bar{g}_2 = \frac{\partial \bar{r}}{\partial \alpha_2}$$

NOTE: Since the derivative of the position vector has been taken with respect to the parametric co-ordinate, α_i, rather than the arc length parameter, s_i, then the ~~unitary~~ *base vectors*, $\bar{g}_i$, are not of unit magnitude.

The *unit tangent vectors*, $\bar{e}_i$, to the space curves forming the surface are given by

$$\bar{e}_i = \frac{\partial \bar{r}}{\partial s_i} \qquad i = 1,2$$

It will be convenient, however, to retain the ~~unitary~~ base vector system for the present; the unit vector system will be discussed in a later section.

2.2.1. The Differential Surface Area

With reference to Fig. 2.1.-1., it will be observed that the area of the differential surface formed by $\overline{g}_1 d\alpha_1$ and $\overline{g}_2 d\alpha_2$ can be obtained quantatively.

i.e.:
$$d\overline{A} = \overline{g}_1 d\alpha_1 \times \overline{g}_2 d\alpha_2 = d_1\overline{r} \times d_2\overline{r}$$
$$= \frac{\partial \overline{r}}{\partial \alpha_1} d\alpha_1 \times \frac{\partial \overline{r}}{\partial \alpha_2} d\alpha_2$$
$$= (\overline{g}_1 \times \overline{g}_2)\, d\alpha_1\, d\alpha_2$$

Since both tangent vectors are in the plane tangent to the surface, the surface area (as a vector quantity) will be perpendicular to both, i.e., normal to the surface at the point common to $\overline{g}_1 d\alpha_1$ and $\overline{g}_2 d\alpha_2$.

Hence,
$$d\overline{A} \equiv d\overline{A}_n = dA_n \overline{e}_3 = (\overline{g}_1 \times \overline{g}_2) d\alpha_1\, d\alpha_2$$
$$= |\overline{g}_1 \times \overline{g}_2|\, d\alpha_1\, d\alpha_2 \overline{e}_3$$

where $\overline{e}_3$ *is a unit vector, normal to the surface.*

Then,
$$dA_n \overline{e}_3 = |\overline{g}_1 \times \overline{g}_2| d\alpha_1 d\alpha_2\, \overline{e}_3$$
so
$$dA_n = |\overline{g}_1 \times \overline{g}_2|\, d\alpha_1 d\alpha_2 = (d\overline{A}_n \cdot d\overline{A}_n)^{\frac{1}{2}}$$
or
$$dA_n = \left[|\overline{g}_1||\overline{g}_2| \quad \text{Sin}\,\phi \quad d\alpha_1 d\alpha_2\right]$$

2.3. THE FIRST FUNDAMENTAL FORM

The arc length, measured in the surface can be prescribed as:

$$d\overline{r} \cdot d\overline{r} = ds^2 \qquad \text{(see § 1.2.)}$$

This is called the *Fundamental Metric Form*. Expanding this gives:

$$ds^2 = d\bar{r}\cdot d\bar{r} = \left(\frac{\partial \bar{r}}{\partial \alpha_1} d\alpha_1 + \frac{\partial \bar{r}}{\partial \alpha_2} d\alpha_2\right)\cdot\left(\frac{\partial \bar{r}}{\partial \alpha_1} d\alpha_1 + \frac{\partial \bar{r}}{\partial \alpha_2} d\alpha_2\right)$$

$$= \left[\left(\frac{\partial \bar{r}}{\partial \alpha_1}\cdot\frac{\partial \bar{r}}{\partial \alpha_1}\right) d\alpha_1^2 + \frac{\partial \bar{r}}{\partial \alpha_1}\cdot\frac{\partial \bar{r}}{\partial \alpha_2} d\alpha_1 d\alpha_2 + \frac{\partial \bar{r}}{\partial \alpha_2}\cdot\frac{\partial \bar{r}}{\partial \alpha_1} d\alpha_2 d\alpha_1 + \left(\frac{\partial \bar{r}}{\partial \alpha_2}\cdot\frac{\partial \bar{r}}{\partial \alpha_2}\right) d\alpha_2^2\right]$$

or, as scalar products are commutative,

$$ds^2 = \left(\frac{\partial \bar{r}}{\partial \alpha_1}\cdot\frac{\partial \bar{r}}{\partial \alpha_1}\right) d\alpha_1^2 + 2\,\frac{\partial \bar{r}}{\partial \alpha_1}\cdot\frac{\partial \bar{r}}{\partial \alpha_2} d\alpha_1 d\alpha_2 + \left(\frac{\partial \bar{r}}{\partial \alpha_2}\cdot\frac{\partial \bar{r}}{\partial \alpha_2}\right) d\alpha_2^2$$

$$= \bar{g}_1\cdot\bar{g}_1\, d\alpha_1^2 + 2\bar{g}_1\cdot\bar{g}_2\, d\alpha_1 d\alpha_2 + \bar{g}_2\cdot\bar{g}_2\, d\alpha_2^2$$

Hence, ds^2 represents a Positive Definite Quadratic Form. Representing $\bar{g}_i \cdot \bar{g}_j$ as g_{ij}, then:

$$\text{(I)} \quad ds^2 = g_{11} d\alpha_1^2 + 2g_{12} d\alpha_1 d\alpha_2 + g_{22}\, d\alpha_2^2$$

which is referred to as the *First Fundamental Form of the Surface*.

NOTE: The scalars g_{11}, g_{12}, g_{22} are frequently given in alternate notation, in many standard works on the subject. These are shown here, in order of frequency of usage.

$$g_{11} = E = A_1^2 = H_1^2$$

$$g_{12} = F$$

$$g_{22} = G = A_2^2 = H_2^2$$

2.3.1. Special Cases of the First Fundamental Form

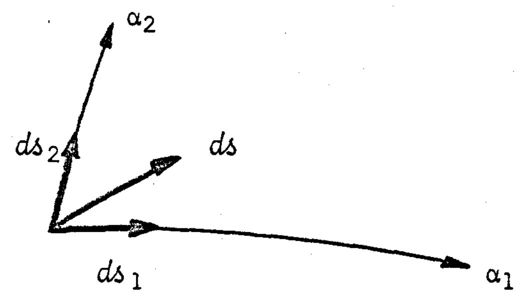


Fig. 2.3.1.-1.

Considering Fig. 2.3.1.-1., it is seen that:

a) if ds is along α_1, then $ds \equiv ds_1$, in which case,

α_2 = constant, or $d\alpha_2 = 0$

then $ds_1^2 = g_{11}\, d\alpha_1^2$

or $ds_1 = \sqrt{g_{11}}\, d\alpha_1 = g_1 d\alpha_1$

b) if ds is along α_2, then $ds \equiv ds_2$, in which case,

α_1 = constant, or $d\alpha_1 = 0$

then $ds_2^2 = g_{22} d\alpha_2^2$

or $ds_2 = \sqrt{g_{22}}\, d\alpha_2 = g_2 d\alpha_2$

where, in (a) and (b) above, $g_i \equiv \sqrt{g_{ii}} \equiv |\bar{g}_i|$
The radical is assumed positive, always. If the parametric lines, $\bar{r}\,(\alpha_1,\ \alpha_2 = \text{constant})$ and $\bar{r}(\alpha_1 = \text{constant},\ \alpha_2)$ are orthogonal, then $\phi = \frac{\pi}{2}$ and $g_{12} = 0 = \bar{g}_1 \cdot \bar{g}_2$. In such a case,

$$ds^2 = g_{11} d\alpha_1^2 + g_{22} d\alpha_2^2$$

2.3.2. The Surface Area as a Positive Definite

The magnitude, $|\bar{g}_1 \times \bar{g}_2| = |\bar{g}_1|\,|\bar{g}_2| \operatorname{Sin}\phi$ can be transformed through the use of the identity $\operatorname{Sin}^2\phi + \operatorname{Cos}^2\phi = 1$, as follows:

$$|\bar{g}_1 \times \bar{g}_2| = |\bar{g}_1|\,|\bar{g}_2| \operatorname{Sin}\phi = g_1 g_2 \sqrt{1 - \operatorname{Cos}^2\phi}$$

or

$$(\bar{g}_1 \times \bar{g}_2)\cdot(\bar{g}_1 \times \bar{g}_2) = g_{11} g_{22}\,(1 - \operatorname{Cos}^2\phi)$$

however, as $\operatorname{Cos}\phi = \dfrac{\bar{g}_1}{g_1}\cdot\dfrac{\bar{g}_2}{g_2} = \dfrac{\bar{g}_1\cdot\bar{g}_2}{g_1 g_2} = \dfrac{g_{12}}{g_1 g_2}$

then $$(\bar{g}_1 \times \bar{g}_2)\cdot(\bar{g}_1 \times \bar{g}_2) \equiv (\bar{g}_1 \times \bar{g}_2)^2 = g_1 g_2 \left(1 - \frac{g_{12}}{g_1 g_2}\right)$$

Thus, $$|\bar{g}_1 \times \bar{g}_2| = \left[g_{11} g_{22} - g_{12}^2\right]^{\frac{1}{2}}$$ because $g_{11} = \bar{g}_1 \cdot \bar{g}_1 = g_1^2$ and $g_{22} = \bar{g}_2 \cdot \bar{g}_2 = g_2^2$

Now, $|\bar{g}_1 \times \bar{g}_2|$ represents the surface area subtended by parametric increments $\Delta\alpha_1 = 1$, $\Delta\alpha_2 = 1$ (see § 2.2.1.) and thus, since

$$|\bar{g}_1 \times \bar{g}_2| = \left|\frac{\partial\bar{r}}{\partial\alpha_1} \times \frac{\partial\bar{r}}{\partial\alpha_2}\right| > 0$$

so $$\text{Area} \equiv A = \left[g_{11}g_{22} - g_{12}^2\right]^{\frac{1}{2}} > 0$$

thus, $$g = |g_{ij}| \equiv \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \equiv (g_{11}g_{22} - g_{12}^2) > 0$$

is always a Positive Definite quantity.

NOTE: The introduction of the relationship

$$\text{Cos } \phi = \frac{\overline{g}_1}{g_1} \cdot \frac{\overline{g}_2}{g_2} = \frac{g_{12}}{g_1 g_2}$$ comes directly from

the fundamental definition,

$$\overline{g}_1 \cdot \overline{g}_2 = |\overline{g}_1||\overline{g}_2| \text{ Cos } \phi = g_1 g_2 \text{ Cos } \phi.$$

The angle between the two vectors is thus conveniently specified by

$$\phi = \text{Cos}^{-1}\left(\frac{g_{12}}{g_1 g_2}\right) = \text{Cos}^{-1}\left[\frac{\left(\frac{\partial \overline{r}}{\partial \alpha_1} \cdot \frac{\partial \overline{r}}{\partial \alpha_2}\right)}{\sqrt{\left(\frac{\partial \overline{r}}{\partial \alpha_1} \cdot \frac{\partial \overline{r}}{\partial \alpha_1}\right)\left(\frac{\partial \overline{r}}{\partial \alpha_2} \cdot \frac{\partial \overline{r}}{\partial \alpha_2}\right)}}\right]$$

2.4. THE CURVATURE OF A SURFACE AND MEUSNIER'S THEOREM

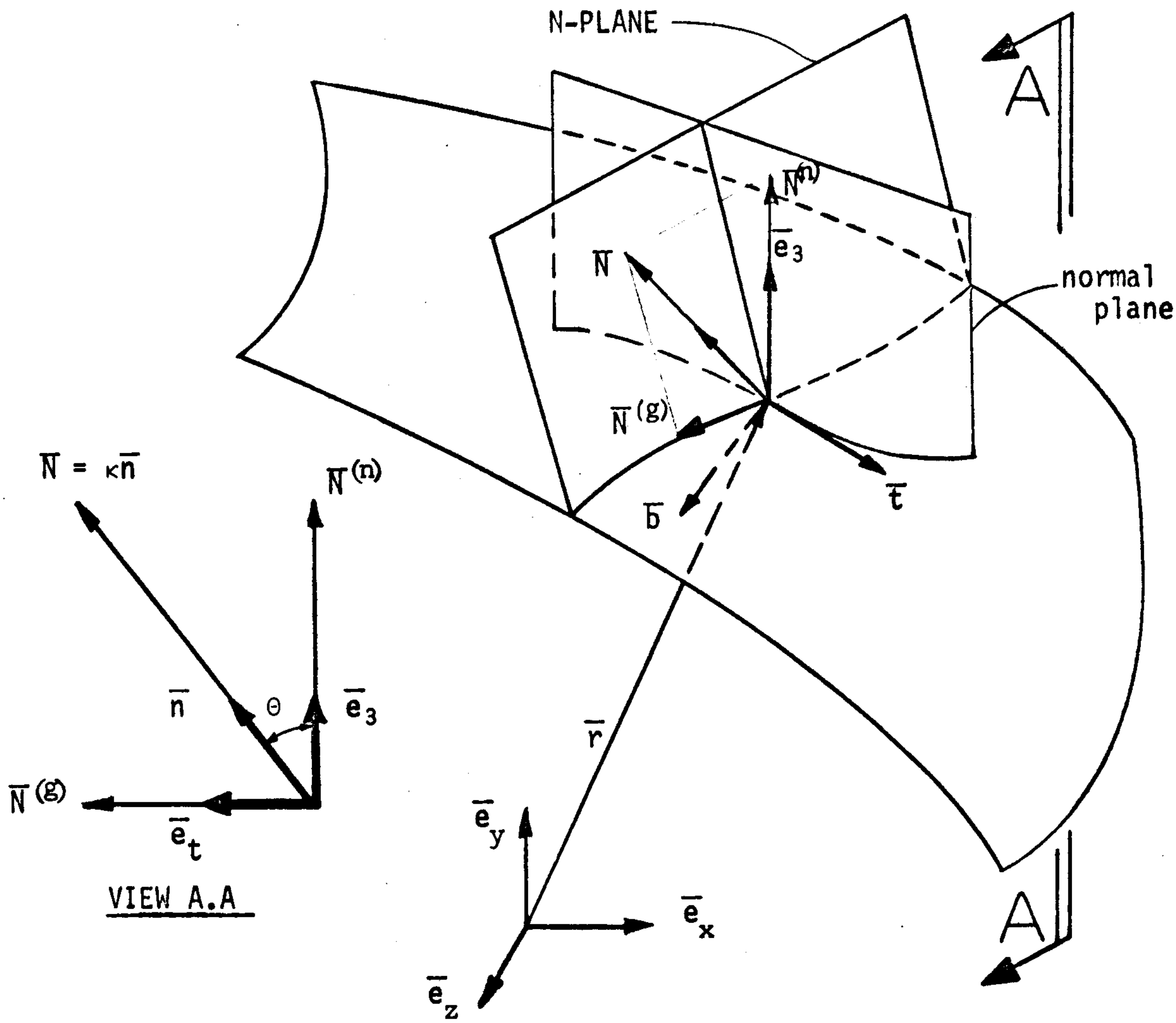


Fig. 2.4.-1.

From Fig. 2.4.-1. (VIEW A.A) $\overline{N}^{(g)}$ describes the rate of rotation of the projected curve on the tangent plane. This is referred to as the *Geodesic Curvature*. $\overline{N}^{(n)}$ may be described as "the curvature for the curve whose tangent is common to the tangent of the normal section". This is referred to as the *Normal Curvature*.

NOTE: A *normal section* of a surface at a given point contains the normal at that point. Such a section will trace out a curve on the surface, the Principal Normal (§ 1.3.) of which, is parallel to the surface normal ..... the normal section (defined by EULER in 1760) being a planar curve.

Then,

$$\overline{N} = \overline{N}^{(n)} + \overline{N}^{(g)}$$

so

$$\kappa\overline{n} = \kappa^{(n)}\,\overline{e}_3 + \overline{N}^{(g)}$$

where $\overline{e}_3$ is a unit vector in the direction of $\overline{N}^{(n)}$, as it is the surface (unit) normal.

Scalar premultiplication by $\overline{e}_3$ gives:

$$\kappa\overline{e}_3\cdot\overline{n} = \kappa^{(n)}\,\overline{e}_3\cdot\overline{e}_3 + \overline{e}_3\cdot\overline{N}^{(g)}$$

or

$$\kappa\overline{e}_3\cdot\overline{n} = \kappa^{(n)}\,\overline{e}_3\cdot\overline{e}_3 = \kappa^{(n)} \qquad \text{..... \{2.4.-1.\}}$$

$$(\overline{e}_3\cdot\overline{N}^{(g)} = 0 \text{ as } \overline{e}_3 \perp \overline{N}^{(g)})$$

However, as $\overline{e}_3\cdot\overline{n} = \text{Cos}\ \Theta$

then

$$\kappa\ \text{Cos}\ \Theta = \kappa^{(n)}$$

Saying $\kappa^{(n)} = \frac{1}{R_n}$, where R_n is the normal radius, then

$$\frac{1}{R}\ \text{Cos}\ \Theta = \frac{1}{R_n}$$

or

$$R_n\ \text{Cos}\ \Theta = R$$

This is known as MEUSNIER's Theorem, after MEUSNIER, in 1785.

With reference again to Fig. 2.4.-1.,

$$\overline{N} = \kappa\overline{n} = \frac{d^2\overline{r}}{ds^2}$$

and

$$\overline{N} = \kappa^{(n)}\overline{e}_3 + \kappa^{(g)}\overline{e}_t$$

where $\overline{N}^{(g)} \equiv \kappa^{(g)}\ \overline{e}_t$, $\overline{e}_t$ being a unit surface tangent vector. Solving for the normal curvature of the surface, associated with the direction $\overline{t} = \frac{d\overline{r}}{ds}$ of the curve, by expanding (2.4.-1.) yields:

$$\kappa^{(n)} = \kappa\overline{e}_3\cdot\overline{n} = \overline{e}_3\cdot\frac{d\overline{t}}{ds} \quad \text{since} \quad \frac{d\overline{t}}{ds} = \overline{N} = \kappa\overline{n}$$

However, as $\overline{e}_3\cdot\overline{t} = 0$

so

$$\frac{d\overline{e}_3}{ds}\cdot\overline{t} + \overline{e}_3\cdot\frac{d\overline{t}}{ds} = 0$$

or

$$\overline{e}_3\cdot\frac{d\overline{t}}{ds} = -\frac{d\overline{e}_3}{ds}\cdot\overline{t}$$

and hence, the normal curvature is given by:

$$\kappa^{(n)} = \overline{e}_3\cdot\frac{d\overline{t}}{ds} = -\frac{d\overline{e}_3}{ds}\cdot\overline{t} = -\frac{d\overline{e}_3}{ds}\cdot\frac{d\overline{r}}{ds}$$

or

$$\kappa^{(n)} = -\frac{d\overline{r}}{ds}\cdot\frac{d\overline{e}_3}{ds}$$

$$= -\frac{d\overline{r}\cdot d\overline{e}_3}{ds^2} = -\frac{d\overline{r}\cdot d\overline{e}_3}{d\overline{r}\cdot d\overline{r}}$$

Referring to $d\overline{r}\cdot d\overline{e}_3$ as II, the *Second Fundamental Form*, and recognizing $d\overline{r}\cdot d\overline{r}$ as I, the First Fundamental Form, then

$$\kappa^{(n)} = -\frac{II}{I} = \frac{I_2}{I_1}$$

2.5. THE SECOND FUNDAMENTAL FORM

From the definition of the Second Fundamental Form, (II, above),

$$II = d\bar{r}\cdot d\bar{e}_3 \quad = d\bar{e}_3\cdot d\bar{r}$$

Expansion of this reveals, in a manner analogous to § 2.3.,

$$d\bar{r}\cdot d\bar{e}_3 = \left(\frac{\partial\bar{r}}{\partial\alpha_1}d\alpha_1 + \frac{\partial\bar{r}}{\partial\alpha_2}d\alpha_2\right)\cdot\left(\frac{\partial\bar{e}_3}{\partial\alpha_1}d\alpha_1 + \frac{\partial\bar{e}_3}{\partial\alpha_2}d\alpha_2\right)$$

since $\dfrac{\partial\bar{r}}{\partial\alpha_i} \equiv \bar{g}_i$, then:

$$d\bar{r}\cdot d\bar{e}_3 = (\bar{g}_1 d\alpha_1 + \bar{g}_2 d\alpha_2)\cdot\left(\frac{\partial\bar{e}_3}{\partial\alpha_1}d\alpha_1 + \frac{\partial\bar{e}_3}{\partial\alpha_2}d\alpha_2\right)$$

$$= \left[\bar{g}_1\cdot\frac{\partial\bar{e}_3}{\partial\alpha_1}d\alpha_1^2 + \bar{g}_1\cdot\frac{\partial\bar{e}_3}{\partial\alpha_2}d\alpha_1 d\alpha_2 + \bar{g}_2\cdot\frac{\partial\bar{e}_3}{\partial\alpha_1}d\alpha_2 d\alpha_1 + \bar{g}_2\cdot\frac{\partial\bar{e}_3}{\partial\alpha_2}d\alpha_2^2\right]$$

Referring to $\bar{g}_i\cdot\dfrac{\partial\bar{e}_3}{\partial\alpha_j}$ as b_{ij}, then:

$$d\bar{r}\cdot d\bar{e}_3 = b_{11}d\alpha_1^2 + (b_{12} + b_{21})d\alpha_1 d\alpha_2 + b_{22}\, d\alpha_2^2$$

Now, from the identity $\bar{g}_i\cdot\bar{e}_3 = 0 = \dfrac{\partial\bar{r}}{\partial\alpha_i}\cdot\bar{e}_3$(due to the perpendicularity of $\dfrac{\partial\bar{r}}{\partial\alpha_i}$ and $\bar{e}_3$), then by differentiating with respect to α_j (i, j = 1,2):

$$\frac{\partial}{\partial\alpha_j}\left(\frac{\partial\bar{r}}{\partial\alpha_i}\cdot\bar{e}_3\right) = \frac{\partial}{\partial\alpha_j}(0) = 0$$

$$\frac{\partial^2 \overline{r}}{\partial \alpha_j \, \partial \alpha_i} \cdot \overline{e}_3 + \frac{\partial \overline{r}}{\partial \alpha_i} \cdot \frac{\partial \overline{e}_3}{\partial \alpha_j} = 0$$

so

$$\frac{\partial \overline{r}}{\partial \alpha_i} \cdot \frac{\partial \overline{e}_3}{\partial \alpha_j} = - \frac{\partial^2 \overline{r}}{\partial \alpha_j \partial \alpha_i} \cdot \overline{e}_3 \qquad \ldots\ldots \ \{2.5.-1.\}$$

For $i = 1$, $j = 2$, {2.5.-1.} gives

$$\frac{\partial \overline{r}}{\partial \alpha_1} \cdot \frac{\partial \overline{e}_3}{\partial \alpha_2} \equiv b_{12} = - \frac{\partial^2 \overline{r}}{\partial \alpha_2 \partial \alpha_1} \cdot \overline{e}_3 \qquad \ldots\ldots \ \{2.5.-2.\}$$

For $i = 2$, $j = 1$, {2.5.-1.} gives

$$\frac{\partial \overline{r}}{\partial \alpha_2} \cdot \frac{\partial \overline{e}_3}{\partial \alpha_1} \equiv b_{21} = - \frac{\partial^2 \overline{r}}{\partial \alpha_1 \partial \alpha_2} \cdot \overline{e}_3 \qquad \ldots\ldots \ \{2.5.-3.\}$$

Employing Nicholas BERNOULLI's condition:

$$\frac{\partial^2 \phi}{\partial \alpha_1 \partial \alpha_2} = \frac{\partial^2 \phi}{\partial \alpha_2 \partial \alpha_1}$$

which is extended to

$$\frac{\partial^2 \overline{\xi}}{\partial \alpha_1 \partial \alpha_2} = \frac{\partial^2 \overline{\xi}}{\partial \alpha_2 \partial \alpha_1}$$

where ϕ and $\overline{\xi}$ are arbitrary scalar and vector point-functions, respectively. Then, comparing {2.5.-2.} and {2.5.-3.} ,

$$- \frac{\partial^2 \overline{r}}{\partial \alpha_2 \partial \alpha_1} \cdot \overline{e}_3 = - \frac{\partial^2 \overline{r}}{\partial \alpha_1 \partial \alpha_2} \cdot \overline{e}_3$$

and so $\boxed{b_{12} = b_{21}}$

Therefore, the Second Fundamental Form assumes the (positive definite) quadratic form:

$$\boxed{II = d\vec{r}\cdot d\vec{e}_3 = b_{11}d\alpha_1^2 + 2b_{12}d\alpha_1 d\alpha_2 + b_{22}d\alpha_2^2}$$

As a consequence of this form,

$$\kappa^{(n)} = -\frac{II}{I} = -\frac{b_{11}d\alpha_1^2 + 2b_{12}d\alpha_1 d\alpha_2 + b_{22}d\alpha_2^2}{g_{11}d\alpha_1^2 + 2g_{12}d\alpha_1 d\alpha_2 + g_{22}d\alpha_2^2}$$

NOTE: Frequently, in the literature of the subject,

I is referred to as I_1

II is referred to as $-I_2$

so that $\kappa^{(n)} = \frac{I_2}{I_1}$

Also, frequent representation of the Second Fundamental Form are:

$$II = e d\alpha_1^2 + 2f d\alpha_1 d\alpha_2 + g d\alpha_2^2 \qquad \text{(American)}$$

$$II = L d\alpha_1^2 + 2M d\alpha_1 d\alpha_2 + N d\alpha_2^2 \qquad \text{(British, German)}$$

2.5.1. Positive Definite Quantities In General

If, for any curve, $ds>0$ (or $ds^2>0$), then ds is said to be a *Positive Definite Quantity*. As an example, consider the expression for the First Fundamental Form:

$$ds^2 = d\vec{r}\cdot d\vec{r} = (g_{11}d\alpha_1^2 + 2g_{12}d\alpha_1 d\alpha_2 + g_{22}d\alpha_2^2)$$

Operating on this yields

$$ds^2 = \left\{ \frac{1}{g_{11}} \left[g_{11}^2 d\alpha_1^2 + 2g_{11}g_{12}d\alpha_1 d\alpha_2 + g_{11}g_{22}d\alpha_2^2 + g_{12}^2 d\alpha_2^2 \right] - \frac{g_{12}^2}{g_{11}} d\alpha_2^2 \right\}$$

$$= \left[\frac{1}{g_{11}} \left(g_{11}^2 d\alpha_1^2 + 2g_{11}g_{12}d\alpha_1 d\alpha_2 + g_{12}^2 d\alpha_2^2 \right) + \frac{g_{11}g_{22} - g_{12}^2}{g_{11}} d\alpha_2^2 \right]$$

$$= \frac{1}{g_{11}} \left[(g_{11}d\alpha_1 + g_{12}d\alpha_2)^2 + (g_{11}g_{22} - g_{12}^2)\, d\alpha_2^2 \right]$$

and this whole quantity must be greater than zero, since:

$(g_{11}d\alpha_1 + g_{12}d\alpha_2)^2 > 0$ for g_{11}, g_{12} real

$(g_{11}g_{22} - g_{12}^2)d\alpha_2^2 > 0$ for $d\alpha_2$ real, [as $(g_{11}g_{22} - g_{12}^2) > 0$; § 2.3.2.]

Thus: 1) $ds^2 > 0$

2) $(g_{11}d\alpha_1 + g_{12}d\alpha_2) > 0$

3) $(g_{11}g_{22} - g_{12}^2) > 0$

Similarly, it may be shown that

$$d\bar{r} \cdot d\bar{e}_3 > 0$$

and $(b_{11}b_{22} - b_{12}^2) > 0$, etc.

Thus, any relationship developed for the First Fundamental Form is also valid for the Second Fundamental Form.

2.6. PRINCIPAL NORMAL CURVATURE AND DIRECTIONS

Recalling from § 2.5.:

$$\kappa^{(n)} = - \frac{b_{11}d\alpha_1^2 + 2b_{12}d\alpha_1 d\alpha_2 + b_{22}d\alpha_2^2}{g_{11}d\alpha_1^2 + 2g_{12}d\alpha_1 d\alpha_2 + g_{22}d\alpha_2^2} = \frac{I_2}{I_1}$$

($-I_2$ is used in place of II for convenience, here)

Then this may be written as

$$\kappa^{(n)} = - \frac{b_{11}\lambda^2 + 2b_{12}\lambda + b_{22}}{g_{11}\lambda^2 + 2g_{12}\lambda + g_{22}} = \frac{I_2}{I_1} = \kappa^{(n)}(\lambda)$$

where $\lambda = \frac{d\alpha_1}{d\alpha_2}$ = "Slope" of Normal Section in the surface. (NB! Since no lengths are involved, this is not the "slope" in the true geometric sense.)

Now, in order that $\kappa^{(n)}$ may have an extremum value with respect to the direction λ, it is necessary that the first variation of the expression for $\kappa^{(n)}$ (with respect to λ) vanish; i.e., at extremum values of $\kappa^{(n)}$, $\kappa^{(n)}$ must be stationary with respect to λ.

Hence, $$\delta\kappa^{(n)} = 0 = \frac{\partial\kappa^{(n)}}{\partial\lambda}\delta\lambda$$

since $\delta\kappa^{(n)} = \delta\left[\frac{I_2}{I_1}\right]$, this (above) becomes:

$$\delta\kappa^{(n)} = \delta\left[\frac{I_2}{I_1}\right] = \frac{I_1\delta I_2 - I_2\delta I_1}{I_1^2} = 0$$

so, $$\frac{1}{I_1^2}\left[I_1 \frac{\partial I_2}{\partial\lambda}\delta\lambda - I_2 \frac{\partial I_1}{\partial\lambda}\delta\lambda \right] = 0$$

thus, for $\frac{1}{I_1^2} \neq 0$,

$$\left[I_1 \frac{\partial I_2}{\partial \lambda} - I_2 \frac{\partial I_1}{\partial \lambda}\right] \delta\lambda = 0$$

or

$$I_1 \frac{\partial I_2}{\partial \lambda} - I_2 \frac{\partial I_1}{\partial \lambda} = 0$$

as $\delta\lambda \neq 0$, being an arbitrary variation.

thus,

$$I_1 \frac{\partial I_2}{\partial \lambda} = I_2 \frac{\partial I_1}{\partial \lambda}$$

or

$$\frac{\left[\frac{\partial I_2}{\partial \lambda}\right]}{\left[\frac{\partial I_1}{\partial \lambda}\right]} = \frac{I_2}{I_1} = \kappa^{(n)}$$

Now,

$$\frac{\partial I_2}{\partial \lambda} = - \frac{\partial}{\partial \lambda} (b_{11}\lambda^2 + 2b_{12}\lambda + b_{22})$$

$$= - (2b_{11}\lambda + 2b_{12})$$

$$= - 2(b_{11}\lambda + b_{12})$$

and similarly, $\frac{\partial I_1}{\partial \lambda} = 2(g_{11}\lambda + g_{12})$

Thus, it is found that the *extremal normal curvature* in the direction of $\lambda = \frac{\partial \alpha_1}{\partial \alpha_2}$ is:

$$\kappa^{(n)} = - \frac{b_{11}\lambda + b_{12}}{g_{11}\lambda + g_{12}}$$

This might be written as, more generally:

$$\kappa^{(n)} = - \left[\frac{b_{11}\lambda^2 + 2b_{12}\lambda + b_{22}}{g_{11}\lambda^2 + 2g_{12}\lambda + g_{22}}\right] \overset{+}{=} - \left[\frac{b_{11}\lambda + b_{12}}{g_{11}\lambda + g_{12}}\right]$$

where the first expression yields $\kappa^{(n)}$ for any direction of the normal section; the second expression yields $\kappa^{(n)}$ which is valid only for the directions λ...... where $\kappa^{(n)}$ possess extremum values.

The expansion of $I_1\frac{\partial I_2}{\partial\lambda} - I_2\frac{\partial I_1}{\partial\lambda} = 0$ gives:

$$I_1\frac{\partial I_2}{\partial\lambda} - I_2\frac{\partial I_1}{\partial\lambda} = \left\{ [g_{11}\lambda^2 + 2g_{12}\lambda + g_{22}]\,[-2(b_{11}\lambda + b_{12})] + [b_{11}\lambda^2 + 2b_{12}\lambda + b_{22}]\,[2(g_{11}\lambda + g_{12})] \right\} = 0$$

Expansion and re-arrangement reveals:

$$[-(g_{11}\lambda + g_{12})(b_{11}\lambda + b_{12})\lambda \;-\; (g_{11}\lambda + g_{12})(b_{11}\lambda + b_{12})$$
$$+\; (b_{11}\lambda + b_{12})(g_{11}\lambda + g_{12})\lambda \;+\; (b_{11}\lambda + b_{22})(g_{11}\lambda + g_{22})] = 0$$

or:

$$-\frac{b_{11}\lambda + b_{12}}{g_{11}\lambda + g_{12}} + \frac{b_{12}\lambda + b_{22}}{g_{12}\lambda + g_{22}} = 0 \qquad \ldots\ldots \{2.6.\text{-}1.\}$$

As the first term in {2.6.-1.} is equal to $\kappa^{(n)}$,

then

$$-\frac{b_{11}\lambda + b_{12}}{g_{11}\lambda + g_{12}} = -\frac{b_{12}\lambda + b_{22}}{g_{12}\lambda + g_{22}} = \kappa^{(n)} \qquad \ldots\ldots \{2.6.\text{-}2.\}$$

Hence, one additional form of $\kappa^{(n)}$ is obtained ({2.6.-2.}) for the case in which the normal curvature assumes the extremum value.

Re-writing {2.6.-2.}, the result is:

$$\left.\begin{aligned} (g_{12}\lambda + g_{22})\kappa^{(n)} &+ (b_{12}\lambda + b_{22}) = 0 \\ (g_{11}\lambda + g_{12})\kappa^{(n)} &+ (b_{11}\lambda + b_{12}) = 0 \end{aligned}\right\} \qquad \ldots\ldots \{2.6.\text{-}3.\}$$

This set of equations ({2.6.-3.}) will be called *the quadratic equations for principal curvatures and principal directions.*

2.6.1. Principal Direction of Normal Curvatures

If the set of equations, {2.6.-3.}, is manipulated for solution, it becomes immediately apparent that the set is *degenerate;* i.e., solution for $\kappa^{(n)}$ as a unique value fails, and $\kappa^{(n)} = \frac{|0|}{|D|}$ is obtained, where

$$|D| = \begin{vmatrix} (g_{12}\lambda + g_{22}) & (b_{12}\lambda + b_{22}) \\ (g_{11}\lambda + g_{12}) & (b_{11}\lambda + b_{12}) \end{vmatrix} \quad \text{.....\{2.6.1.-1.\}}$$

A nontrivial solution may still exist, however, iff the solution for $\kappa^{(n)}$ can be made to assume the indeterminate form: $\kappa^{(n)} = \frac{|0|}{|0|}$.
In such a case, it is essential that $|D| = 0$. Expanding the determinant, as given by {2.6.1.-1.}, and setting the result equal to zero, reveals:

$$(g_{12}\lambda + g_{22})(b_{11}\lambda + b_{12}) - (g_{11}\lambda + g_{12})(b_{12}\lambda + b_{22}) = 0$$

Further expansion, upon carrying out the products, shows:

$$[g_{12}b_{11}\lambda^2 + (g_{22}b_{11} + g_{12}b_{12})\lambda + g_{22}b_{12} - g_{11}b_{12}\lambda^2$$

$$- (g_{12}b_{12} + g_{11}b_{22})\lambda - g_{12}b_{22}] = 0$$

Collecting terms to give a quadratic in λ,

$$(g_{12}b_{11} - g_{11}b_{12})\lambda^2 + (g_{22}b_{11} - g_{11}b_{22})\lambda + (g_{22}b_{12} - g_{12}b_{22}) = 0$$

.....{2.6.1.-2.}

From the theory of equations, if the roots are λ_1, λ_2, then a quadratic equation in λ appears as

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

or $$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0 \qquad \ldots\ldots \quad \{2.6.1.-3.\}$$

The solution of equation {2.6.1.-2.} will give the two directions for which the normal curvature, $\kappa^{(n)}$, assumes an extremum value. Avoiding the complicated procedure of solving {2.6.1.-2.} in terms of g_{ij}, b_{ij}, the root λ_2 is given an arbitrary variational designation, $\frac{\delta\alpha_1}{\delta\alpha_2}$ (and $\lambda_1 = \frac{d\alpha_1}{d\alpha_2}$).

Now, considering two infinitesimal surface vectors, $d\overline{r}$ and $\delta\overline{r}$, at an angle ϕ , one to the other:

$$d\overline{r}\cdot\delta\overline{r} = |d\overline{r}||\delta\overline{r}| \quad \text{Cos}\ \phi$$

or $$\text{Cos}\ \phi = \frac{d\overline{r}\cdot\delta\overline{r}}{|d\overline{r}||\delta\overline{r}|} = \frac{1}{ds\,\delta s}\, d\overline{r}\cdot\delta\overline{r} \qquad \ldots\ldots \quad \{2.6.1.-4.\}$$

since $|d\overline{r}| = ds$, so $|\delta\overline{r}| = \delta s$.

now, as $$d\overline{r} = \overline{g}_1 d\alpha_1 + \overline{g}_2 d\alpha_2$$

then $$\delta\overline{r} = \overline{g}_1\delta\alpha_1 + \overline{g}_2\delta\alpha_2$$

and $$d\overline{r}\cdot\delta\overline{r} = \Big[g_{11}d\alpha_1\delta\alpha_1 + g_{12}(d\alpha_1\delta\alpha_2 + \delta\alpha_1 d\alpha_2) + g_{22}d\alpha_2\delta\alpha_2\Big] \qquad \ldots\ldots \quad \{2.6.1.-5.\}$$

Thus, from {2.6.1.-4.} and {2.6.1.-5.} ,

$$g_{11}d\alpha_1\delta\alpha_1 + g_{12}(d\alpha_1\delta\alpha_2 + \delta\alpha_1 d\alpha_2) + g_{22}d\alpha_2\delta\alpha_2 = ds\,\delta s \quad \text{Cos}\ \phi$$

multiplication by $(d\alpha_1\delta\alpha_1)^{-1}$ gives:

$$\left[g_{11} + g_{12}\left(\frac{\delta\alpha_2}{\delta\alpha_1} + \frac{d\alpha_2}{d\alpha_1}\right) + g_{22}\left(\frac{d\alpha_2}{d\alpha_1}\ \frac{\delta\alpha_2}{\delta\alpha_1}\right)\right] = \left[\frac{ds}{d\alpha_1}\ \frac{\delta s}{\delta\alpha_1}\ \text{Cos}\ \phi\right]$$

$$\left[g_{11} + g_{12}\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) + g_{22}\left(\frac{1}{\lambda_1\lambda_2}\right)\right] = \frac{ds}{d\alpha_1}\frac{\delta s}{\delta\alpha_1} \text{ Cos } \phi$$

$$\left[g_{11} + g_{12}\left(\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}\right) + g_{22}\left(\frac{1}{\lambda_1\lambda_2}\right)\right] = \frac{ds}{d\alpha_1}\frac{\delta s}{\delta\alpha_1} \text{ Cos } \phi \quad \ldots\ldots \quad \{2.6.1.-6.\}$$

Writing equation {2.6.1.-2.} in the form of {2.6.1.-3.} , to obtain expressions for $(\lambda_1 + \lambda_2)$ and $(\lambda_1\lambda_2)$ reveals:

$$\left.\begin{aligned} (\lambda_1 + \lambda_2) &= \frac{g_{11}b_{22} - g_{22}b_{11}}{g_{12}b_{11} - g_{11}b_{12}} \\ (\lambda_1\lambda_2) &= \frac{g_{22}b_{12} - g_{12}b_{22}}{g_{12}b_{11} - g_{11}b_{12}} \end{aligned}\right\} \quad \ldots\ldots \quad \{2.6.1.-7.\}$$

Substitution of {2.6.1.-7.} into {2.6.1.-6.} then shows:

$$\left[g_{11} + g_{12}\left(\frac{g_{11}b_{22} - g_{22}b_{11}}{g_{22}b_{12} - g_{12}b_{22}}\right) + g_{22}\left(\frac{g_{12}b_{11} - g_{11}b_{12}}{g_{22}b_{12} - g_{12}b_{22}}\right)\right] = \frac{ds}{d\alpha_1}\frac{\delta s}{\delta\alpha_1} \text{ Cos } \phi$$

or simplifying,

$$\left[g_{11} + \frac{1}{g_{22}b_{12} - g_{12}b_{22}} [g_{12}(g_{11}b_{22} - g_{22}b_{11}) + g_{22}(g_{12}b_{11} - g_{11}b_{12})]\right] = \frac{ds}{d\alpha_1}\frac{\delta s}{\delta\alpha_1} \text{ Cos } \phi$$

$$\left[g_{11} + \frac{1}{g_{22}b_{12} - g_{12}b_{22}} [g_{11}(g_{12}b_{22} - g_{22}b_{12})]\right] = \frac{ds}{d\alpha_1}\frac{\delta s}{\delta\alpha_1} \text{ Cos } \phi$$

$$g_{11} + g_{11}(-1) = 0 = \frac{ds}{d\alpha_1}\frac{\delta s}{\delta\alpha_1} \text{ Cos } \phi$$

Thus, for $\frac{ds}{d\alpha_1}\frac{\delta s}{\delta\alpha_1} \neq 0$,

$$\text{Cos } \phi = 0$$

Therefore, it is established that *the directions of Principal Normal Curvatures are such that they are always orthogonal to each other.* This was established by EULER, in 1760.

These directions, λ_1 and λ_2, may be obtained quantitatively from equation {2.6.1.-2.} but no useful purpose is served by this; it is sufficient to know the directions as related to each other (orthogonal, as proved).

2.6.2. Principal Curvatures

If the set of equations ({2.6.-3.}) originally found is re-written, so as to place λ in the position of a variable, instead of $\kappa^{(n)}$, then:

$$\left.\begin{aligned} (g_{12}\kappa^{(n)} - b_{12})\lambda + (g_{22}\kappa^{(n)} - b_{22}) &= 0 \\ (g_{11}\kappa^{(n)} - b_{11})\lambda + (g_{12}\kappa^{(n)} - b_{12}) &= 0 \end{aligned}\right\} \quad \ldots\ldots \{2.6.2.\text{-}1.\}$$

Following the procedure of the previous section (§ 2.6.1.), it is observed that an attempt to solve set {2.6.2.-1.} for a unique value of λ fails, unless the determinant of the system is equal to zero.

Setting $|D| = 0$ and expanding, yields:

$$(g_{12}\kappa^{(n)} - b_{12})^2 - (g_{11}\kappa^{(n)} - b_{11})(g_{22}\kappa^{(n)} - b_{22}) = 0$$

Further expansion, and grouping to obtain a quadratic in $\kappa^{(n)}$, gives:

$$[(g_{12}^2 - g_{11}g_{22})(\kappa^{(n)})^2 + (g_{11}b_{22} + g_{22}b_{11} - 2g_{12}b_{12})\kappa^{(n)} + (b_{12}^2 - b_{11}b_{22})] = 0 \quad \ldots\ldots \{2.6.2.\text{-}2.\}$$

referring to $(g_{11}g_{22} - g_{12}^2)$ as $|g|$ (see §2.3.2.)

and thus, to $(b_{11}b_{22} - b_{12}^2)$ as $|b|$

then {2.6.2.-2.} becomes, upon changing to standard form:

$$\left[(\kappa^{(n)})^2 - \frac{1}{|g|}(g_{11}b_{22} + g_{22}b_{11} - 2g_{12}b_{12})\kappa^{(n)} + \frac{|b|}{|g|}\right] = 0 \quad \text{....\{2.6.2.-3.\}}$$

a solution is thus obtained by considering {2.6.2.-3.} to be of the form

$$(\kappa^{(n)})^2 + 2C_1\kappa^{(n)} + C_2 = 0 \quad \text{..... \{2.6.2.-4.\}}$$

$$\left.\begin{aligned} \text{where } -2C_1 &= \left[\frac{g_{11}b_{22} + g_{22}b_{11} - 2g_{12}b_{12}}{|g|}\right] \\ C_2 &= \frac{|b|}{|g|} \end{aligned}\right\} \quad \text{..... \{2.6.2.-5.\}}$$

Thus, for roots $\kappa_1^{(n)}$ and $\kappa_2^{(n)}$, from the theory of equations:

$$-2C_1 = -(\kappa_1^{(n)} + \kappa_2^{(n)})$$

and

$$C_2 = (\kappa_1^{(n)}\ \kappa_2^{(n)})$$

and so, the *invariant coefficients* emerge as:

$$C_1 = \frac{1}{2}(\kappa_1^{(n)} + \kappa_2^{(n)})$$

which is called the SOPHIE GERMAIN *Curvature* or *Mean Curvature*, and

$$C_2 = (\kappa_1^{(n)}\ \kappa_2^{(n)})$$

which is called the GAUSSIAN *Curvature* or *Total Curvature*.

NOTE: The SOPHIE GERMAIN Curvature is named after that author's work in 1831 ; the GAUSSIAN curvature is named for GAUSS, in 1827, yet it was first found by EULER in 1760.

Thus, a solution for the curvatures appears quantitatively as (solving {2.6.2.-4.})

$$\kappa_1^{(n)} = C_1 + \sqrt{C_1^2 - C_2}$$

$$\kappa_2^{(n)} = C_1 - \sqrt{C_1^2 - C_2}$$

where C_1 and C_2 are given in terms of primitive quantities by {2.6.2.-5.}

2.7. CONJUGATE DIRECTIONS

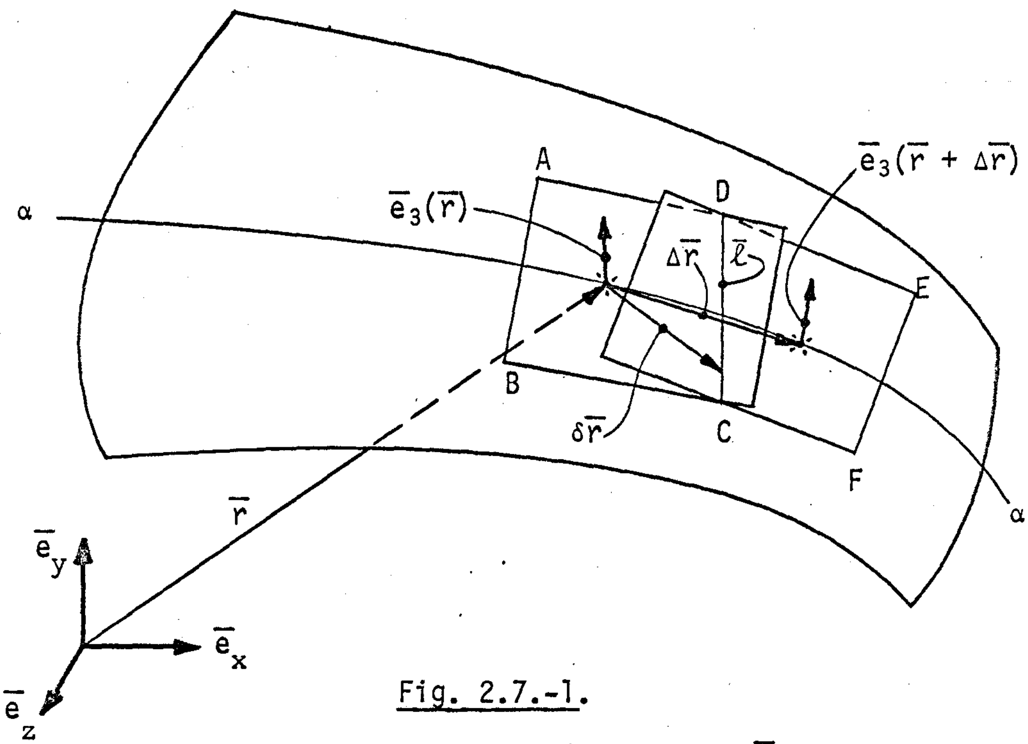


Fig. 2.7.-1.

Conjugate directions at a given point, $\bar{r}$, on a surface are defined as follows: Let $\bar{r}$ and $(\bar{r} + \Delta\bar{r})$ be two neighbouring points on a surface. If the tangent planes to the surface at $\overline{r}$ and $(\overline{r} + \Delta\overline{r})$ intersect, forming a line, $\overline{\mathcal{L}}$, then the limiting directions of line $\Delta\overline{r}$ and line $\overline{\mathcal{L}}$ (as $\Delta\overline{r}$ approaches zero) are called the *conjugate directions at* $\overline{r}$.

Considering Fig. 2.7.-1., it is observed that $\delta\overline{r}$ traces out the line of intersection of planes ABCD and CDEF, where the former represents the tangent plane at $\overline{r}$ and the latter represents the tangent plane at $\overline{r} + \Delta\overline{r}$. Thus, $\delta\overline{r}$ will be orthogonal to both $\overline{e}_3(\overline{r})$ and $\overline{e}_3(\overline{r} + \Delta\overline{r})$ in the limit. Assuming that second-order differential terms are negligible, then

$$\overline{e}_3(\overline{r} + \Delta\overline{r}) = \overline{e}_3 + d\overline{e}_3$$

Hence, $\delta\overline{r}$ must be orthogonal to both $\overline{e}_3$ and $(\overline{e}_3 + d\overline{e}_3)$ in the limit.

i.e. $$\lim_{\Delta\overline{r}\to 0} [\delta\overline{r}\cdot \overline{e}_3] = \lim_{\Delta\overline{r}\to 0} [\delta\overline{r}\cdot(\overline{e}_3 + d\overline{e}_3)] = 0$$

thus, $$\lim_{\Delta\overline{r}\to 0} [\delta\overline{r}\cdot d\overline{e}_3] = 0 \qquad \text{..... \{2.7.-1.\}}$$

where {2.7.-1.} is the necessary and sufficient condition for conjugate directions.

Two curves then form a conjugate system if:

$$\left.\begin{aligned} \delta\overline{r}\cdot d\overline{e}_3 &= 0 \\ d\overline{r}\cdot\delta\overline{e}_3 &= 0 \end{aligned}\right\} \qquad \text{..... \{2.7.-2.\}}$$

Conjugate systems need not be orthogonal systems; however, in such a case that the conjugate and orthogonal systems are identical, then:

$$\delta\overline{r}\cdot d\overline{r} = 0$$

If the first member of {2.7.-2.} is expanded:

i.e.: $\delta\overline{r}\cdot d\overline{e}_3 = 0$

by employing: $\delta\overline{r} = \overline{g}_1\delta\alpha_1 + \overline{g}_2\delta\alpha_2$

and $$d\overline{e}_3 = \frac{\partial\overline{e}_3}{\partial\alpha_1}d\alpha_1 + \frac{\partial\overline{e}_3}{\partial\alpha_2}d\alpha_2$$

then, $$\delta\overline{r}\cdot d\overline{e}_3 = \left[(\overline{g}_1\delta\alpha_1 + \overline{g}_2\delta\alpha_2)\bullet\left(\frac{\partial\overline{e}_3}{\partial\alpha_1}d\alpha_1 + \frac{\partial\overline{e}_3}{\partial\alpha_2}d\alpha_2\right)\right] = 0$$

or $$\left[\left(\overline{g}_1\cdot\frac{\partial\overline{e}_3}{\partial\alpha_1}\right)\delta\alpha_1 d\alpha_1 + \left(\overline{g}_1\cdot\frac{\partial\overline{e}_3}{\partial\alpha_2}\right)\delta\alpha_1 d\alpha_2\right.$$
$$\left. + \left(\overline{g}_2\cdot\frac{\partial\overline{e}_3}{\partial\alpha_1}\right)\delta\alpha_2 d\alpha_1 + \left(\overline{g}_2\cdot\frac{\partial\overline{e}_3}{\partial\alpha_2}\right)\delta\alpha_2 d\alpha_2\right] = 0 \quad \ldots\ldots \{2.7.-3.\}$$

or, using $\overline{g}_i\bullet\frac{\partial\overline{e}_3}{\partial\alpha_j} = b_{ij}$ (see § 2.5.), then {2.7.-3.} becomes:

$$b_{11}\delta\alpha_1 d\alpha_1 + b_{12}\delta\alpha_1 d\alpha_2 + b_{21}\delta\alpha_2 d\alpha_1 + b_{22}\delta\alpha_2 d\alpha_2 = 0$$

but as $b_{12} = b_{21}$ (see § 2.5.), then

$$b_{11}\delta\alpha_1 d\alpha_1 + b_{12}(\delta\alpha_1 d\alpha_2 + \delta\alpha_2 d\alpha_1) + b_{22}\delta\alpha_2 d\alpha_2 = 0 \quad \ldots\ldots \{2.7.-4.\}$$

which is the general equation, specifying conjugate directions.

NOTE: If the second member of {2.7.-2.} is developed in the same manner as the first, the result is observed to be identical (this is obvious, as $\delta\alpha_i d\alpha_j = d\alpha_j\,\delta\alpha_i$). Conjugate directions were first discovered by DUPIN, in 1813.

2.8. THE EQUATION OF RODRIGUES

The necessary and sufficient condition that a curve on a surface be a *line of curvature*, can be determined in the following way.

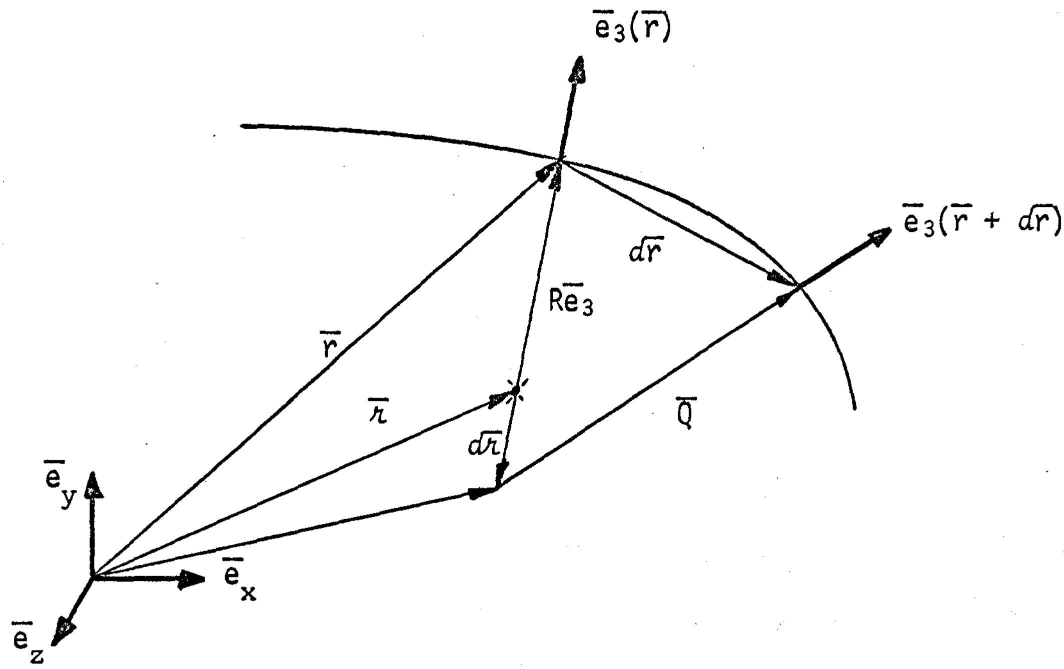


Fig. 2.8.-1.

If $\bar{e}_3$ is the surface unit normal at $\bar{r}$, and R the principal radius of a curvature of the normal section, then the corresponding centre of curvature, $\bar{\rho} = \bar{r} - R\bar{e}_3$

Saying $\qquad R\,(\bar{r} + d\bar{r}) = R + dR$

and $\qquad \bar{e}_3\,(\bar{r} + d\bar{r}) = \bar{e}_3 + d\bar{e}_3$

then $\qquad \bar{Q} = [R + dR]\,[\bar{e}_3 + d\bar{e}_3]$

$$= R\bar{e}_3 + Rd\bar{e}_3 + dR\bar{e}_3 + dRd\bar{e}_3$$

neglecting second-order differential terms,

$$\overline{Q} \doteq R\overline{e}_3 + Rd\overline{e}_3 + dR\overline{e}_3 = R\overline{e}_3 + d[R\overline{e}_3]$$

From $\overline{\pi} = \overline{r} - R\overline{e}_3$

then, $d\overline{\pi} = d\overline{r} - d[R\overline{e}_3]$

$= d\overline{r} - dR\overline{e}_3 - Rd\overline{e}_3$

or $d\overline{r} - d\overline{\pi} - dR\overline{e}_3 - Rd\overline{e}_3 = 0$

but, $d\overline{\pi} = -dR\overline{e}_3$ (Fig. 2.8.-1.)

and, as $d\overline{r}$ is parallel to $d\overline{e}_3$ (to the first order of approximation), since for principal directions, they are coplanar, then:

$$d\overline{r} - Rd\overline{e}_3 = 0$$

or, as $\frac{1}{R} = \kappa^{(n)}$

$$\kappa^{(n)} d\overline{r} - d\overline{e}_3 = 0 \qquad \text{..... \{2.8.-1.\}}$$

which is RODRIGUES' *equation*, after Olinde RODRIGUES (1794-1851) in 1815.

2.8.1. Lines of Curvature and Conjugate Systems

Since lines of curvature are orthogonal (§ 2.6.1.), then

$$\delta\overline{r} \cdot d\overline{r} = 0$$

where $\delta\overline{r}$ and $d\overline{r}$ are segments of the lines of curvature.

But as RODRIGUES' Equation specifies the necessary and sufficient condition for a line in the surface to be a line of principal curvature, then a substitution of the orthogonality condition into RODRIGUES' equation yields:

$$\delta\overline{r}\cdot R d\overline{e}_3 = 0$$

so $$\delta\overline{r}\cdot d\overline{e}_3 = 0$$

Thus, *lines of curvature form a conjugate system* as well as an orthogonal one, as {2.8.-1.} is identical to {2.7.-2.}

2.8.2. Parametric Lines and Conjugate Systems

Parametric lines would form a conjugate system, if they satisfied the general requirement:

$$b_{11}\delta\alpha_1 d\alpha_1 + b_{12}(\delta\alpha_1 d\alpha_2 + \delta\alpha_2 d\alpha_1) + b_{22}\delta\alpha_2 d\alpha_2 = 0 \quad \ldots\ldots \quad \{2.8.2.\text{-}1.\}$$

(see {2.7.-4.})

If the arbitrary line segments,

$$d\overline{r} = \overline{g}_1 d\alpha_1 + \overline{g}_2 d\alpha_2$$

and $$\delta\overline{r} = \overline{g}_1 \delta\alpha_1 + \overline{g}_2 \delta\alpha_2$$

are constrained to be in the directions of $\overline{r}(\alpha_1)$ and $\overline{r}(\alpha_2)$, respectively:

VIZ: $$d\overline{r} = \overline{g}_1 d\alpha_1 \,, \qquad (d\alpha_2 = 0) \quad \ldots\ldots\ldots \quad \{2.8.2.\text{-}2.\}$$

$$\delta\overline{r} = \overline{g}_2 \delta\alpha_2 \,, \qquad (\delta\alpha_1 = 0) \quad \ldots\ldots \quad \{2.8.2.\text{-}3.\}$$

then $d\alpha_2$ and $\delta\alpha_1$ must vanish simultaneously for a system of parametric curves.

Substitution of {2.8.2.-2.} and {2.8.2.-3.} in {2.8.2.-1.} thus reveals:

$$b_{12} d\alpha_1 \delta\alpha_2 = 0$$

which, for $d\alpha_1\delta\alpha_2 \neq 0$, must reduce to:

$$b_{12} = 0 = b_{21} \qquad (\text{as } b_{12} = b_{21})$$

i.e.: $$\frac{\partial \bar{r}}{\partial \alpha_1} \cdot \frac{\partial \bar{e}_3}{\partial \alpha_2} = 0 = \frac{\partial \bar{r}}{\partial \alpha_2} \cdot \frac{\partial \bar{e}_3}{\partial \alpha_1}$$ (by definition of b_{12} and b_{21})

This is, therefore, the necessary and sufficient condition to be satisfied for parametric lines to form a conjugate system.

NOTE: Recall that the requisite condition for parametric lines to form an orthogonal system was given by the (analogous) expression:

$$g_{12} = 0 = g_{21}$$

2.8.3. Principal Coordinates

In the case that the parametric lines are *both* orthogonal and conjugate; i.e., that $g_{12} = 0$ and $b_{12} = 0$, they are then lines of curvature. Or, again, lines of curvature must satisfy RODRIGUES' Equation, thus necessitating that both the orthogonality and conjugation conditions be enforced.

If parametric lines are lines of curvature, they are referred to as *principal coordinates*.

2.9. THE CESÀRO-BURALI-FORTI VECTOR and KINEMATIC SURFACE THEORY

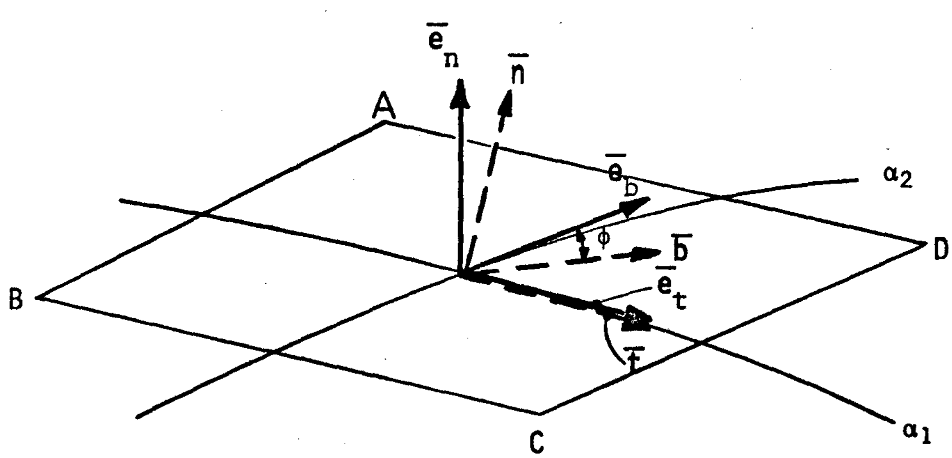


Fig. 2.9.-1.

Fig. 2.9.-1. shows the familiar FRENET Triad ($\bar{t}$, $\bar{n}$, $\bar{b}$) (§ 1.4.), together with the RIBAUCOUR Triad ($\bar{e}_t$, $\bar{e}_b$, $\bar{e}_n = \bar{e}_3$), and the infinitesimal tangent plane ABCD at a point on a surface.

The distinction between the two types of triad is as follows: the FRENET Triad employs $\bar{t}$, the tangent to a curve in space and $\bar{n}$, the normal to the curve (and to $\bar{t}$) which is defined according to § 1.2. and §1.3.. The binormal, $\bar{b}$, is defined by $\bar{t}$ and $\bar{n}$. In the RIBAUCOUR Triad, the tangent $\bar{e}_t$ is tangent to the "space curve" represented by a parametric line (α_1) in the surface; thus, the tangent $\bar{e}_t$ and the tangent $\bar{t}$ are identical. The normal $\bar{e}_3$, however, is defined as the normal to the surface tangent plane at the point of contact, and is thus not the same as $\bar{n}$ of the FRENET Triad. The normal $\bar{e}_3$ is usually defined with the aid of the cross-product of $\bar{e}_t$ with another vector in the tangent plane; a convenient choice for this other tangent-plane vector is, of course, the tangent to the other parametric line. In this way, the normal $\bar{e}_3$ always maintains a position on the "outside" of the surface. It is quite possible for $\bar{e}_3$ and $\bar{n}$ to be oriented in different (general) directions. The fundamental difference between the two triad systems, then, is that the FRENET Triad prescribes both $\bar{t}$ and $\bar{n}$ as independently-obtained quantities and $\bar{n}$ as a "*curve* normal" while the RIBAUCOUR Triad prescribes $\bar{e}_t$ as the only independantly-obtained quantity and $\bar{e}_3$ as a "*surface* normal". In the latter system, once $\bar{e}_3$ has been obtained (through $\bar{e}_t$ and another tangent-plane vector), then the binormal, $\bar{e}_b$, is defined: $\bar{e}_b = \bar{e}_3 \times \bar{e}_t$. The binormal, $\bar{e}_b$, naturally resides in the tangent plane as well. Since, in both systems the binormal is defined via the other two members of the triad, then $\overline{e}_b$ and $\overline{b}$ do not describe the same vector.

The CESÀRO-BURALI-FORTI Vector in the RIBAUCOUR Triad is best defined by comparison: it is precisely analogous to the DARBOUX Vector (§ 1.7.) of the FRENET Triad. That is, the CESÀRO-BURALI-FORTI Vector specifies the (relative) rotation of the RIBAUCOUR Triad, per unit arc length, as the triad moves along a line in the surface. This, of course, specifies the (relative) orientation of the surface itself.

Designating the CESÀRO-BURALI-FORTI Vector as $\overline{C}$, then with reference to Fig. 2.9.-1., the relation of $\overline{C}$ to the DARBOUX Vector can be given:

$$\overline{C} = \overline{D} + \frac{d\phi}{ds}\overline{e}_t = \tau\overline{t} + \kappa\overline{b} + \frac{d\phi}{ds}\overline{e}_t$$

$$= \left(\tau + \frac{d\phi}{ds}\right)\overline{e}_t + \kappa\overline{b} \qquad (\text{as } \overline{t} = \overline{e}_t) \qquad \ldots\ldots \qquad \{2.9.\text{-}1.\}$$

But it is desired to express $\overline{C}$ solely in the RIBAUCOUR system. Realizing that $\overline{b}$ must lie in the $\{\overline{e}_3, \overline{e}_b\}$ plane, as it is perpendicular to $\overline{t}$ (or $\overline{e}_t$), then:

$$\overline{b} = (\overline{b}\cdot\overline{e}_3)\,\overline{e}_3 + (\overline{b}\cdot\overline{e}_b)\,\overline{e}_b$$

$$= \text{Cos}\left(\frac{\pi}{2} + \phi\right)\overline{e}_3 + (\text{Cos}\ \phi)\,\overline{e}_b$$

$$= (\text{Sin}\ \phi)\overline{e}_3 + (\text{Cos}\ \phi)\overline{e}_b$$

Referring to $(\tau + \frac{d\phi}{ds})$ as $\kappa^{(t)}$, then {2.9.-1.} becomes:

$$\overline{C} = \kappa^{(t)}\overline{e}_t + (\kappa\ \text{Cos}\ \phi)\overline{e}_b + (\kappa\ \text{Sin}\ \phi)\overline{e}_3$$

or, redefining terms,

$$\overline{C} = \kappa^{(t)}\,\overline{e}_t + \kappa^{(n)}\,\overline{e}_b + \kappa^{(3)}\,\overline{e}_3 \qquad \ldots\ldots \quad \{2.9.-2.\}$$

where $\kappa^{(t)} = [\tau + \frac{d\phi}{ds}] \equiv$ *Geodesic Torsion* (BONNET, 1845)

$\kappa^{(n)} = [\kappa \text{ Cos } \phi] \equiv$ *Normal Curvature*

$\kappa^{(3)} = [\kappa \text{ Sin } \phi] \equiv$ *Geodesic Curvature* (BONNET, 1848)

The vector, $\overline{C}$, is thus the *Kinematic Rotation Vector* (of the RIBAUCOR Triad) of the parametric line in the surface under consideration.

Calling the curvature in the $\{\overline{e}_t, \overline{e}_b\}$ tangent-plane, *Pure Curvature*, $\overline{\kappa} = [\kappa^{(t)}\overline{e}_t + \kappa^{(n)}\overline{e}_b] = \overline{e}_3 \times \overline{C} \times \overline{e}_3$ ...... {2.9.-3.}

$$= \overline{e}_3 \times \frac{\partial \overline{e}_3}{\partial s} \quad \text{as} \quad \frac{\partial \overline{e}_3}{\partial s} = \overline{C} \times \overline{e}_3$$

Therefore, $\overline{C} = \overline{\kappa} + \kappa^{(3)}\overline{e}_3$ ...... {2.9.-4.}

The CESÀRO-BURALI-FORTI vector is named for Ernesto CESARO (1859-1906) in 1896 and Cesare BURALI-FORTI (1861-1931) in 1912. The RIBAUCOUR triad derives its name from the work of Albert RIBAUCOUR (1845-1923) in 1872-1875.

2.9.1. Classification of Surface Curves By Means of the CESÀRO-BURALI-FORTI Vector (KINEMATIC CLASSIFICATION)

Various types of surface curves may be identified by means of the fact that they cause certain curvature components, $\kappa^{(i)}$ (i = t, n, 3), to vanish.

2.9.1.1. If $\kappa^{(3)} = 0$: (LIOUVILLE's Criterion, 1884)

Then $\overline{C} = \kappa^{(t)}\overline{e}_t + \kappa^{(n)}\overline{e}_b = \overline{\kappa}$

Such a curve is called a *geodesic* and is the "shortest curve (between two neighbouring points) in the surface". The curve is produced by a normal section in the surface.

2.9.1.2. If $\kappa^{(t)} = 0$:

Then $\overline{C} = \kappa^{(n)}\overline{e}_b + \kappa^{(3)}\overline{e}_3$

This surface curve specifies a *principal line of curvature*, or the curve whose consecutive normals intersect (a planar curve). In this case, the vector $\overline{C}$ is perpendicular to the tangent, $\overline{e}_t$. Thus,

$$\overline{C}\cdot\overline{e}_t = 0$$

Hence, for principal lines of curvature,

$$\frac{d\overline{e}_3}{ds} = \overline{C} \times \overline{e}_3 = (\kappa^{(n)}\,\overline{e}_b + \kappa^{(3)}\overline{e}_3) \times \overline{e}_3 \quad \text{...\{2.9.1.2.-1.\}}$$

or

$$\frac{d\overline{e}_3}{ds} = \kappa^{(n)}\overline{e}_t$$

This will be recognized as RODRIGUES' Equation, when re-written as

$$d\overline{e}_3 = \kappa^{(n)}\, ds\, \overline{e}_t \quad \text{..... \{2.9.1.2.-2.\}}$$

or, as $\overline{e}_t = \frac{d\overline{r}}{ds}$, then $ds\,\overline{e}_t = d\overline{r}$

so {2.9.1.2.-2.} becomes

$$d\overline{e}_3 = \kappa^{(n)}\, d\overline{r}$$

$$d\overline{e}_3 - \kappa^{(n)}\, d\overline{r} = 0$$

or $$\kappa^{(n)}\, d\bar{r} - d\bar{e}_3 = 0 \qquad \ldots\ldots \qquad \{2.9.1.2.-3.\}$$

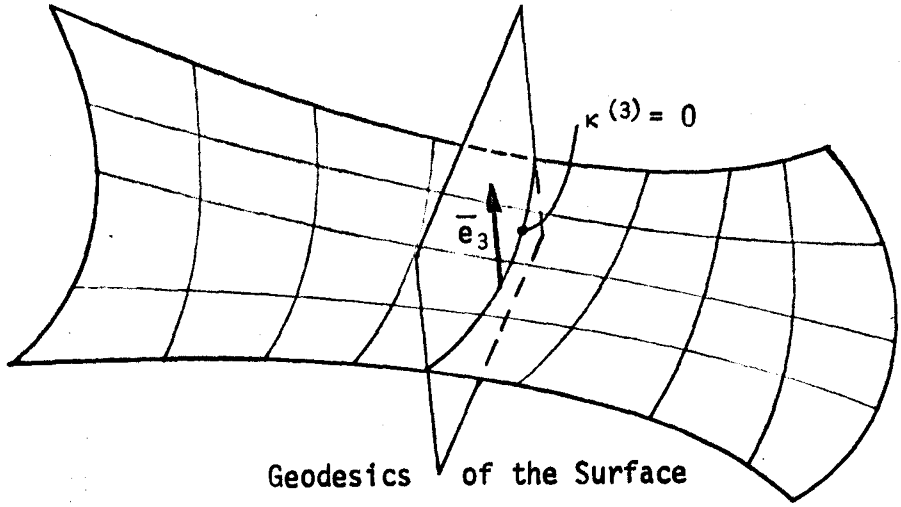
A comparison of {2.9.1.2.-3.} with {2.8.-1.} shows that only a difference of notation exists. The two are otherwise identical.

2.9.1.3. If $\kappa^{(n)} = 0$:

Then $$\bar{C} = \kappa^{(t)}\bar{e}_t + \kappa^{(3)}\bar{e}_3$$

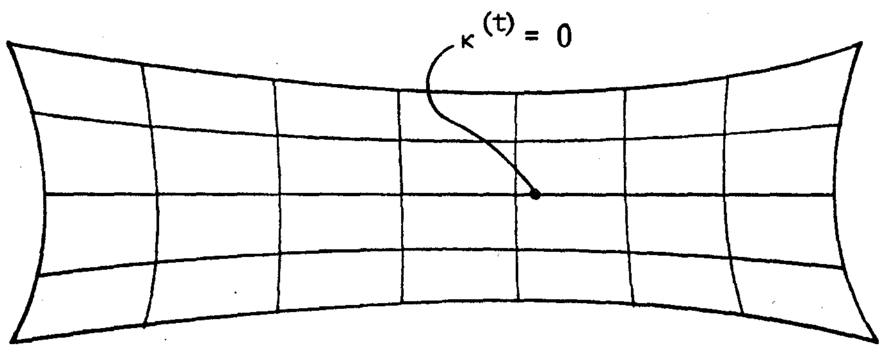
Such a curve, which exhibits no normal curvature, is called an *asymptotic line to the surface.*

NOTE: Equation {2.9.1.2.-1.} illustrates the use of the CESÀRO-BURALI-FORTI Vector. The derivative with respect to the arc length, of a unit vector (the change of the unit vector per unit change of arc length) must be a 'rotational change' of that vector. It is self-evident that a *unit* vector, having constant magnitude, has no rate of change of its magnitude; the rate of change of such a vector is thus prohibited from having a component in the direction of the vector itself. Therefore, it is obvious that the increment of the unit vector must be perpendicular to that vector and thus, may be given as a cross-product of a vector (prescribing rotation) and the unit vector in question. The CESÀRO-BURALI-FORTI vector is, of course, the vector which prescribes the arc-rate of rotation.



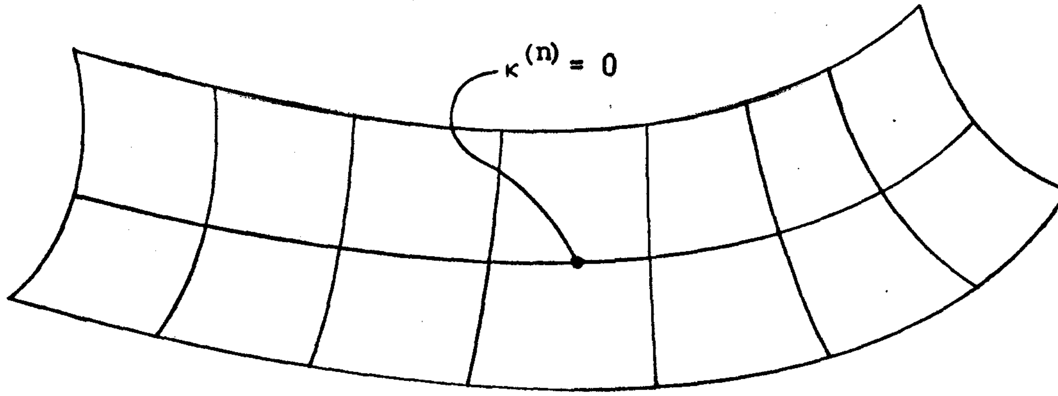
Geodesics of the Surface

Fig. 2.9.1.1.- 1.



Principal Lines of Curvature

Fig. 2.9.1.1.- 2.



Asymptotic Lines

Fig. 2.9.1.1.- 3.

Thus, $$\frac{d\bar{e}_i}{ds} \equiv \bar{C} \times \bar{e}_i$$

where $\bar{e}_i$ is any unit vector, or any vector of constant magnitude which is fixed in the mobile RIBAUCOUR Triad.

In the case that the vector to be differentiated is not of constant magnitude;

(say) $$\bar{\xi} = \bar{\xi}(s) = \xi\bar{e}_i, \quad (\xi \equiv \xi(s) = |\bar{\xi}|):$$

then $$\frac{d\bar{\xi}}{ds} = \frac{d}{ds}(\xi\,\bar{e}_i) = \frac{d\xi}{ds}\bar{e}_i + \xi\frac{d\bar{e}_i}{ds}$$

or $$\frac{d\bar{\xi}}{ds} = \frac{d\xi}{ds}\bar{e}_i + \xi(\bar{C} \times \bar{e}_i)$$

as might be expected.

2.10. PARAMETRIC COORDINATES COINCIDENT WITH PRINCIPAL LINES OF CURVATURE

Recalling the equation for the directions of principal curvature, {2.6.1.-5.}

$$(g_{12}b_{11} - g_{11}b_{12})\lambda^2 + (g_{22}b_{11} - g_{11}b_{22})\lambda + (g_{22}b_{12} - g_{12}b_{22}) = 0$$

..... {2.10.-1.}

or since $\lambda \equiv \frac{d\alpha_1}{d\alpha_2}$, then {2.10.-1.} becomes

$$\Big[(g_{12}b_{11} - g_{11}b_{12})d\alpha_1^2 + (g_{22}b_{11} - g_{11}b_{22})\,d\alpha_1 d\alpha_2$$

$$+ (g_{22}b_{12} - g_{12}b_{22})d\alpha_2^2\Big] = 0$$ ..... {2.10.-2.}

A) Now, for parametric line α_1, as the line of principal curvature; then

$$d\alpha_1 = \text{arbitrary}$$

$$d\alpha_2 = 0 \quad (\text{as } \alpha_2 = \text{constant})$$

so, $d\alpha_1 d\alpha_2 = 0$ and $(-g_{11}b_{12}\, d\alpha_1^2) = 0$ from {2.10.-2.}

B) For parametric line α_2, as the line of principal curvature; then

$$d\alpha_2 = \text{arbitrary}$$

$$d\alpha_1 = 0 \quad (\text{as } \alpha_1 = \text{constant})$$

so, $d\alpha_1 d\alpha_2 = 0$ and $(g_{22}b_{12}\, d\alpha_2^2) = 0$ from {2.10.-2.}

Thus, in general, for parametric coordinates as principal lines of curvature, the following equation is satisfied:

$$d\alpha_1 d\alpha_2 = 0$$

which is equivalent to the equation of conjugate directions.

Now since $\bar{g}_1 \cdot \bar{g}_1 \equiv g_{11} > 0$

$\bar{g}_2 \cdot \bar{g}_2 \equiv g_{22} > 0$

and since $(-g_{11}b_{12}\, d\alpha_1^2) = 0$ (condition "A")

then $b_{12} = 0 \ldots (\text{as } g_{11}d\alpha_1^2 \neq 0)$ .... {2.10.-3.}

and since $(g_{22}b_{12}\, d\alpha_2^2) = 0$ (condition "B")

then $b_{12} = 0 \ldots (\text{as } g_{22}d\alpha_2^2 \neq 0)$ .... {2.10.-3.}

Therefore, for lines of curvature as parametric coordinates (or vice-versa), it is necessary (and sufficient) that the following conditions be satisfied:

$$\left.\begin{aligned} g_{12} &= 0 \\ b_{12} &= 0 \end{aligned}\right\} \quad \text{....... \{2.10.-4.\}}$$

The first member of {2.10.-4.} represents the orthogonality condition of coordinates; the second represents the condition for conjugate directions of parametric coordinates.

NOTE: This sections shows direct agreement with §(2.8.1., 2.8.2., 2.8.3.), where the same results were developed by more intuitive (but less rigorous) arguments.

As a consequence of {2.10.-4.}, the expression for the curvature, $\kappa^{(n)}$ (§ 2.5.), reduces to:

$$\kappa^{(n)} = \frac{b_{11}d\alpha_1^2 + b_{22}d\alpha_2^2}{g_{11}d\alpha_1^2 + g_{22}d\alpha_2^2}$$

$$= \frac{b_{11}d\alpha_1^2 + b_{22}d\alpha_2^2}{ds^2}$$

or,

$$\kappa^{(n)} = b_{11}\left(\frac{d\alpha_1}{ds}\right)^2 + b_{22}\left(\frac{d\alpha_2}{ds}\right)^2 \qquad \text{...... \{2.10.-5.\}}$$

Equation {2.10.-5.} prescribes the curvature of any arbitrary surface curve (for α_1, α_2 as principal lines of curvature), as shown in Fig. 2.10.-1.

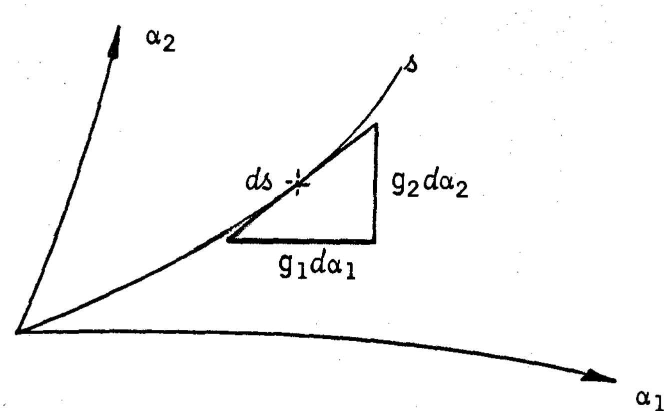


Fig. 2.10.-1.

The normal curvature, $\kappa^{(n)}$, of the parametric line $[d\alpha_1$ = arbitrary, $d\alpha_2 = 0$ $(\alpha_2$ = constant)] is:

$$\kappa_1^{(n)} = \frac{b_{11}}{g_{11}}$$

The normal curvature, $\kappa^{(n)}$, of the parametric line $[d\alpha_2$ = arbitrary, $d\alpha_1 = 0$ $(\alpha_1$ = constant)] is:

$$\kappa_2^{(n)} = \frac{b_{22}}{g_{22}}$$

where the subscript of the term $\kappa_i^{(n)}$ $(i = 1,2)$ refers to the line, with which the curvature is associated.

2.10.1. EULER's Theorem

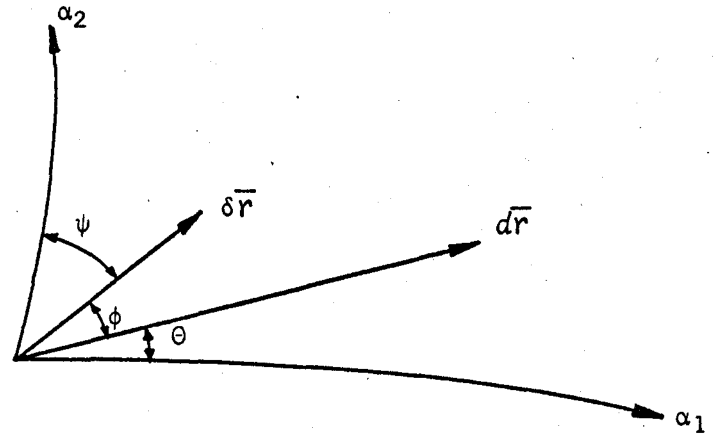


Fig. 2.10.1.-1.

With reference to Fig. 2.10.1.-1; for the two vectors $d\bar{r}$ and $\delta\bar{r}$,

$$\cos\phi = \frac{d\bar{r}\cdot\delta\bar{r}}{|d\bar{r}|\,|\delta\bar{r}|} = \frac{d\bar{r}\cdot\delta\bar{r}}{ds\,\delta s}$$

so, $$\text{Cos}\ \phi = \frac{1}{ds\,\delta s}(\bar{g}_1 d\alpha_1 + \bar{g}_2 d\alpha_2)\cdot(\bar{g}_1 \delta\alpha_1 + \bar{g}_2 \delta\alpha_2)$$

$$= \frac{1}{ds\,\delta s}[g_{11} d\alpha_1 \delta\alpha_1 + g_{12}(d\alpha_1 \delta\alpha_2 + \delta\alpha_1 d\alpha_2) + g_{22} d\alpha_2 \delta\alpha_2]$$

or, $$\text{Cos}\ \phi = g_{11}\frac{d\alpha_1}{ds}\frac{\delta\alpha_1}{\delta s} + g_{12}\left[\frac{d\alpha_1}{ds}\frac{\delta\alpha_2}{\delta s} + \frac{\delta\alpha_1}{\delta s}\frac{d\alpha_2}{ds}\right] + g_{22}\frac{d\alpha_2}{ds}\frac{\delta\alpha_2}{\delta s}$$

A) Consider the special case that $d\bar{r}$ is coincident with parametric line α_1:

VIZ: [$d\alpha_1$ = arbitrary, $d\alpha_2 = 0$ (α_2 = constant)]

i.e.: $\Theta = 0$, Fig. 2.10.1.-1.

Then, $$g_{12} = \bar{g}_1\cdot\bar{g}_2 = 0$$

and $$\text{Cos}\ \phi = \left[g_{11}\frac{d\alpha_1}{ds}\frac{\delta\alpha_1}{\delta s} + g_{22}\frac{d\alpha_2}{ds}\frac{\delta\alpha_2}{\delta s}\right] \quad \text{.......}\ \{2.10.1.\text{-}1.\}$$

then as $ds_1^2 = d\bar{r}_1\cdot d\bar{r}_1$ (Subscript indicates "line 1")

or $$\left(\frac{d\alpha_1}{ds}\right)^2 = \frac{1}{g_{11}}, \text{ so } \frac{d\alpha_1}{ds_1} = \frac{1}{\sqrt{g_{11}}} \equiv \frac{1}{g_1}$$

if $\delta s \to ds_1$, then $\frac{\delta\alpha_1}{\delta s} \to \frac{d\alpha_1}{ds} = \frac{1}{g_1}$

if $\delta s \to ds_2$, then $\frac{\delta\alpha_2}{\delta s} \to 0$ (as $\delta\alpha_2 = 0$)

Hence, from {2.10.1.-1.}

$$\text{Cos}\ \phi = g_{11}\frac{d\alpha_1}{ds}\frac{\delta\alpha_1}{\delta s} = g_{11}\frac{d\alpha_1}{ds}\frac{1}{g_1}$$

or $$\text{Cos}\ \phi = g_1\frac{d\alpha_1}{ds}$$

or again, $$\frac{d\alpha_1}{ds} = \frac{\text{Cos}\ \phi}{g_1} \quad \text{.......}\ \{2.10.1.\text{-}2.\}$$

B) Consider the special case that $\delta\overline{r}$ is coincident with parametric line α_2:

VIZ: [$d\alpha_1 = 0$ (α_1 = constant), $d\alpha_2$ = arbitrary]

i.e.: $\psi = 0$, Fig. 2.10.1.-1.

Then, by a process precisely the same as for case "A" (above):

$$\mathrm{Cos}\ (\pi - \phi)\ g_{22} \frac{d\alpha_2}{ds} \frac{\delta\alpha_2}{\delta s}$$

where $\delta s^2 = g_{22}\ \delta\alpha_2^2$ (as $\delta\alpha_1 = 0$)

so $$\frac{\delta\alpha_2}{\delta s} = \frac{1}{g_2}$$

and finally, $$\frac{d\alpha_2}{ds} = \frac{\mathrm{Sin}\ \phi}{g_2} \qquad \ldots\ldots \quad \{2.10.1.-3.\}$$

Employing {2.10.1.-2.} and {2.10.1.-3.}, in

$$\kappa^{(n)} = b_{11} \left(\frac{d\alpha_1}{ds}\right)^2 + b_{22} \left(\frac{d\alpha_2}{ds}\right)^2 \qquad (\{2.10.-5.\})$$

then $$\kappa^{(n)} = \frac{b_{11}}{g_{11}} \mathrm{Cos}^2\ \phi + \frac{b_{22}}{g_{22}} \mathrm{Sin}^2\ \phi$$

or, as $$\kappa_1^{(n)} \equiv \frac{b_{11}}{g_{11}}, \quad \kappa_2^{(n)} \equiv \frac{b_{22}}{g_{22}} \qquad (\S\ 2.10.)$$

then: $$\kappa^{(n)} = \kappa_1^{(n)} \mathrm{Cos}^2\ \phi + \kappa_2^{(n)}\ \mathrm{Sin}^2\ \phi$$

which is EULER's Theorem in DUPIN's form.

It is thus observed that through the use of the two important theorems:

i.e.: $\kappa^{(n)} = \kappa_1^{(n)} \text{Cos}^2 \phi + \kappa_2^{(n)} \text{Sin}^2 \phi$ ....EULER

$\kappa^{(n)} = \kappa \text{ Cos } \Theta$ ....MEUSNIER

then the curvatures in all directions at a point on the surface may be evaluated.

The EULER Theorem is named in honour of the great mathematician, Leonard EULER (1707-1783), for his work in 1760.

2.10.2. DUPIN's Indicatrix

From EULER's Theorem (§ 2.10.1.)

i.e.: $\kappa^{(n)} = \kappa_1^{(n)} \text{Cos}^2 \phi + \kappa_2^{(n)} \text{Sin}^2 \phi$

and from $\kappa_i^{(n)} \equiv \frac{1}{R_i}$

then $\frac{1}{R} = \frac{1}{R_1} \text{ Cos}^2 \phi + \frac{1}{R_2} \text{ Sin}^2 \phi$

or $1 = \frac{R}{R_1} \text{ Cos}^2 \phi + \frac{R}{R_2} \text{ Sin}^2 \phi$

which, by comparison to the "standard form":

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

is observed to represent the equation of an ellipse.

Thus, $x = \sqrt{R} \text{ Cos } \phi$, $y = \sqrt{R} \text{ Sin } \phi$

Hence, an ellipse may be constructed to prescribe the EULER Theorem:

VIZ:

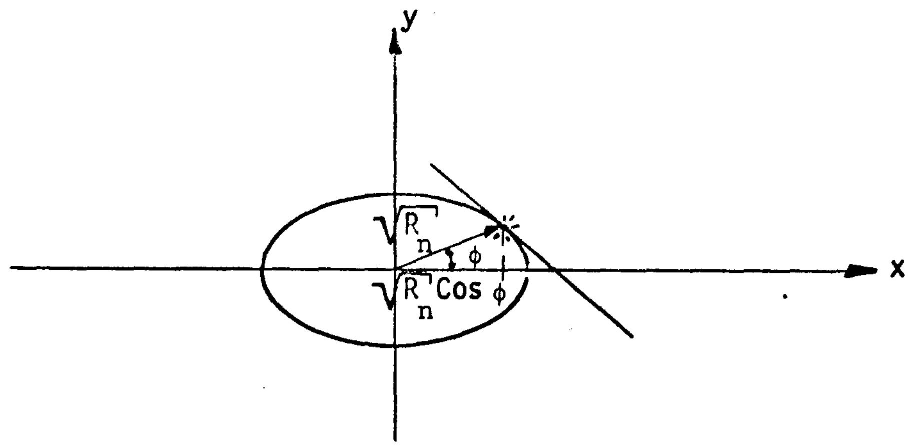


Fig. 2.10.2.-1.

This ellipse is known as DUPIN's Indicatrix, after Francois-Pierre-Charles Baron DUPIN (1784-1873) in 1813. (Discovered in 1807)

2.11. THE DIRECTED DERIVATIVE IN THE SURFACE

The total directed derivative being given as $\frac{d}{d\overline{r}}$, then the directed derivative in the surface is:

$$\frac{\partial}{\partial \overline{r}} = \overline{e}_3 \times \frac{d}{d\overline{r}} \times \overline{e}_3 \qquad \ldots\ldots \qquad \{2.11.\text{-}1.\}$$

Intuitive conceptual justification of the above is accomplished kinematically: the directed derivative may be considered to obey the manipulative and conceptual postulates of vector algebra. Thus, $\overline{e}_3 \times \frac{d}{d\overline{r}} = \overline{\xi}$ is a pseudo-vector, perpendicular to $\overline{e}_3$ and " $\frac{d}{d\overline{r}}$ ". Hence, $\frac{\partial}{\partial \overline{r}} = \overline{e}_3 \times \frac{d}{d\overline{r}} \times \overline{e}_3$ is therefore perpendicular to $\overline{\xi}$ and to $\overline{e}_3$ thus permitting it to lie only in the plane of the surface. No formal proof of this plausibility argument is considered necessary, for a rigorous discussion of the directed derivative itself is beyond the intended scope of this work.

Expanding {2.11.-1.}

$$\frac{\partial}{\partial \bar{r}} \equiv \bar{e}_3 \times \frac{d}{d\bar{r}} \times \bar{e}_3$$

$$\equiv \bar{e}_3 \cdot \bar{e}_3 \frac{d}{d\bar{r}} - \bar{e}_3 \cdot \frac{d}{d\bar{r}} \cdot \bar{e}_3 \qquad \text{(non-operative on } \bar{e}_3\text{)}$$

NO

$$\equiv \frac{d}{d\bar{r}} - (\bar{e}_3 \cdot \bar{e}_i \frac{\partial}{\partial s_i}) \bar{e}_3 \qquad \text{(sum on } i = 1, 2, 3. \text{ Cartesian Base System)}$$

$$\equiv \frac{d}{d\bar{r}} - \delta_{3i} \frac{\delta}{\delta s_i} \bar{e}_3$$

where δ_{3i} is the KRONECKER DELTA

recall:

$$\begin{Bmatrix} \delta_{ij} = 1 & \text{for } i = j \\ \delta_{ij} = 0 & \text{for } i \neq j \end{Bmatrix}$$

so

$$\frac{\partial}{\partial \bar{r}} \equiv \frac{d}{d\bar{r}} - \frac{\partial}{\partial s_n} \bar{e}_3 = \frac{d}{d\bar{r}} - \bar{e}_3 \frac{\partial}{\partial s_3}$$

but

$$\frac{d}{d\bar{r}} \equiv \bar{e}_1 \frac{\partial}{\partial s_1} + \bar{e}_2 \frac{\partial}{\partial s_2} + \bar{e}_3 \frac{\partial}{\partial s_3}$$

thus,

$$\frac{\partial}{\partial \bar{r}} \equiv \bar{e}_1 \frac{\partial}{\partial s_1} + \bar{e}_2 \frac{\partial}{\partial s_2}$$

Thus, the surface directed derivative is given by:

$$\frac{\partial}{\partial \bar{r}} \equiv \bar{e}_i \,.\, \frac{\partial}{\partial s_i} \qquad \text{sum on } i = 1,2.$$

NOTE: Although the directed derivative is given, using an arc length parameter for the normal direction derivative $(\frac{\partial}{\partial s_3})$, it is to be realized that since the normal direction is represented by a straight-line coordinate, then $\frac{\partial}{\partial s_3} \equiv \frac{\partial}{\partial \alpha_3}$.

Generally, since

$$\frac{\partial}{\partial \bar{r}} \equiv \bar{e}_i \frac{\partial}{\partial s_i} \equiv \bar{e}_i \frac{\partial \alpha_i}{\partial s_i} \frac{\partial}{\partial \alpha_i}$$

so,

$$\frac{\partial}{\partial \bar{r}} \equiv \bar{e}_i g_i \frac{\partial}{\partial \alpha_i} \equiv \bar{g}_i \frac{\partial}{\partial \alpha_i}$$

then the above-mentioned condition may be interpreted as:

$$\bar{g}_3 = g_3 \bar{e}_3 = \bar{e}_3 \quad , \quad \text{or } |\bar{g}_3| = g_3 = 1$$

whereas $\quad |\bar{g}_i| = g_i \neq 1, \quad i = 1,2$ (in general).

2.11.1. The Idemfactor in Two Dimensions

The idemfactor (identity tensor) in three dimensions may be given as a function of rectangular coordinates:

$$\frac{d}{d\bar{r}} \bar{r} \equiv \frac{d\bar{r}}{d\bar{r}} = (\bar{e}_1 \frac{\partial}{\partial s_1} + \bar{e}_2 \frac{\partial}{\partial s_2} + \bar{e}_3 \frac{\partial}{\partial s_3})\bar{r}$$

$$= \bar{e}_1 \frac{\partial \bar{r}}{\partial s_1} + \bar{e}_2 \frac{\partial \bar{r}}{\partial s_2} + \bar{e}_3 \frac{\partial \bar{r}}{\partial s_3}$$

$$= \bar{e}_1 \bar{e}_1 + \bar{e}_2 \bar{e}_2 + \bar{e}_3 \bar{e}_3$$

where $\frac{d\bar{r}}{d\alpha_3} \equiv \frac{d\bar{r}}{ds_3} = \bar{e}_3$ (§ 2.11.)

so, $$\frac{d\bar{r}}{d\bar{r}} = \bar{\bar{I}}$$

The idemfactor in two dimensions for rectangular surface coordinates is given by:

$$\frac{\partial\bar{r}}{\partial\bar{r}} = \bar{e}_1\bar{e}_1 + \bar{e}_2\bar{e}_2 = \frac{d\bar{r}}{d\bar{r}} - \bar{e}_3\bar{e}_3$$

Referring to the planar idemfactor as $\bar{\bar{I}}$,

then $$\bar{\bar{I}} = \frac{d\bar{r}}{d\bar{r}} = \frac{\partial\bar{r}}{\partial\bar{r}} + \bar{e}_3\bar{e}_3 = \bar{\bar{I}} + \bar{e}_3\bar{e}_3$$

or $$\bar{\bar{I}} = \bar{\bar{I}} - \bar{e}_3\bar{e}_3 = \frac{\partial\bar{r}}{\partial\bar{r}}$$

2.11.2. The First Fundamental Form

Using the results of §2.11.1., the First Fundamental Form may be obtained directly, as follows:

$$d\bar{r}\cdot d\bar{r} = d\bar{r}d\bar{r}: \frac{\partial\bar{r}}{\partial\bar{r}} = d\bar{r}d\bar{r}: (\bar{\bar{I}} - \bar{e}_3\bar{e}_3)$$
$$= d\bar{r}d\bar{r}: (\bar{e}_1\bar{e}_1 + \bar{e}_2\bar{e}_2)$$
$$= d\bar{r}\cdot(\bar{e}_1\bar{e}_1 + \bar{e}_2\bar{e}_2)\cdot d\bar{r}$$
$$= [(d\bar{r}\cdot\bar{e}_1)(\bar{e}_1\cdot d\bar{r}) + (d\bar{r}\cdot\bar{e}_2)(\bar{e}_2\cdot d\bar{r})]$$

also, $d\bar{r}\cdot d\bar{r} = d\bar{r}\cdot\frac{\partial\bar{r}}{\partial\bar{r}}\cdot(d\bar{r}\cdot\frac{\partial\bar{r}}{\partial\bar{r}}) = d\bar{r}\cdot(\frac{\partial\bar{r}}{\partial\bar{r}}\cdot\frac{\bar{r}\partial}{\partial\bar{r}})\cdot d\bar{r} = \bar{\bar{g}} : d\bar{r}d\bar{r}$

expansion of this reveals:

$$d\bar{r}d\bar{r}: \frac{\partial\bar{r}}{\partial\bar{r}} = \Big([(ds_1\bar{e}_1 + ds_2\bar{e}_2 + d\alpha_3\bar{e}_3)\cdot\bar{e}_1$$
$$+ \bar{e}_1\cdot(ds_1\bar{e}_1 + ds_2\bar{e}_2 + d\alpha_3\bar{e}_3)]$$

$$+ [(ds_1\,\bar{e}_1 + ds_2\,\bar{e}_2 + d\alpha_3\,\bar{e}_3)\cdot\bar{e}_2$$

$$+ \bar{e}_2\cdot(ds_1\,\bar{e}_1 + ds_2\,\bar{e}_2 + d\alpha_3\,\bar{e}_3)]\Big)$$

$$= \Big[ds_1\,(\bar{e}_1\cdot\bar{e}_1 ds_1 + \bar{e}_1\cdot\bar{e}_2\,ds_2)$$

$$+ ds_2(\bar{e}_1\cdot\bar{e}_2\,ds_1 + \bar{e}_2\cdot\bar{e}_2 ds_2)\Big]$$

and as $\bar{e}_i ds_i = \bar{g}_i d\alpha_i$, then

$$d\bar{r}d\bar{r}:\frac{\partial\bar{r}}{\partial\bar{r}} = g_{11}d\alpha_1^2 + 2g_{12}d\alpha_1 d\alpha_2 + g_{22}d\alpha_2^2 = d\bar{r}d\bar{r}:\bar{\bar{g}}$$

which is I, the First Fundamental Form.

2.11.3. The Second Fundamental Form

In a manner similar to §2.11.2., the Second Fundamental Form may be obtained directly, as follows:

$$d\bar{r}\cdot d\bar{e}_3 = d\bar{r}d\bar{r}:\frac{\partial\bar{e}_3}{\partial\bar{r}} = d\bar{r}\cdot\frac{\partial\bar{e}_3}{\partial\bar{r}}\cdot d\bar{r} \quad \text{(by symmetry; see § 2.12.1.)}$$

$$= (ds_i\,\bar{e}_i)\cdot\left(\bar{g}_1\frac{\partial\bar{e}_3}{\partial\alpha_1} + \bar{g}_2\frac{\partial\bar{e}_3}{\partial\alpha_2}\right)\cdot(ds_j\,\bar{e}_j)$$

(sum on i,j = 1,2,3)

$$= \left[\left(\frac{\partial\bar{e}_3}{\partial\alpha_1}\cdot\bar{g}_1\right)d\alpha_1^2 + \left(\frac{\partial\bar{e}_3}{\partial\alpha_1}\cdot\bar{g}_2 + \frac{\partial\bar{e}_3}{\partial\alpha_2}\cdot\bar{g}_1\right) d\alpha_1 d\alpha_2 + \left(\frac{\partial\bar{e}_3}{\partial\alpha_2}\cdot\bar{g}_2\right)d\alpha_2^2\right]$$

and since $\frac{\partial\bar{e}_3}{\partial\alpha_i}\cdot\bar{g}_j = b_{ij} = b_{ji} = \frac{\partial\bar{e}_3}{\partial\alpha_j}\cdot\bar{g}_i$

then $$d\bar{r}d\bar{r}:\frac{\partial\bar{e}_3}{\partial\bar{r}} = b_{11}d\alpha_1^2 + 2b_{12}d\alpha_1 d\alpha_2 + b_{22}d\alpha_2^2 = d\bar{r}d\bar{r}:\bar{\bar{b}}$$

which is II, the Second Fundamental Form.

2.11.4. The Third Fundamental Form

In a manner again similar to § 2.11.2., the third fundamental form may be expressed as:

$$d\bar{e}_3 \cdot d\bar{e}_3 = d\bar{r}d\bar{r}: \left[\frac{\partial \bar{e}_3}{\partial \bar{r}} \cdot \frac{\partial \bar{e}_3}{\partial \bar{r}}\right] = \left[d\bar{r}\cdot\frac{\partial \bar{e}_3}{\partial \bar{r}}\right] \cdot \left[d\bar{r}\cdot\frac{\partial \bar{e}_3}{\partial \bar{r}}\right]$$

$$= \left[\left(\frac{\partial \bar{e}_3}{\partial \alpha_1} \cdot \frac{\partial \bar{e}_3}{\partial \alpha_1}\right) d\alpha_1^2 + 2\left(\frac{\partial \bar{e}_3}{\partial \alpha_1} \cdot \frac{\partial \bar{e}_3}{\partial \alpha_2}\right) d\alpha_1 d\alpha_2 + \left(\frac{\partial \bar{e}_3}{\partial \alpha_2} \cdot \frac{\partial \bar{e}_3}{\partial \alpha_2}\right) d\alpha_2^2\right] = d\bar{r}d\bar{r}:\bar{\bar{a}}$$

referring to $\frac{\partial \bar{e}_3}{\partial \alpha_i} \cdot \frac{\partial \bar{e}_3}{\partial \alpha_j}$ as a_{ij}

then $$d\bar{r}d\bar{r}: \left[\frac{\partial \bar{e}_3}{\partial \bar{r}} \cdot \frac{\partial \bar{e}_3}{\partial \bar{r}}\right] = a_{11}d\alpha_1^2 + 2a_{12}d\alpha_1 d\alpha_2 + a_{22}d\alpha_2^2 = d\bar{r}d\bar{r}:\bar{\bar{a}}$$

which is III, the Third Fundamental Form. (Sometimes, I_3)

NOTE: This form does not have a broad usage, but is employed as a preliminary to later developments.

2.12. INVARIANTS OF THE SURFACE TENSORS

Referring to the quantity $\frac{\partial \bar{r}}{\partial \bar{r}}$ as the First Surface Tensor, since the First Fundamental Form is produced from it, and to $\frac{\partial \bar{e}_3}{\partial \bar{r}}$ as the Second Surface Tensor, for a similar reason, then the invariants of these tensors may be investigated.

Note that $\frac{\partial \bar{r}}{\partial \bar{r}} = \frac{\partial \bar{r}}{\partial \bar{r}} \cdot \frac{\bar{r}\partial}{\partial \bar{r}} = \bar{\bar{g}}$ and $\frac{\partial \bar{e}_3}{\partial \bar{r}} = \frac{\partial \bar{e}_3}{\partial \bar{r}} \cdot \frac{\bar{r}\partial}{\partial \bar{r}} = \bar{\bar{b}}$.

No useful information being produced from the invariants of the First Surface Tensor (or Surface Metric Tensor), attention is directed to the Second Surface Tensor.

2.12.1. The Vector Invariant of the Second Surface Tensor

Denoting the vector invariant as $\left[\frac{\partial \bar{e}_3}{\partial \bar{r}}\right]_v = \overline{\overline{b}}_v = \overline{\overline{1}} \overset{\bullet}{x} \overline{\overline{b}}$

then $$\left[\frac{\partial \bar{e}_3}{\partial \bar{r}}\right]_v = \bar{\bar{1}} \overset{\bullet}{x} \frac{\partial \bar{e}_3}{\partial \bar{r}} = \frac{\partial}{\partial \bar{r}} \times \bar{e}_3$$

$$= \left[\bar{e}_1 \frac{\partial}{\partial s_1} + \bar{e}_2 \frac{\partial}{\partial s_2} + \bar{e}_3 \frac{\partial}{\partial \alpha_3}\right] \times \bar{e}_3$$

$$= \bar{e}_1 \times \frac{\partial \bar{e}_3}{\partial s_1} + \bar{e}_2 \times \frac{\partial \bar{e}_3}{\partial s_2} + \bar{e}_3 \times \frac{\partial \bar{e}_3}{\partial \alpha_3}$$

$\left(\text{as } \frac{\partial}{\partial \bar{r}} \times \bar{\xi} \equiv \bar{g}_i \times \frac{\partial \bar{\xi}}{\partial \alpha_i}, \text{ sum on } i = 1, 2, 3\right)$

$$= \bar{e}_1 \times \frac{\partial \bar{e}_3}{\partial s_1} + \bar{e}_2 \times \frac{\partial \bar{e}_3}{\partial s_2}$$

$\left(\frac{\partial \bar{e}_3}{\partial \alpha_3} = 0\right.$, as α_3 is a straight-line coordinate)

now, specifying the relationships: $\bar{e}_1 = \bar{e}_2 \times \bar{e}_3 E^{-1}$

$\bar{e}_2 = \bar{e}_3 \times \bar{e}_1 E^{-1}$

where $E = \left[\bar{e}_1 \cdot \bar{e}_2 \times \bar{e}_3\right]$

then $$\left[\frac{\partial \bar{e}_3}{\partial \bar{r}}\right]_v = E^{-1}\left[(\bar{e}_2 \times \bar{e}_3) \times \frac{\partial \bar{e}_3}{\partial s_1} + (\bar{e}_3 \times \bar{e}_1) \times \frac{\partial \bar{e}_3}{\partial s_2}\right]$$

$$= E^{-1}\left[\left(\bar{e}_2 \cdot \frac{\partial \bar{e}_3}{\partial s_1}\right) \bar{e}_3 - \left(\bar{e}_3 \cdot \frac{\partial \bar{e}_3}{\partial s_1}\right) \bar{e}_2 + \left(\bar{e}_3 \cdot \frac{\partial \bar{e}_3}{\partial s_2}\right) \bar{e}_1 - \left(\bar{e}_1 \cdot \frac{\partial \bar{e}_3}{\partial s_2}\right) \bar{e}_3\right]$$

and, as $\bar{e}_3 \perp \frac{\partial \bar{e}_3}{\partial s_i}$, then $\bar{e}_3 \cdot \frac{\partial \bar{e}_3}{\partial s_i} = 0 \quad (i = 1, 2)$

so
$$\left[\frac{\partial \bar{e}_3}{\partial \bar{r}}\right]_v = E^{-1}\left[\bar{e}_2 \cdot \frac{\partial \bar{e}_3}{\partial s_1} - \bar{e}_1 \cdot \frac{\partial \bar{e}_3}{\partial s_2}\right] \bar{e}_3$$

$$= E^{-1}\left[\frac{\partial \bar{r}}{\partial s_2} \cdot \frac{\partial \bar{e}_3}{\partial s_1} - \frac{\partial \bar{r}}{\partial s_1} \cdot \frac{\partial \bar{e}_3}{\partial s_2}\right] \bar{e}_3$$

$$= E^{-1}\left[\frac{1}{g_1 g_2}\left(\frac{\partial \bar{r}}{\partial \alpha_2} \cdot \frac{\partial \bar{e}_3}{\partial \alpha_1}\right) - \frac{1}{g_1 g_2}\left(\frac{\partial \bar{r}}{\partial \alpha_1} \cdot \frac{\partial \bar{e}_3}{\partial \alpha_2}\right)\right] \bar{e}_3$$

$$= \frac{\bar{e}_3}{E g_1 g_2}(b_{21} - b_{12}) \qquad \text{(see § 2.5.)}$$

$$= 0 \qquad \text{as } b_{12} = b_{21}$$

This serves to establish that $\frac{\partial \bar{e}_3}{\partial \bar{r}}$ is a *symmetric* tensor. The vanishing of the vector invariant is the necessary and sufficient condition for symmetry of the tensor. Hence, the conjugate tensor $\frac{\bar{e}_3 \partial}{\partial \bar{r}} \equiv \left[\frac{\partial \bar{e}_3}{\partial \bar{r}}\right]_c$ and the original tensor are identical -- a fact which has been employed in § 2.11.3. and § 2.11.4. without a detailed explanation being given in that section

2.12.2. The Second Scalar Invariant of the Second Surface Tensor and The HAMILTON-CAYLEY Equation of Surfaces

The expression for the Second Scalar invariant $\left[\frac{\partial \bar{e}_3}{\partial \bar{r}}\right]_2^{(s)}$, of the tensor, $\frac{\partial \bar{e}_3}{\partial \bar{r}}$, is:

$$\left[\frac{\partial \bar{e}_3}{\partial \bar{r}}\right]_2^{(s)} = \frac{1}{2!}\left(\frac{\partial \bar{e}_3}{\partial \bar{r}} \begin{matrix} \times \\ \times \end{matrix} \frac{\partial \bar{e}_3}{\partial \bar{r}} : \bar{\bar{1}}\right)$$

To obtain a meaningful expression from this, it is first necessary that $\frac{\partial \bar{e}_3}{\partial \bar{r}}$ be expanded in some form.

say $$\frac{\partial \bar{e}_3}{\partial \bar{r}} = \left(\frac{\partial \bar{r}}{\partial \bar{r}} \cdot \frac{\bar{e}_3 \partial}{\partial \bar{r}}\right) = \frac{\partial \bar{r}}{\partial \bar{r}} \cdot \frac{\partial \bar{e}_3}{\partial \bar{r}}$$

now $$\frac{\partial \bar{e}_3}{\partial \bar{r}} = \bar{e}_1 \frac{\partial \bar{e}_3}{\partial s_1} + \bar{e}_2 \frac{\partial \bar{e}_3}{\partial s_2} \qquad \text{(as in § 2.12.1)}$$

yet $$\frac{\partial \bar{e}_3}{\partial s_i} = \bar{C}_i \times \bar{e}_3 \qquad \text{(§2.9. ff.)}$$

where $\bar{C}_i$ represents the CESÀRO-BURALI-FORTI Vector for the triad, the tangent of which is tangent to parametric line α_i (hereafter referred to as 'the CESÀRO-BURALI-FORTI Vector for line α_i')

Then, with reference to {2.9.-2.}

$$\frac{\partial \bar{e}_3}{\partial s_1} = \bar{C}_1 \times \bar{e}_3 = (\kappa_1^{(t)} \bar{e}_1 + \kappa_1^{(n)} \bar{e}_*^{1} + \kappa_1^{(3)} \bar{e}_3) \times \bar{e}_3$$

also $$\frac{\partial \bar{e}_3}{\partial s_2} = \bar{C}_2 \times \bar{e}_3 = (\kappa_2^{(t)} \bar{e}_2 + \kappa_2^{(n)} \bar{e}_*^{2} + \kappa_2^{(3)} \bar{e}_3) \times \bar{e}_3$$

The vectors $\bar{e}_*^{1}$ and $\bar{e}_*^{2}$ are the binormals to lines α_1 and α_2 (respectively), and the unit normal, $\bar{e}_3$, is naturally common to both triads, being a *surface* normal. This system of employing two separate triads, one for line α_1 and one for line α_2, facilitates both a conceptual appreciation of the situation and the mathematics itself. The non-orthogonal case is shown in Fig. 2.12.2.-1., in order to illustrate the two separate dextral triads. Note that only one triad is distinct in three directions; expressions involving vector directions of both triads must necessarily be resolved into the directions of one triad during the process of extracting components.

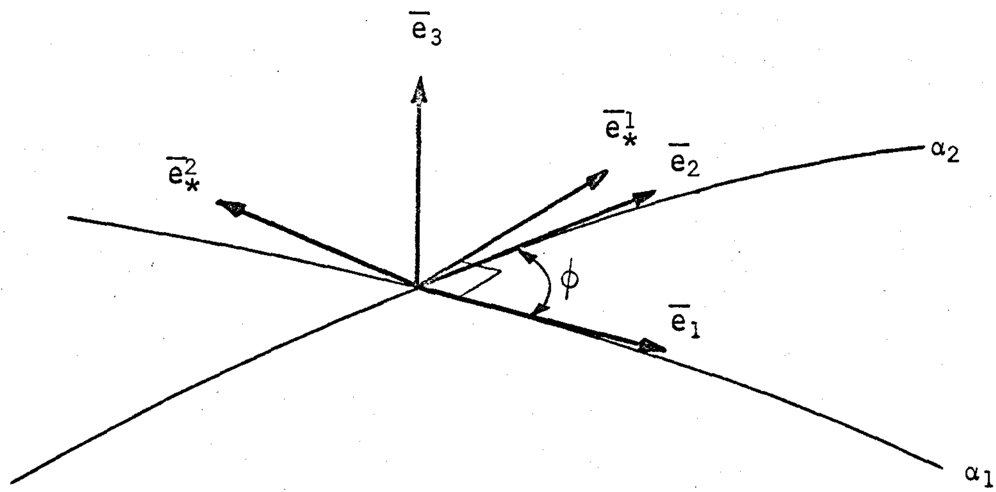


Fig. 2.12.2.-1.

The symbolism previously employed will now be altered, so as to form a consistent system with the tensorial approach.

Thus, *define:*

$\kappa_1^{(n)} \equiv \kappa_{11}$ .... *Normal Curvature, line 1*

$\kappa_1^{(t)} \equiv \kappa_{12}$ .... *Geodesic Torsion, line 1*

$\kappa_1^{(3)} \equiv \kappa_{13}$ .... *Geodesic Curvature, line 1.*

$\kappa_2^{(n)} \equiv \kappa_{22}$ .... *Normal Curvature, line 2*

$\kappa_2^{(t)} \equiv \kappa_{21}$ .... *Geodesic Torsion, line 2*

$\kappa_2^{(3)} \equiv \kappa_{23}$ .... *Geodesic Curvature, line 2*

then,

$$\overline{C}_1 = \kappa_{12}\overline{e}_1 + \kappa_{11}\overset{1}{\overline{e}_*} + \kappa_{13}\overline{e}_3 \qquad \text{..... \{2.12.2.-1.\}}$$

$$\overline{C}_2 = \kappa_{21}\overline{e}_2 + \kappa_{22}\overset{2}{\overline{e}_*} + \kappa_{23}\overline{e}_3 \qquad \text{..... \{2.12.2.-2.\}}$$

Then, in 'operator' form, for rigid vectors:

$$\frac{\partial}{\partial s_i}(\quad) \equiv \overline{C}_i \times (\quad) \qquad i = 1,2$$

For the case under consideration at present, then

(1) $$\frac{\partial \overline{e}_3}{\partial s_1} = \overline{C}_1 \times \overline{e}_3 = (\kappa_{12}\overline{e}_1 + \kappa_{11}\overset{1}{\overline{e}_*} + \kappa_{13}\overline{e}_3) \times \overline{e}_3$$

$$= -\kappa_{12}\overset{1}{\overline{e}_*} + \kappa_{11}\overline{e}_1$$

(2) $$\frac{\partial \overline{e}_3}{\partial s_2} = \overline{C}_2 \times \overline{e}_3 = (\kappa_{21}\overline{e}_2 + \kappa_{22}\overset{2}{\overline{e}_*} + \kappa_{23}\overline{e}_3) \times \overline{e}_3$$

$$= -\kappa_{21}\overset{2}{\overline{e}_*} + \kappa_{22}\overline{e}_2$$

thus,

$$\frac{\partial \overline{e}_3}{\partial \overline{r}} = [\overline{e}_1(-\kappa_{12}\overset{1}{\overline{e}_*} + \kappa_{11}\overline{e}_1) + \overline{e}_2(-\kappa_{21}\overset{2}{\overline{e}_*} + \kappa_{22}\overline{e}_2)]$$

Note: equations (1) and (2) above are known as the RODRIGUES-WEINGARTEN Formulas for vector differentiation.

from which form, the second scalar invariant may be obtained without difficulty:

$$\frac{1}{2!}\left(\frac{\partial \bar{e}_3}{\partial \bar{r}} \begin{matrix} \times \\ \times \end{matrix} \frac{\partial \bar{e}_3}{\partial \bar{r}} : \bar{\bar{1}}\right)$$

$$= (\bar{e}_1 \times \bar{e}_2 \cdot \bar{e}_3)\ [(\kappa_{12}\overset{1}{\bar{e}_*} \times \kappa_{21}\overset{2}{\bar{e}}) + (\kappa_{22}\bar{e}_2 \times \kappa_{12}\overset{1}{\bar{e}_*})$$
$$+ (\kappa_{21}\overset{2}{\bar{e}_*} \times \kappa_{11}\bar{e}_1) + (\kappa_{11}\bar{e}_1 \times \kappa_{22}\bar{e}_2)]\cdot\bar{e}_3 \quad ...\{2.12.2.-3.\}$$

This expression is better left in the present form, as expansion thereof yields only a more complex representation.

However, in order to demonstrate the significance of {2.12.2.-3.} , the case of *orthogonal parametric lines* (α_1,α_2) is considered:

if $\quad \phi = \frac{\pi}{2}$ (see Fig. 2.12.2.-1.)

then, $\quad \overset{1}{\bar{e}_*} = \bar{e}_2, \quad \overset{2}{\bar{e}_*} = -\bar{e}_1$

hence, $\quad \bar{C}_1 = \kappa_{12}\bar{e}_1 + \kappa_{11}\bar{e}_2 + \kappa_{13}\bar{e}_3$

$$\bar{C}_2 = \kappa_{21}\bar{e}_2 - \kappa_{22}\bar{e}_1 + \kappa_{23}\bar{e}_3$$

and so $\quad \frac{\partial \bar{e}_3}{\partial \bar{r}} = [\bar{e}_1(\kappa_{11}\bar{e}_1 - \kappa_{12}\bar{e}_2) + \bar{e}_2(\kappa_{21}\bar{e}_1 + \kappa_{22}\bar{e}_2)]$

or $\quad \frac{\partial \bar{e}_3}{\partial \bar{r}} = \begin{bmatrix} + \kappa_{11}\bar{e}_1\bar{e}_1 - \kappa_{12}\bar{e}_1\bar{e}_2 \\ \\ + \kappa_{21}\bar{e}_2\bar{e}_1 + \kappa_{22}\bar{e}_2\bar{e}_2 \end{bmatrix}$

as $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ are mutually perpendicular in this case, then $\bar{e}_1 \times \bar{e}_2 \cdot \bar{e}_3 = 1$, and {2.12.2.-3.} becomes:

$$\frac{1}{2I}\left(\frac{\partial\bar{e}_3}{\partial\bar{r}} \;\times\!\!\times\; \frac{\partial\bar{e}_3}{\partial\bar{r}} : \bar{\bar{I}}\right) = (\kappa_{11}\kappa_{22} + \kappa_{12}\kappa_{21}) \qquad \ldots\ldots \quad \{2.12.2.\text{-}4.\}$$

$$= \begin{vmatrix} +\ \kappa_{11} & -\ \kappa_{12} \\ +\ \kappa_{21} & +\ \kappa_{22} \end{vmatrix} = \left|\frac{\partial\bar{e}_3}{\partial\bar{r}}\right| \equiv |\bar{\bar{\kappa}}|$$

where $|(\ )|$ represents the absolute value of the quantity within the brackets, whether vector or tensor.

Proceeding one step further in specialization, if these (orthogonal) parametric lines are also coincident with the principal lines of curvature, then the geodesic torsions vanish {§2.9.1.2. ff.},

i.e.: $\kappa_{12} = 0 = \kappa_{21}$

then $$\bar{\bar{\kappa}} = \kappa_{11}\bar{e}_1\bar{e}_1 + \kappa_{22}\,\bar{e}_2\bar{e}_2 = \frac{\partial\bar{e}_3}{\partial\bar{r}}$$

The First Scalar Invariant of $\bar{\bar{\kappa}}$, is then:

$$\bar{\bar{\kappa}}_1^{(s)} = \kappa_{11} + \kappa_{22} \qquad \ldots\ldots \{a\}$$

and the Second Scalar Invariant is given by {2.12.2.-3.}, as:

$$\bar{\bar{\kappa}}_2^{(s)} = \kappa_{11}\kappa_{22} \qquad \ldots\ldots \{b\}$$

also, recalling

$$\bar{\bar{I}} = \frac{d\bar{r}}{d\bar{r}} = \bar{e}_1\bar{e}_1 + \bar{e}_2\bar{e}_2 + \bar{e}_3\bar{e}_3 \qquad \ldots\ldots\ldots \{c\}$$

Then, the relation existing between {a} , {b} and {c} (above) may be expressed in the convenient form:

$$\frac{\partial \bar{e}_3}{\partial \bar{r}} \cdot \frac{\bar{e}_3 \partial}{\partial \bar{r}} - (\kappa_{11} + \kappa_{22}) \frac{\partial \bar{e}_3}{\partial \bar{r}} + \kappa_{11}\kappa_{22} \frac{\partial \bar{r}}{\partial \bar{r}} = 0$$

or as $\dfrac{\partial \bar{e}_3}{\partial \bar{r}} = \dfrac{\bar{e}_3 \partial}{\partial \bar{r}} = \bar{\bar{\kappa}}$

then $$\bar{\bar{\kappa}} \cdot \bar{\bar{\kappa}} - \bar{\bar{\kappa}}_1^{(S)} \bar{\bar{\kappa}} + \bar{\bar{\kappa}}_2^{(S)} \bar{\bar{1}} = 0$$

or $$\bar{\bar{\kappa}} \cdot \bar{\bar{\kappa}} - (\bar{\bar{\kappa}} : \bar{\bar{1}}) \bar{\bar{\kappa}} + |\bar{\bar{\kappa}}| \; \bar{\bar{1}} = 0 \qquad \ldots\ldots \quad \{2.12.2.-5.\}$$

This equation, {2.12.2.-5.} , is known as the HAMILTON-CAYLEY equation for surfaces. It may be stated as: "The surface tensor, $\bar{\bar{\kappa}} = \dfrac{\partial \bar{e}_3}{\partial \bar{r}}$, satisfies its own (SEGNER) Eigenvalue equation". Hence, the SEGNER Eigenvalue equation for principal directions of $\bar{\bar{\kappa}}$ might be given as:

$$\lambda_{\alpha\alpha}^2 - (\bar{\bar{\kappa}} : \bar{\bar{1}}) \lambda_{\alpha\alpha} + |\bar{\bar{\kappa}}| = 0$$

The HAMILTON-CAYLEY equation may also be reduced to the scalar form by taking a double dot product with $d\bar{r}d\bar{r}$.

$$\kappa_{11}\kappa_{22} d\bar{r} \cdot d\bar{r} - (\kappa_{11} + \kappa_{22}) \, d\bar{r} \cdot d\bar{e}_3 + d\bar{e}_3 \cdot d\bar{e}_3 = 0$$

or $$|\bar{\bar{\kappa}}| \; d\bar{r} \cdot d\bar{r} - \bar{\bar{\kappa}} : \bar{\bar{1}} \; d\bar{r} \cdot d\bar{e}_3 + d\bar{e}_3 \cdot d\bar{e}_3 = 0$$

or again, as $d\bar{r} \cdot d\bar{r} = I$, etc.,

$$(\, |\bar{\bar{\kappa}}| \,) \, I - (\bar{\bar{\kappa}} : \bar{\bar{1}}) II + III = 0$$

This equation permits a solution for one scalar invariant in terms of the other two. The HAMILTON-CAYLEY equation is named for Sir William Rowan HAMILTON (1805-1860), for his work in 1853, and for Arthur CAYLEY's (1821-1895) work in 1859. The SEGNER eigenvalue equation derives its name from Johann Andreas von SEGNER (1704-1777), in 1755.

2.13. THE SURFACE AND ITS SPHERICAL IMAGE

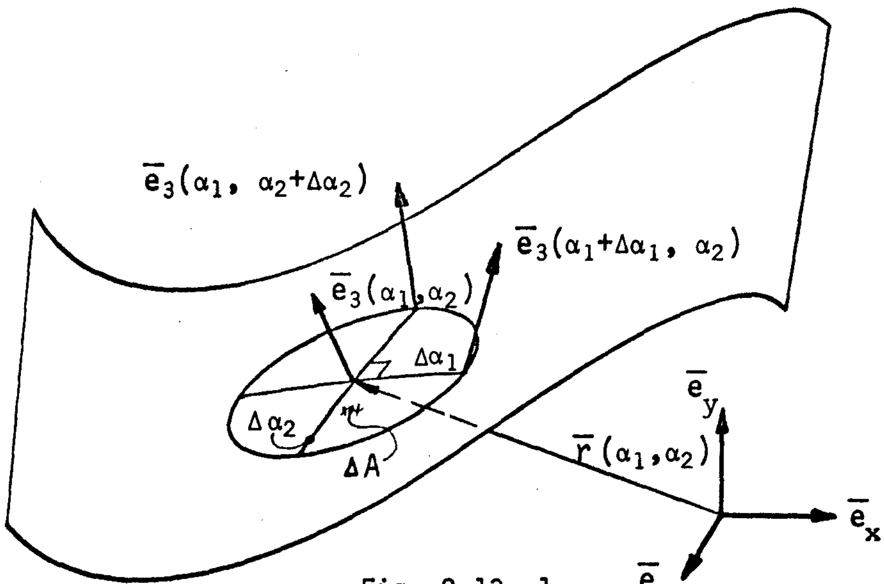
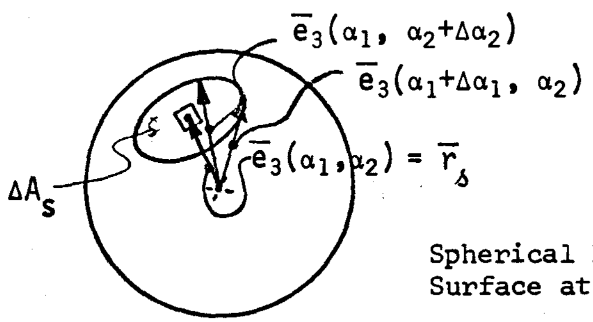


Fig. 2.13.-1.



Spherical Image of Surface at $\bar{r}(\alpha_1,\alpha_2)$

Fig. 2.13.-2.

The total curvature, κ_g , due to RODRIGUES (1815) and GAUSS (1827), is (from Fig. 2.13.-1. and 2.13.-2.):

$$\kappa_g = \lim_{\Delta A \to 0} \left[\frac{\Delta A_s}{\Delta A}\right] = \frac{dA_s}{dA}$$

where $dA = \bar{e}_3 \cdot \left(\frac{\partial \bar{r}}{\partial \alpha_1} d\alpha_1 \times \frac{\partial \bar{r}}{\partial \alpha_2} d\alpha_2\right)$ (§2.2.1.)

as $d\bar{A} = dA\,\bar{e}_3$, or $dA = d\bar{A}\cdot\bar{e}_3$

and where $dA_s = \bar{e}_3 \cdot \left(\frac{\partial \bar{e}_3}{\partial \alpha_1} d\alpha_1 \times \frac{\partial \bar{e}_3}{\partial \alpha_2} d\alpha_2\right)$

Hence, $$\kappa_g = \left[\frac{\left(\frac{\partial \bar{e}_3}{\partial \alpha_1} \times \frac{\partial \bar{e}_3}{\partial \alpha_2} \cdot \bar{e}_3\right)}{\left(\frac{\partial \bar{r}}{\partial \alpha_1} \times \frac{\partial \bar{r}}{\partial \alpha_2} \cdot \bar{e}_3\right)}\right] = \frac{dA_s}{dA}$$

now as $$\frac{\partial \bar{e}_3}{\partial \alpha_1} = \frac{\partial s_1}{\partial \alpha_1}\frac{\partial \bar{e}_3}{\partial s_1} = g_1 \frac{\partial \bar{e}_3}{\partial s_1} = g_1 \bar{C}_1 \times \bar{e}_3$$

then $$\frac{\partial \bar{e}_3}{\partial \alpha_1} = g_1(\bar{\kappa}_1 + \kappa_{13}\bar{e}_3) \times \bar{e}_3 \qquad \text{(from \{2.9.-4.\})}$$

$$= g_1\bar{\kappa}_1 \times \bar{e}_3 \qquad \text{as } \bar{e}_i \times \bar{e}_i = 0$$

Similarly, $$\frac{\partial \bar{e}_3}{\partial \alpha_2} = g_2\bar{\kappa}_2 \times \bar{e}_3$$

Then
$$\frac{\partial \bar{e}_3}{\partial \alpha_1} \times \frac{\partial \bar{e}_3}{\partial \alpha_2} = g_1 g_2\,(\bar{\kappa}_1 \times \bar{e}_3) \times (\bar{\kappa}_2 \times \bar{e}_3)$$
$$= g_1 g_2[(\bar{\kappa}_1 \times \bar{e}_3\cdot\bar{e}_3)\bar{\kappa}_2 - (\bar{\kappa}_1 \times \bar{e}_3\cdot\bar{\kappa}_2)\bar{e}_3]$$
$$= g_1 g_2[-\bar{\kappa}_1 \times \bar{e}_3\cdot\bar{\kappa}_2]\bar{e}_3$$
$$= g_1 g_2[\bar{\kappa}_1 \times \bar{\kappa}_2\cdot\bar{e}_3]\bar{e}_3$$
$$= g_1 g_2(\bar{\kappa}_1 \times \bar{\kappa}_2)\cdot\bar{e}_3\bar{e}_3$$

so
$$\frac{\partial \bar{e}_3}{\partial \alpha_1} \times \frac{\partial \bar{e}_3}{\partial \alpha_2}\cdot\bar{e}_3 = g_1 g_2(\bar{\kappa}_1 \times \bar{\kappa}_2)\cdot\bar{e}_3\bar{e}_3\cdot\bar{e}_3$$
$$= g_1 g_2(\bar{\kappa}_1 \times \bar{\kappa}_2)\cdot\bar{e}_3$$

Now, as $$\frac{\partial \bar{r}}{\partial \alpha_1} \times \frac{\partial \bar{r}}{\partial \alpha_2} = \left(\left|\frac{\partial \bar{r}}{\partial \alpha_1}\right| \left|\frac{\partial \bar{r}}{\partial \alpha_2}\right| \operatorname{Sin}\phi\right)\bar{e}_3 \qquad \{\phi = \operatorname{Cos}^{-1}(\bar{e}_1\cdot\bar{e}_2)\}$$

then $$\kappa_g = \frac{g_1 g_2(\bar{\kappa}_1 \times \bar{\kappa}_2)\cdot\bar{e}_3}{g_1 g_2 \operatorname{Sin}\phi\ \bar{e}_3\cdot\bar{e}_3} = \frac{\bar{\kappa}_1 \times \bar{\kappa}_2\cdot\bar{e}_3}{\operatorname{Sin}\phi}$$

and finally, $$\kappa_g = \frac{1}{\text{Sin } \phi} \overline{\kappa_1} \times \overline{\kappa_2} \cdot \overline{e_3} \quad \ldots \quad \{2.13.-1.\}$$

Expanding {2.13.1.} is accomplished by means of the definitions of $\overline{\kappa_1}$ and $\overline{\kappa_2}$:

VIZ: $$\overline{\kappa_1} \times \overline{\kappa_2} = (\kappa_{12}\overline{e_1} + \kappa_{11}\overline{e_*}^1) \times (\kappa_{21}\overline{e_2} + \kappa_{22}\overline{e_*}^2)$$

$$= [\kappa_{12}\kappa_{21} (\overline{e_1} \times \overline{e_2}) + \kappa_{12}\kappa_{22} (\overline{e_1} \times \overline{e_*}^2)$$

$$+ \kappa_{11}\kappa_{21}(\overline{e_*}^1 \times \overline{e_2}) + \kappa_{11}\kappa_{22}(\overline{e_*}^1 \times \overline{e_*}^2)]$$

realizing that: $$\overline{e_1} \times \overline{e_2} = \text{Sin } \phi \ \overline{e_3}$$

$$\overline{e_1} \times \overline{e_*}^2 = \text{Sin } (\tfrac{\pi}{2} + \phi) \ \overline{e_3} = \text{Cos } \phi \ \overline{e_3}$$

$$\overline{e_*}^1 \times \overline{e_2} = -\text{Sin } (\tfrac{\pi}{2} - \phi) \ \overline{e_3} = - \text{Cos } \phi \ \overline{e_3}$$

$$\overline{e_*}^1 \times \overline{e_*}^2 = \text{Sin } \phi \ \overline{e_3}$$

then $$\overline{\kappa_1} \times \overline{\kappa_2} = [(\kappa_{12}\kappa_{21} + \kappa_{11}\kappa_{22}) \text{ Sin } \phi$$

$$+ (\kappa_{12}\kappa_{22} - \kappa_{11}\kappa_{21}) \text{ Cos } \phi] \ \overline{e_3}$$

so, $$\kappa_g = [\kappa_{11}\kappa_{22} + \kappa_{12}\kappa_{21} + (\kappa_{12}\kappa_{22} - \kappa_{11}\kappa_{21}) \text{ Cot } \phi] \ \ldots \ \{2.13.-2.\}$$

which is, then, the total (GAUSSIAN) curvature for the surface at the point (α_1,α_2), where α_1 and α_2 are at an angle, ϕ.

In the case that $\phi = \frac{\pi}{2}$ (orthogonal parametric lines), then {2.13.-2.} becomes:

$$\kappa_g = (\kappa_{11}\kappa_{22} + \kappa_{12}\kappa_{21}) \quad \ldots\ldots \quad \{2.13.-3.\}$$

(compare with {2.12.2.-4.})

2.13.1. BONNET's Theorem

Having previously established, in § 2.12.1., that $\frac{\partial \overline{e}_3}{\partial \overline{r}}$ is a symmetric tensor and that consequently (as a criterion), the vector invariant $\left[\frac{\partial \overline{e}_3}{\partial \overline{r}}\right]_v$ vanishes, it is then possible to express this condition in terms of the curvature components.

From $$\left[\frac{\partial \overline{e}_3}{\partial \overline{r}}\right]_v = \frac{\partial}{\partial \overline{r}} \times \overline{e}_3 = 0$$

then an expansion reveals:

$$\frac{\partial}{\partial \overline{r}} \times \overline{e}_3 = \overline{e}_1 \times \frac{\partial \overline{e}_3}{\partial s_1} + \overline{e}_2 \times \frac{\partial \overline{e}_3}{\partial s_2} = 0$$

or $$\overline{e}_1 \times (\overline{C}_1 \times \overline{e}_3) + \overline{e}_2 \times (\overline{C}_2 \times \overline{e}_3) = 0$$

$$\overline{e}_1 \times [(\overline{\kappa}_1 + \kappa_{13}\overline{e}_3) \times \overline{e}_3] + \overline{e}_2 \times [(\overline{\kappa}_2 + \kappa_{23}\overline{e}_3) \times \overline{e}_3] = 0$$

$$\overline{e}_1 \times (\overline{\kappa}_1 \times \overline{e}_3) + \overline{e}_2 \times (\overline{\kappa}_2 \times \overline{e}_3) = 0$$

so $$(\overline{e}_1 \cdot \overline{e}_3)\overline{\kappa}_1 - (\overline{e}_1 \cdot \overline{\kappa}_1)\overline{e}_3 + (\overline{e}_2 \cdot \overline{e}_3)\overline{\kappa}_2 - (\overline{e}_2 \cdot \overline{\kappa}_2)\overline{e}_3 = 0$$

or $$-(\overline{e}_1 \cdot \overline{\kappa}_1 + \overline{e}_2 \cdot \overline{\kappa}_2)\overline{e}_3 = 0 \qquad (\text{as } \overline{e}_i \cdot \overline{e}_3 = 0,\ i = 1,2)$$

thus, as $-\overline{e}_3 \neq 0$,

$$\overline{e}_1 \cdot \overline{\kappa}_1 + \overline{e}_2 \cdot \overline{\kappa}_2 = 0$$

$$\overline{e}_1 \cdot (\kappa_{12}\overline{e}_1 + \kappa_{11}\overline{e}_2) + \overline{e}_2 \cdot (\kappa_{21}\overline{e}_2 - \kappa_{22}\overline{e}_1) = 0$$

(for orthogonal parametric lines)

thus, expanding the above reveals:

$$\kappa_{12} + \kappa_{21} = 0 \qquad \ldots\ldots \quad \{2.13.1.-1.\}$$

which is BONNET's Theorem for orthogonal parametric lines.

The Theorem derives its name from the work of Ossian-Pierre BONNET (1819-1892) in 1856.

CHAPTER 3

Three Fundamental Equations of Surfaces

3.1. THE INTEGRABILITY CONDITION

In the ordinary calculus, a form of the following type may occur:

$$d\psi = A_x dx + A_y dy \equiv \overline{A}\cdot d\overline{r}$$

where $d\psi$, *in general*, does not represent a total differential of some function, ψ. However, if $d\psi$ does represent a total differential of some function, then (*and only then*):

$$d\psi = d\psi(\overline{r}) = d\overline{r}\cdot\frac{\partial\psi}{\partial\overline{r}} = d\overline{r}\cdot\overline{A}$$

or

$$d\overline{r}\cdot\left(\frac{\partial\psi}{\partial\overline{r}} - \overline{A}\right) = 0$$

so that, as $d\overline{r} \neq 0$, $\dfrac{\partial\psi}{\partial\overline{r}} - \overline{A} = 0$ ..... {3.1.-1.}

then

$$\frac{\partial}{\partial\overline{r}} \times \frac{\partial\psi}{\partial\overline{r}} = \frac{\partial}{\partial\overline{r}} \times \overline{A}$$

$$0 = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}$$

or

$$\frac{\partial A_x}{\partial y} = \frac{\partial A_y}{\partial x}$$ ..... {3.1.-2.}

then, as $A_x = \frac{\partial\psi}{\partial x}$, $Ay = \frac{\partial\psi}{\partial y}$ (from {3.1.-1.})

then $$\frac{\partial^2\psi}{\partial y\partial x} = \frac{\partial^2\psi}{\partial x\partial y} \qquad \ldots\ldots \qquad \{3.1.-3.\}$$

{3.1.-2.} is usually referred to as the *Integrability Condition of* CLAIRAUT (1743), as well as {3.1.-3.}. The latter equation is, however, sometimes known as the Nicholas BERNOULLI equation .

3.1.1. Geometric Interpretation of the Integrability Condition

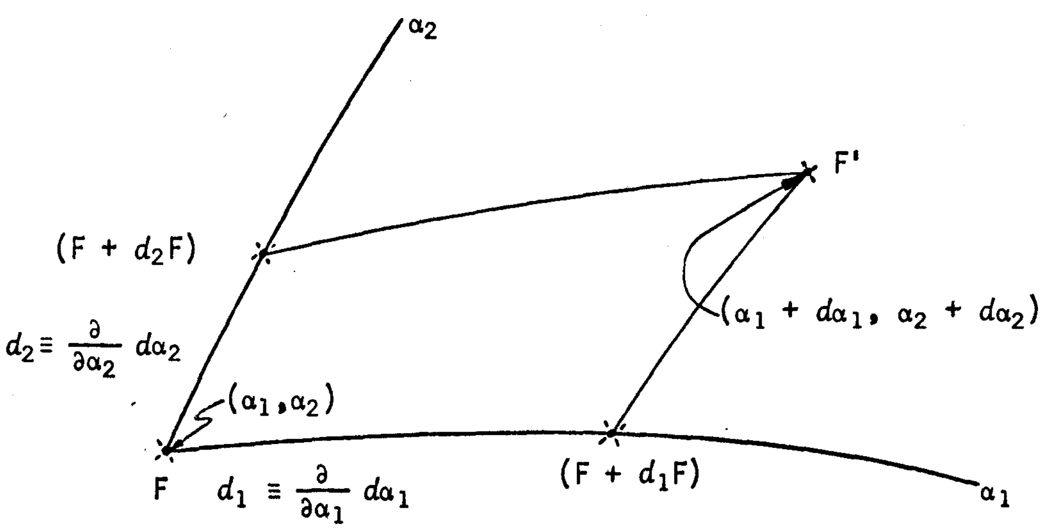


Fig. 3.1.1.-1.

The value of a point-function, F', at some point $(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2)$ in the surface, referenced to the value of the function, F, at the point (α_1,α_2), may be determined in two ways. Translating the function F from (α_1,α_2) to $(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2)$ over two infinitesimal "paths in the surface", then with reference to Fig. 3.1.1.-1.:

for Path 1: $F_1' = F + d_1F + d_2\ (F + d_1F)$

for Path 2: $F_2' = F + d_2F + d_1\ (F + d_2F)$

In order that the surface function, F, may remain "single-valued" it is necessary that the function F has the same value at the point $(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2)$, regardless of the paths traversed , i.e.:

$$F_1' = F_2'$$

or $$F + d_1F + d_2(F + d_1F) = F + d_2F + d_1(F + d_2F)$$

so $$d_2 d_1\ F = d_1 d_2 F \qquad \ldots\ldots \qquad \{3.1.1.-1.\}$$

substituting for the symbolic d_1 and d_2 :

$$d\alpha_1 \frac{\partial}{\partial \alpha_1} \equiv d_1\ , \qquad d\alpha_2 \frac{\partial}{\partial \alpha_2} \equiv d_2$$

then {3.1.1.-1.} becomes

$$\left(\frac{\partial}{\partial \alpha_1}\frac{\partial F}{\partial \alpha_2} - \frac{\partial}{\partial \alpha_2}\frac{\partial F}{\partial \alpha_1}\right) d\alpha_1 d\alpha_2 = 0$$

or $$\frac{\partial}{\partial \alpha_1}\frac{\partial F}{\partial \alpha_2} - \frac{\partial}{\partial \alpha_2}\frac{\partial F}{\partial \alpha_1} = 0 \qquad \ldots\ldots \qquad \{3.1.1.-2.\}$$

which is the Integrability Condition for the surface function, F.

This concept may be expressed in several other ways; for conceptual clarity, two of these are offered here.

A) The value of a point-function, F', at the point $(\alpha_1 + d\alpha_1, \alpha_2 + d\alpha_2)$ must be *unique*, regardless of the 'path' taken from some other point (α_1, α_2) to the point in question.

B) The value at a point, as determined by passing around a closed loop, from the point over the surface and back to the point, must be the same as that value which was existing for the point before the loop was made, i.e.: $d_1 d_2 F - d_2 d_1 F = 0$.

In keeping with the kinematic approach, {3.1.1.-2.} may be expressed in terms of the arc length derivatives, as an alternative to the parametric coordinate derivatives.

Employing the substitutions:

$$ds_1 = g_1 d\alpha_1 \;;\; \text{so } \frac{\partial}{\partial \alpha_1} \equiv g_1 \frac{\partial}{\partial s_1}$$
$$ds_2 = g_2 d\alpha_2 \;;\; \text{so } \frac{\partial}{\partial \alpha_2} \equiv g_2 \frac{\partial}{\partial s_2}$$

then, {3.1.1.-2.} becomes

$$g_1 \frac{\partial g_2}{\partial s_1} \left(\frac{\partial F}{\partial s_2}\right) + g_1 g_2 \frac{\partial^2 F}{\partial s_1 \partial s_2} - g_2 \frac{\partial g_1}{\partial s_2} \left(\frac{\partial F}{\partial s_1}\right) - g_1 g_2 \frac{\partial^2 F}{\partial s_2 \partial s_1} = 0$$

or $$\frac{1}{g_2}\left(\frac{\partial g_2}{\partial s_1}\right)\frac{\partial F}{\partial s_2} + \frac{\partial^2 F}{\partial s_1 \partial s_2} - \frac{1}{g_1}\left(\frac{\partial g_1}{\partial s_2}\right)\frac{\partial F}{\partial s_1} - \frac{\partial^2 F}{\partial s_2 \partial s_1} = 0 \quad \ldots \quad \{3.1.1.\text{-}3.\}$$

referring to $\frac{1}{g_2}\left(\frac{\partial g_2}{\partial s_1}\right)$ as γ_1

and to $\frac{1}{g_1}\left(\frac{\partial g_1}{\partial s_2}\right)$ as γ_2

then

$$\gamma_1 = \frac{1}{g_2}\left(\frac{\partial g_2}{\partial s_1}\right) \equiv \frac{\partial(\ln g_2)}{\partial s_1}$$

$$\gamma_2 = \frac{1}{g_1}\left(\frac{\partial g_1}{\partial s_2}\right) \equiv \frac{\partial(\ln g_1)}{\partial s_2}$$

and so, {3.1.1.-3.} appears as:

$$\boxed{\frac{\partial^2 F}{\partial s_1 \partial s_2} - \frac{\partial^2 F}{\partial s_2 \partial s_1} + \gamma_1 \frac{\partial F}{\partial s_2} - \gamma_2 \frac{\partial F}{\partial s_1} = 0} \quad \ldots\ldots \quad \{3.1.1.\text{-}4.\}$$

or,

$$\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\frac{\partial F}{\partial s_2} - \left(\frac{\partial}{\partial s_2} + \gamma_2\right)\frac{\partial F}{\partial s_1} = 0$$

which is the kinematic Integrability Condition for a point-function, F. This relation has general validity, as F may be either a scalar or vector (etc.) point-function.

3.2. THE GAUSS EQUATION AND THE MAINARDI-CODAZZI EQUATIONS FOR SURFACES, IN THE CASE OF ORTHOGONAL PARAMETRIC LINES

From the Integrability Condition, {3.1.1.-4.} , by setting the arbitrary function, F, equal to the position vector, $\overline{r}$, the following result is obtained.

$$\frac{\partial^2 \overline{r}}{\partial s_1 \partial s_2} - \frac{\partial^2 \overline{r}}{\partial s_2 \partial s_1} + \gamma_1 \frac{\partial \overline{r}}{\partial s_2} - \gamma_2 \frac{\partial \overline{r}}{\partial s_1} = 0 \quad \ldots\ldots \quad \{3.2.-1.\}$$

Equation {3.2.-1.} specifies the closing of the infinitesimal surface parallelogram, in accordance with § 3.1.1.

Rewriting {3.2.-1.} yields:

$$\frac{\partial}{\partial s_1}\left(\frac{\partial \overline{r}}{\partial s_2}\right) - \frac{\partial}{\partial s_2}\left(\frac{\partial \overline{r}}{\partial s_1}\right) + \gamma_1 \frac{\partial \overline{r}}{\partial s_2} - \gamma_2 \frac{\partial \overline{r}}{\partial s_1} = 0$$

and as $\frac{\partial \overline{r}}{\partial s_i} = \overline{e}_i$, then the above reduces to

$$\frac{\partial \overline{e}_2}{\partial s_1} - \frac{\partial \overline{e}_1}{\partial s_2} + \gamma_1 \overline{e}_2 - \gamma_2 \overline{e}_1 = 0 \quad \ldots\ldots \quad \{3.2.-2.\}$$

recalling the CESÀRO-BURALI-FORTI Vectors:

$$\left.\begin{aligned} \overline{C}_1 &= \kappa_{12}\overline{e}_1 + \kappa_{11}\overset{1}{\overline{e}}_* + \kappa_{13}\overline{e}_3 \\ \overline{C}_2 &= \kappa_{21}\overline{e}_2 + \kappa_{22}\overset{2}{\overline{e}}_* + \kappa_{23}\overline{e}_3 \end{aligned}\right\} \ldots\ldots\ldots \text{(arbitrary } \alpha_1, \alpha_2)$$

or

$$\left.\begin{aligned} \overline{C}_1 &= \kappa_{12}\overline{e}_1 + \kappa_{11}\overline{e}_2 + \kappa_{13}\overline{e}_3 \\ \overline{C}_2 &= \kappa_{21}\overline{e}_2 - \kappa_{22}\overline{e}_1 + \kappa_{23}\overline{e}_3 \end{aligned}\right\} \ldots\ldots\ldots (\alpha_1 \perp \alpha_2)$$

Thus, {3.2.-2.} becomes

$$\overline{C}_1 \times \overline{e}_2 - \overline{C}_2 \times \overline{e}_1 + \gamma_1\overline{e}_2 - \gamma_2\overline{e}_1 = 0$$

$$(\kappa_{12}\overline{e}_1 + \kappa_{11}\overline{e}_2 + \kappa_{13}\overline{e}_3) \times \overline{e}_2 - (\kappa_{21}\overline{e}_2 - \kappa_{22}\overline{e}_1 + \kappa_{23}\overline{e}_3) \times \overline{e}_1 + \gamma_1\overline{e}_2 - \gamma_2\overline{e}_1 = 0$$

$$\kappa_{12}\overline{e}_3 - \kappa_{13}\overline{e}_1 + \kappa_{21}\overline{e}_3 - \kappa_{23}\overline{e}_2 + \gamma_1\overline{e}_2 - \gamma_2\overline{e}_1 = 0$$

$$-(\kappa_{13} + \gamma_2)\overline{e}_1 + (-\kappa_{23} + \gamma_1)\overline{e}_2 + (\kappa_{12} + \kappa_{21})\overline{e}_3 = 0 \quad \ldots\ldots \{3.2.-3.\}$$

As the vector directions $\overline{e}_1$, $\overline{e}_2$, $\overline{e}_3$ are independent for the case of orthogonal parametric lines, then {3.2.-3.} is satisfied iff the following conditions are true:

a) $-(\kappa_{13} + \gamma_2) = 0$ or $\gamma_2 = -\kappa_{13}$

b) $-\kappa_{23} + \gamma_1 = 0$ or $\gamma_1 = \kappa_{23}$

c) $\kappa_{12} + \kappa_{21} = 0$ (BONNET's Theorem)

Hence, for the case of orthogonal parametric lines, the Integrability Condition may be given as

$$\frac{\partial^2 F}{\partial s_1 \partial s_2} - \frac{\partial^2 F}{\partial s_2 \partial s_1} + \kappa_{13}\frac{\partial F}{\partial s_1} + \kappa_{23}\frac{\partial F}{\partial s_2} = 0 \quad \ldots\ldots \quad \{3.2.-4.\}$$

where $$\kappa_{13} = -\frac{\partial(\ln g_1)}{\partial s_2} \equiv -\gamma_2$$

$$\kappa_{23} = \frac{\partial(\ln g_2)}{\partial s_1} \equiv \gamma_1$$

A more general case is now considered, still within the framework of orthogonal parametric lines. Let the Integrability Condition be applied to any arbitrary vector, $\overline{v} = \overline{v}(s)$. The vector

$\overline{V}$ is understood to satisfy only the condition of being a (single-valued) point-function of the surface; thus, it is a completely arbitrary surface vector.

From {3.2.-4.}

$$\frac{\partial^2\overline{V}}{\partial s_1 \partial s_2} - \frac{\partial^2\overline{V}}{\partial s_2 \partial s_1} + \kappa_{13}\frac{\partial\overline{V}}{\partial s_1} + \kappa_{23}\frac{\partial\overline{V}}{\partial s_2} = \overline{0}$$

$$\frac{\partial}{\partial s_1}\left(\frac{\partial\overline{V}}{\partial s_2}\right) - \frac{\partial}{\partial s_2}\left(\frac{\partial\overline{V}}{\partial s_1}\right) + \kappa_{13}\frac{\partial\overline{V}}{\partial s_1} + \kappa_{23}\frac{\partial\overline{V}}{\partial s_2} = \overline{0}$$

expanding, and considering $\overline{V} = v\,\overline{e}_v$, then:

$$\left\{\frac{\partial}{\partial s_1}\left[\frac{\partial v}{\partial s_2}\overline{e}_v + \overline{C}_2 \times \overline{V}\right] - \frac{\partial}{\partial s_2}\left[\frac{\partial v}{\partial s_1}\overline{e}_v + \overline{C}_1 \times \overline{V}\right] + \kappa_{13}\left[\frac{\partial v}{\partial s_1}\overline{e}_v + \overline{C}_1 \times \overline{V}\right] + \kappa_{23}\left[\frac{\partial v}{\partial s_2}\overline{e}_v + \overline{C}_2 \times \overline{V}\right]\right\} = 0$$

$$\Bigg[\frac{\partial^2 v}{\partial s_1 \partial s_2}\overline{e}_v + \frac{\partial v}{\partial s_2}\overline{C}_1 \times \overline{e}_v + \frac{\partial(\overline{C}_1 \times \overline{V})}{\partial s_1} - \frac{\partial^2 v}{\partial s_2 \partial s_1}\overline{e}_v - \frac{\partial v}{\partial s_1}\overline{C}_2 \times \overline{e}_v - \frac{\partial(\overline{C}_1 \times \overline{V})}{\partial s_2} + \kappa_{13}\frac{\partial v}{\partial s_1}\overline{e}_v + \kappa_{13}\overline{C}_1 \times \overline{V} + \kappa_{23}\frac{\partial v}{\partial s_2}\overline{e}_v + \kappa_{23}\overline{C}_2 \times \overline{V}\Bigg] = \overline{0} \qquad \text{.......} \quad \{3.2.-5.\}$$

the block of terms,

$$\left[\frac{\partial^2 v}{\partial s_1 \partial s_2} - \frac{\partial^2 v}{\partial s_2 \partial s_1} + \kappa_{13}\frac{\partial v}{\partial s_1} + \kappa_{23}\frac{\partial v}{\partial s_2}\right]\overline{e}_v$$

vanishes identically, as this represents the Integrability Condition, operating on (scalar) v. Expanding the remainder of {3.2.-5.} and collecting terms yields:

$$\left(\frac{\partial \overline{C}_2}{\partial s_1} \times \overline{v} + \left[\overline{C}_2 \times (\overline{C}_1 \times \overline{v})\right] - \frac{\partial \overline{C}_1}{\partial s_2} \times \overline{v} - \left[\overline{C}_1 \times (\overline{C}_2 \times \overline{v})\right] + \kappa_{13}\overline{C}_1 \times \overline{v} + \kappa_{23}\overline{C}_2 \times \overline{v} \right) = 0$$

or
$$\left\{ \left[\frac{\partial \overline{C}_2}{\partial s_1} - \frac{\partial \overline{C}_1}{\partial s_2} + \kappa_{13}\overline{C}_1 + \kappa_{23}\overline{C}_2 \right] \times \overline{v} + \overline{C}_2 \times (\overline{C}_1 \times \overline{v}) - \overline{C}_1 \times (\overline{C}_2 \times \overline{v}) \right\} = 0 \quad \ldots\{3.2.\text{-}6.\}$$

now, as the permutable vector triple product sum is equal to zero,

i.e.: $\left[\overline{C}_1 \times (\overline{v} \times \overline{C}_2)\right] + \left[\overline{v} \times (\overline{C}_2 \times \overline{C}_1)\right] + \left[\overline{C}_2 \times (\overline{C}_1 \times \overline{v})\right] = 0$

then {3.2.-6.} becomes, upon substitution of this identity,

$$\left[\frac{\partial \overline{C}_2}{\partial s_1} - \frac{\partial \overline{C}_1}{\partial s_2} + \kappa_{13}\overline{C}_1 + \kappa_{23}\overline{C}_2 \right] \times \overline{v} + (\overline{C}_2 \times \overline{C}_1) \times \overline{v} = 0$$

or
$$\left[\frac{\partial \overline{C}_2}{\partial s_1} - \frac{\partial \overline{C}_1}{\partial s_2} + \kappa_{13}\overline{C}_1 + \kappa_{23}\overline{C}_2 + (\overline{C}_2 \times \overline{C}_1) \right] \times \overline{v} = 0$$

Referring to the larger factor in the above cross-product equation as $\overline{A}$, then the equation is represented as:

$$\overline{A} \times \overline{v} = 0 \quad \ldots\ldots \quad \{3.2.\text{-}7.\}$$

The conditions, under which {3.2.-7.} will be satisfied, are:

a) $\overline{A}$ is parallel to $\overline{v}$

b) $\overline{v} = 0$

c) $\overline{A} = 0$

Both a) and b) are not allowable conditions, as $\overline{v}$ is to be an arbitrary vector. Therefore, the remaining possibility manifests itself (retranslating $\overline{A}$ to its original form) as:

$$\frac{\partial \overline{C}_2}{\partial s_1} - \frac{\partial \overline{C}_1}{\partial s_2} + \kappa_{13}\overline{C}_1 + \kappa_{23}\overline{C}_2 + \overline{C}_2 \times \overline{C}_1 = 0 \quad \ldots\ldots \quad \{3.2.\text{-}8.\}$$

This equation contains both the GAUSS and MAINARDI-CODAZZI equations, in combined form.

Expanding {3.2.-8.}, by carrying out the differentiations (and the cross-product) requires that the CESÀRO-BURALI-FORTI vectors be employed again:

VIZ:

$$\frac{\partial \bar{C}_2}{\partial s_1} = \frac{\partial}{\partial s_1}(\kappa_{21}\bar{e}_2 - \kappa_{22}\bar{e}_1 + \kappa_{23}\bar{e}_3)$$

$$= \left[\frac{\partial \kappa_{21}}{\partial s_1}\bar{e}_2 + \kappa_{21}\bar{C}_1 \times \bar{e}_2 - \frac{\partial \kappa_{22}}{\partial s_1}\bar{e}_1 - \kappa_{22}\bar{C}_1 \times \bar{e}_1 + \frac{\partial \kappa_{23}}{\partial s_1}\bar{e}_3 + \kappa_{23}\bar{C}_1 \times \bar{e}_3\right]$$

$$= \left[-\frac{\partial \kappa_{22}}{\partial s_1}\bar{e}_1 + \frac{\partial \kappa_{21}}{\partial s_1}\bar{e}_2 + \frac{\partial \kappa_{23}}{\partial s_1}\bar{e}_3 + (\kappa_{11}\kappa_{23} - \kappa_{13}\kappa_{21})\bar{e}_1 - (\kappa_{13}\kappa_{22} + \kappa_{12}\kappa_{23})\bar{e}_2 + (\kappa_{12}\kappa_{21} + \kappa_{11}\kappa_{22})\bar{e}_3\right]$$

and similarly,

$$\frac{\partial \bar{C}_1}{\partial s_2} = \left[\frac{\partial \kappa_{12}}{\partial s_2}\bar{e}_1 + \frac{\partial \kappa_{11}}{\partial s_2}\bar{e}_2 + \frac{\partial \kappa_{13}}{\partial s_2}\bar{e}_3 + (\kappa_{13}\kappa_{21} - \kappa_{11}\kappa_{23})\bar{e}_1 + (\kappa_{12}\kappa_{23} + \kappa_{13}\kappa_{22})\bar{e}_2 - (\kappa_{12}\kappa_{21} + \kappa_{11}\kappa_{22})\bar{e}_3\right]$$

Substitution of these results, together with the expansion of the cross-product term, into {3.2.-8.} yields (after algebraic simplification):

$$\Big\{\left[-\frac{\partial \kappa_{22}}{\partial s_1} - \frac{\partial \kappa_{12}}{\partial s_2} + (\kappa_{11} - \kappa_{22})\kappa_{23} - (\kappa_{21} - \kappa_{12})\kappa_{13}\right]\bar{e}_1$$

$$+ \left[\frac{\partial \kappa_{21}}{\partial s_1} - \frac{\partial \kappa_{11}}{\partial s_2} + (\kappa_{21} - \kappa_{12})\kappa_{23} + (\kappa_{11} - \kappa_{22})\kappa_{13}\right]\bar{e}_2$$

$$+ \left[\frac{\partial \kappa_{23}}{\partial s_1} - \frac{\partial \kappa_{13}}{\partial s_2} + \kappa_{12}\kappa_{21} + \kappa_{11}\kappa_{22} + \kappa_{13}^2 + \kappa_{23}^2 \right] \bar{e}_3 \Bigg\} = 0 \quad \ldots\ldots \{3.2.\text{-}9.\}$$

Since the vector directions are independent, {3.2.-9.} is satisfied iff:

$$\frac{\partial \kappa_{22}}{\partial s_1} + \frac{\partial \kappa_{12}}{\partial s_2} + (\kappa_{22} - \kappa_{11})\kappa_{23} + (\kappa_{21} - \kappa_{12})\kappa_{13} = 0 \quad \ldots\ldots \quad \{3.2.\text{-}10.\}$$

$$\frac{\partial \kappa_{21}}{\partial s_1} - \frac{\partial \kappa_{11}}{\partial s_2} + (\kappa_{21} - \kappa_{12})\kappa_{23} - (\kappa_{22} - \kappa_{11})\kappa_{13} = 0 \quad \ldots\ldots \quad \{3.2.\text{-}11.\}$$

$$\frac{\partial \kappa_{23}}{\partial s_1} - \frac{\partial \kappa_{13}}{\partial s_2} + \kappa_{12}\kappa_{21} + \kappa_{11}\kappa_{22} + \kappa_{13}^2 + \kappa_{23}^2 = 0 \quad \ldots\ldots \quad \{3.2.\text{-}12.\}$$

Equations {3.2.-10.} and {3.2.-11.} are known as the MAINARDI-CODAZZI *Equations of Surfaces* and {3.2.-12.} is called the GAUSS *Equation*, for orthogonal parametric lines α_1, α_2.

If the parametric coordinates are coincident with the principal lines of curvature, then the geodesic torsions vanish ($\kappa_{12} = 0 = \kappa_{21}$) and equations {3.2.-10.}, {3.2.-11.} and {3.2.-12.} reduce to (respectively):

$$\frac{\partial \kappa_{22}}{\partial s_1} + \kappa_{23}\kappa_{22} - \kappa_{23}\kappa_{11} = 0 \quad \ldots\ldots \quad \{3.2.\text{-}13.\}$$

$$\frac{\partial \kappa_{11}}{\partial s_2} + \kappa_{13}\kappa_{22} - \kappa_{13}\kappa_{11} = 0 \quad \ldots\ldots \quad \{3.2.\text{-}14.\}$$

$$\frac{\partial \kappa_{23}}{\partial s_1} - \frac{\partial \kappa_{13}}{\partial s_2} + \kappa_{11}\kappa_{22} + \kappa_{13}^2 + \kappa_{23}^2 = 0 \quad \ldots\ldots \quad \{3.2.\text{-}15.\}$$

These equations {3.2.-10.} to {3.2.-15.} are of primary importance in the Differential Geometry of Surfaces. The relationships thus established between curvatures and their rates of change (with respect to the arc length parameters) provide, in numerous instances, the only means by which useful expressions may be gleaned from complex developments.

The MAINARDI-CODAZZI equations are named after Gaspare Angelo MAINARDI (1800-1879) in 1856 and Delfino CODAZZI (1824-1873) in 1860.* The GAUSS equation is so called, after GAUSS in 1827.

3.3. THE GAUSS EQUATION AND THE MAINARDI-CODAZZI EQUATIONS FOR SURFACES, IN THE CASE OF NON-ORTHOGONAL PARAMETRIC LINES.

If a vector, $\bar{r}$, in its transfer from a point $\bar{r}$ to another point, $\bar{r} + \Delta\bar{r}$, is independent of path, then this transfer or displacement is called *Integrable Directional Transfer*, after Gerhard HESSENBERG in 1925. This is also known as *Integrable Linear Transfer* and *Integrable Parallel Displacement* [in the sense of Tullio LEVI-CIVITA's (1873-1941) parallel displacement, 1917].

Kinematically, such an integrable directional transfer can be represented by the model of a rigid body which is always in contact with the tangent plane of the surface and where the tangent vector, $\bar{e}$, of the path of motion always coincides with the vector $\bar{r}$, fixed in the body and maintaining the same direction as $\bar{e}$.

In such a case, the CESÀRO-BURALI-FORTI vectors, $\bar{C}_1$ and $\bar{C}_2$, do not prescribe the integrable direction, but rather prescribe a direction which differs from the integrable, by a rotation about $\bar{e}_3$ at each point. Thus, if a body is translated along an arbitrary line, the rate of change of $\bar{e}$ with respect to the arc length s_1, will be prescribed by (with reference to Fig. 3.3.-1.):

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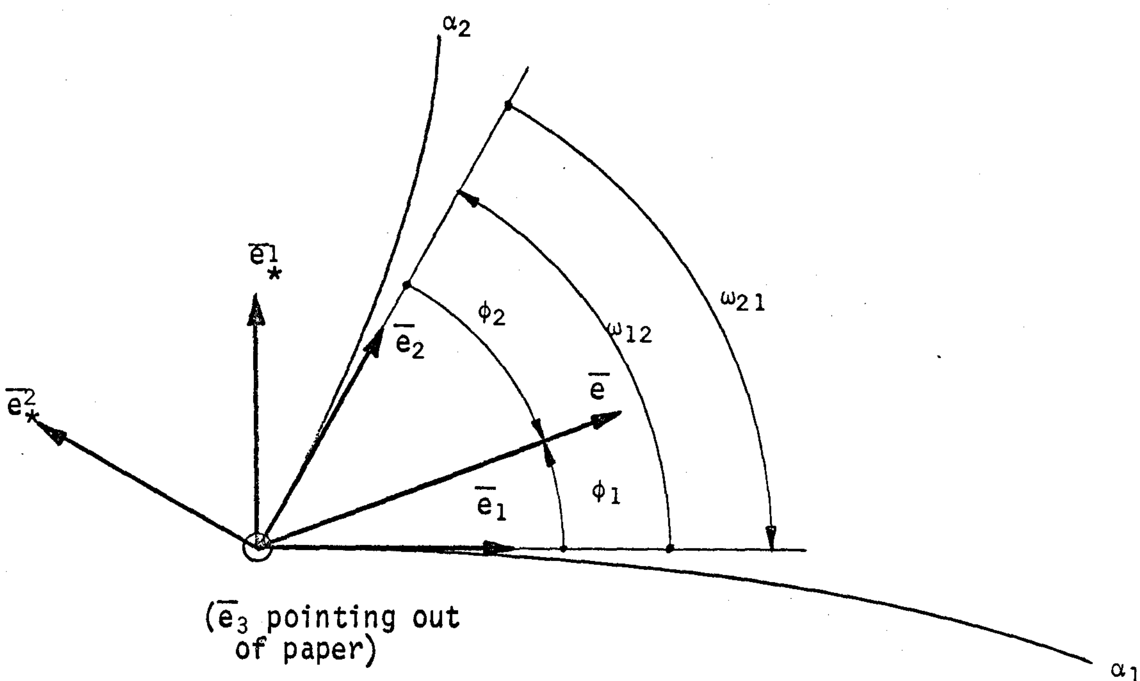
* Since Karl PETERSON obtained essentially the same result in 1853, these equations are really the "PETERSON-MAINARDI" equations.

$$\frac{\partial \overline{e}}{\partial s_1} = (\overline{C}_1 + \frac{\partial \phi_1}{\partial s_1}\overline{e}_3) \times \overline{e}$$



($\overline{e}_3$ pointing out of paper)

Fig. 3.3.-1.

This is actually a mathematical statement describing the fact that α_1 and α_2(as families of lines) do not, in general, meet at a constant angle at different points in the shell surface.

Similarly, for the body translated along the same arbitrary line, the rate of change of $\overline{e}$ with respect to the arc length s_2, will be prescribed by:

$$\frac{\partial \overline{e}}{\partial s_2} = (\overline{C}_2 + \frac{\partial \phi_2}{\partial s_2}\overline{e}_3) \times \overline{e}$$

Therefore, introducing the notation from Fig. 3.3.-1.,

$$\left[\phi_1 - \phi_2\right] = \omega_{12} = - \omega_{21}$$

then for the derivatives which occur frequently in the course of evaluation of expressions in the two different triads,

$$\frac{\partial \bar{e}_2}{\partial s_1} = (\bar{C}_1 + \frac{\partial \omega_{12}}{\partial s_1}\bar{e}_3) \times \bar{e}_2 \equiv \bar{\Omega}_1 \times \bar{e}_2$$

and
$$\frac{\partial \bar{e}_1}{\partial s_2} = (\bar{C}_2 + \frac{\partial \omega_{21}}{\partial s_2}\bar{e}_3) \times \bar{e}_1 \equiv \bar{\Omega}_2 \times \bar{e}_1$$

Obviously, for such derivatives as $\frac{\partial \bar{e}_i}{\partial s_i}$, $\frac{\partial \bar{e}^i_*}{\partial s_i}$ and $\frac{\partial \bar{e}_3}{\partial s_i}$ $(i = 1,2)$, the additive terms, $\frac{\partial \omega_{12}}{\partial s_i}$ or $\frac{\partial \omega_{21}}{\partial s_i}$ do not occur. This is easily seen from the fact that the angle of intersection of the parametric lines has no effect on the rate of change of the unit vectors of a triad, with respect to *its own* arc length parameter.

Now, from the Integrability Condition, {3.1.1.-4.}, by setting the arbitrary function, F, equal to the position vector, $\bar{r}$, the following result is obtained:

$$\frac{\partial^2 \bar{r}}{\partial s_1 \partial s_2} - \frac{\partial^2 \bar{r}}{\partial s_2 \partial s_1} + \gamma_1 \frac{\partial \bar{r}}{\partial s_2} - \gamma_2 \frac{\partial \bar{r}}{\partial s_1} = 0$$

or
$$\frac{\partial}{\partial s_1}(\bar{e}_2) - \frac{\partial}{\partial s_2}(\bar{e}_1) + \gamma_1 \bar{e}_2 - \gamma_2 \bar{e}_1 = 0$$

expanding gives:

$$\bar{\Omega}_1 \times \bar{e}_2 - \bar{\Omega}_2 \times \bar{e}_1 + \gamma_1 \bar{e}_2 - \gamma_2 \bar{e}_1 = 0$$

or
$$(\bar{C}_1 + \frac{\partial \omega_{12}}{\partial s_1}\bar{e}_3) \times \bar{e}_2 - (\bar{C}_2 + \frac{\partial \omega_{21}}{\partial s_2}\bar{e}_3) \times \bar{e}_1 + \gamma_1 \bar{e}_2 - \gamma_2 \bar{e}_1 = 0$$

$$\left\{ \left[\kappa_{12}\bar{e}_1 + \kappa_{11}\bar{e}^1_* + (\kappa_{13} + \frac{\partial \omega_{12}}{\partial s_1})\bar{e}_3 \right] \times \bar{e}_2 - \left[\kappa_{21}\bar{e}_2 + \kappa_{22}\bar{e}^2_* + (\kappa_{23} + \frac{\partial \omega_{21}}{\partial s_2})\bar{e}_3 \right] \times \bar{e}_1 + \gamma_1 \bar{e}_2 - \gamma_2 \bar{e}_1 \right\} = 0 \quad \ldots\{3.3.-1.\}$$

Extracting components is easily accomplished by taking the dot product of this equation with $\bar{e}_1$, $\bar{e}_*^1$ and $\bar{e}_3$, respectively.

a) scalar multiplication of {3.3.-1.} by $\bar{e}_1$ shows:

$$\Big[\kappa_{12}\bar{e}_1 \times \bar{e}_2 \cdot \bar{e}_1 + \kappa_{11}\bar{e}_*^1 \times \bar{e}_2 \cdot \bar{e}_1 + (\kappa_{13} + \frac{\partial\omega_{12}}{\partial s_1})\, \bar{e}_3 \times \bar{e}_2 \cdot \bar{e}_1$$

$$- \kappa_{21}\bar{e}_2 \times \bar{e}_1 \cdot \bar{e}_1 - \kappa_{22}\bar{e}_*^2 \times \bar{e}_1 \cdot \bar{e}_1 - (\kappa_{23} + \frac{\partial\omega_{21}}{\partial s_2})\, \bar{e}_3 \times \bar{e}_1 \cdot \bar{e}_1$$

$$+ \gamma_1\bar{e}_2 \cdot \bar{e}_1 - \gamma_2\bar{e}_1 \cdot \bar{e}_1 \Big] = 0$$

so $$\left[\left(\kappa_{13} + \frac{\partial\omega_{12}}{\partial s_1}\right) \text{Cos}\ (\omega_{12} + \frac{\pi}{2}) + \gamma_1 \text{ Cos } \omega_{12} - \gamma_2\right] = 0$$

or $$\left[-\left(\kappa_{13} + \frac{\partial\omega_{12}}{\partial s_1}\right) \text{Sin } \omega_{12} + \gamma_1 \text{Cos}\omega_{12} - \gamma_2\right] = 0 \quad \text{....\{3.3.-2.\}}$$

b) scalar multiplication of {3.3.-1} by $\bar{e}_*^1$ shows:

$$\Big[\kappa_{12}\bar{e}_1 \times \bar{e}_2 \cdot \bar{e}_*^1 + \kappa_{11}\bar{e}_*^1 \times \bar{e}_2 \cdot \bar{e}_*^1 + (\kappa_{13} + \frac{\partial\omega_{12}}{\partial s_1})\, \bar{e}_3 \times \bar{e}_2 \cdot \bar{e}_*^1$$

$$-\kappa_{21}\bar{e}_2 \times \bar{e}_1 \cdot \bar{e}_*^1 - \kappa_{22}\bar{e}_*^2 \times \bar{e}_1 \cdot \bar{e}_*^1 - (\kappa_{23} + \frac{\partial\omega_{21}}{\partial s_2})\, \bar{e}_3 \times \bar{e}_1 \cdot \bar{e}_*^1$$

$$+ \gamma_1\bar{e}_2 \cdot \bar{e}_*^1 - \gamma_2\bar{e}_1 \cdot \bar{e}_*^1 \Big] = 0$$

so $$\left[\left(\kappa_{13} + \frac{\partial\omega_{12}}{\partial s_1}\right) \text{Cos } \omega_{12} - \left(\kappa_{23} + \frac{\partial\omega_{21}}{\partial s_2}\right) + \gamma_1 \text{Sin } \omega_{12}\right] = 0 \quad \text{.....\{3.3.-3.\}}$$

c) scalar multiplication of {3.3.-1.} by $\bar{e}_3$ shows:

$$\Big[\kappa_{12}\bar{e}_1 \times \bar{e}_2 \cdot \bar{e}_3 + \kappa_{11}\bar{e}_*^1 \times \bar{e}_2 \cdot \bar{e}_3 + (\kappa_{13} + \frac{\partial\omega_{12}}{\partial s_1})\, \bar{e}_3 \times \bar{e}_2 \cdot \bar{e}_3$$

$$- \kappa_{21}\bar{e}_2 \times \bar{e}_1 \cdot \bar{e}_3 - \kappa_{22}\bar{e}_*^2 \times \bar{e}_1 \cdot \bar{e}_3 - (\kappa_{23} + \frac{\partial\omega_{21}}{\partial s_2})\, \bar{e}_3 \times \bar{e}_1 \cdot \bar{e}_3$$

$$+ \gamma_1\bar{e}_2 \cdot \bar{e}_3 - \gamma_2\bar{e}_1 \cdot \bar{e}_3 \Big] = 0$$

so $\left[(\kappa_{12} + \kappa_{21}) \operatorname{Sin} \omega_{12} + (-\kappa_{11} + \kappa_{22}) \operatorname{Cos} \omega_{12}\right] = 0$ ..... {3.3.-4.}

From a) and b), the parameters γ_1 and γ_2 are defined. Re-writing {3.3.-3.};

$$\gamma_1 = \frac{1}{\operatorname{Sin} \omega_{12}} \left[\kappa_{23} + \frac{\partial \omega_{21}}{\partial s_2} - \left(\kappa_{13} + \frac{\partial \omega_{12}}{\partial s_1} \right) \operatorname{Cos} \omega_{12} \right] \quad \text{.... \{3.3.-5.\}}$$

using this as a replacement variable in {3.3.-2.}, the result emerges as:

$$\gamma_2 = \frac{1}{\operatorname{Sin} \omega_{12}} \left[\left(\kappa_{23} + \frac{\partial \omega_{21}}{\partial s_2} \right) \operatorname{Cos} \omega_{12} - \left(\kappa_{13} + \frac{\partial \omega_{12}}{\partial s_1} \right) \right] \quad \text{.. \{3.3.-6.\}}$$

(NOTE that for $\omega_{12} = \frac{\pi}{2}$; $\operatorname{Cos} \omega_{12} = 0$, $\operatorname{Sin} \omega_{12} = 1$ and $\frac{\partial \omega_{12}}{\partial s_1} = 0 = \frac{\partial \omega_{21}}{\partial s_2}$, in which case, $\gamma_1 = \kappa_{23}$ and $\gamma_2 = -\kappa_{13}$ which is correct for the orthogonal case.)

From c), the resulting expression is seen to be:

$$\kappa_{12} + \kappa_{21} = (\kappa_{11} - \kappa_{22}) \operatorname{Cot} \omega_{12} \quad \text{..... \{3.3.-7.\}}$$

which is recognized as BONNET's Theorem in non-orthogonal coordinates

Having thus prescribed γ_1 and γ_2 , let the Integrability Condition now be applied to some arbitrary vector, $\overline{v} = \overline{v}(s)$, as was done for the "orthogonal" case. Then:

$$\frac{\partial^2 \overline{v}}{\partial s_1 \partial s_2} - \frac{\partial^2 \overline{v}}{\partial s_2 \partial s_1} + \gamma_1 \frac{\partial \overline{v}}{\partial s_2} - \gamma_2 \frac{\partial \overline{v}}{\partial s_1} = 0$$

or $$\left\{ \frac{\partial}{\partial s_1} \left[\frac{\partial \overline{v}}{\partial s_2} \overline{e}_v + \left(\overline{C}_2 + \frac{\partial \phi_2}{\partial s_2} \overline{e}_3 \right) \times \overline{v} \right] \right.$$

$$- \frac{\partial}{\partial s_2}\left[\frac{\partial v}{\partial s_1}\bar{e}_v + (\bar{C}_1 + \frac{\partial \phi_1}{\partial s_1}\bar{e}_3) \times \bar{v}\right] + \gamma_1\left[\frac{\partial v}{\partial s_2}\bar{e}_v + (\bar{C}_2 + \frac{\partial \phi_2}{\partial s_2}\bar{e}_3) \times \bar{v}\right]$$

$$-\gamma_2\left[\frac{\partial v}{\partial s_1}\bar{e}_v + (\bar{C}_1 + \frac{\partial \phi_1}{\partial s_1}\bar{e}_3) \times \bar{v}\right]\Bigg\} = 0 \quad \text{....\{3.3.-8.\}}$$

Referring to $\bar{C}_i + \frac{\partial \phi_i}{\partial s_i}\bar{e}_3$ as $\bar{C}_i'$ for convenience, then expanding {3.3.-8.}

$$\Bigg[\frac{\partial^2 v}{\partial s_1 \partial s_2}\bar{e}_v + \frac{\partial v}{\partial s_2}(\bar{C}_1' \times \bar{e}_v) + \frac{\partial \bar{C}_2'}{\partial s_1} \times \bar{v} + \left[\bar{C}_2' \times \frac{\partial v}{\partial s_1}\bar{e}_v\right]$$

$$+ \bar{C}_2' \times (\bar{C}_1' \times \bar{v}) - \frac{\partial^2 v}{\partial s_2 \partial s_1}\bar{e}_v - \frac{\partial v}{\partial s_1}(\bar{C}_2' \times \bar{e}_v) - \frac{\partial \bar{C}_1'}{\partial s_2} \times \bar{v}$$

$$-\left[\bar{C}_1' \times \frac{\partial v}{\partial s_2}\bar{e}_v\right] - \bar{C}_1' \times (\bar{C}_2' \times \bar{v}) + \gamma_1 \frac{\partial v}{\partial s_2}\bar{e}_v + \gamma_1 \bar{C}_2' \times \bar{v}$$

$$- \gamma_2 \frac{\partial v}{\partial s_1}\bar{e}_v - \gamma_2 \bar{C}_1' \times \bar{v}\Bigg] = 0$$

This reduces, through the integrability condition operating on scalar v and through algebraic summation, to:

$$\Bigg[\frac{\partial \bar{C}_2'}{\partial s_1} \times \bar{v} + \left[\bar{C}_2' \times (\bar{C}_1' \times \bar{v})\right] - \frac{\partial \bar{C}_1'}{\partial s_2} \times \bar{v} - \left[\bar{C}_1' \times (\bar{C}_2' \times \bar{v})\right]$$

$$+ \gamma_1 \bar{C}_2' \times \bar{v} - \gamma_2 \bar{C}_1' \times \bar{v}\Bigg] = 0 \quad \text{.....} \quad \text{\{3.3.-9.\}}$$

The permutable cross-product sum being equal to zero, permits the substitution:

$$\left[(\bar{C}_2' \times \bar{C}_1') \times \bar{v}\right] = \left[\bar{C}_2' \times (\bar{C}_1' \times \bar{v})\right] - \left[\bar{C}_1' \times (\bar{C}_2' \times \bar{v})\right]$$

hence, {3.3.-9.} becomes:

$$\left[\frac{\partial \overline{C}_2'}{\partial s_1} - \frac{\partial \overline{C}_1'}{\partial s_2} + \gamma_1 \overline{C}_2' - \gamma_2 \overline{C}_1' + \overline{C}_2' \times \overline{C}_1'\right] \times \overline{v} = 0$$

which, for arbitrary $\overline{v}$, is satisfied iff

$$\frac{\partial \overline{C}_2'}{\partial s_1} - \frac{\partial \overline{C}_1'}{\partial s_2} + \gamma_1 \overline{C}_2' - \gamma_2 \overline{C}_1' + \overline{C}_2' \times \overline{C}_1' = 0 \qquad \ldots\ldots \qquad \{3.3.-10.\}$$

returning to the original form of $\overline{C}_i'$, and regrouping, then {3.3.-10.} becomes:

$$\left[\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\left(\overline{C}_2 + \frac{\partial \phi_2}{\partial s_2}\overline{e}_3\right) - \left(\frac{\partial}{\partial s_2} + \gamma_2\right)\left(\overline{C}_1 + \frac{\partial \phi_1}{\partial s_1}\overline{e}_3\right) + \overline{C}_2 \times \overline{C}_1 + \frac{\partial \phi_2}{\partial s_2}\overline{e}_3 \times \overline{C}_1 + \overline{C}_2 \times \frac{\partial \phi_1}{\partial s_1}\overline{e}_3\right] = 0$$

$$\left[\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\overline{C}_2 - \left(\frac{\partial}{\partial s_2} + \gamma_2\right)\overline{C}_1 + \left(\frac{\partial}{\partial s_1} + \gamma_1\right)\frac{\partial \phi_2}{\partial s_2}\overline{e}_3 - \left(\frac{\partial}{\partial s_2} + \gamma_2\right)\frac{\partial \phi_1}{\partial s_1}\overline{e}_3 - \overline{C}_1 \times \overline{C}_2 - \frac{\partial \phi_2}{\partial s_2}\overline{C}_1 \times \overline{e}_3 + \frac{\partial \phi_1}{\partial s_1}\overline{C}_2 \times \overline{e}_3\right] = 0 \qquad \ldots \qquad \{3.3.-11.\}$$

now, as

$$\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\frac{\partial \phi_2}{\partial s_2}\overline{e}_3 \equiv \left[\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\frac{\partial \phi_2}{\partial s_2}\right]\overline{e}_3 + \frac{\partial \phi_2}{\partial s_2}\overline{C}_1 \times \overline{e}_3$$

$$\left(\frac{\partial}{\partial s_2} + \gamma_2\right)\frac{\partial \phi_1}{\partial s_1}\overline{e}_3 \equiv \left[\left(\frac{\partial}{\partial s_2} + \gamma_2\right)\frac{\partial \phi_1}{\partial s_1}\right]\overline{e}_3 + \frac{\partial \phi_1}{\partial s_1}\overline{C}_2 \times \overline{e}_3$$

then {3.3.-11.} reduces to:

$$\left[\!\!\left[\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\overline{C}_2 - \left(\frac{\partial}{\partial s_2} + \gamma_2\right)\overline{C}_1 + \left[\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\frac{\partial \phi_2}{\partial s_2}\right]\overline{e}_3 - \left[\left(\frac{\partial}{\partial s_2} + \gamma_2\right)\frac{\partial \phi_1}{\partial s_1}\right]\overline{e}_3 - \overline{C}_1 \times \overline{C}_2\right]\!\!\right] = 0 \qquad \ldots\ldots \qquad \{3.3.-12.\}$$

as $\left[\phi_1 - \phi_2\right] = \omega_{12}$ (from Fig. 3.3.-1.)

then $$\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\frac{\partial \phi_2}{\partial s_2} = \left(\frac{\partial}{\partial s_1} + \gamma_1\right)\left[\frac{\partial \phi_1}{\partial s_2} - \frac{\partial \omega_{12}}{\partial s_2}\right]$$

Hence, in {3.3.-12.}, the integrability condition operating on ϕ_1 sums to zero, and the result is:

$$\left\{\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\overline{C}_2 - \left(\frac{\partial}{\partial s_2} + \gamma_2\right)\overline{C}_1 - \left[\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\frac{\partial \omega_{12}}{\partial s_2}\right]\overline{e}_3 - \overline{C}_1 \times \overline{C}_2\right\} = 0 \quad \ldots\ldots \{3.3.-13.\}$$

referring to the operator

$$\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\frac{\partial}{\partial s_2}(\ ) \equiv \left(\frac{\partial}{\partial s_2} + \gamma_2\right)\frac{\partial}{\partial s_1}(\ ) \quad \text{as} \quad D(\ )$$

then {3.3.-13.} becomes

$$\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\overline{C}_2 - \left(\frac{\partial}{\partial s_2} + \gamma_2\right)\overline{C}_1 = \overline{C}_1 \times \overline{C}_2 + [D\ \omega_{12}]\ \overline{e}_3 \quad \ldots\ldots \{3.3.-14.\}$$

This equation contains both the GAUSS and MAINARDI-CODAZZI equations, in compound form. Extraction of these equations is possible in two forms: A) Operational Form

and B) Component Form

These two forms will be discussed separately as follows.

A) Operational Form

Using the identity,

$$\overline{C}_i = \overline{\kappa}_i + \kappa_{i3}\overline{e}_3 \equiv \left[\overline{e}_3 \times \frac{\partial \overline{e}_3}{\partial s_i} + \kappa_{i3}\ \overline{e}_3\right],$$

as a replacement expression for $\overline{C}_i$ in {3.3.-14.}, and recalling that

$$\overline{\kappa}_1 \times \overline{\kappa}_2 = \kappa_g \operatorname{Sin} \omega_{12}\ \overline{e}_3 \qquad \text{(from \{2.13.-1.\})},$$

then {3.3.-14.} becomes, after some minor manipulation:

$$\left\{\left[\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\kappa_{23} - \left(\frac{\partial}{\partial s_2} + \gamma_2\right)\kappa_{13} + \kappa_g \text{ Sin } \omega_{12} - (D\omega_{12})\right]\bar{e}_3 + \bar{e}_3 \times \left[\frac{\partial}{\partial s_1}\left(\frac{\partial \bar{e}_3}{\partial s_2}\right) + \gamma_1 \frac{\partial \bar{e}_3}{\partial s_2} - \frac{\partial}{\partial s_2}\left(\frac{\partial \bar{e}_3}{\partial s_1}\right) - \gamma_2 \frac{\partial \bar{e}_3}{\partial s_1}\right]\right\} = 0$$

The second factor of this equation (above) vanishes, as it represents the integrability condition, operating on $\bar{e}_3$. Thus;

$$\left[\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\kappa_{23} - \left(\frac{\partial}{\partial s_2} + \gamma_2\right)\kappa_{13} + \kappa_g \text{ Sin } \omega_{12} - D\omega_{12}\right]\bar{e}_3 = 0 \quad \text{...\{3.3.-15.\}}$$

$$\text{or} \quad \left[\left(\frac{\partial}{\partial s_2} + \gamma_2\right)\kappa_{13} - \left(\frac{\partial}{\partial s_1} + \gamma_1\right)\kappa_{23} + D\omega_{12}\right] = \left[\kappa_g \text{ Sin } \omega_{12}\right] \text{.....} \quad \text{\{3.3.-16.\}}$$

which is the GAUSS *Equation* in operational form, for non-orthogonal parametric lines.

Realizing that the above is the $\bar{e}_3$ - component of equation {3.3.-13.}, then by subtracting {3.3.-15.} from {3.3.-13.}, the result is:

$$\left[\left(\frac{\partial}{\partial s_1} + \gamma_1\right)\bar{\kappa}_2 - \left(\frac{\partial}{\partial s_2} + \gamma_2\right)\bar{\kappa}_1\right] = \left[2\kappa_g \text{ Sin } \omega_{12}\right]\bar{e}_3 \quad \text{.....} \quad \text{\{3.3.-17.\}}$$

which contains both the MAINARDI-CODAZZI *Equations* in operational form, for non-orthogonal parametric lines.

B) Component Form

Returning to {3.3.-14.}, and expanding in full, using the component form of the CESÀRO-BURALI-FORTI Vectors, the results appear as (taking the dot products with $\bar{e}_1$, $\overset{1}{\bar{e}_*}$, $\bar{e}_3$):

$$\left\{\left[\frac{\partial \kappa_{12}}{\partial s_2} - \kappa_{11}\kappa_{23} + \gamma_2\kappa_{12}\right] + \left[\frac{\partial \kappa_{22}}{\partial s_1} + \kappa_{13}\kappa_{21} + \gamma_1\kappa_{22}\right] \text{Sin } \omega_{12}\right.$$

$$- \left[\frac{\partial \kappa_{21}}{\partial s_1} - \kappa_{13}\kappa_{22} + \gamma_1\kappa_{21} \right] \text{Cos } \omega_{12} \Bigg\} = 0 \qquad \ldots\ldots \qquad \{3.3.\text{-}18.\}$$

$$\Bigg\{ \left[\frac{\partial \kappa_{11}}{\partial s_2} + \kappa_{12}\kappa_{23} + \gamma_2\kappa_{11} \right] - \left[\frac{\partial \kappa_{21}}{\partial s_1} - \kappa_{13}\kappa_{22} + \gamma_1\kappa_{21} \right] \text{Sin } \omega_{12}$$

$$- \left[\frac{\partial \kappa_{22}}{\partial s_1} + \kappa_{13}\kappa_{21} + \gamma_1\kappa_{22} \right] \text{Cos } \omega_{12} \Bigg\} = 0 \qquad \ldots\ldots \qquad \{3.3.\text{-}19.\}$$

$$\Bigg\{ \left[\frac{\partial \kappa_{13}}{\partial s_2} - \frac{\partial \kappa_{23}}{\partial s_1} + \gamma_2\kappa_{13} - \gamma_1\kappa_{23} \right] - \left[\kappa_{11}\kappa_{22} + \kappa_{12}\kappa_{21} \right] \text{Sin } \omega_{12}$$

$$+ \left[\kappa_{11}\kappa_{21} - \kappa_{12}\kappa_{22} \right] \text{Cos } \omega_{12} \Bigg\} = 0 \qquad \ldots\ldots \qquad \{3.3.\text{-}20.\}$$

where equations {3.3.-18.} and {3.3.-19.} are the trigonometric (expanded) form of the MAINARDI-CODAZZI equations for non-orthogonal parametric lines; equation {3.3.-20.} is the trigonometric form of the GAUSS equation for non-orthogonal parametric lines.

These equations reduce, for $\omega_{12} = \frac{\pi}{2}$, to the forms as given for the case of orthogonal parametric lines.

NOTE: In the case that the arbitrary vector, $\overline{v}$, is not a function of the arc length, the developments of §3.2. and §3.3. still hold. In fact, the development is somewhat simplified in the case that $\overline{v} \neq \overline{v}(s)$; this will be easily seen from an inspection of the preliminary work in either section (up to {3.2.-8.} for §3.2., and to {3.3.-14.} for §3.3.).

BOOK II. THIN ELASTIC SHELLS

CHAPTER 4

The Kinematics of Deformation

4.1. DEFINITIONS

Shells are defined to be bodies, the third dimension ("thickness") of which is very small in comparison to the other two dimensions.

The Middle Surface is the locus of points which are equidistant from the two bounding surfaces of the shell.

4.2 GEOMETRY OF THE SHELL

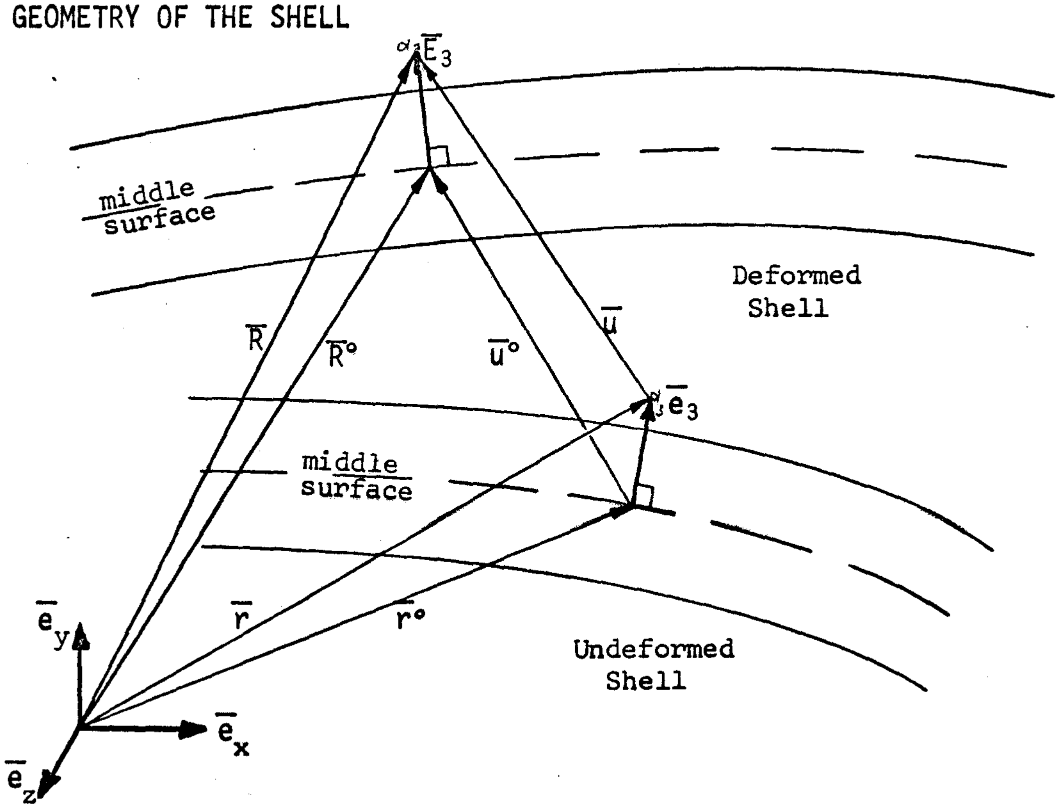


Fig. 4.2.-1.

4.3. THE BASE VECTOR SYSTEM FOR THE DEFORMED MIDDLE SURFACE

Recalling the base vector system for the undeformed (middle) surface:

$$\bar{g}_i = \frac{\partial \bar{r}^\circ}{\partial \alpha_i} = \frac{\partial s_i}{\partial \alpha_i} \cdot \frac{\partial \bar{r}^\circ}{\partial s_i} = g_i\ \bar{e}_i \qquad \text{(no sum, } i = 1,2)$$

where the position vector $\bar{r}^\circ$ is now used in place of $\bar{r}$, *when referring to the middle surface.* (The vector $\bar{r}$ thus retains its status as an arbitrary vector, describing any point within the shell).

Then, to the above may be added (with reference to Fig. 4.2.-1.),

$$\bar{g}_n = \frac{\partial}{\partial \alpha_3}(\bar{r}) = \frac{\partial}{\partial \alpha_3}(\bar{r}^\circ + \alpha_3\ \bar{e}_3)$$

$$= \frac{\partial \bar{r}^\circ}{\partial \alpha_3} + \frac{\partial \alpha_3}{\partial \alpha_3}\bar{e}_3 + \alpha_3 \frac{\partial \bar{e}_3}{\partial \alpha_3}$$

$$= \bar{e}_3$$

since α_3 is a straight-line coordinate, therefore $\frac{\partial e_i}{\partial \alpha_3} = 0$ $(i = 1,2,3)$.

Now, for the deformed middle surface,

$$\bar{R}^\circ = \bar{r}^\circ + \bar{u}^\circ \qquad \text{from Fig. 4.2.-1.}$$

then as $\quad ds^2 = d\bar{r}^\circ \cdot d\bar{r}^\circ \equiv I \quad$ (see §2.3.)

so $\quad dS^2 = d\bar{R}^\circ \cdot d\bar{R}^\circ = [d(\bar{r}^\circ + \bar{u}^\circ) \cdot d(\bar{r}^\circ + \bar{u}^\circ)]$

$$= d\bar{r}^\circ \cdot d\bar{r}^\circ + 2\ d\bar{r}^\circ \cdot d\bar{u}^\circ + d\bar{u}^\circ \cdot d\bar{u}^\circ \quad \ldots\{4.3.\text{-}1.\}$$

expanding {4.3.-1.} by the introduction of:

$$d\bar{r}^\circ = \frac{\partial \bar{r}^\circ}{\partial \alpha_1} d\alpha_1 + \frac{\partial \bar{r}^\circ}{\partial \alpha_2} d\alpha_2 = \bar{g}_1 d\alpha_1 + \bar{g}_2 d\alpha_2$$

$$d\bar{u}^\circ = \frac{\partial \bar{u}^\circ}{\partial \alpha_1} d\alpha_1 + \frac{\partial \bar{u}^\circ}{\partial \alpha_2} d\alpha_2$$

then,

$$dS^2 = \bar{G}_1 \cdot \bar{G}_1 d\alpha_1^2 + 2\bar{G}_1 \cdot \bar{G}_2 d\alpha_1 d\alpha_2 + \bar{G}_2 \cdot \bar{G}_2 d\alpha_2^2$$

$$= G_{11} d\alpha_1^2 + 2G_{12} d\alpha_1 d\alpha_2 + G_{22} d\alpha_2^2$$

where $\bar{G}_i = \dfrac{\partial \bar{R}^\circ}{\partial \alpha_i}$, and

$$G_{11} = g_{11} + 2\bar{g}_1 \cdot \frac{\partial \bar{u}^\circ}{\partial \alpha_1} + \frac{\partial \bar{u}^\circ}{\partial \alpha_1} \cdot \frac{\partial \bar{u}^\circ}{\partial \alpha_1} = (g_{11} + \delta g_{11})$$

$$G_{12} = g_{12} + \bar{g}_1 \cdot \frac{\partial \bar{u}^\circ}{\partial \alpha_2} + \bar{g}_2 \cdot \frac{\partial \bar{u}^\circ}{\partial \alpha_1} + \frac{\partial \bar{u}^\circ}{\partial \alpha_1} \cdot \frac{\partial \bar{u}^\circ}{\partial \alpha_2} = (g_{12} + \delta g_{12})$$

$$G_{22} = g_{22} + 2\bar{g}_2 \cdot \frac{\partial \bar{u}^\circ}{\partial \alpha_2} + \frac{\partial \bar{u}^\circ}{\partial \alpha_2} \cdot \frac{\partial \bar{u}^\circ}{\partial \alpha_2} = (g_{22} + \delta g_{22})$$

Thus, $\bar{G}_1$ and $\bar{G}_2$ define the base vector system for the deformed middle surface.

NOTE: Because of the complexity of the expressions, as exemplified above, all future discussion will assume *orthogonal parametric coordinates for the undeformed shell.* Naturally, this precludes that in the deformed state, the coordinates cannot be orthogonal, being deformed (by detrusion) from the original state.

4.4. THE UNIT VECTOR SYSTEM FOR THE DEFORMED MIDDLE SURFACE

The *base vector* associated with parametric line α_i has been given, in §4.3., as:

$$\bar{G}_i = \frac{\partial \bar{R}^\circ}{\partial \alpha_i}$$

Hence, the *unit vector* associated with the line α_i is

$$\overline{E}_i = \frac{\overline{G}_i}{G_i} = \frac{1}{G_i}\frac{\partial \overline{R}^\circ}{\partial \alpha_i} = \frac{\partial \overline{R}^\circ}{\partial S_i} \qquad (i = 1,2)$$

where dS_i represents the deformed arc length parameter, as before.

Expanding the original expression for $\overline{G}_i$:

$$\overline{G}_i = \frac{\partial \overline{R}^\circ}{\partial \alpha_i} = \left[\frac{\partial \overline{r}^\circ}{\partial \alpha_i} + \frac{\partial \overline{u}^\circ}{\partial \alpha_i}\right]$$

$$= \overline{g}_i + \left[\frac{\partial s_i}{\partial \alpha_i} \quad \frac{\partial \overline{u}^\circ}{\partial s_i}\right]$$

$$= \overline{g}_i + g_i \frac{\partial \overline{u}^\circ}{\partial s_i}$$

$$= g_i \left(\overline{e}_i + \frac{\partial \overline{u}^\circ}{\partial s_i}\right) \text{ (no sum, } i = 1,2) \qquad \ldots\ldots \{4.4.-1.\}$$

Now, expanding $\frac{\partial \overline{u}^\circ}{\partial s_i}$:

$$\frac{\partial \overline{u}^\circ}{\partial s_1} = \frac{\partial}{\partial s_1}(u_1^\circ\, \overline{e}_1 + u_2^\circ\, \overline{e}_2 + u_3^\circ\, \overline{e}_3)$$

$$= \frac{\partial u_1^\circ}{\partial s_1}\overline{e}_1 + u_1^\circ\left[\overline{C}_1 \times \overline{e}_1\right] + \frac{\partial u_2^\circ}{\partial s_1}\overline{e}_2 + u_2^\circ\left[\overline{C}_1 \times \overline{e}_2\right]$$

$$+ \frac{\partial u_3^\circ}{\partial s_1}\overline{e}_3 + u_3^\circ\left[\overline{C}_1 \times \overline{e}_3\right]$$

using the component form of the CESÀRO-BURALI-FORTI Vectors, carrying out the cross-products and regrouping, gives:

$$\frac{\partial \overline{u}^\circ}{\partial s_1} = \left(\frac{\partial u_1^\circ}{\partial s_1} - u_2^\circ\kappa_{13} + u_3^\circ\kappa_{11}\right)\overline{e}_1 + \left(\frac{\partial u_2^\circ}{\partial s_1} + u_1^\circ\kappa_{13} - u_3^\circ\kappa_{12}\right)\overline{e}_2$$

$$+\left(\frac{\partial u_3^\circ}{\partial s_1} - u_1^\circ\kappa_{11} + u_2^\circ\kappa_{12}\right)\overline{e}_3$$

By a similar procedure,

$$\frac{\partial \overline{u}^{\circ}}{\partial s_2} = \left(\frac{\partial u_1^{\circ}}{\partial s_2} - u_2^{\circ}\kappa_{23} + u_3^{\circ}\kappa_{21}\right)\overline{e}_1 + \left(\frac{\partial u_2^{\circ}}{\partial s_2} + u_1^{\circ}\kappa_{23} + u_3^{\circ}\kappa_{22}\right)\overline{e}_2$$

$$+ \left(\frac{\partial u_3^{\circ}}{\partial s_2} - u_1^{\circ}\kappa_{21} - u_2^{\circ}\kappa_{22}\right)\overline{e}_3$$

Introducing the notation

$$\left.\begin{aligned}
\phi_{11} &= \left[\frac{\partial u_1^{\circ}}{\partial s_1} - u_2^{\circ}\kappa_{13} + u_3^{\circ}\kappa_{11}\right] = \frac{\partial \overline{u}^{\circ}}{\partial s_1}\cdot\overline{e}_1 \\
\phi_{12} &= \left[\frac{\partial u_2^{\circ}}{\partial s_1} + u_1^{\circ}\kappa_{13} - u_3^{\circ}\kappa_{12}\right] = \frac{\partial \overline{u}^{\circ}}{\partial s_1}\cdot\overline{e}_2 \\
\phi_{13} &= \left[\frac{\partial u_3^{\circ}}{\partial s_1} - u_1^{\circ}\kappa_{11} + u_2^{\circ}\kappa_{12}\right] = \frac{\partial \overline{u}^{\circ}}{\partial s_1}\cdot\overline{e}_3 \\
\phi_{21} &= \left[\frac{\partial u_1^{\circ}}{\partial s_2} - u_2^{\circ}\kappa_{23} + u_3^{\circ}\kappa_{21}\right] = \frac{\partial \overline{u}^{\circ}}{\partial s_2}\cdot\overline{e}_1 \\
\phi_{22} &= \left[\frac{\partial u_2^{\circ}}{\partial s_2} + u_1^{\circ}\kappa_{23} + u_3^{\circ}\kappa_{22}\right] = \frac{\partial \overline{u}^{\circ}}{\partial s_2}\cdot\overline{e}_2 \\
\phi_{23} &= \left[\frac{\partial u_3^{\circ}}{\partial s_2} - u_1^{\circ}\kappa_{21} - u_2^{\circ}\kappa_{22}\right] = \frac{\partial \overline{u}^{\circ}}{\partial s_2}\cdot\overline{e}_3
\end{aligned}\right\} \quad \ldots\ldots \quad \{4.4.-2.\}$$

where ϕ_{ij} $(i \neq j)$ is interpreted kinematically as the rotation of $\overline{e}_i$ towards $\overline{e}_j$ (and about the axis $\overline{e}_s = \overline{e}_i x \overline{e}_j$), during the process of deformation; the terms ϕ_{ii} represent longitudinal dilations in the direction $\overline{e}_i$.

Then,

$$\frac{\partial \overline{u}^{\circ}}{\partial s_1} = \phi_{11}\overline{e}_1 + \phi_{12}\overline{e}_2 + \phi_{13}\overline{e}_3$$

$$\frac{\partial \overline{u}^{\circ}}{\partial s_2} = \phi_{21}\overline{e}_1 + \phi_{22}\overline{e}_2 + \phi_{23}\overline{e}_3$$

Using this result in the expression {4.4.-1.} for $\overline{G}_i$, then the result appears as:

$$\overline{G}_i = g_i \, [\overline{e}_i + \phi_{i1}\overline{e}_1 + \phi_{i2}\overline{e}_2 + \phi_{i3}\overline{e}_3]$$

thus,
$$\overline{G}_1 = g_1 \, [(1 + \phi_{11})\overline{e}_1 + \phi_{12}\overline{e}_2 + \phi_{13}\overline{e}_3]$$

$$\overline{G}_2 = g_2 \, [\phi_{21}\overline{e}_1 + (1 + \phi_{22})\overline{e}_2 + \phi_{23}\overline{e}_3]$$

therefore, as $\overline{E}_i = \dfrac{\overline{G}_i}{|\overline{G}_i|} \equiv \dfrac{\overline{G}_i}{G_i}$, the problem reduces to an evaluation of G_i.

Hence, from
$$G_i = [\overline{G}_i \cdot \overline{G}_i]^{\frac{1}{2}}$$

$$G_1 = g_1 \, [1 + 2\phi_{11} + \phi_{11}^2 + \phi_{12}^2 + \phi_{13}^2]^{\frac{1}{2}}$$

$$G_2 = g_2 \, [1 + 2\phi_{22} + \phi_{22}^2 + \phi_{21}^2 + \phi_{23}^2]^{\frac{1}{2}}$$

Making the following simplification in notation:

$$[1 + 2\phi_{11} + \phi_{11}^2 + \phi_{12}^2 + \phi_{13}^2]^{-\frac{1}{2}} \equiv m_1$$

$$[1 + 2\phi_{22} + \phi_{22}^2 + \phi_{21}^2 + \phi_{23}^2]^{-\frac{1}{2}} \equiv m_2$$

then $\overline{E}_1 = \dfrac{\overline{G}_1}{G_1} = m_1 \, [(1 + \phi_{11})\overline{e}_1 + \phi_{12}\overline{e}_2 + \phi_{13}\overline{e}_3]$

and $\overline{E}_2 = \dfrac{\overline{G}_2}{G_2} = m_2[\phi_{21}\overline{e}_1 + (1 + \phi_{22})\overline{e}_2 + \phi_{23}\overline{e}_3]$

Now, an assumption is made for the "linear" theory of shells, which is known as KIRCHHOFF's Hypothesis, after KIRCHHOFF in 1876. This fundamental postulate asserts that: *normals to the surface before deformation remain normals to the surface after deformation,* and *undergo* *no axial dilatation.* Mathematically, this is expressed by saying that the deformed surface normal may be expressed as:

$$\overline{E}_3 = \overline{E}_1 \times \overline{E}_2$$

According to the expressions given for $\overline{E}_1$ and $\overline{E}_2$ above, then

$$\overline{E}_3 = m_1 m_2 [(\phi_{12}\phi_{23} - \phi_{13} - \phi_{13}\phi_{22})\overline{e}_1 + (\phi_{13}\phi_{21} - \phi_{23} - \phi_{23}\phi_{11})\overline{e}_2$$

$$+ (1 + \phi_{11} + \phi_{22} + \phi_{11}\phi_{22} - \phi_{12}\phi_{21})\overline{e}_3]$$

Obviously, to obtain any useful results, an approximation must be made, with regard to the relative size of the ϕ-terms. Since the deformations are small, (certainly, any $\phi_{ij} \ll 1$) then the quadratic terms $(\phi_{ij}\phi_{rs})$ may be neglected, when compared to the linear terms. Consequently,

$$m_1 \doteq \frac{1}{1+\phi_{11}} ; \quad m_2 \doteq \frac{1}{1+\phi_{22}}$$

then

$$\overline{E}_1 = \overline{e}_1 + m_1\phi_{12}\overline{e}_2 + m_1\phi_{13}\overline{e}_3$$

$$\overline{E}_2 = m_2\phi_{21}\overline{e}_1 + \overline{e}_2 + m_2\phi_{23}\overline{e}_3$$

$$\overline{E}_3 = -m_1\phi_{13}\overline{e}_1 - m_2\phi_{23}\overline{e}_2 + \overline{e}_3$$

It is to be emphasized, however, that unlike $(\overline{e}_1, \overline{e}_2, \overline{e}_3)$, *the set* $(\overline{E}_1, \overline{E}_2, \overline{E}_3)$ *does not define an orthogonal vector triple,* due to the detrusion incurred in the $\overline{E}_1 - \overline{E}_2$ - plane. Hence, accepting $\overline{E}_3$ to be defined as above, then $\overline{E}^1_*$ and $\overline{E}^2_*$ may be defined by the cross-product with $\overline{E}_3$. *See next page

Thus

$$\overline{E}_*^1 = \overline{E}_3 \times \overline{E}_1 = - m_1\phi_{12}\overline{e}_1 + \overline{e}_2 + m_2\phi_{23}\overline{e}_3$$

$$\overline{E}_*^2 = \overline{E}_3 \times \overline{E}_2 = - \overline{e}_1 + m_2\phi_{21}\overline{e}_2 - m_1\phi_{13}\overline{e}_3$$

and the entire set of unit vectors for the deformed configuration may then be given in terms of the parameters of the undeformed system.

$$\left.\begin{aligned} \overline{E}_1 &= \overline{e}_1 + m_1\phi_{12}\overline{e}_2 + m_1\phi_{13}\overline{e}_3 \\ \overline{E}_*^1 &= - m_1\phi_{12}\overline{e}_1 + \overline{e}_2 + m_2\phi_{23}\overline{e}_3 \\ \overline{E}_2 &= m_2\phi_{21}\overline{e}_1 + \overline{e}_2 + m_2\phi_{23}\overline{e}_3 \\ \overline{E}_*^2 &= - \overline{e}_1 + m_2\phi_{21}\overline{e}_2 - m_1\phi_{13}\overline{e}_3 \\ \overline{E}_3 &= - m_1\phi_{13}\overline{e}_1 - m_2\phi_{23}\overline{e}_2 + \overline{e}_3 \end{aligned}\right\} \quad \ldots\ldots \quad \{4.4.\text{-}3.\}$$

It is to be noted, from {4.4.-3.}, that any unit vector in the deformed system may be expressed as the corresponding unit vector in the undeformed system, plus its first variation. That is, in general: (as a first-order approximation)

$$\overline{E}_i = \overline{e}_i + \delta\overline{e}_i \qquad i = 1,2,3$$

and

$$\overline{E}_*^i = \overline{e}_*^i + \delta\overline{e}_*^i \qquad i = 1,2$$

Therefore,

$$\left.\begin{aligned} \delta\overline{e}_1 &= m_1(\phi_{12}\overline{e}_2 + \phi_{13}\overline{e}_3) \\ \delta\overline{e}_2 &= m_2(\phi_{21}\overline{e}_1 + \phi_{23}\overline{e}_3) \\ \delta\overline{e}_3 &= - m_1\phi_{13}\overline{e}_1 - m_2\phi_{23}\overline{e}_2 \end{aligned}\right\} \quad \ldots\ldots \quad \{4.4.\text{-}4.\}$$

and because $\overline{e}_*^1 = \overline{e}_2$ and $\overline{e}_*^2 = - \overline{e}_1$ (orthogonal coordinates in the undeformed state), then to those above, may be added:

$$-\delta\overline{e}_*^2 \equiv \delta\overline{e}_1 = - m_2\phi_{21}\overline{e}_2 + m_1\phi_{13}\overline{e}_3$$

* Contradictory with the former statements

If $\overline{e}_*^1 = \overline{e}_2$ and $\delta\overline{e}_*^1 = \delta\overline{e}_2 \Rightarrow \overline{E}_*^1 = \overline{e}_2 + \delta\overline{e}_2 = \overline{E}_2$

Although $\overline{e}_*^1 = \overline{e}_2$, $\delta\overline{e}_*^1 \neq \delta\overline{e}_2$

$$\delta\overline{e}_{*}^{1} \equiv \delta\overline{e}_2 = - m_1\phi_{12}\overline{e}_1 + m_2\phi_{23}\overline{e}_3$$ *see former note

NOTE: A comparison of the two possible forms for $\delta\overline{e}_1$ and $\delta\overline{e}_2$ then reveals that

$$m_1\phi_{12} + m_2\phi_{21} = 0$$

This is an identity which must hold true, in order that the results as given above will remain valid.

Therefore, a tensor, $\delta\overline{\overline{E}}$, may be constructed which will prescribe the total variation of the triad $\{\overline{e}_1, \overline{e}_2, \overline{e}_3\}$, necessary to produce the triad $\{\overline{E}_1, \overline{E}_2, \overline{E}_3\}$ of the deformed surface. Defining this tensor in terms of the original triad, say:

$$\delta\overline{e}_i = \overline{e}_i \cdot \delta\overline{\overline{E}}$$

in which case, any element may be defined as

$$\overline{e}_i \cdot \delta\overline{\overline{E}} \cdot \overline{e}_j = \overline{e}_j \cdot \delta\overline{e}_i \qquad i,j = 1,2,3$$

Hence,

$$\delta\overline{\overline{E}} = \begin{bmatrix} 0\ \overline{e}_1\overline{e}_1 + m_1\phi_{12}\overline{e}_1\overline{e}_2 + m_1\phi_{13}\overline{e}_1\overline{e}_3 \\ + m_2\phi_{21}\overline{e}_2\overline{e}_1 + 0\ \overline{e}_2\overline{e}_2 + m_2\phi_{23}\overline{e}_2\overline{e}_3 \\ - m_1\phi_{13}\overline{e}_3\overline{e}_1 - m_2\phi_{23}\overline{e}_3\overline{e}_2 + 0\ \overline{e}_3\overline{e}_3 \end{bmatrix} \quad \ldots\ldots \quad \{4.4.-5.\}$$

This (or any other) tensor may be expressed as the sum of two other tensors of the same rank. This is advantageous for the kinematic interpretation of $\delta\overline{\overline{E}}$.

Specifying that: $\delta\bar{\bar{E}} = \delta\bar{\bar{E}}_a + \delta\bar{\bar{E}}_r$ ..... {4.4.-6.}

where $\delta\bar{\bar{E}}_a$ is chosen to be the antisymmetric part of $\delta\bar{\bar{E}}$, and $\delta\bar{\bar{E}}_r$ is the remaining part,

then by inspection of {4.4.-5.},

$$\delta\bar{\bar{E}}_a = \left[\begin{array}{l} 0\ \bar{e}_1\bar{e}_1 + 0\ \bar{e}_1\bar{e}_2 + m_1\phi_{13}\bar{e}_1\bar{e}_3 \\ +\ 0\ \bar{e}_2\bar{e}_1 + 0\ \bar{e}_2\bar{e}_2 + m_2\phi_{23}\bar{e}_2\bar{e}_3 \\ -\ m_1\phi_{13}\bar{e}_3\bar{e}_1 - m_2\phi_{23}\bar{e}_3\bar{e}_2 + 0\ \bar{e}_3\bar{e}_3 \end{array}\right] \quad \ldots\ldots \quad \{4.4.-7.\}$$

This requires that, from {4.4.-5.},

$$\delta\bar{\bar{E}}_r = \left[\begin{array}{l} 0\ \bar{e}_1\bar{e}_1 + m_1\phi_{12}\bar{e}_1\bar{e}_2 + 0\ \bar{e}_1\bar{e}_3 \\ +\ m_2\phi_{21}\bar{e}_2\bar{e}_1 + 0\ \bar{e}_2\bar{e}_2 + 0\ \bar{e}_2\bar{e}_3 \\ +\ 0\ \bar{e}_3\bar{e}_1 + 0\ \bar{e}_3\bar{e}_2 + 0\ \bar{e}_3\bar{e}_3 \end{array}\right] \quad \ldots\ldots \quad \{4.4.-8.\}$$

It is then observed that since the entire variation, $\delta\bar{e}_i$, of any vector dealt with here is of a rotational nature (since $\bar{e}_i \cdot \delta\bar{e}_i = 0$, or $\delta\bar{e}_i$ may be given by $\delta\bar{\xi} \times \bar{e}_i$), then *the tensor* $\delta\bar{\bar{E}}_a$ *represents the rigid-body rotation of the triad* $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$. This is self-evident, as $\delta\bar{\bar{E}}_a$ is totally antisymmetric. *The tensor* $\delta\bar{\bar{E}}_r$ *respresents the relative rotation of* $\bar{e}_1$ *and* $\bar{e}_2$ *(about* $\bar{e}_3$*)*, as the components found in this tensor specify the detrusion in the tangent plane of the middle surface.

In purely kinematic terms, it is instructive to construct a rotational tensor, $\delta\bar{\bar{\phi}}$, which will prescribe the variations, $\delta\bar{e}_i$, as the cross-product of a rotational vector (contained within the tensor), and the unit vector, $\bar{e}_i$.

Thus, say: $\delta\overline{e}_i = \delta\overline{\phi}_i \times \overline{e}_i$

and hence, $\delta\overline{\phi}_i = \overline{e}_i \cdot \delta\,\overline{\overline{\phi}}$

From the previously-given expressions for $\delta\overline{e}_i$, then:

$$\delta\overline{\phi}_1 = -m_1\phi_{13}\overline{e}_2 + m_1\phi_{12}\overline{e}_3$$
$$\delta\overline{\phi}_2 = m_2\phi_{23}\overline{e}_1 - m_2\phi_{21}\overline{e}_3$$
$$\delta\overline{\phi}_3 = m_2\phi_{23}\overline{e}_1 - m_1\phi_{13}\overline{e}_2$$

and hence,

$$\delta\overline{\overline{\phi}} = \begin{bmatrix} 0\ \overline{e}_1\overline{e}_1 - m_1\phi_{13}\overline{e}_1\overline{e}_2 + m_1\phi_{12}\overline{e}_1\overline{e}_3 \\ + m_2\phi_{23}\overline{e}_2\overline{e}_1 + 0\ \overline{e}_2\overline{e}_2 - m_2\phi_{21}\overline{e}_2\overline{e}_3 \\ + m_2\phi_{23}\overline{e}_3\overline{e}_1 - m_1\phi_{13}\overline{e}_3\overline{e}_2 + 0\ \overline{e}_3\overline{e}_3 \end{bmatrix} \ldots\ldots \qquad \{4.4.-9.\}$$

which is the kinematic rotation tensor, thus specifying $\delta\overline{e}_i$, as:

$$\overline{e}_i \cdot \delta\overline{\overline{\phi}} \times \overline{e}_i = \delta\overline{\phi}_i \times \overline{e}_i = \delta\overline{e}_i$$

4.5. THE CESÀRO-BURALI-FORTI VECTORS FOR THE DEFORMED MIDDLE SURFACE

Recalling the CESÀRO-BURALI-FORTI Vectors for the undeformed case (non-orthogonal coordinates)

$$\overline{C}_1 = \kappa_{12}\overline{e}_1 + \kappa_{11}\overline{e}^1_* + \kappa_{13}\overline{e}_3$$
$$\overline{C}_2 = \kappa_{21}\overline{e}_2 + \kappa_{22}\overline{e}^2_* + \kappa_{23}\overline{e}_3$$

then by analogy to: $\overline{E}_i = \overline{e}_i + \delta\overline{e}_i$

it is said that: $\overline{C}^\dagger_i = \overline{C}_i + \delta\overline{C}_i \qquad (i = 1,2)$

where $\overline{C}^\dagger_i$ represents the CESÀRO-BURALI-FORTI Vector for the deformed suface.

Extending this analogy to the logical conclusion, say:

$$\overline{C}_1^{\dagger} = K_{12}\overline{E}_1 + K_{11}\overline{E}_*^1 + K_{13}\overline{E}_3$$

$$\overline{C}_2^{\dagger} = K_{21}\overline{E}_2 + K_{22}\overline{E}_*^2 + K_{23}\overline{E}_3$$

Then $K_{ij} = \kappa_{ij} + \delta\kappa_{ij}$

This effectively postulates that any quantity in the deformed configuration can be represented by the corresponding quantity in the undeformed configuration, plus its (first) variation. The variational increment is thus considered as "the increment produced by the existence of the state of deformation".

Then $$\overline{C}_1^{\dagger} = \overline{C}_1 + \delta\overline{C}_1 = \Big[(\kappa_{12} + \delta\kappa_{12})(\overline{e}_1 + \delta\overline{e}_1) + (\kappa_{11}+\delta\kappa_{11})(\overline{e}_2 + \delta\overline{e}_2) + (\kappa_{13}+ \delta\kappa_{13})(\overline{e}_3 + \delta\overline{e}_3)\Big]$$

Expanding, and neglecting second-order terms (products of variations), which are considered very small in comparison to the "first variations", then:

$$\overline{C}_1^{\dagger} = [\kappa_{12}\overline{e}_1 + \kappa_{12}\delta\overline{e}_1 + \delta\kappa_{12}\overline{e}_1 + \kappa_{11}\overline{e}_2 + \kappa_{11}\delta\overline{e}_2 + \delta\kappa_{11}\overline{e}_2 + \kappa_{13}\overline{e}_3 + \kappa_{13}\delta\overline{e}_3 + \delta\kappa_{13}\overline{e}_3]$$

now, as $\delta\overline{C}_1 = \overline{C}_1^{\dagger} - \overline{C}_1$

so $$\delta\overline{C}_1 = [\kappa_{12}\delta\overline{e}_1 + \delta\kappa_{12}\overline{e}_1 + \kappa_{11}\delta\overline{e}_2 + \delta\kappa_{11}\overline{e}_2 + \kappa_{13}\delta\overline{e}_3 + \delta\kappa_{13}\overline{e}_3] \quad \ldots\ldots \quad \{4.5.\text{-}1.\}$$

or $$\delta\overline{C}_1 = \delta(\kappa_{12}\overline{e}_1) + \delta(\kappa_{11}\overline{e}_2) + \delta(\kappa_{13}\overline{e}_3)$$

A substitution of previously-obtained values ({4.4.-4.}) for $\delta\overline{e}_i$ into {4.5.-1.}, then reveals

$$\delta\overline{C}_1 = [(\delta\kappa_{12} - m_1\kappa_{11}\phi_{12} - m_1\kappa_{13}\phi_{13})\overline{e}_1 + (\delta\kappa_{11} + m_1\kappa_{12}\phi_{12} - m_2\kappa_{13}\phi_{23})\overline{e}_2 + (\delta\kappa_{13} + m_1\kappa_{12}\phi_{13} + m_2\kappa_{11}\phi_{23})\overline{e}_3] \quad \ldots\ldots \quad \{4.5.-2.\}$$

and by a similar procedure,

$$\delta\overline{C}_2 = [(-\delta\kappa_{22} + m_2\kappa_{21}\phi_{21} - m_1\kappa_{23}\phi_{23})\overline{e}_1 + (\delta\kappa_{21} + m_2\kappa_{22}\phi_{21} - m_2\kappa_{23}\phi_{23})\overline{e}_2 + (\delta\kappa_{23} + m_2\kappa_{21}\phi_{23} - m_1\kappa_{22}\phi_{13})\overline{e}_3] \quad \ldots\ldots \quad \{4.5.-3.\}$$

However, {4.5.-2.} and {4.5.-3.} are not particularly useful forms of the variational expressions, as $\delta\kappa_{ij}$ remains undefined in terms of any primitive quantities (i.e.: $\delta\kappa_{ij}$ is defined only symbolically, at present).

4.5.1. The Curvature Variations in Terms of the Primitive Quantities

From the basic kinematic concept,

$$\frac{\partial \overline{E}_i}{\partial S_j} = \overline{C}_j^{\dagger} \times \overline{E}_i$$

then the curvatures in the deformed system may be obtained by taking the dot product with $(\overline{C}_j^{\dagger} \times \overline{E}_i)$, thus causing all but one component (the desired one) to vanish.

For example, to obtain the expression for K_{13}:

$$\frac{\partial \overline{E}_1}{\partial S_1} \cdot \overline{E}_*^1 = \overline{C}_1^{\dagger} \times \overline{E}_1 \cdot \overline{E}_*^1$$

$$= (K_{12}\overline{E}_1 + K_{11}\overline{E}_*^1 + K_{13}\overline{E}_3) \cdot \overline{E}_*^1$$

$$= K_{13}$$

thus $K_{13} = \kappa_{13} + \delta\kappa_{13} = \frac{\partial \overline{E}_1}{\partial S_1} \cdot \overline{E}^1_*$ ..... {4.5.1.-1.}

Recalling that $\overline{E}_1 = \overline{e}_1 + m_1\phi_{12}\overline{e}_2 + m_1\phi_{13}\overline{e}_3$

$$\overline{E}^1_* = - m_1\phi_{12}\overline{e}_1 + \overline{e}_2 + m_2\phi_{23}\overline{e}_3$$

and that $$\frac{\partial}{\partial S_i} = \frac{\partial s_i}{\partial S_i} \frac{\partial}{\partial s_i} = \frac{\partial \alpha_i}{\partial S_i} \frac{\partial s_i}{\partial \alpha_i} \frac{\partial}{\partial s_i}$$

or $$\frac{\partial}{\partial S_i} = \frac{1}{G_i} g_i \frac{\partial}{\partial s_i} = m_i \frac{\partial}{\partial s_i}$$

where $m_i \doteq \frac{1}{1 + \phi_{ii}}$ (see §4.4.)

then {4.5.1.-1.} may be written, in expanded form,

as:

$$K_{13} = \left[m_1 \frac{\partial}{\partial s_1} (\overline{e}_1 + m_1\phi_{12}\overline{e}_2 + m_1\phi_{13}\overline{e}_3)\right] \cdot (-m_1\phi_{12}\overline{e}_1 + \overline{e}_2 + m_2\phi_{23}\overline{e}_3)$$

Carrying out the differentiation, and neglecting third-and-higher-order terms, the result appears as:

$$K_{13} = m_1 \left[\kappa_{13} - m_2\kappa_{11}\phi_{23} + m_1\frac{\partial \phi_{12}}{\partial s_1} - m_1\kappa_{12}\phi_{13}\right] \quad ..... \quad \{4.5.1.-2.\}$$

Then, from $\delta\kappa_{13} = K_{13} - \kappa_{13}$, a subtraction of κ_{13} from {4.5.1.-2.} yields the final result:

$$\delta\kappa_{13} = - m_1 \left[\kappa_{13}\phi_{11} + m_2\kappa_{11}\phi_{23} - m_1\frac{\partial \phi_{12}}{\partial s_1} + m_1\kappa_{12}\phi_{13}\right]$$

Employing a similar procedure for all other K_{ij}, and thus, $\delta\kappa_{ij}$:

$$K_{11} = - \frac{\partial \overline{E}_1}{\partial S_1} \cdot \overline{E}_3 \quad , \quad K_{12} = - \frac{\partial \overline{E}_3}{\partial S_1} \overline{E}^1_*$$

$$K_{21} = \frac{\partial \overline{E}^2_*}{\partial S_2} \cdot \overline{E}_3 \;,\; K_{22} = - \frac{\partial \overline{E}_2}{\partial S_2} \cdot \overline{E}_3 \;,\; K_{23} = \frac{\partial \overline{E}_2}{\partial S_2} \cdot \overline{E}^2_*$$

then the result is obtained:

$$\left.\begin{aligned}
\delta\kappa_{11} &= m_1\left[-m_1\frac{\partial\phi_{13}}{\partial s_1} - m_1\kappa_{12}\phi_{12} + m_2\kappa_{13}\phi_{23} - \kappa_{11}\phi_{11}\right]\\
\delta\kappa_{12} &= m_1\left[m_2\frac{\partial\phi_{23}}{\partial s_1} + m_1\kappa_{11}\phi_{12} + m_1\kappa_{13}\phi_{13} - \kappa_{12}\phi_{11}\right]\\
\delta\kappa_{13} &= m_1\left[m_1\frac{\partial\phi_{12}}{\partial s_1} - m_2\kappa_{11}\phi_{23} - m_1\kappa_{12}\phi_{13} - \kappa_{13}\phi_{11}\right]\\
\delta\kappa_{21} &= m_2\left[-m_1\frac{\partial\phi_{13}}{\partial s_2} - m_2\kappa_{22}\phi_{21} + m_2\kappa_{23}\phi_{23} - \kappa_{21}\phi_{22}\right]\\
\delta\kappa_{22} &= m_2\left[-m_2\frac{\partial\phi_{23}}{\partial s_2} + m_2\kappa_{21}\phi_{21} - m_1\kappa_{23}\phi_{13} - \kappa_{22}\phi_{22}\right]\\
\delta\kappa_{23} &= m_2\left[-m_2\frac{\partial\phi_{21}}{\partial s_2} - m_2\kappa_{21}\phi_{23} + m_1\kappa_{22}\phi_{13} - \kappa_{23}\phi_{22}\right]
\end{aligned}\right\}\quad \{4.5.1.-3.\}$$

Having thus obtained a somewhat cumbersome set of results, the approximations

$$m_1 = \frac{1}{1+\phi_{11}} \doteq 1, \qquad m_2 = \frac{1}{1+\phi_{22}} \doteq 1$$

would be useful. This is more than justified, due to the relative size of the longitudinal dilatations, ϕ_{11} and ϕ_{22} and the number 1 (i.e.: $1 >> \phi_{11}, \phi_{22}$). Application of this approximation to {4.5.1.-3.} reveals:

$$\left.\begin{aligned}
\delta\kappa_{11} &= -\frac{\partial\phi_{13}}{\partial s_1} - \kappa_{12}\phi_{12} + \kappa_{13}\phi_{23}\\
\delta\kappa_{12} &= \frac{\partial\phi_{23}}{\partial s_1} + \kappa_{11}\phi_{12} + \kappa_{13}\phi_{13}\\
\delta\kappa_{13} &= \frac{\partial\phi_{12}}{\partial s_1} - \kappa_{11}\phi_{23} - \kappa_{12}\phi_{13}\\
\delta\kappa_{21} &= -\frac{\partial\phi_{13}}{\partial s_2} - \kappa_{22}\phi_{21} + \kappa_{23}\phi_{23}
\end{aligned}\right\}\quad \ldots\ldots \quad \{4.5.1.-4.\}$$

$$\left.\begin{aligned}\delta\kappa_{22} &= -\frac{\partial\phi_{23}}{\partial s_2} + \kappa_{21}\phi_{21} - \kappa_{23}\phi_{13}\\ \delta\kappa_{23} &= -\frac{\partial\phi_{21}}{\partial s_2} - \kappa_{21}\phi_{23} + \kappa_{22}\phi_{13}\end{aligned}\right\}$$

which is a considerable simplification, as witnessed by a comparison of {4.5.1.-4.} with {4.5.1.-3.}

If this set of results({4.5.1.-4.}) is substituted into {4.5.-2.} and {4.5.-3.}, still holding valid the approximation that $m_1 \doteq 1$, $m_2 \doteq 1$, then the expressions for the variations of the CESÀRO-BURALI-FORTI Vectors result:

$$\delta\overline{C}_1 = \frac{\partial\phi_{23}}{\partial s_1}\,\overline{e}_1 - \frac{\partial\phi_{13}}{\partial s_1}\,\overline{e}_2 + \frac{\partial\phi_{12}}{\partial s_1}\,\overline{e}_3$$

$$\delta\overline{C}_2 = \frac{\partial\phi_{23}}{\partial s_2}\,\overline{e}_1 - \frac{\partial\phi_{13}}{\partial s_2}\,\overline{e}_2 - \frac{\partial\phi_{21}}{\partial s_2}\,\overline{e}_3$$

Then, as $\overline{C}_i^{\dagger} = \overline{C}_i + \delta\overline{C}_i$, the CESÀRO-BURALI-FORTI Vectors for the deformed configuration are realized:

$$\overline{C}_1^{\dagger} = \left(\kappa_{12} + \frac{\partial\phi_{23}}{\partial s_1}\right)\overline{e}_1 + \left(\kappa_{11} - \frac{\partial\phi_{13}}{\partial s_1}\right)\overline{e}_2 + \left(\kappa_{13} + \frac{\partial\phi_{12}}{\partial s_1}\right)\overline{e}_3$$

$$\overline{C}_2^{\dagger} = \left(-\kappa_{22} + \frac{\partial\phi_{23}}{\partial s_2}\right)\overline{e}_1 + \left(\kappa_{21} - \frac{\partial\phi_{13}}{\partial s_2}\right)\overline{e}_2 + \left(\kappa_{23} - \frac{\partial\phi_{21}}{\partial s_2}\right)\overline{e}_3$$

4.6. THE DEFORMATION OF PARALLEL SURFACES

Surfaces which are at a constant distance from the middle surface are referred to as *parallel surfaces*, and are prescribed by α_3 = constant.

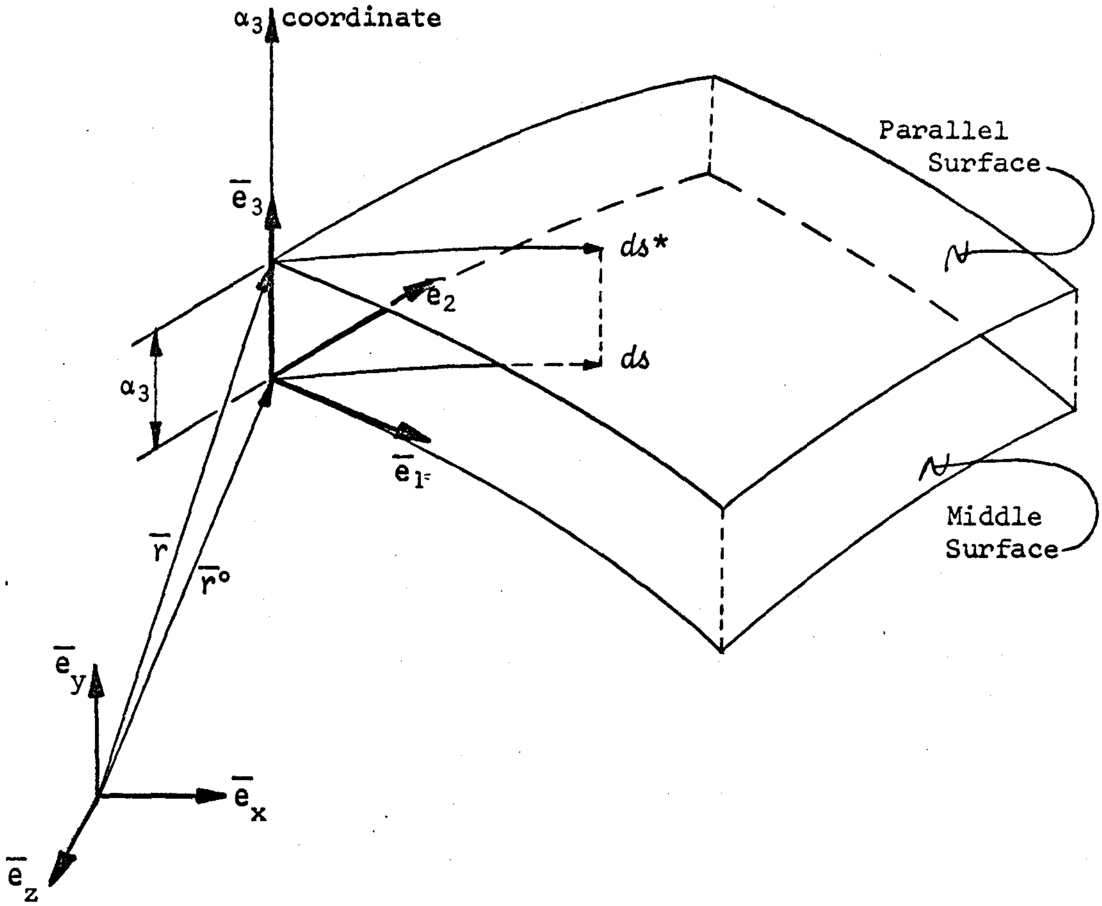


Fig. 4.6.-1.

The position vector, $\overline{r}$, to any parallel surface may be described as (with reference to Fig. 4.6.-1.):

$$\overline{r} = \overline{r}^{\circ} + \alpha_3\overline{e}_3$$

A differential line segment in the parallel surface is thus given by:

$$d\overline{r} = d(\overline{r}^{\circ} + \alpha_3\overline{e}_3) = d\overline{r}^{\circ} + d\alpha_3\overline{e}_3 + \alpha_3 d\overline{e}_3$$

The metric measure in the parallel surface, corresponding to the same parametric increment in the middle surface is:

$$[ds^*]^2 = d\overline{r}\cdot d\overline{r}$$

or $$[ds^*]^2 = (d\overline{r}^\circ + d\alpha_3\overline{e}_3 + \alpha_3 d\overline{e}_3)\cdot(d\overline{r}^\circ + d\alpha_3\overline{e}_3 + \alpha_3 d\overline{e}_3)$$

where ds^* is used to represent $ds(\overline{r})$, as distinguished from ds, which could now be referred to as $ds(\overline{r}^\circ)$ (with respect to the notation of deformed surface).

Expanding the above expression for $[ds^*]^2$:

$$\begin{aligned}[ds^*]^2 = [&d\overline{r}^\circ\cdot d\overline{r}^\circ + 2d\alpha_3 d\overline{r}^\circ\cdot\overline{e}_3 + \alpha_3 d\overline{r}^\circ\cdot d\overline{e}_3 + d\alpha_3^2\\ &+ d\alpha_3\overline{e}_3\cdot d\overline{e}_3 + \alpha_3 d\overline{e}_3\cdot d\overline{r}^\circ + \alpha_3 d\alpha_3\, d\overline{e}_3\cdot\overline{e}_3\\ &+ \alpha_3^2 d\overline{e}_3\cdot d\overline{e}_3]\end{aligned}$$

so $$[ds^*]^2 = [ds + \delta(ds)]^2 = ds^2 + 2ds\,\delta(ds) + [\delta(ds)]^2$$

The directed derivative operator for the parallel surface may now be obtained. Consider a displacement function, $\overline{u} = \overline{u}(\overline{r})$

i.e.: $\overline{u} = \overline{u}_i\ \overline{e}_i$ sum on $i = 1,2,3$

then $$d\overline{u} = d\alpha_i\frac{\partial\overline{u}}{\partial\alpha_i}, \qquad d\overline{e}_3 = d\alpha_i\frac{\partial\overline{e}_3}{\partial\alpha_i}$$

recalling, for $i=1,2$,

$$\frac{\partial\overline{u}}{\partial\alpha_i} = \frac{\partial s_i}{\partial\alpha_i}\ \frac{\partial\overline{u}}{\partial s_i} = g_i\ \frac{\partial\overline{u}}{\partial s_i} \qquad \text{sum on } i = 1,2$$

or $$\frac{\partial\overline{u}}{\partial\alpha_i} = g_i\left[\frac{\partial u_j}{\partial s_i}\,\overline{e}_j + u_j\,\frac{\partial\overline{e}_j}{\partial s_i}\right] \qquad \begin{pmatrix}\text{sum on } j = 1,2\\ \text{no sum on } i\end{pmatrix}$$

so $\quad \frac{\partial \overline{u}}{\partial \alpha_i} = g_i \left[\frac{\partial u_j}{\partial s_i} \overline{e}_j + u_j \overline{C}_i \times \overline{e}_j \right]$

Thus, $\quad d\overline{u} = \Big[g_1 \mathcal{D}_1 \overline{u} + g_2 \mathcal{D}_2 \overline{u} + g_3 \mathcal{D}_3 \overline{u}$

$$+ \alpha_3 \left(d\alpha_1 g_1 \kappa_{11} \overline{e}_1 + d\alpha_2 g_2 \kappa_{22} \overline{e}_2 \right) \cdot \left(\overline{e}_1 \mathcal{D}_1 \overline{u} + \overline{e}_2 \mathcal{D}_2 \overline{u} + \overline{e}_3 \mathcal{D}_3 \overline{u} \right) \Big] \quad \ldots . \{4.6.-1.\}$$

where $\mathcal{D}_i$ $(i = 1,2,3)$ is the particular differential operator, the exact form of which is being sought.

Expanding {4.6.-1.}, and regrouping,

$$d\overline{u} = g_1 (1 + \alpha_3 \kappa_{11}) d\alpha_1 \mathcal{D}_1 \overline{u} + g_2 (1 + \alpha_3 \kappa_{22}) d\alpha_2 \mathcal{D}_2 \overline{u} + d\alpha_3 \mathcal{D}_3 \overline{u} \quad ..\{4.6.-2.\}$$

(since $g_3 = 1$, as per §2.11.)

Now, expanding $d\overline{u}$ as

$$d\overline{u} = d\alpha_1 \frac{\partial \overline{u}}{\partial \alpha_1} + d\alpha_2 \frac{\partial \overline{u}}{\partial \alpha_1} + d\alpha_3 \frac{\partial \overline{u}}{\partial \alpha_3}$$

and substituting in {4.6.-2.}, then

$$\left\{ d\alpha_1 \left[\frac{\partial \overline{u}}{\partial \alpha_1} - g_1 (1 + \alpha_3 \kappa_{11}) \mathcal{D}_1 \overline{u} \right] + d\alpha_2 \left[\frac{\partial \overline{u}}{\partial \alpha_2} - g_2 (1 + \alpha_3 \kappa_{22}) \mathcal{D}_2 \overline{u} \right] + d\alpha_3 \left[\frac{\partial \overline{u}}{\partial \alpha_3} - \mathcal{D}_3 \overline{u} \right] \right\} = 0 \quad \ldots\ldots \quad \{4.6.-3.\}$$

or, by referring to the coefficients of $d\alpha_i$ as $\overline{\xi}_i$, then {4.6.-3.} is expressible as:

$$\overline{\xi}_1 d\alpha_1 + \overline{\xi}_2 d\alpha_2 + \overline{\xi}_3 d\alpha_3 = 0 \quad \ldots\ldots \quad \{4.6.-4.\}$$

This admits physical interpretation as a closed spatial triangle. Hence, the component vectors are coplanar. For three vectors to sum to zero in a plane, the conclusion may thus be drawn that they are *not linearly independent;* condition {4.6.-4.} is then satisfied for two cases:

a) if $\bar{\xi}_1 d\alpha_1 = -(\bar{\xi}_2 d\alpha_2 + \bar{\xi}_3 d\alpha_3)$

b) if $\bar{\xi}_1 = 0 = \bar{\xi}_2 = 0 = \bar{\xi}_3$

Since a) is a special condition, then b) is the only acceptable solution for the general problem. This requires that the following be true:

$$\frac{\partial \bar{u}}{\partial \alpha_1} = g_1(1 + \alpha_3\kappa_{11})\mathcal{D}_1\bar{u}$$

$$\frac{\partial \bar{u}}{\partial \alpha_2} = g_2(1 + \alpha_3\kappa_{22})\mathcal{D}_2\bar{u}$$

$$\frac{\partial \bar{u}}{\partial \alpha_3} = \mathcal{D}_3\bar{u}$$

and so, in operator form:

$$\left.\begin{aligned}
\mathcal{D}_1(\ ) &\equiv \left[\frac{1}{g_1(1 + \alpha_3\kappa_{11})}\right]\frac{\partial}{\partial \alpha_1}(\ ) \equiv \frac{1}{1 + \alpha_3\kappa_{11}}\frac{\partial}{\partial s_1}(\ ) \\
\mathcal{D}_2(\ ) &\equiv \left[\frac{1}{g_2(1 + \alpha_3\kappa_{22})}\right]\frac{\partial}{\partial \alpha_2}(\ ) \equiv \frac{1}{1 + \alpha_3\kappa_{22}}\frac{\partial}{\partial s_2}(\ ) \\
\mathcal{D}_3(\ ) &\equiv \frac{\partial}{\partial \alpha_3}(\ ) \equiv \frac{\partial}{\partial s_3}
\end{aligned}\right\} \quad \{4.6.-5.\}$$

referring to $\frac{1}{1 + \alpha_3\kappa_{ii}}$ as a_i $(i = 1,2)$, then the directed derivative for a parallel surface is given by:

$$\boxed{\frac{d}{d\bar{r}} \; \frac{\partial}{\partial \bar{r}} \equiv a_1\bar{e}_1\frac{\partial}{\partial s_1} + a_2\bar{e}_2\frac{\partial}{\partial s_2} + \bar{e}_3\frac{\partial}{\partial \alpha_3}} \quad \ldots\ldots \quad \{4.6.-6.\}$$

That terminology is in contradiction with the definitions in pp. 55-56

The relationships given by {4.6.-5.} also serve to define the arc lengths for a parallel surface in terms of the arc lengths of the middle surface:

VIZ: $$ds_1^* = g_1(1 + \alpha_3\kappa_{11})d\alpha_1 = (1 + \alpha_3\kappa_{11})ds_1$$

$$ds_2^* = g_2(1 + \alpha_3\kappa_{22})d\alpha_2 = (1 + \alpha_3\kappa_{22})ds_2$$

$$ds_3^* = d\alpha_3 = ds_3$$

The last of these three relations is seen to be physically justifiable from the fact that α_3 is a straight-line coordinate, and hence, its "arc length" (= linear length) is not affected by changes in position, relative to the middle surface.

NOTE: For practical applications of the above; since α_3 is always equal to, or less than, $^h/_2$ (where h is the shell thickness) and since $\kappa_{ii} = \frac{1}{R_{ii}}$ (where R_{ii} is the radius of curvature in the direction of α_i), then $\alpha_3\kappa_{ii} << 1$, and

$$ds_1^* \doteq ds_1, \; ds_2^* \doteq ds_2$$

i.e.: $$\frac{1}{1 + \alpha_3\kappa_{ii}} \doteq 1$$

This is known as LOVE's *First Approximation*, after Augustus Edward Hough LOVE (1863-1940), in 1888 and in 1927.

4.6.1. The Strain Tensor for a Parallel Surface

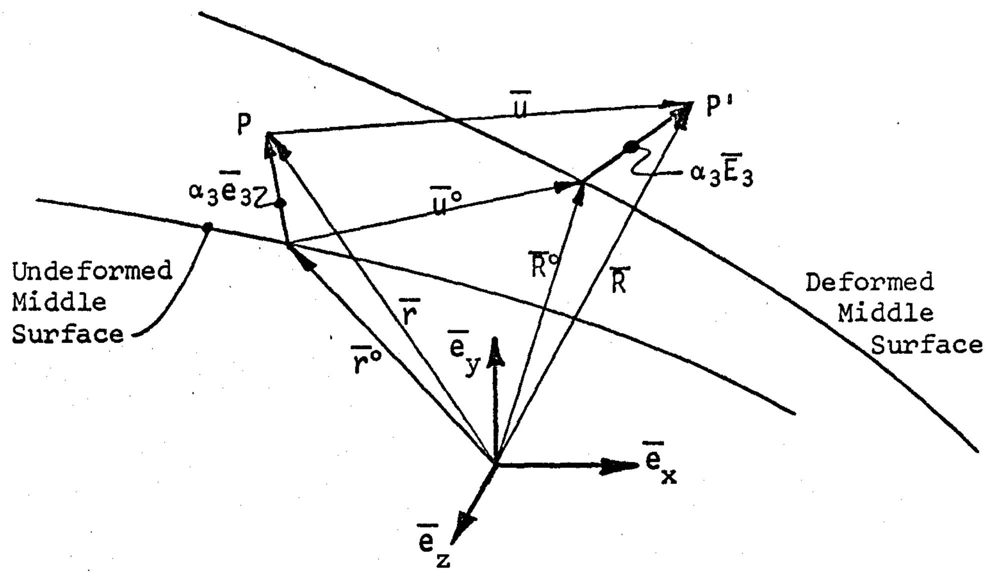


Fig. 4.6.1.-1.

With reference to Fig. 4.6.1.-1., P is a point on a parallel surface of the shell, distant from the middle surface by the amount α_3; P' is the same point, in the deformed configuration of the shell.

From the kinematics of deformation, and in accordance with KIRCHHOFF's Hypothesis:

$$\alpha_3\overline{e}_3 + \overline{u} = \overline{u}^\circ + \alpha_3\overline{E}_3$$

or $$\overline{u} = \overline{u}^\circ + \alpha_3\overline{E}_3 - \alpha_3\overline{e}_3 = \overline{u}^\circ + \alpha_3(\overline{E}_3 - \overline{e}_3)$$

so, as $\overline{E}_i = \overline{e}_i + \delta\overline{e}_i$ (§4.4)

then $$\overline{u} = \overline{u}^\circ + \alpha_3\delta\overline{e}_3$$

Then, the *displacement gradient*, or *deformation tensor* will be given by:

$$\frac{\partial \bar{u}}{\partial \bar{r}} = \frac{\partial \bar{u}^{\circ}}{\partial \bar{r}} + \frac{\partial}{\partial \bar{r}}(\alpha_3 \delta \bar{e}_3) \quad \ldots\ldots \quad \{4.6.1.-1.\}$$

where the operator $\frac{\partial}{\partial \bar{r}}$ is as defined by {4.6.-6.}

The term $\frac{\partial \bar{u}^{\circ}}{\partial \bar{r}}$ is readily evaluated, since

$$\frac{\partial \bar{u}^{\circ}}{\partial \bar{r}} = a_1 \bar{e}_1 \frac{\partial \bar{u}^{\circ}}{\partial s_1} + a_2 \bar{e}_2 \frac{\partial \bar{u}^{\circ}}{\partial s_2} + \bar{e}_3 \frac{\partial \bar{u}^{\circ}}{\partial \alpha_3}$$

(recall: $a_i = \frac{1}{1 + \alpha_3 \kappa_{ii}}$)

and $\frac{\partial \bar{u}^{\circ}}{\partial \alpha_3} = 0$, plus the fact that $\frac{\partial \bar{u}^{\circ}}{\partial s_1}$ and $\frac{\partial \bar{u}^{\circ}}{\partial s_2}$ have been previously evaluated. (§4.4). Hence, multiplying the appropriate quantities by $a_1 \bar{e}_1$ and $a_2 \bar{e}_2$, the expression for $\frac{\partial \bar{u}^{\circ}}{\partial \bar{r}}$ can be immediately written as:

$$\frac{\partial \bar{u}^{\circ}}{\partial \bar{r}} = \begin{bmatrix} a_1\phi_{11}\bar{e}_1\bar{e}_1 + a_1\phi_{12}\bar{e}_1\bar{e}_2 + a_1\phi_{13}\bar{e}_1\bar{e}_3 \\ + a_2\phi_{21}\bar{e}_2\bar{e}_1 + a_2\phi_{22}\bar{e}_2\bar{e}_2 + a_2\phi_{23}\bar{e}_2\bar{e}_3 \\ + 0\ \bar{e}_3\bar{e}_1 \quad + 0\ \bar{e}_3\bar{e}_2 \quad + 0\ \bar{e}_3\bar{e}_3 \end{bmatrix} \quad \ldots\ldots \quad \{4.6.1.-2.\}$$

where $\phi_{ij} = \frac{\partial \bar{u}^{\circ}}{\partial s_i} \cdot \bar{e}_j$ ($i = 1,2;\ j = 1,2,3$) have been previously defined by {4.4.-2.}.

Then, the term $\frac{\partial}{\partial \bar{r}}(\alpha_3 \delta \bar{e}_3)$ must be evaluated. This is accomplished, as follows:

writing $\delta \bar{e}_3$ in the original form of $(\bar{E}_3 - \bar{e}_3)$, then $$\frac{\partial}{\partial \bar{r}}(\alpha_3 \delta \bar{e}_3) = \frac{\partial}{\partial \bar{r}}[\alpha_3(\bar{E}_3 - \bar{e}_3)]$$

$$= \frac{\partial \alpha_3}{\partial \bar{r}}(\bar{E}_3 - \bar{e}_3) + \alpha_3 \frac{\partial}{\partial \bar{r}}(\bar{E}_3 - \bar{e}_3)$$

recalling the operator $\frac{\partial}{\partial \bar{r}}$ to be defined by {4.6.-6.}, then the expansion of the above shows:

$$\frac{\partial}{\partial \bar{r}}(\alpha_3 \delta \bar{e}_3) = \left\{ \alpha_3 \left[a_1 \bar{e}_1 \frac{\partial}{\partial s_1}(\bar{E}_3 - \bar{e}_3) + a_2 \bar{e}_2 \frac{\partial}{\partial s_2}(\bar{E}_3 - \bar{e}_3) \right] + \bar{e}_3 \frac{\partial}{\partial \alpha_3}[\alpha_3(\bar{E}_3 - \bar{e}_3)] \right\} \quad \ldots\ldots \quad \{4.6.1.-3.\}$$

examining the first term:

$$a_1 \bar{e}_1 \frac{\partial}{\partial s_1}(\bar{E}_3 - \bar{e}_3) = a_1 \bar{e}_1 \left[\frac{\partial \bar{E}_3}{\partial s_1} - \frac{\partial \bar{e}_3}{\partial s_1} \right]$$

$$= a_1 \bar{e}_1 \left[\frac{\partial S_1}{\partial s_1} \frac{\partial \bar{E}_3}{\partial S_1} - \frac{\partial \bar{e}_3}{\partial s_1} \right]$$

considering that $$\frac{\partial S_1}{\partial s_1} \frac{\partial}{\partial S_1} \equiv \frac{\partial \alpha_1}{\partial s_1} \frac{\partial S_1}{\partial \alpha_1} \frac{\partial}{\partial S_1}$$

$$= \frac{1}{G_1} \; g_1 \frac{\partial}{\partial S_1}$$

$$= m_1 \frac{\partial}{\partial S_1}$$

then $$a_1 \bar{e}_1 \frac{\partial}{\partial s_1}(\bar{E}_3 - \bar{e}_3) = a_1 \bar{e}_1 \left[m_1 \frac{\partial \bar{E}_3}{\partial S_1} - \frac{\partial \bar{e}_3}{\partial s_1} \right]$$

$$= a_1 \bar{e}_1 [m_1 \bar{C}_1^{\dagger} \times \bar{E}_3 - \bar{C}_1 \times \bar{e}_3]$$

however as $\bar{C}_1^{\dagger} \times \bar{E}_3 = (\bar{C}_1 + \delta \bar{C}_1) \times (\bar{e}_3 + \delta \bar{e}_3)$,

$$a_1 \bar{e}_1 \frac{\partial}{\partial s_1}(\bar{E}_3 - \bar{e}_3) = a_1 \bar{e}_1 [m_1(\bar{C}_1 + \delta \bar{C}_1) \times (\bar{e}_3 + \delta \bar{e}_3) - \bar{C}_1 \times \bar{e}_3]$$

$$= a_1 \bar{e}_1 [m_1 \bar{C}_1 \times \bar{e}_3 + m_1 \delta \bar{C}_1 \times \bar{e}_3 + m_1 \bar{C}_1 \times \delta \bar{e}_3 + m_1 \delta \bar{C}_1 \times \delta \bar{e}_3 - \bar{C}_1 \times \bar{e}_3]$$

$$= a_1 m_1 \bar{e}_1 [-\phi_{11} \bar{C}_1 \times \bar{e}_3 + \delta \bar{C}_1 \times \bar{e}_1 + \bar{C}_1 \times \delta \bar{e}_3 + \delta \bar{C}_1 \times \bar{e}_3]$$

and, if $m_1 \doteq 1$ (to maintain consistency with former developments) and the second-order variation, $\delta\overline{C}_1 \times \delta\overline{e}_3$, is neglected as being small in comparison with the first-order quantities, then the result is:

$$a_1\overline{e}_1 \frac{\partial}{\partial\delta_1}(\overline{E}_3 - \overline{e}_3) = a_1\overline{e}_1(\delta\overline{C}_1 \times \overline{e}_3 + \overline{C}_1 \times \delta\overline{e}_3) \quad \ldots\ldots\ldots \{4.6.1.\text{-}4.\}$$

A similar procedure shows

$$a_2\overline{e}_2 \frac{\partial}{\partial\delta_2}(\overline{E}_3 - \overline{e}_3) = a_2\overline{e}_2(\delta\overline{C}_2 \times \overline{e}_3 + \overline{C}_2 \times \delta\overline{e}_3) \quad \ldots\ldots\ldots \{4.6.1.\text{-}5.\}$$

and the final term is quickly evaluated as:

$$\begin{aligned}\overline{e}_3 \frac{\partial}{\partial\alpha_3}[\alpha_3(\overline{E}_3 - \overline{e}_3)] &= \overline{e}_3 \frac{\partial\alpha_3}{\partial\alpha_3}(\overline{E}_3 - \overline{e}_3) + \alpha_3\overline{e}_3 \frac{\partial}{\partial\alpha_3}(\overline{E}_3 - \overline{e}_3)\\ &= \overline{e}_3(\overline{E}_3 - \overline{e}_3)\\ &= \overline{e}_3\delta\overline{e}_3 \quad \ldots\ldots \quad \{4.6.1.\text{-}6.\}\end{aligned}$$

Replacing {4.6.1.-4.} to {4.6.1.-6.} in {4.6.1.-3.}, the result appears as:

$$\begin{aligned}\frac{\partial}{\partial\overline{r}}[\alpha_3\delta\overline{e}_3] = [&a_1\alpha_3\overline{e}_1(\delta\overline{C}_1 \times \overline{e}_3 + \overline{C}_1 \times \delta\overline{e}_3)\\ &+ a_2\alpha_3\overline{e}_2(\delta\overline{C}_2 \times \overline{e}_3 + \overline{C}_2 \times \delta\overline{e}_3) + \alpha_3\, \overline{e}_3\delta\overline{e}_3] \ldots \{4.6.1.\text{-}7.\}\end{aligned}$$

expressing the final term, $\alpha_3\overline{e}_3\delta\overline{e}_3$, as $\alpha_3\overline{e}_3\delta\overline{\phi}_3 \times \overline{e}_3$ (in accordance with §4.4.), then {4.6.1.-7.} may be written in the convenient, kinematic form:

$$\begin{aligned}\frac{\partial}{\partial\overline{r}}[\alpha_3\delta\overline{e}_3] = [&(a_1\alpha_3\overline{e}_1\delta\overline{C}_1 + a_2\alpha_3\overline{e}_2\delta\overline{C}_2 + \overline{e}_3\delta\overline{\phi}_3) \times \overline{e}_3\\ &+ (a_1\alpha_3\overline{e}_1\overline{C}_1 + a_2\alpha_3\overline{e}_2\overline{C}_2) \times \delta\overline{e}_3\,] \quad \ldots\ldots \quad \{4.6.1.\text{-}8.\}\end{aligned}$$

If the expressions for $\overline{C}_i$, $\delta\overline{C}_i$, $\overline{e}_i$, $\delta\overline{e}_i$, $\delta\overline{\phi}_3$ (in terms of the primitive quantities) are substituted into {4.6.1.-8.}, the result appears as:

$$\begin{aligned}\frac{\partial}{\partial\overline{r}}(\alpha_3\delta\overline{e}_3) = \Big[& a_1\alpha_3\left(-\frac{\partial\phi_{13}}{\partial s_1} - \kappa_{12}\phi_{12} + \kappa_{13}\phi_{23}\right)\overline{e}_1\overline{e}_1 \\ & + a_1\alpha_3\left(-\frac{\partial\phi_{23}}{\partial s_1} - \kappa_{13}\phi_{13}\right)\overline{e}_1\overline{e}_2 \\ & + a_1\alpha_3(\kappa_{11}\phi_{13})\,\overline{e}_1\overline{e}_3 \\ & + \alpha_2\alpha_3\left(-\frac{\partial\phi_{13}}{\partial s_2} + \kappa_{23}\phi_{23}\right)\overline{e}_2\overline{e}_1 \\ & + a_2\alpha_3\left(-\frac{\partial\phi_{23}}{\partial s_2} + \kappa_{21}\phi_{21} - \kappa_{23}\phi_{13}\right)\overline{e}_2\overline{e}_2 \\ & + a_2\alpha_3(\kappa_{22}\phi_{23})\,\overline{e}_2\overline{e}_3 \\ & - \phi_{13}\overline{e}_3\overline{e}_1 - \phi_{23}\overline{e}_3\overline{e}_2 + 0\,\overline{e}_3\overline{e}_3\Big] \qquad \ldots\ldots \quad \{4.6.1.-9.\}\end{aligned}$$

Superimposing the results of {4.6.1.-2.} and {4.6.1.-9.}, the deformation tensor results:

$$\overline{\overline{u}} = \frac{\partial\overline{u}}{\partial\overline{r}} = \begin{bmatrix} u_{11}\overline{e}_1\overline{e}_1 + u_{12}\overline{e}_1\overline{e}_2 + u_{13}\overline{e}_1\overline{e}_3 \\ + u_{21}\overline{e}_2\overline{e}_1 + u_{22}\overline{e}_2\overline{e}_2 + u_{23}\overline{e}_2\overline{e}_3 \\ + u_{31}\overline{e}_3\overline{e}_1 + u_{32}\overline{e}_3\overline{e}_2 + u_{33}\overline{e}_3\overline{e}_3 \end{bmatrix} = u_{rs}\overline{e}_r\overline{e}_s \qquad (r,s = 1,2,3)$$

where:

$$u_{11} = a_1\left[\phi_{11} + \alpha_3\left(-\frac{\partial\phi_{13}}{\partial s_1} - \kappa_{12}\phi_{12} + \kappa_{13}\phi_{23}\right)\right]$$

$$u_{12} = a_1\left[\phi_{12} + \alpha_3\left(-\frac{\partial\phi_{23}}{\partial s_1} - \kappa_{13}\phi_{13}\right)\right]$$

$$u_{13} = a_1\left[\phi_{13} + \alpha_3(\kappa_{11}\phi_{13})\right] = a_1\phi_{13}(1 + \alpha_3\kappa_{11}) = \phi_{13}$$

$$u_{21} = a_2\left[\phi_{21} + \alpha_3\left(-\frac{\partial\phi_{13}}{\partial s_2} + \kappa_{23}\phi_{23}\right)\right]$$

$$u_{22} = a_2\left[\phi_{22} + \alpha_3\left(-\frac{\partial\phi_{23}}{\partial s_2} + \kappa_{21}\phi_{21} - \kappa_{23}\phi_{13}\right)\right]$$

$$u_{23} = a_2\left[\phi_{23} + \alpha_3(\kappa_{22}\phi_{23})\right] = a_2\phi_{23}(1 + \alpha_3\kappa_{22}) = \phi_{23}$$

$$\left.\begin{aligned} u_{31} &= -\phi_{13} \\ u_{32} &= -\phi_{23} \\ u_{33} &= 0 \end{aligned}\right\} \text{ note that } u_{13} + u_{31} = 0,\ u_{23} + u_{32} = 0$$

Then, the *strain tensor*, $\bar{\bar{\varepsilon}}$, *for the parallel surface* may be constructed:

$$\bar{\bar{\varepsilon}} = \frac{1}{2}\left[\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}\partial}{\partial \bar{r}}\right] = \frac{1}{2}\left[\nabla\bar{u} + \bar{u}\nabla\right]$$

which is the linear strain tensor, as obtained from the displacement gradient. The additional accuracy of the nonlinear strain tensor,

$$\bar{\bar{\varepsilon}}^{*} = \frac{1}{2}\left[\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}\partial}{\partial \bar{r}} + \frac{\partial \bar{u}}{\partial \bar{r}} \cdot \frac{\bar{u}\partial}{\partial \bar{r}}\right]$$

is not considered to be warranted here, due to the fact that comparable approximations have been made already, in an effort to reach this point.

NOTE: $\frac{\bar{u}\partial}{\partial \bar{r}}$ is the *conjugate tensor* to $\frac{\partial \bar{u}}{\partial \bar{r}}$.

The strain tensor is thus given by (in symbolic notation):

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} u_{11}\bar{e}_1\bar{e}_1 + \frac{1}{2}(u_{12} + u_{21})\,\bar{e}_1\bar{e}_2 + 0\,\bar{e}_1\bar{e}_3 \\ + \frac{1}{2}(u_{21} + u_{12})\,\bar{e}_2\bar{e}_1 + u_{22}\bar{e}_2\bar{e}_2 + 0\,\bar{e}_2\bar{e}_3 \\ + \quad 0\,\bar{e}_3\bar{e}_1 \quad + 0\,\bar{e}_3\bar{e}_2 + 0\,\bar{e}_3\bar{e}_3 \end{bmatrix} = \varepsilon_{ij}\,\bar{e}_i\bar{e}_j \quad \text{(sum on } i,j = 1,2,3\text{)}$$

where $\varepsilon_{13} = \varepsilon_{31}$ and $\varepsilon_{23} = \varepsilon_{32}$ vanish due to the algebraic summation of components (as noted above), and ε_{33} is zero identically.

If written in full, to show the form of the strain tensor in terms of the primitive quantities, then:

$$\bar{\bar{\varepsilon}} = \left[a_1\phi_{11} + a_1\alpha_3\left(-\frac{\partial\phi_{13}}{\partial s_1} - \kappa_{12}\phi_{12} + \kappa_{13}\phi_{23}\right)\right]\bar{e}_1\bar{e}_1$$
$$+ \frac{1}{2}\left[a_1\phi_{12} + a_2\phi_{21} + a_1\alpha_3\left(-\frac{\partial\phi_{23}}{\partial s_1} - \kappa_{13}\phi_{13}\right) + a_2\alpha_3\left(-\frac{\partial\phi_{13}}{\partial s_2} + \kappa_{23}\phi_{23}\right)\right]\bar{e}_1\bar{e}_2$$
$$+ \frac{1}{2}\left[a_1\phi_{12} + a_2\phi_{21} + a_1\alpha_3\left(-\frac{\partial\phi_{23}}{\partial s_1} - \kappa_{13}\phi_{13}\right) + a_2\alpha_3\left(-\frac{\partial\phi_{13}}{\partial s_2} + \kappa_{23}\phi_{23}\right)\right]\bar{e}_2\bar{e}_1$$
$$+ \left[a_2\phi_{22} + a_2\alpha_3\left(-\frac{\partial\phi_{23}}{\partial s_2} + \kappa_{21}\phi_{21} - \kappa_{23}\phi_{13}\right)\right]\bar{e}_2\bar{e}_2 \quad .$$

However, a comparison of the factors of α_3 (as found in the above) with the set of expressions given by {4.5.1.-4.}, shows that the strain tensor may be written as:

$$\bar{\bar{\varepsilon}} = \left[a_1\phi_{11} + a_1\alpha_3\,(\delta\kappa_{11})\right]\bar{e}_1\bar{e}_1$$
$$+ \frac{1}{2}\left[a_1\phi_{12} + a_2\phi_{21} + a_1\alpha_3\,(-\delta\kappa_{12} + \kappa_{11}\phi_{12}) + a_2\alpha_3(\delta\kappa_{21} + \kappa_{22}\phi_{21})\right]\bar{e}_1\bar{e}_2$$
$$+ \frac{1}{2}\left[a_1\phi_{12} + a_2\phi_{21} + a_1\alpha_3\,(-\delta\kappa_{12} + \kappa_{11}\phi_{12}) + a_2\alpha_3(\delta\kappa_{21} + \kappa_{22}\phi_{21})\right]\bar{e}_2\bar{e}_1$$
$$+ \left[a_2\phi_{22} + a_2\alpha_3\,(\delta\kappa_{22})\right]\bar{e}_2\bar{e}_2$$

which may be simplified (algebraically) to give (writing a_1 and a_2 in full):

$$\bar{\bar{\varepsilon}} = \left\{\left[\frac{\phi_{11} + \alpha_3\delta\kappa_{11}}{1 + \alpha_3\kappa_{11}}\right]\bar{e}_1\bar{e}_1 + \left[\frac{\phi_{12} + \phi_{21}}{2} + \frac{\alpha_3}{2}\left(\frac{\delta\kappa_{21}}{1 + \alpha_3\kappa_{22}} - \frac{\delta\kappa_{12}}{1 + \alpha_3\kappa_{11}}\right)\right]\bar{e}_1\bar{e}_2\right.$$
$$+ \left[\frac{\phi_{12} + \phi_{21}}{2} + \frac{\alpha_3}{2}\left(\frac{\delta\kappa_{21}}{1 + \alpha_3\kappa_{22}} - \frac{\delta\kappa_{12}}{1 + \alpha_3\kappa_{11}}\right)\right]\bar{e}_2\bar{e}_1$$
$$\left. + \left[\frac{\phi_{22} + \alpha_3\delta\kappa_{22}}{1 + \alpha_3\kappa_{22}}\right]\bar{e}_2\bar{e}_2\right\} \qquad \text{....... \{4.6.1.-10.\}}$$

which is the final form of the strain tensor for the parallel surface, a result obtained by John SCHROEDER, in 1964.

If LOVE's first approximation is invoked, then {4.6.1.-10.} reduces to

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} (\phi_{11} + \alpha_3\delta\kappa_{11})\,\bar{e}_1\bar{e}_1 + \frac{1}{2}\left[\phi_{12} + \phi_{21} + \alpha_3(\delta\kappa_{21}-\delta\kappa_{12})\right]\,\bar{e}_1\bar{e}_2 \\ + \frac{1}{2}\left[\phi_{12} + \phi_{21} + \alpha_3(\delta\kappa_{21}-\delta\kappa_{12})\right]\bar{e}_2\bar{e}_1 + (\phi_{22} + \alpha_3\delta\kappa_{22})\,\bar{e}_2\bar{e}_2 \end{bmatrix} \quad \{4.6.1.-11.\}$$

However, it is strongly advised that this form, {4.6.1.-11.}, be employed with caution, as the approximation is dependent *directly* upon shell thickness and shallowness.

Either form {4.6.1.-10.} or {4.6.1.-11.} will reduce, for $\alpha_3 = 0$, to the *strain tensor for the middle surface*, $\bar{\bar{\varepsilon}}^\circ$.

i.e.: $$\bar{\bar{\varepsilon}}^\circ = \begin{bmatrix} \phi_{11}\;\bar{e}_1\bar{e}_1 + \frac{1}{2}(\phi_{12} + \phi_{21})\,\bar{e}_1\bar{e}_2 \\ + \frac{1}{2}\;(\phi_{12} + \phi_{21})\,\bar{e}_2\bar{e}_1 + \phi_{22}\bar{e}_2\bar{e}_2 \end{bmatrix}$$

Hence, $\phi_{11} \equiv \varepsilon^\circ_{11}$, $\frac{1}{2}(\phi_{12} + \phi_{21}) \equiv \varepsilon^\circ_{12} = \varepsilon^\circ_{21}$ (for the symmetric tensor), and $\phi_{22} \equiv \varepsilon^\circ_{22}$. This symbolism aids in the recognition of the various quantities, in future developments.

CHAPTER 5

The Compatibility Equations for the Strained Middle Surface

5.1. THE KINEMATIC COMPATIBILITY EQUATIONS

The local integrability condition, which was previously given in general form as

$$\frac{\partial^2 F}{\partial s_1 \partial s_2} - \frac{\partial^2 F}{\partial s_2 \partial s_1} + \gamma_1 \frac{\partial F}{\partial s_2} - \gamma_2 \frac{\partial F}{\partial s_1} = 0 \quad \text{(see \{3.1.1.-4.\})}$$

where

$$\gamma_1 = \frac{1}{g_2} \frac{\partial g_2}{\partial s_1} = \frac{\partial (\ln g_2)}{\partial s_1}$$

$$\gamma_2 = \frac{1}{g_1} \frac{\partial g_1}{\partial s_2} = \frac{\partial (\ln g_1)}{\partial s_2}$$

expresses the local independence of the integral of the function F from its path of integration. In this representation, the function F is understood to be any arbitrary scalar or directed quantity, as a point-function of the surface. Hence, this equation may be considered to represent a necessary condition to be satisfied, if F is to be a function of the surface.

Inasmuch as this equation has been developed (§3.1.1.) for an arbitrary surface, it is then applicable to the deformed surface as a particular case of interest. Therefore, in the notation pertaining to that region, {3.1.1.-4.} appears as:

$$\frac{\partial^2 F}{\partial S_1 \partial S_2} - \frac{\partial^2 F}{\partial S_2 \partial S_1} + \Gamma_1 \frac{\partial F}{\partial S_2} - \Gamma_2 \frac{\partial F}{\partial S_1} = 0 \quad \ldots\ldots \quad \{5.1.-1.\}$$

where $$\Gamma_1 = \frac{1}{G_2}\frac{\partial G_2}{\partial S_1} = \frac{\partial(\ell n G_2)}{\partial S_1}$$

$$\Gamma_2 = \frac{1}{G_1}\frac{\partial G_1}{\partial S_2} = \frac{\partial(\ell n G_1)}{\partial S_2}$$

This is obviously the same equation as before, with the exception that it now refers to deformed surfaces.

The operation of the integrability condition upon the function F, must prescribe the relationships necessarily existing between the defining parameters of the surface (ϕ_{ij}, $\delta\kappa_{ij}$, etc.) for the middle surface in the deformed configuration. The relationships found to exist between such parameters, via the same operation for the case of the undeformed surface, revealed the GAUSS and MAINARDI-CODAZZI Equations (§3.3.); such an operation for the deformed surface must, therefore, yield a similar result. Thus, it will be shown that the GAUSS and MAINARDI-CODAZZI Equations for the deformed surface, expressed in terms of the parameters of the undeformed surface actually represent the *Equations of Kinematic Compatibility of Strains* in the middle surface of the shell.*

Prescribing the arbitrary function, F, to be any vector $\bar{V} = \bar{V}(S) \equiv V\,\bar{E}_V$ associated with the surface, then {5.1.-1.} becomes:

$$\frac{\partial^2\bar{V}}{\partial S_1\partial S_2} - \frac{\partial^2\bar{V}}{\partial S_2\partial S_1} + \Gamma_1\frac{\partial\bar{V}}{\partial S_2} - \Gamma_2\frac{\partial\bar{V}}{\partial S_1} = 0 \qquad ..... \qquad \{5.1.-2.\}$$

The vector differentiation is accomplished with the aid of the modified CESÀRO-BURALI-FORTI Vectors:

* Particularly pertinent to the kinematic development, is the paper of LÖBELL, F., 1929. (Also pertinent are the papers of HESSENBERG, G., 1925 and LÖBELL, F., 1927.)

$$\overline{\Omega}_1^{\dagger} = \overline{C}_1^{\dagger} + \frac{\partial \Phi_1}{\partial S_1} \overline{E}_3 = K_{12}\overline{E}_1 + K_{11}\overline{E}_*^1 + (K_{13} + \frac{\partial \Phi_1}{\partial S_1})\overline{E}_3$$

$$\overline{\Omega}_2^{\dagger} = \overline{C}_2^{\dagger} + \frac{\partial \Phi_2}{\partial S_2} \overline{E}_3 = K_{21}\overline{E}_2 + K_{22}\overline{E}_*^2 + (K_{23} + \frac{\partial \Phi_2}{\partial S_2})\overline{E}_3$$

The additive terms $\frac{\partial \Phi_1}{\partial S_1}$ and $\frac{\partial \Phi_2}{\partial S_2}$ represent the rates of change of the angle Φ_1 between $\overline{E}_1$ and $\overline{V}$ and the angle Φ_2 between $\overline{E}_2$ and $\overline{V}$, in order to preserve an integrable direction (of HESSENBERG), independent of any particular choice of integrable displacement.

NOTE: The use of such additive terms to maintain an integrable direction has been discussed in §3.3.

The expansion of {5.1.-2.} then proceeds as follows:

$$\left\{ \frac{\partial}{\partial S_1}\left[\frac{\partial V}{\partial S_2}\overline{E}_V + \overline{\Omega}_2^{\dagger} \times \overline{V}\right] - \frac{\partial}{\partial S_2}\left[\frac{\partial V}{\partial S_1}\overline{E}_V + \overline{\Omega}_1^{\dagger} \times \overline{V}\right] + \Gamma_1\left[\frac{\partial V}{\partial S_2}\overline{E}_V + \overline{\Omega}_2^{\dagger} \times \overline{V}\right] - \Gamma_2\left[\frac{\partial V}{\partial S_1}\overline{E}_V + \overline{\Omega}_1^{\dagger} \times \overline{V}\right] \right\} = 0$$

or, carrying out the second differentiation,

$$\Big\{ \frac{\partial^2 V}{\partial S_1 \partial S_2}\overline{E}_V + \frac{\partial V}{\partial S_2}\left[\overline{\Omega}_1^{\dagger} \times \overline{E}_V\right] + \frac{\partial \overline{\Omega}_2^{\dagger}}{\partial S_1} \times \overline{V} + \left(\overline{\Omega}_2^{\dagger} \times \frac{\partial V}{\partial S_1}\overline{E}_V\right)$$

$$+\left[\overline{\Omega}_2^{\dagger} \times (\overline{\Omega}_1^{\dagger} \times \overline{V})\right] - \frac{\partial^2 V}{\partial S_2 \partial S_1}\overline{E}_V - \frac{\partial V}{\partial S_1}\left[\overline{\Omega}_2^{\dagger} \times \overline{E}_V\right] - \frac{\partial \overline{\Omega}_1^{\dagger}}{\partial S_2} \times \overline{V}$$

$$-\left(\overline{\Omega}_1^{\dagger} \times \frac{\partial V}{\partial S_2}\overline{E}_V\right) - \left[\overline{\Omega}_1^{\dagger} \times (\overline{\Omega}_2^{\dagger} \times \overline{V})\right] + \Gamma_1\frac{\partial V}{\partial S_2}\overline{E}_V + \left[\Gamma_1\overline{\Omega}_2^{\dagger} \times \overline{V}\right]$$

$$- \Gamma_2\frac{\partial V}{\partial S_1}\overline{E}_V - \left[\Gamma_2\overline{\Omega}_1^{\dagger} \times \overline{V}\right] \Big\} = 0$$

re-grouping:

$$\left[\frac{\partial^2 V}{\partial S_1 \partial S_2} - \frac{\partial^2 V}{\partial S_2 \partial S_1} + \Gamma_1 \frac{\partial V}{\partial S_2} - \Gamma_2 \frac{\partial V}{\partial S_1}\right] \overline{E}_V + \frac{\partial V}{\partial S_2} \overline{\Omega}_1^{\dagger} \times \overline{E}_V$$

$$+ \frac{\partial \overline{\Omega}_2^{\dagger}}{\partial S_1} \times \overline{V} + \left[\overline{\Omega}_2^{\dagger} \times \frac{\partial V}{\partial S_1}\right] \overline{E}_V + \left[\overline{\Omega}_2^{\dagger} \times (\overline{\Omega}_1^{\dagger} \times \overline{V})\right] - \frac{\partial V}{\partial S_1} \overline{\Omega}_2^{\dagger} \times \overline{E}_V$$

$$- \frac{\partial \overline{\Omega}_1^{\dagger}}{\partial S_2} \times \overline{V} - \left[\overline{\Omega}_1^{\dagger} \times \frac{\partial V}{\partial S_2}\right] \overline{E}_V - \left[\overline{\Omega}_1^{\dagger} \times (\overline{\Omega}_2^{\dagger} \times \overline{V})\right] + \Gamma_1 \overline{\Omega}_2^{\dagger} \times \overline{V}$$

$$- \Gamma_2 \overline{\Omega}_1^{\dagger} \times \overline{V} = 0$$

which therefore reduces, through the integrability condition operating on (scalar) V, and through algebraic summation, to:

$$\left[\frac{\partial \overline{\Omega}_2^{\dagger}}{\partial S_1} \times \overline{V} + \left[\overline{\Omega}_2^{\dagger} \times (\overline{\Omega}_1^{\dagger} \times \overline{V})\right] - \frac{\partial \overline{\Omega}_1^{\dagger}}{\partial S_2} \times \overline{V} - \left[\overline{\Omega}_1^{\dagger} \times (\overline{\Omega}_2^{\dagger} \times \overline{V})\right] + \Gamma_1 \overline{\Omega}_2^{\dagger} \times \overline{V} - \Gamma_2 \overline{\Omega}_1^{\dagger} \times \overline{V}\right] = 0 \qquad \ldots\ldots \qquad \{5.1.-3.\}$$

Since the sum of permuted cross-products vanishes

i.e.: $\left[\overline{\Omega}_1^{\dagger} \times (\overline{\Omega}_2^{\dagger} \times \overline{V})\right] + \left[\overline{V} \times (\overline{\Omega}_1^{\dagger} \times \overline{\Omega}_2^{\dagger})\right] + \left[\overline{\Omega}_2^{\dagger} \times (\overline{V} \times \overline{\Omega}_1^{\dagger})\right] = 0$

then the following substitution is employed:

$$(\overline{\Omega}_2^{\dagger} \times \overline{\Omega}_1^{\dagger}) \times \overline{V} = \left[\overline{\Omega}_2^{\dagger} \times (\overline{\Omega}_1^{\dagger} \times \overline{V})\right] - \left[\overline{\Omega}_1^{\dagger} \times (\overline{\Omega}_2^{\dagger} \times \overline{V})\right]$$

and hence, {5.1.-3.} becomes

$$\left[\frac{\partial \overline{\Omega}_2^{\dagger}}{\partial S_1} - \frac{\partial \overline{\Omega}_1^{\dagger}}{\partial S_2} + \Gamma_1 \overline{\Omega}_2^{\dagger} - \Gamma_2 \overline{\Omega}_1^{\dagger} + \overline{\Omega}_2^{\dagger} \times \overline{\Omega}_1^{\dagger}\right] \times \overline{V} = 0$$

This is satisfied for any *arbitrary* $\overline{V}$, if and only if:

$$\frac{\partial \overline{\Omega}_2^{\dagger}}{\partial S_1} - \frac{\partial \overline{\Omega}_1^{\dagger}}{\partial S_2} + \Gamma_1 \overline{\Omega}_2^{\dagger} - \Gamma_2 \overline{\Omega}_1^{\dagger} + \overline{\Omega}_2^{\dagger} \times \overline{\Omega}_1^{\dagger} = 0 \qquad \ldots\ldots \qquad \{5.1.-4.\}$$

Re-writing the expressions for $\overline{\Omega}_i^{\dagger}$ in the form $\overline{C}_i^{\dagger} + \frac{\partial \Phi_i}{\partial S_i} \overline{E}_3$, and re-grouping, then {5.1.-4.} becomes

$$\Bigg\{ \left(\frac{\partial}{\partial S_1} + \Gamma_1\right) \overline{C}_2^{\dagger} - \left(\frac{\partial}{\partial S_2} + \Gamma_2\right) \overline{C}_1^{\dagger} + \left(\frac{\partial}{\partial S_1} + \Gamma_1\right)\left(\frac{\partial \Phi_2}{\partial S_2} \overline{E}_3\right)$$

$$- \left(\frac{\partial}{\partial S_2} + \Gamma_2\right)\left(\frac{\partial \Phi_1}{\partial S_1} \overline{E}_3\right) - \overline{C}_1^{\dagger} \times \overline{C}_2^{\dagger} - \left[\frac{\partial \Phi_2}{\partial S_2} \overline{C}_1^{\dagger} \times \overline{E}_3\right] + \left[\frac{\partial \Phi_1}{\partial S_1} \overline{C}_2^{\dagger} \times \overline{E}_3\right] \Bigg\} = 0 \quad \ldots \{5.1.-5.\}$$

Now, as

$$\left(\frac{\partial}{\partial S_1} + \Gamma_1\right)\left(\frac{\partial \Phi_2}{\partial S_2} \overline{E}_3\right) \equiv \left[\left(\frac{\partial}{\partial S_1} + \Gamma_1\right) \frac{\partial \Phi_2}{\partial S_2}\right] \overline{E}_3 + \frac{\partial \Phi_2}{\partial S_2}\left[\overline{C}_1^{\dagger} \times \overline{E}_3\right]$$

and

$$\left(\frac{\partial}{\partial S_2} + \Gamma_2\right)\left(\frac{\partial \Phi_1}{\partial S_1} \overline{E}_3\right) \equiv \left[\left(\frac{\partial}{\partial S_2} + \Gamma_2\right) \frac{\partial \Phi_1}{\partial S_1}\right] \overline{E}_3 + \frac{\partial \Phi_1}{\partial S_1}\left[\overline{C}_2^{\dagger} \times \overline{E}_3\right]$$

then {5.1.-5.}is reduced to:

$$\Bigg\{ \left(\frac{\partial}{\partial S_1} + \Gamma_1\right) \overline{C}_2^{\dagger} - \left(\frac{\partial}{\partial S_2} + \Gamma_2\right) \overline{C}_1^{\dagger} + \left[\left(\frac{\partial}{\partial S_1} + \Gamma_1\right) \frac{\partial \Phi_2}{\partial S_2}\right] \overline{E}_3$$

$$- \left[\left(\frac{\partial}{\partial S_2} + \Gamma_2\right) \frac{\partial \Phi_1}{\partial S_1}\right] \overline{E}_3 - \overline{C}_1^{\dagger} \times \overline{C}_2^{\dagger} \Bigg\} = 0 \qquad \ldots\ldots \qquad \{5.1.-6.\}$$

Since the quantity $(\Phi_1 - \Phi_2)$ represents the angle subtended by $\overline{E}_1$ and $\overline{E}_2$, then let this angle be denoted by χ_{12}.

Hence,

$$\Phi_2 = \Phi_1 - \chi_{12} \qquad \ldots\ldots \qquad \{5.1.-7.\}$$

A substitution of {5.1.-7.} into {5.1.-6.} then yields

$$\Bigg\{ \left(\frac{\partial}{\partial S_1} + \Gamma_1\right) \overline{C}_2^{\dagger} - \left(\frac{\partial}{\partial S_2} + \Gamma_2\right) \overline{C}_1^{\dagger} + \left[\left(\frac{\partial}{\partial S_1} + \Gamma_1\right) \frac{\partial \Phi_1}{\partial S_2}\right] \overline{E}_3$$

$$- \left[\left(\frac{\partial}{\partial S_2} + \Gamma_2\right) \frac{\partial \Phi_1}{\partial S_1}\right] \overline{E}_3 - \left[\left(\frac{\partial}{\partial S_1} + \Gamma_1\right) \frac{\partial \chi_{12}}{\partial S_2}\right] \overline{E}_3 - \overline{C}_1^{\dagger} \times \overline{C}_2^{\dagger} \Bigg\} = 0$$

where it is observed that the integrability condition, operating on Φ_1, vanishes. Hence:

$$\left(\frac{\partial}{\partial S_1} + \Gamma_1\right)\overline{C}_2^{\dagger} - \left(\frac{\partial}{\partial S_2} + \Gamma_2\right)\overline{C}_1^{\dagger} - \left[\left(\frac{\partial}{\partial S_1} + \Gamma_1\right)\frac{\partial \chi_{12}}{\partial S_2}\right]\overline{E}_3 - \overline{C}_1^{\dagger} \times \overline{C}_2^{\dagger} = 0 \quad \{5.1.\text{-}8.\}$$

Now, referring to the operator

$$\left(\frac{\partial}{\partial S_1} + \Gamma_1\right)\frac{\partial}{\partial S_2}(\quad) \equiv \left(\frac{\partial}{\partial S_2} + \Gamma_2\right)\frac{\partial}{\partial S_1}(\quad) \quad \text{as} \quad D^{\dagger}(\quad)$$

Then {5.1.-8.} appears in final form as:

$$\left(\frac{\partial}{\partial S_1} + \Gamma_1\right)\overline{C}_2^{\dagger} - \left(\frac{\partial}{\partial S_2} + \Gamma_2\right)\overline{C}_1^{\dagger} - D^{\dagger}(\chi_{12})\overline{E}_3 + \overline{C}_2^{\dagger} \times \overline{C}_1^{\dagger} = 0 \quad \ldots\ldots \quad \{5.1.\text{-}9.\}$$

This equation must now be expanded to the full component form, in order that the three Equations of Compatibility may be extracted. Expanding to the full form, and taking the dot product with $\overline{E}_1$, $\overline{E}_*^1$ and $\overline{E}_3$ (as these three vector directions are *unique*), then the resulting component equations appear as (respectively):

$$\left[\left(\frac{\partial K_{12}}{\partial S_2} - K_{11}K_{23} + \Gamma_2 K_{12}\right) + \left(\frac{\partial K_{22}}{\partial S_1} + K_{13}K_{21} + \Gamma_1 K_{22}\right)\text{Sin }\chi_{12}\right.$$
$$\left. - \left(\frac{\partial K_{21}}{\partial S_1} - K_{13}K_{22} + \Gamma_1 K_{21}\right)\text{Cos }\chi_{12}\right] = 0 \quad \ldots\ldots \quad \{5.1.\text{-}10.\}$$

$$\left[\left(\frac{\partial K_{11}}{\partial S_2} + K_{12}K_{23} + \Gamma_2 K_{11}\right) - \left(\frac{\partial K_{21}}{\partial S_1} - K_{13}K_{22} + \Gamma_1 K_{21}\right)\text{Sin }\chi_{12}\right.$$
$$\left. - \left(\frac{\partial K_{22}}{\partial S_1} + K_{13}K_{21} + \Gamma_1 K_{22}\right)\text{Cos }\chi_{12}\right] = 0 \quad \ldots\ldots \quad \{5.1.\text{-}11.\}$$

$$\left[\left(\frac{\partial K_{13}}{\partial S_2} - \frac{\partial K_{23}}{\partial S_1} + \Gamma_2 K_{13} - \Gamma_1 K_{23}\right) - \left(K_{11}K_{22} + K_{12}K_{21}\right)\text{Sin }\chi_{12}\right.$$
$$\left. + \left(K_{11}K_{21} - K_{12}K_{22}\right)\text{Cos }\chi_{12}\right] = 0 \quad \ldots\ldots \quad \{5.1.\text{-}12.\}$$

These equations will now be written in terms of the kinematic parameters of the undeformed surface, via the following transformations

$$\left.\begin{aligned} K_{11} &= \kappa_{11} + \delta\kappa_{11} \\ K_{12} &= \kappa_{12} + \delta\kappa_{12} \\ K_{13} &= \kappa_{13} + \delta\kappa_{13} + \frac{\partial\chi_{12}}{\partial S_1} \\ K_{21} &= \kappa_{21} + \delta\kappa_{21} \\ K_{22} &= \kappa_{22} + \delta\kappa_{22} \\ K_{23} &= \kappa_{23} + \delta\kappa_{23} + \frac{\partial\chi_{21}}{\partial S_2} \end{aligned}\right\} \quad \ldots\ldots \quad \{5.1.-13.\}$$

where all $\delta\kappa_{ij}$ are as defined by {4.5.1.-4.}. The terms $\frac{\partial\chi_{12}}{\partial S_1}$ and $\frac{\partial\chi_{21}}{\partial S_2}$ ($\chi_{12} = -\chi_{21}$) appear as the additive quantities, specifying the rate of change of the angle between $\overline{E}_1$ and $\overline{E}_2$. However, as χ_{12} may also be expressed as the angle between $\overline{e}_1$ and $\overline{e}_2$ (of the undeformed system), plus the change of this angle due to the detrusional rotations, then:

$$\chi_{12} = -\chi_{21} = \left[\frac{\pi}{2} - (\phi_{12} + \phi_{21})\right]$$

thus,

$$\frac{\partial\chi_{12}}{\partial S_1} = -\frac{\partial(\phi_{12} + \phi_{21})}{\partial S_1} \quad \ldots\ldots \quad \{5.1.-14.\}$$

$$\frac{\partial\chi_{21}}{\partial S_2} = +\frac{\partial(\phi_{12} + \phi_{21})}{\partial S_2}$$

Also, for the transformation of {5.1.-10.}, {5.1.-11.} and {5.1.-12.}, the following approximations are made, in order to retain consistency with former developments:

$$\frac{\partial}{\partial S_i} \equiv \frac{\partial s_i}{\partial S_i} \frac{\partial}{\partial s_i} \equiv \frac{g_i}{G_i} \frac{\partial}{\partial s_i} \doteq \frac{\partial}{\partial s_i}$$

and $\quad \text{Cos}\ \chi_{12} = \text{Cos}\left[\frac{\pi}{2} - (\phi_{12} + \phi_{21})\right] = \text{Sin}\ (\phi_{12} + \phi_{21}) \doteq (\phi_{12} + \phi_{21})$

$$\text{Sin}\ \chi_{12} = \text{Sin}\left[\frac{\pi}{2} - (\phi_{12} + \phi_{21})\right] = \text{Cos}\ (\phi_{12} + \phi_{21}) \doteq 1$$

(as $(\phi_{12} + \phi_{21})$ is a very small angle)

and finally, for Γ_1 and Γ_2,

$$\Gamma_1 = \gamma_1 + \delta\gamma_1 = \frac{1}{G_2} \frac{\partial G_2}{\partial S_1} \doteq \frac{1}{g_2} \frac{\partial g_2}{\partial s_1} = \gamma_1$$

$$\Gamma_2 = \gamma_2 + \delta\gamma_2 = \frac{1}{G_1} \frac{\partial G_1}{\partial S_2} \doteq \frac{1}{g_1} \frac{\partial g_1}{\partial s_2} = \gamma_2$$

thus, $\quad \gamma_1 = \kappa_{23}\ , \quad \gamma_2 = -\ \kappa_{13} \qquad$ (see §3.2.)

and $\delta\gamma_1 = 0 = \delta\gamma_2$ if the same order of approximation is enforced throughout all developments.

The replacement of {5.1.-13.}, {5.1.-14.} and the approximation listed above in {5.1.-10.} then produces the result:

$$\Bigg\{ \frac{\partial\kappa_{12}}{\partial s_2} + \frac{\partial(\delta\kappa_{12})}{\partial s_2} - \kappa_{11}\kappa_{23} - \kappa_{11}\delta\kappa_{23} + \kappa_{11}\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2} + \kappa_{23}\delta\kappa_{11} + \frac{\partial\kappa_{22}}{\partial s_1}$$

$$+ \frac{\partial(\delta\kappa_{22})}{\partial s_1} + \kappa_{13}\kappa_{21} + \kappa_{13}\delta\kappa_{21} + \kappa_{21}\delta\kappa_{13} - \kappa_{21}\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}$$

$$+ \gamma_1\kappa_{22} + \gamma_1\delta\kappa_{22} - (\phi_{12} + \phi_{21})\Bigg[\frac{\partial\kappa_{21}}{\partial s_1} + \frac{\partial(\delta\kappa_{21})}{\partial s_1} - \kappa_{13}\kappa_{22}$$

$$- \kappa_{13}\delta\kappa_{22} - \kappa_{22}\delta\kappa_{13} + \kappa_{22} \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1} + \gamma_1\kappa_{21} + \gamma_1\delta\kappa_{21}\Bigg]\Bigg\} = 0$$

(where the second-order variations have been neglected.)

Deducting from this equation, those terms which sum to zero by virtue of the MAINARDI-CODAZZI relations for the undeformed surface ({3.2.-10.}, {3.2.-11.}, {3.2.-12.}), and neglecting terms of the fourth order and higher, then the final form of the first Compatibility Equation appears as:

$$\Bigg\{ \frac{\partial(\delta\kappa_{22})}{\partial s_1} + \frac{\partial(\delta\kappa_{12})}{\partial s_2} - \kappa_{11}\left[\delta\kappa_{23} + \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2}\right] + \kappa_{21}\left[\delta\kappa_{13} - \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}\right]$$

$$+ \kappa_{13}\left[\delta\kappa_{21} - \delta\kappa_{12} + \kappa_{22}(\phi_{12} + \phi_{21})\right] + \kappa_{23}\left[\delta\kappa_{22} - \delta\kappa_{11} - \kappa_{21}(\phi_{12} + \phi_{21})\right]$$

$$- (\phi_{12} + \phi_{21}) \frac{\partial\kappa_{21}}{\partial s_1} \Bigg\} = 0 \qquad \ldots\ldots \qquad \{5.1.-15.\}$$

Similarly, the same substitution processes produce the other two Compatibility Equations (from {5.1.-11.} and {5.1.-12.}) as follows:

$$\Bigg\{ \frac{\partial(\delta\kappa_{11})}{\partial s_2} - \frac{\partial(\delta\kappa_{21})}{\partial s_1} + \kappa_{12}\left[\delta\kappa_{23} + \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2}\right] + \kappa_{22}\left[\delta\kappa_{13} - \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}\right]$$

$$+ \kappa_{13}\left[\delta\kappa_{22} - \delta\kappa_{11} - \kappa_{21}(\phi_{12} + \phi_{21})\right] + \kappa_{23}\left[\delta\kappa_{12} - \delta\kappa_{21} - \kappa_{22}(\phi_{12} + \phi_{21})\right]$$

$$- (\phi_{12} + \phi_{21}) \frac{\partial\kappa_{22}}{\partial s_1} \Bigg\} = 0 \qquad \ldots\ldots \qquad \{5.1.-16.\}$$

$$\Bigg\{ \frac{\partial(\delta\kappa_{23})}{\partial s_1} - \frac{\partial(\delta\kappa_{13})}{\partial s_2} + \kappa_{11}\delta\kappa_{22} + \kappa_{22}\delta\kappa_{11} + \kappa_{23}\left[\delta\kappa_{23} + \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2}\right]$$

$$+ \kappa_{13}\left[\delta\kappa_{13} - \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}\right] + \kappa_{12}\left[\delta\kappa_{21} - \delta\kappa_{12} + (\kappa_{11} + \kappa_{22})(\phi_{12} + \phi_{21})\right]$$

$$+ \frac{\partial^2(\phi_{12} + \phi_{21})}{\partial s_2 \partial s_1} - \kappa_{23} \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2} \Bigg\} = 0 \quad \ldots\ldots \quad \{5.1.\text{-}17.\}$$

Equations {5.1.-15.}, {5.1.-16.} and{5.1.-17.} are therefore, the *Equations of Compatibility of Strains* in the middle surface, for the case of orthogonal parametric lines in the undeformed configuration of the shell.

If these orthogonal parametric lines are coincident with the lines of principal curvature, then the geodesic torsions, κ_{12} and κ_{21}, vanish (§2.10.). Consequently, the Compatibility Equations simplify to the following forms:

$$\Bigg\{ \frac{\partial(\delta\kappa_{22})}{\partial s_1} + \frac{\partial(\delta\kappa_{12})}{\partial s_2} - \kappa_{11}\left[\delta\kappa_{23} + \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2}\right] + \kappa_{23}[\delta\kappa_{22} - \delta\kappa_{11}]$$

$$+ \kappa_{13}\,[\delta\kappa_{21} - \delta\kappa_{12} + \kappa_{22}(\phi_{12} + \phi_{21})] \Bigg\} = 0 \quad \ldots\ldots \quad \{5.1.\text{-}18.\}$$

$$\Bigg\{ \frac{\partial(\delta\kappa_{11})}{\partial s_2} - \frac{\partial(\delta\kappa_{21})}{\partial s_1} + \kappa_{22}\left[\delta\kappa_{13} - \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}\right] + \kappa_{13}\,[\delta\kappa_{22} - \delta\kappa_{11}]$$

$$+ \kappa_{23}\,[\delta\kappa_{12} - \delta\kappa_{21} - \kappa_{11}(\phi_{12} + \phi_{21})] \Bigg\} = 0 \quad \ldots\ldots \quad \{5.1.19.\}$$

$$\Bigg\{ \frac{\partial(\delta\kappa_{23})}{\partial s_1} - \frac{\partial(\delta\kappa_{13})}{\partial s_2} + \kappa_{11}\delta\kappa_{22} + \kappa_{22}\delta\kappa_{11} + \kappa_{23}\delta\kappa_{23}$$

$$+ \kappa_{13}\left[\delta\kappa_{13} - \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}\right] + \frac{\partial^2(\phi_{12} + \phi_{21})}{\partial s_2 \partial s_1} \Bigg\} = 0 \quad \ldots \quad \{5.1.\text{-}20.\}$$

5.2. THE SAINT-VENANT COMPATIBILITY EQUATIONS

A unique strain tensor, $\bar{\bar{\varepsilon}}$, is defined by a single-valued, prescribed, displacement function, $\bar{u}$, by the relationship

$$\bar{\bar{\varepsilon}} = \frac{1}{2}\left[\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}\partial}{\partial \bar{r}}\right] \quad \ldots\ldots \quad \{5.2.-1.\}$$

That is to say, so long as the prescribed $\bar{u}$ is a continuous, single-valued vector point-function (apart from arbitrary rigid-body displacements), then the strain tensor is unique.

If, however, it is considered that in {5.2.-1.}, the strain tensor $\bar{\bar{\varepsilon}}$ is prescribed and it is $\bar{u}$ which is sought, then there must exist certain relations between the components ε_{ij}, in order that it might have been produced from a single-valued $\bar{u}$. A prescribed, single-valued $\bar{u}$ thus defines a unique $\bar{\bar{\varepsilon}}$, but the converse is not true. Obviously, the relations between ε_{ij} must emanate from {5.2.-1.}, yet such relations must not contain $\bar{u}$ explicitly. Consequently, the condition to be imposed upon $\bar{\bar{\varepsilon}}$ which guarantees the existence of the single-valued unique displacement field is obtainable from the strain tensor definition {5.2.-1.} by a formal elimination of $\bar{u}$ from this relationship.

This is accomplished by taking the "double curl" of $\bar{\bar{\varepsilon}}$

VIZ: $$\frac{\partial}{\partial \bar{r}} \times \bar{\bar{\varepsilon}} \times \frac{\partial}{\partial \bar{r}} = \frac{\partial}{\partial \bar{r}} \times \frac{1}{2}\left[\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}\partial}{\partial \bar{r}}\right] \times \frac{\partial}{\partial \bar{r}} = 0$$

which is equal to zero, regardless of the actual value of $\bar{u}$, as Curl Grad $\bar{u} \equiv \frac{\partial}{\partial \bar{r}} \times \frac{\partial \bar{u}}{\partial \bar{r}} = 0$. It is then obvious that if the strain tensor

$\bar{\bar{\varepsilon}}$ is prescribed and satisfies the equation

$$\bar{\bar{Q}} = \frac{\partial}{\partial \bar{r}} \times \bar{\bar{\varepsilon}} \times \frac{\partial}{\partial \bar{r}} = 0$$

then this equation will, in turn, prescribe the relations existing between all ε_{ij}, such that $\bar{\bar{\varepsilon}}$ is defined by a continuous, single-valued displacement field.

For example, for a rigid-body displacement

$$\bar{U} = \bar{u}^* + \bar{\phi}^* \times \bar{r}$$

where $\bar{\phi}^* \neq \bar{\phi}^*(\bar{r})$ denotes a small rotation vector and $\bar{u}^* \neq \bar{u}^*(\bar{r})$ denotes a constant displacement, then

$$\bar{\bar{\varepsilon}} = \frac{1}{2}\left[\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}\partial}{\partial \bar{r}}\right] = \frac{1}{2}\left[\frac{\partial \bar{u}^*}{\partial \bar{r}} + \frac{\bar{u}^*\partial}{\partial \bar{r}}\right] + \frac{\partial}{\partial \bar{r}}\left[(\bar{\phi}^* \times \bar{r})\right]$$

$$+ \left[(\bar{\phi}^* \times \bar{r})\frac{\partial}{\partial \bar{r}}\right]$$

or $$\bar{\bar{\varepsilon}} = \left[\frac{\partial \bar{\phi}^*}{\partial \bar{r}} \times \bar{r}\right] + \left[\bar{\phi}^* \times \frac{\partial \bar{r}}{\partial \bar{r}}\right] + \left[\frac{\phi^*\partial}{\partial \bar{r}} \times \bar{r}\right] + \left[\bar{\phi}^* \times \frac{\bar{r}\partial}{\partial \bar{r}}\right]$$

$$= \left[\bar{\phi}^* \times \frac{\partial \bar{r}}{\partial \bar{r}}\right] + \left[\frac{\partial \bar{r}}{\partial \bar{r}} \times \bar{\phi}^*\right]$$

$$= \left[\bar{\phi}^* \times \bar{\bar{I}}\right] + \left[\bar{\bar{I}} \times \bar{\phi}^*\right] = 0$$

This demonstrates that the rigid-body displacement has no effect upon the strain tensor -- a result which is intuitively obvious, in any case.

The tensor

$$\bar{\bar{Q}} = \frac{\partial}{\partial \bar{r}} \times \bar{\bar{\varepsilon}} \times \frac{\partial}{\partial \bar{r}} = 0$$

is called the *Local Kinematic* SAINT-VENANT *Compatibility Tensor*. This tensor is symmetric (i.e., $\bar{Q} \equiv \bar{Q}_c$), a fact which will be presently shown to lead to interesting results. The symmetry, although somewhat obvious, may be demonstrated as follows: Consider any symmetric tensor, $\bar{\bar{\xi}}$, *not necessarily composed of the gradient of a vector plus its conjugate* (as is the strain tensor), but simply any symmetric tensor. Then, representing $\bar{\bar{\xi}}$ as the sum of two other tensors, one of which is the conjugate of the other:

$$\text{(say)} \qquad \bar{\bar{\xi}} = \bar{\bar{a}} + \bar{\bar{a}}_c$$

then
$$\frac{\partial}{\partial \bar{r}} \times \bar{\bar{\xi}} \times \frac{\partial}{\partial \bar{r}} = \frac{\partial}{\partial \bar{r}} \times (\bar{\bar{a}} + \bar{\bar{a}}_c) \times \frac{\partial}{\partial \bar{r}}$$

$$= \left[\frac{\partial}{\partial \bar{r}} \times \bar{\bar{a}}\right] \times \frac{\partial}{\partial \bar{r}} + \left[\frac{\partial}{\partial \bar{r}} \times \bar{\bar{a}}_c\right] \times \frac{\partial}{\partial \bar{r}}$$

also
$$\left(\frac{\partial}{\partial \bar{r}} \times \bar{\bar{\xi}} \times \frac{\partial}{\partial \bar{r}}\right)_c = -\left[-\left(\frac{\partial}{\partial \bar{r}}\right) \times \bar{\bar{\xi}}_c \times \left(\frac{\partial}{\partial \bar{r}}\right)\right]$$

$$= \left(\frac{\partial}{\partial \bar{r}} \times \bar{\bar{a}}_c\right) \times \frac{\partial}{\partial \bar{r}} + \left(\frac{\partial}{\partial \bar{r}} \times \bar{\bar{a}}\right) \times \frac{\partial}{\partial \bar{r}}$$

thus, if $\bar{\bar{\xi}}$ is a symmetric tensor, then $(\frac{\partial}{\partial \bar{r}} \times \bar{\bar{\xi}} \times \frac{\partial}{\partial \bar{r}})$ will also be a symmetric tensor.

Returning to the original tensor under consideration

VIZ:
$$\bar{Q} = Q_{ij}\, \bar{e}_i \bar{e}_j = \frac{\partial}{\partial \bar{r}} \times \bar{\bar{\varepsilon}} \times \frac{\partial}{\partial \bar{r}} = 0 \qquad (i,j = 1,2,3)$$

it is seen that if the strain tensor for a parallel surface were subjected to the application of the operator $\frac{\partial}{\partial \bar{r}} \times (\;)$ from both sides, and the quantity α_3(see §4.6.) were set equal to zero, then the resulting equations would be the Compatibility Equations for middle-surface strains.

It has been demonstrated that for a parallel surface of the shell ($\alpha_3 \neq 0$), the directed derivative assumes the form:

$$\frac{\partial}{\partial \bar{r}} = a_1\bar{e}_1 \frac{\partial}{\partial s_1} + a_2\bar{e}_2 \frac{\partial}{\partial s_2} + \bar{e}_3 \frac{\partial}{\partial \alpha_3} \qquad (\{4.6.-6.\})$$

$$\text{where } a_1 = \frac{1}{1 + \alpha_3\kappa_{11}},\ a_2 = \frac{1}{1 + \alpha_3\kappa_{22}}$$

The strain tensor for a parallel surface has also been calculated ({4.6.1.-10.}), and is given by

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} [a_1(\phi_{11} + \alpha_3\delta\kappa_{11})]\,\bar{e}_1\bar{e}_1 + \frac{1}{2}[\phi_{12} + \phi_{21} + \alpha_3(a_2\delta\kappa_{21} - a_1\delta\kappa_{12})]\,\bar{e}_1\bar{e}_2 \\ + \frac{1}{2}[\phi_{12} + \phi_{21} + \alpha_3(a_2\delta\kappa_{21} - a_1\delta\kappa_{12})]\,\bar{e}_2\bar{e}_1 + [a_2(\phi_{22} + \alpha_3\delta\kappa_{22}]\,\bar{e}_2\bar{e}_2 \end{bmatrix}$$

or, retaining the symbolic form for the present,

$$\bar{\bar{\varepsilon}} = \varepsilon_{11}\bar{e}_1\bar{e}_1 + \varepsilon_{12}\,\bar{e}_1\bar{e}_2 + \varepsilon_{21}\bar{e}_2\bar{e}_1 + \varepsilon_{22}\bar{e}_2\bar{e}_2$$

where ε_{ij} are given by the corresponding coefficients of the tensor directions in the expanded form (above).

Applying the directed derivative in cross-product to $\bar{\bar{\varepsilon}}$ (in symbolic form) and evaluating the vector derivatives with the aid of the CESÀRO-BURALI-FORTI Vectors i.e.: $\overline{C}_1 = \kappa_{12}\overline{e}_1 + \kappa_{11}\overline{e}_2 + \kappa_{13}\overline{e}_3$ (Orthogonal Parametric

$\overline{C}_2 = -\kappa_{22}\overline{e}_1 + \kappa_{21}\overline{e}_2 + \kappa_{23}\overline{e}_3$ Lines)

then a tensor $\bar{\bar{P}}$ will result, where

$$\bar{\bar{P}} = P_{ij}\ \overline{e}_i\ \overline{e}_j = \frac{\partial}{\partial \overline{r}} \times \bar{\bar{\varepsilon}}$$

One further application of the directed derivative will then yield the desired result,

$$\bar{\bar{Q}} = Q_{ij}\ \overline{e}_i\overline{e}_j = \left[\bar{\bar{P}} \times \frac{\partial}{\partial \overline{r}}\right] = \left[\frac{\partial}{\partial \overline{r}} \times \bar{\bar{\varepsilon}} \times \frac{\partial}{\partial \overline{r}}\right] = 0$$

A typical operation of the first step (to obtain $\bar{\bar{P}}$) is as follows:

$$a_1\overline{e}_1 \frac{\partial}{\partial s_1} \times \varepsilon_{21}\overline{e}_2\overline{e}_1 = a_1\overline{e}_1 \times \frac{\partial(\varepsilon_{21}\overline{e}_2\overline{e}_1)}{\partial s_1}$$

$$= \left[a_1\overline{e}_1 \times \frac{\partial \varepsilon_{21}}{\partial s_1}\overline{e}_2\overline{e}_1 + a_1\overline{e}_1 \times \varepsilon_{21}\frac{\partial \overline{e}_1}{\partial s_1}\overline{e}_1 + a_1\overline{e}_1 \times \varepsilon_{21}\overline{e}_2\frac{\partial \overline{e}_1}{\partial s_1}\right]$$

$$= \left[a_1\frac{\partial \varepsilon_{21}}{\partial s_1}(\overline{e}_1 \times \overline{e}_2)\overline{e}_1 + a_1\varepsilon_{21}\overline{e}_1 \times (\overline{C}_1 \times \overline{e}_1)\overline{e}_1 + a_1\varepsilon_{21}(\overline{e}_1 \times \overline{e}_2)(\overline{C}_1 \times \overline{e}_1)\right]$$

$$= a_1\left[\frac{\partial \varepsilon_{21}}{\partial s_1}\overline{e}_3\overline{e}_1 - \varepsilon_{21}\kappa_{12}\overline{e}_2\overline{e}_1 - \varepsilon_{21}\kappa_{11}\overline{e}_3\overline{e}_3 + \varepsilon_{21}\kappa_{13}\overline{e}_3\overline{e}_2\right]$$

Performing twelve of such operations and accumulating the coefficients of the tensor dyads, reveals the tensor:

$$\bar{\bar{P}} = \begin{bmatrix} + P_{11}\bar{e}_1\bar{e}_1 + P_{12}\bar{e}_1\bar{e}_2 + P_{13}\bar{e}_1\bar{e}_3 \\ + P_{21}\bar{e}_2\bar{e}_1 + P_{22}\bar{e}_2\bar{e}_2 + P_{23}\bar{e}_2\bar{e}_3 \\ + P_{31}\bar{e}_3\bar{e}_1 + P_{32}\bar{e}_3\bar{e}_2 + P_{33}\bar{e}_3\bar{e}_3 \end{bmatrix}$$

where:

$$P_{11} = -a_2(\varepsilon_{11}\kappa_{12} + \varepsilon_{21}\kappa_{22}) - \frac{\partial\varepsilon_{21}}{\partial\alpha_3}$$

$$P_{12} = -a_2(\varepsilon_{12}\kappa_{21} + \varepsilon_{22}\kappa_{22}) - \frac{\partial\varepsilon_{22}}{\partial\alpha_3}$$

$$P_{13} = 0$$

$$P_{21} = a_1(\varepsilon_{11}\kappa_{11} - \varepsilon_{21}\kappa_{12}) + \frac{\partial\varepsilon_{11}}{\partial\alpha_3}$$

$$P_{22} = a_1(\varepsilon_{12}\kappa_{11} - \varepsilon_{22}\kappa_{12}) + \frac{\partial\varepsilon_{12}}{\partial\alpha_3}$$

$$P_{23} = 0$$

$$P_{31} = \left(a_1\left[\varepsilon_{11}\kappa_{13} + \frac{\partial\varepsilon_{21}}{\partial s_1} - \varepsilon_{22}\kappa_{13}\right] + a_2\left[-\frac{\partial\varepsilon_{11}}{\partial s_2} + \varepsilon_{12}\kappa_{23} + \varepsilon_{21}\kappa_{23}\right]\right)$$

$$P_{32} = \left(a_1\left[\varepsilon_{12}\kappa_{13} + \varepsilon_{21}\kappa_{13} + \frac{\partial\varepsilon_{22}}{\partial s_1}\right] + a_2\left[-\varepsilon_{11}\kappa_{23} - \frac{\partial\varepsilon_{12}}{\partial s_2} + \varepsilon_{22}\kappa_{23}\right]\right)$$

$$P_{33} = \left(a_1[\varepsilon_{22}\kappa_{12} - \varepsilon_{21}\kappa_{11}] + a_2[\varepsilon_{12}\kappa_{22} + \varepsilon_{11}\kappa_{21}]\right)$$

Taking the transposed curl of $\bar{\bar{P}}$,

i.e.: $\bar{\bar{P}} \times \frac{\partial}{\partial\bar{r}}$

produces the tensor, $\bar{\bar{Q}}$. Performing the twenty-seven operations required, and accumulating coefficients, the result is:

$$\frac{\partial}{\partial \bar{r}} \times \bar{\bar{\varepsilon}} \times \frac{\partial}{\partial \bar{r}} = \bar{Q} = \begin{bmatrix} + Q_{11}\bar{e}_1\bar{e}_1 + Q_{12}\bar{e}_1\bar{e}_2 + Q_{13}\bar{e}_1\bar{e}_3 \\ + Q_{21}\bar{e}_2\bar{e}_1 + Q_{22}\bar{e}_2\bar{e}_2 + Q_{23}\bar{e}_2\bar{e}_3 \\ + Q_{31}\bar{e}_3\bar{e}_1 + Q_{32}\bar{e}_3\bar{e}_2 + Q_{33}\bar{e}_3\bar{e}_3 \end{bmatrix} = 0$$

where (in terms of the former coefficients, P_{ij}):

$$Q_{11} = a_2 [P_{12}\kappa_{22} - P_{33}\kappa_{21} + P_{11}\kappa_{21}] + \frac{\partial P_{12}}{\partial \alpha_3}$$

$$Q_{12} = a_1 [P_{33}\kappa_{11} + P_{12}\kappa_{12} - P_{11}\kappa_{11}] - \frac{\partial P_{11}}{\partial \alpha_3}$$

$$Q_{13} = \Big(a_1 \Big[-P_{11}\kappa_{13} - \frac{\partial P_{12}}{\partial \delta_1} + P_{22}\kappa_{13} - P_{32}\kappa_{11} \Big] + a_2 \Big[P_{31}\kappa_{21} + \frac{\partial P_{11}}{\partial \delta_2} - P_{12}\kappa_{23} - P_{21}\kappa_{23} \Big]\Big)$$

$$Q_{21} = a_2 [P_{21}\kappa_{21} + P_{22}\kappa_{22} - P_{33}\kappa_{22}] + \frac{\partial P_{22}}{\partial \alpha_3}$$

$$Q_{22} = a_1 [P_{22}\kappa_{12} - P_{21}\kappa_{11} - P_{33}\kappa_{12}] - \frac{\partial P_{21}}{\partial \alpha_3}$$

$$Q_{23} = \Big(a_1 \Big[-P_{12}\kappa_{13} - P_{21}\kappa_{13} - \frac{\partial P_{22}}{\partial \delta_1} + P_{32}\kappa_{12}\Big] + a_2 \Big[P_{11}\kappa_{23} + \frac{\partial P_{21}}{\partial \delta_2} - P_{22}\kappa_{23} + P_{31}\kappa_{22}\Big]\Big)$$

$$Q_{31} = a_2 \Big[P_{32}\kappa_{22} + P_{31}\kappa_{21} - \frac{\partial P_{33}}{\partial \delta_2}\Big] + \frac{\partial P_{32}}{\partial \alpha_3}$$

$$Q_{32} = a_1 \Big[P_{32}\kappa_{12} - P_{31}\kappa_{11} + P_{33}\kappa_{12} + \frac{\partial P_{33}}{\partial \delta_1}\Big] - \frac{\partial P_{31}}{\partial \alpha_3}$$

$$Q_{33} = \Big(a_1 \Big[P_{12}\kappa_{11} - P_{22}\kappa_{12} - \frac{\partial P_{32}}{\partial \delta_1} - P_{31}\kappa_{13} \Big] + a_2 \Big[-P_{11}\kappa_{21} - P_{21}\kappa_{22} - P_{32}\kappa_{23} + \frac{\partial P_{31}}{\partial \delta_2} + P_{33}\kappa_{21}\Big]\Big)$$

Since $\bar{Q}$ is a zero-tensor, and since the tensor dyads, $\bar{e}_i\bar{e}_j$, are unique, then each of the coefficients, Q_{ij}, must vanish separately in order for $\bar{Q}$ to vanish. This produces 9 scalar equations (as components of the tensor), of which only 6 are unique, as the tensor $\bar{Q}$ is symmetric.

Substituting the values of a_1, a_2, and P_{ij} into the expressions for the coefficients in the tensor $\bar{\bar{Q}}$ and setting each such coefficient equal to zero, reveals the following results:

(1) From $Q_{11} = 0$

$$\kappa_{21}[(\phi_{12} + \phi_{21})(2\kappa_{22} - \kappa_{11}) + 2(\delta\kappa_{21} - \delta\kappa_{12}) + 2\kappa_{12}(\phi_{22} - 2\phi_{11})] = 0 \quad \ldots\{5.2.-2.\}$$

(2) From $Q_{12} = 0$

$$\Big[(\kappa_{11} - \kappa_{22}) [(\kappa_{11} - \kappa_{22})(\phi_{12} + \phi_{21}) - (\delta\kappa_{12} + \delta\kappa_{21})] + \kappa_{21}[2\kappa_{11}(\phi_{22} - \phi_{11}) + 2\kappa_{22}\phi_{11} - 2(\delta\kappa_{11} + \delta\kappa_{22}) + \kappa_{12}(\phi_{12} + \phi_{21})] \Big] = 0 \quad \ldots\ldots \quad \{5.2.-3.\}$$

(3) From $Q_{13} = 0$

$$\Bigg\{ \frac{\partial(\delta\kappa_{22})}{\partial s_1} + \frac{1}{2}\left[\frac{\partial(\delta\kappa_{12})}{\partial s_2} - \frac{\partial(\delta\kappa_{21})}{\partial s_2}\right] + \frac{1}{2}(\kappa_{11} - \kappa_{22})\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2}$$

$$+ \kappa_{13}[\delta\kappa_{21} - \delta\kappa_{12} + \frac{1}{2}(\kappa_{22} - \kappa_{11})(\phi_{12} + \phi_{21})]$$

$$+ \kappa_{23}[\delta\kappa_{22} - \delta\kappa_{11} + \kappa_{21}(\phi_{12} + \phi_{21})] - \phi_{11}\frac{\partial\kappa_{21}}{\partial s_2}$$

$$+ \kappa_{21}\left[\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1} - 2\frac{\partial\phi_{11}}{\partial s_2} + 2\kappa_{13}\phi_{11}\right]$$

$$+ \kappa_{11}[\kappa_{23}(\phi_{11} - \phi_{22}) - \frac{\partial\phi_{22}}{\partial s_1}]$$

$$+ \frac{1}{2}(\phi_{12} + \phi_{21})\left[\frac{\partial\kappa_{21}}{\partial s_1} - \frac{\partial\kappa_{22}}{\partial s_2}\right]\Bigg\} = 0 \quad \ldots\ldots \quad \{5.2.-4.\}$$

(4) From $Q_{21} = 0$

$$\Big[(\kappa_{11} - \kappa_{22})\ [(\kappa_{11} - \kappa_{22})(\phi_{12} + \phi_{21}) - (\delta\kappa_{12} + \delta\kappa_{21})]$$
$$+ \kappa_{12}[2\kappa_{22}(\phi_{22} - \phi_{11}) - 2\kappa_{11}\phi_{22} + 2(\delta\kappa_{11} + \delta\kappa_{22})$$
$$+ \kappa_{21}(\phi_{12} + \phi_{21})]\Big] = 0 \quad \ldots\ldots \quad \{5.2.\text{-}5.\}$$

(5) From $Q_{22} = 0$

$$\kappa_{12}[(\phi_{12} + \phi_{21})(2\kappa_{11} - \kappa_{22}) + 2(\delta\kappa_{21} - \delta\kappa_{12}) + 2\kappa_{21}(2\phi_{22} - \phi_{11})] = 0 \quad \{5.2.\text{-}6.\}$$

(6) From $Q_{23} = 0$

$$\Big\{ \frac{\partial(\delta\kappa_{11})}{\partial s_2} + \frac{1}{2}\left[\frac{\partial(\delta\kappa_{12})}{\partial s_1} - \frac{\partial(\delta\kappa_{21})}{\partial s_1}\right] + \frac{1}{2}(\kappa_{22} - \kappa_{11})\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}$$
$$+ \kappa_{13}[\delta\kappa_{22} - \delta\kappa_{11} + \kappa_{12}(\phi_{12} + \phi_{21})] + \phi_{22}\frac{\partial\kappa_{12}}{\partial s_1}$$
$$+ \kappa_{23}[\delta\kappa_{12} - \delta\kappa_{21} + \frac{1}{2}(\kappa_{22} - \kappa_{11})(\phi_{12} + \phi_{21})]$$
$$+ \kappa_{12}\left[-\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2} + 2\frac{\partial\phi_{22}}{\partial s_1} + 2\kappa_{23}\phi_{22}\right]$$
$$+ \kappa_{22}\left[\kappa_{13}(\phi_{11} - \phi_{22}) - \frac{\partial\phi_{11}}{\partial s_2}\right]$$
$$- \frac{1}{2}(\phi_{12} + \phi_{21})\left[\frac{\partial\kappa_{11}}{\partial s_1} + \frac{\partial\kappa_{12}}{\partial s_2}\right]\Big\} = 0 \quad \ldots\ldots \quad \{5.2.\text{-}7.\}$$

(7) From $Q_{31} = 0$

$$\Big\{ \frac{\partial(\delta\kappa_{22})}{\partial s_1} + \frac{1}{2}\left[\frac{\partial(\delta\kappa_{12})}{\partial s_2} - \frac{\partial(\delta\kappa_{21})}{\partial s_2}\right] + \frac{1}{2}(\kappa_{11} - \kappa_{22})\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2}$$
$$+ \kappa_{13}[\delta\kappa_{21} - \delta\kappa_{12} + (\kappa_{22} - \kappa_{11})(\phi_{12} + \phi_{21})] + (\phi_{22} - \phi_{11})\frac{\partial\kappa_{21}}{\partial s_2}$$

$$+ \kappa_{23}[\delta\kappa_{22} - \delta\kappa_{11} + \kappa_{21}(\phi_{12} + \phi_{21}) - \kappa_{22}\phi_{22}] - \phi_{22}\frac{\partial\kappa_{22}}{\partial s_1}$$

$$+ \kappa_{21}\left[\frac{1}{2}\ \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1} - 2\frac{\partial\phi_{11}}{\partial s_2} + \kappa_{13}(\phi_{11} - \phi_{22}) + \frac{\partial\phi_{22}}{\partial s_2}\right]$$

$$\left. + \kappa_{11}[\kappa_{23}\phi_{11} - \frac{\partial\phi_{22}}{\partial s_1}] + \frac{1}{2}(\phi_{12} + \phi_{21})\left[\frac{\partial\kappa_{11}}{\partial s_2} - \frac{\partial\kappa_{22}}{\partial s_2}\right]\right\} = 0 \quad \ldots\ldots \{5.2.-8.\}$$

(8) From $Q_{32} = 0$

$$\left\{\frac{\partial(\delta\kappa_{11})}{\partial s_2} + \frac{1}{2}\left[\frac{\partial(\delta\kappa_{12})}{\partial s_1} - \frac{\partial(\delta\kappa_{21})}{\partial s_1}\right] + \frac{1}{2}(\kappa_{22} - \kappa_{11})\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}\right.$$

$$+ \kappa_{13}[\delta\kappa_{22} - \delta\kappa_{11} + \kappa_{12}(\phi_{12} + \phi_{21}) + \kappa_{11}\phi_{11}] + (\phi_{11} - \phi_{22})\frac{\partial\kappa_{21}}{\partial s_1}$$

$$+ \kappa_{23}[\delta\kappa_{12} - \delta\kappa_{21} + (\kappa_{22} - \kappa_{11})(\phi_{12} + \phi_{21})] - \kappa_{22}\left[\kappa_{13}\phi_{22} + \frac{\partial\phi_{11}}{\partial s_2}\right]$$

$$+ \kappa_{12}\left[-\frac{1}{2}\ \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2} + 2\frac{\partial\phi_{22}}{\partial s_1} + \kappa_{23}(\phi_{22} - \phi_{11}) - \frac{\partial\phi_{11}}{\partial s_1}\right]$$

$$\left. + \frac{1}{2}(\phi_{12} + \phi_{21})\left[\frac{\partial\kappa_{22}}{\partial s_1} - \frac{\partial\kappa_{11}}{\partial s_1}\right] - \phi_{11}\frac{\partial\kappa_{11}}{\partial s_2}\right\} = 0 \quad \ldots\ldots \{5.2.-9.\}$$

(9) From $Q_{33} = 0$

$$\left\{\kappa_{11}\delta\kappa_{22} + \kappa_{22}\delta\kappa_{11} + 2\kappa_{13}\kappa_{23}(\phi_{12} + \phi_{21}) + \frac{\partial^2\phi_{11}}{\partial s_2^2} + \frac{\partial^2\phi_{22}}{\partial s_1^2}\right.$$

$$+ \kappa_{12}[\delta\kappa_{21} - \delta\kappa_{12} + (\phi_{12} + \phi_{21})(\kappa_{22} - \kappa_{11}) + 3\kappa_{21}\phi_{11} + \kappa_{12}\phi_{22}]$$

$$+ \kappa_{13}\left[\kappa_{13}(\phi_{11} - \phi_{22}) + \frac{3}{2}\ \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1} - 2\frac{\partial\phi_{11}}{\partial s_2} + \frac{\partial\phi_{22}}{\partial s_2}\right]$$

$$+ \kappa_{23}\left[\kappa_{23}(\phi_{22} - \phi_{11}) - \frac{3}{2}\ \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2} + 2\frac{\partial\phi_{22}}{\partial s_1} - \frac{\partial\phi_{11}}{\partial s_1}\right]$$

$$+ (\phi_{22} - \phi_{11})\left[\frac{\partial\kappa_{13}}{\partial s_2} + \frac{\partial\kappa_{23}}{\partial s_1}\right] + (\phi_{12} + \phi_{21})\left[\frac{\partial\kappa_{13}}{\partial s_1} - \frac{\partial\kappa_{23}}{\partial s_2}\right]$$

$$\left. - \frac{1}{2}\ \frac{\partial^2(\phi_{12} + \phi_{21})}{\partial s_1\partial s_2} - \frac{1}{2}\ \frac{\partial^2(\phi_{12} + \phi_{21})}{\partial s_2\partial s_1}\right\} = 0 \quad \ldots\{5.2.-10.\}$$

Each of these equations ({5.2.-2.} to {5.2.-10.}) is a "compatiblity equation" in the sense that each prescribes a relationship,differential or otherwise, which must exist between the components of the original strain tensor. However, equations {5.2.-2.} (from $Q_{11} = 0$), {5.2.-3.} (from $Q_{12} = 0$), {5.2.-5.} (from $Q_{21} = 0$) and {5.2.-6.} (from $Q_{22} = 0$) are algebraic equations and are thus classified as identities.* Therefore, *equations* {5.2.-4.} (from $Q_{13} = 0$) {5.2.-7.} (from $Q_{23} = 0$) *and* {5.2.-10.} (from $Q_{33} = 0$) *are the Compatibility Equations of Middle Surface Strains* which have been sought. As was previously noted, the equation resulting from $Q_{ij} = 0$ will express the same relationship as the equation resulting from $Q_{ji} = 0$, due to the symmetry of the tensor $\bar{Q}$. This gives rise to the following interesting result.

Any component, Q_{ij}, may be set equal to any other component, Q_{rs}, since each has the value of zero; in most cases, the result of setting one component equal to another would yield merely a combined form of results which have already been obtained ({5.2.-2.} to {5.2.-10.}). However, in the case of the components which are equal by symmetry considerations, the setting of one equal to the other might reasonably be expected to produce a result which is not a combined form of both. (This is anticipated by virtue of the fact that the forms of such components are quite similar, yet not identical).

Pursuing this investigation produces the following results:

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* A substitution of the more primitive forms of the quantities employed in these equations causes the equations to vanish identically.

From $Q_{12} = Q_{21}$

$$\kappa_{11}\phi_{11} - \kappa_{22}\phi_{22} = 0 \qquad \ldots\ldots \qquad \{5.2.-11.\}$$

From $Q_{13} = Q_{31}$

$$\left\{ \frac{1}{2}\left[\frac{\partial\kappa_{21}}{\partial s_1} - \frac{\partial\kappa_{11}}{\partial s_2} + \kappa_{13}(\kappa_{11} - \kappa_{22})\right](\phi_{12} + \phi_{21}) + \left[\frac{\partial\kappa_{22}}{\partial s_1} + \frac{\partial\kappa_{12}}{\partial s_2} - \kappa_{23}(\kappa_{11} - \kappa_{22}) + \kappa_{13}\kappa_{21}\right]\phi_{22} + \kappa_{21}\left[\frac{1}{2}\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1} - \frac{\partial\phi_{22}}{\partial s_2} + \kappa_{13}\phi_{11}\right] \right\} = 0 \qquad \ldots\ldots \qquad \{5.2.-12.\}$$

From $Q_{23} = Q_{32}$

$$\left\{ \left[\frac{\partial\kappa_{21}}{\partial s_1} - \frac{\partial\kappa_{11}}{\partial s_2} + \kappa_{13}(\kappa_{11} - \kappa_{22}) + \kappa_{23}\kappa_{21}\right]\phi_{11} + \frac{1}{2}\left[\frac{\partial\kappa_{22}}{\partial s_1} + \frac{\partial\kappa_{12}}{\partial s_2} - \kappa_{23}(\kappa_{11} - \kappa_{22})\right](\phi_{12} + \phi_{21}) + \kappa_{21}\left[\frac{\partial\phi_{11}}{\partial s_1} - \frac{1}{2}\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2} + \kappa_{23}\phi_{22}\right] \right\} = 0 \qquad \ldots\ldots \qquad \{5.2.-13.\}$$

One further equation suggests itself, from the fact that the forms of {5.2.-2.} and {5.2.-6.} are similar (although these are not equal by symmetry). Consequently,

From $Q_{11} = Q_{22}$

$$(\kappa_{11} - \kappa_{22})(\phi_{12} + \phi_{21}) + \kappa_{12}(\phi_{11} - \phi_{22}) = 0 \qquad \ldots\ldots \qquad \{5.2.-14.\}$$

Equations {5.2.-12.} and {5.2.-13.}, through the use of {5.2.-14.} (as the primary substitution) and the application of previously-developed transformations, produce the set of equations:

$$\frac{\partial \kappa_{21}}{\partial s_1} - \frac{\partial \kappa_{11}}{\partial s_2} + \kappa_{13}(\kappa_{11} - \kappa_{22}) + \kappa_{23}(\kappa_{21} - \kappa_{12}) = 0 \quad ..... \quad \{5.2.\text{-}15.\}$$

$$\frac{\partial \kappa_{22}}{\partial s_1} + \frac{\partial \kappa_{12}}{\partial s_2} - \kappa_{23}(\kappa_{11} - \kappa_{22}) + \kappa_{13}(\kappa_{21} - \kappa_{12}) = 0 \quad ..... \quad \{5.2.\text{-}16.\}$$

It is then observed that equations {5.2.-15.} and {5.2.-16.} are identical with equations {3.2.-11.} and {3.2.-10.} (respectively), which are the MAINARDI-CODAZZI Equations for the undeformed surface. Thus, *the* SAINT-VENANT *compatibility equations contain the* MAINARDI-CODAZZI *equations for the undeformed surface implicitly,* as the transformation identities requisite to comply with the symmetry condition of the tensor $\bar{\bar{Q}}$ (or its zero value).

In the case that the orthogonal parametric lines are coincident with the principal lines of curvature (i.e.: the geodesic torsions, κ_{12} and κ_{21}, vanish), then a cursory inspection of {5.2.-12.} and {5.2.-13.} shows that the appropriate MAINARDI-CODAZZI equations appear without further manipulation. For, in such a case, it is observed that the MAINARDI-CODAZZI equations are not coupled.

It is to be noted that the expression

$$\frac{\partial}{\partial \bar{r}} \times \bar{\bar{\varepsilon}} \times \frac{\partial}{\partial \bar{r}} = 0 \quad ..... \quad \{5.2.\text{-}17.\}$$

may be considered as an integrability condition. If, for example, an infinitesimal displacement, $\bar{u}$, is considered as*

$$d\bar{u} = d\bar{u}_o + d\bar{r}\cdot\bar{\bar{\varepsilon}} + d\bar{r}\cdot\bar{\bar{\phi}}$$

* See Appendix A for a discussion of the kinematic representation of deformation.

(where $d\overline{u}_o$ represents rigid-body translation, $\overline{\overline{\varepsilon}}$ denotes the strain tensor, and $\overline{\overline{\phi}}$ designates the rotation tensor) then the displacement $\overline{u}$ may be found as

$$\overline{u} = \overline{u}_o + \int_{\overline{r}_0}^{\overline{r}} d\overline{r}\cdot\overline{\overline{\varepsilon}} + \int_{\overline{r}_0}^{\overline{r}} d\overline{r}\cdot\overline{\overline{\phi}}$$

or

$$\overline{u} = \overline{u}_0 + \int_{\overline{r}_0}^{\overline{r}} \left(d\overline{r}\cdot\overline{\overline{\varepsilon}}\right) - \frac{1}{2}\int_{\overline{r}_0}^{\overline{r}} \left(d\overline{r} \times \frac{\partial \times \overline{u}}{\partial \overline{r}} \right)$$

Thus, $d\overline{r}\cdot\overline{\overline{\varepsilon}}$ must be an integrable differential form, and the requirement {5.2.-17.} specifies this.

5.3. A COMPARISON STUDY OF THE COMPATIBILITY EQUATIONS OBTAINED BY VARIOUS AUTHORS

The methods employed and the results obtained for compatibility equations by various authors will now be considered, with a view toward the extablishment of the position of the results of §5.1. and §5.2., relative to the "standard" works on the subject. The authors selected for purposes of comparison are: GOL'DENVEIZER, NOVOZHILOV, REISSNER and VLASOV. There is a multiplicity of authors who deal with the question of compatibility, but the four mentioned above are selected for the reason that they deal with this question at approximately the same (unsophisticated and detailed) level of discussion.

5.3.1. The Compatibility Equations of GOL'DENVEIZER

GOL'DENVEIZER, in 1953, produced a set of compatibility equations by applying the integrability condition to two separate vectors, $\overline{U}$ and $\overline{\Omega}$, where he termed the former, the "vector of elastic displacement" and the latter, the "vector of elastic rotation". $\overline{U}$, in the notation used in this work, is the displacement vector of Chapter 4, $\overline{u} = \overline{u}^{\circ} + \alpha_3\overline{e}_3$, while $\overline{\Omega}$ could be expressed as $[-\phi_{23}\overline{e}_1 + \phi_{13}\overline{e}_2 + \frac{1}{2}(\phi_{12} - \phi_{21})\overline{e}_3]$. Applying the integrability condition as a mathematical, rather than a physical criterion,

i.e.:
$$\frac{\partial}{\partial\beta}\frac{\partial\overline{U}}{\partial\alpha} - \frac{\partial}{\partial\alpha}\frac{\partial\overline{U}}{\partial\beta} = 0$$

and
$$\frac{\partial}{\partial\beta}\frac{\partial\overline{\Omega}}{\partial\alpha} - \frac{\partial}{\partial\alpha}\frac{\partial\overline{\Omega}}{\partial\beta} = 0$$

GOL'DENVEIZER then obtained six "equations of compatibility". He noted however, that only the first three of these ({5.3.1.-1.}, {5.3.1.-2.},{5.3.1.-3.}, below) are equations of compatibility, as the remaining three are identities. These six equations appear as:

$$\frac{\partial}{\partial\alpha}(B\chi_2) + \frac{\partial A}{\partial\beta}\tau^{(2)} - \frac{\partial}{\partial\beta}(A\tau^{(1)}) - \frac{\partial B}{\partial\alpha}\chi_1 + AB\left[\frac{\xi_2}{R_1'} + \frac{\zeta_1}{R_{12}}\right] = 0 \quad \ldots\ldots \quad \{5.3.1.-1.\}$$

$$\frac{\partial}{\partial\alpha}(B\tau^{(2)}) - \frac{\partial A}{\partial\beta}\chi_2 + \frac{\partial}{\partial\beta}(A\chi_1) - \frac{\partial B}{\partial\alpha}\tau^{(1)} - AB\left[\frac{\zeta_1}{R_2'} + \frac{\zeta_2}{R_{12}}\right] = 0 \quad \ldots \quad \{5.3.1.-2.\}$$

$$AB\left[\frac{\chi_2}{R_1'} + \frac{\chi_1}{R_2'} + \frac{\tau^{(1)} - \tau^{(2)}}{R_{12}}\right] - \frac{\partial}{\partial\alpha}(B\zeta_2) + \frac{\partial}{\partial\beta}(A\zeta_1) = 0 \quad \ldots\ldots \quad \{5.3.1.-3.\}$$

$$-\frac{\partial}{\partial\alpha}(B\omega^{(2)}) + \frac{\partial A}{\partial\beta}\varepsilon_2 - \frac{\partial}{\partial\beta}(A\varepsilon_1) + \frac{\partial B}{\partial\alpha}\omega^{(1)} + AB\zeta_1 = 0 \quad \ldots\ldots \quad \{5.3.1.-4.\}$$

$$\frac{\partial}{\partial\alpha}(B\varepsilon_2) + \frac{\partial A}{\partial\beta}\omega^{(2)} - \frac{\partial}{\partial\beta}(A\omega^{(1)}) - \frac{\partial B}{\partial\alpha}\varepsilon_1 + AB\zeta_2 = 0 \quad \ldots\ldots \quad \{5.3.1.-5.\}$$

$$\tau^{(2)} + \tau^{(1)} - \frac{\omega^{(2)}}{R_1'} - \frac{\omega^{(1)}}{R_2'} + \frac{\varepsilon_1 - \varepsilon_2}{R_{12}} = 0 \quad \ldots\ldots \quad \{5.3.1.-6.\}$$

If these six equations are transformed into the notation used in this work*, they appear respectively as:

$$\frac{\partial(\delta\kappa_{22})}{\partial s_1} + \frac{\partial(\delta\kappa_{12})}{\partial s_2} + \kappa_{13}[\delta\kappa_{21} - \delta\kappa_{12} + \kappa_{22}(\phi_{12} + \phi_{21})]$$
$$+ \kappa_{23}[\delta\kappa_{22} - \delta\kappa_{11} + \kappa_{12}(\phi_{12} + \phi_{21})] - (\phi_{12} + \phi_{21})\frac{\partial\kappa_{21}}{\partial s_1}$$
$$+ \kappa_{21}[\delta\kappa_{13} - \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}] - \kappa_{11}[\delta\kappa_{23} + \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2}] = 0 \quad \{5.3.1.-7.\}$$

$$\frac{\partial(\delta\kappa_{11})}{\partial s_2} - \frac{\partial(\delta\kappa_{21})}{\partial s_1} + \kappa_{13}[\delta\kappa_{22} - \delta\kappa_{11} + \kappa_{21}(\phi_{12} + \phi_{21})]$$
$$+\kappa_{23}[\delta\kappa_{12} - \delta\kappa_{21} - \kappa_{11}(\phi_{12} + \phi_{21})] + (\phi_{12} + \phi_{21})\frac{\partial\kappa_{12}}{\partial s_2}$$
$$+\kappa_{12}[\delta\kappa_{23} + \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2}] + \kappa_{22}[\delta\kappa_{13} - \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}] = 0 \quad \{5.3.1.-8.\}$$

$$\frac{\partial(\delta\kappa_{23})}{\partial s_1} - \frac{\partial(\delta\kappa_{13})}{\partial s_2} + \kappa_{13}\delta\kappa_{13} + \kappa_{11}\delta\kappa_{22} + \kappa_{22}\delta\kappa_{11}$$
$$+\kappa_{12}[(\kappa_{11} + \kappa_{22})(\phi_{12} + \phi_{21}) + \delta\kappa_{21} - \delta\kappa_{12}] + \frac{\partial^2(\phi_{12} + \phi_{21})}{\partial s_1 \partial s_2}$$
$$+ \kappa_{23}[\delta\kappa_{23} + \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2}] = 0 \quad \ldots\ldots \quad \{5.3.1.-9.\}$$

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* See Appendix B for notation transformations.

$$\kappa_{23}(\phi_{12} + \phi_{21}) + \kappa_{13}(\phi_{11} - \phi_{22}) + \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1} - \frac{\partial \phi_{11}}{\partial s_2} - \delta\kappa_{13} = 0 \quad \ldots\{5.3.1.-10.\}$$

$$\kappa_{13}(\phi_{12} + \phi_{21}) + \kappa_{23}(\phi_{22} - \phi_{11}) - \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2} + \frac{\partial \phi_{22}}{\partial s_1} - \delta\kappa_{23} = 0 \quad \ldots\{5.3.1.-11.\}$$

$$(\kappa_{11} - \kappa_{22})(\phi_{12} + \phi_{21}) - (\delta\kappa_{12} + \delta\kappa_{21}) + \kappa_{12}(\phi_{11} - \phi_{22}) = 0 \quad \ldots\{5.3.1.-12.\}$$

A comparison of the kinematic compatibility equations, as obtained in §5.1., with these equations of GOL'DENVEIZER shows the following correspondence.

1) Equation {5.1.-15.}, through the use of BONNET's Theorem {2.13.1.-1}, becomes identical with {5.3.1.-7.} above.

2) Equation {5.1.-16.}, through the use of MAINARDI-CODAZZI Equation {3.2.-10.}, becomes identical with {5.3.1.-8.}, above.

3) Equation {5.1.-17.}, through the use of the Integrability Condition {3.2.-4.} (operating on $\phi_{12} + \phi_{21}$), becomes idential with {5.3.1.-9.} above.

A comparison of the SAINT-VENANT compatibility equations, as obtained in §5.2., with these equations of GOL'DENVEIZER shows the following correspondence.

1) Equation {5.2.-4.} (from $Q_{13} = 0$), through the use of BONNET's Theorem {2.13.1.-1.} and transformation identities {5.3.1.-12.} and {5.2.-14.}, becomes identical with {5.3.1.-7.} above.

2) Equation {5.2.-8.} (from $Q_{31} = 0$), through the use of BONNET's Theorem {2.13.1.-1.}, transformation identities {5.3.1.-12.} and {5.2.-14.}, and MAINARDI-CODAZZI Equation {3.2.-11.}, becomes identical with {5.3.1.-7.} above.

3) Equation {5.2.-7.} (from $Q_{23} = 0$), through the use of BONNET's Theorem {2.13.1.-1.} and transformation identities {5.2.-4.} and {5.3.1.-12.}, becomes identical with {5.3.1.-8.} above.

4) Equation {5.2.-9.} (from $Q_{32} = 0$), through the use of BONNET's Theorem {2.13.1.-1.}, transformation identities {5.3.1.-12.} and {5.2.-4.}, and MAINARDI-CODAZZI Equation {3.2.-10.}, becomes identical with {5.3.1.-8.} above.

5) Equation {5.2.-10.} (from $Q_{33} = 0$), through the use of transformation identities {5.3.1.-10.} and {5.3.1.-11.}, and the Integrability Condition {3.2.-4.} (operating on $\phi_{12} + \phi_{21}$), as well as BONNET's Theorem {2.13.1.-1.}, becomes identical to {5.3.1.-9.} above.

It is therefore concluded that the kinematic equations, the SAINT-VENANT equations and the GOL'DENVEIZER equations all represent different forms of the same Compatibility Equations for the Strained Middle Surface of a shell.

NOTE: GOL'DENVEIZER's equations of 1953 agree with his results of 1939, at which time he obtained equations of a different form by applying formal variational procedures to the MAINARDI-CODAZZI and GAUSS Equations.

5.3.2. The Compatibility Equations of NOVOZHILOV

NOVOZHILOV, in 1951, produced a set of compatibility equations by applying the integrability condition (as a purely mathematical criterion), separately, to the vectors, $\overline{R}$, $\overline{e}'_n$, $\overline{e}'_1$, and $\overline{e}'_2$. In the notation used in this work, these vectors would be written as $\overline{R}$, $\overline{E}_3$, $\overline{E}_1$

and $\overline{E}_2$, respectively. From the twelve scalar equations (some of which, are identities) which result from the operation of the integrability condition on the four vectors, he concluded that groups of terms in some equations were linear multiples of groups in other equations. Setting these groups equal by eliminating the linear multiples, he then obtained several identities (which are not given) and three equations shown below ({5.3.2.-1.}, {5.3.2.-2.} and {5.3.2.-3.}).

$$\frac{\partial}{\partial \alpha_2}(A_1 \kappa_1) - \kappa_2 \frac{\partial A_1}{\partial \alpha_2} - \frac{\partial (A_2 \tau)}{\partial \alpha_1} - \tau \frac{\partial A_2}{\partial \alpha_1} + \frac{\omega}{R_1} \frac{\partial A_2}{\partial \alpha_1} - \frac{1}{R_2}\left[\frac{\partial (A_1 \varepsilon_1)}{\partial \alpha_2} - \frac{\partial (A_2 \omega)}{\partial \alpha_1} - \varepsilon_2 \frac{\partial A_1}{\partial \alpha_2}\right] = 0 \qquad \ldots\ldots \quad \{5.3.2.\text{-}1.\}$$

$$\frac{\partial}{\partial \alpha_1}(A_2 \kappa_2) - \kappa_1 \frac{\partial A_2}{\partial \alpha_1} - \frac{\partial (A_1 \tau)}{\partial \alpha_2} - \tau \frac{\partial A_1}{\partial \alpha_2} + \frac{\omega}{R_2} \frac{\partial A_1}{\partial \alpha_2} - \frac{1}{R_1}\left[\frac{\partial (A_2 \varepsilon_2)}{\partial \alpha_1} - \varepsilon_1 \frac{\partial A_2}{\partial \alpha_1} - \frac{\partial (A_1 \omega)}{\partial \alpha_2}\right] = 0 \qquad \ldots\ldots \quad \{5.3.2.\text{-}2.\}$$

$$\frac{\kappa_1}{R_1} + \frac{\kappa_2}{R_2} + \frac{1}{A_1 A_2}\left\{ \frac{\partial}{\partial \alpha_1} \frac{1}{A_1}\left[A_2 \frac{\partial \varepsilon_2}{\partial \alpha_1} + \frac{\partial A_2}{\partial \alpha_1}(\varepsilon_2 - \varepsilon_1) - \frac{1}{2} A_1 \frac{\partial \omega}{\partial \alpha_2} - \frac{\partial A_1}{\partial \alpha_2}\omega\right] + \frac{\partial}{\partial \alpha_2}\frac{1}{A_2}\left[A_1 \frac{\partial \varepsilon_1}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2}(\varepsilon_1 - \varepsilon_2) - \frac{1}{2} A_2 \frac{\partial \omega}{\partial \alpha_1} - \frac{\partial A_2}{\partial \alpha_1}\omega\right]\right\} = 0 \qquad \ldots\ldots \quad \{5.3.2\text{-}3.\}$$

If these three equations are transformed into the notation employed in this work*, they appear respectively as:

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* See Appendix B for notation transformations.

$$\left\{ \frac{\partial(\delta\kappa_{11})}{\partial s_2} + \frac{\partial(\delta\kappa_{12})}{\partial s_1} + \kappa_{13}[\delta\kappa_{22} - \delta\kappa_{11}] + 2\kappa_{23}[\delta\kappa_{12} - \frac{1}{2}\kappa_{11}(\phi_{12} + \phi_{21})] \right.$$

$$- (\phi_{12} + \phi_{21}) \frac{\partial\kappa_{11}}{\partial s_1} - \kappa_{11} \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1} - \kappa_{22} \left[\frac{\partial\phi_{11}}{\partial s_2} \right.$$

$$\left. \left. + \kappa_{13}(\phi_{22} - \phi_{11}) - \kappa_{23}(\phi_{12} + \phi_{21}) - \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1} \right] \right\} = 0 \quad \{5.3.2.-4.\}$$

$$\left\{ \frac{\partial(\delta\kappa_{22})}{\partial s_1} + \frac{\partial(\delta\kappa_{12})}{\partial s_2} + \kappa_{23}[\delta\kappa_{22} - \delta\kappa_{11}] + 2\kappa_{13}[- \delta\kappa_{12} + \frac{1}{2}(\kappa_{11} - \kappa_{22}) \times \right.$$

$$\times (\phi_{12} + \phi_{21})] - (\phi_{12} + \phi_{21}) \frac{\partial\kappa_{11}}{\partial s_2} - \kappa_{11} \left[\kappa_{23}(\phi_{22} - \phi_{11}) \right.$$

$$\left. \left. + \kappa_{13}(\phi_{12} + \phi_{21}) + \frac{\partial\phi_{22}}{\partial s_1} \right] \right\} = 0 \quad \ldots\ldots \quad \{5.3.2.-5.\}$$

$$\left\{ \kappa_{22}\delta\kappa_{11} + \kappa_{11}\delta\kappa_{22} + \kappa_{23}\frac{\partial\phi_{22}}{\partial s_1} - \kappa_{13}\frac{\partial\phi_{11}}{\partial s_2} + \frac{\partial}{\partial s_1} \left[\frac{\partial\phi_{22}}{\partial s_1} + \kappa_{23}(\phi_{22} - \phi_{11}) \right. \right.$$

$$\left. - \frac{1}{2} \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2} + \kappa_{13}(\phi_{12} + \phi_{21}) \right] + \frac{\partial}{\partial s_2} \left[\frac{\partial\phi_{11}}{\partial s_2} \right.$$

$$\left. \left. - \kappa_{13}(\phi_{11} - \phi_{22}) - \frac{1}{2} \frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1} - \kappa_{23}(\phi_{12} + \phi_{21}) \right] \right\} = 0 \quad \{5.3.2.-6.\}$$

NOTE: The absence of any geodesic torsions, κ_{12} or κ_{21} , is easily explained by the fact the NOVOZHILOV *considers only the case that the orthogonal parametric lines are coincident with the lines of principal curvature.* The geodesic torsions vanish for such a case, as previously noted.

A comparison of the kinematic compatibility equations, as obtained in §5.1., with these equations of NOVOZHILOV shows the following correspondence (where κ_{12} and κ_{21} are set equal to zero in the kinematic system, so as to be comparable to NOVOZHILOV's system).

1) Equation {5.1.-18.}, through the use of transformation identities {5.3.1.-10.} and {5.3.1.-12.}, and MAINARDI-CODAZZI Equation {3.2.-14.}, becomes identical with {5.3.2.-5.} above.

2) Equation {5.1.-19.}, through the use of transformation identities {5.3.1.-11.} and {5.3.1.-12.}, and MAINARDI-CODAZZI Equation {3.2.-13.}, becomes identical with {5.3.2.-4.} above.

3) Equation {5.1.-20.}, through the use of transformation identities {5.3.1.-10.} and {5.3.1.-11.}, and GAUSS Equation {3.2.-15.}, becomes identical with {5.3.2.-6.} above.

A direct comparison of the SAINT-VENANT compatibility equations with these equations as obtained by NOVOZHILOV will not be undertaken, as it has been shown that the kinematic equations and the SAINT-VENANT equations differ only in form. Hence, as the kinematic equations agree with NOVOZHILOV's equations, so must the SAINT-VENANT equations.

5.3.3. The Compatibility Equations of REISSNER

REISSNER, in 1965, produced a set of compatibility equations by showing that the coefficients of four stress functions in his "work equation" must vanish. The four expressions so obtained are his compatibility equations, which appear as

$$\left[\kappa_{21} - \kappa_{12} + \frac{1}{R_{12}}(\varepsilon_{11} - \varepsilon_{22}) + \frac{\varepsilon_{12}}{R_{22}} - \frac{\varepsilon_{21}}{R_{11}}\right] = \frac{1}{\alpha_1\alpha_2}\left[(\alpha_1\gamma_1)_{,2} - (\alpha_2\gamma_2)_{,1}\right] \quad \ldots\ldots \quad \{5.3.3.-1.\}$$

$$\Bigg\{\left[\frac{1}{\alpha_2}\left((\alpha_1\varepsilon_{11})_{,2} - (\alpha_2\varepsilon_{21})_{,1} - \alpha_{1,2}\varepsilon_{22} - \alpha_{2,1}\varepsilon_{12}\right)\right]_{,2}$$

$$+\left[\frac{1}{\alpha_1}\left((\alpha_2\varepsilon_{22})_{,1} - (\alpha_2\varepsilon_{12})_{,2} - \alpha_{2,1}\varepsilon_{11} - \alpha_{1,2}\varepsilon_{21}\right)\right]_{,1}$$

$$+ \alpha_1\alpha_2\left[\frac{\kappa_{11}}{R_{22}} + \frac{\kappa_{22}}{R_{11}} - \frac{1}{R_{12}}(\kappa_{12} + \kappa_{21})\right]\Bigg\}$$

$$= \left[\frac{\alpha_1\gamma_2}{R_{11}} - \frac{\alpha_1\gamma_1}{R_{12}}\right]_{,2} + \left[\frac{\alpha_2\gamma_1}{R_{22}} - \frac{\alpha_2\gamma_2}{R_{12}}\right]_{,1} \quad \ldots\ldots \quad \{5.3.3.-2.\}$$

$$\Bigg\{\frac{1}{\alpha_1\alpha_2}\left[(\alpha_2\kappa_{22})_{,1} - (\alpha_1\kappa_{12})_{,2} - \alpha_{2,1}\kappa_{11} - \alpha_{1,2}\kappa_{21}\right]$$

$$- \frac{1}{\alpha_1\alpha_2}\left[(\alpha_2\varepsilon_{22})_{,1} - (\alpha_1\varepsilon_{12})_{,2} - \alpha_{2,1}\varepsilon_{11} - \alpha_{1,2}\varepsilon_{21}\right]\frac{1}{R_{11}}$$

$$- \frac{1}{\alpha_1\alpha_2}\left[(\alpha_1\varepsilon_{11})_{,1} - (\alpha_2\varepsilon_{21})_{,1} - \alpha_{1,2}\varepsilon_{22} - \alpha_{2,1}\varepsilon_{12}\right]\frac{1}{R_{12}}\Bigg\}$$

$$= \left[\frac{1}{R_{12}^2} - \frac{1}{R_{11}R_{22}}\right]\gamma_1 \quad \ldots\ldots \quad \{5.3.3.-3.\}$$

$$\Bigg\{\frac{1}{\alpha_1\alpha_2}\left[(\alpha_1\kappa_{11})_{,2} - (\alpha_2\kappa_{21})_{,1} - \alpha_{1,2}\kappa_{22} - \alpha_{2,1}\kappa_{12}\right]$$

$$- \frac{1}{\alpha_1\alpha_2}\left[(\alpha_1\varepsilon_{11})_{,2} - (\alpha_2\varepsilon_{21})_{,1} - \alpha_{1,2}\varepsilon_{22} - \alpha_{2,1}\varepsilon_{12}\right]\frac{1}{R_{22}}$$

$$- \frac{1}{\alpha_1\alpha_2}\left[(\alpha_2\varepsilon_{22})_{,1} - (\alpha_1\varepsilon_{12})_{,2} - \alpha_{2,1}\varepsilon_{11} - \alpha_{1,2}\varepsilon_{21}\right]\frac{1}{R_{12}}\Bigg\}$$

$$= \left[\frac{1}{R_{12}^2} - \frac{1}{R_{11}R_{22}} \right] \gamma_2 \qquad ..... \qquad \{5.3.3.-4.\}$$

If these four equations are transformed into the notation used in this work* and, as REISSNER requires (for comparisons), "make the assumption of no transverse shear deformation, that is, set $\gamma_1 = \gamma_2 = 0$ ...", then the equations appear respectively as:

$$\left[\delta\kappa_{21} + \delta\kappa_{12} + (\kappa_{22} - \kappa_{11})(\phi_{12} + \phi_{21}) + \kappa_{12}(\phi_{11} - \phi_{22})\right] = 0 .. \{5.3.3.-5.\}$$

$$\left\{ \frac{\partial}{\partial s_2}\left[\kappa_{13}(\phi_{22} - \phi_{11}) - \kappa_{23}(\phi_{12} + \phi_{21}) + \frac{\partial \phi_{11}}{\partial s_2} - \frac{1}{2}\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}\right] \right.$$

$$+ \frac{\partial}{\partial s_1}\left[\kappa_{13}(\phi_{12} + \phi_{21}) + \kappa_{23}(\phi_{22} - \phi_{11}) + \frac{\partial \phi_{22}}{\partial s_1} - \frac{1}{2}\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2}\right]$$

$$\left. + \kappa_{22}\delta\kappa_{11} + \kappa_{11}\delta\kappa_{22} + \kappa_{12}(\delta\kappa_{12} - \delta\kappa_{21}) \right\} = 0 \qquad ..... \qquad \{5.3.3.-6.\}$$

$$\left\{ \frac{\partial(\delta\kappa_{22})}{\partial s_1} + \frac{\partial(\delta\kappa_{12})}{\partial s_2} + \kappa_{13}[\delta\kappa_{21} - \delta\kappa_{12} + \frac{1}{2}(\kappa_{11} + \kappa_{22})(\phi_{12} + \phi_{21}) \right.$$

$$+ \kappa_{23}(\delta\kappa_{22} - \delta\kappa_{11}) - \frac{1}{2}(\phi_{12} + \phi_{21})\frac{\partial \kappa_{21}}{\partial s_1} - \kappa_{21}\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}$$

$$- \frac{1}{2}(\phi_{12} + \phi_{21})\frac{\partial \kappa_{11}}{\partial s_2} - \kappa_{11}[\kappa_{23}(\phi_{22} - \phi_{11}) + \kappa_{13}(\phi_{12} + \phi_{21}) + \frac{\partial \phi_{22}}{\partial s_1}]$$

$$\left. - \kappa_{12}[\kappa_{13}(\phi_{22} - \phi_{11}) - \kappa_{23}(\phi_{12} + \phi_{21}) + \frac{\partial \phi_{11}}{\partial s_2}] \right\} = 0 \qquad \{5.3.3.-7.\}$$

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* See Appendix B for notation transformations.

$$\left\{\frac{\partial(\delta\kappa_{11})}{\partial s_2} - \frac{\partial(\delta\kappa_{21})}{\partial s_1} - \kappa_{23}[\delta\kappa_{21} - \delta\kappa_{12} + \frac{1}{2}(\kappa_{22} + \kappa_{11})(\phi_{12} + \phi_{21}) + \kappa_{13}(\delta\kappa_{22} - \delta\kappa_{11}) + \frac{1}{2}(\phi_{12} + \phi_{21})\frac{\partial\kappa_{12}}{\partial s_2} + \kappa_{12}\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2} - \frac{1}{2}(\phi_{12} + \phi_{21})\frac{\partial\kappa_{22}}{\partial s_1} - \kappa_{22}[\kappa_{13}(\phi_{22} - \phi_{11}) - \kappa_{23}(\phi_{12} + \phi_{21}) + \frac{\partial\phi_{11}}{\partial s_2}] - \kappa_{12}[\kappa_{23}(\phi_{22} - \phi_{11}) + \kappa_{13}(\phi_{12} + \phi_{21}) + \frac{\partial\phi_{22}}{\partial s_1}]\right\} = 0 \qquad \{5.3.3.\text{-}8.\}$$

REISSNER concludes that this system of equations "differs from the system of three such equations as given by NOVOZHILOV .... [but] .... agrees with four equations which may be obtained from the six equations .... of GOL'DENVEIZER's text".

This would not appear to be the case, for a comparison of REISSNER's equations {5.3.3.-5.} to {5.3.3.-8.} with GOL'DENVEIZER's equations {5.3.1.-7.}, {5.3.1.-8.}, {5.3.1.-9.} and {5.3.1.-12.} (by means of MAINARDI-CODAZZI Equations {3.2.-10.} and {3.2.-11.}, and transformation identities {5.3.1.-10.} and {5.3.1.-11.}) shows that although the terms appear to be the same, there exist differences in the signs of the terms. For example, compare REISSNER's equation {5.3.3.-5.} to GOL'DENVEIZER's equation {5.3.1.-12.}. It is noted that these expressions are identical, save for the sign of the term $\kappa_{12}(\phi_{11} - \phi_{22})$; all attempts to transform the forms, in order to eliminate the differences, fail.

Furthermore, it is known that GOL'DENVEIZER's and NOVOZHILOV's results *do* agree, except for the fact that NOVOZHILOV considers the parametric coordinates to be coincident with the lines of principal curvature; i.e., the results of GOL'DENVEIZER simplify to those of NOVOZHILOV for $\kappa_{12} = 0 = \kappa_{21}$.

It is therefore concluded that the kinematic compatibility equations and the SAINT-VENANT compatibility equations do not agree with the equations as obtained by REISSNER. It is speculated that since these two former results agree with each other and with GOL'DENVEIZER's results, there may well be typographical errors in REISSNER's paper.

5.3.4. The Compatibility Equations of VLASOV

VLASOV, in 1949, obtained a set of compatibility equations by a method analogous to that as employed by NOVOZHILOV. That is, VLASOV "eliminates the displacement terms" from the expressions for the longitudinal strain and detrusion terms, by differentiation and grouping to obtain similar quantities. (His results are not shown here for that reason). Furthermore, as VLASOV considers (very thoroughly) only the case that the parametric lines are coincident with the lines of principal curvature*, his results and NOVOZHILOV's are therefore exactly equivalent.

VLASOV noted that his results did not agree with GOL'DENVEIZER's 1939 results, as "the quantities .... that we [VLASOV] have determined

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* In the supplement to the English edition of VLASOV's work, the author considers the equations of compatibility on the basis of the vanishing RIEMANN-CHRISTOFFEL Curvature Tensor for EUCLIDIAN space. Such a discussion is beyond the intended scope of this work.

have a different geometrical sense and differ from the analogous quantities derived by A. L. GOL'DENVEIZER ... ". However, if the transformation identities, as enumerated for NOVOZHILOV's equations, were invoked, then VLASOV's results could be shown to become identical with GOL'DENVEIZER's (for the case that the latter's equations are reduced by setting $\kappa_{12} = 0 = \kappa_{21}$, of course).

It is therefore concluded that the compatibility equations, as obtained by the kinematic approach, the SAINT-VENANT method, GOL'DENVEIZER, NOVOZHILOV and VLASOV all agree, even though the agreement must be obtained through a multiplicity of transformations in which all conceptual significance is destroyed. The equations of REISSNER evidently agree with none of the above-mentioned results, and attempts to rectify the situation fail -- leaving only the conclusion that REISSNER's paper must be plagued by typographical errors.

CHAPTER 6

The General Force and Moment Equilibrium Equations

6.1. THE FUNDAMENTAL SYSTEM

In order to obtain the criteria for equilibrium, a general form of CAUCHY's analysis is pursued, as follows.

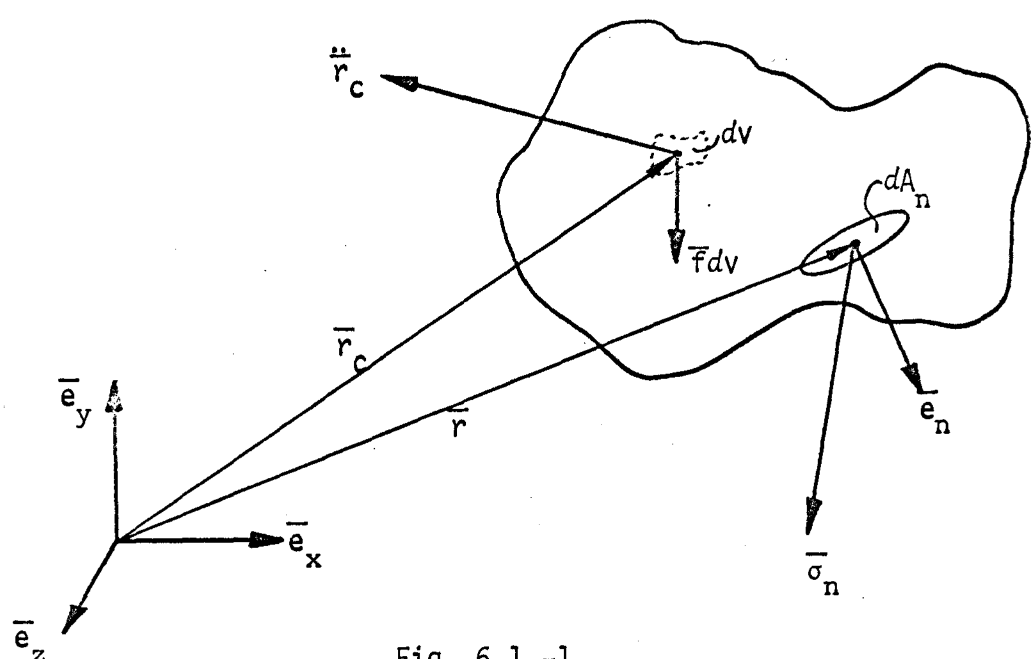

Fig. 6.1.-1.

With reference to Fig. 6.1.-1., it will be observed that the elemental section of the continuum is considered to be in a state of dynamic equilibrium if

$$- \int_m dm\ddot{\bar{r}}_c + \int_v \bar{f}dv + \int_{A_n} \bar{\sigma}_n \, dA_n = 0$$

This may be expressed, through the use of the transformation $dm = \rho dv$, as:

$$\int_V (\bar{f} - \rho\ddot{\bar{r}}_c)dv + \int_{A_n} \bar{\sigma}_n dA_n = 0 \qquad \ldots\ldots \qquad \{6.1.-1.\}$$

By virtue of CAUCHY's Relation,

VIZ: $\bar{e}_n \cdot \bar{\bar{\sigma}} = \bar{\sigma}_n = \sigma_{n1}\bar{e}_1 + \sigma_{n2}\bar{e}_2 + \sigma_{n3}\bar{e}_3$

(where $\bar{e}_n$ is any surface normal, not generally identical with $\bar{e}_3$)

then {6.1.-1.}may be given in the form

$$\int_V (\bar{f} - \rho\ddot{\bar{r}}_c)dv + \int_{A_n} \bar{e}_n \cdot \bar{\bar{\sigma}} dA_n = 0 \qquad \ldots\ldots \qquad \{6.1.-2.\}$$

where $\bar{f} = f_1\bar{e}_1 + f_2\bar{e}_2 + f_3\bar{e}_3$ represents the body force intensity,

$\rho = \frac{dm}{dv}$ denotes the mass density

$\ddot{\bar{r}}_c = \frac{d^2\bar{r}}{dt^2}$ designates the absolute acceleration of the centre of mass,

and $\bar{\bar{\sigma}} = [\bar{e}_1\bar{\sigma}_1 + \bar{e}_2\bar{\sigma}_2 + \bar{e}_3\bar{\sigma}_3] = \bar{e}_i\bar{\sigma}_i$ (sum on i = 1,2,3)

$$= \begin{bmatrix} \sigma_{11}\bar{e}_1\bar{e}_1 + \sigma_{12}\bar{e}_1\bar{e}_2 + \sigma_{13}\bar{e}_1\bar{e}_3 \\ +\sigma_{21}\bar{e}_2\bar{e}_1 + \sigma_{22}\bar{e}_2\bar{e}_2 + \sigma_{23}\bar{e}_2\bar{e}_3 \\ +\sigma_{31}\bar{e}_3\bar{e}_1 + \sigma_{32}\bar{e}_3\bar{e}_2 + \sigma_{33}\bar{e}_3\bar{e}_3 \end{bmatrix} = \sigma_{ij}e_ie_j$$

(sum on i,j=1,2,3)

represents the *stress tensor* which specifies the state of stress acting on the elemental section of the continuum under consideration (given arbitrarily in terms of the coordinates $\overline{e}_1$, $\overline{e}_2$, $\overline{e}_3$).

Equation {6.1.-2.} can be re-arranged somewhat, to give

$$\int_V (\overline{f} - \rho\ddot{\overline{r}}_c)dv + \int_{A_n} (\overline{e}_n dA_n)\cdot\overline{\overline{\sigma}} = 0$$

or, as $\overline{e}_n dA_n = d\overline{A}_n$

then

$$\int_V (\overline{f} - \rho\ddot{\overline{r}}_c)dv + \int_{A_n} d\overline{A}_n\cdot\overline{\overline{\sigma}} = 0 \qquad \ldots\ldots \qquad \{6.1.-3.\}$$

Employing the GAUSS Divergence Theorem, which in operator form, is given by (among other forms)

$$\int_{A_n} d\overline{A}_n\cdot(\quad) \equiv \int_V \frac{\partial}{\partial\overline{r}}\cdot(\quad)dv$$

then equation {6.1.-3.} becomes

$$\int_V (\overline{f} - \rho\ddot{\overline{r}}_c)dv + \int_V \frac{\partial\cdot\overline{\overline{\sigma}}}{\partial\overline{r}}\,dv = 0$$

or

$$\int_V (\overline{f} - \rho\ddot{\overline{r}}_c + \frac{\partial\cdot\overline{\overline{\sigma}}}{\partial\overline{r}})dv = 0 \qquad \ldots\ldots \qquad \{6.1.-4.\}$$

(This equation will be presently employed as the basis of the Force Equilibrium Equations.)

Returning to equation {6.1.-1.},

i.e., $$-\int_m dm\ddot{\overline{r}}_c + \int_V \overline{f}dv + \int_{A_n} \overline{\sigma}_n dA_n = 0$$

and employing EULER's Law of Dynamic Equilibrium, it may be said that (with reference to Fig. 6.1.-1.):

$$-\int_m \overline{r}_c \times dm\ddot{\overline{r}}_c + \int_V \overline{r}_c \times \overline{f}dv + \int_{A_n} \overline{r}_n \times \overline{\sigma}_n dA_n = 0$$

Introducing $dm = \rho dv$ as before, then

$$\int_V \overline{r}_c \times (\overline{f} - \rho\ddot{\overline{r}}_c)dv + \int_{A_n} \overline{r}_n \times \overline{\sigma}_n dA_n = 0$$

or $$\int_V \overline{r}_c \times (\overline{f} - \rho\ddot{\overline{r}}_c)dv - \int_{A_n} \overline{\sigma}_n \times \overline{r}_n \, dA_n = 0$$

which, as $\overline{e}_n \cdot \overline{\overline{\sigma}} = \overline{\sigma}_n$, becomes

$$\int_V \overline{r}_c \times (\overline{f} - \rho\ddot{\overline{r}}_c)dv - \int_{A_n} \overline{e}_n \cdot \overline{\overline{\sigma}} \times \overline{r}_n \, dA_n = 0$$

or $$\int_V \overline{r}_c \times (\overline{f} - \rho\ddot{\overline{r}}_c)dv - \int_{A_n} (\overline{e}_n dA_n) \cdot (\overline{\overline{\sigma}} \times \overline{r}_n) = 0$$

so $$\int_V \overline{r}_c \times (\overline{f} - \rho\ddot{\overline{r}}_c)dv - \int_{A_n} d\overline{A}_n \cdot (\overline{\overline{\sigma}} \times \overline{r}_n) = 0 \; ..... \qquad \{6.1.-5.\}$$

Once this form ({6.1.-5.}) has been obtained, the GAUSS Divergence Theorem becomes applicable and equation {6.1.-5.} becomes

$$\int_V \overline{r}_c \times (\overline{f} - \rho\ddot{\overline{r}}_c)dv - \int_V \frac{\partial}{\partial\overline{r}} \cdot (\overline{\overline{\sigma}} \times \overline{r})dv = 0$$

or $$\int_V [\overline{r}_c \times (\overline{f} - \rho\ddot{\overline{r}}_c) - \frac{\partial}{\partial\overline{r}} \cdot (\overline{\overline{\sigma}} \times \overline{r})]dv = 0 \quad \ldots\ldots \quad \{6.1.-6.\}$$

(This equation forms the basis of the Moment Equilibrium Equation.)

Now having obtained equations {6.1.-4.} and {6.1.-6.}, the specification is made that the problem to be considered, will be a *static* problem. This causes {6.1.-4.} to reduce to

$$\int_V (\overline{f} + \frac{\partial\cdot\overline{\overline{\sigma}}}{\partial\overline{r}})dv = 0 \quad \ldots\ldots \quad \{6.1.-7.\}$$

and {6.1.-6.} to reduce to

$$\int_V [\overline{r}_c \times \overline{f} - \frac{\partial}{\partial\overline{r}} \cdot (\overline{\overline{\sigma}} \times \overline{r})]dv = 0 \quad \ldots\ldots \quad \{6.1.-8.\}$$

Finally, for "engineering" problems, the body force (self-weight) will be neglected for the present, and considered as an applied boundary load, after a solution has been obtained. This forces a further reduction of {6.1.-7.} and {6.1.-8.}, respectively, to:

$$\int_V \frac{\partial\cdot\overline{\overline{\sigma}}}{\partial\overline{r}} \; dv = 0 \quad \ldots\ldots \quad \{6.1.-9.\}$$

and $$\int_V - \frac{\partial}{\partial\overline{r}} \cdot (\overline{\overline{\sigma}} \times \overline{r})dv = 0 \quad \ldots\ldots \quad \{6.1.-10.\}$$

Thus, equation {6.1.-9.}, above, prescribes the conditions necessarily existing in the continuum for the equilibrium of stress resultants and will therefore yield the "Force Equilibrium Equations". Equation {6.1.-10.}, above, prescribes the conditions necessarily existing in the continuum for the equilibrium of the stress couples about some (*arbitrary*) point and therefore will supply the "Moment Equilibrium Equations".

6.2. THE FORCE EQUILIBRIUM EQUATIONS

A segment of the shell with boundaries α_1, $(\alpha_1 + ds_2)$, α_2, $(\alpha_2 + ds_1)$, $\alpha_3 = \frac{h}{2}$, $\alpha_3 = -\frac{h}{2}$ is now considered, as in Fig. 6.2.-1. Since the volume integration has only one integral, the limits of which are definite (the integration over the thickness, h, in the α_3-direction), then equation {6.1.-9.} may be given in the following convenient form.

$$\int_{s_1^*}\int_{s_2^*}\int_{\alpha_3=-h/2}^{\alpha_3 = h/2} \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{r}} \; d\alpha_3 ds_2^* ds_1^* = 0 \qquad \ldots\ldots \qquad \{6.2.-1.\}$$

It is to be noted that the shell segment itself (Fig. 6.2.-1.) is infinitesimal in two dimensions and small but finite in the third; the element of this segment which is under consideration is, of course, infinitesimal in all three dimensions.

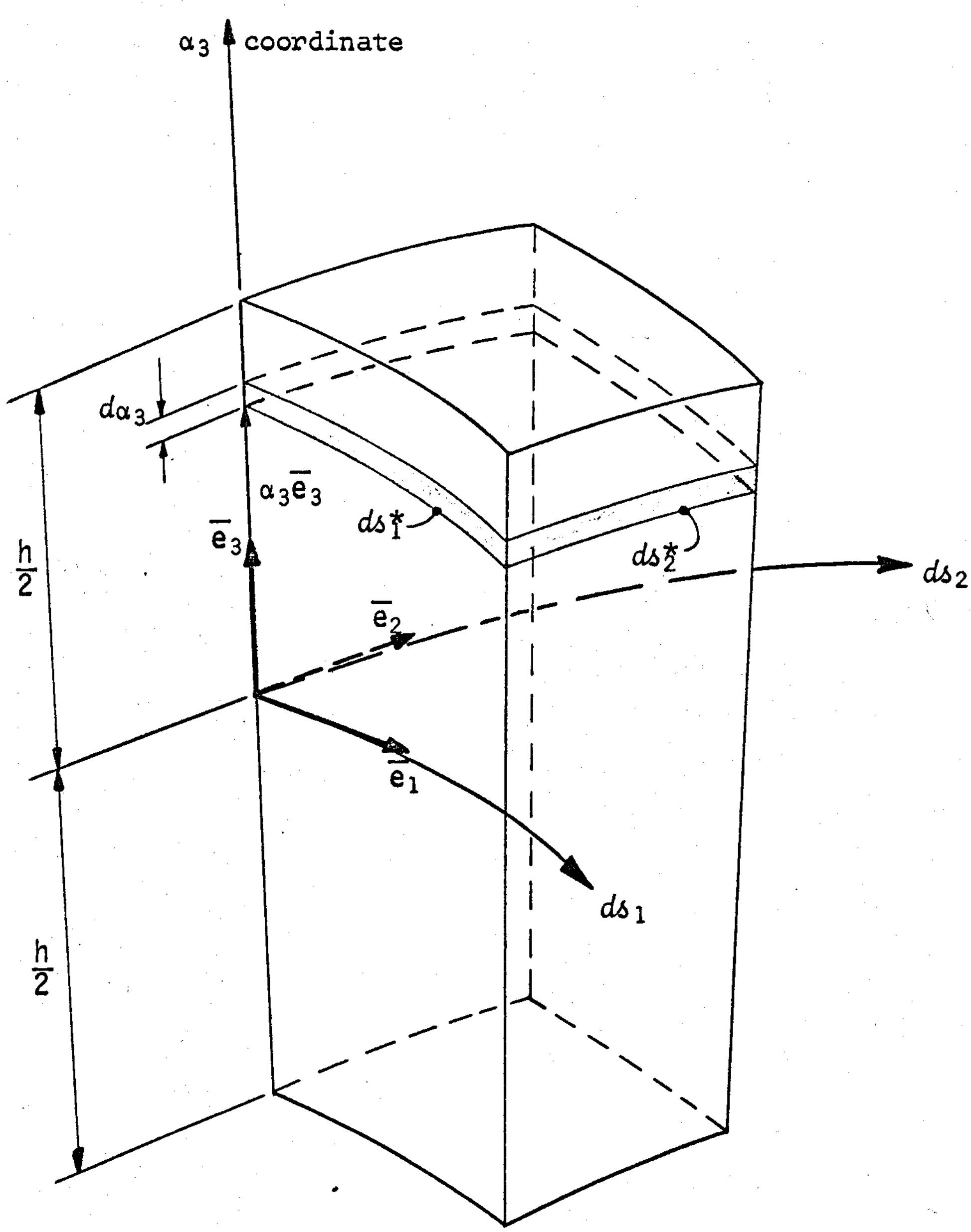

Fig. 6.2.-1.

From previous investigations of the local geometry of the shell (§ 4.6.) it is known that

$$\left.\begin{aligned} ds_1^* &= (1 + \alpha_3\kappa_{11})ds_1 \\ ds_2^* &= (1 + \alpha_3\kappa_{22})ds_2 \end{aligned}\right\} (\{4.6.-7.\})$$

Equation {6.2.-1.}, upon substitution of {4.6.-7.} then becomes

$$\int_{s_1}\int_{s_2}\int_{\alpha_3=-h/2}^{\alpha_3=h/2} \frac{\partial\cdot\bar{\bar{\sigma}}}{\partial\bar{r}} (1 + \alpha_3\kappa_{11})(1+ \alpha_3\kappa_{22})d\alpha_3 ds_2 ds_1 = 0 \qquad \{6.2.-2.\}$$

And, as stated above, since this integration takes place over indefinite limits of "s_1" and "s_2" -- which are not functions of α_3-- then the integrand of the "area integral"

VIZ: $\int_{s_1}\int_{s_2} (. . . .)ds_2 ds_1 = 0$

must vanish separately, in order that {6.2.-2.} be satisfied.

This requires that

$$\int_{\alpha_3=-h/2}^{\alpha_3=h/2} \frac{\partial\cdot\bar{\bar{\sigma}}}{\partial\bar{r}} (1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})d\alpha_3 = 0 \qquad \{6.2.-3.\}$$

This will be written, in the following discussion as

$$\int_{\alpha_3} \frac{\partial\cdot\bar{\bar{\sigma}}}{\partial\bar{r}} (1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})d\alpha_3 = 0 \qquad \{6.2.-4.\}$$

the limits of the integration being understood to be $\alpha_3 = - h/2$ to $\alpha_3 = + h/2$.

Having previously established (§ 4.6.) that for surfaces other than the middle surface, the directed derivative is given as

$$\frac{\partial}{\partial \bar{r}} \equiv \frac{\bar{e}_1}{1+\alpha_3\kappa_{11}}\frac{\partial}{\partial s_1} + \frac{\bar{e}_2}{1+\alpha_3\kappa_{22}}\frac{\partial}{\partial s_2} + \bar{e}_3\frac{\partial}{\partial \alpha_3} \qquad (\{4.6.-6.\})$$

then with the aid of this expression, {6.2.-4.} appears as

$$\int_{\alpha_3} \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{r}} \pi_1\pi_2 \, d\alpha_3 = \int_{\alpha_3}\Big(\pi_2\bar{e}_1 \cdot \frac{\partial}{\partial s_1}\left[\bar{e}_1\bar{\sigma}_1 + \bar{e}_2\bar{\sigma}_2 + \bar{e}_3\bar{\sigma}_3\right]$$
$$+ \pi_1\bar{e}_2 \cdot \frac{\partial}{\partial s_2}\left[\bar{e}_1\bar{\sigma}_1 + \bar{e}_2\bar{\sigma}_2 + \bar{e}_3\bar{\sigma}_3\right]$$
$$+ \pi_1\pi_2\bar{e}_3 \cdot \frac{\partial}{\partial \alpha_3}\left[\bar{e}_1\bar{\sigma}_1 + \bar{e}_2\bar{\sigma}_2 + \bar{e}_3\bar{\sigma}_3\right]\Big) d\alpha_3 = 0 \cdots\cdots \{6.2.-5.\}$$

where the short-form notation

$$\pi_1 = (1 + \alpha_3\kappa_{11}), \quad \pi_2 = (1 + \alpha_3\kappa_{22})$$

has been employed for convenience.

The CESÀRO-BURALI-FORTI Vectors will be required for the vector differentiations in the expansion of {6.2.-5.}:

(recall) $\frac{\partial \bar{e}_\beta}{\partial s_r} = \bar{C}_r \times \bar{e}_\beta$ (no sum. $r = 1,2$; $\beta = 1,2,3$)

where
$$\bar{C}_1 = \kappa_{12}\,\bar{e}_1 + \kappa_{11}\bar{e}_2 + \kappa_{13}\bar{e}_3$$
$$\bar{C}_2 = -\kappa_{22}\bar{e}_1 + \kappa_{21}\bar{e}_2 + \kappa_{23}\bar{e}_3$$

for the case of orthogonal parametric lines.

NOTE: Obviously, an expression such as $\frac{\partial \bar{e}_\beta}{\partial \alpha_3} = 0$, as α_3 is a straight-line coordinate, and its DARBOUX Vector (§ 1.7.) is consequently zero.

Expanding {6.2.-5.} with the aid of the CESÀRO-BURALI-FORTI Vectors then yields

$$\int_{\alpha_3} \frac{\partial \cdot \bar{\bar{\sigma}}}{\partial \bar{r}} \pi_1 \pi_2 d\alpha_3 = \int_{\alpha_3} \left(\left(\pi_2 \left[\frac{\partial \bar{\sigma}_1}{\partial s_1} - \kappa_{13}\bar{\sigma}_2 + \kappa_{11}\bar{\sigma}_3 \right] + \pi_1 \left[\kappa_{23}\bar{\sigma}_1 + \frac{\partial \bar{\sigma}_2}{\partial s_2} + \kappa_{22}\bar{\sigma}_3 \right] + \pi_1\pi_2 \left[\frac{\partial \bar{\sigma}_3}{\partial \alpha_3} \right] \right) \right) d\alpha_3 = 0$$

Further expansion into the component form of the tensor $\bar{\bar{\sigma}}$ is accomplished by expanding $\bar{\sigma}_\beta$ as

$$\bar{\sigma}_\beta = \sigma_{\beta\gamma}\, \bar{e}_\gamma \quad \text{(sum on } \gamma = 1,2,3)$$

and performing the requisite differentiations. Grouping terms as coefficients of the vector directions, $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$, yields the equation

$$\begin{aligned}\int_{\alpha_3} \Bigg\{ & \Bigg[\frac{\partial \sigma_{11}}{\partial s_1} \pi_2 - \kappa_{13}(\sigma_{12} - \sigma_{21})\, \pi_2 + \kappa_{11}(\sigma_{13} + \sigma_{31})\pi_2 + \kappa_{23}(\sigma_{11} - \sigma_{22})\, \pi_1 \\ & \quad + \kappa_{22}\sigma_{31}\pi_1 + \kappa_{21}\sigma_{32}\pi_1 + \frac{\partial \sigma_{21}}{\partial s_2} \pi_1 + \frac{\partial \sigma_{31}}{\partial \alpha_3} \pi_1\pi_2 \Bigg] \bar{e}_1 \\ & + \Bigg[\kappa_{13}(\sigma_{11} - \sigma_{22})\pi_2 + \kappa_{11}\sigma_{32}\pi_2 + \frac{\partial \sigma_{12}}{\partial s_1} \pi_2 + \kappa_{23}(\sigma_{12} + \sigma_{21})\pi_1 \\ & \quad + \kappa_{22}(\sigma_{23} + \sigma_{32})\pi_1 - \kappa_{12}\sigma_{13}\pi_2 + \frac{\partial \sigma_{22}}{\partial s_2} \pi_1 + \frac{\partial \sigma_{32}}{\partial \alpha_3} \pi_1\pi_2 \Bigg] \bar{e}_2 \\ & + \Bigg[\kappa_{11}(\sigma_{33} - \sigma_{11})\pi_2 + \kappa_{22}(\sigma_{33} - \sigma_{22})\pi_1 - \kappa_{13}\sigma_{23}\pi_2 + \kappa_{23}\sigma_{13}\pi_1 \\ & \quad + \kappa_{12}\sigma_{12}\pi_2 - \kappa_{21}\sigma_{21}\pi_1 + \frac{\partial \sigma_{13}}{\partial s_1} \pi_2 + \frac{\partial \sigma_{23}}{\partial s_2} \pi_1 + \frac{\partial \sigma_{33}}{\partial \alpha_3} \pi_1\pi_2 \Bigg] \bar{e}_3 \Bigg\} d\alpha_3 = 0\end{aligned}$$

..... {6.2.-6.}

Now, as the vectors themselves are not functions of α_3, and the integration is distributive to each term; and since all three vector directions are unique, then {6.2.-6.} reveals three scalar expressions (the coefficients of the vectors) which must each vanish separately, after integration over α_3.

The integration of these coefficients is straightforward, except for those terms which contain derivatives of the stresses. Consider, for example, the integration of a term such as

$$\int_{\alpha_3} \frac{\partial\sigma_{11}}{\partial s_1}\pi_2 d\alpha_3 = \int_{\alpha_3} \frac{\partial\sigma_{11}}{\partial s_1}(1 + \alpha_3\kappa_{22})d\alpha_3$$

This expands to

$$\int_{\alpha_3} \frac{\partial\sigma_{11}}{\partial s_1}\pi_2 d\alpha_3 = \int_{\alpha_3} \frac{\partial}{\partial s_1}[\sigma_{11}(1 + \alpha_3\kappa_{22})]d\alpha_3 - \int_{\alpha_3} \sigma_{11}\frac{\partial(1 + \alpha_3\kappa_{22})}{\partial s_1}d\alpha_3$$

$$= \frac{\partial}{\partial s_1}\int_{\alpha_3} \sigma_{11}(1 + \alpha_3\kappa_{22})d\alpha_3 - \int_{\alpha_3} \sigma_{11}\alpha_3\frac{\partial\kappa_{22}}{\partial s_1}d\alpha_3$$

$$= \frac{\partial}{\partial s_1}\int_{\alpha_3} \sigma_{11}(1 + \alpha_3\kappa_{22}) - \frac{\partial\kappa_{22}}{\partial s_1}\int_{\alpha_3} \sigma_{11}\alpha_3 d\alpha_3$$

[as $\kappa_{22} \neq \kappa_{22}(\alpha_3)$]

Three equations thus result from {6.2.-6.}, and are given below

<u>From the $\bar{e}_1$ - direction</u>

$$\frac{\partial}{\partial s_1}\int_{\alpha_3} \sigma_{11}(1 + \alpha_3\kappa_{22})d\alpha_3 - \frac{\partial\kappa_{22}}{\partial s_1}\int_{\alpha_3} \sigma_{11}\alpha_3 d\alpha_3 + \kappa_{11}\int_{\alpha_3} \sigma_{13}(1 + \alpha_3\kappa_{22})d\alpha_3$$
$$+ \kappa_{23}\int_{\alpha_3} \sigma_{11}(1 + \alpha_3\kappa_{11})d\alpha_3 - \kappa_{23}\int \sigma_{22}(1 + \alpha_3\kappa_{11})d\alpha_3$$
$$+ \kappa_{21}\int_{\alpha_3} \sigma_{32}(1 + \alpha_3\kappa_{11})d\alpha_3 + \frac{\partial}{\partial s_2}\int_{\alpha_3} \sigma_{21}(1 + \alpha_3\kappa_{11})d\alpha_3 - \frac{\partial\kappa_{11}}{\partial s_2}\int_{\alpha_3} \sigma_{21}\alpha_3 d\alpha_3$$
$$+ \frac{\partial}{\partial\alpha_3}\int_{\alpha_3} \sigma_{31}(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})d\alpha_3 = 0 \quad \ldots\ldots \quad \{6.2.-7.\}$$

From the $\bar{e}_2$-direction

$$\frac{\partial}{\partial s_1}\int_{\alpha_3} \sigma_{12}(1+\alpha_3\kappa_{22})d\alpha_3 - \frac{\partial\kappa_{22}}{\partial s_1}\int_{\alpha_3} \sigma_{12}\alpha_3 d\alpha_3 + \kappa_{13}\int_{\alpha_3} \sigma_{11}(1+\alpha_3\kappa_{22})d\alpha_3$$
$$-\kappa_{13}\int_{\alpha_3} \sigma_{22}(1+\alpha_3\kappa_{22})d\alpha_3 + 2\kappa_{23}\int_{\alpha_3} \sigma_{21}(1+\alpha_3\kappa_{11})d\alpha_3$$
$$+\kappa_{22}\int_{\alpha_3} \sigma_{23}(1+\alpha_3\kappa_{11})d\alpha_3 - \kappa_{12}\int \sigma_{13}(1+\alpha_3\kappa_{22})d\alpha_3$$
$$+\frac{\partial}{\partial s_2}\int_{\alpha_3} \sigma_{22}(1+\alpha_3\kappa_{11})d\alpha_3 - \frac{\partial\kappa_{11}}{\partial s_2}\int_{\alpha_3} \sigma_{22}\alpha_3 d\alpha_3$$
$$+\frac{\partial}{\partial\alpha_3}\int_{\alpha_3} \sigma_{32}(1+\alpha_3\kappa_{11})(1+\alpha_3\kappa_{22})d\alpha_3 = 0 \quad \ldots\ldots \quad \{6.2.-8.\}$$

From the $\bar{e}_3$-direction

$$-\kappa_{11}\int_{\alpha_3} \sigma_{11}(1+\alpha_3\kappa_{22})d\alpha_3 - \kappa_{22}\int_{\alpha_3} \sigma_{22}(1+\alpha_3\kappa_{11})d\alpha_3 - \kappa_{13}\int_{\alpha_3} \sigma_{32}(1+\alpha_3\kappa_{22})d\alpha_3$$
$$+\kappa_{23}\int_{\alpha_3} \sigma_{31}(1+\alpha_3\kappa_{11})d\alpha_3 + \kappa_{12}\int \sigma_{12}(1+\alpha_3\kappa_{22})d\alpha_3 - \kappa_{21}\int_{\alpha_3} \sigma_{21}(1+\alpha_3\kappa_{22})d\alpha_3$$
$$+\frac{\partial}{\partial s_1}\int_{\alpha_3} \sigma_{13}(1+\alpha_3\kappa_{22})d\alpha_3 - \frac{\partial\kappa_{22}}{\partial s_1}\int_{\alpha_3} \sigma_{13}\alpha_3 d\alpha_3$$
$$+\frac{\partial}{\partial s_2}\int_{\alpha_3} \sigma_{23}(1+\alpha_3\kappa_{11})d\alpha_3 - \frac{\partial\kappa_{11}}{\partial s_2}\int_{\alpha_3} \sigma_{23}\alpha_3 d\alpha_3$$
$$+\frac{\partial}{\partial\alpha_3}\int_{\alpha_3} \sigma_{33}(1+\alpha_3\kappa_{11})(1+\alpha_3\kappa_{22})d\alpha_3 = 0 \quad \ldots\ldots \quad \{6.2.-9.\}$$

The use of the MAINARDI-CODAZZI equations ({3.2.-10.}, {3.2.-11.}) permits these three equations above to be written in a more convenient form, as follows.

$$\left\{\frac{\partial}{\partial s_1}\int_{\alpha_3} \sigma_{11}(1+\alpha_3\kappa_{22})d\alpha_3 + \frac{\partial}{\partial s_2}\int_{\alpha_3} \sigma_{21}(1+\alpha_3\kappa_{11})d\alpha_3 + \kappa_{11}\int_{\alpha_3} \sigma_{13}(1+\alpha_3\kappa_{22})d\alpha_3\right.$$

$$- \kappa_{23}\int_{\alpha_3} \sigma_{22}(1 + \alpha_3\kappa_{11})d\alpha_3 + \kappa_{21}\int_{\alpha_3} \sigma_{23}(1+ \alpha_3\kappa_{11})d\alpha_3 + \kappa_{23}\int_{\alpha_3} \sigma_{11}(1 + \alpha_3\kappa_{22})d\alpha_3$$

$$- \kappa_{13}\int_{\alpha_3} \sigma_{12}(1 + \alpha_3\kappa_{22})d\alpha_3 - \kappa_{13}\int_{\alpha_3} \sigma_{21}(1+ \alpha_3\kappa_{11})d\alpha_3 + \left(\frac{\partial\kappa_{12}}{\partial s_2} + 2\kappa_{21}\kappa_{13}\right)\int_{\alpha_3} \sigma_{11}\alpha_3 d\alpha_3$$

$$\left. - \left(\frac{\partial\kappa_{21}}{\partial s_1} + 2\kappa_{21}\kappa_{23}\right)\int_{\alpha_3} \sigma_{12}\alpha_3 d\alpha_3 + \sigma_{31}(1+ \alpha_3\kappa_{11})(1+ \alpha_3\kappa_{22})\Big|_{\alpha_3}\right\} = 0 \quad \{6.2.-10.\}$$

$$\left\{\frac{\partial}{\partial s_1}\int_{\alpha_3} \sigma_{12}(1 + \alpha_3\kappa_{22})d\alpha_3 + \frac{\partial}{\partial s_2}\int_{\alpha_3} \sigma_{22}(1 + \alpha_3\kappa_{11})d\alpha_3 + \kappa_{13}\int_{\alpha_3} \sigma_{11}(1+\alpha_3\kappa_{22})d\alpha_3\right.$$

$$+ \kappa_{23}\int_{\alpha_3} \sigma_{21}(1 + \alpha_3\kappa_{11})d\alpha_3 + \kappa_{22}\int_{\alpha_3} \sigma_{23}(1 + \alpha_3\kappa_{11})d\alpha_3 - \kappa_{12}\int_{\alpha_3} \sigma_{13}(1+\alpha_3\kappa_{22})d\alpha_3$$

$$- \kappa_{13}\int_{\alpha_3} \sigma_{22}(1 + \alpha_3\kappa_{11})d\alpha_3 + \kappa_{23}\int_{\alpha_3} \sigma_{12}(1 + \alpha_3\kappa_{22})d\alpha_3$$

$$+ \left(\frac{\partial\kappa_{12}}{\partial s_2} + 2\kappa_{21}\kappa_{13}\right)\int_{\alpha_3} \sigma_{12}\alpha_3 d\alpha_3 - \left(\frac{\partial\kappa_{21}}{\partial s_1} + 2\kappa_{21}\kappa_{23}\right)\int_{\alpha_3} \sigma_{22}\alpha_3 d\alpha_3$$

$$\left. + \sigma_{32}(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})\Big|_{\alpha_3}\right\} = 0 \quad \ldots\ldots \quad \{6.2.-11.\}$$

$$\left\{\frac{\partial}{\partial s_1}\int_{\alpha_3} \sigma_{13}(1 + \alpha_3\kappa_{22})d\alpha_3 + \frac{\partial}{\partial s_2}\int_{\alpha_3} \sigma_{23}(1 + \alpha_3\kappa_{11})d\alpha_3 - \kappa_{11}\int_{\alpha_3} \sigma_{11}(1 + \alpha_3\kappa_{22})d\alpha_3\right.$$

$$- \kappa_{22}\int_{\alpha_3} \sigma_{22}(1 + \alpha_3\kappa_{11})d\alpha_3 + \kappa_{12}\int_{\alpha_3} \sigma_{12}(1 + \alpha_3\kappa_{22})d\alpha_3 - \kappa_{21}\int_{\alpha_3} \sigma_{21}(1 + \alpha_3\kappa_{11})d\alpha_3$$

$$+ \kappa_{23}\int_{\alpha_3} \sigma_{13}(1 + \alpha_3\kappa_{22})d\alpha_3 - \kappa_{13}\int_{\alpha_3} \sigma_{23}(1 + \alpha_3\kappa_{11})d\alpha_3$$

$$+ \left(\frac{\partial\kappa_{12}}{\partial s_2} + 2\kappa_{21}\kappa_{13}\right)\int_{\alpha_3} \sigma_{13}\alpha_3 d\alpha_3 - \left(\frac{\partial\kappa_{21}}{\partial s_1} + 2\kappa_{21}\kappa_{23}\right)\int_{\alpha_3} \sigma_{23}\alpha_3 d\alpha_3$$

$$\left. + \sigma_{33}(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})\Big|_{\alpha_3}\right\} = 0 \quad \ldots\ldots \quad \{6.2.-12.\}$$

These three equations, {6.2.-10.}, {6.2.-11.}, {6.2.-12.} above, are the equations of "Force Equilibrium" for any general shell.

Now, the quantities which appear in these equations admit physical interpretation, for the most part. With reference to Fig. 6.1.-1., it will be observed that for a unit width of section at the middle surface (as considered), then a term such as

$$\int_{\alpha_3} \sigma_{11}(1 + \alpha_3\kappa_{22})d\alpha_3$$

represents the integration of the stress σ_{11} over the elemental area of the section. Hence, this integral represents the *stress resultant* for the stress σ_{11}, taken over the cross-section. Referring to this integral as $F_{11}(\sigma)$, then the remainder of the quantities follows, to give the result:

$$\left.\begin{aligned}
F_{11}(\sigma) &= \int_{\alpha_3} \sigma_{11}(1 + \alpha_3\kappa_{22})d\alpha_3 \\
F_{12}(\sigma) &= \int_{\alpha_3} \sigma_{12}(1 + \alpha_3\kappa_{22})d\alpha_3 \\
F_{13}(\sigma) &= \int_{\alpha_3} \sigma_{13}(1 + \alpha_3\kappa_{22})d\alpha_3 \\
F_{21}(\sigma) &= \int_{\alpha_3} \sigma_{21}(1 + \alpha_3\kappa_{11})d\alpha_3 \\
F_{22}(\sigma) &= \int_{\alpha_3} \sigma_{22}(1 + \alpha_3\kappa_{11})d\alpha_3 \\
F_{23}(\sigma) &= \int_{\alpha_3} \sigma_{23}(1 + \alpha_3\kappa_{11})d\alpha_3
\end{aligned}\right\} \quad \ldots\ldots \quad \{6.2.-13.\}$$

Furthermore, there is a physical interpretation for terms such as

$$\sigma_{3i}\,(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})\Big|_{\alpha_3} \equiv \sigma_{3i}\,(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})\Big|_{-h/2}^{+h/2}$$

Such terms are seen to be the *boundary forces* acting on the shell. This is easily seen by the interpretation of σ_{3i} itself -- the stress on the surface of the element, the normal to which is $\overline{e}_3$, and acting in the i-direction. Then, as the quantity $[(1 + \alpha_3\kappa_{11}) \times (1 + \alpha_3\kappa_{22})]$ represents the surface areas (when $\alpha_3 = h/2$, $\alpha_3 = -h/2$ are inserted as limits), the entire quantity $\sigma_{3i}\,(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})\Big|_{\alpha_3}$ becomes the "algebraic sum of the boundary forces", or the *net boundary forces*. Referring to such terms as P_i, then

$$\begin{aligned} P_1 &= \sigma_{31}(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})\Big|_{\alpha_3} \\ P_2 &= \sigma_{32}(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})\Big|_{\alpha_3} \\ P_3 &= \sigma_{33}(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})\Big|_{\alpha_3} \end{aligned} \qquad \ldots.\{6.2.\text{-}14.\}$$

Hence, {6.2.-13.} and {6.2.-14.} allow the equations of Force Equilibrium to be written in final form as:

$$\left\{ \frac{\partial F_{11}(\sigma)}{\partial s_1} + \frac{\partial F_{21}(\sigma)}{\partial s_2} + \kappa_{11}F_{13}(\sigma) - \kappa_{13}\left[F_{12}(\sigma) + F_{21}(\sigma) - 2\kappa_{21}\int_{\alpha_3} \sigma_{11}\alpha_3 d\alpha_3 \right] \right.$$

$$+ \kappa_{23}\left[F_{11}(\sigma) - F_{22}(\sigma) - 2\kappa_{21}\int_{\alpha_3} \sigma_{12}\alpha_3 d\alpha_3 \right] + \kappa_{21}F_{23}(\sigma) + \frac{\partial \kappa_{12}}{\partial s_2}\int_{\alpha_3} \sigma_{11}\alpha_3 d\alpha_3$$

$$\left. - \frac{\partial \kappa_{21}}{\partial s_1}\int_{\alpha_3} \sigma_{12}\alpha_3 d\alpha_3 + P_1 \right\} = 0 \qquad \ldots.. \qquad \{6.2.\text{-}15.\}$$

$$\left\{ \frac{\partial F_{12}(\sigma)}{\partial s_1} + \frac{\partial F_{22}(\sigma)}{\partial s_2} + \kappa_{13}\left[F_{11}(\sigma) - F_{22}(\sigma) + 2\kappa_{21}\int_{\alpha_3} \sigma_{12}\alpha_3 d\alpha_3 \right] \right.$$

$$+ \kappa_{23}\left[F_{21}(\sigma) + F_{12}(\sigma) - 2\kappa_{21}\int_{\alpha_3} \sigma_{22}\alpha_3 d\alpha_3 \right] + \kappa_{22}F_{23}(\sigma) - \kappa_{12}F_{13}(\sigma)$$

$$+ \frac{\partial \kappa_{12}}{\partial s_2} \int_{\alpha_3} \sigma_{12}\alpha_3 d\alpha_3 - \frac{\partial \kappa_{21}}{\partial s_1} \int_{\alpha_3} \sigma_{22}\alpha_3 d\alpha_3 + P_2 \Bigg\} = 0 \quad \ldots\ldots \quad \{6.2.\text{-}16.\}$$

$$\Bigg\{ \frac{\partial F_{13}(\sigma)}{\partial s_1} + \frac{\partial F_{23}(\sigma)}{\partial s_2} - \kappa_{13}\left[F_{23}(\sigma) - 2\kappa_{21} \int_{\alpha_3} \sigma_{13}\alpha_3 d\alpha_3 \right]$$

$$+ \kappa_{23}\left[F_{13}(\sigma) - 2\kappa_{21}\int_{\alpha_3} \sigma_{23}\alpha_3 d\alpha_3\right] - \kappa_{11}F_{11}(\sigma) - \kappa_{22}F_{22}(\sigma)$$

$$+ \kappa_{12}\left[F_{12}(\sigma) + F_{21}(\sigma)\right] + \frac{\partial \kappa_{12}}{\partial s_2}\int_{\alpha_3} \sigma_{13}\sigma_3 d\alpha_3$$

$$- \frac{\partial \kappa_{21}}{\partial s_1}\int_{\alpha_3} \sigma_{23}\alpha_3 d\alpha_3 + P_3 \Bigg\} = 0 \quad \ldots\ldots \quad \{6.2.\text{-}17.\}$$

A comparison of the form of these three equations with the kinematic compatibility equations ({5.1.-15.}, {5.1.-16.}, {5.1.-17.}) shows that the two forms display an exceptional similarity. *This is usually referred to as the statico-geometrical analogy in shell theory.*

The similarity is even more pronounced in the case that the orthogonal parametric lines are coincident with the lines of principal curvature. In this case, the geodesic torsions, κ_{12} and κ_{21}, vanish and the reduced form of the kinematic compatibility equations ({5.1.-18.}, {5.1.-19.}, {5.1.-20.}) may be compared to the reduced form of the force equilibrium equations, below.

$$\frac{\partial F_{11}(\sigma)}{\partial s_1} + \frac{\partial F_{21}(\sigma)}{\partial s_2} - \kappa_{13}\left[F_{12}(\sigma) + F_{21}(\sigma)\right] + \kappa_{23}\left[F_{11}(\sigma) - F_{22}(\sigma)\right]$$

$$+ \kappa_{11}F_{13}(\sigma) + P_1 = 0 \quad \ldots\ldots \quad \{6.2.\text{-}18.\}$$

$$\frac{\partial F_{12}(\sigma)}{\partial s_1} + \frac{\partial F_{22}(\sigma)}{\partial s_2} + \kappa_{13}\left[F_{11}(\sigma) - F_{22}(\sigma)\right] + \kappa_{23}\left[F_{12}(\sigma) + F_{21}(\sigma)\right]$$

$$+ \kappa_{22}F_{23}(\sigma) + P_2 \quad = 0 \qquad \ldots\ldots \qquad \{6.2.-19.\}$$

$$\frac{\partial F_{13}(\sigma)}{\partial s_1} + \frac{\partial F_{23}(\sigma)}{\partial s_2} - \kappa_{13}F_{23}(\sigma) + \kappa_{23}F_{13}(\sigma)$$

$$- \kappa_{11}F_{11}(\sigma) - \kappa_{22}F_{22}(\sigma) + P_3 \quad = 0 \qquad \ldots\ldots \qquad \{6.2.-20.\}$$

6.3. THE MOMENT EQUILIBRIUM EQUATIONS

Commencing with the previously-developed relationship

$$\int_V - \frac{\partial}{\partial \overline{r}} \cdot (\overline{\overline{\sigma}} \times \overline{r})dv = 0 \qquad (\{6.1.-10.\})$$

this may be expanded immediately as

$$- \int_V \left[\frac{\partial \cdot \overline{\overline{\sigma}}}{\partial \overline{r}} \times \overline{r} + \frac{\partial}{\partial \overline{r}} \cdot \overline{\overline{\sigma}} \times \underline{\overline{r}} \right] dv = 0 \qquad \ldots\ldots \qquad \{6.3.-1.\}$$

where the underscored quantity indicates that the directed derivative operates only on that quantity.

However, $\quad \frac{\partial}{\partial \overline{r}} \cdot \overline{\overline{\sigma}} \times \underline{\overline{r}} = \overline{\overline{\sigma}}_c \cdot \frac{\partial \times \overline{r}}{\partial \overline{r}}$

where the subscript c denotes the conjugate tensor, as usual.

The symmetry of the stress tensor requires that

$$\overline{\overline{\sigma}} = \overline{\overline{\sigma}}_c$$

and so, $$\frac{\partial}{\partial \overline{r}} \cdot \overline{\overline{\sigma}} \times \underline{\overline{r}} = \overline{\overline{\sigma}}_c \cdot \frac{\partial \times \overline{r}}{\partial \overline{r}} = \overline{\overline{\sigma}}\,{}_{,}\cdot \frac{\partial \times \overline{r}}{\partial \overline{r}}$$

However, the term $\frac{\partial \times \overline{r}}{\partial \overline{r}}$ is the vector invariant of the identity tensor, $\frac{\partial \overline{r}}{\partial \overline{r}} = \overline{\overline{1}}$, and is consequently equal to zero. Therefore, {6.3.-1.} reduces to

$$-\int_V \frac{\partial \cdot \overline{\overline{\sigma}}}{\partial \overline{r}} \times \overline{r}\, dv = 0$$

or

$$\int_V \overline{r} \times \frac{\partial \cdot \overline{\overline{\sigma}}}{\partial \overline{r}}\, dv = 0 \qquad \ldots\ldots \qquad \{6.3.-2.\}$$

Now, the position vector $\overline{r}$ to a parallel surface may be considered to be the sum of two other vectors: $\overline{r}^{\circ}$, which locates a point in the middle surface, plus the normal vector, $\alpha_3\overline{e}_3$, from the middle surface to the parallel surface. Thus,

$$\overline{r} = \overline{r}^{\circ} + \alpha_3\overline{e}_3$$

Equation {6.3.-2.} then becomes

$$\int_V (\overline{r}^{\circ} + \alpha_3\, \overline{e}_3) \times \frac{\partial \cdot \overline{\overline{\sigma}}}{\partial \overline{r}}\, dv = 0 \qquad \ldots\ldots \qquad \{6.3.-3.\}$$

By precisely the same argument that was advanced for the Force Equilibrium equation, i.e., since the integration is definite only over the limits of α_3, then {6.3.-3.} reduces to (see §6.2.)

$$\int_{\alpha_3} (\overline{r}^{\circ} + \alpha_3\overline{e}_3) \times \frac{\partial \cdot \overline{\overline{\sigma}}}{\partial \overline{r}}\, \pi_1\pi_2 d\alpha_3 = 0 \qquad \ldots\ldots \qquad \{6.3.-4.\}$$

Since, however, $\overline{r}^\circ \neq \overline{r}^\circ\,(\alpha_3)$ then {6.3.-4.} may be given as

$$r^\circ \times \int_{\alpha_3} \frac{\partial \cdot \overline{\overline{\sigma}}}{\partial \overline{r}} \pi_1\pi_2 d\alpha_3 + \int_{\alpha_3} \alpha_3\overline{e}_3 \times \frac{\partial \cdot \overline{\overline{\sigma}}}{\partial \overline{r}} \pi_1\pi_1 d\alpha_3 = 0$$

The first term of this equation vanishes, as the integral itself vanishes, being the "Force Equilibrium Equation" (see {6.2.-5.}). This reveals the Moment Equilibrium equation in its most succinct form, as

$$\int_{\alpha_3} \alpha_3\overline{e}_3 \times \frac{\partial \cdot \overline{\overline{\sigma}}}{\partial \overline{r}} \pi_1\pi_2 d\alpha_3 = 0 \qquad .... \qquad \{6.3.-5.\}$$

Considerable effort in the expansion of this expression is saved, by considering that the quantity $\frac{\partial \cdot \overline{\overline{\sigma}}}{\partial \overline{r}} \pi_1\pi_2$ has been expanded in the course of obtaining the Force Equilibrium equations. Thus, taking the cross-product of $\alpha_3\overline{e}_3$ with the expanded form of $\frac{\partial \cdot \overline{\overline{\sigma}}}{\partial \overline{r}} \pi_1\pi_2$(i.e., equation {6.2.-6.}), and retaining the result as the integrand of $\int_{\alpha_3} (\quad) d\alpha_3$, then {6.3.-5.} appears as

$$\int_{\alpha_3}\Bigg\{ -\alpha_3 \Bigg[\kappa_{13}(\sigma_{11} - \sigma_{22})\pi_2 + \kappa_{11}\sigma_{32}\pi_2 + \frac{\partial \sigma_{12}}{\partial s_1}\pi_2 + \kappa_{23}(\sigma_{12} + \sigma_{21})\pi_1 + \kappa_{22}(\sigma_{23} + \sigma_{32})\pi_1 - \kappa_{12}\sigma_{13}\pi_2 + \frac{\partial \sigma_{22}}{\partial s_2}\pi_1 + \frac{\partial \sigma_{32}}{\partial \alpha_3}\pi_1\pi_2 \Bigg] \overline{e}_1 + \alpha_3 \Bigg[\frac{\partial \sigma_{11}}{\partial s_1}\pi_2 - \kappa_{13}(\sigma_{12} - \sigma_{21})\pi_2 + \kappa_{11}(\sigma_{13} + \sigma_{31})\pi_2 + \kappa_{23}(\sigma_{11} - \sigma_{22})\pi_1 + \kappa_{22}\sigma_{31}\pi_1 + \kappa_{21}\sigma_{23}\pi_1 + \frac{\partial \sigma_{21}}{\partial s_2}\pi_1 + \frac{\partial \sigma_{31}}{\partial \alpha_3}\pi_1\pi_2 \Bigg] \overline{e}_2 \Bigg\} = 0 \quad \{6.3.-6.\}$$

The third term originally found in the expansion of $\frac{\partial \cdot \overline{\overline{\sigma}}}{\partial \overline{r}}\pi_1\pi_2$, naturally vanishes as it was in the $\overline{e}_3$-direction, and the cross-product is taken with $\overline{e}_3$.

Since the vector directions $\overline{e}_1$ and $\overline{e}_2$ are independent of α_3 (and of each other), then for {6.3.-6.} to vanish, the coefficients of $\overline{e}_1$ and $\overline{e}_2$ must vanish individually. Therefore, performing the integration of these coefficients and setting the results equal to zero (as for the Force Equilibium equations) reveals the two following equations.

$$- \Bigg\{ \frac{\partial}{\partial s_1} \int_{\alpha_3} \alpha_3 \sigma_{12}(1 + \alpha_3\kappa_{22}) d\alpha_3 + \frac{\partial}{\partial s_2} \int_{\alpha_3} \alpha_3\sigma_{22}(1 + \alpha_3\kappa_{11}) d\alpha_3 + \kappa_{13} \int_{\alpha_3} \alpha_3\sigma_{11}(1 + \alpha_3\kappa_{22}) d\alpha_3$$
$$+ \kappa_{23} \int_{\alpha_3} \alpha_3\sigma_{21}(1 + \alpha_3\kappa_{11}) d\alpha_3 - \int_{\alpha_3} \sigma_{23}(1 + \alpha_3\kappa_{11}) d\alpha_3 - \kappa_{12} \int_{\alpha_3} \alpha_3\sigma_{13}(1 + \alpha_3\kappa_{22}) d\alpha_3$$
$$- \kappa_{13} \int_{\alpha_3} \alpha_3\sigma_{22}(1 + \alpha_3\kappa_{11}) d\alpha_3 + \kappa_{23} \int_{\alpha_3} \alpha_3\sigma_{12}(1 + \alpha_3\kappa_{22}) d\alpha_3$$
$$+ \left(\frac{\partial\kappa_{12}}{\partial s_2} + 2\kappa_{21}\kappa_{13}\right) \int_{\alpha_3} \alpha_3^2\sigma_{12} d\alpha_3 - \left(\frac{\partial\kappa_{21}}{\partial s_1} + 2\kappa_{21}\kappa_{23}\right) \int_{\alpha_3} \alpha_3^2\sigma_{22} d\alpha_3$$
$$+ \alpha_3\sigma_{32}(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22}) \Big|_{\alpha_3} \Bigg\} \overline{e}_1 = 0 \qquad \ldots\ldots \qquad \{6.3.-7.\}$$

$$\Bigg\{ \frac{\partial}{\partial s_1} \int_{\alpha_3} \alpha_3\sigma_{11}(1 + \alpha_3\kappa_{22}) d\alpha_3 + \frac{\partial}{\partial s_2} \int_{\alpha_3} \alpha_3\sigma_{21}(1 + \alpha_3\kappa_{11}) d\alpha_3 - \int_{\alpha_3} \sigma_{13}(1 + \alpha_3\kappa_{22}) d\alpha_3$$
$$- \kappa_{23} \int_{\alpha_3} \alpha_3\sigma_{22}(1 + \alpha_3\kappa_{11}) d\alpha_3 + \kappa_{21} \int_{\alpha_3} \alpha_3\sigma_{23}(1 + \alpha_3\kappa_{11}) d\alpha_3 + \kappa_{23} \int_{\alpha_3} \alpha_3\sigma_{11}(1 + \alpha_3\kappa_{22}) d\alpha_3$$
$$- \kappa_{13} \int_{\alpha_3} \alpha_3\sigma_{12}(1 + \alpha_3\kappa_{22}) d\alpha_3 - \kappa_{13} \int_{\alpha_3} \alpha_3\sigma_{21}(1 + \alpha_3\kappa_{11}) d\alpha_3$$
$$+ \left(\frac{\partial\kappa_{12}}{\partial s_2} + 2\kappa_{21}\kappa_{13}\right) \int_{\alpha_3} \alpha_3^2\sigma_{11} d\alpha_3 - \left(\frac{\partial\kappa_{21}}{\partial s_1} + 2\kappa_{21}\kappa_{23}\right) \int_{\alpha_3} \alpha_3^2\sigma_{12} d\alpha_3$$
$$+ \alpha_3\sigma_{31}(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22}) \Big|_{\alpha_3} \Bigg\} \overline{e}_2 = 0 \qquad \ldots\ldots \qquad \{6.3.-8.\}$$

where the form of these equations has been altered, by means of the MAINARDI-CODAZZI Equations ({3.2.-10.}, {3.2.-11.}), as was done for the Force Equilibrium equations.

Now, just as the quantities $\int_{\alpha_3} \sigma_{ij}(1 + \alpha_3\kappa_{rr})d\alpha_3$ were observed to bear physical interpretation as stress resultants for the case of the Force Equilibrium equations; so the quantities in the two Moment Equilibrium equations above, allow a similar interpretation.

Consider Fig. 6.3.-1., below. Here, the infinitesimal element of the shell is shown removed from the "semi-infinitesimal" segment of the shell, as shown in Fig. 6.1.-1. The stress vectors $\overline{\sigma}_i$ are shown applied to the element, although only $\overline{\sigma}_1 = \sigma_{11}\overline{e}_1 + \sigma_{12}\overline{e}_2 + \sigma_{13}\overline{e}_3$ is shown in detail, to avoid confusion.

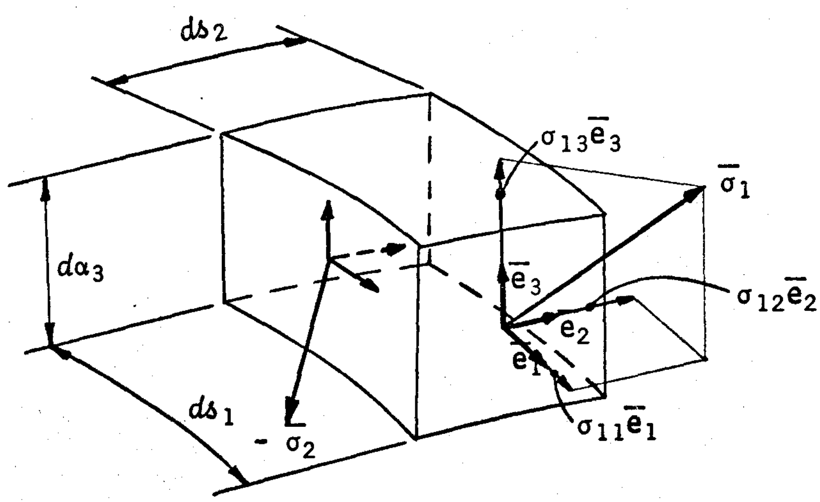

Fig. 6.3.-1.

With reference to the system of Fig. 6.3.-1., it is observed that a quantity such as

$$\int_{\alpha_3} \alpha_3 \sigma_{11}(1 + \alpha_3 \kappa_{22}) d\alpha_3$$

may be considered as

$$\int_{\alpha_3} (\alpha_3) \; [\sigma_{11}(1 + \alpha_3 \kappa_{22})] d\alpha_3$$

This quantity is thus observed to be a *stress couple,* as it is a moment, $\sigma_{11}(1 + \alpha_3 \kappa_{22})$ being multiplied by a distance (α_3) and integrated over the region of action of σ_{11}. A similar interpretation is possible for the remaining integrals of the same form; referring to such stress couples as $M_{ij}(\sigma)$, and retaining consistent vector notation for the subscripts, then

$$\begin{aligned}
M_{11}(\sigma)\overline{e}_1 &= -\int_{\alpha_3} \alpha_3 \sigma_{12}(1 + \alpha_3 \kappa_{22}) d\alpha_3 \; \overline{e}_1 \\
M_{12}(\sigma)\overline{e}_2 &= \int_{\alpha_3} \alpha_3 \sigma_{11}(1 + \alpha_3 \kappa_{22}) d\alpha_3 \; \overline{e}_2 \\
M_{13}(\sigma)\overline{e}_3 &= \int_{\alpha_3} \alpha_3 \sigma_{13}(1 + \alpha_3 \kappa_{22}) d\alpha_3 \; \overline{e}_3 \\
M_{21}(\sigma)\overline{e}_1 &= -\int_{\alpha_3} \alpha_3 \sigma_{22}(1 + \alpha_3 \kappa_{11}) d\alpha_3 \; \overline{e}_1 \\
M_{22}(\sigma)\overline{e}_2 &= \int_{\alpha_3} \alpha_3 \sigma_{21}(1 + \alpha_3 \kappa_{11}) d\alpha_3 \; \overline{e}_2 \\
M_{23}(\sigma)\overline{e}_3 &= \int_{\alpha_3} \alpha_3 \sigma_{23}(1 + \alpha_3 \kappa_{11}) d\alpha_3 \; \overline{e}_3
\end{aligned} \qquad \{6.3.\text{-}9.\}$$

where $\overline{M}_1(\sigma) = M_{11}(\sigma)\overline{e}_1 + M_{12}(\sigma)\overline{e}_2 + M_{13}(\sigma)\overline{e}_3$

and $\overline{M}_2(\sigma) = M_{21}(\sigma)\overline{e}_1 + M_{22}(\sigma)\overline{e}_2 + M_{23}(\sigma)\overline{e}_3$

There also exist quantities which have the form

$$\alpha_3\ \sigma_{3i}(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})\Big|_{\alpha_3} \qquad (i = 1,2)$$

Such quantites are seen to represent the algebraic sum of the moments caused directly by the boundary forces (when the limits, $\alpha_3 = h/2$, $\alpha_3 = -h/2$ are inserted). This is quite obvious, since the term σ_{3i} represents the components of the vector $\overline{\sigma}_3 = \sigma_{31}\overline{e}_1 + \sigma_{32}\overline{e}_2 + \sigma_{33}\overline{e}_3$ in the tangent plane at the boundary (for i=1,2 as above), while the quantity $(1 + \alpha_3\kappa_{11})(1 + \alpha_3\kappa_{22})$ represents the area of the surface at the boundary (for a unit element of area at the middle surface). Denoting such terms as this by the symbolism, M_i, then equations {6.3.-7.} and {6.3.-8.} may be written as follows.

$$\frac{\partial M_{11}(\sigma)}{\partial s_1} + \frac{\partial M_{21}(\sigma)}{\partial s_2} - \kappa_{13}\left[M_{12}(\sigma) + M_{21}(\sigma) + 2\kappa_{21}\int_{\alpha_3} \alpha_3^2\sigma_{12}d\alpha_3\right]$$

$$+ \kappa_{23}\left[M_{11}(\sigma) - M_{22}(\sigma) + 2\kappa_{21}\int_{\alpha_3} \alpha_3^2\sigma_{22}d\alpha_3\right] + \kappa_{12}M_{13}(\sigma)$$

$$+ F_{23}(\sigma) - \frac{\partial\kappa_{12}}{\partial s_2}\int_{\alpha_3} \alpha_3^2\sigma_{12}d\alpha_3 + \frac{\partial\kappa_{21}}{\partial s_1}\int_{\alpha_3} \alpha_3^2\sigma_{22}d\alpha_3 - M_2 = 0$$

..... {6.3.-10.}

$$\frac{\partial M_{12}(\sigma)}{\partial s_1} + \frac{\partial M_{22}(\sigma)}{\partial s_2} + \kappa_{13}\left[M_{11}(\sigma) - M_{22}(\sigma) + 2\kappa_{21}\int_{\alpha_3} \alpha_3^2\sigma_{11}d\alpha_3\right]$$

$$+ \kappa_{23}\left[M_{12}(\sigma) + M_{21}(\sigma) - 2\kappa_{21}\int_{\alpha_3} \alpha_3^2\sigma_{12}d\alpha_3\right] + \kappa_{21}M_{23}(\sigma) - F_{13}(\sigma)$$

$$+ \frac{\partial\kappa_{12}}{\partial s_2}\int_{\alpha_3} \alpha_3^2\sigma_{11}d\alpha_3 - \frac{\partial\kappa_{21}}{\partial s_1}\int_{\alpha_3} \alpha_3^2\sigma_{12}d\alpha_3 + M_1 = 0$$

..... {6.3.-11.}

The close similarity between the form of these two equations and that of the kinematic compatibility equations is another illustration of the statico-geometrical analogy. As was noted for the Force Equilibrium equations, the analogy is even more pronounced for the case that the orthogonal parametric lines are coincident with the lines of principal curvature. In such a case, the Moment Equilibrium equations, {6.3.-10.} and {6.3.-11.} above, reduce to the following equations.

$$\frac{\partial M_{11}(\sigma)}{\partial s_1} + \frac{\partial M_{21}(\sigma)}{\partial s_2} - \kappa_{13}\left[M_{12}(\sigma) + M_{21}(\sigma)\right]$$

$$+ \kappa_{23}\left[M_{11}(\sigma) - M_{22}(\sigma)\right] + F_{23}(\sigma) - \mathcal{M}_2 = 0 \; \ldots\ldots \qquad \{6.3.\text{-}12.\}$$

$$\frac{\partial M_{12}(\sigma)}{\partial s_1} + \frac{\partial M_{22}(\sigma)}{\partial s_2} + \kappa_{13}\left[M_{11}(\sigma) - M_{22}(\sigma)\right]$$

$$+ \kappa_{23}\left[M_{12}(\sigma) + M_{21}(\sigma)\right] - F_{13}(\sigma) + \mathcal{M}_1 = 0 \; \ldots\ldots \qquad \{6.3.\text{-}13.\}$$

NOTE: The conventional symbolism for the stress couples, as used in many works on shell theory*

VIZ: $M_{\alpha\beta}(\sigma) = \int_{\alpha_3} \alpha_3 \sigma_{\alpha\beta} \; (1 + \alpha_3 \kappa_{ii})\, d\alpha_3$

has the inherent disadvantage that physical interpretation of such terms is exceptionally difficult. The direct notation, as employed in this work, permits immediate recognition of the physical significance of these terms.

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* See page 187.

6.4. THE CONSTITUTIVE COMPATIBILITY CONDITIONS

Since the expression for the strain tensor, $\bar{\bar{\varepsilon}}$, showed the tensor to be two-dimensional (§4.6.1.), it becomes evident that a contradiction exists between this tensor and the stress resultant tensor

$$\bar{\bar{F}}(\sigma) = \bar{\bar{F}}_{ij}(\sigma)\,\bar{e}_i\bar{e}_j = \int_{\alpha_3} \sigma_{ij}(1 + \alpha_3\kappa_{rr})d\alpha_3\;\bar{e}_i\bar{e}_j \qquad \text{(sum: i=1,2; j=1,2,3)}$$

where all $F_{ij}(\sigma)$ are as given by {6.2.-13.}. This contradiction arises over two terms, $F_{13}(\sigma)$ and $F_{23}(\sigma)$ -- or in reality, over the two terms which form the basis of $F_{13}(\sigma)$ and $F_{23}(\sigma)$, namely σ_{13} and σ_{23}. The strain tensor, as given by {4.6.1.-10} implies that such terms should not exist; the equilibrium equations, as given by {6.2.-15}, {6.2.-16.} and {6.2.-17.} contend that such terms must exist for equilibrium to be satisfied.

The most convenient resolution of this dilemma would appear to be as follows. Since the introduction of a three-dimensional strain tensor would contradict KIRCHHOFF's Hypothesis, then assume that the strain tensor is two-dimensional, and that the kinematic KIRCHHOFF Hypothesis remains valid. On the other hand, σ_{13} and σ_{23} will continue to exist, for satisfaction of the equilibrium equations. Thus, the appropriate solution to the contridiction is simply to admit that it exists and to say that the strain tensor is a good (linear) approximation to the true state of strain -- and that the terms neglected are small in comparison to the terms retained. This is, naturally, justified by the fact that the transverse strains would always be much less than the surface strains.

The Constitutive Compatibility Conditions, or the stress resultants and stress couples in terms of the strain parameter relations, can then be found in the following way: since ε_{ij} (i,j=1,2) is known, then $F_{ij}(\sigma)$ and $M_{ij}(\sigma)$ (i,j=1,2) can be found directly, Once these are known, $F_{i3}(\sigma)$ (i=1,2) may be determined, using the Equilibrium Equations as the vehicle of evaluation.

Proceeding in accordance with the above, the expression for $F_{11}(\sigma)$ is obtained as follows.

By definition: $$F_{11}(\sigma) = \int_{\alpha_3} \sigma_{11}(1 + \alpha_3\kappa_{22})\, d\alpha_3 \quad \ldots\ldots \quad \{6.4.-1.\}$$

From the stress-strain relation for an isotropic, homogeneous HOOKEAN (linearly elastic) material,

VIZ: $$\bar{\bar{\sigma}} = 2\mu\bar{\bar{\varepsilon}} + \lambda(\bar{\bar{\varepsilon}}:\bar{\bar{I}})\ \bar{\bar{I}}$$

or $$\sigma_{ij}\bar{e}_i\bar{e}_j = 2\mu\varepsilon_{ij}\bar{e}_i\bar{e}_j + \lambda(\bar{\bar{\varepsilon}}:\bar{\bar{I}})\delta_{ij}\bar{e}_i\bar{e}_j$$

where μ and λ are the usual CAUCHY-LAMÉ elastic constants

then $$\sigma_{11}\bar{e}_1\bar{e}_1 = [2\mu\varepsilon_{11} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})]\ \bar{e}_1\bar{e}_1 \quad \ldots\ldots \quad \{6.4.-2.\}$$

This introduces another unknown, ε_{33}. This problem is quickly overcome, however, by making the assumption that $\sigma_{33} = 0$, an assumption consistent with the discussion of the contradiction (above). That is, for surface structures, the cross-sectional surface stresses are much larger than the stresses normal to the middle surface.

Thus $\sigma_{33} = 0 = 2\mu\varepsilon_{33} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$

or $$\varepsilon_{33} = -\frac{\lambda(\varepsilon_{11} + \varepsilon_{22})}{2\mu + \lambda}$$

or, as $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$; $\mu = \dfrac{E}{2(1+\nu)}$

where ν is POISSON's Ratio

then $$\varepsilon_{33} = -\left(\frac{\nu}{1-\nu}\right)(\varepsilon_{11} + \varepsilon_{22}) \quad \ldots\ldots \quad \{6.4.-3.\}$$

Consequently, from {6.4.-2.} and {6.4.-3.},

$$\sigma_{11} = 2\mu\varepsilon_{11} + \frac{\nu E}{1-\nu^2}(\varepsilon_{11} + \varepsilon_{22})$$

$$= \frac{E}{1-\nu}\varepsilon_{11} + \frac{\nu E}{1-\nu^2}(\varepsilon_{11} + \varepsilon_{22})$$

so $$\sigma_{11} = \frac{E}{1-\nu^2}[\varepsilon_{11} + \nu\varepsilon_{22}] \quad \ldots\ldots \quad \{6.4.-4.\}$$

Equation {6.4.-4.}, in conjunction with {6.4.-1.}, gives

$$F_{11}(\sigma) = \int_{\alpha_3}\left[\frac{E}{1-\nu^2}(\varepsilon_{11} + \varepsilon_{22})(1 + \alpha_3\kappa_{22})\right]d\alpha_3$$

or $$F_{11}(\sigma) = \left\{\frac{E}{1-\nu^2}\int_{\alpha_3}\varepsilon_{11}(1 + \alpha_3\kappa_{22})d\alpha_3 + \frac{\nu E}{1-\nu^2}\int_{\alpha_3}\varepsilon_{22}(1 + \alpha_3\kappa_{22})d\alpha_3\right\} \quad \ldots\ldots \quad \{6.4.-5.\}$$

Substituting the expressions for ε_{11} and ε_{22}, as given by the strain tensor ({4.6.1.-10.})

VIZ: $$\varepsilon_{11} = \frac{1}{1 + \alpha_3\kappa_{11}}(\phi_{11} + \alpha_3\delta\kappa_{11})$$

$$\varepsilon_{22} = \frac{1}{1 + \alpha_3\kappa_{22}}(\phi_{22} + \alpha_3\delta\kappa_{22})$$

into equation {6.4.-5.} above, reveals

$$F_{11}(\sigma) = \left\{ \frac{E}{1-\nu^2} \int_{\alpha_3} (\phi_{11} + \alpha_3 \delta\kappa_{11}) \frac{1 + \alpha_3\kappa_{11}}{1 + \alpha_3\kappa_{22}} d\alpha_3 + \frac{\nu E}{1-\nu^2} \int_{\alpha_3} (\phi_{22} + \alpha_3 \delta\kappa_{22}) d\alpha_3 \right\}$$

Carrying out the integration, and evaluating between the limits of α_3 (+ h/2, - h/2) produces

$$F_{11}(\sigma) = \frac{Eh}{1-\nu^2} \left\{ \phi_{11} + \nu\phi_{22} + \frac{h^2}{12}\left[\kappa_{11}(\kappa_{11}\phi_{11} - \kappa_{22}\phi_{22}) + \delta\kappa_{11}(\kappa_{22} - \kappa_{11})\right] + \frac{h^4\kappa_{11}^2}{80}\left[\kappa_{11}(\kappa_{11}\phi_{11} - \kappa_{22}\phi_{22}) + \delta\kappa_{11}(\kappa_{22} - \kappa_{11})\right] \right\}$$

By a similar procedure, the other terms, $F_{12}(\sigma)$, $F_{21}(\sigma)$ and $F_{22}(\sigma)$ are evaluated as

$$F_{12}(\sigma) = \mu h \left[(\phi_{12} + \phi_{21}) + \frac{h^2}{12} \delta\kappa_{12}(\kappa_{11} - \kappa_{22}) + \frac{h^4\kappa_{11}^2}{80} \delta\kappa_{12}(\kappa_{11} - \kappa_{22}) \right]$$

$$F_{21}(\sigma) = \mu h \left[(\phi_{12} + \phi_{21}) + \frac{h^2}{12} \delta\kappa_{21}(\kappa_{11} - \kappa_{22}) + \frac{h^4\kappa_{22}^2}{80} \delta\kappa_{21}(\kappa_{11} - \kappa_{22}) \right]$$

$$F_{22}(\sigma) = \frac{Eh}{1-\nu^2} \Big(\phi_{11} + \nu\phi_{22} + \frac{h^2}{12} [\kappa_{22}(\kappa_{22}\phi_{22} - \kappa_{11}\phi_{11}) + \delta\kappa_{22}(\kappa_{22} - \kappa_{11})] + \frac{h^4\kappa_{22}^2}{80} [\kappa_{22}(\kappa_{22}\phi_{22} - \kappa_{11}\phi_{11}) + \delta\kappa_{22}(\kappa_{22} - \kappa_{11})] \Big)$$

and without any essential difference in procedure, the stress couples appear as

$$M_{11}(\sigma) = \frac{\mu h^3}{12} [\delta\kappa_{12} - \delta\kappa_{21} - \kappa_{22}(\phi_{12} + \phi_{21})] + \frac{\mu h^5}{80} \kappa_{11}\delta\kappa_{12}(\kappa_{22} - \kappa_{11})$$

$$M_{12}(\sigma) = \left\{\frac{Eh^3}{12(1-\nu^2)} \left[\phi_{11}(\kappa_{22} - \kappa_{11}) + \delta\kappa_{11} + \nu\delta\kappa_{22}\right] + \frac{Eh^5\kappa_{11}}{80(1-\nu^2)} (\kappa_{11}\phi_{11} - \delta\kappa_{11})(\kappa_{22} - \kappa_{11})\right\}$$

$$M_{21}(\sigma) = \left\{\frac{Eh^3}{12(1-\nu^2)} \left[\phi_{22}(\kappa_{22} - \kappa_{11}) - \delta\kappa_{22} - \nu\delta\kappa_{11}\right] + \frac{Eh^5\kappa_{22}}{80(1-\nu^2)} (\kappa_{22}\phi_{22} - \delta\kappa_{22})(\kappa_{22} - \kappa_{11})\right\}$$

$$M_{22}(\sigma) = \frac{\mu h^3}{12} \left[\delta\kappa_{21} - \delta\kappa_{12} + \kappa_{11}(\phi_{12} + \phi_{21})\right] + \frac{\mu h^5}{80} \kappa_{22}\delta\kappa_{21}(\kappa_{22} - \kappa_{11})$$

Considering the insignificance of such terms as $\frac{h^4\kappa_{ii}}{80}$ in the expressions for $F_{ij}(\sigma)$ and of such terms as $\frac{\mu h^5}{80}$ or $\frac{Eh^5\kappa_{jj}}{80(1 - \nu^2)}$ in the expressions for $M_{ij}(\sigma)$ (with respect to the other terms in these expressions) and considering also, for the case of $F_{ij}(\sigma)$, that $\kappa_{11}\phi_{11} = \kappa_{22}\phi_{22}$ ({5.2.-11.}), then these expressions above reduce to the following.

$$F_{11}(\sigma) = \frac{Eh}{1-\nu^2} \left[\phi_{11} + \nu\phi_{22} + \frac{h^2}{12} \delta\kappa_{11}(\kappa_{22} - \kappa_{11})\right]$$

$$F_{12}(\sigma) = \mu h \left[(\phi_{12} + \phi_{21}) - \frac{h^2}{12} \delta\kappa_{12}(\kappa_{22} - \kappa_{11})\right]$$

$$F_{21}(\sigma) = \mu h \left[(\phi_{12} + \phi_{21}) - \frac{h^2}{12} \delta\kappa_{21}(\kappa_{22} - \kappa_{11})\right]$$

$$F_{22}(\sigma) = \frac{Eh}{1-\nu^2} \left[\phi_{22} + \nu\phi_{11} - \frac{h^2}{12}\delta\kappa_{22}(\kappa_{22} - \kappa_{11})\right]$$

$$M_{11}(\sigma) = \frac{\mu h^3}{12}[\delta\kappa_{12} - \delta\kappa_{21} - \kappa_{22}(\phi_{12} + \phi_{21})]$$

$$M_{12}(\sigma) = \frac{Eh^3}{12(1-\nu^2)}[\delta\kappa_{11} + \nu\delta\kappa_{22} + \phi_{11}(\kappa_{22} - \kappa_{11})]$$

$$M_{21}(\sigma) = \frac{Eh^3}{12(1-\nu^2)}[-\delta\kappa_{22} - \nu\delta\kappa_{11} + \phi_{22}(\kappa_{22} - \kappa_{11})]$$

$$M_{22}(\sigma) = \frac{\mu h^3}{12}[\delta\kappa_{21} - \delta\kappa_{12} + \kappa_{11}(\phi_{12} + \phi_{21})]$$

The remaining stress resultants, $F_{13}(\sigma)$ and $F_{23}(\sigma)$ may be obtained, if desired, by the substitution of the expressions for $F_{ij}(\sigma)$ and $M_{ij}(\sigma)$ above, into Equilibrium equations {6.2.-15.} and {6.2.-16.} or {6.3.-10.} and {6.3.-11.}.

NOTE: Often, in the literature of the subject, the indirect approach leads to the naming of the stress couples in the following manner.

$$M_{ij}(\sigma) = \int_{\alpha_3} \alpha_3\sigma_{ij}(1 + \alpha_3\kappa_{rr})d\alpha_3$$

Thus, the following correspondence exists:

This Work	Other Authors
$+ M_{11}(\sigma)$	$- M_{12}(\sigma)$
$+ M_{12}(\sigma)$	$+ M_{11}(\sigma)$
$+ M_{13}(\sigma)$	$+ M_{13}(\sigma)$
$+ M_{21}(\sigma)$	$- M_{22}(\sigma)$
$+ M_{22}(\sigma)$	$+ M_{21}(\sigma)$
$+ M_{23}(\sigma)$	$+ M_{23}(\sigma)$

CHAPTER 7

Conclusions

The development of the general theory of thin elastic shells via the direct kinematic method provides the foundation for the derivation of the equations of local compatibility of middle-surface strains. This method is seen to provide a set of conceptually-motivated compatibility equations which do not require the use of special techniques or *a priori* knowledge in their formulation. Thus, the "synthetic" approach to the development of such equations is eliminated and the general theory of shells benefits from increased coherence as a result. To the author's knowledge, such equations have not been derived by the kinematic approach before this time. Some difficulty was originally encountered in the development of these equations, in the form of extraneous terms, the existence of which was not justifiable. However, subsequent analysis showed that these terms arose from the use of expressions which were too accurate; that is, expressions had been employed which were accurate beyond the limits of the original basic assumption that the *linear* shell theory would be employed as the foundation of the work. Such expressions were then corrected so as to conform to the "linear theory" hypothesis.

In order to compare the kinematic compatibility equations with the equations developed by other authors, two things were necessary. First, a "standard of comparison" which was independent of the kinematic method and the method of another author was required. Second, it was necessary to have available a set of transformation identities which would relate the various quantities employed in the equations of compatibility. The Saint-Venant approach to compatibility provided both these requisites. Although it is a formal technique, it was nevertheless, invaluable for the information produced. It was noted that the Saint-Venant approach yielded results which contained the Mainardi-Codazzi equations of surfaces implicitly. To the author's knowledge, the equations of compatiblity of strains in the middle surface of a thin elastic shell have never been developed by the Saint-Venant approach before.

Using the identities provided by the Saint-Venant method (and others), the compatibility equations as developed by Gol'denveizer, Novozhilov, Reissner and Vlasov were compared to the kinematic equations, the Saint-Venant equations, and therefore, to each other. It was seen that after a multitude of transformations, in which all conceptual significance was destroyed, all the different sets of equations of compatibility agreed (within the scope of the linear theory), except the equations as given by Reissner. The difference being one of algebraic signs, however, it was concluded that Reissner's equations must contain typographical errors (of signs).

A general comparison of the kinematic method with other methods of shell analysis was undertaken and was appended (Appendix B) to the main discussion. This was thought to be of value to those who are not at all familiar with the kinematic approach. It was shown in this discussion, that the kinematic method maintained at least as high a standard of accuracy (or, in many cases, a higher one) as did any of the other methods considered. The 1959 paper of Koiter was employed as the vehicle, by means of which the comparison was carried out.

APPENDIX A

A.1. THE FUNDAMENTAL DEFINITIONS

The Continuum

A continuum represents a continuous distribution of structureless matter. The very concept is, therefore, a *macroscopic* notion.

Homogeneity of Continua

A continuum is homogeneous if the physical properties (or physical constitution) thereof are independent of position, $\overline{r}$.

Isotropy of Continua

A continuum is isotropic at any point, $\overline{r}$, if the physical properties are independent of direction (orientation) at this point. A continuum, the physical properties of which *are* dependent upon direction at any point, is referred to as a "non-isotropic", "aelotropic", or "anisotropic" continuum.

Stress

If a very small force, $\Delta\overline{F}$, acts on a very small area, ΔA, then the state of stress experienced by the element of area is defined to be

$$\overline{\sigma} = \lim_{\Delta A \to 0} \left[\frac{\Delta\overline{F}}{\Delta A} \right] = \frac{d\overline{F}}{dA}$$

The stress distribution, i.e., the way in which the stress vector $\overline{\sigma}$ is distributed, over infinitesimal distances, $d\alpha$, is assumed to be essentially *linear*. This disallows the existence of any discontinuous functions. Then,

$$d\overline{F}\,(\overline{\sigma}_\alpha) = \overline{\sigma}_\alpha(\overline{r}_c)\,d\alpha$$

[or in words] if the stress vector at the centre of mass is multiplied by the differential area of the face upon which the stress is acting, the stress resultant for that face is produced.

A.2. THE GENERAL NATURE OF TENSORS

A tensor of any order is a multilinear vector form which remains invariant under a rotation of coordinates.

It is observed that a tensor is thus defined in terms of vectors (i.e., a tensor is a "vector form"). No attempt will be made to define a vector in more primitive terms.

Since tensors are multiply-directed quantities, this gives rise to the following "classification".

SCALAR: a tensor of the 0^{th} order
VECTOR: a tensor of the 1^{st} order
DYADIC: a tensor of the 2^{nd} order
TRIADIC: a tensor of the 3^{rd} order
⋮
POLYADIC: a tensor of the n^{th} order

Employing the EINSTEINIAN Summation Convention for repeated indices, these forms are represented in direct notation as follows:

SCALAR: T

VECTOR: $\overline{T} = T_\alpha \overline{e}_\alpha$

DYADIC: $\overline{\overline{T}} = T_{\alpha\beta} \overline{e}_\alpha \overline{e}_\beta$

TRIADIC: $\overline{\overline{\overline{T}}} = \overset{3}{\overline{T}} = T_{\alpha\beta\gamma} \overline{e}_\alpha \overline{e}_\beta \overline{e}_\gamma$

.
.
.
.

POLYADIC: $\overset{n}{\overline{T}} = T_{\alpha\beta\gamma\delta\ldots\ldots} \; \overline{e}_\alpha \overline{e}_\beta \overline{e}_\gamma \overline{e}_\delta \; \ldots\ldots$

The subject of elasticity in general (and shell theory in particular) deals with tensors which are primarily of the second order, i.e., dyadics. Accordingly, the following section deals exclusively with such quantities.

A.3. PARTICULAR DYADICS OF INTEREST

A.3.1. The Identity Tensor

The identity tensor ("idemfactor", or "eigentensor") is defined as

$$\overline{\overline{1}} = \overline{e}_i \; \overline{e}_i$$

This definition naturally arises, since the identity tensor must be a tensor, the fundamental property of which is to reproduce any quantity taken in (dot) product with it. For example, any vector $\overline{v}$ could be written as

$$\bar{v} = v_\alpha \bar{e}_\alpha = (\bar{v}\cdot\bar{e}_\alpha)\ \bar{e}_\alpha$$
$$= \bar{v}\cdot\bar{e}_\alpha\bar{e}_\alpha$$
$$= \bar{v}\cdot(\bar{e}_\alpha\bar{e}_\alpha) = \bar{v}\cdot\bar{\bar{1}} = \bar{v}$$

thus, $\bar{\bar{1}} = \bar{e}_1\bar{e}_1 + \bar{e}_2\bar{e}_2 + \bar{e}_3\bar{e}_3$ for the summation over three directions in *Cartesian* space. Unless otherwise specified, repeated indices in this discussion will be assumed to sum over three directions.

The identity tensor reproduces also, any dyadic taken in dot product with it,

$$\bar{\bar{T}} = \bar{\bar{T}}\cdot\bar{\bar{1}} = \bar{\bar{1}}\cdot\bar{\bar{T}} = \bar{\bar{T}}$$

thus, $$\bar{\bar{1}} = \bar{e}_i\bar{e}_j\delta_{ij} = \bar{e}_i\bar{e}_i$$

where δ_{ij} is the (simplified) KRONECKER DELTA, defined as:

$$\delta_{ij} = 0,\ i \neq j$$
$$\delta_{ij} = 1,\ i = j$$

A.3.2. Conjugate Dyadics

If a dyadic, $\bar{\bar{T}}$, is given as

$$\bar{\bar{T}} = T_{ij}\bar{e}_i\bar{e}_j$$

then the conjugate dyadic is defined as

$$\bar{\bar{T}}_c = T_{ji}\bar{e}_i\bar{e}_j$$

OR $$\bar{\bar{T}}_c = T_{ij}\bar{e}_j\bar{e}_i$$

The conjugate of a dyadic is thus precisely analogous to the "transpose" of a matrix. It follows obviously that

$$(\bar{\bar{T}}_c)_c = (T_{ji}\bar{e}_i\bar{e}_j)_c = T_{ij}\bar{e}_i\bar{e}_j = \bar{\bar{T}}$$

A.3.3. Symmetric Dyadics

A dyadic is defined to be symmetric iff

$$\bar{\bar{T}} = \bar{\bar{T}}_c$$

or, as $\bar{\bar{T}} = T_{ij}\bar{e}_i\bar{e}_j$ and $\bar{\bar{T}}_c = T_{ji}\bar{e}_i\bar{e}_j$

then for symmetry,

$$\bar{\bar{T}} - \bar{\bar{T}}_c = 0 = (T_{ij} - T_{ji})\ \bar{e}_i\bar{e}_j$$

which requires that the components, T_{ij} and T_{ji}, be equal.

A.3.4. Antisymmetric Dyadics

A dyadic is defined to be antisymmetric iff

$$\bar{\bar{T}} = -\ \bar{\bar{T}}_c$$

or, $\bar{\bar{T}} + \bar{\bar{T}}_c = 0 = (T_{ij} + T_{ji})\ \bar{e}_i\bar{e}_j$

which requires that the components, $(+T_{ij})$ and $(-T_{ji})$ be equal. It is to be observed that in the case that $i = j$, the above requirement is that $(+T_{ii})$ must be equal to $(-T_{ii})$. This is possible only for $T_{ii} = 0$; therefore the principal diagonal of the nonion form of an antisymmetric tensor vanishes.

The "nonion form" of a dyadic is simply its representation in fully-expanded form, written as an array for convenient use of such familiar matrix terms as "principal diagonal", etc.; the nonion form of a general tensor, $\bar{\bar{T}}$, is as follows.

$$\bar{\bar{T}} = \begin{bmatrix} T_{11}\bar{e}_1\bar{e}_1 + T_{12}\bar{e}_1\bar{e}_2 + T_{13}\bar{e}_1\bar{e}_3 \\ + T_{21}\bar{e}_2\bar{e}_1 + T_{22}\bar{e}_2\bar{e}_2 + T_{23}\bar{e}_2\bar{e}_3 \\ + T_{31}\bar{e}_3\bar{e}_1 + T_{32}\bar{e}_3\bar{e}_2 + T_{33}\bar{e}_3\bar{e}_3 \end{bmatrix}$$

A.3.5. Resolution of a Dyadic

Any dyadic may be resolved into two other dyadics of particular interest, namely a symmetric and an antisymmetric dyadic. Denoting the symmetric part of $\bar{\bar{T}}$ as $\bar{\bar{T}}^{(s)}$ and the antisymmetric part as $\bar{\bar{T}}^{(a)}$, then consider the identity

$$\bar{\bar{T}} \equiv \frac{1}{2}[\bar{\bar{T}} + \bar{\bar{T}}_c] + \frac{1}{2}[\bar{\bar{T}} - \bar{\bar{T}}_c]$$

as $[\bar{\bar{T}} + \bar{\bar{T}}_c]_c = [\bar{\bar{T}}_c + \bar{\bar{T}}] = [\bar{\bar{T}} + \bar{\bar{T}}_c]$

then this part of the tensor is symmetric. Similarly, as $[\bar{T} - \bar{T}_c]_c = [\bar{T}_c - \bar{T}] = -[\bar{T} - \bar{T}_c]$ then this part of the tensor is antisymmetric.

Then $$\bar{\bar{T}} = \bar{\bar{T}}^{(s)} + \bar{\bar{T}}^{(a)}$$

where $$\bar{\bar{T}}^{(s)} = \frac{1}{2}[\bar{\bar{T}} + \bar{\bar{T}}_c]$$

$$\bar{\bar{T}}^{(a)} = \frac{1}{2}[\bar{\bar{T}} - \bar{\bar{T}}_c]$$

A.4. THE ALGEBRA OF DYADICS

Any dyadic, $\bar{D}$, may be considered to be the sum of the juxtaposition of vector pairs, say $\bar{m}_i\bar{n}_i = \bar{\bar{D}}$. (This holds as long as the dyadic $\bar{D}$ is real, as the real number system is closed under multiplication and is, in fact, a field.) This representation affords one explanation for the various products, while the summation notation provides another. The two are, naturally, equivalent -- but one or the other may be more useful for some particular purpose; accordingly, both are discussed here.

A.4.1. Single Products of Dyadics

A.4.1.1. Dot Product

a) with a vector

Direct: $\bar{\bar{D}}\cdot\bar{v} = (\bar{m}\,\bar{n})\cdot\bar{v} = \bar{m}(\bar{n}\cdot\bar{v}) = (\bar{n}\cdot\bar{v})\,\bar{m}$

Summation: $\bar{\bar{D}}\cdot\bar{v} = D_{ij}\bar{e}_i\bar{e}_j\cdot\; v_k\bar{e}_k$

$= D_{ij}v_k\bar{e}_i(\bar{e}_j\cdot\bar{e}_k) = D_{ij}v_k\bar{e}_i\delta_{jk}$

$= D_{ij}v_j\bar{e}_i$ RESULT: A vector

b) with a dyadic

Direct: $\bar{\bar{D}}\cdot\bar{\bar{T}} = (\bar{m}\,\bar{n})\cdot(\bar{p}\,\bar{q}) = \bar{m}(\bar{n}\cdot\bar{p})\bar{q} = (\bar{n}\cdot\bar{p})\bar{m}\,\bar{q}$

Summation: $\bar{\bar{D}}\cdot\bar{\bar{T}} = D_{ij}\bar{e}_i\bar{e}_j\cdot T_{rs}\bar{e}_r\bar{e}_s$

$= D_{ij}T_{rs}\bar{e}_i(\bar{e}_j\cdot\bar{e}_r)\bar{e}_s = D_{ij}T_{rs}\bar{e}_i\bar{e}_s\delta_{jr}$

$= D_{ij}T_{js}\bar{e}_i\bar{e}_s$ RESULT: A dyadic

A.4.1.2. Cross Product

a) with a vector

Direct: $\bar{\bar{D}} \times \bar{v} = (\bar{m}\,\bar{n}) \times \bar{v} = \bar{m}(\bar{n} \times \bar{v})$

Summation: $\bar{\bar{D}} \times \bar{v} = D_{ij}\bar{e}_i\bar{e}_j \times v_k\bar{e}_k$

$= D_{ij}v_k\bar{e}_i(\bar{e}_j \times \bar{e}_k) = D_{ij}v_k\bar{e}_iE_{jkr}\bar{e}_r$

$= D_{ij}v_kE_{jkr}\bar{e}_i\bar{e}_r$ RESULT: A dyadic

NOTE: The symbol $E_{\alpha\beta\gamma}$ is used here to denote the LEVI-CIVITA Three-Index Density Function. It is defined as:

$$E_{\alpha\beta\gamma} = 0 \text{ for } \alpha = \beta \text{ or } \beta = \gamma \text{ or } \alpha = \gamma$$

$= +1$ for $\alpha \neq \beta$, $\beta \neq \gamma$, $\alpha \neq \gamma$, and cyclic order is maintained

$= -1$ for $\alpha \neq \beta$, $\beta \neq \gamma$, $\alpha \neq \gamma$, and cyclic order is not maintained.

b) with a dyadic

Direct: $\bar{\bar{D}} \times \bar{\bar{T}} = (\bar{m}\,\bar{n}) \times (\bar{p}\,\bar{q}) = \bar{m}\,(\bar{n} \times \bar{p})\bar{q}$

Summation: $\bar{\bar{D}} \times \bar{\bar{T}} = D_{ij}\bar{e}_i\bar{e}_j \times T_{rs}\bar{e}_r\bar{e}_s$

$= D_{ij}T_{rs}\bar{e}_i(\bar{e}_j \times \bar{e}_r)\bar{e}_s = D_{ij}T_{rs}\bar{e}_i E_{jru}\bar{e}_u\bar{e}_s$

$= D_{ij}T_{rs}E_{jru}\bar{e}_i\bar{e}_u\bar{e}_s$ RESULT: A Triadic

A.4.2. Double Products of Dyadics

A.4.2.1. Double Dot Product

Direct: $\bar{\bar{D}}:\bar{\bar{T}} = (\bar{m}\,\bar{n}):(\bar{p}\,\bar{q}) = (\bar{m}\cdot\bar{p})(\bar{n}\cdot\bar{q})$

Summation: $\bar{\bar{D}}:\bar{\bar{T}} = D_{ij}\bar{e}_i\bar{e}_j:T_{rs}\bar{e}_r\bar{e}_s$

$= D_{ij}T_{rs}(\bar{e}_i\cdot\bar{e}_r)(\bar{e}_j\cdot\bar{e}_s) = D_{ij}T_{rs}\delta_{ir}\delta_{js}$

$= D_{ij}T_{ij}$ RESULT: A Scalar

A.4.2.2. Double Cross Product

Direct: $\bar{\bar{D}} \overset{\times}{\times} \bar{\bar{T}} = (\bar{m}\,\bar{n}) \overset{\times}{\times} (\bar{p}\,\bar{q}) = (\bar{m} \times \bar{p})(\bar{n} \times \bar{q})$

Summation: $\bar{\bar{D}} \underset{\times}{\overset{\times}{}} \bar{\bar{T}} = D_{ij}\bar{e}_i\bar{e}_j \underset{\times}{\overset{\times}{}} T_{rs}\bar{e}_r\bar{e}_s$

$$= D_{ij}T_{rs}(\bar{e}_i \times \bar{e}_r)(\bar{e}_j \times \bar{e}_s) = D_{ij}T_{rs}E_{iru}\bar{e}_u E_{jsv}\bar{e}_v$$

$$= D_{ij}T_{rs}E_{iru}E_{jsv}\bar{e}_u\bar{e}_v$$

$$= D_{ij}T_{rs}\delta^{iru}_{jsv}\bar{e}_u\bar{e}_v \qquad \text{RESULT: A Dyadic}$$

where $\delta^{\alpha\beta\gamma}_{\mu\nu\pi}$ is the Generalized KRONECKER DELTA

(see any standard work on Tensor analysis)

A.4.2.3. Mixed Dot and Cross Product

Direct: $\bar{\bar{D}} \overset{\bullet}{\times} \bar{\bar{T}} = (\bar{m}\,\bar{n})\overset{\bullet}{\times}(\bar{p}\,\bar{q}) = (\bar{m}\cdot\bar{p})(\bar{n} \times \bar{q})$

Summation: $\bar{\bar{D}} \overset{\bullet}{\times} \bar{\bar{T}} = D_{ij}\bar{e}_i\bar{e}_j \overset{\bullet}{\times} T_{rs}\bar{e}_r\bar{e}_s$

$$= D_{ij}T_{rs}(\bar{e}_i\cdot\bar{e}_r)(\bar{e}_j \times \bar{e}_s) = D_{ij}T_{rs}\delta_{ir}E_{jsu}\bar{e}_u$$

$$= D_{ij}T_{is}E_{jsu}\bar{e}_u \qquad \text{RESULT: A Vector}$$

Similarly,

Direct: $\bar{\bar{D}} \underset{\bullet}{\overset{\times}{}} \bar{\bar{T}} = (\bar{m}\,\bar{n})\underset{\bullet}{\overset{\times}{}}(\bar{p}\,\bar{q}) = (\bar{m} \times \bar{p})(\bar{n}\cdot\bar{q}) = (\bar{n}\cdot\bar{q})(\bar{m} \times \bar{p})$

Summation: $\bar{\bar{D}} \underset{\bullet}{\overset{\times}{}} \bar{\bar{T}} = D_{ij}\bar{e}_i\bar{e}_j \underset{\bullet}{\overset{\times}{}} T_{rs}\bar{e}_r\bar{e}_s$

$$= D_{ij}T_{rs}(\bar{e}_i \times \bar{e}_r)(\bar{e}_j\cdot\bar{e}_s) = D_{ij}T_{rs}E_{irv}\bar{e}\delta_{js}$$

$$= D_{ij}T_{rj}E_{irv}\bar{e}_v \qquad \text{RESULT: A Vector}$$

NOTE: From the above discussion, it is evident that the double products are commutative,

i.e. $\bar{\bar{D}}:\bar{\bar{T}} = \bar{\bar{T}}:\bar{\bar{D}}$

$$\bar{\bar{D}} \underset{\times}{\overset{\times}{}} \bar{\bar{T}} = \bar{\bar{T}} \underset{\times}{\overset{\times}{}} \bar{\bar{D}}$$

A.5. DYADIC INVARIANTS

A.5.1. The First Scalar Invariant

The first scalar invariant, $\bar{\bar{D}}_s^{(1)}$, of a dyadic, $\bar{\bar{D}}$, is defined to be

$$\bar{\bar{D}}_s^{(1)} = \bar{\bar{D}}:\bar{\bar{1}}$$

Thus
$$\bar{\bar{D}}_s^{(1)} = D_{ij}\bar{e}_i\bar{e}_j:\bar{e}_r\bar{e}_r = D_{ij}(\bar{e}_i\cdot\bar{e}_r)(\bar{e}_j\cdot\bar{e}_r)$$
$$= D_{ij}\delta_{ir}\delta_{jr} = D_{rr}$$

or $\bar{\bar{D}}_s^{(1)} = (D_{11} + D_{22} + D_{33})$ in expanded form. In *matric* form, $\bar{\bar{D}}_s^{(1)}$ is equal to the sum of the elements of the principal diagonal, and is known as the *trace* of the matrix.

A.5.2. The Second Scalar Invariant

The second scalar invariant, $\bar{\bar{D}}_s^{(2)}$, of a dyadic, $\bar{\bar{D}}$, is defined to be

$$\bar{\bar{D}}_s^{(2)} = \frac{1}{2!}\,\bar{\bar{D}} \overset{\times}{\times} \bar{\bar{D}}:\bar{\bar{1}}$$

or
$$\bar{\bar{D}}_s^{(2)} = \begin{vmatrix} D_{22} & D_{23} \\ D_{32} & D_{33} \end{vmatrix} + \begin{vmatrix} D_{11} & D_{12} \\ D_{31} & D_{33} \end{vmatrix} + \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix}$$

In matric form, $\bar{\bar{D}}_s^{(2)}$ represents the sum of the *minors* of the matrix, *expanded about the principal diagonal.*

A.5.3. The Third Scalar Invariant

The third scalar invariant, $\bar{\bar{D}}_s^{(3)}$, of a dyadic, $\bar{\bar{D}}$, is defined to be

$$\bar{\bar{D}}_s^{(3)} = \frac{1}{3!}\,\bar{\bar{D}} \overset{\times}{\times} \bar{\bar{D}}:\bar{\bar{D}}$$

or, $$\bar{\bar{D}}_s^{(3)} = \begin{vmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{vmatrix}$$

In matric form, $\bar{D}_s^{(3)}$ represents the *full determinant* of the matrix.

A.5.4. The Vector Invariant

The vector invariant, $\bar{D}_v$ (sometimes, $\bar{D}_x$), of a dyadic, $\bar{D}$, is defined as

$$\bar{\bar{D}}_v = \bar{\bar{1}} \overset{\bullet}{x} \bar{\bar{D}} \quad (= - \bar{\bar{D}} \overset{x}{\bullet} \bar{\bar{1}})$$

If $\bar{\bar{D}} = \bar{e}_i\bar{D}_i = \bar{e}_1\bar{D}_1 + \bar{e}_2\bar{D}_2 + \bar{e}_3\bar{D}_3$ (the "trinomial" form of the dyadic), then,

$$\bar{\bar{D}}_v = \bar{\bar{1}} \overset{\bullet}{x} \bar{e}_i\bar{D}_i = \bar{e}_i x \bar{D}_i$$

or $$\bar{\bar{D}}_v = (\bar{e}_1 \times \bar{D}_1 + \bar{e}_2 \times \bar{D}_2 + \bar{e}_3 \times \bar{D}_3)$$

It is observed that if the dyadic were expressed as the juxtaposition of two vectors

$$\text{(say) } \bar{\bar{D}} = \bar{m}\,\bar{n} \quad \text{as before}$$

then
$$\bar{\bar{D}}_v = \bar{\bar{1}} \times \bar{m}\,\bar{n}$$
$$= \bar{\bar{1}} \times [\tfrac{1}{2}(\bar{m}\,\bar{n} + \bar{n}\,\bar{m}) + \tfrac{1}{2}(\bar{m}\,\bar{n} - \bar{n}\,\bar{m})]$$
$$= \tfrac{1}{2}(\bar{m} \times \bar{n} + \bar{n} \times \bar{m}) + \tfrac{1}{2}(\bar{m} \times \bar{n} - \bar{n} \times \bar{m})$$

The first term vanishes as $\bar{n} \times \bar{m} = - \bar{m} \times \bar{n}$

so $\bar{\bar{D}}_v = \frac{1}{2}(\bar{m} \times \bar{n} - \bar{n} \times m) = \frac{1}{2}(2\bar{m} \times \bar{n}) = (\bar{m} \times \bar{n})$

Thus, the vector invariant is obtained solely from the *antisymmetric* part of the dyadic. Thus, this is a criterion for dyadic symmetry -- if the tensor is symmetric, the antisymmetric part does not exist and consequently, *for symmetric dyadics, the vector invariant vanishes.*

By a similar argument, it is easily seen that *for an antisymmetric dyadic, the first scalar invariant vanishes.*

A.6. THE LAGRANGIAN FORM OF TAYLOR'S SERIES EXPANSION FOR A POINT-FUNCTION

In the differential calculus, the TAYLOR's Series expansion is usually developed as

$$F(x + \Delta x) = \left[1 + \frac{1}{1!} \frac{d}{dx} (\Delta x) + \frac{1}{2!} \frac{d^2}{dx^2} (\Delta x)^2 + \ldots + \frac{1}{n!} \frac{d^n}{dx^n} (\Delta x)^n \right] F(x)$$

However, there is a direct one-to-one correspondence between this representation and

$$e^z = 1 + \frac{1}{1!} z + \frac{1}{2!} z^z + \ldots . + \frac{1}{n!} z^n$$

(where e is the base of natural logarithms)

such that if $z = \frac{d}{dx} (\Delta x)$

and $\left[\frac{d}{dx} (\Delta x) \right]^n$ is understood to signify $\frac{d^n}{dx^n} (\Delta x)^n$

then $e^{\left[\frac{d}{dx} \Delta x\right]} = \left[1 + \frac{1}{1!} \frac{d}{dx} (\Delta x) + \frac{1}{2!} \frac{d^2}{dx^2} (\Delta x)^2 + \ldots + \frac{1}{n!} \frac{d^n}{dx^n} (\Delta x)^n \right]$

and therefore, it may be said that

$$F(x + \Delta x) = e^{\left[\frac{d}{dx} \Delta x\right]} F(x)$$

or, as $\frac{d}{dx}$ is non-operative re: Δx, this might be written, to avoid ambiguity, as

$$F(x + \Delta x) = e^{\left[\Delta x \frac{d}{dx}\right]} F(x) \qquad \ldots\ldots \qquad \{A.6.-1.\}$$

A.6.1. The Expansion for a Scalar Point-Function

Given a scalar point-function, $F(x,y,z)$ and any parametric variable, t, such that

$\alpha = \alpha(t)$, $\alpha = \{x,y,z\}$, then

$$F(t + \Delta t) = e^{\left[\Delta t \frac{d}{dt}\right]} F(t)$$

Now, since

$$\frac{d}{dt} \equiv \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z}$$

so $$\Delta t \frac{d}{dt} \equiv \Delta t \frac{dx}{dt}\frac{\partial}{\partial x} + \Delta t \frac{dy}{dt}\frac{\partial}{\partial y} + \Delta t \frac{dz}{dt}\frac{\partial}{\partial z}$$

With reference to the usual concepts of EUCLIDIAN three-space, it may be concluded that $\frac{\Delta x}{\Delta y} = \frac{dx}{dy}$, as a "pseudo-geometric (or affine) proportionality".

Then $$\Delta t \frac{d}{dt} \equiv \Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} + \Delta z \frac{\partial}{\partial z} \equiv \overline{r}\cdot\frac{\partial}{\partial \overline{r}}$$

where $\overline{r} = x\overline{e}_x + y\overline{e}_y + z\overline{e}_z$

Therefore, for any *scalar* point-function, $F(t)$

$$F(t + \Delta t) = e^{\left[\Delta \overline{r}\cdot\frac{\partial}{\partial \overline{r}}\right]} F(t) \qquad \ldots\ldots \qquad \{A.6.1.-1.\}$$

A.6.2. The Expansion for a Vector Point-Function

Having established {A.6.1.-1.} for scalar point-functions, then it is but a simple step to specify that scalar point-function in terms of a constant vector in dot-product with a vector point-function.

VIZ: (say) $F(t) = \bar{a}\cdot\bar{F}'(t)$ $[\bar{a} \neq \bar{a}(\bar{r})]$

From this, it follows that

$$F(t + \Delta t) = e^{\left[\Delta\bar{r}\cdot\frac{\partial}{\partial\bar{r}}\right]} [\bar{a}\cdot\bar{F}'(t)]$$

$$= \bar{a}\cdot\left[e^{\left[\Delta\bar{r}\cdot\frac{\partial}{\partial\bar{r}}\right]} \bar{F}'(t)\right]$$

Therefore, the TAYLOR's Series expansion for a *vector* point-function appears as

$$\bar{F}'(t + \Delta t) = e^{\left[\Delta\bar{r}\cdot\frac{\partial}{\partial\bar{r}}\right]} \bar{F}'(t)$$

A.7. THE LINEAR THEORY OF STRAIN

It is assumed that the deformation of a continuous medium is homogeneous; i.e., infinitesimal vectors, $d\bar{r}$, may deform to infinitesimal vectors, $d\bar{R}$, but not to infinitesimal (or finite) curves (see Fig. A.7.-1.).

In Fig. A.7.-1., the quantities $(d\bar{r}\cdot\bar{\bar{\varepsilon}})$ and $(d\bar{r}\cdot\bar{\bar{\phi}})$ are the components of $d\bar{u}$, parallel to and perpendicular to $d\bar{R}$, respectively.

From Fig. A.7.-1., it is seen that

$$d\bar{u} = \bar{u}\,(\bar{r} + d\bar{r}) - \bar{u}(\bar{r})$$

Expanding $\bar{u}(\bar{r} + d\bar{r})$ as a TAYLOR's Series expansion:

$$\bar{u}\,(\bar{r} + d\bar{r}) = e^{\left[d\bar{r}\cdot\frac{\partial}{\partial\bar{r}}\right]} \bar{u}(\bar{r})$$

$$= \left[1 + d\bar{r}\cdot\frac{\partial}{\partial\bar{r}} + \frac{1}{2!}\left(d\bar{r}\cdot\frac{\partial}{\partial\bar{r}}\right)^2 + \ldots\right]\bar{u}(\bar{r})$$

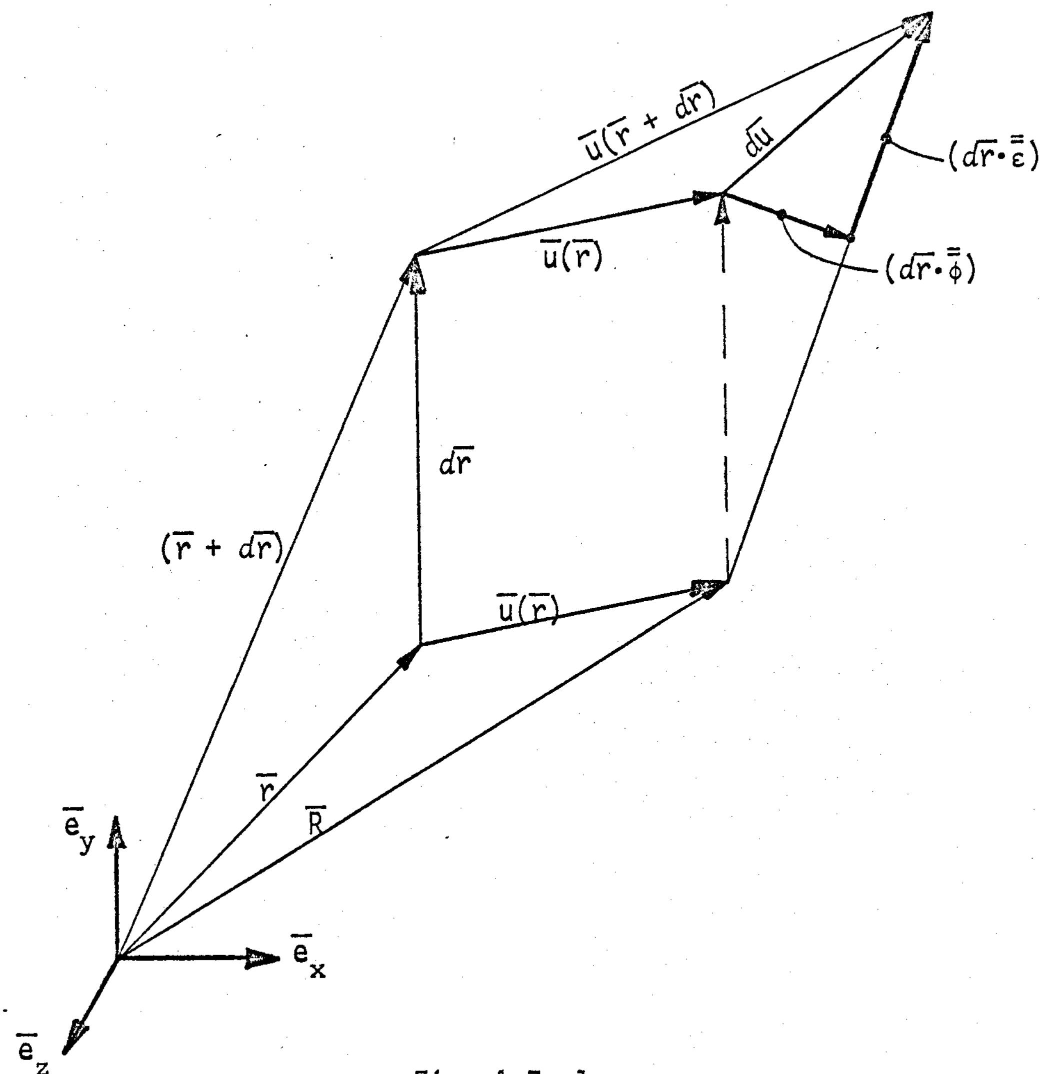

Fig. A.7.-1.

Assuming a first-order approximation to be sufficiently accurate for the linear theory, then

$$\bar{u}\ (\bar{r} + d\bar{r}) = \bar{u}(\bar{r}) + d\bar{r}\cdot \frac{\partial \bar{u}}{\partial \bar{r}}$$

Therefore,

$$d\bar{u} = [\bar{u}(\bar{r} + d\bar{r}) - \bar{u}(\bar{r})] = d\bar{r}\cdot\frac{\partial \bar{u}}{\partial \bar{r}}$$

i.e.:

$$d\bar{u} = d\bar{r}\cdot\frac{\partial \bar{u}}{\partial \bar{r}}$$

which represents the total relative displacement of $d\bar{r}$.

Referring to $\bar{\bar{u}} = \frac{\partial \bar{u}}{\partial \bar{r}}$ as the *deformation tensor* (sometimes: *displacement gradient*), then by allowing $\bar{\bar{u}}$ to be decomposed into its symmetric and antisymmetric parts (see §A.3.5.), then the total relative displacement is composed of two parts:

$$d\bar{r}\cdot\frac{\partial \bar{u}}{\partial \bar{r}} = d\bar{r}\cdot\left\{\frac{1}{2}\left[\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}\partial}{\partial \bar{r}}\right] + \frac{1}{2}\left[\frac{\partial \bar{u}}{\partial \bar{r}} - \frac{\bar{u}\partial}{\partial \bar{r}}\right]\right\}$$

i.e., the relative straining displacement is given by

$$\frac{1}{2}\, d\bar{r}\cdot\left[\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\bar{u}\partial}{\partial \bar{r}}\right] \equiv d\bar{r}\cdot\bar{\bar{\varepsilon}}$$

where $\bar{\bar{\varepsilon}}$ is the *strain tensor* and the relative (rigid-body) rotational displacement is given by

$$\frac{1}{2}\, d\bar{r}\cdot\left[\frac{\partial \bar{u}}{\partial \bar{r}} - \frac{\bar{u}\partial}{\partial \bar{r}}\right] \equiv d\bar{r}\cdot\bar{\bar{\phi}}$$

where $\bar{\bar{\phi}}$ is the *rotation tensor*.

Recognizing that the form of the rotation tensor is analogous to

$$\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A}\cdot\bar{C})\bar{B} - (\bar{A}\cdot\bar{B})\bar{C}$$

then $$d\bar{r}\cdot\bar{\bar{\phi}} = \frac{1}{2}\left[d\bar{r} \times \left(\bar{u} \times \frac{\partial}{\partial \bar{r}}\right)\right]$$

thus, the relative rotation of $d\bar{r}$, i.e., $d\bar{r}\cdot\bar{\bar{\phi}}$, may be given as

$$d\bar{r}\cdot\bar{\bar{\phi}} = \frac{1}{2}\frac{\partial \times \bar{u}}{\partial \bar{r}} \times d\bar{r}$$

Then, in summary,

a) the relative rigid-body rotation of $d\bar{r}$ during deformation is given by

$$d\bar{r}\cdot\bar{\bar{\phi}} = \frac{1}{2}\frac{\partial \times \bar{u}}{\partial \bar{r}} \times d\bar{r}$$

b) the relative straining displacement of $d\bar{r}$ during deformation is given by

$$d\bar{r}\cdot\bar{\bar{\varepsilon}} = \frac{1}{2}\, d\bar{r}\cdot\left[\frac{\partial\bar{u}}{\partial\bar{r}} + \frac{\bar{u}\partial}{\partial\bar{r}}\right]$$

$$= d\bar{r}\cdot\frac{\partial\bar{u}}{\partial\bar{r}} - \frac{1}{2}\frac{\partial \times \bar{u}}{\partial\bar{r}} \times d\bar{r}$$

A.8. THE SEGNER EIGENVALUE EQUATION FOR DYADICS

Consider first, the development for a vector rather than a dyadic. Although somewhat trivial, this serves to clarify the basic conceptions.

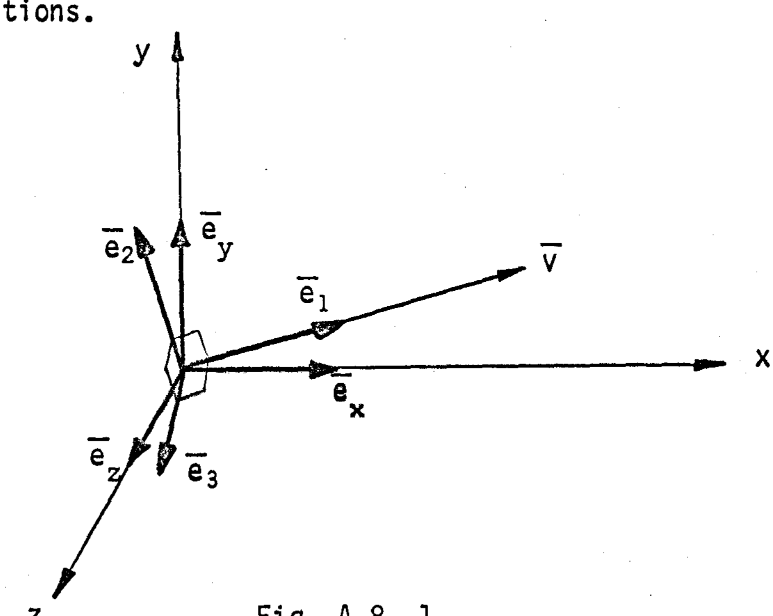

Fig. A.8.-1.

In Fig. A.8.-1., the vector $\overline{v}$ may be given as

$$\overline{v} = v_\alpha \overline{e}_\alpha \qquad \text{(sum on } \alpha = x,y,z)$$

It is desired to represent this vector ($\overline{v}$) in such a form that it will have its components *maximized* in one direction and *minimized* in the others. Hence, a new coordinate system (say, the $\overline{e}_1$, $\overline{e}_2$, $\overline{e}_3$ system) is sought which is different from the present ($\overline{e}_x$, $\overline{e}_y$, $\overline{e}_z$) system.

This is a minimum-maximum or *extremum* problem, since $\overline{v}$ has its maximum along $\overline{e}_1$ (arbitrarily chosen from $\overline{e}_1$, $\overline{e}_2$ and $\overline{e}_3$) and therefore, its minimum components along $\overline{e}_2$ and $\overline{e}_3$.

Now,

$$\overline{v} = v_\alpha \overline{e}_\alpha = (\overline{v}\cdot\overline{e}_\alpha)\ \overline{e}_\alpha$$

(for Cartesian-base unit vectors)

For an extremum value of v_α,

$$\delta v_\alpha = 0$$

so

$$\delta(\overline{v}\cdot\overline{e}_\alpha) = 0$$

or

$$\delta\overline{v}\cdot\overline{e}_\alpha + \overline{v}\cdot\delta\overline{e}_\alpha = 0$$

However, the whole vector, $\overline{v}$, is an invariant quantity, so

$$\delta\overline{v} = 0$$

Thus,

$$\overline{v}\cdot\delta\overline{e}_\alpha = 0 \qquad \ldots\ldots \qquad \{A.8.-1.\}$$

The constraint condition enforced for the variation is

$$(\overline{e}_\alpha + \delta\overline{e}_\alpha)\cdot(\overline{e}_\alpha + \delta\overline{e}_\alpha) = \overline{e}_\alpha\cdot\overline{e}_\alpha$$

or, neglecting second-order variations,

$$\overline{e}_\alpha\cdot\overline{e}_\alpha + \overline{e}_\alpha\cdot\delta\overline{e}_\alpha + \delta\overline{e}_\alpha\cdot\overline{e}_\alpha = \overline{e}_\alpha\cdot\overline{e}_\alpha \qquad \ldots\ldots \qquad \{A.8.-2.\}$$

In words, this could be stated as: the variation of $\overline{e}_\alpha$ must not change its length; this implies a rotational variation. Thus, from {A.8.-2.},

$$\overline{e}_\alpha \cdot \delta\overline{e}_\alpha + \delta\overline{e}_\alpha \cdot \overline{e}_\alpha = 2\overline{e}_\alpha \cdot \delta\overline{e}_\alpha = 0$$

so $$\overline{e}_\alpha \cdot \delta\overline{e}_\alpha = 0 \qquad \ldots\ldots \qquad \{A.8.-3.\}$$

Such a constraint condition would be formally referred to as *the constraint condition on the variation of the axis of reference.* The rotational nature of the variation would allow $\delta\overline{e}_\alpha$ to be expressed as $\delta\overline{\phi} \times \overline{e}_\alpha$, if desired -- thus indicating that $\delta\overline{e}_\alpha$ has no component in the direction of $\overline{e}_\alpha$.

Examining {A.8.-1.} and{A.8.-3.}, it is seen that these are proportional equations.

Therefore, $$\overline{v} \cdot \delta\overline{e}_\alpha = \lambda_\alpha \, \overline{e}_\alpha \cdot \delta\overline{e}_\alpha$$

where the surplus scalar factor, λ_α, is called the eigenvalue (or "characteristic value").

Then $$(\overline{v} = \lambda_\alpha \overline{e}_\alpha) \cdot \delta\overline{e}_\alpha = 0$$

and for nontrivial variations, $\delta\overline{e}_\alpha$, then

$$\overline{v} - \lambda_\alpha \overline{e}_\alpha \qquad \text{(no sum on } \alpha \text{ now)}$$

so $$\overline{v} = \lambda_\alpha \overline{e}_\alpha \qquad \text{(no sum)}$$

The eigenvalue λ_α thus prescribes the extremum for the direction $\overline{e}_\alpha$.

Having thus described the underlying basis of the extremizing process, this procedure will be now employed for a second-order tensor; in this case the result will not be so obvious as for the above.

Any dyadic can be given as

$$\bar{\bar{T}} = T_{ij}\bar{e}_i\bar{e}_j \qquad \text{(as before)}$$

and the trace of its matrix form, as

$$T_{ii} = \bar{e}_i\cdot\bar{\bar{T}}\cdot\bar{e}_i$$

According to EULER's Extremal Property for the Principal Directions of Tensors, the main diagonal terms assume extremal values.

i.e., $T_{\alpha\alpha}$ = extremum, so $\delta T_{\alpha\alpha} = 0$

It is noted that $\delta T_{\alpha\alpha} = 0$ is a necessary, but not a sufficient condition for an extremal value. It may then be said that the necessary condition for the extremal value of $T_{\alpha\alpha}$ is the stationary value:

$$\delta T_{\alpha\alpha} = \delta(\bar{e}_\alpha\cdot\bar{\bar{T}}\cdot\bar{e}_\alpha) = 0$$

or

$$\delta\bar{e}_\alpha\cdot\bar{\bar{T}}\cdot\bar{e}_\alpha + \bar{e}_\alpha\cdot\delta\bar{\bar{T}}\cdot\bar{e}_\alpha + \bar{e}_\alpha\cdot\bar{\bar{T}}\cdot\delta\bar{e}_\alpha = 0$$

or, as $\bar{\bar{T}}$ vanishes ($\bar{\bar{T}}$ itself being invariant) then

$$\delta\bar{e}_\alpha\cdot\bar{\bar{T}}\cdot\bar{e}_\alpha + \bar{e}_\alpha\cdot\bar{\bar{T}}\cdot\delta\bar{e}_\alpha = 0 \qquad \ldots\ldots \qquad \{A.8.-4.\}$$

Iff the tensor is symmetric ($\bar{\bar{T}} = \bar{\bar{T}}_c$), then {A.8.-4.} becomes

$$2\bar{e}_\alpha\cdot\bar{\bar{T}}\cdot\delta\bar{e}_\alpha = 0$$

or

$$\bar{e}_\alpha\cdot\bar{\bar{T}}\cdot\delta\bar{e}_\alpha = 0 \qquad \ldots\ldots \qquad \{A.8.-5.\}$$

The constraint condition on the variation is again given as the "restriction to variation of rotation"

i.e.

$$\bar{e}_\alpha\cdot\delta\bar{e}_\alpha = 0 \qquad \ldots\ldots \qquad \{A.8.-6.\}$$

Now, as {A.8.-5.} and {A.8.-6.} are proportional equations, then

$$\bar{e}_\alpha\cdot\bar{\bar{T}}\cdot\delta\bar{e}_\alpha = \lambda_{\alpha\alpha}\,(\bar{e}_\alpha\cdot\delta\bar{e}_\alpha)$$

where $\lambda_{\alpha\alpha}$ is the eigenvalue, as before.

Then $$(\bar{e}_\alpha \cdot \bar{\bar{T}} - \lambda_{\alpha\alpha}\bar{e}_\alpha)\cdot\delta\bar{e}_\alpha = 0$$

and consequently, for nontrivial $\delta\bar{e}_\alpha$,

$$\bar{e}_\alpha \cdot \bar{\bar{T}} - \lambda_{\alpha\alpha}\bar{e}_\alpha = 0 \qquad \ldots\ldots \qquad \{A.8.-7.\}$$

Now, it is always possible to say

$$\bar{e}_\alpha = \bar{e}_\alpha \cdot \bar{\bar{I}}$$

so, {A.8.-7.} becomes

$$\bar{e}_\alpha \cdot \bar{\bar{T}} - \lambda_{\alpha\alpha}\bar{e}_\alpha \cdot \bar{\bar{I}} = 0$$

or $$\bar{e}_\alpha \cdot [\bar{\bar{T}} - \lambda_{\alpha\alpha}\bar{\bar{I}}] = 0 \qquad \ldots\ldots \qquad \{A.8.-8.\}$$

where $(\bar{\bar{T}} - \lambda_{\alpha\alpha}\bar{\bar{I}})$ may be referred to as the *eigentensor.*

It is now desired to solve for both $\bar{e}_\alpha$ and $\lambda_{\alpha\alpha}$, but one more relationship is required in order that the number of equations be equal to the number of unknowns (determinate form). This relationship is actually available from the criterion $\bar{e}_\alpha \cdot \bar{e}_\alpha = 1$. Then, proceeding,

$$\bar{e}_\alpha \cdot \bar{\bar{T}} = \bar{e}_\alpha \cdot (\bar{e}_x\bar{T}_x + \bar{e}_y\bar{T}_y + \bar{e}_z\bar{T}_z)$$

$$= \ell_\alpha^x\bar{T} + \ell_\alpha^y\bar{T} + \ell_\alpha^z\bar{T}$$

where ℓ_α^β is the cosine obtained from the product

$$\bar{e}_\beta \cdot \bar{e}_\alpha = \mathrm{Cos}\ \psi = \ell_\alpha^\beta$$

Accordingly, {A.8.-8.} becomes

$$(e_\alpha \cdot \bar{\bar{T}} - \lambda_{\alpha\alpha}\bar{e}_\alpha) = [\ell_\alpha^x\bar{T}_x + \ell_\alpha^y\bar{T}_y + \ell_\alpha^z\bar{T}_z - \lambda_{\alpha\alpha}(\ell_\alpha^x\bar{e}_x + \ell_\alpha^y\bar{e}_y + \ell_\alpha^z\bar{e}_z)]$$

$$= [\ell_\alpha^x(\bar{T}_x - \lambda_{\alpha\alpha}\bar{e}_x) + \ell_\alpha^y(\bar{T}_y - \lambda_{\alpha\alpha}\bar{e}_y) + \ell_\alpha^z(\bar{T}_z - \lambda_{\alpha\alpha}\bar{e}_z)]$$

$$= (\ell_\alpha^x\bar{A}_x + \ell_\alpha^y\bar{A}_y + \ell_\alpha^z\bar{A}_z) = 0 \qquad \ldots\ldots \qquad \{A.8.-9.\}$$

where $\bar{A}_i = (\bar{T}_i - \lambda_{\alpha\alpha}\bar{e}_i)$

From this ({A.8.-9.}), it is observed that for the three vectors, $\overline{A}_i$, to sum to zero, then they must satisfy the physical interpretation of a closed, spatial triangle. Hence, these are three coplanar vectors, and for such a case, the relationship (expressing zero volume)

$$\overline{A}_x \cdot \overline{A}_y \times \overline{A}_z = 0$$

is valid. In original form, this appears as

$$(\overline{T}_x - \lambda_{\alpha\alpha}\overline{e}_x)\cdot(\overline{T}_y - \lambda_{\alpha\alpha}\overline{e}_y) \times (\overline{T}_z - \lambda_{\alpha\alpha}\overline{e}_z) = 0$$

or, expanding,

$$\begin{aligned}[&\overline{T}_x\cdot\overline{T}_y \times \overline{T}_z - \lambda_{\alpha\alpha}(\overline{e}_x\cdot\overline{T}_y \times \overline{T}_z + \overline{T}_x\cdot\overline{e}_y \times \overline{T}_z + \overline{T}_x\cdot\overline{T}_y \times \overline{e}_z) \\ &+ \lambda_{\alpha\alpha}^2(\overline{e}_x\cdot\overline{e}_y \times \overline{T}_z + \overline{e}_y\cdot\overline{e}_z \times \overline{T}_x + \overline{e}_z\cdot\overline{e}_x \times \overline{T}_y) \\ &- \lambda_{\alpha\alpha}^3(\overline{e}_x\cdot\overline{e}_y \times \overline{e}_z)] = 0 \quad \ldots\ldots \quad \{A.8.-10.\}\end{aligned}$$

For the Cartesian unit vectors, $\overline{e}_x\cdot\overline{e}_y \times \overline{e}_z = 1$ and if $\overline{e}_x$ be represented by $\overline{e}_y \times \overline{e}_z$ (etc.) in the coefficient of $\lambda_{\alpha\alpha}$, and allowing $\overline{e}_x \times \overline{e}_y$ to be represented by $\overline{e}_z$ (etc.) in the coefficient of $\lambda_{\alpha\alpha}^2$, the A.8.-10. becomes

$$\begin{aligned}\Big(&- \lambda_{\alpha\alpha}^3 + \lambda_{\alpha\alpha}^2[\overline{e}_x\cdot\overline{T}_x + \overline{e}_y\cdot\overline{T}_y + \overline{e}_z\cdot\overline{T}_z] \\ &+ \lambda_{\alpha\alpha}[(\overline{e}_y \times \overline{e}_z)\cdot(\overline{T}_y \times \overline{T}_z) + (\overline{e}_z \times \overline{e}_x)\cdot(\overline{T}_z \times \overline{T}_x) \\ &+ (\overline{e}_x \times \overline{e}_y)\cdot(\overline{T}_x \times \overline{T}_y)] + [\overline{T}_x\cdot\overline{T}_y \times \overline{T}_z]\Big) = 0 \quad \ldots\ldots \quad \{A.8.-11.\}\end{aligned}$$

However, the coefficients of $\lambda_{\alpha\alpha}^n$ are seen to be the expanded form of the scalar invariants of the tensor, $\overline{\overline{T}}$.

That is;

$$\bar{e}_x \cdot \bar{T}_x + \bar{e}_y \cdot \bar{T}_y + \bar{e}_z \cdot \bar{T}_z = \bar{\bar{1}} : \bar{\bar{T}} = \bar{\bar{T}}_s^{(1)}$$

$$(\bar{e}_y \times \bar{e}_z) \cdot (\bar{T}_y \times \bar{T}_z) + (\bar{e}_z \times \bar{e}_x) \cdot (\bar{T}_z \times \bar{T}_x) + (\bar{e}_x \times \bar{e}_y) \cdot (\bar{T}_x \times \bar{T}_y)$$

$$= \frac{1}{2!} \; \bar{\bar{T}} \, {}^{x}_{x} \, \bar{\bar{T}} : \bar{\bar{1}} = \bar{\bar{T}}_s^{(2)}$$

$$\bar{T}_x \cdot \bar{T}_y \times \bar{T}_z = \frac{1}{3!} \; \bar{\bar{T}} \, {}^{x}_{x} \, \bar{\bar{T}} : \bar{\bar{T}} = \bar{\bar{T}}_s^{(3)}$$

Therefore, equation {A.8.-11.} may be written as

$$\lambda_{\alpha\alpha}^{3} - \bar{\bar{T}}_s^{(1)} \lambda_{\alpha\alpha}^{2} + \bar{\bar{T}}_s^{(2)} \lambda \quad - \bar{\bar{T}}_s^{(3)} = 0$$

which is the SEGNER Eigenvalue Equation. Hence, $\bar{\bar{T}}_s^{(i)}$ are referred to as the invariants of the Eignevalue Equation, and $\bar{e}_\alpha$ are the invariant directions.

NOTE: The quantity $\lambda_{\alpha\alpha}$ is sometimes referred to as the EULER-LAGRANGE Multiplier. Generally, in the literature, the above development is presented as:

if F = extremum, then $\delta F^* = 0$

where $F^* = F + \lambda\phi$; the λ being defined as above, and ϕ being the constraint condition(s). In this case, the constraint condition would be

$$\phi = 0 = (1 - \bar{e}_\alpha \cdot \bar{e}_\alpha)$$

or $$\bar{e}_\alpha \cdot \bar{e}_\alpha = 1$$

APPENDIX B

B.1. NOTATION TRANSFORMATIONS

B.1.1.

GOL'DENVEIZER	vs.	This Author
$\omega^{(1)}$	....	$\frac{1}{2}(\phi_{12} + \phi_{21}) \equiv \varepsilon^{\circ}_{12}$
$\omega^{(2)}$	....	$-\frac{1}{2}(\phi_{12} + \phi_{21}) \equiv -\varepsilon^{\circ}_{12}$
γ_1	....	ϕ_{13}
γ_2	....	ϕ_{23}
ω_1	....	ϕ_{12}
ω_2	....	ϕ_{21}
ω	....	$(\phi_{12} + \phi_{21})$
δ	....	$\frac{1}{2}(\phi_{21} - \phi_{12})$
ε_α	....	$\phi_{\alpha\alpha}\ (\equiv \varepsilon^{\circ}_{\alpha\alpha})$
χ_1	....	$\delta\kappa_{11} + \frac{1}{2}\kappa_{12}(\phi_{12} + \phi_{21})$
χ_2	....	$\delta\kappa_{22} - \frac{1}{2}\kappa_{21}(\phi_{12} + \phi_{21})$
$\tau^{(1)}$	....	$-\delta\kappa_{12} + \frac{1}{2}\kappa_{11}(\phi_{12} + \phi_{21})$
$\tau^{(2)}$	....	$-\delta\kappa_{21} - \frac{1}{2}\kappa_{22}(\phi_{12} + \phi_{21})$
ζ_1	....	$-\delta\kappa_{13} + \frac{1}{2}\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_1}$
ζ_2	....	$-\delta\kappa_{23} - \frac{1}{2}\frac{\partial(\phi_{12} + \phi_{21})}{\partial s_2}$
$\frac{1}{R'_\alpha}$	....	$\kappa_{\alpha\alpha}$

B.1.2. NOVOZHILOV vs. This Author

NOVOZHILOV		This Author
ε_1	....	$\phi_{11}(\equiv \varepsilon^\circ_{11})$
ε_2	....	$\phi_{22}(\equiv \varepsilon^\circ_{22})$
ω_1	....	ϕ_{12}
ω_2	....	ϕ_{21}
ω	....	$(\phi_{12} + \phi_{21}) \equiv 2\varepsilon^\circ_{12}$
κ_1	....	$\delta\kappa_{11}$
κ_2	....	$\delta\kappa_{22}$
τ_1	....	$-\delta\kappa_{12} + \kappa_{11}\phi_{12}$
τ_2	....	$\delta\kappa_{21} + \kappa_{22}\phi_{21}$
τ	.... either	$-\delta\kappa_{12} + \kappa_{11}(\phi_{12} + \phi_{21})$ *
	or	$\delta\kappa_{21} + \kappa_{22}(\phi_{12} + \phi_{21})$ *
$\frac{1}{R_\alpha}$	....	$\kappa_{\alpha\alpha}$
$\overline{e}_\alpha$	....	$\overline{e}_\alpha$
$\overline{e}'_\alpha$	....	$\overline{E}_\alpha$

NOTE: These two expressions for τ are identical for the case (as considered by NOVOZHILOV) that $\kappa_{12} = 0 = \kappa_{21}$. See equation {5.2.-3.} when $\kappa_{12} = 0 = \kappa_{21}$.

B.1.3.

REISSNER	vs.	This Author
ε_{11}	....	$\phi_{11} (\equiv \varepsilon^{\circ}_{11})$
ε_{12}	....	$\frac{1}{2}(\phi_{12} + \phi_{21}) \equiv \varepsilon^{\circ}_{12}$
ε_{21}	....	$\frac{1}{2}(\phi_{12} + \phi_{21}) \equiv \varepsilon^{\circ}_{12}$
ε_{22}	....	$\phi_{22} \ (\equiv \varepsilon^{\circ}_{22})$
β_1	....	$-\phi_{13}$
β_2	....	$-\phi_{23}$
κ_{11}	....	$\delta\kappa_{11} + \frac{1}{2}\kappa_{12}(\phi_{12} + \phi_{21})$
κ_{12}	....	$-\delta\kappa_{12} + \frac{1}{2}\kappa_{11}(\phi_{12} + \phi_{21})$
κ_{21}	....	$\delta\kappa_{21} + \frac{1}{2}\kappa_{22}(\phi_{12} + \phi_{21})$
κ_{22}	....	$\delta\kappa_{22} - \frac{1}{2}\kappa_{21}(\phi_{12} + \phi_{21})$
α_1	....	g_1
α_2	....	g_2
$\frac{1}{\alpha_1}(\ )_{,1}$	....	$\frac{\partial}{\partial s_1}(\ )$
$\frac{1}{\alpha_2}(\ )_{,2}$	....	$\frac{\partial}{\partial s_2}(\ )$
$\frac{1}{R_{\alpha\beta}}$	....	$\kappa_{\alpha\beta}$
ω	....	$\frac{1}{2}(\phi_{12} - \phi_{21})$
γ_1 and γ_2	....	Zero†

† NOTE: γ_1 and γ_2 are set equal to zero by REISSNER for comparison with other works. When γ_1 and γ_2 are not equal to zero, they have no counterpart in this work.

B.1.4. KOITER vs. This Author

KOITER		This Author
ε_1	....	$\phi_{11}(\equiv \varepsilon^{\circ}_{11})$
ε_2	....	$\phi_{22}(\equiv \varepsilon^{\circ}_{22})$
ψ	....	$(\phi_{12} + \phi_{21}) \equiv 2\varepsilon^{\circ}_{12}$
ϕ_1	....	$-\ \phi_{13}$
ϕ_2	....	$-\ \phi_{23}$
Ω	....	$\frac{1}{2}(\phi_{12} - \phi_{21})$
$\frac{1}{R_1}$	....	κ_{11}
$\frac{1}{R_2}$	....	κ_{22}
T	....	$-\ \kappa_{12}$
u	....	u°_1
v	....	u°_2
w	....	$-u^{\circ}_3$

NOTE: KOITER's quantities κ_1, κ_2 and τ represent the "change of curvature" of $\frac{1}{R_1}$, $\frac{1}{R_2}$ and T, respectively. This is thoroughly discussed in §B.2.

B.2. A GENERAL COMPARISON OF THE RESULTS PRODUCED BY THE DIRECT KINEMATIC METHOD WITH THE RESULTS OBTAINED BY THE NONKINEMATIC METHODS OF OTHER AUTHORS

The classic paper of KOITER, in 1959, discussed the results obtained in some thirteen different books and papers, with respect to (primarily) the expressions for the changes of curvature. KOITER noted that no less than ten different expressions had been put forth by the thirteen papers -- each set of results having been derived under the assumptions of the same (linear) theory. His object in the paper was, as the title specifies, to provide "a *consistent* first approximation in the general theory of thin elastic shells" [italics mine]; the expressions for the curvature changes simply provide a convenient vehicle which facilitates the comparison with other authors.

This author has taken the liberty of reproducing KOITER's tabulated results in Table B.2.1., and appending his results to the list (at the end of the original list). The table is given in KOITER's notation, in deference to that author, but it may be re-converted to the notation employed in this work, through the use of §B.1.4. above.

In a note above the table, KOITER explains the meaning of the tabulated quantities by the following statement:

"The entrances in this table indicate the corrections $\Delta\kappa_1$, $\Delta\kappa_2$, $\Delta\tau$ which must be <u>added</u> to our expressions for κ_1 , κ_2, τ in order to obtain the expressions in the cited references. Where necessary, adjustments for sign and/or a numerical factor 2 have been made to achieve conformity with our notation. Essential differences in the sense of paras. 2.5. and 3.5. are marked by an asterisk. References employing the lines of curvature as parametric curves are marked by a small circle".

It can be observed, from Table B.2.1., that the corresponding results of this present work differ from KOITER's results for all "curvature change" expressions by a multiple (which is a numerical factor times a curvature) of the shear strain between the orthogonal parametric curves (i.e., ψ). KOITER has shown that such a difference is not an "essential difference", for as he states in §2.5. of his paper:

> "In particular, it is therefore permissible ..... to add to the expressions for the physical components of the changes of curvature and torsion (κ_1, κ_2 and τ) terms of the type ε/R (where ε is any of the middle surface strains ε_1, ε_2 or ψ, and R is any radius of curvature or torsion of the middle surface R_1, R_2 or T), multiplied by a numerical factor, provided this factor is not large compared to unity.".

Such a conclusion, as given by KOITER, is implicit in the kinematic development of the deformation parameters (Chapter 4.), although with (admittedly) less stringent mathematical criteria as a basis. Consider §4.4.; the variations of the unit vectors

AUTHORS AND REFERENCES	$\Delta\kappa_1$	$\Delta\kappa_2$	$\Delta\tau$
LOVE, 1888 °	-	-	$+\frac{1}{4}\psi\left(\frac{1}{R_2}-\frac{3}{R_1}\right)$
LAMB, 1891 ° REISSNER, 1942 ° WLASSOW [VLASOV], 1949 ° OSGOOD and JOSEPH, 1950 ° HAYWOOD and WILSON, 1958 °	$-\frac{\varepsilon_1}{R_1}$	$-\frac{\varepsilon_2}{R_2}$	$-\frac{1}{4}\psi\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$
REISSNER, 1941 °	-	-	$+\frac{1}{2}\Omega\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$ (*)
KOITER, 1945 °	$+\frac{2\varepsilon_1+\varepsilon_2}{R_1}$	$+\frac{\varepsilon_1+2\varepsilon_2}{R_2}$	$+\frac{1}{4}\psi\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$
GOLDENVEIZER, 1953	-	-	$+\frac{\varepsilon_1+\varepsilon_2}{T}+\frac{1}{4}\psi\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$
COHEN, 1955	$-\frac{\varepsilon_1}{R_1}+\frac{\psi}{2T}$	$-\frac{\varepsilon_2}{R_2}+\frac{\psi}{2T}$	$-\frac{\varepsilon_1+\varepsilon_2}{2T}+\frac{1}{4}\psi\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{W}{T}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$
COHEN, 1959	$-\frac{\varepsilon_1}{R_1}+\frac{\psi}{2T}$	$-\frac{\varepsilon_2}{R_2}+\frac{\psi}{2T}$	$-\frac{\varepsilon_1+\varepsilon_2}{2T}-\frac{1}{4}\psi\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$
KNOWLES and REISSNER, 1957	$-\frac{\Omega}{T}$ (*)	$+\frac{\Omega}{T}$ (*)	$+\frac{1}{2}\Omega\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$ (*)
KNOWLES and REISSNER, 1958	$-\frac{\varepsilon_1}{R_1}-\frac{\psi}{2T}$	$-\frac{\varepsilon_2}{R_2}-\frac{\psi}{2T}$	$-\frac{\varepsilon_1+\varepsilon_2}{2T}-\frac{1}{4}\psi\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$
McLEAN, 1966	$+\frac{\psi}{2T}$	$+\frac{\psi}{2T}$	$-\frac{1}{4}\psi\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$

TABLE B.2.-1.

$\overline{e}_1$, $\overline{e}_2$, $\overline{e}_3$, as a consequence of the process of deformation, were developed and given by the following relations.

$$\delta\overline{e}_1 = m_1(\phi_{12}\overline{e}_2 + \phi_{13}\overline{e}_3)$$

$$\delta\overline{e}_2 = m_2(\phi_{21}\overline{e}_1 + \phi_{23}\overline{e}_3)$$

$$\delta\overline{e}_3 = -m_1\phi_{13}\overline{e}_1 - m_2\phi_{23}\overline{e}_2$$

$$\delta\overline{e}^1_* = -\, m_1\phi_{12}\overline{e}_2 + m_2\phi_{23}\overline{e}_3$$

$$\delta\overline{e}^2_* = m_2\phi_{21}\overline{e}_2 - m_1\phi_{13}\overline{e}_3$$

$$\left.\begin{aligned} \text{where } m_1 &= \frac{1}{1+\phi_{11}} \equiv \frac{1}{1+\varepsilon^{\circ}_{11}} \\ \text{and } \quad m_2 &= \frac{1}{1+\phi_{22}} \equiv \frac{1}{1+\varepsilon^{\circ}_{22}} \end{aligned}\right\} \text{(as before)}$$

It was then noted that since, for the *orthogonal* triad $\{\overline{e}_1, \overline{e}_2, \overline{e}_3\}$,

$$\overline{e}_1 = -\,\overline{e}^2_*$$

and $$\overline{e}_2 = \overline{e}^1_*$$

then consequently,

$$\delta\overline{e}_1 = -\delta\overline{e}^2_*$$

and $$\delta\overline{e}_2 = \delta\overline{e}^1_*$$

Either one of these two equalities was then seen to reduce to

$$m_1\phi_{12} = -m_2\phi_{21} \qquad \ldots\ldots \qquad \{B.2.-1.\}$$

This result was not pursued further in §4.4., for the reason that when it is carried to its logical conclusion, it allows a "free-choice" to be made for the forms of the terms in the quantities $\delta\kappa_{ij}$ -- a choice which was not desired at that time, if the kinematical forms were to be obtained with no *a priori* prejudice.

Now, however, if equation {B.2.-1.} is subjected to close scrutiny, in the light of the procedures employed in Chapter 4., the following ensues.

From $$m_1\phi_{12} = -m_2\phi_{21}$$

it is observed that the approximations which were subsequently employed in Chapter 4

i.e., that $$m_1 = \frac{1}{1 + \varepsilon^{\circ}_{11}} = 1 \text{ (as } \varepsilon^{\circ}_{11} < < 1)$$

and $$m_2 = \frac{1}{1 + \varepsilon^{\circ}_{22}} = 1 \text{(as } \varepsilon^{\circ}_{22} < < 1)$$

produce the relationship

$$\phi_{12} = - \phi_{21} \qquad \ldots\ldots \qquad \{B.2.-2.\}$$

Before proceeding further in this discussion, it is important to note that such approximations as $m_1 = 1$, etc., are not additional approximations to the linear theory but are rather necessary ones, required for the purpose of maintaining all expressions at the required level of accuracy. That is, if such approximations were not made, the mathematical operations would accumulate terms in the various expressions which would be far beyond the level of accuracy warranted (or allowed!) by the initial restrictions of the linear theory.

Returning to equation {B.2.-2.}, it is seen that this result affords two "physical" interpretations. The first is that, although ϕ_{12} and ϕ_{21} are separately non-negligible, the combination of the two as $(\phi_{12} + \phi_{21})$ may be considered as negligible. This approach has the inherent disadvantage that the resulting forms of various expressions would not reveal the position occupied by the quantity $(\phi_{12} + \phi_{21})$, thereby destroying (in part) the conceptual unity. The second approach is to consider {B.2.-2.} literally in the form in which it is shown above. That is, consider {B.2.-2.} to specify the fact that ϕ_{12} may be replaced, at any time, by $-\phi_{21}$ with negligible error resulting from the substitution. This, then, implies

$$\phi_{12} = \frac{1}{2}(\phi_{12} - \phi_{21}) = \frac{1}{4}(\phi_{12} - 3\phi_{21}) = (\text{etc.}) = -\phi_{21} \qquad \{B.2.-3.\}$$

which explains why a "free-choice" for the form of terms would then be possible.

If this result ({B.2.-3.}, above) were employed in the expressions for $\delta\kappa_{ij}$, then all such expressions would agree in form with the corresponding results obtained by KOITER.

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