THE COMPLEMENTARY ENERGY PRINCIPLE IN FINITE ELASTICITY

THE COMPLEMENTARY POTENTIAL ENERGY PRINCIPLE IN FINITE ELASTIC DEFORMATIONS

by

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SCOPE AND CONTENTS:

This thesis establishes the general Principle of Complementary Potential Energy for the finite deformations of an elastic continuum, in which the Lagrange stress tensor is employed as the stress variable. It is demonstrated that constitutive relations, formulated in terms of the Lagrange stress tensor and the deformation gradient, will admit inversion. Consequently, the present theorem and the theorem proposed by LEVINSON are established as valid principles. The complementary strain energy density of the present theorem, however, is shown to be independent of rigid displacements, in contrast to that of the LEVINSON formulation. The general Principle is reduced to the form appropriate to finite elastic systems, and it is established that the present theorem reduces to, and therefore contains as a special case, the LIBOVE Theorem.

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This work is dedicated to my wife, Ann, in recognition of the sacrifices which she has made during the past five years, to the end that this goal might be achieved.

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NOTATION

F_1, F_2	force magnitudes
U_1, U_2	displacement magnitudes
$\overline{F}_1, \overline{F}_2$	force vectors
$\overline{U}_1, \overline{U}_2$	displacement vectors
$\delta\overline{U}_i$	virtual displacement of a point "i"
δW	virtual work of all forces in a finite system
$\overline{T}_n$	a stress vector on a surface S
$\overline{F}$	body force intensity per unit (undeformed) volume
$\overline{F}_i^{(a)}$	assigned force vector
$\overline{F}_i^{(b)}$	bilateral constraining force vector
$\overline{F}_i^{(c)}$	unilateral constraining force vector
δW	virtual work of assigned and bilateral forces
$\overline{T}_n^{(a)}$	assigned stress vector
$\overline{T}_n^{(b)}$	bilateral constraining stress vector
$\overline{T}_n^{(c)}$	unilateral constraining stress vector
$\overline{F}^{(i)}$	inertial or apparent force
t	time parameter
m	mass
$\overline{R}$	position vector, undeformed configuration
$\overline{r}$	position vector, deformed configuration
$\ddot{\overline{r}}$	material acceleration vector
P_0	a point located by $\overline{R}$
B_0	a region of the undeformed continuum

P	a point located by $\bar{r}$
B	a region of the deformed continuum
$\bar{U}$	the displacement of a point at $\bar{R}$
$\theta^1,\theta^2,\theta^3$	parametric coordinates at P_0 and P
dV	a volume element of B_0
dv	a volume element of B
$d\bar{R}_i$	a line element along θ^i in B_0
$\bar{G}_i$	a base vector at P_0
$\{\bar{G}_i\}$	the basis spanning the space $\bar{R}$
$\bar{G}^i$	a base vector reciprocal to $\bar{G}_i$
$\{\bar{G}^i\}$	the basis reciprocal to $\{\bar{G}_i\}$
δ_i^j	the KRONECKER Delta
$d\bar{S}_i$	a surface element on the surface θ^i = Constant in B_0
dS_i	the magnitude of $d\bar{S}_i$
$\frac{\partial}{\partial \bar{R}}$	the directed derivative
$d\bar{r}_i$	a line element along θ^i in B
$\bar{g}_i$	a base vector at P
$\{\bar{g}_i\}$	the basis spanning the space $\bar{r}(\bar{R})$
$\bar{g}^i$	a base vector reciprocal to $\bar{g}_i$
$\{\bar{g}^i\}$	the basis reciprocal to $\{\bar{g}_i\}$
$d\bar{s}_i$	a surface element on the surface θ^i = Constant in B
ds_i	the magnitude of $d\bar{s}_i$
ρ_0	the mass density at $\bar{R}$
ρ	the mass density at $\bar{r}$
$d\bar{R}$	an arbitrary line element at P_0

$d\bar{r}$	an arbitrary line element at P
$\bar{\bar{1}}$	the identity tensor
$\bar{\bar{E}}$	the strain tensor
G_{ij}	components of the metric tensor in $\bar{R}$
g_{ij}	components of the metric tensor in $\bar{r}$
E_{ij}	covariant components of the strain tensor $\bar{\bar{E}}$
$\bar{t}_i$	a stress vector on a surface (θ^i) or $(\theta^i + d\theta^i)$
$\bar{\bar{\sigma}}$	an arbitrary stress tensor
$\bar{\sigma}_n$	an arbitrary stress vector on a surface defined by $\bar{e}_n$
$\bar{\sigma}_n^{(-)}$	a stress vector on a surface defined by $-\bar{e}_n$
$\Delta\bar{f}$	a force vector on a surface $\Delta\bar{s}$
$\bar{t}_i$	a force vector corresponding to the stress $\bar{t}_i$
$d\bar{f}$	the net force acting on the elemental volume dv
$\bar{\gamma}$	body force intensity per unit (deformed) volume
$\bar{U}^{\#}$	the displacement $\bar{U}$ plus $\delta\bar{U}$
η	a small real number
$\bar{Z}$	a vector such that $\eta\bar{Z} = \delta\bar{U}$
$\bar{e}^i$	a unit vector collinear with $\bar{g}^i$
$\bar{\bar{\tau}}$	the CAUCHY-GREEN "true" stress tensor
τ^{ij}	contravariant components of $\bar{\bar{\tau}}$
$\bar{\tau}_i$	a force vector on a surface of normal $\bar{g}^i$
$\bar{\bar{S}}$	the TREFFTZ stress tensor
S^{ij}	contravariant components of $\bar{\bar{S}}$
$\bar{N}$	a unit normal to the surface S of B_0
$\bar{T}_n$	a stress vector on S

S_t	the portion of S on which $\overline{T}_n$ is prescribed
S_u	the portion of S on which $\overline{U}$ is prescribed
U_0^S	the strain energy density function
U^S	the (total) strain energy of a volume V
V^S	the potential function for surface stresses
Q^S	the sum of U^S and V^S ("total potential energy")
$\overline{T}_n^*$	the prescribed stresses on S_t
$\overline{U}^*$	the prescribed displacements on S_u
Π^S	the functional of the Potential Energy Principle
U_1^C	the classical complementary strain energy density function
Π^C	the functional of the Complementary Potential Energy Principle
$\underset{\sim}{\Pi}^C$	infinitesimal version of Π
W^C	the complementary strain energy of the CROTTI Theorem
W^C	complementary work for a particle
H^C	the LIBOVE complementary strain energy
Q^C	the LIBOVE total complementary (potential) energy
$\overline{r}_i^*$	a point of prescribed location after deformation
$\overline{F}_i^r$	the discrete force at $\overline{r}_i^*$
$\overline{\overline{C}}$	the CAUCHY-GREEN (left) deformation tensor
Ω	the MASUR complementary strain energy density function
$\overline{r}^*$	prescribed location field ($\overline{r}^* = \overline{R} + \overline{U}^*$)
$U_j\|_i$	"covariant" displacement derivative
$U^j\|_i$	as above
T^{ij}	contravariant components of the Lagrange stress tensor
W_0^C	the Levinson complementary strain energy density function

$\overline{\overline{\Gamma}}$	the deformation gradient
$\overline{\overline{U}}$	the displacement gradient or Lagrange strain tensor
$\overline{\overline{T}}$	the Lagrange stress tensor
Ψ	a potential (energy density) function
$\overline{\overline{\zeta}}, \overline{\overline{k}}, \overline{\overline{h}}, \overline{\overline{\ell}}$	tensor-valued functions of tensors
U_0^c	the present complementary strain energy density function
$\overline{\overline{\Gamma}}^{-1}$	the tensor reciprocal to $\overline{\overline{\Gamma}}$
$\overline{K}$	a constant vector
$\overline{\overline{V}}$	the left stretch tensor
$\overline{\overline{W}}$	the right stretch tensor
$\overline{\overline{\Phi}}$	the finite rotation tensor
$\overline{\overline{B}}$	the CAUCHY-GREEN (right) deformation tensor
$\underset{\sim}{\overline{\overline{E}}}$	a strain measure based on $\overline{\overline{B}}$
$\overline{\overline{D}}$	a mixed deformation tensor
$\overline{\overline{\Gamma}}^*$	a tensor which differs from $\overline{\overline{\Gamma}}$ by a rotation
$\overline{E}_1, \overline{E}_2$	unit vectors in a Cartesian frame
β	a numerical parameter
$\{a,b\}$	a set of real positive or real negative numbers
$\overline{\overline{F}}$	a solenoidal tensor
$\overline{\overline{w}}, \overline{\overline{y}}$	tensor-valued functions of tensors
j, m	tensor-valued functions of tensors
$\overline{f}$	a function of $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$
ζ	an energy density function
$\overline{\overline{T}}$	a stress tensor
U^c	the total complementary energy (volume integral of U_0^c)

F_r	a redundant force in a finite elastic system
T^c	the MASUR formulation of the CROTTI energy
L	the undeformed length of a unidimensional element
A	the undeformed area of a unidimensional element
$\bar{R}_1, \bar{R}_2$	the location of the end-points of a unidimensional element
τ, S, T	stress magnitudes in a unidimensional continuum
U, Γ	displacement-related magnitudes (unidimensional)
μ, λ, γ	material constants
σ	a stress parameter related to S
ψ, ξ, ϕ	stress parameters related to T
e	elongation of a unidimensional element (LIBOVE)
U_t	elongation of a unidimensional element (this work)
C, E, V	displacement-related magnitudes (unidimensional)
α	a material constant
$\bar{\bar{X}}$	a general stress-related tensor variable
$\bar{\bar{Y}}$	a general displacement-related tensor variable

INTRODUCTION

This work establishes the general Principle of Complementary Potential Energy, for the case of finite deformations of a general elastic continuum, in which the Lagrange stress tensor is employed as the stress tensor variable.

It is demonstrated that the nature of the Lagrange stress tensor is such that constitutive relations which are formulated in terms of this tensor and the deformation gradient will admit inversion. The form of the Principle which has been proposed by LEVINSON, based on the hypothesis of a similar constitutive inversion in terms of the Lagrange stress tensor and the Lagrange strain tensor (or displacement gradient), is examined. It is then established that, as the constitutive inversion has been shown to be possible, LEVINSON'S Theorem is a valid formulation of the Complementary Potential Energy Principle. The complementary strain energy density function of the LEVINSON Theorem, constructed as a Legendre transformation on the conjugate tensor variables of that theorem (i.e., the Lagrange stress and displacement gradient), is examined, and it is determined that this energy density is a function of the rigid-body rotations of the displacement field. The complementary strain energy density of the present theorem, constructed as a Legendre transformation on the Lagrange stress tensor and the deformation gradient, is then also examined, and it is established that this energy density function is independent of any

rigid displacements. Thus, the complementary strain energy density of the present theorem satisfies all pertinent requirements in order to qualify as the basis for the constitutive relations of an elastic medium. It is noted, however, that due to the nature of the Lagrange tensor variables in the LEVINSON formulation, a constitutive relation based on an energy density which is a function of rigid rotations does not constitute an error in the theorem. It is, at most, an inconvenience. Yet, as the present theorem avoids this condition, the Lagrange stress tensor and the deformation gradient are considered to represent the appropriate conjugate tensor variables for the formulation of the Principle.

The admissibility conditions for the stress tensor are pronounced, and CAUCHY'S second axiom of motion is developed in a direct tensorial form which does not require the explicit use of reciprocal base vectors of the deformed configuration. Consequently, both CAUCHY axioms of motion (in this case, stress and stress-couple or "moment" equilibrium) are established in a form which is amenable to a stress function approach to the problem. The relationship of the present statement of the Principle to the other formulations, as given by MASUR, LEVINSON and REISSNER, is examined, and it is demonstrated that the four formulations arise as a consequence of the various possible expressions for the strain energy density function.

The general form of the Principle is reduced to the form appropriate to finite elastic systems, or systems of discrete forces. In this case, the complementary strain energy of the LEVINSON Theorem

is established as the energy of the CROTTI Theorem (often denoted as "CASTIGLIANO'S Principle"), and consequently, LEVINSON'S energy is shown to represent the true finite counterpart of the complementary strain energy of the infinitesimal theory. Thus, it is established that LEVINSON'S Theorem represents the generalization of the CROTTI Theorem to continuous stress systems, in finite elasticity. The present theorem is proven to reduce directly to, and therefore to contain as a special case, the LIBOVE Theorem for finite elastic systems: also, the present theorem admits a simple relationship to the CROTTI Theorem, while retaining the property of independence of rigid rotations. Thus, the present theorem is shown to represent the direct generalization of the LIBOVE Theorem to continuous stress systems.

It has been necessary, in order to avoid confusion, to refer to the theorem constructed in terms of the Lagrange stress tensor and the displacement gradient as "LEVINSON'S Theorem", while denoting the theorem formulated in terms of the Lagrange stress tensor and the deformation gradient as "the present theorem" (a policy which is followed throughout this work). However, it is proposed that the terminology "LEVINSON'S Theorem" be employed hereafter to describe the Complementary Potential Energy Principle in which the Lagrange stress tensor is employed with <u>either</u> kinematic variable, as LEVINSON was the first to propose the construction of the Principle in terms of this stress tensor.

CHAPTER I

The Energy Principles

I.1. THE VIRTUAL WORK PRINCIPLE

The general Principle of Virtual Work, which provides the foundation upon which much of the structure of non-relativistic mechanics may be constructed, yields for an elastic continuum, the Potential Energy Principle and its dual, the Complementary Potential Energy Principle.

I.1.1. Historical and Intuitive Development

The most primitive mathematical criterion of equilibrium of two gravitational forces, F_1 and F_2, acting upon a rigid horizontal lever, may have been known to ARISTOTELES (or "ARISTOTLE": 384-322 B.C.), and would then have been expressed in the traditional Hellenic dimensionless form

$$\frac{F_1}{F_2} = \frac{U_2}{U_1} \qquad \{I.1.1.-1\}$$

In this relation, U_1 and U_2 denote the vertical displacements of the lever at the points of application of the forces F_1 and F_2, respectively.

This relationship could be expressed as a "work" equation,

$$F_1U_1 = F_2U_2 \qquad \{I.1.1.-2\}$$

or in contemporary vector form,

$$\overline{F}_1 \cdot \overline{U}_1 = \overline{F}_2 \cdot \overline{U}_2 \qquad \{1.1.1.-3\}$$

and it is noted that a form similar to that of {1.1.1.-2} was well known to HERON (or "HERO": *circa* 60 A.D.), a Hellenistic mathematician and engineer who possessed a considerable mastery of the "work principle".

The concept of *Virtual Work*, produced by *actual* forces and *virtual* displacements, was employed by Galileo GALILEI† (1564-1642), René DESCARTES de PERRON (1596-1650), Evangelista TORRICELLI (1608-1647), Christiaan HUYGENS de ZUYLEN (1629-1695), and others. It was not until 1715, however, that the first mathematical formulation of the Virtual Work Principle for a discrete force system was given.* Johann BERNOULLI (1667-1748), in a letter to Pierre VARIGNON (1654-1722), gave a formulation which would appear as

$$\overline{F}_i \cdot \delta\overline{U}_i = 0 \qquad \{1.1.1.-4\}$$

in contemporary vector summation terminology. The system of discrete forces is represented here as $\overline{F}_i$, and $\delta\overline{U}_i$ denotes the corresponding set of compatible virtual displacements of the body. Inherent in this formulation is the requirement that $\overline{F}_i \equiv \overline{F}_i(\overline{R})$ represents a set of forces, each of which is a continuous function of the space, $\overline{R}$.

Referring to the quantity $\overline{F}_i \cdot \delta\overline{U}_i$ as the Virtual Work δW,

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† References are given chronologically in the bibliography.
* Refer to Historical Notes, page C-19, Appendix C.

then

$$\delta W = \overline{F}_i \cdot \delta \overline{U}_i = 0 \qquad \{1.1.1.-5\}$$

In the event that the system under consideration is a (hypothetical) *rigid* continuum with applied surface stresses $\overline{T}_n$ and a body force intensity $\overline{F}$ per unit volume, then the above relation could be given as

$$\delta W = \iint_S \overline{T}_n \cdot \delta \overline{U} \, dS + \iiint_V \overline{F} \cdot \delta \overline{U} \, dV = 0 \qquad \{1.1.1.-6\}$$

where S designates the bounding surface of the volume V of the continuum, and is the surface upon which the applied stresses $\overline{T}_n$ are assumed to act.

Jacob BERNOULLI (1654-1705) and later Jean LeRond D'ALEMBERT (1717-1783) postulated that each force $\overline{F}_i$ could be resolved into two distinct, separate parts: the applied or "assigned" force and the constraining force. Joseph Jean-Baptiste FOURIER (1768-1830) postulated, following BERNOULLI and D'ALEMBERT, that any force $\overline{F}_i$ could be represented as a superposition of forces:

$$\overline{F}_i = \overline{F}_i^{(a)} + \overline{F}_i^{(b)} + \overline{F}_i^{(c)} \qquad \{1.1.1.-7\}$$

where $\overline{F}_i^{(a)}$ denotes the assigned force, $\overline{F}_i^{(b)}$ denotes the bilateral or "reversible" constraining force which is a continuous function of $\overline{R}$, and $\overline{F}_i^{(c)}$ designates the unilateral or "irreversible" constraining force which is a discontinuous function of $\overline{R}$.

In 1798, FOURIER further postulated his celebrated Fourier

Inequality which stated, essentially, that the virtual work performed by the unilateral constraining forces is always equal to, or greater than, zero. Thus,

$$\bar{F}_i^{(c)} \cdot \delta\bar{U}_i \geq 0 \qquad \{1.1.1.-8\}$$

in which case,

$$\bar{F}_i^{(a)} \cdot \delta\bar{U}_i + \bar{F}_i^{(b)} \cdot \delta\bar{U}_i = -\bar{F}_i^{(c)} \cdot \delta\bar{U}_i \leq 0 \qquad \{1.1.1.-9\}$$

or, introducing the terminology δW to denote all virtual work *except* that which is produced by the unilateral constraining forces, then

$$\delta W = \bar{F}_i^{(a)} \cdot \delta\bar{U}_i + \bar{F}_i^{(b)} \cdot \delta\bar{U}_i \leq 0 \qquad \{1.1.1.-10\}$$

For the case of the rigid continuum, the equivalent assumption appears as

$$\bar{T}_n = \bar{T}_n^{(a)} + \bar{T}_n^{(b)} + \bar{T}_n^{(c)} \qquad \{1.1.1.-11\}$$

and it follows that

$$\delta W = \iint_S \bar{T}_n^{(a)} \cdot \delta\bar{U}\, dS + \iint_S \bar{T}_n^{(b)} \cdot \delta\bar{U}\, dS + \iint_S \bar{T}_n^{(c)} \cdot \delta\bar{U}\, dS + \iiint_V \bar{F} \cdot \delta\bar{U}\, dV \qquad \{1.1.1.-12\}$$

and so,

$$\delta W = \iint_S \bar{T}_n^{(a)} \cdot \delta\bar{U}\, dS + \iint_S \bar{T}_n^{(b)} \cdot \delta\bar{U}\, dS + \iiint_V \bar{F} \cdot \delta\bar{U}\, dV \leq 0 \qquad \{1.1.1.-13\}$$

since

$$\iint_S \overline{T}_n^{(c)} \cdot \delta\overline{U}\, dS \geq 0 \qquad \{1.1.1.-14\}$$

in accordance with the Fourier Postulate.

However, whether the "discrete force" form or the "rigid continuum" form of this relationship is employed, the fact remains that for bilateral systems (systems without unilateral constraining forces), the displacement $\delta\overline{U}$ and the reversed displacement $-\delta\overline{U}$ must yield the same result. Therefore, both

$$\delta W = \overline{F}_i^{(a)} \cdot \delta\overline{U} + \overline{F}_i^{(b)} \cdot \delta\overline{U} \leq 0 \qquad \{1.1.1.-15\}$$

and $$\delta W = \overline{F}_i^{(a)} \cdot (-\delta\overline{U}) + \overline{F}_i^{(b)} \cdot (-\delta\overline{U}) \leq 0$$

i.e. $$\delta W = -\left[\overline{F}_i^{(a)} \cdot \delta\overline{U} + \overline{F}_i^{(b)} \cdot \delta\overline{U}\right] \leq 0 \qquad \{1.1.1.-16\}$$

must be true. This is, of course, generally possible if and only if

$$\delta W = \overline{F}_i^{(a)} \cdot \delta\overline{U} + F_i^{(b)} \cdot \delta\overline{U} = 0 \qquad \{1.1.1.-17\}$$

This result represents the celebrated Virtual Work Principle for bilateral constraint systems. In this formulation, it is assumed that the system is static or moving with a constant velocity, and consists only of a rigid body (or system of bodies). The corresponding result for the rigid continuum would appear as

$$\delta W = \iint_S \overline{T}_n^{(a)} \cdot \delta\overline{U}\, dS + \iint_S \overline{T}_n^{(b)} \cdot \delta\overline{U}\, dS + \iiint_V \overline{F} \cdot \delta\overline{U}\, dV = 0 \qquad \{1.1.1.-18\}$$

where the distributed stresses act in the capacity of the discrete forces of the previous representation. It is observed that this equation may be given as

$$\delta W = \iint_S \bar{T}_n \cdot \delta\bar{U}\, dS + \iiint_V \bar{F} \cdot \delta\bar{U}\, dV = 0 \qquad \{1.1.1.-19\}$$

if it is stipulated that $\bar{T}_n$ is to specify only a bilateral stress system.

In the event that the system is not a static (or constant velocity) one, the effects of acceleration, in the form of apparent forces, must be considered. Following the concepts of HUYGENS, Jacob BERNOULLI (especially), EULER, D'ALEMBERT, and Lazare CARNOT, the apparent or inertia forces $\bar{F}^{(i)}$ are considered "ordinary" forces, such as $\bar{F}^{(a)}$ and $\bar{F}^{(b)}$, and are considered equipollent to the reversed material acceleration for the given unit mass of the system. Thus, for a system of constant mass,

$$\bar{F}^{(i)} = -\frac{d}{dt}\left[m\frac{d\bar{r}}{dt}\right] = -m\ddot{\bar{r}} \qquad \{1.1.1.-20\}$$

defines the apparent or inertia force in terms of the mass m and the material acceleration $\ddot{\bar{r}}$ of the body with respect to an inertial frame of reference.

The Virtual Work Principle for discrete forces acting on a rigid body under bilateral constraint conditions therefore becomes, in the general form

$$\delta W = \bar{F}_j^{(a)} \cdot \delta\bar{U}_j + \bar{F}_j^{(b)} \cdot \delta\bar{U}_j + \bar{F}_j^{(i)} \cdot \delta\bar{U}_j = 0 \qquad \{1.1.1.-21\}$$

and the corresponding result for a rigid continuum appears as

$$\delta W = \iint_S \bar{T}_n^{(a)} \cdot \delta\bar{U}\, dS + \iint_S \bar{T}_n^{(b)} \cdot \delta\bar{U}\, dS + \iiint_V \bar{F} \cdot \delta\bar{U}\, dV + \iiint_m (-dm\, \ddot{\bar{r}} \cdot \delta\bar{U}) = 0 \qquad \{1.1.1.\text{-}22\}$$

The foregoing discussion represents the historical and intuitive approach to the Virtual Work Principle for (hypothetical) non-deformable bodies.

1.1.2. Formal Development

The fundamental postulate of the Virtual Work Principle is as follows.

For a non-holonomic, rheonomic system, the necessary and sufficient condition for a dynamic state of the system to exist is that: the work performed by the force system of the dynamic state, over any admissible virtual displacements which are imposed on that state, is equal to or less than zero: $\delta W \leq 0$.

In the enunciation of this principle, a non-holonomic system is defined as one in which the forces are polygenic (i.e., cannot be derived from a single scalar function); a holonomic system is, therefore, one in which the forces are monogenic. A rheonomic system is defined as one in which the total energy of the system is not constant with respect to time; a system in which the total energy is conserved is then called scleronomic. Furthermore, the apparent or inertia force of the dynamic state is postulated, following Jacob

BERNOULLI, to be equipollent to the reversed material acceleration for the given unit mass with respect to an inertial frame of reference.

An admissible virtual displacement shall be construed to signify, in the most general sense, a displacement which is:

(a) hypothetical

(b) of such a magnitude that terms of the order of its square (or higher) may be considered negligible, in comparison with the quantity itself

(c) instantaneously applied.

The most general (non-holonomic and rheonomic) system, as given in the postulate, may be subject to reduction to a more restricted system, if desired. The statement of the principle remains exactly as given above for systems which are: non-holonomic and scleronomic, holonomic and rheonomic, and holonomic and scleronomic. In conjunction with the latter case, if an admissible virtual displacement, in addition to satisfying the criteria (a), (b) and (c) above, is also required to be

(d) compatible with the kinematic constraints of the system

(e) bilateral or "reversible"

then the general statement appears as follows.

For a holonomic, scleronomic system, the necessary and sufficient condition for a dynamic state of the system to exist is that: the work performed by the force system of the dynamic state,

over any admissible virtual displacements which are imposed on that state, must vanish: $\delta W = 0$.

Since this statement of the Virtual Work Principle restricts the system to a holonomic one, and since the apparent or inertia force is polygenic, such a force should be excluded from the "force system" mentioned in the general statement. That is, the system under consideration should be a static or constant velocity one. However, as a time-integration procedure may be implemented to avoid this difficulty in the case of the apparent force, such forces will be admitted: the "scleronomic and holonomic" system will then be understood to include systems which are "scleronomic and reducible-to-holonomic".

The Virtual Work Principle, as given above for admissible virtual displacements [which satisfy all five conditions (a) to (e)], may be expressed mathematically for a continuum, as follows. Consider, in Fig. I.1.2.-1 below, a point P_0 in the undeformed body† B_0. This point is denoted as P in the deformed body B (after B_0 has been subjected to a displacement field $\overline{U}$); thus P_0 and P represent the same material point in two different states or configurations of the body. At P_0 and at P, there exists a system of curvilinear parametric coordinates θ^1, θ^2, θ^3 which define an elemental volume dV at P_0 and and dv at P. The elements dV and dv are, therefore, infinitesimal curvilinear parallelepipeds which are bounded by the surfaces

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† The "body" denotes an arbitrary, finite region of the space occupied by a continuum at some time, t.

$(\theta^i, \theta^i+d\theta^i)$ = Constant.

Before an examination of the forces acting on the element dv may be conducted, it is necessary to establish certain (differential) geometrical relationships of the system shown in Fig. 1.1.2.-1.

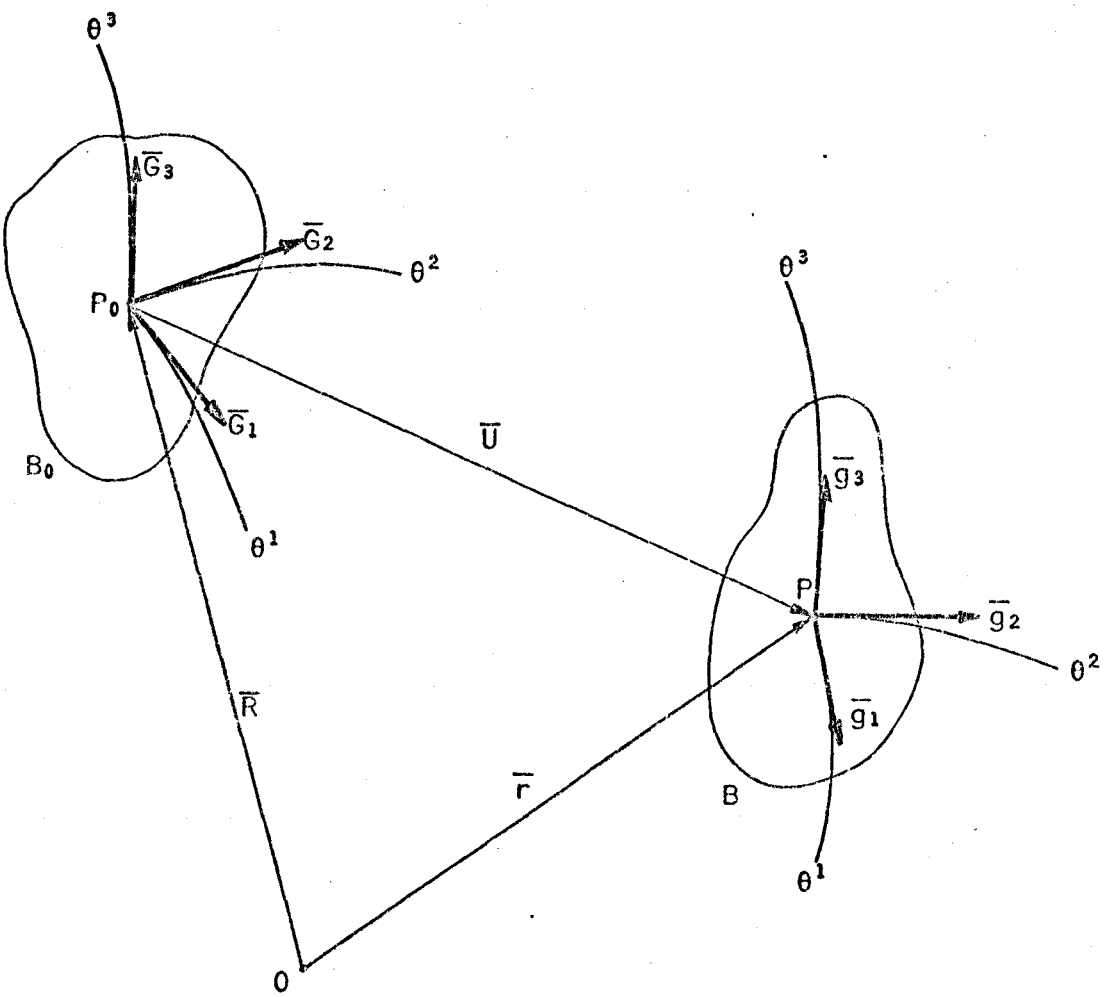

FIGURE 1.1.2.-1

A line element along parametric coordinate θ^i in the undeformed configuration B_0 may be represented as $d\overline{R}_i$, where

$$d\bar{R}_i = \frac{\partial \bar{R}}{\partial \theta^{\underline{i}}} d\theta^{\underline{i}} = \bar{G}_{\underline{i}} \, d\theta^{\underline{i}} \qquad \{1.1.2.-1\}$$

where the underscored indices indicate *no summation* on those indices (indices not so designated are summed over the range 1, 2, 3, as usual). The independent base vectors $\bar{G}_i$ of the directed base $\{\bar{G}_i\}$ are thus prescribed as

$$\bar{G}_i = \frac{\partial \bar{R}}{\partial \theta^i} \qquad \{1.1.2.-2\}$$

in terms of the position vector $\bar{R}$ and coordinate θ^i. The reciprocal base vectors $\bar{G}^i$ define the set of independent base vectors which must, by definition, satisfy the relationship

$$\bar{G}^i \cdot \bar{G}_j = \delta^i_j \qquad \{1.1.2.-3\}$$

where δ^i_j denotes the KRONECKER delta. Therefore, the reciprocal directed base is specified explicitly as:

$$\left.\begin{aligned} \bar{G}^1 &= \frac{1}{\sqrt{G}} \bar{G}_2 \times \bar{G}_3 \\ \bar{G}^2 &= \frac{1}{\sqrt{G}} \bar{G}_3 \times \bar{G}_1 \\ \bar{G}^3 &= \frac{1}{\sqrt{G}} \bar{G}_1 \times \bar{G}_2 \end{aligned}\right\} \qquad \{1.1.2.-4\}$$

where $$\sqrt{G} = \bar{G}_1 \cdot \bar{G}_2 \times \bar{G}_3 = \bar{G}_1 \times \bar{G}_2 \cdot \bar{G}_3 \qquad \{1.1.2.-5\}$$

A differential surface element $d\bar{S}_1$ on the surface θ^1 = Constant may then be obtained as

$$d\bar{S}_1 = d\bar{R}_2 \times d\bar{R}_3 = (\bar{G}_2 \times \bar{G}_3)d\theta^2 d\theta^3$$

or $$d\bar{S}_1 = \bar{G}^1\sqrt{G}d\theta^2 d\theta^3$$

Similarly, $$d\bar{S}_2 = \bar{G}^2\sqrt{G}d\theta^1 d\theta^3$$

and $$d\bar{S}_3 = \bar{G}^3\sqrt{G}d\theta^1 d\theta^2$$

} {I.1.2.-6}

for the surfaces θ^2 and θ^3. The magnitude of any surface element (the differential area) may be found directly from {I.1.2.-6} as

$$dS_i = | d\bar{S}_i | = \sqrt{GG^{\underline{ii}}}\, d\theta^j d\theta^k \quad (i \neq j \neq k) \qquad \{I.1.2.-7\}$$

where $$G^{\underline{ii}} = \bar{G}^{\underline{i}} \cdot \bar{G}^{\underline{i}} \qquad \{I.1.2.-8\}$$

An element of volume dV may then be calculated as

$$dV = d\bar{R}_1 \cdot d\bar{R}_2 \times d\bar{R}_3 = (\bar{G}_1 \cdot \bar{G}_2 \times \bar{G}_3)d\theta^1 d\theta^2 d\theta^3$$

or, by virtue of {I.1.2.-5},

$$dV = \sqrt{G}d\theta^1 d\theta^2 d\theta^3 \qquad \{I.1.2.-9\}$$

Finally, the directed derivative is expressed as

$$\frac{\partial}{\partial \bar{R}}(\ \ ) \equiv \bar{G}^i \frac{\partial}{\partial \theta^i}(\ \ ) \qquad \{I.1.2.-10\}$$

and is sometimes denoted as $\nabla(\ \ )$, which is then called the HAMILTONIAN form of the operator.

A line element along parametric coordinate θ^i in the deformed body B may be represented as $d\bar{r}_i$, where

$$d\bar{r}_i = \frac{\partial \bar{r}}{\partial \theta^{\underline{i}}} d\theta^{\underline{i}} = \bar{g}_{\underline{i}} d\theta^{\underline{i}} \qquad \{1.1.2.-11\}$$

and the independent base vectors $\bar{g}_i$ of the directed base $\{\bar{g}_i\}$ are therefore prescribed as

$$\bar{g}_i = \frac{\partial \bar{r}}{\partial \theta^i} \qquad \{1.1.2.-12\}$$

or, since $$\bar{r} = \bar{R} + \bar{U} \qquad \{1.1.2.-13\}$$

then $$\bar{g}_i = \frac{\partial (\bar{R} + \bar{U})}{\partial \theta^i} = \bar{G}_i + \frac{\partial \bar{U}}{\partial \theta^i} \qquad \{1.1.2.-14\}$$

The independent reciprocal base vectors $\bar{g}^i$ satisfy the requirement

$$\bar{g}^i \cdot \bar{g}_j = \delta^i_j \qquad \{1.1.2.-15\}$$

and therefore appear, similar to the undeformed case, as

$$\left.\begin{aligned} \bar{g}^1 &= \frac{1}{\sqrt{g}} \bar{g}_2 \times \bar{g}_3 \\ \bar{g}^2 &= \frac{1}{\sqrt{g}} \bar{g}_3 \times \bar{g}_1 \\ \bar{g}^3 &= \frac{1}{\sqrt{g}} \bar{g}_1 \times \bar{g}_2 \end{aligned}\right\} \qquad \{1.1.2.-16\}$$

where $$\sqrt{g} = \bar{g}_1 \cdot \bar{g}_2 \times \bar{g}_3 = \bar{g}_1 \times \bar{g}_2 \cdot \bar{g}_3 \qquad \{1.1.2.-17\}$$

A differential surface element $d\bar{s}_1$ on the surface θ^1 = Constant may then be obtained as

$$d\bar{s}_1 = d\bar{r}_2 \times d\bar{r}_3 = (\bar{g}_2 \times \bar{g}_3)\, d\theta^2 d\theta^3$$

or $$d\bar{s}_1 = \bar{g}^1 \sqrt{g}\, d\theta^2 d\theta^3$$

Similarly, $$d\bar{s}_2 = \bar{g}^2 \sqrt{g}\, d\theta^1 d\theta^3$$

and $$d\bar{s}_3 = \bar{g}^3 \sqrt{g}\, d\theta^1 d\theta^2$$ } {I.1.2.-18}

Finally, an element of volume dv is calculated as

$$dv = d\bar{r}_1 \cdot d\bar{r}_2 \times d\bar{r}_3 = (\bar{g}_1 \cdot \bar{g}_2 \times \bar{g}_3)\, d\theta^1 d\theta^2 d\theta^3 \qquad \{I.1.2.-19\}$$

or, by virtue of {I.1.2.-17},

$$dv = \sqrt{g}\, d\theta^1 d\theta^2 d\theta^3 \qquad \{I.1.2.-20\}$$

Several important relationships may be constructed to relate the deformed and the undeformed states. If ρ_0 and ρ denote the mass densities in the undeformed and the deformed states, respectively, then

$$\rho_0 = \frac{dm}{dV} \quad \text{and} \quad \rho = \frac{dm}{dv} \qquad \{I.1.2.-21\}$$

Consequently, if mass is conserved in the deformation, as is assumed here, then

$$\rho_0 dV = \rho dv = dm \qquad \{I.1.2.-22\}$$

or $$\rho_0 \sqrt{G} = \rho \sqrt{g} \qquad \{I.1.2.-23\}$$

which is known as the Equation of Continuity. This may also be

expressed as

$$\frac{1}{\sqrt{G}}dV = \frac{1}{\sqrt{g}}dv = d\theta^1 d\theta^2 d\theta^3 \qquad \{1.1.2.-24\}$$

which is an implicit form of the Equation of Continuity.

Another relation between the two states is the comparison of line elements in the undeformed and the deformed configurations. An arbitrary line element $d\bar{r}$ in the deformed state may be given as

$$d\bar{r} = \bar{g}_i d\theta^i \qquad \{1.1.2.-25\}$$

and in the undeformed state, as $d\bar{R}$, where

$$d\bar{R} = \bar{G}_i d\theta^i \qquad \{1.1.2.-26\}$$

Then the magnitudes $d\bar{r} \cdot d\bar{r}$ and $d\bar{R} \cdot d\bar{R}$, representing the squares of $|d\bar{r}|$ and $|d\bar{R}|$, respectively, differ by the amount

$$d\bar{r} \cdot d\bar{r} - d\bar{R} \cdot d\bar{R} = d\bar{R} \cdot \frac{\partial \bar{r}}{\partial \bar{R}} \cdot \frac{\bar{r}\partial}{\partial \bar{R}} \cdot d\bar{R} - d\bar{R} \cdot d\bar{R}$$

where $\frac{\bar{r}\partial}{\partial \bar{R}}$ represents the tensor conjugate to $\frac{\partial \bar{r}}{\partial \bar{R}}$, or $\frac{\bar{r}\partial}{\partial \bar{R}} = \left(\frac{\partial \bar{r}}{\partial \bar{R}}\right)_c$ where the subscript "c" denotes the conjugate.

The previous relation may be written in a form in which the dot products with $d\bar{R}$ are extracted, i.e.,

$$d\bar{r} \cdot d\bar{r} - d\bar{R} \cdot d\bar{R} = d\bar{R} \cdot \left[\frac{\partial \bar{r}}{\partial \bar{R}} \cdot \frac{\bar{r}\partial}{\partial \bar{R}} - \bar{\bar{I}}\right] \cdot d\bar{R} \qquad \{1.1.2.-27\}$$

or $$d\bar{r} \cdot d\bar{r} - d\bar{R} \cdot d\bar{R} = d\bar{R} \cdot (2\bar{\bar{E}}) \cdot d\bar{R} \qquad \{1.1.2.-28\}$$

where $$\bar{\bar{1}} = \frac{\partial \bar{R}}{\partial \bar{R}} \qquad \{1.1.2.-29\}$$

denotes the identity tensor, and

$$\bar{\bar{E}} = \frac{1}{2}\left[\frac{\partial \bar{r}}{\partial \bar{R}} \cdot \frac{\bar{r}\partial}{\partial \bar{R}} - \bar{\bar{1}}\right] \qquad \{1.1.2.-30\}$$

denotes the strain tensor for the deformation. Therefore, since

$$\frac{\partial \bar{r}}{\partial \bar{R}} = \frac{\partial \theta^i}{\partial \bar{R}} \frac{\partial \bar{r}}{\partial \theta^i} = \bar{G}^i \frac{\partial \bar{r}}{\partial \theta^i} = \bar{G}^i \bar{g}_i$$

so that $$\frac{\partial \bar{r}}{\partial \bar{R}} \cdot \frac{\bar{r}\partial}{\partial \bar{R}} = \bar{G}^i \bar{g}_i \cdot \bar{g}_j \bar{G}^j = (\bar{g}_i \cdot \bar{g}_j)\bar{G}^i \bar{G}^j = g_{ij} \bar{G}^i \bar{G}^j$$

and $$\bar{\bar{1}} = \bar{\bar{1}} \cdot \bar{\bar{1}} = \frac{\partial \bar{R}}{\partial \bar{R}} \cdot \frac{\partial \bar{R}}{\partial \bar{R}} = \frac{\partial \bar{R}}{\partial \bar{R}} \cdot \frac{\bar{R}\partial}{\partial \bar{R}} = \bar{G}^i \bar{G}_i \cdot \bar{G}_j \bar{G}^j$$

so that $$\bar{\bar{1}} = (\bar{G}_i \cdot \bar{G}_j)\bar{G}^i \bar{G}^j = G_{ij} \bar{G}^i \bar{G}^j$$

where $$g_{ij} = \bar{g}_i \cdot \bar{g}_j \quad \text{and} \quad G_{ij} = \bar{G}_i \cdot \bar{G}_j$$

then $$\bar{\bar{E}} = E_{rs} \bar{G}^r \bar{G}^s = \frac{1}{2}(g_{rs} - G_{rs})\bar{G}^r \bar{G}^s \qquad \{1.1.2.-31\}$$

where $$E_{rs} = \frac{1}{2}(g_{rs} - G_{rs})$$

Also, $$E_{ij} = \bar{G}_i \bar{G}_j : \bar{\bar{E}} = \bar{G}_j \bar{G}_i : \bar{\bar{E}} \qquad \{1.1.2.-32\}$$

and the functions E_{ij} are the symmetric covariant components of the strain tensor $\bar{\bar{E}}$, referred to the reciprocal directed base.

The forces acting on the element dv in the deformed state may now be examined (the element dV, being in the undeformed state, is of

little interest at present). Consider the element dv as shown in Fig. I.I.2.-2.

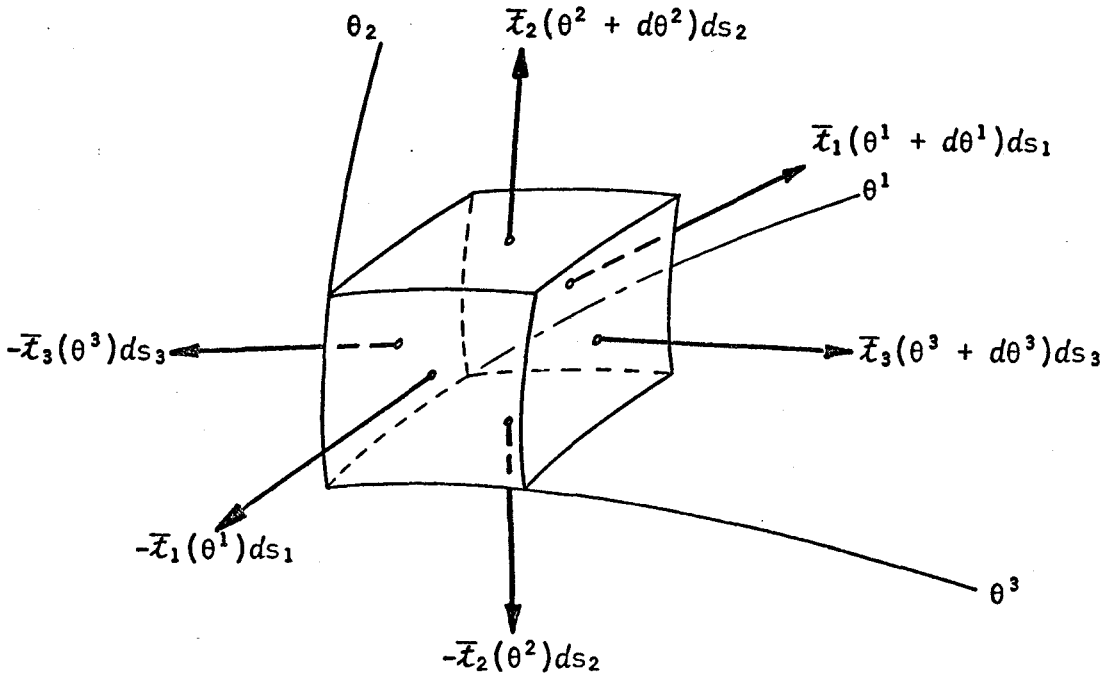

FIGURE I.I.2.-2

The forces on the surfaces θ^i = Constant are denoted as $-\bar{\mathcal{T}}_{\underline{i}}(\theta^{\underline{i}})ds_{\underline{i}}$, where the surface area ds_i is given as

$$ds_i = |d\bar{s}_i| = \sqrt{g g^{\underline{ii}}}\, d\theta^j d\theta^k \quad ; \quad j \neq i \neq k \qquad \{I.I.2.-33\}$$

and $-\bar{\mathcal{T}}_{\underline{i}}(\theta^{\underline{i}})$ indicates the stress vector on the surface, an external normal to which is $-\bar{g}^i$. This follows directly from the CAUCHY Stress Principle,

$$\bar{\sigma}_n = \bar{e}_n \cdot \bar{\bar{\sigma}} \qquad \{1.1.2.-34\}$$

where $\bar{\bar{\sigma}}$ represents a stress tensor, and $\bar{\sigma}_n$ denotes the stress vector on a surface, the unit normal to which is $\bar{e}_n$. That is, if $\bar{\sigma}_n^{(-)}$ denotes the stress vector on a surface, the unit normal to which is $-\bar{e}_n$, then

$$\bar{\sigma}_n^{(-)} = (-\bar{e}_n) \cdot \bar{\bar{\sigma}} = -(\bar{e}_n \cdot \bar{\bar{\sigma}}) = -\bar{\sigma}_n \qquad \{1.1.2.-35\}$$

Similarly, the forces on the surfaces $\theta^i+d\theta^i$ = Constant are denoted as $\bar{\mathcal{T}}_{\underline{i}}(\theta^{\underline{i}}+d\theta^{\underline{i}})ds_{\underline{i}}$, and $\bar{\mathcal{T}}_{\underline{i}}(\theta^{\underline{i}}+d\theta^{\underline{i}})$ thus indicates the stress vector on the surface, an external normal to which is $\bar{g}^i$. The stress vectors represent, of course, a true measure of the stress, being defined in general for a force $\Delta\bar{f}$ on a *deformed* surface area Δs, as

$$\bar{\mathcal{T}} = \lim_{\Delta s \to 0} \frac{\Delta \bar{f}}{\Delta s} = \frac{d\bar{f}}{ds} \qquad \{1.1.2.-36\}$$

Considering {1.1.2.-33}, the forces on the θ^i- surfaces may be written as

$$-\bar{\mathcal{T}}_{\underline{i}}(\theta^{\underline{i}})ds_{\underline{i}} = -\bar{\mathcal{T}}_{\underline{i}}(\theta^{\underline{i}})\sqrt{gg^{\underline{ii}}}\, d\theta^j d\theta^k = -\bar{t}_{\underline{i}}(\theta^{\underline{i}})d\theta^j d\theta^k \qquad \{1.1.2.-37\}$$

where $$\bar{t}_i = \sqrt{gg^{\underline{ii}}}\,\bar{\mathcal{T}}_{\underline{i}} \qquad \{1.1.2.-38\}$$

Similarly, for the $(\theta^i+d\theta^i)$ - surfaces, the forces on these surfaces may be given as

$$\bar{\mathcal{T}}_{\underline{i}}(\theta^{\underline{i}} + d\theta^{\underline{i}})\, ds_{\underline{i}} = \bar{t}_{\underline{i}}(\theta^{\underline{i}} + d\theta^{\underline{i}})\, d\theta^j d\theta^k \qquad \{1.1.2.-39\}$$

where $\bar{t}_i$ is prescribed by {1.1.2.-38}.

The superposition of forces for the surfaces $(\theta^i+d\theta^i)$ = Constant then appears as

$$d\bar{f}_i = \left[-\bar{t}_{\underline{i}}(\theta^{\underline{i}}) + \bar{t}_{\underline{i}}(\theta^{\underline{i}} + d\theta^{\underline{i}})\right] d\theta^j d\theta^k \qquad \{1.1.2.-40\}$$

or, denoting $\bar{t}_{\underline{i}}(\theta^{\underline{i}})$ as $\bar{t}_i$ and employing a TAYLOR series expansion,

$$d\bar{f}_i = \left[-\bar{t}_i + \bar{t}_i + \frac{\partial \bar{t}_i}{\partial \theta^{\underline{i}}} d\theta^{\underline{i}} + \frac{1}{2}\frac{\partial}{\partial \theta^{\underline{i}}}\left(\frac{\partial \bar{t}_i}{\partial \theta^{\underline{i}}}\right)(d\theta^{\underline{i}})^2 + \ldots\right] d\theta^j d\theta^k$$

Neglecting infinitesimals of the second order or higher, then

$$d\bar{f}_i = \frac{\partial \bar{t}_i}{\partial \theta^{\underline{i}}} d\theta^{\underline{i}} d\theta^j d\theta^k \qquad \{1.1.2.-41\}$$

Consequently, the superposition of all these forces acting on the elemental volume dv appears as

$$d\bar{f} = d\bar{f}_1 + d\bar{f}_2 + d\bar{f}_3$$

so
$$d\bar{f} = \left[\frac{\partial \bar{t}_1}{\partial \theta^1} + \frac{\partial \bar{t}_2}{\partial \theta^2} + \frac{\partial \bar{t}_3}{\partial \theta^3}\right] d\theta^1 d\theta^2 d\theta^3$$

or
$$d\bar{f} = \frac{\partial \bar{t}_i}{\partial \theta^i} d\theta^1 d\theta^2 d\theta^3 \qquad \{1.1.2.-42\}$$

Two other forces are present in the element, namely the body force and the apparent or inertia force (considered in the capacity of an ordinary force, in conjunction with the Jacob BERNOULLI concept). These forces, like $d\bar{f}$, are considered to act, in the limit,

at the centroid of the elemental volume. Denoting the body force as $\bar{\gamma}_0\, dv$, where $\bar{\gamma}_0$ represents the body force intensity in the deformed volume, and the apparent force as $-\ddot{\bar{r}}\, dm$ for the element of constant mass, then

$$d\bar{f} + \bar{\gamma}_0\, dv + (-\ddot{\bar{r}}\, dm) = \frac{\partial \bar{t}_i}{\partial \theta^i} d\theta^1 d\theta^2 d\theta^3 + \bar{\gamma}_0\, dv + (-\ddot{\bar{r}}\, dm) \qquad \{1.1.2.-43\}$$

represents the superposition of all forces acting on the element.

Since $$dv = \sqrt{g}\, d\theta^1 d\theta^2 d\theta^3$$

and $$dm = \rho dv = \rho\sqrt{g}\, d\theta^1 d\theta^2 d\theta^3$$

as previously noted, then {1.1.2.-43} may be given as

$$d\bar{f} + \bar{\gamma}_0\, dv + (-\ddot{\bar{r}}\, dm) = \left[\frac{\partial \bar{t}_i}{\partial \theta^i} + \bar{\gamma}_0\sqrt{g} + (-\ddot{\bar{r}}\, \rho\sqrt{g})\right] d\theta^1 d\theta^2 d\theta^3 \qquad \{1.1.2.-44\}$$

Consider now that the deformed configuration is subjected to an admissible virtual displacement, $\delta\bar{U}$. That is, let it be assumed that the elemental volume, originally located (in the undeformed state) at $\bar{R}$, at time $t_0 = 0$, instead of being subjected to the displacement $\bar{U}(\theta^1,\theta^2,\theta^3,t)$, was instead subjected to the displacement $\bar{U}^{\#}(\theta^1,\theta^2,\theta^3,t)$, where

$$\bar{U}^{\#} = \bar{U} + \delta\bar{U} \qquad \{1.1.2.-45\}$$

That the admissible virtual displacement $\delta\bar{U}$ must be instantaneous and

compatible with the kinematic constraints of the system, is readily observed from {1.1.2.-45}, for $\overline{U}^{\#} = \overline{U} + \delta\overline{U}$ and $\overline{U}$ must represent two kinematically possible displacements during the same time, t.

From {1.1.2.-45}, $\delta\overline{U}$ is given as

$$\delta\overline{U} = \overline{U}^{\#} - \overline{U}$$

and this may be written as

$$\delta\overline{U} = \overline{U}^{\#} - \overline{U} = \eta\overline{Z} \qquad \{1.1.2.-46\}$$

where η represents a real number, such that $(\eta^2, \eta^3, \ldots) \ll \eta$, as required by part (b) of the definition of an admissible virtual displacement. Thus, the virtual displacement $\delta\overline{U}$ may be identified with the (first) variation of the displacement $\overline{U}$, in accordance with the tenets of the calculus of variations.

It is noted that this restriction on the magnitude of $\delta\overline{U}$ in no way restricts the magnitude of the displacement field $\overline{U}$, which may be arbitrarily large (but finite).

The work done by the elemental volume dv, in the course of the virtual displacement $\delta\overline{U}$, under the actual forces previously given, represents the virtual work of the element, or

$$\delta W(dv) = \left[\frac{\partial \overline{t}_i}{\partial \theta^i} + \overline{f}\sqrt{g} + (-\ddot{\overline{r}}\,\rho\sqrt{g}) \right] \cdot \delta\overline{U}\, d\theta^1 d\theta^2 d\theta^3 \qquad \{1.1.2.-47\}$$

This may be written, since dv and dV denote the same elemental volume in different states, as

$$\delta W(dV) = \frac{1}{\sqrt{G}}\left[\frac{\partial t_i}{\partial \theta^i} + \gamma_0\sqrt{g} + (-\ddot{\bar{r}}\,\rho\sqrt{g})\right]\cdot\delta\bar{U}\,dV \qquad \{1.1.2.-48\}$$

by virtue of {1.1.2.-24}. The virtual work performed by the entire body is then found as the integral of {1.1.2.-48} over the undeformed volume V (or the integral of {1.1.2.-47} over the deformed volume v, if so desired). Thus, the virtual work for the body is obtained as

$$\delta W = \iiint_V \left[\frac{1}{\sqrt{G}}\frac{\partial \bar{t}_i}{\partial \theta^i} + \bar{F} + (-\rho_0\ddot{\bar{r}})\right]\cdot\delta\bar{U}\,dV \qquad \{1.1.2.-49\}$$

where

$$\bar{F} = \sqrt{\frac{g}{G}}\,\gamma_0$$

denotes the body force intensity per unit volume of the undeformed continuum, and ρ_0 represents the mass density in that configuration, as specified by {1.1.2.-21}.

Finally, as $\delta\bar{U}$ is an admissible virtual displacement, satisfying all criteria (a) to (e) as previously specified, then the primary form of the Principle of Virtual Work emerges as

$$\delta W = \iiint_V \left[\frac{1}{\sqrt{G}}\frac{\partial \bar{t}_i}{\partial \theta^i}\cdot\delta\bar{U} + \bar{F}\cdot\delta\bar{U} + (-\rho_0\ddot{\bar{r}}\cdot\delta\bar{U})\right]dV = 0 \qquad \{1.1.2.-50\}$$

Equation {1.1.2.-50} represents the necessary and sufficient condition that the deformed state of the continuum be an actual dynamic state.

The Virtual Work Principle is frequently written in an alternate form. Noting that

$$\bar{t}_i = \sqrt{gg^{ii}}\,\bar{\underline{t}}_i$$

as previously defined, then consider that the stress vector $\bar{\mathcal{T}}_i$, which prescribes the true stress on the surface which is defined by a normal $\bar{g}^i$, is determined as

$$\bar{\mathcal{T}}_i = (\bar{e}^i \cdot \bar{g}_j)\bar{\tau}_j \qquad \{1.1.2.-51\}$$

where $$\bar{e}^i = \frac{\bar{g}^i}{|\bar{g}^i|} = \frac{\bar{g}^i}{\sqrt{g^{ii}}} \qquad \{1.1.2.-52\}$$

Thus, $\bar{e}^i$ represents a *unit* normal vector to the surface, or a unit vector which is collinear with $\bar{g}^i$. Consequently, from {1.1.2.-51},

$$\bar{\mathcal{T}}_i = (\bar{e}^i \cdot \bar{g}_j)\bar{\tau}_j = \bar{e}^i \cdot (\bar{g}_j \bar{\tau}_j) = \bar{e}^i \cdot \bar{\bar{\tau}} \qquad \{1.1.2.-53\}$$

and the quantity $\bar{\bar{\tau}}$ is then known as the CAUCHY-GREEN "true" stress tensor

$$\bar{\bar{\tau}} = \bar{g}_i \bar{\tau}_i = \tau^{ij} \bar{g}_i \bar{g}_j \qquad \{1.1.2.-54\}$$

referred to the directed base $\{\bar{g}_i\}$ of the deformed state. The symmetric contravariant components τ^{ij} thus provide a true measure of the state of stress in the continuum.

It follows from the above, that

$$\bar{\mathcal{T}}_i = \frac{1}{\sqrt{g^{ii}}} \tau^{ij} \bar{g}_j \qquad \{1.1.2.-55\}$$

so the vector $\bar{t}_i$ may be expressed as

$$\bar{t}_i = \sqrt{g}(\sqrt{g^{ii}}\bar{\mathcal{T}}_i) = \sqrt{g}\,\tau^{ij}\bar{g}_j \qquad \{1.1.2.-56\}$$

Now, another stress tensor $\overline{\overline{S}}$ is defined, relative to the true stress tensor $\overline{\overline{\tau}}$, such that each component S^{ij} of $\overline{\overline{S}}$ (referred to the directed base $\{\overline{G}_i\}$ of the undeformed state) is related to τ^{ij} as

$$S^{ij}dV = \tau^{ij}dv \qquad \{1.1.2.-57\}$$

or $$S^{ij} = \frac{dv}{dV}\tau^{ij} = \sqrt{\frac{g}{G}}\,\tau^{ij} \qquad \{1.1.2.-58\}$$

where $$\overline{\overline{S}} = S^{ij}\overline{G}_i\overline{G}_j \quad \text{or} \quad S^{ij} = \overline{G}^i\overline{G}^j : \overline{\overline{S}} \qquad \{1.1.2.-59\}$$

The quantities S^{ij} thus describe the state of stress in the deformed state of the continuum, but the measure of the stress magnitude is given *per unit area of the undeformed state.* Thus, S^{ij}, known as the TREFFTZ stress components, provide a correct measure of the stress when considered with the proper base system: however, as the scalar magnitudes of S^{ij} differ from those of τ^{ij}, then S^{ij} is not called a "true" stress.

The vector $\overline{t}_i$ may now be written as

$$\overline{t}_i = \sqrt{g}\,\tau^{ij}\overline{g}_j = \sqrt{G}\,S^{ij}\overline{g}_j \qquad \{1.1.2.-60\}$$

and the first term in the integrand of the Virtual Work Principle {1.1.2.-50} then becomes

$$\frac{1}{\sqrt{G}}\frac{\partial \overline{t}_i}{\partial \theta^i}\cdot\delta\overline{U} = \frac{1}{\sqrt{G}}\frac{\partial}{\partial \theta^i}\left(\sqrt{G}\,S^{ij}\overline{g}_j\right)\cdot\delta\overline{U}$$

$$= \left[\frac{\partial}{\partial \overline{R}}\cdot\left(S^{ij}\overline{G}_i\overline{g}_j\right)\right]\cdot\delta\overline{U}$$

This expression may be written as

$$\frac{1}{\sqrt{G}}\frac{\partial \bar{t}_i}{\partial \theta^i}\cdot\delta\bar{U} = \frac{\partial}{\partial \bar{R}}\cdot(S^{ij}\bar{G}_i\bar{g}_j\cdot\delta\bar{U}) - S^{ij}\bar{g}_j\cdot\frac{\partial(\delta\bar{U})}{\partial\theta^i} \qquad \{1.1.2.-61\}$$

and the Virtual Work Principle then appears as

$$\delta W = \iiint_V \left[\frac{\partial}{\partial \bar{R}}\cdot(S^{ij}\bar{G}_i\bar{g}_j\cdot\delta\bar{U}) - S^{ij}\bar{g}_j\cdot\frac{\partial(\delta\bar{U})}{\partial\theta^i} + \bar{F}\cdot\delta\bar{U} + (-\rho_0\ddot{\bar{r}}\cdot\delta\bar{U})\right]dV = 0$$

Transforming the first term of the integral by means of the GAUSS Divergence Theorem, then

$$\delta W = \iint_S \bar{N}\cdot(S^{ij}\bar{G}_i\bar{g}_j)\,dS + \iiint_V \left[-S^{ij}\bar{G}_j\cdot\frac{\partial(\delta\bar{U})}{\partial\theta^i} + \bar{F}\cdot\delta\bar{U} + (-\rho_0\ddot{\bar{r}}\cdot\delta\bar{U})\right]dV = 0$$

$\{1.1.2.-62\}$

where S represents the bounding surface of the undeformed volume V, and $\bar{N} = N_i\bar{G}^i$ denotes a unit normal to the surface S. Now, as

$$\bar{N}\cdot S^{ij}\bar{G}_i\bar{g}_j = N_i S^{ij}\bar{g}_j = \sqrt{\frac{g}{G}}\,N_i\tau^{ij}\bar{g}_j$$

then

$$\bar{N}\cdot S^{ij}\bar{G}_i\bar{g}_j = \bar{T}_n \qquad \{1.1.2.-63\}$$

where $\bar{T}_n$ denotes the stress vector on the deformed surface, the unit normal to which (in its undeformed position) is $\bar{N}$. The stress

vector $\overline{\overline{T}}_n$ thus measures the stress per unit undeformed area of the continuum.

Furthermore, from the definition of the strain tensor {1.1.2.-31}, it is observed that

$$2\overline{\overline{E}} = (g_{ij} - G_{ij})\overline{G}^i\overline{G}^j = (\overline{g}_i \cdot \overline{g}_j - \overline{G}_i \cdot \overline{G}_j)\overline{G}^i\overline{G}^j$$

Now, as $\delta\overline{G}_i = \delta\overline{G}^i = \overline{0}$ for the base vectors of the undeformed state, then

$$2\delta\overline{\overline{E}} = \delta(\overline{g}_i \cdot \overline{g}_j)\overline{G}^i\overline{G}^j = (\delta\overline{g}_i \cdot \overline{g}_j + \overline{g}_i \cdot \delta\overline{g}_j)\overline{G}^i\overline{G}^j$$

and since $$\overline{g}_i = \overline{G}_i + \frac{\partial\overline{U}}{\partial\theta^i}$$

then $$2\delta\overline{\overline{E}} = \left[\delta\left(\frac{\partial\overline{U}}{\partial\theta^i}\right) \cdot \overline{g}_j + \overline{g}_i \cdot \delta\left(\frac{\partial\overline{U}}{\partial\theta^j}\right)\right]\overline{G}^i\overline{G}^j$$

However, from the definition of $\delta\overline{U}$ as

$$\delta\overline{U} = \overline{U}^{\#} - \overline{U} = \eta\overline{Z}$$

then $$\frac{\partial(\delta\overline{U})}{\partial\theta^i} = \frac{\partial(\eta\overline{Z})}{\partial\theta^i} = \eta\frac{\partial\overline{Z}}{\partial\theta^i} = \delta\left[\frac{\partial\overline{U}}{\partial\theta^i}\right]$$

Therefore, since $\overline{\overline{S}} = \overline{\overline{S}}_c$ (or $S^{ij} = S^{ji}$, in component form), the first term in the volume integrand of {1.1.2.-62} becomes

$$S^{ij}\overline{g}_j \cdot \frac{\partial(\delta\overline{U})}{\partial\theta^i} = S^{ij}\delta E_{ij} = \overline{\overline{S}} : \delta\overline{\overline{E}} \qquad \{1.1.2.-64\}$$

and the virtual displacement $\delta\overline{U}$ is now identified with the formal variation of the displacement $\overline{U}$. The Virtual Work Principle then becomes, in final form

$$\delta W = \iint_S \overline{T}_n \cdot \delta\overline{U}\, dS + \iiint_V \left[-\overline{\overline{S}} : \delta\overline{\overline{E}} + \overline{F} \cdot \delta\overline{U} + (-\rho_0 \ddot{\overline{r}} \cdot \delta\overline{U})\right] dV = 0 \qquad \{1.1.2.-65\}$$

For static systems, in which the body force is neglected -- a situation of particular interest in the analysis of deformable bodies -- the Virtual Work Principle appears as

$$\delta W = \iint_S \overline{T}_n \cdot \delta\overline{U}\, dS - \iiint_V \overline{\overline{S}} : \delta\overline{\overline{E}}\, dV = 0 \qquad \{1.1.2.-66\}$$

In the entire discussion of the Virtual Work Principle, the volume V and the associated bounding surface S have defined a hypothetical "body", which was in reality, an arbitrary finite region of the geometric space occupied by a continuum at some time, t. Now, in the event that the volume V and the surface S prescribe an actual boundary (i.e., a true body), then the Virtual Work Principle is still valid, as no restrictions have been imposed on V or S. In this case, however, it may be advantageous to consider the surface S as the sum of two surface areas, or

$$S = S_t + S_u$$

where S_t defines that part of the boundary on which there exist

prescribed stresses $\overline{T}_n^*$, and S_u defines that part of the boundary on which there exist prescribed displacements $\overline{U}^*$.

Then, $\overline{T}_n = \overline{T}_n^*$ on S_t and $\delta\overline{U} = \delta\overline{U}^* = \overline{0}$ on S_u, as there can be no variation of a *prescribed* kinematic condition if $\delta\overline{U}$ is to be compatible with the kinematic constraints of the system. Consequently, the Virtual Work Principle may be given in the form

$$\delta W = \iint_{S_t} \overline{T}_n^* \cdot \delta\overline{U}\, dS - \iiint_V \overline{\overline{S}} : \delta\overline{\overline{E}}\, dV = 0 \qquad \{1.1.2.-67\}$$

This relation is, of course, also true for the hypothetical body (the region of space occupied by the continuum), but is of less significance in such a case.

1.2. THE POTENTIAL ENERGY PRINCIPLE

Consider the Virtual Work Principle in the form of {1.1.2.-66}, for a static or constant velocity system without body force, which may then be written as

$$\iiint_V \overline{\overline{S}} : \delta\overline{\overline{E}}\, dV = \iint_S \overline{T}_n \cdot \delta\overline{U}\, dS \qquad \{1.2.-1\}$$

Now, if the integrand $\overline{\overline{S}} : \delta\overline{\overline{E}}$ may be represented by the total variation of a single scalar function, then the body is said to be *elastic*. In this case, the Virtual Work Principle yields the Principle of Stationary Potential Energy, or briefly, the Potential Energy Principle. That is, if there exists a function U_0^S, which is known as the internal energy density function or the *strain energy density*

function, such that $\delta U_0^S = \overline{\overline{S}} : \delta\overline{\overline{E}}$, then the continuum is elastic†.

Since $\overline{\overline{E}} = E_{ij}\overline{G}^i\overline{G}^j$, then the tensor (or directed) derivative with respect to $\overline{\overline{E}}$ is defined in terms of the base which is reciprocal to $\{\overline{G}^i\}$, namely $\{\overline{G}_i\}$. Thus,

$$\frac{\partial}{\partial\overline{\overline{E}}}(\ ) \equiv \overline{G}_i\overline{G}_j \frac{\partial}{\partial E_{ij}}(\ ) \qquad \{1.2.-2\}$$

and the elasticity of the continuum is then prescribed by the relation

$$\delta U_0^S = \partial\overline{\overline{E}} : \frac{\partial U_0^S}{\partial\overline{\overline{E}}} = \overline{\overline{S}} : \delta\overline{\overline{E}} \qquad \{1.2.-3\}$$

Since U_0^S is a scalar quantity, then

$$\delta\overline{\overline{E}} : \frac{\partial U_0^S}{\partial\overline{\overline{E}}} = \frac{\partial U_0^S}{\partial\overline{\overline{E}}} : \delta\overline{\overline{E}} \qquad \{1.2.-4\}$$

and {1.2.-3} may be written as

$$\delta U_0^S = \frac{\partial U_0^S}{\partial\overline{\overline{E}}} : \delta\overline{\overline{E}} = \overline{\overline{S}} : \delta\overline{\overline{E}} \qquad \{1.2.-5\}$$

It then follows from {1.2.-5} that the constitutive relation for an elastic material is given, in terms of the strain tensor and the TREFFTZ stress tensor, as

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† This requirement is considered further, in Chapter II.

$$\overline{\overline{S}} = \frac{\partial U_0^S}{\partial \overline{\overline{E}}} \qquad \{1.2.-6\}$$

or, in terms of components

$$S^{ij} = \overline{G}^i\overline{G}^j : \overline{\overline{S}} = \overline{G}^i\overline{G}^j : \frac{\partial U_0^S}{\partial \overline{\overline{E}}} = \frac{\partial U_0^S}{\partial E_{ij}} \qquad \{1.2.-7\}$$

Employing {1.2.-5} in {1.2.-1}, the latter assumes the form

$$\iiint_V \delta U_0^S \, dV = \iint_S \overline{T}_n \cdot \delta\overline{U} \, dS \qquad \{1.2.-8\}$$

Recalling that

$$dV = \sqrt{G} \, d\theta^1 d\theta^2 d\theta^3 = (\overline{G}_1 \cdot \overline{G}_2 \times \overline{G}_3) d\theta^1 d\theta^2 d\theta^3$$

and that $\delta\overline{G}_i = \overline{0}$, then

$$\delta(\sqrt{G} \, d\theta^1 d\theta^2 d\theta^3) = 0 \qquad \{1.2.-9\}$$

so it follows that

$$\iiint_V \delta U_0^S \, dV = \delta \iiint_V U_0^S \, dV \qquad \{1.2.-10\}$$

The total strain energy for the body is then denoted as U^S, where

$$U^S = \iiint_V U_0^S \, dV \qquad \{1.2.-11\}$$

since it is assumed that $U^S = 0$ in the undeformed configuration, as implied by the absence of a constant in equation {1.2.-10}.

From {1.2.-10} and {1.2.-11}, it is observed that {1.2.-8} may be written as

$$\delta U^S = \iint_S \bar{T}_n \cdot \delta\bar{U}\, dS$$

or as

$$\delta U^S - \iint_S \bar{T}_n \cdot \delta\bar{U}\, dS = 0 \qquad \{1.2.-12\}$$

which is the primary form of the Potential Energy Principle. Equation {1.2.-12} specifies the condition required for the existence of an equilibrium state of the deformed configuration of an elastic continuum.

If the surface forces can be derived from a potential energy function V^S, such that

$$\iint_S \bar{T}_n \cdot \delta\bar{U}\, dS = -\delta V^S \qquad \{1.2.-13\}$$

then the Principle ensures the equilibrium of the continuum if the functional $(U^S + V^S)$ assumes a stationary value, or

$$\delta Q^S = 0 \qquad \{1.2.-14\}$$

where $Q^S = U^S + V^S$ represents the total potential energy of the system.

Returning to equation {1.2.-12}, it is seen that a formal total variation of $\bar{T}_n \cdot \bar{U}$ yields

$$\delta(\bar{T}_n \cdot \bar{U}) = \delta\bar{T}_n \cdot \bar{U} + \bar{T}_n \cdot \delta\bar{U} \qquad \{1.2.-15\}$$

and {1.2.-12} therefore becomes

$$\delta U^s - \iint_S \delta(\bar{T}_n \cdot \bar{U})\, dS + \iint_S \delta\bar{T}_n \cdot \bar{U}\, dS = 0 \qquad \{1.2.-16\}$$

or

$$\delta U^s - \delta\iint_S \bar{T}_n \cdot \bar{U}\, dS + \iint_S \delta\bar{T}_n \cdot \bar{U}\, dS = 0 \qquad \{1.2.-17\}$$

The virtual displacement $\delta\bar{U}$ is thus identified with the formal variation of the vector $\bar{U}$ (as was previously implemented in the derivation of equation {1.1.2.-64}. However, as noted earlier, this is entirely permissible due to the precise definition of an "admissible virtual displacement" in §1.1.2., above.

Considering now, that the surface S may be represented as the sum of S_t and S_u, as previously defined, and noting that $\delta\bar{T}_n^* = \bar{0}$ on S_t and $\delta\bar{U}^* = \bar{0}$ on S_u, then {1.2.-17} may be written as

$$\delta U^s - \delta\iint_S \bar{T}_n \cdot \bar{U}\, dS + \delta\iint_{S_u} \bar{T}_n \cdot \bar{U}^*\, dS = 0 \qquad \{1.2.-18\}$$

or

$$\delta U^s - \delta\iint_{S_t} \bar{T}_n^* \cdot \bar{U}\, dS = 0 \qquad \{1.2.-19\}$$

Therefore, the Potential Energy Principle shows that equilibrium of the continuum is ensured if the functional

$$\Pi^S = \iiint_V u_0^S \, dV - \iint_{S_t} \bar{T}_n^* \cdot \bar{U} \, dS \qquad \{1.2.-20\}$$

assumes a stationary value.

The EULER-LAGRANGE equations, obtained from the vanishing of the first variation of Π^S, are the stress equilibrium equations

$$\frac{\partial \bar{t}_i}{\partial \theta^i} = \bar{0} \qquad \{1.2.-21\}$$

or, as $\bar{t}_i = \sqrt{G}\, S^{ij} \bar{g}_j$,

$$\frac{\partial (S^{ij} \bar{g}_j)}{\partial \theta^i} = \bar{0} \qquad \{1.2.-22\}$$

and the natural boundary condition is that the stress vector on S_t must be the prescribed stress vector $\bar{T}_n^*$.

Since the virtual displacement, as an entity, has been replaced in the Potential Energy Principle by the variation of the displacement field $\bar{U}$, it is necessary that $\bar{U}$ be subjected to certain restrictions, in order that $\delta\bar{U}$ may retain all the properties originally postulated for an admissible virtual displacement. This is reflected in the "general statement" of the Potential Energy Principle, as follows.

For admissible displacement fields, i.e., those fields which satisfy continuity and which are consistent with the prescribed displacements, the stresses which are predicted from these fields via the constitutive relation of the elastic medium will not satisfy equilibrium, nor be consistent with the prescribed stresses, except

for the admissible displacement field which coincides with the true one. For the admissible displacement field which *does* coincide with the true one, the functional Π^S assumes a stationary value.

Since the strain tensor $\overline{\overline{E}}$ is a derived function of $\overline{U}$, the above general statement may also be phrased in terms of "admissible states of strain" as well as "admissible displacement fields".

1.3. THE COMPLEMENTARY POTENTIAL ENERGY PRINCIPLE

The Principle of Stationary Complementary Potential Energy is a consequent of the Potential Energy Principle, obtained by means of a transformation executed on the variables employed. The Complementary Potential Energy Principle is, however, unlike the Potential Energy Principle, a "mathematical entity" which does not represent a real physical work principle, although the quantities employed in its formulation possess dimensions of work.

Consider the functional Π^S of the Potential Energy Principle

$$\Pi^S = \iiint_V U_0^S \, dV - \iint_{S_t} \overline{T}_n^* \cdot \overline{U} \, dS$$

or, in the form of {1.2.-18}

$$\Pi^S = \iiint_V U_0^S \, dV - \iint_S \overline{T}_n \cdot \overline{U} \, dS + \iint_{S_u} \overline{T}_n \cdot \overline{U}^* \, dS \qquad \{1.3.-1\}$$

where

$$\delta U_0^S = \overline{\overline{S}} : \delta \overline{\overline{E}} \qquad \{1.3.-2\}$$

Now, if $\overline{\overline{E}}$ may be expressed as a function of $\overline{\overline{S}}$, then a Legendre† transformation on U_0^s defines a complementary strain energy density function U_1^c

$$U_1^c = \overline{\overline{S}} : \overline{\overline{E}} - U_0^s \qquad \{1.3.-3\}$$

Therefore, a new functional Π^c may be constructed from the functional Π^s and the transformation {1.3.-3}. That is,

$$\Pi^c = \iiint_V \left[\overline{\overline{S}} : \overline{\overline{E}} - U_1^c\right] dV - \iint_S \overline{T}_n \cdot \overline{U}\, dS + \iint_{S_u} \overline{T}_n \cdot \overline{U}^*\, dS \qquad \{1.3.-4\}$$

where the constitutive relation of the elastic continuum is now prescribed to be

$$\overline{\overline{E}} = \frac{\partial U_1^c}{\partial \overline{\overline{S}}} = \overline{G}^i \overline{G}^j \frac{\partial U_1^c}{\partial S^{ij}} \qquad \{1.3.-5\}$$

Since $\overline{\overline{E}}$ may be expressed as

$$\overline{\overline{E}} = \frac{1}{2}\left[\overline{g}_i \cdot \overline{g}_j - \overline{G}_i \cdot \overline{G}_j\right] \overline{G}^i \overline{G}^j$$

as before, then the expansion of this expression in terms of $\{\overline{G}_i\}$ and $\overline{U}$ is

$$\overline{\overline{E}} = \frac{1}{2}\left[\overline{G}_i \cdot \frac{\partial \overline{U}}{\partial \theta^j} + \overline{G}_j \cdot \frac{\partial \overline{U}}{\partial \theta^i} + \frac{\partial \overline{U}}{\partial \theta^i} \cdot \frac{\partial \overline{U}}{\partial \theta^j}\right] \overline{G}^i \overline{G}^j$$

† Refer to Historical Notes, page C-19, Appendix C.

Therefore, $\overline{\overline{S}} : \overline{\overline{E}}$ is obtained as

$$\overline{\overline{S}} : \overline{\overline{E}} = S^{ij}\overline{G}_j \cdot \frac{\partial \overline{U}}{\partial \theta^i} + \frac{1}{2} S^{ij} \frac{\partial \overline{U}}{\partial \theta^i} \cdot \frac{\partial \overline{U}}{\partial \theta^j} \qquad \{1.3.-6\}$$

or, employing the relation

$$\overline{G}_i = \overline{g}_i - \frac{\partial \overline{U}}{\partial \theta^i}$$

in {1.3.-6}, then

$$\overline{\overline{S}} : \overline{\overline{E}} = S^{ij}\overline{g}_j \cdot \frac{\partial \overline{U}}{\partial \theta^i} - \frac{1}{2} S^{ij} \frac{\partial \overline{U}}{\partial \theta^i} \cdot \frac{\partial \overline{U}}{\partial \theta^j} \qquad \{1.3.-7\}$$

The functional Π^c then becomes

$$\Pi^c = \iiint_V \left[S^{ij}\overline{g}_j \cdot \frac{\partial \overline{U}}{\partial \theta^i} - \frac{1}{2} S^{ij} \frac{\partial \overline{U}}{\partial \theta^i} \cdot \frac{\partial \overline{U}}{\partial \theta^j} - U_i^c \right] dV$$

$$- \iint_S \overline{T}_n \cdot \overline{U}\, dS + \iint_{S_u} \overline{T}_n \cdot \overline{U}^*\, dS \qquad \{1.3.-8\}$$

The first term in the volume integrand may be transformed as follows:

$$S^{ij}\overline{g}_j \cdot \frac{\partial \overline{U}}{\partial \theta^i} = S^{ij}\overline{G}_i\overline{g}_j : \frac{\partial \overline{U}}{\partial \overline{R}}$$

$$= \frac{\partial}{\partial \overline{R}} \cdot (S^{ij}\overline{G}_i\overline{g}_j \cdot \overline{U}) - \frac{\partial \cdot (S^{ij}\overline{G}_i\overline{g}_j)}{\partial \overline{R}} \cdot \overline{U} \qquad \{1.3.-9\}$$

Therefore, employing the GAUSS Divergence Theorem under the assumption of equilibrium in the continuum, the functional Π^c assumes the final, irreducible form

$$\Pi^c = -\iiint_V \left[U_1^c + \frac{1}{2} S^{ij} \frac{\partial \overline{U}}{\partial \theta^i} \cdot \frac{\partial \overline{U}}{\partial \theta^j} \right] dV + \iint_{S_u} \overline{T}_n \cdot \overline{U}^* \, dS \qquad \{1.3.-10\}$$

This functional, as noted by LEVINSON, does not provide a suitable foundation for the Complementary Potential Energy Principle, by virtue of the explicit presence of the displacement terms in the first integral.

If the restrictions of small-deformation theory are applied to the functional {1.3.-10}, then the result appears as

$$\underset{\sim}{\Pi}^c = -\iiint_V U_1^c \, dV + \iint_{S_u} \overline{T}_n \cdot \overline{U}^* \, dS$$

which demonstrates that the formulation of the functional in terms of $\overline{\overline{S}}$ and $\overline{\overline{E}}$ *does* provide a suitable foundation for the Principle in the theory of infinitesimal elasticity.

The formulation of the Complementary Potential Energy Principle requires that restrictions be imposed upon the state of stress, similar to the restrictions placed on the displacement field in the Potential Energy Principle. This is reflected in the "general statement" of the Complementary Potential Energy Principle, as follows.

For admissible states of stress, i.e., those stress states

which satisfy equilibrium and are consistent with the prescribed stresses, the states of strain which are predicted from these stress states via the constitutive relation of the elastic medium will not satisfy compatibility nor be consistent with the prescribed displacements, except for the admissible stress state which coincides with the true one. For the admissible stress state which *does* coincide with the true one, the functional Π^c assumes a stationary value.

1.4. THE DUAL PRINCIPLES

From the foregoing developments of the dual Principles of Potential Energy and Complementary Potential Energy from the Virtual Work Principle, the following information may be obtained.

The Potential Energy Principle is a real work principle, in which displacements or displacement-related quantities are taken as the independent variables. The *a priori* satisfaction of prescribed kinematic conditions is required for the admissibility of displacement fields or displacement-related quantities, and the true displacement field is then selected from the set of all admissible ones by virtue of the fact that the functional Π^s assumes a stationary value for the true field. The Potential Energy Principle may thus be said to represent a "minimal principle of displacements".

Once the true displacement field has been obtained, the state of stress may be found from this field in conjunction with the constitutive relation of the elastic material.

The Complementary Potential Energy Principle is a mathematical

entity, expressed in terms of quantities which have dimensions of work, in which stresses or stress-related functions are employed as the independent variables. The *a priori* satisfaction of prescribed stress conditions is required for the admissibility of stress states, and the true stress state is then selected from all admissible ones by virtue of the fact that the functional Π^c assumes a stationary value for the true state. The Complementary Potential Energy Principle may thus be said to represent a "maximal principle of stresses".

Once the true state of stress has been obtained, the state of strain may be found from this stress state in conjunction with the constitutive relation of the elastic material.

In order to establish which of the dual principles may yield the most satisfactory result in a given situation, it is necessary to consider, in addition to the obvious *a priori* requirements of each, the following pertinent fact. Since the solution of engineering problems is accomplished very infrequently in analytical form, it is most likely that the solution will require the use of approximate procedures. Therefore, the appropriate Principle should be selected under this criterion.

For example, if a solution for a displacement field is obtained as an approximation by means of the RAYLEIGH-RITZ Trial Function Method within the framework of the Potential Energy Principle, then the results will likely be a close approximation to the true solution. However, the state of stress must be calculated from the

constitutive relation, which necessitates the use of the displacement *gradient*. Since the accuracy of the results will generally deteriorate rapidly with differentiation, the approximate stress state solution will not be of the same order of accuracy as the solution for the displacements.

It may therefore be concluded that the Complementary Potential Energy Principle is an advantageous principle in the analysis of elastic media, for both infinitesimal and finite deformations.

The establishment of a strict Complementary Potential Energy Principle (i.e., in which displacement quantities are not explicity present in conjunction with the complementary strain energy density) for the case of finite deformations, representing the verification and extension of the principle proposed by LEVINSON, forms the core of this work, as presented in the following chapters.

CHAPTER II

The Complementary Potential Energy Principle In Finite Elastic Deformations

2.1. RECENT DEVELOPMENTS

The most recent comprehensive statement regarding the status of the Complementary Potential Energy Principle for the case of finite deformations has been given by WASHIZU, in 1968. He has said, in part, that "..... coupling of the displacements with the stress components in finite deformation problems complicates the derivation of the principle of stationary complementary [potential] energy .....; the principle can no longer be expressed purely in terms of stress components". WASHIZU has further observed that "..... these [complementary] principles play important roles in the small displacement theory of elasticity. However, extensions of these principles to the finite displacement theory of elasticity are not found successful .....".

These statements may be considered as a succinct description of the state of the Complementary Potential Energy Principle at the time at which they were written, assuming that the works of MASUR and of LEVINSON were unknown to WASHIZU. The developments which led WASHIZU to these conclusions represent the history of the Principle, the salient points of which are as follows.

The fundamental concepts of complementary potential energy

and the long and tortuous development of those concepts by earlier engineers and mathematicians have been well documented in the papers by ORAVAS and McLEAN, in 1966. Subsequent to those developments, the first earnest attempt to extend the Complementary Potential Energy Principle from the infinitesimal case to the case of finite deformations appears to have been made by C.-T. WANG in 1949. In this paper, however, the author has constructed a special method to deal with the analysis of thin plates and shells, subject to many restrictive assumptions. Consequently, the approach cannot properly be termed a form of the *Principle*: it does, however, represent a method of analysis which is the "complement" of the Potential Energy Principle for a particular class of problems which contain certain aspects of finite deformation.

The first approach dealing with the finite extension of the Complementary Potential Energy Principle, in the true form of a principle, was constructed for the one-dimensional case by LANGHAAR, in 1953. In his paper, LANGHAAR stated "CASTIGLIANO'S Theorem" as

$$U_i = \frac{\partial U^s}{\partial F_i} \qquad \{2.1.-1\}$$

where F_i represents the scalar value of the discrete force $\overline{F}_i$ at a point "i" (i = 1,2, ..., n) in the system

$U_i = U_i(F_1, F_2, \ldots, F_n)$ denotes the components of the displacement $\overline{U}_i$ (at point "i") in the direction of $\overline{F}_i$

U^s represents the strain energy of the system.

LANGHAAR observed that {2.1.-1} is valid only for linearly elastic

materials, and subsequently constructed a Legendre transformation in order to define the complementary strain energy W^C as

$$W^C = \overline{F}_i \cdot \overline{U}_i - U^S \qquad \{2.1.-2\}$$

where the summation takes place over the n degrees of freedom of the finite system. Ultimately, he developed the "generalization of CASTIGLIANO'S Theorem",

$$U = \frac{\partial W^C}{\partial F} \qquad \{2.1.-3\}$$

for a force $\overline{F} = F\overline{e}_f$ and a corresponding displacement $U = \overline{U} \cdot \overline{e}_f$, thus establishing the validity of the CROTTI Theorem for the case of finite deformations. LANGHAAR entered into a short discussion of the Complementary Potential Energy Principle for a continuum, but restricted the discussion to the consideration of 'finite-but-small' displacements, for which there is "no appreciable change in geometry". Consequently, the contribution of LANGHAAR to the finite deformation formulation of the Complementary Potential Energy Principle lies in re-exposing the Legendre transformation {2.1.-2} to the profession, and in the development of the CROTTI Theorem {2.1.-3} for the case of finite deformations.

In 1953, REISSNER presented a very general variational theorem for the finite deformations of elastic continua. This theorem, although valid, is formulated in terms of the strain tensor and the TREFFTZ stress tensor, and is therefore subject to the observations made by WASHIZU, as given above. It is important to note, however,

that this theorem possesses an advantage over many other variational theorems, in that no subsidiary conditions or "admissibility conditions" need be imposed upon the variables. In the same paper, REISSNER presented the theorem in its specific form for the analysis of plates, but subject to the restriction that the finite deformations be 'small'.

The next significant contribution to the Complementary Potential Energy Principle for finite deformations was made by LIBOVE, in 1962. LIBOVE postulated that since the work W of a force $\overline{F}$, accrued in the displacement of a particle ("a") from a position $\overline{R}_a$ to a position $\overline{r}_a = \overline{R}_a + \overline{U}_a$ is defined as

$$W = \int_{\overline{R}_a}^{\overline{r}_a} \overline{F} \cdot d\overline{r} \qquad \{2.1.-4\}$$

then the "complementary work" W^c for the same particle could be constructed, *by analogy*, as

$$W^c = \int_A^B \overline{r} \cdot d\overline{F} \qquad \{2.1.-5\}$$

where the integration takes place from "state A" to "state B" of the system. Applying this result to an elastic body (one for which the strain energy density function exists), upon which n discrete forces $\overline{F}_i$ are assumed to act, LIBOVE constructed a type of complementary energy H^c as the sum of the complementary work of all the applied forces in the finite system:

$$H^c = \int_A^B \bar{r}_i \cdot d\bar{F}_i \qquad \{2.1.-6\}$$

where $\bar{r}_i$ represents the point of application of the force $\bar{F}_i$ as the system deforms from state A to state B. Finally, LIBOVE obtained a quantity which he called the "total complementary energy" Q^c: this quantity is defined from H^c as

$$Q^c = H^c - \bar{r}_j^* \cdot \bar{F}_j^{(r)} \qquad \{2.1.-7\}$$

where $\bar{r}_j^*$ denotes the position, after deformation, of any points in the system which have a *prescribed location* and $\bar{F}_j^{(r)}$ represents the reaction force at prescribed point $\bar{r}_j^*$. Thus, LIBOVE constructed his complementary energy principle by requiring that the functional Q^c be stationary when an admissible state coincides with the true state of the system, or

$$\delta Q^c = \delta(H^c - \bar{r}_j^* \cdot \bar{F}_j^{(r)}) = 0 \qquad \{2.1.-8\}$$

where $\delta\bar{r}_j^* = \bar{0}$. It is observed that the quantity H^c in the LIBOVE Theorem, contrary to LIBOVE'S suggestion, does not represent the finite counterpart of the complementary strain energy of the infinitesimal formulation[†]. That is, although H^c represents a type of complementary energy, it is not equivalent to the energy W^c of the CROTTI Theorem. Furthermore, since the LIBOVE Theorem has as its foundation,

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[†] The energy H^c is further examined in Chapter IV.

the work analogue for a particle and several subsequent definitions, it is possible neither to prove nor to disprove the theorem on its own merit: external verification which does not employ the analogue is required.

In 1962, a paper by PIPES was published, in which the work analogue was stated but was never employed. This paper is quite similar to the 1953 paper of LANGHAAR, and serves to re-establish LANGHAAR'S results.

The next significant contribution to the Complementary Potential Energy Principle for the case of finite deformations was made by MASUR, in 1965. MASUR, in a brief discussion of LIBOVE'S work, established the complementary strain energy density function Ω by means of a Legendre transformation on the TREFFTZ stress tensor $\overline{\overline{S}}$ and the CAUCHY-GREEN deformation tensor $\overline{\overline{C}}$, as

$$\Omega = \frac{1}{2}\overline{\overline{S}} : \overline{\overline{C}} - U_0^S \qquad \{2.1.-9\}$$

where $\overline{\overline{C}}$ is related to the strain tensor $\overline{\overline{E}}$ by the relationship $\overline{\overline{C}} = 2\overline{\overline{E}} + \overline{\overline{1}}$ (as will be discussed in greater detail, presently) and U_0^S represents the strain energy density, as before. MASUR then noted that the constitutive relation for the elastic continuum, in terms of Ω, appears as

$$\overline{\overline{C}} = 2\frac{\partial \Omega}{\partial \overline{\overline{S}}} \qquad \{2.1.-10\}$$

since $\Omega = \Omega(\overline{\overline{S}})$, as {2.1.-9} is a Legendre transformation. Finally,

he obtained the Complementary Potential Energy functional $\Pi^c(\bar{\bar{S}})$ as

$$\Pi^c(\bar{\bar{S}}) = \iiint_V \left[\bar{\bar{S}} : \frac{\partial \Omega}{\partial \bar{\bar{S}}} + \Omega\right] dV - \iint_{S_u} \bar{T}_n \cdot \bar{r}^* \, dS \qquad \{2.1.-11\}$$

where $\bar{r}^* = \bar{R} + \bar{U}^*$ denotes the location of any regions of prescribed displacement $\bar{U}^*$ on the surface S_u, and $\bar{T}_n = \bar{N} \cdot S^{ij}\bar{G}_i\bar{g}_j = N_i S^{ij}\bar{g}_j$ as before. Thus, MASUR verified LIBOVE'S Theorem, extended it from a one-dimensional theorem to a principle valid for an elastic continuum, and avoided a functional form which would be subject to comments such as those given by WASHIZU. It is noted, however, that the complementary strain energy density function Ω appears both alone and as a tensor derivative, and that admissible stress states $\bar{\bar{S}}$ must satisfy the equilibrium equation which cannot be expressed in terms of $\bar{\bar{S}}$ alone.

The latest significant contribution to the Complementary Potential Energy Principle was made by LEVINSON, in 1965. LEVINSON constructed a complementary strain energy density function W_0^c as a Legendre transformation on the Lagrange stress tensor components T^{ij} and the Lagrange strain tensor (or displacement gradient) components $U_j|_i$. Employing the notation of GREEN and ZERNA or GREEN and ADKINS, these tensor components may be defined as follows:

$$U_j|_i = \bar{G}_j \cdot \frac{\partial \bar{U}}{\partial \theta^i} \qquad \{2.1.-12\}$$

$$U^j|_i = \bar{G}^j \cdot \frac{\partial \bar{U}}{\partial \theta^i} \qquad \{2.1.-13\}$$

thus, $$\frac{\partial \bar{U}}{\partial \theta^i} = U_j|_i \, \bar{G}^j = U^j|_i \, \bar{G}_j \qquad \{2.1.-14\}$$

The Lagrange stress tensor components are then defined from the TREFFTZ stress tensor or the CAUCHY-GREEN stress tensor components, as

$$T^{ij} = S^{im}\left[\delta^j_m + U^j|_m\right] = \sqrt{\frac{g}{G}}\,\tau^{im}\left[\delta^j_m + U^j|_m\right] \qquad \{2.1.-15\}$$

or $$T^{ij} = S^{im}\,\bar{g}_m \cdot \bar{G}^j$$

The variational strain energy density may then be shown to be

$$\delta U_0^S = S^{ij}\delta E_{ij} = T^{ij}\delta U_j|_i \qquad \{2.1.-16\}$$

(as will be discussed later in this work), and the constitutive relation of the elastic continuum exists in the form

$$T^{ij} = \frac{\partial U_0^S}{\partial U_j|_i} \qquad \{2.1.-17\}$$

Now, LEVINSON postulated that if the constitutive relation {2.1.-17} could be inverted, so that the components $U_j|_i$ could be expressed as a function of T^{rs} (a disputed point), then a Legendre transformation could be constructed to define the complementary strain energy density function $W_0^C = W_0^C(T^{rs})$ as

$$W_0^C = T^{ij}U_j|_i - U_0^S \qquad \{2.1.-18\}$$

where the constitutive equation, in terms of W_0^C, now appears as

$$U_j|_i = \frac{\partial W_0^c}{\partial T^{ij}} \qquad \{2.1.-19\}$$

LEVINSON then proved that, if this constitutive inversion is possible, the Complementary Potential Energy functional $\Pi^c(T^{ij})$ appears in the form

$$\Pi^c(T^{ij}) = -\iiint_V W_0^c \, dV + \iint_{S_u} \overline{T}_n \cdot \overline{U}^* \, dS \qquad \{2.1.-20\}$$

where components T^{ij} of admissible stress states are required to satisfy the equation of stress equilibrium,

$$T^{ij}|_i = \left[S^{ir}(\delta_r^j + U^j|_r)\right]\Big|_i = 0 \qquad \{2.1.-21\}$$

and the equation of stress-couple or "moment" equilibrium

$$T^{im}\overline{G}_m \cdot \overline{g}^j = T^{jn}\overline{G}_n \cdot \overline{g}^i \qquad \{2.1.-22\}$$

where the latter is an alternate statement of the symmetry of the TREFFTZ stress tensor, expressed in terms of T^{ij}.

Thus, if the constitutive inversion {2.1.-19} is possible, LEVINSON'S Theorem provides the most attractive form of the Principle, since W_0^c appears alone in the volume integral of the functional, and comments such as those given by WASHIZU are no longer applicable. Also, the volume integral of W_0^c can be shown, for discrete force systems, to be identical with the complementary energy W^c of the CROTTI Theorem: thus, the volume integral of W_0^c represents the

"finite counterpart" of the complementary strain energy of the infinitesimal formulation. Although this constitutive inversion, contrary to some expressed opinions, is *not* generally impossible (due to the particular nature of the Lagrange stress tensor components T^{ij}),† it is most unfortunate that the complementary strain energy density W_0^c of the LEVINSON formulation is a function of rigid-body rotations in the displacement field (see Chapter III).

Therefore, at the time of the writing of WASHIZU'S comprehensive work, four different approaches existed, regarding the Complementary Potential Energy Principle for the case of finite deformations. Briefly, these approaches may be summarized as follows.

1. The LANGHAAR-PIPES Approach

   This is an approach characterized by the extension of the CROTTI Theorem to the case of finite deformations of one-dimensional elastic bodies or elastic systems of a finite number of degrees of freedom. Any attempted generalization of the theorem to include the finite deformations of elastic continua is, at best, uncertain, and has usually resulted in the construction of a functional containing displacement quantities in addition to the complementary strain energy density function, in the volume integral.

† The supposed impossibility of such an inversion has been stated by the anonymous Reviewer of the proposed paper "Complementary Variational Principles in Elastodynamics" by B. H. Karnopp.

2. The REISSNER Approach

This approach establishes the validity of the Complementary Potential Energy Principle for the case of finite deformations of an elastic continuum, considering the strain tensor and the TREFFTZ stress tensor as the conjugate variables. Therefore, this theorem is subject to the remarks made by WASHIZU, concerning the coupling of the displacements with the stress components. It is noted, however, that the theorem requires the imposition of no subsidiary conditions on the variables.

3. The MASUR Approach

This approach formulates the Complementary Potential Energy Principle for an elastic continuum in terms of the TREFFTZ stress tensor and the CAUCHY-GREEN deformation tensor, thus avoiding the coupling of stress and displacement components. However, the complementary strain energy density function appears both alone and as a tensor derivative in the volume integral, and the stress tensor must satisfy admissibility conditions (equilibrium conditions) which cannot be expressed in terms of the stress tensor alone. The MASUR

Theorem reduces to, and therefore contains as a special case, the LIBOVE *Theorem, when the continuum is assumed to be unidimensional.*

4. The LEVINSON Approach

This approach postulates the inversion of the constitutive relation of an elastic continuum in terms of the Lagrange stress and the Lagrange strain (displacement gradient) tensor components. From this inversion, it follows that the complementary strain energy density function, defined by a Legendre transformation on these Lagrange variables, is not found in conjunction with displacement quantities in the functional. Furthermore, the volume integral of W_0^c can be shown to be identical with the complementary strain energy W^c of the CROTTI theorem, in the case of discrete force systems. Thus, the LEVINSON Theorem appears to be the most advantageous form of the Principle. However, the complementary strain energy density W_0^c is unfortunately a function of rigid-body rotations in the displacement field. Although this condition does not constitute an error in the formulation, it does place a restriction upon its applicability, since energy densities which are functions of

rigid-body rotation must be employed with caution. It is noted, however, that LEVINSON'S formulation depends only on stresses, and not explicitly on rotations.

The investigation of LEVINSON'S Theorem has generated this work, in which it is demonstrated that the Lagrange stress tensor and the *deformation* gradient define the appropriate conjugate tensors for the formulation of the Complementary Potential Energy Principle (not the Lagrange stress tensor and the *displacement* gradient, as proposed by LEVINSON). It is demonstrated that the complementary strain energy density, defined by the former set of Lagrange variables (above) is not a function of rigid-body rotations, and is therefore a suitable function to serve in the constitutive relations of an elastic continuum. This complementary strain energy density is shown to be quite similar to the energy density Ω of the MASUR theorem, if expressed in terms of $\overline{\overline{S}}$ and $\overline{\overline{C}}$, differing only by a constant factor in part of the expression.

It is also shown that the present formulation reduces to, and therefore contains as a special case, the LIBOVE Theorem, when the continuum is assumed to be unidimensional. Since the complementary strain energy density W_0^c of the LEVINSON Theorem corresponds to the energy W^c of the CROTTI Theorem, and as W_0^c is a function of rigid-body rotations, it is shown that W^c has no simple extension to a general continuum in the case of finite deformations. That is, the theorems of CROTTI and MENABREA cannot be generalized to a continuum in a simple form in the finite case, in contrast to the case of

infinitesimal deformations.

2.2. THE COMPLEMENTARY POTENTIAL ENERGY PRINCIPLE IN FINITE ELASTICITY

The Virtual Work Principle, prior to the (usual) representation formulated in terms of the strain tensor $\overline{\overline{E}}$ and the TREFFTZ stress tensor $\overline{\overline{S}}$, appeared as

$$\delta W = \iiint_V \left[\frac{1}{\sqrt{G}} \frac{\partial \overline{t}_i}{\partial \theta^i} \cdot \delta \overline{U} + \overline{F} \cdot \delta \overline{U} + (-\rho_0 \ddot{\overline{r}} \cdot \delta \overline{U}) \right] dV = 0 \qquad \{2.2.-1\}$$

which is the primary form of the Principle and is, of course, valid for both infinitesimal and finite deformations. Equation {2.2.-1} may be expressed in terms of variables other than $\overline{\overline{S}}$ and $\overline{\overline{E}}$, namely the Lagrange stress tensor $\overline{\overline{T}}$ and the displacement gradient or Lagrange strain tensor $\overline{\overline{U}}$, which are defined as follows.

With reference to Fig. 1.1.2.-1, the deformation gradient $\overline{\overline{\Gamma}}$ is defined as

$$\overline{\overline{\Gamma}} = \frac{\partial \overline{r}}{\partial \overline{R}} = \frac{\partial (\overline{R} + \overline{U})}{\partial \overline{R}} = \frac{\partial \overline{R}}{\partial \overline{R}} + \frac{\partial \overline{U}}{\partial \overline{R}} \qquad \{2.2.-2\}$$

or

$$\overline{\overline{\Gamma}} = \overline{\overline{I}} + \overline{\overline{U}} \qquad \{2.2.-3\}$$

where

$$\overline{\overline{I}} = \frac{\partial \overline{R}}{\partial \overline{R}} = \overline{G}^i \overline{G}_i = \overline{G}_j \overline{G}^j \qquad \{2.2.-4\}$$

represents the self-conjugate identity tensor,

and

$$\overline{\overline{U}} = \frac{\partial \overline{U}}{\partial \overline{R}} = \overline{G}^i \frac{\partial \overline{U}}{\partial \theta^i} \qquad \{2.2.-5\}$$

represents the displacement gradient. However, as $\overline{\overline{\Gamma}}$ may also be given in the form

$$\overline{\overline{\Gamma}} = \frac{\partial \overline{r}}{\partial \overline{R}} = \overline{G}^{i} \frac{\partial \overline{r}}{\partial \theta^{i}} \qquad \{2.2.-6\}$$

then $$\overline{\overline{\Gamma}} = \overline{\overline{1}} + \overline{\overline{U}} = \overline{G}^{i} \overline{g}_{i} \qquad \{2.2.-7\}$$

The Lagrange strain tensor $\overline{\overline{U}}$ is referred to the directed base $\{\overline{G}^{i}\}$, so that

$$\overline{\overline{U}} = U_{ij} \overline{G}^{i} \overline{G}^{j} \qquad \{2.2.-8\}$$

or $$U_{ij} = \overline{G}_{i} \overline{G}_{j} : \overline{\overline{U}} \qquad \{2.2.-9\}$$

and a component U_{ij} of the tensor $\overline{\overline{U}}$ is thus seen to be identical with a component $U_{j}|_{i}$ of the LEVINSON formulation. Now, the Lagrange stress tensor, referred to the directed base $\{\overline{G}_{i}\}$, i.e.,

$$\overline{\overline{T}} = T^{ij} \overline{G}_{i} \overline{G}_{j} \qquad \{2.2.-10\}$$

is defined from the TREFFTZ stress tensor $\overline{\overline{S}}$ and the deformation gradient $\overline{\overline{\Gamma}}$ by means of the mapping

$$\overline{\overline{S}} \cdot \overline{\overline{\Gamma}} = \overline{\overline{T}} \qquad \{2.2.-11\}$$

Since $\overline{\overline{S}}$ and $\overline{\overline{T}}$ have algebraic structures of the same type, and since operations are preserved in the mapping, {2.2.-11} is actually a homomorphism (see Appendix B).

From this definition, it is observed that $\overline{\overline{T}}$ correctly

represents a stress tensor, since the tensor $\overline{\overline{S}}$ represents a measure of the state of stress, and the tensor $\overline{\overline{\Gamma}}$ is dimensionless. In fact, a component $T^{ij} = \overline{G}^i\overline{G}^j : \overline{\overline{T}}$ denotes the magnitude of stress in the deformed continuum, but measured in terms of the undeformed continuum and referred to the directed base $\{\overline{G}_i\}$ of the undeformed reference state of the continuum.

Employing the trinomial form of $\overline{\overline{\Gamma}}$ from {2.2.-7} in {2.2.-11}, $\overline{\overline{T}}$ may be written as

$$\overline{\overline{T}} = \overline{\overline{S}} \cdot \overline{\overline{\Gamma}} = S^{ij}\overline{G}_i\overline{G}_j \cdot \overline{G}^r\overline{g}_r = S^{ir}\overline{G}_i\overline{g}_r \qquad \{2.2.-12\}$$

and therefore, from the dot product with $\overline{G}^m$,

$$\overline{G}^m \cdot \overline{\overline{T}} = T^{mn}\overline{G}_n = S^{mr}\overline{g}_r \qquad \{2.2.-13\}$$

and so

$$T^{mn} = S^{mr}\overline{g}_r \cdot \overline{G}^n \qquad \{2.2.-14\}$$

which establishes that the components T^{ij} of the tensor $\overline{\overline{T}}$ are identical with the components T^{ij} of LEVINSON'S formulation† (as desired).

From {2.2.-13}, it is observed that the form $S^{mr}\overline{g}_r$ suggests that the force vector $\overline{t}_i$ in the Virtual Work Principle may now be represented in terms of $\overline{\overline{T}}$ as well as $\overline{\overline{S}}$ or (originally) $\overline{\overline{\tau}}$. Consequently, from {1.1.2.-60} and {2.2.-13},

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† GREEN and ADKINS denote these tensor components as t^{ij}, defined by the relation $t^{ij}\overline{G}_j = S^{im}\overline{g}_m$.

$$\bar{t}_i = \sqrt{g}\tau^{im}\bar{g}_m = \sqrt{G}S^{ir}\bar{g}_r = \sqrt{G}T^{ij}\bar{G}_j \qquad \{2.2.-15\}$$

or

$$\bar{t}_i = \sqrt{G}\,\bar{T}_i = \sqrt{G}\,\bar{G}^i \cdot \bar{\bar{T}} \qquad \{2.2.-16\}$$

and the first term in the integrand of {2.2.-1} becomes

$$\frac{1}{\sqrt{G}}\frac{\partial \bar{t}_i}{\partial \theta^i} \cdot \delta\bar{U} = \frac{1}{\sqrt{G}}\frac{\partial}{\partial \theta^i}(\sqrt{G}\,\bar{G}^i \cdot \bar{\bar{T}}) \cdot \delta\bar{U}$$

$$= \bar{G}^i \cdot \frac{\partial \bar{\bar{T}}}{\partial \theta^i} \cdot \delta\bar{U}$$

$$= \frac{\partial \cdot \bar{\bar{T}}}{\partial \bar{R}} \cdot \delta\bar{U} \qquad \{2.2.-17\}$$

However,

$$\frac{\partial \cdot \bar{\bar{T}}}{\partial \bar{R}} \cdot \delta\bar{U} = \frac{\partial}{\partial \bar{R}} \cdot (\bar{\bar{T}} \cdot \delta\bar{U}) - \bar{\bar{T}} : \frac{\partial(\delta\bar{U})}{\partial \bar{R}} \qquad \{2.2.-18\}$$

and, since $\delta\bar{G}^i = \delta\bar{G}_j = \bar{0}$ for the vectors of the directed bases of the reference configuration, then

$$\frac{\partial(\delta\bar{U})}{\partial \bar{R}} = \delta\left[\frac{\partial \bar{U}}{\partial \bar{R}}\right] = \delta\bar{\bar{U}} \qquad \{2.2.-19\}$$

as previously established in §1.1.2.. Consequently,

$$\frac{1}{\sqrt{G}}\frac{\partial \bar{t}_i}{\partial \theta^i} \cdot \delta\bar{U} = \frac{\partial}{\partial \bar{R}} \cdot (\bar{\bar{T}} \cdot \delta\bar{U}) - \bar{\bar{T}} : \delta\bar{\bar{U}} \qquad \{2.2.-20\}$$

and the Virtual Work Principle appears as

$$\delta W = \iiint_V \left[\frac{\partial}{\partial \bar{R}} \cdot (\bar{\bar{T}} \cdot \delta \bar{U}) - \bar{\bar{T}} : \delta \bar{\bar{U}} + \bar{F} \cdot \delta \bar{U} + (-\rho_0 \ddot{\bar{r}} \cdot \delta \bar{U}) \right] dV = 0 \qquad \{2.2.-21\}$$

Employing the GAUSS Divergence Theorem, this may be expressed as

$$\delta W = \iiint_V [-\bar{\bar{T}} : \delta \bar{\bar{U}} + \bar{F} \cdot \delta \bar{U} + (-\rho_0 \ddot{\bar{r}} \cdot \delta \bar{U})] dV + \iint_S \bar{T}_n \cdot \delta \bar{U} \, dS = 0 \qquad \{2.2.-22\}$$

where $\bar{T}_n$ designates the stress vector on the surface S, the unit normal to which (referred to its undeformed position) is $\bar{N} = N_i \bar{G}^i$. Thus,

$$\bar{T}_n = \bar{N} \cdot \bar{\bar{T}} \qquad \{2.2.-23\}$$

or

$$\bar{T}_n = N_i T^{ij} \bar{G}_j = N_i S^{im} \bar{g}_m \qquad \{2.2.-24\}$$

as previously specified in terms of $\bar{\bar{S}}$ by {1.1.2.-63}.

For systems of constant or vanishing velocity in the absence of body force, the Principle may be written in the form of {1.2.-1}, i.e.,

$$\iiint_V \bar{\bar{T}} : \delta \bar{\bar{U}} \, dV = \iint_S \bar{T}_n \cdot \delta \bar{U} \, dS \qquad \{2.2.-25\}$$

Now, if the variational form $\bar{\bar{T}} : \delta \bar{\bar{U}}$ may be represented by the total variation of a single scalar function (so that $\bar{\bar{T}} : \delta \bar{\bar{U}}$ is no longer a variational *form*, but rather a total variation), then the body is

said to be elastic, and the Virtual Work Principle yields the Potential Energy Principle. That is, if there exists a function Ψ, such that

$$\delta\Psi = \overline{\overline{T}} : \delta\overline{\overline{U}} \qquad \{2.2.-26\}$$

then the continuum is elastic. However, since

$$\begin{aligned}
\delta\Psi &= \overline{\overline{T}} : \delta\overline{\overline{U}} \\
&= \overline{\overline{S}} \cdot \overline{\overline{I}} : \delta\overline{\overline{U}} \\
&= S^{ij}\overline{G}_i\overline{g}_j : \delta\left[\frac{\partial\overline{U}}{\partial\overline{R}}\right] \\
&= S^{ij}(\overline{G}_i \cdot \overline{G}^k)\overline{g}_j \cdot \delta\left[\frac{\partial\overline{U}}{\partial\theta^k}\right] \\
&= S^{ij}\overline{g}_j \cdot \delta\left[\frac{\partial\overline{U}}{\partial\theta^i}\right] \\
&= \overline{\overline{S}} : \delta\overline{\overline{E}} \qquad \{2.2.-27\}
\end{aligned}$$

then Ψ is seen to be identical with the strain energy density function $U_0^S = U_0^S(\overline{\overline{E}}) = U_0^S(\overline{\overline{U}})$, or

$$\delta\Psi = \delta U_0^S = \overline{\overline{S}} : \delta\overline{\overline{E}} = \overline{\overline{T}} : \delta\overline{\overline{U}} \qquad \{2.2.-28\}$$

Furthermore, since the CAUCHY-GREEN deformation tensor $\overline{\overline{C}}$ is related to $\overline{\overline{E}}$ as (see also, Chapter III)

$$\overline{\overline{E}} = \frac{1}{2}(\overline{\overline{C}} - \overline{\overline{I}}) \qquad \{2.2.-29\}$$

and as the deformation gradient $\overline{\overline{\Gamma}}$ is related to $\overline{\overline{U}}$ as

$$\overline{\overline{\Gamma}} = \overline{\overline{I}} + \overline{\overline{U}} \qquad \{2.2.-30\}$$

then it follows that

$$\tfrac{1}{2}\delta\overline{\overline{C}} = \delta\overline{\overline{E}} \qquad \{2.2.-31\}$$

and

$$\delta\overline{\overline{\Gamma}} = \delta\overline{\overline{U}} \qquad \{2.2.-32\}$$

since the identity tensor $\overline{\overline{I}}$ is composed of base vectors $\overline{G}_i$ and $\overline{G}^i$, the variation of which, vanish.

Therefore, the variation of the strain energy density δU_0^S may be expressed in four forms, each of which suggests a different set of conjugate tensor variables for the eventual Legendre transformation which will define a complementary strain energy density function. That is, δU_0^S may be expressed as

$$\delta U_0^S = \overline{\overline{S}} : \delta\overline{\overline{E}} \qquad \text{(REISSNER)} \qquad \{2.2.-33\}$$

$$= \tfrac{1}{2}\overline{\overline{S}} : \delta\overline{\overline{C}} \qquad \text{(MASUR)} \qquad \{2.2.-34\}$$

$$= \overline{\overline{T}} : \delta\overline{\overline{U}} \qquad \text{(LEVINSON)} \qquad \{2.2.-35\}$$

$$= \overline{\overline{T}} : \delta\overline{\overline{\Gamma}} \qquad \text{(This Work)} \qquad \{2.2.-36\}$$

Returning to the consideration of the strain energy density variation in the form $\overline{\overline{T}} : \delta\overline{\overline{U}}$, it is seen that, if U_0^S exists (as postulated), then the elasticity of the continuum is prescribed by the relationship

$$\delta U_0^S = \delta\overline{\overline{U}} : \frac{\partial U_0^S}{\partial \overline{\overline{U}}} = \overline{\overline{T}} : \delta\overline{\overline{U}} \qquad \{2.2.-37\}$$

Since U_0^S is a scalar function, then the double dot product of its derivative with respect to $\overline{\overline{U}}$ is commutative, and {2.2.-37} may be written as

$$\delta U_0^S = \frac{\partial U_0^S}{\partial \overline{\overline{U}}} : \delta\overline{\overline{U}} = \overline{\overline{T}} : \delta\overline{\overline{U}} \qquad \{2.2.-38\}$$

Thus, by transposition,

$$\left[\frac{\partial U_0}{\partial \overline{\overline{U}}} - \overline{\overline{T}}\right] : \delta\overline{\overline{U}} = 0 \qquad \{2.2.-39\}$$

and, as $\delta\overline{\overline{U}} \neq \overline{\overline{0}}$ in general, then the elasticity of the continuum is prescribed by the relationship

$$\overline{\overline{T}} = \frac{\partial U_0^S}{\partial \overline{\overline{U}}} \qquad \{2.2.-40\}$$

which follows from the *postulate* of the existence of δU_0^S as the total variation of the function U_0^S.

Consider now, however, that the existence of U_0^S is not postulated: assume instead, that the constitutive relation of the continuum is specified in some form

$$\overline{\overline{S}} = \overline{\overline{f}}(\overline{\overline{E}}) = \overline{\overline{k}}(\overline{\overline{C}}) \qquad \{2.2.-41\}$$

or

$$\overline{\overline{T}} = \overline{\overline{h}}(\overline{\overline{U}}) = \overline{\overline{\ell}}(\overline{\overline{\Gamma}}) \qquad \{2.2.-42\}$$

where $\overline{\overline{f}}$, $\overline{\overline{k}}$, $\overline{\overline{h}}$ and $\overline{\overline{\ell}}$ represent tensor-valued functions of the tensors $\overline{\overline{E}}$, $\overline{\overline{C}}$, $\overline{\overline{U}}$ and $\overline{\overline{\Gamma}}$, respectively. Then it is still possible to write

$$\delta U_0^S = \overline{\overline{T}} : \delta \overline{\overline{U}} \qquad \{2.2.-43\}$$

but in this case, δU_0^S is only a *variational form*, and is not necessarily the total variation of a quantity U_0^S (i.e., U_0^S does not necessarily exist, even though δU_0^S exists as a variational form). The conditions under which δU_0^S does represent the total variation of a function U_0^S are therefore of primary importance, for it is under these conditions that the continuum will be elastic.

In order to deduce the conditions which guarantee the existence of U_0^S as a function which is independent of the intermediate states in its evaluation (i.e., a state function), the implications of the original postulate of the existence of U_0^S are examined in detail. In this way, the desired conditions are obtained in the form of a relationship between the tensors $\overline{\overline{T}}$ and $\overline{\overline{U}}$ or $\overline{\overline{S}}$ and $\overline{\overline{E}}$, which are now assumed to be specified.

If U_0^S is postulated to exist as a state function, then it follows that

$$\overline{\overline{T}} = \frac{\partial U_0^S}{\partial \overline{\overline{U}}} \qquad \{2.2.-44\}$$

as established above. Now, as $\overline{\overline{T}} = \overline{\overline{h}}(\overline{\overline{U}})$, then the operator $\frac{\partial}{\partial \overline{\overline{U}}}(\ \ )$ may be applied to this relation as a pre-operator, to yield

$$\frac{\partial \overline{\overline{T}}}{\partial \overline{\overline{U}}} = \frac{\partial}{\partial \overline{\overline{U}}}\left[\frac{\partial U_0^S}{\partial \overline{\overline{U}}}\right] = \frac{\partial^2 U_0^S}{\partial \overline{\overline{U}} \partial \overline{\overline{U}}} \qquad \{2.2.-45\}$$

or

$$\frac{\partial \overline{\overline{T}}}{\partial \overline{\overline{U}}} = \frac{\partial^2 U_0^S}{\partial U_{ij} \partial U_{mn}} \overline{G}_i \overline{G}_j \overline{G}_m \overline{G}_n \qquad \{2.2.-46\}$$

Similarly, applying the same operator to equation {2.2.-44} as a post-operator, yields

$$\frac{\overline{\overline{T}}\partial}{\partial\overline{\overline{U}}} = \left\{\left[\frac{\partial u_0^s}{\partial\overline{\overline{U}}}\right]\frac{\partial}{\partial\overline{\overline{U}}}\right\} \qquad \{2.2.-47\}$$

or

$$\frac{\overline{\overline{T}}\partial}{\partial\overline{\overline{U}}} = \frac{\partial^2 u_0^s}{\partial U_{rs}\partial U_{pq}}\overline{G}_p\overline{G}_q\overline{G}_r\overline{G}_s \qquad \{2.2.-48\}$$

Therefore, from {2.2.-46} and {2.2.-48},

$$\frac{\partial\overline{\overline{T}}}{\partial\overline{\overline{U}}} - \frac{\overline{\overline{T}}\partial}{\partial\overline{\overline{U}}} = \frac{\partial^2 u_0^s}{\partial U_{ij}\partial U_{mn}}\overline{G}_i\overline{G}_j\overline{G}_m\overline{G}_n - \frac{\partial^2 u_0^s}{\partial U_{rs}\partial U_{pq}}\overline{G}_p\overline{G}_q\overline{G}_r\overline{G}_s$$

or

$$\frac{\partial\overline{\overline{T}}}{\partial\overline{\overline{U}}} - \frac{\overline{\overline{T}}\partial}{\partial\overline{\overline{U}}} = \left[\frac{\partial^2 u_0^s}{\partial U_{pq}\partial U_{rs}} - \frac{\partial^2 u_0^s}{\partial U_{rs}\partial U_{pq}}\right]\overline{G}_p\overline{G}_q\overline{G}_r\overline{G}_s \qquad \{2.2.-49\}$$

Now, as u_0^s is required to be a state function (continuous and single-valued), then

$$\frac{\partial^2 u_0^s}{\partial U_{pq}\partial U_{rs}} = \frac{\partial^2 u_0^s}{\partial U_{rs}\partial U_{pq}} \qquad \{2.2.-50\}$$

so that {2.2.-49} becomes

$$\frac{\partial\overline{\overline{T}}}{\partial\overline{\overline{U}}} - \frac{\overline{\overline{T}}\partial}{\partial\overline{\overline{U}}} = \overline{\overline{0}} \qquad \{2.2.-51\}$$

or

$$\frac{\partial\overline{\overline{T}}}{\partial\overline{\overline{U}}} = \frac{\overline{\overline{T}}\partial}{\partial\overline{\overline{U}}} \qquad \{2.2.-52\}$$

In component form, this appears from

$$\overline{G}^i\overline{G}^j\overline{G}^m\overline{G}^n :: \frac{\partial\overline{\overline{T}}}{\partial\overline{\overline{U}}} = \overline{G}^i\overline{G}^j\overline{G}^m\overline{G}^n :: \frac{\overline{\overline{T}}\partial}{\partial\overline{\overline{U}}}$$

as $$\frac{\partial T^{mn}}{\partial U_{ij}} = \frac{\partial T^{ij}}{\partial U_{mn}} \qquad \{2.2.-53\}$$

and it is noted that a similar form may be constructed in terms of $\overline{\overline{S}}$ and $\overline{\overline{E}}$. Also, since $\overline{\overline{\Gamma}}$ differs from $\overline{\overline{U}}$ only by $\overline{\overline{I}}$ and $\overline{\overline{C}}$ differs from $\overline{\overline{E}}$ only by the factor $\frac{1}{2}$ and the tensor $\overline{\overline{I}}$, then {2.2.-52} implies that

$$\frac{\partial \overline{\overline{T}}}{\partial \overline{\overline{\Gamma}}} = \frac{\overline{\overline{T}}\partial}{\partial \overline{\overline{\Gamma}}} \qquad \{2.2.-54\}$$

and the corresponding condition in terms of $\overline{\overline{S}}$ and $\overline{\overline{E}}$, namely

$$\frac{\partial \overline{\overline{S}}}{\partial \overline{\overline{E}}} = \frac{\overline{\overline{S}}\partial}{\partial \overline{\overline{E}}} \qquad \{2.2.-55\}$$

also implies that

$$\frac{\partial \overline{\overline{S}}}{\partial \overline{\overline{C}}} = \frac{\overline{\overline{S}}\partial}{\partial \overline{\overline{C}}} \qquad \{2.2.-56\}$$

Thus, {2.2.-52} and {2.2.-54} are equivalent forms, as are {2.2.-55} and {2.2.-56}, and all four are equivalent in the sense that each one prescribes the condition (in different variables) which guarantees the existence of U_0^S as a state function. However, for simplicity, the discussion will be continued in terms of $\overline{\overline{T}}$ and $\overline{\overline{U}}$ (and $\overline{\overline{S}}$ and $\overline{\overline{E}}$), it being understood that $\overline{\overline{\Gamma}}$ and $\overline{\overline{C}}$ are also appropriate variables, if desired.

Therefore, the strain energy density U_0^S is guaranteed to exist as a state function, if the relationship

$$\frac{\partial \overline{\overline{T}}}{\partial \overline{\overline{U}}} = \frac{\overline{\overline{T}}\partial}{\partial \overline{\overline{U}}} \qquad \{2.2.-52\}$$

i.e., $$\frac{\partial \Gamma^{mn}}{\partial U_{ij}} = \frac{\partial T^{ij}}{\partial U_{mn}} \qquad \{2.2.-53\}$$

exists between the tensors $\overline{\overline{U}}$ and $\overline{\overline{T}} = \overline{\overline{h}}(\overline{\overline{U}}) = \overline{\overline{\ell}}(\overline{\overline{\Gamma}})$; or equivalently, if the relationship

$$\frac{\partial \overline{\overline{S}}}{\partial \overline{\overline{E}}} = \frac{\overline{\overline{S}}\partial}{\partial \overline{\overline{E}}} \qquad \{2.2.-55\}$$

i.e., $$\frac{\partial S^{mn}}{\partial E_{ij}} = \frac{\partial S^{ij}}{\partial E_{mn}} \qquad \{2.2.-57\}$$

exists between the tensors $\overline{\overline{E}}$ and $\overline{\overline{S}} = \overline{\overline{f}}(\overline{\overline{E}}) = \overline{\overline{k}}(\overline{\overline{C}})$.

These relationships are often denoted as the "conditions of exactness", under which

$$dU_0^S = \overline{\overline{T}} : d\overline{\overline{U}} = \overline{\overline{T}} : d\overline{\overline{\Gamma}} \qquad \{2.2.-58\}$$

or $$dU_0^S = \overline{\overline{S}} : d\overline{\overline{E}} = \frac{1}{2}\overline{\overline{S}} : d\overline{\overline{C}} \qquad \{2.2.-59\}$$

become "exact" or "perfect" differentials, such that the state function U_0^S may be evaluated as

$$U_0^S = \int_{\overline{\overline{U}}} \overline{\overline{T}} : d\overline{\overline{U}} = \int_{\overline{\overline{\Gamma}}} \overline{\overline{T}} : d\overline{\overline{\Gamma}} \qquad \{2.2.-60\}$$

or $$U_0^S = \int_{\overline{\overline{E}}} \overline{\overline{S}} : d\overline{\overline{E}} = \int_{\overline{\overline{C}}} \frac{1}{2}\overline{\overline{S}} : d\overline{\overline{C}} \qquad \{2.2.-61\}$$

For this reason, these same relationships are also known as "integrability conditions"

In either case ({2.2.-52} or {2.2.-55}, above), the strain

energy density U_0^S is assured to exist as a state function; the quantity δU_0^S represents the total variation of U_0^S; the strain energy density may be evaluated as shown in {2.2.-60} or {2.2.-61}; and the continuum is said to be elastic. Thus, the postulate of the existence of U_0^S as a state function implies the relationship {2.2.-52} or {2.2.-55}: conversely, the existence of either of these relationships for a continuum of specified constitutive relation defines {2.2.-58} or {2.2.-59} to be an exact or perfect differential, from which it follows that U_0^S is guaranteed to exist as a state function.

The Virtual Work Principle then becomes, for an elastic continuum,

$$\iiint_V \delta U_0^S \, dV = \iint_S \overline{T}_n \cdot \delta \overline{U} \, dS \qquad \{2.2.-62\}$$

or

$$\delta U^S - \iint_S \overline{T}_n \cdot \delta \overline{U} \, dS = 0 \qquad \{2.2.-63\}$$

where

$$U^S = \iiint_V U_0^S \, dV \qquad \{2.2.-64\}$$

as before, and U_0^S is considered to be a function of $\overline{\overline{T}}$ and $\overline{\overline{U}}$ (or $\overline{\overline{\Gamma}}$), as defined by {2.2.-60}. The absence of a constant in {2.2.-63} indicates that $U^S = 0$ in the undeformed state, as previously observed. Thus, equation {2.2.-63} represents the primary form of the Potential Energy Principle.

The equilibrium of the continuum is then ensured if the functional

$$\Pi^S = \iiint_V U_0^S \, dV - \iint_{S_t} \overline{\overline{T}}_n^* \cdot \overline{U} \, dS \qquad \{2.2.-65\}$$

assumes a stationary value. This is the final form of the Potential Energy Principle, obtained from {2.2.-63} by the process of §1.2.. The EULER-LAGRANGE equations, obtained from the vanishing of the first variation of Π^S, are the stress equilibrium equations,

$$\frac{\partial \cdot \overline{\overline{T}}}{\partial \overline{R}} = \overline{0} \qquad \{2.2.-66\}$$

and the natural boundary condition is that the stress vector $\overline{\overline{T}}_n = \overline{N} \cdot \overline{\overline{T}}$ on S_t must be equal to the prescribed stress vector $\overline{\overline{T}}_n^*$.

Consider now, the complementary strain energy density function U_0^C, defined for an elastic continuum by the Legendre transformation on the tensor variables $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$, as

$$U_0^C = \overline{\overline{T}} : \overline{\overline{\Gamma}} - U_0^S \qquad \{2.2.-67\}$$

where $\overline{\overline{\Gamma}}$ is thus required to be expressed as a function of $\overline{\overline{T}}$ (as will be further discussed in Chapter III), similar to the requirement of the LEVINSON Theorem that $\overline{\overline{U}}$ must be expressed as a function of $\overline{\overline{T}}$. It is noted that U_0^C *always exists as a state function if* U_0^S *exists as a state function, without the necessity of a constitutive inversion*†: however, in such a case, U_0^C is guaranteed to exist only as a function of the independent variable of U_0^S (namely $\overline{\overline{U}}$, $\overline{\overline{\Gamma}}$, $\overline{\overline{E}}$ or

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† See Appendix A for proof of this statement.

$\overline{\overline{C}}$), and not necessarily as a function of $\overline{\overline{T}}$ or $\overline{\overline{S}}$. Consequently, it is noted that the existence of U_0^S guarantees the existence of U_0^C in terms of the same independent variable.

If $\overline{\overline{\Gamma}}$ is expressed as a function of $\overline{\overline{T}}$, then the Legendre transformation 2.2.-67 yields

$$dU_0^C = \overline{\overline{\Gamma}} : d\overline{\overline{T}} + \overline{\overline{T}} : d\overline{\overline{\Gamma}} - dU_0^S$$

or

$$dU_0^C = \overline{\overline{\Gamma}} : d\overline{\overline{T}}$$

since it is required that $dU_0^S = \overline{\overline{T}} : d\overline{\overline{\Gamma}}$ in order that the strain energy density may exist. Therefore, the constitutive relation for the continuum now appears, in terms of U_0^C, as

$$\overline{\overline{\Gamma}} = \frac{\partial U_0^C}{\partial \overline{\overline{T}}} \qquad \{2.2.-68\}$$

and it follows that the relationship

$$\frac{\partial \overline{\overline{\Gamma}}}{\partial \overline{\overline{T}}} = \frac{\overline{\overline{\Gamma}} \partial}{\partial \overline{\overline{T}}} \qquad \{2.2.-69\}$$

i.e.,

$$\frac{\partial \Gamma_{mn}}{\partial T^{ij}} = \frac{\partial \Gamma_{ij}}{\partial T^{mn}} \qquad \{2.2.-70\}$$

exists between the tensors $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}} = \overline{\overline{L}}(\overline{\overline{T}})$.

Employing the Legendre transformation to define U_0^S as

$$U_0^S = \overline{\overline{T}} : \overline{\overline{\Gamma}} - U_0^C \qquad \{2.2.-71\}$$

then the functional {2.2.-65} of the Potential Energy Principle becomes the functional Π^c of the Complementary Potential Energy Principle,

$$\Pi^c = \iiint_V [\overline{\overline{T}} : \overline{\overline{\Gamma}} - U_0^c] dV - \iint_{S_t} \overline{T}_n^* \cdot \overline{U}\, dS \qquad \{2.2.-72\}$$

This may be written in the form

$$\Pi^c = \iiint_V [\overline{\overline{T}} : \overline{\overline{\Gamma}} - U_0^c] dV - \iint_{S_t} \overline{T}_n^* \cdot \overline{U}\, dS - \iint_{S_u} \overline{T}_n \cdot (\overline{U} - \overline{U}^*)\, dS$$

since $\overline{U} = \overline{U}^*$ on S_u. Therefore,

$$\Pi^c = \iiint_V [\overline{\overline{T}} : \overline{\overline{\Gamma}} - U_0^c] dV - \iint_S \overline{T}_n \cdot \overline{U}\, dS + \iint_{S_u} \overline{T}_n \cdot \overline{U}^*\, dS \qquad \{2.2.-73\}$$

Now, since

$$\overline{\overline{T}} : \overline{\overline{\Gamma}} = \overline{\overline{T}} : \overline{\overline{I}} + \overline{\overline{T}} : \overline{\overline{U}} \qquad \{2.2.-74\}$$

and since

$$\overline{\overline{U}} = \frac{\partial \overline{U}}{\partial \overline{R}}$$

then $$\overline{\overline{T}} : \overline{\overline{U}} = \overline{\overline{T}} : \frac{\partial \overline{U}}{\partial \overline{R}} = \frac{\partial}{\partial \overline{R}} \cdot (\overline{\overline{T}} \cdot \overline{U}) - \frac{\partial \cdot \overline{\overline{T}}}{\partial \overline{R}} \cdot \overline{U} \qquad \{2.2.-75\}$$

and under the assumption of equilibrium in the continuum,

$$\frac{\partial \cdot \overline{\overline{T}}}{\partial \overline{R}} = \overline{0} \qquad \{2.2.-76\}$$

so that $$\overline{\overline{T}} : \overline{\overline{\Gamma}} = \overline{\overline{T}} : \overline{\overline{I}} + \frac{\partial}{\partial \overline{R}} \cdot (\overline{\overline{T}} \cdot \overline{U}) \qquad \{2.2.-77\}$$

Thus, the functional Π^c appears as

$$\Pi^c = \iiint_V \left[\overline{\overline{T}} : \overline{\overline{I}} + \frac{\partial}{\partial \overline{R}} \cdot (\overline{\overline{T}} \cdot \overline{U}) - U_0^c \right] dV - \iint_S \overline{T}_n \cdot \overline{U}\, dS + \iint_{S_u} \overline{T}_n \cdot \overline{U}^*\, dS$$

From the GAUSS Divergence Theorem, it is seen that

$$\iiint_V \frac{\partial}{\partial \overline{R}} \cdot (\overline{\overline{T}} \cdot \overline{U})\, dV = \iint_S \overline{N} \cdot \overline{\overline{T}} \cdot \overline{U}\, dS = \iint_S \overline{T}_n \cdot \overline{U}\, dS$$

so the Complementary Potential Energy functional assumes the form

$$\Pi^c = \iiint_V [\overline{\overline{T}} : \overline{\overline{I}} - U_0^c] dV + \iint_{S_u} \overline{T}_n \cdot \overline{U}^*\, dS \qquad \{2.2.-78\}$$

Or, since the first scalar invariant of $\overline{\overline{T}}$ may be expressed as

$$\overline{\overline{T}} : \overline{\overline{I}} = \overline{\overline{T}} : \frac{\partial \overline{R}}{\partial \overline{R}} = \frac{\partial}{\partial \overline{R}} \cdot (\overline{\overline{T}} \cdot \overline{R}) - \frac{\partial \cdot \overline{\overline{T}}}{\partial \overline{R}} \cdot \overline{R}$$

which, under the assumption of equilibrium in the continuum, becomes

$$\overline{\overline{T}} : \overline{\overline{I}} = \frac{\partial}{\partial \overline{R}} \cdot (\overline{\overline{T}} \cdot \overline{R})$$

then the functional {2.2.-78} may also be given as

$$\Pi^c = - \iiint_V U_0^c\, dV + \iint_S \overline{T}_n \cdot \overline{R}\, dS + \iint_{S_u} \overline{T}_n \cdot \overline{U}^*\, dS \qquad \{2.2.-79\}$$

which reduces to the final form

$$\Pi^c = -\iiint_V U_0^c \, dV + \iint_{S_u} \bar{T}_n \cdot \bar{r}^* \, dS \qquad \{2.2.-80\}$$

where $\bar{r}^* = \bar{R} + \bar{U}^*$, as in the MASUR formulation.

If admissible Lagrange stress states are required to satisfy the equation of equilibrium {2.2.-76} and the stress boundary conditions, then the EULER-LAGRANGE equations are the conditions of compatibility for the deformation:

$$\frac{\partial^2 \bar{U}}{\partial \theta^i \partial \theta^j} = \frac{\partial^2 \bar{U}}{\partial \theta^j \partial \theta^i} \qquad \{2.2.-81\}$$

and the natural boundary condition is that, on S_u, the displacement is the prescribed displacement $\bar{U}^*$.

In either form of the functional ({2.2.-78} or {2.2.-80} above), coupled displacements and stress components are not found in conjunction with the complementary strain energy density U_0^c. In the first form, U_0^c is found only with the first scalar invariant of the Lagrange stress tensor; in the second form (the MASUR form), the function U_0^c is found alone. Consequently, these (equivalent) functionals provide a suitable foundation for the Complementary Potential Energy Principle in finite elasticity. The Lagrange stress tensor $\bar{\bar{T}}$ and the deformation gradient $\bar{\bar{\Gamma}}$ therefore represent the appropriate conjugate tensor variables for the formulation of the Principle: the comments of WASHIZU are no longer applicable in this case.

Thus, for the admissible stress state which coincides with the true stress state, the functional Π^c, as given by {2.2.-78} or {2.2.-80}, assumes a stationary value. This is the Complementary Potential Energy Principle for the case of finite deformations of an elastic continuum.

It is now necessary to establish the fact that U_0^c is a function only of the "stretching" portion of the displacement field, and not of the rigid-body rotation (as this is not at all obvious from the formulation). Also, it is advantageous to examine the nature of the constitutive inversion required for the operation of the Principle, and to establish in greater detail, the admissibility conditions for the Lagrange stress tensor $\overline{\overline{T}}$. These, and other pertinent topics, including a comparison of U_0^c of the present theorem with Ω of the MASUR Theorem, W_0^c of the LEVINSON Theorem, and U_1^c of the classical theorem, are treated in the following chapter.

CHAPTER III

Continued Investigation

of

The Complementary Potential Energy Principle

3.1. THE COMPLEMENTARY STRAIN ENERGY DENSITY

The complementary strain energy density function U_0^c, as formulated in terms of the Lagrange variables $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$, must be established as a function which is independent of rigid-body rotations in the displacement field, if the present formulation of the Principle is to be of general applicability. In order to establish this fact, the tensors employed in the previous chapter are first examined in greater detail.

Consider the homomorphism

$$\overline{\overline{S}} \cdot \overline{\overline{\Gamma}} = \overline{\overline{T}} \qquad \{3.1.-1\}$$

in which $\overline{\overline{\Gamma}}$ maps $\overline{\overline{S}}$ onto $\overline{\overline{T}}$, as previously given by {2.2.-1}. Now, if $\overline{\overline{\Gamma}}$ is invertible, then the homomorphism {3.1.-1} becomes a bijective homomorphism or an isomorphism. That is, if there always exists a tensor $\overline{\overline{\Gamma}}^{-1}$, such that

$$\overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}^{-1} = \overline{\overline{\Gamma}}^{-1} \cdot \overline{\overline{\Gamma}} = \overline{\overline{I}} \qquad \{3.1.-2\}$$

then it is always possible to construct the inverse transformation

$$\overline{\overline{T}} \cdot \overline{\overline{\Gamma}}^{-1} = \overline{\overline{S}} \cdot \overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}^{-1} = \overline{\overline{S}} \cdot \overline{\overline{I}} = \overline{\overline{S}} \qquad \{3.1.-3\}$$

where $\overline{\overline{\Gamma}}^{-1}$ maps $\overline{\overline{T}}$ onto $\overline{\overline{S}}$. In this case, the mappings {3.1.-1} and {3.1.-3} are said to be isomorphic.

Now, from the definition of $\overline{\overline{\Gamma}}$ in trinomial form,

$$\overline{\overline{\Gamma}} = \overline{G}^i \overline{g}_i \qquad \{3.1.-4\}$$

and as $$\overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}^{-1} = \overline{G}^i \overline{g}_i \cdot \overline{\overline{\Gamma}}^{-1} = \overline{\overline{I}}$$

then the inverse tensor $\overline{\overline{\Gamma}}^{-1}$ is seen to be

$$\overline{\overline{\Gamma}}^{-1} = \overline{g}^j \overline{G}_j \qquad \{3.1.-5\}$$

so that $$\overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}^{-1} = \overline{G}^i \overline{g}_i \cdot \overline{g}^j \overline{G}_j = \overline{G}^i \overline{G}_i = \overline{\overline{I}}$$

or $$\overline{\overline{\Gamma}}^{-1} \cdot \overline{\overline{\Gamma}} = \overline{g}^j \overline{G}_j \cdot \overline{G}^i \overline{g}_i = \overline{g}^i \overline{g}_i = \overline{\overline{I}}$$

as required by the definition.

Thus, $\overline{\overline{\Gamma}}^{-1}$ always exists (or $\overline{\overline{\Gamma}}$ is always invertible), except in the case that $\overline{r} = \overline{K}$, where $\overline{K}$ is a constant vector. However, although this is a mathematical possibility, it is, of course, a physical absurdity, as it implies that $\overline{U} = \overline{K} - \overline{R}$, which is a specification that the entire continuum must collapse into one point.

Since $\overline{\overline{\Gamma}}^{-1}$ always exists, then $\overline{\overline{\Gamma}}$ may be expressed as the (dot) product of two other tensors, by virtue of the polar decomposition theorem. Thus,

$$\overline{\overline{\Gamma}} = \overline{\overline{V}} \cdot \overline{\overline{\Phi}} \qquad \{3.1.-6\}$$

where $\overline{\overline{V}}$ represents a symmetric, positive-definite tensor known as the left stretch tensor of the deformation, and $\overline{\overline{\Phi}}$ denotes an orthogonal tensor known as the (finite) rotation tensor. The decomposition of $\overline{\overline{\Gamma}}$ may also be written as

$$\overline{\overline{\Gamma}} = \overline{\overline{\Phi}} \cdot \overline{\overline{W}} \qquad \{3.1.-7\}$$

where $\overline{\overline{\Phi}}$ again denotes the rotation tensor, as above, and $\overline{\overline{W}}$ represents a symmetric, positive-definite tensor known as the right stretch tensor of the deformation.

Since $\overline{\overline{\Phi}}$ is an orthogonal tensor, then by definition,

$$\overline{\overline{\Phi}} \cdot \overline{\overline{\Phi}}_c = \overline{\overline{\Phi}}_c \cdot \overline{\overline{\Phi}} = \overline{\overline{I}} \qquad \{3.1.-8\}$$

or

$$\overline{\overline{\Phi}}_c \equiv \overline{\overline{\Phi}}^{-1} \qquad \{3.1.-9\}$$

and it follows from the above decompositions that

$$\overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}_c = \overline{\overline{V}} \cdot \overline{\overline{\Phi}} \cdot (\overline{\overline{V}} \cdot \overline{\overline{\Phi}})_c = \overline{\overline{V}} \cdot \overline{\overline{\Phi}} \cdot \overline{\overline{\Phi}}_c \cdot \overline{\overline{V}}_c = \overline{\overline{V}} \cdot \overline{\overline{I}} \cdot \overline{\overline{V}}_c$$

or

$$\overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}_c = \overline{\overline{V}} \cdot \overline{\overline{V}}_c$$

and similarly,

$$\overline{\overline{\Gamma}}_c \cdot \overline{\overline{\Gamma}} = \overline{\overline{W}}_c \cdot \overline{\overline{\Phi}}_c \cdot \overline{\overline{\Phi}} \cdot \overline{\overline{W}} = \overline{\overline{W}}_c \cdot \overline{\overline{I}} \cdot \overline{\overline{W}} = \overline{\overline{W}}_c \cdot \overline{\overline{W}}$$

However, as $\overline{\overline{V}}$ and $\overline{\overline{W}}$ are symmetric, then $\overline{\overline{V}}=\overline{\overline{V}}_c$ and $\overline{\overline{W}}=\overline{\overline{W}}_c$, so the above expressions become

$$\overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}_c = \overline{\overline{V}} \cdot \overline{\overline{V}} \qquad \{3.1.-10\}$$

$$\overline{\overline{\Gamma}}_c \cdot \overline{\overline{\Gamma}} = \overline{\overline{W}} \cdot \overline{\overline{W}} \qquad \{3.1.-11\}$$

so that $\overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}_c$ and $\overline{\overline{\Gamma}}_c \cdot \overline{\overline{\Gamma}}$ are functions only of the "stretching" portion of the deformation.

It is noted that since

$$\overline{\overline{\Gamma}} = \overline{\overline{V}} \cdot \overline{\overline{\Phi}} = \overline{\overline{\Phi}} \cdot \overline{\overline{W}}$$

then $\overline{\overline{\Gamma}} \cdot \overline{\overline{\Phi}}_c = \overline{\overline{V}} \cdot \overline{\overline{\Phi}} \cdot \overline{\overline{\Phi}}_c = \overline{\overline{V}} \cdot \overline{\overline{I}} = \overline{\overline{V}} = \overline{\overline{\Phi}} \cdot \overline{\overline{W}} \cdot \overline{\overline{\Phi}}_c$

so that $\overline{\overline{V}}$ and $\overline{\overline{W}}$ are related through the "similarity transformations"

$$\overline{\overline{V}} = \overline{\overline{\Phi}} \cdot \overline{\overline{W}} \cdot \overline{\overline{\Phi}}_c \qquad \{3.1.-12\}$$

and

$$\overline{\overline{W}} = \overline{\overline{\Phi}}_c \cdot \overline{\overline{V}} \cdot \overline{\overline{\Phi}} \qquad \{3.1.-13\}$$

Now, from the definition of the strain tensor $\overline{\overline{E}}$, as

$$\overline{\overline{E}} = \frac{1}{2}\left[\frac{\partial \overline{r}}{\partial \overline{R}} \cdot \frac{\overline{r}\partial}{\partial \overline{R}} - \overline{\overline{I}}\right] \qquad \{3.1.-14\}$$

then it is seen that

$$\overline{\overline{E}} = \frac{1}{2}(\overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}_c - \overline{\overline{I}}) = \frac{1}{2}(\overline{\overline{V}} \cdot \overline{\overline{V}} - \overline{\overline{I}}) \qquad \{3.1.-15\}$$

so that the strain tensor is defined by means of the quadratic $\overline{\overline{V}} \cdot \overline{\overline{V}}$ (sometimes denoted as $\overline{\overline{V}}^2$) of the left stretch tensor. In terms of components, since

$$\overline{\overline{\Gamma}}_c = \overline{g}_j \overline{G}^j \qquad \{3.1.-16\}$$

then $$\bar{\bar{\Gamma}} \cdot \bar{\bar{\Gamma}}_c = G^i \bar{g}_i \cdot \bar{g}_j \bar{G}^j = g_{ij} \bar{G}^i \bar{G}^j \qquad \{3.1.-17\}$$

and as $$\bar{\bar{I}} = \bar{G}^i \bar{G}_i = \bar{G}_j \bar{G}^j = G_{ij} \bar{G}^i \bar{G}^j \qquad \{3.1.-18\}$$

then $\bar{\bar{E}}$ may be given as

$$\bar{\bar{E}} = \frac{1}{2}(g_{ij} - G_{ij}) \bar{G}^i \bar{G}^j \qquad \{3.1.-19\}$$

which agrees with the earlier definition, constructed as {1.1.2.-31}.

Denoting $\bar{\bar{V}} \cdot \bar{\bar{V}}$ as the (left) CAUCHY-GREEN Deformation Tensor $\bar{\bar{C}}$, and $\bar{\bar{W}} \cdot \bar{\bar{W}}$ as the (right) CAUCHY-GREEN Deformation Tensor $\bar{\bar{B}}$, so that

$$\bar{\bar{C}} = \bar{\bar{\Gamma}} \cdot \bar{\bar{\Gamma}}_c = \bar{\bar{V}} \cdot \bar{\bar{V}} \qquad \{3.1.-20\}$$

and $$\bar{\bar{B}} = \bar{\bar{\Gamma}}_c \cdot \bar{\bar{\Gamma}} = \bar{\bar{W}} \cdot \bar{\bar{W}} \qquad \{3.1.-21\}$$

then the strain tensor $\bar{\bar{E}}$ is given as

$$\bar{\bar{E}} = \frac{1}{2}(\bar{\bar{C}} - \bar{\bar{I}}) \qquad \{3.1.-22\}$$

Since $\bar{\bar{\Gamma}}$ may be written as

$$\bar{\bar{\Gamma}} = \bar{\bar{I}} + \bar{\bar{U}}$$

then $$\bar{\bar{\Gamma}}_c = \bar{\bar{I}}_c + \bar{\bar{U}}_c = \bar{\bar{I}} + \bar{\bar{U}}_c \qquad \{3.1.-23\}$$

and the tensor $\bar{\bar{C}}$ appears in the form

$$\bar{\bar{C}} = (\bar{\bar{I}} + \bar{\bar{U}}) \cdot (\bar{\bar{I}} + \bar{\bar{U}}_c) = \bar{\bar{I}} + \bar{\bar{U}} + \bar{\bar{U}}_c + \bar{\bar{U}} \cdot \bar{\bar{U}}_c \qquad \{3.1.-24\}$$

Thus, in terms of the displacement gradient,

$$\bar{\bar{E}} = \frac{1}{2}(\bar{\bar{U}} + \bar{\bar{U}}_c + \bar{\bar{U}} \cdot \bar{\bar{U}}_c) \qquad \{3.1.-25\}$$

or

$$\bar{\bar{E}} = \frac{1}{2}\left[\bar{G}^i \frac{\partial \bar{U}}{\partial \theta^i} + \frac{\partial \bar{U}}{\partial \theta^j}\bar{G}^j + \bar{G}^k \frac{\partial \bar{U}}{\partial \theta^k} \cdot \frac{\partial \bar{U}}{\partial \theta^m}\bar{G}^m\right] \qquad \{3.1.-26\}$$

which has components of the form

$$E_{ij} = \bar{G}_i\bar{G}_j : \bar{\bar{E}} = \frac{1}{2}\left[\bar{G}_j \cdot \frac{\partial \bar{U}}{\partial \theta^i} + \bar{G}_i \cdot \frac{\partial \bar{U}}{\partial \theta^j} + \frac{\partial \bar{U}}{\partial \theta^i} \cdot \frac{\partial \bar{U}}{\partial \theta^j}\right] \qquad \{3.1.-27\}$$

It is of interest to note that the tensor $\bar{\bar{E}}$ is only one of several possible measures of the state of strain. The strain is often expressed in terms of the (right) CAUCHY-GREEN Tensor $\bar{\bar{B}}$, as

$$\underset{\sim}{\bar{\bar{E}}} = \frac{1}{2}(\bar{\bar{B}} - \bar{\bar{1}}) \qquad \{3.1.-28\}$$

However, from {3.1.-13} and {3.1.-21}, it is observed that

$$\bar{\bar{B}} = \bar{\bar{W}} \cdot \bar{\bar{W}} = \bar{\bar{\Phi}}_c \cdot \bar{\bar{V}} \cdot \bar{\bar{\Phi}} \cdot \bar{\bar{\Phi}}_c \cdot \bar{\bar{V}} \cdot \bar{\bar{\Phi}}$$

or

$$\bar{\bar{B}} = \bar{\bar{\Phi}}_c \cdot \bar{\bar{V}} \cdot \bar{\bar{V}} \cdot \bar{\bar{\Phi}} = \bar{\bar{\Phi}}_c \cdot \bar{\bar{C}} \cdot \bar{\bar{\Phi}} \qquad \{3.1.-29\}$$

Therefore,

$$\underset{\sim}{\bar{\bar{E}}} = \frac{1}{2}(\bar{\bar{\Phi}}_c \cdot \bar{\bar{C}} \cdot \bar{\bar{\Phi}} - \bar{\bar{1}})$$

but this may be written as

$$\underset{\sim}{\bar{\bar{E}}} = \bar{\bar{\Phi}}_c \cdot \left[\frac{1}{2}(\bar{\bar{C}} - \bar{\bar{1}})\right] \cdot \bar{\bar{\Phi}} \qquad \{3.1.-30\}$$

since

$$\bar{\bar{\Phi}}_c \cdot \bar{\bar{1}} \cdot \bar{\bar{\Phi}} = \bar{\bar{\Phi}}_c \cdot \bar{\bar{\Phi}} = \bar{\bar{1}}$$

Consequently,

$$\underset{\sim}{\overline{\overline{E}}} = \overline{\overline{\Phi}}_c \cdot \overline{\overline{E}} \cdot \overline{\overline{\Phi}} \qquad \{3.1.-31\}$$

and $\underset{\sim}{\overline{\overline{E}}}$ and $\overline{\overline{E}}$ are thus observed to be entirely equivalent measures of the state of strain in the continuum, the difference between the two being merely one of basis.

Examining the foregoing results, it is seen that the necessary and sufficient condition for a pure rigid displacement (rigid-body translation and/or rotation) is that

$$\overline{\overline{C}} = \overline{\overline{I}} \qquad \{3.1.-32\}$$

from which it follows that

$$\overline{\overline{V}} = \overline{\overline{W}} = \overline{\overline{I}} \qquad \{3.1.-33\}$$

and

$$\overline{\overline{\Gamma}} = \overline{\overline{\Phi}} \quad , \quad \overline{\overline{\Gamma}}_c = \overline{\overline{\Phi}}_c \qquad \{3.1.-34\}$$

In the event that a displacement represents a "pure stretch" (without rigid-body rotation), then

$$\overline{\overline{\Phi}} = \overline{\overline{\Phi}}_c = \overline{\overline{I}} \qquad \{3.1.-35\}$$

from which it follows that

$$\overline{\overline{\Gamma}} = \overline{\overline{\Gamma}}_c = \overline{\overline{V}} = \overline{\overline{W}} \qquad \{3.1.-36\}$$

Finally, in the event of a zero displacement field,

$$\overline{\overline{U}} = \overline{\overline{0}} \qquad \{3.1.-37\}$$

from which it follows that

$$\overline{\overline{\Gamma}} = \overline{\overline{\Gamma}}_c = \overline{\overline{V}} = \overline{\overline{W}} = \overline{\overline{\Phi}} = \overline{\overline{\Phi}}_c = \overline{\overline{I}} \qquad \{3.1.-38\}$$

and all deformation mappings become, in this case, automorphisms.

Consider now, the complementary strain energy density function U_0^C, defined previously as

$$U_0^C = \overline{\overline{T}} : \overline{\overline{\Gamma}} - U_0^S \qquad \{3.1.-39\}$$

The strain energy density function U_0^S is a function only of the stretching portion of the deformation, as may be readily observed from the formulation of this quantity as

$$U_0^S = \int_{\overline{\overline{E}}} \overline{\overline{S}} : d\overline{\overline{E}} = \frac{1}{2}\int_{\overline{\overline{C}}} \overline{\overline{S}} : d\overline{\overline{C}} \qquad \{3.1.-40\}$$

Of course, it is also true that

$$U_0^S = \int_{\overline{\overline{U}}} \overline{\overline{T}} : d\overline{\overline{U}} = \int_{\overline{\overline{\Gamma}}} \overline{\overline{T}} : d\overline{\overline{\Gamma}} \qquad \{3.1.-41\}$$

but in this case, it is not so obvious that U_0^S is independent of rigid deformations: however, as all four forms of U_0^S in {3.1.-40} and {3.1.-41} have been shown to be equivalent, then {3.1.-40} serves to establish the point.

Consequently, the nature of $\overline{\overline{T}} : \overline{\overline{\Gamma}}$ determines the nature of U_0^C (as might have been anticipated). Examining this quantity, it is seen that, from the definition of $\overline{\overline{T}}$,

$$\overline{\overline{T}} : \overline{\overline{\Gamma}} = \overline{\overline{S}} \cdot \overline{\overline{\Gamma}} : \overline{\overline{\Gamma}} \qquad \{3.1.-42\}$$

and employing the trinomial form of $\overline{\overline{\Gamma}}$,

$$\bar{\bar{S}}\cdot\bar{\bar{\Gamma}}:\bar{\bar{\Gamma}} = \bar{\bar{S}}\cdot\bar{G}^r\bar{g}_r:\bar{G}^s\bar{g}_s$$

$$= \bar{G}^r\cdot\bar{\bar{S}}_c\bar{g}_r:\bar{G}^s\bar{g}_s$$

$$= \bar{G}^r\cdot\bar{\bar{S}}_c\cdot\bar{G}^s\bar{g}_r\cdot\bar{g}_s$$

$$= \bar{\bar{S}}_c:\bar{G}^r\bar{G}^s\bar{g}_r\cdot\bar{g}_s \qquad \{3.1.-43\}$$

However, the second member of this double dot product may be written as

$$\bar{G}^r\bar{G}^s\bar{g}_r\cdot\bar{g}_s = \bar{G}^r(\bar{g}_r\cdot\bar{g}_s)\bar{G}^s$$

or $$\bar{G}^r\bar{G}^s\bar{g}_r\cdot\bar{g}_s = \bar{G}^r\bar{g}_r\cdot\bar{g}_s\bar{G}^s = \bar{\bar{\Gamma}}\cdot\bar{\bar{\Gamma}}_c$$

so {3.1.-43} becomes

$$\bar{\bar{S}}\cdot\bar{\bar{\Gamma}}:\bar{\bar{\Gamma}} = \bar{\bar{S}}_c:\bar{\bar{\Gamma}}\cdot\bar{\bar{\Gamma}}_c \qquad \{3.1.-44\}$$

Since $\bar{\bar{S}}$ is symmetric, this result appears as

$$\bar{\bar{S}}\cdot\bar{\bar{\Gamma}}:\bar{\bar{\Gamma}} = \bar{\bar{S}}:\bar{\bar{\Gamma}}\cdot\bar{\bar{\Gamma}}_c \qquad \{3.1.-45\}$$

and the function in question assumes the form

$$\bar{\bar{T}}:\bar{\bar{\Gamma}} = \bar{\bar{S}}:\bar{\bar{\Gamma}}\cdot\bar{\bar{\Gamma}}_c = \bar{\bar{S}}:\bar{\bar{V}}\cdot\bar{\bar{V}} \qquad \{3.1.-46\}$$

Therefore, as the stress tensor $\bar{\bar{S}} = \bar{\bar{f}}(\bar{\bar{E}})$ and the left stretch tensor $\bar{\bar{V}}$ are completely independent of any rigid displacements, then $\bar{\bar{T}}:\bar{\bar{\Gamma}}$ (and hence, U_0^c) is also independent of such displacements and is thus a function only of the stretching portion of the deformation, as required. Consequently, it is concluded that U_0^c represents a function

which provides a suitable foundation for the constitutive relations of the elastic continuum.

The above analysis may be employed to demonstrate that the complementary strain energy density w_0^c of the LEVINSON formulation is a function of rigid-body rotations in the displacement field. The function w_0^c has been defined as

$$w_0^c = T^{ij} U_j|_i - U_0^s \qquad \{3.1.-47\}$$

which, in direct form, is given as

$$w_0^c = \overline{\overline{T}} : \overline{\overline{U}} - U_0^s \qquad \{3.1.-48\}$$

This may be written, expanding $\overline{\overline{T}}$, as

$$w_0^c = \overline{\overline{S}} \cdot \overline{\overline{\Gamma}} : \overline{\overline{U}} - U_0^s \qquad \{3.1.-49\}$$

or, as $\overline{\overline{U}} = \overline{\overline{\Gamma}} - \overline{\overline{I}}$, then

$$w_0^c = \overline{\overline{S}} \cdot \overline{\overline{\Gamma}} : \overline{\overline{\Gamma}} - \overline{\overline{S}} \cdot \overline{\overline{\Gamma}} : \overline{\overline{I}} - U_0^s \qquad \{3.1.-50\}$$

or

$$w_0^c = U_0^c - \overline{\overline{S}} \cdot \overline{\overline{\Gamma}} : \overline{\overline{I}} \qquad \{3.1.-51\}$$

Since the complementary strain energy density U_0^c of the present formulation has been shown to be independent of any rigid-body displacements, it is seen that the nature of w_0^c hinges on the nature of the term $\overline{\overline{S}} \cdot \overline{\overline{\Gamma}} : \overline{\overline{I}}$. Expressing $\overline{\overline{\Gamma}}$ as $\overline{\overline{V}} \cdot \overline{\overline{\Phi}}$ by virtue of the polar decomposition theorem, then

$$\bar{\bar{S}}\cdot\bar{\bar{\Gamma}}:\bar{\bar{1}} = \bar{\bar{S}}\cdot\bar{\bar{V}}\cdot\bar{\bar{\Phi}}:\bar{\bar{1}}$$

$$= \bar{\bar{D}}\cdot\bar{\bar{\Phi}}:\bar{\bar{1}} \qquad \{3.1.-52\}$$

where $\bar{\bar{D}}$ represents the tensor $\bar{\bar{S}}\cdot\bar{\bar{V}}$ (which is generally unsymmetric, even though both $\bar{\bar{S}}$ and $\bar{\bar{V}}$ are symmetric, as $\bar{\bar{S}}\cdot\bar{\bar{V}}$ is not generally commutative). Therefore,

$$\bar{\bar{S}}\cdot\bar{\bar{\Gamma}}:\bar{\bar{1}} = \bar{\bar{D}}\cdot\bar{\bar{\Phi}}:\bar{\bar{1}} = \bar{\bar{D}}_c:\bar{\bar{\Phi}} \qquad \{3.1.-52\}$$

where the form $\bar{\bar{D}}_c:\bar{\bar{\Phi}}$ is obtained for simplicity of discussion.

Consider now, two displacement fields $\bar{U}$ and $\bar{U}^*$, such that

$$\bar{\bar{\Gamma}} = \frac{\partial(\bar{R}+\bar{U})}{\partial\bar{R}} = \bar{\bar{V}}\cdot\bar{\bar{\Phi}} \qquad \{3.1.-53\}$$

and

$$\bar{\bar{\Gamma}}^* = \frac{\partial(\bar{R}+\bar{U}^*)}{\partial\bar{R}} = \bar{\bar{V}}\cdot\bar{\bar{\Phi}}^* \qquad \{3.1.-54\}$$

Thus, these two displacement fields give rise to the same state of stretch, $\bar{\bar{V}}$, and differ only by a rigid-body rotation. Now, as

$$\bar{\bar{\Gamma}}^*\cdot\bar{\bar{\Gamma}}^*_c = \bar{\bar{\Gamma}}\cdot\bar{\bar{\Gamma}}_c = \bar{\bar{V}}\cdot\bar{\bar{V}}$$

then

$$\bar{\bar{S}}^* = \bar{\bar{S}} \quad , \quad \bar{\bar{D}}^* = \bar{\bar{D}}$$

so that

$$\bar{\bar{S}}\cdot\bar{\bar{\Gamma}}:\bar{\bar{1}} = \bar{\bar{D}}_c:\bar{\bar{\Phi}} \qquad \{3.1.-55\}$$

and

$$\bar{\bar{S}}^*\cdot\bar{\bar{\Gamma}}^*:\bar{\bar{1}} = \bar{\bar{D}}_c:\bar{\bar{\Phi}}^* \qquad \{3.1.-56\}$$

Therefore, as $\bar{\bar{D}}_c:\bar{\bar{\Phi}}$ and $\bar{\bar{D}}_c:\bar{\bar{\Phi}}^*$ are generally different quantities, then it must be concluded that the energy density W_0^c is a function

both of the stretching portion and of the rigid-body rotation of the displacement field. Moreover, it appears that no special cases of particular value exist, in which this problem is avoided. Even in the two-dimensional cartesian case, in which the rotation tensor assumes a type of anti-symmetry, the problem still exists. That is, if $\bar{E}_1$ and $\bar{E}_2$ denote a self-reciprocal orthonormal base, then $\bar{\bar{\Phi}}$ may be represented in this bidimensional space as

$$\bar{\bar{\Phi}} = \begin{bmatrix} \Phi_{11}\bar{E}_1\bar{E}_1 & + \Phi_{12}\bar{E}_1\bar{E}_2 \\ + \Phi_{21}\bar{E}_2\bar{E}_1 & + \Phi_{22}\bar{E}_2\bar{E}_2 \end{bmatrix} = \beta \begin{bmatrix} a\bar{E}_1\bar{E}_1 & + b\bar{E}_1\bar{E}_2 \\ - b\bar{E}_2\bar{E}_1 & + a\bar{E}_2\bar{E}_2 \end{bmatrix}$$

where $\beta = (a^2 + b^2)^{-\frac{1}{2}}$, and the tensor $\bar{\bar{\Phi}}$ is then orthogonal for any values of a, b (a, b are both real and positive or both real and negative). Then,

$$\bar{\bar{D}}_c : \bar{\bar{\Phi}} = \beta(D_{11}a - D_{12}b + D_{21}b + D_{22}a)$$

and even if $\bar{\bar{D}}$ should happen to be symmetric,

$$\bar{\bar{D}}_c : \bar{\bar{\Phi}} = \beta a(D_{11} + D_{22})$$

which is still a function of a, b -- of which there are an infinite number of combinations. Only if b=0, i.e., if $\bar{\bar{\Phi}}=\bar{\bar{I}}$, will $\bar{\bar{D}}_c : \bar{\bar{\Phi}}$ become constant.

It is therefore concluded that the energy density U_0^c is preferable to W_0^c in the formulation of the Principle, as the former is independent of any rigid displacements, while the latter is generally not.

3.2. THE CONDITIONS OF ADMISSIBILITY

An admissible Lagrange stress tensor $\overline{\overline{T}}$ must satisfy the first CAUCHY Axiom of motion (in this case, the stress equilibrium equation),

$$\frac{\partial \cdot \overline{\overline{T}}}{\partial \overline{R}} = \overline{0} \qquad \{3.2.-1\}$$

as previously established in the development of the Complementary Potential Energy Principle. However, this is not the only condition of admissibility which must be imposed on $\overline{\overline{T}}$; this tensor must also satisfy a condition which is analogous to the symmetry condition of the TREFFTZ stress tensor or the CAUCHY-GREEN stress tensor. That is, $\overline{\overline{T}}$ must satisfy the second CAUCHY Axiom of motion (in this case, the stress couple or "moment" equilibrium condition).

Since the symmetry of the stress tensor $\overline{\overline{S}}$ exists as a result of this Axiom, it is possible to deduce the equivalent conditions in terms of $\overline{\overline{T}}$ which follow from this symmetry. From the expansion of $\overline{\overline{T}}$ in terms of $\overline{\overline{S}}$,

$$\overline{\overline{T}} = T^{ij}\overline{G}_i\overline{G}_j = S^{mn}\overline{G}_m\overline{G}_n \cdot \overline{G}^r\overline{g}_r = S^{mr}\overline{G}_m\overline{g}_r \qquad \{3.2.-2\}$$

the component S^{ij} is obtained as

$$S^{ij} = \overline{G}^i \cdot T^{rs}\overline{G}_r\overline{G}_s \cdot \overline{g}^j$$

or

$$S^{ij} = T^{is}\overline{G}_s \cdot \overline{g}^j \qquad \{3.2.-3\}$$

Therefore, since $\overline{\overline{S}}$ is symmetric, $S^{ij}=S^{ji}$, and the equivalent condition in terms of the components of $\overline{\overline{T}}$ is

$$T^{im}\bar{G}_m \cdot \bar{g}^j = T^{jn}\bar{G}_n \cdot \bar{g}^i \qquad \{3.2.-4\}$$

as obtained by LEVINSON. Although this result represents the desired condition which must be satisfied by the components of $\bar{\bar{T}}$, the explicit presence of the reciprocal base vectors $\{\bar{g}^i\}$ of the basis of the deformed state indicates that some difficulty may be encountered in the attempted application of the condition. Noting that

$$T^{im}\bar{G}_m = \bar{G}^i \cdot \bar{\bar{T}} \qquad \{3.2.-5\}$$

then it is possible to write {3.2.-4} in the direct form

$$\bar{G}^i \cdot \bar{\bar{T}} \cdot \bar{g}^j = \bar{g}^i \cdot \bar{\bar{T}} \cdot \bar{G}^j \qquad \{3.2.-6\}$$

but this does little to alleviate the situation.

Considering, however, that the symmetry of $\bar{\bar{S}}$ is a *derived* condition which follows as a consequence of the stress couple equilibrium condition, this latter condition is examined directly. With reference to Fig. 1.1.2.-2, the equilibrium of stress couples in the elemental volume dv is prescribed by the relationship

$$\bar{g}_1 \times \bar{t}_1 + \bar{g}_2 \times \bar{t}_2 + \bar{g}_3 \times \bar{t}_3 = \bar{0}$$

or

$$\bar{g}_i \times \bar{t}_i = \bar{0} \qquad \{3.2.-7\}$$

However, as $\bar{t}_i$ is related to $\bar{\bar{T}}$ as

$$\bar{t}_i = \sqrt{G}\,\bar{T}_i = \sqrt{G}\,T^{ij}\bar{G}_j \qquad \{3.2.-8\}$$

as given by {2.2.-15}, then {3.2.-7} becomes

$$\bar{g}_i \times \sqrt{G}\, T^{ij} \bar{G}_j = \bar{0}$$

or
$$\bar{g}_i \times T^{ij} \bar{G}_j = \bar{0} \qquad \{3.2.-9\}$$

Now, since
$$\bar{g}_i = \bar{G}_i + \frac{\partial \bar{U}}{\partial \theta^i}$$

then
$$\bar{G}_i \times T^{ij} \bar{G}_j + \frac{\partial \bar{U}}{\partial \theta^i} \times T^{ij} \bar{G}_j = \bar{0} \qquad \{3.2.-10\}$$

but
$$\bar{G}_i \times T^{ij} \bar{G}_j = \bar{G}_i \cdot \bar{\bar{I}} \times T^{ij} \bar{G}_j$$

$$= \bar{\bar{I}} \mathbin{\dot{\times}} T^{ij} \bar{G}_i \bar{G}_j$$

$$= \bar{\bar{I}} \mathbin{\dot{\times}} \bar{\bar{T}} \qquad \{3.2.-11\}$$

Similarly,
$$\frac{\partial \bar{U}}{\partial \theta^i} \times T^{ij} \bar{G}_j = \bar{G}_i \cdot \frac{\partial \bar{U}}{\partial \bar{R}} \times T^{ij} \bar{G}_j$$

$$= \bar{G}_i \cdot \bar{\bar{U}} \times T^{ij} \bar{G}_j$$

$$= \bar{\bar{U}} \mathbin{\dot{\times}} \bar{\bar{T}} \qquad \{3.2.-12\}$$

Therefore, the condition of stress couple or moment equilibrium appears as

$$\bar{g}_i \times T^{ij} \bar{G}_j = (\bar{\bar{I}} + \bar{\bar{U}}) \mathbin{\dot{\times}} \bar{\bar{T}} = \bar{0} \qquad \{3.2.-13\}$$

or, since $\bar{\bar{\Gamma}} = \bar{\bar{I}} + \bar{\bar{U}}$, simply as

$$\bar{\bar{\Gamma}} \mathbin{\dot{\times}} \bar{\bar{T}} = \bar{0} \qquad \{3.2.-14\}$$

Equation {3.2.-14} thus represents, in terms of the conjugate variables $\bar{\bar{T}}$ and $\bar{\bar{\Gamma}}$, the second condition which must be satisfied by an admissible Lagrange stress tensor.

It is of interest to note that this condition may be expressed in another form, which may prove to be convenient in particular circumstances. From {3.2.-14},

$$\bar{\bar{\Gamma}} \overset{\cdot}{\times} \bar{\bar{T}} = \frac{\partial \bar{r}}{\partial \bar{R}} \overset{\cdot}{\times} \bar{\bar{T}} = \bar{0}$$

but

$$\frac{\partial \bar{r}}{\partial \bar{R}} \overset{\cdot}{\times} \bar{\bar{T}} = \frac{\partial \cdot \bar{\bar{T}}}{\partial \bar{R}} \times \bar{r} - \frac{\partial}{\partial \bar{R}} \cdot (\bar{\bar{T}} \times \bar{r}) \qquad \{3.2.-15\}$$

and, by virtue of the stress equilibrium equation, the first member of the right-hand side of {3.2.-15} must vanish. Therefore, the two admissibility conditions for $\bar{\bar{T}}$ may be stated as

(1)
$$\frac{\partial \cdot \bar{\bar{T}}}{\partial \bar{R}} = \bar{0} \qquad \{3.2.-1\}$$

(2)
$$\frac{\partial \cdot (\bar{\bar{T}} \times \bar{r})}{\partial \bar{R}} = \bar{0} = \bar{\bar{\Gamma}} \overset{\cdot}{\times} \bar{\bar{T}} \qquad \{3.2.-16\}$$

Consequently, any solenoidal tensor $\bar{\bar{F}}$, for which $\bar{\bar{F}} \times \bar{r}$ is also solenoidal, is an admissible stress function tensor in this formulation.

3.3. THE CONSTITUTIVE RELATION

It is advantageous to examine the general nature of the constitutive relation in terms of the Lagrange stress tensor $\bar{\bar{T}}$ and the deformation gradient $\bar{\bar{\Gamma}}$, as it has been suggested that, although $\bar{\bar{T}}$ may be expressed as a function of $\bar{\bar{\Gamma}}$, an inversion of this relation is always impossible. This, however, is not the case, as will be demonstrated in what follows.

Consider first, the argument which has been proposed in order to show that $\overline{\overline{\Gamma}}$ (or $\overline{\overline{U}}$) can never be obtained as a function of a stress tensor.

Any stress tensor (say, $\overline{\overline{\sigma}}$) which is expressed as a function of the state of strain may be written as

$$\overline{\overline{\sigma}} = \overline{\overline{f}}(\overline{\overline{\Gamma}}) \qquad \{3.3.-1\}$$

since $\overline{\overline{\Gamma}}$ represents one of the primary quantities of the deformation, from which other quantities may be derived. However, $\overline{\overline{\sigma}}$ is not a function of $\overline{\overline{\Gamma}}$ *per se*, so that although {3.3.-1} is true, a more explicit statement of this relation would appear as

$$\overline{\overline{\sigma}} = \overline{\overline{k}}(\overline{\overline{\Gamma}}, \overline{\overline{\Gamma}}_c) \qquad \{3.3.-2\}$$

or, even more explicitly,

$$\overline{\overline{\sigma}} = \overline{\overline{h}}(\overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}_c) = \overline{\overline{\ell}}(\overline{\overline{\Gamma}}_c \cdot \overline{\overline{\Gamma}}) \qquad \{3.3.-3\}$$

i.e.,
$$\overline{\overline{\sigma}} = \overline{\overline{h}}(\overline{\overline{C}}) = \overline{\overline{\ell}}(\overline{\overline{B}}) \qquad \{3.3.-4\}$$

Or, in words, although $\overline{\overline{\sigma}}$ is a function of $\overline{\overline{\Gamma}}$ in the formal sense, it is actually not a function of the <u>entire</u> deformation gradient: rather, $\overline{\overline{\sigma}}$ is a function only of a <u>part</u> of $\overline{\overline{\Gamma}}$, namely the symmetric positive-definite (quadratic) part $\overline{\overline{C}} = \overline{\overline{V}} \cdot \overline{\overline{V}}$ or $\overline{\overline{B}} = \overline{\overline{W}} \cdot \overline{\overline{W}}$. Consequently, it follows from {3.3.-4} that an inversion of the type

$$\overline{\overline{C}} = \overline{\overline{h}}^{-1}(\overline{\overline{\sigma}}) = \overline{\overline{w}}(\overline{\overline{\sigma}}) \qquad \{3.3.-5\}$$

or
$$\overline{\overline{B}} = \overline{\overline{\ell}}^{-1}(\overline{\overline{\sigma}}) = \overline{\overline{y}}(\overline{\overline{\sigma}}) \qquad \{3.3.-6\}$$

is generally not impossible, so that $\overline{\overline{C}}$ or $\overline{\overline{B}}$ may be expressed as a function of the stress tensor. This inversion may be written explicitly in terms of $\overline{\overline{\Gamma}}$, as

$$\overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}_c = \overline{\overline{w}}(\overline{\overline{\sigma}}) \qquad \{3.3.-7\}$$

or

$$\overline{\overline{\Gamma}}_c \cdot \overline{\overline{\Gamma}} = \overline{\overline{y}}(\overline{\overline{\sigma}}) \qquad \{3.3.-8\}$$

so that $\overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}_c$ or $\overline{\overline{\Gamma}}_c \cdot \overline{\overline{\Gamma}}$ may be expressed as a function of $\overline{\overline{\sigma}}$. However, here, the process stops. Since there exists an infinite number of deformation gradients $\overline{\overline{\Gamma}}^*$, each one differing from the others only by a rigid rotation, such that

$$\overline{\overline{\Gamma}}^* \cdot \overline{\overline{\Gamma}}_c^* = \overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}_c = \overline{\overline{w}}(\overline{\overline{\sigma}}) \qquad \{3.3.-9\}$$

or

$$\overline{\overline{\Gamma}}_c^* \cdot \overline{\overline{\Gamma}}^* = \overline{\overline{\Gamma}}_c \cdot \overline{\overline{\Gamma}} = \overline{\overline{y}}(\overline{\overline{\sigma}}) \qquad \{3.3.-10\}$$

then it is possible only to obtain $\overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}_c$ or $\overline{\overline{\Gamma}}_c \cdot \overline{\overline{\Gamma}}$ as a function of $\overline{\overline{\sigma}}$, not $\overline{\overline{\Gamma}}$ itself. Therefore, a constitutive inversion is generally not impossible for $\overline{\overline{B}}$ or $\overline{\overline{C}}$, but is always impossible for $\overline{\overline{\Gamma}}$ or $\overline{\overline{U}}$, in terms of the stress tensor $\overline{\overline{\sigma}}$.

The above argument is entirely true: however, it happens not to be applicable to the Lagrange stress tensor of the present formulation. The major premise in this argument is that the stress tensor $\overline{\overline{\sigma}}$ is not a function of the entire deformation gradient (from which it follows that $\overline{\overline{\Gamma}}$ cannot be recovered from $\overline{\overline{\sigma}}$). However, the Lagrange stress tensor $\overline{\overline{T}}$, by virtue of its particular nature, *is* a function of the entire deformation gradient, from which it follows that recovery

of $\overline{\overline{\Gamma}}$ from $\overline{\overline{T}}$ is not a general impossibility. That is, from the relationship

$$\overline{\overline{T}} = \overline{\overline{S}} \cdot \overline{\Gamma} \qquad \{3.3.-11\}$$

since

$$\overline{\overline{S}} = \overline{k}(\overline{C}) \qquad \{3.3.-12\}$$

then

$$\overline{\overline{T}} = \overline{h}(\overline{C},\overline{\Gamma}) \qquad \{3.3.-13\}$$

However, once $\overline{\Gamma}$ is known to be present explicitly in its entirety, then it is possible to express $\overline{\overline{T}}$ as

$$\overline{\overline{T}} = \overline{j}(\overline{\Gamma}) \qquad \{3.3.-14\}$$

since

$$\overline{\overline{C}} = \overline{\ell}(\overline{\Gamma}) \qquad \{3.3.-15\}$$

Therefore, it is not always impossible to obtain $\overline{\Gamma}$ from the Lagrange stress tensor as

$$\overline{\Gamma} = \overline{L}(\overline{\overline{T}}) \qquad \{3.3.-16\}$$

or to obtain $\overline{U}$ (as in LEVINSON'S Theorem) as

$$\overline{U} = \overline{M}(\overline{\overline{T}}) \qquad \{3.3.-17\}$$

Of course, it is true that there exist numerous relations of the form {3.3.-14} which cannot be inverted to yield the form {3.3.-16} (as is true for constitutive relations in terms of <u>any</u> tensors), but the significant point here, is that such an inversion is not *a priori* impossible.

This conclusion is further evidenced by the form in which the second condition of admissibility is obtained. Recalling {3.2.-14}, it is seen that this condition

$$\overline{\overline{\Gamma}} : \overline{\overline{T}} = \overline{0} \qquad \{3.3.-18\}$$

prescribes the existence of a relationship of the form

$$\overline{f}(\overline{\overline{\Gamma}}, \overline{\overline{T}}) = \overline{0} \qquad \{3.3.-19\}$$

Therefore, by the implicit-function theorem,

$$d\overline{f} = d\overline{\overline{\Gamma}} : \frac{\partial \overline{f}}{\partial \overline{\overline{\Gamma}}} + d\overline{\overline{T}} : \frac{\partial \overline{f}}{\partial \overline{\overline{T}}} = \overline{0}$$

or

$$d\overline{f} = d\overline{\overline{T}} : \left[\frac{\partial \overline{\overline{\Gamma}}}{\partial \overline{\overline{T}}} : \frac{\partial \overline{f}}{\partial \overline{\overline{\Gamma}}} + \frac{\partial \overline{f}}{\partial \overline{\overline{T}}} \right] = \overline{0}$$

so

$$\frac{\partial \overline{\overline{\Gamma}}}{\partial \overline{\overline{T}}} : \frac{\partial \overline{f}}{\partial \overline{\overline{\Gamma}}} = - \frac{\partial \overline{f}}{\partial \overline{\overline{T}}} \qquad \{3.3.-20\}$$

so that, if $\dfrac{\partial \overline{f}}{\partial \overline{\overline{\Gamma}}}$ is nonsingular, such that

$$\frac{\partial \overline{f}}{\partial \overline{\overline{\Gamma}}} \cdot \left[\frac{\partial \overline{f}}{\partial \overline{\overline{\Gamma}}} \right]^{-1} = \overline{\overline{I}} \qquad \{3.3.-21\}$$

then {3.3.-20} becomes

$$\frac{\partial \overline{\overline{\Gamma}}}{\partial \overline{\overline{T}}} = - \frac{\partial \overline{f}}{\partial \overline{\overline{T}}} \cdot \left[\frac{\partial \overline{f}}{\partial \overline{\overline{\Gamma}}} \right]^{-1} \qquad \{3.3.-22\}$$

and $\overline{\overline{\Gamma}}$ is found to be a function of the Lagrange stress tensor $\overline{\overline{T}}$. Thus, the implicit-function theorem shows that, given a function of $\overline{\overline{\Gamma}}$ and $\overline{\overline{T}}$ in the form of {3.3.-19}, it is possible to express $\overline{\overline{\Gamma}}$ as a function of $\overline{\overline{T}}$ when {3.3.-21} holds.

It should be noted that there exists one set of conditions, under which inversion is not possible; namely, when the state of stress is identically zero, or

$$\overline{\overline{T}} = \overline{\overline{S}} = \overline{\overline{0}} \qquad \{3.3.-23\}$$

In this case, $\overline{\overline{T}}$ and $\overline{\overline{S}}$ are identically zero, but it is necessary only that $\overline{\overline{\Gamma}}$ be orthogonal (not necessarily zero) in order to produce this condition. Consequently, in this case, inversion is not possible. However, as this represents a situation of relatively little importance in the mechanics of deformable media, it is considered to impose only a trivial restriction upon the use of the Complementary Potential Energy Principle.

It is perhaps advantageous, at this point, to discuss the constitutive relation in terms of $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$ (and its inversion) in a different context, for a constitutive relation based on these variables is somewhat unusual. Consider first, the more familiar form of constitutive equation,

$$\overline{\overline{S}} = \overline{\overline{\gamma}}_0(\overline{\overline{C}}) = \overline{\overline{\gamma}}_0(\overline{\overline{V}} \cdot \overline{\overline{V}}) \qquad \{3.3.-24\}$$

in which a state of stress $\overline{\overline{S}}$ is prescribed as a function of the symmetric, positive-definite atretch quadratic, $\overline{\overline{C}} = \overline{\overline{V}} \cdot \overline{\overline{V}}$. The material being elastic, $\overline{\overline{S}}$ and $\overline{\overline{C}}$ are related by means of a strain energy density function U_0^S, as

$$\overline{\overline{S}} = \frac{\partial U_0^S}{\partial \overline{\overline{C}}} \qquad \{3.3.-25\}$$

where U_0^S is a function only of the stretching portion of the

deformation, and not of any rigid-body displacements. When the particular form of {3.3.-24} admits inversion, then $\overline{\overline{S}}$ and $\overline{\overline{C}}$ are further related through a complementary strain energy density function $\zeta(\zeta = 2\Omega)$, as

$$\overline{\overline{C}} = \frac{\partial \zeta}{\partial \overline{\overline{S}}} \qquad \{3.3.-26\}$$

where ζ, defined by a Legendre transformation on $\overline{\overline{S}}$ and $\overline{\overline{C}}$, is also a function only of the stretching portion of the deformation.

Consider now, a constitutive relation of the form

$$\overline{\overline{T}} = \overline{\overline{h}}(\overline{\overline{V}}) \qquad \{3.3.-27\}$$

in which a state of stress $\overline{\overline{T}}$ is prescribed as a function of the symmetric, positive-definite stretch tensor, $\overline{\overline{V}}$. The material being elastic, $\overline{\overline{T}}$ and $\overline{\overline{V}}$ are related by means of a strain energy density function U_0^S, as

$$\overline{\overline{T}} = \frac{\partial U_0^S}{\partial \overline{\overline{V}}} \qquad \{3.3.-28\}$$

where U_0^S is a function only of the stretching portion of the deformation, and not of any rigid-body displacements. When the particular form of {3.3.-27} admits inversion, then $\overline{\overline{T}}$ and $\overline{\overline{V}}$ are further related through a complementary strain energy density function U_0^C, as

$$\overline{\overline{V}} = \frac{\partial U_0^C}{\partial \overline{\overline{T}}} \qquad \{3.3.-29\}$$

where U_0^C, defined by a Legendre transformation on $\overline{\overline{T}}$ and $\overline{\overline{V}}$, is also a function only of the stretching portion of the deformation.

However, as the stretch $\overline{\overline{V}}$ is an exceedingly difficult quantity to isolate for use as a tensor variable, then let $\overline{\overline{V}}$ be mapped onto $\overline{\overline{\Gamma}}$ by $\overline{\overline{\Phi}}$ (as given by {3.1.-6}) and similarly, let $\overline{\overline{T}}$ be mapped onto $\overline{\overline{\mathrm{T}}}$ by $\overline{\overline{\Phi}}$, so that the constitutive relation now appears as

$$\overline{\overline{\mathrm{T}}} = \overline{h}(\overline{\overline{\Gamma}}) \qquad \{3.3.-30\}$$

The elasticity of the material is then prescribed as

$$\overline{\overline{\mathrm{T}}} = \frac{\partial U_0^S}{\partial \overline{\overline{\Gamma}}} \qquad \{3.3.-31\}$$

where U_0^S *remains a function only of the stretching portion of the deformation.* When {3.3.-30} (or {3.3.-27}) admits inversion, then $\overline{\overline{\mathrm{T}}}$ and $\overline{\overline{\Gamma}}$ are further related, as before:

$$\overline{\overline{\Gamma}} = \frac{\partial U_0^C}{\partial \overline{\overline{\mathrm{T}}}} \qquad \{3.3.-32\}$$

where U_0^C, defined by a Legendre transformation on $\overline{\overline{\mathrm{T}}}$ and $\overline{\overline{\Gamma}}$, also remains a function only of the *stretching* portion of the deformation.

This, in principle, represents the rationale of the somewhat unusual constitutive relations of the present theorem, and of the similar relations of LEVINSON'S Theorem (in terms of $\overline{\overline{\mathrm{T}}}$ and $\overline{\overline{\mathrm{U}}}$). In conjunction with the latter theorem, it is observed that the absence of a simple mapping of $\overline{\overline{V}}$ onto $\overline{\overline{U}}$ accounts for the failure of the complementary strain energy density function of that theorem W_0^C to retain its independence of rigid displacements, in contrast to the energy density U_0^C of the present formulation.

3.4. FOUR ENERGY DENSITY FUNCTIONS

As noted in Chapter II, there exist four sets of conjugate tensor variables, by means of which it is possible to define the strain energy density function U_0^S. Consequently, as each of these sets suggests a different set of variables for use in the construction of the Legendre transformation which defines the complementary strain energy density, then four different complementary functions may be produced.

These functions, written in terms of their original variables, appear as follows.

$$U_1^C = \overline{\overline{S}} : \overline{\overline{E}} - U_0^S \qquad \text{(Classical, REISSNER)} \quad \{3.4.-1\}$$

$$\Omega = \frac{1}{2}\overline{\overline{S}} : \overline{\overline{C}} - U_0^S \qquad \text{(MASUR)} \quad \{3.4.-2\}$$

$$W_0^C = \overline{\overline{T}} : \overline{\overline{U}} - U_0^S \qquad \text{(LEVINSON)} \quad \{3.4.-3\}$$

$$U_0^C = \overline{\overline{T}} : \overline{\overline{\Gamma}} - U_0^S \qquad \text{(This Work)} \quad \{3.4.-4\}$$

The functionals which are constructed from these Legendre transformations appear as, respectively:

$$\Pi^C(\overline{\overline{S}}) = -\iiint_V \left[U_1^C + \frac{1}{2}\overline{\overline{S}} : \overline{\overline{U}} \cdot \overline{\overline{U}}_c\right] dV + \iint_{S_u} \overline{T}_n \cdot \overline{U}^* \, dS \qquad \{3.4.-5\}$$

(Classical)

$$\Pi^C(\overline{\overline{S}}) = -\iiint_V \left[\Omega + \overline{\overline{S}} : \frac{\partial \Omega}{\partial \overline{\overline{S}}}\right] dV + \iint_{S_u} \overline{T}_n \cdot \overline{r}^* \, dS \qquad \{3.4.-6\}$$

(MASUR)

$$\Pi^C(\overline{\overline{T}}) = -\iiint_V W_0^C \, dV + \iint_{S_u} \overline{T}_n \cdot \overline{U}^* \, dS \qquad \{3.4.-7\}$$

(LEVINSON)

$$\Pi^c(\bar{\bar{T}}) = -\iiint_V U_0^c \, dV + \iint_{S_u} \bar{T}_n \cdot \bar{r}^* \, dS \qquad \{3.4.-8\}$$

(This Work)

and this latter functional has the alternate form

$$\Pi^c(\bar{\bar{T}}) = -\iiint_V \left[U_0^c - \bar{\bar{T}} : \bar{\bar{I}}\right] dV + \iint_{S_u} \bar{T}_n \cdot \bar{U}^* \, dS \qquad \{3.4.-9\}$$

(This Work)

where all functionals are shown in the form of a "maximum principle", for purposes of comparison.

Examining the complementary strain energy densities again, in terms of $\bar{\bar{S}}$ and $\bar{\bar{C}}$ or $\bar{\bar{T}}$ and $\bar{\bar{\Gamma}}$ only, the following relationships are observed.

$$U_1^c = \frac{1}{2}\bar{\bar{S}} : \bar{\bar{C}} - \frac{1}{2}\bar{\bar{S}} : \bar{\bar{I}} - U_0^s \qquad \text{(Classical)}$$

$$\Omega = \frac{1}{2}\bar{\bar{S}} : \bar{\bar{C}} - U_0^s \qquad \text{(MASUR)}$$

$$W_0^c = \bar{\bar{T}} : \bar{\bar{\Gamma}} - \bar{\bar{T}} : \bar{\bar{I}} - U_0^s \qquad \text{(LEVINSON)}$$

$$U_0^c = \bar{\bar{T}} : \bar{\bar{\Gamma}} - U_0^s \qquad \text{(This Work)}$$

It is, of course, possible to express the last two functions in terms of $\bar{\bar{S}}$, but this is misleading, as $\bar{\bar{S}}$ is not the stress variable in the functionals which emanate from W_0^c and U_0^c ({3.4.-7}, {3.4.-8} and {3.4.-9}).

From the functionals {3.4.-5} to {3.4.-9}, by reducing these forms to a common form, the relationship between the various energy density functions is found to be as follows

$$u_1^c + \frac{1}{2}\bar{\bar{S}}:\bar{\bar{U}}\cdot\bar{\bar{U}}_c + \bar{\bar{S}}:\bar{\bar{\Gamma}} = \Omega + \bar{\bar{S}}:\frac{\partial\Omega}{\partial\bar{\bar{S}}} \qquad \{3.4.-10\}$$

$$= w_0^c + \bar{\bar{T}}:\bar{\bar{I}} \qquad \{3.4.-11\}$$

$$= u_0^c \qquad \{3.4.-12\}$$

Since the terms of the classical expression may be written in the form

$$\bar{\bar{S}}:\bar{\bar{\Gamma}} + \frac{1}{2}\bar{\bar{S}}:\bar{\bar{U}}\cdot\bar{\bar{U}}_c = \bar{\bar{S}}:(\bar{\bar{\Gamma}} + \frac{1}{2}\bar{\bar{U}}\cdot\bar{\bar{U}}_c)$$

$$= \bar{\bar{S}}:(\bar{\bar{I}} + \bar{\bar{U}} + \frac{1}{2}\bar{\bar{U}}\cdot\bar{\bar{U}}_c)$$

$$= \bar{\bar{S}}:(\bar{\bar{I}} + \bar{\bar{E}}) \qquad \{3.4.-13\}$$

(since $\bar{\bar{S}}$ is symmetric), then the above relationships may be given as

$$u_1^c + \bar{\bar{S}}:(\bar{\bar{I}} + \bar{\bar{E}}) = \Omega + \bar{\bar{S}}:\frac{\partial\Omega}{\partial\bar{\bar{S}}} \qquad \{3.4.-14\}$$

$$= w_0^c + \bar{\bar{T}}:\bar{\bar{I}} \qquad \{3.4.-15\}$$

$$= u_0^c \qquad \{3.4.-16\}$$

It may be concluded that if the TREFFTZ stress tensor is employed as the stress variable, then the Classical expression results in a coupling of stress and displacement terms in the functional. MASUR'S formulation, however, avoids this difficulty, and is therefore the appropriate formulation of the Principle in terms of the TREFFTZ stress tensor.

If the Lagrange stress tensor is employed as the stress

variable, then the LEVINSON expression results in a functional of simple form, which is distinctly advantageous: yet, the complementary strain energy density W_0^c is a function of rigid displacements. The present formulation, however, avoids this difficulty, and is therefore the appropriate formulation of the Principle in terms of the Lagrange stress tensor.

It is noted once again, that the dependence of W_0^c on rigid displacements (rigid-body rotation) does *not* constitute an error in the LEVINSON formulation -- merely an inconvenience. The LEVINSON Theorem is equally as applicable as the present theorem, and constitutive relations in terms of $\overline{\overline{T}}$, $\overline{\overline{U}}$ and W_0^c are no more unusual than those in terms of $\overline{\overline{T}}$, $\overline{\overline{\Gamma}}$ and U_0^c (as in the present theorem), due to the nature of the Lagrange stress tensor. The only difference is that W_0^c must be employed with caution, as it does not remain constant under rotation in the course of the displacement.

CHAPTER IV

The Complementary Potential Energy Principle For Unidimensional Continua

4.1. SYSTEMS OF DISCRETE FORCES

The discrete-force formulation of the Complementary Potential Energy Principle for finite deformations represents an important special case of the general formulation of the Principle. This case, in addition to its importance in the area of structural mechanics, contains the Theorems of CROTTI and MENABREA. Consequently, it is advantageous to examine the reduction of the general Principle to this case, and in so doing, to demonstrate that the CROTTI Theorem ("CASTIGLIANO'S Theorem") has no convenient extension to a general continuum in finite-deformation theory, in contrast to the results of the infinitesimal theory. It is shown, in fact, that the CROTTI Theorem, which is usually considered as "equivalent" to the Complementary Potential Energy Principle in infinitesimal elasticity, leads to the LEVINSON formulation as a direct generalization in finite elasticity. Therefore, the true "finite counterpart" of the complementary strain energy of the infinitesimal formulation is established as a function of rigid displacements, and must be employed with caution.

Consider the functional Π^c of the general formulation of the Principle in the form {2.2.-80}, i.e.,

$$\Pi^c = -\iiint_V u_0^c \, dV + \iint_{S_u} \bar{T}_n \cdot \bar{r}^* \, dS \qquad \{4.1.-1\}$$

Denoting the (total) complementary strain energy, i.e. the volume integral of the density u_0^c, as U^c;

$$U^c = \iiint_V u_0^c \, dV \qquad \{4.1.-2\}$$

then the functional appears in the form

$$\Pi^c = -U^c + \iint_{S_u} \bar{T}_n \cdot \bar{r}^* \, dS \qquad \{4.1.-3\}$$

Now, if the stress vector $\bar{T}_n = \bar{N} \cdot \bar{\bar{T}}$ on the surface S represents a number of very small areas of very high stress magnitude concentration, then in the limit, $\bar{r}^*$ prescribes the location of certain specified points at which there exist "discrete forces": or, the surface integral in {4.1.-3} is represented by a summation

$$\iint_{S_u} \bar{T}_n \cdot \bar{r}^* \, dS \rightarrow \bar{r}_i^* \cdot \bar{F}_i^{(r)} \qquad \{4.1.-4\}$$

where $\bar{F}_i^{(r)}$ denotes the discrete force located at the prescribed position $\bar{r}_i^*$ $(i = 1,2,\ldots,n)$. In this case, the functional becomes

$$\Pi^c = -U^c + \bar{r}_i^* \cdot \bar{F}_i^{(r)} \qquad \{4.1.-5\}$$

Therefore, comparing {4.1.-5} and {2.1.-7}, the functional Π^c is seen to be equal to the (negative) functional $-Q^c$ of the LIBOVE Theorem. The difference in signs arises as a result of the fact that

the present theorem is obtained as a maximum principle by a Legendre transformation on a minimum principle (the Potential Energy Principle), while LIBOVE'S Theorem is constructed as a minimum principle.

Consequently, the complementary strain energy U^c of the present formulation and the energy H^c of the LIBOVE Theorem represent the same quantity: or, in the case of a discrete force system,

$$\int_{\overline{F}} \overline{r}_i \cdot d\overline{F}_i = H^c \equiv U^c = \iiint_V U_0^c \, dV \qquad \{4.1.-6\}$$

Thus, the present theorem contains the LIBOVE Theorem as a special case for finite elastic systems, and the complementary strain energy U^c of the present theorem represents the direct generalization of H^c of the LIBOVE Theorem to a continuous stress system.

Employing the relationships between the various energy densities, as given in §3.4., it is also possible to say

$$\int_{\overline{F}} \overline{r}_i \cdot d\overline{F}_i = H^c = \iiint_V [w_0^c + \overline{\overline{T}} : \overline{\overline{1}}] dV \qquad \{4.1.-7\}$$

$$= \iiint_V \left[\Omega + \overline{\overline{S}} : \frac{\partial \Omega}{\partial \overline{\overline{S}}}\right] dV \qquad \{4.1.-8\}$$

$$= \iiint_V [U_1^c + \overline{\overline{S}} : (\overline{\overline{1}} + \overline{\overline{E}})] dV \qquad \{4.1.-9\}$$

which relates the energy densities w_0^c, Ω and U_1^c to the LIBOVE energy H^c for discrete force systems. The first of the above relationships may be transformed, in a manner similar to that shown for {2.2.-79}, as follows. From

$$\iiint_V \bar{\bar{T}} : \bar{\bar{1}} \, dV = \iint_S \bar{T}_n \cdot \bar{R} \, dS$$

which is valid under the assumption of equilibrium in the continuum, then the surface integral may be interpreted as a summation in the case of discrete forces, and {4.1.-7} appears as

$$\int_{\bar{F}} (\bar{r}_i - \bar{R}_i) \cdot d\bar{F}_i = \iiint_V w_0^c \, dV$$

or

$$\int_{\bar{F}} \bar{U}_i \cdot d\bar{F}_i = \iiint_V w_0^c \, dV \qquad \{4.1.-10\}$$

However, recalling the complementary strain energy defined by LANGHAAR, as

$$w^c = \bar{F}_i \cdot \bar{U}_i - U^s \qquad \{4.1.-11\}$$

which, as a Legendre transformation, may be expressed as

$$dw^c = \bar{U}_i \cdot d\bar{F}_i \qquad \{4.1.-12\}$$

then it is seen that {4.1.-10} may be written as

$$\int_{\bar{F}} \bar{U}_i \cdot d\bar{F}_i = w^c = \iiint_V w_0^c \, dV \qquad \{4.1.-13\}$$

Therefore, the LEVINSON formulation of the Complementary Potential Energy Principle represents the direct generalization of the CROTTI Theorem to a continuous stress system, and the complementary strain energy w^c, considered as the volume integral of the LEVINSON density w_0^c, represents the true "finite counterpart" of the complementary

strain energy of the infinitesimal theory. As noted previously, however, W_0^c is a function of rigid displacements, and therefore the CROTTI Theorem does not admit a simple extension to a general continuum in finite elasticity, in contrast to the infinitesimal theory.

The constitutive relations of the general formulation, relating $\overline{\overline{\Gamma}}$, $\overline{\overline{T}}$ and U_0^c (or $\overline{\overline{U}}$, $\overline{\overline{T}}$ and W_0^c),

$$\overline{\overline{\Gamma}} = \frac{\partial U_0^c}{\partial \overline{\overline{T}}} \qquad \{4.1.-14\}$$

$$\overline{\overline{U}} = \frac{\partial W_0^c}{\partial \overline{\overline{T}}} \qquad \{4.1.-15\}$$

i.e.,
$$\overline{\overline{U}} = \frac{\partial W_0^c}{\partial \overline{\overline{T}}} = \frac{\partial U_0^c}{\partial \overline{\overline{T}}} - \overline{\overline{I}} \qquad \{4.1.-16\}$$

become statements of the CROTTI Theorem, when a discrete force system is considered. In this case, {4.1.-16} becomes, for complementary strain energies W^c and U^c (representing the volume integrals of the densities W_0^c and U_0^c respectively),

$$U_i = \frac{\partial W^c}{\partial F_i} = \frac{\partial}{\partial F_i}(U^c - \overline{R}_j \cdot \overline{F}_j) \qquad \{4.1.-17\}$$

for a force $\overline{F}_i = F_i \overline{E}$ (where $\overline{E}$ denotes a unit vector) and the corresponding component of displacement $U_i = \overline{U}_i \cdot \overline{E}$, since the surface integral of $\overline{T}_n = \overline{N} \cdot \overline{\overline{T}}$ bears interpretation as a discrete force. The relation of the two densities, previously established as

$$U_0^c = W_0^c + \overline{\overline{T}} : \overline{\overline{I}} \qquad \{4.1.-18\}$$

thus leads to the relation

$$U^c = H^c = W^c + \bar{R}_i \cdot \bar{F}_i \qquad \{4.1.-19\}$$

for the total energies, in the case of discrete force systems.

For finite elastic systems containing redundant (statically indeterminate) forces, then

$$0 = \frac{\partial W^c}{\partial F_r} = \frac{\partial}{\partial F_r}(U^c - \bar{R}_i \cdot \bar{F}_i) \qquad \{4.1.-20\}$$

which represents the MENABREA Principle for a redundant force $\bar{F}_r = F_r \bar{E}$.

These theorems may also be stated in terms of the complementary strain energy of the MASUR formulation. From the relationship

$$W_0^c + \bar{\bar{T}} : \bar{\bar{1}} = \Omega + \bar{\bar{S}} : \frac{\partial \Omega}{\partial \bar{\bar{S}}} \qquad \{4.1.-21\}$$

then the CROTTI Theorem may be given as

$$U_i = \frac{\partial}{\partial F_i}(T^c - \bar{R}_j \cdot \bar{F}_j) \qquad \{4.1.-22\}$$

where

$$T^c = \iiint_V \left[\Omega + \bar{\bar{S}} : \frac{\partial \Omega}{\partial \bar{\bar{S}}}\right] dV \qquad \{4.1.-23\}$$

Similarly, the MENABREA Theorem for redundant forces appears as

$$0 = \frac{\partial}{\partial F_r}(T^c - \bar{R}_j \cdot \bar{F}_j) \qquad \{4.1.-24\}$$

It is observed that, in the case of discrete force systems, the energy formulation in terms of W^c provides a simpler and more direct basis for the Theorems of CROTTI and MENABREA than does the

formulation in terms of U^c. However, as the system becomes generalized to a continuous stress system, the energy density U_0^c is more appropriate than W_0^c, the latter being a function of rigid-body rotation.

This problem does not arise in the infinitesimal theory, due to the nature of the definition of the strain tensor in that theory. In fact, in such a case, the LEVINSON density W_0^c becomes identical with the energy density U_1^c of the Classical Theory (hence the statement that the CROTTI Theorem is "equivalent" to the Complementary Potential Energy Principle in infinitesimal elasticity). This is easily established from {3.4.-11}, which states that

$$U_1^c + \frac{1}{2}\overline{\overline{S}}:\overline{\overline{U}}\cdot\overline{\overline{U}}_c + \overline{\overline{S}}:\overline{\overline{\Gamma}} = W_0^c + \overline{\overline{T}}:\overline{\overline{1}} \qquad \{4.1.-25\}$$

Now, since $\overline{\overline{T}}:\overline{\overline{1}}$ may be written as

$$\overline{\overline{T}}:\overline{\overline{1}} = \overline{\overline{S}}\cdot\overline{\overline{\Gamma}}:\overline{\overline{1}} = \overline{\overline{S}}:\overline{\overline{\Gamma}} \qquad \{4.1.-26\}$$

for symmetric $\overline{\overline{S}}$, then

$$U_1^c + \frac{1}{2}\overline{\overline{S}}:\overline{\overline{U}}\cdot\overline{\overline{U}}_c = W_0^c \qquad \{4.1.-27\}$$

and as $\overline{\overline{U}}\cdot\overline{\overline{U}}_c$ is precisely the "quadratic" portion of the strain tensor which is neglected in the infinitesimal theory, then

$$U_1^c = W_0^c$$

in infinitesimal elasticity (only). Consequently, the situation is simplified considerably in this case.

4.2. THE UNIDIMENSIONAL CONTINUUM

Consider now, for purposes of comparison with the well-known results of the infinitesimal theory, a unidimensional continuum. Specifically, consider a prismatic rod of length L and cross-sectional area A in the undeformed state (the "truss member" discussed by LIBOVE), subjected to axial forces $\overline{F}$ and $-\overline{F}$ at the end points. The coordinate description of the rod being arbitrary, one end point will be considered located at $\overline{R}_1 = \overline{K}$ (where $\overline{K}$ is a constant vector) at which point the force $-\overline{F}$ will be applied, and the other end point will then be denoted as $\overline{R}_2$, where

$$\overline{R}_2 = \overline{K} + \overline{L} = \overline{K} + L\overline{G}_1 \qquad \{4.2.-1\}$$

at which point the force $\overline{F}$ will be applied. Thus, the axis of the rod is chosen as the θ^1-coordinate, and, as implied by {4.2.-1}, the base vector $\overline{G}_1$ denotes a unit vector. Therefore, any point on the rod axis is specified as

$$\overline{R} = \overline{K} + \theta^1\overline{G}_1 \qquad \{4.2.-2\}$$

The base vector system $\{\overline{G}_i\}$ is now constructed as an orthonormal basis, so that

$$\overline{G}_i = \overline{G}^i \qquad \{4.2.-3\}$$

and

$$G_{ij} = \overline{G}_i \cdot \overline{G}_j = \overline{G}^i \cdot \overline{G}_j = \delta_{ij} \qquad \{4.2.-4\}$$

with

$$\overline{G}_1 \times \overline{G}_2 = \overline{G}_3 \qquad \{4.2.-5\}$$

which specifies the basis as a dextral system.

The displacement field $\overline{U}$ for the rod is the usual one-

dimensional field $\bar{U} = U_1\bar{G}^1$, produced by the action of the axial forces at the ends of the rod. Thus,

$$\bar{U} = U_1\bar{G}^1 = U^1\bar{G}_1 = U_1\bar{G}_1 \qquad \{4.2.-6\}$$

and $$U^1|_1 = U_1|_1 = \frac{\partial U_1}{\partial \theta^1} = \bar{G}_1 \cdot \frac{\partial \bar{U}}{\partial \theta^1} \qquad \{4.2.-7\}$$

since $\bar{G}_1$ is not a function of the parametric coordinates. That is,

$$\bar{G}_1 \cdot \frac{\partial \bar{U}}{\partial \theta^j} = 0 \quad ; \quad j \neq 1 \qquad \{4.2.-8\}$$

$$\bar{G}_i \cdot \frac{\partial \bar{U}}{\partial \theta^1} = 0 \quad ; \quad i \neq 1 \qquad \{4.2.-9\}$$

in accordance with the usual postulates of one-dimensional analysis.

Therefore, the position vector $\bar{r}$ is prescribed as

$$\bar{r} = \bar{R} + \bar{U} = \bar{K} + (\theta^1 + U^1)\bar{G}_1 \qquad \{4.2.-10\}$$

and the basis $\{\bar{g}_i\}$ of the deformed configuration appears as

$$\begin{aligned} \bar{g}_1 &= \left[1 + \frac{\partial U_1}{\partial \theta^1}\right]\bar{G}_1 \\ \bar{g}_2 &= \bar{G}_2 \\ \bar{g}_3 &= \bar{G}_3 \end{aligned} \qquad \{4.2.-11\}$$

Or, as the deformation gradient is obtained as

$$\bar{\bar{\Gamma}} = (1 + U_{11})\bar{G}_1\bar{G}_1 \qquad \{4.2.-12\}$$

with $$\bar{\bar{U}} = U_{11}\bar{G}_1\bar{G}_1 = \frac{\partial U_1}{\partial \theta^1}\bar{G}_1\bar{G}_1 \qquad \{4.2.-13\}$$

then $$\bar{g}_1 = (1 + U_{11})\bar{G}_1 \qquad \{4.2.-14\}$$

and $\bar{g}_2 = \bar{G}_2$, $\bar{g}_3 = \bar{G}_3$, as given above. From the foregoing, it follows that

$$\sqrt{G} = \bar{G}_1 \cdot \bar{G}_2 \times \bar{G}_3 = 1 \qquad \{4.2.-15\}$$

and $$\sqrt{g} = \bar{g}_1 \cdot \bar{g}_2 \times \bar{g}_3 = 1 + U_{11} \qquad \{4.2.-16\}$$

so that $$\sqrt{\frac{g}{G}} = 1 + U_{11} \qquad \{4.2.-17\}$$

and the components of the stress tensors $\bar{\bar{\tau}}$ and $\bar{\bar{S}}$ are related, from {1.1.2.-57}, as

$$(1 + U_{11})\tau^{11} = S^{11} \qquad \{4.2.-18\}$$

other components being assumed to vanish. Furthermore, from the definition of the Lagrange stress tensor as

$$\bar{\bar{T}} = \bar{\bar{S}} \cdot \bar{\bar{\Gamma}} \qquad \{4.2.-19\}$$

then $$T^{11} = S^{11}(1 + U_{11}) \qquad \{4.2.-20\}$$

and all three types of stress components are related as

$$(1 + U_{11})^2\tau^{11} = (1 + U_{11})S^{11} = T^{11} \qquad \{4.2.-21\}$$

For simplicity in this one-dimensional case, T^{11} may be denoted as T, S^{11} as S, U_{11} as U, etc., so that {4.2.-21} appears as

$$(1 + U)^2\tau = (1 + U)S = T \qquad \{4.2.-22\}$$

Now, for a unit normal $\overline{N}$, prescribing the surface upon which the force $\overline{F}$ (or $-\overline{F}$) is defined to act, then

$$\overline{N} = -\overline{G}_1 \quad \text{at} \quad \overline{R} = \overline{K} \qquad \{4.2.-23\}$$

$$\overline{N} = \overline{G}_1 \quad \text{at} \quad \overline{R} = \overline{K} + \overline{L} \qquad \{4.2.-24\}$$

so that

$$\overline{F} = \iint_S \overline{T}_n \, dS = \iint_S \overline{G}_1 \cdot \overline{\overline{T}} \, dS \qquad \{4.2.-25\}$$

which, in this case, becomes

$$\overline{F} = F\overline{G}_1 = \iint_A T\overline{G}_1 \, dA = \left[\iint_A T \, dA\right]\overline{G}_1 \qquad \{4.2.-26\}$$

or

$$F = \iint_A T \, dA \qquad \{4.2.-27\}$$

Since T is constant over the cross-section in this case, then {4.1.-27} becomes simply

$$F = TA \qquad \{4.2.-28\}$$

The strain tensor $\overline{\overline{E}}$ is obtained from $\overline{\overline{U}}$ and $\overline{\overline{U}}_c$, as given by {3.1.-25}, and appears as

$$\overline{\overline{E}} = E_{11}\overline{G}_1\overline{G}_1 = \left[U_{11} + \tfrac{1}{2}(U_{11})^2\right]\overline{G}_1\overline{G}_1 \qquad \{4.2.-29\}$$

or

$$\overline{\overline{E}} = E_{11}\overline{G}_1\overline{G}_1 = \left[U + \tfrac{1}{2}U^2\right]\overline{G}_1\overline{G}_1 \qquad \{4.2.-30\}$$

Now, if a linear constitutive relation, of the form

$$\overline{\overline{S}} = 2\mu\overline{\overline{E}} + \lambda(\overline{\overline{E}} : \overline{\overline{I}})\overline{\overline{I}} \qquad \{4.2.-31\}$$

exists between $\overline{\overline{S}}$ and $\overline{\overline{E}}$, then

$$S = (2\mu + \lambda)E_{11} \qquad \{4.2.-32\}$$

or

$$S = \gamma\left[U + \frac{1}{2}U^2\right] \qquad \{4.2.-33\}$$

where

$$\gamma = 2\mu + \lambda \qquad \{4.2.-34\}$$

Denoting S/γ as σ, then {4.2.-33} yields U as

$$U = (1 + 2\sigma)^{\frac{1}{2}} - 1 \qquad \{4.2.-35\}$$

and the relationship of the various stress components may be written in the form

$$(1 + 2\sigma)\tau = (1 + 2\sigma)^{\frac{1}{2}}S = T \qquad \{4.2.-36\}$$

so that T and U are both represented in terms of σ (or σ appears in the capacity of a parameter).

From {4.2.-36}, S and T are seen to be related as

$$T = S\left[1 + \frac{2S}{\gamma}\right]^{\frac{1}{2}} \qquad \{4.2.-37\}$$

or

$$2S^3 + \gamma S^2 - \gamma T^2 = 0 \qquad \{4.2.-38\}$$

and from this result, with {4.2.-33}, U and T are seen to be related as

$$T = \frac{\gamma}{2}(U)(U + 1)(U + 2) \qquad \{4.2.-39\}$$

or

$$\gamma U^3 + 3\gamma U^2 + 2\gamma U - 2T = 0 \qquad \{4.2.-40\}$$

The complementary strain energy density U_0^c may now be evaluated.

Recalling that U_0^c is given as

$$U_0^c = \overline{\overline{T}} : \overline{\overline{\Gamma}} - U_0^s \qquad \{4.2.-41\}$$

then $\overline{\overline{T}} : \overline{\overline{\Gamma}}$ is found, in this case, as

$$\overline{\overline{T}} : \overline{\overline{\Gamma}} = T\Gamma_{11} = T(1 + U) \qquad \{4.2.-42\}$$

However, as

$$\Gamma_{11} = (1 + U) = 1 + [(1 + 2\sigma)^{\frac{1}{2}} - 1] = (1 + 2\sigma)^{\frac{1}{2}} \qquad \{4.2.-43\}$$

then from {4.2.-36} and {4.2.-43}

$$\overline{\overline{T}} : \overline{\overline{\Gamma}} = [S(1 + 2\sigma)^{\frac{1}{2}}](1 + 2\sigma)^{\frac{1}{2}}$$

$$= S(1 + 2\sigma) \qquad \{4.2.-44\}$$

and this result may be given completely in terms of S, as

$$\overline{\overline{T}} : \overline{\overline{\Gamma}} = S\left[1 + \frac{2S}{Y}\right] \qquad \{4.2.-45\}$$

The remaining term in the expression for U_0^c, namely the strain energy U_0^s, is found as

$$U_0^s = \int \overline{\overline{T}} : d\overline{\overline{\Gamma}} \qquad \{4.2.-46\}$$

or so

$$U_0^s = \int \overline{\overline{S}} : d\overline{\overline{E}} \qquad \{4.2.-47\}$$

or as either of the other two possible representations, as discussed in §2.2.. In the first case, U_0^s is obtained as

$$U_0^S = \int T \, d[(1+2\sigma)^{\frac{1}{2}}]$$

$$= \int S(1+2\sigma)^{\frac{1}{2}}(1+2\sigma)^{-\frac{1}{2}} \, d\sigma$$

$$= \int \frac{S}{\gamma} \, dS$$

$$= \frac{S^2}{2\gamma} \qquad \{4.2.-48\}$$

This result, of course, could also have been obtained from the integration of $\overline{\overline{S}} : d\overline{\overline{E}}$, as

$$U_0^S = \int \overline{\overline{S}} : d\overline{\overline{E}}$$

$$= \int \gamma E_{11} \, dE_{11}$$

$$= \frac{1}{2}\gamma E_{11}^2 = \frac{S^2}{2\gamma}$$

since the construction of the strain energy density function in all four sets of variables must lead to the same result.

From the expressions for $\overline{\overline{T}} : \overline{\overline{\Gamma}}$ and U_0^S, the complementary strain energy density is obtained as

$$U_0^C = S\left[1 + \frac{2S}{\gamma}\right] - \frac{S^2}{2\gamma}$$

$$= \frac{3S^2}{2\gamma} + S$$

$$= S\left[1 + \frac{3S}{2\gamma}\right] \qquad \{4.2.-49\}$$

This result may be expressed in terms of T, from the solution of {4.2.-38} or {4.2.-40}: if

$$\psi = \left[\frac{T^2}{\gamma^2} - \frac{1}{27}\right]^{1/2} \qquad \{4.2.-50\}$$

and $$A = \left[\frac{T}{\gamma} + \psi\right]^{1/3} \quad ; \quad B = \left[\frac{T}{\gamma} - \psi\right]^{1/3} \qquad \{4.2.-51\}$$

with $$\xi = -\frac{1}{2}[A + B + (A - B)\sqrt{-3}] \qquad \{4.2.-52\}$$

then U may be given, in terms of T, as

$$U = \xi - 1 \qquad \{4.2.-53\}$$

and S is obtained from {4.2.-33} as

$$S = \frac{\gamma}{2}[U(U + 2)] = \frac{\gamma}{2}(\xi^2 - 1) \qquad \{4.2.-54\}$$

In simpler form, if

$$\cos \phi = \frac{3\sqrt{3}\, T}{\gamma} \qquad \{4.2.-55\}$$

then $$\xi = \frac{2}{\sqrt{3}} \cos \frac{\phi}{3} \qquad \{4.2.-56\}$$

and U and S follow as given by {4.2.-53} and {4.2.-54}. Verifying this result, the constitutive relation may be examined in the form

$$\frac{\partial u_0^c}{\partial T} = \frac{\partial \sigma}{\partial T} \frac{\partial u_0^c}{\partial \sigma} \qquad \{4.2.-57\}$$

since σ is employed in the capacity of a parameter in the formulation of u_0^c. Since

$$\frac{\partial \sigma}{\partial T} = \frac{(1 + 2\sigma)^{1/2}}{\gamma(1 + 3\sigma)} \qquad \{4.2.-58\}$$

and $$\frac{\partial u_0^c}{\partial \sigma} = \gamma(1 + 3\sigma) \qquad \{4.2.-59\}$$

then $$\frac{\partial U_0^c}{\partial T} = (1 + 2\sigma)^{\frac{1}{2}} = U + 1 = \Gamma_{11} \qquad \{4.2.-60\}$$

as desired. Similarly, the energy density W_0^c of the LEVINSON formulation, i.e.,

$$W_0^c = \overline{\overline{T}} : \overline{\overline{U}} - U_0^s \qquad \{4.2.-61\}$$

may be obtained from the expressions for $\overline{\overline{T}}$, $\overline{\overline{U}}$ and U_0^s as given above. From {4.2.-35} and {4.2.-36},

$$\overline{\overline{T}} : \overline{\overline{U}} = \gamma\sigma(1 + 2\sigma)^{\frac{1}{2}}[(1 + 2\sigma)^{\frac{1}{2}} - 1]$$

$$= S\left[1 + \frac{2S}{\gamma}\right] - S\left[1 + \frac{2S}{\gamma}\right]^{\frac{1}{2}} \qquad \{4.2.-62\}$$

so that W_0^c becomes

$$W_0^c = S\left[1 + \frac{3S}{2\gamma}\right] - S\left[1 + \frac{2S}{\gamma}\right]^{\frac{1}{2}} \qquad 4.2.-63$$

The volume integral of the density U_0^c yields the total complementary energy U^c, or

$$U^c = \iiint_V U_0^c \, dV = \iiint_V S\left[1 + \frac{3S}{2\gamma}\right] dV \qquad \{4.2.-64\}$$

so $$U^c = \int_0^L \left[\iint_A S\left[1 + \frac{3S}{2\gamma}\right] dA\right] d\theta^1$$

and, as S is constant over the cross-sectional area and the length in this case, then

$$U^c = H^c = S\left[1 + \frac{3S}{2\gamma}\right] AL \qquad \{4.2.-65\}$$

This expression represents the total complementary energy for a uni-dimensional member subjected to axial forces.

LIBOVE has obtained a corresponding expression, under the assumption that the elongation, e, (the total displacement) of the bar is a function of the total force F applied to the end points. In this case, he obtained the result

$$H^c = FL + \int e \, dF \qquad \{4.2.-66\}$$

or, in the case of "a bar which obeys Hooke"s law", he obtained

$$H^c = FL + \frac{F^2 L}{2AE} \qquad \{4.2.-67\}$$

where E denotes the modulus of elasticity (corresponding to γ, above). LIBOVE observed that "It is important to note that [H^c] is not the same as the definition usually given for the complementary energy of a bar. The usual definition consists only of the second part ..... While the usual definition is sufficient for trusses with negligible rotations [small changes of geometry], the extra term FL in the new definition is needed for application to trusses whose bars undergo large rotations [large changes of geometry]. It will become clear through the later discussion that when the rotations are assumed to be vanishingly small, the extra terms in the new definition reduce to a constant, which may be dropped from the complementary energy expression, thus bringing the new complementary-energy theory into agreement with the conventional one".

It is important to note that LIBOVE'S explanation of the nature of the energy H^c is in error. The term FL has no relation

whatever to the finite aspect of the theory and, in fact, remains constant for all deformations of the member -- finite or infinitesimal. It is the energy W^c of the LEVINSON formulation which reduces to the "conventional" expression, not the energy H^c of the LIBOVE Theorem (or its generalization, the energy U^c of the present theorem). The term FL arises as a natural consequence of the conjugate variables which are selected for the formulation of the theorem. In the general case, FL corresponds to the term which marks the difference between the functions W^c and U^c: namely, the volume integral of $\overline{\overline{T}} : \overline{\overline{I}}$. In the case of discrete forces, this difference is represented, as noted by {4.1.-19}, as $\overline{R}_i \cdot \overline{F}_i$. This is easily established by considering the original specification of the system:

$$\overline{R}_1 = \overline{K} \quad \text{and} \quad \overline{F}_1 = -\overline{F} \quad \text{at} \quad \overline{R}_1 \qquad \{4.2.-68\}$$

$$\overline{R}_2 = \overline{K} + L \quad \text{and} \quad \overline{F}_2 = \overline{F} \quad \text{at} \quad \overline{R}_2 \qquad \{4.2.-69\}$$

Thus, the term $\overline{R}_i \cdot \overline{F}_i$ is found as

$$\begin{aligned} \overline{R}_i \cdot \overline{F}_i &= \overline{K} \cdot (-\overline{F}) + (\overline{K} + \overline{L}) \cdot \overline{F} \\ &= -\overline{K} \cdot \overline{F} + \overline{K} \cdot \overline{F} + \overline{F} \cdot \overline{L} \\ &= F\overline{G}_1 \cdot L\overline{G}_1 \\ &= FL \end{aligned} \qquad \{4.2.-70\}$$

and this term remains unchanged (in form, not magnitude), whether the deformation is finite or infinitesimal. Thus, the term FL in LIBOVE'S formulation is a function of the choice of conjugate variables and is not a term which arises as a result of the finite nature of the

deformation.

The second term in LIBOVE'S expression, $\frac{F^2L}{2AE}$, agrees with the "conventional" expression because it is precisely that. A linear force-displacement relation considered with a linear stress-strain relation implies a strain tensor of the form $\overline{\overline{E}} = \frac{1}{2}(\overline{\overline{U}} + \overline{\overline{U}}_c)$ in which the term $\overline{\overline{U}} \cdot \overline{\overline{U}}_c$ is neglected. Consequently, the complementary strain energy is obtained as a result of the infinitesimal theory. Of course, the use of a linear force-displacement and stress-strain relation may well be a very reasonable approximation in the analysis of engineering problems, due to the nature of most 'engineering materials', but it is essential to note that in such a case, the theory is not a true "finite theory". Rather, it becomes a theory of *infinitesimal elasticity*, in which the equations of equilibrium are considered in terms of the *deformed geometry*, or a "hybrid theory".

From the expression {4.2.-65} for the complementary strain energy U^c, it is seen that

$$\frac{\partial U^c}{\partial F} = L\left[1 + \frac{2S}{Y}\right]^{1/2}$$

$$= L(U + 1) \qquad \{4.2.-71\}$$

However, as

$$U = \frac{\partial U_1}{\partial \theta^1} = \frac{U_1}{L} \qquad \{4.2.-72\}$$

in this case, due to the nature of the force system, then

$$\frac{\partial U^c}{\partial F} = U_t + L = \ell \qquad \{4.2.-73\}$$

where ℓ represents the length of the member after deformation and U_t $(=U_1)$ denotes the total elongation of the member. This result agrees with the corresponding general statement, previously given as

$$U_i = \frac{\partial W^c}{\partial F_i} = \frac{\partial}{\partial F_i}(U^c - \bar{R}_j \cdot \bar{F}_j) \qquad \{4.2.-74\}$$

Here, $\bar{R}_j \cdot \bar{F}_j$ has been established as FL, so

$$U_t = \frac{\partial W^c}{\partial F} = \frac{\partial}{\partial F}(U^c - FL)$$

or

$$U_t = \frac{\partial W^c}{\partial F} = \frac{\partial U^c}{\partial F} - L \qquad \{4.2.-75\}$$

and it is seen that {4.2.-73} and {4.2.-75} represent the same relationship, as required.

4.3. THE STRESS-STRETCH RELATION

It is instructive to examine the relationship of the Lagrange stress to the state of stretch, as discussed in §3.3.. The unidimensional continuum provides a convenient vehicle by means of which this may be accomplished, due to the simplicity of the expressions in this case.

As noted in §3.3., a state of stress $\bar{\bar{T}}$ may be expressed in terms of the state of stretch $\bar{\bar{V}}$, as

$$\bar{\bar{T}} = \bar{\bar{h}}(\bar{\bar{V}}) \qquad \{4.3.-1\}$$

which is mapped by $\bar{\bar{\Phi}}$ into a relationship between the Lagrange stress tensor and $\bar{\bar{\Gamma}}$,

$$\overline{\overline{T}} = \overline{\overline{h}}(\overline{\overline{\Gamma}}) \qquad \{4.3.-2\}$$

In the case under consideration at present,

$$\overline{\overline{\Gamma}} = (1 + U_{11})\overline{G}_1\overline{G}_1 \qquad \{4.3.-3\}$$

and

$$\overline{\overline{\Phi}} = \overline{\overline{1}} \qquad \{4.3.-4\}$$

so the relationship to be examined will appear in the form

$$\overline{\overline{T}} = \overline{\overline{h}}(\overline{\overline{V}}) \qquad \{4.3.-5\}$$

Since a linear constitutive relation (in terms of the TREFFTZ stress $\overline{\overline{S}}$ and the strain $\overline{\overline{E}}$) has been assumed for purposes of discussion above, it will be assumed that this relation is still enforced: the corresponding relationship in terms of $\overline{\overline{T}}$ and $\overline{\overline{V}}$ will then be deduced. Since

$$\overline{\overline{E}} = \frac{1}{2}(\overline{\overline{C}} - \overline{\overline{1}})$$

then

$$\overline{\overline{C}} = 2\overline{\overline{E}} + \overline{\overline{1}} = \overline{\overline{\Gamma}} \cdot \overline{\overline{\Gamma}}_c = \overline{\overline{V}} \cdot \overline{\overline{V}}$$

or, denoting C_{11} as C, Γ_{11} as Γ, etc., in this case,

$$C = 2E + 1 = \Gamma^2 = V^2 \qquad \{4.3.-6\}$$

The relation of S to E, previously given as

$$S = (2\mu + \lambda)E = \gamma E \qquad \{4.3.-7\}$$

yields {4.3.-6} as

$$C = (1 + 2\sigma) = V^2 \qquad \{4.3.-8\}$$

where $\sigma = S/\gamma$, as before. Now, as

$$T = \gamma\sigma(1 + 2\sigma)^{\frac{1}{2}} = \gamma\sigma V \qquad \{4.3.-9\}$$

and as

$$\sigma = \frac{1}{2}(V^2 - 1) \qquad \{4.3.-10\}$$

then the stress-stretch relation is obtained as

$$T = \alpha V(V^2 - 1) \qquad \{4.3.-11\}$$

where $\alpha = \gamma/2$ denotes the material constant.

This relationship, produced by a linear constitutive relation (in terms of $\overline{\overline{S}}$ and $\overline{\overline{E}}$), is seen to be nonlinear. Of course, this result was anticipated, since a linear relation of $\overline{\overline{S}}$ to $\overline{\overline{E}}$ is a linear relation of $\overline{\overline{S}}$ to the stretch *quadratic*, $\overline{\overline{V}} \cdot \overline{\overline{V}}$.

The cubic form of the relation of T to V, with multiple values at the zero-stress state ($V = -1, 0, +1$) raises the question of the single-valuedness of the inversion $V = f(T)$. However, as may be observed from {4.3.-6},

$$|V| \geq 1 \qquad \{4.3.-12\}$$

so that the multi-valued region $-1 < V < +1$ presents no obstacle whatever. Since the relationship {4.3.-11} is monotonic in the regions $V \leq -1$ and $V \geq +1$ (everywhere except $-1 < V < +1$), and since $|V| \geq 1$, then it is of no consequence that the relationship is multi-valued in the region $-1 < V < +1$, as this region is inadmissible.

The Lagrange stress T may thus be prescribed in terms of any function of the stretch V which is deemed appropriate to the constitution of the material under consideration. It is not necessary that the function be single-valued in the interval $-1 < V < +1$, since this region is an inadmissible one.

Comparing the expressions for T in terms of V and T in terms of U (as given by {4.2.-39}),

$$T = \alpha V(V^2 - 1) = \alpha\Gamma(\Gamma^2 - 1) \qquad \{4.3.-11\}$$

$$T = \alpha U(U + 1)(U + 2) \qquad \{4.3.-13\}$$

it is observed that the latter admits zero-stress state values of $U = -2, -1, 0$. However, as the region $-2 < U < 0$ is inadmissible, then the inversion $U = k(T)$ presents no difficulty.

It is of interest to note that equation {4.3.-13} shows the order of approximation which is implicit in the assumption that the force is a linear function of the (total) displacement. In this case, U is considered to be negligible with respect to 1, and T is assumed to be

$$T = \alpha U(U + 1)(U + 2) \triangleq 2\alpha U = \gamma U \qquad \{4.3.-14\}$$

It then follows that

$$\iiint_V T\, dV = \iiint_V \gamma U\, dV \qquad \{4.3.-15\}$$

or

$$FL = \gamma A U_t \qquad \{4.3.-16\}$$

so $$U_t = \frac{FL}{A\gamma} \qquad \{4.3.-17\}$$

which is the familiar expression of the infinitesimal theory.

Both zero-stress state conditions (in terms of V and of U) follow from the criterion of pure rigid deformation

$$\overline{\overline{C}} = \overline{\overline{1}} \qquad \{4.3.-18\}$$

or, here, $$C = 1 \qquad \{4.3.-19\}$$

and from the relation of C, V and U,

$$C = V^2 = (U + 1)^2 = 2E + 1 \qquad \{4.3.-20\}$$

Thus, the Lagrange stress $\overline{\overline{T}}$ may be prescribed in terms of $\overline{\overline{\Gamma}}$ (or $\overline{\overline{V}}$) as in the present theorem, or in terms of $\overline{\overline{U}} = \overline{\overline{\Gamma}} - \overline{\overline{1}}$ as in the LEVINSON formulation, according to the constitution of the material, and the Complementary Potential Energy Principle for finite deformations assumes the forms given in §3.4.

CHAPTER V

Conclusion

5.1. CONCLUSIONS

This work establishes the general Principle of Complementary Potential Energy for the case of finite deformations of an elastic continuum, in which the Lagrange stress tensor $\overline{\overline{T}}$ and the deformation gradient $\overline{\overline{\Gamma}}$ appear as the appropriate conjugate variables. This study, which was generated by an investigation of LEVINSON'S Theorem, serves to establish several important results regarding the Principle and its formulation, and also relates the various functionals and different tensor variables which have been proposed for the construction of the Principle.

The formulation of the Principle by means of the Lagrange stress tensor and the Lagrange strain tensor or displacement gradient $\overline{\overline{U}}$ has been demonstrated to be a valid construction of the Principle. The inversion of the constitutive relation

$$\overline{\overline{T}} = \overline{h}(\overline{\overline{U}}) = \overline{h}(\overline{\overline{\Gamma}} - \overline{\overline{I}}) \qquad \{5.1.-1\}$$

to express $\overline{\overline{U}}$ in terms of $\overline{\overline{T}}$, as

$$\overline{\overline{U}} = \overline{\overline{\Gamma}} - \overline{\overline{I}} = \overline{\ell}(\overline{\overline{T}}) \qquad \{5.1.-2\}$$

has been established as *not always impossible*, due to the explicit dependence of the Lagrange stress tensor on $\overline{\overline{\Gamma}}$. Therefore, LEVINSON'S

formulation of the Principle has been shown to be a valid one, in contrast to the previously-held opinions that it could never be valid as a consequence of the supposed impossibility of constitutive relations of the form {5.1.-2}.

As a matter of interest, it is noted that the direct tensor formulation of this work, in which the Lagrange stress tensor is constructed as

$$\overline{\overline{T}} = \overline{\overline{S}} \cdot \overline{\overline{\Gamma}}$$

demonstrates with almost deceptive simplicity that $\overline{\overline{T}}$ is a function of $\overline{\overline{\Gamma}}$ *in its entirety* (the fundamental requirement of inversion). The original definition of this tensor, however, as first given by GREEN and ADKINS in the indirect form

$$T^{ij} = S^{im}(\delta^j_m + U^j|_m)$$

does little to demonstrate this fact. Even in the semi-indirect form

$$T^{ij}\overline{G}_j = S^{im}\overline{g}_m$$

the relationship appears to be only a "change of basis", as noted by GREEN and ADKINS. This, of course, is true: but the far-reaching consequences of this "change of basis", such as the nature of the tensor $\overline{\overline{T}}$, as established in this work, are of far greater importance than the change of basis, *per se*.

As previously noted, the fact that {5.1.-2} is not always impossible cannot be construed to mean that such a relation is always possible. That is, constitutive relations in terms of $\overline{\overline{T}}$ and $\overline{\overline{U}}$ may

certainly be constructed in such a form that inversion is not possible, as is true for constitutive relations in terms of $\overline{\overline{S}}$ and $\overline{\overline{C}}$ (or $\overline{\overline{S}}$ and $\overline{\overline{E}}$, etc.). The important point, however, is that $\overline{\overline{T}}$ and $\overline{\overline{U}}$ form a set of conjugate variables, the nature of which does not preclude inversion. This may also be said of the sets of variables $\{\overline{\overline{T}},\overline{\Gamma}\}$, $\{\overline{\overline{S}},\overline{\overline{C}}\}$ and $\{\overline{\overline{S}},\overline{\overline{E}}\}$, but not, for example, of the sets $\{\overline{\overline{S}},\overline{\Gamma}\}$, $\{\overline{\overline{S}},\overline{\overline{U}}\}$, $\{\overline{\tau},\overline{\Gamma}\}$ or $\{\overline{\overline{\tau}},\overline{\overline{U}}\}$, for the reason discussed in Chapter III.

The validity of LEVINSON'S formulation having been established, it was demonstrated that the complementary strain energy W^c, representing the volume integral of the density function W_0^c of this theorem, was identical with the energy W^c of the CROTTI Theorem for discrete force systems or "finite elastic systems". Thus, it was shown that the LEVINSON Theorem represents the direct generalization of the CROTTI Theorem to continuous stress systems. That is, the energy W^c of LEVINSON'S formulation has been established as the true finite counterpart of the complementary strain energy of the infinitesimal theory, as the function W^c reduces to the volume integral of U_1^c in the infinitesimal case, and W^c therefore appears as the energy function in both the CROTTI Theorem and the Complementary Potential Energy Principle in both (finite and infinitesimal) cases. Consequently, LEVINSON'S Theorem was found to be the most advantageous form of the Complementary Potential Energy Principle for finite deformations.

Briefly, the functional of LEVINSON'S Theorem is given as

$$\Pi^c = -\iiint_V W_0^c \, dV + \iint_{S_u} \overline{T}_n \cdot \overline{U}^* \, dS \qquad \{5.1.-3\}$$

where the Lagrange stress tensor must satisfy the conditions of admissibility

1) $$\frac{\partial \cdot \overline{\overline{T}}}{\partial \overline{R}} = \overline{0} \qquad \{5.1.-4\}$$

2) $$\overline{\overline{\Gamma}} \times \overline{\overline{T}} = \overline{0} \qquad \{5.1.-5\}$$

Further investigation of the nature of the complementary strain energy density function W_0^c of the LEVINSON formulation, defined as

$$W_0^c = \overline{\overline{T}} : \overline{\overline{U}} - U_0^s \qquad \{5.1.-6\}$$

revealed, however, that this energy density is generally a function of rigid-body rotations in the displacement field. This does not constitute an error in the LEVINSON formulation. In fact, it is a result which is not even inconsistent with the nature of the other variables employed in the formulation, since $\overline{\overline{T}}$ and $\overline{\overline{U}}$ are both functions of $\overline{\overline{\Gamma}}$ and hence, of the finite rotation tensor $\overline{\overline{\Phi}}$. Constitutive relations in the form

$$\overline{\overline{U}} = \frac{\partial W_0^c}{\partial \overline{\overline{T}}} \qquad \{5.1.-7\}$$

are therefore entirely permissible for an energy density W_0^c which is a function of rigid rotations, due to the nature of the variables $\overline{\overline{T}}$ and $\overline{\overline{U}}$. However, the original constitutive relation of the elastic material in terms of the same variables, namely

$$\overline{\overline{T}} = \frac{\partial U_0^s}{\partial \overline{\overline{U}}} \qquad \{5.1.-8\}$$

employs an energy density (the strain energy density U_0^S) which is not a function of rigid rotations. Consequently, it was considered that a constitutive relation based on a function U_0^S, which is independent of $\overline{\overline{\Phi}}$, which inverts to yield a constitutive relation based on a function W_0^C, which is dependent on $\overline{\overline{\Phi}}$, is not particularly advantageous. Such an inversion, it was felt, would necessarily require caution in its use and would be less convenient than a relation based on U_0^S which inverts to yield a relation which is also independent of rigid rotations. For this reason, it was stated that the CROTTI Theorem has no convenient generalization to continuous stress systems: the generalization is LEVINSON'S Theorem, but this requires the use of the function W_0^C which is dependent upon rigid rotations in the finite case (although not in the infinitesimal case).

Subsequent investigation revealed that it was possible to construct the Complementary Potential Energy Principle in a form which retained the advantages of the Lagrange stress formulation, but which led to an energy density which was not a function of rigid rotations. In this case (the present theorem), the Lagrange stress tensor $\overline{\overline{T}}$ and the deformation gradient $\overline{\overline{\Gamma}}$ were found to be the appropriate conjugate variables for the formulation of the Principle. In this regard, the construction of the present theorem in terms of $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$ instead of $\overline{\overline{T}}$ and $\overline{\overline{U}}$ is analogous to MASUR'S construction of the theorem in terms of $\overline{\overline{S}}$ and $\overline{\overline{C}}$ rather than $\overline{\overline{S}}$ and $\overline{\overline{E}}$. Both instances are seen to involve a change of the displacement-related variable only by the addition of the identity tensor $\overline{\overline{I}}$, for the purpose of securing a more

advantageous form of expression.

It has been established that constitutive relations may be constructed in terms of $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$ and that the inversion of such relations is not generally impossible, due to the nature of the stress tensor $\overline{\overline{T}}$ (this fact is established; it then follows that the same is true for $\overline{\overline{T}}$ and $\overline{\overline{U}}$, as stated above). Thus, the inversion of the relation

$$\overline{\overline{T}} = \overline{k}(\overline{\overline{\Gamma}}) = \overline{k}(\overline{\overline{U}} + \overline{\overline{1}}) \qquad \{5.1.-9\}$$

to express $\overline{\overline{\Gamma}}$ in terms of $\overline{\overline{T}}$, as

$$\overline{\overline{\Gamma}} = \overline{\overline{U}} + \overline{\overline{1}} = \overline{\overline{\gamma}}(\overline{\overline{T}}) \qquad \{5.1.-10\}$$

has been demonstrated to be not always impossible. However, as previously observed, an inversion may be impossible because of its particular form -- as is true for constitutive relations in terms of any tensor variables. It was shown that relations in terms of $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$ bear interpretation as a relation of the state of stress to the state of stretch, and the rationale of such relations is therefore more easily constructed in terms of these variables than in terms of $\overline{\overline{T}}$ and $\overline{\overline{U}}$.

The complementary strain energy density function U_0^c of the present theorem, defined as

$$U_0^c = \overline{\overline{T}} : \overline{\overline{\Gamma}} - U_0^s \qquad \{5.1.-11\}$$

has been established as an energy density which is not a function of rigid rotations. Thus, the constitutive relation

$$\overline{\overline{T}} = \frac{\partial u_0^s}{\partial \overline{\overline{\Gamma}}} \qquad \{5.1.-12\}$$

and its inversion

$$\overline{\overline{\Gamma}} = \overline{\overline{U}} + \overline{\overline{I}} = \frac{\partial u_0^c}{\partial \overline{\overline{T}}} \qquad \{5.1.-13\}$$

are both based on energy density functions which are independent of rigid rotations.

The Complementary Potential Energy Principle of the present formulation has been shown to reduce directly to, and therefore to contain as a special case, the LIBOVE Theorem, in the case of systems of discrete forces or finite elastic systems. Thus, the present theorem represents the direct generalization of the LIBOVE Theorem to continuous stress systems. Although the complementary strain energy U^c of the present theorem, representing the volume integral of the density u_0^c, has been established as being different from the energy W^c of the CROTTI Theorem (for this is LEVINSON'S energy), it has been demonstrated that the CROTTI Theorem may still be expressed in terms of U^c in a reasonably simple form. Thus, the present formulation eliminates the dependence of the complementary strain energy function on rigid rotations without sacrificing the CROTTI Theorem for discrete force systems.

The conditions of admissibility have been established for the Lagrange stress tensor $\overline{\overline{T}}$ of the present formulation, in a form which is amenable to a stress-function solution to the problem, and which does not require the explicit use of reciprocal base vectors of the

deformed state. It has been noted that the admissibility conditions of the present theorem are identical with the admissibility conditions of the LEVINSON Theorem, as the stress tensor variable remains the same in both formulations, although the displacement-related tensor variable differs by the identity tensor.

The unidimensional continuum has been examined, and the results have been compared to the results of the infinitesimal theory. The erroneous interpretation of the nature and the origin of the term $\bar{R}_i \cdot \bar{F}_i$ in the discrete-force formulation, as given by LIBOVE, has been corrected. It has been shown that this term arises as a result of the particular conjugate variables employed, and is not peculiar to finite deformations.

The complementary strain energy density functions proposed by other authors have been examined, and the relationships between these functions have been constructed. It has also been demonstrated that the four different formulations which have been discussed, representing four different sets of conjugate tensor variables, have arisen as a result of the fact that the strain energy density function U_0^s may be constructed in these different variables as the same function. Specifically, it has been shown that

$$\delta U_0^s = \bar{\bar{X}} : \delta \bar{\bar{Y}} \qquad \{5.1.-14\}$$

where $\{\bar{\bar{X}},\bar{\bar{Y}}\} = \{\bar{\bar{S}},\bar{\bar{E}}\}$, $\{\bar{\bar{S}},\frac{1}{2}\bar{\bar{C}}\}$, $\{\bar{\bar{T}},\bar{\bar{U}}\}$, $\{\bar{\bar{T}},\bar{\bar{\Gamma}}\}$. Each of these four sets of variables has then been shown to yield a different formulation of the complementary strain energy density function, by means of a

Legendre transformation on the variables of the set. The first set, $\{\bar{\bar{S}},\bar{\bar{E}}\}$, generates the Classical (and the REISSNER) formulation, in which the complementary energy density U_1^c is found: this formulation results in the coupling of stress and displacement components, as previously noted. The set $\{\bar{\bar{S}},\frac{1}{2}\bar{\bar{C}}\}$ generates the MASUR formulation, in which the energy density Ω is found: this formulation avoids the difficulties of the Classical form of the Principle, and is the most advantageous form of the theorem in terms of the TREFFTZ stress tensor. The set $\{\bar{\bar{T}},\bar{\bar{U}}\}$ generates the LEVINSON formulation, in which the energy density W_0^c is found: this formulation has been shown to represent the true finite counterpart of the infinitesimal theory, and would be the most expedient of all four representations, were it not for the fact that W_0^c is a function of rigid rotations. The set $\{\bar{\bar{T}},\bar{\bar{\Gamma}}\}$ generates the present theorem, in which the energy density U_0^c is found: this formulation avoids the construction of a complementary strain energy which is a function of rigid rotations; also, the present formulation reduces to the LIBOVE Theorem for finite elastic systems and retains a simple relation to the energy W^c of the CROTTI Theorem. Thus, the present theorem represents the most advantageous form of the Complementary Potential Energy Principle in terms of the Lagrange stress tensor.

In summary, the following facts and their subsequent conclusions have been established by this work. The nature of the Lagrange stress tensor, as determined by the direct tensor formulation, has been established as such that constitutive relations in terms of $\bar{\bar{T}}$

and $\overline{\overline{U}}$ do admit inversion. Consequently, it is concluded that LEVINSON'S Theorem is valid, contrary to previous arguments regarding the impossibility of such inversions (and the subsequent impossibility of LEVINSON'S formulation). It has also been established that LEVINSON'S complementary strain energy represents the energy function which is appropriate to the CROTTI Theorem, and which reduces to the classical expression in the case of infinitesimal deformations. Thus, it is concluded that LEVINSON'S complementary strain energy represents the true finite extension of the complementary strain energy of the infinitesimal formulation. The LEVINSON energy has been shown to be a function of rigid rotations in the displacement field, and a new complementary strain energy density function has been defined by $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$, to re-formulate the Principle in such a way that this characteristic of the energy is avoided. In addition, the latter formulation has been demonstrated to be the direct generalization of LIBOVE'S Theorem, and has been shown to retain a simple relationship to the CROTTI Theorem. Therefore, it is concluded that the formulation of the Principle in terms of $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$ represents the most advantageous form of the Principle in terms of the Lagrange stress tensor, and that the CROTTI Theorem has no convenient generalization to continuous stress systems in finite elasticity, in contrast to the case of infinitesimal elasticity. The relationship of all four complementary energy density functions proposed for the construction of the Principle has been examined. From this, it is concluded that all are valid, and that four such functions exist as a consequence of the four possible

definitions of the strain energy density in terms of four different sets of tensor variables.

5.2. RECOMMENDATIONS FOR FUTURE RESEARCH

Since this work has established the validity of the Complementary Potential Energy Principle in which the Lagrange stress tensor appears as the stress variable (LEVINSON'S or the present Theorem), it is hoped that future research in this area may be devoted to the application of the Theorem to problems of engineering interest.

The primary objective of future investigations will doubtless be the construction of constitutive relations in terms of $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$ (or $\overline{\overline{T}}$ and $\overline{\overline{U}}$) which describe the material properties of the system under investigation. Once such relations have been constructed, the Theorem may be employed to advantage, due to the simplicity of the functional form. Certainly, the one-dimensional or "structural" case represents the point at which investigation is likely to begin, for two reasons. First, this case is one of considerable practical importance, as the majority of structures which are currently being designed are "linear element" structures. Second, this case represents the simplest area of application of the theorem. In this regard, the theorem should be investigated for simple cases of idealized two-dimensional elements which are usually considered in the same context

as one-dimensional elements: e.g., linear elements subjected to flexural loading systems ("beam elements"). This case, representing one of great practical value, is one of the simplest cases of a continuous stress distribution which is not constant throughout the member. LIBOVE has examined this case by means of his formulation of the theorem, but since his theorem is restricted to the consideration of discrete forces, he was forced to consider the stress distribution in a flexural member as a superposition of discrete forces. The present theorem, which represents the generalization of the LIBOVE Theorem to continuous stress distributions, is not subject to this restriction, and is therefore more appropriate to the analysis of such a system. The results of such an investigation would be quite valuable, particularly in the event that the order of approximation of LIBOVE'S results could be established for particular classes of problems. In this event, it would be possible to predict the cases in which the simpler formulation of the theorem (LIBOVE'S Theorem) could be employed without significant error.

The theorem will probably prove to be of greatest value in the true two-dimensional case, in the analysis of thin plates, thin shallow shells, and plane strain problems in general. In this regard, a detailed investigation of stress function tensors which satisfy the conditions of admissibility would be of considerable value. Although a great deal of information regarding stress function tensors (and displacement functions) has been accumulated in terms of the familiar representations of these quantities, such as the CAUCHY-GREEN Tensor,

the TREFFTZ Tensor and the strain tensor, very little is known of such functions in terms of the variables of the present formulation.

Since it will doubtless be difficult to obtain solutions to all but the simplest problems in analytic form, as is true for elasticity problems in general and finite elasticity problems in particular, the subject of approximation techniques should be investigated with regard to their application to the present theorem. Approximations in the constitutive relation should be examined, in order to obtain the simplest possible meaningful relations for particular situations.

Since the "physical significance" of a relationship between $\overline{\overline{S}}$ and $\overline{\overline{E}}$ is more readily interpreted than one between $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$ due to the length of time for which the former set has been in use, the construction of constitutive relations in terms of $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$ might be investigated as follows. If a constitutive relation (invertible) is prescribed in terms of $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$, and this relationship is assumed to be the "fundamental" relation for the material, then the relationship between $\overline{\overline{S}}$ and $\overline{\overline{E}}$ which follows from this fundamental relation may be constructed as a "secondary" relation, for purposes of comparison. The two constitutive relations will, of course, specify the same condition in terms of two different sets of variables: the relation of $\overline{\overline{T}}$ to $\overline{\overline{\Gamma}}$ (or $\overline{\overline{U}}$, if desired) will be employed in the Complementary Potential Energy Principle, while the equivalent relation of $\overline{\overline{S}}$ to $\overline{\overline{E}}$ will provide a comparison to the better-known forms of constitutive relations.

APPENDIX A

The Existence of the Complementary Strain Energy Density Function

Consider the complementary strain energy density function U_0^c, defined for an elastic continuum (one for which the strain energy density U_0^s exists as a state function) from the function U_0^s and the Lagrange variables $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$, by means of a general contact transformation

$$U_0^c = \overline{\overline{T}} : \overline{\overline{\Gamma}} - U_0^s \qquad \text{(A-1)}$$

The finite nature of this definition of U_0^c (as opposed to the differential nature of the definition of U_0^s) shows that no ancillary conditions are required for the existence of U_0^c as a state function. That is, *the complementary strain energy density exists as a state function if the strain energy density exists as a state function.* This assertion, which has been the subject of some dispute, may be proven as follows.

From the definition of U_0^c, consider the total differential of this quantity, as

$$dU_0^c = d(\overline{\overline{T}} : \overline{\overline{\Gamma}}) - dU_0^s \qquad \text{(A-2)}$$

or

$$dU_0^c = \overline{\overline{T}} : d\overline{\overline{\Gamma}} + \overline{\overline{\Gamma}} : d\overline{\overline{T}} - dU_0^s \qquad \text{(A-3)}$$

However, as

$$dU_0^s = \overline{\overline{T}} : d\overline{\overline{\Gamma}}$$

by virtue of {2.2.-60}, then

$$dU_0^c = \overline{\overline{\Gamma}} : d\overline{\overline{T}} \tag{A-4}$$

This is the point at which the dispute regarding the existence of U_0^c usually arises. It has been frequently noted that "equation (A-4) shows that the constitutive relation must be inverted, i.e., $\overline{\overline{\Gamma}}$ must be expressed as a function of $\overline{\overline{T}}$, in order that U_0^c may exist". This is not true. The inversion of the constitutive relation is required for another property of U_0^c, namely its *independence* from U_0^s in its definition, but the inversion is not a prerequisite for the *existence* of the complementary strain energy density.

To prove this, proceed from (A-4) as follows, noting that no inversions have been postulated: $\overline{\overline{\Gamma}}$ is still considered as the independent tensor variable, and $\overline{\overline{T}} = \overline{\overline{\ell}}(\overline{\overline{\Gamma}})$.

From $$dU_0^c = \overline{\overline{\Gamma}} : d\overline{\overline{T}} \tag{A-5}$$

since $\overline{\overline{T}} = \overline{\overline{\ell}}(\overline{\overline{\Gamma}})$, then

$$dU_0^c = \overline{\overline{\Gamma}} : \left[d\overline{\overline{\Gamma}} : \frac{\partial \overline{\overline{T}}}{\partial \overline{\overline{\Gamma}}} \right]$$

or $$dU_0^c = \overline{\overline{\Gamma}} : \frac{\overline{\overline{T}}\partial}{\partial \overline{\overline{\Gamma}}} : d\overline{\overline{\Gamma}}$$

Now, this may be written as

$$dU_0^c = \overline{\overline{\xi}} : d\overline{\overline{\Gamma}} \tag{A-6}$$

where $$\overline{\overline{\xi}} = \overline{\overline{\Gamma}} : \frac{\overline{\overline{T}}\partial}{\partial \overline{\overline{\Gamma}}} = \frac{\partial \overline{\overline{T}}}{\partial \overline{\overline{\Gamma}}} : \overline{\overline{\Gamma}} \tag{A-7}$$

Since $\overline{\overline{\xi}}$ is a function solely of $\overline{\overline{\Gamma}}$, then u_0^c exists as a state function if $\overline{\overline{\xi}} : d\overline{\overline{\Gamma}}$ is an exact or perfect differential. Thus, u_0^c exists as a state function if

$$\frac{\partial \overline{\overline{\xi}}}{\partial \overline{\overline{\Gamma}}} = \frac{\overline{\overline{\xi}}\partial}{\partial \overline{\overline{\Gamma}}} \tag{A-8}$$

which is the condition under which (A-6) becomes exact. In component form, this condition appears as

$$\frac{\partial \xi^{mn}}{\partial \Gamma_{pq}} = \frac{\partial \xi^{pq}}{\partial \Gamma_{mn}} \tag{A-9}$$

where a component ξ^{rs} is given from (A-7), as

$$\xi^{rs} = \Gamma_{ij} \frac{\partial T^{ij}}{\partial \Gamma_{rs}} \tag{A-10}$$

Therefore, u_0^c exists as a state function if

$$\frac{\partial}{\partial \Gamma_{pq}} \left[\Gamma_{ij} \frac{\partial T^{ij}}{\partial \Gamma_{mn}} \right] = \frac{\partial}{\partial \Gamma_{mn}} \left[\Gamma_{ij} \frac{\partial T^{ij}}{\partial \Gamma_{pq}} \right] \tag{A-11}$$

$$\text{or} \quad \frac{\partial \Gamma_{ij}}{\partial \Gamma_{pq}} \frac{\partial T^{ij}}{\partial \Gamma_{mn}} + \Gamma_{ij} \frac{\partial^2 T^{ij}}{\partial \Gamma_{pq} \partial \Gamma_{mn}} = \frac{\partial \Gamma_{ij}}{\partial \Gamma_{mn}} \frac{\partial T^{ij}}{\partial \Gamma_{pq}} + \Gamma_{ij} \frac{\partial^2 T^{ij}}{\partial \Gamma_{mn} \partial \Gamma_{pq}}$$

$$\delta_i^p \delta_j^q \frac{\partial T^{ij}}{\partial \Gamma_{mn}} + \Gamma_{ij} \frac{\partial^2 T^{ij}}{\partial \Gamma_{pq} \partial \Gamma_{mn}} = \delta_i^m \delta_j^n \frac{\partial T^{ij}}{\partial \Gamma_{pq}} + \Gamma_{ij} \frac{\partial^2 T^{ij}}{\partial \Gamma_{mn} \partial \Gamma_{pq}}$$

$$\text{so} \quad \frac{\partial T^{pq}}{\partial \Gamma_{mn}} + \Gamma_{ij} \frac{\partial^2 T^{ij}}{\partial \Gamma_{pq} \partial \Gamma_{mn}} = \frac{\partial T^{mn}}{\partial \Gamma_{pq}} + \Gamma_{ij} \frac{\partial^2 T^{ij}}{\partial \Gamma_{mn} \partial \Gamma_{pq}}$$

This may be written in the form

$$\Gamma_{ij}\left[\frac{\partial^2 T^{ij}}{\partial\Gamma_{pq}\partial\Gamma_{mn}} - \frac{\partial^2 T^{ij}}{\partial\Gamma_{mn}\partial\Gamma_{pq}}\right] = \frac{\partial T^{mn}}{\partial\Gamma_{pq}} - \frac{\partial T^{pq}}{\partial\Gamma_{mn}} \qquad \text{(A-12)}$$

However, the right-hand side of equation (A-12) must vanish identically, since it is necessary that

$$\frac{\partial T^{mn}}{\partial\Gamma_{pq}} = \frac{\partial T^{pq}}{\partial\Gamma_{mn}} \qquad \text{(A-13)}$$

in order that the strain energy density may exist as a state function. Therefore, (A-12) becomes

$$\Gamma_{ij}\left[\frac{\partial^2 T^{ij}}{\partial\Gamma_{pq}\partial\Gamma_{mn}} - \frac{\partial^2 T^{ij}}{\partial\Gamma_{mn}\partial\Gamma_{pq}}\right] = 0 \qquad \text{(A-14)}$$

which is generally satisfied, as $\Gamma_{ij} \neq 0$, if

$$\frac{\partial^2 T^{ij}}{\partial\Gamma_{pq}\partial\Gamma_{mn}} - \frac{\partial^2 T^{ij}}{\partial\Gamma_{mn}\partial\Gamma_{pq}} = 0$$

i.e.

$$\frac{\partial^2 T^{ij}}{\partial\Gamma_{pq}\partial\Gamma_{mn}} = \frac{\partial^2 T^{ij}}{\partial\Gamma_{mn}\partial\Gamma_{pq}} \qquad \text{(A-15)}$$

for any component T^{ij} of $\overline{\overline{T}}$.

But equation (A-15) is nothing more than the specification that $\overline{\overline{T}}$ must be a continuous function of $\overline{\overline{\Gamma}}$, a specification which is implicitly required by {2.2.-54} in any case. (The essential difference in equations (A-13) and (A-15) is of interest: the former represents a set of relations between the components of $\overline{\overline{T}}$ and $\overline{\overline{\Gamma}}$ while the latter is merely a specification of continuity -- if the former is assured to exist!)

Consequently, the conditions {2.2.-52-56} which guarantee the existence of U_0^S also guarantee the existence of U_0^C, or: *the complementary strain energy density exists as a state function if the strain energy density exists as a state function*; constitutive inversion is not required.

It is of interest to note that this statement is also true for the complementary strain energy density U_1^C, formulated in terms of the strain tensor and the TREFFTZ stress tensor as

$$U_1^C = \overline{\overline{S}} : \overline{\overline{E}} - U_0^S \tag{A-16}$$

as previously given by {1.3.-3}, where $\overline{\overline{E}}$ represents the independent tensor variable and $\overline{\overline{S}} = \overline{\overline{\gamma}}_0(\overline{\overline{E}})$. In this case, the condition [corresponding to (A-15)] which guarantees the existence of U_1^C as a state function, appears as

$$\frac{\partial^2 S^{ij}}{\partial E_{mn} \partial E_{pq}} = \frac{\partial^2 S^{ij}}{\partial E_{pq} \partial E_{mn}} \tag{A-17}$$

since it was required that

$$\frac{\partial S^{mn}}{\partial E_{pq}} = \frac{\partial S^{pq}}{\partial E_{mn}} \tag{A-18}$$

in order to guarantee the existence of U_0^S. It is observed here, that if $\overline{\overline{S}}$ is a linear function of $\overline{\overline{E}}$, such that

$$\overline{\overline{S}} = \overset{4}{\overline{H}} : \overline{\overline{E}} \tag{A-19}$$

where $$\overset{4}{\overline{H}} = H^{rsuv} \overline{G}_r \overline{G}_s \overline{G}_u \overline{G}_v$$

is a fourth-order tensor known as the HOOKE tensor, then a component $S^{ij} = \bar{G}^i \bar{G}^j : \bar{\bar{S}}$ is obtained as

$$S^{ij} = H^{ijuv} E_{uv}$$

and therefore, the strain energy density U_0^s is guaranteed to exist if, by (A-18),

$$\frac{\partial}{\partial E_{pq}} (H^{mnrs} E_{rs}) = \frac{\partial}{\partial E_{mn}} (H^{pqrs} E_{rs})$$

or, as $\overset{4}{H} \neq \overset{4}{H}(\bar{\bar{E}})$,

$$H^{mnrs} \delta^p_r \delta^q_s = H^{pqrs} \delta^m_r \delta^n_s$$

so

$$H^{mnpq} = H^{pqmn} \qquad \text{(A-20)}$$

This is often called the "third type of symmetry" of the HOOKE Tensor; the first type ($H^{mnpq} = H^{nmpq}$) and the second type ($H^{mnpq} = H^{mnqp}$) follow from the symmetry of $\bar{\bar{S}}$ and $\bar{\bar{E}}$, respectively. Noting that the first derivative of $\bar{\bar{S}}$ with respect to $\bar{\bar{E}}$ is a constant in this linear case, then (A-17) is satisfied identically, and U_1^c exists as the state function specified by (A-16).

It is noted that in the above developments, the complementary strain energy density is assured to exist as a function of $\bar{\bar{T}}$ (or $\bar{\bar{U}}$) or $\bar{\bar{E}}$ (or $\bar{\bar{C}}$) if the strain energy density exists as a state function. However, the complementary strain energy density does not necessarily exist as a function of $\bar{\bar{T}}$ or $\bar{\bar{S}}$. If a constitutive inversion is possible, then U_0^c (or W_0^c) or U_1^c (or Ω), respectively, exists as a function of the state of stress $\bar{\bar{T}}$ or $\bar{\bar{S}}$.

APPENDIX B

Mapping

DEFINITION of an Ordered Pair

An ordered pair is defined to be a pair of elements, "a" and "b", in which the order is fixed. Denoting such a pair as (a,b),

$$(a,b) \neq (b,a) \tag{B-1}$$

unless "a" and "b" are identical elements.

CARTESIAN PRODUCT

If $a \varepsilon A$ and $b \varepsilon B$, then the Cartesian product of set A and set B is the set of all ordered pairs, in which the first element (or pre-element) is a member of set A and the second element (or post-element) is a member of set B: i.e., the set of all ordered pairs (a,b). Denoting the Cartesian product of A and B as $A \times B$, then

$$A \times B = \{(a,b) \mid a \varepsilon A,\ b \varepsilon B\} \tag{B-2}$$

Since $A \times B$ is as defined above, and

$$B \times A = \{(b,a) \mid a \varepsilon A,\ b \varepsilon B\} \tag{B-3}$$

then

$$A \times B \neq B \times A \tag{B-4}$$

as might be anticipated from (B-1).

RELATION of Sets

A relation, R, of a set A to a set B is a propositional

function for an ordered pair of unspecified elements, such as (a,b) where $a \epsilon A$ and $b \epsilon B$, which yields a meaningful statement.

An alternate definition of a relation may be constructed in product form, as follows. A relation, R, of a set A to a set B is a subset of $A \times B$.

Thus, a relation of a set A to a set B is any rule which assigns to one or more elements of A, a correspondence with one or more elements of B.

From the product form of the definition, it may be observed that there exists a definite order in the relation R, as R is a subset of $A \times B$, and $A \times B \neq B \times A$, as established by (B-4).

DOMAIN and RANGE

The set of all elements of the set A which are found as pre-elements of ordered pairs in R, is called the domain of R. The set of all elements of the set B which are found as post-elements of ordered pairs in R, is called the range of R. Thus, if D denotes the domain, and R represents the range of R, then

$$D = \{a \mid a \epsilon A, (a,b) \epsilon R\} \qquad \text{(B-5)}$$

$$R = \{b \mid b \epsilon B, (a,b) \epsilon R\} \qquad \text{(B-5)}$$

MAPPING

A particular type of relation between sets has become fundamental for all aspects of mathematics. This type of relation is known as mapping, and has its primitive roots in the obscure LEIBNIZ definition of the properties of curves.

By definition, a mapping of a set A into a set B (or, "a function of A into B") is any rule of correspondence or propositional function, the variables of which are the elements of sets A and B, which assigns a unique element of B to each element of A. (Note that this definition does not require a unique element of B to be assigned to a unique element of A. This is a special type of mapping which will be discussed presently.)

Symbolically, the statement that "f maps A into B" appears as

$$f : A \rightarrow B \tag{B-7}$$

An alternate definition of mapping may be constructed in the form of a product specification, as follows. A mapping of set A into set B is any subset $A \times B$, in which every element $a \in A$ appears in only one ordered pair, and in which the set of ordered pairs of the function (or graphs of the function) are distinguished from the rule of correspondence itself.

The property which is characteristic of a mapping is the fact that any prescribed element of its domain corresponds to a uniquely-determined element of its range. For the mapping $f : A \rightarrow B$, the set A is called the domain of the function f, and B is called the co-domain of the function f. The range of the mapping is then defined by the elements in B which correspond to (or are assigned to) elements in A. Thus, if $a \in A$, then

$$R = \{f(a) \mid a \in A\} \tag{B-8}$$

defines the range of the mapping. This set, which is a subset of B, may be written as

$$R = f(A) \subset B \tag{B-9}$$

since the set of all f(a) defines the function f of {a}, and A={a}.

Since the unique element $b \in B$ is determined by the mapping

$$f : A \rightarrow B$$

and is given as

$$b = f(a) \tag{B-10}$$

then f(a) is called the image of $a \in A$. It follows that the image of a set $S \subset P$ is defined by the set of all images of $s \in S$. That is,

$$f(S) = f\{s \mid s \in S\}$$

so,

$$f(S) = \{f(s) \mid s \in S\} \tag{B-11}$$

This conclusion follows from (B-8) and (B-9) above, for the set $A \subset B$.

Again, it should be noted that, although a unique element $b \in B$ is determined by $f : A \rightarrow B$, it does not necessarily follow that a unique element $a \in A$ may be determined by the "reverse" mapping $F : B \rightarrow A$. That is, it is possible that more than one element $a \in A$ corresponds to the element $b \in B$.

PRE-IMAGES and IDENTITY MAPPING

Any domain element of the mapping $f : A \rightarrow B$ (i.e., any element $a \in A$) which has as its image the element $b \in B$, is called the pre-image of b. That is, just as b = f(a) is called the image of $a \in A$, so $a \in A$ is called the pre-image or "inverse image" of $b \in B$.

The pre-image of a set $S \subset P$ is the set of all pre-images of the elements of S, and is prescribed for the mapping

$$f : A \rightarrow S \tag{B-12}$$

as $$f^{-1}(S) = \{a \mid a \in A,\ f(a) \in S\} \tag{B-13}$$

It is possible, but not necessary, that

$$f^{-1}(S) = A \tag{B-14}$$

If (B-14) is true, a particular type of mapping exists which is more restrictive than the general type specified by (B-12).

The mapping which causes each element of a set A to produce itself as its image is called identity mapping. In this case, the operating function f is denoted as I, and the general expression

$$f : A \rightarrow B$$

becomes $$I : A \rightarrow A \tag{B-15}$$

Thus, $$I(a) = a \tag{B-16}$$

and the range R may be given as

$$R = \{I(a) \mid a \in A\} \tag{B-17}$$

which reduces, because of (B-16), to

$$R = \{a \mid a \in A\} = A \tag{B-18}$$

It also follows, from (B-12) and (B-13), that

$$I^{-1}(A) = \{a \mid a \in A,\ I(a) \in A\} \tag{B-19}$$

which reduces to

$$I^{-1}(A) = \{a \mid a \varepsilon A\} = A \qquad \text{(B-20)}$$

Finally,

$$I(A) = R = I^{-1}(A) = A \qquad \text{(B-21)}$$

INVERSE MAPPING

Considering the mapping of X into Y,

$$f : X \rightarrow Y \qquad \text{(B-22)}$$

that is, for $x \varepsilon X$ and $y \varepsilon Y$, considering the entire set of images

$$f(x) = y \qquad \text{(B-23)}$$

the question arises as to whether an inverse set of relations exists, such that

$$g(y) = x \qquad \text{(B-24)}$$

for all $x \varepsilon X$ and $y \varepsilon Y$, or in other words,

$$g : Y \rightarrow X \qquad \text{(B-25)}$$

That (B-24) follows from (B-23) [or (B-25) follows from (B-22] is obviously not ALWAYS true, since it is possible that a unique image, y, may be obtained from more than one element x, as noted previously. Consequently, the inversion (B-25) would require that the domain be specified by Y, some elements of which do not possess a unique image in the range g(Y).

Conversely, if it is possible to eschew a non-unique correspondence of the type described above, then it is seen that an inversion of (B-22) into (B-25) may be possible. (This will be considered below.)

ONTO MAPPING

A mapping, such as

$$f : A \rightarrow B$$

is said to be "onto", if every element $b \in B$ is the image of at least one element $a \in A$.

That is, B contains no elements which are not images of the elements of A.

ONE-TO-ONE MAPPING

A mapping, such as

$$f : A \rightarrow B$$

is said to be "one-to-one", if every image $b \in B$ has a unique pre-image $a \in A$.

From the foregoing considerations of onto and one-to-one mapping, it is observed that if the mapping

$$f : A \rightarrow B$$

is both onto and one-to-one, then

(i) every element $a \in A$ has a unique image $b \in B$, where $f(a) = b$

(ii) every element $b \in B$ has a unique pre-image $a \in A$

(iii) no elements of B exist which are not images of the elements of A

Consequently, it may be concluded that iff a mapping is both onto and one-to-one, then it is a reversible mapping (the inverse mapping exists).

In such a case,

$$f : X \rightarrow Y \tag{B-22}$$

admits the inverse mapping

$$g : Y \rightarrow X \tag{B-25}$$

or $$f^{-1} : Y \rightarrow X \qquad (f^{-1} \equiv g) \tag{B-26}$$

Or, in terms of the entire set of images,

$$f(x) = y \tag{B-23}$$

admits the set of pre-images

$$g(y) \equiv f^{-1}(y) = x \tag{B-27}$$

The onto and one-to-one correspondence rules may therefore be considered as the necessary and sufficient conditions for the reversibility of a mapping (or the existence of an inverse mapping).

EQUAL MAPPINGS

Two mappings,

$$f : A \rightarrow B \quad \text{and} \quad g : A \rightarrow B \tag{B-28}$$

are defined as equal mappings iff

$$f(a) = g(a) \tag{B-29}$$

for all $a \epsilon A$. Or, briefly, the two mappings are equal iff

$$f(A) = g(A) \tag{B-30}$$

COMPOSITE MAPPINGS

Two mappings, e.g.,

$$f : A \rightarrow B \quad \text{and} \quad g : B \rightarrow C \tag{B-31}$$

may be expressed as a composite mapping

$$(g \circ f) : A \rightarrow C \tag{B-32}$$

where the symbol $\circ$ defines an operator such that

$$(g \circ f)(a) \equiv g[f(a)] \tag{B-33}$$

Since, in general

$$g[f(a)] \neq f[g(a)] \tag{B-34}$$

then it will be assumed that, in general,

$$(g \circ f)(\ ) \neq (f \circ g)(\ ) \tag{B-35}$$

THEOREM I.

For any mapping,

$$f : A \rightarrow B \tag{B-36}$$

and the identity mapping,

i.e., $$I : B \rightarrow B \tag{B-37}$$

then $$(I \circ f)(\ ) = f(\ ) \tag{B-38}$$

PROOF: By definition of the composite operator,

$$(I \circ f)(a) \equiv I[f(a)] \tag{B-39}$$

but $$I[f(a)] = f(a) \tag{B-40}$$

as is required for identity mapping.

Therefore,

$$(I \circ f)(\ ) = f(\ ) \tag{B-41}$$

THEOREM 2.

If two mappings,

$$f : A \rightarrow B \quad \text{and} \quad g : B \rightarrow C \tag{B-42}$$

are both onto and one-to-one, then the composite mapping

$$(g \circ f) : A \rightarrow C \tag{B-43}$$

is both onto and one-to-one.

PROOF: To demonstrate that (B-43) is one-to-one if both members of (B-42) are one-to-one, consider two domain elements a_r, $a_s \in A$, such that $a_r \neq a_s$. Then

$$f(a_r) \neq f(a_s)$$

since f is one-to-one. However,

$$f(a_r),\ f(a_s) \in B$$

and therefore, are distinct domain elements of g. Thus,

$$g[f(a_r)] \neq g[f(a_s)] \tag{B-44}$$

as g is one-to-one. This implies that

$$(g \circ f)(a_r) \neq (g \circ f)(a_s) \tag{B-45}$$

by definition of the composition (B-31).

Therefore, $(g \circ f)(\ )$ is one-to-one.

To demonstrate that (B-43) is onto if both members of (B-42) are onto, it is necessary to establish that every element of C is the image of some element of A.

If $c \in C$, then there exists some $b \in B$, such that

$$g(b) = c$$

as g is onto.

Similarly, there exists some $a \varepsilon A$, such that

$$f(a) = b$$

as f is onto.

Substitution then yields

$$c = g(b) = g[f(a)] \tag{B-46}$$

But

$$g[f(a)] \equiv (g \circ f)(a) \tag{B-47}$$

by definition of the composite mapping.

Thus, $a \varepsilon A$ is a pre-image of $c \varepsilon C$, and since c is arbitrary, then every element of C is the image of some element $a \varepsilon A$, under $(g \circ f)(\ )$.

Therefore, the mapping $(g \circ f) : A \rightarrow C$ is onto mapping.

THEOREM 3.

If the mapping

$$f : A \rightarrow B \tag{B-48}$$

is one-to-one and onto, then

$$f^{-1} \circ f = I \tag{B-49}$$

defines the identity mapping on set A

PROOF: The mapping f possesses an inverse mapping f^{-1}, as f is both one-to-one and onto.

Now, $(f^{-1} \circ f)(a) = f^{-1}[f(a)]$

by definition of the composition. Also,

$$f^{-1}[f(a)] = a$$

by definition of the inverse f^{-1}.

Therefore, $$(f^{-1} \circ f)(a) = a \qquad \text{(B-50)}$$

Also, $$I(a) = a \qquad \text{(B-51)}$$

by definition of the identity mapping.

Therefore, $$(f^{-1} \circ f)(a) = I(a) \qquad \text{(B-52)}$$

This holds for all $a \varepsilon A$, since f is defined to be onto. Therefore,

$$(f^{-1} \circ f) = I$$

for the set A, or

$$(f^{-1} \circ f) : A \Longrightarrow I : A \rightarrow A \qquad \text{(B-53)}$$

ALGEBRAIC STRUCTURE OF SETS

In analysis, it frequently becomes necessary to combine elements of sets; this process cannot be treated adequately by mapping functions, since such functions may require that several elements fuse to form a new element (as in the case of composite mappings). This results, of course, in a loss of identity of the original elements, which defeats the purpose of the mapping in this particular case.

For problems of this type, the concept of a product, e.g. the Cartesian product, is quite useful.

Since the combination of two elements may be generally considered as an operation between two elements, combinations of this sort will be called "binary operations".

DEFINITION: A binary operation on a set A represents any rule or process which orders a unique element of A to each ordered pair of elements of A.

A binary operation on a set A is a mapping of $A \times A$ into A, or

$$* : A \times A \rightarrow A \tag{B-54}$$

That is, relation (B-54) states that the operator $*$ maps $A \times A$ into A.

If $a, b, c \in A$, relation (B-54) specifies the set of operations

$$a * b = c \tag{B-55}$$

thus demonstrating that $*$ is an operation on set A which assigns the element c to the ordered pair (a,b). Element c is then the image of (a,b) under $*$, and in strict mapping notation is represented as

$$c = * ((a,b)) \tag{B-56}$$

However, it is customary to denote the image of (a,b) under $*$ as $a * b$, as given by (B-55). This product notation will be employed in what follows.

COMMUTATIVE Operations

If, for all $a, b \in A$

$$a * b = b * a \tag{B-57}$$

in an operation $*$ on A, then the operation $*$ is said to be commutative.

ASSOCIATIVE Operations

If, for all $a, b, c \in A$

$$(a * b) * c = a * (b * c) \tag{B-58}$$

in an operation $*$ on A, then the operation $*$ is called associative.

DISTRIBUTIVE Operations

If, for all $a, b, c \in A$

$$a \circ (b * c) = (a \circ b) * (a \circ c) \tag{B-59}$$

in the operations $\circ$ and $*$ on A, then the operation $\circ$ is called pre-distributive (or "left-distributive") under $*$.

Analogously, $\circ$ is called post-distributive (or "right-distributive") under $*$, if

$$(a * b) \circ c = (a \circ c) * (b \circ c) \qquad \text{(B-60)}$$

for all a, b, c$\in$A, in the operations $\circ$ and $*$ on A.

If the operation $\circ$ is commutative, then either type of distributivity (above) implies the other. In this case, $\circ$ is simply said to be distributive under $*$.

Similarly, if for all a, b, c$\in$A,

$$a * (b \circ c) = (a * b) \circ (a * c) \qquad \text{(B-61)}$$

in the operations $*$ and $\circ$ on A, then the operation $*$ is called pre-distributive under $\circ$.

Analogously, $*$ is called post-distributive under $\circ$, if

$$(a \circ b) * c = (a * c) \circ (b * c) \qquad \text{(B-62)}$$

for all a, b, c$\in$A, in the operations $\circ$ and $*$ on A.

If the operation $*$ is commutative, then either type of distributivity (above) implies the other. In this case, $*$ is simply said to be distributive under $\circ$.

If $*$ and $\circ$ are both commutative, and if (B-59) and (B-61) [or (B-60) and (B-62)] hold simultaneously, then the former is the dual of the latter, and conversely.

As may be inferred from the above discussions, binary operations require the *a priori* prescription of the set on which the operations

are defined.

Thus, the definition of an algebraic structure may be constructed as follows.

The prescription of a set, together with the definition of one or more operations acting on this set, will be called an algebraic structure.

The general abstract treatment of algebraic structures was initiated by the German mathematician, Günther GRASSMANN in 1844.

GROUPS

One such algebraic structure, which was initiated by the French mathematician Evariste GALOIS and the Norwegian mathematician Niels Henrik ABEL, in connection with nonlinear algebraic equations, is called "group theory".

The properties of groups derive from generalizations of the fundamental laws of arithmetic.

In order to examine the theory of groups, it is expedient to define two quantifying symbols, as follows.

UNIVERSAL QUANTIFIER:

Denote by $\forall$, the universal quantifier, which is defined to mean "for all" or "for every", and is employed in an operational capacity.

e.g., $\forall\, x \in P$ signifies "for all elements x in the set P", or "for every element x in the set P"

$\forall\, a,b,c \in A \mid a * b = c$ signifies "for all elements a,b,c, in A, $a * b = c$".

EXISTENTIAL QUANTIFIER

Denote by $\exists$, the existential quantifier, which is defined to mean "for some" or "for at least one", or "there exists at least one", and is employed in an operational capacity similar to $\forall$.

e.g., $\exists x \in P \mid f(x) = a$ signifies "there exists at least one element x in the set P, for which f(x) = a"

$\exists P \mid Q \subset P$ signifies "for some set P, Q is a subset of P", or "there exists at least one set P such that P contains Q as a subset".

DEFINITION of a Group

The algebraic structure which consists of the prescribed set G and the defined operation $*$, written $(G, *)$, is called a group iff it satisfies the following conditions.

(a) G is closed with respect to $*$

i.e., $(a * b) \in G$ is unique for all $a, b \in G$

or $\forall a, b \in G \mid (a * b) \in G$ is unique

Thus, $\forall a, b, c \in G \mid (a * b) \neq (a * c);\ b \neq c$

(b) The operator $*$ is associative in G,

or $\forall a, b, c \in G \mid a * (b * c) = (a * b) * c$

(c) There exists an identity element i in G, such that the operation $*$ with i has no effect on any element $a \in G$,

or $\exists i \in G \mid i * a = a * i = a, \forall a \in G$

(d) There exists an inverse element a^{-1} in G, such that the operation $*$ of a^{-1} with $a \in G$ yields the identity element $i \in G$,

or $\exists a^{-1} \in G \mid a^{-1} * a = a * a^{-1} = i, \forall a \in G$

ABELIAN Group

If the algebraic structure (G, $*$) defines a group in which the operation $*$ is commutative, then (G, $*$) is called an ABELIAN Group. In this case, then,

$$\forall a, b \in G \mid a * b = b * a$$

THEOREM 4.

A group possesses a unique identity element.

THEOREM 5.

For all a, b, $c \in G$, $a * b = a * c$ implies $b = c$. Similarly, $b * a = c * a$ implies $b = c$. (This is known as the cancellation law for groups.)

THEOREM 6.

Each element of the set G in a group possesses a unique inverse.

THEOREM 7.

If a, x, $b \in G$ and (G, $*$) defines a group, then the equation $a * x = b$ possesses a unique solution in G.

NOTE: If a, $b \in G$ but $x \notin G$, then $a * x$ is undefined. Consequently, the relation $a * x = b$ cannot exist: thus, a, x, $b \in G$ if $a * x = b$.

SUBGROUPS

If $S \subset G$ where (S, $*$) denotes a group, then (S, $*$) is called a subgroup of (G, $*$) under the (identical) operation $*$.

THEOREM 8.

The structure (S, $*$) is a subgroup of (G, $*$) if $S \subset G$, such that

1. S is closed under the operation *

2. $a^{-1} \varepsilon S$, $\forall a \varepsilon S$ or, the element which is the inverse of $a \varepsilon S$ is also a member of the set S (for all $a \varepsilon S$).

SPECIAL GROUPS

1. TRANSFORMATION Groups

The set F of all one-to-one mappings of a set into another set is a group with respect to an operation *, if the following conditions are satisfied.

(a) If $f, g \varepsilon F$ then $(f * g) \varepsilon F$ is unique

(b) If $f \varepsilon F$, then $f * i = i * f = f$, $i \varepsilon F$

(c) If $f \varepsilon F$, then $f^{-1} \varepsilon F$

A structure (F, *) which satisfies these conditions is called a transformation group. It is observed that $\forall f \varepsilon F$ must be onto transformations in order that condition (c) be satisfied.

Since the operation * is not generally commutative, it is anticipated that the group (F, *) will not generally be Abelian.

2. PERMUTATION Groups

A one-to-one mapping of a set onto itself is called a permutation, and any group of such mappings is known as a permutation group.

RINGS

A consideration of two binary operations on the elements of a set leads to the concept of a ring.

DEFINITION of a Ring

The algebraic structure which consists of a set R and the two binary operations * and ∘, written (R, *, ∘), such that

(a) $(R, *)$ is an Abelian group

(b) $\circ$ is associative in R

(c) $\circ$ is distributive (both pre- and post-distributive) under $*$ in R

is called a Ring.

The concept of a ring exhibits the absence of three properties of the algebraic structure $(R, \circ)$, namely

(a) commutativity of the operator $\circ$

(b) existence of an identity element $u \in R$ under $\circ$

(c) existence of an inverse element $r^{-1} \in R$, $\forall r \in R$ under $\circ$

Any algebraic structure $(R, \circ)$ which satisfies one or more of these three conditions yields a ring $(R, *, \circ)$ which is of a restricted nature.

A ring which satisfies (a) above, i.e., a ring in which the operation $\circ$ is commutative in R, is called a commutative ring.

A ring which satisfies (b) above, i.e., a ring in which an identity element $u \in R$ exists under $\circ$, is called a unitary ring. (The identity element is denoted as u and is called "unity" to avoid confusion with $i \in R$, which is the identity element under $*$).

Thus, in a unitary ring,

$$a \circ u = u \circ a = a , \quad \forall a \in R \tag{B-63}$$

The unitary ring then possesses two identity elements, unity $u \in R$ and identity $i \in R$, and in general, $u \neq i$, as may be observed from the following theorems.

THEOREM 9.

In any ring $(R, *, \circ)$

$$\left.\begin{aligned} a * i &= a \\ a \circ i &= i \end{aligned}\right\} \forall a \epsilon R$$

and

THEOREM 10.

In any unitary ring $(R, *, \circ)$, which contains at least two elements in R,

$$i \neq u$$

An important algebraic structure is a ring which is commutative and unitary and which possesses one additional property: the property of integral domain.

DEFINITION of Integral Domain

The commutative unitary ring $(R, *, \circ)$ which contains more than two elements in R constitutes an integral domain if it possesses the following property.

For any two elements $a, b \epsilon R$, such that

$$a \circ b = i$$

the condition $a = i$ or $b = i$ is valid.

THEOREM 11.

If $a, b \epsilon R$ in the ring $(R, *, \circ)$, then

$$(a \circ b)^{-1} = a \circ b^{-1} \qquad \text{(B-64)}$$

THEOREM 12.

A commutative unitary ring which contains at least three elements is an integral domain iff, for any elements a, b, c of the

ring, such that $a \neq i$, the relation

$$a \circ b = a \circ c$$

implies that $b = c$

THEOREM 13. (CONVERSE OF THEOREM 12.)

In an integral domain $(R, *, \circ)$ containing elements $a, b, c \in R$ in which $a \neq i$, the relation

$$a \circ b = a \circ c$$

implies that $b = c$

Finally, the last absent property of a ring $(R, *, \circ)$, namely the existence of inverse elements $r^{-1} \in R$ $(\forall r \in R)$ under $\circ$, leads to the concept of a "field".

DEFINITION of a Field

A commutative unitary ring $(R, *, \circ)$ containing at least two elements is said to be a field if every element $a \in R$, except i, has an inverse under $\circ$. That is, if $a^{-1} \in R$; $\forall a \in R$ $(a \neq i)$ under $\circ$.

Therefore, in summary, a field is an algebraic structure $(F, *, \circ)$ containing at least two elements $a, b \in F$, iff

$(F, *)$	$(F, \circ)$
(a) is associative	(a) is associative
(b) has an identity, i	(b) has a unity, u
(c) has $a^{-1} \in F$, $\forall a \in F$	(c) has $a^{-1} \in F$, $\forall a \in F$, $a \neq i$
(d) is commutative	(d) is commutative

where $\circ$ is distributive (pre- and post-distributive) under $\circ$.

THEOREM 14.

Every field is an integral domain.

ALGEBRAIC STRUCTURE OF MAPPINGS

Mapping functions permit a comparison between two sets. Thus, the question naturally arises as to whether the concept of mapping may be extended to sets in combination with operations, such as groups, rings and fields.

It is intuitively evident that any comparison between any objects which consist of component parts, deals with the "size" and "shape" of these composite objects. Such objects are then said to be "the same" or "equivalent" only if the "size" measures and the "shape" measures are identical for both (or all) objects under consideration.

Algebraic structures are compared in a manner analogous to the above. Thus, the "size" of two sets, considered as the objects, implies a measure of their elements, and the "shape" implies their binary operations.

DEFINITION of a MORPHISM

A mapping

$$f : A \rightarrow B \tag{B-65}$$

of a set A into a set B, which preserves prescribed operations, is called a morphism.

HOMOMORPHISM

Two algebraic structures A and B are said to be homomorphic, if there exists an operation-preserving mapping

$$f : A \rightarrow B \tag{B-65}$$

which is not necessarily one-to-one or onto. The homomorphism of A and B is then denoted by $A \doteq B$ (sometimes $A \sim B$).

Thus, in homomorphism:

(a) every element $b \varepsilon B$ appears at least once as an image

(b) every operation between a, $c \varepsilon A$ holds for the corresponding image elements b, $d \varepsilon B$.

Very explicitly, in terms of $*$ and $\circ$, a homomorphism may be defined as follows, employing the symbolism of (B-65).

A homomorphism with respect to $*$ and $\circ$ of a set A into a set B, denoted as "(A, $*$; B, $\circ$) - homomorphism", is a transformation A under $*$ into B under $\circ$ such that, for every a, $c \varepsilon A$ and every f(a), $f(c) \varepsilon B$,

$$f(a * c) = f(a) \circ f(c) \qquad \text{(B-66)}$$

It is observed that, under homomorphic mapping, the identity maps into the identity, and the inverses map into inverses.

Furthermore, the homomorphic image of a group is itself a group, and the homomorphic image of a ring is itself a ring.

ENDOMORPHISM

If a homomorphism maps a set A into itself,

$$f : A \rightarrow A$$

then the mapping is endomorphic, and the relationship

$$f : A \rightarrow B, \quad B \subseteq A \qquad \text{(B-67)}$$

defines an endomorphism. (Recall that $B \subseteq A$ defines a subset, not a proper subset, relation).

AUTOMORPHISM

If an endomorphism is one-to-one, then it is known as an automorphism.

ISOMORPHISM

If a homomorphism

$$f : A \rightarrow B$$

is one-to-one and onto, then it is called an isomorphism.

That is, a reversible homomorphism, sometimes written as

$$f : A \longleftrightarrow B \tag{B-68}$$

to indicate that $f : A \rightarrow B$

and $f^{-1} : B \rightarrow A$

is called an isomorphism.

This may be expressed in terms of $*$ and $\circ$, as shown above for homomorphism, if so desired.

When the $(A, *; B, \circ)$ - homomorphism is reversible, it is known as an $(A, *; B, \circ)$ - isomorphism, and is usually written as

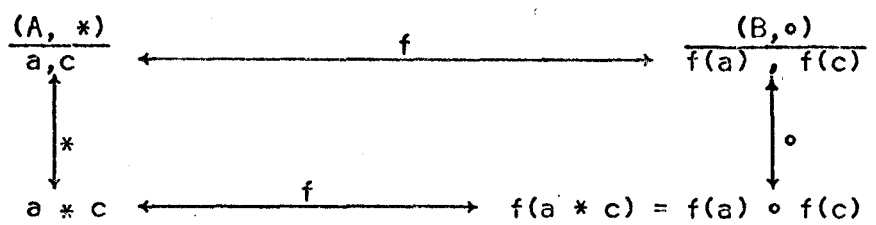
$$(A, *) \cong (B, \circ) \tag{B-69}$$

or

$$A \cong B \tag{B-70}$$

thus denoting that $(A, *)$ and $(B, \circ)$ are isomorphic.

The following diagram of isomorphic transformation is quite beneficial:



The diagram for homomorphism would be exactly as above, except that the operations f(a) and f(c) would not be reversible (as they are above, for isomorphism).

i.e. for homomorphism, $f(a) = b$

for isomorphism, $f(a) = b$, $f^{-1}(b) = a$

as shown on the diagram by reversible directions for f(a), f(c).

The operators $*$ and $\circ$ are always "reversible", as

$* ((a,c)) \Longleftrightarrow a * c$, etc.

THEOREM 15.

If (G, $*$) represents a group which is isomorphic to the algebraic structure (S, $\circ$), then (S, $\circ$) is also a group.

THEOREM 16.

If (S, $\circ$) represents a group, then the set F of all functions from some set A to (S, $\circ$), constituting an algebraic structure (F, $\circ$) is also a group.

LINEAR TRANSFORMATIONS

If the algebraic structures discussed above represent directed spaces, then the morphisms are called linear transformations, linear operators, or transformation tensors.

TOPOLOGY OF MAPPING

Geometry may be considered to be a study of equivalence relations on the set of all geometric figures in some space. The equivalence relation in each case is determined by the allowable transformations: such transformations may involve shrinking, flexure, stretching and twisting, and will produce two topologically equivalent

geometric transformations. In the event that the transformations prescribe rigid displacement, the relations are congruences.

It becomes necessary to formulate the topological equivalence remote from the descriptive point of view. In order to institute such a formulation, the concept of a neighbourhood is useful.

DEFINITION of a Neighbourhood (Euclidian Plane)

Let P be a point in the Euclidian Plane and let d be any positive real number. Then the set of all points in the plane, such that the distance to any point from P is less than d, is called a d-neighbourhood of P.

Clearly, the concept of a d-neighbourhood facilitates the establishment of a precise notion of "nearness", through the choice of the positive number d.

DEFINITION of the Continuity of Mapping

A mapping

$$f : A \rightarrow B$$

is continuous at a point $P \in A$ if, given any d-neighbourhood D of P, there exists a d*-neighbourhood D* of P, such that $D^* \subset D$. Then f is said to be continuous iff it is continuous $\forall P \in A$.

In terms of "nearness" or "proximity", a function is continuous at a point P, if regardless of how near one approaches f(P) in the range, all points of some disc of centre P in the domain possess images which are sufficiently "near" to f(P).

DEFINITION of Homeomorphic Mapping

If the mapping

$$f : A \rightarrow B$$

is one-to-one and onto, i.e., if

$$f : A \longleftrightarrow B$$

such that both f and f^{-1} are continuous, then the mapping is called a homeomorphism. The sets A and B are then known as homeomorphic or topologically equivalent.

It is easily demonstrated that the set of all homeomorphisms of a plane forms a group with respect to mapping composition. However, such a development requires as a foundation, the concept of a d-neighbourhood, and the development is therefore restricted to Euclidian geometry.

APPENDIX C

Bibliography

NOTE: For a more complete bibliography of the earlier works, see the papers by ORAVAS and McLEAN (this author) in 1966.

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HISTORICAL NOTES

The Legendre Transformation (page 38)

The contact transformation known as the Legendre Transformation should be denoted as the "Euler-Legendre Transformation", as Leonhard EULER presented the first formal development in 1732. The original concept of such a transformation is due to Gottfried-Wilhelm Freiherr von LEIBNIZ, who applied it to the discussion of a circle in a letter to HUYGENS (November 6, 1673) and to the discussion of general curves in a letter to H. OLDENBURG (July 15, 1674).

The Virtual Work Principle (page 5)

Although DESCARTES was the first to sketch the analytic form of the Virtual Work Principle, Johann BERNOULLI was the first to provide a true analytical formulation thereof. BERNOULLI wrote two important letters to Pierre VARIGNON: the first of these was written August 12, 1714, regarding the work of RENAU and particularly the latter's book on ship mechanics. In this letter, the word "energy" was first employed to denote the product of force and displacement. The second letter, written February 26, 1715, contained the first analytical formulation of the Virtual Work Principle (this date was later misquoted in Varignon's book, "Nouvelle Mécanique" -- published posthumously 1725, Paris -- as January 26, 1717).