BENDING OF SANDWICH BEAMS AND COLUMNS

BENDING OF SANDWICH BEAMS AND COLUMNS

BY

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SCOPE AND CONTENTS:

A detailed synopsis of the state-of-the-art in the field of the Structural Analysis of sandwich beams is presented. Deficiencies, inaccuracies, lack of clarity, and the imposition of unnecessary assumptions of behaviour found in the related bibliography are presented in a comparative fashion. A method of analysis with obvious advantages over the others studied in this thesis is derived, and its use is suggested. The presentation of all methods of analysis is made under the most general cases of dimensions and loadings to make them as applicable as possible to the common cases encountered in sandwich components for the building industry.

Experimental work carried out on several materials with some potential to be used in sandwich members for buildings and the tests carried out on some sandwich beams and beam-columns are reported.

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NOTATION

A	Area factor in Shear Stiffness
a	Initial bowing at mid-height
a_n	Fourier Coefficient
B	General representation of the abscissa of a boundary
b	Width of the Section
b_{1n}, b_{2n}	Fourier Coefficients
C	Constant of integration; also point of application of concentrated forces and/or moments along the span
c	Core thickness
c_{1n}, c_{2n}	Fourier coefficients
D	Operator indicating derivative with respect to the abscissa; also point indicating the location of the Reference Level
d	Distance between centroids of skins, also differential operator
d_1, d_2	Distances from Reference Level to the centroids of the top and bottom skins respectively
d_{1n}, d_{2n}	Fourier Coefficients
E_c	Modulus of Elasticity of the core material
E_{comp}	Modulus of elasticity under compressive stresses

E_f Modulus of Elasticity of the skins, when they are equal to each other

E_{Ten} Modulus of elasticity under tensile stresses

E_1, E_2 Modulus of Elasticity of top and bottom skins respectively

(EI) Total Bending Stiffness of the section

$(EI)_c$ Bending stiffness of the core

$(EI)_d$ Part of the total bending stiffness consisting in the product of the areas of the skins times their corresponding modulus of elasticity times the squares of the distances from their centroids to the Reference Level

$(EI)_f$ Bending stiffness of the skins with respect to their own centroids

$(EI)_{f1}$, $(EI)_{f2}$ Bending stiffness of the top and bottom skin respectively with respect to their own centroid

e Eccentricity of the end thrusts, measured with respect to the Reference Level

e_{1n}, e_{2n} Fourier Coefficients

F Net axial force in each skin, the direct effect of end thrusts excluded, also, general function developable in Fourier series

F_1, F_2,... Functions of the parameters λ and g

f Function defined in a finite interval

f_{1n}, f_{2n} Fourier coefficients

G	Shear modulus of the core material
g	Parameter
g_1, g_2	Distances from the neutral axis to the interfaces of the core with the top and bottom skins respectively.
H	Point indicating the location of the neutral axis
h	Total thickness
i, j	Integers
L	Span
L_i	Portion of the span between concentrated forces and/or moments and other such points or supports
ℓ	Subscript indicating portion to the left; also, span (Section 8.2 only)
M	Total bending moment
M'	Total shear force
M_d	Portion of the total bending moment consisting in the net axial force in one skin (effect of end thrusts excluded) multiplied by the distance between the centroids of the skins
M_s	Portion of the total bending moment, taken by the skins when bending with respect to their own centroids

M_{f1}, M_{f2}	Portion of the total bending moment taken by skin 1 and skin 2 respectively when bending with respect to their own centroids
M_1, M_2	Portions of the total bending moment as defined in Section 4.2
M_1', M_2'	Portions of the total shear force as defined in Section 4.2
M_ℓ, M_r	Bending moment along the portions ℓ and r of the span, respectively
M_ℓ', M_r'	Shear force along the portions ℓ and r of the span, respectively
n	Integer; also, summation counter
P	End thrust
P_c	Critical loading
q	Distributed loading
q_n	Fourier coefficient
q_1, q_2	Portions of the total distributed loading, as defined in Section 4.2
r	Factor defined in Section 8.2; also, subscript indicating "portion to the right"
s	Short writing for $\frac{n\pi}{L}$
t	Skins thickness, when they are equal to each other
t_1, t_2	Thicknesses of top and bottom skins respectively
U	Total Strain Energy

U_i	Portion of the total strain energy as defined in each section
u_1, u_2	Net horizontal displacements of the centroids of the top and bottom skins respectively
V	Loss of potential energy of the applied loads
V_i	Portion of the loss of potential energy of the applied loads as defined in each section
v	Vertical deflection
v_ℓ, v_r	Vertical deflection for the portions ℓ and r of the span, respectively
v_1, v_2	Portions of the total vertical deflection as defined in each section
W	Concentrated load
w_1, w_2	Net horizontal displacements of the top and bottom skin respectively with respect to the net displacement of the whole section when end thrusts act.
x	Abscissas
α	Constant defined by $\sqrt{(EI)AG/(EI)_d(EI)_f}$
β, β_1, β_2	Constant Arguments
γ	Shear deformation of the core
δ	Variational operator
λ	Parameter
ρ	Radius of curvature

σ^{ult}_{comp}	Ultimate compressive strength
σ^{ult}_{Tens}	Ultimate tensile strength
τ_c	Shear stress in the core
μ	Constant defined by $[(EI)/(EI)_d]^2 AG$
μ_w	Constant defined by $[(\alpha L/2)/(\alpha L/2 - Tanh\ \alpha L/2)]\mu$
Ω	Total Potential Energy
∂	Operator used in partial differentiation

DEFINITIONS

Very thin skins	Those with negligible stiffness in bending about their own axes, and sufficiently thin for d to be equated with c
Thin skins	Those with negligible stiffness in bending about their own axes, but not so thin that d and c may be equated
Thick skins	Those with significant stiffness in bending about their own axes, and too thick for d and c to be equated
Reference Level	Centroid of the elastically transformed section
Neutral Axis	Location of the points in the section with no horizontal displacements

CHAPTER I

INTRODUCTION

1.1 FOREWORD

Sandwich Construction has been very successfully used in the field of Structural Aeronautics. This concept satisfies many of the requirements for the production of industrialized components of buildings. The most important ones are structural as well as architectural efficiency (its components carry out other functions besides the structural one). Besides, services may be embedded in the panels in the factory. In addition, transportation is cheaper owing to the lower weights involved, and so are other structural components as dead loads are smaller than in buildings with monolitic panels.

Because of the obvious merits of using sandwich types of building components their applicability to building design has been studied more extensively in recent years. However, not too much of the previous research related to this subject may be readily applied to building design because of the lack of generality with respect to support conditions, loads, dimensions of the sections and combinations of materials. In addition, many of the relevant references either contain inconsistencies, or unjustified and often unnecessary assumptions or are not presented in a clear manner. One of the main goals of this work is to present some of the published methods of analysis in a more clear, general and rational way. In a comparative fashion, the assumptions of behaviour made by each author and the implication of these assumptions will be detailed.

Because of the above mentioned lack of generality a method of analysis was formulated to provide a more general form of analysis than others to be found in the related bibliography. This generalized form of analysis was obtained by two different methods by simply working through the mathematics on the most general pattern of deformations and taking care to avoid the inclusion of arbitrary assumptions of behaviour. It is suggested here that this method is simpler for the designer to use because it only employs concepts of ordinary beam theory and does not require a great deal of interpretation. Correspondingly, the variables dealt with are deflections and moments, which are of immediate practical use as compared with other methods where the variables cannot be interpreted directly with respect to physical behaviour. This method described above was obtained from generalizations and corrections to the work by Hartsock(12)*, and is presented in detail in Section 4.4 and Chapter VII.

In this study some initial investigations were done regarding the use of new materials in sandwich panel construction. To establish the potential of some of these new materials, theoretical and experimental analyses of their properties as well as

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* References are listed at the end.

trials of fabrication methods were included. This portion of the research program is described in Appendix A. The results of some tests of sandwich beams and sandwich beam-columns are also presented.

For the most part this thesis presents a detailed theoretical study of only the basic flexural behaviour of sandwich construction. Aside from the study of bending of sandwich beams, there is no suggestion that this is an exhaustive treatise on the subject of sandwich construction. Such aspects as wrinkling instability, thermal warp, ultimate strength and two-directional bending are not included. It is hoped the material presented in this thesis will aid designers understanding and analysing the basic behaviour of sandwich construction. In considering some of the above mentioned aspects, an understanding of the basic behaviour will undoubtedly be a necessary first step.

1.2 LITERATURE REVIEW

The earliest reference of the concept of sandwich construction is traced back to 1849 by Allen[1], but the first relatively spectacular success of the idea was in the design of the *Mosquito Bomber* in 1943[1,21]. Since then the concept has been studied extensively mainly by aeronautical investigators and for this reason most of the literature available on the subject deals with sandwich members with *thin* or *very thin* skins*, which

*According to the common use of the term (see Allen[1], for instance), a sandwich section is said to have *very thin* skins when (1) The bending stiffness of the skins with respect to their own centroids is negligible as compared with the total stiffness of the section and (2) The distance between those centroids may be approximately equalled to the core thickness. A sandwich in which the approximation (1) is valid but where approximation (2) may not be made without introducing considerable errors is said to have *thin* skins. Finally, in a sandwich with *thick* skins, neither approximation (1) nor (2) is applicable.

are most commonly used in airplane design.

Several of the references [1,12,18] indicate the limitations for application of the proposed methods of analysis but others do not, even though the assumptions used imply the analysis will be accurate only for cases of thin or very thin skins. A typical example is the applicability of the Navier-Bernouilli principle, which says that *originally plane sections will remain plane after deformation.* It will be shown in Section 4.6 that the application of such a principle is justifiable only for cases with thin or very thin skins or where the core has a very high shear stiffness. In addition, it will be shown that the particular formulas as presented by the authors using this assumption are applicable only to cases with symmetric conditions of loading and support. Again, this limitation has not been clearly defined. From the list of investigators applying this principle to facilitate use of the Equivalent I-Beam method, using *elastic transformation of the section* (see Chapter III), the following were consulted for this work:

Allen[1], who is the only author warning about the non-applicability of his formula for cases with non-symmetric loading and support conditions.

Hughes and Wajda[14], who include final formulas for particular cases of loading and support and for symmetric sections (those having identical skins). The formulas as presented in the paper by the latter authors have some printing errors, unclear treatment and inconsistent conclusions as it will be shown in Section 2.2 in this work.

Darvas uses the above mentioned method in two papers. One of them [(6)] included the effect of shear deformation in the core while the other [(5)] neglected this effect using the assumption that *strips of the whole sandwich originally plane and perpendicular to the Neutral Axis of the section* remain both plane and perpendicular to the Neutral Axis after deformation.*

Smolenski and Krokosky[(23)] use the same method to consider shear deformations of the core while Leabu[(17)] neglects this aspect in his study of thermal warp problems as do Ellis and Cummings[(9)] in their analysis of concrete sandwich panels**. The U.S. Forest Products Laboratory (FPL)[(24)] also uses this equivalent I-beam method. Even though they apply the results to the study of sandwich panels with thick plywood skins, they neglect the local bending stiffness of the skins in the expression for the total stiffness of the section.

The way in which Hoff's solution was originally presented in reference (13) makes it applicable only to the case of very thin skins. This is contrary to what Hoff states and results from the study of the assumptions made in the derivations of

*The Neutral Axis of the section as defined by Darvas in his article and in other papers by some authors coincides with the "Reference Level" or "Centroidal Axis" to be defined in Section 2.3 of this thesis. The definition of Neutral Axis to be used in here will be given in Section 2.7.

**The words panel and beam will be used interchangeably in here for members with unidirectional bending. The word panel is used by some authors (1,3) in the sense of plate (two-directional bending).

the solution. His original solution was presented for a case with a symmetrical section under a particular system of loads.

O'Dell and Graham[19] when assuming that the whole shear force is taken by the core, imply that the results they obtain are applicable only to cases with very thin skins. Hoff used a strain energy approach while O'Dell and Graham used the equivalent I-beam method.

Concerning thick skins, Hartsock[12] has presented a method to solve the problem for a sandwich beam with a generality which is sufficient for many problems. He considers dissimilar skins and his solution is good only for the cases of a simply supported sandwich beam with mid-span concentrated load and uniformly distributed loading. However, he does not identify this limitation. The generalised methods presented in section 4.4 and Chapter VIII are based on Hartsock's basic idea of splitting the total applied moment in two parts but the treatment, form and applicability of the solution in those Sections are different from those presented in his paper.

Allen[1] presents three methods of analysis. The first one, which is applicable only to problems with thin and very thin skins, uses the method of the equivalent I-beam with allowance for shear deformations of the core and is applicable to symmetrical cases of loading and support. In the second method he used the ordinary beam theory in a way which lacks clarity, even though it coincides with the exact *solution** for cases with symmetrical loads and support conditions as applied to three particular cases of loading. The third method uses Strain Energy principles as applied to a deformed section in which some arbitrary conditions of behaviour are imposed to obtain a solution which is again applicable only to some cases of symmetrical loads and support. He presented this third method for a section with identical skins but in his study of plates in the same book he includes formulae with allowance for sections with dissimilar skins. Being the most general case, these formulae for plates were studied to generalize his theory for beams.

Hoff's approach mentioned above uses the principle of the total potential Energy as a stationary function by using as variables the horizontal displacements of one of the skins and the vertical deflection. His solution was obtained for a cantilever having identical skins and a concentrated load at the free end. Contrary to his claims it is valid only for sections with very thin skins. Hughes and Wajda(14) give the same final result as obtained by Hoff with some errors, as commented earlier. Allen's Strain Energy method arbitrarily assumes that the ratio of the shear deformation of the core to the slope of the member is constant and so his solution depends upon

* A solution will be said to be *exact* in this work when it is the result of applying valid mathematical operations and structural principles to a problem having been set-up with no arbitrary assumptions of behaviour and in which only the factors previously proven to be numerically negligible are discarded.

the determination of a function (the vertical deflection) and a parameter (the ratio mentioned above) while Hoff's solution depends upon two functions (the ones described above). The change of a function to a parameter as suggested by Allen should facilitate the solution of the mathematical problem. However, as will be shown, it does not happen in this case.

Fisher[10] sets up the expression for the strain energy stored in the beam as a function of the externally applied bending moments and shear forces. Then he applies Castigliano's first theorem to find the deflection at mid-span for three particular systems of load; mid-span concentrated load, two equal concentrated loads at the thirds of the span*, and uniformly distributed load. The main assumption in his derivations is that the whole applied moment is taken by the net elongation and contraction of the skins. This is equivalent to neglecting the local bending stiffness of the skins and so his results are applicable only to members having thin skins. His idea of applying strain energy principles once the strains have been expressed as functions of the applied moments and shear forces was used to solve the much more general problem described in Chapter VII.

Several authors, for example Doherthy et al[7] and Benjamin[3] use an approach originally derived by H.W. March of

* A rigorous reworking of this formula revealed that it was derived for two equal quarter point loads, in contradiction to Fisher's claim.

the Forest Products Laboratory of the United States. According to their description of the method, it seems to coincide with the one developed by Fisher, but no verification of this was made. The paper by Doherthy et al gives an unrecognisable value for the shear stiffness of the section while the paper by Fisher gives (in his Fig. 1) a distribution of shear stresses in the core which is inconsistent with the assumed distribution of axial stresses in the same figure. Both apply formulae obtained for symmetric sections to members having non-identical skins. The fact that both papers have inconsistencies on the same point may be easily understood from the fact that each includes the other as its first reference. Their formulae are commented on in more detail in section 3.4.

Pfeifer and Hanson[20] study the limiting cases in which the core is either very stiff and the sandwich beam works as an I-beam or the core is very weak and the skins act independently. The conclusion from the experimental results is that the behaviour of sandwich beams, either with or without shear connectors, lies somewhere between those two limiting cases.

Only a few of the above mentioned papers derive or give formulae applicable to sections with different skins. Hartsock[12] and Darvas[5] include this aspect for sandwich beams but the latter's work is not applicable for cases with thick skins. Allen[1] includes this feature in his study of plates. Doherthy et al[7] and Fisher[10] also include non-symmetric sections but they treat them in an unclear way as mentioned in more detail in Section 3.4.

Hansen and Curtis[4,11] give some consideration to ultimate strength of concrete sandwich panels. Pfeifer and Hanson[20] and Hummel[15] study the effect of shear connectors only from the experimental view point. Dundrová et al[8] and Allen[1] study the two-directional bending with the use of strain energy methods while Darvas[6] does it by using transformed areas in the two directions. Skattum[22] also uses strain energy in his dynamic analysis of coupled shear walls and applies the results to the problem of a sandwich beam. His work is not analysed in this thesis.

1.3 SCOPE AND CONTENTS

After the general view of the state-of-the-art made in the previous section, and due to the many inconsistencies, lack of generality and the many unjustified assumptions found in the related bibliography, the following Chapters present a detailed analysis of the methods that have been used. Many of those have been generalised to cover wider possibilities of loadings and dimensions of the elements of the section than was originally the case. Comparisons between them are also made. The detailed methods are presented as follows.

Chapter II presents the elements of the theory of bending of sandwich beams and defines the problem to be studied in this thesis. It also includes a list of the general assumptions related to the defined problem to be solved in this work.

Chapter III includes the methods which are based on the assumption that plane sections remain plane after deformation, with the effect of neglecting the shear deformation of the core being presented first. The method having allowance made for the shear deformation of the core is then presented and the effect of the various shear stiffnesses proposed by several authors is then included in a comparative manner.

Chapter IV contains Allen's and Hartsock's applications of ordinary beam theory to the elements of the section. The latter was generalized to accommodate more general conditions of loading and support. Also the set up of the problem is presented in a different, but still equivalent way. Some comments about the problem in defining boundary conditions are made for both methods.

Chapter V makes use of the principle that the *Total Potential Energy stored in a deflected member is a stationary function.* This is a generalization of the work by Hoff to cover wider possibilities of loading (especially non-symmetric loading systems), support conditions and dimensions of the elements of the section. The original work by Hoff is commented on and compared with the more general findings obtained. Variational calculus and the Rayleigh-Ritz method are presented as possibilities for solution of the mathematical problem set-up as indicated above. Some comments are made about boundary conditions.

Chapter VI presents Allen's strain energy method by following the derivations he made for sandwich plates rather than sandwich beams because in his chapter on sandwich beams he considered only cases with identical skins. The implications of Allen's assumptions of behaviour are discussed and three possibilities to solve the mathematical problem are proposed.

Chapter VII contains a method of analysis based on the Application of the *Principle of Least Work* to the basic pattern of deformations described in Hartsock's generalised solution (Section 4.4). This method is especially recommended owing to the simplicity of the derivations and its ready application to the most general cases of loading, support and dimensions of the section.

Chapter VIII contains some comments about the formulas compared by Hughes and Wajda[14] and also provides a list of conclusions and recommendations for analysis and future research.

Appendix A contains some analyses made on materials not too commonly used in sandwich construction for the building industry, and it includes the results and some theoretical considerations concerning some tests made on them to determine some of their structural properties. Appendix B presents the description and results of some tests performed on sandwich panels with thick skins under transverse and beam column loadings.

CHAPTER II

PRINCIPLES

2.1 INTRODUCTION

A sandwich beam is a flexural member composed of three elements which are bonded together. The two external ones are *skins* which are made of very strong materials to take most of the bending stresses due to their very high moment of inertia created by holding them apart from each other by means of a much weaker *core*. The latter element is usually made of a very light material in order to decrease the self-weight load. Materials with high thermal insulation properties are usually preferred for both, aeronautical and architectural purposes for obvious reasons.

Fig. 2.1 shows a section of a sandwich beam having two skins of thicknesses t_1 and t_2 respectively and a core with a thickness c. The width of the section is b and the distance d joining the centroids of the skins is given by

$$d = c + \frac{t_1 + t_2}{2}$$

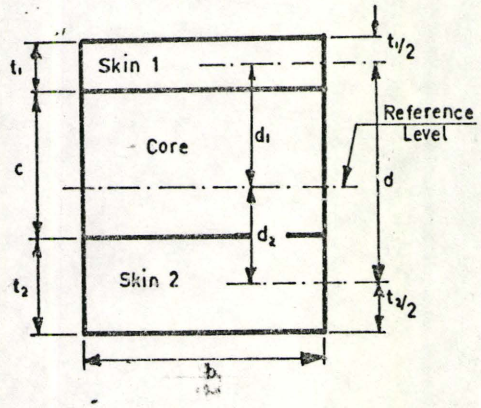

FIG. 2.1

Section of the Sandwich Beam

The main role of the core, in addition to holding the skins apart from each other, consists of taking most of the shear forces. A structure constructed this way is very effective, but the introduction of a weak material in the core may not be made without penalty. In fact, the shear deformation of the core may be very large and therefore must be considered.

2.2 DEFINITION OF THE PROBLEM AND BASIC ASSUMPTIONS

The problem to be studied in this work is the static structural behaviour of sandwich panels subjected to uni-directional bending. The skins are in general made of different materials and their thicknesses are in general different and not necessarily small compared with the core thickness.

The general assumptions used throughout this thesis are the following ones.

1. *The constituent materials of both skins and core behave in a linear elastic manner* when subjected to stresses below certain values (limit of proportionality). Hence the applied loads are such that these stresses are not exceeded anywhere in the beam for this study.
2. *Deflections and slopes are small* and so the second derivative of the deflection may be equalled to the curvature, i.e.

$$\frac{1}{\rho(x)} = \frac{v''(x)}{\{1 + [v'(x)]\}^{3/2}} \simeq v''(x)$$

where $\rho(x)$ is the radius of curvature at the abscisa x, $v(x)$ is the deflection at the same section and primes stand for derivatives with respect to x.

3. *All longitudinal vertical sections deform identically.* That is, there is bending in only one direction.

4. *The core and the skins are not deformed significantly in their shortest dimensions* (thicknesses). This means that the geometry of the section is not affected due to axial stresses in the vertical direction. This is usually accepted since in beams the direct vertical stresses due to transverse loading are much smaller than the longitudinal axial stresses and the shear stresses caused by those loadings. Under concentrated loading this condition may not be satisfied in the locality of the loads but the overall behaviour will not be affected appreciably. Zahn[25] considers these non-negligible vertical stresses with the use of a non-linear theory to find stresses in the neighborhood of concentrated loads.

5. All materials in the section are *homogeneous*. That is, the structural properties are the same at every point in any one direction. *Anisotropic* materials are not banned.

6. *Shear deformations of the skins are neglected* because of the very high span-to-depth ratio and because of the high shear modulus of the generally used skin materials. This does not imply that shear stresses in the skins are neglected.

7. *The contribution of the core to the bending stiffness of the section is negligible* because the Young's modulus of the core is so small compared to the skins' that the stresses required to cause strains compatible with those in the skins

are negligible. This assumption implies that the problem of bending of a sandwich beam having a stiff core (or the problem of I-shaped or homogeneous rectangular beams) may not be regarded* as a limit particular case of sandwich construction by the analytical approaches presented in this study when the core material is made very stiff. Allen[1] suggests a device to consider cases where core materials have significant bending stiffness resulting from high modulus of elasticity.

8. The bonding of skins and core is perfect. Hence, no slip exists at any section in the skins-core interface.

2.3 BENDING STIFFNESS

The bending stiffness of the section shown in Fig. 2.1 is given by the following expression, where E_1, E_2 and E_c are the moduli of elasticity of skins 1 and 2 and core respectively and the other symbols are as defined in Fig. 2.1.

$$(EI) = (EI)_d + (EI)_f + (EI)_c \qquad (2.1)$$

where $$(EI)_d = b(E_1 t_1 d_1^2 + E_2 t_2 d_2^2) \qquad (2.2.a)$$

$$(EI)_f = (EI)_{f1} + (EI)_{f2} \qquad (2.2.b)$$

$$(EI)_{fi} = \frac{b}{12} E_i t_i^3 \qquad i = 1,2 \qquad (2.2.c)$$

$$(EI)_c = \frac{bE_c}{3}\left[\left(d_1 - \frac{t_1}{2}\right)^3 + \left(d_2 - \frac{t_2}{2}\right)^3\right] \qquad (2.2.d)$$

In expressions (2.2.a) and (2.2.d), the terms d_1 and d_2

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* The case of a homogeneous rectangular beam however can be obtained as a particular case of sandwich panels by making the core thickness c indefinitely small. The depth of the team will be $t_1 + t_2$ and b its width.

define the location of a *reference level* (shown in Fig. 2.1) defined as the centroid of the elastically transformed section. From equilibrium considerations of the net force acting in the section it may be easily found that

$$d_1 = \frac{\frac{1}{2}\, c(c+t_1)+t_2E_2d}{t_1E_1+cE_c+t_2E_2} \quad \text{and} \quad d_2 = \frac{\frac{1}{2}\, c(c+t_2)+t_1E_1d}{t_1E_1+cE_c+t_2E_2} \qquad (2.3)$$

If the neutral axis of the section were defined as the surface such that the axial stresses due to bending (total minus net axial stress in cases with applied thrusts) vanish, d_1 and d_2 as given by formulas (2.3) would define its location in the section with reference to the centroids of the skins. This definition of neutral axis is found in most papers dealing with the problem of dissimilar skins in sandwich panels (see Hartsock[12] or Darvas[5] for instance). The actual meaning of the level defined by Equations (2.3) is just the location of the centroid of the transformed section and it will be referred to as *Reference Level*. In view of the fact that the meaning of neutral axis as defined above will disappear when the axial stresses in the core are neglected, it will not be used in this work. In these cases all points in the core are assumed to have zero axial stresses and in some methods, such as those not incorporating the Navier-Bernouill principle, the possibility exists of having extra points with zero axial stress in the skins or even of having no such points anywhere in the core. Another definition of neutral axis will be given in the next pages.

The first term on the right hand side of Equation (2.1) is invariably dominant. In cases with thin or very thin skins the second term may be neglected. In fact, that is the way *thin* and *very thin* skins were defined in Chapter I. For cases with different skins the term $(EI)_f$ should be evaluated and compared with $(EI)_d$, the decision of keeping it or neglecting it being made only after that comparison is made. For cases with identical skins, Allen[1] proved that, when $d/t > 5.77$ (where $t = t_1 = t_2$), the second term, $(EC)_f$, is less than 1% of $(EI)_d$ and so $(EI)_f$ may be neglected for practical purposes.

Concerning the last term on the right hand side of Equation (2.1), the same process as above should be done for each particular case before deciding whether to neglect it or not. In practical sandwich members, however, the modulus of elasticity is almost always much larger for the skins than for the core. Hence the term $(EI)_c$ is negligible in almost every case. Allen[1] found that, for symmetric sections, this term represents less than 1% of $(EI)_d$ when

$$G \frac{E_f}{E_c} \frac{t}{c} \left(\frac{d}{c}\right)^2 > 100$$

where $t = t_1 = t_2$ as before and $E_f = E_1 = E_2$ (symmetric section). The above condition may be unsatisfied in particular cases and it would seem wise to keep that term in the general derivations. However, this makes the analysis a little too cumbersome and it will not be considered here based on the fact that in this study* and in most practical cases the dimensions of the section and particularly the properties of the materials make it negligible. Allen[1] makes some suggestions to transform his solution using ordinary beam theory by keeping $(EI)_c$ in the expression for (EI), but using a transformed value of the shear modulus of the core. His suggested solution is applicable here.

As a result of neglecting $(EI)_c$ (and hence the longitudinal axial stresses in the core) throughout this thesis, the case of a sandwich beam with a very stiff core is not a limiting case of the theories developed. It was thought to be important to reaffirm that I-beams or rectangular sandwich beams with high ratios of E_c/E_f must be analysed in a different way.

After these considerations are accepted, Equations (2.1) and (2.3) become

$$(EI) = (EI)_d + (EI)_f \tag{2.4}$$

and $$d_1 = \frac{t_2E_2d}{t_1E_1 + t_2E_2} \qquad d_2 = \frac{t_1E_1d}{t_1E_1 + t_2E_2} \tag{2.5}$$

2.4 SIGN CONVENTIONS

The following sign convention will be used throughout this thesis:

The horizontal displacements and distances along the abscisas are positive when measured to the right while deflections and other vertical ordinates are positive when measured

* See Section 2.2 for types of panels studied.

downward.

External bending moments are positive when they produce tension in the bottom fibres of the beam. Correspondingly, they produce negative curvature. The external shear forces are positive when they produce a positive slope in the deflected beam (Total External shear force will always equal the derivative of the external bending moment with no change in sign). No sign convention is used for applied forces. Their directions are arbitrarily assumed. Answers resulting in negative values indicate that the opposite direction to that shown is correct.

It also follows that once the sign convention is stated as above for the horizontal displacements, axial stresses and strains are positive when tensile and negative when compressive. Shear stresses and strains will be positive (arbitrarily) when they act as shown in Fig. 2.2, which would contribute to a positive slope $v'(x)$.

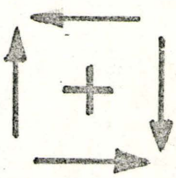
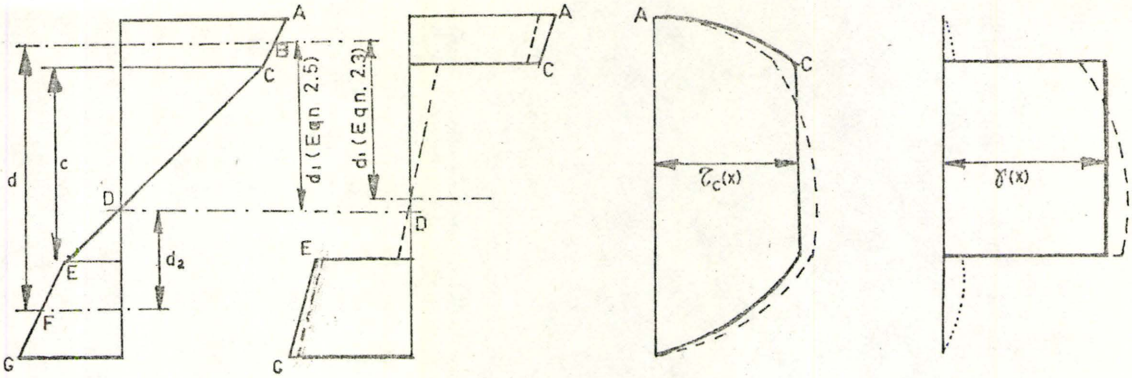


FIG. 2.2

Sign Convention for Shear Stresses and Strains

2.5 DISTRIBUTION OF SHEAR STRESSES IN THE SECTION

Fig. 2.3 shows the distribution of axial strains and stresses and shear stresses and strains on a section of a sandwich beam. It will be observed that the axial strains in the core are not neglected (Fig. 2.3.a) because they are



(a) Axial Strains

(b) Axial Stresses

(c) Shear Stresses

(d) Shear Strains

FIG. 2.3

Axial and Shear Stresses and Strains

necessary to comply with the compatibility of deformations in the section. Not withstanding this, the axial stresses in the core (Fig. 2.3.b) are neglected because of the very low modulus of elasticity of the core. The dashed lines in Fig. (2.3.b) show in an exaggerated way the effect that would be obtained if those stresses had been considered. The continuous heavy line in Figure 2.3.c illustrates the distribution of shear stresses in the section, where it may be observed that shear stresses are not neglected in the skins. The width, b, times the area between the parabola AB and the straight line with the same ends is the portion of the shear force taken by the top skin when bending about its centroid. The same thing is applicable to the bottom skin. When the deflection of each skin is $v(x)$, the total shear force applied to the section may then be shown to be

$$M'(x) = bd\tau_c(x) - (EI)_f v'''(x)$$

where $M'(x)$ is the shear force at the section $x, \tau_c(x)$ is the shear stress in the core (Fig. 2.3.c) and the negative sign in the second term on the right hand side was introduced to comply with the sign convention.

But the shear stress in the core, $\tau_c(x)$, may be expressed as the product

$$\tau_c(x) = G\gamma(x)$$

where G is the shear modulus of the core and $\gamma(x)$ is the shear strain. So

$$M'(x) = bdG\gamma(x) - (EI)_f v'''(x) \qquad (2.6)$$

The dashed lines in Fig. 2.3.c show the effect of including the axial stresses in the core. Finally, Fig. 2.3.d shows the shear strains in the section, and it may be observed that the shear strains in the skins (dotted lines) are much smaller than the ones in the core because of the much higher shear modulus of the skins materials compared to the core material. The dashed lines for shear strains in the core corresponds, as above, to the case in which axial stresses in the core are included.

These figures show the hypothetical model to be studied in this work. The modulus of elasticity of the core is assumed to be zero, therefore, stresses are neglected but strains are permitted. The shear modulus of the skins is taken as being infinitely large so that shear strains in the skins are neglected but shear stresses are not.

In problems concerning sections with thin skins, the second term in Equation (2.6) vanishes and the expression

$$M'(x) = bdG\gamma(x)$$

is valid. In cases of very thin skins the approximation $d \approx c$ would yield

$$M'(x) = bcG\gamma(x)$$

2.6 COMPATIBILITY OF DEFORMATIONS

Once the properties of the constituent elements are defined and restricted to the specified limits, the deformed section at the abscisa x may be drawn as shown in Fig. 2.4, where every variable is shown in its positive direction as defined by the sign convention in section 2.4.

A small length of the beam, dx, is shown in its original position and in its position after deformations and displacements have taken place. If the deformed section appears to have no curvature, it is only due to its very small length. This curvature could be in the shown case positive or negative.

As its name implies, the reference level as defined by

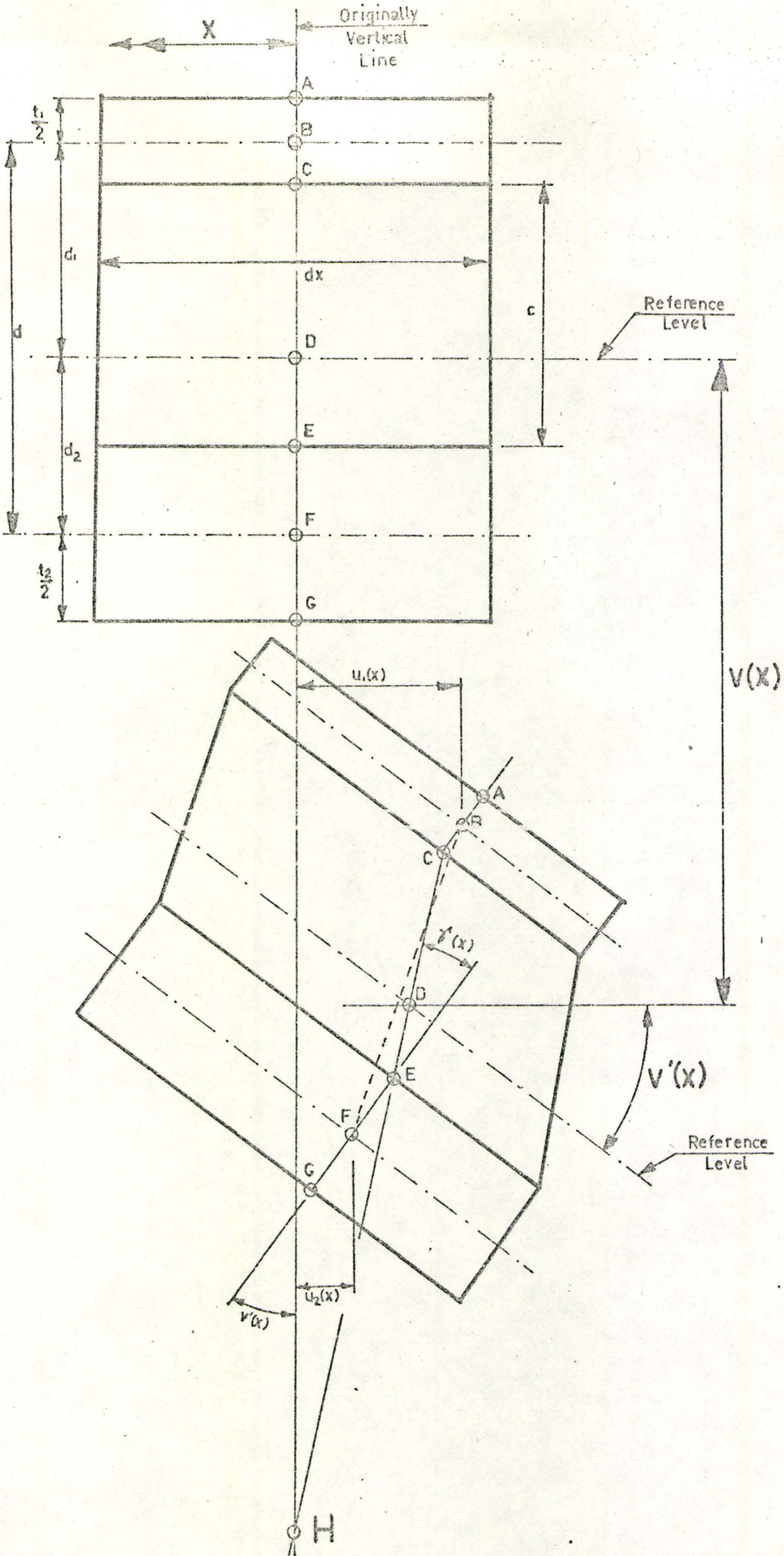


FIG. 2.4

Deformations and Displacements

equations (2.5) was used to measure vertical deflections while the horizontal displacements of the skins, $u_1(x)$ and $u_2(x)$, are given by the movement in the horizontal direction of the centroids of the skins in the deformed section. The originally vertical line ABCDEFG deformed to a polygonal having the lines ABC and EFG perpendicular to the deformed reference level because of the fact that shear deformations of the skins were neglected. The portion CDE is a straight line because the axial stresses in the core were neglected and the shear stresses and strains along the core are both constant (Section 2.5). A very important relationship may be obtained from Figure 2.4 relating the shear deformation of the core $\gamma(x)$, the horizontal displacements of the centroids of the skins, $u_1(x)$ and $u_2(x)$, and the slope of the member, $v'(x)$. Simply with geometry it may be shown that

$$\gamma(x) = \frac{d}{c}[v'(x) - \frac{u_1(x) - u_2(x)}{d}] \qquad (2.7)$$

This expression will be used several times in this work.

2.7 DEFINITION OF NEUTRAL AXIS

A final word concerning Fig. 2.4 is related to the definition of neutral axis to be used in this work. In this study, the location defined by the point H in Fig. 2.4, which means the intersection of the line CDE representing the deformed core and the originally vertical line drawn through the undisturbed section, is defined as the neutral axis of the section. This seems to be an arbitrary definition and in fact it really is, but for many practical cases (depending upon the section properties and the applied loads), the point H lies within the core and represents the point of the core with no horizontal displacements. This definition seems to be a logical one and it is used by the authors using strain energy methods based on displacements (see Hoff[12] or Allen[1] for instance). They invariable draw H within the core, which is (as will be proven later in this Section) correct only for particular cases of loading, support and section properties.

Darvas[5], Hartsock[12] and others define the neutral axis as the points with neither tensile nor compressive stresses. By their assumption (also made in this study), that axial stresses in the core are negligible, leads to the absurd result that every point in the core is on the neutral axis. Moreover, it is clear that for sandwich beams having cores so weak that they are incapable of transmitting much shear force, the two skins will act relatively independently and two extra points with neither tensile nor compressive stresses could exist. All of these authors use as neutral axis what has been called in here the Reference Level.

By the use of elementary concepts of elasticity, the net strains in the skins (axial strains at their centroids) may be found to be $u_1'(x)$ and $u_2'(x)$, where primes stand for derivatives with respect to x and $u_1(x)$ and $u_2(x)$ are the displacements as shown in Fig. 2.4. These strains are positive if tensile and negative if compressive.

By equilibrium considerations of the forces acting perpendicular to the section it may be shown that

$$\frac{u_1'(x)}{d_1} + \frac{u_2'(x)}{d_2} = -\frac{Pd}{(EI)_d} \qquad (2.8.a)$$

where P is the net axial force acting on the section and the other terms are as defined earlier in this work.

If $w_1(x)$ and $w_2(x)$ are defined as the displacements of the centroids of the skins minus the net displacement of the section due to the compressive action of end thrusts P, it follows that

$$\frac{w_1'(x)}{d_1} + \frac{w_2'(x)}{d_2} = 0 \qquad (2.8.b)$$

In problems involving no end thrusts P, there is no point in making this distinction since $u_i(x) = w_i(x)$ $(i = 1,2)$. The use of displacements $w_i(x)$ as defined above is very common, where the portion of the axial stresses due to end thrusts does not affect the bending behaviour. From Equation (2.8)

$$\frac{w_1(x)}{d_1} + \frac{w_2(x)}{d_2} = C \qquad (2.9.a)$$

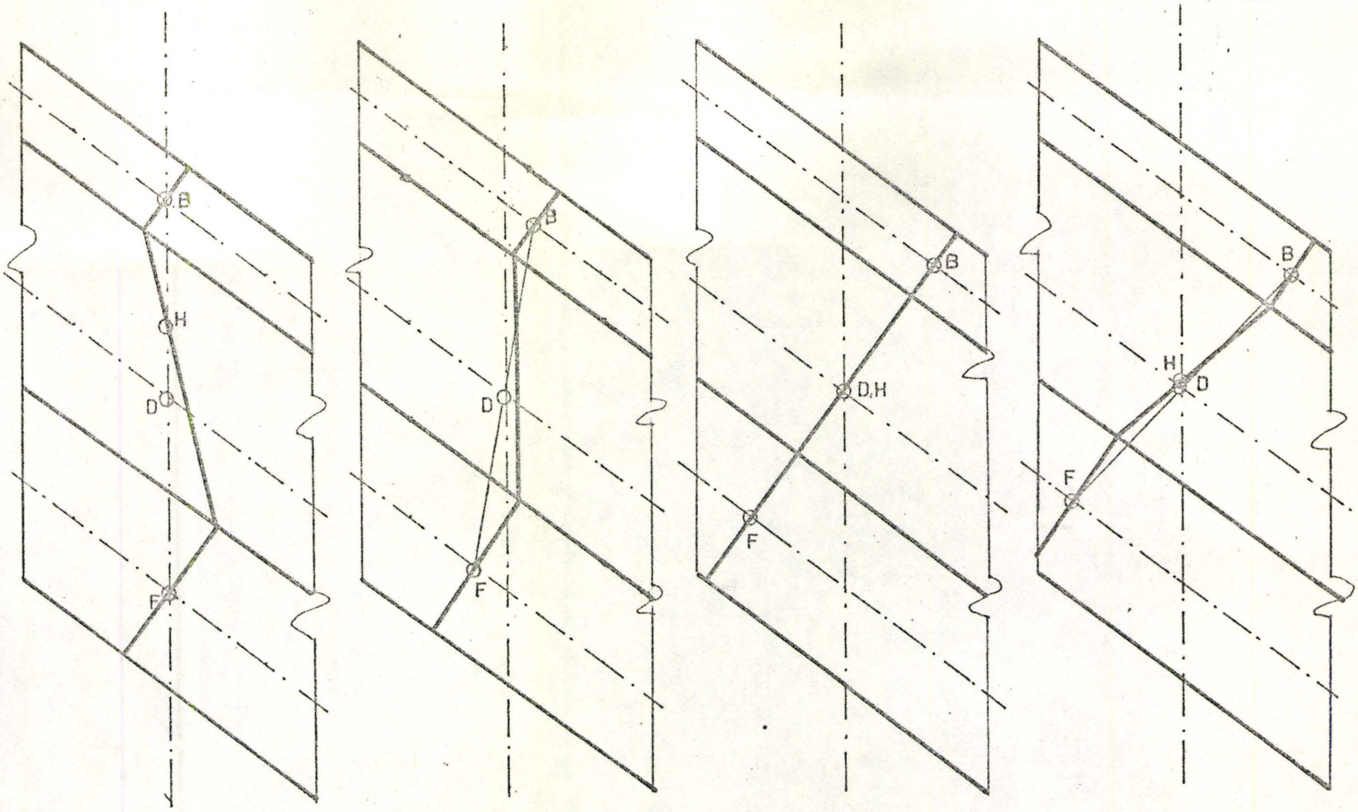
and

$$\frac{u_1(x)}{d_1} + \frac{u_2(x)}{d_2} = C - \frac{Pd}{(EI)_d}\,x \qquad (2.9.b)$$

where the constant C will be shown to vanish only in cases in which at least one section of the panel remains undeformed. This case occurs for beams with at least one built-in end, or in problems concerning symmetric conditions of loading and support or in cases where an infinitely stiff insert in the core does not allow any relative movement of the skins at the section where it is located. These considerations have been identified because every problem that has appeared in the related bibliography on the subject is of the kind having $C = 0$. In those cases the sketching of Fig. 2.4 is simple. In effect, the line joining the centroids of the skins (BF) crosses the reference level at its intersection with the originally vertical line [if $u_i(x)$ are replaced by $w_i(x)$, of course].

The difference in using $u_i(x)$ or $w_i(x)$ in the derivations is equivalent to the difference in considering either the total stresses or only bending stresses in a beam-column problem. If $w_i(x)$ is used, the net stresses due to end thrusts have to be added to the bending stresses $E_i w_i'(x)$ if the total stresses are of interest. The only method in this work using the actual displacements $u_i(x)$ is Hoff's generalized solution. These notes are worthwhile to include in here because several of the references consulted use misleading definitions of these stresses, especially when defining the neutral axis.

According to the above considerations, the deformed section for those particular cases in which $C = 0$ may be easily sketched. Figure 2.5 includes four very representative possibilities. Figure 2.5.a show the hypothetical limiting case where the core material is so weak that it cannot transmit



(a) Limiting Case (Maximum shear deformation in the core)

(b) N.A. at Infinity (Positive shear in the core)

(c) Plane Section Remaining Plane (No shear in the core)

(d) (Negative Shear in the core)

FIG. 2.5

Possible Deformed Sections

shearing forces and so only the skins act structurally to support the loads. Figure 2.5.b shows a case in which the location of the neutral axis is at an infinite distance from the section. Finally, Figures 2.5c and 2.5d show two interesting possibilities. At a particular section of the member it could happen that the effect of the applied loads would produce deformed sections as shown. These are interesting findings because Allen[1] said that the case shown in Figure 2.5.c would be a limiting case and it corresponds to a sandwich beam having a core material infinitely stiff in shear. It does not seem reasonable to apply conclusions obtained from a model structure having an infinitely weak core material under axial stresses (his basic conditions of deformability of the core material are the same as the ones adopted in this work), to a core material infinitely stiff under shear stresses. At least it is very difficult to think of a material having a very small Young's modulus and a very large shear modulus. These possibilities will be further analysed in Chapter VI.

All conclusions obtained in the previous paragraph are applicable only to the cases with C = 0. For cases having non-symmetric loading and support conditions, no rigid inserts and no built-in ends (and it could be added in here, for statically indeterminate structures and for overhangs, with or without loads), the analysis of limiting cases for shear deformations of the core and the location of the neutral axis is too complicated and will be omitted in here. The methods of analysis to be derived in section 4.4 and Chapter V and VII to provide the *exact solution*, do not require the use of the concept of neutral axis.

CHAPTER III

EQUIVALENT I-BEAM

3.1 INTRODUCTION

As was mentioned earlier, sandwich construction was initially used almost exclusively for aeronautical purposes and most studies were made for sandwich panels with thin or very thin skins. In these cases the solutions obtained by assuming that the Navier-Bernoulli principle is valid produces the *exact solution**. For Sandwich sections having thick skins, this assumption is not valid and the following derivations must be regarded as approximate. They are included mainly for the purpose of comparisons.

Once the Navier-Bernoulli Principle is assumed to be applicable, the concept of elastic transformation of the sections is invariably applied by every author. In order to keep consistency in the presentation of all formulae in this work, the components of the sandwich section will not be transformed to a uniform material but the section will be kept as it was originally. It is however very easy to prove that the mathematical treatment of the section as it will be done in here is perfectly

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* The meaning of the term *exact solution* used throughout this work was defined in a footnote in Section 1.2.

equivalent to the transformed section concept.

The assumption of applicability of the Navier-Bernouilli principle will imply basic distributions of axial stresses and strains in the section which differs from the ones shown in Figures 2.3.a and b. The polygonal ACDEG will now become a straight line and the same thing may be said with respect to Figure 2.4, in which case that line will be perpendicular to the reference level if the shear deformation of the core is neglected (as shown in Figure 2.5.c).

The assumption mentioned above stating that plane sections remain plane after bending deformations* may be easily understood to be applicable for cases having thin or very thin skins. In fact, the local stiffness of these skins may be neglected, and hence the lines AC and EG in Figure 2.3.a can be rotated about B and F respectively with no change in stress while the line CE may always be rotated without introducing forces either because of the assumption $E_c=0$. It will be shown later on that the *exact solution* coincides with the ones assuming the Navier-Bernouilli principle to be applicable when $(EI)_6 \to 0$ in cases with $C=0$ (see Section 2.7) and when the shear deformations of the core are considered (See Section 3.3).

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* When shear deformations are considered, these can not conform to the Navier-Bernouilli principle as originally stated but only in cases with very thin skins.

3.2 A METHOD IN WHICH SHEAR DEFORMATIONS OF THE CORE ARE NEGLECTED

The basic assumption is that plane sections perpendicular to the Reference level remain both, plane and perpendicular to the Reference Level after deformation. The assumption that plane sections remain place (Navier-Bernouilli principle) permits the use of the Equivalent I-beam approach while the condition of no variation of the angle of the section with the Reference Level is equivalent to saying that shear deformations of the core are negligible.

The basic differential equation in this method is the very well known one,

$$v''(x) = -\frac{M(x)}{(EI)} \tag{3.1}$$

where $v(x)$ is the vertical deflection of the section located at the abscisa x, $M(x)$ is the external bending moment at that section, (EI) is the total bending stiffness as defined in Equations (2.4) and the primes denote derivatives with respect to x. The negative sign was incorporated to comply with the sign conventions adopted throughout this work and specified in Section 2.4.

The two boundary conditions necessary for the solution of Equation (3.1) are related with the deflections at two sections or with the slope at one section and the deflection at the same or other sections of the beam.

Except for the fact that Darvas[5] used transformed sections and neglected $(EI)_f$ in his derivations, this is the method he recommended. Leabu[17] also used formulas as obtained from Equation (3.1) when studying the problem of thermal warp for sandwich panels with concrete skins (in which case the skins were necessarily thick). Pfeifer and Hanson[20] used this method to evaluate minimum values of deflection vs. load. Maximum values were obtained from the assumption that the skins act independently. Their conclusion is that test results fall somewhere between these two limiting curves, they being closer to the first curve in panels with shear-connectors and to the second one in sandwich members having very weak cores and no shear connectors.

3.3 EQUIVALENT I-BEAM FORMULAE WITH SHEAR DEFORMATION NEGLECTED

Because differential equation (3.1) coincides with the commonly used formula for bending of homogeneous beams, formulas for some particular cases of loading may be obtained from elementary beam theory. For purposes of comparisons, the following loading cases are included for a simply supported beam of span L.

A. Uniformly distributed load q

$$v(x) = \frac{q}{24(EI)}(L^3 - 2Lx^2 + x^3)x, \quad v(L/2) = \frac{5}{384}\frac{qL^4}{(EI)}$$

B. Concentrated load W at a distance a from the left support

$$v(x) = \begin{cases} \dfrac{Wb}{6(EI)L}\,[2L(L-x)-b^2-(L-x)^2]\,x & 0<x<a \\ \dfrac{Wa}{6(EI)L}\,[2Lb-b^2-(L-x)^2]\,(L-x) & a<x<b \end{cases} \qquad \text{where } a+b = L$$

C. Concentrated load W at mid-span

$$v(x) = \frac{W}{48(EI)}(3L^2-4x^2)\,x \qquad 0<x<L/2 \qquad v(L/2) = \frac{WL^3}{48(EI)}$$

D. Moment $M(0)$ applied at the left support.

$$v(x) = \frac{M(0)}{6(EI)L}\,(L^2-3Lx+x^2)\,x$$

E. Eccentric end thrusts P with eccentricity e (equal at both ends) measured from the Reference Level (Secondary moments considered).

$$v(x) = e\left[\frac{\cos\beta(\frac{L}{2}-x)}{\cos\beta\frac{L}{2}} - 1\right] \qquad v(L/2) = e[\sec\beta\tfrac{L}{2} - 1] \text{ with } \beta^2 = \frac{P}{(EI)}$$

F. Critical axial load.

$$P_c = \frac{\pi^2(EI)}{L^2}$$

3.4 A METHOD IN WHICH SHEAR DEFORMATIONS OF THE CORE ARE CONSIDERED

Most of the practical examples of sandwich construction involve the use of core materials having a very low shear modulus, which means that the shear component of the deflection should not be discarded. The method presented in this section accounts for shear deformations.

In addition to the general assumptions given in Section 2.2, this method requires the application of the Navier-Bernouilli principle to the bending deformations of the section. The basic differential equation describing the deflected shape is

$$v''(x) = -\frac{M(x)}{(EI)} + \frac{M''(x)}{AG} \tag{3.2}$$

which may also be written as

$$v(x) = v_1(x) + v_2(x) \tag{3.2.a}$$

$$v_1''(x) = -\frac{M(x)}{(EI)} \tag{3.2.b}$$

$$v_2'(x) = \frac{M'(x)}{AG} \tag{3.2.c}$$

where $v_1(x)$ is the deflection due to bending and $v_2(x)$ is the deflection caused by the shear forces $M'(x)$. G is the shear modulus of the core material and A is a factor quite similar to the one used in ordinary beam theory. The product AG is usually referred to as the shear stiffness of the sandwich beam.

The differential equations above may be easily solved once $M(x)$ and $M'(x)$ are known. The two arbitrary constants may be found from the boundary conditions which define the deflections at two points or the deflection and the slope at two points (not necessarily different). When considering slopes it must be borne in mind that the slope $v'(x)$ does not have to be continuous in some particular cases of loading. In fact, the application of a concentrated load producing a sudden change in the shear force $M'(x)$, will also produce [See Equation (3.2.c)] a sudden change in the slope $v'(x)$ (The beam then has infinite curvature at the point of application of a concentrated load).

As happens for homogeneous beams, the consideration of shear deformations involves an inconsistency. As a matter of fact, the distribution of shear stresses at the section, obtained from the axial stresses found assuming the Navier-Bernouilli principle, produces deformations in the section which do not conform to the above mentioned principle. The effects of this inconsistency are generally disregarded in homogeneous beam theory.

Several different values for A in Equation (3.2.c) are found in the related bibliography. O'Dell and Graham[19] use $A = \frac{2}{3}bc$, which is equivalent to saying that the skins do not take any shear, the shear stresses are zero at the top and bottom fibres of the core and the distribution of shear stresses along the core is parabolic. These three implications are absurd.

In the formulas for deflections given by Darvas[6], he used $A = bc$ even though in his formulas for shear stresses in the core, the total shear force $M'(x)$ was divided by the area bd, which is in contradiction with the value of A he used.

Most of the authors consulted use $A = bd$ in their derivations. Hughes and Wajda[14] give final formulae for deflections for two particular cases of loading by the use of three different methods of analysis. Those formulae appear with several errors which will be commented on in detail in Section 8.2. The value they give for A seems to be d instead of bd, but checks made on their graphs showed that the reason for this is that they took $b=1''$ in their particular problems, even though they mentioned that the width b to be used was 2 inches.

McCavour[18], Kuenzi[16], Smolenski and Kroksky[23] and Darvas[5] also used $A=bd$. The work in references (23) and (5) deserves special comment. In the setting up of their equations, they state that, for more generality, they neglected neither the axial stresses in the core nor the local bending stiffness of the facings. Then, when studying the effect of shear deformations, they took the total shear force at the section as bd times the constant shear stress in the core. This is in contradiction with the two considerations above (See Figure 2.3).

Several authors use a strain energy approach which is attributed to H.W. March*. According to their description, it is based in assumptions of behaviour which apply only to sandwich members having thin and very thin skins depending upon the value of A used. The final formulae coincide with the ones being dealt with in this section and it is therefore appropriate to include comments on the way some authors use them. Even though the derivations of those formulae did not rely on the I-beam similitude, the final formulae are, as said above, exactly the same and the implications of their results may be included here.

Doherthy et al[7] use March's approach and they give final formulas coinciding with formulae (3.2), but they take

$$(EI) = \frac{bE_{\delta}}{12} (h^3 - c^3)$$

$$A = \frac{2bh}{3c} \frac{h^3 - c^3}{h^2 - c^2}$$

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* The papers by March, which are quoted very often in related articles are: March, H.W., "Effects of Shear Deformation in the Core of a Flat Rectangular Sandwich Panel", Forest Products Laboratory Report No.1583, U.S. Dept. of Agriculture, Madison, Wisconsin, 1948 and "Flexural Rigidity of a Rectangular Strip of Sandwich Construction", Forest Products Lab. Report No.1505, U.S. Dept. of Agriculture, Madison, Wisconsin, 1944. Unfortunately, neither of these were available for this work.

where $h = c + t_1 + t_2$ is the total thickness of the section and E_b represents the modulus of elasticity of the skins. They use transformed areas in cases with non-identical skins. The first of these expressions coincides with the stiffness found in Section 2.3 for the case of a section with identical skins, but they say that this expression may be used for cases with dissimilar skins, in which case E_b would be the Young's modulus of the transformed section. They do not say what value of b should be taken in this case. The expression for A above is not clear.

Fisher[10] obtains deflections by using Castigliano's First Theorem as applied to a member with thin skins, and the formulae he provides for three different load cases* coincide with the results obtained from Equation (3.2) with $A = bd^2/c$. This value for A will be discussed later, but it may be said here that it coincides with the value to be obtained in the *exact solution*. His formulas may be said to be exact for cases with thin skins. He did not use the conditions shown in his Figure 1, where he shows shear and axial stresses which are incompatible from the viewpoint of the equilibrium of stresses in the section. In fact, even though he neglected the axial stresses in the core, his distribution of shear stresses is parabolic along the core (See Figure 2.3.c), which seems to contradict the previous statement. Fortunately, as said before, he did not use this inconsistency in his derivations. For the "verification of the accuracy" of his theory, he needed the

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* One of those formulas does not correspond to the loading case specified by Fisher as was pointed out earlier in this work.

moduli of elasticity of the skins, E_f, and the shear modulus, G, of the core. The first value was obtained from the bending tests of homogeneous samples as given by Doherthy et al[7]. The shear modulus of the core was obtained from other tests carried out by the same authors. These tests were performed on sandwich beams having five different combinations of skins. The value of the shear modulus was obtained from the formulas studied by the other authors who used different expressions than Fisher* for A and (EI). In synthesis, to verify his formula, Fisher used the values of G derived by applying other formulae to the results of sandwich beam tests. The differences between experimental and theoretical results he obtained (of the order of 10% for sandwich members having very thin aluminum skins) could be attributed to the irrational way to determine G as described above.

Benjamin[3] used March's approach and his results coincide with formula (3.2) with $A = bc$ which, as will be shown in Section 4.6, is applicable only for sandwich beams having very thin skins.

In Section 4.6 it will be proven that the *exact solution* to the problem corresponding to the basic assumptions in Chapter II, and for cases with thin skins, coincides with equations (3.2)

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* The value Fisher finds for (EI) is what has been called $(EI)_d$ in this thesis, while his expression for A is bd^2/c. The values used by Doherthy et al were discussed earlier in this section.

where

$$A = bd^2/c \tag{3.3}$$

In problems involving sections with thick skins, the solutions given by Equations (3.2) are no longer exact but it will be proven in Section 3.6 that expression (3.3) provides the best approximation to the *exact solution*. The Forest Products Laboratory Wood Handbook[24] recommends use of expression (3.3) and Allen[1] also arrived at this conclusion. He derives this expression by studying the shear deformation in the core separately from the bending situation where plane sections are considered to remain plane. Expression (3.3) will be derived by other methods later on in this work.

3.5 EQUIVALENT I-BEAM FORMULAE WITH SHEAR DEFORMATION CONSIDERED

Deflections may be found for particular cases of loading where shear deformations are considered by the use of Equation (3.2). Deflections for some common cases of loading in a simply supported beam of span L are included below. For the sake of brevity, step by step derivations are not included.

A. Uniformly distributed load q.

$$v(x) = \frac{q}{24(EI)}(L^3 - 2L^2x + x^3)x + \frac{q}{2AG}(L-x)x$$

$$v(L/2) = \frac{5}{384}\frac{qL^4}{(EI)} + \frac{qL^2}{8AG} \tag{3.4}$$

B. Concentrated load W at a distance a from the left support.

$$v(x) = \begin{cases} \dfrac{Wb}{6(EI)L}\,[2L(L-x) - b^2 - (L-x)^2]\,x + \dfrac{Wb}{LAG}\,x & 0<x<a \\[2ex] \dfrac{Wa}{6(EI)L}\,[2Lb - b^2 - (L-x)^2]\,(L-x) + \dfrac{Wa}{LAG}(L-x) & a<x<b \end{cases}$$

where $a + b = L$

C. Concentrated Load W at mid-span.

$$v(x) = \frac{W}{48(EI)}\,(3L^2 - 4x^2)\,x + \frac{W}{2AG} \qquad 0<x<L/2$$

$$v(L/2) = \frac{WL^3}{48(EI)} + \frac{WL}{4AG} \tag{3.5}$$

D. Moment $M(0)$ applied at the left end.

$$v(x) = \frac{M(0)}{6(EI)L}\,(L^2 - 3Lx + x^2)\,x$$

The above expression would coincide with the one found when the shear deformation of the core was neglected. The reason for this is that, if shear is constant in Equation (3.2) for all values $0<x<L$, no shear component of deflection appears (The derivative of shear vanishes).

E. Eccentric end thrusts P with eccentricity e (equal at both ends), measured from the Reference Level (Secondary moments considered).

$$v(x) = e\left[\frac{\cos \beta(\frac{L}{2} - x)}{\cos \frac{\beta L}{2}} - 1\right]$$

$$v(L/2) = e\left[\sec \frac{\beta L}{2} - 1\right]$$

with $$\beta^2 = \frac{P}{(EI)(1-P/AG)}$$

F. Critical load Pc

$$\therefore \quad Pc = \frac{\pi^2 (EI)}{L^2 + \pi^2 \frac{(EI)}{AG}} \qquad (3.6)$$

3.6 CONCLUSIONS

It will be proven later on in this work that the *exact solution* coincides with the one obtained from the method described in Section 3.4 for cases in which the local bending stiffness of the skins is negligible and expression (3.3) is used for A. Methods which apply the Navier-Bernouilli principle are the most popular for the structural analysis of sandwich panels. Therefore, this chapter consists of a generalization of these methods to account for the effect of thick skins. All of the methods presented in this chapter have some arbitrary assumptions of behaviour such as applicability of the Navier-

Bernouilli principle. Some of them (when shear deformation was considered) did not take into account an inconsistency in the way the shear stresses were *accommodated* in the section.

CASE STUDY		MID-SPAN DEFLECTION	
		Mid-Span Load	U.D.L.
No shear deformation		.1822	.1457
Shear Deformation Considered	Exact Solution	.3561	.3006
	$A = bd^2/c$	.3839	.3475
	$A = bd^2/c \quad (EI)_f = 0$	.4521	.4101
	$A = bd$	.5100	.4736
	$A = bc$	.7150	.6876
	$A = 2bc/3$	.9811	.9654

TABLE 3.1

Mid-Span Deflections for Two Cases of Load

Table 3.1 presents a comparison of the deflections at mid-span that would be obtained by using these methods in the case of a simply supported sandwich beam with thick dissimilar skins subjected to a mid-span concentrated load in one case and to a uniformly distributed load in the other*. The first line in

---

* The particular cases of loading and dimensions studied will be used as the basis for most comparisons in this thesis. It consists of a simply supported sandwich beam spanning 8 ft. and having a section with width $b=16''$, $c=1''$, $t_1=1/2''$, $t_2=3/4''$, $E_1=2.25\times10^6 psi$, $E_2=1.75\times10^6 psi$ and $G=600 psi$, corresponding to a panel with unreinforced mortar skins and expanded polystyrene core material. The values for the applied loading in this example are: Uniformly distributed loading q=53.33plf, corresponding to a wind pressure of 40 psf. Mid-span concentrated load W=212.13 lb. such that it produces the same mid-span moment as the uniform wind pressure of 40 psf.

Table 3.1 was calculated by using the Equivalent I-beam method with no allowance made for shear deformations of the core. The second line was evaluated with the use of the *exact solution* to be derived in Chapter VII. The remaining values were obtained by the use of the Equivalent I-beam method which considers shear deformations with different values of A. The effect of neglecting the local bending stiffness of the skins is also included.

Figure 3.1 shows a plot of mid-span deflection versus an eccentrically applied end thrust P* for the same section studied above and including the same cases studied in Table 3.1. Also included is an extra case which illustrates the effect of neglecting secondary bending moments. The panel is assumed to be originally perfectly flat.

Table 3.1 and Figure 3.1 both show that, for the loading cases studied, all formulas which consider shear deformations of the core over-estimate deflections. The effect is the largest when $A = \frac{2}{3} bc$ and the smallest when $A = bd^2/c$. The formula which neglects the shear deformation of the core under-estimates the deflections. It can also be seen from Figure 3.1 that for small values of end thrusts (when both, shear forces and secondary moments are small) all formulas give a very good approximation of the deflection.

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*The eccentricity, e, of the end thrust P in this particular case was taken as $e=d_1$, which is equivalent to saying that the end thrusts are applied at the centroid of skin 1 at both ends of the simply supported beam.

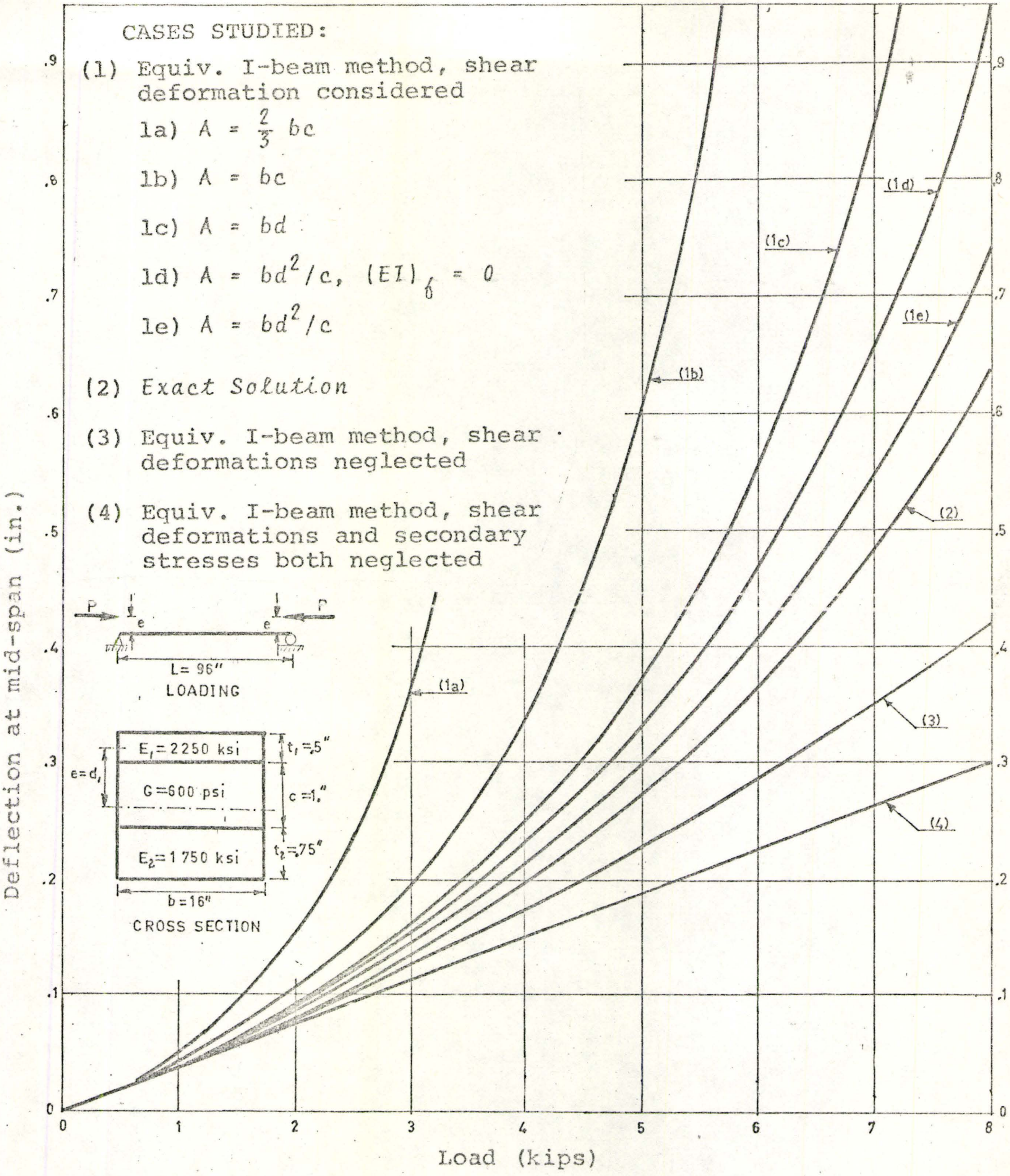

Figure 3.1

Effect of Several Assumptions for a Beam-Column Load

CHAPTER IV

APPLICATION OF ORDINARY BEAM THEORY

4.1 INTRODUCTION

In Chapters II and III it was pointed out that the common use of core materials having a low shear modulus makes it necessary to take into account the shear deformation of the core in practical sandwich beams. It was also mentioned that the bending stiffness of the skins with respect to their own centroids should not be neglected for sandwich beams with thick skins. Even though Equations (3.2) consider both aspects, it may be seen from Equation (3.2.c) that, when concentrated loads act on the beam, a sudden change in the shear deformation exists at the points of application of these loads. This in turn leads to the conclusion that the curvatures at these sections become infinite. This is understandable for sandwich members having skins which are so thin that their local bending stiffness is not capable of resisting this bending. This situation does not apply to sandwich beams having thick skins because they are unable to have infinite curvature at any point.

The inconsistency of using the Navier-Bernouilli principle when studying the compatibility of shear deformations of the core makes the results only approximate. In order to obtain the *exact solution*, the Navier-Bernouilli principle will not be used in any of the remaining methods to be studied in this work.

Two approaches which are based entirely on the application of ordinary beam theory are studied here. Allen's method(1) is not developed step by step but just commented on. Hartsock's method(12) is redeveloped so as to present a more general solution, but comments about the way that he solved the problem are also included. Both notations were changed from their original form to have a more compact terminology in this thesis.

4.2 ALLEN'S APPLICATION OF ORDINARY BEAM THEORY

The solution by Allen(1) is based on the following hypotheses of structural behaviour: "... a sandwich beam with a total load $q[q(x)]$ undergoes two distinct sets of displacements, $w_1[v_1(x)]$ and $w_2[v_2(x)]$. The first represents the ordinary bending deflection associated with a shear force $Q_1[M_1'(x)]$ which is shared between the faces [skins] and the core. The second represents the shear deflection of the core due to $Q_1[M_1'(x)]$. The faces [skins] participate in this extra deflection by bending about their own axes. In doing so, they support an extra shear force $Q_2[M_2'(x)]$. The sum of $Q_1[M_1'(x)]$ and $Q_2[M_2'(x)]$ is the shear force applied to the beam."

The portions of the applied loads related to the corresponding deflections are given by Allen as:

$$q_1(x) = -\frac{dM_1'(x)}{dx},\ M_1'(x) = \frac{dM_1(x)}{dx},\ M_1(x) = -(EI)\ v_1''(x) \qquad (4.1.a)$$

$$q_2(x) = -\frac{dM_2'(x)}{dx},\ M_2'(x) = \frac{dM_2(x)}{dx},\ M_2(x) = -(EI)_b\ v_2''(x) \qquad (4.1.b)$$

$$q_1(x) + q_2(x) = q(x) \qquad (4.2.a)$$

$$M_1'(x) + M_2'(x) = M'(x) \qquad (4.2.b)$$

$$M_1(x) + M_2(x) = M(x) \qquad (4.2.c)$$

$$v_1(x) + v_2(x) = v(x) \qquad (4.2.d)$$

where $q(x)$, $M'(x)$ and $v(x)$ are as defined before, $M(x)$ is the total applied bending moment and the values with sub-indexes 1 and 2 are associated with $v_1(x)$ and $v_2(x)$.

From the relationships given above and after some algebraic manipulations Allen arrives at the following differential equations

$$(D^2 - \alpha^2)M_1'(x) = -\alpha^2 M'(x) \qquad (4.3.a)$$

$$v_1'''(x) = -\frac{M_1'(x)}{(EI)} \qquad (4.3.b)$$

$$v_2'(x) = -\frac{M_1'(x)}{\alpha^2 (EI)_b} \qquad (4.3.c)$$

where $$\alpha^2 = \frac{(EI)AG}{(EI)_b (EI)_d}, \tag{4.4}$$

$$A = bd^2/c \tag{3.3}$$

and $$D = \frac{d}{dx}$$

Equations (4.3) are very easy to solve and the next step is to find the six arbitrary constants involved in the solution of those equations [two for $M_1'(x)$, three for $v_1(x)$ and one for $v_2(x)$. Then, the vertical deflection $v(x)$ may be found from equation (4.2.d).

4.3 COMMENTS ON BOUNDARY CONDITIONS FOR ALLEN'S APPLICATION OF ORDINARY BEAM THEORY

The difficulty with the method as derived by Allen is in finding six suitable boundary conditions to evaluate the arbitrary constants for Equations (4.3). That difficulty arises because not enough is known about the values of those functions at the boundaries. For the problems he solves (uniformly distributed load and concentrated forces at the two quarter points and at mid-span), four boundary conditions may be found in terms of the total deflection $v(x)$ and the values of the moment $M(x)$ and shear $M'(x)$ at the boundaries (See Section 4.5). In fact, for the case consisting of a simply supported beam of span L, these can be easily obtained as follows:

1. $v(0) = 0$ (Arbitrary, implies choice of origin of coordinate axis at the left end of beam).

2. $v'(L/2) = 0$ (Symmetry, if the slope is continuous).

3. $v''(0) = 0$ (The skins deflect the same way the whole member does, so that the moment acting on the skins is given by $-(EI)_\delta\, v''(x)$ and, unless end moments are applied to the skins, its value at the boundary 0 must vanish. See further comments in Section 4.5).

4. $v'''(L/2) = -\frac{W}{2(EI)_\delta}$ (W is the mid-span concentrated load. At $L/2$, because of symmetric loading and support conditions, the core is not deformed in shear and so the whole shear force must be taken by the skins. Then condition (4) is clear because the shear force taken by the skins is given by $-(EI)_\delta\, v'''(x)$. In the cases with uniformly distributed load and four point loading, the right hand side of condition (4) is zero).

If different variables had been used, these four boundary conditions would suffice to the complete solution of the problem. That is not the case with Allen's solution because six boundary conditions are required in his method. The boundary conditions as stated by Allen* are the following:

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* Notation and coordinate system are both changed.

The three conditions required for $v_1(x)$ are:

(1) $v_1(0) = 0$ (Arbitrary)

(2) $v_1'(L/2) = 0$ (Symmetry)

(3) $v_1'''(L/2) = 0$ (Symmetry)

where the words within parentheses are directly quoted from Allen. Condition (1) defines the arbitrary choice of the origin of the coordinates. Condition (2) is generally accepted but should be proved because of the following arguments: The symmetry of loading and support conditions coupled with the fact that the slope of the beam $v'(x)$ must be a continuous function leads to the condition $v'(L/2) = 0$ mentioned earlier in this Section. The same thing may not be said so easily about $v_1'(L/2)$ because, although the symmetry of the corresponding load $q_1(x)$ may be reasonably assumed, its continuity should be proven. Condition (3) is less evident If the symmetry of $q_1(x)$ were accepted, the continuity of $v_1'''(x)$ is even less evident than the continuity of $v_1'(x)$, because it is related to shear forces while $v_1'(x)$ is related to slopes.

The condition Allen gives for $v_2(x)$ is

(4) $v_2(0) = 0$ (Arbitrary)

which is dictated by the coordinate system.

Concerning the two required conditions for $M_1'(x)$, nothing is known about the boundary values of this function or of its derivatives. Therefore, other functions are required to find them. Condition (5) is stated as

$$(5) \quad M(L/2) = -(EI)\, v_1''(L/2) - (EI)_6\, v_2''(L/2)$$

where $M(L/2)$ is known in the cases of loading solved by Allen.

The sixth condition could have been found from the fact that no bending moments were applied at the supports. It would have taken the form

$$M(0) = -(EI) v_1''(0) - (EI)_6\, v_2''(0) = 0.$$

However, Allen did not solve the problem this way because the beam he studied had symmetric overhangs with no load. For the inner span he could find only five constants with the boundary conditions shown above. He then solves the problem for the overhang where six additional constants must be found, but he could find only four boundary conditions as follows:

$$v_1 = v_2 = v_1'' = v_2'' = 0$$

at the free end of the overhang. The first of these conditions is arbitrary because the coordinate system for the overhang is located at the free end. The second one is a result of the first one and the two last conditions "...arise because M_1 and M_2 are assumed to vanish separately at the free end."

[It is interesting to remark at this point that if a problem is to be solved with no overhang, only one condition is still necessary (there were five before) and any linear combination of those two last conditions would produce the right condition. The vanishing of both v_1'' and v_2'' independently at the end could not be used any longer because seven conditions would be obtained while only six were required].

The three conditions still missing (one for the loaded span and two for the overhangs) are obtained from the continuity of the functions v_1', v_2' and v_1'' at the support. The continuity of v_2'', v_1''' and v_1^{iv} provides conditions which are linearly dependent upon previous conditions according to Allen.

To be able to compare these comments with Allen's original solution, Table 4.1 contains the corresponding notations, where the values omitted are either unchanged or not used.

DESCRIPTION	ALLEN'S	THIS THESIS
Deflections	w_1, w_2, w	$v_1(x)$, $v_2(x)$, $v(x)$
Shear Forces	Q_1, Q_2, Q	$M_1'(x)$, $M_2'(x)$, $M'(x)$
Stiffnesses	D, EI_f	(EI), $(EI)_6$
Argument	a	α

TABLE 4.1

Comparison with Allen's Notation

In addition, comparison of the formulas in this work with Allen's must take into account that the origins of co-ordinates are different.

An interesting fact about Allen's solution is that it is applicable only to sections with thick skins. If the approximation $(EI)_f = 0$ is made in his equations, no useable results are obtained from them. This is likely the reason he provided separate solutions for cases with thin and very thin skins. Other methods studied in this thesis will be applicable to sandwich sections with any *size* of skins.

Formulae for the particular cases of loads studied for other methods are not included because it seems very difficult to identify what boundary conditions Allen would choose. This situation arises because he gives no criteria for the evaluation of these. The results obtained by him for the cases of uniformly distributed load q and mid-span concentrated load W appear in his book and are not included here. Checks made on these show that they coincide with the *exact solution* to be presented in Section 4.5 and Chapters V and VII. However, as Allen warns, his method can not be used for cases with non-symmetric loading and support conditions or without rigid inserts in the core.

4.4 GENERALIZATION OF HARTSOCK'S SOLUTION

Hartsock[12] developed a method to analyse the structural behaviour of sandwich structures based on the state of stresses at the deformed section. These stresses were found as functions of the bending moments acting on the skins and the bending moment resulting from the product of the net axial force in the skins multiplied by the distance between their centroids. These two moments, $M_f(x)$ and $M_d(x)$ respectively, add up to the total applied moment $M(x)$ so that

$$M(x) = M_f(x) + M_d(x) \tag{4.5}$$

where $M(x)$ is the total external bending moment at the section x and $M_d(x)$ and $M_f(x)$ are as defined above.

Once the moments $M_f(x)$ and $M_d(x)$ are expressed as functions of the curvature $v''(x)$ produced by them, a differential equation relating $M(x)$ and $v''(x)$ results. The relationship of $M_f(x)$ with the curvature $v''(x)$ is easy to find because the centroids of the skins deflect the same way as the panel does due to the assumption that the core is not deformable in its shortest dimension (Assumption 4 in Section 2.2) and that shear deformations in the skins are neglected (Assumption 6 in Section 2.2). Thus the realtionship

$$M_f(x) = - (EI)_f \; v''(x) \tag{4.6}$$

may be written at once. $(EI)_f$ is the local bending stiffness of the skins as defined by Expression (2.2.b), $v''(x)$ is the curvature at the abscisa x, and the negative sign was introduced to comply with the sign convention followed throughout this work.

Concerning the relationship between the moment $M_d(x)$ and the curvature $v''(x)$, it is not so simple to find. First, by definition

$$M_d(x) = F(x)d \tag{4.7}$$

where d is the distance between the centroids of the skins and $F(x)$ is the axial force acting on one of the skins. The net compression due to any end thrust P is not included. If the cross section, Young's modulus and net axial strain of the skin i are bt_i, E_i and $w_i'(x)$ respectively, the expression for the net axial force $F(x)$ in each skin is given by

$$F(x) = - bt_1E_1w_1'(x) = + bt_2E_2w_2'(x) \tag{4.8}$$

where the signs were introduced because positive values of $M(x)$ require positive values of $w_2'(x)$, which is tension in the bottom skin and consequently negative values of $w_1'(x)$ are required.

The problem is now reduced to express $w_1'(x)$ [or $w_2'(x)$] as a function of the curvature $v''(x)$ and the moment $M(x)$. Figure 2.4 shows a deformed section of the sandwich member under load and Equation (2.7) is applicable. In this formula $u_i(x)$ may be replaced by $w_i(x)$ because only the difference of the displacements is included. The effect of the thrust strains in the skins, being the same for both skins, cancels out. Differentiating Equation (2.7) once gives

$$\gamma'(x) = \frac{d}{c}[v''(x) - \frac{w_1'(x) - w_2'(x)}{d}]. \tag{4.9}$$

In Chapter II it was proven that

$$\frac{w_1'(x)}{d_1} + \frac{w_2'(x)}{d_2} = 0. \tag{2.8.b}$$

If $w_2'(x)$ is now replaced in Equation (4.9) and the resulting equation is solved for $w_1'(x)$, the result is

$$w_1'(x) = d_1[v''(x) - \frac{c}{d}\gamma'(x)]. \tag{4.10}$$

This expression coincides with the one found by Hartsock from a figure which is not too clear. For comparisons with Hartsock's work, Table 4.2, presenting the equivalence of notations is useful.

The only thing left is to find an expression to relate the shear strain $\gamma(x)$ in the core to the applied bending moment $M(x)$ and curvature $v''(x)$. The distribution of shear stresses along the section is shown in Figure 2.3, where $\tau_c(x)$ may be easily shown to be given by

$$\tau_c(x) = \frac{F'(x)}{b}. \tag{4.11}$$

DESCRIPTION	*HARTSOCK'S*	*THIS THESIS*
Bending Moments	M, MC, MO	$M(x), M_d(x), M_\delta(x)$
Force in skins	F	$F(x)$
Dimensions	c_1, c_2, D, DC	d_1, d_2, d, c
Angle Subtended by arc dx	$d\theta$	$v''(x)dx$
Axial deformation of skins	δx_1	$w_1'(x)dx$
Deflection	y	$v(x)$
Skins cross sections	a_1, a_2	bt_1, bt_2
Core Shear Stress	Ssc	$G\gamma(x)$
Component of Shear Force neglecting bending of skins	VC	$M_d'(x)$
Relative rotation of skins due to core shear deformation	$d\beta$	$c\gamma'(x)dx/d$
Shear Stiffness	bD^2G/DC	AG
Bending Stiffnesses	E_1IC, E_1IO, E_1I	$(EI)_d, (EI)_\delta, (EI)$

TABLE 4.2

Comparison with Hartsock's Notation

Relations (4.7) and (4.11) yield

$$\tau_c(x) = \frac{M_d'(x)}{bd}$$

but $\tau_c(x)$ is also equal to $G\gamma(x)$ and so

$$\gamma(x) = \frac{M_d'(x)}{bdG} \tag{4.12}$$

By substituting Equation (4.12) into Equation (4.10) and the resulting expression in (4.8) and then in (4.7) gives

$$M_d(x) = -\ bd_{t1}d_1E_1[v''(x) - \frac{M_d''(x)}{AG}] \tag{4.13}$$

where $$A = bd^2/c \tag{3.3}$$

Before Equation (4.13) is usable, $M_d''(x)$ in the right hand side must be expressed as a function of the given moment $M(x)$ and the sought after curvature $v''(x)$. Using Equations (4.5) and (4.6)

$$M_d''(x) = M''(x) - M_b''(x) = M''(x) + (EI)_b\ v^{iv}(x)$$

and then Equation (4.13) becomes

$$M_d(x) = -\ bdt_1d_1E_1\{v''(x) - \frac{1}{AG}[M''(x) + (EI)_b\ v^{iv}(x)]\}.$$

This equation is further simplified by the fact that

$$bdt_1d_1E_1 = bdt_2d_2E_2 = (EI)_d \qquad (4.14)$$

therefore

$$M_d(x) = -(EI)_d\{v''(x) - \frac{1}{AG}[M''(x) + (EI)_\delta v^{iv}(x)]\} \qquad (4.15)$$

Finally, (4.15) and (4.6) in (4.5) yield:

$$M(x) = -[(EI)_\delta + (EI)_d]v''(x) + \frac{(EI)_d}{AG}[M''(x) + (EI)_\delta v^{iv}(x)]$$

and after further manipulation becomes

$$D^2(D^2-\alpha^2)v(x) = -[\frac{D^2}{(EI)_\delta} - \frac{\alpha^2}{(EI)}]M(x) \qquad (4.16)$$

where

$$\alpha^2 = \frac{(EI)AG}{(EI)_d(EI)_\delta} \quad \text{and} \quad D = \frac{d}{dx}$$

Equation (4.16) is a differential equation relating the externally applied bending moments and the vertical deflection $v(x)$ [or the curvature $v''(x)$] by means of properties of the section exclusively. From the derivations it was clear that no assumptions of behaviour different from the general ones stated in Chapter I were applied. The differential equation (4.16) is then applicable to any case of loading and support condition. This will be corroborated in Chapter VII.

Equation (4.16) was not obtained by Hartsock. Instead, he made the integrations as soon as they appeared and, in order to do so, he had to define his boundary conditions early in his work. The boundary conditions he applied correspond only to symmetric cases of loading and support. Besides, he indirectly used the fact that the applied bending moment $M(x)$ was a polynomial of the second degree in x with no constant term. This makes his solution applicable only to symmetric concentrated loading and uniformly distributed loads. He also studies the thermal warp problem, but he sets up the equations from independent considerations and does not use the formulae found before in his work.

Hartsock's treatment is so obscure that McCavour[18], who listed Hartsock's reference[12] as the first reference in his paper comments in the introduction that for sandwich panels with thick skins, "since a rational method of analysis has not been developed for this kind of facing [skins], the panels must be designed by testing".

4.5 COMMENTS ON BOUNDARY CONDITIONS FOR THE GENERALIZED HARTSOCK'S SOLUTION

Equation (4.16) could be easily applied to more general cases of symmetric loading. The four boundary conditions required for the solution could be taken as follows:

1) Deflection $v(B)$ is given (usually zero) at one of the supports. B represents the abscisa of the boundary in which the condition is given.

2) Slope $v'(B)$ is zero at the point of symmetry B.

3) It depends only on the applied end moments. There exist several possibilities as shown in Figure 4.1.

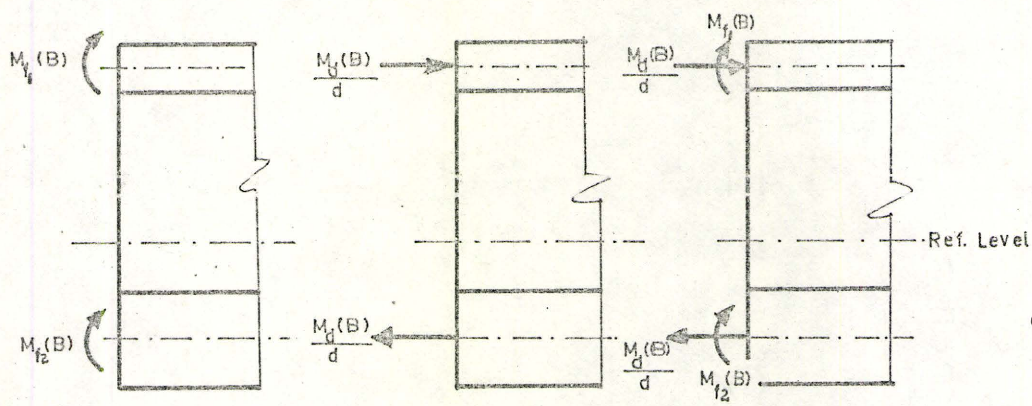

FIGURE 4.1

Possible Cases of Applied End Moments

Case (a) in Figure 4.1 is only of theoretical interest because no practical situation which duplicates these end moments is envisioned (See further comments in Section 5.4). The compatibility of the deformations of the skins require that

$$M_{\delta 1}(B) = \frac{E_1 t_1^3}{E_1 t_1^3 + E_2 t_2^3} M_\delta(B) \text{ and } M_{\delta 2}(B) = \frac{E_2 t_2^3}{E_1 t_1^3 + E_2 t_2^3} M_\delta(B)$$

where $M_\delta(B)$ is the total end moment of the form shown in Figure 4.1.a at the boundary B.

The required boundary condition in case (a) would be

$$v''(B) = - \frac{M_\delta(B)}{(EI)_\delta}$$

at the boundary B.

Cases (b) and (c) are of more practical interest especially in beam-column problems. Case (b) represents the application of end moments $M_d(B)$ at the boundary B by means of a couple acting on the gravity centers of the skins. Since no applied moment of the form of $M_\delta(B)$ exists, the required boundary condition would be

$$v''(B) = 0$$

at the boundary B.

Case (c) represents a case in which both moments, $M_d(B)$ and $M_\delta(B)$ are acting and the corresponding boundary condition will be like the one in case (a). A very practical particular case of this end load condition is the case of beam-columns with rigid inserts in the core at the ends or built-in ends of beams. This condition also occurs at points of symmetry. At those sections there is no shear deformation of the core because a line perpendicular to the skins remains perpendicular to these after deformation. Besides, at that particular section the Navier-Bernouilli principle is valid so that the Equivalent I-beam approach which neglects shear deformations of the core, studied earlier in this work, is exactly applicable. If $M(B)$ is the total applied end moment at the boundary B,

$$M(B) = M_d(B) + M_b(B)$$

and the section is kept undeformed at the ends, the corresponding boundary condition would be:

$$v''(B) = - \frac{M(B)}{(EI)}$$

at the boundary B. It will be seen in Chapter VII, however, that the condition to use in this case must be

$$v'''(B) = - \frac{M'(B)}{(EI)_b}$$

instead, which means that the total shear force is taken by the skins.

4) Concerning the fourth boundary condition, it is also simple to find it. At the section of symmetry, built-in ends or ends with rigid inserts, no shear deformation of

the core exists. With no shear deformation in the core $[\gamma(B) = 0]$, the shear stress in the core will also be zero $[\tau_c(B) = 0]$ and so the whole externally applied shear force is taken by the skins. The above condition may be re-stated as

$$v'''(B) = -\frac{M'(B)}{(EI)_\delta}$$

at the boundary B. [It corresponds to a distribution of shear stresses as shown in Figure 2.3.c, but with $\tau_c(B) = 0$].

For non-symmetric loading and support conditions with single spans and no concentrated loads along the span, the boundary conditions will be shown later to be as follows:

1) Deflection has to be defined at one support at least.
2) Either the deflection is defined at any other section or the slope is known at any of the boundaries.
3) Curvature is zero at a boundary where no end moments of the form $M_\delta(B)$ act and is $-M_\delta(B)/(EI)_\delta$ when they are acting. If there exists a restriction of the kind of a built-in end, or an end with a rigid insert, or a section of symmetry, the condition is as studied above for these cases.
4) Set-up in the same way as condition (3).

Conditions for continuous beams and segments of the beam under concentrated loads (when no symmetry may be invoqued) will be studied in Chapter VII.

No formulae for particular cases of loading are included here because the solution coincides with the *exact solution* to be found in Chapter VII, where the analysis of these formulae is also presented.

4.6 CONCLUSIONS

The generalization of Hartsock's method produces a solution which will be called *exact* here because no arbitrary assumptions of behaviour other than the standard ones specified in Chapter II were imposed. This method may be shown to coincide with the Equivalent I-beam Approach where shear deformations of the core are considered. In fact, Equation (4.16) may be expressed as follows

$$(EI)_f \, v^{iv}(x) - \frac{(EI)AG}{(EI)_d} v''(x) = - M''(x) + \frac{AG}{(EI)_d} M(x) .$$

In cases where $(EI)_f$ is negligible [and consistently, as $(EI)_f \to 0$, $(EI)_d \to (EI)$], the last equation becomes

$$v''(x) = - \frac{M(x)}{(EI)} + \frac{M''(x)}{AG} \qquad (3.2)$$

and the equivalence with the method studied in Section 3.4 is then proven. This shows that the method of the Equivalent I-Beam considering shear deformations of the core provides the exact solution for sandwich beams built-up with thin (and of course also very thin) skins if the expression for A is taken as

$$A = bd^2/c \tag{3.3}$$

for the case with thin skins and

$$A = bd \simeq bc$$

for the case with very thin skins.

Also, if $v(x)$ and $M(x)$ are split into components

$$v(x) = v_1(x) + v_2(x)$$

and

$$M(x) = M_1(x) + M_2(x)$$

and these are substituted into Equation (4.16), appropriate manipulations of this formula by the use of expressions (4.1) and (4.2) produces Allen's equation (4.3.a), which may then also be called *exact* for the cases studied. Unfortunately, it seems complicated to verify the correctness of the boundary conditions as stated by Allen by substitutions in the boundary conditions commented on for Hartsock's generalized solution (for example, six boundary conditions must be obtained from only four).

Finally, observing that a uniform beam can be considered as the limiting sandwich beam in which the core thickness tends to vanish, the limit when c tends to zero in formula (4.16) may be easily shown to coincide with the very well known formula

$$v''(x) = -\frac{M(x)}{(EI)} .$$

As said before, it is not possible to consider a homogeneous beam as the sandwich beam in which the core becomes very stiff. At least it is not valid in the case derived here.

Another interesting limiting case is the one in which the core is so weak that it can not transmit any shearing forces from skin to skin (See Figure 2.5.a). In this case, the substitution $G \to 0$ in (4.16) leads to

$$v^{iv}(x) = -\frac{M''(x)}{(EI)_6}$$

or $$v''(x) = -\frac{M(x)}{(EI)_6} ,$$

which indicates that the total bending moment $M(x)$ is taken by the skins acting independently.

CHAPTER V

GENERALIZATION OF HOFF'S SOLUTION

5.1 INTRODUCTION

The analytical methods studied in Chapter IV produce the *exact solution* to the behaviour of the type of sandwich panel specified in Chapter II. However, Allen's solution is not too clear in its derivations and so many arbitrary assumptions of behaviour are involved that its applicability to more general cases of loading and support seems complicated. Hartsock's generalized solution, on the other hand, does not provide the necessary boundary conditions for the solution of the differential equation which was shown to be right for every flexural loading situation. These are not easy to find in many cases and the solution as presented in Chapter IV would not be easy to use.

If Strain Energy Methods are applied, quite simple solutions may be derived and the values of the unknown variables at the boundaries may be *obtained* from the analysis. This chapter and Chapters VI and VII present methods of analysis based on Strain Energy Principles. The first two methods are generalizations of the work by Hoff(13) and Allen(1) to account for more general loading and support conditions and to make the results applicable to sandwich members with thick dissimilar skins. The third method is an application of the Strain Energy Principles to the problem as set up in Section 4.4.

5.2 DESCRIPTION OF HOFF'S METHOD

Hoff[13] solved the problem of bending of a cantilever sandwich beam having very thin identical skins* and subjected to a concentrated load at the free end. This work attempts to generalise his approach to consider the bending problem of a sandwich section having thick dissimilar skins and subjected to much more general loading conditions.

Hoff assumed that the deformation of the section under load could be split in two parts, one of them being the deformation due exclusively to axial displacements of the centroids of the skins with no bending allowed and the other being the deformation due to bending of the skins with no horizontal displacements of their centroids permitted. In order to maintain uniformity, the pattern of deformations of the section is taken as shown in Figure 2.4. A compatible solution could also be obtained by using Hoff's superposition of deformations.

The variables used by Hoff are the vertical deflection $v(x)$ and the horizontal displacement of the top skin $u_1(x)$. The

*Contrary to what he states, it will be shown later that the fact that he did not neglect the local bending stiffness of the skins does not necessarily mean that his analysis is applicable to cases with thick skins because his approximations are inconsistent.

latter is called u in his paper and $u_2(x)$ is missing because in the particular case solved by him, having a built-in end, creates a relationship between $u_1(x)$ and $u_2(x)$ as shown by equation (2.9.b) with $C = P = 0$. Moreover, as a result of the skins being identical in his case $(d_1 = d_2)$, the displacements $u_1(x)$ and $u_2(x)$ are equal and opposite.

The physical model of the sandwich beam used by Hoff in his derivations consists in "... a beam, having two thin but not infinitesimal faces [skins], and a core attached to the median lines of the faces [skins]. The extensional rigidity of the core is very small compared to that of the faces [skins], and its portion extending between $y = c/2$ and $y = (c+t)/2$ [the inner halves of the skins] does not store strain energy". By "... thin but not infinitesimal faces [skins], ...", according to his derivations, Hoff means that their thicknesses are negligible compared with the core thickness* but the local stiffness of the skins, $(EI)_{\delta}$ is not negligible compared to their stiffness $(EI)_d$ with respect to the reference level, which in his case is in the middle of the core. These two statements seem to be contradictory. The ratio $(EI)_{\delta}/(EI)_d$ depends on $(t/d)^2$ while $(d-c)/d$ is just t/d. The first ratio, being the square of a small number, is generally smaller than the second one, which is only of the first power.

* He does not state this directly but indirectly when equalling the shear deformation of the core to the rotation with respect to the reference level of the line joining the median lines of the skins [$\gamma(x)$ = Angle EFB in Figure 2.4].

It is important to mention again that the centroidal axis defined by d_1 and d_2 (Section 2.3) may not be taken as either the neutral axis or as the surface whose points have no horizontal displacements. In the case solved by Hoff, this is permissible only because he used identical skins where one of the ends of the beam was embedded in a very stiff wall (case $C=0$, Section 2.7) and no thrusts, P, were acting. For a general case with non-symmetric loading and support conditions, or when no rigid inserts occur along the span being considered, or in cases studying overhangs with or without loading, it is wrong to assume that the centroidal line may be taken as the axis with no points moving horizontally. Finally, if the neutral axis is taken as the surface whose points have displacements equal to the horizontal movement due to elastic shortening of the beam-column when acted upon by an end thrust, P, the deflection $v(x)$ is not affected but the displacements $u_1(x)$ and $u_2(x)$ change.

5.3 STRAIN ENERGY

The shear strains (unlike stresses) in the skins are neglected and the very low modulus of elasticity of the core results in a negligible contribution to the bending stiffness of the section. This implies that the only strain energies to be considered are the ones due to elongation and local bending of the skins and shear strains in the core. The expression for the total strain energy stored in the member is set up as follows:

1) Net elongation of the skins.

Since the axial displacements at the centroids of the skins are $u_1(x)$ and $u_2(x)$, the net strains at the skins are $u_1'(x)$ and $u_2'(x)$. These in the basic expression for strain energy

$$U = \frac{E}{2}\int_{Vol.} [Strain]^2 \; d\ Vol.$$

permit finding the strain energy due only to elongation of the skins as

$$U_1 = \frac{bt_1E_1}{2}\int_L [u_1'(x)]^2 dx + \frac{bt_2E_2}{2}\int_L [u_2'(x)]^2 dx \tag{5.1}$$

where L stands for the whole length of the member under consideration. No relationship between $u_1(x)$ and $u_2(x)$ was imposed and so this value will be the real ones. (These displacements are then measured from the originally vertical line and not from the already displaced line corresponding to the net compression due to end thrusts P).

2) Bending deformation of the skins.

The strain energy associated with the local bending deformation of the skins only is given by

$$U_2 = \frac{(EI)_{\delta}}{2}\int_L [v''(x)]^2 dx \tag{5.2}$$

where their deflection $v(x)$ is the same as for the whole panel.

3) Shear deformation of the core

The shear strain in the core from Equation (2.7) in the expression

$$U = \frac{G}{2} \int_{Vol.} [Shear\ Strain]^2 d\ Vol.$$

gives

$$U_3 = \frac{AG}{2} \int_L [v'(x) - \frac{u_1(x) - u_2(x)}{d}]^2 dx \qquad (5.3)$$

where

$$A = bd^2/c \qquad (3.3)$$

Thus the total strain energy, U, stored in the member may be obtained by adding up Equations (5.1), (5.2) and (5.3).

$$U = \frac{1}{2} \int_L [bt_1E_1u_1'^2 + bt_2E_2u_2'^2 + (EI)_f v''^2 + AG(v' - \frac{u_1 - u_2}{d})^2] dx \qquad (5.4)$$

5.4 LOSS OF POTENTIAL ENERGY OF THE APPLIED LOADS

The strain energy as found above is an exclusive function of the displacements $u_1(x)$, $u_2(x)$ and $v(x)$, where no loading terms are included on its expression. Hence, the loss of the potential energy of the applied loads will have to be considered in order to find the sought after relationship between displacements and loading. In order to find the loss of potential energy of the applied loads, any system of forces acting on the member has to be assumed. The loading system shown in Figure 5.1 will be used in order to maintain the generality of loading. The kinds of loading which are missing

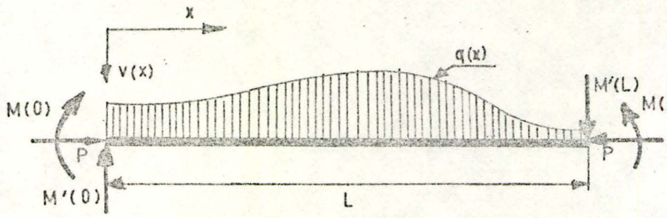

FIGURE 5.1

Loading System

are concentrated forces along the span and, less important, concentrated moments and shearing forces along the span. Inclusion of these types of loading is not absolutely necessary because, in case they do appear in a particular problem, it could be solved by parts where each portion between concentrated forces or moments has displacements $u_{i1}(x)$, $u_{i2}(x)$ and $v_i(x)$ and a length L_i. The strain energy stored in the whole member would then be given by the sum of the strain energies of the several portions,

$$U = \sum_i U_i(u_{i1}, u_{i2}, v_i, L_i).$$

The potential energy lost by the loads in Figure 5.1

are found as follows:

1) Loss of potential energy of $q(x)$

$$V_1 = -\int_0^L q(x)v(x)dx \tag{5.5.a}$$

2) Loss of potential energy of the end shear forces $M'(0)$ and $M'(L)$

$$V_2 = M'(0)v(0) - M'(L)v(L) \tag{5.5.b}$$

3) Loss of potential energy of end thrust P. This consists of two parts:

a) The loss of energy is partially due to the decrease in the distance from 0 to L owing to the curvature of the deflected member. It may be easily found(1) that this loss is given by

$$V_{3a}' = -\frac{P}{2}\int_0^L [v'(x)]^2 dx \tag{5.5.c}$$

b) The remaining loss of energy is due to the elastic shortening of the beam. To find this expression it is necessary to define the way in which end thrusts P are applied. The analysis being made here includes the application of end moments $M(0)$ and $M(L)$ (the loss of energy of these will be found next) and so it seems reasonable to apply the end thrusts at such a location that they do not produce any end moment. To solve a problem with eccentric end thrusts, these will be assumed to be applied with no eccentricity and then end moments Pe, (e being the eccentricity) will be applied.

The definition of the reference level or centroidal axis as used in this work suggests that, if a load is applied right at the centroid of the transformed section, the axial stresses produced in both skins are the same and so there is no contribution to bending. This being the case, the end thrusts will be assumed to act as shown in Figure 5.2. They do not act directly on the core but rather on the skins in order to be

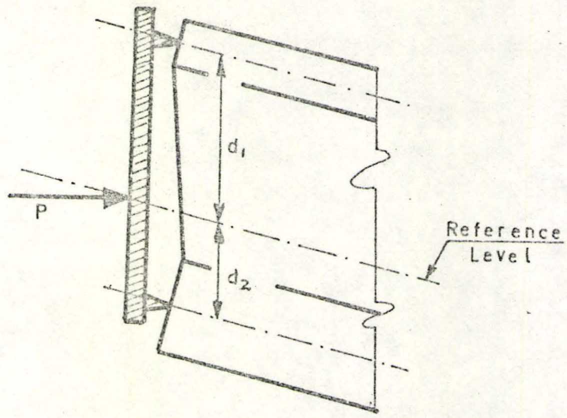

FIGURE 5.2

Application of End Thrusts

consistent with the assumption that the core does not contribute with any axial stresses. This loss of potential energy of the thrusts P may then be found as

$$V_{3b} = -\frac{P}{d}\{d_2[u_1(0) - u_1(L)] + d_1[u_2(0) - u_2(L)]\} \quad (5.5.d)$$

V_{3b} could have also been found as

$$-\frac{1}{2}\frac{P^2L}{b(t_1E_1 + t_2E_2)}$$

where, because it is independent of all displacement variables, it would drop out of the derivations when obtaining the variation of the total potential energy. If the last expression were used instead of (5.5.d), it would be equivalent to saying the the displacements $u_1(x)$ and $u_2(x)$ were not defined with respect to the originally vertical line joining the centroids of the skins in the undeflected section but rather with respect to this line in the location it would have if only the thrusts P were acting [That is, $w_i(x)$ would be used instead of $u_i(x)$]. The location of those two lines will differ by the amount representing the net axial shortening of the length L due to the axial loads P. Expression (5.5.d) will be used here to provide a more general definition of the variables $u_1(x)$ and $u_2(x)$.

4) Loss of potential energy of end moments $M(0)$ and $M(L)$.

This is a very delicate matter. In ordinary beams it would be quite true to give the loss of energy of the end moments as

$$- M(0)v'(0) + M(L)v'(L)$$

where the loss of energy corresponds to the applied moment times the angles they rotate through when the beam deflects. In general, an applied end moment may be thought of as being composed of a couple acting on the centroids of the skins plus end moments acting independently about the centroids of the skins as shown in Figure 4.1, where

$M(B) = M_{\delta}(B) + M_{d}(B)$.

The general assumptions of behaviour stated in Chapter II require that both skins deflect equally and this implies that the moments $M_{\delta 1}(B)$ and $M_{\delta 2}(B)$, at the boundary B being considered (0 or L), be related as follows:

$$M_{\delta 1}(B) = \frac{t_1^3 E_1 \; M_{\delta}(B)}{t_1^3 E_1 + t_2^3 E_2}$$

$$M_{\delta 2}(B) = \frac{t_2^3 E_2 \; M_{\delta}(B)}{t_1^3 E_1 + t_2^3 E_2}$$

$$M_{\delta}(B) = M_{\delta 1}(B) + M_{\delta 2}(B)$$

In other words, $M_{\delta 1}(B)$ and $M_{\delta 2}(B)$ must be proportional to the local bending stiffnesses of the corresponding skins.

The loss of energy of these moments is given by

$$V_{4a} = - M_{\delta}(0)v'(0) + M_{\delta}(L)v'(L) \qquad (5.5.e)$$

The loss of potential energy of the end moments acting as couples, $M_d(B)$ is

$$V_{4b} = - \frac{M_d(0)}{d} [u_1(0) - u_2(0)] + \frac{M_d(L)}{d} [u_1(L) - u_2(L)] \qquad (5.5.f)$$

Now it is clear that defining the loss of potential energy of the applied end moments in the way that at first glance seems to be logical, that is,

$$- M(0)v'(0) + M(L)v'(L),$$

is equivalent to saying that the whole external moment $M(B)$ is applied as shown in Figure 4.1.a. This implies that $M_{\delta}(B)$ equals $M(B)$ and $M_d(B)$ is zero at the boundary B. Even though it is theoretically possible to apply the end moments this way, it has no practical use since most of the cases involving the application of end moments correspond to beam-column problems, where the applied end moments are more likely of the form shown in Figure 4.1.b or are a combination of both, $M_{\delta}(B)$ and $M_d(B)$. This latter case corresponds to built-in ends and to sandwich beams having rigid inserts at the ends.

In this case of rigid inserts at the end (shown in

Figure 5.3), the moment $M_f(B)$ taken by the skins and the couple $M_d(B)$ may be shown to be:

$$M_f(B) = \frac{(EI)_f}{(EI)} M(B) \qquad (5.6.a)$$

$$M_d(B) = \frac{(EI)_d}{(EI)} M(B) \qquad (5.6.b)$$

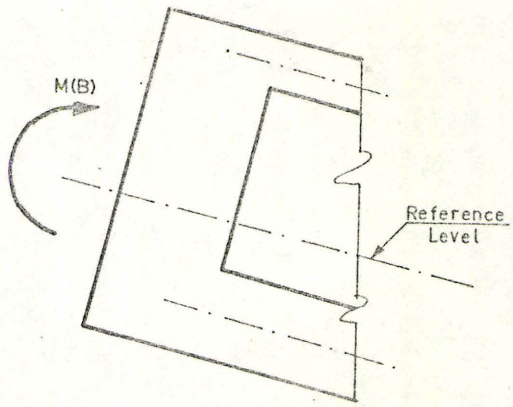

Figure 5.3
Effect of Rigid Inserts

Expressions (5.6) are applicable to any problem having no shear deformation of the core at the boundary B.

The bending behaviour of a sandwich beam subjected to external end moments has not been studied by any of the authors whose papers were reviewed for this work. Even though many of them give formulae for buckling loads, none of them indicates how the axial load should be applied in order not to introduce end moments due to eccentricity of the applied thrusts. Hummel[15] gives some experimental curves for sandwich beam-columns and stressed-skin beam-columns* and states that "...The load/deflection curve for these tests [eccentric compression of beam-columns with sandwich and

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* Stressed-skin panels either do not have or do not count on a core material to transmit shear forces and hold the skins apart. Longitudinal webs of stiff materials are used instead.

stressed-skin sections] agrees very well with the theoretical load/deflection curve as given by the secant formula". He does not give any indication about that formula. This is the only part of his paper mentioning theoretical analysis. All his other findings are of the kind "... the results of the tests fall somewhere between the theoretical case where the faces [skins] act independently and the theoretical case where they act as a unit, ...". No reference to this *Secant Formula* was found in any of the other references.*

The cases of loading studied so far are sufficient for many problems but there is still a possibility missing. It was mentioned earlier that the derivations in this section were being made for a case of loading having no concentrated forces or moments applied along the span L. If a problem having forces or moments of this kind is to be solved, it was also mentioned that the problem should be solved by parts, with each length between concentrated forces or moments being taken as the span L_i. In considering cases like this (see Figure 5.4), special care must be exercised in the evaluation of the boundary values of the shear forces and moments. When considering the value of the shear force $M'_\ell(L_\ell)$ for the right

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* Sections 3.3, 3.5 and 7.6 include, in their paragraphs E, formulas containing trigonometric *Secant* (or *cosine*) functions for the problem mentioned by Hummel. It is likely that he was referring to one of these in his paper, but unfortunately, no check on this could be made because the only graph he presents shows the load versus deflection curve for a stressed-skin panel. The use of an I-beam approach is recommended in his case and probably that is the formula he used.

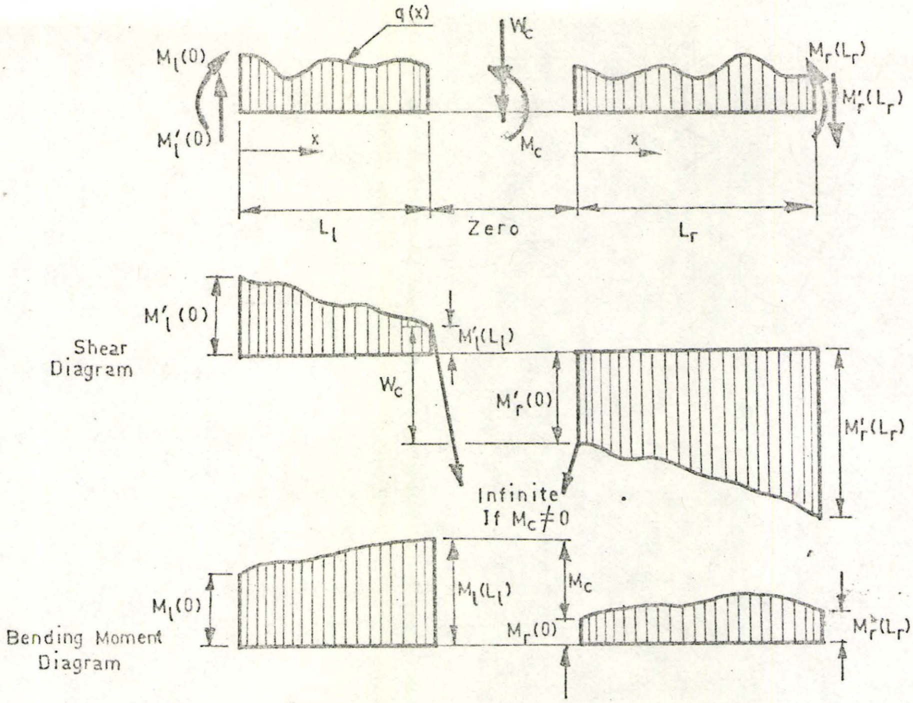

Figure 5.4

End Values of Shears and Moments in Loading Cases Having Concentrated Forces and Moments Along the Span

hand end of the length L_ℓ, this should be taken as the shear force immediately to the left of the point C having the concentrated load W_C. The same thing applies to the shear $M_r'(0)$ at the left end of span L_r.

Concerning the boundary values for the moments, a similar approach could be taken but an extra problem arises. The distribution of the moments $M_\ell(L_\ell)$ or $M_r(0)$ in terms of the values for M_δ and M_d have to be known independently to find the potential energy of the applied loads and so, knowledge of the values of the total end moments $M_{\ell}(L_{\ell})$ and $M_{r}(0)$ is necessary but not sufficient to obtain their contribution to the energy loss. The continuity of the slope $v'_{\ell}(L_{\ell}) = v'_{r}(0)$ and of the displacements of the centroids of the skins $u_{\ell 1}(L_{\ell}) = u_{r1}(0)$ and $u_{\ell 2}(L_{\ell}) = u_{r2}(0)$ will cause those losses of potential to cancel out when the operations are performed for the whole beam. No further analysis is then necessary for the moments caused by the remaining part of the beam at the boundary C. The way the external moment M_c is applied, however, does have to be indicated.

5.5 FURTHER COMMENTS ON THE BOUNDARY LOADS

In this work it has been insisted that the way the external loads are applied be defined. Saint-Venant's principle*, although originally derived for homogeneous beams, is quite often applied to other structural systems without demonstration of its validity.

Alwar[2] studied the applicability of Saint-Venant's principle by testing cantilever sandwich beams having a concentrated load at their free ends. The concentrated load was applied by resting it on the top skin and by hanging it from the centroid of the bottom skin. He found differences in

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* Stresses due to statically equivalent systems of loading are approximately the same at sections located at a distance from the points of application of the loads equal to or greater than the largest dimension of the section.

stresses in the core in the order of 40 per cent at distances greater than specified by Saint-Venant. His main conclusion is that, in general, Saint-Venant principle is not applicable to sandwich construction.

The differences in deflections produced by equal end moments of the forms M_δ and M_d are unacceptably large. Even if Saint-Venant's principle were applicable, it specifies that stresses, not deflections, are not too different when statically equivalent load systems act. It is quite evident that large differences in the curvatures exist at the boundaries of a simply supported beam having end moments M_δ or M_d. Hence, the deflections are very different too. However, as an interesting finding, the large differences in stresses (a measure of these being given by the second and third derivatives of the deflection) do seem to attenuate at distances over the one specified by Saint-Venant's principle. Figure 5.5 shows comparative plots of the deflection and its first three derivatives for a beam column having the same properties as the beam studied in section 3.6 and having an end thrust $P = 2{,}000\ \ell bs.$ acting on the centroid of the top skin (eccentricity e equals d_1).

Large differences in deflection $v(x)$ can be observed everywhere in the range 0 to $L/2$ plotted in Figure 5.5.a. The differences in the slope are still quite large but not as marked as for the deflections. The differences in the curvature v'' and the third derivative v''' can be seen to gradually

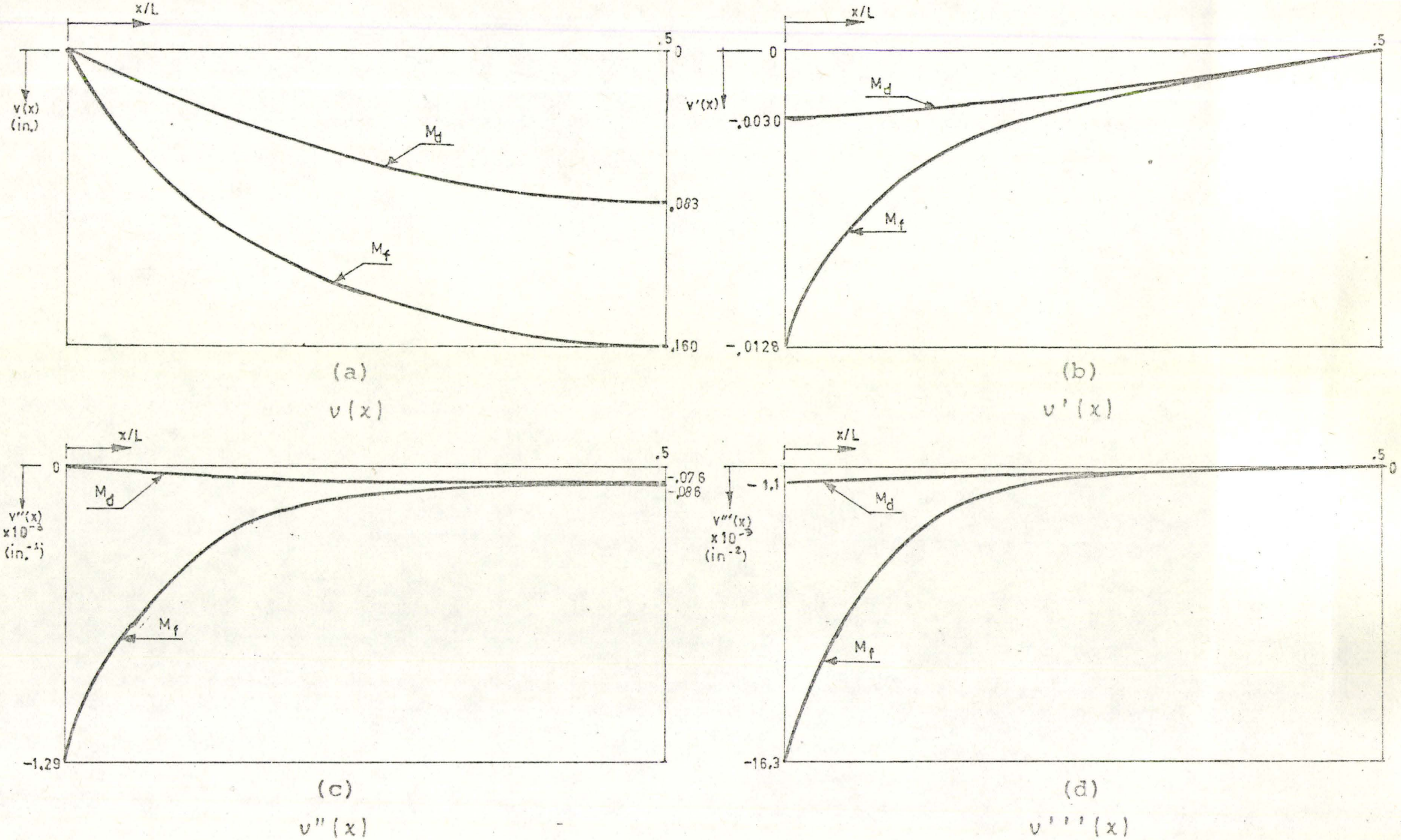

FIGURE 5.5

COMPARISON OF DEFLECTION AND THE FIRST THREE DERIVATIVES FOR TWO STATICALLY EQUIVALENT LOAD SYSTEMS disappear for abscisas near the center. As said before, the example shown in Figure 5.5 was calculated for a sandwich beam-column having rather thick skins. In cases with thinner skins, the difference would be greatly increased.

5.6 TOTAL POTENTIAL ENERGY

The total potential energy stored in the system may then be given by:

$$\Omega = U + V$$

$$\therefore\ \Omega = \frac{1}{2}\int_0^L \{bt_1E_1u_1'^2 + bt_2E_2u_2'^2 + (EI)_\delta v''^2 + AG(v' - \frac{u_1-u_2}{d})^2\}dx$$

$$- \int_0^L qv\,dx + M'(0)v(0) - M'(L)v(L)$$

$$- \frac{P}{2}\int_0^L v'^2 dx - P[d_2 \frac{u_1(0) - u_1(L)}{d} + d_1 \frac{u_2(0) - u_2(L)}{d}]$$

$$- M_\delta(0)v'(0) + M_\delta(L)v'(L) - M_d(0)\frac{u_1(0) - u_2(0)}{d}$$

$$+ M_d(L)\frac{u_1(L) - u_2(L)}{d} \qquad (5.7)$$

This expression was constructed from expressions (5.4) and (5.5) where, for the sake of brevity, the functions $v(x)$, $u_1(x)$, $u_2(x)$, $q(x)$ and their derivatives were expressed without any explicit indication that they are functions of x.

By using the general theorem stating that *the total potential energy of any system has a stationary value when that system is in equilibrium, and a minimum value where the equilibrium is stable*, a relationship between the variables $u_1(x)$, $u_2(x)$ and $v(x)$ and the applied loads $q(x)$, $M'(0)$, $M'(L)$, P, $M_f(0)$, $M_f(L)$, $M_d(0)$ and $M_d(L)$ may be obtained. For the solution of the mathematical problem of obtaining values of the functions $v(x)$, $u_1(x)$ and $u_2(x)$ satisfying the theorem just stated, several approaches may be used and some of them will be presented next.

5.7 SOLUTION BY CALCULUS OF VARIATIONS

The general theorem of Mechanics stated in the previous section may be interpreted as follows: If $v(x)$, $u_1(x)$ and $u_2(x)$ are the functions indicating the *actual* displacements undergone by the deformed structure and very small arbitrary variations $\delta v(x)$, $\delta u_1(x)$ and $\delta u_2(x)$ are introduced in these by any external means, the *change* in the total potential energy due to the very small *change* in the deformed configuration must be zero. This fact expressed in mathematical form is

$$\delta\Omega = \int_0^L \{bt_1E_1u_1'\delta u_1' + bt_2E_2u_2'\delta u_2' + (EI)_f v''\delta v''$$

$$+ AG(v' - \frac{u_1 - u_2}{d})(\delta v' - \frac{\delta u_1 - \delta u_2}{d}) - q\delta v$$

$$- Pv'\delta v'\}dx + M'(0)\delta v(0) - M'(L)\delta v(L)$$

$$- P[d_2 \frac{\delta u_1(0) - \delta u_1(L)}{d} - d_1 \frac{\delta u_2(0) - \delta u_2(L)}{d}]$$

$$- M_\delta(0)\delta v'(0) + M_\delta(L)\delta v'(L) - M_d(0) \frac{\delta u_1(0) - \delta u_2(0)}{d}$$

$$+ M_d(L) \frac{\delta u_1(L) - \delta u_2(L)}{d} = 0 \qquad (5.8)$$

The mathematical process for finding $\delta\Omega$ as it is described in Equation (5.8) is quite right for the very common case in which the applied loads $q(x)$, $M'(0)$, $M'(L)$, P, $M_\delta(0)$, $M_\delta(L)$, $M_d(0)$ and $M_d(L)$ do not depend upon the deformed configuration but rather are constant. It has been mentioned earlier, however, that the method being derived here should be applicable to portions of the beam bounded by points of application of concentrated forces and/or moments. In this case the applied loads at the boundaries 0 and L are the reactions from the next portion of the beam and there is no reason to believe they are independent of the displacements. The *missing term* in Equation (5.8) may be easily found. The variations in the applied loads due to the change in the deformed configuration produce the *missing term* as:

$$\int_0^L \delta q v dx - \frac{1}{2} \delta P \int_0^L v'^2 dx + \delta M'(0)v(0) - \delta M'(L)v(L)$$

$$- \delta M_\delta(0)v'(0) + \delta M_\delta(L)v'(L)$$

$$- \delta M_d(0) \frac{u_1(0) - u_2(0)}{d} + \delta M_d(L) \frac{u_1(L) - u_2(L)}{d}$$

$$- \delta P[d_2 \frac{u_1(0) - u_1(L)}{d} - d_1 \frac{u_2(0) - u_2(L)}{d}] \qquad (5.9)$$

It is not evident before hand that this term must vanish in the case in which L does not represent the whole span but rather a portion of the beam between concentrated forces W_c or moments M_c. In fact, it does not vanish. A term like the one above must appear for each portion L_i of the total span. If the external loads on the whole structure are assumed to be constant [independent of the deformed configuration given by $v(x)$, $u_1(x)$ and $u_2(x)$] the variation $\delta q(x)$ and δP could be easily understood to be zero. By the same argument, the values M', M_s and M_d at the left hand end of the first portion of the span and at the right hand end of the last span must also be zero. The analysis then has to concentrate on the values of M', M_s and M_d at the points c having concentrated forces and/or moments. When evaluating the so-called *missing terms* for two neighbouring portions of the span L_ℓ and L_r, the terms

$$- \delta M'_\ell(L_\ell) v_\ell(L_\ell) + \delta M'_r(0) v_r(0) \qquad (5.10)$$

will appear along with all the others, where the subindexes ℓ and r refer to values calculated for the portions of the span at the left and the right of the point C respectively (Point C is where the concentrated forces W_c and/or moments M_c act, as shown in Figure 5.4).

From Figure 5.4 it is evident that

$$M'_{\ell}(L_{\ell}) + W_c = M'_{r}(0) \qquad (5.11)$$

and so the terms in (5.10) may be re-written as follows:

$$- \delta M'_{\ell}(L_{\ell})v_{\ell}(L_{\ell}) + [\delta M'_{\ell}(L_{\ell}) + \delta W_c]v_2(0).$$

These terms vanish if the applied force W_c does not depend on the deformed configuration ($\delta W_c = 0$) and the *actual* vertical deflection $v(x)$ is continuous $[v_{\ell}(L_{\ell}) = v_r(0)]$.

The same analysis may be performed in terms including δM_s and δM_d. The condition for those terms to vanish are the constant value of the concentrated moment M_c externally applied at point C, the continuity of the slope $v'(x)$ (no infinite curvature is allowed) and the continuity of the horizontal displacements $u_1(x)$ and $u_2(x)$ of the centroids of the skins (the skins do not break).

It is important to make clear that in cases like the one just studied, Equation (5.8) should be of the form

$$\delta\Omega = \sum_i \delta\Omega_i = 0$$

where each term $\delta\Omega_i$ has the form of the terms at the left hand side of equation (5.8) with subscripts i in the variables and loads involved.

In order to obtain something meaningful from Equation (5.8), integration by parts must be performed in the terms involving variations of the derivatives of $v(x)$, $u_1(x)$ and $u_2(x)$. After those integrations by parts are carried out and some extra algebraic manipulations are performed, Equation (5.8) is found to be equivalent to

$$\int_0^L \{ [(EI)_\delta v^{iv} - (AG - P)v'' + AG \frac{u_1' - u_2'}{d} - q]\delta v$$

$$- [bd^2 t_1 E_1 \frac{u_1''}{d} + AG(v' - \frac{u_1 - u_2}{d})] \frac{\delta u_1}{d}$$

$$- [bd^2 t_2 E_2 \frac{u_2''}{d} - AG(v' - \frac{u_1 - u_2}{d})] \frac{\delta u_2}{d} \} dx$$

$$+ [(EI)_\delta v'''(0) - (AG - P)v'(0) + AG \frac{u_1(0) - u_2(0)}{d}$$

$$+ M'(0)]\delta v(0) - [(EI)_\delta v'''(L) - (AG - P)v'(L)$$

$$+ AG \frac{u_1(L) - u_2(L)}{d} + M'(L)]\delta v(L)$$

$$- [(EI)_\delta v''(0) + M_\delta(0)]\delta v'(0) + [(EI)_\delta v''(L) + M_\delta(L)]\delta v'(L)$$

$$- [Pd_2 + M_d(0) + bd^2 t_1 E_1 \frac{u_1'(0)}{d}] \frac{\delta u_1(0)}{d}$$

$$+ [Pd_2 + M_d(L) + bd^2 t_1 E_1 \frac{u_1'(L)}{d}] \frac{\delta u_1(L)}{d}$$

$$+ [Pd_1 + M_d(0) - bd^2 t_2 E_2 \frac{u_2'(0)}{d}] \frac{\delta u_2(0)}{d}$$

$$- [Pd_1 + M_d(L) - bd^2 t_2 E_2 \frac{u_2'(L)}{d}] \frac{\delta u_2(L)}{d} = 0. \qquad (5.12)$$

where, as in Equations (5.7), (5.8) and (5.9), for the sake of brevity, no explicit indication of the dependency on x was made in the variables involved.

Since the imposed variations $\delta v(x)$, $\delta u_1(x)$ and $\delta u_2(x)$ are arbitrary, each term in (5.12) must vanish separately. The condition for the integral to be zero is the vanishing of its three terms within brackets. This conclusion could be arrived at by giving zero values to two of the three increments involved and a non-zero value to the third one (arbitrarily chosen). The following three differential equations then result

$$(EI)_0 v^{iv}(x) - (AG - P)v''(x) + AG \frac{u_1'(x) - u_2'(x)}{d} = q(x) \qquad (5.13.a)$$

$$bd^2 t_1 E_1 \frac{u_1''(x)}{d} + AG[v'(x) - \frac{u_1(x) - u_2(x)}{d}] = 0 \qquad (5.13.b)$$

and $$bd^2 t_2 E_2 \frac{u_2''(x)}{d} - AG[v'(x) - \frac{u_1(x) - u_2(x)}{d}] = 0 \qquad (5.13.c)$$

The vanishing of the eight terms outside of the integral sign requires that either the function whose variation is involved be given (and so no variation is allowed) or that the term within parentheses vanishes. The eight boundary conditions thus produced are

Either $(EI)_{\delta} v'''(B) - (AG - P)v'(B) + AG \dfrac{u_1(B) - u_2(B)}{d} = - M'(B)$

or $v(B)$ given (5.14.a)

Either $(EI)_{\delta} v''(B) = - M_{\delta}(B)$ or $v'(B)$ given (5.14.b)

Either $bd^2 t_1 E_1 \dfrac{u_1'(B)}{d} = - Pd_2 - M_d(B)$ or $u_1(B)$ given (5.14.c)

Either $bd^2 t_2 E_2 \dfrac{u_2'(B)}{d} = Pd_1 + M_d(B)$ or $u_2(B)$ given (5.14.d)

where B represents 0 and L and so each condition in (5.14) must be satisfied at both boundaries.

Equations (5.13) represent a set of three simultaneous differential equations with constant coefficients which are not too difficult to solve. It can be shown that to completely define the solutions of these, eight boundary conditions are needed. Therefore, relations (5.14) (eight in total) are sufficient.

It is interesting to analyse Equations (5.13) and (5.14) because they contain a great deal of useful information about the structural behaviour of sandwich beams. Some observation of properties derived from them follow.

If Equations (5.13.b) and (5.13.c) are added up, then

$$t_1 E_1 u_1''(x) + t_2 E_2 u_2''(x) = 0$$

but from formulas (2.5) it may be easily found that

$$\frac{t_1 E_1}{t_2 E_2} = \frac{d_2}{d_1}$$

and so the expression above may be written as

$$\frac{u_1''(x)}{d_1} + \frac{u_2''(x)}{d_2} = 0 \qquad (5.15)$$

which, integrated once, would produce equation (2.8.a) after the constant of integration is found with the use of boundary conditions (5.14.c) and (5.14.d). The implications and interpretation of such an equation were already commented on in Section 2.7.

Equation (5.13.b) may be written as

$$-\,bt_1 E_1 [u_1'(x + dx) - u_1'(x)] = (bdx)\{\frac{d}{c}[v'(x) - \frac{u_1'(x) - u_2'(x)}{d}]G\}$$

and this result may be interpreted as follows. The term on the left side represents the difference of net axial forces in the top skin at both sides of a differential element dx long while the right side is the shear force in the core which is necessary to satisfy equilibrium. The term inside the second brackets is the shear stress in the core, as shown in formula (2.7) and bdx is the area in which that shear stress is acting. Equation (5.13.c) represents the same thing for the bottom skin.

5.8 COMMENTARY ON BOUNDARY CONDITIONS IN HOFF'S GENERALIZED SOLUTION

Expression (5.14.a) may be written as

$$bd\{\frac{d}{c}[v'(B) - \frac{u_1(B) - u_2(B)}{d}]G\} - (EI)_6 v'''(B) = M'(B) - Pv'(B)$$

and, as formula (2.7) shows it, the first term on the left side represents the total shear force taken by the core plus the linearly varying distribution of shear stresses as shown in Figure 2.3. The second term is the shear taken by the skins when bending about their own centroids. The terms on the right side represent the applied shear force at the boundary $B(0$ or $L)$ and the component of the axial force perpendicular to the deflected beam respectively. The value of the latter is significant only when the slopes are not too small because it corresponds to the consideration of secondary bending stresses created by the thrust P. This boundary condition is applicable at ends where the deflection is not given.

The boundary condition implied by expression (5.14.b) is that, if the slope at a boundary is not given, the curvature at that end depends exclusively on the moment externally applied about the centroids of the skins.

Expression (5.14.c) indicates that the total internal force in the top skin at the boundary, $bt_1E_1u_1'(x)$, must equal the negative of the total external axial force, $Pd_2/d + M_d(B)/d$. The last boundary condition calls for the same requirement with respect to the bottom skin. The strict way in which each of these conditions should have been presented is to specify that, at each boundary the total force (external plus internal) must vanish. This is the reason that the external loads appear in all the boundary conditions with the opposite sign in the right hand side.

The considerations discussed above are applicable for the ends of the beam but, if concentrated forces and/or moments are applied, the analysis would have been made in the following way: Three differential equations like (5.13) would have been obtained for each portion L_i of the member, in which the variables would now be $u_{1i}(x)$, $u_{2i}(x)$ and $v_i(x)$ but a difference would exist in the boundary conditions. For the first portion of the span, boundary conditions as in (5.14) may be applied but for the left end of the second span those conditions are replaced by the continuity of $v(x)$, $v'(x)$, $u_1(x)$ and $u_2(x)$ at the common point of both portions, while at its right end the conditions are again as in (5.14). The same thing applies to all the partial spans.

Formulas (5.13) and (5.14) coincide with the ones found by Hoff once the loading and dimensions by Hoff are replaced.

The only exception is the value of the constant A mentioned earlier in this chapter.

Hoff states in his book that his solution was verified by experiments, but he neither gives any clue about the way those experiments were carried out nor describes the dimensions and materials used in those tests. He says that he obtained results which were satisfactorily close to the theoretical predictions. The possible reason for this is that he may have tested sandwich beams having very thin skins, for which case his theory is exact. The solution of equations (5.13) may be carried out by eliminating the displacement functions $u_1(x)$ and $u_2(x)$ from these equations by a process involving differentiations and algebraic substitutions as follows:

Equations (5.13.b) and (5.13.c) give

$$\frac{u_1''(x) - u_2''(x)}{d} = -\frac{AG}{(EI)_d}\left[v'(x) - \frac{u_1(x) - u_2(x)}{d}\right] \tag{5.16}$$

after the identity

$$\frac{1}{(EI)_d} = \frac{1}{bd^2}\left(\frac{1}{t_1E_1} + \frac{1}{t_2E_2}\right) \tag{5.17}$$

is shown. If (5.13.a) is differentiated once and the resulting equation is solved simultaneously with (5.16) the result is

$$(EI)_6 v^{V}(x) - (AG-P)v'''(x) + \frac{(AG)^2}{(EI)_d}\left[v'(x) - \frac{u_1(x) - u_2(x)}{d}\right] = q'(x). \tag{5.18}$$

The elimination of $\frac{u_1(x) - u_2(x)}{d}$ from (5.13.a) and (5.18) produces

$$D^2\{D^4 - [\alpha^2 - P/(EI)_6]D^2 - \alpha^2 P/(EI)\}v(x) = \frac{1}{(EI)_6}[D^2 - \frac{AG}{(EI)_d}]q(x)$$

(5.19)

where

$$\alpha^2 = \frac{(EI)AG}{(EI)_6(EI)_d}$$

The variables $u_1(x)$ and $u_2(x)$ may now be found from (5.13). Even though they are not too useful after all coefficients in $v(x)$ are evaluated, they have to be obtained in every case because they appear in the boundary conditions for $v(x)$.

If the bending moment,

$$M(x) = M(0) + M'(0)x + Pv(x) - \int_0^x (x-\eta)q(\eta)d\eta, \quad (5.20)$$

due to the load system shown in Figure 5.1, is replaced in Equation (4.16), the result is

$$\{D^4 - [\alpha^2 - P/(EI)_6]D^2 - \alpha^2 P/(EI)\}v(x) = \frac{\alpha^2}{(EI)}[M(0) + M'(0)x]$$

$$+ \frac{1}{(EI)_6}[q(x) + \frac{AG}{(EI)_d}\int_0^x (x-\eta)q(\eta)d\eta]. \quad (5.21)$$

which, if derived twice with respect to x, coincides with Equation (5.19). In brief, if the loading system used in the derivation of Equation (5.13) is replaced in the second derivative of Equation (4.16), Equation (5.19) is produced. Equation (5.19) had previously been obtained from the elimination of $u_1(x)$ and $u_2(x)$ from Differential Equations (5.13). Thus the generalized form of Hartsock's solution and the generalized form of Hoff's solution are equivalent. The former has the advantage of greater generality and simplicity while the latter provides the boundary conditions, which as said before are not always easily obtained.

Formulae for some particular systems of loading are derived in Section 7.4 by using the method of analysis proposed in this thesis, which is exactly equivalent to Hoff's generalised solution for the loading case shown in Figure 5.1. Therefore, these will not be developed independently for Hoff's generalized solution.

5.9 RAYLEIGH-RITZ METHOD

This is the method most commonly used for the solution of the mathematical problem of finding the values of the deflection and the other displacements once the expression for the total potential energy is set-up. This is done as follows.

If the load $q(x)$ is expressed in its trigonometric Fourier form and suitable Fourier expressions* for $u_1(x)$, $u_2(x)$

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* By suitable Fourier expressions for $u_1(x)$, $u_2(x)$ and $v(x)$ it is meant those satisfying at least some of the boundary conditions of the problem, especially those related directly to values of the functions $u_1(x)$, $u_2(x)$ and $v(x)$ and, if possible, their first derivatives but not necessarily higher order derivatives. If the chosen functions are such that they satisfy all the boundary conditions, the solution converges much faster.[13]

and $v(x)$ can be found, these functions could be replaced in (5.7) and the integrations performed (the fact that the trigonometric eigen-functions in the Fourier series constitute an orthogonal set of functions, makes the integration a very simple one). The requirements for Ω to be a stationary function will now imply that its partial derivative with respect to every one of the coefficients of the Fourier expansions for $u_1(x)$, $u_2(x)$ and $v(x)$ should vanish. In this way an algebraic equation may be obtained for each of those coefficients. The integrations and subsequent differentiations are very simple as said above because of the orthogonality of the functions $Sin \frac{n\pi x}{L}$ and $Cos \frac{n\pi x}{L}$.

$$\int_0^L Sin \frac{i\pi x}{L} Sin \frac{j\pi x}{L} dx = \int_0^L Cos \frac{i\pi x}{L} Cos \frac{j\pi x}{L} dx = \begin{cases} 0 & i \neq j \\ L/2 & i = j. \end{cases} \quad (5.22)$$

Also the partial derivatives of Ω with respect to each of the coefficients of the Fourier expansions of $u_1(x)$, $u_2(x)$ and $v(x)$ provide linear algebraic equations containing only the coefficients of the three functions corresponding to the mode considered. That is, the partial derivative of Ω with the n^{th} coefficient of the Fourier expansion of $v(x)$, for instance, gives a linear algebraic equation containing the n^{th} coefficients of $u_1(x)$, $u_2(x)$ and $v(x)$ at most.

By Fourier expansion of a function $f(x)$ which is defined in a finite interval (the span L in this case), it is meant the Fourier expansion of a function $F(x)$ which is defined for the range from minus infinity to plus infinity and is such that in the finite interval in which the former function $f(x)$ is defined, it coincides with the latter function. It is possible to find an infinite number of functions $F(x)$ satisfying the conditions above and which are developable in a Fourier series (the conditions for this may be found in any text on the subject). This means that a function $f(x)$ defined in an interval $0<x<L$ has many possible Fourier expansions. The most commonly chosen functions $F(x)$ have period $2L$, where half the period contains the given function $f(x)$ while the other half is constructed arbitrarily in one of two ways. If it is constructed arbitrarily as the mirror image of the given function $f(x)$, it yields a *Cosine* series whereas an anti-symmetric choice for the other half of the period produces a Fourier expansion which is a *Sine* series. Figure 5.6 shows these two ways to construct $F(x)$.

The function $q(x)$ may be expressed as a *Sine* or a *Cosine* series or as any linear combination of these. The same thing can be said with respect to the variables $u_1(x)$, $u_2(x)$ and $v(x)$, but it is recommended to choose a series such that each of its eigenfunctions satisfies most of the boundary conditons, because the convergence is much faster. For instance, the zero deflection at the supports of a simply supported beam

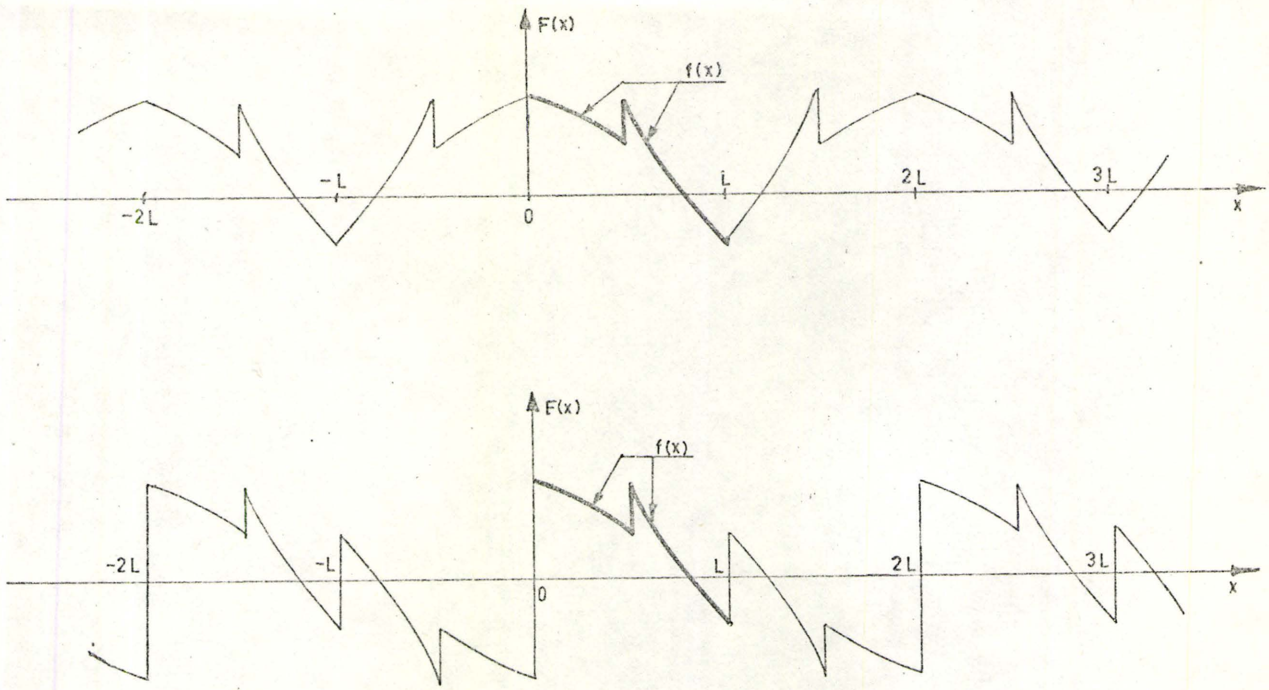

FIGURE 5.6

Cosine and *Sine* Series for $F(x)$

suggests the use of a *Sine* series for $v(x)$ (when one of the supports is chosen as the origin). In order to make operations simpler once the eigenfunctions for $v(x)$ are chosen to be sinusoidal, it is recommended to choose a *Sine* series for $q(x)$ and *Cosine* series for $u_1(x)$ and $u_2(x)$. This is the way the derivations are to be made here, but it is clear that there are other possibilities. Therefore, by choice,

$$q(x) = \sum_{n=1}^{\infty} q_n \sin\frac{n\pi x}{L}, \qquad v(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L},$$

$$u_i(x) = \sum_{n=1}^{\infty} b_{in} \cos\frac{n\pi x}{L} \qquad i = 1,2.$$

where the coefficients q_n are given by

$$q_n = \frac{1}{L}\int_0^L q(x)\sin\frac{n\pi x}{L}\,dx$$

and the coefficients a_n, b_{1n} and b_{2n} are to be found.

Substituting these values into equation (5.7) and performing integrations as in (5.23) produces

$$\Omega = \sum_{n=1}^{\infty}\{\frac{1}{4} bLs^2[t_1E_1b_{1n}^2 + t_2E_2b_{2n}^2] + \frac{1}{4}Ls^4(EI)_f a_n^2$$

$$+ \frac{1}{4} AGL\,[s^2a_n^2 - 2sa_n\frac{b_{1n} - b_{2n}}{d} + (\frac{b_{1n} - b_{2n}}{d})^2] - \frac{1}{2} Lq_na_n$$

$$- \frac{1}{4} PLs^2a_n^2 - \frac{P}{d}\{d_2[1-(-1)^n]b_{1n} + d_1[1-(-1)^n]b_{2n} - sM_f(0)a_n$$

$$+ (-1)^n sM_f(L)a_n - \frac{M_d(0)-(-1)^nM_d(L)}{d}(b_{1n} - b_{2n})\} \qquad (5.23)$$

where $s = \frac{n\pi}{L}$.

The principle of the stationary value of Ω may now be applied by saying that

$$\frac{\partial\Omega}{\partial a_n} = \frac{\partial\Omega}{\partial b_{1n}} = \frac{\partial\Omega}{\partial b_{2n}} = 0 \qquad n = 1,2,3,\ldots \tag{5.24}$$

Equations (5.24) are easily found to produce

$$s[s^2(EI)_\delta + (AG-P)]a_n - \frac{AG}{d}(b_{1n} - b_{2n}) = \frac{q_n}{s} + \frac{2}{L}[M_\delta(0) - (-1)^n M_\delta(L)] \tag{5.25.a}$$

$$dsAGa_n - (AG + bd^2s^2b_1E_1)b_{1n} - AGb_{2n} = -\frac{2d}{L}\{M_d(0) - (-1)^n M_d(L) + Pd_2[1-(-1)^n]\} \tag{5.25.b}$$

$$dsAGa_n - AGb_{1n} - (AG - bd^2s^2t_2E_2)b_{2n} = -\frac{2d}{L}\{M_d(0) - (-1)^n M_d(L) - Pd_1[1 - (-1)^n]\} \tag{5.25.c}$$

The solution of the simultaneous Equations (5.25) yields the values of the coefficients a_n, b_{1n} and b_{2n} which, once replaced in the Fourier expansion for $v(x)$, $u_1(x)$ and $u_2(x)$ above, provide the solution of the problem. This will be the *exact solution* when all terms are considered in the Fourier expansions.

As it was seen in Section 3.3, the second and third derivatives of the function $v(x)$ are necessary for the evaluation of axial stresses and shear stresses. If these are of a special interest to the designer, he can evaluate these derivatives by differentiating $v(x)$ in a term by term fashion with only one exception. When end moments $M_6(B)$ are applied at the boundaries or along the span, the third derivative of $v(x)$ with respect to x gives a divergent series because the function $F(x)$ corresponding to the second derivative (curvature) is not continuous at the boundaries*. This is an obvious disadvantage of the method for that particular case of loading. However, this is not the case with any other load condition. An advantage of the method is that the solution does not have to be split in parts when concentrated loads and moments act along the span. Thus, a more general system of loads than the one shown in Figure 5.1 could be studied. Another advantage is the simplicity of the theoretical solution where the three differential equations in the variational calculus approach with their eight boundary conditions became simply three

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* One of the conditions for a Fourier Series to be differentiable term by term is its continuity for the whole range $-\infty<x<\infty$. See, for instance, Whittaker, E.T. and Watson, G.N., "Modern Analysis", p. 169.

algebraic equations. The numerical evaluations, however, involve more calculations and the use of an electronic computer is almost necessary.

Rayleigh's method of taking only one term of the Fourier expansions for the variables involved is often used. Even though the accuracy in the numerical values of the involved variables is usually acceptable, the higher order derivatives (second and third mainly), so important in the evaluation of stresses in the section give results which have been shown to be far from the actual results. Therefore, this method is impractical for purposes of stress analysis and design.

CHAPTER VI

ALLEN'S STRAIN ENERGY METHOD*

6.1 INTRODUCTION

Hoff's generalized solution as presented in the previous chapter is an *exact solution* for the problem defined in Chapter II. The solution is not too complicated but it has to be obtained from three simultaneous equations with three unknown dependent functions $u_1(x)$, $u_2(x)$ and $v(x)$.** Therefore, it is worthwhile to investigate solutions which are obtainable from simpler mathematical processes. This chapter presents a method employed by Allen[(1)] in which two constants (only one in problems involving sections with identical skins) are introduced in the analysis with the probable intention of simplifying the solution. The method followed by Allen is described in the following sections.

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* Allen does not claim originality in his derivations. He mentions Williams, Legget and Hopkins as being the first authors using the idealizations employed by him. He also says that he follows March' and Ericksen's derivations. The method studied here will be referred to as Allen's because he presents it in a convenient form for discussion.

** In problems with symmetric loading and support conditions (or with sections having rigid inserts), a relationship between $u_1(x)$ and $u_2(x)$ may be found and the number of differential equations may be reduced to two.

6.2 ASSUMPTIONS OF BEHAVIOUR IN ALLEN'S STRAIN ENERGY METHOD

Allen made the derivations for sandwich beams having identical skins only. In this work all other methods have been generalized to account for dissimilar skins and, in order to make comparisons between them, Allen's theory is also generalized to include non-symmetric sections. This is done by following the derivations he makes in his chapter on sandwich plates and applying them to sandwich beams.

The model of deformations of the section used by Allen is shown in Figure 6.1. The only difference from the general model described in Figure 2.4 results from Allen's two main assumptions in his theory. One of them says that after loading, the angle formed by an originally vertical line in the core with the vertical line is a constant λ times the slope $v'(x)$. This is equivalent to saying that the shear deformation of the core $\gamma(x)$ and the slope $v'(x)$ are related by

$$\gamma(x) = (1-\lambda)v'(x) \qquad (6.1)$$

The second assumption states that the location of the neutral axis, defined by g_1 and g_2 in Figure 6.1, is fixed along the length of the beam.

The point H could lie outside of the core but it was drawn inside just to make the derivations clearer. Allen invariably draws it inside the core and he does not warn about the possibility of H being outside of the core.

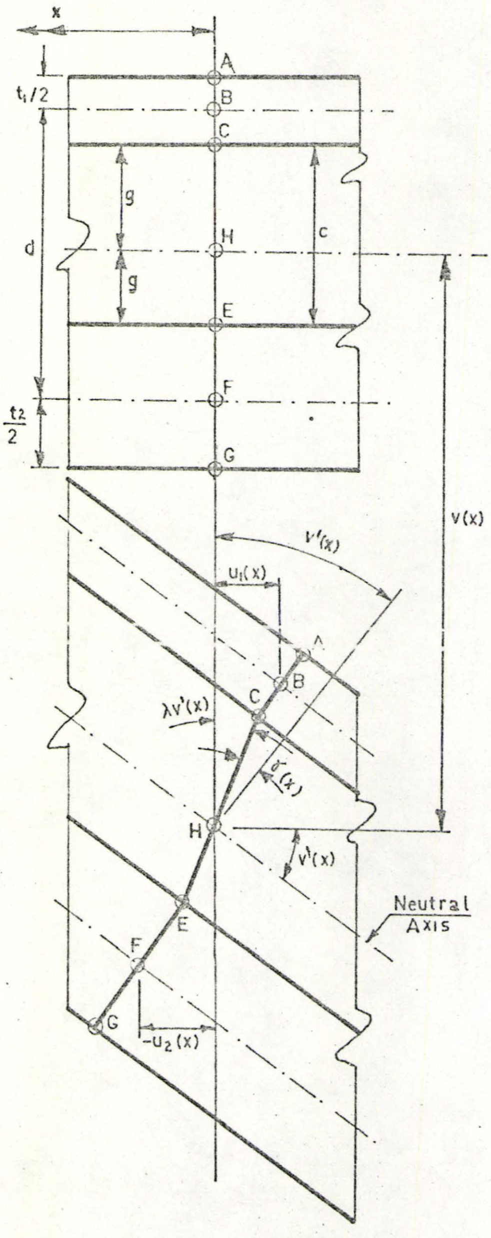

FIGURE 6.1

Allen's Deformed Section

Because of the way Allen made his derivations for sandwich beams having identical skins, his solution is applicable only to symmetric loading and support conditions. This deficiency is improved, however, when the neutral axis is left free and not restrained to a particular point. Thus the solution as presented here is applicable to any loading system.

6.3 IMPLICATIONS OF ALLEN'S ASSUMPTIONS

Allen defines his neutral axis as the family of points in the core with no horizontal displacements. From the *exact* theory developed in the previous chapter it may be easily found that the location of the neutral axis as defined above is determined by the distance $g_1(x)$ measured (positive when downwards) from the interface of the core and the top skin such that

$$\left\{\frac{[c-g_1(x)]u_1(x) + g_1(x)u_2(x)}{cv'(x)} + \frac{d-c}{c}\, g_1(x) - \frac{t_1}{2}\right\}v'(x) = 0 \qquad (6.2)$$

where c, d, $g_1(x)$, $v'(x)$ and t_1 are as defined in Figure 6.1 and $u_1(x)$ and $u_2(x)$ are the displacements of the median lines of the skins as defined in Figure 2.4. Since $v'(x)$ is not zero for all values of x, the sum of the terms enclosed in the brackets must vanish for all values of x. Therefore, the location of the neutral axis $g_1(x)$ is obtained by equalling the terms inside the brackets to zero. It is evident that, in general, $g_1(x)$ will be a function of x but, if it is required to be a constant, this implies that the proportionality

$$(c-g_1)u_1(x) + g_1 \quad u_2(x) = c(\frac{t_1}{2} - \frac{d-c}{c} g_1)v'(x) \tag{6.3}$$

must exist for all values of x.

The implications of Allen's assumption, expressed in equation (6.1), may be studied by equalling the *exact* shear deformation expressed by formula (2.7) with Equation (6.1). Thus

$$\gamma(x) = \frac{d}{c}[v'(x) - \frac{u_1(x) - u_2(x)}{d}] = (1-\lambda)v'(x)$$

produces

$$u_1(x) - u_2(x) = [d - (1-\lambda)c]v'(x) \tag{6.4}$$

Also, expressions (6.3) and (6.4) produce

$$u_1(x) = (\lambda g_1 + \frac{t_1}{2})v'(x), \tag{6.5.a}$$

$$u_2(x) = -(\lambda g_2 + \frac{t_2}{2})v'(x) \tag{6.5.b}$$

and

$$\frac{u_1(x)}{\lambda g_1 + \frac{t_1}{2}} + \frac{u_2(x)}{\lambda g_2 + \frac{t_2}{2}} = 0 \tag{6.5.c}$$

Equations (6.5) show that linear relationships exist (according to Allen's assumptions) for the three displacements involved. These relationships will be used later on.

Concerning Allen's definition of neutral axis, it is felt that he should have pointed out that it could fall out of the core because, in cases like that, his definition loses its sense.

In order to keep consistency, his definition of the neutral axis should have been made as the intersection of two lines as was specified in Chapter II of this work. The neutral axis can in fact fall outside of the core as shown in Figure 2.5. Case (a) corresponds to $\lambda = -\frac{t_1 + t_2}{2c}$, which is the minimum. Case (b) corresponds to $\lambda = 0$ or $g_1 = \infty$. Case (c) corresponds to $\lambda = 1$ or $g_1 = d_1 - \frac{t_1}{2}$. Case (d) corresponds to $\lambda > 1$, which is opposite to Allen's statement limiting the value of λ to a maximum value of 1.

6.4 VARIATION OF ALLEN'S PARAMETER λ

Concerning his definition of the constant λ, it is worthwhile to study it in more detail.

From expression (6.3)

$$\lambda(x) = 1 - \frac{d}{c}\left[1 - \frac{u_1(x) - u_2(x)}{dv'(x)}\right] \qquad (6.6)$$

where the function in the right hand side was called $\lambda(x)$ to make explicit that Allen's λ is really a function of x, and that his assumption that it is constant is just an approximation. Since the terms on the right side can be obtained from Hoff's generalized solution presented in the previous chapter, the function $\lambda(x)$ is easily evaluated for every case of load. To indicate the variation of this supposedly constant function, Figures 6.2 and 6.3 were plotted to show the variation over half the length for the beam described in Figure 6.2. The loading conditions are also described there.

From Figure 6.2 it is apparent that end moments of the form M_d produce values of $\lambda(x)$ which may be greater than 1, as opposed to what Allen states. It is also apparent that for moments of the form M_δ, the assumption λ = constant is unsatisfactory. A third very interesting observation is the way the function $\lambda(x)$ tends to become constant when the thrusts P approach the critical loading regardless of the type of transverse loading. It may be proven by using Hoff's generalized solution that when the thrust P reaches the critical value, $\lambda(x)$ is really constant. It may also be proven that any other kind of loading produces non-constant functions $\lambda(x)$ unless critical values of the end thrusts P are also acting. As an interesting result, Allen's approximation is *exact* only for buckling analysis.

Figure 6.3 shows that the approximation of $\lambda(x)$ being constant is not too bad for uniformly distributed load but is

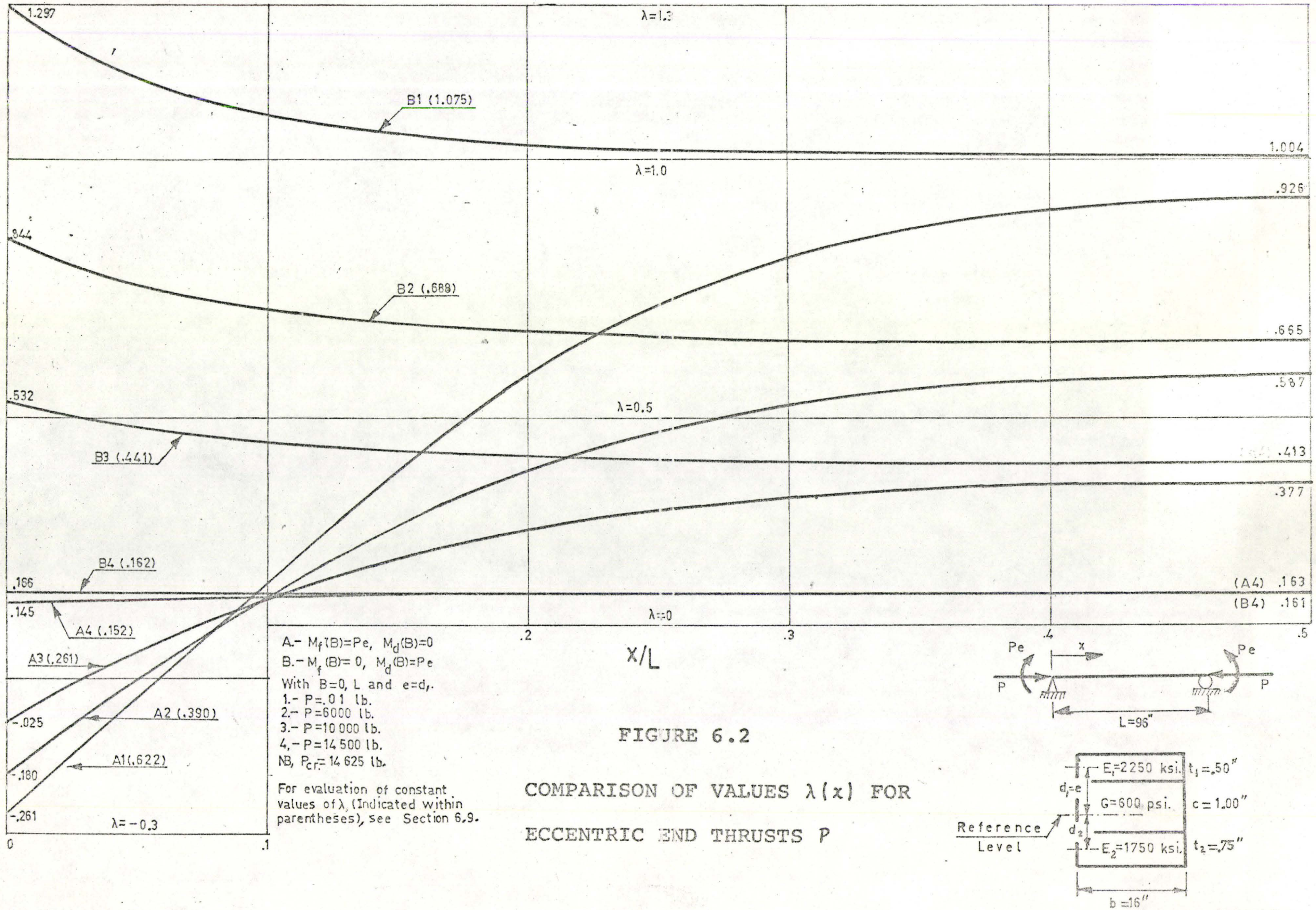

FIGURE 6.2

COMPARISON OF VALUES λ(x) FOR ECCENTRIC END THRUSTS P

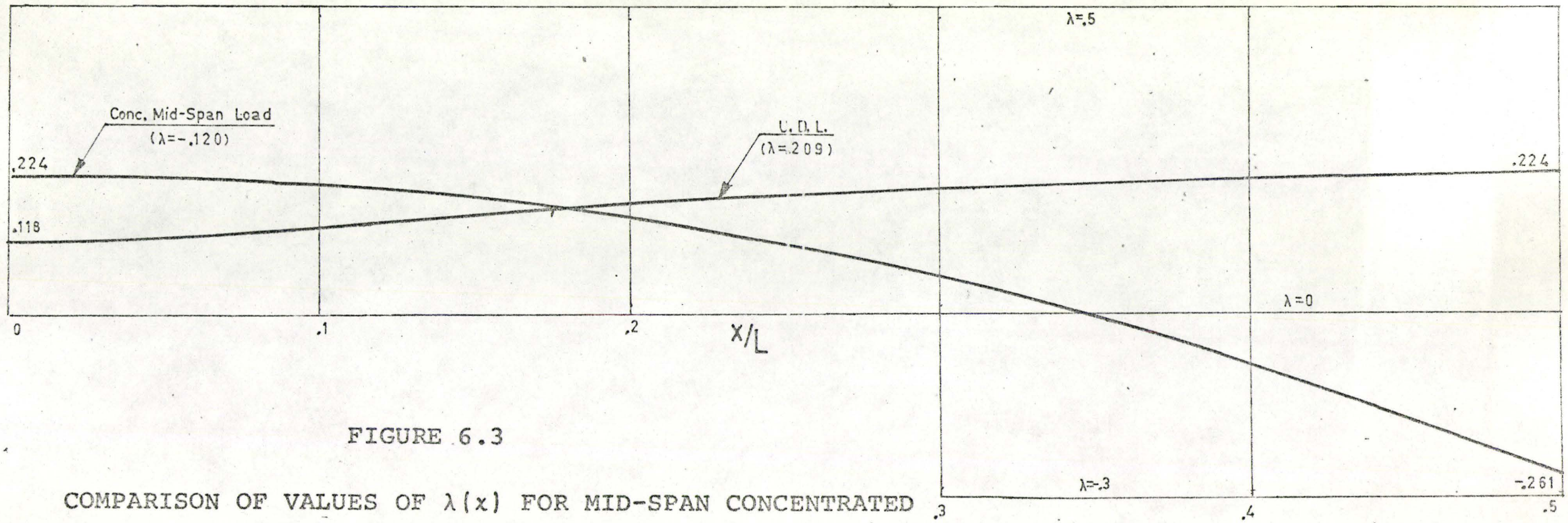

FIGURE 6.3

COMPARISON OF VALUES OF $\lambda(x)$ FOR MID-SPAN CONCENTRATED LOAD W AND UNIFORMLY DISTRIBUTED LOAD q

far from valid for concentrated loading at mid-span. For the latter case $\lambda(x)$ has a sudden change of slope at $x = L/2$ because of the sudden change in shear forces.

Incidentally, in several parts of his book Allen says that in his Section 7.7 he would prove that the assumption of $\lambda(x)$ being constant was valid for certain cases of load. No such proof was found.

Allen states that the limiting values of λ are

$$\lambda_{minimum} = - \frac{t_1 + t_2}{2c} \qquad \textit{corresponding to Figure 2.5.a}$$

$$\lambda_{maximum} = + 1 \qquad \textit{corresponding to Figure 2.5.c}$$

Allen justifies the value for $\lambda_{maximum}$ by saying that it corresponds to a sandwich section having a core material which is infinitely stiff in shear. This may look to be correct but it must be remembered that Allen made all his derivations based on the main assumption that the core was very weak. As a result he was able to neglect some terms in the same way as was done in this work. By no means is the case of a section with a stiff core a particular case of his derivations. It is very difficult to think of a material being simultaneously very weak for bending stresses and very stiff for shear stresses. Besides, values equal to and greater than 1 were obtained for core materials which were very weak in shear. These values depend mainly on the applied loads.

The obvious lack of meaning of Allen's constant λ makes it of little use.

6.5 STRAIN ENERGY

Referring to Figure 6.1, the strain energy stored in each of the components of the section may be found to be as follows:

1. Strain energy due to net elongation of the skins.

$$U_1 = \frac{b}{2}[t_1E_1(g_1\lambda + \frac{t_1}{2})^2 + t_2E_2(g_2\lambda + \frac{t_2}{2})^2]\int_0^L [v''(x)]^2 dx$$

2. Strain energy due to local bending of the skins.

$$U_2 = \frac{1}{2}(EI)_6 \int_0^L [v''(x)]^2 dx$$

3. Shear strain energy in the core.

$$U_3 = \frac{1}{2} F_2(\lambda)\int_0^L [v'(x)]^2 dx$$

with $\quad F_2(\lambda) = bcG(1-\lambda)^2 \qquad (6.7)$

In the same way as for the other methods, the shear strain energy stored in the skins is neglected along with the bending strain energy stored in the core.

The total strain energy stored in the member is then given by

$$U = \frac{1}{2}\int_0^L \{F_1(\lambda,g_1)[v''(x)]^2 + F_2(\lambda)[v'(x)]^2\}\,dx \qquad (6.8)$$

where $F_1(\lambda,g_1) = b[t_1E_1(g_1\lambda + \frac{t_1}{2})^2 + t_2E_2(g_2\lambda + \frac{t_2}{2})^2] + (EI)_\delta .$

$$(6.9)$$

6.6 LOSS OF POTENTIAL ENERGY OF THE APPLIED LOADS.

If the same externally applied loads as the ones considered for Hoff's generalized solution (Figure 5.1) are used, the loss of the potential energy of $q(x)$, $M'(0)$, $M'(L)$, $M_\delta(0)$ and $M_\delta(L)$ are as indicated in formula (5.5) but now the losses for $M_d(0)$, $M_d(L)$ and P have different expressions. These may be easily obtained by substituting Equation (6.5) into Equations (5.5.d) and (5.5.f) so that

$$V_{3b} = -\frac{P}{d}\{d_2(g_1\lambda + \frac{t_1}{2})[v'(0) - v'(L)]$$

$$- d_1(g_1\lambda + \frac{t_2}{2})[v'(0) - v'(L)]$$

$$= -PF_3(\lambda,g_1)[v'(0) - v'(L)] \qquad (6.10)$$

where $F_3(\lambda,g_1) = \frac{1}{d}[d_2(g_1\lambda + \frac{t_1}{2}) - d_1(g_2\lambda + \frac{t_2}{2})]. \qquad (6.11)$

And finally,

$$V_{4b} = -\frac{M_d(0)}{d}\,[(g_1\lambda + \frac{t_1}{2}) + (g_2\lambda + \frac{t_2}{2})]\,v'(0)$$

$$+\frac{M_d(L)}{d}\,[(g_1\lambda + \frac{t_1}{2}) + (g_1\lambda + \frac{t_2}{2})]\,v'(L)$$

$$= -F_4(\lambda)\,[M_d(0)v'(0) - M_d(L)v'(L)] \qquad (6.12)$$

with $$F_4(\lambda) = \frac{c}{d}(\lambda + \frac{t_1 + t_2}{2c}). \qquad (6.13)$$

The loss of potential energy of the applied loads may now be obtained by adding expressions (5.5) [Except for (5.5.d) and (5.5.f)], (6.10) and (6.12).

$$V = -\int_0^L q(x)v(x)\,dx + M'(0)v(0) - M'(L)v(L) - \frac{P}{2}\int_0^L [v'(x)]^2\,dx$$

$$- PF_3(\lambda,g_1)[v'(0) - v'(L)]$$

$$- M_\delta(0)v'(0) + M_\delta(L)v'(L) - F_4(\lambda)[M_d(0)v'(0) - M_d(L)v'(L)]$$

thus $$V = -\int_0^L \{q(x)v(x) + \frac{P}{2}[v'(x)]^2\}\,dx + M'(0)v(0) - M'(L)v(L)$$

$$- F_5(\lambda,g_1)v'(0) + F_6(\lambda,g_1)v'(L) \qquad (6.14)$$

with $$F_5(\lambda,g_1) = PF_3(\lambda,g_1) + M_\delta(0) + F_4(\lambda)M_d(0) \qquad (6.15.a)$$

and $F_6(\lambda, g_1) = PF_3(\lambda, g_1) + M_\delta(L) + F_4(\lambda)M_d(L)$ (6.15.b)

6.7 TOTAL POTENTIAL ENERGY

As before, the total potential energy is obtained by adding the strain energy and the loss of potential energy. Hence Equation (6.8) plus Equation (6.18) give

$$\Omega = \int_0^L \{\frac{1}{2} F_1 v''^2 + \frac{F_2 - P}{2} v'^2 - qv\}dx + M'(0)v(0) - M'(L)v(L)$$

$$- F_5 v'(0) + F_6 v'(L), \qquad (6.16)$$

and this function must be stationary.

There are as before several possibilities for applying the principle of the minimum potential energy. The approaches following the Rayleigh-Ritz' procedure are usually simpler but the methods provided by the Calculus of Variations furnish a great deal of useful information. Even though Allen invariably uses a Rayleigh-Ritz approach, it was thought to be interesting to include the Variational Calculus approach to facilitate comparison with Hoff's generalized method. These methods are studied next.

6.8 VARIATIONAL CALCULUS APPROACH

In order to simplify the calculations somewhat, a parameter $g = \lambda g_1$ will be used instead of g_1 in the derivations in the same way as Allen suggests. The function $v(x)$ and the parameters λ and g must be such that, when they are subjected to small arbitrary variations $\delta v(x)$ and $\delta\lambda$ and δg, the total potential energy Ω remains unchanged. Thus

$$\delta\Omega = \int_0^L [F_1 v''\delta v'' + (F_2-P)v'\delta v' - q\delta v]dx + M'(0)\delta v(0)$$

$$- M'(L)\delta v(L) - F_5\delta v'(0) + F_6\delta v'(L)$$

$$- \delta\lambda\{F_{5\lambda}v'(0) - F_{6\lambda}v'(L) - \frac{1}{2}\int_0^L [F_{1\lambda}v''^2 + F_{2\lambda}v'^2]dx\}$$

$$- \delta g\{F_{5g}v'(0) - F_{6g}v'(L) - \frac{1}{2}F_{1g}\int_0^L v''^2 dx\} = 0 \qquad (6.17)$$

where

$$F_{1\lambda} = \frac{\partial F_1}{\partial\lambda} = 2bct_2E_2(g_2\lambda + \frac{t_2}{2}) \qquad (6.17.a)$$

$$F_{1g} = \frac{\partial F_1}{\partial g} = 2b[t_1E_1(g_1\lambda + \frac{t_1}{2}) - t_2E_2(g_2\lambda + \frac{t_2}{2})] \qquad (6.17.b)$$

$$F_{2\lambda} = \frac{\partial F_2}{\partial\lambda} = -2bcG(1-\lambda) \qquad (6.17.c)$$

$$F_{3\lambda} = \frac{\partial F_3}{\partial\lambda} = -\frac{cd_1}{d} \qquad (6.17.d)$$

$$F_{3g} = \frac{\partial F_3}{\partial g} = 1 \qquad (6.17.e)$$

$$F_{4\lambda} = \frac{\partial F_4}{\partial\lambda} = \frac{c}{d} \qquad (6.17.f)$$

$$F_{5\lambda} = \frac{\partial F_5}{\partial \lambda} = \frac{c}{d}\,[M_d(0) - d_1 P] \tag{6.17.g}$$

$$F_{5g} = \frac{\partial F_5}{\partial g} = P \tag{6.17.h}$$

$$F_{6\lambda} = \frac{\partial F_6}{\partial \lambda} = \frac{c}{d}\,[M_d(L) - d_1 P] \tag{6.17.i}$$

and

$$F_{6g} = \frac{\partial F_6}{\partial g} = P \tag{6.17.j}$$

If integration by parts of the first integral in Equation (6.17) is performed to eliminate the variations of the derivatives of $v(x)$ and if formulae (6.17.a) to (6.17.j) are substituted into (6.17), it yields

$$\begin{aligned}
\delta\Omega = & \int_0^L [F_1 v^{iv} - (F_2 - P)v'' - q]\,\delta v\,dx - \{\tfrac{c}{d}[M_d(0)v'(0) - M_d(L)v'(L)] \\
& - \frac{c d_1 P}{d}\,[v'(0) - v'(L)] - \frac{1}{2}\int_0^L [F_{1\lambda} v''^2 + F_{2\lambda} v'^2]\,dx\}\,\delta\lambda \\
& - \{P[v'(0) - v'(L)] - \frac{1}{2} F_{1g}\int_0^L v''^2 dx\}\,dg \\
& + [F_1 v'''(0) - (F_2 - P)v'(0) + M'(0)]\,\delta v(0) \\
& - [F_1 v'''(L) - (F_2 - P)v'(L) + M'(L)]\,\delta v(L) - [F_1 v''(0) + F_5]\,\delta v'(0) \\
& + [F_1 v''(L) + F_6]\,\delta v'(L) = 0
\end{aligned} \tag{6.18}$$

The arbitrariness of the variations $\delta\lambda$, δg and $\delta v(x)$ in Equation (6.18) produces two algebraic equations for λ and g,

$$F_{1g}(g)\int_0^L [v''(x)]^2 dx = 2P[v'(0) - v'(L)] \tag{6.19.a}$$

and

$$\int_0^L \{F_{1\lambda}(\lambda,g)[v''(x)]^2 + F_{2\lambda}(\lambda)[v'(x)]^2\}dx = \frac{2c}{d}[M_d(0)v'(0) - M_d(L)v'(L)] - \frac{2cd_1P}{d}[v'(0) - v'(L)], \tag{6.19.b}$$

one fourth order differential equation for $v(x)$,

$$D^2[D^2 - \frac{F_2(\lambda) - P}{F_1(\lambda,g)}]v(x) = \frac{q(x)}{F_1(\lambda,g)}, \tag{6.20}$$

and four boundary conditions:

Either $v'''(0) - \frac{F_2(\lambda)-P}{F_1(\lambda,g)}v'(0) = -\frac{M'(0)}{F_1(\lambda,g)}$ or $v(0)$ given (6.21.a)

" $v'''(L) - \frac{F_2(\lambda)-P}{F_1(\lambda,g)}v'(L) = -\frac{M'(L)}{F_1(\lambda,g)}$ " $v(L)$ " (6.21.b)

" $v''(0) = -\frac{F_5(\lambda,g)}{F_1(\lambda,g)}$ " $v'(0)$ " (6.21.c)

" $v''(L) = -\frac{F_6(\lambda,g)}{F_1(\lambda,g)}$ " $v'(L)$ " (6.21.d)

A comparison of formulae (6.19), (6.20) and (6.21) with (5.13) and (5.14) seems to indicate that Allen's assumptions do simplify the mathematical problem. In effect, two of the three differential equations in (5.13) become algebraic (algebraic equations are generally easier to solve) and consistently, the corresponding boundary conditions disappear. From a practical viewpoint, however, Allen's solution is set up in a way which is almost impossible to solve because of the difficulty in solving the algebraic equations. The mathematical procedure to solve the set of equations (6.19) to (6.21) is described below.

Differential Equation (6.20) may be easily solved once $q(x)$ is given. The result is the deflection $v(x)$ expressed as a function of λ, g_1 and four constants that will be known (also as functions of λ and g_1) once boundary conditions (6.21) are applied. The function $v(x)$ is then differentiated twice and the second powers of its first two derivatives must be integrated along the span of the beam L. The resulting functions of λ and g_1 are substituted into Equations (6.19) and then those two algebraic equations must be solved. The solution of Equations (6.20) will contain trigonometric functions if $F_2(\lambda)$ is less than P, polynomial functions if $F_2(\lambda)$ is equal to P and hyperbolic functions if $F_2(\lambda)$ is greater than P. This implies that any one of the three possibilities has to be assumed before solving Equation (6.20) and, after algebraic Equations (6.19) are solved, a verification of the value of $F_2(\lambda)$ versus P has to be made. If the assumed sign of $F_2(\lambda)-P$ does not agree with the final finding, one of the two remaining assumptions has to be tried and the problem solved again. The complicated form of the algebraic equations makes it almost compulsory to use iterative methods and this is not feasible without the help of an electronic computer.

A possibility for simplifying the solution to the mathematical problem arises if V_{3b} is taken as

$$V_{3b} = -\frac{1}{2}\frac{P^2 L}{b(t_1E_1 + t_2E_2)}$$

instead of formula (5.5.d). This is equivalent to stating that the net shortening of the member does not affect its bending behaviour. This was discussed in Section 5.4. This change implies first that the neutral axis now is defined as the surface where its points have an axial strain equal to the net axial strain in the whole section rather than zero horizontal displacements. Secondly it is implied that the calculated axial stresses in the skins will have to be added to the stress in the section due to end thrusts to obtain the actual values of the stresses. Boundary conditions (5.14.c) and (5.14.d) and Equation (6.19.a) are the only ones affected as a result of the disappearance of the terms accompanying P. The other functions are not affected because $u_1(x)$ and $u_2(x)$ always are either subtracted one from the other (the net displacements in the skins are affected by P but their differences are not) or they appear in a second derivative (the strains caused by the thrusts

P are constant and so their derivatives vanish). Hence it is evident that the deflection $v(x)$ is not affected by the net stresses caused by P. In cases with no axial thrusts acting, logically, the same simplifications take place. In both cases, Equation (6.19.a) becomes:

$$F_{1g}(\lambda, g_1) = 0$$

which implies that

$$g_1 = \frac{2cd_1\lambda - d_2t_1 + d_1t_2}{2d\lambda} \tag{6.23}$$

and the problem is less difficult to solve.

The calculations for a simply supported beam having a uniformly distributed load, a mid-span concentrated load, end thrusts P and symmetric moments $M_d(0) = M_d(L)$ and $M_\delta(0) = M_\delta(L)$ were performed using a computer to solve the complicated algebraic equations for λ. Equation (6.23) was used, therefore g_1 does not have to be considered as an independent parameter any longer. The resulting constant values of λ for the particular cases described in Figure 6.2 are described in both, Figures 6.2 and 6.3 along with the variation of the actual function $\lambda(x)$ as obtained by applying Hoff's generalized solution. It is not worthwhile to include the derivations here because of their length and complication.

6.9 THE GENERAL APPLICATION OF THE RAYLEIGH RITZ METHOD

In view of the difficulty in solving the mathematical problem set up in Section 6.8, the use of the very simple application of the Rayleigh-Ritz method seems to be recommendable. The set up of this method is described below.

If the distributed load $q(x)$ may be expressed in a *Sine* Fourier Series,

$$q(x) = \sum_{n=1}^{\infty} q_n \, Sin \, \frac{n\pi x}{L} ,$$

the solution $v(x)$ may be stated to have the form

$$v(x) = \sum_{n=1}^{\infty} a_n \, Sin \, \frac{n\pi x}{L} .$$

Considering the orthogonality of the eigenfunctions $Sin \frac{n\pi x}{L}$ and $Cos \frac{n\pi x}{L}$, Equation (6.15) may be written as follows:

$$\Omega = \sum_{n=1}^{\infty} \{\frac{1}{2} F_1 s^4 \frac{L}{2} a_n^2 + \frac{1}{2}(F_2 - P) s^2 \frac{L}{2} a_n^2 - \frac{L}{2} q_n a_n - s[F_5 - (-1)^n F_6] a_n\},$$

where $s = \frac{n\pi}{L}$.

The total potential energy will be a minimum if

$$\frac{\partial\Omega}{\partial a_n} = \frac{\partial\Omega}{\partial\lambda} = \frac{\partial\Omega}{\partial g} = 0 \qquad n = 1,2,3,\ldots$$

Hence,

$$\frac{\partial\Omega}{\partial a_n} = \frac{L}{2}\{s^2[s^2F_1 + (F_2-P)]a_n - q_n\} - s[F_5-(-1)^n F_6] = 0$$

and

$$a_n = \frac{\frac{2s}{L}[F_5(\lambda,g)-(-1)^n F_6(\lambda,g)] + q_n}{s^2[s^2F_1(\lambda,g) + F_2(\lambda)-P]}, \qquad n = 1,2,3,.. \tag{6.24.a}$$

$$\frac{\partial\Omega}{\partial\lambda} = \sum_{n=1}^{\infty} s\{\frac{1}{2}\frac{L}{2}s[s^2F_{1\lambda}(\lambda,g)+F_{2\lambda}(\lambda)]a_n - [F_{5\lambda}-(-1)^n F_{6\lambda}]\}a_n = 0 \tag{6.24.b}$$

$$\frac{\partial\Omega}{\partial g} = \sum_{n=1}^{\infty} s\{\frac{1}{2}\frac{L}{2}s^3F_{1g}(\lambda,g)a_n - [1-(-1)^n]P\}a_n = 0 \tag{6.24.c}$$

From Equation (6.24.c), and using Equation (6.17.b), g_1 may be obtained as a function of the coefficients a_n in an expression having the form

$$g = \lambda g_1 = \frac{\sum_{n=1}^{\infty}(C_{1n}a_n + C_{2n})a_n}{\sum_{n=1}^{\infty}(d_{1n}a_n + d_{2n})a_n} \tag{6.25}$$

where coefficients c_{1n}, c_{2n}, d_{1n} and d_{2n} are all exclusively dependent on the properties of the sandwich beam and thus may be always found. By substituting the appropriate functions from Equation (6.17) into Equation (6.24.b), an expression of the form

$$\lambda = \frac{\sum_{n=1}^{\infty} (e_{1n}a_n + e_{2n})a_n}{\sum_{n=1}^{\infty} (f_{1n}a_n + f_{2n})a_n} \tag{6.26}$$

is obtained, where the coefficients e_{1n}, e_{2n}, f_{1n} and f_{2n} depend upon the properties of the beam and also upon g.

The solution becomes very complicated because g (or g_1) is a function of the unknown coefficients a_n, and λ is a function of g and of all the coefficients a_n. The solution requires the use of iterative methods with two independent parameters, λ and g, satisfying two conditions, and as a result is very difficult.

As it was said in Section 6.8, the solution of the problem may be simplified by changing the definition of the neutral axis. In that case the term accompanying P in Equation (6.24.c) disappears and this equation becomes

$$\frac{1}{2}\,\frac{L}{2}\,F_{1g}(\lambda, g_1) \sum_{n=1}^{\infty} s^3 a_n^2 = 0.$$

Because the summation is always positive, $F_{1g}(\lambda, g_1)$ must be equal to zero, which leads to expression (6.23) again.

In this case, the problem must be solved as follows. First, the number of terms to be used in series expansion of $v(x)$ has to be defined (the method is no longer exact). Then a value for λ has to be assumed so that coefficients a_n may be obtained from (6.24.a) and substituted into (6.26) to check the assumed value of λ. The values of the assumed and the calculated λ may be made to converge by an iterative process, and once a satisfactory value of λ is obtained, the coefficients a_n may be substituted into the expression for $v(x)$ to obtain the final solution. Computer programs to solve the problem as explained above were used to check the accuracy of the method. Answers quite close to the exact method studied in the previous section were obtained by using 10 to 50 terms. Cases with distributed loads require fewer terms than cases with concentrated loads and end moments.

6.10 THE RAYLEIGH-RITZ METHOD AS APPLIED BY ALLEN

The methods presented in Sections 6.8 and 6.9 are impractical because of the difficulty in obtaining answers because the mathematical problem is very cumbersome to solve. Allen uses a method of analysis which is valid for simply supported beams having distributed loads and end thrusts with no eccentricity. His method is explained as follows. If an axial thrust P with no eccentricity and a sinusoidaly varying distributed load of the form

$$q(x) = q_n \, Sin \, sx,$$

with $s = \frac{n\pi}{L}$ act on a simply supported sandwich beam, it may be found from Equation (6.24.a)*(if only the bending contribution of the end thrusts is considered) that the deflection $v(x)$ may be expressed as

$$v(x) = a_n \, Sin \, sx.$$

Allen then replaces $v(x)$ in the expression for the total potential energy (6.15) and finds a_n, λ_n and g_n from

$$\frac{\partial \Omega}{\partial a_n} = \frac{\partial \Omega}{\partial \lambda_n} = \frac{\partial \Omega}{\partial g_n} = 0 \qquad n = 1, 2, 3, \ldots$$

where the subindex n was added to λ and g in order to clarify that the values of λ and g to be obtained are applicable only to the nth mode. Allen does not make that distinction.

The two last equations above produce

$$g_{1n} = \frac{2cd_{1\lambda n} - d_2 t_1 + d_1 t_2}{2d\lambda_n} \qquad (6.27.a)$$

and

$$\lambda_n = \frac{1 - \dfrac{t_1 + t_2}{2c} \dfrac{(EI)_d}{AG} s^2}{1 + \dfrac{(EI)_d}{AG} s^2}. \qquad (6.27.b)$$

and the first one yields

$$s^2 [s^2 F_1(\lambda_n, g_n) + F_2(\lambda_n) - P] a_n = q_n$$

---

* This is more strict than Allen's original form. He states that the sinusoidal deflected shape "may be assumed".

or $$a_n = \frac{q_n}{s^2[s^2F_1(\lambda_n, g_n) + F_2(\lambda_n) - P]} \quad . \qquad (6.28)$$

The deflection at mid-span, a_n, and the parameters λ_n and g_n being now known permits the solution to be constructed and it is the exact mathematical solution to the problem as set up earlier in this chapter under the loads described above.

To generalize it to a general distributed load given by

$$q(x) = \sum_{n=1}^{\infty} q_n \, Sin \, sx,$$

with

$$s = \frac{n\pi}{L},$$

Allen follows the same steps again. First the solution is "assumed" to be of the form

$$v(x) = \sum_{n=1}^{\infty} a_n \, Sin \, sx.$$

These values are substituted into expression (6.16) and then he applies

$$\frac{\partial\Omega}{\partial a_n} = \frac{\partial\Omega}{\partial\lambda_n} = \frac{\partial\Omega}{\partial g_n} = 0 \qquad n = 1,2,3,\ldots$$

The last two of these equations give expressions similar to Equations (6.27). The first equation gives formula (6.28) as before. With those three values known, the solution is easily constructed.

6.11 CONCLUSIONS

It was shown in this chapter that Allen's assumptions simplify the problem of bending of sandwich beams under certain conditions of loading and if his mathematical methods are followed. Calculations were performed for several loading cases in order to compare the accuracy of the solution. The results are commented on below.

Figure 6.4 shows a plot of the deflection for half the span of a simply supported beam subjected to uniformly distributed loading. The intensity of this load and the properties of the beam under consideration are also indicated. The solutions are obtained by the Equivalent I-beam approach (both with and without including shear deformations of the core) and the *exact solution* as presented in Chapter V are also included as a basis for comparisons. A close agreement between Allen's Strain Energy method and the *exact solution* is observed, the maximum difference (at mid-span) being 1.4 percent.

The convergence of both Rayleigh-Ritz methods studied in Sections 6.9 and 6.10 is very fast. The solution plotted in Figure 6.4 was calculated with 50 terms and the first 8 digits coincide in both approaches and in the variational

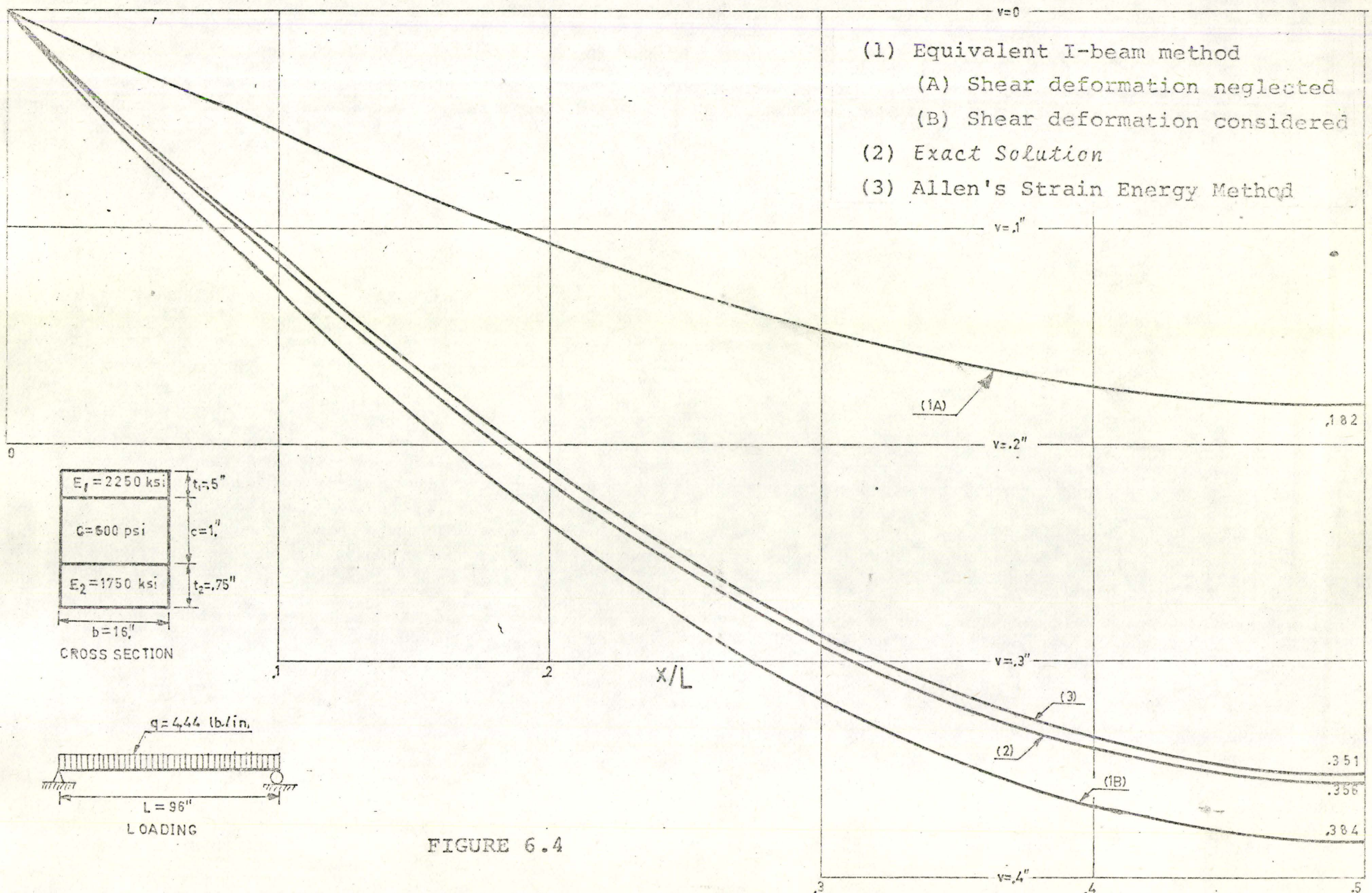

FIGURE 6.4

DEFLECTION COMPARISONS FOR UNIFORMLY DISTRIBUTED LOAD calculus approach. A much smaller number of terms would very likely have produced sufficiently accurate answers. From this viewpoint, the method as presented by Allen is an improvement (for the particular cases of loading discussed earlier) over the generalized form of Hoff's method studied in Chapter V. For other kinds of loading the same thing can not be said.

Figure 6.5 presents the plot of the deflection for the same beam as before but subjected to a mid-span concentrated load. If the exact mathematical solution of Allen's problem is used (either Section 6.8 or 6.9), the difference from the *exact solution* is 5.0 percent at mid-span. This difference is due just to the assumptions that $\lambda(x)$ and g_1 are constant. If Allen's application of the Rayleigh-Ritz method is used*, a difference of 10.6 percent from the *exact solution* is obtained, and this difference is not acceptable. The coincidence of the methods outlined in Sections 6.8 and 6.9 is in this case almost perfect (50 terms were used in all series as before).

In cases with eccentric end thrusts, where the end moments are assumed to be of the form $M_d(0) = M_d(L) = Pe$, shown in Figure 6.6, the agreement between the results of all formulas (the Equivalent I-beam method considering shear deformations of the core excepted) is very good due to the very small values of

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* In his book, Allen applies his method only to distributed loads and end thrusts. The way he presented his theory, however, does not seem to ban its application to other loading systems.

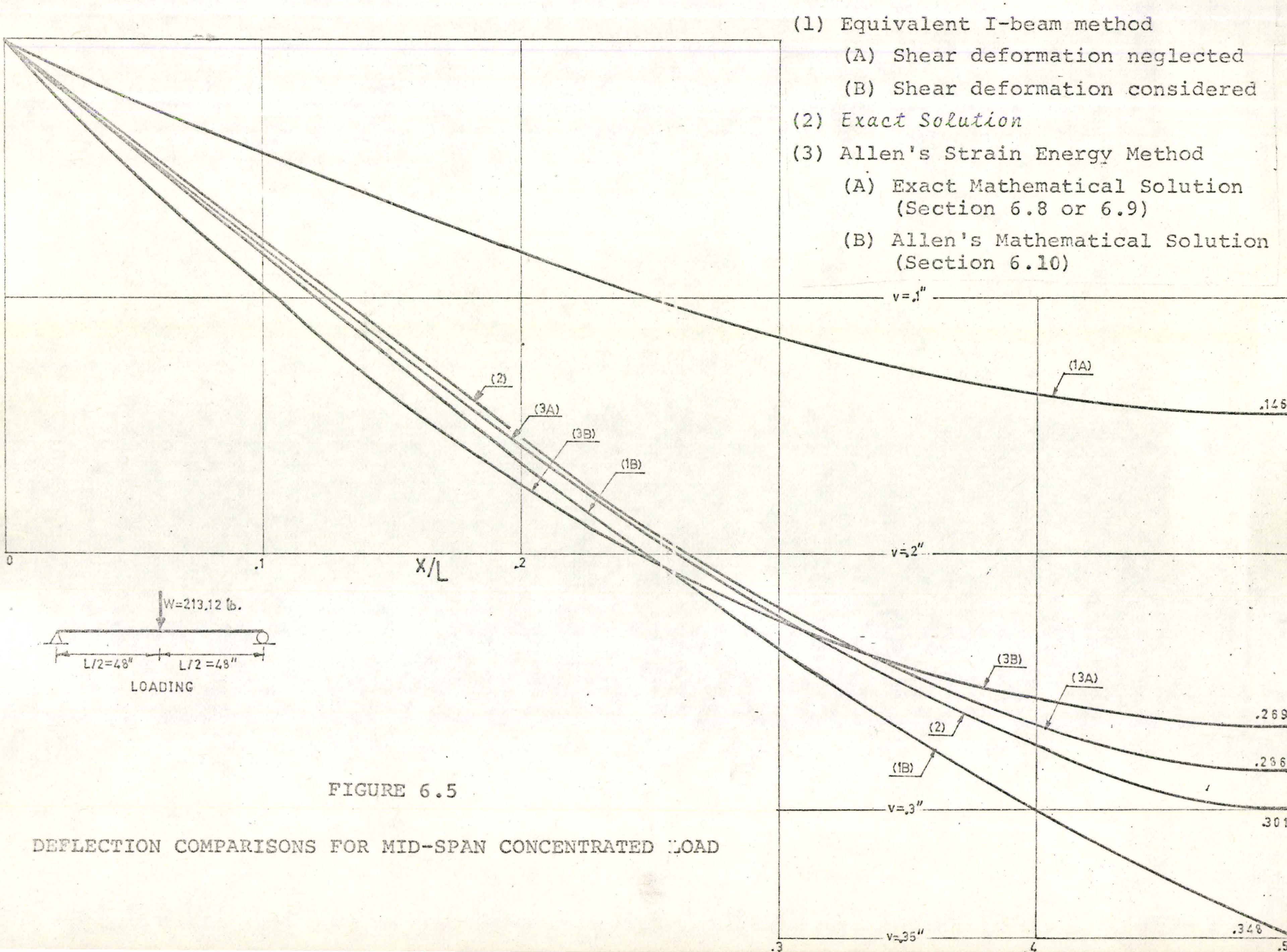

FIGURE 6.5

DEFLECTION COMPARISONS FOR MID-SPAN CONCENTRATED LOAD

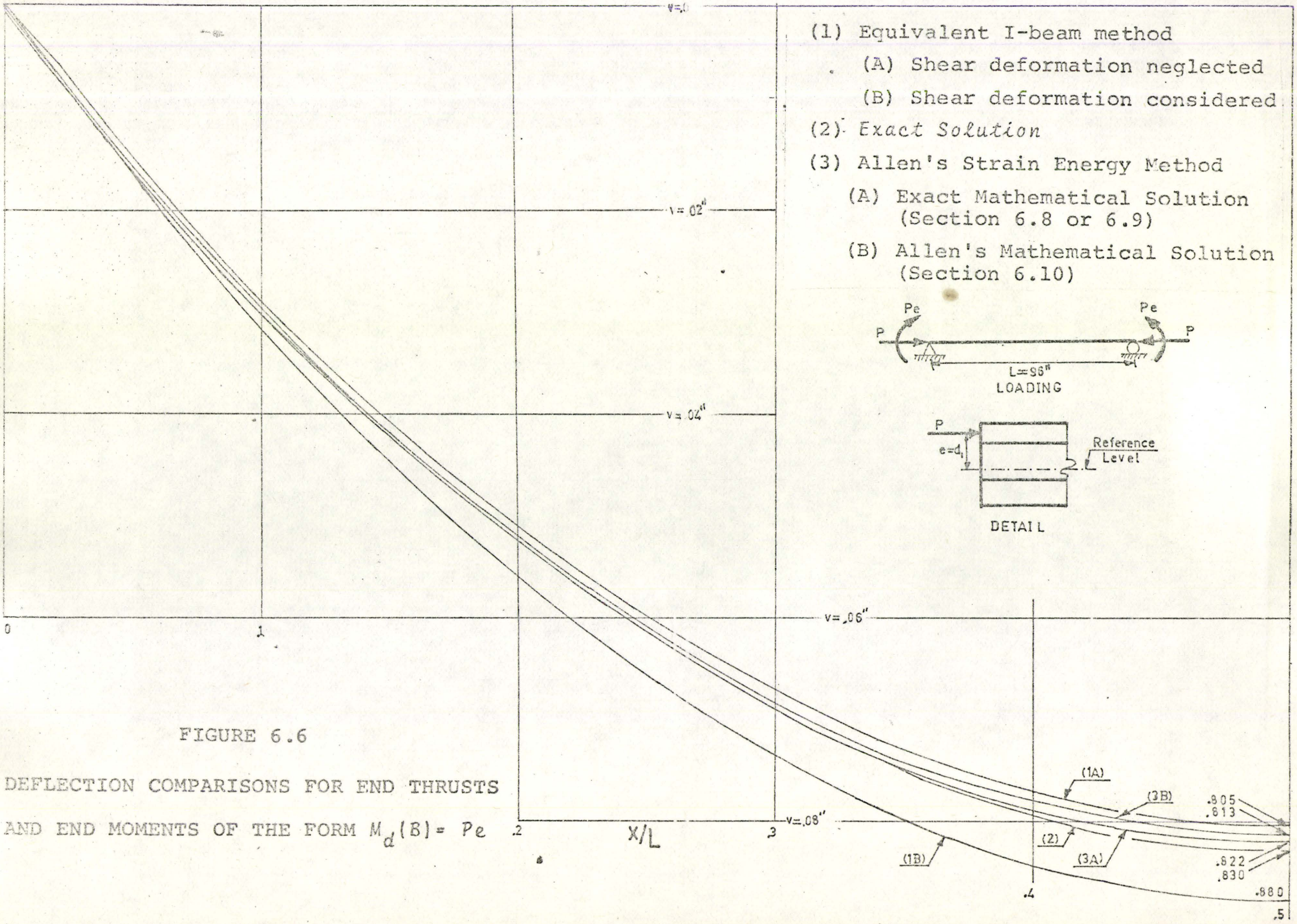

FIGURE 6.6

DEFLECTION COMPARISONS FOR END THRUSTS AND END MOMENTS OF THE FORM $M_d(B) = Pe$

shear force. In fact, the only shear forces acting are due to the secondary moments, which are very small. A difference between Allen's method described in Section 6.10 and the *exact solution* of his method also exists in this case.

As a final result, the accuracy of Allen's Strain Energy Method is quite good for some cases of loading and not so good for others. The loss in accuracy (although small in some cases) is a penalty for simpler mathematical methods and so it represents an improvement over Hoff's generalized solution, but only for some cases of loading. The non-applicability to cases involving concentrated loads limits its usefulness.

CHAPTER VII

PROPOSED METHOD OF ANALYSIS

7.1 INTRODUCTION

Hoff's generalized solution presented in Chapter V, gives an exact solution to the problem corresponding and limited to the assumptions specified in Chapter II. As was mentioned, the solution of the mathematical problem involving three unknown displacements (in some cases only two) was unnecessarily complicated because only one of those was of an immediate interest to the engineer. Once the deflection is known, the other structural aspects can be determined. Besides, the introduction of all types of applied loads had to be included for the sake of generality. This is not very practical since in most cases only some of the loads appear and so the handling of all the others makes the solution unnecessarily complicated. Finally, the application of intermediate concentrated moments and forces creates complicated problems for this type of solution.

Allen's Strain Energy method as presented in the previous chapter is an attempt to simplify the solution a little by introducing two parameters to replace two of the three unknowns. However, this simplification is made at the expense of accuracy because arbitrary assumptions of behaviour have to be imposed. Also it was noted that the solution is simplified only for two cases of load and that for all the others, the solution to the mathematical problem is much more difficult than it was originally.

The method presented in this chapter makes use of a function (the bending moment) that in itself may represent every kind of loading and support condition as long as they affect the bending behaviour of the member. The same generality considered in Hoff's generalized solution may be acquired by the single introduction of the bending moment function. Equilibrium considerations are used to express two of the three displacements in the section as functions of the bending moment $M(x)$ and the curvature $v''(x)$ and so the final differential equation contains exclusively these two functions. The relationships used are taken from the generalized form of Hartsock's solution discussed in Section 4.3.

7.2 STRAIN ENERGY

The strain energy stored in the beam when the external bending moment $M(x)$ is acting may be separated as follows:

1) The strain energy of the net strains in the skins is

$$U_1 = \frac{bt_1E_1}{2}\int_0^L [u_1'(x)]^2 dx + \frac{bt_2E_2}{2}\int_0^L [u_2'(x)]^2 dx,$$

but $u_i'(x)$ may be obtained as a direct function of the applied moment $M(x)$ and the curvature $v''(x)$ at the section x.

The internal forces acting at the centroids of the skins may be regarded as consisting of two distinct parts. The first part consists of forces acting in the same directions such that they add up to the applied end thrust P, and are distributed in such a way that the stresses produced by them are the same in both skins. This force will not affect the bending behaviour at all because the deformation introduced by them in a portion dx of the beam does not produce any curvature.* The second part consists of equal forces acting in opposite directions, and they are given by

$$F(x) = \frac{M_d(x)}{d} = \frac{M(x) + (EI)_\delta v''(x)}{d} ,$$

and therefore, the strains $w_1'(x)$ and $w_2'(x)$ are directly obtained as

$$w_1'(x) = -\frac{M(x) + (EI)_\delta v''(x)}{bdt_1E_1} \text{ and } w_2'(x) = \frac{M(x) + (EI)_\delta v''(x)}{bdt_2E_2} . \qquad (7.1)$$

where negative signs were introduced to comply with the sign conventions (Section 2.4).

The same expressions could have been obtained by the

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* These forces are defined as being applied to the section x, not to the ends of the beam, which means that the secondary moments $Pv(x)$ are not being discarded.

use of expressions (4.5), (4.6), (4.7) and (4.8). Furthermore, the identity

$$d_1 t_1 E_1 = d_2 t_2 E_2$$

in equation (7.1) produces

$$\frac{w_1'(x)}{d_1} + \frac{w_2'(x)}{d_2} = 0 \qquad (2.8.c)$$

which agrees with what had been found and commented on in Section 2.7.

The implication of splitting the forces in the skins into two parts is that, when the solution is obtained, the actual strains in the skins have to be obtained by adding algebraically the value obtained for $w_i'(x)$ to the net strain produced by the axial thrust, P, which is given by

$$\frac{P}{b(t_1 E_1 + t_2 E_2)} .$$

Substitution of the values for strain from Equation 7.1 into the expression for the strain energy gives

$$U_1 = \frac{1}{2bd^2} \left(\frac{1}{t_1 E_1} + \frac{1}{t_2 E_2}\right) \int_0^L [M(x) + (EI)_0 v''(x)]^2 dx \qquad (7.2)$$

where the strain energy due to the elastic shortening of the member subjected to end thrusts P was not included since it does not affect the bending behaviour of the member. If included, it would consist of a constant which would drop out of the derivations when differentiations were involved.

It may be proven that

$$\frac{1}{(EI)_d} = \frac{1}{bd^2}\left(\frac{1}{t_1E_1} + \frac{1}{t_2E_2}\right) \tag{5.17}$$

and so

$$U_1 = \frac{1}{2(EI)_d}\int_0^L [M(x) + (EI)_b v''(x)]^2 dx. \tag{7.3}$$

2) The strain energy due to the local bending of the skins was found before to be

$$U_2 = \frac{(EI)_b}{2}\int_0^L [v''(x)]^2 dx \tag{5.2}$$

3) The final portion of the strain energy is the shear strain energy stored in the core. Equations (4.12), (4.5) and (4.6) yield

$$\gamma(x) = \frac{M'(x) + (EI)_b v'''(x)}{bdG}. \tag{7.4}$$

Therefore,

$$U_3 = \frac{1}{2AG}\int_0^L [M'(x)+(EI)_s v'''(x)]^2 dx \qquad (7.5)$$

with $\qquad A = bd^2/c.$

The shear strain energy stored in the skins and the energy of bending strains in the core are neglected to be consistent with the assumptions in Chapter II and to be comparable with all the other methods studied here. The total strain energy stored in the member is then given by

$$U = \int_0^L \{\frac{1}{2(EI)_d}[M(x)+(EI)_s v''(x)]^2 + \frac{(EI)_s}{2}[v''(x)]^2$$

$$+ \frac{1}{2AG}[M'(x)+(EI)_s v'''(x)]^2\}dx \qquad (7.6)$$

7.3 APPLICATION OF THE PRINCIPLE OF LEAST WORK (SUBJECTED TO AN EXTERNAL CONSTRAINT)

The relationships between the strains used in the evaluation of formula (7.6) [this is, $\omega_i'(x)$, $\gamma(x)$ and local bending strains in the skins] and the applied moment $M(x)$ and the curvature $v''(x)$ at the section x were obtained with the exclusive use of equilibrium considerations. No conditions of compatibility of deformations in the section were required.

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* The problem is internally statically indeterminate and the moment $M_s(x)$ has been chosen as the redundant. Because of the constitutive relationship (4.6) $[M_s(x) = -(EI)_s v''(x)]$ it is permissible to use the curvature $v''(x)$ as the unknown variable. The external constraint arises from the independence of the externally applied bending moment $M(x)$.

Expression (7.6) thus indicates the strain energy for a beam having an applied load $M(x)$ and any curvature $v''(x)$. Regardless of the value of the curvature $v''(x)$, the stresses at the section are in equilibrium with the applied loads represented by $M(x)$. From all possible deformed shapes, indicated by the curvature $v''(x)$, the actual one is, according to the *Principle of Least Work*, the one which corresponds to the minimum strain energy.

Equation (7.6) was set up in a way which does not indicate zero strain energy before the beam has been deformed. This is not the case when the strain energy is expressed as an exclusive function of the displacement functions [Equation (5.4), for instance].

The strain energy as given by Equation (7.6) must then be a minimum, which means that any very small variation $\delta v''(x)$ introduced in the curvature does not change the value of U [the variation $\delta v''(x)$ is made along a *horizontal tangent* to the *curve* U versus $v''(x)$]. Therefore,

$$\delta U = (EI)_f \int_0^L \{ \frac{1}{(EI)_d} [M(x) + (EI)_f v''(x)] \delta v''(x) + v''(x) \delta v''(x)$$

$$+ \frac{1}{AG} [M'(x) + (EI)_f v'''(x)] \delta v'''(x) \} dx = 0 \qquad (7.7)$$

It may be noted that in the derivation of Equation (7.7) the terms,

$$\frac{\partial M(x)}{\partial v''(x)} \text{ and } \frac{\partial M'(x)}{\partial v'''(x)}$$

were not included. This does not mean that secondary moments are being neglected. It means that the bending moment $M(x)$ is prescribed to be independent of the curvature $v''(x)$ and the shear force $M'(x)$ is prescribed to be independent of $v'''(x)$. This derivation does not exclude bending moments of the form $Pv(x)$ nor does it exclude the corresponding shear forces $Pv'(x)$. Even moments depending on the slope, if any kind of load producing them could be thought of, would be automatically included.

Going back to Equation (7.7), its last term may be integrated once by parts and replaced to obtain

$$\delta U = (EI)_6 \int_0^L \left[\frac{M+(EI)_6 v''}{(EI)_d} + v'' - \frac{M''+(EI)_6 v^{iv}}{AG}\right]\delta v'' dx$$

$$- \frac{(EI)_6}{AG}\{[M'(0)+(EI)_6 v'''(0)]\delta v''(0)$$

$$- [M'(L)+(EI)_6 v'''(L)]\delta v''(L)\} = 0 \qquad (7.8)$$

The arbitrariness of the variation $\delta v''(x)$ imposed on the curvature $v''(x)$ requires that each of the terms in Equation (7.8) vanish separately. Applied to the integral function, this produces

$$\frac{M(x) + (EI)_\delta v''(x)}{(EI)_d} + v''(x) - \frac{M''(x)+(EI)_\delta v^{iv}(x)}{AG} = 0$$

which may be written in a compact form as

$$(D^2-\alpha^2)v''(x) = -\frac{1}{(EI)_\delta}[D^2 - \frac{AG}{(EI)_d}]M(x) \qquad (7.9)$$

with

$$\alpha^2 = \frac{(EI)AG}{(EI)_d(EI)_\delta}$$

where D stands for $\frac{d}{dx}$ as before. This equation coincides perfectly with the one found in Section 4.3, which was obtained by the use of equilibrium and compatibility of deformations at the section.

The terms outside the integral produce the boundary condition of

Either $M'(B)+(EI)_\delta v'''(B) = 0$ or $v''(B)$ given (7.10)

where B represents both, 0 and L.

7.4 FORMULAE FOR SOME PARTICULAR CASES OF LOADING

It is interesting to study some particular loading systems on simply supported sandwich beams.

A. Uniformly distributed load of intensity q. When the expression for the bending moment,

$$M(x) = \frac{1}{2} q(L-x)x,$$

is replaced in equation (7.9), and the boundary conditions 1) $v(0) = 0$, 2) $v''(0) = 0$, 3) $v'(L/2) = 0$ [or $v(L) = 0$] and 4) $v'''(L/2) = 0$ [or $v''(L) = 0$], are imposed in the solution, the following expression may be obtained

$$v(x) = \frac{q}{24(EI)}(L^3-2Lx^2+x^3)x + [\frac{(EI)_d}{(EI)}]^2 \frac{q}{AG}[\frac{Cosh\ \alpha(\frac{L}{2} - x) - 1}{\alpha^2\ Cosh\ \frac{\alpha L}{2}}$$

$$+ \frac{1}{2}(L - x)x], \tag{7.11.a}$$

where $\alpha^2 = \frac{(EI)AG}{(EI)_d(EI)_\delta}$ and $A = bd^2/c$

and the maximum deflection may be easily found to be

$$v(L/2) = \frac{5qL^4}{384(EI)} + \frac{qL^2}{8\mu} \quad \text{with} \quad \mu = [\frac{(EI)}{(EI)_d}]^2 AG \tag{7.11.b}$$

B. Concentrated load W at a distance a from the left support.

$$v(x) = \begin{cases} A_2x + A_4x^3 + A_6\ Sinh\ \alpha x & 0<x<a \quad (7.12.a) \\ B_1+B_2x+B_3x^2+B_4x^3+B_5\ Cosh\ \alpha x + B_6\ Sinh\ \alpha x & a<x<L \quad (7.12.b) \end{cases}$$

with $A_2 = \frac{aW(2L^2-3aL+a^2)}{6(EI)L} + \frac{aW}{\mu L}$ $\quad A_4 = -\frac{bW}{6(EI)L}$ $\quad A_6 = -\frac{W\ Sinh\ \alpha b}{\alpha\mu\ Sinh\ \alpha L}$

$$B_1 = -\frac{a^3 W}{6(EI)} - \frac{aW}{\mu}, \quad B_2 = \frac{a(2L^2+a^2)W}{6(EI)L} + \frac{aW}{\mu L}, \quad B_3 = -\frac{aW}{2(EI)},$$

$$B_4 = \frac{aW}{6(EI)L}, \quad B_5 = -\frac{W \sinh \alpha a}{\alpha\mu}, \quad B_6 = \frac{W \sinh \alpha a \cosh \alpha L}{\alpha\mu \sinh \alpha L}$$

$$\mu = [\frac{(EI)}{(EI)_d}]^2 AG, \quad A = bd^2/c \text{ and } a + b = L.$$

C. Mid-span concentrated load W.

$$v(x) = \frac{W}{48(EI)}(3L^2-4x^2)x + \frac{(EI)_d WL}{4\alpha^2(EI)(EI)_\delta}[\frac{2x}{L} - \frac{\sinh \alpha x}{\frac{\alpha L}{2}\cosh \frac{\alpha L}{2}}]$$

$$0<x<L/2 \qquad (7.13.a)$$

$$v(L/2) = \frac{WL^3}{48(EI)} + \frac{WL}{4\mu_W} \text{ with } \mu_W = \frac{\frac{\alpha L}{2}\mu}{\frac{\alpha L}{2} - \tanh \frac{\alpha L}{2}},$$

$$\mu = [\frac{(EI)}{(EI)_d}]^2 AG \text{ and } A = bd^2/c \qquad (7.13.b)$$

D. End Moment $M(0)$ applied at the left end. The expression for the bending moment,

$$M(x) = \frac{M(0)}{L}(L-x),$$

replaced in differential equation (7.9) yields the solution,

$$v(x) = \frac{M(0)}{6(EI)L}(2L^2-3Lx+x^2)x + B_5[\frac{x}{L} - 1 + \frac{\sinh \alpha(L-x)}{\sinh \alpha L}], \qquad (7.14.a)$$

with the use of the boundary conditions

1) $v(0) = 0$, 2) $v(L) = 0$ and 3) $v''(L) = 0$

The application of the fourth boundary condition (to define B_5) requires knowledge about how bending moment $M(0)$ is applied. If the portion $M_\delta(0)$ of the total moment $M(0)$ is given, the fourth boundary condition is

$$v''(0) = - M_\delta(0)/(EI)_\delta$$

and the corresponding solution is (7.14.a) with

$$B_5 = \frac{1}{\alpha^2} \left[\frac{M(0)}{(EI)} - \frac{M_\delta(0)}{(EI)_\delta}\right] \qquad (7.14.b)$$

If the curvature of the member is not given at the boundary, the mathematical formulation of the problem [Boundary condition (7.10)] produces results where the shear stress in the core is zero at the boundary. This is equivalent to the existence of a physical restriction to relative movement of the skins at the boundary, and the fourth condition is

$$v'''(0) = \frac{M(0)}{L(EI)_\delta}$$

which produces

$$B_5 = - \frac{M(0)}{L\mu\alpha} \, Tanh \; \alpha L \qquad (7.14.c)$$

E. Eccentric end thrust P with eccentricity e measured from the reference level defined by d_1 and d_2. In this case,

$$M(x) = P[e + v(x)]$$

and the solution of Equation (7.9) may be easily found to be

$$v(x) = (e-B_4)\frac{\cosh \beta_1(\frac{L}{2}-x)}{\cosh \frac{\beta_1 L}{2}} + B_4 \frac{\cos \beta_2(\frac{L}{2}-x)}{\cos \frac{\beta_2 L}{2}} - e \qquad (7.15.a)$$

where

$$\begin{bmatrix} \beta_1{}^2 \\ \\ \beta_2{}^2 \end{bmatrix} = \sqrt{\frac{1}{4}[\alpha^2 - P/(EI)_\delta]^2 + \alpha^2 P/(EI)} \pm \frac{1}{2}[\alpha^2 - P/(EI)_\delta]$$

after applying the boundary conditions

1) $v(0) = 0$, 2) $v'(L/2) = 0$, 3) $v'''(L/2) = 0$.

The application of the fourth boundary condition, as before, calls for extra knowledge of the way in which the loads are applied. If the portion $M_\delta(0)$ of the total end moment Pe is given, the fourth condition is

$$v''(0) = -\frac{M_\delta(0)}{(EI)_\delta}$$

which yields

$$B_4 = \frac{\beta_1^2 e + M_\delta(0)/(EI)_\delta}{\beta_1^2 + \beta_2^2}. \qquad (7.15.b)$$

Of particular importance is the case where $M_\delta(0)$ is zero, which corresponds to the case shown in Figure 5.3.b. In this situation

$$B_4 = \frac{\beta_1^2 e}{\beta_1^2 + \beta_2^2} \text{ and } v(x) = e\{\frac{1}{\beta_1^2 + \beta_2^2}[\beta_2^2 \frac{Cosh\ \beta_1(\frac{L}{2}-x)}{Cosh\ \frac{\beta_1 L}{2}}$$

$$+ \beta_1^2 \frac{Cos\ \beta_2(\frac{L}{2}-x)}{Cos\ \frac{\beta_2 L}{2}}] - 1\}. \qquad (7.15.c)$$

In the case with rigid inserts at the ends, the condition

$$Pv'(0) + (EI)_\delta v'''(0) = 0$$

is applied instead and it yields

$$B_4 = \frac{e\beta_1[P+\beta_1^2(EI)_\delta]Tanh\ \frac{\beta_1 L}{2}}{\beta_1[P+\beta_1^2(EI)_\delta]Tanh\ \frac{\beta_1 L}{2} + \beta_2[P-\beta_2^2(EI)_\delta]Tan\ \frac{\beta_2 L}{2}} \qquad (7.15.d)$$

F. Buckling load for a pin-ended strut. The critical load for a pin-ended strut may be easily found to be

$$P_c = \frac{\frac{\pi^2}{L^2} + \alpha^2}{\frac{\pi^2}{L^2} + \frac{AG}{(EI)_d}} \frac{\pi^2}{L^2}(EI)_\delta \tag{7.16}$$

which coincides with the expression found by Allen in his application of ordinary beam theory and also in his chapter using strain energy principles. It was mentioned before (and shown in Figure 6.2) that his assumption of a constant value for $\lambda(x)$ was true for (and only for) buckling loads.

G. Uniformly distributed load q, mid-span concentrated load W, end thrusts P and end moments of the form $M_d(0) = M_d(L) = Pe$ acting simultaneously. This quite general problem arises often in the study of beam-columns. The solution may be easily found to be given by

$$v(x) = B_1 + B_2x + B_3x^2 + B_4 \cosh \beta_1 x + B_5 \sinh \beta_1 x + B_6 \cos \beta_2 x$$

$$+ B_7 \sin \beta_2 x \qquad 0<x<L/2 \tag{7.17}$$

with $B_3 = q/2P$, $B_4 = \dfrac{\alpha^2 Pe}{\beta_1^2(\beta_1^2 + \beta_2^2)(EI)} - \dfrac{2[\beta_2^2 - P/(EI)_\delta]}{\beta_1^2(\beta_1^2+\beta_2^2)} B_3$,

$$B_6 = \frac{2B_3}{\beta_2^2} + \frac{\beta_1^2}{\beta_2^2}B_4, \quad B_1 = B_4 - B_6, \quad B_2 = -\frac{qL+W}{2P},$$

$$F_2'(L/2) = \frac{[\beta_1^2 + P/(EI)_b]W}{2P(\beta_1^2 + \beta_2^2)}, \quad F_1'(L/2) = \frac{\beta_2^2}{\beta_1^2} F_2'(L/2) - \frac{W}{2\beta_1^2 (EI)_b},$$

$$B_5 = [\frac{F_1'(L/2)}{\beta_1} - B_4 \, Sinh \, \frac{\beta_1 L}{2}] / Cosh \, \frac{\beta_1 L}{2},$$

$$B_7 = [\frac{F_2'(L/2)}{\beta_2} + B_6 \, Sin \, \frac{\beta_2 L}{2}] / Cos \, \frac{\beta_2 L}{2},$$

$$\alpha^2 = \frac{(EI)AG}{(EI)_d (EI)_b} \quad \begin{bmatrix} \beta_1^2 \\ \beta_2^2 \end{bmatrix} = \sqrt{\frac{1}{4}[\alpha^2 - P/(EI)_b] + \alpha^2 P/(EI)}$$

$$\pm \frac{1}{2} [\alpha^2 - P/(EI)_b], \qquad P \neq 0.$$

This solution is important because, if transverse loading acts simultaneously with end thrusts and secondary moments are to be considered, no superposition of independent results is allowable.

H. Mid-span concentrated load W, end thrusts P and end moments of the form $M_d(0) = M_d(L) = Pe$ acting on a simply supported beam column which is initially bowed. In order to show how the method of analysis being studied in this chapter is applicable to cases like this one, the initial bowing will be assumed to be parabolic with amplitude a (measured positive when in the same direction as the eccentricity e). The initial deformed shape may then be expressed as

$$4a(1 - \frac{x}{L})\frac{x}{L} .$$

The bending moment is now given by

$$M(x) = \frac{W}{2}x + P[e - 4a(1 - \frac{x}{L})\frac{x}{L} + v(x)],$$

which, when substituted into equation (7.9), produces the differential equation

$$\{D^4 - [\alpha^2 - P/(EI)_\delta]D^2 - \alpha^2 P/(EI)\}v(x) = [\frac{\alpha^2}{(EI)} e - \frac{8a}{L^2(EI)_\delta}]P$$

$$- \frac{\alpha^2}{(EI)} [\frac{4aP}{L} - \frac{W}{2}]x + \frac{4a\alpha^2 P}{L^2(EI)} x^2 .$$

The application of boundary conditions

$$v(0) = v'(L/2) = v''(0) = 0, \quad v'''(0) = -\frac{W}{2(EI)_\delta}$$

yields the solution

$$v(x) = B_1 + B_2 x + B_3 x^2 + B_4 \cosh \beta_1 x + B_5 \sinh \beta_1 x + B_6 \cos \beta_2 x$$

$$+ B_7 \sin \beta_2 x \qquad (7.18)$$

with

$$B_3 = -\frac{4a}{L^2}, \quad B_2 = \frac{W}{2P} - \frac{4a}{L}, \quad B_1 = \frac{8a(EI)}{PL^2} - e, \quad B_4 = -\frac{\beta_2^{\,2}B_1 + 2B_3}{\beta_1^{\,2} + \beta_2^{\,2}},$$

$$B_6 = -B_1 - B_4, \quad B_5 = \left[\frac{F_1'(L/2)}{\beta_1} - B_4\, Sinh\,\frac{\beta_1 L}{2}\right]/Cosh\,\frac{\beta_1 L}{2},$$

$$B_7 = \left[\frac{F_2'(L/2)}{\beta_2} + B_6\, Sin\,\frac{\beta_2 L}{2}\right]/Cos\,\frac{\beta_2 L}{2},$$

$$F_1'(L/2) = -\left[\frac{W}{2(EI)_\delta} + \beta_2^{\,2}(B_2 + B_3 L)\right]/(\beta_1^{\,2} + \beta_2^{\,2}),$$

$$F_2'(L/2) = \left[\beta_1^{\,2}F_1'(L/2) + \frac{W}{2(EI)_\delta}\right]/\beta_2^{\,2},$$

$$\begin{bmatrix}\beta_1^{\,2}\\ \beta_2^{\,2}\end{bmatrix} = \sqrt{\tfrac{1}{4}[\alpha^2 - P/(EI)_\delta]^2 + \alpha^2 P/(EI)} \pm \tfrac{1}{2}[\alpha^2 - P/(EI)_\delta],$$

$$\alpha^2 = \frac{AG(EI)}{(EI)_d(EI)_\delta}, \quad A = bd^2/c$$

7.5 THE RAYLEIGH-RITZ METHOD

If the bending moment function is expressed in its Fourier form,

$$M(x) = \sum_{n=1}^{\infty} M_n\, Sin\; sx,$$

with

$$M_n = \frac{2}{L}\int_0^L M(x)\, Sin\, sx\, dx \quad \text{and} \quad s = \frac{n\pi}{L},$$

the solution may be stated to be of the form

$$v(x) = \sum_{n=1}^{\infty} a_n\, Sin\, sx.$$

If both are replaced in Equation (7.6), the result is

$$U = \frac{L}{2}\{[\frac{1}{(EI)_d} + \frac{s^2}{AG}]\frac{[M_n - s^2(EI)_f a_n]^2}{2} + \frac{s^4(EI)_f}{2} a_n^2\}.$$

It may then be obtained that

$$\frac{\partial U}{\partial a_n} = - s^2(EI)_f \frac{L}{2}\{[\frac{1}{(EI)_d} + \frac{s^2}{AG}][M_n - s^2(EI)_f a_n] - s^2 a_n\} = 0$$

which, after some manipulations, produces

$$a_n = \frac{s^2 + \frac{AG}{(EI)_d}}{s^2 + \alpha^2}\,\frac{M_n}{s^2(EI)_f}, \qquad (7.19)$$

which is very easy to use, as may be seen from the examples in the following section.

7.6 FORMULAE FOR PARTICULAR CASES OF LOADING USING THE RAYLEIGH-RITZ METHOD

All the cases to be considered correspond to a simply supported beam.

A. Uniformly distributed load of intensity q.

$$M(x) = \frac{1}{2} q(L-x)x = \sum_{n=1}^{\infty} M_n \sin sx, \text{ thus } M_n = \frac{4q}{Ls^3}, \quad n=1,3,5,\ldots$$

and then, with the result of

$$a_n = \frac{s^2 + \frac{AG}{(EI)_d}}{s^2 + \alpha^2} \frac{4q}{Ls^5 (EI)_0}, \quad n=1,3,5,\ldots$$

the solution $v(x)$ is now easily constructed.

B. Concentrated load W at a distance a from the left support.

$$M(x) = \begin{cases} \frac{bW}{L} x & 0<x<a \\ \frac{aW}{L} (L-x) & a<x<L, \end{cases} \quad \text{thus, } M_n = \frac{2WL}{(sL)^2} \sin sa, \quad n=1,2,3,\ldots$$

The coefficients a_n are easily obtained from Equation (7.19). In this case the simplicity of the calculations is remarkable compared to the calculations in the Variational Calculus approach.

C. Mid-span concentrated load W. The replacement of $a = L/2$ in the formula above yields

$$M_n = \frac{2W}{s^2 L}(-1)^{\frac{n-1}{2}} \qquad n=1,3,5,\ldots$$

which, after substitution in Equation (7.19), yields

$$a_n = \frac{s^2 + \frac{AG}{(EI)_d}}{s^2 + \alpha^2} \frac{2W(-1)^{\frac{n-1}{2}}}{s^4 L(EI)_b}. \qquad n=1,3,5,\ldots$$

Of particular interest is the use of only one term of the Fourier expansion. This is equivalent to assuming a sinusoidal deflection (Rayleigh's method),

$$v(L/2) = a_1 = \frac{s^2 + \frac{AG}{(EI)_d}}{s^2 + \alpha^2} \frac{2W}{s^4 L(EI)_b},$$

which may be rewritten as

$$v(L/2) = \frac{2LW}{\pi^2 (EI)_d} [\frac{(EI)_d}{AG} + \frac{L^2}{\pi^2}]. \qquad (7.20)$$

D. Buckling load for a pin-ended strut.

$M(x) = Pv(x)$ and so $M_n = Pa_n$.

This value in Equation (7.19) yields

$$a_n = \frac{s^2 + \frac{AG}{(EI)_d}}{s^2 + \alpha^2} \frac{Pa_n}{s^2 (EI)_b}$$

or

$$[\frac{s^2 + \frac{AG}{(EI)_d}}{s^2 + \alpha^2} \frac{P}{s^2 (EI)_b} - 1] a_n = 0$$

which means that, either $a_n = 0$ and there is no deflection, or

$$P_n = \frac{s^2 + \alpha^2}{s^2 + \frac{AG}{(EI)_d}} s^2 (EI)_b$$

whose first mode $(n=1)$ coincides with expression (7.16).

The loading cases studied above are representative of the possibilities of the Rayleigh-Ritz method as applied to the approach being studied in this chapter.

CHAPTER VIII

OBSERVATIONS, CONCLUSIONS AND RECOMMENDATIONS

8.1 INTRODUCTION

Throughout this thesis, an underlying theme which has developed is that the state of development of sandwich panel analysis as applied to civil engineering structures is not adequate. It has been shown that most of the references studied include inconsistencies, or are not presented in a clear manner, or impose unnecessary assumptions of behaviour. Many of these deficiencies were pointed out earlier, and it is not necessary to recount them again.

There is, however, a contribution to the realm of structural analysis of sandwich panels that has served as a major reference to many other works. At the beginning of this study, this reference caused considerable trouble because of its treatment of the subject. It seems worthwhile to include it here because it serves the purpose of showing the state-of-the-art and because it is a major reference, its particular deficiencies should be pointed out. Even though many authors have studied the work by Hughes and Wajda[14], none of them has pointed out the deficiencies.

8.2 COMMENTS ON HUGHES AND WAJDA'S FORMULAS

Hughes and Wajda(14) do not make step by step derivations but simply compare three methods of analysis*. In their paper they compare the mid-span deflection of a simply supported sandwich beam with a concentrated load at mid-span. Predictions given by three methods of analysis are compared with experimental results.

The first formula they give is "The engineering formula of bending with allowance made for the deflections due to shearing deformations at the core", which coincides with the Equivalent I-Beam method with consideration of shear deformations (Section 3.3). In fact, the formula they give for the mid-span deflection is (with the notation changed to the one used here):

$$v(L/2) = W\left[\frac{L^3}{48(EI)} + \frac{L}{4dG}\right] \qquad (8.1)$$

while the one obtained by solving the problem as presented in Section 3.3 is:

$$v(L/2) = W\left[\frac{L^3}{48(EI)} + \frac{L}{4AG}\right] \qquad (3.5)$$

with

$$A = bd^2/c \qquad (3.3)$$

---

* They state that those are three "different" methods of analysis. It will be proven here that they are just the result of different approximations from the same method.

From a comparison of these two formulas it is apparent that they took A equal to bd*, which is applicable for sandwich panels with very thin skins as was proven earlier. Since the experiments they carried out were performed on sandwich beams with thin skins, formula (8.1) should have provided a solution very close to the *exact solution.*

The second formula they give was obtained by using strain energy principles applying Rayleigh's method. The result they give is

$$v(L/2) = \frac{2LW}{\pi^2 (EI)} \left[\frac{(EI)_d}{AG} + \frac{L^2}{\pi^2}\right] \qquad (8.2)$$

with

$$A = bd^2/c \qquad (3.3)$$

where its coincidence with formula (7.20) for cases with thin skins, [Implying that $(EI) = (EI)_d$], is clear.

The third formula given is similar to formula (7.13.b). Table 8.1 shows a comparison of Hughes and Wajda's formula 3 with formula (7.13.b), rewritten in a different form to make comparisons easier

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* The term b is missing in some of their formulas but checks made on their calculations show that they took $b=1$ inch, even though they state the width b for their beams was 2 inches.

Exact Solution

$$\left.\begin{aligned}
v(L/2) &= r\,\frac{L^3 W}{48(EI)},\\
\text{where } r &= 1 + Q\,\frac{(EI)_d}{(EI)_f},\\
Q &= \frac{3}{(\alpha L/2)^2}\left[1 - \frac{\tanh \frac{\alpha L}{2}}{\frac{\alpha L}{2}}\right],\\
\alpha^2 &= \frac{AG}{\bar{D}}\\
\text{and } \bar{D} &= \frac{(EI)_d (EI)_f}{(EI)}
\end{aligned}\right\} \quad (8.3.a)$$

Hughes and Wajda's

$$\left.\begin{aligned}
v(L/2) &= r\,\frac{L^3}{48(EI)},\\
r &= 1 + Q\,\frac{(EI)_d}{(EI)_f},\\
Q &= \frac{3}{(\alpha \ell)^2}\left[1 + \frac{\tanh \alpha \ell}{\alpha L}\right],\\
\alpha^2 &= \frac{cG}{\bar{D}}\\
\bar{D} &= \frac{(EI)_d (EI)_f}{(EI)}
\end{aligned}\right\} \quad (8.3.b)$$

TABLE 8.1

COMPARISON OF HUGHES AND WAJDA'S FORMULA 3 WITH HOFF'S

Hughes and Wajda do not say where they obtained their formulae (8.3.b), but they mention an article by Hoff in their references. This would explain the similarity in the presentation of their formulae with Hoff's. Hoff's formula takes $A = bc$ (see Section 5.2) as do Hughes and Wajda.

The differences between Equations (8.3.a) and (8.3.b) are discussed next. First, in Equation (8.3.b), A equal to bc is used instead of bd^2/c This is perfectly acceptable for cases with very thin skins. Secondly, they introduce lower case and capital L symbols with no stated meaning. The correct solution yields $L/2$ instead of ℓ and L. In addition, the sign of the second term in Q is also wrong. Finally, the load W is missing in the expression for $v(L/2)$. Some of these must be printing errors but it is not possible to know which of them are.

As a synthesis, all their formulas are variations of the *exact solution* and result from using different approximations. Their formula 1 [(8.1) here] neglects $(EI)_\delta$ and takes $A = bd$, both approximations being applicable to members with very thin skins. Formula 2 [(8.2) here] neglects $(EI)_\delta$ and uses $A = bd^2/c$. This is acceptable for members with thin skins. Their formula 3 [(8.3.b) here], if corrected, corresponds to the effect of taking $A = bc$. This fact makes it applicable to sections with very thin skins, regardless of the fact that they did not neglect $(EI)_\delta$.

Formulas 1 and 3 should then produce almost the same results while Formula 2 produces differences with the former ones solely because the method used to solve the mathematical problem is approximate. These comments are confirmed by Figure 8.1, which shows the results of mid-span deflection over load plotted for different core thicknesses, c, for the three formulas as originally presented. The results for the corrected Formula 3, a reproduction of their curve attributed to Formula 3 (not corresponding to either their formula 3 or to the corrected one), the *exact solution* and their experimental curve are also shown. The properties of the beam and loading system are also illustrated in Figure 8.1.

Their conclusion with reference to their Figure 21 (which shows the dashed curves in Figure 8.1) is that "The experimental curve takes approximately the same shape as the theoretical curves and lies about midway between the theoretically predicted deflection curves of Equations 2 and 3 [(8.2) and (8.3.b) here]. It is considered that deflections can best be predicted by employing Equation 2[(8.2)], which is more directly applicable than Equation 3 [(8.3.b)]; Equation 1 [(8.1)] over estimates deflections".

Equation (8.2) is derived using the same basic assumptions as Equation (8.1) with only the definition of the term A being different. However, for the thin and very thin skins used, this difference in definition does not create significantly

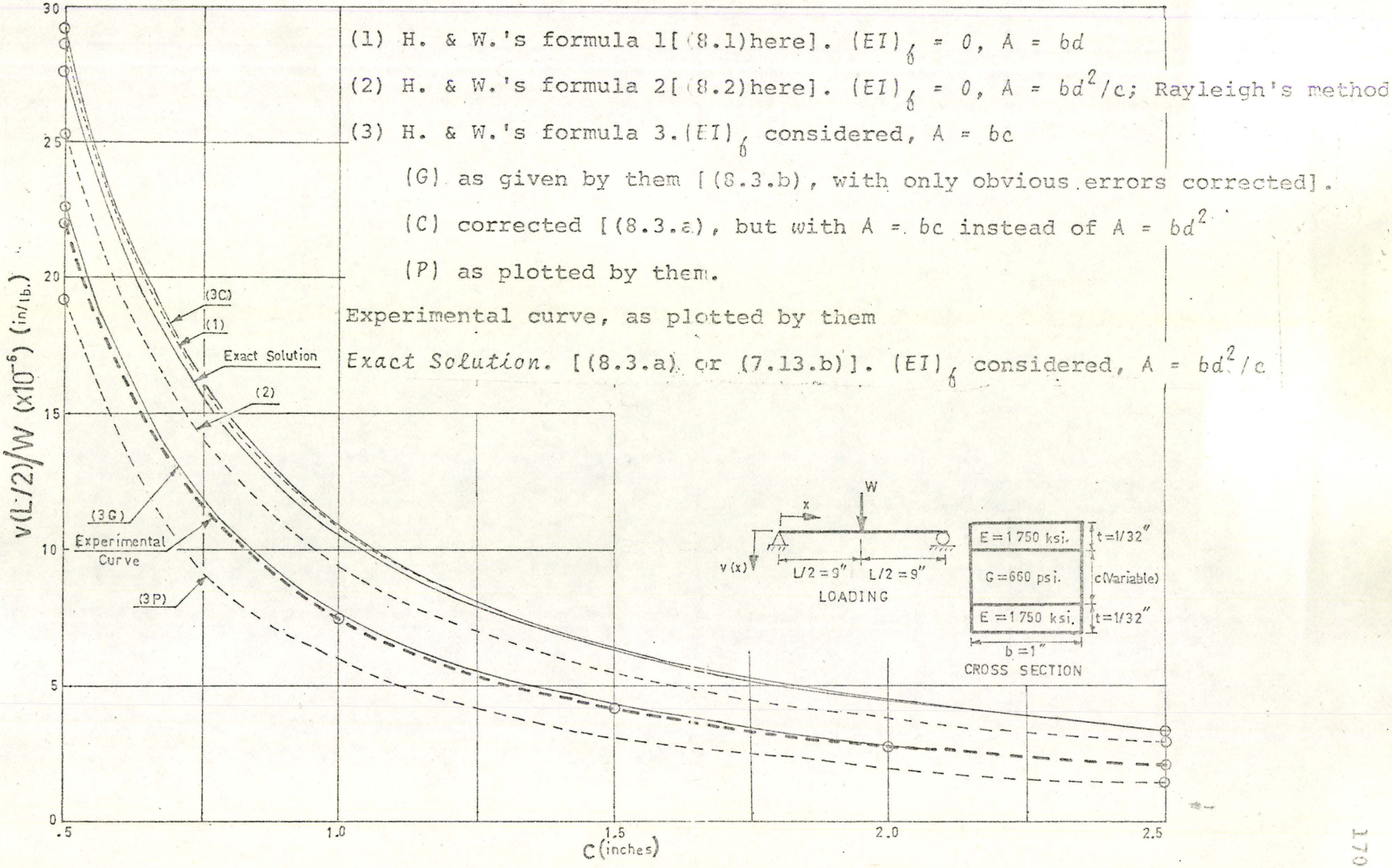

FIGURE 8.1

Hughes and Wajda's Formulas

different results. Therefore, the major difference between the two formulas, is that equation (8.2) employed an assumed deflected shape instead of using the strict mathematical solution. Hence it is absurd to conclude that Equation (8.2) is more accurate than Equation (8.1) just because it corresponds more closely to the experimental results. In fact, if more terms of the Fourier expansion had been taken, (Rayleigh-Ritz method), it would produce results that are very close to the results of Equation (8.1).

It can be observed from Figure 8.1 that curves (1) [from (8.1)], (3c) [from (8.3.a), but with $A = bc$] and the *exact solution* [from (8.3.a) or (7.13.b)] produce very close results. This means that the terms $(EI)_6$ and the difference in values of A are not too important because the skins are in all cases either thin or very thin.

Coincidentialy, their experimental curve coincides quite well with the curve that they should have obtained from their uncorrected formula (8.3.b), and this is a surprising result because that formula has many errors. Fortunately, they seem to have made mistakes in plotting that formula because, otherwise, they could have concluded that it was the most accurate one.

Surprisingly, from the references studied, none of the several authors using this paper as a reference has pointed out the errors and unjustified conclusions above.

8.3 CONCLUSIONS

There exists a lack of complete understanding of the structural behaviour of sandwich panels on the part of many of the authors dealing with the subject. The previous section and many of the comments earlier in this thesis serve as evidence for the above statement. Many of the references studied for this work present inconsistencies, inaccuracies, printing errors, assumptions of behaviour, the validity and/or the justification of which are not confirmed, and, quite often, derivations which are not clear.

Three methods of analysis (the ones presented in Section 4.4 and Chapters V and VII) were proven to provide the *exact solution* for the problem set up in Chapter II, and they were proven to be equivalent. The first two of these are not very easy to apply because of lack of information about the necessary boundary conditions in the first one and lengthy mathematical treatment in the second one. The solution as presented in Chapter VII is quite simple, is more general and is at least as accurate as any solution studied for this work*. In the author's opinion the merits of this method have been clearly indicated and it is recommended to be used in the analysis and design of sandwich panels.

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* Some authors do not neglect the bending stiffness of the core and this would make their solutions more accurate if they had not inconsistently made the distribution of shear stresses in the core to be constant.

The solution proposed in Chapter VII is not verified by experiments in this work. Several authors have stated that their theoretical results, which are just particular cases of the general solution in Chapter VII, agree with experimental studies. Since these methods are just particular cases of the general solution and are the result of imposing extra arbitrary assumptions of behaviour, some experimental verification of the applicability of Equation (7.9) may be thought to have been achieved already. The need for verification of its applicability under other systems of loading is pointed out in Section 8.5.

8.4 RECOMMENDATIONS FOR ANALYSIS

It is recommended that differential Equation (7.9) be applied to every case of analysis of sandwich beams having thick identical or dissimilar skins complying with the assumptions specified in Section 2.2. Boundary conditions (7.10) and the two arbitrary conditions defining the deflections or deflection and slope (easily prescribed in each case) completely define the problem. For some problems it could be simpler to employ the solution described in Section 7.5 which, while still corresponding to the *exact solution*, uses a different mathematical method to obtain the answer. If checks of the error introduced by taking only a finite number of terms are made, the size of the error may be controlled within acceptable limits. The last method is particularly simple to apply if an electronic computer is available.

In cases where the skins are thin, meaning that their local bending stiffness is negligible, the Equivalent I-Beam method with allowance for the shear deformation of the core described in Section 3.4, with

$$A = bd^2/c, \qquad (3.3)$$

may be used to obtain the exact solution. To check whether the skins are really thin, the criteria given in Section 2.3 are valid.

When the skins are very thin, the same comments are applicable and A may be taken as bd or bc. However, it is very difficult to give criteria to tell whether the skins are very thin or not. Fortunately, this distinction does not seem to be necessary any longer. The simplification provided by the fact that the skins are very thin is not worthwhile and it is recommended here to use expression (3.3) above in every case.

8.5 RECOMMENDATIONS FOR FUTURE RESEARCH

1) The theory developed in Chapter VII should be experimentally verified, particularly for unsymmetric loading conditions on sandwich panels with thick dissimilar skins. In the event that the agreement between the experimental results and theoretical predictions was not adequate, the general assumptions of behaviour specified in Chapter II would have to be revised.

2) The theoretical developments in Chapter VII should be

extended to cover two-directional bending. This extension should not be very complicated.

3) Other aspects of structural behaviour like wrinkling instability, differential thermal and shrinkage warping, creep, and ultimate strength should be studied. It is suggested that the method proposed in Chapter VII should form the basis for this future work.

4) The applicability of Saint-Venant's principle to some loading systems on sandwich panels should be studied experimentally.

5) An attempt should be made to study the case of sandwich members with core material which contributes significantly to the bending stiffness.

6) The effect of concentrated loads in the neighbourhood of their points of application should also be studied. An assumption made in all references on sandwich construction (this thesis included), is that the core is not deformed in the direction of its thickness. This assumption should not enforced in this case.

7) The applicability of derivations in Chapter VII to sandwich members with honeycomb cores should be studied experimentally.

APPENDIX A

TESTS ON MATERIALS

A.1 INTRODUCTION

An effort was made to identify and make a preliminary study of several materials with potential for use in sandwich components for the building industry. Some of these materials are new and others, while having been used before in the building industry, have not been considered as structural materials. A preliminary experimental study of some of the most relevant structural properties of those materials was carried out.

A.2 CLASSIFICATION OF THE MATERIALS STUDIED

Two skin materials: unreinforced mortar and commercially available gypsum, and three core materials: Styrofoam®, Polystyrene-Concrete and Sawdust-Concrete, were studied for this report. Because all but one of these materials were used in the construction of several sandwich panels, a roman number indicating the corresponding panel will be used to identify the materials. In addition, the letters SC, ST, S or C will be placed before the roman number above to indicate whether the material was intended to be used in the compressive skin, in the tensile skin, in both skins, or in the core, respectively. Finally, numbers 1,2,3,... will be used to differentiate between repetitions of the same test.

The materials for panel I were later reconstructed, and they are represented as IA. One core material, Sawdust-Concrete, was not used in any panel and has no special classification.

A.3 DESCRIPTION OF MATERIALS

A.3.1 Skin Materials

The skin materials I, IA, II, IV and V were unreinforced mortar cast with the following proportions by weight:

$$\text{Sand/Cement} = 2.5$$

$$\text{Water/Cement} = 0.5$$

Materials I and IA both had entrained air, obtained by adding .008 gr. of DEREX^{R} air entrainment agent per lb. of cement. The air entrainer was added to the water. The mortar for the compressive skin was always vibrated, but the tensile skins were not vibrated except for Panel V. It was observed that vibration produced segregation of the components in the freshly placed polystyrene-concrete core. However, the compressive skin was invariably cast first so that there was no problem in vibrating it. The sand used in the mortars was dried indoors for an average of seven days. The cement used was Normal Portland Cement.

The materials corresponding to the tensile and compressive skins were cast from different batches for panels I, II and IV. For panel V both skins were cast from the same batch. For the reconstructed material IA there is no distinction between compressive and tensile skins.

The skin material III was gypsum, commercially available in 8 ft. by 4 ft. by 1/2 inch boards.

A.3.2 Core Materials

The core materials I, IA, II and IV were a mix called here *Polystyrene-Concrete*, prepared as follows: Commercially available polystyrene beads of diameters in the order of 1/16 inch, were mixed with a cement paste in the volumetric proportion of 80 per cent of beads and 20 per cent of cement paste (Equivalent to a polystyrene/cement paste weight ratio of 6.24 to 100). The cement paste was prepared with a water/cement ratio of 0.5. Air entrainer (DEREX^R) had previously been added to the water in materials I and IA in the proportions specified by the manufacturer but no air entrainment was used in any of the other batches. Polystyrene-Concrete could never be vibrated because of segregation problems. Compaction was obtained by firmly pressing the mix into the mold.

Expanded Polystyrene boards (Styrofoam^R), commercially available in dimensions of 8 ft. by 4 ft. by 1 inch were used as core materials III and V.

A third core material investigated for this work was *Sawdust-Concrete*. This material was studied in an attempt to produce a satisfactory core material using natural vegetable fibres as the bulk component. The proportionings were tried as follows. Sawdust was added to a cement paste with a water/cement ratio of 0.5. It was observed that the sawdust absorbed much of the water from the cement paste and so extra water had to be added. In the first trial the final cement/water/sawdust ratio was

a) 3.038/3.137/1

by weight, A 12 inch by 6 inch diameter cylinder was cast with this material and two days later, when it was removed from the mold, it collapsed. The material still looked saturated and showed no signs of hardening. It had compacted over 2 inches.

For the others, the sawdust was saturated beforehand. Another five samples were cast (b in a 12 inch by 6 inch diameter cylinder and the other four in 2 inch cubes). These were prepared with the following proportions of cement/water/sawdust in weight:

b) 2.385/3.856/1

c) 4.000/4.600/1

d) 5.000/5.000/1

e) 1.370/3.350/1

f) 1.910/3.870/1

These samples were left in the forms for seven days. The last four showed no cohesion at all when they were taken out of the forms. The one in the cylinder, b, did not show much cohesion, but at least maintained its shape (a finger, however, could go through with no difficulty). The cylinder was the only sample thought to be worthwhile keeping to be tested. One month after being cast, it was dry. The crust which had formed on the outside was quite hard, whereas the inside was dry but remained very soft (It could be desintegrated with the fingers or by blowing on it).

A.4 TESTS FOR COMPRESSIVE PROPERTIES

Cylinders for compressive tests were cast for the mortars, the polystyrene-concrete and the sawdust-concrete. All of these were the standard 6 inch diameter by 12 inch high cylinders with the exception of four samples for panel I (two of each batch), two of the cylinders for skin materials IA and a preliminary test on the polystyrene-concrete. These were cast in 3 inch diameter by 6 inch cylinders. The sample of sawdust-cement kept to be tested was six inches diameter but only 4 1/2 inches high.

The tests for the compressive properties of gypsum board were performed on samples built up from eight 12 inch by 4 inch by 1/2 inch pieces of board which were joined together to form a 12 inch by 4 inch by 4 inch prism. The attachment was made with clamps for one specimen and by glueing these pieces together with COLMA-DUR[R] for another specimen.

Some of the compressive tests were performed in a 120 kip capacity TINIUS OLSEN[R] Universal Testing Machine at a low speed (.05 in/min) whereas the others were performed in a 300 kip capacity machine at a higher speed. In the first case deflections were measured as the relative displacement of the loading bridge of the machine with respect to its base. It was found later that those readings did not provide an accurate measure of the total shortening of the compression test specimens. Therefore, no value for modulus of elasticity are listed in Table A.1 for these tests. The deformations of the specimens tested in the larger machine were measured using a dial gauge (.0001 inch per division) between the spherical seat and the base. This provided a reasonably accurate reading of strain. However, some variations between similar specimens would be expected because, as a result of eccentricity of load or local weakness of the cylinder it is often found that one side deforms more than the other. The results for strength and modulus of elasticity are contained in Table A.1.

Table A.2 contains the compression test results for the core materials. The strains and hence the modulus of elasticity were obtained using displacements of the machine loading bridge as described above. However, it was decided that the very high deformations of the specimens sufficiently minimized the errors associated with the measurement tech-

Test	E_{comp} (ksi)	E_{comp} (Avg.) (ksi)	σ^{ult}_{comp} (psi)	σ^{ult}_{comp} (Avg.) (psi)	Comments 1	Comments 2
SC-I-1	-		2770	2750	B*	B
SC-I-2	-		2730		B*	B
ST-I-1	-		2580	2375	B	B
ST-I-2	-		2170		B	B
S-IA-1	-		6200	6240	B*	B
S-IA-2	3340	3205	6380		B*	A
S-IA-3	3070		6100		B*	A
SC-II-1	1800		-		A*	A
SC-II-2	2320	2245	5720	5680	A*	A
SC-II-3	2170		5640		A*	A
ST-II-1	2440	2260	4410	4870	A	A
ST-II-2	2080		5330		A	A
S-III-1	84.4	76.4	480	480	C	C
S-III-2	68.4		480		C	C
SC-IV-1	1750		4740		A*	A
ST-IV-1	1885	1760	4370	3990	A	A
ST-IV-2	1635		3610		A	A
S-V-1	2380	2355	5500	5725	A*	A
S-V-2	2330		5950		A*	A

Comments
1A Unreinforced mortar with no air entrainer.
1B Unreinforced mortar with air entrainer.
1C Gypsum.
* Mortar had been vibrated.
2A 12" x 6" cylinders were used in the test.
2B 6" x 3" cylinders were used in the test. No usable E_c values found.
2C Built-up prisms 12" x 4" x 4" were used in the test.

TABLE A.1

COMPRESSIVE TEST RESULTS FOR SKIN MATERIALS

Test	E_{comp}	E_{comp} (Avg.)	σ^{ult}_{comp}	σ^{ult}_{comp} (Avg.)	Comments			
	(ksi)	(ksi)	(psi)	(psi)	1	2	3	4
Preliminary	12.5		88.0			*	A	
C-I-1	57.6	56.5	131.5	132.9	*		B	
C-I-2	55.4		134.3		*		B	
C-IA-1	11.0	-	36.4	27.2			B	
C-IA-2	4.7		18.1				B	
C-IA-3	4.1	4.0	18.4	19.8		*	B	
C-IA-4	4.0		21.3			*	B	
C-II-1	26.5	24.7	73.7	65.3				
C-II-2	23.0		57.0					
C-IV-1	61.4	57.0	166.5	174.3				
C-IV-2	52.7		182.2					
Sawdust-cement	14.7		260.0					*

1. After failure it was learned that voids inside (due to the compaction bar) had reduced the effective area considerably. Only 90% of the area of the cylinders was used for the calculation of both, E_{comp} and $\sigma_{ult\ comp}$.
2. Cylinders 6" x 3" were used, instead of the standard size 12" x 6".
3. Mix had air entrained (.008 gr. per lb. of cement).
   3A. - It was added to the beads-cement paste.
   3B. - It was added to the water.
4. Cylinder was only 4 1/2" high (x 6" diameter).

TABLE A.2

COMPRESSIVE TESTS RESULTS FOR CORE MATERIALS nique to make this values worth printing.

The results in Tables A.1 and A.2 are discussed in the last section of this Appendix.

A.5 TESTS FOR TENSILE PROPERTIES

Two types of tests were performed to evaluate tensile properties of skin materials. Firstly, direct tensile tests were performed for gypsum. The results obtained, however, are not quite reliable due mainly to the lack of appropriate equipment for this kind of tests. The tensile strengths and one measured value of the modulus of elasticity are shown in Table A.3 and the sample used in the test is shown in Figure A.1.

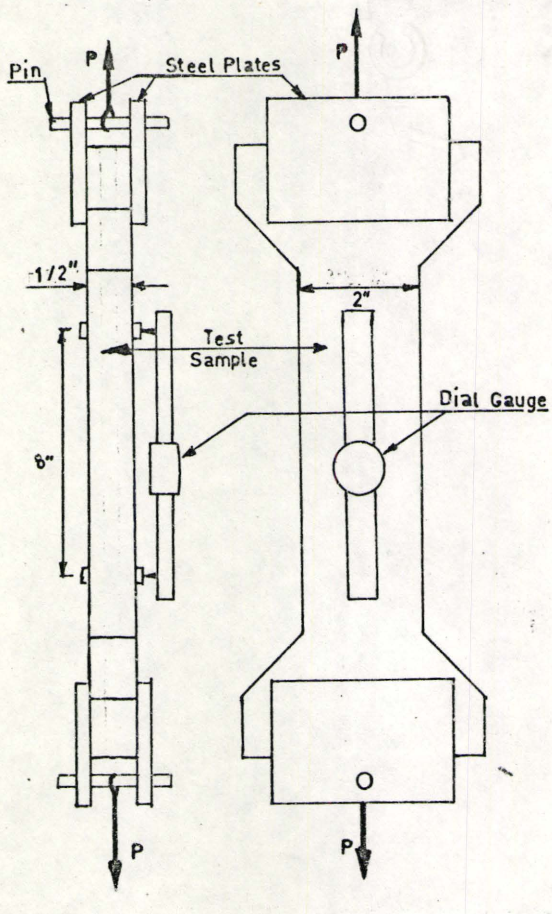

FIGURE A.1

Gypsum Sample for Tensile Test

Test	E_{ten}	σ_{ten}^{ult}	Comments
	(ksi)	(psi)	1
S-IA-1		643	C
S-III-1	128	207	A
S-III-2		86	A
ST-IV-1		355	B
ST-IV-2		348	B
S-V-1		427	C
S-V-2		430	C

Comments: 1A Direct test.

1B Indirect test, mid-span concentrated load.

1C Indirect test, two equal concentrated loads at the thirds of the span.

TABLE A.3

TENSILE TESTS RESULTS

The second type of test was indirect. To evaluate both, the tensile modulus of elasticity and the ultimate strength in tension, beams were cast from some of the mortars corresponding to the skins working under tension. From the load-deflection behaviour of those beams, simple formulas yield the tensile modulus of elasticity and the ultimate tensile strength once the compressive modulus of elasticity was known from direct compressive tests. Unfortunately, difficulties were encountered in obtaining meaningful load-deflection relationships. Therefore, only the indirect tensile strengths (modulus of rupture), calculated using a constant value of $E_{comp} = E_{ten}$ are included in Table A.3.

If tests of this kind are intended, it is recommended here to measure strains at the top and the bottom of the mortar beam with the use of electric resistant strain gauges in a constant moment region instead of load-deflection relationships. It is also recommended here to avoid the use of mid-span concentrated loading systems in these tests.

The results of these tests, shown in Table A.3, are discussed in Section A.7.

A.6 TESTS FOR SHEAR PROPERTIES

Two types of shear tests were tried. In the first one, a 20 inch by 20 inch by 3 1/2 inch sample of polystyrene-concrete was cast in a mold having very stiff sides and hinges in the four corners. Figure A.2 shows a schematic description of the test, which consists of deforming the originally square sample into a romboidal shape. The shear modulus and the ultimate shear strength of the material being tests can be obtained from the slope of the graph correlating the applied forces and the angular deformation of the sample at the corners by simple methods*. Unfortunately, the adhesion of the sample to the mould was weak and the

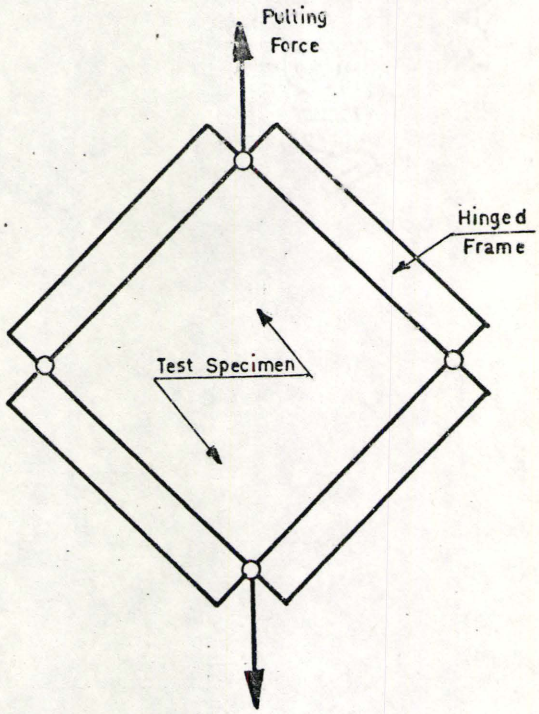

FIGURE A.2

FIRST TYPE OF SHEAR TEST

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* See, for instance, Allen, Howard G., "Analysis and Design of Structural Sandwich Panels", Pergamon Press, p. 255 or Penzien, Joseph and Didriksson, Theodor, "Effective Shear Modulus of Honeycomb Cellular Structure", AIAA Journal, Vol. 2, No. 3, March 1964, pp. 531-535.

sample did not deform into a romsoidal shape as required. In addition, the hinges used were not sufficiently strong and thus the test results are not trustworthy and are not included here.

The second type of shear test was more direct. Two blocks of the material to be tested were glued to an apparatus as shown in Figure A.3. Shear forces were created by applying a vertical load downward on the middle plate. The dimensions of the blocks were such that, in the tests made on polystyrene-concrete, bending deformations must also be taken into account.

The theoretical problem could be idealized as a beam of span $2L$ with both ends built-in, being acted upon by a mid-span concentrated load W. In that case the deflection at the mid-span, $v(L)$, would be given by

$$v(L) = \frac{WL^3}{24EI} + \frac{WL}{2AG} \tag{A.1}$$

where E is the modulus of elasticity of the constituent material, I is the moment of inertia of the section, L is the *length* of one block, G is the shear modulus and A is the equivalent shear area. It was felt that the use of a constant distribution of shear stresses represents more accurately the case here studies and, consistently, A was taken as

$$A = bh \tag{A.2}$$

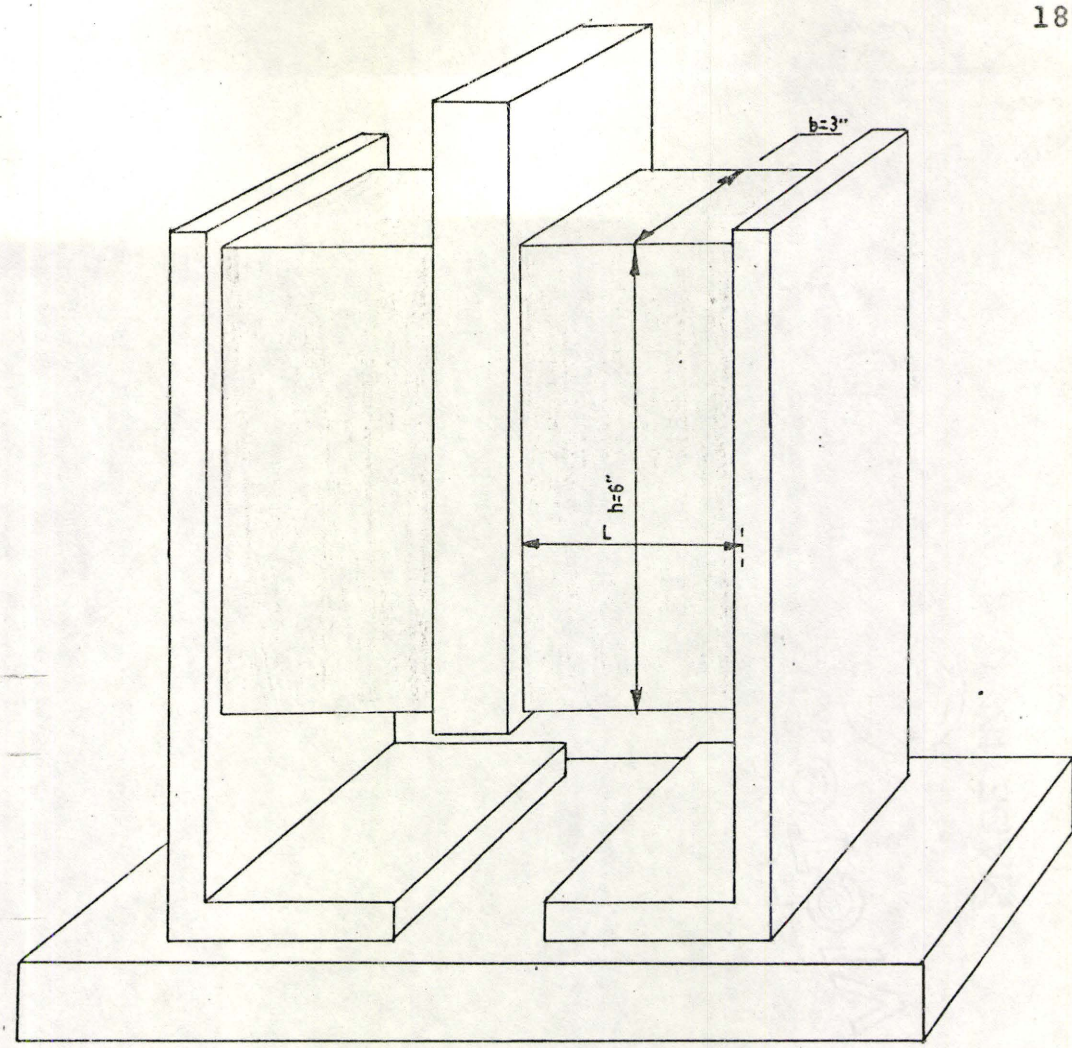

FIGURE A.3

SECOND TYPE OF SHEAR TEST

for the purposes of the calculations in this case. The values of I and A in expression (A.1) give

$$G = \frac{(W/2bh)/[v(L)/L]}{1 - \frac{L^2}{Eh^2}(W/2bh)/[v(L)/L]} \qquad \text{(A.3.a)}$$

where the slope $W/v(L)$ is obtained by tests, E has been previously found experimentally and all other terms are known dimensions in each case.

In some tests (1 inch thick polystyrene boards), the ratio L/h was 1/6 and it may be easily proven that, under those circumstances, the bending deformations are under the one per cent of the shear deflection. For this case, the simple equation

$$G = \frac{W/2bh}{v(L)/L} \qquad \text{(A.3.b)}$$

will be used instead of (A.3.a). For this reason no direct compressive tests to obtain the modulus of elasticity E_{comp} were performed for plain polystyrene.

Deflections were measured by attaching dial gauges with magnetic bases to the sides of the middle plate with the tips of the gauges resting on the base platform of the testing machine. Unfortunately, in some of the tests (indicated in table A.4), readings were taken at only one side of the apparatus and the results are not very reliable. In

Test	E used (ksi)	G found (psi)	G (Avg.) (psi)	$\tau_{max.}$ (psi)	$\tau_{max.}$ (Avg.) (psi)	Comment 1	2	3	4
C-I-2	56.5	28600	22700	29.6	29.1	B	A	B	
C-I-3		16800		28.5		B	A	B	
C-IA-1	4.05	1460	1360	7.9	8.0	B	B	A	
C-IA-2		1260		8.0		B	B	A	
C-III-1	-	102		2.30		A	B	B	*
C-III-2	-	127		3.59		A	B	B	*
C-III-3	-	592		12.8		A	B	B	
C-IV-1	57.0	14700	22100	45.8	46.2	B	B	B	
C-IV-2		29400		46.7		B	B	B	
C-V-1	-	405	384	12.8	13.7	A	B	A	
C-V-2		362		14.6		A	B	A	

Comments:
- 1A Ratio $h/L = 6$ and so, E is not necessary in the evaluation of G.
- 1B Ratio $h/L = 2$.
- 2A Deflections were measured at only one side of the sample.
- 2B Deflections were measured at both sides of the sample.
- 3A Loading was transmitted to the middle plate through a 1/2 inch by 1/2 inch bar.
- 3B Loading was directly applied to the middle plate.
- 4 The shearing strength of the glue was exceeded and the sample blocks slid but did not break. Glue used was other than COLMA-DUR[R].

TABLE A.4

SHEAR TESTS RESULTS

fact, checks made later showed large differences in deflections on opposite sides of the apparatus. Use of two dial gauges gave reasonable average results.

The glue used to attach the sample blocks to the apparatus was COLMA-DUR[R] (Registered trade mark by Sika Products), in most samples and other kinds of commercially available glue in two of the samples. This second glue did not produce good results (Drying time is of the order of months).

The results presented in table A.4 are discussed in the next section.

A.7 CONCLUSIONS

This appendix has presented the results of some attempts to identify and study the properties of suitable materials for sandwich panel construction. The results of the tests, some comments on the requirements for the materials and suggestions for improved testing techniques are presented below.

A.7.1 Skin Materials.

a) Concrete Mortar: One of the most common skin materials is concrete mortar. It satisfies many of the requirements very well in that it is readily produced, is fairly easily made, has a low cost, and physically has high fire resistance, good durability, high compressive strength and high sound proofing characteristics. The major deficiency of concrete mortar for skins is related to its relatively low tensile strength for situations where high bending moments may be encountered. The tensile strength may be improved considerably by reinforcing with wire mesh (Ferro-Cement), or including fine randomly mixed lengths of wire. The thermal insulating properties of the concrete mortar skins are not good so that the core must be counted on for most of the thermal resistance.

Table A.1 contains the compressive test results for the concrete. A large variation in strength between nominally similar mixes is apparent. It is obvious that mixing and curing operations were not accurately reproduced. However, the duplicated strength tests for each mix are reasonably close so that it is concluded that the strength values indicated are representative of the real strength. The procedure for determining the compressive modulus of elasticity was discussed earlier. From experience with normal concrete these values appear to be slightly low. However, there is not too much research literature available to compare the properties of concrete mortar. Since low amounts of aggregate (sand) should result in a decreased modulus of elasticity these values are accepted as being reasonable.

Table A.3 contains the tensile strength results. No tensile modulus of elasticity was found so it is assumed to be equal to the value from the compressive tests.

b) Gypsum Board: The compressive and tensile properties of 1/2 inch thick gypsum board are shown in Tables A.1 and A.3 respectively. Considerable variation in results was observed. The tests were not repeated to obtain confirmed values because panel III (the only one using gypsum) failed due to improper glueing so that no values were needed for predicting its behaviour. However, even the minimal amount of work done on this material indicates its value as a structural component. It has many of the advantages of concrete mortar and in some cases will be less expensive and easier to use. The fact that this material has largely been used to satisfy only some of the architectural requirements is a waste of a considerable contribution to strength.

<u>A.7.2 Core Materials.</u>

a) Polystyrene: Low density polystyrene sheets are widely used as thermal insulation and have the advantage of lightness and being available in a convenient form. Polystyrene has the disadvantages of very low fire resistance, low sound insulation, low strength, special bonding requirements and, for some applications, high cost. Polystyrene (Density 1.5 pcf.) was studied here to see if it had sufficient strength to create composite action between the skins and to see if the fabrication techniques could satisfy the bonding problem.

The only structural property needed for the theoretical analysis was the shear modulus. Table A.4 contains these.

b) Polystyrene-Concrete: The polystyrene-concrete should have high thermal insulating values as well as providing significant fire resistance and sound insulation, good bonding properties and greatly improved strength as compared to plain polystyrene. The increased weight is a decided disadvantage and some improvement may be achieved through use of a better graded selection of beads.

It was found that the properties of the polystyrene-concrete varied considerably from mix to mix. Aside from possible differences in mix proportions and different curing, it was concluded that the amount of compaction could have the largest influence on the strength and elastic properties. The mix tended to segregate if vibrated,therefore, it was pressed into the molds. Table A.2 contains results which show a difference in compressive strengths where one mix (C-IV) is 10 times the strength of a similar one (C-IA). Table A.4 contains the results of the shear tests. The values of shear strength and shear modulus vary in a manner which corresponds to the compressive tests.

The conclusion is that if the polystyrene-concrete is properly compacted, its structural properties are more than adequate for its intended use.

c) Sawdust-Concrete: The trials with sawdust-concrete were performed to study the possibility of using natural vegetable fibres. A conclusion reached is that the materials should not be highly water absorbent, which implies either careful selection of materials or pre-treatment to seal them. The possibility of using charred sawdust or other vegetable fibres seems to be worth considering. TDR Engineering Developments Ltd., Toronto*, has patented a process for burning wheat to create a light weight aggregate.

A.7.3. Comments on Test Procedures.

a) Compression Tests: For accurate measurements of deformations, the method recommended by ASTM**, where one dial gauge is clamped to the cylinder, should be replaced by a system which measures deformations on opposite sides of the cylinder. Use of a mechanical indicator or electric resistance strain gauges directly on the cylinder is suggested. The latter would be especially applicable when small specimens are tested.

b) Tensile Tests: For mortar the tensile strength and tensile modulus of elasticity should be found from properly shaped specimens where bending and stress concentrations are eliminated. This direct tensile test is probably most representative of the conditions in the sandwich panel. Another measure of tensile properties may be obtained by testing mortar beams

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* "Wheat as an Aggregate", Concrete, Vol. 6, No. 1, January 1972, London, England, p. 36.

**Hedderich, H. F. and Artuso, J. F., "Testing Concrete", Chapter 12 of "Concrete Construction Handbook", Edited by Joseph J. Waddell, Mc Graw-Hill Book Co., p. 12-16.

where the compressive modulus of elasticity is known. In both cases strains would best be measured by using electric resistance strain gauges. Loading systems producing regions with constant moment are recommended instead of the commonly used mid-span concentrated loadings.

c) Shear Tests: Concerning shear tests, the second type described earlier in this appendix and shown in Figure A.3 is considered to give quite accurate results. In order to avoid the complication of considering bending deformations, the dimensions of the sample blocks should be changed in such a way that the height-to-length ratio becomes at least of 5. If the first shear test described earlier in this appendix is to be used, care must be taken to design sufficiently strong hinges and also very important, to glue the fresh material to the hinged form. COLMA-FIX^{R} is especially recommended to glue fresh concrete (No matter what aggregate is used) to the steel form.

A third type of shear test that could be used consists in the torsion of a cylinder of the material to be tested. A system of loadings could be easily devised with the use of pulleys to convert unidirectional forces (Gravity, or produced by a jack) into torques. Tensile load cells would have to be used to account for friction effects.

A.7.4. Résumé.

The properties of materials found here are used in the analysis of the panels which is presented in Appendix B. Because of inexperience with testing techniques and other problems it is suggested that the values presented here be regarded as preliminary results and as such must be confirmed by subsequent repetition of tests.

APPENDIX B

TESTS ON SANDWICH BEAMS AND BEAM-COLUMNS

B.1 INTRODUCTION

Five panels were constructed to be failed under beam-column loading. Some unaffected pieces of these were later failed under mid-span concentrated beam loading. The descriptions and results of these tests are presented in the following sections.

B.2 DESCRIPTION OF TEST METHODS FOR BEAM-COLUMN LOADING

The tests with beam-column loadings were performed as follows. Eccentric loads distributed along the width of the panels were applied at the centroidal line of the compressive skin so that $e = d_1$, according to the notation used throughout this thesis. This loading was applied through an apparatus as shown in Figure B.1.

The load cell attachments were made with weldings but, for Panel V, two re-placeable rings were used instead because it was learned later that the weldings had very likely affected the load cell. As will be discussed later, the calibration of the load cell for the second, third and fourth tests makes it impossible to quote exact load levels.

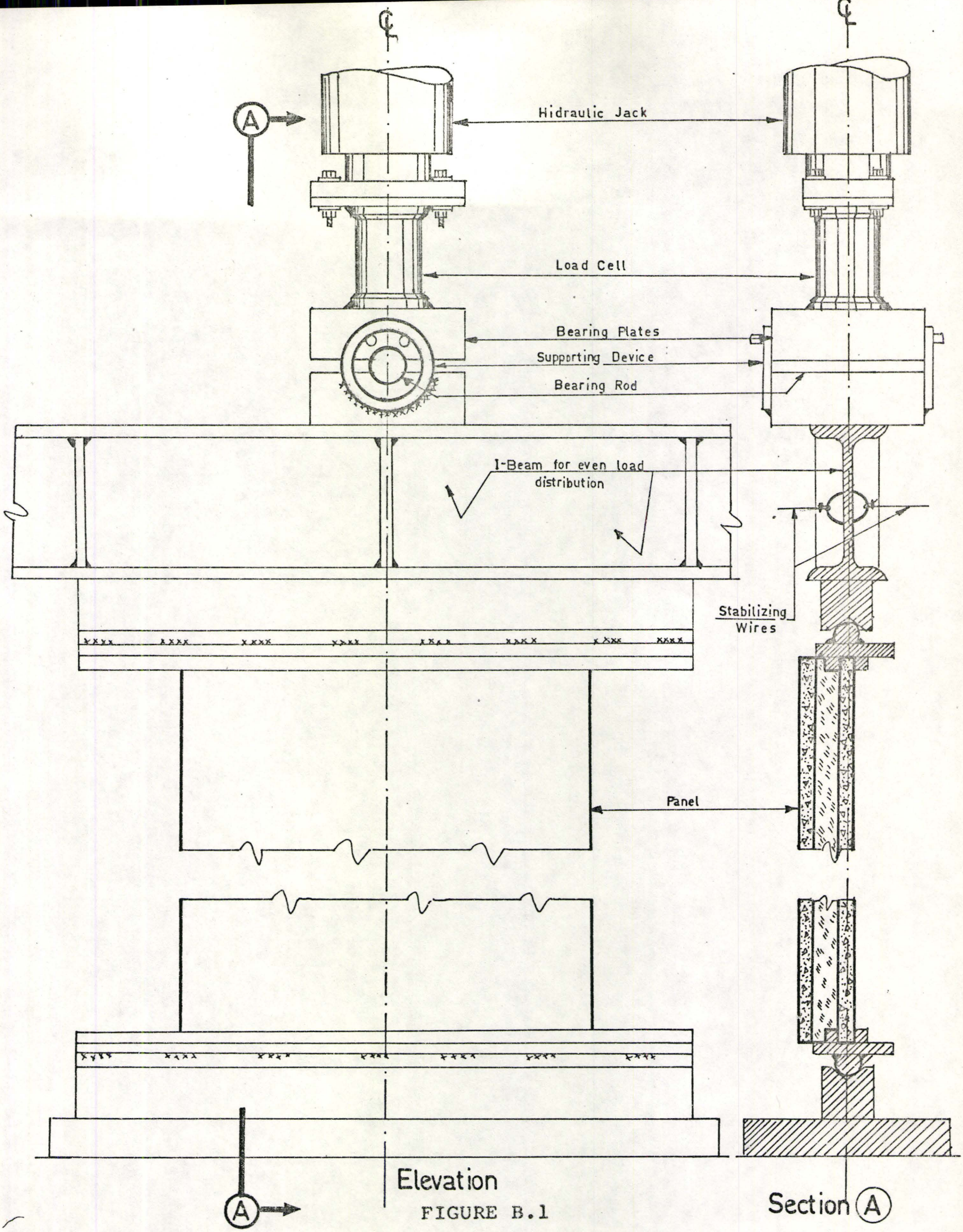
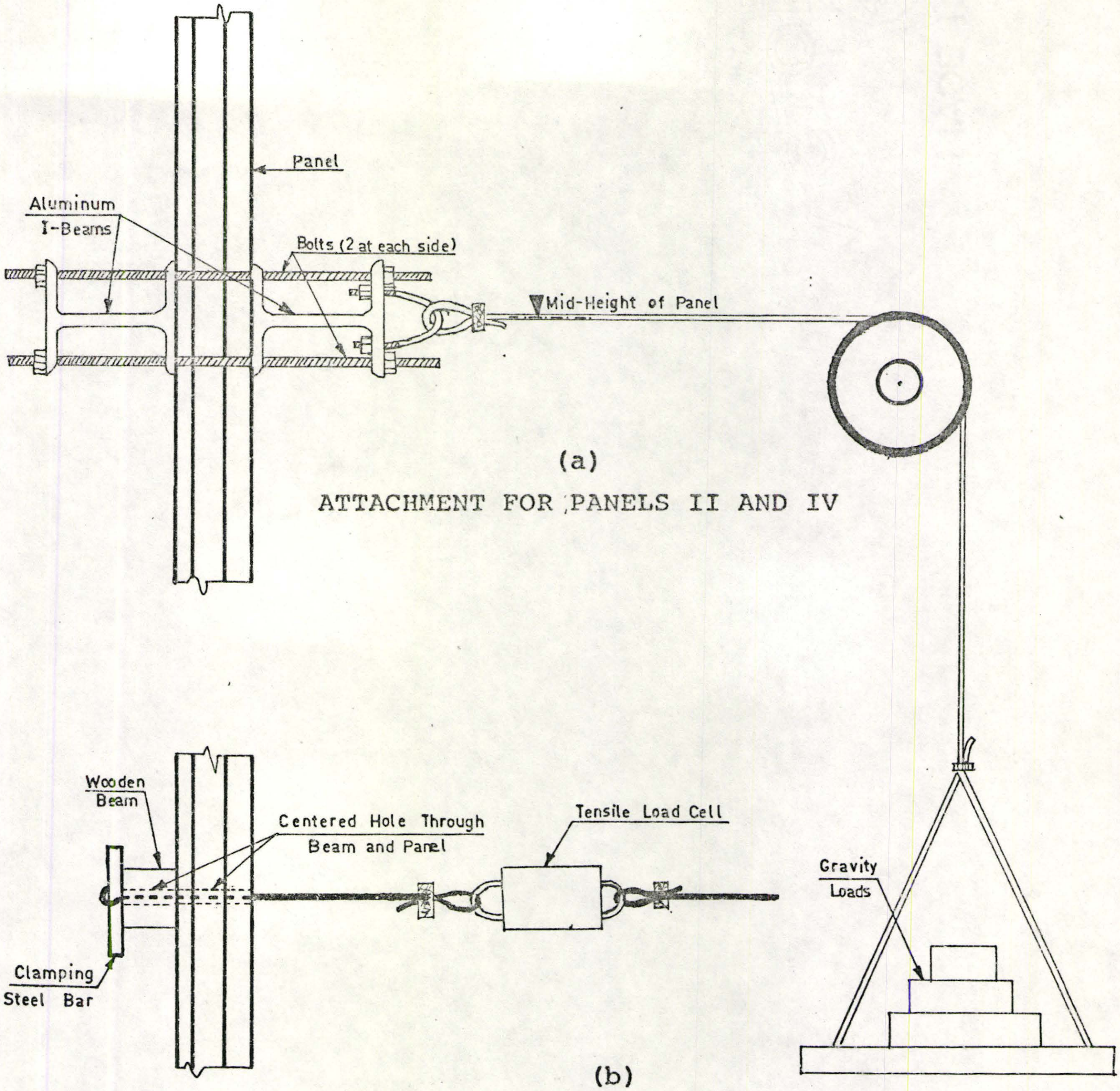


FIGURE B.1

APPARATUS FOR APPLICATION OF ECCENTRIC END THRUSTS

In Panels I and III only the eccentric end load described above was applied. In the remaining panels, a concentrated lateral load was applied at mid-height. That load was applied by the use of gravity forces turned horizontally with the use of a pulley, as shown in Figure B.2. The attachment of the rope to the mid-height of the panel was made as shown in Figure B.2.a for panels II and IV. In Panel V it was considered better to drill a hole through the center of the panel and to transmit the lateral mid-height concentrated load through a wooden beam at the opposite side, as shown in Figure B.2.b. A tensile load cell was used in the latter test to exclude the friction effect from the pulley, but none was used in the tests for panels II and IV. In these cases it was intended to measure the friction effect after the test, but this was not done. From the results of lateral loading for panel V approximate values for friction were obtained. An exact reproduction of the conditions for tests of panels II and IV was not attempted because other problems made these results only approximate.

Three dial gauges were used to measure lateral deflections at the mid-height and at the two quarter points of the height. In panels II to V, a dial gauge was set at the very top of the panel to measure possible rotations of the panel as a whole about the bottom. In the case of Panel I, the dial gauges were all placed at its middle vertical line, but be-



(a)

ATTACHMENT FOR PANELS II AND IV

(b)

ATTACHMENT FOR PANEL V

FIGURE B.2

APPARATUS FOR APPLICATION OF MID-HEIGHT CONCENTRATED LATERAL LOADING cause some gauges were damaged when the sandwich panel failed it was decided that for the remaining panels these would be set at the sides of the panel. Overhanging wooden pieces had been clamped to the panels for the purpose of measuring those deflections. The distances of the final location of the tips of the gauges to the center line of the panels were not measured.

B.3 TEST RESULTS FOR BEAM-COLUMN LOADING

B.3.1 Panel I

Panel I was poured with two 1/2 inch thick unreinforced mortar skins and had a 1 inch thick core made out of Polystyrene-Concrete. These materials were described in Appendix A. A wooden form was used, the flatness of which had been checked before with an optic level. The maximum deformation found in the form was in the order of 1/8 inch out of the tangent plane and it was not possible to correct it any further from there. The compressive skin was cast first, vibrated and then made even. The core was cast after three to four hours. It was not possible to vibrate it because, when an attempt was made, the beads tended to rise to the top, which, if allowed, would produce an undesirable segregation. The third layer, cast after three or four hours, could not be vibrated either. The free surface of the last cast skin produced free water in large amounts, which made the levelling of that surface difficult to accomplish. The reason for the excesive amount of water is unknown.

The panel was cured for seven days by pouring water on it twice a day without taking it off the form. It is possible that this procedure produced differential shrinkage in the skins, which would cause the panel to bow initially. However, no measurements were taken to determine any initial bowing.

Panel I had entrained air in its skins as well as in its core material. No other panel was built with air-entrained materials.

Panel I was tested to failure when it was forty days old. The eccentric load, measured with the use of the load cell shown in Figure B.1, was increased by 124 lbs., which corresponded to the weight of the apparatus hanging from the cell. The deflections of the top and bottom quarters of the height, which should have coincided, did not. It was thought at the time that the reason for this was that the panel had rotated as a whole with respect to the bottom edge line. Results obtained with the other panels show that this was not the reason. The graph for Eccentric End Load P versus Mid-Height Deflection $v(L/2)$ is presented in Figure B.3.

In order to evaluate the theoretical Load versus Mid-Height Deflection curves, the properties of the constituent materials are required. It was mentioned in Appendix A the great difficulty experienced in obtaining meaningful values for those properties. The values of the Young's moduli as

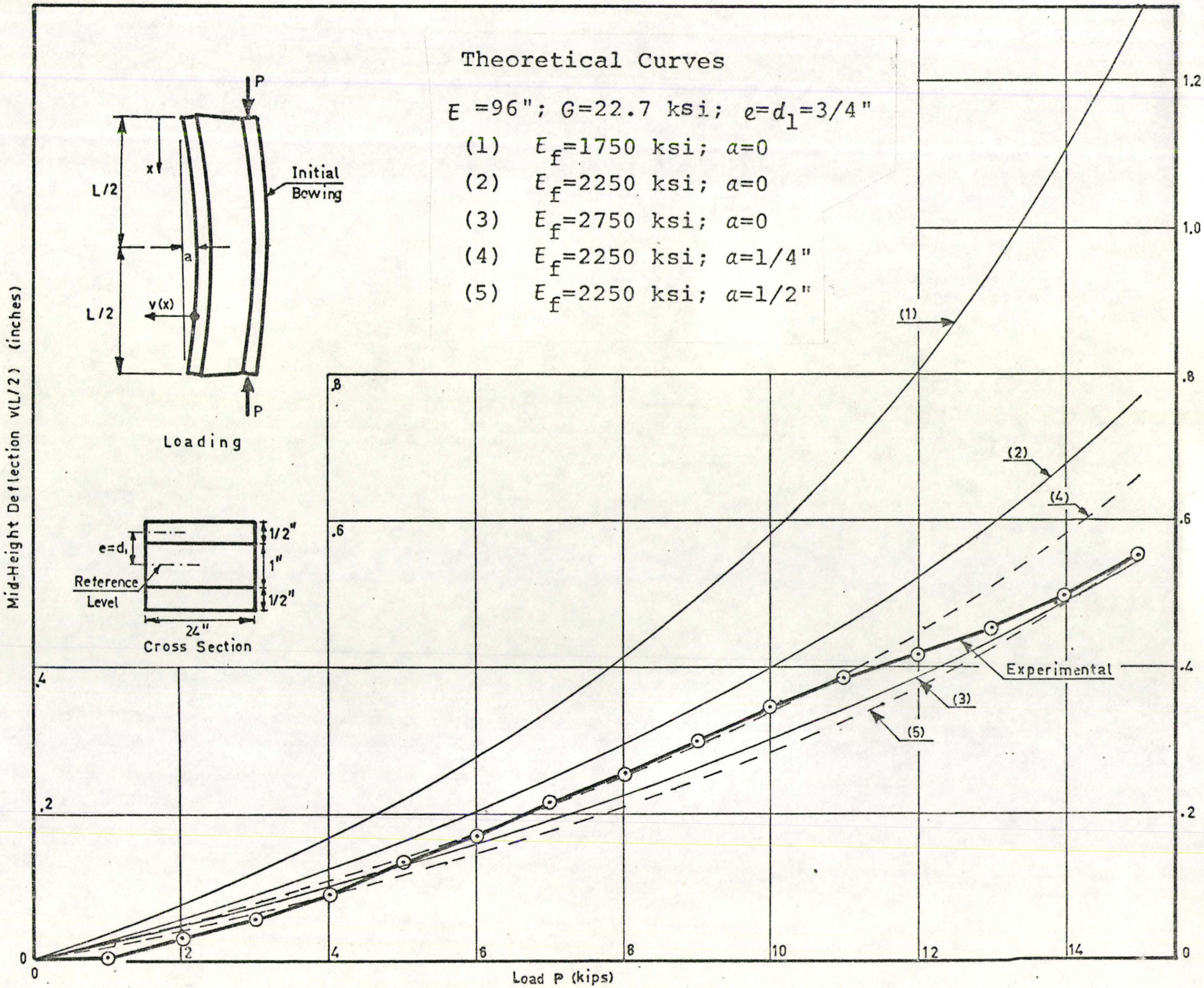


FIGURE B.3

LOAD VERSUS DEFLECTION CURVES FOR PANEL I found in Appendix A for tension and compression of the skin materials were too small. In order to weigh the importance of those numbers, theoretical Load-versus-Deflection curves were also plotted in Figure B.3 with three different values of the modulus of elasticity of the skins (Assumed equal in tension and compression). These values are 1750 ksi, 2250 ksi and 2750 ksi [Curves (1), (2) and (3) respectively]. The very large differences in results suggest that very accurate tests to predict the Young's moduli of the skins are necessary. The same thing cannot be said about the shear modulus of the core when its value is a large number. In fact, for this panel, variations in the shear modulus, G, of the order of the 50 percent produced variations of less than the one percent in deflections. Had the core material been a foamed plastics (Modulus of Shear of the order of 500 to 1000 psi), an exact evaluation of its value would have been required. A representative value of the shear modulus, which was used for these calculations, is 22.6 ksi.

Figure B.3 also shows that, when the loading was of approximately 11 kips, slightly larger forces were required to produce comparable deflections, that is, the panel became stiffer. No reason is known for this. No reason is known for the difference in deflections at the quarter points of the height either. The assumed rotation as a whole was shown not to exist in later panels. Finally, flaws in the manufacture of the panel (Initial bowing, mainly), could have influenced the results. Using E_f = 2250 ksi, the theoretical results for maximum initial bowing of 1/8 inch and 1/4 inch are shown as curves (4) and (5) respectively in Figure B.3.

In addition, it was concluded that the load cell used was not 100 percent trustworthy, because after re-welding for testing later panels it was found to produce large variations in readings.

Because of the uncertainty involved in determining the modulus of elasticity of the concrete skins, the amount of initial bowing and the discrepancy in load cell calibration, it is not possible to make direct conclusions about the elastic analysis. However, in terms of ultimate capacity, the later calibrations of the welded load cell showed that it always underestimated the actual load. Therefore, the recorded failure load of 15.1 kips is a conservative value. This corresponds to an Eccentric Load Capacity of 7.55 kips per foot of wall.

It is interesting to study the tensile stresses which appear in Table B.1 for the skins of the panel for the conditions in Figure B.3. The most reasonable values for concrete tensile strength coincide with the curve which comes closest to the experimental curve.

Case	Assumed E_f (ksi)	Assumed Initial Bowing a (in)	Stresses at Failure (psi)	
			Compressive	Tensile
1	1750	0	3032	1740
2	2250	0	2262	1009
3	2750	0	2115	824
4	2250	1/8	2045	779
5	2250	1/4	1788	522

TABLE B.1

THEORETICAL STRESSES AT FAILURE FOR CASES CONSIDERED IN FIGURE B.3.

B.3.2 Panel II

Panel II was cast in a steel form to avoid the initial unflatness. The dimensions of the panel were (as in Panel I) 8 feet by 2 feet, but the thickness of its tensile skin was made 3/4 inch instead of 1/2 inch. In the same way as Panel I, only the compressive skin could be vibrated.

The tests on Panel II included the application of Mid-Height Concentrated Lateral Loading. The attachments were made as shown in Figure B.2.a but, unfortunately, the lack of foresight caused that, when the I-beams to hold the lateral loading acting on the panel were tightened together, too much tightening started visible cracking of the panel at mid-height.

In addition, the welding spots holding the load cell failed before the test was carried out. The load cell fell down with the heavy attachments and this could have affected it. The welding spots were replaced with continuous weld and the resulting high temperatures could have also affected the load cell. As a result, it was learned after testing Panel IV that the results from the load cell did not agree with the initial calibration. When the stiff plates used to attach the load cell to the other loading apparatus were kept in place during the calibration, it was observed that, depending upon the placement, the initial zero was shifted by between more than 1000 lbs. and less than 2000 lbs. After many recalibrations, a recalibration curve for an initial zero shift of 1500 lbs. was used to calculate the loading from the recorded load cell readings from Panel II. Therefore it is suggested that the reconstructed load values will be within 500 lbs. of the actual values. Using these corrected values it was possible to have a fair idea of the stresses acting on the panel at the failure load. With an eccentric axial load of P = 2350 lbs., an estimated value of horizontal load W = 160 lbs., and taking the moduli E_f = 2250 ksi and G = 22.6 ksi, it was found that the maximum axial stresses in the skins at failure were 488 psi in compression and 280 psi in tension. These small values are reasonable when the previous cracking is considered.

B.3.3 Panel III

Panel III was constructed by glueing 1/2 inch thick gypsum skins on both sides of an 1 inch thick Expanded Polystyrene sheet. A commercially available adhesive recommended by the distributor was used. The structural strength of that glue was very poor and the test did not produce satisfactory results. When the load had some value near 800 lbs., the tensile skin went away from the core at the bottom of the panel. Prior to this, large amounts of creep were observed and the deflections were disproportionately large. The test was carried out two months after the construction of the panel.

Because the initial shift of the calibration curve was recorded in this case, the correction to the calibration of the load cell produces an accuracy in the evaluation of the applied thrusts which is sufficient to gain some idea of the behaviour of this panel. Figure B.4 shows a plot of Mid-Height Lateral Deflection versus Applied End Thrusts for the tests performed.

Test 1 shows the first curve obtained. The high creep may be observed at the points with $P = 570$ lbs. and $P = 840$ lbs., where the test was delayed for approximately 15 minutes. When the loading reached 800 lbs., the bottom of the tensile skin had very visibly pulled away from the core indicating that the glue had not worked properly. Most of the creep mentioned above was due to slip at the skin

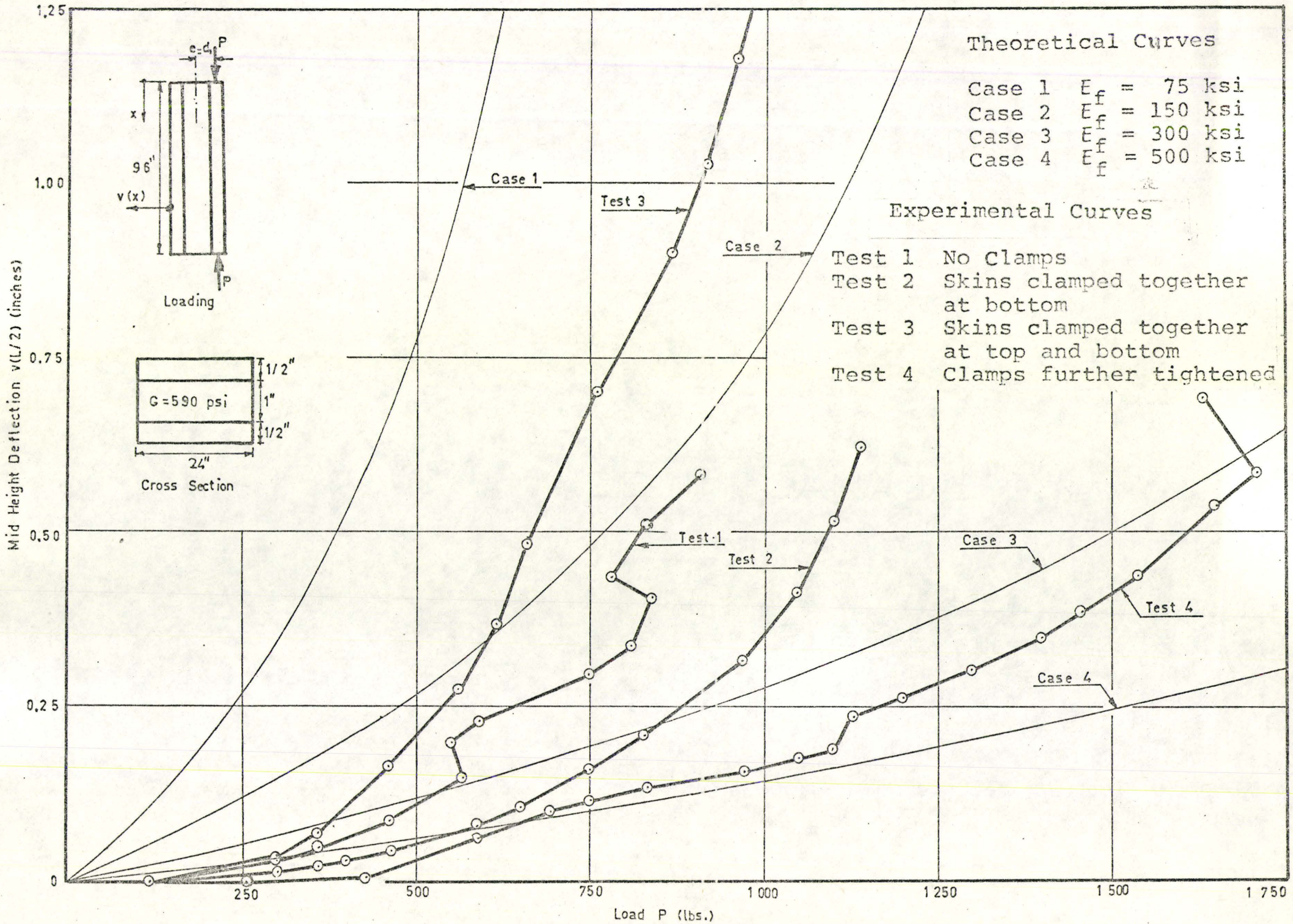


FIGURE B.4

LOAD VERSUS DEFLECTION CURVES FOR PANEL III core interfaces.

A clamp was applied at the bottom of the panel to correct that problem and Curve Test 2 was obtained. When the load reached values nearing 1140 lbs., the tensile skin went away from the core at the top and both ends were clamped after unloading. Tests 3 and 4 were performed with both ends clamped. In the case of Test 4, the clamps were tightened more than for Test 3.

Four curves evaluated with the theory developed in Chapter VII are also included. They were calculated from four different values of the elastic modulus of the skins (75, 150, 300 and 500 ksi). As before, the necessity of accurately evaluating the modulus of elasticity is evident.

A beam test of a length of Gypsum board gave a calculated modulus of elasticity of 230 ksi. It can be seen that, at low loads, all the tests give results near the theoretical curves corresponding to similar cases. However, as creep and delamination occurred, the deflections increased rapidly. For Test 4 it is suggested that the very tight clamping significantly increased the bonding.

The shear strength of the glue used was experimentally found to be approximately 3.7 psi two weeks after it had been applied. With proper bonding, this type of panel could be used in loading bearing situations.

B.3.4 Panel IV

Panel IV was 16 inches wide and in other respects was the same as Panel II, where the thickness of the tensile skin was 3/4 inch, and the constituent materials were unreinforced mortar and Polystyrene-Concrete.

Eccentric thrusts were applied as explained for the other panels until an axial load that was thought to be small was reached. Horizontal loading was then applied and afterwards, end thrusts were gradually increased until structural collapse occurred.

After the test was performed it was realized that the readings of the load cell were very different when the mounting plates were attached to it. The initial calibration had been performed for the load cell with no attachments of any kind and therefore a recalibration with the plates on the load cell was necessary to be able to evaluate the loads that had been applied to the panel.

The new calibration showed that the end thrust applied before loading laterally was not as small as was intended. In fact, it could have reached a value as high as 1700 lbs. Figure B.5 shows a plot of the deflections at midheight versus applied end thrusts. The fact that no significant displacement took place at the top edge of the beam-column implies that the observed differences in the deflections at the top and bottom quarter points can not be explained as a rotation of the panel as a whole with respect to the

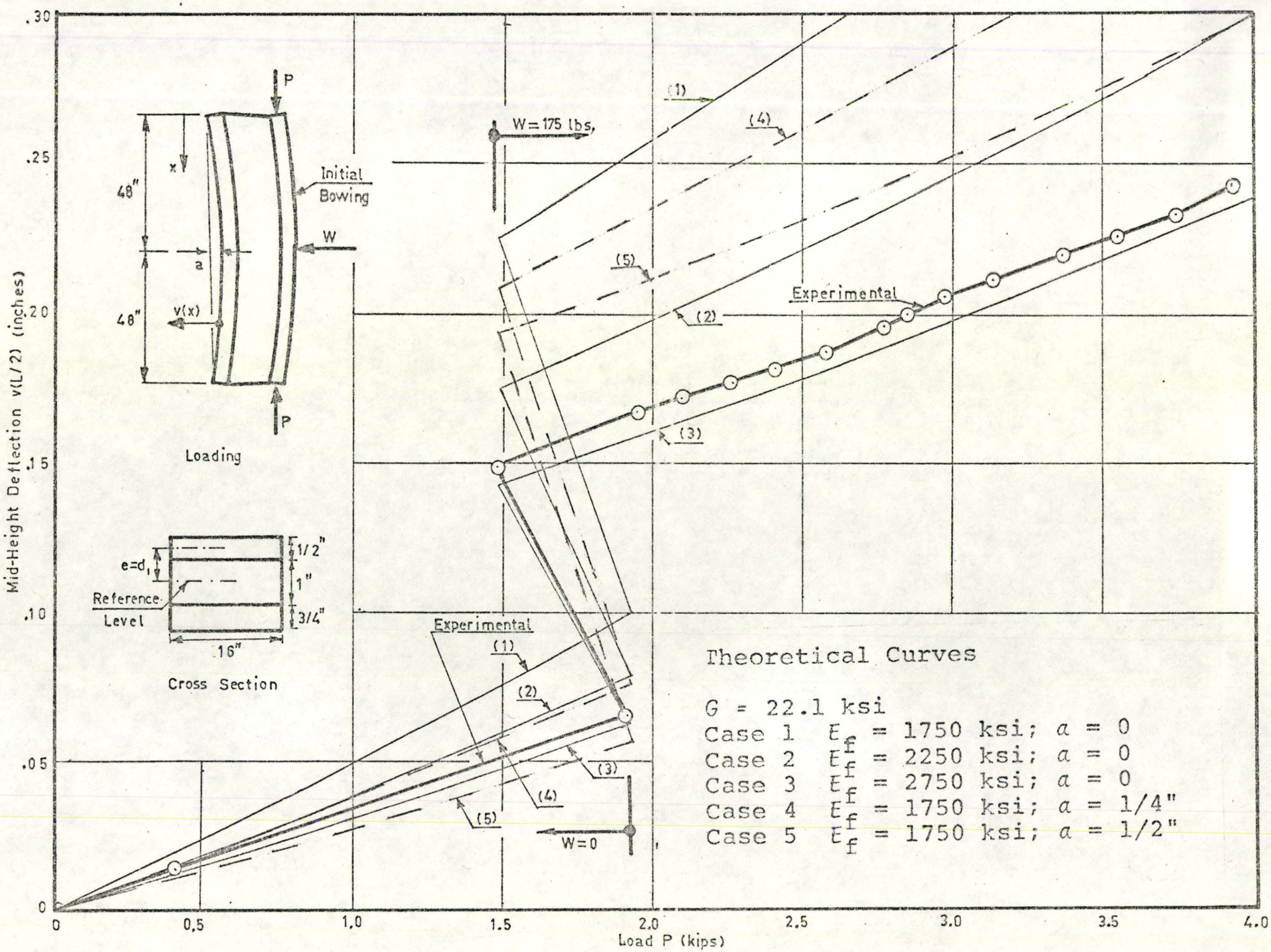


FIGURE B.5

LOAD VERSUS DEFLECTION CURVES FOR PANEL IV bottom edge. Possible reasons for the large differences could be varying eccentricities due to imperfections or twisting of the panel under load.

Five trials to find theoretical curves to compare with the experimental graphs in Figure B.5 were made. Three of them were made with material properties similar to those commented on in Appendix A and the other two trials showed the effect of different initial shapes of parabolical bowing with 1/4 inch and 1/2 inch amplitude at mid-height, with curvature towards the opposite side from the eccentricity. This was done because the likelihood of having initial bowing in panel IV was indicated by measuring of these conditiona for Panel V. No bowing measurements were taken for Panel IV.

Panel IV failed when the vertical load was increased to 3.9 kips while the horizontal load was maintained at approximately 175 lbs. It is interesting to note that the tensile stresses at failure, shown in Table B.2, are all in the

Case	Assumed E_f (psi)	Assumed Initial Bowing a (in)	Stresses at Failure (psi)	
			Compressive	Tensile
1	1750	0	1111	586
2	2250	0	1078	561
3	2750	0	1059	548
4	1750	1/4	1001	493
5	1750	1/2	891	401

TABLE B.2

THEORETICAL STRESSES AT FAILURE FOR CASES CONSIDERED IN FIGURE B.5.

range of generally acknowledged tensile strength for 4000 psi concrete. The test results follow the same basic shape of the predicted behaviour and by choosing an appropiate modulus of elasticity and/or an initial bowing the agreement could be spectacular.

The ultimate capacity of this panel was 3 kips eccentric axial load per foot length of wall with a horizontal concentrated load giving a maximum bending moment equivalent to a 33 lb/ft^2 wind loading.

B.3.5 Panel V

Panel V was also 8 feet by 16 inch by 2½ inch and with skins similar to Panel IV, but this time the core material was an 1 inch thick Expanded Polystyrene board. COL-MA-FIX^R (Registered Trade Mark by SIKA Products) was used to glue the fresh mortar to the core. In this panel unlike the previous ones, both skins were vibrated.

This panel was intended to be tested under eccentric thrusts only but the behaviour under such a load was very strange. In fact, the panel was even deflecting in the wrong direction. It was learned that this strange behaviour was due to initial bowing which was probably due to differential shrinkage. The application of lateral loading was then introduced. A hole was drilled in the centre of the panel and the lateral loading was applied as shown in Figure B.2.b. The load cell for measuring the end thrusts and its attachments were replaced because it was learned after the failure of Panel IV that the cell used formerly was not providing satisfactory results. A tensile load cell was used to measure the lateral load. Deflections at mid-height and the corresponding loadings were recorded and their plot is shown in Figure B.6. The sequence of loading was: Vertical loading to 1.04 kips; 15 lbs. horizontal load; decrease vertical load to 300 lbs; increase horizontal load to 65 lbs; increase vertical load to 560 lbs.; increase horizontal load to 125 lbs.; gradually increase vertical load to 3.42 kips; and finally the horizontal loading was increased to failure at 175 lbs.

As happened with the other panels, deflections at the two quarter points of the height were very different from eachother. As said before, the likely reason for this is unsymmetrical initial bowing or some misalignment which would cause twisting.

For the evaluation of the theoretical curves, the following data were used: E_f = 1750 ksi [Curve (1)] and E_f = 2355 ksi [Curve (2)], G = 385 psi, and a = 9/32 inch (measured).

It may be observed that, even though large differences between the absolute experimental and theoretical values of the mid-height deflections exist, the general slopes of both curves are approximately the same. Experimental deflections were measured from a state in which the panel had P = 1050 lbs. and W = 15 lbs.

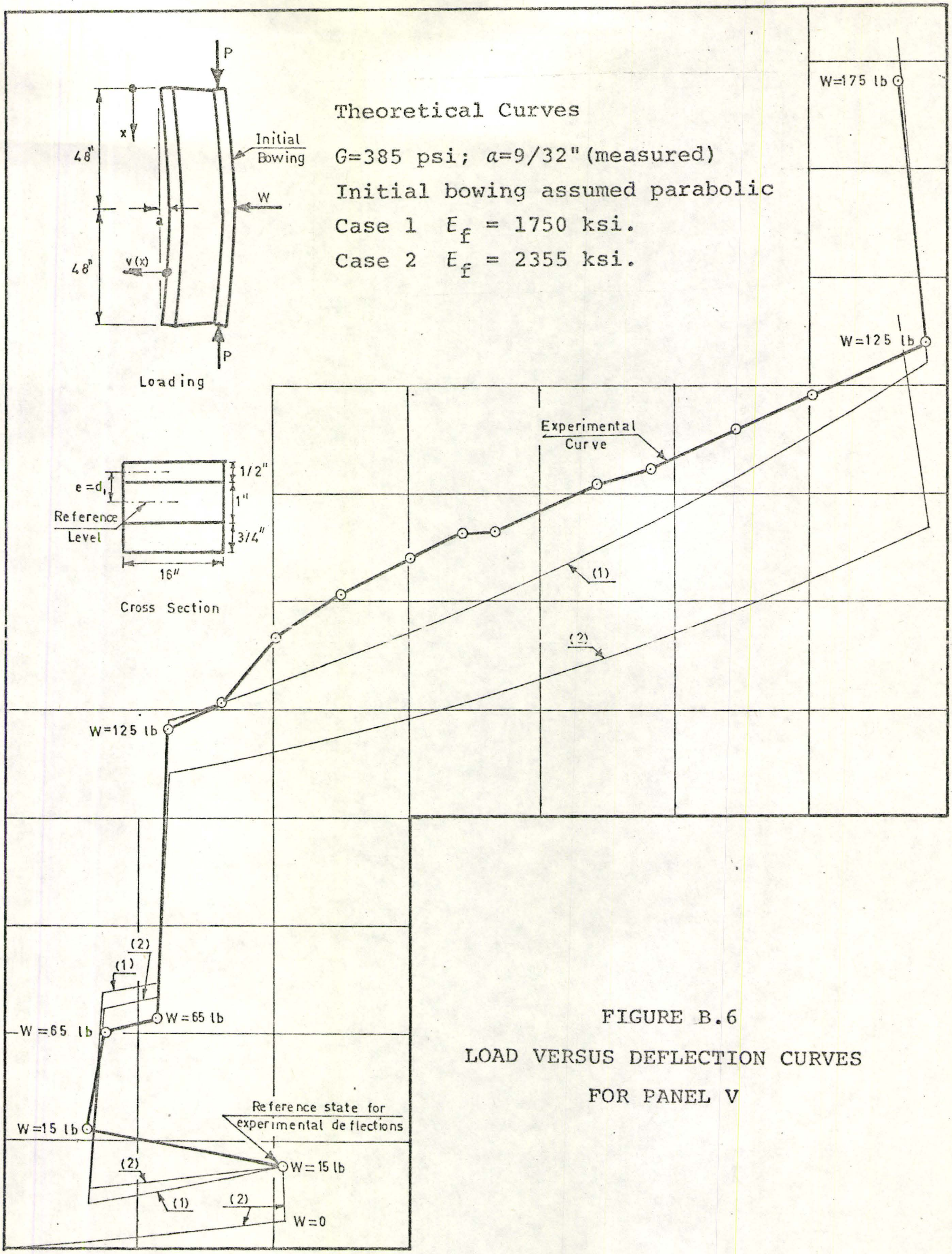


FIGURE B.6

LOAD VERSUS DEFLECTION CURVES

FOR PANEL V

The ultimate capacity of this panel was 2560 lbs. eccentric axial load per foot of wall length and a concentrated horizontal mid-height force which produces a maximum bending moment equivalent to a wind loading of 33 psf. No break down in bonding between the skins and the Polystyrene core was observed. The stresses in the skins at the failure loading are shown in Table B.3.

Case	Assumed E_f (ksi)	Assumed Initial Bowing a (in)	Stresses at Failure (psi)	
			Compressive	Tensile
1	1750	9/32	1180	862
2	2355	9/32	1162	870

TABLE B.3

THEORETICAL STRESSES AT FAILURE FOR CASES CONSIDERED IN FIGURE B.6.

B.4 BEAM LOADING TESTS

Undamaged portions of panels II, IV and V from the beam-column tests described earlier were used in some beam loading tests. These tests consisted of application of mid-span concentrated loads on simply supported beams. The load was applied by using a TINIUS OLSEN[R] Universal Testing Machine. Deflections at Mid-span were recorded. The ratios $v(L/2)/W$ (Mid-span deflection over mid-span concentrated load) were evaluated for one beam from Panel II, and two from each of Panels IV and V. Table B.4 presents these values

Test	t_1	t_2	b	c	L	E_f used	G used	$v(L/2)/W$ ($\times 10^{-6}$ in/lb)	
	(inches)					(ksi)	(psi)	Theoret.	Experim.
II	1/2	3/4	24	1	45	2250	22600	70.3	75.2
IV-1	1/2	3/4	16	1	30	1750	22600	25.7	28.0
IV-2	1/2	3/4	16	1	16	1750	22600	5.0	21.2
V-1	1/2	3/4	16	1	40	2355	385	329.	221.
V-2	1/2	3/4	16	1	40	2355	385	329.	396.

TABLE B.4

BEAM LOADING TEST RESULTS

and, for purposes of comparisons, it also includes the theoretical predictions of these values and the properties of the beams used in the theoretical evaluations.

The inconsistencies in results are large. Again twisting of the panel and local crushing at the load point and supports could account for some of this inconsistency.

B.5 SUMMARY AND CONCLUSIONS

The program of experiments described in this appendix was aimed at studying the performance of sandwich panels constructed with materials common to the building industry. The loadings applied also correspond to the common cases in buildings.

Unfortunately, the results were not entirely satisfactory, owing mainly to the lack of experience in dealing with problems of this kind. In fact, the relative importance of several of the factors involved was not known when the tests were being carried out. Some of the most important aspects will be discussed in the following paragraphs.

The flatness of the panels was always thought to be important because of the secondary moments involved. Even though the forms were ensured to be as flat as possible, the initial bowing of the hardened panel was not measured until Panel V. In the case of this panel, the initial bowing (probably due to differential shrinkage of the skins) was noticed because of the fact that deflections initially occurred in the opposite direction to that expected.

The necessity for accurate load control is obvious.

The dial gauges to measure deflections at the three quarter points of the height of the panel were placed at the center line in the case of Panel I. The collapse of the panel, however, broke several of them and, for the sake of economy, it was decided to set them out of the center lines in the remaining cases. The deflections were measured in these by clamping little pieces of wood to the panels and measuring the displacements of these points. Any twisting of the panel as a unit would produce wrong results and this very likely happened. It is suggested here to use dial gauges at *both* sides of the panel for future tests. The average of these should produce more reliable results than from only one reading.

The problems found in the evaluation of the properties of the materials, already commented on in Appendix A, made it impossible to have direct comparisons with the theories developed earlier in this thesis. Only hypothetical theoretical curves could be plotted.

These difficulties plus other problems accounted for earlier detracted from the overall usefulness of the test results. However, some generalities can still be drawn.

The high strength of the sandwich panels with only 1/2 inch thick unreinforced mortar skins and a 1 inch thick Polystyrene-concrete core is satisfactory for many purposes. With the exception of panels II and III, the capacities of the panels were encouraging in terms of considering their practical application. The test of Panel II gave warning of the low strength of those panels to transverse compression. Local cracking of the skins resulted from clamping a loading beam in place. Loadings of this type must be designed for.

Panel III made clear the suitability of Gypsum-Expanded Polystyrene panels when an effective bonding is achieved. COLMA-DUR^R is especially recommended for future trials with those materials.

Panels IV and V were subjected to lateral loading equivalent to those produced by strong winds (30 to 40 psf. uniform pressure) and they were still able to withstand eccentric end thrusts over the values commonly found for load-supporting walls in low-rise buildings.

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