AN ANALYSIS OF MATERIALS OF DIFFERENTIAL TYPE

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by

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SCOPE AND CONTENTS:

An investigation of general Materials of Differential Type [MDT], and Motions With Constant Stretch History [MCSH] is presented. Rivlin-Ericksen tensors $\bar{\bar{A}}_n$ are shown to result from a Taylor series expansion of the relative strain tensor $\bar{\bar{C}}_t(\tau)$. Internal constraint in MDT is discussed. General Solutions of Motions of Differential Type are worked out. Dynamically possible stresses are found for certain irrotational motions. Theorems regarding necessary and sufficient conditions for MCSH are proved. A class of MCSH is introduced, and an approximate MCSH is suggested. Necessary equations regarding gradients of a scalar-valued tensor function are derived.

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Bijoy K. Misra

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NOTATION

$\bar{\bar{A}} = \bar{A}_r\bar{B}_r$	is a second order tensor considered as tensor product of vector spaces $\{\bar{A}_r\}$ and $\{\bar{B}_r\}$
$\bar{\bar{A}}^T = \bar{B}_r\bar{A}_r$	Transpose of tensor $\bar{\bar{A}}$
$\bar{r}$	Position vector of particle R in current configuration $\bar{\chi}$ at time t
$\bar{R}$	Position vector of Particle R in reference configuration $\bar{\kappa}$ at time t_0
$\bar{\xi}$	Position vector of particle R in any configuration at time $\tau < t$
$d\bar{r}$	Material element in current configuration.
$d\bar{R}$	Material element in reference configuration.
x^k	Curvilinear coordinates in current configuration
x_k	Rectangular cartesian coordinates in current configuration
X^K	Curvilinear coordinates in reference configuration
X_K	Rectangular cartesian coordinates in reference configuration
ξ^k	Generalised curvilinear coordinates in any configuration at time $\tau < t$
$[\bar{\bar{A}}]$	Matrix of second order tensor $\bar{\bar{A}}$

$\|\bar{\bar{A}}\| = \|[\bar{\bar{A}}]\|$	Determinant of $[\bar{\bar{A}}]$
$\bar{\bar{I}}:\bar{\bar{A}} = \bar{\bar{A}}:\bar{\bar{I}}$	Trace of tensor $\bar{\bar{A}}$
$\bar{I} = \bar{\bar{I}}$	Identity tensor
$\overset{(n)}{\bar{\bar{A}}}$	n^{th} material time derivative of tensor $\bar{\bar{A}}$
$\overset{p}{\bar{A}}$	p^{th} spatial derivative of position vector $\bar{A}$ [which will be a tensor of order (p+1)]
$\bar{\bar{W}}$	is a skew tensor if $\bar{\bar{W}}^T = -\bar{\bar{W}}$
$\bar{\bar{F}} = \dfrac{\partial \bar{r}}{\partial \bar{R}}$	Deformation gradient tensor
$\bar{\bar{C}} = \bar{\bar{F}}\cdot\bar{\bar{F}}^T$	Strain tensor or Green deformation tensor
$\bar{g}_k$	Base vectors in current configuration
$\bar{G}_K$	Base vectors in reference configuration
$g_{ij} = \bar{g}_i \cdot \bar{g}_j$	Components of metric tensor in current configuration

$\bar{\bar{L}} = \bar{\bar{L}}_1(t) = \frac{d}{d\tau} \bar{\bar{F}}_t(\tau)\Big|_{\tau=t}$ - Spatial Velocity Gradient tensor

$\bar{\bar{Q}}$	Orthogonal tensor such that $\bar{\bar{Q}}\cdot\bar{\bar{Q}}^T = \bar{I}$
$\bar{\bar{\sigma}}_d$	Determinate stress
v	Volume in current configuration
V	Volume in reference configuration
$\rho_{\bar{\chi}}$	Density of fluid in current configuration $\bar{\chi}$

$m = \int_{\bar{\chi}(R)} \rho_{\bar{\chi}} dv$	a non-negative measure known as *mass* which is absolutely continuous in the body. [The integration is carried out over the configuration $\bar{\chi}(R)$]
$\binom{n}{r}$	Binomial Coefficient = $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
MDT	Materials of Differential Type
MCSH	Motions with Constant Stretch History

Repeated indices imply Einsteinian Summation convention unless the indices are underlined in which case summation is suspended.

In general, *tensor* refers to a second order three dimensional tensor.

INTRODUCTION

DEVELOPMENTS IN MATERIALS OF DIFFERENTIAL TYPE

Rivlin and Ericksen's Work

RIVLIN [21]* solved correctly a number of problems for large elastic deformations. ERICKSEN [4,5] set out to determine all deformations possible in every isotropic incompressible perfectly elastic body. His list was not exhaustive, and additions have been made by FOSDICK [7], and by SINGH and PIPKIN [25]. A final list of six families of motions [one of homogeneous deformations] is shown to be the solutions of the *simplest* type possible by MÜLLER [12]. These families of motions have been generalised for simple materials by CARROLL [1], and by FOSDICK [6].

The first extensive work in Materials of Differential Type where the constitutive relations took into account all order time rates of the deformation tensor was put forward by RIVLIN and ERICKSEN [22]. The Rivlin-Ericksen tensors $\bar{\bar{A}}_n$ (n = 1, 2....) were defined as time derivatives of the strain history tensor $\bar{\bar{C}}_t^t(s)$

$$\bar{\bar{A}}_n(t) = (-1)^n \frac{d^n}{ds^n} \bar{\bar{C}}_t^t(s)\big|_{s=0} \qquad (I.1)$$

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* Superscripts in brackets refer to references on page 129.

Representation Equations were proposed for symmetric isotropic tensors based on the algebraic principle of linear independence. In three dimensions, it was shown that stress was a function of the first two Rivlin-Ericksen tensors $\bar{\bar{A}}_1$ and $\bar{\bar{A}}_2$. An appropriate constitutive equation for stress was suggested using a linear combination of any six linearly independent symmetric functions of $\bar{\bar{A}}_1$ and $\bar{\bar{A}}_2$.

While the Material of Differential Type (the Rivlin-Ericksen Fluid) was more general in nature than the simple fluid, it was not the most general since it still could not account for gradual stress relaxation phenomenon. But within this definition, if a class of flows could be found wherein effects associated with *relaxation* did not occur, it would serve as a very general flow. Such a flow was defined by COLEMAN [3].

Coleman and Noll's Analysis

COLEMAN defined a *Substantially Stagnant Motion* as one where strain history $\bar{\bar{C}}_t^t(s)$ of a particle P was merely rotated along the path of P for all t, $-\infty < t < \infty$, and s, $0 < s < \infty$. Thus

$$\bar{\bar{C}}_t^t(s) = \bar{\bar{Q}}^T(t) \cdot \bar{\bar{C}}_0^0(s) \cdot \bar{\bar{Q}}(t) \qquad (I.2)$$

for all orthogonal $\bar{\bar{Q}}(t)$ such that

$$\bar{\bar{Q}}(0) = \bar{\bar{I}}.$$

According to COLEMAN, for such flows, "The memory of a simple fluid, no matter how elaborate it may be, is left very little to remember". All known exact solutions for motions of simple fluids were included in Substantially Stagnant Motions. Viscometric Flows formed a class of Substantially Stagnant Motions.

COLEMAN defined a *Steady Helical Flow** as one where the velocity field $\bar{v}(\bar{r})$ had the contravariant components

$$\{v^1(\bar{r}),\ v^2(\bar{r}),\ v^3(\bar{r})\} = \{0,\ w(x^1),\ u(x^1)\} \qquad (I.3)$$

This flow works out to be a general Viscometric Flow and includes Couette Flow, Poiseuille Flow and other shear flows.

The significance of Viscometric Flows is that many such motions could be *non-homogeneous but controllable* for simple fluids. A list of such flows is given by PIPKIN [19] who opines that his list is not exhaustive. In Viscometric Flows, the rate of deformation gradient tensor $\bar{\bar{L}}_1$ is such that

$$(\bar{\bar{L}}_1)^2 = \bar{\bar{0}}$$

due to which only three viscometric functions are sufficient to determine the general behaviour of the fluid [31]. Also, the

---

* In a steady flow, the local time rate of change of velocity is zero everywhere, i.e. $\frac{\partial \bar{v}}{\partial t} = \bar{0}$.

second rate of deformation gradient

$$\bar{\bar{L}}_2 = \bar{\bar{0}}$$

for all Viscometric Flows. It turns out that all Substantially Stagnant Motions other than Viscometric Flows are *homogeneous* [Section 9], and therefore, are controllable in every homogeneous incompressible material [34].

In the same paper, COLEMAN provides a representation for strain tensor $\bar{\bar{C}}_t^t(s)$ in terms of the Rivlin-Ericksen tensors $\bar{\bar{A}}_n$. This has been shown to be a natural consequence in the present work [Section 4]. COLEMAN has also considered the inclusion of thermodynamic variables in constitutive equations for incompressible simple fluids.

NOLL [16] has called Substantially Stagnant Motions *Motions with Constant Stretch History*, and provided a definition in terms of the deformation gradient, as

$$\bar{\bar{F}}_0(\tau) = e^{\tau\bar{\bar{M}}}\cdot\bar{\bar{Q}}(\tau), \quad \bar{\bar{Q}}(0) = \bar{\bar{I}} \tag{I.4}$$

where $\bar{\bar{M}}$ is a constant tensor and $\bar{\bar{Q}}(\tau)$ is an orthogonal tensor function.

Motions with Constant Stretch History [16] have been classified into three categories:

1. MCSH I for which $\bar{\bar{M}}^2 = \bar{\bar{0}}$
2. MCSH II for which $\bar{\bar{M}}^2 \neq \bar{\bar{0}}$ but $\bar{\bar{M}}^3 = \bar{\bar{0}}$
3. MCSH III for which $\bar{\bar{M}}^n \neq \bar{\bar{0}}$ for all n = 1,2,3....

All Viscometric Flows fall into MCSH I. The generalised Viscometric Flow has been named *Curvilineal Flow*. An example of MCSH II is a flow with a velocity field given by

$$\left.\begin{aligned} v_1 &= 0 \\ v_2 &= k_1 x_1 \\ v_3 &= k_2 x_1 + k_3 x_2 \end{aligned}\right\} \qquad \text{(I.5)}$$

where k_1, k_2 and k_3 are constants.

All steady extensions [NOLL, 15] belong to MCSH III. NOLL [15] has shown controllability of steady extension of a box, and of a circular cylinder.

WANG [27] has proved that for any MCSH, the first three Rivlin-Ericksen tensors - $\bar{\bar{A}}_1$, $\bar{\bar{A}}_2$ and $\bar{\bar{A}}_3$ determine the strain tensor $\bar{\bar{C}}_t^t(s)$ uniquely. Therefore, for any simple material in a MCSH, the stress tensor is a function of the first three Rivlin-Ericksen tensors alone.

According to WANG strain tensor $\bar{\bar{C}}_t(\tau)$ becomes

$$\bar{\bar{C}}_t(\tau) = e^{-s\bar{\bar{L}}_1(t)} \cdot e^{-s\bar{\bar{L}}_1^T(t)} \qquad \text{(I.6)}$$

where

$$\bar{\bar{L}}_1(t) = \bar{\bar{Q}}^T(t) \cdot \bar{\bar{M}}_t \cdot \bar{\bar{Q}}(t) .*$$

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* Refer to equation 3.4 for relation between $\bar{\bar{L}}(t)$ and $\bar{\bar{M}}(t)$.

Huilgol's Analysis

HUILGOL [8,9] has provided extensive insight into the study of MCSH. He has worked out generalised expressions for Viscometric Flows, and obtained MCSH II, and MCSH III by superpositions of MCSH I. Hence the names *Doubly Superposed Viscometric Flow* and *Triply Superposed Viscometric Flows* for MCSH II and MCSH III respectively.

It is to be noted that only homogeneous MCSH I may be superposed to obtain MCSH II and MCSH III. Hence, superposition of two homogeneous Viscometric Flows whose velocity fields are given by

$$\left.\begin{array}{lcl} v_1 = 0 & & v_1 = 0 \\ v_2 = 0 & \text{and} & v_2 = k_1 x_1 \\ v_3 = k_2 x_1 + k_3 x_2 & & v_3 = 0 \end{array}\right\} \qquad (I.7)$$

leads to a MCSH II given by (I.5).

It has already been stated that in general, Viscometric Flows could be non-homogeneous in which case no superposition would lead to MCSH II or MCSH III.

In private communication, HUILGOL has proposed a non-homogeneous velocity field which, according to him, is an example of a Doubly Superposed Viscometric Flow. The velocity field is

$$\left.\begin{array}{l} v^1 = 0 \\ v^2 = f(x^1) \\ v^3 = x^2 g(x^1) \end{array}\right\} \qquad (I.8)$$

for which, the rate of deformation tensor $\bar{\bar{L}}_1$ works out to be

$$[\bar{\bar{L}}_1] = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial f}{\partial x^1}\sqrt{\frac{g_{22}}{g_{11}}} & 0 & 0 \\ x^2\frac{\partial g}{\partial x^1}\sqrt{\frac{g_{33}}{g_{11}}} & g(x^1)\sqrt{\frac{g_{33}}{g_{22}}} & 0 \end{bmatrix} \qquad (I.9)$$

Since $\bar{\bar{L}}_1$ is not a constant tensor, (I.8) does not represent a MCSH according to HUILGOL'S previous definition [8, (2.10)].

NOLL'S equation (2.4)[16] for a MCSH gives

$$\bar{\bar{F}}_t(\tau) = \bar{\bar{Q}}^T(t)\cdot e^{(\tau-t)\bar{\bar{M}}}\cdot\bar{\bar{Q}}(\tau)$$

$$\therefore \frac{\partial}{\partial\tau}\bar{\bar{F}}_t(\tau) = \bar{\bar{Q}}^T(t)\cdot\bar{\bar{M}}\cdot e^{(\tau-t)\bar{\bar{M}}}\cdot\bar{\bar{Q}}(\tau) + \bar{\bar{Q}}^T(t)\cdot e^{(\tau-t)\bar{\bar{M}}}\cdot\dot{\bar{\bar{Q}}}(\tau)$$

At $\tau = t$, we get

$$\bar{\bar{L}}_1(t) = \bar{\bar{Q}}^T(t)\cdot\bar{\bar{M}}\cdot\bar{\bar{Q}}(t) + \bar{\bar{Q}}^T(t)\cdot\dot{\bar{\bar{Q}}}(t)$$

$$= \bar{\bar{M}}_t + \bar{\bar{Z}}(t) \qquad (I.10)$$

where

$$\bar{\bar{M}}_t = \bar{\bar{Q}}^T(t)\cdot\bar{\bar{M}}\cdot\bar{\bar{Q}}(t),$$

and

$$\bar{\bar{Z}}(t) = \bar{\bar{Q}}(t)\cdot\dot{\bar{\bar{Q}}}(t)$$

by definition. Again, for MCSH,

$$\bar{\bar{C}}_t^t(s) = (\bar{\bar{Q}}^T(t)\cdot e^{-s\bar{\bar{M}}}\cdot\bar{\bar{Q}}(t))\cdot(\bar{\bar{Q}}^T(t)\cdot e^{-s\bar{\bar{M}}^T}\cdot\bar{\bar{Q}}(t))$$

$$= e^{-s\bar{\bar{Q}}^T(t)\cdot\bar{\bar{M}}\cdot\bar{\bar{Q}}(t)}\cdot e^{-s\bar{\bar{Q}}^T(t)\cdot\bar{\bar{M}}^T\cdot\bar{\bar{Q}}(t)}$$

$$= e^{-s\bar{\bar{M}}_t}\cdot e^{-s\bar{\bar{M}}_t^T} \qquad \text{(I.11)}$$

In the works of both WANG [27] and HUILGOL [8], MCSH has been defined as a motion for which

$$\bar{\bar{C}}_t^t(s) = e^{-s\bar{\bar{L}}_1(t)}\cdot e^{-s\bar{\bar{L}}_1^T(t)} \qquad \text{(I.12)}$$

From (I.10), (I.11) and (I.12), it is seen that WANG and HUILGOL have dealt with MCSH where $\bar{\bar{Z}}(t) = \bar{\bar{0}}$. The conditions under which this is true is discussed in section 8 of the present work.

ZAHORSKI [30] has shown that two Simple Shearing Steady Viscometric Flows can be superposed to provide a pure shearing flow belonging to MCSH III.

The two Simple Shearing Flows are given by

$$v_1^{(1)} = k_1x_2 \qquad\qquad v_1^{(2)} = 0$$

$$v_2^{(1)} = 0 \qquad \text{and} \qquad v_2^{(2)} = k_1x_1$$

$$v_3^{(1)} = 0 \qquad\qquad v_3^{(2)} = 0$$

where

k_1 = constant.

Superposition of the two flows provides a pure shearing flow given by

$$v_1 = k_2 x_2$$

$$v_2 = k_2 x_1$$

$$v_3 = 0$$

where

k_2 = constant.

In case of an incompressible Reiner-Rivlin fluid, the determinate stress tensor for steady pure shearing flow is the arithmetic mean of determinate stress tensors in the two Simple Shearing flows if $k_1 = 2k_2$. Additional conditions have been derived to generalise the flow to an incompressible simple fluid.

The extensional flow considered by ZAHORSKI is the same as that due to NOLL [15].

Scope and Content of Present Work

The present investigation is intended to be expository in nature, and will serve much purpose if it helps in understanding Motions of Differential Type and those with Constant Stretch History.

The motivation to the investigation was due to the following realisations:

1. For Materials of Differential Type, there are many internal constraint functions each of which lead to a special form for *determinate stress* $\bar{\bar{\sigma}}_d$ (only the condition of incompressibility leads to $\bar{\bar{\sigma}}_d = \bar{\bar{\sigma}} + p\bar{I}$).

2. Many steady Motions of Differential Type can be derived from a velocity field described as

$$v_i = a_{ij}e^{x_j} \qquad \text{(I.13)}$$

where a_{ij} are constants. (I.13) includes the known controllable motions as special cases. Investigation of flows is done in Sections 7, 10, and 11.

3. Necessary and sufficient conditions are required to determine MCSH. Two fundamental theorems have been proved in section 9 for this purpose.

4. The class of Viscometric Flows is a first order MCSH. It includes non-homogeneous motions some of which are controllable. The rest of steady MCSH are homogeneous motions.

In the course of this investigation, certain interesting velocity fields have resulted. The motions resulting from such velocity fields are yet to be classified.

The necessary tensor relations regarding convergence and gradients of tensor functions have been developed in Appendix A.

CHAPTER I

MATERIALS AND MOTIONS

1. Definition of Continuous Media

By a continuous medium is meant a set of points called material points denoted by R, with a measure called mass m. At each time t, these points have positions

$$\bar{r} = \bar{\chi}(R,t) \qquad (1.1)$$

in space, and occupy a certain region. The mapping $\bar{\chi}$ is assumed to be one to one. The mass m induces at each time t a measure which is assumed to be absolutely continuous in space and time. Therefore, it has a density

$$\rho_{\bar{\chi}} = \rho(\bar{r},t) \; . \qquad (1.2)$$

While $\bar{r}$ defines the current configuration of the body, the reference configuration is defined by

$$\bar{R} = \bar{\kappa}(R,t_0) \qquad (1.3)$$

where t_0 is the time at which R occupies the reference configuration so that

$$\bar{r} = \bar{\chi}(\bar{\kappa}^{-1}(\bar{R}),t) = \bar{\chi}_{\bar{\kappa}}(\bar{R},t) \tag{1.4}$$

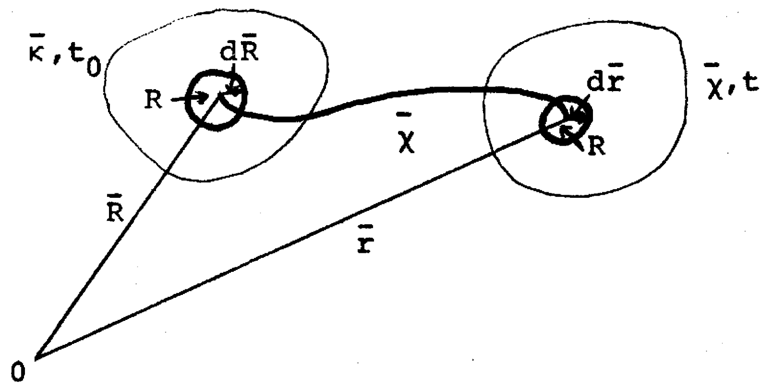

FIGURE 1

A body $\mathcal{B}$ is a smooth manifold of elements P,X,Z.... called particles. The configurations of $\mathcal{B}$ are the elements of a set of one-to-one mappings of $\mathcal{B}$ into a three dimensional Euclidean point space $\mathcal{E}$.

The mappings of body configurations from the reference configuration for a fixed time t are called deformations.

For each real time t, we may define

$$\bar{\chi}^{t}_{\bar{\kappa}}(s) = \bar{\chi}_{\bar{\kappa}}(t-s) \qquad \text{when } s \geqslant 0 \tag{1.5}$$

and call it the *history of deformation* up to time $t \geqslant 0$.

We have

$$\bar{\chi}^{t}_{\bar{\kappa}}(0) = \bar{\chi}_{\bar{\kappa}}(t) \tag{1.6}$$

$\chi^{-t}_{\kappa}(s)$ is undefined for $s < 0$.

A basic motivation to studies in Continuum Mechanics is provided by the following two questions:

1. When a continuous medium is subjected to a specified system of forces, what would the resulting deformed configuration be like?

2. What forces would be necessary to create a specified deformation configuration in a continuous medium?

Before any attempt is made to answer the above questions, in every continuous medium a fundamental assumption is necessary in the form of a definite relationship between the stress components at a point of the body and some known function of the deformation components at that point of the body. Such a relationship is provided by a *response functional* or *constitutive functional*. Such constitutive relations can be defined only for an ideal medium. Within specified ranges, it is possible to predict the behaviour of a natural medium.

In the present work, whereever necessary, appropriate constitutive relations are assumed since we are interested only in motions, and whenever possible, the stresses required to cause them.

2. Material

Definition of *Body* will apply to *Material*. Also, while referring to *Material*, statements regarding its constitutive property and the range of its applicability will be understood. At present, only mechanical properties are considered in the constitutive relation.

<u>a. Homogeneity and Isotropy:</u>

Two particles are *materially isomorphic* if and only if their response to deformation histories, described with respect to suitable reference configurations, is identical. A body $\mathcal{B}$ is called *materially uniform* if all of its particles are materially isomorphic to each other. If it is possible to choose a single reference configuration $\bar{R}(\cdot) = \bar{\kappa}$ for the whole body $\mathcal{B}$ so that the response functional is the same for all particles, the body is said to be *homogeneous*.

Therefore, a body is called *homogeneous* with respect to a reference state $\bar{R} = \bar{\kappa}(R,t_0)$ if and only if its constitutive equation is invariant under transformations of the form

$$\bar{R} \to \bar{R}^* = \bar{R} + \bar{Y} \qquad (2.1)$$

where $\bar{Y}$ is an arbitary vector [13].

A homogeneous material is called *isotropic* with respect to its reference position $\bar{R} = \bar{\kappa}(R,t_0)$ if and only if for any point Y and any rotation $\bar{\bar{R}}$, the transformation $\Psi(\bar{Y},\bar{\bar{R}})$ defined by

$$\Psi:\bar{R} \to \bar{R}^* = \bar{Y} + (\bar{R}-\bar{Y})\cdot\bar{\bar{R}} \qquad (2.2)$$

leaves the constitutive equation invariant.

A material which is not necessarily homogeneous is called isotropic if the invariance under $\Psi(\bar{Y},\bar{\bar{R}})$ holds for each point Y in the limit $\bar{R} \to \bar{Y}$.

Unless otherwise stated, we shall assume isotropy and homogeneity in all materials considered.

b. Simple Material

A simple material is one in which the stress $\bar{\bar{\sigma}}(t)$ at any time t is completely determined by the history of its deformation gradient $\bar{\bar{F}}(t)$ [34].

$$\bar{\bar{\sigma}}(t) = \overset{\infty}{\underset{s=0}{\bar{\bar{\mathcal{C}}}}} (\bar{\bar{F}}(t)) \tag{2.3}$$

where $\bar{\bar{\mathcal{C}}}$ is the constitutive functional obeying the Principle of Material Frame Indifference [13]

$$\bar{\bar{Q}}_0^T \cdot \overset{\infty}{\underset{s=0}{\bar{\bar{\mathcal{C}}}}} (\bar{\bar{F}}^t(s)) \cdot \bar{\bar{Q}}_0 = \overset{\infty}{\underset{s=0}{\bar{\bar{\mathcal{C}}}}} (\bar{\bar{F}}^t(s) \cdot \bar{\bar{Q}}(s)) \tag{2.4}$$

where $\bar{\bar{Q}}_0 = \bar{\bar{Q}}(0)$ = orthogonal tensor function.

A material is called *Elastic* if it is simple, and if the stress at time t depends only on the local configuration at time t, and not on the entire past history of motion. Thus, elastic materials are simple materials with a perfect memory of a very special and limited kind.

For an elastic material, the constitutive relation is of the form

$$\bar{\bar{\sigma}} = \bar{\bar{g}}(\bar{\bar{F}}) \tag{2.5}$$

where $\bar{\bar{g}}$ is the *response function.*

c. Simple Fluid:

A simple fluid is one in which the response functional $\bar{\bar{C}}$ remains invariant under all changes of reference configuration [14].

The constitutive equation is given as

$$\bar{\bar{\sigma}} = -p(\rho)\bar{\bar{I}} + \mathop{\bar{\bar{S}}}_{s=0}^{\infty} (\bar{\bar{G}}(s), \rho) \tag{2.6}$$

where $\rho_{\underline{X}} = \rho$ is the density of the fluid; and $\bar{\bar{G}}(s) = \bar{\bar{C}}(s) - \bar{\bar{I}}$, $\bar{\bar{C}}$ being the strain tensor.

Response functional $\bar{\bar{S}}$ satisfies the isotropy relation

$$\bar{\bar{Q}}_0^T \cdot \mathop{\bar{\bar{S}}}_{s=0}^{\infty} (\bar{\bar{G}}(s), \rho) \cdot \bar{\bar{Q}}_0 = \mathop{\bar{\bar{S}}}_{s=0}^{\infty} (\bar{\bar{Q}}_0^T \cdot \bar{\bar{G}}(s) \cdot \bar{\bar{Q}}_0, \rho) \tag{2.7}$$

For a perfect fluid, the determinate stress $\bar{\bar{\sigma}}_d$ is a zero tensor.

$$\bar{\bar{\sigma}}_d = \bar{\bar{\sigma}} + p\bar{\bar{I}} = \bar{\bar{0}} \tag{2.8}$$

All fluids where stress $\bar{\bar{\sigma}}$ is a function of the stretching tensor $\bar{\bar{D}}$ are called *Stokesian Fluids*. A *Newtonian Fluid* corresponds to

$$\bar{\bar{\sigma}}_d = \mu\bar{\bar{A}}_1 \tag{2.9}$$

where

$$\mu = \text{constant}$$

and

$$\bar{\bar{A}}_1 = \text{First Rivlin-Ericksen tensor.}$$

For a *Reiner-Rivlin Fluid*, $\bar{\bar{\sigma}}_d$ depends non-linearly on $\bar{\bar{A}}_1$.

$$\bar{\bar{\sigma}}_d = \gamma\bar{\bar{I}} + \gamma_1\bar{\bar{A}}_1 + \gamma_2(\bar{\bar{A}}_1)^2 \tag{2.10}$$

where, for the incompressible case, γ_1 and γ_2 are scalar functions of $\bar{\bar{I}}:\bar{\bar{A}}_1{}^2$ and $\bar{\bar{I}}:\bar{\bar{A}}_2{}^3$, and γ is determined by some normalisation convention such as $\bar{\bar{I}}:\bar{\bar{\sigma}}_d = \bar{\bar{0}}$.

d. Materials of Grade n:

A Material of Grade n is one for which the stress is a function of n gradients of deformation, so that the constitutive equation may be written in the form

$$\bar{\bar{\sigma}}(t) = \overset{\infty}{\underset{s=0}{\bar{\bar{\mathcal{D}}}}}(\bar{\bar{F}}(t-s),\ \bar{\bar{\bar{F}}}(t-s),\ldots\overset{n}{\bar{\bar{F}}}(t-s) \tag{2.11}$$

where $\overset{n}{\bar{\bar{F}}}(t-s) = \dfrac{\partial^n \bar{\xi}}{\partial \bar{R}^n}$,

subject to the isotropy condition

$$\bar{\bar{Q}}_0^T \cdot \bar{\bar{\sigma}}(t) \cdot \bar{\bar{Q}}_0 = \overset{\infty}{\underset{s=0}{\bar{\bar{\mathcal{D}}}}} (\bar{\bar{F}}*(t-s), \ \dot{\bar{\bar{F}}}*(t-s), \ \ldots, \ \overset{n}{\bar{\bar{F}}}*(t-s))$$

where

$$\bar{\bar{F}}*(t-s) = \bar{\bar{F}}(t-s) \cdot \bar{\bar{Q}}_s$$

$$\dot{\bar{\bar{F}}}*(t-s) = \dot{\bar{\bar{F}}}(t-s) \cdot \bar{\bar{Q}}_s$$

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$$\overset{n}{\bar{\bar{F}}}*(t-s) = \overset{n}{\bar{\bar{F}}}(t-s) \cdot \bar{\bar{Q}}_s$$

e. Materials of Differential Type:

Materials of Differential Type are those for which stress is a function of the first spatial gradients of velocity, acceleration, ..., $(n-1)^{th}$ acceleration (when the n^{th} acceleration is zero), so that the constitutive equation may be written in the form

$$\bar{\bar{\sigma}}_d = \bar{\bar{G}}(\bar{\bar{A}}_1, \bar{\bar{A}}_2, \ldots, \bar{\bar{A}}_n) \qquad (2.12)$$

$\bar{\bar{A}}_1, \bar{\bar{A}}_2, \ldots, \bar{\bar{A}}_n$ being the Rivlin-Ericksen tensors. The isotropy condition is given by

$$\bar{\bar{Q}}^T \cdot \bar{\bar{G}}(\bar{\bar{A}}_1, \bar{\bar{A}}_2, \ldots, \bar{\bar{A}}_n) \cdot \bar{\bar{Q}} = \bar{\bar{G}}(\bar{\bar{Q}}^T \cdot \bar{\bar{A}}_1 \cdot \bar{\bar{Q}}, \bar{\bar{Q}}^T \cdot \bar{\bar{A}}_2 \cdot \bar{\bar{Q}}, \ldots, \bar{\bar{Q}}^T \cdot \bar{\bar{A}}_n \cdot \bar{\bar{Q}})$$

RIVLIN and ERICKSEN [22] provide representation for $\bar{\bar{G}}$ as a function of $\bar{\bar{A}}_1$ and $\bar{\bar{A}}_2$ alone as

$$\begin{aligned}\bar{\bar{\sigma}}_d = \gamma\bar{\bar{I}} &+ \alpha_1\bar{\bar{A}}_1 + \alpha_2\bar{\bar{A}}_2 + \alpha_3(\bar{\bar{A}}_1)^2 + \alpha_4(\bar{\bar{A}}_2)^2 + \alpha_5(\bar{\bar{A}}_1 \cdot \bar{\bar{A}}_2 + \bar{\bar{A}}_2 \cdot \bar{\bar{A}}_1) \\ &+ \alpha_6((\bar{\bar{A}}_1)^2 \cdot \bar{\bar{A}}_2 + \bar{\bar{A}}_2 \cdot (\bar{\bar{A}}_1)^2) + \alpha_7(\bar{\bar{A}}_1 \cdot (\bar{\bar{A}}_2)^2 + (\bar{\bar{A}}_2)^2 \cdot \bar{\bar{A}}_1) \\ &+ \alpha_8((\bar{\bar{A}}_1)^2 \cdot (\bar{\bar{A}}_2)^2 + (\bar{\bar{A}}_2)^2 \cdot (\bar{\bar{A}}_1)^2) \qquad (2.13)\end{aligned}$$

In the incompressible case, $\alpha_1, \alpha_2, \ldots, \alpha_8$ are scalar functions of the eight mixed invariants $\bar{\bar{I}}:(\bar{\bar{A}}_1)^2$, $\bar{\bar{I}}:(\bar{\bar{A}}_1)^3$, $\bar{\bar{I}}:(\bar{\bar{A}}_2)^3$, $\bar{\bar{I}}:(\bar{\bar{A}}_1 \cdot \bar{\bar{A}}_2)$, $\bar{\bar{I}}:((\bar{\bar{A}}_1)^2 \cdot \bar{\bar{A}}_2)$, $\bar{\bar{I}}:(\bar{\bar{A}}_1 \cdot (\bar{\bar{A}}_2)^2)$ and $\bar{\bar{I}}:((\bar{\bar{A}}_1)^2 \cdot (\bar{\bar{A}}_2)^2)$ which form the *function basis* [26].

f. Materials of Rate Type:

Constitutive equation of a Material of Rate Type [34] is given as

$$\overset{(p)}{\bar{\bar{\sigma}}} = \bar{\bar{H}}(\bar{\bar{\sigma}}, \dot{\bar{\bar{\sigma}}}, \ldots, \overset{(p-1)}{\bar{\bar{\sigma}}}; \bar{\bar{F}}, \dot{\bar{\bar{F}}}, \ldots, \overset{(n)}{\bar{\bar{F}}}) \qquad (2.14)$$

on the assumption that for each prescribed sufficiently smooth function $\bar{\bar{F}}(t)$ and prescribed initial data

$$\bar{\bar{\sigma}}(t_0), \dot{\bar{\bar{\sigma}}}(t_0), \ldots, \overset{(p-1)}{\bar{\bar{\sigma}}}(t_0), \qquad (2.15)$$

the differential equation (2.14) has a unique solution $\bar{\bar{\sigma}}(t)$.

NOLL'S *Hygrosteric Material* [13] is an example of Material of Rate Type of the first order.

g. Differential Materials of Grade n :

A Differential Material of Grade n may be defined by combining (2.11) and (2.12). The constitutive equation may be written as

$$\bar{\bar{\sigma}}(t) = \overset{\infty}{\underset{s=0}{\bar{\bar{F}}}} (\bar{\bar{F}}(t-s), \overset{\equiv}{F}(t-s), \ldots, \overset{n}{\bar{\bar{F}}}(t-s); \bar{\bar{A}}_1, \bar{\bar{A}}_2, \ldots, \bar{\bar{A}}_n) \qquad (2.16)$$

subject to the isotropy condition

$$\bar{\bar{Q}}_0^T \cdot \bar{\bar{\sigma}}(t) \cdot \bar{\bar{Q}}_0 = \overset{\infty}{\underset{s=0}{\bar{\bar{F}}}} (\bar{\bar{F}}(t-s) \cdot \bar{\bar{Q}}_0, \ldots, \overset{n}{\bar{\bar{F}}}(t-s) \cdot \bar{\bar{Q}}_0; \bar{\bar{Q}}_0^T \cdot \bar{\bar{A}}_1 \cdot \bar{\bar{Q}}_0, \ldots, \bar{\bar{Q}}_0^T \cdot \bar{\bar{A}}_n \cdot \bar{\bar{Q}}_0)$$

Materials of Rate Type and Differential Materials of Grade n are two broad categories of materials within the definitions of which all solids and fluids can be defined.

h. Incompressibility:

A material is incompressible if it is susceptible only to motions in which the density at each material point is constant in time. Incompressibility is a restrictive condition characterizing a certain material property. Therefore it is a constitutive assumption. Forces are required to maintain such constraints within a material. It is a basic assumption of Mechanics that the forces maintaining a constraint do no work when the material is deformed by external forces.

The Principle of Determinism [App. B1] for incompressible simple materials yields that the stress is determined by the deformation history only to within a hydrostatic pressure p; hence

$$\bar{\bar{\sigma}}_d = \bar{\bar{\sigma}} + p\bar{\bar{I}} \tag{2.17}$$

where $\bar{\bar{\sigma}}_d$ is known as the *determinate stress*.

3. Motion

The time-sequence of mappings of reference configuration $\bar{\chi}_{\underline{\kappa}}(\bar{R},t)$ is called the *motion* of the body.

a. Rigid Body Motion:

If, for all times t, the position vectors $\bar{r}(R)$ of all points in a material body remain fixed with respect to a frame of reference attached to the material, the motion is a *rigid body motion*.

With respect to a fixed reference configuration all material points undergo the same rotation and linear displacement.

Euler's Laws of Motion are sufficient to determine the forces necessary to cause such motions.

b. Homogeneous Motion:

Consider a point R in a deforming material, whose position vector at present time t is $\bar{r}$ and at any time $\tau < t$, it is $\bar{\xi}$.

By the *Postulate of Affine Connection* [33],

$$d\bar{\xi} = d\bar{r} \cdot \frac{\partial \bar{\xi}}{\partial \bar{r}} \tag{3.1}$$

If the elements of the tensor $\frac{\partial \bar{\xi}}{\partial \bar{r}} = \bar{\bar{F}}_t(\tau)$ are independent of position $\bar{\xi}$ (they could be functions of time), then

$$\bar{\xi} = \bar{r} \cdot \bar{\bar{F}}_t(\tau) + \bar{C}(t) \tag{3.2}$$

where $\bar{C}(t)$ is a function of t only.

All motions represented by (3.2) are known as *homogeneous motions*. Such motions are important as they can always be caused in simple materials by application of suitable surface tractions alone.

Defining the spatial velocity gradient $\bar{\bar{L}}_1(t)$ as

$$\bar{\bar{L}}_1(t) = \frac{\partial \bar{v}}{\partial \bar{r}} = \frac{d}{d\tau} \bar{\bar{F}}_t(\tau)\Big|_{\tau=t} \tag{3.3}$$

where $\bar{v}(R,t)$ is the velocity at R at time t, it follows from

(3.2) that for all homogeneous motions,

$$\left.\frac{d\bar{\xi}}{d\tau}\right|_{\tau=t} = \bar{r}\cdot\frac{d}{d\tau}\bar{\bar{F}}_t(\tau)\Big|_{\tau=t} + \left.\frac{d\bar{c}}{d\tau}\right|_{\tau=t}$$

or,

$$\left.\begin{aligned} \bar{v} &= \bar{r}\cdot\bar{\bar{L}}_1(t) + \dot{\bar{c}} \\ &- - - - - - - - - \\ n^{th} \text{ acceleration} = \bar{a}_n &= \bar{r}\cdot\bar{\bar{L}}_{n+1}(t) + \overset{(n+1)}{\bar{c}} \end{aligned}\right\} \tag{3.4}$$

c. Motions of Differential Type:

All motions in which $\bar{\bar{F}}_t(\tau)$ is determined by $\bar{\bar{L}}_1(t)$, $\bar{\bar{L}}_2(t)$, ..., $\bar{\bar{L}}_n(t)$; s shall be referred to as Motions of Differential type

$$\bar{\bar{F}}_t(\tau) = \bar{\bar{f}}(\bar{\bar{L}}_1(t), \bar{\bar{L}}_2(t), \ldots, \bar{\bar{L}}_n(t); s) \tag{3.5}$$

Kinematic relations and examples of Motions of Differential Type have been worked out in sections 4, 5 and 7. Such motions may be homogeneous or non-homogeneous.

d. Motions with Constant Stretch History:

Following NOLL [16], Motions with Constant Stretch History are such that

$$\bar{\bar{F}}_t(t-s) = e^{-s\bar{\bar{M}}_t} \tag{3.6}$$

MCSH is a homogeneous Motion [Section 9] of Differential Type, and a necessary and sufficient condition for MCSH subject to (8.15) is given by

$$\bar{\bar{L}}_n(t) = (\bar{\bar{L}}_1(t))^n, \quad n = 1,2,.... \tag{3.7}$$

But when $\bar{\bar{L}}_n(t) = \bar{\bar{0}} = (\bar{\bar{L}}_1(t))^n$, the MCSH may not be homogeneous. Such an example is the first order MCSH, i.e., the Viscometric Flow where $\bar{\bar{L}}_2 = \bar{\bar{0}} = (\bar{\bar{L}}_1)^2$. A general curvilineal flow [13,14] is viscometric but not necessarily homogeneous.

Materials and Motions of Differential Type and MCSH will be fully developed in the course of this investigation. Other materials and motions, when mentioned, will follow the definitions provided in this chapter.

CHAPTER II

MATERIALS OF DIFFERENTIAL TYPE

4. Basic Kinematics

The position vector of a typical point of a continuum referred to a fixed system of rectangular cartesian axes, at time t, may be designated by

$$\bar{r} = \bar{\chi}_{\bar{\kappa}}(\bar{R}, t) \tag{4.1}$$

The deformation gradient relative to the reference configuration $\bar{\kappa}$ is given by

$$\bar{\bar{F}} = \frac{\partial \bar{r}}{\partial \bar{R}} \tag{4.2}$$

The function $\bar{\chi}$ which is a mapping function from the reference configuration $\bar{\kappa}$ to the current configuration $\bar{\chi}$ $(\bar{\chi}: \bar{R} \to \bar{r})$ is such that no volume element in the reference configuration becomes zero or negative in the present configuration.

$$\frac{\partial v}{\partial V} = |\bar{\bar{F}}| > 0 \tag{4.3}$$

The spatial velocity gradient = $\bar{\bar{L}}(t)$ is defined by

$$\bar{\bar{L}}(t) = \frac{\partial\bar{v}}{\partial\bar{r}} \tag{4.4}$$

The *Rate of Deformation* tensor is given by

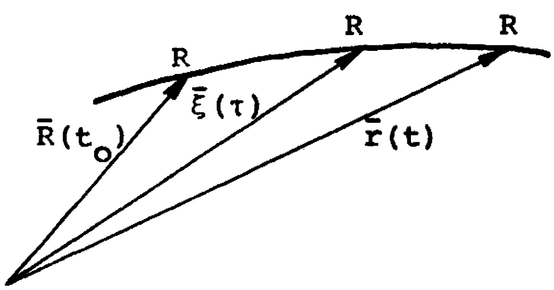
$$\left.\begin{aligned} \bar{\bar{D}}(t) &= \tfrac{1}{2}(\bar{\bar{L}}(t) + \bar{\bar{L}}^T(t)) = \tfrac{1}{2}\left(\frac{\partial\bar{v}}{\partial\bar{r}} + \frac{\bar{v}\partial}{\partial\bar{r}}\right) \\ &\text{\textit{Spin} or \textit{Vorticity} tensor is defined as} \\ \bar{\bar{W}}(t) &= \tfrac{1}{2}(\bar{\bar{L}}(t) - \bar{\bar{L}}^T(t)) = \tfrac{1}{2}\left(\frac{\partial\bar{v}}{\partial\bar{r}} - \frac{\bar{v}\partial}{\partial\bar{r}}\right) \end{aligned}\right\} \tag{4.5}$$

so that

$$\bar{\bar{L}}(t) = \bar{\bar{D}}(t) + \bar{\bar{W}}(t) \tag{4.6}$$

where $\bar{\bar{D}}(t)$ is a symmetric tensor and $\bar{\bar{W}}(t)$ is skew.

If the current configuration is taken to be the reference configuration, one may refer the position of a particle at s time-units ago to the present configuration.



$$\bar{R}(R,t_0) = \bar{\kappa}(R,t_0)$$

$$\bar{\xi}(R,\tau) = \bar{\chi}^{-1}_{\bar{\kappa}}(R,\tau)$$

$$\bar{r}(R,t) = \bar{\chi}_{\bar{\kappa}}(R,t)$$

FIGURE 2

Let the position vector of particle P at time $\tau = t-s$ be $\bar{\xi}$.

The relative deformation gradient $\bar{\bar{F}}_t(\tau)$ is defined as

$$\bar{\bar{F}}_t(\tau) = \bar{\bar{F}}_t(t-s) = \bar{\bar{F}}_t^t(s) = \frac{\partial \bar{\xi}}{\partial \bar{r}} \qquad (4.7)$$

Spatial velocity gradient $\bar{\bar{L}}(t)$ is defined as the value of the time derivative of relative deformation gradient at $\tau = t$:

$$\bar{\bar{L}}(t) = \dot{\bar{\bar{F}}}_t(t) = \frac{d}{d\tau} \bar{\bar{F}}_t(\tau)\Big|_{\tau=t} = -\frac{d}{ds} \bar{\bar{F}}_t^t(s)\Big|_{s=0} = -\dot{\bar{\bar{F}}}_t^t(0) \qquad (4.8)$$

since

$$\frac{d}{d\tau}() = -\frac{d}{ds}()$$

Denoting $\bar{\bar{L}}(t)$ as $\bar{\bar{L}}_1(t)$, the n^{th} time derivative of $\bar{\bar{F}}_t(\tau)$ at $\tau = t$ can be calculated as

$$\bar{\bar{L}}_n(t) = (-1)^n \frac{d^n}{ds^n} \bar{\bar{F}}_t^t(s)\Big|_{s=0} = \overset{(n)}{\bar{\bar{F}}}_t(t) = (-1)^n \overset{(n)}{\bar{\bar{F}}}{}_t^t(0) \qquad (4.9)$$

From (4.2),

$$\frac{d}{dt} \bar{\bar{F}}(t) = \frac{d}{dt} \frac{\partial \bar{r}}{\partial \bar{R}} = \frac{\partial \bar{v}}{\partial \bar{R}} = \frac{\partial \bar{r}}{\partial \bar{R}} \cdot \frac{\partial \bar{v}}{\partial \bar{r}} = \bar{\bar{F}}(t) \cdot \bar{\bar{L}}_1(t) \qquad (4.10)$$

Differentiating $\bar{\bar{F}}(t)$ n times with respect to t,

$$\frac{d^n\bar{\bar{F}}(t)}{dt^n} = \overset{(n)}{\bar{\bar{F}}}{}_0^{\,t}(0) = \bar{\bar{F}}(t)\cdot\bar{\bar{L}}_n(t) \tag{4.11}$$

(4.11) results by differentiating $\frac{d^{n-1}\bar{\bar{F}}(t)}{dt^{n-1}} = \bar{\bar{F}}(t)\cdot\bar{\bar{L}}_{n-1}(t)$ with respect to t. Differentiating $\bar{\bar{F}}(t)\cdot\bar{\bar{L}}_{n-1}(t)$ with respect to t and comparing with (4.11) leads to

$$\bar{\bar{L}}_n(t) = \bar{\bar{L}}_1(t)\cdot\bar{\bar{L}}_{n-1}(t) + \dot{\bar{\bar{L}}}_{n-1}(t) \tag{4.12}$$

Also, differentiating right hand and left hand sides of (4.10) separately with respect to t and equating the results,

$$\bar{\bar{L}}_2(t) = (\bar{\bar{L}}_1(t))^2 + \dot{\bar{\bar{L}}}_1(t)$$

$$\bar{\bar{L}}_3(t) = (\bar{\bar{L}}_1(t))^3 + \dot{\bar{\bar{L}}}_1(t)\cdot\bar{\bar{L}}_1(t) + 2\bar{\bar{L}}_1(t)\cdot\dot{\bar{\bar{L}}}_1(t) + \ddot{\bar{\bar{L}}}_1(t)$$

$$\bar{\bar{L}}_4(t) = (\bar{\bar{L}}_1(t))^4 + \dot{\bar{\bar{L}}}_1(t)\cdot(\bar{\bar{L}}_1(t))^2 + \bar{\bar{L}}_1(t)\cdot\dot{\bar{\bar{L}}}_1(t)\cdot\bar{\bar{L}}_1(t)$$

$$+ (\bar{\bar{L}}_1(t))^2\cdot\dot{\bar{\bar{L}}}_1(t) + \ddot{\bar{\bar{L}}}_1(t)\cdot\bar{\bar{L}}_1(t) + 2\bar{\bar{L}}_1(t)\cdot\ddot{\bar{\bar{L}}}_1(t)$$

$$+ 3(\bar{\bar{L}}_1(t))^2 + \dddot{\bar{\bar{L}}}_1(t)$$

- -

$$\bar{\bar{L}}_n(t) = \bar{\bar{\mathfrak{f}}}(\bar{\bar{L}}_1(t), \dot{\bar{\bar{L}}}_1(t), \ddot{\bar{\bar{L}}}_1(t), \ldots, \overset{(n-1)}{\bar{\bar{L}}_1}(t)) \tag{4.13}$$

It is to be noted in (4.13) that $\bar{\bar{L}}_1(t) = \bar{\bar{0}}$ leads to $\bar{\bar{L}}_n(t) = \bar{\bar{0}}$. But if $\dot{\bar{\bar{L}}}_1(t) = \bar{\bar{0}}$, all higher order material time derivatives of $\bar{\bar{L}}_1(t)$ would also be $\bar{\bar{0}}$. So that

$$\bar{\bar{L}}_n(t) = (\bar{\bar{L}}_1(t))^n \tag{4.14}$$

The invertible linear transformation $\bar{\bar{F}}_t(\tau)$ has two unique product decompositions called *polar decompositions*

$$\left.\begin{aligned} \bar{\bar{F}}_t(\tau) &= \bar{\bar{U}}_t(\tau)\cdot\bar{\bar{R}}_t(\tau) \\ &= \bar{\bar{R}}_t(\tau)\cdot\bar{\bar{V}}_t(\tau) \end{aligned}\right\} \tag{4.15}$$

in which $\bar{\bar{R}}_t(\tau)$ is orthogonal, and $\bar{\bar{U}}_t(\tau)$ and $\bar{\bar{V}}_t(\tau)$ are symmetric and positive definite.

Differentiating (4.15-1) with respect to τ,

$$\dot{\bar{\bar{F}}}_t(\tau) = \dot{\bar{\bar{U}}}_t(\tau)\cdot\bar{\bar{R}}_t(\tau) + \bar{\bar{U}}_t(\tau)\cdot\dot{\bar{\bar{R}}}_t(\tau) \tag{4.16}$$

At $s = 0$, i.e. $\tau = t$

$$\bar{\bar{U}}_t(t) = \bar{\bar{R}}_t(t) = \bar{\bar{I}}$$

From (4.16),

$$\bar{\bar{L}}_1(t) = \dot{\bar{\bar{U}}}_t(t) + \dot{\bar{\bar{R}}}_t(t) \tag{4.17}$$

N.B.! In [11], HUILGOL'S condition (4.2) is equivalent to the present equation (4.14).

Also, since $\bar{\bar{R}}_t(\tau)$ is orthogonal,

$$\bar{\bar{R}}_t(\tau)\cdot\bar{\bar{R}}_t^T(\tau) = \bar{\bar{I}}$$

so that

$$\dot{\bar{\bar{R}}}_t(\tau)\cdot\bar{\bar{R}}_t^T(\tau) + \bar{\bar{R}}_t(\tau)\cdot\dot{\bar{\bar{R}}}_t^T(\tau) = \bar{\bar{0}}$$

$\therefore$ At $\tau = t$,

$$\dot{\bar{\bar{R}}}_t(t) + \dot{\bar{\bar{R}}}_t^T(t) = \bar{\bar{0}}$$

so that $\dot{\bar{\bar{R}}}_t(t)$ is a skew tensor.

Comparison of (4.6) and (4.17) leads to

$$\bar{\bar{D}}(t) = \dot{\bar{\bar{U}}}_t(t);\ \bar{\bar{W}}(t) = \dot{\bar{\bar{R}}}_t(t) \qquad (4.18)$$

Differentiating (4.16) with respect to τ and calculating its value at $\tau = t$;

$$\bar{\bar{L}}_2(t) = \ddot{\bar{\bar{U}}}_t(t) + 2\dot{\bar{\bar{U}}}_t(t)\cdot\dot{\bar{\bar{R}}}_t(t) + \ddot{\bar{\bar{R}}}_t(t) = \bar{\bar{D}}_2(t) + \bar{\bar{W}}_2(t)$$

— — — — — — — — — — — — — — — — — —

$$\bar{\bar{L}}_n(t) = \overset{(n)}{\bar{\bar{U}}}_t(t) + \overset{(n)}{\bar{\bar{R}}}_t(t) + \sum_{i=1}^{n-1}\binom{n}{i}\overset{(n-i)}{\bar{\bar{U}}}\cdot\overset{(i)}{\bar{\bar{W}}} = \bar{\bar{D}}_n(t) + \bar{\bar{W}}_n(t) \qquad (4.19)$$

Consider the effect of change of frame on deformation gradient $\bar{\bar{F}}$. The frame transformation is given as

$$\bar{r}^*(t^*) = \bar{C}(t) + \bar{r}\cdot\bar{\bar{Q}}(t)$$

where

$$t^* = (t-a)$$

and $\bar{\bar{Q}}(t)$ is an orthogonal tensor.

$$\bar{\bar{F}}^* = \frac{\partial \bar{r}^*}{\partial \bar{R}^*} = \frac{\partial \bar{r}}{\partial \bar{R}^*}\cdot\frac{\partial \bar{r}^*}{\partial \bar{r}} = \frac{\partial \bar{r}}{\partial \bar{R}^*}\cdot\bar{\bar{Q}}(t)$$

Keeping the reference configuration same for both frames, i.e., $\bar{R}^*(\cdot) = \bar{\bar{R}}(\cdot)$

$$\frac{\partial\,()}{\partial \bar{R}^*} = \frac{\partial\,()}{\partial \bar{R}}$$

$$\therefore \quad \bar{\bar{F}}^* = \frac{\partial \bar{r}}{\partial \bar{R}}\cdot\bar{\bar{Q}}(t) = \bar{\bar{F}}\cdot\bar{\bar{Q}}(t) \qquad (4.20)$$

Consider the transformation of relative deformation gradient $\bar{\bar{F}}_t(\tau)$:

$$\bar{\bar{F}}^*_{t*}(\tau^*) = \frac{\partial \bar{\xi}^*}{\partial \bar{r}^*} = \frac{\partial \bar{r}}{\partial \bar{r}^*}\cdot\frac{\partial \bar{\xi}}{\partial \bar{r}}\cdot\frac{\partial \bar{\xi}^*}{\partial \bar{\xi}} \qquad (4.21)$$

If

$$\bar{\xi}^* = \bar{C}(\tau) + \bar{\xi}\cdot\bar{\bar{Q}}(\tau)$$

$$\left.\begin{aligned} &\frac{\partial\bar{\xi}^*}{\partial\bar{\xi}} = \bar{\bar{Q}}(\tau) \\ &\text{also} \\ &\frac{\partial()}{\partial\bar{r}^*} = \frac{\partial\bar{r}}{\partial\bar{r}^*}\cdot\frac{\partial()}{\partial\bar{r}} = \bar{\bar{Q}}^T(t)\cdot\frac{\partial()}{\partial\bar{r}} \end{aligned}\right\} \qquad (4.22)$$

Combining (4.22) and (4.21),

$$\bar{\bar{F}}^*_{t^*}\ (\tau^*) = \bar{\bar{Q}}^T(t)\cdot\bar{\bar{F}}_t(\tau)\cdot\bar{\bar{Q}}(\tau)$$

Any second order tensor $\bar{\bar{T}}$ is defined to be frame-indifferent when

$$\bar{\bar{T}}^* = \bar{\bar{Q}}^T(t)\cdot\bar{\bar{T}}\cdot\bar{\bar{Q}}(t)$$

Evidently, $\bar{\bar{F}}$ is not frame indifferent; but $\bar{\bar{F}}_t(\tau)$ is frame-indifferent only if the orthagonal tensor $\bar{\bar{Q}}$ is not a function of time. Change of frame leads to

$$\bar{\bar{L}}^* = \bar{\bar{D}}^* + \bar{\bar{W}}^* = \bar{\bar{Q}}^T(t)\cdot\bar{\bar{L}}\cdot\bar{\bar{Q}}(t) + \bar{\bar{Q}}^T(t)\cdot\dot{\bar{\bar{Q}}}\ (t)$$

$$= \bar{\bar{Q}}^T(t)\cdot\bar{\bar{D}}\cdot\bar{\bar{Q}}(t) + (\bar{\bar{Q}}^T(t)\cdot\bar{\bar{W}}\cdot\bar{\bar{Q}}(t) + \bar{\bar{Q}}^T(t)\cdot\dot{\bar{\bar{Q}}}(t)) \qquad (4.23)$$

where $\bar{\bar{Q}}^T(t)\cdot\dot{\bar{\bar{Q}}}^T(t)$ is a skew tensor.

$$\left.\begin{aligned} \therefore \quad \bar{\bar{D}}^* &= \bar{\bar{Q}}^T(t)\cdot\bar{\bar{D}}\cdot\bar{\bar{Q}}(t) \\ \bar{\bar{W}}^* &= \bar{\bar{Q}}^T(t)\cdot\bar{\bar{W}}\cdot\bar{\bar{Q}}(t) + \bar{\bar{Q}}^T(t)\cdot\dot{\bar{\bar{Q}}}(t) \end{aligned}\right\} \qquad (4.24)$$

Velocity gradient $\bar{\bar{L}}_1$ is frame-indifferent only if $\bar{\bar{Q}}$ is a constant orthogonal tensor independent of time t, i.e., $\dot{\bar{\bar{Q}}} = \bar{\bar{0}}$

$\bar{\bar{U}}$ remains unchanged under a change of frame, whereas $\bar{\bar{R}}$ changes to

$$\bar{\bar{R}}^* = \bar{\bar{R}}\cdot\bar{\bar{Q}}(t)$$

and $\bar{\bar{V}}$ becomes

$$\bar{\bar{V}}^* = \bar{\bar{Q}}^T(t)\cdot\bar{\bar{V}}\cdot\bar{\bar{Q}}$$

Therefore, while stretching tensor $\bar{\bar{U}}$ and rotation tensor $\bar{\bar{R}}$ are not frame-indifferent, stretching tensor $\bar{\bar{V}}$ is frame-indifferent.

Strain Tensor and Rivlin-Ericksen Tensors

Let $(dS)^2$ and $(ds)^2$ be the squared elements of the arc lengths associated with particle P at times t_0 and t respectively.

$$(ds)^2 - (dS)^2 = d\bar{r}\cdot d\bar{r} - d\bar{R}\cdot d\bar{R} = d\bar{R}\cdot\frac{\partial\bar{r}}{\partial\bar{R}}\cdot d\bar{R}\cdot\frac{\partial\bar{r}}{\partial\bar{R}} - d\bar{R}\cdot d\bar{R}$$

$$= d\bar{R}\cdot\frac{\partial\bar{r}}{\partial\bar{R}}\cdot\frac{\bar{r}\partial}{\partial\bar{R}}\cdot d\bar{R} - d\bar{R}\cdot\bar{\bar{I}}\cdot d\bar{R}$$

$$= d\bar{R}\ d\bar{R}:(\bar{\bar{C}} - \bar{\bar{I}})$$

where $\bar{\bar{C}}$ is defined as the strain tensor given by

$$\bar{\bar{C}} = \frac{\partial\bar{r}}{\partial\bar{R}}.\frac{\bar{r}\partial}{\partial\bar{R}} = \bar{\bar{F}}\cdot\bar{\bar{F}}^T \qquad (4.25)$$

The above description also conforms to an element of arc $d\bar{R}$ at $\bar{R}$ deforming to an element of arc $d\bar{r}$ at $\bar{r}$.

Then, *stretch* $\lambda_{(\bar{N})}$ in the direction of $\bar{N}$ is defined by

$$\lambda_{(\bar{N})} = \frac{\text{final length of arc}}{\text{initial length of arc}} = \left(\frac{d\bar{r}\cdot d\bar{r}}{d\bar{R}\cdot d\bar{R}}\right)^{1/2} = \left(\frac{ds^2}{dS^2}\right)^{1/2} = \frac{ds}{dS}$$

Differentiating (4.25) with respect to t,

$$\frac{d\bar{\bar{C}}}{dt} = \frac{d}{dt}\left(\frac{\partial\bar{r}}{\partial\bar{R}}.\frac{\bar{r}\partial}{\partial\bar{R}}\right) = \frac{\partial\bar{r}}{\partial\bar{R}}.\frac{\partial\bar{v}}{\partial\bar{r}}.\frac{\bar{r}\partial}{\partial\bar{R}} + \frac{\partial\bar{r}}{\partial\bar{R}}.\frac{\bar{v}\partial}{\partial\bar{r}}.\frac{\bar{r}\partial}{\partial\bar{R}}$$

$$= 2\bar{\bar{F}}\cdot\bar{\bar{D}}\cdot\bar{\bar{F}}^T \qquad (4.26)$$

Defining *relative strain tensor* $\bar{\bar{C}}_t(\tau)$ as

$$\bar{\bar{C}}_t(\tau) = \frac{\partial\bar{\xi}}{\partial\bar{r}}.\frac{\bar{\xi}\partial}{\partial\bar{r}} = \bar{\bar{F}}_t(\tau)\cdot\bar{\bar{F}}_t^T(\tau), \qquad (4.27)$$

and differentiating it n times with respect to τ

$$\overset{(n)}{\bar{\bar{C}}}_t(\tau) = \sum_{i=0}^{n} \binom{n}{i} \overset{(n-i)}{\bar{\bar{F}}}_t(\tau) \cdot \overset{(i)T}{\bar{\bar{F}}}_t(\tau)$$

$$= \bar{\bar{F}}_t(\tau) \cdot (2\bar{\bar{D}}_n + \sum_{i=1}^{n-1} \binom{n}{i} \bar{\bar{D}}_{n-i} \cdot \bar{\bar{D}}_i) \cdot \bar{\bar{F}}_t^T(\tau) \tag{4.28}$$

The value of $\overset{(n)}{\bar{\bar{C}}}_t(\tau)$ at $\tau = t$ is defined as the n^{th} Rivlin-Ericksen tensor [22]

$$\left.\begin{aligned} \bar{\bar{A}}_n &= \bar{\bar{L}}_n + \bar{\bar{L}}_n^T + \sum_{i=1}^{n-1} \binom{n}{i} \bar{\bar{L}}_{n-i} \cdot \bar{\bar{L}}_i^T \\ &= 2\bar{\bar{D}}_n + \sum_{i=1}^{n-1} \binom{n}{i} \bar{\bar{D}}_{n-i} \cdot \bar{\bar{D}}_i \end{aligned}\right\} \tag{4.29}$$

<u>Physical Components of Deformation Gradient $\bar{\bar{F}}$</u>

By the Postulate of Affine Connection

$$d\bar{r} = d\bar{R} \cdot \frac{\partial \bar{r}}{\partial \bar{R}} = d\bar{R} \cdot \bar{\bar{F}} \tag{4.30}$$

In a general curvilinear co-ordinate system, $d\bar{r}$ may be written as

$$d\bar{r} = dx^r \, \bar{g}_r = dx^1 \, \bar{g}_1 + dx^2 \, \bar{g}_2 + dx^3 \, \bar{g}_3 \tag{4.31}$$

where dx^r are the contravariant components relative to the natural basis $\{\bar{g}_r\}$.

Similarly, $d\bar{R}$ may be written as

$$d\bar{R} = dX^U \bar{G}_U = dX^1 \bar{G}_1 + dX^2 \bar{G}_2 + dX^3 \bar{G}_3 \qquad (4.32)$$

where dX^U are the contravariant components relative to the natural basis $\{\bar{G}_U\}$.

Applying (4.31) and (4.32) to (4.30),

$$dx^r \bar{g}_r = dX^U \bar{G}_U \cdot \bar{G}^T \frac{\partial x^m}{\partial X^T} \bar{g}_m \qquad (4.33)$$

since

$$\frac{\partial ()}{\partial \bar{R}} = \bar{G}^T \frac{\partial ()}{\partial X^T}$$

From (4.33), since $\bar{G}_U \cdot \bar{G}^T = \delta_U^T$ [Kronecker delta];

$$\left.\begin{aligned} dx^r \bar{g}_r &= dX^U \frac{\partial x^m}{\partial X^U} \bar{g}_m \\ &= dX^U F^m_{;U} \bar{g}_m \end{aligned}\right\} \qquad (4.34)$$

where $\frac{\partial x^m}{\partial X^U} = F^m_{;U}$ are the components of $\bar{\bar{F}}$.

Let $\bar{e}_n$ and $\bar{E}_S$ be the dimensionless unit vectors parallel to $\bar{g}_n$ and $\bar{G}_S$ respectively.

magnitude of $\bar{g}_n$ is

$$|\bar{g}_n| = \sqrt{\bar{g}_{\underline{n}} \cdot \bar{g}_{\underline{n}}} = \sqrt{g_{\underline{nn}}}$$

magnitude of $\bar{G}_S$ is (4.35)

$$|\bar{G}_S| = \sqrt{\bar{G}_{\underline{S}} \cdot \bar{G}_{\underline{S}}} = \sqrt{G_{\underline{SS}}}$$

By definition, physical components of a vector at any point are the vector components parallel to the covariant unit base vectors. Hence, physical components $dx^{<r>}$ of $d\bar{r}$ are

$$dx^{<r>} = dx^r \sqrt{g_{\underline{rr}}}$$

so that,

$$d\bar{r} = dx^{<r>} \bar{e}_r \qquad (4.36)$$

and

$$dX^{<N>} = dX^N \sqrt{G_{\underline{NN}}}$$

so that,

$$d\bar{R} = d\bar{X}^{<N>} \bar{E}_N$$

From (4.34) and (4.36),

$$dx^r \sqrt{g_{\underline{rr}}}\; \bar{e}_r \;=\; dX^U \frac{\partial x^m}{\partial X^U} \sqrt{g_{\underline{mm}}}\; \bar{e}_m$$

or

$$dx^{<r>} \bar{e}_r \;=\; \frac{dX^{<U>}}{\sqrt{G_{\underline{UU}}}} \frac{\partial x^m}{\partial X^U} \sqrt{g_{\underline{mm}}}\; \bar{e}_m$$

or

$$\left[dx^{<r>} - \frac{dX^{<U>}}{\sqrt{G_{\underline{UU}}}} \frac{\partial x^r}{\partial X^U} \sqrt{g_{\underline{rr}}}\,\right] \bar{e}_r \;=\; \bar{0}.$$

Since $\{\bar{e}_r\}$ is a linearly independent set of base vectors, then

$$dx^{<r>} = dX^{<U>} F^r_{\;;U} \sqrt{\frac{g_{\underline{rr}}}{G_{\underline{UU}}}}$$

$$= dX^{<U>} F^{<rU>} \tag{4.37}$$

where

$$F^{<rU>} = F^r_{\;;U} \sqrt{\frac{g_{\underline{rr}}}{G_{\underline{UU}}}}$$

are defined as the *physical components of the tensor* $\bar{\bar{F}}$.

If $\bar{R}$ and $\bar{r}$ be referred to the same basis $\{\bar{g}_r\}$, then

$$F^{<rU>} = F^{r}_{;U} \sqrt{\frac{g_{\underline{\underline{rr}}}}{g_{\underline{\underline{UU}}}}}$$

A Basic Equation in Thermodynamics [33]

The *Thermodynamic Energy-balance Field Equation* or *Fourier-Kirchhoff-Neumann Equation* is given by

$$\rho s - \rho \frac{de}{dt} + \bar{\bar{\sigma}}:\bar{\bar{D}} - \frac{\partial \cdot \bar{h}}{\partial \bar{r}} = 0 \qquad (4.38)$$

where s = body heating per unit mass

e = internal energy density

$\bar{h}$ = density of surface heating flux

The free energy density $\psi = e - \theta\eta$ (4.39)

Where

θ = temperature

η = entropy density

so that

$$\dot{\psi} = \dot{e} - \dot{\theta}\eta - \theta\dot{\eta} \qquad (4.40)$$

(4.30) may also be written as

$$\rho \frac{de}{dt} = \rho s + \bar{\bar{\sigma}}:\bar{\bar{D}} - \frac{\partial \cdot \bar{h}}{\partial \bar{r}} \qquad (4.41)$$

Clausius - Duhem Inequality is given by

$$\theta\rho\dot{\eta} \geq -\theta \frac{\partial}{\partial \bar{r}} \cdot (\frac{\bar{h}}{\theta}) + \rho s \tag{4.42}$$

also,

$$\frac{\partial}{\partial \bar{r}} \cdot \frac{\bar{h}}{\theta} = \frac{1}{\theta} \frac{\partial \cdot \bar{h}}{\partial \bar{r}} + \frac{\partial \theta}{\partial \bar{r}} \cdot \frac{\partial}{\partial \theta} \frac{\bar{h}}{\theta} = \frac{1}{\theta} \frac{\partial \cdot \bar{h}}{\partial \bar{r}} - \frac{\partial \theta}{\partial \bar{r}} \cdot \frac{\bar{h}}{(\theta)^2} \tag{4.43}$$

From (4.34) and (4.35)

$$\theta\rho\dot{\eta} \geq -\frac{\partial \cdot \bar{h}}{\partial \bar{r}} + \frac{\partial \theta}{\partial \bar{r}} \cdot \frac{\bar{h}}{\theta} + \rho s \tag{4.44}$$

From (4.32),

$$\theta\rho\dot{\eta} = \rho\dot{e} - \rho\dot{\theta}\eta - \rho\dot{\psi} \tag{4.45}$$

Substituting for $\rho\dot{e}$ from (4.33) in (4.36),

$$\rho\theta\dot{\eta} = \rho s + \bar{\bar{\sigma}}:\bar{\bar{D}} - \frac{\partial \cdot \bar{h}}{\partial \bar{r}} - \rho(\dot{\psi} + \dot{\theta}\eta) \tag{4.46}$$

From (4.35) and (4.38),

$$\rho s + \bar{\bar{\sigma}}:\bar{\bar{D}} - \frac{\partial \cdot \bar{h}}{\partial \bar{r}} + \rho(\dot{\psi} + \dot{\theta}\eta) \geq -\frac{\partial \cdot \bar{h}}{\partial \bar{r}} + \frac{\partial \theta}{\partial \bar{r}} \cdot \frac{\bar{h}}{\theta} + \rho s$$

or

$$-\rho(\dot{\psi} + \dot{\theta}\eta) + \bar{\bar{\sigma}}:\bar{\bar{D}} - \frac{\partial \theta}{\partial \bar{r}} \cdot \frac{\bar{h}}{\theta} \geq 0 \tag{4.47}$$

(4.47) represents the *Energy Production Inequality*

5. Taylor Series Expansion of Strain Tensor

Let $d\bar{\xi}$ and $d\bar{r}$ be elements of arc lengths of a particle P in motion, at times τ and t respectively so that

$$t - \tau = s$$

If

$$(ds_1)^2 = d\bar{\xi}\cdot d\bar{\xi}$$

and

$$(ds)^2 = d\bar{r}\cdot d\bar{r}, \tag{5.1}$$

for s close to zero, the difference between $(ds_1)^2$ and $(ds)^2$ may be expanded into a Taylor Series of the form

$$(ds_1)^2-(ds)^2 = \frac{d}{dt}(ds)^2(\tau-t) + \frac{1}{2!}\frac{d^2}{dt^2}(ds)^2(\tau-t)^2 + \dots$$

$$= \sum_{i=1}^{n} \frac{(-1)^i}{i!} \overset{(i)}{(ds)^2}(s)^i + R_\eta(\xi) \tag{5.2}$$

For an expansion of the form (5.2) to exist, it is necessary that all time derivatives of the squared elements of arc length must exist at $\tau = t$. Such a regular function as $(ds_1)^2$ has only one expansion in powers of s given by (5.2).

$R_n(\xi)$, the remainder after n terms, tends to zero as $n \to \infty$ and/or $s \to 0$. At present we consider $R_n(\xi)$ to be zero.

$$\frac{d}{dt}(ds)^2 = \frac{d}{dt}(d\bar{r}\cdot d\bar{r}) = d\bar{R}\cdot\frac{\partial\bar{v}}{\partial\bar{R}}\cdot d\bar{r} + d\bar{r}\cdot d\bar{r}\cdot\frac{\partial\bar{v}}{\partial\bar{r}}$$

$$= 2\, d\bar{r}\cdot\bar{\bar{D}}\cdot d\bar{r}$$

$$= 2\, d\bar{R}\, d\bar{R}:(\bar{\bar{F}}\cdot\bar{\bar{D}}\cdot\bar{\bar{F}}^T) \tag{5.3}$$

from (4.26) and (5.3),

$$\left.\begin{aligned} \dot{\overline{(ds)^2}} &= d\bar{R}\, d\bar{R} : \dot{\bar{\bar{C}}} \\ &- - - - - - - - \\ \overset{(i)}{\overline{(ds)^2}} &= d\bar{R}\, d\bar{R} : \overset{(i)}{\bar{\bar{C}}} \end{aligned}\right\} \tag{5.4}$$

Pre-multiplying equation (4.28) by $\bar{\bar{F}}(t)\cdot(\bar{\bar{F}}_t(\tau))^{-1}$ and post-multiplying by $(\bar{\bar{F}}_t^T(t))^{-1}\cdot\bar{\bar{F}}^T(t)$.

$$\overset{(i)}{\bar{\bar{C}}} = \bar{\bar{F}}(t)\cdot(2\bar{\bar{D}}_i + \sum_{q=1}^{i-1}\binom{i}{q}\bar{\bar{D}}_{i-q}\cdot\bar{\bar{D}}_q)\cdot\bar{\bar{F}}^T(t) \tag{5.5}$$

from (5.2), (5.4) and (5.5),

$$(ds_1)^2-(ds)^2 = d\bar{R}\, d\bar{R}: \sum_{i=1}^{n}\frac{(-1)^i}{i!}\{\bar{\bar{F}}\cdot(2\bar{\bar{D}}_i + \sum_{q=1}^{i-1}\binom{i}{q}\bar{\bar{D}}_{i-q}\cdot\bar{\bar{D}}_q)\cdot\bar{\bar{F}}^T\}(s)^i \tag{5.6}$$

Also,

$$(ds_1)^2-(ds)^2 = d\bar{\xi}\cdot d\bar{\xi} - d\bar{r}\cdot d\bar{r}$$

$$= d\bar{R}\ d\bar{R}:\ (\bar{\bar{F}}\cdot\bar{\bar{C}}_t(\tau)\cdot\bar{\bar{F}}^T - \bar{\bar{F}}\cdot\bar{\bar{I}}\cdot\bar{\bar{F}}^T) \tag{5.7}$$

From (5.6) and (5.7),

$$\bar{\bar{C}}_t(\tau) = \bar{\bar{I}} + \sum_{i=1}^{n} \frac{(-1)^i}{i!} (2\bar{\bar{D}}_i + \sum_{q=1}^{i-1} \binom{i}{q} \bar{\bar{D}}_{i-q}\cdot\bar{\bar{D}}_q)(s)^i \tag{5.8}$$

therefore, from (4.29) and (5.8), we get

$$\bar{\bar{C}}_t(\tau) = \bar{\bar{I}} + \sum_{i=1}^{n} (-1)^i \frac{(s)^i}{i!} \bar{\bar{A}}_i(t) \tag{5.9}$$

This proves the *Coleman Lemma* [3,(1.17)].

(5.9) serves as a definition for Motions of Differential Type. The limit $s \to 0$ conforms to the definition of Materials of Differential Type. From (5.9), stresses in Materials of Differential Type can be determined from the n Rivlin-Ericksen tensors $\bar{\bar{A}}_n$.

From Appendix A.2, if $\sum_{n=0}^{\infty} \bar{\bar{A}}_n^* (\tau - t)^n$ converges in a non-zero interval about $t = \tau$ and hence represents a function, say $\bar{\bar{C}}_t(\tau)$, in that interval then

$$\bar{\bar{C}}_t(\tau) = \sum_{n=0}^{\infty} \bar{\bar{A}}_n^* (\tau-t)^n \tag{5.10}$$

Differentiating both sides of (5.10) k times and setting $\tau = t$,

$$\overset{=}{C}_t^{(k)}(t) = k!\,\overset{=}{A}_k^{*} \qquad (k = 0,1,2,\ldots) \tag{5.11}$$

From (5.10) and (5.11),

$$\overset{=}{C}_t(\tau) = \sum_{n=0}^{\infty} \frac{\overset{=}{C}_t^{(n)}(t)}{n!}(\tau-t)^n \tag{5.12}$$

Let us define $\overset{=}{A}_n(t)$ as

$$\overset{=}{A}_n(t) = \left.\frac{d^n\overset{=}{C}_t(\tau)}{d\tau}\right|_{\tau=t} = \overset{=}{C}_t^{(n)}(t) \tag{5.13}$$

From (5.12) and (5.13),

$$\overset{=}{C}_t(\tau) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\overset{=}{A}_n(t)\;(t-\tau)^n \tag{5.14}$$

In expanding (5.14), the remainder after N terms is

$$\overset{=}{R}_N(\tau) = \frac{\left.\overset{=}{C}_t^{N}(\tau)\right|_{\tau=\tau^1}}{N!}(\tau-t)^N \tag{5.15}$$

Where τ^1 is some point between τ and t.

If it is postulated that $\overset{=}{A}_N = \overset{=}{0}$, and $\overset{=}{A}_0 = \overset{=}{I}$, from (5.14) and (5.15),

$$\overset{=}{C}_t(\tau) = \overset{=}{I} + \sum_{n=1}^{N-1} \frac{(-1)^n}{n!}\overset{=}{A}_n(t-\tau)^n \tag{5.16}$$

(5.16) and (5.9) lead to the same result.

In defining Materials of Differential Type by equation (5.9) it is being said that while in a simple material stress $\bar{\bar{\sigma}}(t)$ is determined by the values of strain $\bar{\bar{C}}_t^t(s)$ for all $s \geq 0$, for MDT only a very short part of the history of strain $\bar{\bar{C}}_t(\tau)$ has an influence on the stress as s must be very near zero for equation (5.16) to be true.

6. Internal Constraints in Materials of Differential Type

The condition of incompressibility applies an internal constraint to the deforming body. A simple internal constraint is defined by a scalar valued function $\gamma(\bar{\bar{F}})$ of a tensor variable $\bar{\bar{F}}$. A particle P in a body is subject to the constraint defined by $\gamma(\bar{\bar{F}})$ if the possible motions of the body are restricted to those motions for which

$$\gamma\{\bar{\bar{F}}(\tau)\} = 0; \quad -\infty < \tau < \infty \tag{6.1}$$

A constraint (6.1) is a constitutive equation and hence is subject to the Principle of Material Frame Indifference. Thus (6.1) may also be written as

$$\gamma\{\bar{\bar{C}}(\tau)\} = 0, \qquad -\infty < \tau < \infty \tag{6.2}$$

Incompressibility demands that only isochoric motions be allowed. From (4.3), we get

$$|\bar{\bar{F}}(\tau)| = 1$$

and hence,

$$|\bar{\bar{C}}(\tau)| = 1$$

$\therefore$ A constraint function for incompressibility is

$$\lambda(\bar{\bar{C}}) = |\bar{\bar{C}}| - 1 \tag{6.3}$$

Forces causing deformation in a body do work. Mechanical power is defined by

$$P = \bar{\bar{\sigma}}:\bar{\bar{D}} \tag{6.4}$$

In a body subject to an internal constraint, a certain force system is necessary to maintain the constraint. Conceivably infinitely many different systems of such forces may suffice to maintain any given constraint. Usually, a simplifying assumption is made that forces maintaining the constraints do no work.

In equation (4.47), if thermodynamic variables are neglected, then

$$\bar{\bar{\sigma}}:\bar{\bar{D}} \geqslant 0 \tag{6.5}$$

Let $\bar{\bar{N}}$ be such a tensor that

$$\bar{\bar{N}}:\bar{\bar{D}} = 0 \tag{6.6}$$

Addition of (6.6) to (6.5) does not change the energy-production or energy-balance inequalities if $\bar{\bar{N}}$ has the units of stress. Since (6.6) means stress $\bar{\bar{N}}$ does no work in deformation, it may be assumed to be maintaining the internal constraint which, in (6.3), is incompressibility.

Since $\bar{\bar{N}}$ is not a function of deformation of the body, i.e.

$$\bar{\bar{N}} \neq \bar{\bar{N}}(\bar{\bar{D}})$$

it cannot be determined from the mechanical constitutive equations. Thus the *Principle of Determinism* of stress for simple materials subject to internal constraints [34] is stated as in a simple body subject to internal constraints, stress $\bar{\bar{\sigma}}(t)$ is determined by the history $\bar{\bar{F}}(\tau)$ only to within a stress $\bar{\bar{N}}$ that does no work in any motion satisfying the constraint.

Incompressible simple materials are simpler than compressible simple materials in two ways: not only are all possible deformations isochoric but also the stress is determined from the deformation only to within a hydrostatic pressure [34]. Thus the response of an incompressible material in a given isochoric deformation generally is not the same as that of a compressible material in precisely the same deformation. For example, simple shear is a deformation which may be undergone by both incompressible and compressible simple bodies; but the stress system in the compressible bodies may not be read off from the solution of the incompressible bodies where the arbitrary pressure makes possible greater variety in the solution. More over, various non-homogeneous isochoric deformations which may not be maintained as states of equilibrium for arbitrary isotropic compressible simple bodies free from the action of body force, are amenable to exact solutions for all incompressible isotropic simple bodies subjected to suitable tractions applied on the boundary. [1,6,7,12].

Theorem

The stress system $\bar{\bar{N}}_i$ $(i = 1,2,\ldots,n)$ required to maintain an internal constraint in a material of differential type obeys the relation

$$\sum_{i=0}^{n-1} \bar{\bar{N}}_i : \bar{\bar{A}}_{i+1} = 0 \qquad (6.7)$$

Where $\bar{\bar{A}}_i$ are Rivlin-Ericksen tensors.

Proof

The constraint is given as

$$\lambda(\bar{\bar{C}}, \dot{\bar{\bar{C}}}, \ldots, \overset{(n)}{\bar{\bar{C}}}) = 0, \qquad n = 0, 1, 2, \ldots. \qquad (6.8)$$

where

$$\bar{\bar{C}} = \overset{(0)}{\bar{\bar{C}}}, \quad \dot{\bar{\bar{C}}} = \overset{(1)}{\bar{\bar{C}}}, \text{ etc.}$$

Differentiating (6.8) with respect to time t

$$\frac{d\bar{\bar{C}}}{dt}:\frac{\partial\lambda}{\partial\bar{\bar{C}}} + \frac{d\dot{\bar{\bar{C}}}}{dt}:\frac{d\lambda}{d\dot{\bar{\bar{C}}}} + \cdots + \frac{d\overset{(n)}{\bar{\bar{C}}}}{dt}:\frac{\partial\lambda}{\partial\overset{n}{\bar{\bar{C}}}} = 0 \tag{6.9}$$

Since $\bar{\bar{C}}$ is assumed to have only n time derivatives;

$$\frac{d\overset{(n)}{\bar{\bar{C}}}}{dt} = \bar{\bar{0}}.$$

Therefore, (6.9) may be written as

$$\sum_{i=0}^{n-1} \overset{(i+1)}{\bar{\bar{C}}} : \frac{\partial\lambda}{\partial\overset{(i)}{\bar{\bar{C}}}} = 0 \tag{6.10}$$

From (5.5) and (6.10)

$$\sum_{i=0}^{n-1}\{\bar{\bar{F}}\cdot\{2\bar{\bar{D}}_{i+1} + \sum_{k=1}^{i}\binom{i+1}{k}\bar{\bar{D}}_{i+1-k}\cdot\bar{\bar{D}}_k\}\cdot\bar{\bar{F}}^T :$$

$$\frac{\partial\lambda}{\partial(\bar{\bar{F}}\cdot\{2\bar{\bar{D}}_i + \sum_{k=1}^{i-1}\binom{i}{k}\bar{\bar{D}}_{i-k}\cdot\bar{\bar{D}}_k\}\cdot\bar{\bar{F}}^T)}\} = 0$$

or

$$\sum_{i=0}^{n=1} \{2\bar{\bar{D}}_{i+1} + \sum_{k=1}^{i} \binom{i+1}{k} \bar{\bar{D}}_{i+1-k} \cdot \bar{\bar{D}}_k\} :$$

$$\{\bar{\bar{F}}^T \cdot \frac{\partial \lambda}{\partial (\bar{\bar{F}} \cdot \{2\bar{\bar{D}}_i + \sum_{k=1}^{i-1} \binom{i}{k} \bar{\bar{D}}_{i-k} \cdot \bar{\bar{D}}_k\} \cdot \bar{\bar{F}}^T)} \cdot \bar{\bar{F}}\} = 0 \tag{6.11}$$

Let $\bar{\bar{N}}_i$ be defined as

$$\bar{\bar{N}}_i = q(\bar{\bar{F}}^T \cdot \frac{\partial \lambda}{\partial (\bar{\bar{F}} \cdot \{2\bar{\bar{D}}_i + \sum_{k=1}^{i-1} \binom{i}{k} \bar{\bar{D}}_{i-k} \cdot \bar{\bar{D}}_k\} \cdot \bar{\bar{F}}^T)} \cdot \bar{\bar{F}}) \tag{6.12}$$

where q is a scalar constant. From (4.29, (6.11) and (6.12),

$$\sum_{i=0}^{n-1} \bar{\bar{N}}_i : \bar{\bar{A}}_{i+1} = 0 \qquad \text{Q.E.D.}$$

NB! 1. for n=0, (6.11) yields

$$\bar{\bar{D}} : \bar{\bar{N}} = 0;$$

and (6.12) is of the form

$$\bar{\bar{N}} = q\, \bar{\bar{F}}^T \cdot \frac{\partial \lambda}{\partial \bar{\bar{C}}} \cdot \bar{\bar{F}},$$

which are true for simple materials. ($\bar{\bar{A}}_o = \bar{\bar{I}}$)

2. $(\bar{\bar{F}} \cdot \bar{\bar{D}} \cdot \bar{\bar{F}}^T) : \bar{\bar{T}} = (\bar{\bar{F}} \cdot \bar{\bar{D}} \cdot \bar{\bar{F}}^T \cdot \bar{\bar{T}}) : \bar{\bar{I}} = \bar{\bar{I}} : (\bar{\bar{F}} \cdot \bar{\bar{D}} \cdot \bar{\bar{F}}^T \cdot \bar{\bar{T}}) = \bar{\bar{F}} : (\bar{\bar{D}} \cdot \bar{\bar{F}}^T \cdot \bar{\bar{T}})$

$= (\bar{\bar{D}} \cdot \bar{\bar{F}}^T \cdot \bar{\bar{T}}) : \bar{\bar{F}} = (\bar{\bar{D}} \cdot \bar{\bar{F}}^T \cdot \bar{\bar{T}} \cdot \bar{\bar{F}}) : \bar{\bar{I}} = \bar{\bar{I}} : (\bar{\bar{D}} \cdot \bar{\bar{F}}^T \cdot \bar{\bar{T}} \cdot \bar{\bar{F}})$

$= \bar{\bar{D}} : (\bar{\bar{F}}^T \cdot \bar{\bar{T}} \cdot \bar{\bar{F}})$

hence equation (6.11).

If the internal constraint is incompressibility,

$$\lambda = |\bar{\bar{C}}| - 1 \tag{6.13}$$

Then equation (6.11) is of the form

$$\bar{\bar{D}} : \bar{\bar{N}} = 0$$

where

$$\bar{\bar{N}} = q\ \bar{\bar{F}}^T \cdot \frac{\partial\lambda}{\partial\bar{\bar{C}}} \cdot \bar{\bar{F}} \quad ,$$

the other derivatives of λ being 0.

From (A.1.12),

$$\frac{\partial\lambda}{\partial\bar{\bar{C}}} = \frac{\partial}{\partial\bar{\bar{C}}}\ (|\bar{\bar{C}}| - 1) \quad = \quad |\bar{\bar{C}}|\ \{(\bar{\bar{C}})^{-1}\}^T$$

$$\therefore \qquad \bar{\bar{N}} = q\ \bar{\bar{F}}^T\ |\bar{\bar{C}}| \cdot (\bar{\bar{F}}^T)^{-1} \cdot (\bar{\bar{F}})^{-1} \cdot \bar{\bar{F}} = q\ \bar{\bar{I}} \tag{6.14}$$

since

$$|\bar{\bar{C}}| = 1.$$

$\therefore$ In incompressible Materials of Differential Type, stress is determined by the deformation history only to within an arbitrary pressure q.

Consider another set of internal constraints given by

$$\lambda_n = |[\bar{\bar{F}} \cdot \{2\bar{\bar{D}}_n + \sum_{i=1}^{n-1} \binom{n}{i} \bar{\bar{D}}_{n-i} \cdot \bar{\bar{D}}_i\} \cdot \bar{\bar{F}}^T]| - K_n = 0 \tag{6.15}$$

$$n = 0, 1, 2 \ldots .$$

where

$$K_n = |\bar{\bar{F}}\cdot\bar{\bar{A}}_n\cdot\bar{\bar{F}}^T| \qquad \text{are constants.}$$

From (6.12), and (A.1.12)

$$\bar{\bar{N}}_n = q\left\{\bar{\bar{F}}^T\cdot \frac{\partial(|[\bar{\bar{F}}\cdot\{2\bar{\bar{D}}_n + \sum_{i=1}^{n-1}\binom{n}{i}\bar{\bar{D}}_{n-i}\cdot\bar{\bar{D}}_i\}\cdot\bar{\bar{F}}^T]| - K_n)}{\partial(\bar{\bar{F}}\cdot\{2\bar{\bar{D}}_n + \sum_{i=1}^{n-1}\binom{n}{i}\bar{\bar{D}}_{n-i}\cdot\bar{\bar{D}}_i\}\cdot\bar{\bar{F}}^T)}\right\}$$

$$= q(\bar{\bar{F}}^T\cdot|\bar{\bar{F}}\cdot\bar{\bar{A}}_n\cdot\bar{\bar{F}}^T|\{(\bar{\bar{F}}\cdot\bar{\bar{A}}_n\cdot\bar{\bar{F}}^T)^{-1}\}^T\cdot\bar{\bar{F}})$$

$$= q\ K_n\ \bar{\bar{F}}^T\cdot(\bar{\bar{F}}^T)^{-1}\cdot(\bar{\bar{A}}_n^{-1})^T\cdot\bar{\bar{F}}^{-1}\cdot\bar{\bar{F}}$$

$$= q\ K_n(\bar{\bar{A}}_n^{-1})^T \tag{6.16}$$

Let

$$q\ K_n = p_n$$

be constant. Since $\bar{\bar{A}}_n$ is symmetric,

$$\bar{\bar{N}}_n = p_n\bar{\bar{A}}_n^{-1} \tag{6.17}$$

Therefore, for an internal constraint of the type(6.15)in a Material of Differential Type, the stress system $\bar{\bar{N}}_i$ required to maintain the internal constraint is given by(6.17).

Further properties of the internal constraint described by (6.8) follow.

λ, as an explicit function of only $\bar{\bar{C}}$ and its n time derivatives, satisfies the following conditions:

$$\left.\begin{aligned} \frac{d\lambda}{dt} &= 0 \\ \frac{d}{dt}\frac{\partial\lambda}{\partial\bar{\bar{C}}} &= 0 \\ &\text{- - - - - - - -} \\ \frac{d}{dt}\frac{\partial\lambda}{\partial\overset{(n-1)}{\bar{\bar{C}}}} &= 0 \end{aligned}\right\} \qquad (6.18)$$

Then the n time derivatives of $\lambda(\bar{\bar{C}}, \dot{\bar{\bar{C}}}, \ldots, \overset{(n)}{\bar{\bar{C}}})$ are given by (6.9) and the following:

Differentiating (6.9) with respect to t,

$$\frac{d^2\bar{\bar{C}}}{dt^2} : \frac{\partial\lambda}{\partial\bar{\bar{C}}} + \frac{d\bar{\bar{C}}}{dt} : \frac{d}{dt}\frac{\partial\lambda}{\partial\bar{\bar{C}}} + \frac{d^2\dot{\bar{\bar{C}}}}{dt^2} : \frac{\partial\lambda}{\partial\dot{\bar{\bar{C}}}} + \frac{d\dot{\bar{\bar{C}}}}{dt} : \frac{d}{dt}\frac{\partial\lambda}{\partial\dot{\bar{\bar{C}}}} + \ldots$$

$$+ \frac{d^2\overset{(n-2)}{\bar{\bar{C}}}}{dt^2} : \frac{\partial\lambda}{\partial\overset{(n-2)}{\bar{\bar{C}}}} + \frac{d\overset{(n-2)}{\bar{\bar{C}}}}{dt} : \frac{d}{dt}\frac{\partial\lambda}{\partial\overset{(n-2)}{\bar{\bar{C}}}} = 0 \qquad (6.19)$$

Applying (6.18) to (6.19),

$$\frac{d^2\overset{\equiv}{C}}{dt^2} : \frac{\partial\lambda}{\partial\overset{\equiv}{C}} + \frac{d^2\dot{\overset{\equiv}{C}}}{dt^2} : \frac{\partial\lambda}{\partial\dot{\overset{\equiv}{C}}} + \cdots + \frac{d^2\overset{(n-2)}{\overset{\equiv}{C}}}{dt^2} : \frac{\partial\lambda}{\partial\overset{(n-2)}{\overset{\equiv}{C}}} = 0$$

The last three equations are

$$\left.\begin{aligned}
&\frac{d^{n-2}\overset{\equiv}{C}}{dt^{n-2}} : \frac{\partial\lambda}{\partial\overset{\equiv}{C}} + \frac{d^{n-2}\dot{\overset{\equiv}{C}}}{dt^{n-2}} : \frac{\partial\lambda}{\partial\dot{\overset{\equiv}{C}}} + \frac{d^{n-2}\ddot{\overset{\equiv}{C}}}{dt^{n-2}} : \frac{\partial\lambda}{\partial\ddot{\overset{\equiv}{C}}} = 0\\
&\frac{d^{n-1}\overset{\equiv}{C}}{dt^{n-1}} : \frac{\partial\lambda}{\partial\overset{\equiv}{C}} + \frac{d^{n-1}\dot{\overset{\equiv}{C}}}{dt^{n-1}} : \frac{\partial\lambda}{\partial\dot{\overset{\equiv}{C}}} = 0\\
&\frac{d^{n}\overset{\equiv}{C}}{dt^{n}} : \frac{\partial\lambda}{\partial\overset{\equiv}{C}} = 0
\end{aligned}\right\} \quad (6.20)$$

(6.20) may also be written as

$$\left.\begin{aligned}
&\overset{(n)}{\overset{\equiv}{C}} : \frac{\partial\lambda}{\partial\overset{\equiv}{C}} = 0\\
&\overset{(n-1)}{\overset{\equiv}{C}} : \frac{\partial\lambda}{\partial\overset{\equiv}{C}} + \overset{(n)}{\overset{\equiv}{C}} : \frac{\partial\lambda}{\partial\dot{\overset{\equiv}{C}}} = 0\\
&\overset{(n-2)}{\overset{\equiv}{C}} : \frac{\partial\lambda}{\partial\overset{\equiv}{C}} + \overset{(n-1)}{\overset{\equiv}{C}} : \frac{\partial\lambda}{\partial\dot{\overset{\equiv}{C}}} + \overset{(n)}{\overset{\equiv}{C}} : \frac{\partial\lambda}{\partial\ddot{\overset{\equiv}{C}}} = 0
\end{aligned}\right\} \quad (6.21)$$

Since

$$\frac{d\overset{(n)}{\overset{\equiv}{C}}}{dt} = 0,$$

integration with respect to t gives

$$\overset{(n)}{\underline{\underline{C}}} = K_1 \bar{\bar{I}} \tag{6.22}$$

where K_1 is a constant with respect to time, and

$$\frac{d\bar{\bar{I}}}{dt} = \frac{\partial \bar{r}}{\partial t} \cdot \frac{\partial \bar{\bar{I}}}{\partial \bar{r}} = \bar{\bar{0}}$$

since, from *RICCI Lemma* [33], $\frac{\partial \bar{\bar{I}}}{\partial \bar{r}} = \bar{\bar{0}}$.

Substituting (6.22) in (6.21-1),

$$k_1 \bar{\bar{I}} : \frac{\partial \lambda}{\partial \bar{\bar{C}}} = 0$$

or,

$$\bar{\bar{I}} : \frac{\partial \lambda}{\partial \bar{\bar{C}}} = 0 \tag{6.23}$$

Also,

$$\frac{d}{dt} \left[\overset{(n-1)}{\underline{\underline{C}}} : \frac{\partial \lambda}{\partial \bar{\bar{C}}} \right] = \frac{d \overset{(n-1)}{\underline{\underline{C}}}}{dt} : \frac{\partial \lambda}{\partial \bar{\bar{C}}} + \overset{(n-1)}{\underline{\underline{C}}} : \frac{d}{dt} \frac{\partial \lambda}{\partial \bar{\bar{C}}}$$

$$= \overset{(n)}{\underline{\underline{C}}} : \frac{\partial \lambda}{\partial \bar{\bar{C}}} \qquad \text{(due to 6.18)} \tag{6.24}$$

From (6.22), (6.23) and (6.24),

$$\frac{d}{dt}\left[\overset{(n-1)}{\bar{\bar{C}}} : \frac{\partial\lambda}{\partial\bar{\bar{C}}}\right] = 0 \tag{6.25}$$

Integration of (6.25) gives

$$\overset{(n-1)}{\bar{\bar{C}}} : \frac{\partial\lambda}{\partial\bar{\bar{C}}} = k_2 \tag{6.26}$$

where k_2 is a constant of time.

From (6.21-2) and (6.26),

$$k_2 + k_1\bar{\bar{I}} : \frac{\partial\lambda}{\partial\bar{\bar{C}}} = 0$$

or

$$\bar{\bar{I}} : \frac{\partial\lambda}{\partial\bar{\bar{C}}} = -\frac{k_2}{k_1} \tag{6.27}$$

Consider

$$\frac{d}{dt}\left[\overset{(n-2)}{\bar{\bar{C}}} : \frac{\partial\lambda}{\partial\bar{\bar{C}}}\right] = \frac{d\overset{(n-2)}{\bar{\bar{C}}}}{dt} : \frac{\partial\lambda}{\partial\bar{\bar{C}}} + \overset{(n-2)}{\bar{\bar{C}}} : \frac{d}{dt}\frac{\partial\lambda}{\partial\bar{\bar{C}}} \tag{6.28}$$

From (6.18) and (6.28),

$$\frac{d}{dt}\left[\overset{(n-2)}{\bar{\bar{C}}} : \frac{\partial\lambda}{\partial\bar{\bar{C}}}\right] = \overset{(n-1)}{\bar{\bar{C}}} : \frac{\partial\lambda}{\partial\bar{\bar{C}}} = k_2 \qquad \text{(from 6.26).}$$

$$\therefore \quad \overset{(\underline{n}-2)}{\bar{\bar{C}}} : \frac{\partial \lambda}{\partial \bar{\bar{C}}} = k_2 t + k_3 \tag{6.29}$$

where k_3 is another constant of time.

Also,

$$\frac{d}{dt}\left[\overset{(\underline{n}-1)}{\bar{\bar{C}}} : \frac{\partial \lambda}{\partial \dot{\bar{\bar{C}}}}\right] = \overset{(\underline{n})}{\bar{\bar{C}}} : \frac{\partial \lambda}{\partial \dot{\bar{\bar{C}}}} = - k_2 \qquad \text{(from 6.27)}$$

$$\therefore \quad \overset{(\underline{n}-1)}{\bar{\bar{C}}} : \frac{\partial \lambda}{\partial \dot{\bar{\bar{C}}}} = - k_2 t + k_4 \tag{6.30}$$

Substitution of (6.22), (6.30) and (6.29) in (6.21-3) gives

$$k_3 + k_4 + k_1 \bar{\bar{I}} : \frac{\partial \lambda}{\partial \ddot{\bar{\bar{C}}}} = 0$$

or

$$\bar{\bar{I}} : \frac{\partial \lambda}{\partial \ddot{\bar{\bar{C}}}} = - \frac{k_3 + k_4}{k_1} \tag{6.31}$$

Generalising (6.23), (6.27) and (6.31),

$$\bar{\bar{I}} : \frac{\partial \lambda}{\partial \overset{(\underline{n})}{\bar{\bar{C}}}} = \frac{k_n^*}{k_1}$$

where

$$\overset{(\underline{n})}{\bar{\bar{C}}} = k_1 \bar{\bar{I}}, \tag{6.32}$$

$$\bar{\bar{I}} : \frac{\partial\lambda}{\partial\bar{\bar{C}}} = 0$$

and

$$k_n^* = \text{constants},\ n = 1,2,\ldots.$$

Equation (6.22) is a sufficient representation for the n^{th} time derivative of strain tensor $\bar{\bar{C}}$. It is not a necessary form; but by adopting such a form, one can fix the traces of partial derivatives of the internal constraint function λ with respect to its independent variables $\bar{\bar{C}}$, $\dot{\bar{\bar{C}}}$,..., $\overset{(n)}{\bar{\bar{C}}}$ by providing the constants of integration k_n^*, ..., k_1 (constants of time t), with the help of equation (6.32). Subject to (6.18), (6.10) is a set of n equations of the form

$$\sum_{i=0}^{n-j-1} \overset{(i+j+1)}{\bar{\bar{C}}} : \frac{\partial\lambda}{\partial\overset{(i)}{\bar{\bar{C}}}} = 0; \quad j = 0, 1, 2, \ldots, n-1 \tag{6.33}$$

For each equation of (6.10), (6.7) is satisfied as

$$\sum_{i=0}^{n-j-1} \bar{\bar{N}}_i : \bar{\bar{A}}_{i+j+1} = 0, \quad j = 0, 1, 2, \ldots, n-1 \tag{6.34}$$

where $\bar{\bar{N}}_i$ is defined by (6.12).

<u>Interpretation of (6.18)</u>:

Consider the incompressible simple material for which

$$\lambda = |\bar{\bar{C}}| - 1 = 0$$

$$\therefore \quad \frac{\partial \lambda}{\partial \bar{\bar{C}}} = \frac{\partial |\bar{\bar{C}}|}{\partial \bar{\bar{C}}} = |\bar{\bar{C}}| (\bar{\bar{C}}^{-1})^T = \bar{\bar{C}}^{-1} \qquad (6.35)$$

Since

$$|\bar{\bar{C}}| = 1$$

$$\therefore \quad \frac{d}{dt} \left(\frac{\partial \lambda}{\partial \bar{\bar{C}}}\right) = \frac{d}{dt} (\bar{\bar{C}}^{-1}) = 0 \quad \text{(from 6.18).} \qquad (6.35a)$$

Since

$$\bar{\bar{C}} \cdot \bar{\bar{C}}^{-1} = \bar{\bar{I}},$$

$$\frac{d}{dt} (\bar{\bar{C}} \cdot \bar{\bar{C}}^{-1}) = \frac{d}{dt} \bar{\bar{C}} \cdot \bar{\bar{C}}^{-1} + \bar{\bar{C}} \cdot \frac{d}{dt} \bar{\bar{C}}^{-1} = 0$$

$$\therefore \quad \bar{\bar{C}} \cdot \frac{d}{dt} \bar{\bar{C}}^{-1} = - \frac{d}{dt} \bar{\bar{C}} \cdot \bar{\bar{C}}^{-1} \qquad (6.36)$$

Pre-multiplying each side of (6.36) with $\bar{\bar{C}}^{-1}$,

$$\frac{d}{dt} \bar{\bar{C}}^{-1} = - \bar{\bar{C}}^{-1} \cdot \frac{d}{dt} \bar{\bar{C}} \cdot \bar{\bar{C}}^{-1}$$

$$= - (\bar{\bar{F}} \cdot \bar{\bar{F}}^T)^{-1} \cdot 2\bar{\bar{F}} \cdot \bar{\bar{D}} \cdot \bar{\bar{F}}^T \cdot (\bar{\bar{F}} \cdot \bar{\bar{F}}^T)^{-1}$$

$$= - 2 (\bar{\bar{F}}^T)^{-1} \cdot \bar{\bar{D}} \cdot (\bar{\bar{F}})^{-1} \qquad (6.36a)$$

from (6.35a) and (6.36a),

$$\bar{\bar{D}} = \bar{\bar{0}}$$

For the list of controllable deformations in incompressible Elastic Materials [12], the deformation gradient $\bar{\bar{F}}$ is non-zero. Therefore, (6.18) implies that the constants associated with deformations from the reference configuration to the present configuration are independent of time, and that the deformation gradient $\bar{\bar{F}}(t)$ has no time derivative.

For Materials of Differential Type, consider a general set of internal constraint function in equation (6.15) written as

$$\lambda_n = |\overset{(n)}{\bar{\bar{C}}}| - k_n = 0, \quad n = 1, 2, 3, \ldots.$$

$$\therefore \quad \frac{\partial \lambda_n}{\partial \overset{(n-i)}{\bar{\bar{C}}}} = \frac{\partial}{\partial \overset{(n-i)}{\bar{\bar{C}}}} \left(|\overset{(n)}{\bar{\bar{C}}}| - k_n \right)$$

$$= \frac{\partial \overset{(n)}{\bar{\bar{C}}}}{\partial \overset{(n-i)}{\bar{\bar{C}}}} \cdot \frac{\partial |\overset{(n)}{\bar{\bar{C}}}|}{\partial \overset{(n)}{\bar{\bar{C}}}} \tag{6.37}$$

Without much demand, it may be stated that $\{\bar{\bar{C}}, \dot{\bar{\bar{C}}}, \ldots, \overset{(n)}{\bar{\bar{C}}}\}$ form a functionally independent set, and therefore, terms like $\frac{\partial \overset{(n)}{\bar{\bar{C}}}}{\partial \overset{(n-i)}{\bar{\bar{C}}}}$ are all zero except when $i = 0$.

$\therefore$ The only non-zero derivatives in (6.37) are

$$\frac{\partial \lambda_{\underline{i}}}{\partial \overset{(\underline{i})}{\overline{\overline{C}}}} = |\overset{(\underline{i})}{\overline{\overline{C}}}| (\overset{(\underline{i})}{\overline{\overline{C}}}{}^{-1})^T = k_{\underline{i}} (\overset{(\underline{i})}{\overline{\overline{C}}}{}^{-1})^T, \quad \underset{\text{(no sum)}}{i=0,1,\ldots,n} \tag{6.38}$$

Therefore, (6.18) would not be a natural set of assumptions for internal constraint functions (6.15) as time derivatives of (6.38) are non-zero. But (6.23) still holds, and in this case, it is the incompressibility condition.

7. Certain Motions of Differential Type

We will be concerned only with steady flows. The general approach in this chapter would be to describe a velocity field and then to calculate the deformation gradient tensor $\overline{\overline{F}}$ and its time rates $\overline{\overline{L}}_n$ by solving the necessary differential equations.

The most general velocity field in a steady flow may have a functional form

$$v^i = f_i(x^1, x^2, x^3) \tag{7.1}$$

Among the various forms of the function f_i, one which would still be sufficiently general is

$$v^i = a_{ij}\, e^{x^j}, \tag{7.2}$$

that is,

$$\left.\begin{aligned} v^1 &= a_{11}e^{x^1} + a_{12}e^{x^2} + a_{13}e^{x^3} \\ v^2 &= a_{21}e^{x^1} + a_{22}e^{x^2} + a_{23}e^{x^3} \\ v^3 &= a_{31}e^{x^1} + a_{32}e^{x^2} + a_{33}e^{x^3} \end{aligned}\right\} \quad (7.2a)$$

where a_{ij} are constants.

If $\xi^i = x^i(t-s)$, then

$$-\frac{d\xi^i}{ds} = v^i(\xi^i, t-s); \text{ so that } \xi^i|_{s=0} = x^i \quad (7.2b)$$

The differential equation of motion is written as

$$-\frac{d\xi^i}{ds} = a_{ij}e^{\xi^j} \quad (7.3)$$

or

$$\left.\begin{aligned} -\frac{d\xi^1}{ds} &= a_{11}e^{\xi^1} + a_{12}e^{\xi^2} + a_{13}e^{\xi^3} \\ -\frac{d\xi^2}{ds} &= a_{21}e^{\xi^1} + a_{22}e^{\xi^2} + a_{23}e^{\xi^3} \\ -\frac{d\xi^3}{ds} &= a_{31}e^{\xi^1} + a_{32}e^{\xi^2} + a_{33}e^{\xi^3} \end{aligned}\right\} \quad (7.3a)$$

Subject to the initial condition

$$\xi^i|_{s=0} = x^i \quad (7.4)$$

In (7.3), if a_{ij} are symmetric, let them be the elements of a symmetric tensor $\bar{\bar{A}}$.

Consider a change of frame so that

$$\bar{\xi} = \bar{\gamma}\cdot\bar{\bar{Q}}$$

where $\bar{\bar{Q}}$ is a constant orthogonal tensor.

∴ From (7.3)

$$-\frac{d}{ds}(\bar{\gamma}\cdot\bar{\bar{Q}}) = e^{\bar{\gamma}\cdot\bar{\bar{Q}}}\cdot\bar{\bar{A}} = e^{\bar{\gamma}}\cdot\bar{\bar{Q}}\cdot\bar{\bar{A}} \tag{7.5}$$

since

$$e^{\bar{\gamma}\cdot\bar{\bar{Q}}} = e^{\bar{\gamma}}\cdot\bar{\bar{Q}}$$

for all orthogonal $\bar{\bar{Q}}$.

$$\therefore -\frac{d\bar{\gamma}}{ds} = e^{\bar{\gamma}}\cdot\bar{\bar{Q}}\cdot\bar{\bar{A}}\cdot\bar{\bar{Q}}^T = e^{\bar{\gamma}}\cdot\bar{\bar{D}} \tag{7.6}$$

where

$$\bar{\bar{Q}}\cdot\bar{\bar{A}}\cdot\bar{\bar{Q}}^T = \bar{\bar{D}}$$

If $\bar{\bar{Q}}$ be so chosen that $\bar{\bar{D}}$ is a diagonal tensor, (7.6) may be written as

$$-\frac{d\gamma^i}{ds} = d_{\underline{ii}}e^{\gamma^{\underline{i}}} \tag{7.6a}$$

where $d_{\underline{i}\underline{i}}$ are elements of matrix $[\bar{\bar{D}}]$ of tensor $\bar{\bar{D}}$.
Solution to (7.6) is considered in (7.34).
In the equation (7.2a), ξ^i may be substituted as

$$\left.\begin{aligned} & y_i = e^{\xi^i} \\ & \text{so that} \\ & \frac{1}{y_{\underline{i}}}\frac{dy_{\underline{i}}}{ds} = \frac{d\xi^i}{ds} \end{aligned}\right\} \qquad (7.7)$$

Therefore, (7.2a) becomes

$$\left.\begin{aligned} -\frac{dy_1}{ds} &= a_{11}(y_1)^2 + a_{12}y_1y_2 + a_{13}y_1y_3 \\ -\frac{dy_2}{ds} &= a_{21}y_2y_1 + a_{22}(y_2)^2 + a_{23}y_2y_3 \\ -\frac{dy_3}{ds} &= a_{31}y_3y_1 + a_{32}y_3y_2 + a_{33}(y_3)^2 \end{aligned}\right\} \qquad (7.8)$$

In Einsteinian convention,

$$-\frac{dy_i}{ds} = a_{ij}y_iy_j \qquad (7.8a)$$

In the matrix form, (7.8) may be written as

$$
-\begin{Bmatrix} \frac{dy_1}{ds} \\ \frac{dy_2}{ds} \\ \frac{dy_3}{ds} \end{Bmatrix} = \begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \qquad (7.8b)
$$

Let y_i be substituted by $\tilde{y}_i$ so that

$$
\{y_i\} = [b_{ij}]\ \{\tilde{y}_i\} \qquad (7.9)
$$

where $[b_{ij}]$ is an upper-triangular matrix,

$$
[b_{ij}] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} \qquad (7.9a)
$$

Substituting (7.9) and (7.9a) in (7.8b),

$$
-\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} \begin{Bmatrix} d\tilde{y}_1/ds \\ d\tilde{y}_2/ds \\ d\tilde{y}_3/ds \end{Bmatrix}
$$

$$= \begin{bmatrix} b_{11}\tilde{y}_1 + b_{12}\tilde{y}_2 + b_{13}\tilde{y}_3 & 0 & 0 \\ 0 & b_{22}\tilde{y}_2 + b_{23}\tilde{y}_3 & 0 \\ 0 & 0 & b_{33}\tilde{y}_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} \begin{Bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{Bmatrix} \qquad (7.9b)$$

Let

$$(b_{11}\tilde{y}_1 + b_{12}\tilde{y}_2 + b_{13}\tilde{y}_3)a_{11}b_{11} = c_{11}$$

$$(b_{22}\tilde{y}_2 + b_{23}\tilde{y}_3)a_{21}b_{11} = c_{21}$$

$$b_{33}a_{31}b_{11}\,\tilde{y}_3 = c_{31}$$

$$(b_{11}\tilde{y}_1 + b_{12}\tilde{y}_2 + b_{13}\tilde{y}_3)(a_{11}b_{12} + a_{12}b_{22}) = c_{12}$$

$$(b_{22}\tilde{y}_2 + b_{23}\tilde{y}_3)(a_{21}b_{12} + a_{22}b_{22}) = c_{22}$$

$$b_{33}\tilde{y}_3(a_{31}b_{12} + a_{32}b_{22}) = c_{32}$$

$$(b_{11}\tilde{y}_1 + b_{12}\tilde{y}_2 + b_{13}\tilde{y}_3)(a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}) = c_{13}$$

$$(b_{22}\tilde{y}_2 + b_{23}\tilde{y}_3)(a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}) = c_{23}$$

$$b_{33}\tilde{y}_3 \; (a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}) = c_{33}$$

$\therefore$ (7.9b) may be written as

$$= \; [c_{ij}]\{\tilde{y}_j\} \qquad (7.10)$$

A solution to (7.10) may be found if $[c_{ij}]$ is an upper-triangular matrix. This may be so under two conditions:

1. b_{ij} are such that $b_{11} = 0$, and $(a_{31}b_{12} + a_{32}b_{22}) = 0$ for all a_{ij}.

2. If $[a_{ij}]$ is an upper-triangular matrix to start with, i.e., $a_{21} = a_{31} = a_{32} = 0$.

Condition (1) implies the following:

when b_{11} is 0, the first two equations of (7.10) are

$$-\left(b_{12}\frac{d\tilde{y}_2}{ds} + b_{13}\frac{d\tilde{y}_3}{ds}\right) = (b_{12}\tilde{y}_2 + b_{13}\tilde{y}_3)[(a_{11}b_{12} + a_{12}b_{22})\tilde{y}_2 + (a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33})\tilde{y}_3]$$

$$-\left(b_{22}\frac{d\tilde{y}_2}{ds} + b_{23}\frac{d\tilde{y}_3}{ds}\right) = (b_{22}\tilde{y}_2 + b_{23}\tilde{y}_3)[(a_{21}b_{12} + a_{22}b_{22})\tilde{y}_2 + + (a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{23})\tilde{y}_3] \qquad (7.11)$$

The third equation yields

$$-b_{33}\frac{d\tilde{y}_3}{ds} = b_{33}(a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33})(\tilde{y}_3)^2 \qquad (7.12)$$

Solution of (7.12) subject to initial condition

$$\tilde{y}_3 = \frac{e^{x^3}}{b_{33}} \quad \text{at} \quad s=0$$

is

$$\tilde{y}_3 = [b_{33}e^{-x^3} + (a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33})s]^{-1} \qquad (7.13)$$

Also, from (7.11),

$$\left.\begin{aligned} \ln\left[\frac{k_1}{b_{12}\tilde{y}_2 + b_{13}\tilde{y}_3}\right] &= \int[(a_{11}b_{12} + a_{12}b_{22})\tilde{y}_2 + \\ &\quad + (a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33})\tilde{y}_3]ds \\ \ln\left[\frac{k_2}{b_{22}\tilde{y}_2 + b_{23}\tilde{y}_3}\right] &= \int[(a_{21}b_{12} + a_{22}b_{22})\tilde{y}_2 + \\ &\quad + (a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33})\tilde{y}_3]ds \end{aligned}\right\} \qquad (7.14)$$

where k_1 and k_2 are constants to be calculated from the initial condition.

Since $\tilde{y}_3$ is determined from (7.13), (7.14) represents two equations in one unknown $\tilde{y}_2$. Therefore, a unique solution exists for y_2 only if one of the equations of (7.14) is a scalar multiple of the other.

If suitable b_{ij}'s can be found to satisfy the above conditions, y_i can be calculated from (7.9), and substitution in (7.7) will yield ξ^i.

Condition (2) calls for $[a_{ij}]$ to be an upper triangular matrix, so that

$$-\frac{d\xi^1}{ds} = a_{11}e^{\xi^1} + a_{12}e^{\xi^2} + a_{13}e^{\xi^3} \tag{7.15a}$$

$$-\frac{d\xi^2}{ds} = a_{22}e^{\xi^2} + a_{23}e^{\xi^3} \tag{7.15b}$$

$$-\frac{d\xi^3}{ds} = a_{33}e^{\xi^3} \tag{7.15c}$$

From (7.15c) and (7.4),

$$\xi^3 = \ln\left[\frac{e^{x^3}}{a_{33}e^{x^3}s + 1}\right] \tag{7.16}$$

Substitution of ξ^3 in (7.15b) leaves an equation in ξ^2 only which may be solved for ξ^2. Substituting ξ^2 and ξ^3 in (7.15a), an equation in ξ^1 only may be obtained.

It may be noted that for the case (7.6a), using (7.4)

$$\gamma_i = \ln\left[\frac{e^{q_{ik}x^k}}{d_{\underline{ii}}\ e^{q_{jk}x^k} s + 1}\right] \tag{7.17}$$

where q_{jk} are elements of the orthogonal tensor $\bar{\bar{Q}}^T$.

A Particular Solution

Consider a solution to (7.2) subject to a special condition that while the point R moves along its path, its velcoity components along three orthogonal curvilinear coordinate axes are equal to each other.

A solution to (7.3a-1) subject to (7.4) is

$$\left.\begin{aligned}
\xi^1 &= \ln \frac{e^{x^1}}{(a_{11}e^{x^1} + a_{12}e^{x^2} + a_{13}e^{x^3})s + 1} \\
\xi^2 &= \ln \frac{e^{x^2}}{(a_{11}e^{x^1} + a_{12}e^{x^2} + a_{13}e^{x^3})s + 1} \\
\xi^3 &= \ln \frac{e^{x^3}}{(a_{11}e^{x^1} + a_{12}e^{x^2} + a_{13}e^{x^3})s + 1}
\end{aligned}\right\} \tag{7.18}$$

Since $v^1 = v^2 = v^3 = v$

$$(a_{11}e^{x^1} + a_{12}e^{x^2} + a_{13}e^{x^3}) = (a_{21}e^{x^1} + a_{22}e^{x^2} + a_{23}e^{x^3})$$

$$= (a_{31}e^{x^1} + a_{32}e^{x^2} + a_{33}e^{x^3}) \qquad (7.19)$$

Substitution of condition (7.19) in (7.18) shows that (7.18) satisfies both (7.2a-2) and (7.2a-3).

Therefore, a general solution of (7.2) subject to (7.4) and (7.19) is

$$\xi^i = \ln \frac{e^{x^i}}{(sa_{ij}e^{x^i} + 1)} \qquad (7.20)$$

from equation (7.20),

$$[\bar{\bar{F}}_t(\tau)] = \frac{\partial \xi^i}{\partial x^j}$$

$$\therefore\ [\bar{\bar{F}}_t(\tau)] = [\bar{I}] - \frac{s}{sa_{ij}e^{x^j} + 1}\begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

where

$$l_{11} = a_{11}e^{x^1}, \qquad l_{12} = a_{12}e^{x^2}\left(\frac{g_{11}}{g_{22}}\right)^{\frac{1}{2}}$$

$$l_{13} = a_{13}e^{x^3}\left(\frac{g_{11}}{g_{33}}\right)^{\frac{1}{2}} \qquad l_{21} = a_{21}e^{x^1}\left(\frac{g_{22}}{g_{11}}\right)^{\frac{1}{2}}$$

$$l_{22} = a_{22}e^{x^2}, \qquad l_{23} = a_{23}e^{x^3}\left(\frac{g_{22}}{g_{33}}\right)^{\frac{1}{2}}$$

$$l_{31} = a_{31}e^{x^1}\left(\frac{g_{33}}{g_{11}}\right)^{\frac{1}{2}} \qquad l_{32} = a_{32}e^{x^2}\left(\frac{g_{32}}{g_{22}}\right)^{\frac{1}{2}}$$

$$l_{33} = a_{33}e^{x^3}, \tag{7.21}$$

$$[\bar{\bar{L}}_1] = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \tag{7.22a}$$

$$[\bar{\bar{L}}_2] = 2a_{ij}e^{x^j} \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \tag{7.22b}$$

$$= 2v\ [\bar{\bar{L}}_1] \tag{7.22c}$$

$$[\bar{\bar{L}}_3] = 6(v)^2\ [\bar{\bar{L}}_1] \tag{7.22d}$$

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$$[\bar{\bar{L}}_n] = n!(v)^{(n-1)}\ [\bar{\bar{L}}_1] \tag{7.22e}$$

Steady Extensional Motions

The flow represented by (7.6) is now considered in greater detail.

Let the velocity v^i at any point R in a body in motion be given by

$$v^i = a_{\underline{i}}e^{x^{\underline{i}}} - a_{\underline{i}} = a_{\underline{i}}[e^{x^{\underline{i}}} - 1] \tag{7.23}$$

Equation (7.23) ensures that the origin is stationary. a_i's are constants such that

$$a_i e^{x^i} = 0 \tag{7.23a}$$

so that $$\frac{\partial v^i}{\partial x^i} = 0 \tag{7.23b}$$

(7.23b) is the incompressibility criterion and (7.23a) is the condition of incompressibility.

From (7.23a) and (7.23),

$$\Sigma v^i = - \Sigma a_i \qquad (7.23c)$$

so that the sum of velocity components at any point in three orthogonal directions is a constant independent of the coordinates of the point.

The differential equation of motion, following (7.2b) is

$$- \frac{d\xi^i}{ds} = a_{\underline{i}} e^{\xi^{\underline{i}}} - a_i \qquad (7.24)$$

Subject to initial condition (7.4).

Substituting $\xi^i = \ln y_i$ in (7.24)

$$\frac{d\xi^i}{ds} = \frac{dy_{\underline{i}}}{ds} \frac{d}{dy_{\underline{i}}} (\ln y_{\underline{i}}) = \frac{dy_{\underline{i}}}{ds} \frac{1}{y_{\underline{i}}}$$

so that the differential equations of motion become

$$\frac{dy_i}{ds} - a_{\underline{i}} y_{\underline{i}} + a_{\underline{i}} (y_{\underline{i}})^2 = 0 \qquad (i = 1,2,3) \qquad (7.24a)$$

Substituting

$$y_{\underline{i}} = \frac{du_{\underline{i}}/ds}{a_{\underline{i}}u_{\underline{i}}} ,$$

(7.25) becomes

$$\frac{d^2u_i}{ds^2} - a_{\underline{i}} \frac{du_i}{ds} = 0 \qquad (7.24b)$$

A general solution of (7.24b) is of the form

$$u_i = c_1^{(i)} + c_2^{(i)} e^{a_i s}; \quad \text{(no sum with respect to i)}$$

so that

$$y_i = \frac{a_{\underline{i}} c_2^{(i)} e^{a_{\underline{i}} s}}{a_{\underline{i}}(c_1^{(i)} + c_2 e^{a_{\underline{i}} s})} = \frac{c_2^{(i)} e^{a_{\underline{i}} s}}{c_1^{(i)} + c_2^{(i)} e^{a_{\underline{i}} s}}$$

$$\therefore \quad \xi^i = \ln y_i = \ln \left(\frac{c_2^{(i)} e^{a_{\underline{i}} s}}{c_1^{(i)} + c_2^{(i)} e^{a_{\underline{i}} s}} \right) \qquad (7.25)$$

Applying (7.4) to (7.25),

$$x^i = \ln \frac{c_2^{(i)}}{c_1^{(i)} + c_2^{(i)}} ;$$

hence, substituting for $c_1^{(i)}$ in (7.25),

$$\xi^{i} = \ln \frac{e^{a_{\underline{i}} s}}{(e^{-x^{\underline{i}}} - 1 + e^{a_{\underline{i}} s})} \tag{7.25a}$$

Therefore,

$$[\bar{\bar{F}}_t(\tau)] = \left[\frac{\partial \xi^{i}}{\partial x^{j}}\right] = \left[1 - e^{x^{i}} + e^{(a_i s + x^{i})}\right]^{-1} \tag{7.26}$$

$$\therefore\ [\bar{\bar{C}}_t(\tau)] = [\bar{\bar{F}}_t(\tau) \cdot \bar{\bar{F}}_t^{T}(\tau)\ ] = \left[1 - e^{x^{i}} + e^{(a_i s + x^{i})}\right]^{-2} \tag{7.26a}$$

From (4.8),

$$[\bar{\bar{L}}_1] = [a_{\underline{i}} e^{x^{\underline{i}}}]$$

$$[\bar{\bar{A}}_1] = [2a_{\underline{i}} e^{x^{\underline{i}}}] \tag{7.26b}$$

$$[\bar{\bar{L}}_2] = [(a_{\underline{i}})^2\ e^{x^{\underline{i}}}\ (2e^{x^{\underline{i}}} - 1)] \tag{7.26c}$$

$$[\bar{\bar{L}}_3] = [(a_{\underline{i}})^3\ e^{x^{\underline{i}}}(6e^{x^{\underline{i}}} - 6e^{x^{\underline{i}}} + 1)] \tag{7.26d}$$

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$$[\bar{\bar{L}}_n] = [n!\,(\bar{\bar{L}}_1)^n + (n-1)!\ \sum_{i=1}^{n} (n-i)\,a_i\,(\bar{\bar{L}}_1)^{n-1}\ +$$

$$+ (n-2)! \sum_{i=2}^{n} (n-i) \sum_{j=0}^{n-i} (n-i-j)(a_i)^2 (\bar{\bar{L}}_1)^{n-2} + \cdots +$$

$$+ \sum_{i=1}^{n-1} (2)^{n-i} (a_i)^{n-2} (\bar{\bar{L}}_1)^2 + (a_i)^{n-1} \bar{\bar{L}}_1] \tag{7.26e}$$

From equation (4.29), the n Rivlin-Ericksen tensors can be calculated as power series in $\bar{\bar{L}}_1$. Since $\bar{\bar{A}}_1 = 2\bar{\bar{L}}_1$, a knowledge of $\bar{\bar{L}}_1$ is sufficient to determine the Rivlin-Ericksen tensors.

Calculation of Stress Tensor

It is noted that since $[\bar{\bar{F}}_t(\tau)]$ is a diagonal matrix, the velocity field describes a *Steady Extensional Motion* [15]. In this case, only incompressible fluids are dealt with. Since the n Rivlin-Ericksen tensors are also diagonal, from (2.12), it is inferred that the determinate stress matrix $[\bar{\bar{\sigma}}_d]$ will be in the diagonal form. Under these conditions, a representation for $\bar{\bar{\sigma}}$ is of the form [22, (29.15)].

$$\varepsilon\bar{\bar{\sigma}} + \varepsilon_1\bar{\bar{A}}_1 + \varepsilon_2(\bar{\bar{A}}_1)^2 + \varepsilon_3\bar{\bar{I}} = \bar{\bar{0}} \tag{7.27}$$

where

$$\varepsilon = k\,|\bar{\bar{\varepsilon}}|$$

for an arbitrary scalar multiplier k, and

$$[\bar{\bar{E}}] = \begin{bmatrix} \bar{\bar{I}}:(\bar{\bar{A}}_1)^2 & \bar{\bar{I}}:(\bar{\bar{A}}_1)^3 & \bar{\bar{I}}:\bar{\bar{A}}_1 \\ \bar{\bar{I}}:(\bar{\bar{A}}_1)^3 & \bar{\bar{I}}:(\bar{\bar{A}}_1)^4 & \bar{\bar{I}}:(\bar{\bar{A}}_1)^2 \\ \bar{\bar{I}}:\bar{\bar{A}}_1 & \bar{\bar{I}}:(\bar{\bar{A}}_1)^2 & \bar{\bar{I}}:\bar{\bar{I}} \end{bmatrix} \quad (7.27a)$$

$$= \begin{bmatrix} 4(a_i)^2 e^{2x^i} & 8(a_i)^3 e^{3x^i} & 0 \\ 8(a_i)^3 e^{3x^i} & 16(a_i)^4 e^{4x^i} & 4(a_i)^2 e^{2x^i} \\ 0 & 4(a_i)^2 e^{2x^i} & 3 \end{bmatrix}$$

and

$$\varepsilon_i = - k|\bar{\bar{E}}_i|$$

where k is the same arbitrary scalar multiplier, and $[\bar{\bar{E}}_i]$ are obtained by replacing the first, second and third colums of $[\bar{\bar{E}}]$ respectively by the column $\{(\bar{\bar{\sigma}}:\bar{\bar{A}}_1),(\bar{\bar{\sigma}}:\bar{\bar{A}}_1^2),(\bar{\bar{\sigma}}:\bar{\bar{I}})\}$
Let (7.27) be written in the form

$$\bar{\bar{\sigma}} + p\bar{\bar{I}} = \alpha_1\bar{\bar{A}}_1 + \alpha_2(\bar{\bar{A}}_1)^2 \quad (7.28)$$

Substituting for $\bar{\bar{A}}_1$ and $\bar{\bar{A}}_1^{\,2}$ from (7.26b)

$$
\left.\begin{aligned}
(\sigma_{11} - \sigma_{22}) &= 2\alpha_1(a_1 e^{x^1} - a_2 e^{x^2}) + 2\alpha_2[3(a_1)^2 e^{2x^1} \\
&\quad - (a_1)^2 e^{x^1} - 3(a_2)^2 e^{2x^2} + (a_2)^2 e^{x^2}] \\
(\sigma_{22} - \sigma_{33}) &= 2\alpha_1(a_2 e^{x^2} - a_3 e^{x^3}) + 2\alpha_2[3(a_2)^2 e^{2x^2} \\
&\quad - (a_2)^2 e^{x^2} - 3(a_3)^2 e^{2x^3} + (a_3)^2 e^{x^3}]
\end{aligned}\right\} \qquad (7.29)
$$

Cauchy's Equations of Motion [15] are given by

$$\sum_j \sigma_{ij,j} - \rho\psi,_i = \rho\dot{v}_i \qquad (7.30)$$

where

$$\dot{v}_i = \frac{\partial v_i}{\partial t} + \sum_j v_{i,j}\, v_j \qquad (7.30a)$$

Therefore,

$$
\left.\begin{aligned}
(\sigma_{11} - \rho\psi),_1 &= \rho(a_1)^2\, e^{2x^1} \\
(\sigma_{22} - \rho\psi),_2 &= \rho(a_2)^2\, e^{2x^2} \\
(\sigma_{33} - \rho\psi),_3 &= \rho(a_3)^2\, e^{2x^3}
\end{aligned}\right\} \qquad (7.31)
$$

where ψ which is a single valued potential of body force, is regarded as specified in advance.

Equations (7.29) and (7.31) constitute five equations in three unknowns σ_{11}, σ_{22} and σ_{33}. In order that the Steady Extensional Motion be dynamically possible in a general incompressible fluid of Differential Type, it is necessary and sufficient that (7.29) and (7.31) have a solution.

A solution of the form

$$\left.\begin{aligned} \sigma_{\underline{ii}} &= \rho\psi + 2\alpha_1 a_{\underline{i}} e^{x^{\underline{i}}} + 2\alpha_2 (a_{\underline{i}})^2 (3e^{2x^{\underline{i}}} - e^{x^{\underline{i}}}) + f(t) \\ &= \rho a_{\underline{i}} e^{x^{\underline{i}}} + \rho\psi - 6\alpha_2 (a_{\underline{i}})^2 e^{2x^{\underline{i}}} + f(t) \;;\; i = 1,2,3 \end{aligned}\right\} \quad (7.32)$$

where f(t) is a function of time alone, is possible if α_1 and α_2, as calculated from (7.27) and (7.27a), satisfy a further condition

$$\alpha_1 + \alpha_2 a_{\underline{i}} (6e^{x^{\underline{i}}} - 1) \;=\; \frac{\rho}{2}\, a_{\underline{i}} e^{x^{\underline{i}}} \quad (7.33)$$

NB! According to (7.27), (7.27a) and (7.28),

$$|\bar{\bar{E}}| = \bar{\bar{I}}:(\bar{\bar{A}}_1)^2\{3\ \bar{\bar{I}}:(\bar{\bar{A}}_1)^4 - (\bar{\bar{I}}:(\bar{\bar{A}}_1)^2)^2\} - 3(\bar{\bar{I}}:(\bar{\bar{A}}_1)^3)^2$$

$$|\bar{\bar{E}}_1| = \bar{\bar{\sigma}}:\bar{\bar{A}}_1\{3\ \bar{\bar{I}}:(\bar{\bar{A}}_1)^4 - (\bar{\bar{I}}:(\bar{\bar{A}}_1)^2)^2\} - \bar{\bar{I}}:(\bar{\bar{A}}_1)^3\{3\bar{\bar{\sigma}}:(\bar{\bar{A}}_1)^2$$

$$- (\bar{\bar{I}}:(\bar{\bar{A}}_1)^2)(\bar{\bar{\sigma}}:\bar{\bar{I}})\}$$

$$|\bar{\bar{E}}_2| = \bar{I}:(\bar{\bar{A}}_1)^2\{3\bar{\bar{\sigma}}:(\bar{\bar{A}}_1)^2 - (\bar{I}:(\bar{\bar{A}}_1)^2)(\bar{\bar{\sigma}}:\bar{I})\} - 3(\bar{I}:(\bar{\bar{A}}_1)^3)(\bar{\bar{\sigma}}:\bar{I})$$

$$\alpha_1 = -\frac{|\bar{\bar{E}}_1|}{|\bar{\bar{E}}|} ; \qquad \alpha_2 = -\frac{|\bar{\bar{E}}_2|}{|\bar{\bar{E}}|} \tag{7.33a}$$

Also, from (7.33),

$$\frac{|\bar{\bar{E}}_1|}{|\bar{\bar{E}}|} + \frac{|\bar{\bar{E}}_2|}{|\bar{\bar{E}}|} a_{\underline{i}}(6e^{x^{\underline{i}}} - 1) = -\frac{\rho}{2} a_{\underline{i}} e^{x^{\underline{i}}} \tag{7.33b}$$

Equation (7.33b) provides auxiliary conditions to be satisfied by the scalar invariants of stress tensor and the first Rivlin-Ericksen tensor $\bar{\bar{A}}_1$ and its powers, in the form

$$|\bar{\bar{E}}_1| + |\bar{\bar{E}}_2| a_{\underline{i}}(6e^{x^{\underline{i}}} - 1) = -|\bar{\bar{E}}| \frac{\rho}{2} a_{\underline{i}} e^{x^{\underline{i}}} \tag{7.33c}$$

In the <u>second Steady Extensional Motion</u> to be considered, the velocity field is given by

$$v^i = a_{\underline{i}} e^{x^{\underline{i}}} \tag{7.34}$$

Differential equation of motion is

$$-\frac{d\xi^i}{ds} = a_{\underline{i}} e^{\xi^{\underline{i}}} \tag{7.34a}$$

subject to initial condition (7.4).

Following the procedure adopted in the previous example,

$$\xi^i = \{x^i - \ln(1 + a_{\underline{i}} s e^{x^{\underline{i}}})\} \tag{7.35}$$

$$\therefore \quad [\bar{\bar{F}}_t(\tau)] = [1 + a_{\underline{i}} s e^{x^{\underline{i}}}]^{-1}, \qquad i = 1,2,3 \tag{7.35a}$$

$$[\bar{\bar{C}}_t(\tau)] = [1 + a_{\underline{i}} s e^{x^{\underline{i}}}]^{-2} \tag{7.35b}$$

$$[\bar{\bar{L}}_1] = [a_{\underline{i}} e^{x^{\underline{i}}}] \tag{7.35c}$$

$$[\bar{\bar{L}}_2] = [2!(a_{\underline{i}})^2 e^{2x^{\underline{i}}}]$$

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$$[\bar{\bar{L}}_n] = [n!(a_{\underline{i}})^n e^{nx^{\underline{i}}}] \tag{7.35d}$$

$[\bar{\bar{A}}_n]$ can be calculated from (4.29)

$$[\bar{\bar{A}}_1] = [2a_{\underline{i}} e^{x^{\underline{i}}}] = 2[\bar{\bar{L}}_1]$$

$$[\bar{\bar{A}}_2] = [6(a_{\underline{i}})^2 e^{2x^{\underline{i}}}] = 6[\bar{\bar{L}}_1]^2 \tag{7.35e}$$

$$- - - - - - - - - - - - - - - - -$$

Stress tensor as a function of Rivlin-Ericksen tensors $\bar{\bar{A}}_1$ and $\bar{\bar{A}}_2$ will be in the diagonal form since $[\bar{\bar{A}}_1]$ and $[\bar{\bar{A}}_2]$ are diagonal. Therefore, the representation theorem (7.27) will hold.

The constitutive relation yields

$$\left.\begin{aligned} \sigma_{11} - \sigma_{22} &= 2\alpha_1(a_1 e^{x^1} - a_2 e^{x^2}) + 6\alpha_2((a_1)^2 e^{2x^1} - (a_2)^2 e^{2x^2}) \\ \sigma_{22} - \sigma_{33} &= 2\alpha_1(a_2 e^{x^2} - a_3 e^{x^3}) + 6\alpha_2((a_2)^2 e^{2x^2} - (a_3)^2 e^{2x^3} \end{aligned}\right\} \quad (7.36a)$$

From (7.34), (7.30a) and (7.30), the equations of motion yield

$$\begin{aligned} (\sigma_{11} - \rho\psi)_{,1} &= \rho(a_1)^2\, e^{2x^1} \\ (\sigma_{22} - \rho\psi)_{,2} &= \rho(a_2)^2\, e^{2x^2} \\ (\sigma_{33} - \rho\psi)_{,3} &= \rho(a_3)^2\, e^{2x^3} \end{aligned} \quad (7.36b)$$

(7.36a) and (7.36b) represent five equations in the three unknowns $\sigma_{\underline{i}\underline{i}}$.

A solution to (7.36a) and (7.36b) is

$$\sigma_{\underline{i}\underline{i}} = \rho\psi + 2\alpha_1\, a_{\underline{i}} e^{x^{\underline{i}}} + 6\alpha_2(a_{\underline{i}})^2\, e^{2x^{\underline{i}}} + f(t) \quad (7.36c)$$

where f(t) is a known function of time alone, and

$$\frac{\alpha_1}{a_{\underline{i}} e^{x^{\underline{i}}}} + 6\alpha_2 = \frac{\rho}{2} = \text{constant} \tag{7.36d}$$

ρ being the density of the material.

NB! The incompressibility condition for the above example is given as sum of the three components of the velocity at any point R in an orthogonal co-ordinate system is zero, or,

$$\frac{\partial v^i}{\partial x^i} = a_i e^{x^i} = 0$$

Another example of <u>Steady Extensional Motion</u> in incompressible Materials of Differential Type has the velocity field

$$v^i = 2a_{\underline{i}} (x^{\underline{i}})^{1/2} \tag{7.37}$$

Let a_i be such that

$$\frac{a_1}{x^1} + \frac{a_2}{x^2} + \frac{a_3}{x^3} = 0 \tag{7.37a}$$

$$\therefore \quad \frac{\partial v^i}{\partial x^i} = 0 \rightarrow \text{the flow is incompressible} \tag{7.37b}$$

Differential equation of motion is

$$-\frac{d\xi^i}{ds} = 2a_{\underline{i}}(\xi^{\underline{i}})^{1/2} \qquad (7.37c)$$

Subject to initial condition (7.4).

The solution of (7.37c) is

$$\xi^i = (\sqrt{x^i} - a_i s)^2 \qquad (7.38)$$

$$[\bar{\bar{F}}_t(\tau)] = [2(\sqrt{x^{\underline{i}}} - a_{\underline{i}}s)\frac{1}{2}(x^{\underline{i}})^{-1/2}] = [1 - \frac{a_{\underline{i}}s}{\sqrt{x^{\underline{i}}}}], \quad (i = 1,2,3) \qquad (7.38a)$$

$$[\bar{\bar{C}}_t(\tau)] = [1 - \frac{a_{\underline{i}}s}{\sqrt{x^{\underline{i}}}}]^2 \qquad (7.38b)$$

$$[\bar{\bar{L}}_1] = [\frac{a_{\underline{i}}}{\sqrt{x^{\underline{i}}}}] \qquad (7.38c)$$

$$[\bar{\bar{A}}_1] = [\frac{2a_{\underline{i}}}{\sqrt{x^{\underline{i}}}}] \qquad (7.38d)$$

$$[\bar{\bar{A}}_2] = [\frac{2(a_{\underline{i}})^2}{x^{\underline{i}}}] \qquad (7.38e)$$

All other Rivlin-Ericksen tensors, and $\bar{\bar{L}}_2$ etc. are $\bar{\bar{0}}$.

From (7.38b-e),

$$[\bar{\bar{C}}_t(\tau)] = [\bar{\bar{I}}] - s[\bar{\bar{L}}_1 + \bar{\bar{L}}_1^T] + (s)^2 [\bar{\bar{L}}_1 \cdot \bar{\bar{L}}_1^T]$$

$$= [\bar{\bar{I}}] - s[\bar{\bar{A}}_1] + \frac{(s)^2}{2} [\bar{\bar{A}}_2] \qquad (7.38f)$$

Therefore, from[3;(16)],it would appear (7.37) represents a Viscometric Flow. But from (7.38c), this is not so since $[\bar{\bar{L}}_1]^2 \neq [\bar{\bar{0}}]$.

The flow represented by (7.37) is a special flow in which

$$[\bar{\bar{L}}_n] = [\bar{\bar{0}}] \qquad \text{for} \quad n > 1, \qquad (7.38g)$$

but $\quad [\bar{\bar{L}}_1]^n \neq [\bar{\bar{0}}] \qquad \text{for} \quad n = 1,2,....$

For calculation of the stress tensor, constitutive relation (7.28) is valid.

From (7.28) and (7.38d),

$$\left.\begin{aligned} (\sigma_{11} - \sigma_{22}) &= 2\alpha_1\left(\frac{a_1}{\sqrt{x^1}} - \frac{a_2}{\sqrt{x^2}}\right) + 4\alpha_2\left(\frac{(a_1)^2}{x^1} - \frac{(a_2)^2}{x^2}\right) \\ (\sigma_{22} - \sigma_{33}) &= 2\alpha_2\left(\frac{a_2}{\sqrt{x^2}} - \frac{a_3}{\sqrt{x^3}}\right) + 4\alpha_2\left(\frac{(a_2)^2}{x^2} - \frac{(a_3)^2}{x^3}\right) \end{aligned}\right\} \qquad (7.39a)$$

From equations of motion (7.30), and (7.30a),

$$\left.\begin{aligned}(\sigma_{11} - \rho\psi)_{,1} &= 2\rho(a_1)^2\\(\sigma_{22} - \rho\psi)_{,2} &= 2\rho(a_2)^2\\(\sigma_{33} - \rho\psi)_{,3} &= 2\rho(a_3)^2\end{aligned}\right\} \tag{7.39b}$$

A solution to (7.39a) and (7.39b) is given by

$$\sigma_{\underline{i}\underline{i}} = \rho\psi + 2\rho((a_{\underline{i}})^2 x^{\underline{i}}) + 2\alpha_1 \frac{a_{\underline{i}}}{\sqrt{x^{\underline{i}}}} + 4\alpha_2 \frac{(a_{\underline{i}})^2}{\sqrt{x^{\underline{i}}}} + f(t) \tag{7.40}$$

where $f(t)$ is a known function of time alone, and α_1 and α_2 are given by

$$\left.\begin{aligned}\alpha_1 &= \beta_1(x^1x^2x^3)^{1/2}\\\alpha_2 &= \beta_2(x^1x^2x^3)\end{aligned}\right\} \tag{7.40a}$$

where β_1 and β_2 are determined from (7.27a), (7.33a) and (7.40a).

<u>The fourth Steady Extension</u> to be considered has its velocity field as

$$v^i = a_{\underline{i}}(x^{\underline{i}})^2 \tag{7.41}$$

Incompressibility condition is given by

$$2a_{\underline{i}}x^{i} = 0 \tag{7.41a}$$

Differential equation of motion is

$$-\frac{d\xi^{i}}{ds} = a_{\underline{i}}(\xi^{\underline{i}})^{2} \tag{7.41b}$$

subject to initial condition (7.4).
Solution of (7.41b) is

$$\xi^{i} = \{(x^{i})^{-1} + a_{\underline{i}}s\}^{-1} \tag{7.42}$$

So that

$$[\bar{\bar{F}}_{t}(\tau)] = [1 + sa_{\underline{i}}x^{\underline{i}}]^{-2} \qquad (i = 1,2,3) \tag{7.42a}$$

$$[\bar{\bar{C}}_{t}(\tau)] = [1 + sa_{\underline{i}}x^{\underline{i}}]^{-4} \tag{7.42b}$$

$$[\bar{\bar{L}}_{1}] = [2a_{\underline{i}}x^{\underline{i}}] \tag{7.42c}$$

— — — — — — — — — — — — —

$$[\bar{\bar{L}}_{n}] = (n+1)![(a_{\underline{i}})^{n}(x^{\underline{i}})^{n}] \tag{7.42d}$$

$$[\bar{\bar{A}}_{1}] = 4[a_{\underline{i}}x^{\underline{i}}] \tag{7.42e}$$

— — — — — — — — — — — — — —

$$[\bar{\bar{A}}_n] = \frac{(n+3)!}{3!} [(a_{\underline{i}})^n (x^{\underline{i}})^n] \tag{7.42f}$$

The five equations to be solved to determine the components σ_{11}, σ_{22} and σ_{33} of the diagonal stress tensor $\bar{\bar{\sigma}}$ are

$$\left.\begin{aligned}(\sigma_{11} - \sigma_{22}) &= 4\alpha_1(a_1x^1 - a_2x^2) + 20\alpha_2\{(a_1)^2(x^1)^2 - (a_2)^2(x^2)^2\} \\ (\sigma_{22} - \sigma_{33}) &= 4\alpha_2(a_2x^2 - a_3x^3) + 20\alpha_2\{(a_2)^2(x^2)^2 - (a_3)^2(x^3)^2\}\end{aligned}\right\} \tag{7.43a}$$

$$\left.\begin{aligned}(\sigma_{11} - \rho\psi)_{,1} &= \rho(a_1)^2(x^1)^3 \\ (\sigma_{22} - \rho\psi)_{,2} &= \rho(a_2)^2(x^2)^3 \\ (\sigma_{33} - \rho\psi)_{,3} &= \rho(a_3)^2(x^3)^3\end{aligned}\right\} \tag{7.43b}$$

A solution to (7.43a) and (7.43b) is

$$\sigma_{\underline{ii}} = \rho\psi + \frac{\rho}{4} (a_i)^2(x^i)^4 + 4\alpha_1 a_{\underline{i}} x^{\underline{i}} + 20\alpha_2 (a_{\underline{i}})^2 (x^{\underline{i}})^2 + f(t) \tag{7.44}$$

where $f(t)$ is a function of t only, and α_1 and α_2, in addition to satisfying (7.33a), are of the form

$$\left.\begin{aligned} \alpha_1 &= k_1(x^1x^2x^3)^{-1} \\ \alpha_2 &= k_2(x^1x^2x^3)^{-2} \end{aligned}\right\} \qquad (7.44a)$$

where k_1 and k_2 are constants.

<u>A Particular Case</u> of the velocity field (7.2) is now considered where the velocity components at a point R are linear functions of the coordinates of R.

The velocity field is described as

$$\left.\begin{aligned} v^1 &= a_1x^1 + b_1x^2 + c_1x^3 + d_1 \\ v^2 &= a_2x^1 + b_2x^2 + c_2x^3 + d_2 \\ v^3 &= a_3x^1 + b_3x^2 + c_3x^3 + d_3 \end{aligned}\right\} \qquad (7.45)$$

where a_i, b_i, c_i and d_i are constants.

The differential equation of motion is given by

$$-\frac{d\xi^1}{ds} = a_1\xi^1 + b_1\xi^2 + c_1\xi^3 + d_1$$

$$-\frac{d\xi^2}{ds} = a_2\xi^1 + b_2\xi^2 + c_2\xi^3 + d_2$$

$$-\frac{d\xi^3}{ds} = a_3\xi^1 + b_3\xi^2 + c_3\xi^3 + d_3 \qquad (7.46)$$

Subject to the initial condition (7.4).

From (7.46),

$$-ds = \frac{d\xi^1}{a_1\xi^1 + b_1\xi^2 + c_1\xi^3 + d_1} = \frac{d\xi^2}{a_2\xi^1 + b_2\xi^2 + c_2\xi^3 + d_2}$$

$$= \frac{d\xi^3}{a_3\xi^1 + b_3\xi^2 + c_3\xi^3 + d_3} \qquad (7.47)$$

Let

$$\left.\begin{aligned} a_1\xi^1 + b_1\xi^2 + c_1\xi^3 + d_1 &= \alpha_1 \\ a_2\xi^1 + b_2\xi^2 + c_2\xi^3 + d_2 &= \alpha_2 \\ a_3\xi^1 + b_3\xi^2 + c_3\xi^3 + d_3 &= \alpha_3 \end{aligned}\right\} \qquad (7.47a)$$

From (7.47) and (7.47a)

$$\frac{d\xi^1}{\alpha_1} = \frac{d\xi^2}{\alpha_2} = \frac{d\xi^3}{\alpha_3} = -\,ds \qquad (7.47b)$$

$$\therefore \quad \left.\begin{aligned} d\xi^1 &= -\,\alpha_1 ds \\ d\xi^2 &= -\,\alpha_2 ds \\ d\xi^3 &= -\,\alpha_3 ds \end{aligned}\right\} \qquad (7.47c)$$

Let l_1, l_2 and l_3 be some arbitrary constants. Multiplying the three equations of (7.47c) by l_1, l_2 and l_3 respectively and adding,

$$-\,ds = \frac{l_1 d\xi^1 + l_2 d\xi^2 + l_3 d\xi^3}{l_1\alpha_1 + l_2\alpha_2 + l_3\alpha_3} \qquad (7.47d)$$

let l_1, l_2 and l_3 be so chosen that

$$\left.\begin{aligned} l_1a_1 + l_2a_2 + l_3a_3 &= l_1\beta \\ l_1b_1 + l_2b_2 + l_3b_3 &= l_2\beta \\ l_1c_1 + l_2c_2 + l_3c_3 &= l_3\beta \end{aligned}\right\} \qquad (7.48)$$

Substitution of (7.47a) and (7.48) in (7.47d) leads to

$$- ds = \frac{d(l_1\xi^1+l_2\xi^2+l_3\xi^3)}{l_1(a_1\xi^1+b_1\xi^2+c_1\xi^3+d_1)+ l_2(a_2\xi^1+b_2\xi^2+c_2\xi^3+d_2) + +l_3(a_3\xi_1+b_3\xi_2+c_3\xi_3+d_3)}$$

$$= \frac{d(l_1\xi^1+l_2\xi^2+l_3\xi^3)}{\xi^1(l_1a_1+l_2a_2+l_3a_3)+ \xi^2(l_1b_1+l_2b_2+l_3b_3)+ \xi^3(l_1c_1+l_2c_2+l_3c_3)+ + l_1d_1+l_2d_2+l_3d_3}$$

$$= \frac{d(l_1\xi^1 + l_2\xi^2 + l_3\xi^3)}{\beta(l_1\xi^1 + l_2\xi^2 + l_3\xi^3 + r)} \qquad (7.49)$$

where $\beta r = l_1d_1 + l_2d_2 + l_3d_3$

Since (7.48) may be written as

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} l_1 \\ l_2 \\ l_3 \end{Bmatrix} = \begin{bmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix} \begin{Bmatrix} l_1 \\ l_2 \\ l_3 \end{Bmatrix} \qquad (7.50)$$

the choice of l_1, l_2 and l_3 is possible if β is a root of the equation

$$\begin{vmatrix} (a_1 - \beta) & a_2 & a_3 \\ b_1 & (b_2 - \beta) & b_3 \\ c_1 & c_2 & (c_3 - \beta) \end{vmatrix} = 0 \qquad (7.50a)$$

Let the roots of the equation (7.50a), supposed distinct, be $\frac{1}{\lambda_1}$, $\frac{1}{\lambda_2}$, $\frac{1}{\lambda_3}$, and let the corresponding values of l_i and r be

$$l_1^i,\ l_2^i,\ l_3^i,\ r^i \qquad (i = 1,2,3) \qquad (7.50b)$$

Then,

$$-\,ds = \frac{\lambda_i\ d(l_1^i\xi^1 + l_2^i\xi^2 + l_3^i\xi^3)}{(l_1^i\xi^1 + l_2^i\xi^2 + l_3^i\xi^3 + r^i)} \qquad (i = 1,2,3) \qquad (7.51)$$

Integrating both sides of (7.51),

$$-s = \lambda_{\underline{i}} \ln(l_1^{\underline{i}}\xi^1 + l_2^{\underline{i}}\xi^2 + l_3^{\underline{i}}\xi^3 + r^{\underline{i}}) + c_i, \quad (i = 1,2,3) \tag{7.51a}$$

Applying (7.4) to (7.51a),

$$(l_1^i\xi^1 + l_2^i\xi^2 + l_3^i\xi^3 + r^i) = (l_1^{\underline{i}}x^1 + l_2^{\underline{i}}x^2 + l_3^{\underline{i}}x^3 + r^{\underline{i}})e^{-s/\lambda_{\underline{i}}},$$

$$(i = 1,2,3) \tag{7.51b}$$

The three equations of (7.51b) may be solved to determine ξ^i in terms of x^i.

It would then be possible to calculate $[\bar{\bar{F}}_t(\tau)]$, $[\bar{\bar{A}}_1]$, $[\bar{\bar{A}}_2]$ etc., and check the controllability of motion.

<u>A Special Case</u> of (7.45) is

$$\left.\begin{aligned} v^1 &= ax^2 + bx^3 \\ v^2 &= cx^1 + dx^3 \\ v^3 &= ex^1 + fx^2 \end{aligned}\right\} \tag{7.52}$$

where a, ..., f are constants.

The differential equation of motion is

$$\left.\begin{aligned} -\frac{d\xi^1}{ds} &= a\xi^2 + b\xi^3 \\ -\frac{d\xi^2}{ds} &= c\xi^1 + d\xi^3 \\ -\frac{d\xi^3}{ds} &= e\xi^1 + f\xi^2 \end{aligned}\right\} \qquad (7.53)$$

subject to initial condition (7.4).

From (7.53-1),

$$-\frac{d^2\xi^1}{ds^2} = a\,\frac{d\xi^2}{ds} + b\,\frac{d\xi^3}{ds}$$

$$= a(-\,c\xi^1 - d\xi^3) + b(-\,e\xi^1 - f\xi^2)$$

$$= -\,(ac + be)\xi^1 - (ad\xi^3 + bf\xi^2)$$

Differentiating again with respect to s,

$$-\frac{d^3\xi^1}{ds^3} = -\,(ac + be)\,\frac{d\xi^1}{ds} - ad\,\frac{d\xi^3}{ds} - bf\,\frac{d\xi^2}{ds}$$

$$= - (ac + be)\frac{d\xi^1}{ds} + ad(e\xi^1 + f\xi^2) + bf(c\xi^1 + d\xi^3)$$

$$\therefore \frac{d^3\xi^1}{ds^3} - (ac + be + fd)\frac{d\xi^1}{ds} + (ade + bfc)\xi^1 = 0 \qquad (7.53a)$$

Similar third order equations may be constructed for ξ^2 and ξ^3. A solution for (7.53a) is of the form

$$\xi^1 = C_1e^{m_1s} + C_2e^{m_2s} + C_3e^{m_3s} \qquad (7.53b)$$

where m_1, m_2 and m_3 are the roots of the cubic equation

$$m^3 - (ac + be + fd)m + (ade + bfc) = 0$$

and from (7.4) and (7.53b); (7.53c)

$$x^1 = C_1 + C_2 + C_3$$

It would now be possible to calculate $\bar{\bar{A}}_{n'}(n = 1,2,\ldots)$ and check for controllability of motion.

CHAPTER III

MOTIONS WITH CONSTANT STRETCH HISTORY

8. Preliminaries

Consider any four positions $\bar{r}_1(t_1)$, $\bar{r}_2(t_2)$, $\bar{r}_3(t_3)$ and $\bar{r}_4(t_4)$ of a point R in a deforming continuous media, such that the positive time intervals $(t_2 - t_1) = (t_4 - t_3) = s$. The motion of point R is one with a *Constant Stretch History* if the strain tensors $\bar{\bar{C}}_{t_2}(t_1)$ and $\bar{\bar{C}}_{t_4}(t_3)$ are related by an equation of the form

$$\bar{\bar{C}}_{t_4}(t_3) = \bar{\bar{P}}^T(t)\cdot\bar{\bar{C}}_{t_2}(t_1)\cdot\bar{\bar{P}}(t)$$

or,

$$\bar{\bar{C}}_{t_4}^{t_4}(s) = \bar{\bar{P}}^T(t_4)\cdot\bar{\bar{C}}_{t_2}^{t_2}(s)\cdot\bar{\bar{P}}(t_4) \qquad (8.1)$$

where $\bar{\bar{P}}(t_4)$ is an orthogonal tensor.

Equation (8.1) means for an observer moving with the material point R, magnitudes of the principal stretches and changes of direction of principal axes of strain are functions of time lapse s alone and not of present time t.

Due to NOLL [16], deformation gradient has the representation

$$\bar{\bar{F}}_0(\tau) = e^{\tau\bar{\bar{M}}}\cdot\bar{\bar{Q}}(\tau); \quad \bar{\bar{Q}}(0) = \bar{\bar{I}} \tag{8.2}$$

where $\bar{\bar{M}}$ is a constant tensor and $\bar{\bar{Q}}(\tau)$ is an orthogonal tensor function.

Let

$$\bar{\bar{M}}_t = \bar{\bar{Q}}^T(t)\cdot\bar{\bar{M}}\cdot\bar{\bar{Q}}(t)$$

and

$$\bar{\bar{Z}}(t) = \bar{\bar{Q}}^T(t)\cdot\dot{\bar{\bar{Q}}}(t) \tag{8.2a}$$

so that

$$\bar{\bar{L}}_1(t) = \bar{\bar{M}}_t + \bar{\bar{Z}}(t) \tag{8.2b}$$

$$\begin{aligned}\bar{\bar{F}}_t(\tau) = (\bar{\bar{F}}(t))^{-1}\cdot\bar{\bar{F}}(\tau) &= (e^{t\bar{\bar{M}}}\cdot\bar{\bar{Q}}(t))^{-1}\cdot e^{\tau\bar{\bar{M}}}\cdot\bar{\bar{Q}}(\tau)\\ &= \bar{\bar{Q}}^T(t)\cdot e^{-t\bar{\bar{M}}}\cdot e^{\tau\bar{\bar{M}}}\cdot\bar{\bar{Q}}(\tau)\\ &= \bar{\bar{Q}}^T(t)\cdot e^{(\tau-t)\bar{\bar{M}}}\cdot\bar{\bar{Q}}(\tau)\end{aligned} \tag{8.3}$$

$$\bar{\bar{C}}_t(\tau) = \bar{\bar{F}}_t(\tau)\cdot\bar{\bar{F}}_t^T(\tau) = \bar{\bar{Q}}^T(t)\cdot e^{(\tau-t)\bar{\bar{M}}}\cdot\bar{\bar{Q}}(\tau)\cdot\bar{\bar{Q}}^T(\tau)\ e^{(\tau-t)\bar{\bar{M}}^T}\cdot\bar{\bar{Q}}(t)$$

$$= \bar{\bar{Q}}^T(t)\cdot e^{-s\bar{\bar{M}}}\cdot e^{-s\bar{\bar{M}}^T}\cdot\bar{\bar{Q}}(t) \tag{8.4}$$

$$= \bar{\bar{Q}}^T(t)\cdot e^{-s\bar{\bar{M}}}\cdot\bar{\bar{Q}}(t)\cdot\bar{\bar{Q}}^T(t)\cdot e^{-s\bar{\bar{M}}^T}\cdot\bar{\bar{Q}}(t)$$

$$= e^{-s\bar{\bar{Q}}^T(t)\cdot\bar{\bar{M}}\cdot\bar{\bar{Q}}(t)}\cdot e^{-s\bar{\bar{Q}}^T(t)\cdot\bar{\bar{M}}\cdot\bar{\bar{Q}}(t)}$$

$$= e^{-s\bar{\bar{M}}_t}\cdot e^{-s\bar{\bar{M}}_t{}^T} \tag{8.4a}$$

From (I.10).

$$\bar{\bar{D}}(t) = \frac{1}{2}(\bar{\bar{M}}_t + \bar{\bar{M}}_t^T)$$

$$\bar{\bar{W}}(t) = \frac{1}{2}(\bar{\bar{M}}_t - \bar{\bar{M}}_t^T) + \bar{\bar{Z}}(t) \tag{8.5}$$

$\bar{\bar{Z}}(t)$ is a measure of rotation of the basis $\bar{e}_k(t)$ relative to which $\bar{\bar{C}}_t(\tau)$ has components independent of time t.
Let the tensor defining the natural logarithm of $\bar{\bar{M}}$ be $\bar{\bar{N}}$.

$$\therefore \quad \bar{\bar{N}} = \ln\bar{\bar{M}} \tag{8.6a}$$

$$\therefore \quad \bar{\bar{Q}}^T\cdot\bar{\bar{M}}\cdot\bar{\bar{Q}} = \bar{\bar{Q}}^T\cdot e^{\bar{\bar{N}}}\cdot\bar{\bar{Q}}$$

$$= e^{\bar{\bar{Q}}^T\cdot\bar{\bar{N}}\cdot\bar{\bar{Q}}} \tag{8.6b}$$

Taking natural logarithm of both sides,

$$\ln(\bar{\bar{Q}}^T \cdot \bar{\bar{M}} \cdot \bar{\bar{Q}}) = \bar{\bar{Q}}^T \cdot \bar{\bar{N}} \cdot \bar{\bar{Q}} = \bar{\bar{Q}}^T \cdot \ln\bar{\bar{M}} \cdot \bar{\bar{Q}} \tag{8.7}$$

In deriving equation (8.7) use was made of

$$\bar{\bar{Q}}^T \cdot e^{\bar{\bar{M}}} \cdot \bar{\bar{Q}} = e^{\bar{\bar{Q}}^T \cdot \bar{\bar{M}} \cdot \bar{\bar{Q}}}.$$

From (8.2),

$$\bar{\bar{F}}_0(t) = e^{t\bar{\bar{M}}} \cdot \bar{\bar{Q}}(t)$$

or,

$$\bar{\bar{Q}}^T(t) \cdot \bar{\bar{F}}_0(t) = \bar{\bar{Q}}^T(t) \cdot e^{t\bar{\bar{M}}} \cdot \bar{\bar{Q}}(t) = e^{t\bar{\bar{M}}_t}$$

$$\therefore \ \text{In } (\bar{\bar{Q}}^T(t) \cdot \bar{\bar{F}}_0(t)) = t\bar{\bar{M}}_t = t\bar{\bar{Q}}^T(t) \cdot \bar{\bar{M}} \cdot \bar{\bar{Q}}(t)$$

$$\therefore \ \bar{\bar{M}} = \frac{1}{t}\, \bar{\bar{Q}}(t) \cdot \ln(\bar{\bar{Q}}^T(t) \cdot \bar{\bar{F}}_0(t)) \cdot \bar{\bar{Q}}^T(t)$$

$$= \frac{1}{t} \ln(\bar{\bar{Q}}(t) \cdot \bar{\bar{Q}}^T(t) \cdot \bar{\bar{F}}_0(t) \cdot \bar{\bar{Q}}^T(t))$$

$$= \frac{1}{t} \ln(\bar{\bar{F}}_0(t) \cdot \bar{\bar{Q}}^T(t)) \tag{8.8}$$

also, from NOLL [16, (2.15)],

$$\bar{\bar{M}} = \frac{d}{dt} (\bar{\bar{F}}_0(t) \cdot \bar{\bar{Q}}(t))\Big|_{t=0} = \dot{\bar{\bar{F}}}_0(0) + \dot{\bar{\bar{Q}}}(0) \tag{8.9}$$

since

$$\bar{\bar{F}}_0(0) = \bar{\bar{Q}}(0) = \bar{\bar{I}}$$

If the initial values of rate of deformation gradient, $\dot{\bar{\bar{F}}}_0(0) = \bar{\bar{L}}(0)$ and $\dot{\bar{\bar{Q}}}(0)$ are given, then from (8.8) and (8.9),

$$\frac{1}{t}\ln(\bar{\bar{F}}_0(t)\cdot\bar{\bar{Q}}^T(t)) = \bar{\bar{L}}(0) + \dot{\bar{\bar{Q}}}(0)$$

or,

$$\bar{\bar{Q}}(t) = ((\bar{\bar{F}}_0(t))^{-1}\cdot e^{t(\bar{\bar{L}}(0) + \dot{\bar{\bar{Q}}}(0))})^T \tag{8.10}$$

In [3; (5.6)], COLEMAN mentions that $\bar{\bar{Q}}(\ )$ is not uniquely determined by $\bar{\bar{C}}_{()}^{()}(s)$. All quantities necessary to determine $\bar{\bar{Q}}()$ uniquely are shown in (8.10).

If $\bar{\bar{M}}_t$ and $\bar{\bar{M}}_t^T$ commute, i.e.,

$$\bar{\bar{M}}_t\cdot\bar{\bar{M}}_t^T = \bar{\bar{M}}_t^T\cdot\bar{\bar{M}}_t \tag{8.11a}$$

then equation (8.4a) may be written as

$$\bar{\bar{C}}_t(\tau) = e^{-s(\bar{\bar{M}}_t + \bar{\bar{M}}_t^T)} \tag{8.11b}$$

A sufficient condition for $\bar{\bar{M}}_t$ and $\bar{\bar{M}}_t^T$ to commute is that $\bar{\bar{M}}_t$ be symmetric, since

$$\bar{\bar{M}}_t \cdot \bar{\bar{M}}_t^T \;=\; (\bar{\bar{M}}_t)^2 = \bar{\bar{M}}_t^T \cdot \bar{\bar{M}}_t \tag{8.11c}$$

Therefore, (8.11b) may be expanded to

$$\bar{\bar{C}}_t^t(s) = \bar{\bar{I}} + \sum_{i=1}^{N-1} (-1)^i \frac{(s)^i}{i!} (\bar{\bar{M}}_t + \bar{\bar{M}}_t^T)^i$$

$$= \bar{\bar{I}} + \sum_{i=1}^{N-1} (-1)^i \frac{(s)^i}{i!} (2\bar{\bar{M}}_t)^i \tag{8.11d}$$

From (5.9) and (8.11d),

$$\bar{\bar{A}}_n(t) \;=\; (\bar{\bar{M}}_t + \bar{\bar{M}}_t^T)^n \;=\; (2\bar{\bar{M}}_t)^n \tag{8.11e}$$

$$= (\bar{\bar{M}}_t + \bar{\bar{M}}_t)^n$$

$$= (\bar{\bar{M}}_t)^n + (\bar{\bar{M}}_t)^n + \sum_{i=1}^{n-1} \binom{n}{i} (\bar{\bar{M}}_t)^{n-i} \cdot (\bar{\bar{M}}_t)^i \tag{8.11f}$$

In (4.29), if $\bar{\bar{L}}_n(t)$, $n = 1,2,\ldots.$ are symmetric, then

$$\bar{\bar{A}}_n(t) = \bar{\bar{L}}_n(t) + \bar{\bar{L}}_n(t) + \sum_{i=1}^{n-1} \binom{n}{i} \bar{\bar{L}}_{(n-i)} \cdot \bar{\bar{L}}_i \tag{8.11g}$$

A sufficient condition for equality of (8.11f) and (8.11g) is

$$\bar{\bar{L}}_n(t) = (\bar{\bar{M}}_t)^n = (\bar{\bar{L}}_1(t))^n, \; n = 1,2,.... \qquad (8.12)$$

(8.11c) also holds good if

1. $\bar{\bar{M}}_t$ is skew, or

2. $\bar{\bar{M}}_t$ is orthogonal, since for (1)

$$\bar{\bar{M}}_t \cdot \bar{\bar{M}}_t^T = -(\bar{\bar{M}}_t)^2 = \bar{\bar{M}}_t^T \cdot \bar{\bar{M}}_t \qquad (8.13a)$$

and for (2)

$$\bar{\bar{M}}_t \cdot \bar{\bar{M}}_t^T = \bar{\bar{I}} = \bar{\bar{M}}_t^T \cdot \bar{\bar{M}}_t$$

From (8.13a-1), (8.11d) becomes

$$\bar{\bar{C}}_t^t(s) = \bar{\bar{I}}$$

since

$$\bar{\bar{M}}_t + \bar{\bar{M}}_t^T = \bar{\bar{0}} \quad .$$

From (8.13a-2) and (8.3), $\bar{\bar{F}}_t(\tau)$ is an orthogonal tensor.

$$\therefore \quad \bar{\bar{C}}_t(\tau) = \bar{\bar{F}}_t(\tau) \cdot \bar{\bar{F}}_t^T(\tau) = \bar{\bar{I}}$$

Cases for which strain tensor $\bar{\bar{C}}_t^t(s)$ is an identity tensor are not considered here.

From equations (8.3) and (8.2b) it is also noted that if $\bar{\bar{Q}}$ is independent of t so that $\dot{\bar{\bar{Q}}} = \bar{\bar{0}}$,

$$\frac{\partial}{\partial\tau}\bar{\bar{F}}_t(\tau)\Big|_{\tau=t} = (\bar{\bar{Q}}^T\cdot\bar{\bar{M}}\cdot e^{(\tau-t)\bar{\bar{M}}}\cdot\bar{\bar{Q}})\Big|_{\tau=t}$$

or

$$\bar{\bar{L}}_1(t) = \bar{\bar{Q}}^T\cdot\bar{\bar{M}}\cdot\bar{\bar{Q}} = \bar{\bar{M}}_t$$

Similarly,

$$\bar{\bar{L}}_2(t) = (\bar{\bar{M}}_t)^2 = (\bar{\bar{L}}_1(t))^2 \qquad (8.14)$$

$$- - - - - - - - - - - - -$$

$$\bar{\bar{L}}_n(t) = (\bar{\bar{M}}_t)^n = (\bar{\bar{L}}_1(t))^n$$

where

$$(\bar{\bar{L}}_1(t))^n = (\bar{\bar{L}}_1(t))\cdot(\bar{\bar{L}}_1(t)\cdots(\bar{\bar{L}}_1(t)), \text{ (n terms)}$$

Therefore, equation (8.12) holds good in two cases:

1. when the tensor $\bar{\bar{M}}$ as defined in (8.2) is symmetric (8.15)
2. when the orthogonal tensor $\bar{\bar{Q}}$ as defined in (8.2) is independent of time.

9. Necessary and Sufficient Conditions for MCSH

Theorem 1

A Motion with Constant Stretch History can always be found for a motion in which the $(n-1)^{th}$ acceleration gradient $\bar{\bar{L}}_n(t)$ is equal to the n^{th} power of the velocity gradient $\bar{\bar{L}}_1(t)$, for $n = 1,2,\ldots.$

Proof:

The $(n-1)^{th}$ acceleration gradient is defined as

$$\bar{\bar{L}}_n(t) = \frac{d^n}{d\tau^n} \bar{\bar{F}}_t(\tau)\Big|_{\tau=t} = \overset{(n)}{\bar{\bar{F}}_t}(t), \quad n = 1,2,\ldots. \tag{9.1}$$

If $\bar{\bar{L}}_n(t)$ is such that

$$\bar{\bar{L}}_n(t) = (\bar{\bar{L}}_1(t))^n, \quad n = 1,2,\ldots. \tag{9.2}$$

then, from (4.8) and (9.2), (9.1) is satisfied by

$$[\bar{\bar{F}}_t(\tau)] = e^{-s[\bar{\bar{L}}_1(t)]} \tag{9.3}$$

Also, from [16], considering an orthonormal basis $\bar{b}_1(t)$, $\bar{b}_2(t)$, $\bar{b}_3(t)$ which is attached to the material point and which rotates, as the point moves, according to the law

$$\bar{b}_k(t) = \bar{b}_k(0)\cdot\bar{\bar{Q}}(t) \tag{9.4}$$

the definition of a MCSH from (8.3) takes the form

$$[\bar{\bar{F}}_t(\tau)] = e^{-s[\bar{\bar{M}}]} \tag{9.5}$$

Therefore, (9.3) is a MCSH whenever

$$[\bar{\bar{L}}_1(t)] = [\bar{\bar{M}}] \tag{9.6}$$

Two such cases are noted in (8.15). From (9.6), deformation gradient is given as

$$\bar{\bar{F}}_0(\tau) = e^{\tau\bar{\bar{L}}_1(t)} \cdot \bar{\bar{Q}} \tag{9.7}$$

<u>Theorem 2</u>

All Steady Motions with Constant Stretch History obeying

$$[\bar{\bar{L}}_n(t)] = [\bar{\bar{L}}_1(t)]^n$$

except viscometric flows where

$$(\bar{\bar{L}}_1(t))^2 = \bar{\bar{L}}_2(t) = \bar{\bar{0}};$$

are homogeneous motions.

Proof:

Consider a motion in which any position $\bar{\xi}(R,\tau)$ of point R at time τ is related to current position $\bar{r}(R,t)$ as

$$\left.\begin{aligned} \xi_1 &= f_1(x_1, x_2, x_3; s) \\ \xi_2 &= f_2(x_1, x_2, x_3; s) \\ \xi_3 &= f_3(x_1, x_2, x_3; s) \end{aligned}\right\} \tag{9.8}$$

The relative deformation gradient $\bar{\bar{F}}_t(\tau)$ is given as

$$[\bar{\bar{F}}_t(\tau)] = [\frac{\partial \bar{\xi}}{\partial \bar{r}}] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} \tag{9.9}$$

By definition of MCSH, from equations (8.3) and (8.4) both relative deformation tensor $\bar{\bar{F}}_t(\tau)$ and relative strain tensor $\bar{\bar{C}}_t(\tau)$ are independent of position vector $\bar{r}$ so that each element $\frac{\partial f_i}{\partial x_j}$ of (9.9) is a constant with respect to x_j.

Therefore, (9.9) can be integrated to yield

$$\bar{\xi} = \bar{r}\cdot\bar{\bar{F}}_t(\tau) + \bar{C}(t) \tag{9.10}$$

where $\bar{C}(t)$ is a function of time.

(9.10) represents a homogeneous motion. There are two non-homogeneous flows as exceptions to the integration (9.10):

1. A flow in which one of the coordinates remains constant at all times: (planar flow)

$$\left.\begin{aligned} \xi^1 &= x^1 \\ \xi^2 &= x^2 + f(x^1,s) \\ \xi^3 &= x^3 + g(x^1, s) \end{aligned}\right\} \tag{9.11}$$

so that relative deformation gradient $\bar{\bar{F}}_t(\tau)$ is given by

$$[\bar{\bar{F}}_t(\tau)] = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\partial f}{\partial x^1}\sqrt{\frac{g_{22}}{g_{11}}} & 1 & 0 \\ \frac{\partial g}{\partial x^1}\sqrt{\frac{g_{33}}{g_{11}}} & 0 & 1 \end{bmatrix} \tag{9.12}$$

Since the x^1 coordinate of all points in the flow remains constant in time, $f(x^1, s)$ and $g(x^1, s)$ could be non-linear functions of x^1, but still yield a MCSH as

$$[\bar{\bar{L}}_1(t)] = [\bar{\bar{M}}_t] = - \begin{bmatrix} 0 & 0 & 0 \\ \frac{d}{ds}\left(\frac{\partial f}{\partial x^1}\right)\Big|_{s=0}\sqrt{\frac{g_{22}}{g_{11}}} & 0 & 0 \\ \frac{d}{ds}\left(\frac{\partial g}{\partial x^1}\right)\Big|_{s=0}\sqrt{\frac{g_{33}}{g_{11}}} & 0 & 0 \end{bmatrix} \tag{9.13}$$

so that

$$[\bar{\bar{M}}_t]^2 = [\bar{\bar{0}}] = [\bar{\bar{L}}_1(t)]^2$$

Also, if f and g are linear in s (steady flow) so that

$$[\bar{\bar{L}}_2(t)] = [\bar{\bar{0}}],$$

then the general class of flows represented by (9.11) with a velocity field

$$\left.\begin{aligned} v^1 &= 0 \\ v^2 &= \dot{f}(x^1) \\ v^3 &= \dot{g}(x^1) \end{aligned}\right\} \tag{9.14}$$

is a viscometric flow. [3, 16, 31, 34].

2. In a general rectilinear flow under a constant pressure gradient [17], the velocity field may be expressed as

$$\left.\begin{aligned} v^1 &= 0 \\ v^2 &= 0 \\ v^3 &= f(x^1, x^2) \end{aligned}\right\} \tag{9.15}$$

Integration of (9.15) subject to $\xi^i|_{s=0} = x^i$ leads to

$$\left.\begin{aligned} \xi^1 &= x^1 \\ \xi^2 &= x^2 \\ \xi^3 &= x^3 - sf(x^1, x^2) \end{aligned}\right\} \tag{9.16}$$

$$\therefore \quad [\overline{\overline{F}}_t(\tau)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -s\,\dfrac{\partial f}{\partial x^1}\sqrt{\dfrac{g_{33}}{g_{11}}} & -s\,\dfrac{\partial f}{\partial x^2}\sqrt{\dfrac{g_{33}}{g_{22}}} & 0 \end{bmatrix} \tag{9.17}$$

$$[\bar{\bar{L}}_1(t)] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial f}{\partial x^1}\sqrt{\frac{g_{33}}{g_{11}}} & \frac{\partial f}{\partial x^2}\sqrt{\frac{g_{33}}{g_{22}}} & 0 \end{bmatrix} \tag{9.18}$$

so that though $f(x^1, x^2)$ may be a non-linear function in x^1 and x^2, (9.18) obeys the relation

$$[\bar{\bar{L}}_1(t)]^2 = [\bar{\bar{0}}] = [\bar{\bar{L}}_2(t)].$$

Therefore, the velocity field (9.15) is one of viscometric flow.

N.B.! For non-steady Viscometric Flows,

$$[\bar{\bar{L}}_1(t)]^n = [\bar{\bar{0}}] \quad \text{for} \quad n = 2,3,....$$

but $\quad [\bar{\bar{L}}_n(t)] \neq [\bar{\bar{0}}] \quad \text{for} \quad n = 1, 2, 3, ...$

These conditions are reversed in (7.38g) for the flow represented by equation (7.37).

Theorem 3

If $[\bar{\bar{L}}_1(t)]$ in its most general form is given by

$$[\bar{\bar{L}}_1(t)] = [a_{ij}], \quad ij = 1,2,3; \ a_{ij} \text{ are constants}$$

with

$$[\bar{\bar{L}}_n(t)] = [\bar{\bar{L}}_1(t)]^n \neq [\bar{\bar{0}}]; \quad n = 1,2,\ldots.$$

in cartesian coordinates; then a MCSH can always be constructed from a velocity field given by

$$\dot{x}_i = a_{ij}x_j + k_i^1 \quad ,$$

acceleration field by

$$\ddot{x}_i = a_{ik}a_{kj}x_j + k_i^2$$

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the $(n-1)^{th}$ acceleration field by

$$\overset{(n)}{x}_i = a_{ik}a_{ki} \overset{(n \text{ times})}{\cdots\cdots\cdots\cdots} a_{qj}x_j + k_i^n$$

where $k_i^1, k_i^2, \ldots, k_i^n$ are constants.

Proof:

Since

$$[\bar{\bar{L}}_n(t)] = [\bar{\bar{L}}_1(t)]^n, \quad n = 1,2,\ldots.$$

and if

$$[\bar{\bar{L}}_1(t)] = [a_{ij}],$$

then

$$[\bar{\bar{L}}_2(t)] = [a_{ij}]^2 = [a_{ik}a_{kj}] \tag{9.19}$$

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$$[\bar{\bar{L}}_n(t)] = [a_{ij}]^n = [a_{ik}a_{k\,l}\overset{(n\ \text{times})}{\cdots\cdots\cdots}\ a_{qj}]$$

Since $[\bar{\bar{L}}_n(t)]$ is defined as

$$[\bar{\bar{L}}_n(t)] = [\overset{(n)}{\bar{\bar{F}}_t}(t)] = [(-1)^n \frac{d^n}{ds^n} \bar{\bar{F}}^t_t(s)]\Big|_{s=0} \quad ,$$

from (9.19), $[\bar{\bar{F}}_t(\tau)]$ may have the general form

$$[\bar{\bar{F}}_t(\tau)] = [\delta_{ij}] - s[a_{ij}] + \frac{(s)^2}{2!}[a_{ik}a_{kj}] - \frac{(s)^3}{3!}[a_{ik}a_{kl}a_{lj}]$$

$$+ - \ldots. + (-1)^n \frac{(s)^n}{n!} [a_{ik}a_{kl}\overset{(n\ \text{times})}{\cdots\cdots\cdots}\ a_{qj}] + - \ldots.$$

or

$$[\bar{\bar{F}}_t(\tau)] = e^{-s[a_{ij}]} \tag{9.20}$$

Also, since $[\bar{\bar{F}}_t(\tau)] = [\frac{\partial \xi_i}{\partial x_j}]$;

integration of (9.20) leads to

$$\xi_i = e^{-sa_{ij}} x_j + f_i(s) + c_i \tag{9.21}$$

where f_i are functions of s alone and C_i are constants. Differentiating (9.21) with respect to s, and then substituting s = 0,

$$\dot{x}_i = -\frac{d\xi_i}{ds}\Big|_{s=0} = a_{ij}x_j + k_i^1$$

$$\ddot{x}_i = -\frac{d^2\xi_i}{ds^2}\Big|_{s=0} = a_{ik}a_{kj}x_j + k_i^2$$

— — — — — — — — — — — — — — — — — —

$$\overset{(n)}{x}_i = (-1)^n \frac{d^n\xi_i}{ds^n}\Big|_{s=0} = a_{ik}\cdot a_{kl}\overset{(n\ \text{times})}{\cdots\cdots\cdots}a_{qj}x_j + k_i^n \tag{9.22}$$

Q.E.D.

N.B.! This proof suggests that (7.45) represents the velocity field of most general MCSH.

10. Special Solutions

1. As a particular case of (7.52), let the components of the velocity at any point R be linear functions of any two coordinates; so that

$$\left.\begin{aligned} \dot{x}^1 &= ax^2 + bx^3 \\ \dot{x}^2 &= cx^2 + dx^3 \\ \dot{x}^3 &= ex^2 + fx^3 \end{aligned}\right\} \qquad (10.1)$$

where a,...,f are constants.

This flow is amenable to an exact solution for the positions of point R when c = -c, f = c. The resulting differential equation of motion is

$$\left.\begin{aligned} -\frac{d\xi^1}{ds} &= a\xi^2 + b\xi^3 \\ -\frac{d\xi^2}{ds} &= -c\xi^2 + d\xi^3 \\ -\frac{d\xi^3}{ds} &= e\xi^2 + c\xi^3 \end{aligned}\right\} \qquad (10.2)$$

subject to the initial condition

$$\xi^i \Big|_{s=0} = x^i$$

Differentiating (10.2-2) and (10.2-3) with respect to s,

$$-\frac{d^2\xi^2}{ds^2} = -c\,\frac{d\xi^2}{ds} + d\,\frac{d\xi^3}{ds} = -c[c\xi^2 - d\xi^3] + d[-e\xi^2 - c\xi^3]$$

$$= -(c)^2\xi^2 + cd\xi^3 - ed\xi^2 - cd\xi^3$$

or

$$\frac{d^2\xi^2}{ds^2} = ((c)^2 + de)\xi^2 \qquad (10.2a)$$

Similarly,

$$\frac{d^2\xi^3}{ds^2} = ((c)^2 + de)\xi^3 \qquad (10.2b)$$

Let

$$((c)^2 + de) = (k)^2$$

The solution to (10.2a) and (10.2b) is given as

$$\left.\begin{aligned} \xi^2 &= c_1e^{ks} + c_2e^{-ks} \\ \xi^3 &= c_3e^{ks} + c_4e^{-ks} \end{aligned}\right\} \qquad (10.3)$$

where

c_1, c_2, c_3, and c_4 are constants.

The four initial conditions are given by

$$\left.\begin{aligned} \xi^2\Big|_{s=0} &= x^2 \\ \xi^3\Big|_{s=0} &= x^3 \\ -\frac{d\xi^2}{ds}\Big|_{s=0} &= -cx^2 + dx^3 \\ -\frac{d\xi^3}{ds}\Big|_{s=0} &= ex^2 + cx^3 \end{aligned}\right\} \tag{10.3a}$$

Substitution of (10.3a) in (10.3) gives

$$\xi^2 = x^2 \cosh ks + \frac{cx^2 - dx^3}{k} \sinh ks \tag{10.4a}$$

$$\xi^3 = x^3 \cosh ks - \frac{ex^2 + cx^3}{k} \sinh ks \tag{10.4b}$$

Substituting for ξ^2 and ξ^3 in (10.2-1),

$$\xi^1 = x^1 - x^2\left(\frac{a}{k} \sinh ks + \frac{ac - be}{(k)^2} \cosh ks\right) + x^3\left(-\frac{b}{k} \sinh ks + \right.$$

$$\left. + \frac{ad + bc}{(k)^2} \cosh ks\right) \tag{10.4c}$$

$$[\bar{\bar{F}}_t(\tau)] = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

where

$$\left.\begin{aligned}
& f_{11} = 1,\ f_{12} = -\left(\frac{a}{k}\sinh ks + \frac{ac - be}{(k)^2}\cosh ks\right)\sqrt{\frac{g_{11}}{g_{22}}} \\
& f_{13} = \left(-\frac{b}{k}\sinh ks + \frac{ad + bc}{(k)^2}\cosh ks\right)\sqrt{\frac{g_{11}}{g_{33}}},\quad f_{21} = 0 \\
& f_{22} = \cosh ks + \frac{c}{k}\sinh ks;\ f_{23} = -\frac{d}{k}\sinh ks\sqrt{\frac{g_{22}}{g_{33}}} \\
& f_{31} = 0,\ f_{32} = -\frac{e}{k}\sinh ks\sqrt{\frac{g_{33}}{g_{22}}},\ f_{33} = \cosh ks - \frac{c}{k}\sinh ks
\end{aligned}\right\} \quad (10.5a)$$

$$[\bar{\bar{L}}_1(t)] = \begin{bmatrix} 0 & a\sqrt{\frac{g_{11}}{g_{22}}} & b\sqrt{\frac{g_{11}}{g_{33}}} \\ 0 & -c & d\sqrt{\frac{g_{22}}{g_{33}}} \\ 0 & e\sqrt{\frac{g_{33}}{g_{22}}} & c \end{bmatrix} \quad (10.5b)$$

$$\therefore \; \frac{d}{dt} \; [\bar{\bar{L}}_1(t)] = [\bar{\bar{0}}]$$

Also,

$$[\bar{\bar{L}}_2(t)] = \begin{bmatrix} 0 & (be - ac)\sqrt{\dfrac{g_{11}}{g_{22}}} & (ad + bc)\sqrt{\dfrac{g_{11}}{g_{33}}} \\ 0 & (k)^2 & 0 \\ 0 & 0 & (k)^2 \end{bmatrix} = [\bar{\bar{L}}_1(t)]^2 \qquad (10.5c)$$

$$[\bar{\bar{L}}_3(t)] = \begin{bmatrix} 0 & a(k)^2\sqrt{\dfrac{g_{11}}{g_{22}}} & b(k)^2\sqrt{\dfrac{g_{11}}{g_{33}}} \\ 0 & -c(k)^2 & d(k)^2\sqrt{\dfrac{g_{22}}{g_{33}}} \\ 0 & e(k)^2\sqrt{\dfrac{g_{33}}{g_{22}}} & c(k)^2 \end{bmatrix} = [\bar{\bar{L}}_1(t)]^3$$

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$$[\bar{\bar{L}}_n(t)] = [\bar{\bar{L}}_1(t)]^n \qquad (10.5d)$$

Due to Theorem (3), the velocity field (10.2) describes MCSH. In (10.1), if $\dot{x}^1 = 0$, the flow simplifies to

$$[\bar{\bar{F}}_t(\tau)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\cosh ks + \frac{c}{k}\sinh ks) & -\frac{d}{k}\sinh ks\sqrt{\frac{g_{22}}{g_{33}}} \\ 0 & -\frac{e}{k}\sinh ks\sqrt{\frac{g_{33}}{g_{22}}} & (\cosh ks - \frac{c}{k}\sinh ks) \end{bmatrix} \quad (10.6a)$$

$$[\bar{\bar{L}}_1(t)] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -c & d\sqrt{\frac{g_{22}}{g_{33}}} \\ 0 & e\sqrt{\frac{g_{33}}{g_{22}}} & c \end{bmatrix}$$

$$[\bar{\bar{L}}_2(t)] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (k)^2 & 0 \\ 0 & 0 & (k)^2 \end{bmatrix} = [\bar{\bar{L}}_1(t)]^2$$

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$$[\bar{\bar{L}}_n(t)] = [\bar{\bar{L}}_1(t)]^n \quad (10.6b)$$

Hence, the flow is MCSH.

From (10.5b)

$$[\bar{\bar{A}}_1(t)] = [\bar{\bar{L}}_1(t)] + [\bar{\bar{L}}_1^T(t)] =$$

$$= \begin{bmatrix} 0 & a\sqrt{\frac{g_{11}}{g_{22}}} & b\sqrt{\frac{g_{11}}{g_{33}}} \\ a\sqrt{\frac{g_{11}}{g_{22}}} & -2c & d\sqrt{\frac{g_{22}}{g_{33}}} + e\sqrt{\frac{g_{33}}{g_{22}}} \\ b\sqrt{\frac{g_{11}}{g_{22}}} & d\sqrt{\frac{g_{22}}{g_{33}}} + e\sqrt{\frac{g_{33}}{g_{22}}} & 2c \end{bmatrix} \quad (10.7a)$$

Also, from (10.5b) and (10.5c)

$$[\bar{\bar{A}}_2(t)] = [\bar{\bar{L}}_2(t)] + [\bar{\bar{L}}_2^T(t)] + 2[\bar{\bar{L}}_1(t)]\cdot[\bar{\bar{L}}_1^T(t)]$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where

$$a_{11} = 2(a)^2 \frac{g_{11}}{g_{22}} + (b)^2 \frac{g_{11}}{g_{33}}, \quad a_{12} = \{(be - 3ac) + 2bd \frac{g_{22}}{g_{33}}\}\sqrt{\frac{g_{11}}{g_{22}}}$$

$$a_{13} = \{(ad + 3bc) + 2ae \frac{g_{33}}{g_{22}}\}\sqrt{\frac{g_{11}}{g_{33}}}, \quad a_{21} = \{(be - 3ac) \frac{g_{11}}{g_{22}} + $$

$$+ 2bd \frac{g_{11}}{g_{33}}\}\sqrt{\frac{g_{22}}{g_{11}}}$$

$$a_{22} = 2\{2(c)^2 + d(e + d \frac{g_{22}}{g_{33}})\}, \quad a_{23} = 2c(d - e \frac{g_{33}}{g_{22}})\sqrt{\frac{g_{22}}{g_{33}}}$$

$$a_{31} = \{(3bc + ad) \frac{g_{11}}{g_{33}} + 2ae \frac{g_{11}}{g_{22}}\}\sqrt{\frac{g_{33}}{g_{11}}},$$

$$a_{32} = 2c(d \frac{g_{22}}{g_{33}} - e)\sqrt{\frac{g_{33}}{g_{22}}}$$

$$a_{33} = 2\{2(c)^2 + e(2d + e \frac{g_{33}}{g_{22}})\} \qquad (10.7b)$$

From [27], the determinate stress tensor may be calculated for this motion.

N.B.! In (10.2), if $(c)^2 + ef = 0$, the flow is a Doubly Superposed Viscometric Flow. [8].

<u>An Approximate MCSH</u>

Consider an irrotational motion in which

$$\xi^i = x^{\underline{i}} \, e^{-(2a_{\underline{i}}/\sqrt{x^{\underline{i}}})s} \tag{10.8}$$

so that

$$-\frac{d\xi^i}{ds} = x^{\underline{i}} \frac{2a_{\underline{i}}}{\sqrt{x^{\underline{i}}}} e^{-(2a_{\underline{i}}/\sqrt{x^{\underline{i}}})s} = 2a_{\underline{i}}\sqrt{x^{\underline{i}}} \, e^{-(2a_{\underline{i}}/\sqrt{x^{\underline{i}}})s} \tag{10.8a}$$

and at $s = 0$,

$$v^i = 2a_{\underline{i}}\sqrt{x^{\underline{i}}} \tag{10.8b}$$

The initial condition of the flow is the same as (7.37), but in the present case,

$$-\frac{d\xi^i}{ds} = 2a_i \; \xi^{\underline{i}}/\sqrt{x^{\underline{i}}}$$

From (10.8),

$$\frac{\partial \xi^{\underline{i}}}{\partial x^{\underline{i}}} = e^{-2a_{\underline{i}}s/\sqrt{x^{\underline{i}}}} + 2a_{\underline{i}}sx^{\underline{i}} \, \tfrac{1}{2}(x^{\underline{i}})^{-3/2} \, e^{-(2a_{\underline{i}}/\sqrt{x^{\underline{i}}})s}$$

or

$$[\bar{\bar{F}}_t(\tau)] = [(1 + \frac{a_{\underline{i}}s}{\sqrt{x^i}}) \, e^{-(2a_{\underline{i}}/\sqrt{x^{\underline{i}}})s}] \tag{10.9}$$

$$[\bar{\bar{L}}_1(t)] = [(-1)(1 + \frac{a_i s}{\sqrt{x_i}})(-\frac{2a_i}{\sqrt{x_i}})\, e^{-(2a_i/\sqrt{x_i})s}\Big|_{s=0}$$

$$+ (-1)\frac{a_i}{x_i}\, e^{-2a_i s/\sqrt{x_i}}\Big|_{s=0}]$$

$$= [\frac{2a_i}{\sqrt{x_i}} - \frac{a_i}{\sqrt{x_i}}] = [\frac{a_i}{\sqrt{x_i}}] \tag{10.9a}$$

$$[\bar{\bar{L}}_2(t)] = [(1 + \frac{a_i s}{\sqrt{x_i}})(-\frac{2a_i}{\sqrt{x_i}})^2\, e^{-2a_i s/\sqrt{x_i}}\Big|_{s=0}] = [\frac{2a_i}{\sqrt{x_i}}]^2 \tag{10.9b}$$

$$[\bar{\bar{L}}_3(t)] = [(-1)(1 + \frac{a_i s}{\sqrt{x_i}})(-\frac{2a_i}{\sqrt{x_i}})^3\, e^{-2a_i s/\sqrt{x_i}}\Big|_{s=0}] = [\frac{2a_i}{\sqrt{x_i}}]^3 \tag{10.9c}$$

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$$[\bar{\bar{L}}_n(t)] = [2\bar{\bar{L}}_1(t)]^n = [\bar{\bar{A}}_1(t)]^n \tag{10.9d}$$

For a MCSH, $\bar{\bar{L}}_n(t) = [\bar{\bar{L}}_1(t)]^n$. Due to the similarity of equation (10.9d) to this condition, the motion in (10.8) is being referred to as an approximate MCSH.

11. CONCLUSIONS

In this thesis, an analysis of general Motions of Differential Type and those with constant Stretch History has been made.

Since it has been suggested that all controllable motions for elastic materials have been found [12], the attempt in the present work was to search for controllable motions in Materials of Differential Type. As a pre-requisite, certain kinematic, and internal constraint equations in MDT were worked out. An interpretation of Rivlin-Ericksen tensors $\bar{\bar{A}}_n$ starting with a Taylor series expansion of relative strain tensor $\bar{\bar{C}}_t^t(s)$ was made. As long as $\bar{\bar{L}}_1(t)$ was not nil-potent, and hence $\bar{\bar{A}}_1(t)$ too was not nil-potent (considering non-skew $\bar{\bar{L}}_1(t)$),it would always be possible to expand $\bar{\bar{C}}_t(\tau)$ as an infinite power series of $\bar{\bar{A}}_1(t)$. It was proved that motions where $[\bar{\bar{L}}_1(t)]^n = [\bar{\bar{L}}_n(t)] \rightarrow [\bar{\bar{A}}_1(t)]^n = [\bar{\bar{A}}_n(t)]$ were Motions with Constant Stretch History.

The general case of steady Motions of Differential Type was considered to have a velocity field described by $v^i = a_{ij}e^{x^j}$. Some solutions could be found for special cases of this velocity field. Problems concerning irrotational (Steady Extensional) flows were described and solved for stresses.

Motions with Constant Stretch History were introduced as special cases of Motions of Differential Type. NOLL'S definition of MCSH [16] was used. Certain theorems regarding necessary and sufficient conditions for MCSH were proved. A class of MCSH was introduced, and an approximate MCSH was suggested.

After COLEMAN and NOLL'S introduction of Motions with Constant Stretch History [3,15,16], HUILGOL [8,9] has done extensive research on the topic. He has worked out further examples of such motions, and has showed that many classes of MCSH are obtained by superposition of Viscometric Flows. It was observed in the present work that while certain non-homogeneous motions could be classified as Viscometric Flows, the more general Motions with Constant Stretch History were homogeneous. Therefore, confusion could arise in classifying certain types of MCSH as Superposable Viscometric Flows. All examples of Viscometric Flows used by Dr. Huilgol to obtain MCSH on superposition are homogeneous motions.

It is noted in (8.3.3) that for controllable motions, α_i should not be subjected to any restrictions. In motions considered in the present work, the restrictions on α_i in (7.27), (7.34), (7.37) and (7.41) have been stated. If α_i generally satisfy the conditions (7.33), (7.36d), (7.40a) and (7.44a) respectively, then the motions are controllable.

At the end of Chapter II, physical components of deformation gradient tensor $\bar{\bar{F}}$ are introduced. In subsequent solutions of problems, such physical components are considered.

Some tensor relations regarding gradients of scalar valued tensor functions, are derived in Appendix A. Brief introductions to constitutive equations, representation theorems and controllable motions are given in Appendix B.

The question of discovering all possible controllable motions is not closed at present.

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APPENDIX A

TENSOR ANALYSIS

A.1 Gradients of a Scalar-valued Tensor Function

In this section, only second order tensors are considered. Second order tensors occupy a space L of dimension $(n)^2$. Symmetric tensors form a subspace S of L of dimension $\frac{1}{2}n(n+1)$.

Any function whose arguments are tensors in L or S and whose values are scalars, vectors or tensors is called a tensor function.

A scalar-valued isotropic tensor function $E(\bar{\bar{B}})$ of one variable is called an orthogonal invariant or briefly, invariant of $\bar{\bar{B}}$.

Principal invariants $I_k(\bar{\bar{B}})$, $k = 1,2,\ldots,n$ of the tensor $\bar{\bar{B}}$ are defined as the coefficients of the following polynomial in λ:

$$|(\lambda\bar{\bar{I}} + \bar{\bar{B}})| = (\lambda)^n + I_1(\bar{\bar{B}})(\lambda)^{n-1} + \cdots + I_{n-1}(\bar{\bar{B}})\lambda + I_n(\bar{\bar{B}}) \quad (A.1.1)$$

In particular,

$$I_1(\bar{\bar{B}}) = \bar{\bar{I}}:\bar{\bar{B}}$$

$$I_n(\bar{\bar{B}}) = |\bar{\bar{B}}|$$

In 3 dimensions (n = 3),

$$I_1(\bar{\bar{B}}) = I_{\bar{\bar{B}}}, \quad I_2(\bar{\bar{B}}) = II_{\bar{\bar{B}}}, \quad I_3(\bar{\bar{B}}) = III_{\bar{\bar{B}}} \qquad \text{(A.1.2)}$$

where, for the matrix of the tensor $\bar{\bar{B}}$

$$[\bar{\bar{B}}] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

the first principal invariant $I_{\bar{\bar{B}}}$ is given by

$$I_{\bar{\bar{B}}} = b_{11} + b_{22} + b_{33} = \bar{\bar{B}}:\bar{\bar{I}},$$

the second principal invariant $II_{\bar{\bar{B}}}$ is given by

$$II_{\bar{\bar{B}}} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{33} & b_{31} \\ b_{13} & b_{11} \end{vmatrix} = \frac{1}{2!}\bar{\bar{B}}\overset{\times}{\times}\bar{\bar{B}}:\bar{\bar{I}}$$

N.B.! In Sections **A.1** and A.2, no difference is made between a tensor and its matrix.

and the third principal invariant $III_{\bar{\bar{B}}}$ is given by

$$III_{\bar{\bar{B}}} = |\bar{\bar{B}}| = \frac{1}{3!} \bar{\bar{B}} \times \bar{\bar{B}} : \bar{\bar{B}}$$

Another important invariant of $\bar{\bar{B}}$ is defined by

$$I_k^*(\bar{\bar{B}}) = \bar{\bar{I}}:[(\bar{\bar{B}})^k] \qquad \text{(A.1.3)}$$

which are known as the *moments* of $\bar{\bar{B}}$.

The gradient of the scalar-valued tensor function $E(\bar{\bar{B}})$ is defined by

$$\frac{\partial E}{\partial \bar{\bar{B}}} = E_{\bar{\bar{B}}} \qquad \text{(A.1.4)}$$

Consider $\bar{\bar{C}}$ to be another second order tensor.

$$\therefore \frac{d}{ds} E(\bar{\bar{B}} + s\bar{\bar{C}})\Big|_{s=0} = \left\{\frac{\partial(\bar{\bar{B}} + s\bar{\bar{C}})}{\partial s} : \frac{\partial E(\bar{\bar{B}} + s\bar{\bar{C}})}{\partial(\bar{\bar{B}} + s\bar{\bar{C}})}\right\}\Bigg|_{s=0}$$

$$= \left(\frac{\partial \bar{\bar{B}}}{\partial s} + \bar{\bar{C}} + s\,\frac{\partial \bar{\bar{C}}}{\partial s}\right)\Bigg|_{s=0} : \frac{\partial E(\bar{\bar{B}})}{\partial \bar{\bar{B}}}$$

$$= \bar{\bar{C}} : \frac{\partial E(\bar{\bar{B}})}{\partial \bar{\bar{B}}} \qquad \text{(A.1.5)}$$

since $\bar{\bar{B}}$ is not a function of s.

This is another way of introducing the gradient of a scalar-valued tensor function.

If $\bar{\bar{B}}$ is invertible, then the following is true:

$$|[\bar{\bar{B}} + s\bar{\bar{C}}]| = (s)^n|\bar{\bar{B}}||[\frac{1}{s}\,\bar{\bar{I}} + (\bar{\bar{B}})^{-1}\cdot\bar{\bar{C}}]| \tag{A.1.6}$$

Using the equation (A.1.1) for $\lambda = \frac{1}{s}$, and replacing $[\bar{\bar{B}}]$ by $[(\bar{\bar{B}})^{-1}\cdot\bar{\bar{C}}]$, we get

$$|[\bar{\bar{B}} + s\bar{\bar{C}}]| = |\bar{\bar{B}}|[1 + I_1\{(\bar{\bar{B}})^{-1}\cdot\bar{\bar{C}}\}s + \cdots + I_n\{(\bar{\bar{B}})^{-1}\cdot\bar{\bar{C}}\}(s)^n] \tag{A.1.7}$$

or

$$I_n(\bar{\bar{B}} + s\bar{\bar{C}}) = I_n(\bar{\bar{B}})[1 + I_1\{(\bar{\bar{B}})^{-1}\cdot\bar{\bar{C}}\}s + \cdots + I_n\{(\bar{\bar{B}})^{-1}\cdot\bar{\bar{C}}\}(s)^n] \tag{A.1.8}$$

Differentiating (A.1.8) with respect to s and putting s=0,

$$\frac{d}{ds}\{I_n(\bar{\bar{B}} + s\bar{\bar{C}})\}\Big|_{s=0} = I_n(\bar{\bar{B}})[I_n\{(\bar{\bar{B}})^{-1}\cdot\bar{\bar{C}}\}] \tag{A.1.9}$$

From (A.1.5) and (A.1.9),

$$\bar{\bar{C}}:\frac{\partial I_n(\bar{\bar{B}})}{\partial\bar{\bar{B}}} = I_n(\bar{\bar{B}})\ \bar{\bar{I}}:\{(\bar{\bar{B}})^{-1}\cdot\bar{\bar{C}}\} \tag{A.1.10}$$

For any arbitrary $\bar{\bar{C}}$,

$$\bar{\bar{I}}:\{(\bar{\bar{B}})^{-1}\cdot\bar{\bar{C}}\} = \bar{\bar{C}}:\{(\bar{\bar{B}})^{-1}\}^{T},$$

Therefore, (A.1.10) becomes

$$\bar{\bar{C}}:\frac{\partial I_n(\bar{\bar{B}})}{\partial\bar{\bar{B}}} = I_n(\bar{\bar{B}})\ \bar{\bar{C}}:\{(\bar{\bar{B}})^{-1}\}^{T} \tag{A.1.11}$$

Again since $\bar{\bar{C}}$ is arbitrary, (A.1.11) is always true if

$$\frac{\partial I_n(\bar{\bar{B}})}{\partial\bar{\bar{B}}} = I_n(\bar{\bar{B}})\{(\bar{\bar{B}})^{-1}\}^{T}$$

or

$$\frac{\partial|\bar{\bar{B}}|}{\partial\bar{\bar{B}}} = |\bar{\bar{B}}|\{(\bar{\bar{B}})^{-1}\}^{T} \tag{A.1.12}$$

Equation (A.1.1) may also be written as

$$|[\lambda\bar{\bar{I}} + \bar{\bar{B}}]| = \sum_{k=0}^{n}(\lambda)^{n-k}\ I_k(\bar{\bar{B}}) \tag{A.1.13}$$

where

$$I_0(\bar{\bar{B}}) = 1$$

By (A.1.12), we have

$$\frac{\partial}{\partial \bar{\bar{B}}} | [\lambda \bar{I} + \bar{\bar{B}}] | = | [\lambda \bar{I} + \bar{\bar{B}}] | \{ (\lambda \bar{I} + \bar{\bar{B}})^{-1} \}^T \tag{A.1.14}$$

Pre-multiplying both sides of (A.1.14) by $(\lambda \bar{I} + \bar{\bar{B}})^T$,

$$(\lambda \bar{I} + \bar{\bar{B}})^T \cdot \frac{\partial}{\partial \bar{\bar{B}}} | [\lambda \bar{I} + \bar{\bar{B}}] | = | (\lambda \bar{I} + \bar{\bar{B}}] | \bar{I} \tag{A.1.15}$$

since inversion and transpose are interchangeable matrix operations. Substituting (A.1.13) in (A.1.15),

$$(\lambda \bar{I} + \bar{\bar{B}})^T \cdot \sum_{k=0}^{n} (\lambda)^{n-k} \frac{\partial I_k(\bar{\bar{B}})}{\partial \bar{\bar{B}}} = \bar{I} \sum_{k=0}^{n} (\lambda)^{n-k} I_k(\bar{\bar{B}}) \tag{A.1.16}$$

or

$$\bar{I} \cdot \sum_{k=0}^{n} (\lambda)^{(n-k+1)} \frac{\partial I_k(\bar{\bar{B}})}{\partial \bar{\bar{B}}} + \bar{\bar{B}}^T \cdot \sum_{k=0}^{n} (\lambda)^{(n-k)} \frac{\partial I_k(\bar{\bar{B}})}{\partial \bar{\bar{B}}}$$

$$= \bar{I} \sum_{k=0}^{n} (\lambda)^{(n-k)} I_k(\bar{\bar{B}}) \tag{A.1.17}$$

Comparing the coefficients of the powers of λ leads to

$$\frac{\partial I_{k+1}(\bar{\bar{B}})}{\partial \bar{\bar{B}}} = I_k(\bar{\bar{B}}) \bar{I} - \bar{\bar{B}}^T \cdot \frac{\partial I_k(\bar{\bar{B}})}{\partial \bar{\bar{B}}} , \quad k = 0,1,\ldots,n \tag{A.1.18}$$

where $I_0 = 1$ and $I_{n+1} = 0$.

Thus, using induction,

$$\frac{\partial I_{k+1}(\bar{\bar{B}})}{\partial \bar{\bar{B}}} + \bar{\bar{B}}^T \cdot \frac{\partial I_k(\bar{\bar{B}})}{\partial \bar{\bar{B}}} = I_k(\bar{\bar{B}})\ \bar{\bar{I}} \tag{A.1.19}$$

$$\frac{\partial I_k(\bar{\bar{B}})}{\partial \bar{\bar{B}}} + \bar{\bar{B}}^T \cdot \frac{\partial I_{k-1}(\bar{\bar{B}})}{\partial \bar{\bar{B}}} = I_{k-1}(\bar{\bar{B}})\ \bar{\bar{I}} \tag{A.1.20}$$

Premultiplying (A.1.20) by $\bar{\bar{B}}^T$ and subtracting from (A.1.19),

$$\frac{\partial I_{k+1}(\bar{\bar{B}})}{\partial \bar{\bar{B}}} - (\bar{\bar{B}}^T)^2 \cdot \frac{\partial I_{k-1}(\bar{\bar{B}})}{\partial \bar{\bar{B}}} = I_k(\bar{\bar{B}})\ \bar{\bar{I}} - \bar{\bar{B}}^T \cdot I_{k-1}(\bar{\bar{B}})\ \bar{\bar{I}} \tag{A.1.21}$$

Also, from induction, after (A.1.20), we get

$$\frac{\partial I_{k-1}(\bar{\bar{B}})}{\partial \bar{\bar{B}}} + \bar{\bar{B}}^T \cdot \frac{\partial I_{k-2}(\bar{\bar{B}})}{\partial \bar{\bar{B}}} = I_{k-2}(\bar{\bar{B}})\ \bar{\bar{I}} \tag{A.1.22}$$

Pre-multiplying (A.1.22) with $(\bar{\bar{B}}^T)^2$ and adding to (A.1.21),

$$\frac{\partial I_{k+1}(\bar{\bar{B}})}{\partial \bar{\bar{B}}} + (\bar{\bar{B}}^T)^3 \cdot \frac{\partial I_{k-2}(\bar{\bar{B}})}{\partial \bar{\bar{B}}} = I_k(\bar{\bar{B}})\bar{\bar{I}} - \bar{\bar{B}}^T \cdot I_{k-1}(\bar{\bar{B}})\bar{\bar{I}} + (\bar{\bar{B}}^T)^2 \cdot I_{k-2}(\bar{\bar{B}})\bar{\bar{I}} \tag{A.1.23}$$

Generalisation of (A.1.23) yields

$$\frac{\partial I_k(\bar{\bar{B}})}{\partial \bar{\bar{B}}} = \{\sum_{j=0}^{k-1} (-1)^j \{I_{k-j+1}(\bar{\bar{B}})\}(\bar{\bar{B}})^j\}^T \qquad \text{(A.1.24)}$$

Putting k = n+1, since $I_{n+1} = 0$, (A.1.24) yields the theorem of Hamilton and Cayley

$$\bar{\bar{B}} - I_1(\bar{\bar{B}})\{\bar{\bar{B}}\}^{n-1} + I_2(\bar{\bar{B}})\{\bar{\bar{B}}\}^{n-2} - + \cdots + (-1)^n \, I_n(\bar{\bar{B}})\bar{\bar{I}} \qquad \text{(A.1.25)}$$

In three dimensions (n=3), (A.1.24) gives

$$\left.\begin{aligned} \frac{\partial I_{\bar{\bar{B}}}}{\partial \bar{\bar{B}}} &= \bar{\bar{I}} \\ \frac{\partial II_{\bar{\bar{B}}}}{\partial \bar{\bar{B}}} &= I_{\bar{\bar{B}}}\,\bar{\bar{I}} - \bar{\bar{B}}^T \\ \frac{\partial III_{\bar{\bar{B}}}}{\partial \bar{\bar{B}}} &= |\bar{\bar{B}}|\,(\bar{\bar{B}}^{-1})^T = \{(\bar{\bar{B}})^2 - I_{\bar{\bar{B}}}\,\bar{\bar{B}} + II_{\bar{\bar{B}}}\,\bar{\bar{I}}\}^T \end{aligned}\right\} \qquad \text{(A.1.26)}$$

Gradients of moments $I_k^*(\bar{\bar{B}})$ as defined by (A.1.3), may be calculated in a similar manner. Since

$$I_1^*(\bar{\bar{B}}) = I_1(\bar{\bar{B}}) = \bar{\bar{I}}:\bar{\bar{B}}$$

(A.1.24) yields

$$\frac{\partial(\bar{\bar{I}}:\bar{\bar{B}})}{\partial\bar{\bar{B}}} = \bar{\bar{I}} \qquad \text{(A.1.27)}$$

A generalised formula, following [34] is

$$\frac{\partial I_k^*(\bar{\bar{B}})}{\partial\bar{\bar{B}}} = k[(\bar{\bar{B}})^{k-1}]^T \qquad \text{(A.1.28)}$$

A.2 Convergence

We deal with convergence of series containing second order tensors in three space.

Let $\bar{\bar{A}}_0^*$, $\bar{\bar{A}}_1^*,\ldots,\bar{\bar{A}}_n^*$ represent (n+1) second order tensors whose components are given by

$$a_{ik}^0,\ a_{ik}^1,\ldots,a_{ik}^n$$

(with suitable conditions on the metric tensor; we are interested in the magnitudes of the components).

An expression of the form

$$\bar{\bar{A}}_0^* + (\tau-t)\bar{\bar{A}}_1^* + \cdots + (\tau-t)^n\bar{\bar{A}}_n^* + \cdots = \sum_{n=0}^{\infty}(\tau-t)^n\bar{\bar{A}}_n^* \qquad \text{(A.2.1)}$$

represents nine equations in its components

$$a_{ik}^0 + (\tau-t)a_{ik}^1 + \cdots + (\tau-t)^n a_{ik}^n + \cdots = \sum_{n=0}^{\infty}(\tau-t)^n a_{ik}^n \qquad \text{(A.2.2)}$$

Equation (A.1.1) represents a power series and is defined as the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} (\tau-t)^n \,\bar{\bar{A}}_n^* \qquad \text{(A.2.3)}$$

for those values of τ for which the limit exists. For such values of τ, the series is said to converge. Since we are dealing with the matrix of the tensor, and a (nxn) matrix is said to converge when all its n^2 elements converge, then (A.1.3) will converge if

$$\lim_{N\to\infty} \sum_{n=0}^{N} (\tau-t)^n \, a_{ik}^n \qquad \text{(A.2.4)}$$

converges.

Any standard test to determine the convergence of (A.2.4) for values of τ may be employed. We show the ratio test.

Let β_{ik} be defined as

$$\beta_{ik} = \lim_{n\to\infty} \left|\frac{a_{ik}^{n+1}}{a_{ik}^{n}}\right| |(\tau-t)| = L_{ik}|(\tau-t)| \qquad \text{(A.2.5)}$$

where

$$L_{ik} = \lim_{n\to\infty} \left|\frac{a_{ik}^{n+1}}{a_{ik}^{n}}\right|$$

if this limit exists.

It follows that (4.1.1) converges when

$$\left.\begin{aligned} &|\tau - t| < \frac{1}{L_{ik}} \\ &\text{and diverges when} \\ &|\tau - t| > \frac{1}{L_{ik}} \end{aligned}\right\} \qquad (A.2.6)$$

Therefore, when L_{ik} exists and is finite, the intervals of convergence

$$\{(t - \frac{1}{L_{ik}}), \quad (t + \frac{1}{L_{ik}})\} \qquad (A.2.7)$$

are determined symmetrically about the point t such that inside the interval, the series (A.2.1) converges, and outside the interval, it diverges.

The intervals of convergence may coincide for all elements if

$$\lim_{n\to\infty} \left|\frac{a_{ik}^{n+1}}{a_{ik}^{n}}\right| = L$$

is the same for all components a_{ik}.

If any value of L_{ik} is 0, or L = 0, the intervals of convergence include all values of τ. However, if L_{ik} is infinite, or if more generally,

$$\left|\frac{a_{ik}^{(n+1)}}{a_{ik}^{n}}\right|$$

is unbounded, as $n\to\infty$, the series converges only at the point $\tau = t$. In all other cases, namely, when L_{ik}'s exist and are finite, a finite interval of convergence is determined.

A.3 Isotropy

A scalar-valued tensor function $E(\bar{\bar{B}}_1, \bar{\bar{B}}_2, \ldots, \bar{\bar{B}}_n)$ is said to be isotropic if the relation

$$E(\bar{\bar{B}}_1, \bar{\bar{B}}_2, \ldots, \bar{\bar{B}}_n) = E(\bar{\bar{Q}}^T\cdot\bar{\bar{B}}_1\cdot\bar{\bar{Q}}, \bar{\bar{Q}}^T\cdot\bar{\bar{B}}_2\cdot\bar{\bar{Q}}, \ldots, \bar{\bar{Q}}^T\cdot\bar{\bar{B}}_n\cdot\bar{\bar{Q}}) \qquad (A.3.1)$$

holds for all orthogonal tensors $\bar{\bar{Q}}$ and all $\bar{\bar{B}}_n$ in the domain of definition of E.

A tensor-valued function $\bar{\bar{g}}(\bar{\bar{B}}_1, \bar{\bar{B}}_2, \ldots, \bar{\bar{B}}_n)$ is said to be isotropic if the relation

$$\bar{\bar{Q}}^T\cdot\bar{\bar{g}}(\bar{\bar{B}}_1, \bar{\bar{B}}_2, \ldots, \bar{\bar{B}}_n)\cdot\bar{\bar{Q}} = \bar{\bar{g}}(\bar{\bar{Q}}^T\cdot\bar{\bar{B}}_1\cdot\bar{\bar{Q}}, \bar{\bar{Q}}^T\cdot\bar{\bar{B}}_2\cdot\bar{\bar{Q}}, \ldots, \bar{\bar{Q}}^T\cdot\bar{\bar{B}}_n\cdot\bar{\bar{Q}}) \qquad (A.3.2)$$

is satisfied for all orthogonal tensors $\bar{\bar{Q}}$ and all $\bar{\bar{B}}_n$ in the domain of definition of $\bar{\bar{g}}$.

Consider a point R in the reference configuration $\bar{\kappa}$ undergoing deformation with $\bar{r}$ as the position vector in the present configuration. Referring the motion to a configuration $\bar{\kappa}_1$ where the position vector of the point is $\bar{R}_1$,

$$\frac{\partial \bar{r}}{\partial \bar{R}} = \frac{\partial \bar{R}_1}{\partial \bar{R}} \cdot \frac{\partial \bar{r}}{\partial \bar{R}_1} \qquad (A.3.3)$$

$\frac{\partial \bar{R}_1}{\partial \bar{R}} = \bar{\bar{G}}$ changes the reference configuration from $\bar{\kappa}$ to $\bar{\kappa}_1$.

If the response functionals of the point R be the same in both configurations $\bar{\kappa}$ and $\bar{\kappa}_1$, then

$$\bar{\bar{G}}_{\bar{\kappa}_1}(\bar{\bar{F}}_1) = \bar{\bar{G}}_{\bar{\kappa}}(\bar{\bar{G}} \cdot \bar{\bar{F}}_1) \qquad (A.3.4)$$

where

$$\bar{\bar{F}} = \frac{\partial \bar{r}}{\partial \bar{R}} = \frac{\partial \bar{R}_1}{\partial \bar{R}} \cdot \frac{\partial \bar{r}}{\partial \bar{R}_1} = \bar{\bar{G}} \cdot \bar{\bar{F}}_1 \qquad (A.3.4a)$$

Law of Conservation of Mass states

$$\rho_{\bar{\kappa}} = \rho_{\bar{\kappa}_1} \frac{dv_{\bar{\kappa}_1}}{dV_{\bar{\kappa}}}$$

where

$$\frac{dv_{\bar{\kappa}_1}}{dV_{\bar{\kappa}}} = |\bar{\bar{G}}| \ , \tag{A.3.5}$$

$dv_{\bar{\kappa}_1}$ and $dV_{\bar{\kappa}}$ being the volume elements in the configuration $\bar{\kappa}_1$ and the reference configuration $\bar{\kappa}$ respectively.

Isotropy ensures that

$$\rho_{\bar{\kappa}} = \rho_{\bar{\kappa}_1}$$

$$\therefore \quad \frac{dv_{\bar{\kappa}_1}}{dV_{\bar{\kappa}}} = 1 = |\bar{\bar{G}}| \tag{A.3.6}$$

$\bar{\bar{G}}$ belongs to a group called *Unimodular Tensors* to be denoted by $\{\bar{\bar{H}}_i\}$.

The total set of $\bar{\bar{H}}$, $\bar{\bar{H}}_r$, $\bar{\bar{H}}_{r+1}\cdot\bar{\bar{H}}_r$ etc. form the *Isotropy Group*.

A constitutive functional $\bar{\bar{G}}$ defines a solid if its isotropy group is a subgroup of the orthogonal group, i.e., a solid is *isotropic* if the isotropy group of its defining functional is the full orthogonal group.

If there is a change of reference frame,

$$\bar{\bar{G}}(\bar{\bar{F}}^*) = \bar{\bar{G}}(\bar{\bar{F}}\cdot\bar{\bar{Q}}) \tag{A.3.7}$$

Principle of Isotropy of space (from (A.3.4a)) gives

$$\bar{\bar{G}}(\bar{\bar{Q}}^T \cdot \bar{\bar{F}}*) = \bar{\bar{G}}(\bar{\bar{F}} \cdot \bar{\bar{Q}}) \qquad \text{(A.3.8)}$$

From (A.3.7) and (A.3.8),

$$\bar{\bar{G}}(\bar{\bar{Q}}^T \cdot \bar{\bar{F}} \cdot \bar{\bar{Q}}) = \bar{\bar{G}}(\bar{\bar{F}} \cdot \bar{\bar{Q}}) \qquad \text{(A.3.9)}$$

Principle of Material Frame-Indifference States

$$\bar{\bar{G}}(\bar{\bar{F}} \cdot \bar{\bar{Q}}) = \bar{\bar{Q}}^T \cdot \bar{\bar{G}}(\bar{\bar{F}}) \cdot \bar{\bar{Q}} \qquad \text{(A.3.10)}$$

From (A.3.9) and (A.3.10)

$$\bar{\bar{G}}(\bar{\bar{Q}}^T \cdot \bar{\bar{F}} \cdot \bar{\bar{Q}}) = \bar{\bar{Q}}^T \cdot \bar{\bar{G}}(\bar{\bar{F}}) \cdot \bar{\bar{Q}} \qquad \text{(A.3.11)}$$

All isotropic tensor-valued functions obey (A.3.11) for all orthogonal $\bar{\bar{Q}}$.

APPENDIX B

B.1 Constitutive Equations

Any mechanical constitutive relation assumes the following fundamental postulates [14].

1. Principle of Determinism of Stress:

Stress in a body is determined by the history of motion of that body. For a simple material.

$$\bar{\bar{\sigma}}(t) = \mathop{\bar{\bar{\mathfrak{z}}}}_{s=0}^{\infty} (\bar{\chi}_s)$$

2. Principle of Local Action:

In determining the stress at a given point R, the motion outside an arbitrary neighbourhood of R may be disregarded.

$$\bar{\bar{\mathfrak{z}}}(\bar{\chi}, R) = \bar{\bar{\mathfrak{z}}}(\bar{\chi}', R)$$

3. Principle of Material Frame-Indifference:

Constitutive equations must be invariant under changes of frame of reference. (Two observers in relative motion to each other observe the same stress in a given body).

If $\bar{\bar{\sigma}}$ and $\bar{\bar{\sigma}}*$ be the stresses in two reference frames,

$$\bar{\bar{\sigma}}* = \bar{\bar{Q}}^T \cdot \bar{\bar{\sigma}} \cdot \bar{\bar{Q}}; \quad \text{for all orthogonal } \bar{\bar{Q}}$$

According to the Principle of Determinism,

$$\bar{\bar{\sigma}}(R,t) = \bar{\bar{\mathfrak{z}}}(\bar{\chi}^t, R, t) \tag{B.1.1}$$

Principle of Local Action imposes a further restriction in the form

$$\bar{\bar{\sigma}}(R,t) = \bar{\bar{\mathfrak{z}}}(\bar{\chi}^t, (R,R^1), t) \; ; \; R^1 \epsilon N(R, t-s) \tag{B.1.2}$$

where N is an arbitrary neighbourhood.

For a change of frame given by

$$\bar{\chi}*(R^1,\tau) = \bar{C}(\tau) + \bar{\chi}(R^1,\tau) \cdot \bar{\bar{Q}} \tag{B.1.3}$$

Principle of Material Frame Indifference [34] leads to

$$\left.\begin{aligned} &\bar{\bar{\sigma}}(t) = \bar{\bar{\mathfrak{z}}}(\bar{\bar{F}}(t)) \\ &\text{where} \\ &\bar{\bar{Q}}^T \cdot \bar{\bar{\mathfrak{z}}}(\bar{\bar{F}}(t)) \cdot \bar{\bar{Q}} = \bar{\bar{\mathfrak{z}}}(\bar{\bar{F}}(t) \cdot \bar{\bar{Q}}) \end{aligned}\right\} \tag{B.1.4}$$

In the reduced form (4.15), (B.1.4-1) may be written as

$$\bar{\bar{\sigma}}(t) = \bar{\bar{R}}^T(t) \cdot \overset{\infty}{\underset{s=0}{\bar{\bar{G}}}} (\bar{\bar{U}}(t-s)) \cdot \bar{\bar{R}}$$

or (B.1.5)

$$\bar{\bar{R}}(t) \cdot \bar{\bar{\sigma}}(t) \cdot \bar{\bar{R}}^T(t) = \overset{\infty}{\underset{s=0}{\bar{\bar{g}}}} (\bar{\bar{U}}^*_t(t-s);\ \bar{\bar{U}}(t))$$

where

$$\bar{\bar{U}}^*_t(t) = \bar{\bar{R}}(t) \cdot \bar{\bar{U}}_t(t-s) \cdot \bar{\bar{R}}^T(t)$$

For a Material of Differential Type, stress is determined by the n time derivatives of deformation gradient $\bar{\bar{F}}^t(s)$ calculated at s=0. Equation (B.1.5) can be written for Materials of Differential Type as

$$\bar{\bar{R}}(t) \cdot \bar{\bar{\sigma}}(t) \cdot \bar{\bar{R}}^T(t) = \bar{\bar{k}}(\bar{\bar{U}}^*_t(t),\ \dot{\bar{\bar{U}}}^*_t(t), \ldots, \overset{(n)}{\bar{\bar{U}}}{}^*_t(t);\ \bar{\bar{U}}(t)) \qquad \text{(B.1.6)}$$

For

$$\bar{\bar{D}}_t(t) = \dot{\bar{\bar{U}}}_t(t), \text{ and}$$

$$\bar{\bar{D}}^*_t(t) = \bar{\bar{R}}^T(t) \cdot \bar{\bar{D}}_t(t) \cdot \bar{\bar{R}}(t)$$

(B.1.6) becomes

$$\bar{\bar{R}}(t)\cdot\bar{\bar{\sigma}}(t)\cdot\bar{\bar{R}}^T(t) = \bar{\ell}(\bar{\bar{D}}_1^*, \bar{\bar{D}}_2^*, \ldots, \bar{\bar{D}}_n^*; \bar{\bar{U}}) \qquad (B.1.7)$$

Defining (B.1.5) as

$$\bar{\bar{R}}(t)\cdot\bar{\bar{\sigma}}(t)\cdot\bar{\bar{R}}^T(t) = \bar{h}(\bar{\bar{B}}_t^*(t-s); \bar{\bar{B}}(t))$$

where

$$\bar{\bar{B}}_t^*(t-s) = \bar{\bar{R}}(t)\cdot\bar{\bar{B}}_t(t-s)\cdot\bar{\bar{R}}^T(t),$$

(B.1.7) becomes

$$\bar{\bar{R}}(t)\cdot\bar{\bar{\sigma}}(t)\cdot\bar{\bar{R}}^T(t) = \bar{j}(\bar{\bar{A}}_1^*, \bar{\bar{A}}_2^*, \ldots, \bar{\bar{A}}_n^*; \bar{\bar{B}}(t)) \qquad (B.1.8)$$

Applying condition of Isotropy, (B.1.7) and (B.1.8) become

$$\bar{\bar{\sigma}} = \bar{\ell}(\bar{\bar{D}}_1, \bar{\bar{D}}_2, \ldots, \bar{\bar{D}}_n; \bar{\bar{V}})$$

$$\bar{\bar{\sigma}} = \bar{j}(\bar{\bar{A}}_1, \bar{\bar{A}}_2, \ldots, \bar{\bar{A}}_n; \bar{\bar{C}}) \qquad (B.1.9)$$

as the constitutive equations of Materials of Differential Type.

B.2 Representation Theorems:

Representation theorems for isotropic tensors are considered.

An isotropic function is a constitutive function subject to a definite objectivity - symmetry condition. A constitutive relation between any functional and the functions upon which it depends is given by a representation theorem.

Let $\bar{\bar{A}}_1, \ldots, \bar{\bar{A}}_N$; $\bar{\bar{W}}_1, \ldots, \bar{\bar{W}}_M$; and $\bar{v}_1, \ldots, \bar{v}_P$ denote N symmetric tensors, M anti-symmetric tensors and P vectors respectively.

The functions

$$\mathcal{E}(\bar{\bar{A}}_1, \bar{\bar{A}}_2, \ldots, \bar{\bar{A}}_N; \bar{\bar{W}}_1, \bar{\bar{W}}_2, \ldots, \bar{\bar{W}}_M; \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_P)$$

$$\bar{\mathfrak{z}}(\bar{\bar{A}}_1, \bar{\bar{A}}_2, \ldots, \bar{\bar{A}}_N; \bar{\bar{W}}_1, \bar{\bar{W}}_2, \ldots, \bar{\bar{W}}_M; \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_P)$$

$$\bar{\bar{H}}(\bar{\bar{A}}_1, \bar{\bar{A}}_2, \ldots, \bar{\bar{A}}_N; \bar{\bar{W}}_1, \bar{\bar{W}}_2, \ldots, \bar{\bar{W}}_M; \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_P)$$

$$\bar{\bar{Z}}(\bar{\bar{A}}_1, \bar{\bar{A}}_2, \ldots, \bar{\bar{A}}_N; \bar{\bar{W}}_1, \bar{\bar{W}}_2, \ldots, \bar{\bar{W}}_M; \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_P)$$

are said to be scalar valued, vector valued, symmetric tensor valued, and skew tensor valued isotropic functions respectively if

$$\mathcal{E}(\bar{\bar{Q}}^T \cdot \bar{\bar{A}}_i \cdot \bar{\bar{Q}}, \bar{\bar{Q}}^T \cdot \bar{\bar{W}}_p \cdot \bar{\bar{Q}}, \bar{v}_m \cdot \bar{\bar{Q}}) = \mathcal{E}(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m) \qquad \text{(B.2.1)}$$

where

$\mathcal{E}$ is the mapping

$$\mathcal{E}: S^N \times A^M \times V^P \rightarrow R \qquad (B.2.2)$$

where S, A and V denote spaces of symmetric tensors, anti-symmetric tensors and vectors respectively.

$$\bar{\mathcal{G}}(\bar{\bar{Q}}^T \cdot \bar{\bar{A}}_i \cdot \bar{\bar{Q}}, \; \bar{\bar{Q}}^T \cdot \bar{\bar{W}}_p \cdot \bar{\bar{Q}}, \; \bar{v}_m \cdot \bar{\bar{Q}}) = \bar{\mathcal{G}}(\bar{\bar{A}}_i, \; \bar{\bar{W}}_p, \; \bar{v}_m) \cdot \bar{\bar{Q}} \qquad (B.2.3)$$

where $\bar{\mathcal{G}}$ is the mapping

$$\bar{\mathcal{G}}: S^N \times A^M \times V^P \rightarrow V \qquad (B.2.4)$$

$$\bar{\bar{\mathcal{H}}}(\bar{\bar{Q}}^T \cdot \bar{\bar{A}}_i \cdot \bar{\bar{Q}}, \; \bar{\bar{Q}}^T \cdot \bar{\bar{W}}_p \cdot \bar{\bar{Q}}, \; \bar{v}_m \cdot \bar{\bar{Q}}) = \bar{\bar{Q}}^T \cdot \bar{\bar{\mathcal{H}}}(\bar{\bar{A}}_i, \; \bar{\bar{W}}_p, \; \bar{v}_m) \cdot \bar{\bar{Q}} \qquad (B.2.5)$$

where $\bar{\bar{\mathcal{H}}}$ is the mapping

$$\bar{\bar{\mathcal{H}}}: S^N \times A^M \times V^P \rightarrow S \qquad (B.2.6)$$

$$\bar{\bar{\mathcal{Z}}}[\bar{\bar{Q}}^T \cdot \bar{\bar{A}}_i \cdot \bar{\bar{Q}}, \; \bar{\bar{Q}}^T \cdot \bar{\bar{W}}_p \cdot \bar{\bar{Q}}, \; \bar{v}_m \cdot \bar{\bar{Q}}] = \bar{\bar{Q}}^T \cdot \bar{\bar{\mathcal{Z}}}[\bar{\bar{A}}_i, \; \bar{\bar{W}}_p, \; \bar{v}_m] \cdot \bar{\bar{Q}} \qquad (B.2.7)$$

where $\bar{\bar{\mathcal{Z}}}$ is the mapping

$$\bar{\bar{\mathcal{Z}}}: S^N \times A^M \times V^P \rightarrow A \qquad (B.2.8)$$

holds for all orthagonal $\bar{\bar{Q}}$.

A function basis for isotropic invariants of $\bar{\bar{A}}_1$, $\bar{\bar{A}}_2$, ..., $\bar{\bar{A}}_N$; $\bar{\bar{W}}_1$, $\bar{\bar{W}}_2, \ldots, \bar{\bar{W}}_M$; $\bar{v}_1$, $\bar{v}_2, \ldots, \bar{v}_P$ is a set of isotropic invariants $I_1, \ldots, I_r$ such that any isotropic invariant $I_k(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m)$ is a single valued function of $I_1, \ldots, I_r$ [26]. RIVLIN and ERICKSEN [22] have determined such a function basis for isotropic invariants of two symmetric tensors. WANG [29] has employed a different approach to derive a function basis for the general case. His results have been shown to be either insufficient, or invalid in particular cases by SMITH [26] who has generalised RIVLIN and ERICKSEN's approach to derive a function basis for the general case.

Among the generally known representations [29] are

i. CAUCHY'S representation formula for (B.2.1) with N=0, M=0, P = arbitrary

$$E = E(\bar{v}_1 \cdot \bar{v}_1,\ \bar{v}_1 \cdot \bar{v}_2, \ldots,\ \bar{v}_m \cdot \bar{v}_{m-1},\ \bar{v}_m \cdot \bar{v}_m) \qquad (B.2.9)$$

ii. Representation formula for (B.2.1) with N=1, M=0, P=0

$$E = E(I_{\bar{\bar{A}}},\ II_{\bar{\bar{A}}},\ III_{\bar{\bar{A}}}) \qquad (B.2.10)$$

where $I_{\bar{\bar{A}}}$, $II_{\bar{\bar{A}}}$, and $III_{\bar{\bar{A}}}$ are the three principal invariants of $\bar{\bar{A}}_1$.

iii. NOLL'S representation formula for (B.2.1) with N=1, M=0, P=1

$$\mathcal{E} = \mathcal{E}(I_{\bar{\bar{A}}}, II_{\bar{\bar{A}}}, III_{\bar{\bar{A}}}, \bar{v}\cdot\bar{v}, \bar{v}\cdot\bar{\bar{A}}_1\cdot\bar{v}, \bar{v}\cdot(\bar{\bar{A}}_1)^2\cdot\bar{v}) \qquad (B.2.11)$$

iv. NOLL'S representation for (B.2.3) with N=1, M=0, P=1

$$\bar{\mathcal{J}} = \bar{v}\cdot(\mathcal{E}_0\bar{\bar{I}} + \mathcal{E}_1\bar{\bar{A}}_1 + \mathcal{E}_2(\bar{\bar{A}}_1)^2) \qquad (B.2.12)$$

where $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ are scalar valued isotropic functions of $(\bar{\bar{A}}, \bar{v})$

v. RIVLIN-ERICKSEN'S representation formula for (B.2.5) with N=1, M=0, P=0

$$\bar{\bar{\mathcal{A}}} = \mathcal{E}_0\bar{\bar{I}} + \mathcal{E}_1\bar{\bar{A}}_1 + \mathcal{E}_2(\bar{\bar{A}}_1)^2 \qquad (B.2.13)$$

where $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ are scalar valued isotropic functions of $\bar{\bar{A}}$.

vi. RIVLIN-ERICKSEN'S representationfformula for (B.2.1) with N=2, M=0, P=0.

$$\mathcal{E} = \mathcal{E}(\ell_{\bar{\bar{A}}_1}, \bar{\bar{A}}_2) \qquad (B.2.14)$$

where $\ell_{\bar{\bar{A}}_1, \bar{\bar{A}}_2}$ is a set of ten basic invariants of $\bar{\bar{A}}_1, \bar{\bar{A}}_2$.

$$\ell_{\bar{\bar{A}}_1, \bar{\bar{A}}_2} = \{\bar{\bar{I}}:\bar{\bar{A}}_1,\ \bar{\bar{I}}:(\bar{\bar{A}}_1)^2,\ \bar{\bar{I}}:(\bar{\bar{A}}_1)^3,\ \bar{\bar{I}}:\bar{\bar{A}}_2,\ \bar{\bar{I}}:(\bar{\bar{A}}_2)^2,\ \bar{\bar{I}}:(\bar{\bar{A}}_2)^3,$$

$$\bar{\bar{I}}:(\bar{\bar{A}}_1\cdot\bar{\bar{A}}_2),\ \bar{\bar{I}}:(\bar{\bar{A}}_1\cdot(\bar{\bar{A}}_2)^2),\ \bar{\bar{I}}:((\bar{\bar{A}}_1)^2\cdot\bar{\bar{A}}_2),\ \bar{\bar{I}}:((\bar{\bar{A}}_1)^2\cdot(\bar{\bar{A}}_2)^2)\}$$

In the original work, RIVLIN and ERICKSEN used $|\bar{\bar{A}}_2|$ instead of $\bar{\bar{I}}:(\bar{\bar{A}}_2)^3$. However, knowing the other nine invariants, $|\bar{\bar{A}}_2|$ determines $\bar{\bar{I}}:(\bar{\bar{A}}_2)^3$ uniquely.

vii. RIVLIN'S representation formula for (B.2.5) with N=2, M=0, P=0

$$\bar{\bar{H}} = E_0\bar{\bar{I}} + E_1\bar{\bar{A}}_1 + E_2\bar{\bar{A}}_2 + E_3(\bar{\bar{A}}_1)^2 + E_4(\bar{\bar{A}}_2)^2 + E_5(\bar{\bar{A}}_1\cdot\bar{\bar{A}}_2 + \bar{\bar{A}}_2\cdot\bar{\bar{A}}_1)$$
$$+ E_6\{(\bar{\bar{A}}_1)^2\cdot\bar{\bar{A}}_2 + \bar{\bar{A}}_2\cdot(\bar{\bar{A}}_1)^2\} + E_7\{\bar{\bar{A}}_1\cdot(\bar{\bar{A}}_2)^2 + (\bar{\bar{A}}_2)^2\cdot\bar{\bar{A}}_1\}$$
$$+ E_8\{(\bar{\bar{A}}_1)^2\cdot(\bar{\bar{A}}_2)^2 + (\bar{\bar{A}}_2)^2\cdot(\bar{\bar{A}}_1)^2\} \qquad \text{(B.2.15)}$$

where $E_0, \ldots, E_8$ are scalar valued isotropic functions of $(\bar{\bar{A}}_1, \bar{\bar{A}}_2)$.

viii. WANG'S general representation theorem for a scalar valued isotropic function: For a three dimensional space, a complete and irreducible representation for any scalar valued isotropic function $E(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m)$ is given by

$$E(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m) = F(\ell_{\bar{\bar{A}}_1, \ldots, \bar{\bar{A}}_N; \bar{\bar{W}}_1, \ldots, \bar{\bar{W}}_M; \bar{v}_1, \ldots, \bar{v}_P}) \quad (B.2.16)$$

where $\ell_{\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m}$ is formed by all invariants of $\bar{\bar{A}}$, $\bar{\bar{W}}$ and $\bar{v}$ based on any one, two, three or four variables in the argument $(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m)$ to within a permutation of entries.

ix. WANG'S general representation theorem for vector valued Isotropic functions: for a three dimensional space, a complete and irreducible representation for any vector valued isotropic function $\bar{f}(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m)$ is given by

$$\bar{f}(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m) = \sum_{k=1}^{g} f^k(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m)\bar{f}_k(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m) \quad (B.2.17)$$

where $f^1, \ldots, f^g$ are scalar valued isotropic functions and $\{\bar{f}_1, \ldots, \bar{f}_g\}$ is a generating set formed by all generators based on all combinations of none, one, two or three variables in the argument $(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m)$ to within a permutation of entries.

x. WANG'S general representation theorem for symmetric tensor valued isotropic functions: for a three dimensional space, a complete and irreducible representation for any symmetric tensor valued isotropic function $\bar{\bar{H}}[\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m]$ is given by

$$\bar{\bar{H}}(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m) = \sum_{k=1}^{g} H^k(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m)\ \bar{\bar{H}}_k(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m) \qquad (B.2.18)$$

where $\bar{H}^1, \ldots, \bar{H}^g$ are scalar-valued isotropic functions and where $\{\bar{H}_1, \ldots, \bar{H}_g\}$ is a generating set formed by all generators based on all combinations of none, one, two or three variables in the argument $(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m)$ to within a permutation.

xi. WANG'S general representation theorem for anti-symmetric tensor valued isotropic function: for a three dimensional space, a complete and irreducible representation for any anti-symmetric tensor valued isotropic function $\bar{Z}(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m)$ is given by

$$\bar{Z}(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m) = \sum_{k=1}^{g} Z^k(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m)\ \bar{Z}_k(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m) \qquad (B.2.19)$$

where $Z^1, \ldots, Z^g$ are scalar valued isotropic functions, and $(\bar{Z}_1, \ldots, \bar{Z}_g)$ is a generating set formed by all generators based on all combinations of none, one, two or three variables in the argument $(\bar{\bar{A}}_i, \bar{\bar{W}}_p, \bar{v}_m)$ to within a permutation.

As an example of (B.2.17), representation of a vector valued isotropic function $\bar{\mathfrak{z}}$, of two symmetric tensors $\bar{\bar{A}}$ and $\bar{\bar{B}}$, and a vector $\bar{v}$ may be written as

$$\bar{\mathfrak{z}} = \alpha_1\bar{v} + \alpha_2\bar{v}\cdot\bar{\bar{A}} + \alpha_3\bar{v}\cdot(\bar{\bar{A}})^2 + \alpha_4\bar{v}\cdot\bar{\bar{B}} + \alpha_5\bar{v}\cdot(\bar{\bar{B}})^2 + \alpha_6\bar{v}\cdot\bar{\bar{A}}\cdot\bar{\bar{B}}$$

$$+ \alpha_7\bar{v}\cdot\bar{\bar{B}}\cdot\bar{\bar{A}} \qquad (B.2.20)$$

where $\alpha_1, \ldots, \alpha_7$ are scalar valued isotropic functions of $\bar{\bar{A}}$, $\bar{\bar{B}}$ and $\bar{v}$.

xii. SMITH'S representation for a vector valued isotropic function $\bar{\delta}$ of two symmetric tensors $\bar{\bar{A}}$, $\bar{\bar{B}}$ and a vector $\bar{v}$ is given as

$$\bar{\delta} = \beta_1\bar{v} + \beta_2\bar{v}\cdot\bar{\bar{A}} + \beta_3\bar{v}\cdot(\bar{\bar{A}})^2 + \beta_4\bar{v}\cdot\bar{\bar{B}} + \beta_5\bar{v}\cdot(\bar{\bar{B}})^2 + \beta_6\bar{v}\cdot(\bar{\bar{A}}\cdot\bar{\bar{B}} - \bar{\bar{B}}\cdot\bar{\bar{A}}) \tag{B.2.21}$$

where $\beta_1, \ldots, \beta_6$ are scalar valued isotropic functions of $\bar{\bar{A}}$, $\bar{\bar{B}}$ and $\bar{v}$.

B.3 Controllable Motions

Representation theorems for isotropic tensors have been discussed in (B.2). For motions of homogeneous isotropic Materials of Differential Type, determinate stress $\bar{\bar{\sigma}}_d$ may be represented by a functional $\bar{\bar{G}}$ of the form (2.12).

From the first Cauchy Axiom of Motion for continua,

$$\frac{\partial\cdot\bar{\bar{\sigma}}_d}{\partial\bar{r}} + \bar{f} = \rho\,\frac{d\bar{v}}{dt} \tag{B.3.1}$$

where $\bar{f}$ is the external body force; ρ the density of the continuum, and $\bar{v}$, the velocity of motion.

From (2.12) and (B.3.1),

$$\frac{\partial \cdot \bar{\bar{G}}(\bar{\bar{A}}_1, \bar{\bar{A}}_2, \ldots, \bar{\bar{A}}_n)}{\partial \bar{r}} + \bar{f} = \rho \frac{d\bar{v}}{dt} \tag{B.3.2}$$

● If the external body force $\bar{f}$ is a known function of $\bar{r}$ and t, (B.3.2) involves solving a differential equation with suitable initial and boundary conditions to determine the motion $\bar{r} = \bar{\chi}$ $(\bar{R},t)$. All solution of (B.3.2) are dynamically possible motions of the material under consideration.

Assuming incompressibility, and the body force $\bar{f}$ to be conservative, i.e., $\bar{f} = \dfrac{\partial e}{\partial \bar{r}}$ where e is single valued; from (6.14), (2.13) and (B.3.2),

$$\begin{aligned}\frac{\partial}{\partial \bar{r}} \cdot \{&\alpha_1\bar{\bar{A}}_1 + \alpha_2\bar{\bar{A}}_2 + \alpha_3(\bar{\bar{A}}_1)^2 + \alpha_4(\bar{\bar{A}}_2)^2 + \alpha_5(\bar{\bar{A}}_1\cdot\bar{\bar{A}}_2 + \bar{\bar{A}}_2\cdot\bar{\bar{A}}_1) \\ &+ \alpha_6((\bar{\bar{A}}_1)^2\cdot\bar{\bar{A}}_2 + \bar{\bar{A}}_2\cdot(\bar{\bar{A}}_1)^2 + \alpha_7(\bar{\bar{A}}_1\cdot(\bar{\bar{A}}_2)^2 + (\bar{\bar{A}}_2)^2\cdot\bar{\bar{A}}_1) \\ &+ \alpha_8((\bar{\bar{A}}_1)^2\cdot(\bar{\bar{A}}_2)^2 + (\bar{\bar{A}}_2)^2\cdot(\bar{\bar{A}}_1)^2)\} + \frac{\partial(q+e)}{\partial \bar{r}} = \rho\frac{d\bar{v}}{dt}\end{aligned} \tag{B.3.3}$$

where q is an arbitrary pressure defined in (6.14).

Assumption of body force $\bar{f}$ to be conservative does not make the solution of the incompressible material more difficult since in (B.3.3), an adjustment in pressure q takes care of any changes in e so that (q+e) remains a constant.

Given $\bar{\bar{A}}_1$ and $\bar{\bar{A}}_2$, whether (B.3.3) is satisfied will depend in general, upon the functions α_i. However, there are certain forms of $\bar{\bar{A}}_1$ and $\bar{\bar{A}}_2$ such that (B.3.3) is satisfied no matter what α_i be. The motions with these forms of Rivlin-Ericksen tensors $\bar{\bar{A}}_1$ and $\bar{\bar{A}}_2$ can be affected in every homogeneous, incompressible, isotropic Material of Differential Type by application of surface tractions alone. The values of surface tractions and resulting interior stresses in a particular body will of course depend strongly upon the material properties. When the motion of a material is supported by surface traction alone, it has been referred to as a *controllable motion* by SINGH and PIPKIN [25]. In the present work, the phrase has been used in the same context.

APPENDIX C

Calculations resulting in equations (7.26 b-e) are shown in this appendix.

From (7.26),

$$[\bar{\bar{F}}_t(\tau)] = [1 - e^{x^i} + e^{a_i s + x^i}]^{-1}, \quad i = 1,2,3$$

$$\frac{\partial}{\partial s}[\bar{\bar{F}}_t(\tau)] = [(-1)^1(-)(1 - e^{x^{\underline{i}}} + e^{a_{\underline{i}} s + x^{\underline{i}}})^{-2}\, a_{\underline{i}} e^{a_{\underline{i}} s + x^{\underline{i}}}]$$

$$\therefore \quad \frac{\partial}{\partial s}[\bar{\bar{F}}_t(\tau)]\Big|_{s=0} = [\bar{\bar{L}}_1] = [a_{\underline{i}}\, e^{x^{\underline{i}}}] \qquad \text{(C.1)}$$

$$\frac{\partial^2}{\partial s^2}[\bar{\bar{F}}_t(\tau)] = [2(1 - e^{x^{\underline{i}}} + e^{a_{\underline{i}} s + x^{\underline{i}}})^{-3}\,(a_{\underline{i}})^2\, e^{2(a_{\underline{i}} s + x^{\underline{i}})}$$

$$- (1 - e^{x^{\underline{i}}} + e^{a_{\underline{i}} s + x^{\underline{i}}})^{-2}(a_{\underline{i}})^2\, e^{a_{\underline{i}} s + x^{\underline{i}}}]$$

$$\therefore \quad \frac{\partial^2}{\partial s^2}[\bar{\bar{F}}_t(\tau)]\Big|_{s=0} = [\bar{\bar{L}}_2] = [2(a_{\underline{i}})^2\, e^{2x^{\underline{i}}} - (a_{\underline{i}})^2\, e^{x^{\underline{i}}}]$$

$$= [(a_{\underline{i}})^2\, e^{x^{\underline{i}}}(2e^{x^{\underline{i}}} - 1)] \qquad \text{(C.2)}$$

$$\frac{\partial^3}{\partial s^3}[\bar{\bar{F}}_t(\tau)] = [6(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-4}\ (a_{\underline{i}})^3\ e^{3(a_{\underline{i}}s + x_{\underline{i}})}$$

$$- 4(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-3}\ (a_{\underline{i}})^3\ e^{2(a_{\underline{i}}s + x_{\underline{i}})}$$

$$- 2(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-3}\ (a_{\underline{i}})^3\ e^{2(a_{\underline{i}}s + x_{\underline{i}})}$$

$$+ (1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-2}(a_{\underline{i}})^3\ e^{2(a_{\underline{i}}s + x_{\underline{i}})}$$

$$= [6(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-4}\ (a_{\underline{i}})^3\ e^{3(a_{\underline{i}}s + x_{\underline{i}})}$$

$$- 6(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-3}\ (a_{\underline{i}})^3\ e^{2(a_{\underline{i}}s + x_{\underline{i}})}$$

$$+ (1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-2}(a_{\underline{i}})^3\ e^{(a_{\underline{i}}s + x_{\underline{i}})}]$$

$$\therefore\ [\bar{\bar{L}}_3] = [6(a_{\underline{i}})^3\ e^{3x_{\underline{i}}} - 6(a_{\underline{i}})^3\ e^{2x_{\underline{i}}} + (a_{\underline{i}})^3\ e^{x_{\underline{i}}}]$$

$$= [(a_{\underline{i}})^3\ e^{x_{\underline{i}}}\ (6e^{2x_{\underline{i}}} - 6e^{x_{\underline{i}}} + 1)] \qquad (C.3)$$

$$\frac{\partial^4}{\partial s^4}\,[\bar{\bar{F}}_t(\tau)] = [24(1 - e^{x_i} + e^{a_i s + x_i})^{-5}\,(a_i)^4\,e^{4(a_i s + x_i)}$$

$$- 18(1 - e^{x_i} + e^{a_i s + x_i})^{-4}(a_i)^4\,e^{3(a_i s + x_i)}$$

$$- 18(1 - e^{x_i} + e^{a_i s + x_i})^{-4}(a_i)^4\,e^{3(a_i s + x_i)}$$

$$+ 12(1 - e^{x_i} + e^{a_i s + x_i})^{-3}\,(a_i)^4\,e^{2(a_i s + x_i)}$$

$$+ 2(1 - e^{x_i} + e^{a_i s + x_i})^{-3}\,(a_i)^4\,e^{2(a_i s + x_i)}$$

$$- (1 - e^{x_i} + e^{a_i s + x_i})^{-2}\,(a_i)^4\,e^{a_i s + x_i}]$$

$$= [24(1 - e^{x_i} + e^{a_i s + x_i})^{-5}\,(a_i)^4\,e^{4(a_i s + x_i)}$$

$$- 36(1 - e^{x_i} + e^{a_i s + x_i})^{-4}(a_i)^4\,e^{3(a_i s + x_i)}$$

$$+ 14(1 - e^{x_i} + e^{a_i s + x_i})^{-3}(a_i)^4\,e^{2(a_i s + x_i)}$$

$$- (1 - e^{x_i} + e^{a_i s + x_i})^{-2} (a_i)^4 \, e^{a_i s + x_i}]$$

$$\therefore\ [\bar{\bar{L}}_4] = [24(a_i)^4 \, e^{4x_i} - 36(a_i)^4 \, e^{3x_i} + 14(a_i)^4 \, e^{2x_i} - (a_i)^4 \, e^{x_i}]$$

$$= [(a_i)^4 \, e^{x_i}(24e^{3x_i} - 36e^{2x_i} + 14e^{x_i} - 1)] \qquad (C.4)$$

$$\frac{\partial^5}{\partial s^5} [\bar{\bar{F}}_t(\tau)] = [120(1 - e^{x_i} + e^{a_i s + x_i})^{-6} (a_i)^5 \, e^{5(a_i s + x_i)}$$

$$- 96(1 - e^{x_i} + e^{a_i s + x_i})^{-5} (a_i)^5 \, e^{4(a_i s + x_i)}$$

$$- 144(1 - e^{x_i} + e^{a_i s + x_i})^{-5} (a_i)^5 \, e^{4(a_i s + x_i)}$$

$$+ 108(1 - e^{x_i} + e^{a_i s + x_i})^{-4} (a_i)^5 \, e^{3(a_i s + x_i)}$$

$$+ 42(1 - e^{x_i} + e^{a_i s + x_i})^{-4} (a_i)^5 \, e^{3(a_i s + x_i)}$$

$$- 28(1 - e^{x_i} + e^{a_i s + x_i})^{-3} (a_i)^5 \, e^{2(a_i s + x_i)}$$

$$- 2(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-3}(a_{\underline{i}})^5 e^{2(a_{\underline{i}}s + x_{\underline{i}})}$$

$$+ (1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-2}(a_{\underline{i}})^5 e^{a_{\underline{i}}s + x_{\underline{i}}}]$$

$$= [120(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-6}(a_{\underline{i}})^5 e^{5(a_{\underline{i}}s + x_{\underline{i}})}$$

$$- 240(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-5}(a_{\underline{i}})^5 e^{4(a_{\underline{i}}s + x_{\underline{i}})}$$

$$+ 150(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-4}(a_{\underline{i}})^5 e^{3(a_{\underline{i}}s + x_{\underline{i}})}$$

$$- 30(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-3}(a_i)^5 e^{2(a_{\underline{i}}s + x_{\underline{i}})}$$

$$+ (1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-2}(a_{\underline{i}})^5 e^{a_i s + x_{\underline{i}}}]$$

$$\therefore [\bar{\bar{L}}_5] = [120(a_{\underline{i}})^5 e^{5x_{\underline{i}}} - 240(a_{\underline{i}})^5 e^{4x_{\underline{i}}} + 150(a_{\underline{i}})^5 e^{3x_{\underline{i}}}$$

$$- 30(a_i)^5 e^{2x_{\underline{i}}} + (a_{\underline{i}})^5 e^{x_{\underline{i}}}]$$

$$= [(a_{\underline{i}})^5\ e^{x\underline{i}}(120\ e^{4x\underline{i}} - 240\ e^{3x\underline{i}} + 150\ e^{2x\underline{i}}$$

$$- 30\ e^{x\underline{i}} + 1)] \qquad (C.5)$$

Similarly,

$$[\bar{\bar{L}}_6] = [(a_{\underline{i}})^6\ e^{x\underline{i}}(720\ e^{5x\underline{i}} - 1800\ e^{4x\underline{i}} + 1560\ e^{3x\underline{i}}$$

$$- 540\ e^{2x\underline{i}} + 62\ e^{x\underline{i}} - 1)] \qquad (C.6)$$

The general formula for $[\bar{\bar{L}}_n]$ is

$$[\bar{\bar{L}}_n] = n!([\bar{\bar{L}}_1])^n + (n-1)! \sum_{i=1}^{n} (n-i)\ a_i([\bar{\bar{L}}_1])^{n-1}$$

$$+ (n-2)! \sum_{i=2}^{n} (n-i) \sum_{j=0}^{n-1} (n-i-j)(a_i)^2([\bar{\bar{L}}_1])^{n-2}$$

$$+ \ldots + \sum_{i=1}^{n-1} 2^{(n-i)}(a_i)^{(n-2)}([\bar{\bar{L}}_1])^2 + (a_i)^{n-1}[\bar{\bar{L}}_1] \qquad (C.7)$$

From (7.26a),

$$[\bar{\bar{C}}_t(\tau)] = ([1 - e^{x^{i}} + e^{a_{i}s + x^{i}}])^{-2}$$

$$\frac{\partial}{\partial s}\,[\bar{\bar{C}}_t(\tau)] = [2(1 - e^{x^{\underline{i}}} + e^{a_{\underline{i}}s + x^{\underline{i}}})^{-3}\, a_{\underline{i}}\, e^{a_{\underline{i}}s + x^{\underline{i}}}]$$

$$\therefore\ [\bar{\bar{A}}_1] = \frac{\partial}{\partial s}\,[\bar{\bar{C}}_t(\tau)]\Big|_{s=0} = [2a_{\underline{i}}\, e^{x^{\underline{i}}}] \qquad \text{(C.8)}$$

$$\frac{\partial^2}{\partial s^2}\,[\bar{\bar{C}}_t(\tau)] = [6(1 - e^{x^{\underline{i}}} + e^{a_{\underline{i}}s + x^{\underline{i}}})^{-4}(a_{\underline{i}})^2\, e^{2(a_{\underline{i}}s + x^{\underline{i}})}$$

$$- 2(1 - e^{x^{\underline{i}}} + e^{a_{\underline{i}}s + x^{\underline{i}}})^{-3}(a_{\underline{i}})^2\, e^{a_{\underline{i}}s + x^{\underline{i}}}]$$

$$\therefore\ [\bar{\bar{A}}_2] = [6(a_{\underline{i}})^2\, e^{2x^{\underline{i}}} - 2(a_{\underline{i}})^2\, e^{x^{\underline{i}}}] = [2(a_{\underline{i}})^2\, e^{x^{\underline{i}}}(3e^{x^{\underline{i}}} - 1)] \qquad \text{(C.9)}$$

$$\frac{\partial^3}{\partial s^3}\,[\bar{\bar{C}}_t(\tau)] = [24(1 - e^{x^{\underline{i}}} + e^{a_{\underline{i}}s + x^{\underline{i}}})^{-5}(a_{\underline{i}})^3\, e^{3(a_{\underline{i}}s + x^{\underline{i}})}$$

$$- 18(1 - e^{x^{\underline{i}}} + e^{a_{\underline{i}}s + x^{\underline{i}}})^{-4}(a_{\underline{i}})^3\, e^{2(a_{\underline{i}}s + x^{\underline{i}})}$$

$$+ 2(1 - e^{x^{\underline{i}}} + e^{a_{\underline{i}}s + x^{\underline{i}}})^{-3}(a_{\underline{i}})^3\, e^{a_{\underline{i}}s + x^{\underline{i}}}]$$

$$\therefore\ [\bar{\bar{A}}_3] = [24(a_{\underline{i}})^3\ e^{3x_{\underline{i}}} - 18(a_{\underline{i}})^3\ e^{2x_{\underline{i}}} + 2(a_{\underline{i}})^3\ e^{x_{\underline{i}}}]$$

$$= [2(a_{\underline{i}})^3\ e^{x_{\underline{i}}}(12e^{2x_{\underline{i}}} - 9e^{x_{\underline{i}}} + 1)] \qquad \text{(C.10)}$$

$$\frac{\partial^4}{\partial s^4}\ [\bar{\bar{C}}_t(\tau)] = [120(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-6}(a_{\underline{i}})^4\ e^{4(a_{\underline{i}}s + x_{\underline{i}})}$$

$$- 144(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-5}(a_{\underline{i}})^4\ e^{3(a_{\underline{i}}s + x_{\underline{i}})}$$

$$+ 42(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-4}(a_{\underline{i}})^4\ e^{2(a_{\underline{i}}s + x_{\underline{i}})}$$

$$- 2(1 - e^{x_{\underline{i}}} + e^{a_{\underline{i}}s + x_{\underline{i}}})^{-3}(a_{\underline{i}})^4\ e^{a_{\underline{i}}s + x_{\underline{i}}}]$$

$$\therefore\ [\bar{\bar{A}}_4] = [120(a_{\underline{i}})^4\ e^{4x_{\underline{i}}} - 144(a_{\underline{i}})^4\ e^{3x_{\underline{i}}} + 42(a_{\underline{i}})^4\ e^{2x_{\underline{i}}}$$

$$- 2(a_{\underline{i}})^4\ e^{x_{\underline{i}}}]$$

$$= [2(a_{\underline{i}})^4\ e^{x_{\underline{i}}}(60\ e^{3x_{\underline{i}}} - 72\ e^{2x_{\underline{i}}} + 21\ e^{x_{\underline{i}}} - 1)] \qquad \text{(C.11)}$$

A general formula for $[\bar{\bar{A}}_n]$ may be written by combining (4.29) and (C.7).