OPTIMIZATION AND DESIGN

A STUDY OF OPTIMIZATION TECHNIQUES

AND

THEIR APPLICATIONS TO DESIGN

By

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A Thesis

Submitted to the Faculty of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Engineering

McMaster University

September 1964

MASTER OF ENGINEERING (1964)
(Mechanical Engineering)

McMASTER UNIVERSITY
Hamilton, Ontario.

TITLE: A Study of Optimization Techniques and Their Applications to Design

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NUMBER OF PAGES: (vi), 163

SCOPE AND CONTENTS:

This thesis consists of a general survey of existing mathematical optimization techniques and of an attempt to apply such techniques to various design situations. Several analytical and numerical techniques of determining extrema of univariable and multivariable known or unknown functions subjected to equality and inequality constraints are investigated when such extrema lie either in the interior or on the boundary of a given region. Design examples are worked out to illustrate the applicability and the limitations of the methods and the procedure of attacking a problem for solution. Some of these applications are original with the author. An extensive survey of applications of optimization techniques to several design situations is also included. No such design orientated survey seems to have ever been taken. Comments, regarding the applicability of these techniques to various design situations and suggestions for further research are made.

ACKNOWLEDGEMENTS

It is the author's pleasure to acknowledge the encouragement, help and valuable advice from Professor J. N. Siddall and the arrangement of financial assistance from Dr. D. G. Huber. The author is also indebted to the Mills Memorial Library, National Research Council and Rand Corporation for their cooperation in furnishing valuable information.

CONTENTS

INTRODUCTION

1.1 Meaning of Optimization

Optimization is the process of searching for the best under certain prescribed conditions. It may be for maximizing a certain parameter or minimizing yet another. From this it is clear that the word optimization is not new to man. At every step a man has to make a decision and by nature inclined to make one which is best. However, he is not free to do what he likes; forces of nature, customs of society in which he lives, his own skill and abilities all force him to think several times, analyze the whole problem and apply his past experience and knowledge to mould the circumstances in such a fashion that he may arrive at a most favourable decision. This process of determining the conditions which ultimately result in arriving at a best solution is the process of optimization and the decision thus arrived at is known as an optimum decision.

Tools which can be used in arriving at such a decision are many and varied. All aspects of a problem are first to be studied carefully; every possible effort should be made in collecting necessary information about the problem; analysis of the said data in the light of past experience; prediction of future and its implications, all must be studied minutely before taking a final decision.

This problem of decision making has led man to evolve the branch of knowledge which is presently known as "mathematics".

In the process of evolution he has formulated many theories and tried various methods which, in the advent of time, he has either abandoned in favour of what seems to be better or permanently adopted. However, time and again, he has to look back and ascertain whether he can utilise the methods which he has once discarded, considering the knowledge he has acquired in the mean time. And many a time he found that methods once considered of little value proved to be the most useful tools in the future.

One such mathematical tool is a procedure known as "algorithm" bearing the name of the celebrated Muslim mathematician of the 9th century, Alkhwarthimi*, "who may be regarded perhaps as the founder of modern algebra - the name itself came from him".[2] Dealing with the origin of the word algorithm, Oyster Ore[3] writes, "the most influential work in this period is due to Mohammed ibn Musa Alkhowarzimi who lived in the beginning of the 9th century. His books on arithmetic and algebra were widely spread through translations, but by the confusion of the translators the author's name was corrupted into the word algorithm,[4] originally used to denote calculations in Hindu-Arabic numbers and still used in modern times to denote a repeated method of procedure." By this method, "to solve an equation", writes Cajorie[5], "$f(x)=v$, assume for the moment, two values of x, e.g., $x = a$, and $x = b$. Then from $f(a) = A$ and $f(b) = B$, determine the error $v-A = E_a$ and $v-B = E_b$ then the required $x = \frac{bE_a - aE_b}{E_a - E_b}$ is generally a close approximation, but is absolutely accurate whenever $f(x)$ is a linear function of x". The iterative nature of the method made it most suitable for computer applications. After undergoing several changes, at present, the word

*Abu Abdullah Mohammed Ibn-e-Musa Alkhwarthimi[1].

algorithm is used to signify any iterative mathematical procedure[6] and several such procedures are presently known and are extensively used in the solution of problems of optimization. The main difference between the conventional mathematical techniques and the optimization techniques is that the solution of the former is always unique whereas that of the latter is never so. Several feasible solutions are usually possible. The value of the criterion at the optimum point differs very little from the value at a point close to optimum and hence close approximations are always acceptable.

At present many mathematical techniques are known which have been or are being used for optimization of various type problems, and it is very difficult to say which is preferable. In this connection Leitmann[7] writes, "during the past decade there has been a remarkable growth of interest in problems of system optimization and optimal control. And with this interest has come an increasing need of methods useful for rendering a system optimum. One may expect that a particular method is superior to others for the solution of some problems, rarely for all problems".

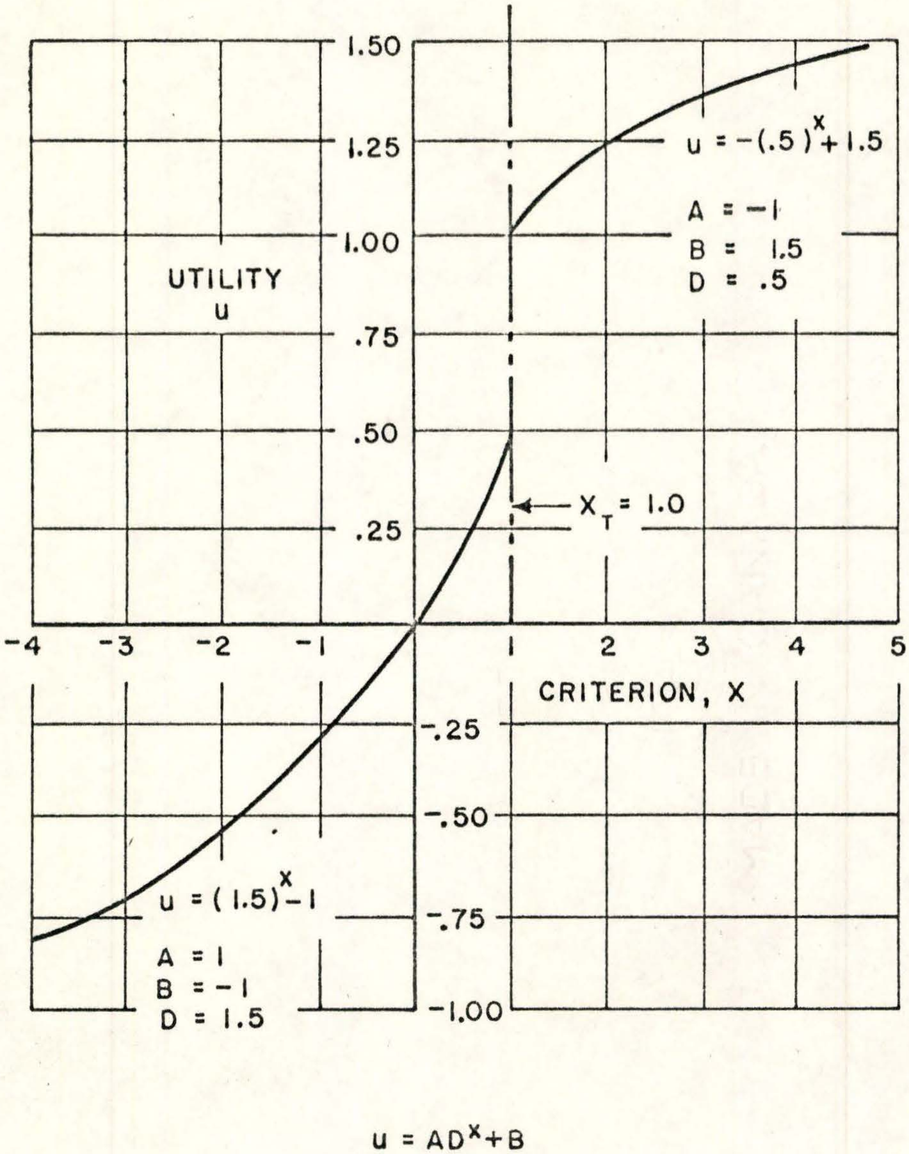
Almost all mathematical tools presently used for optimization are approximate; utility and not precision is the criteria, and as Irwin[8] writes, "a model is neither true nor false. The standard for comparing (mathematical) models is therefore dependent on the situation in which it is used; it is not intrinsic".

Some of the relatively more useful tools in the field of ordinary theory of maxima and minima, variational calculus, mathematical programming and statistics are dealt with in section 2. Applications of these methods in Engineering Design are described in section 3. The author is not aware of any previous complete survey of optimization techniques from the point of view of engineering design. Even the texts on optimization by operations researchers are surprisingly incomplete. Some of the applications given here are original with the author.

1.2 Concept of Utility

Attached to each optimization problem is the utility criteria and hence it is necessary to understand what it conveys to one who is interested in the problem of optimization.

Utility, worth or value is a relative term and depends on various factors. On account of the complex nature of a design, a plan or a decision, utility is usually measured on the basis of the most important variable under certain assumptions. It is the strategy of this variable that gives a value to an item under consideration. However, as soon as circumstances under which this value has been determined vary, the worth or value of the item also changes. No true measure of worth is therefore possible. Certain assumptions must be made and a few allowances must be given if any reasonable measure of worth is desired. In dealing with the problems of optimization in design it is therefore customary to find out the most important variable. This variable is known as pay-off function[9], objective function[10], utility function[11], optimization function[12], response function[13] or criterion[14]. In mechanical design problems the criterion may be life, efficiency, weight, or premissible error. In production design it may be cost or profit and in aerospace it may be either weight or fuel consumption.



EXAMPLE OF DESIGN UTILITY FUNCTION

FIG. 1·2

(Lifson)

In each case this criterion is either to be maximized or minimized.

Usually equipment designs are supposed to satisfy certain prescribed conditions, e.g., specifications, etc., laid down by the users. The utility of an equipment in such cases increases very rapidly at or near the threshold of specifications[15]. On both sides of this point the rate of increase in utility is comparatively very slow. At the zero value of the variable the utility is also zero and it does not reach infinity at any point. Pfanzagl[16] has suggested that this situation could be represented by the function

$$U = A\, D^{x} + B$$

where A, D and B are constants. The characteristic curve for such a situation can be represented by the curve shown in fig. 1.2. The other important design variables are strength, weight, life, serviceability, etc.

Actually the variables of this type are many and hence the shape of the utility curve encountered is always different. The method of tackling the problems are also not always the same. As for assumptions which are next to be considered, it is usually assumed that the dependent variables have an intrinsic worth which can easily be identified.

If minutely studied, worth and utility are not the same*. However, since this difference does not affect the problem under consideration, utility can be defined in a concise though not in a rigorous manner as "that which would satisfy desire[18]". According to Siddall[11] "in some way it is a measure of how well satisfied the users of a design are".

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*Utility is sometimes defined as value or worth of use,[17] whereas value is a more general concept, including utility, aesthetic value, spiritual value, etc.

1.3 Optimization in Design

Usually there are two distinct type of design problems in the field of optimization. One deals with a whole system or a complete machine and the other deals with sub-systems, sub-assemblies of a machine or even the elements of the sub-systems and sub-assemblies. The first type of problem is in essence the true optimization problem and is usually called as primary optimization problem. The second type of problem is often called a sub-optimization problem although terms such as secondary and tertiary optimization are not uncommon.

A sub-optimization design problem is obviously simpler than a primary optimization problem. Since the basic technique is the same it is usual to consider sub-optimization problem for explaining the principles involved. Most of the problems used in section 2 as well as in section 3 are sub-optimization problems. However using digital computers the same methods may be applied for complet primary optimization problems.

1.4 Formulation of Problem

Each design usually has to satisfy three basic requirements. First, a design must meet constraints (specifications, etc.) imposed on the system, machine or component to be produced. Second, it should satisfy certain limiting conditions. Third, it must at the same time be optimum with respect to some criterion. Mathematically these requirements can be expressed as follows:

Criterion $U = U\ (x_1, x_2, x_3 \ldots, x_n)$ a function to be maximized or minimized.

Or $U = U\left[y(x)\right]$ a functional to be maximized or minimized.
Constraining equation $\phi_i = \phi_i (x_1, x_2, . . , x_n) = 0 \quad i = 1, 2, . , m$
Limiting conditions $\underline{B}_j \leqq \bar{\phi}_j (x_1, x_2, . . , x_n) \leqq \bar{B}_j \quad j = 1, 2, . , k$
The formulation of the optimization problem in design is setting up the above mentioned functions, equations, and inequalities using physical laws governing the performance, properties of materials used, and geometric configuration of the equipment.

From the above it is clear that design inherently is an optimization problem. Each designer tries to produce a design which is the best. However, in normal circumstances the factors which influence design are so many that it is almost impossible for any designer to calculate all possible solutions and select the best. The only alternative left for him is to choose a few typical designs which experience has shown him to be of representative character, and choose the best intuitively. This design can therefore be far from optimum in true sense. But, since the computers have entered the field, the situation is altogether changed[19, 20, 21]. A designer, using this powerful tool can search hundreds and thousands of feasible solutions before arriving at the best or optimum solution. However present techniques still limit him to the use of one criterion function or dominant design variable.

MATHEMATICAL TOOLS

2.1 Graphical Methods

Graphical methods are the simplest of all presently known mathematical optimization techniques. However, these methods are applicable to comparatively elementary problems. The ease with which they can be applied has made them popular in the past; and even when more rigorous and sophisticated methods are known, graphical methods are frequently used.

The most important aspect of graphical methods is the formulation of the problem. When formulation is established, the rest of the problem is very simple and involves simple algebro-graphical principles of elementary mathematics. The results are then read directly from graphs.

A graphical method can therefore be described as a problem of formulation of one or more equations based upon physical laws affecting design, properties of materials or geometry of the part to be designed. By varying an independent variable, a graph indicating the relation of the dependent and independent variable can be drawn. If there is only one dependent variable and hence only one equation involved, the problem is the simplest type and the extremum (maximum or minimum) can be read from the graph directly as the highest or the lowest point on the graph.

If there are two equations and hence two curves involved an extremum would occur either at the point of intersection of the two curves or when the tangents to the two curves become parallel. A special case of this occurs when the two tangents coincide; the common tangent gives the extremum.

Simple two-dimensional problems of linear programming and problems of break-even analysis of economics and production management fall under this category. Latta has applied the method of parallel tangents for determining optimum tool life or optimum cutting speed by minimizing cost. Hinkle[23] has used the method of intersection of curves for design of machines for optimum speed consideration. By similar approach the method of selection of most efficient machines for a particular production are described by various authors. A common tangent method of finding extremum is used by Faulkner[24] for solving optimum thrust problem for rockets and for optimum fuel consumption problem for trajectories. A two-dimensional linear programming problem for optimum manufacturing schedule for maximum profit will be described under linear programming.

Many simple problems involving only one curve are described by Hinkle[23] using graphical and graphical cum analytical methods. Hall[25] has dealt with a problem of synthesis of four-bar mechanism for optimum transmission angle; and Wennburg[26] a problem of determination of minimum cost or maximum production either for optimum tool life or optimum cutting speed. Maier[27] has used this technique for designing compression springs for optimum load; and Bowman and Fetter[28] utilised it for solution of optimum lot order size and re-order point.

Problems involving more than two equations such as multi-dimensional linear programming problems, etc., are difficult to solve by this technique. In spite of this difficulty, such problems are frequently solved by this technique by slight variations. By using graphical methods in conjunction with analytical methods, charts can be prepared which are then used for optimum design and these can be handled by less qualified designers, thus reducing the cost of design as well.

A synthesis of four-bar mechanism for optimum force transmission is described by Jenson and Volmer[29] using this approach and Willis[30] has dealt with a problem of light weight gear design by the same method.

2.2 Indirect Methods of Ordinary Theory of Maxima and Minima

Indirect methods, also known as classical or analytical methods, are methods of arriving at the extremum by means of a necessary condition for the extremum. This approach gives comparatively more accurate solution than any other approach. However the method is more complicated and is not easy to apply in each case. In spite of this disadvantage, it is frequently used by the designers because of its accuracy, and because it provides a general solution of a group of problems rather than a single problem, as the case with the easier direct methods.

In the following three sections three cases of this method are dealt with separately.

1. n variables without any constraining equations
2. n variables with m constraining equations

3. Method of Lagrange's Multipliers

However, before proceeding further it is necessary to get a clear concept of the various terms used in the ordinary theory of maxima and minima. Since better understanding of the problem is possible through graphic interpretation, this approach will be used more often. To start with, a case of two-dimensional problems will be considered where the dependent variable y can be expressed as: $y = U(x_1, x_2)$. This function can be represented as a contour map shown in fig. 2.2 where each contour line represents the value of the dependent variable y and the chain line represents the constraining equation $\phi_i = 0$.

A point such as A, B and D as shown in fig. 2.2, which is higher than any other point in its vicinity is known as a local maximum. The highest of these points is point D which is therefore known as the absolute maximum point. On the same reasoning points E, J, G, F and H which are lower than any other point in their vicinity are called local minimum points. Together all these points are known as local extrema. Thus a local extremum is a point which is extreme either in the interior or on the boundary of a suitably defined domain. A point C, which is highest point on the path between ridges in fig. 2.2, is neither a maximum nor a minimum since in its immediate neighbourhood both higher and lower points exist along the ridges. All points such as point C are known as saddle points.

At point B, the derivative of y with respect to both x_1 and x_2 vanishes and hence point B is called a stationary point. A stationary point can therefore be defined as a point where all first order partial derivatives of a function with respect to independent variables vanish. A

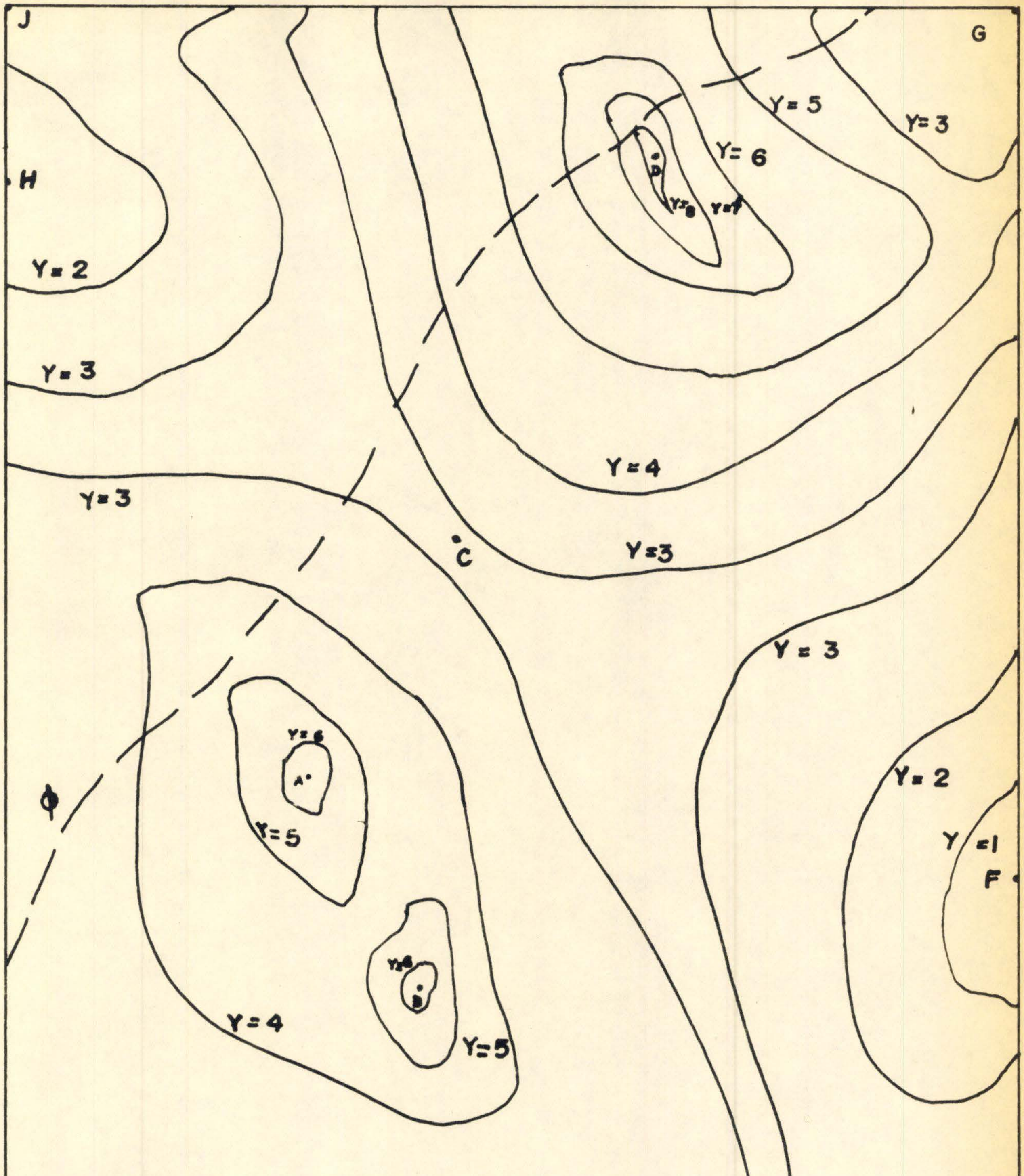

FIG. 2·2

saddle point is also a stationary point, however, it is not an extremum.

In the general problem limit equations may also occur. In fig. 2.2 they could be thought of as fences which cannot be crossed. The basic analytical method of maxima and minima does not handle constraints. It can be extended to handle constraints, as shown in section 2.22 and 2.23; to handle limits as shown in section 2.25; and to handle both simultaneously as shown in section 2.26.

2.21 Case of n Variables Without Any Constraining Equations

The problem of finding an extremum can be described as the problem of finding all local extrema in the interior and on the boundary of a suitably defined region, and the comparison of such extrema for arriving at an absolute extremum. The theorem of Weierstrass, which states, "Every function which is continuous in a closed region possesses a largest and a smallest value either in the interior or on the boundary of the region",[34] guarantees the existence of a solution. Location of the extremum can then be found by the theorem of ordinary calculus which states, "A continuous function U $(x_1, x_2, x_3, \ldots x_n)$ of n independent variables $x_1, x_2, x_3, \ldots x_n$ attains a maximum or minimum in the interior of a region R only at the values of the variables x for which the n partial derivatives $U_{x1}, U_{x2} \ldots U_{xn}$ either vanish simultaneously (a stationary point) or at which one or more derivatives cease to exist (are discontinuous)"[32]. This means that the total differential of the function will also be zero for any arbitrary differential displacements of n variables, i.e.,

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \ldots\ldots + \frac{\partial U}{\partial x_n} dx_n = 0 \qquad (1)$$

The general formulation leads to n simultaneous non-linear algebraic equations to be solved. Techniques of solution of comparatively difficult equations of this type are discussed in 2.42. After satisfying necessary conditions of existence of an extremum, sufficiency conditions can be expressed in terms of Leitmann's[33] notation in the following manner: for a stationary point to be a local maximum

$$D_i < 0 \qquad \text{for } i = 1, 3, 5 \ldots \text{(i.e., odd)}$$

$$\text{and } D_i > 0 \qquad \text{for } i = 2, 4, 6 \ldots \text{(i.e., even)}$$

and for the stationary point to be a local minimum

$$D_i > 0 \qquad \text{for } i = 1, 2, 3, 4, \ldots n$$

where

$$D_i = \begin{vmatrix} U_{x1x1} & U_{x1x2} & U_{x1x3} & \cdots & U_{x1xi} \\ U_{x2x1} & U_{x2x2} & U_{x2x3} & \cdots & U_{x2xi} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ U_{xix1} & U_{xix2} & U_{xix3} & \cdots & U_{xixi} \end{vmatrix} \tag{2}$$

For 1 independent variable (i.e., $i = 1$), $U(x)$ will be a maximum when

$$D_i = U_{x1x1} < 0$$

and would be minimum when

$$D_i = U_{x1x1} > 0$$

which is the fundamental result of the elementary calculus.

Problems falling under the case described above are many and are usually dealt with in almost all books of calculus. A problem of optimum design of a tray for maximum capacity is dealt with in detail by Johnson[34] and a similar problem of an oblique sheet metal tray is dealt with by Sokolnikoff[32]. A problem of maximizing power obtainable from either a d.c. or an a.c. source is described by Lavi[31a].

2.22 Case of n Variables with m Constraining Equations

This is a particular case of the problem dealt with in the preceding section and hence equation (1) will still hold. However, constraining equations $\phi_i = \phi_i(x_1, x_2, x_3 \ldots x_n)$ for $i = 1, 2 \ldots n$ imply that the differential displacements will not be arbitrary any more. Hence differentiating the m constraining equations we can write

$$d\phi_1 = \frac{\partial\phi_1}{\partial x_1} dx_1 + \ldots \frac{\partial\phi_1}{\partial x_n} dx_n = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$d\phi_m = \frac{\partial\phi_m}{\partial x_1} dx + \ldots \frac{\partial\phi_m}{\partial x_n} dx = 0$$

Thus, m-1 linear homogeneous equations in the n differentials dx must be satisfied. The condition for a stationary point would now be that the Jacobian determinant must vanish, i.e., one Jacobian determinant for each n-m independent variables

$$J\left[\frac{U, \phi_1, \phi_2, \phi_3 \ldots, \phi_{m-1}, \phi_m}{x_1, x_2, x_3 \ldots, x_m, x_{m+1}}\right] = 0$$

$$J\left[\frac{U, \phi_1, \phi_2, \phi_3 \ldots, \phi_{m-1}, \phi_m}{x_1, x_2, \ldots, x_m, x_{m+2}}\right] = 0 \qquad (2a)$$

$$\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots$$

$$J\left[\frac{U, \phi_1, \phi_2, \phi_3 \ldots, \phi_{m-1}, \phi_m}{x_1, x_2, x_3 \ldots, x_m, x_n}\right] = 0$$

where

$$J\left[\frac{\lambda_1,\lambda_2,\lambda_3 \ldots \lambda_n}{t_1,t_2,t_3 \ldots t_n}\right] = \begin{vmatrix} \frac{\partial\lambda_1}{\partial t_1} \ldots \frac{\partial\lambda_1}{\partial t_n} \\ \frac{\partial\lambda_2}{\partial t_1} \ldots \frac{\partial\lambda_2}{\partial t_n} \\ \frac{\partial\lambda_n}{\partial t_1} \ldots \frac{\partial\lambda_n}{\partial t_1} \end{vmatrix}$$

where λ_i and t_i are any dependent and independent variables respectively.[35] For graphical interpretation we may refer back to fig. 2.2 and use the two-dimensional problem $y = U\ (x_1,\ x_2)$. The maximum of this function satisfying the constraining equation $\phi\ (x_1,\ x_2) = 0$ would be at point I where slope of $U\ (x_1,\ x_2)$ = constant and $\phi\ (x_1,\ x_2) = 0$ coincide,

i.e., $$-\frac{\partial U/\partial x_1}{\partial U/\partial x_2} = -\frac{\partial\phi/\partial x_1}{\partial\phi/\partial x_2}$$

i.e., $$-\frac{\partial\phi/\partial x_1}{\partial\phi/\partial x_2} + \frac{\partial U/\partial x_1}{\partial U/\partial x_2} = 0$$

or $$\frac{\partial U}{\partial x_1} \times \frac{\partial\phi}{\partial x_2} - \frac{\partial U}{\partial x_2} \times \frac{\partial\phi}{\partial x_1} = 0$$

which is a special case of equation (2) where $n = 2$, $m = 1$. As an example the design of a circular base cylindrical fuel tank to have a maximum volume for a given surface area is explained here.

If r and h denote base radius and altitude of the tank respectively, $U\ (r,\ h) = \pi r^2 h$ and $\phi\ (r,\ h) = 2\pi r^2 + 2\pi r h$. Hence using equation (2a) $\pi \begin{vmatrix} 2hr & r^2 \\ 4r+2h & 2r \end{vmatrix} = 0$ solving this we get $h = 2r$.

2.23 Lagrange's Method of Undetermined Multipliers

Since the derivation as well as solution of the equations is comparatively difficult with the method described in the preceding section, the method of Lagrange's multipliers to be described in this section is commonly used. The method is thus an alternative technique for n variables with m constraining equations.

In the preceding section for a two-dimensional problem the following relation was derived.

$$\frac{\partial U/\partial x_1}{\partial U/\partial x_2} = \frac{\partial \phi/\partial x_1}{\partial \phi/\partial x_2} \qquad \text{i.e. } \frac{\partial U}{\partial x_1} \Big/ \frac{\partial \phi}{\partial x_1} = \frac{\partial U}{\partial x_2} \Big/ \frac{\partial \phi}{\partial x_2}$$

If this ratio is assumed to be equal to some constant $-\lambda$ the above relation can be re-written as

$$\frac{\partial U}{\partial x_1} + \lambda \frac{\partial \phi}{\partial x_1} = 0 \qquad \text{and} \qquad \frac{\partial U}{\partial x_2} + \lambda \frac{\partial \phi}{\partial x_2}$$

This constant λ is known as Lagrange's multiplier and associated with this is the well-known theorem of calculus which states: "If U (x, y) and ϕ(x, y) be differentiable in a neighbourhood N (a, b) of (a, b) and it is assumed that U has a relative extremum there, subject to a constraint on x and y of the form $\phi(x, y) = k$ which defines y as a differentiable function $\phi_{(x)}$ of x in a neighbourhood of $x = a$ and that ϕ(a, b) $\neq 0$ then there must exist a constant λ such that

$$U_1(a,b) + \lambda \phi_1(a,b) = U_2(a,b) + \lambda \phi_2(a,b) \equiv 0 \text{ "}$$ [32]

This can now be extended further to the general case of n variables and m constraints, i.e., to the existence of m constants $\lambda_1, \lambda_2 \ldots \lambda_n$ such that

$$\frac{\partial U}{\partial x_1} + \lambda_1 \frac{\partial \phi_1}{\partial x_1} + \lambda_2 \frac{\partial \phi_2}{\partial x_1} + \cdots\cdots\cdots + \lambda_m \frac{\partial \phi_m}{\partial x_1} = 0$$

$$\frac{\partial U}{\partial x_2} + \lambda_1 \frac{\partial \phi_1}{\partial x_2} + \lambda_2 \frac{\partial \phi_2}{\partial x_2} + \cdots\cdots\cdots + \lambda_m \frac{\partial \phi_m}{\partial x_2} = 0$$

....

$$\frac{\partial U}{\partial x_n} + \lambda_1 \frac{\partial \phi_1}{\partial x_n} + \lambda_2 \frac{\partial \phi_2}{\partial x_n} + \text{——————} + \frac{\partial \phi_m}{\partial x_n} = 0$$

i.e., in summation notation $U_{x_j} + \sum_{i=1}^{m} \lambda_i \phi i_{x_j} = 0.$

for $j = 1, 2, \ldots n$

For the purpose of illustration, a transmission line design problem solved by Asimow[36] using Jacobian determinants will be solved here using Lagrange's multipliers.

Example

Electrical energy is to be transmitted to a distribution station 200 miles from the generating station. The three-phase a.c. transmission line to be designed is to have a most economical transmission voltage 'E', line current 'I' and conductance 'G'. The constant power generated at the generating end 'P' = 50 megawatts, the value of power delivered k_1 = \$.01/kwhr or \$87.5/kwhr., cost of conductor for the whole length of transmission line $k_2 = \$13 \times 10^6$/mho. Incremental cost per kilovolt of the transmission voltage k_3 = \$1000/kv., and rate of imputed charges ascribed to invested capital i = \$.1/ yr/\$

Assuming that all effects except resistance and corona losses are negligible and that the line drop compared to line voltage is too small to be considered we can write

Power loss in the 3 conductor 3 phase line $= 3 \times 10^{-3} I^2 G^{-1}$ kw

Value of this loss $= 3 \times 10^{-3} I^2 G^{-1} k_1$ \$/year

Depreciation at the rate i of the value of conductors

$= 3\, G k_2 i$ \$/year

Imputed charges ascribed to invested capital in incremental cost

$= E k_3 i$ \$/year

Total operating cost $U = 3 \times 10^{-3} I^2 G^{-1} k_1 + 3G\, k_2\, i + E\, k_3 i$ (3)

For arriving at the most economic transmission line design U should be minimized. However, total power that can be transmitted imposes the constraining equation

$$\phi = \sqrt{3}\, E\, I - P = 0 \qquad (4)$$

Further, beyond a critical voltage E_c corona loss becomes appreciable thus imposing a limiting condition

$$\bar{\phi} = E_c - E \geq 0 \qquad (5)$$

But since $E_c = 63.5\, G^{1/2}\, (18.52 - \ln G)$ we can write

$$\bar{\phi} = 63.5\, G^{1/2}\, (18.52 - \ln G) - E \geq 0 \qquad (6)$$

Using Lagrange's Multipliers we can now write the augmented functions and hence the relation

$$\frac{\partial U}{\partial I} + \lambda \frac{\partial \phi}{\partial I} = 0, \quad \frac{\partial U}{\partial E} + \lambda \frac{\partial \phi}{\partial E} = 0, \quad \frac{\partial U}{\partial G} + \lambda \frac{\partial \phi}{\partial G} = 0$$

differentiating equation (3) with respect to I, E, and G we get

$$\frac{\partial U}{\partial I} = 6.10^{-3} k_1 I\, G^{-1}, \quad \frac{\partial U}{\partial E} = k_3 i, \quad \frac{\partial U}{\partial G} = -3 \cdot 10^{-3} I^2 G^{-2} k_1 - 3 i k_2$$

similarly differentiating equation (4) with respect to I, E and G we get

$$\frac{\partial \phi}{\partial I} = \sqrt{3}\, E, \quad \frac{\partial \phi}{\partial E} = \sqrt{3}\, I, \quad \frac{\partial \phi}{\partial G} = 0$$

substituting these values in the above equations we get

$$6 \cdot 10^{-3} k_1\, I\, G^{-1} + \sqrt{3}\, E = 0$$

$$i\, k_3 + \lambda \sqrt{3}\, I = 0$$

$$-3 \cdot 10^{-3}\, I\, k_1 G^{-2} + \sqrt{3}\, i\, k_2 = 0$$

These equations along with the constraining equation provide a set of 4 equations in 4 unknowns λ, I, E and G. Solving these we get

$$\lambda = \frac{-ik_3}{3\ I}$$

$$I = \frac{k_3\ (10^3 \cdot\ i)^{\frac{1}{2}}\ E}{6\ (k_2\ k_1)^{\frac{1}{2}}}$$

$$G = \frac{k_3}{6\ k_2}$$

$$E = 6P(k_2 \cdot k_1)^{\frac{1}{2}} / k_3(3 \cdot 10^3)^{\frac{1}{2}}\ i^{\frac{1}{2}}$$

Substituting numerical values at this stage we get

E=242kv., I=120A, G=0.0311mho, U=484000\$, $\bar{\phi}=3.76>0$

These are the same as obtained by Asimow.

2.24 Zener, Fein and Duffin's Method

The optimization function in design problems can often be expressed by a polynomial function

$$U = \sum_{i=1}^{n} E_i \text{ to be minimized} \tag{7}$$

where E_i can be expressed as an exponential function

$$E_i = a_i \prod_{j=1}^{m} X_j^{\beta_{ij}} \qquad \text{for each } i,\ i=1,\ldots n \tag{8}$$

subject to the condition that

$$0 < X_j < \infty$$

Here a_i and β_{ij} are positive constants.

Solution of this type of problems can be obtained by the methods described in the preceding sections. Zener[37] has recently shown that such problems can be solved by a comparatively simple method whereby the extremum can be found directly without solving for the independent variables provided $n = m + 1$.

Knowing the values of U the first step would be to find n terms product $\prod_{j=1}^{n} E_j^{\alpha_j} = K$, where α_j and K are certain constants and can be determined as follows. By comparison with the given function U, values of β_{ij} can be determined. Values of α_j can then be found by the transformation equation

$$\sum_{i=1}^{n} \alpha_i \beta_{ij} = 0 \qquad \text{for each } j,\ j=1,\ldots,m \tag{9}$$

and the normalization condition

$$\sum_{i=1}^{n} \alpha_i = 1 \tag{10}$$

so that $$U_{opt} = \frac{K}{\prod_{i=1}^{n} (\alpha_i)^{\alpha_i}} \tag{11}$$

The problem thus reduces to the relatively simple one of solving n simultaneous linear algebraic equations. This is further illustrated through an example at the end of this section. A rigorous proof of the validity of the above procedure is given by Fein[38]. Later, Duffin[39] has shown that the restriction imposed by Zener, $n = m + 1$, can be removed. This he has proved by using a dual programme, and thus reduced the method to a more general form. Adopting this approach Zener[40] has further extended this method to a more general case of minimizing as well as maximizing by using perturbation technique. In December 1963 Zener[41] presented yet another extension of his method for optimization of systems in terms of sub-systems. This could be described as follows:

For a system consisting of N overlapping sub-systems having an overlap of not more than one element, if the sub-system is optimized for minimum cost for independent operation, then the common element will have the same weighting exponent in all sub-systems. Calling this exponent as σ and the minimum cost of independent operation of the jth sub-system as m_j, the minimum cost of complete system M can be expressed as

$$M^{\frac{1}{1-\sigma}} = \sum_{J=1}^{N} m_j^{\frac{1}{1-\sigma}} \qquad (12)$$

If we let β_{jk} be the weighting exponent of the jth term in the kth sub-system when it is optimized for independent operation, the weighting exponent of the jth term of the kth sub-system would be

when the whole system is optimized. $$\frac{m^{\frac{1}{1-\sigma}}}{\left[\sum_{k=1}^{N} m^{\frac{1}{1-\sigma}}\right]} \beta_{jk}$$

Example

Assuming that the cost function of a design problem can be expressed as

$$U(x_1,x_2)= a_1x_1 + \frac{a_2}{x_1x_2} + a_3 x_2^2$$

and that this is to be minimized* we can proceed as follows:

Since

$$U=\sum_{i=1}^{3} E_i = a_1 x_1 + \frac{a_2}{x_1x_2} + a_3 x_2^2 \qquad (13)$$

and $E_1 = a_1(x_1^{\beta_{11}} \cdot x_2^{\beta_{12}})$, $E_2= a_2(x_1^{\beta_{21}} \cdot x_2^{\beta_{22}})$, $E_3= a_3(x_1^{\beta_{31}} \cdot x_2^{\beta_{32}})$

$$U=a_1(x_1^{\beta_{11}} \cdot x_2^{\beta_{12}})+a_2(x_1^{\beta_{21}} \cdot x_2^{\beta_{22}})+a_3(x_1^{\beta_{31}} \cdot x_2^{\beta_{32}}) \qquad (14)$$

By comparing equation (13) and (14) we get

$$\beta_{11}=\beta_{22}=-\beta_{21} = 1, \ \beta_{12}=\beta_{31} = 0, \ \beta_{32} = 2$$

Now using the transformation equation we get

$$\beta_{11}\alpha_1+\beta_{21}\alpha_2+\beta_{31}\alpha_3 = 0, \quad \beta_{12}\alpha_1+\beta_{22}\alpha_2+\beta_{32}\alpha_3 = 0$$

Then applying the normalization condition gives

$$\alpha_1+\alpha_2+\alpha_3 = 1$$

Solving the above two sets of equations we get

$$\alpha_1 = \alpha_2 = 2/5, \quad \alpha_3 = 1/5$$

From this

$$K= (a_1 x_1)^{2/5} \left(\frac{a_2}{x_1x_2}\right)^{2/5} (a_3 x_2^2)^{1/5}$$

Giving $$U_{opt}= \frac{(a_1^2 \cdot a_2^2 \cdot a_3)^{1/5}}{(2/5)^{2/5}(2/5)^{2/5}(1/5)^{1/5}} = \frac{5(a_1^2 \cdot a_2^2 \cdot a_3)^{1/5}}{2^{4/5}}$$

Sherwood[42] has used this method for the solution of a gas line design problem and by solving the same problem with the classical method has shown how considerable labour and time can be saved by using this method.

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*From reference 37 with simplification wherever needed.

2.25 Charnes and Cooper's Extension of Zener's Method to Design Under Inequality Constraints

Zener's method, though provides a very strong tool for optimization design problems, does not mention anything about constraints. In actual practice unconstrained design problems are but rare. Charnes and Cooper's[43] extension of the method to inequality constraints is therefore a most welcome step.

In this method the independent variable x_j is replaced by an exponential term e^{u_j}, i.e.,

$$x_j = e^{\ln x_j} = e^{u_j} \tag{15}$$

By substituting this in equation (7) we get

$$U = \sum_{i=1}^{n} a_i \, e^{\sum_{j=1}^{m} \beta_{ij} \ln x_j} = \sum_{i=1}^{n} a_i \, e^{\sum_{j=1}^{m} \beta_{ij} u_j} \tag{16}$$

This indicates that the criterion is a convex function*. Now considering limits or inequality constraints of the type C

$$\underline{B}_k \leq \prod_{j=1}^{m} x_j^{\gamma_{kj}} \leq \overline{B}_k \tag{17}$$

Where $\underline{B}_k$ indicates the lower bound and $\overline{B}_k$ the upper bound on the design respectively. Taking natural logarithms these can be converted to the following linear form

$$\ln \underline{B}_k \leq \sum_{j=1}^{m} \gamma_{kj} u_j \leq \ln \overline{B}_k \tag{18}$$

Our minimization problem is thus reduced to minimize

$$U = \sum_{i=1}^{n} a_i \, e^{\sum_{j=1}^{m} \beta_{ij} u_j} \tag{19}$$

subject to the constraint

$$\sum_{j=1}^{m} \gamma_{kj} u_j - K_k \geq 0$$

*For the definition of convex function see Section 2.5.

This can further be simplified to the following

$$\text{minimize} \quad \sum_{i=1}^{n} a_i \, e^{w_i} \qquad (20)$$

subject to the constraints

$$-\sum_{j=1}^{m} \beta_{ij} + u_j + w_i = 0 \, , \quad \sum_{j=1}^{m} \gamma_{kj} \cdot u_j - K_k \geqq 0 \text{ for } k=1,\ldots,$$

These functions can now be handled by the mathematical programming methods to be described in section 2.5.

Example

A fly wheel for a small light regenerative vehicle is to be designed. The fly wheel absorbs energy by coasting of the vehicle and feeds it back to the drive when power is required.

The dependent variable to be optimized is the amount of stored energy. Constraints on the problem are:

i. Weight must not exceed 150 lbs.

ii. Diameter must not exceed 36 inches.

iii. The speed of rotation must be acceptable to

Design A - a belt-pulley system $\left\{ \right.$ with smallest driver pulley

Design B - a chain-sprocket system. or sprocket.

Assuming a solid disc type fly wheel we can write the following:

$$\text{Energy } U = wk\, d^4 s\, n^2/(4.11744) \text{ to be maximized}$$

$$= A_1 . d^4 . n^2$$

Subject to

$$\frac{\pi}{4} d^2\, w\, s \leq 150 \text{ or } A_2 d^2 \leq v_2$$

$$d \leq 3 \qquad \text{or } d \leq v_3$$

$$\pi\, nD \leq v \qquad \text{or } n \leq v_1$$

(stresses are assumed to be well below safe limit)

$\bar{v}$ = 6000 fpm for belt drive

= 1000 fpm for chain drive

where d is the diameter of the fly wheel, n the rpm of the fly wheel, w the width of the fly wheel, s specific gravity of the material of the fly wheel, $\bar{v}$ the rim velocity, k the velocity fluctuation factor, and D the diameter of the driver pulley or sprocket.

By using equations (15) and (16) this can be expressed as

$$2 \ln d \leq a \qquad \text{where } a = \ln \frac{150 \cdot 4}{\pi \cdot w \cdot s}$$

$$\ln d \leq b \qquad \text{where } b = \ln 3$$

$$\ln n \leq c \qquad \text{where } c = \ln v_1$$

Assuming $u_1 = \ln d$ and $u_2 = \ln n$ the problem would resolve into

$$\text{maximize } U = A\, e^{4u_1 + 2u_2} = A\, e^{u_3}$$

subject to $\phi = u_3 - 4u_1 - 2u_2 = 0$

$$2u_1 \leq a, \; u_1 \leq b, \; u_2 \leq c$$

Introducing slack variables u_4, u_5 and u_6 we can convert the inequalities into equalities so that

$$2u_1 + u_4 = a, \; u_1 + u_5 = b \; \text{ and } u_2 + u_6 = c$$

the problem is now reduced to maximizing a convex function under linear equality constraints and can be solved by using the convex programming methods described under section 2.54.

2.26 Extension of Lagrange's Method to Inequality Constraints

In section 2.23 we have seen that the method of undetermined multipliers can solve the extremization problem if the constraints are equalities. Valentine, Pennisi and Klein* have shown that in case of functions as well as functionals to be maximized or minimized under inequality constraints, the inequality constraints can be transformed into equalities by transforming the given independent variable into a new independent variable and introducing another new variable simultaneously. Applying the Lagrange's technique at this stage, the constrained problem is transformed into an unconstrained problem. Mathematically we can express this as follows:

Let $U = U(x_1, x_2, x_3, \ldots, x_n)$ to be maximized

subject to the inequality constraints

$$\phi_i = \phi_i (x_1, x_2, x_3 \ldots x_n) \leq 0$$

Replacing x_i with h_i^2 and $\phi_i (x_j)$ with $\phi_i (h_j) = k_i^2$ we can write

$$\overline{U} = U(h_j) + \sum_{i=1}^{n} \lambda_i \left[\phi_i (h_j) - k_i^{2} \right] \text{ to be maximized}$$
$$j = 1, \ldots\ldots, n \text{ for each } i$$

Using the results of the section 2.23 we can write the necessary condition for the extreme value as

$$\partial \overline{U} / \partial h = 0 \, , \; \partial \overline{U} / \partial k_i = 0 \, , \; \partial \overline{U} / \partial \lambda_i = 0$$

The method is particularly suited to nonlinear design problems when independent variables involved are few. The fly wheel energy maximization problem of section 2.25 will be used here to further illustrate the method.

In section 2.25 we have formulated the problem as

$U = A_1 d^4 n^2$ to be maximized

* see ref. 43a, 43b & 43c.

subject to $n \leq v_1$

$$A_2 d^2 \leq v_2$$

$$d \leq v_3$$

Writing $d = h^2$ and $n = g^2$ the constraints can be expressed as

$$g^2 - v_1 - k_1^2 = 0, \; A_2 h^4 - v_2 - k_2^2 = 0, \; h^2 - v_3 - k_3^2 = 0$$

Introducing Lagrange's multipliers λ_i we can write the augmented function

$$\bar{U} = A_1 h^8 g^4 + \lambda_1 (g^2 - v_1 - k_1^2) + \lambda_2 (A_2 h^4 - v_2 - k_2^2) + \lambda_3 (h^2 - v_3 - k_3^2)$$

for this to be maximum

$$\frac{\partial \bar{U}}{\partial h} = 8 A_1 h^7 g^4 + 4\lambda_2 A_2 h^3 + 2\lambda_3 h = 0$$

$$\frac{\partial \bar{U}}{\partial g} = 4 A_1 h^8 g^3 + 2\lambda_1 h^2 g = 0$$

$$\frac{\partial \bar{U}}{\partial k_1} = -2\lambda_1 k_1 = 0$$

$$\frac{\partial \bar{U}}{\partial k_2} = -2\lambda_2 k_2 = 0$$

$$\frac{\partial \bar{U}}{\partial k_3} = -2\lambda_3 k_3 = 0$$

$$\frac{\partial \bar{U}}{\partial \lambda_1} = g^2 - v_1 - k_1^2 = 0$$

$$\frac{\partial \bar{U}}{\partial \lambda_2} = A_2 h^4 - v_2 - k_2^2 = 0$$

$$\frac{\partial \bar{U}}{\partial \lambda_3} = h^2 - v_3 - k_3^2 = 0$$

1. Assuming $\lambda_1 = \lambda_2 = \lambda_3 = 0$,

either $h = 0$ or $g = 0$. Since we are interested in positive definite values of h and g this solution is trivial.

2. Assuming $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 \neq 0$

   $h = 0$ which is again trivial.

3. Assuming $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 \neq 0$

   $h = 0$ this is also trivial.

4. Assuming $\lambda_2 = \lambda_3 = 0$ and $\lambda_1 \neq 0$

   $g = 0$ this too is trivial.

5. Assuming $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 = 0$

   we get $k_1 = 0$, $k_2 = 0$, $g^2 = v_1$, $h^4 = v_2/A_2$, $\lambda_1 = -2A_1h^8v_1$,

   $$\lambda_2 = -2A_1h^4g^4/A_2.$$

6. Assuming $\lambda_1 \neq 0$, $\lambda_3 \neq 0$ and $\lambda_2 = 0$

   we get $k_1 = 0$, $k_3 = 0$, $g^2 = v_1$, $h^2 = v_3$, $\lambda_2 = -4A_1h^6g^4$, $\lambda_1 = -2h^8v_1$

7. Assuming $\lambda_2 \neq 0$, $\lambda_3 \neq 0$ and $\lambda_1 = 0$

   $k_2 = 0$, $k_3 = 0$, $A_2h^4 = v_2$, $h^2 = v_3$,

   This means that $A_2v_3^2 = v_2$ which is incompatible.

8. Assuming that $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$

   $k_1 = k_2 = k_3 = 0$ and $g^2 = v_1$, $A_2h^4 = v_2$, $h^2 = v_3$.

   The results are the same as case 7.

From above it is clear that only feasible solutions are 5 and 6. Solution 6 gives $d = v_3$, which is the limiting value, hence 5 is the required solution. Assuming cast iron to be the material of choice s = 451 lbs/cft. If the width w of flywheel is chosen to be 3 inches and the diameter of the driverpulley 4½ inches for belting and diameter of sprocket 3/4" for chain we get the following optimum design parameters:

Case A — d = 1.3 ft. and n = 5090 r.p.m. weight = 128 lbs.

Case B — d = 1.3 ft. and n = 5090 r.p.m. weight = 128 lbs.

Since our assumption of a solid disc type flywheel is against practice, the web width can be slightly changed to accommodate the effect of arms and hub; since weight depends on d and cannot be changed without altering optimum conditions. (H.P. $\underline{\angle}$ 1/4)

2.30 Indirect Methods of Variational Calculus

A vast majority of optimization problems in engineering do not fall under ordinary theory of maxima and minima. In aerospace operations minimizing the gross weight of an n stage missile, in control theory minimizing error or maximizing capacity of sensory devices, in production engineering maximizing productivity or minimizing cost are problems having quantities which depend on a variable running through a set of functions which are determined by a definite choice of these variable functions. Such quantities are known as functionals[44, 45, 46] and the branch of mathematics which deals with finding of maxima and minima of these quantities is known as calculus of variations. For one who is interested in the study of optimization techniques in design, it is therefore necessary to have some background of the theory of calculus of variations. A few important terms in this field will be considered in this section.

A variable U is called a functional depending on a function $y(x)$ such that $U = U\left[y(x)\right]$, if to each function $y(x)$, from a certain class of functions, there corresponds a certain value U.

Variation δy of argument $y(x)$ of a functional $U\left[y(x)\right]$ is the difference of two functions $y(x) - y_o(x)$.

A functional $U\left[y(x)\right]$ is said to be continuous along $y = y(x)$ in the sense of closeness of order k, if for any arbitrary positive number ϵ there exists a quantity $\delta > 0$ such that

$$\left| U\left[y(x)\right] - U\left[y_o(x)\right] \right| < \epsilon \text{ whenever}$$
$$\left| y(x) - y_o(x) \right| < \delta$$

$$\left| y'(x) - y_0'(x) \right| < \delta$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\left| y^{(k)}(x) - y_0^{(k)}(x) \right| < \delta$$

Two curves $y = y(x)$ and $y = y_1(x)$ are close in the sense of closeness of order k, if the absolute value of difference

$$\left| y(x) - y_1(x),\ y'(x) - y_1'(x), \ldots\ldots, y^{(k)}(x) - y_1^{(k)}(x) \right|$$

are small.

A functional $u[y(x)]$ is called a linear functional, if it satisfies the condition

$$U[Cy(x)] = CU[y(x)] \text{ and } U[y_1(x) + y_2(x)] = U[y_1(x)] + U[y_2(x)]$$

where C is a constant.

If an increment $\Delta U = U[y(x) + \delta y] - U[y(x)]$ of a functional is of the form $\Delta U = u[y(x), \delta y] + \alpha[y(x), \delta y] \max.|\delta y|$, where $u[y(x), \delta y]$ is a linear functional in y and $\max.|\delta y|$ is the maximum value of δy, and $\alpha[y(x), \delta y]$ tends to zero whenever $\max.|\delta y|$ tends to zero, then the part of this increment which is linear in δy, e.g., $u[y(x), \delta y]$, is known as the variation of the functional and is designated by δU.

The variation of a functional $U[y(x)]$ is usually expressed as

$$\frac{\partial}{\delta \alpha} U\left[y(x) + \alpha\, \delta y\right] \Big|_{\alpha = 0}$$

If variation of a functional $U[y(x)]$ exists, and if U becomes maximum or minimum along curve $y = y_o(x)$, then δU vanishes along curve $y = y_o(x)$.

2.31 Case of a Functional With Fixed End Points and Without Constraining Equations

For finding the extrema of a functional

$$U[y(x)] = \int_{x_0}^{x_1} F[x, y(x), y'(x)] \; dx \qquad (21)$$

it is assumed that the end points of the curves $y_o = y(x_o)$ and $y_1 = y(x_1)$ are fixed. Various curves $y(x)$ between these fixed ends would give different values of $U[y(x)]$. It is therefore the intention to find that curve $y(x)$ for which $U[y(x)]$ has an extremum. For this the following assumptions are first to be made:

1. That $F[x, y(x), y'(x)]$ viewed as a function of its argument x, y, y' has continuous partial derivatives of second order.

2. That there is a curve $y = y(x)$ with continously turning tangent that minimizes or maximizes $U[y(x)]$. Taking any admissible curve $y = \bar{y}(x)$ close to $y = y(x)$ a single parameter family of curves $y(x,\alpha) = y(x) + \alpha[\bar{y}(x) - y(x)]$ can be set up. Calling $\bar{y}(x) - y(x)$ as $\eta(x)$, the above equation can be written as

$$y(x,\alpha) = y(x) + \alpha\,\eta(x) = Y$$

where $\eta(x_o) = 0$ and $\eta(x_1) = 0$.

Substituting these values in equation (21) we can write

$$U[y(x,\alpha)] = \phi(\alpha) = \int_{x_o}^{x_1} F[(x, y(x) - \alpha\eta(x), y'(x) + \alpha\eta'(x)] \, dx \qquad (22)$$

$$\phi'(\alpha) = \int_{x_o}^{x_1} F(x, Y, Y')\,dx = \int_{x_o}^{x_1} \left[\frac{\partial F}{\partial y}\,\eta(x) + \frac{\partial F}{\partial y'}\,\eta'(x)\right] dx \qquad (23)$$

and hence $$\phi'(0) = \int_{x_o}^{x_1} \left[\frac{\partial F}{\partial y}\,\eta(x) + \frac{\partial F}{\partial y'}\,\eta'(x)\right] dx \qquad (24)$$

For the function $U[y(x,\alpha)]$ to have an extremum the necessary condition is that variation $\delta U = \phi'(0)$ should vanish.

Hence

$$\delta U = \phi'(0) = \int_{x_0}^{x_1} \frac{\delta F}{\delta y}\eta(x) + \frac{\delta F}{\delta y'}\eta'(x) \quad dx = 0$$

Since $\eta(x_0) = \eta(x_1) = 0$ the first term drops out on integration.

Therefore, integrating second terms by parts we can write

$$\int_{x_0}^{x_1} \frac{\delta F}{\delta y}\eta'(x)dx = \frac{\delta F}{\delta y'}\eta(x)\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta(x)\frac{d}{dx}\left(\frac{\delta F}{\delta y'}\right)dx$$

and hence

$$\delta U = \phi'(0) = \int_{x_0}^{x_1} \left[\frac{\delta F}{\delta y}\eta(x) - \frac{d}{dx}\left(\frac{\delta F}{\delta y'}\right)\right]\eta(x) \quad dx$$

$$= \int_{x_0}^{x_1} \eta(x)\left[\frac{\delta F}{\delta y} - \frac{d}{dx}\left(\frac{\delta F}{\delta y'}\right)\right] dx = 0$$

Since $\eta(x) = \bar{y}(x) - y(x)$ is arbitrary and vanishes at the fixed end points the above can only be true if

$$\frac{\delta F}{\delta y} - \frac{d}{dx}\left(\frac{\delta F}{\delta y}\right) = 0$$

or in subscript notation

$$F_y - \frac{d}{dx} F_{y'} = 0 \qquad (25)$$

This is the famous Euler equation and is the necessary though not the sufficient condition for an extremum.

The integral curves $y = y(x, c_1, c_2)$ of the Euler's equation are known as extremals.

Generalizing the above procedure it can be shown that for a functional

$$\phi(y) = F(x, y, y', y'', \ldots\ldots\ldots, y^{(k)})dx \qquad (26)$$

Euler's equation would be

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''}\ldots\ldots(-1)^k \frac{d^k}{dx^k}F_{y^{(k)}} = 0 \qquad (27)$$

and further for a double integral

$$\phi(v) = \iint F(x, y, v, v_n, v_y)dx\, dy \qquad (28)$$

Euler equation can be written as

$$F_v - \frac{\delta F_{v_n}}{\delta x} - \frac{\delta F_{v_y}}{\delta y} = 0 \qquad (29)$$

Example

In designing a rocket tail it is required to determine a curve with given end points, such that by revolving this curve around the x-axis a surface of minimum area can be generated*.

Since the area of the surface of revolution generated by the rotation of the curve $y = y(x)$ about x-axis is

$$A\, y(x) = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\, dx$$

and since the integrand depends only on y and y', the Euler equation has the first integral

$$F - y' F_{y'} = K$$

hence

$$y\sqrt{1+y'^2} - \frac{y \cdot y'^2}{\sqrt{1+y'^2}} = K \quad \text{or} \quad y = K\sqrt{1-y^2}$$

i.e.,

$$y' = \sqrt{\frac{y^2 - K^2}{K^2}}$$

separating variables, we can write

$$dx = K dy \,/ \sqrt{y^2 - K^2}$$

i.e.,

$$x + K_1 = K \ln\left(\frac{y + \sqrt{y^2 - K^2}}{K}\right)$$

This is the same as

$$y = K \cosh\left(\frac{x + K_1}{K}\right)$$

Since this is the equation of a catenary passing through the given end points the required curve is a catenary giving the catenoid as the surface of the required rocket tail. Values of K and K_1 can be found by applying end conditions.

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*Adopted from Reference 46.

2.32 Case of a Functional With Fixed End Points and With Constraints

A functional of the type discussed in preceding section is some times associated with some sort of a constraint. In what is called the isoperimetric problems of the variational calculus it is usually required to maximize or minimize a functional

$$U = \int_{x_0}^{x_1} F(x,y,y')\,dx \qquad \text{subject to the condition} \qquad (30)$$

$$\phi = \int_{x_0}^{x_1} G(x,y,y')\,dx$$

Using Lagrange's method of undetermined multiplier we can write

$$U + \lambda\phi = \int_{x_0}^{x_1} \left[F(x,y,y') \;+\; G(x,y,y')\right] dx$$

The necessary condition for extrema can then be expressed by Euler's equation

$$\frac{\partial(F+\lambda G)}{\partial y} - \frac{d}{dx}\,\frac{\partial(F+\lambda G)}{\partial y'} = 0 \qquad (31)$$

The constants of integration and Lagrange's multipliers can now be determined using end conditions and the constraint ϕs.

2.33 Case of a Functional With Variable End Conditions

After dealing with the case of a functional with fixed end points in the preceding sections we will now consider a more general case where ends are not fixed but terminate on a prescribed curve. Taking the familiar functional

$$U\ y(x) = \int_{x_o}^{x_1} F(x,y,y') \, dx$$

and choosing arbitrarily $\eta(x)$ such that $\eta(x_o) \neq 0$ and $\eta(x_1) \neq 0$ any admissible curve $y(x,\alpha) = y(x) + \alpha\, \eta(x)$ close to $y(x)$ can be chosen. Now substituting $y(x,\alpha)$ into the functional we can write

$$\phi(\alpha) = \int_{x_o}^{x_1} F\left[x, y(x,\alpha), y'(x,\alpha)\right] dx$$

Using Leibniz' formula, when limits are functions of the variable of differentiation, we can write

$$\phi'(\alpha) = F_1 \frac{dy_1}{d\alpha}(\alpha) - F_o \frac{dx_o}{d\alpha}(\alpha) + \int_{x_o(\alpha)}^{x_1(\alpha)} \frac{\partial F}{\partial \alpha} dx \qquad (32)$$

Integrating the last term by parts

$$\int_{x_o(\alpha)}^{x_1(\alpha)} \frac{\partial F}{\partial \alpha} dx = \int_{x_o(\alpha)}^{x_1(\alpha)} \left[\frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'} \cdot \frac{\partial y'}{\partial \alpha}\right] dx$$

$$= \int_{x_o(\alpha)}^{x_1(\alpha)} \left[\frac{\partial F}{\partial y} \eta(x) + \frac{\partial F}{\partial y'} \eta'(x)\right] dx$$

$$= \frac{\partial F}{\partial y} \eta(x) \Big|_{x_o}^{x_1} + \int_{x_o(\alpha)}^{x_1(\alpha)} \left[\frac{\partial F}{\partial y} - \frac{d\ \partial F/\partial y}{dx}\right] \eta(x)\, dx$$

and since $y(x_1,\alpha) = y(x_1) + \alpha\ (x_1)$, the above can be written as

$$\frac{dy_1}{d\alpha} = \left(\frac{dy}{dx}\right)_1 \frac{dx_1(\alpha)}{d\alpha} + \eta(x_1) + \alpha \left(\frac{d\eta}{dx}\right)_1 \cdot \frac{dx_1(\alpha)}{d\alpha}$$

$$= (y' + \alpha\eta') \frac{dx_1}{d\alpha} + \eta(x_1)$$

$$= (y')_1 \frac{dx_1}{d\alpha} + \eta(x_1) \qquad \text{i.e. } \eta(x_1) = \frac{dy_1}{d\alpha} - (y')_1 \frac{dx_1}{d\alpha}$$

similarly $\eta(x_o) = \frac{dy_o}{d\alpha} - (y')_o \frac{dx_o}{d\alpha}$

substituting these values in equation (32) gives

$$\phi'(\alpha) = (F)_1 \frac{dx_1(\alpha)}{d\alpha} - (F)_o \frac{dx_o(\alpha)}{d\alpha} + (F_{y'}) \left[\frac{dy_1}{d\alpha} - (y')_1 \frac{dx_1}{d\alpha} \right] - (F_{y'})_o \left[\frac{dy_o}{d\alpha} - (y')_o \frac{dx_o}{d\alpha} \right]$$

Rearranging and using subscript notation

$$(F_y - y'F_{y'})_1 \frac{dx_1}{d\alpha} - (F_y - y'F_{y'})_o \frac{dx_o}{d\alpha} + (F_{y'})_1 \frac{dy_1}{d\alpha} - (F_{y'})_o \frac{dy_o}{d\alpha} + \int_{x_o(\alpha)}^{x_1(\alpha)} (F_y - \frac{d F_{y'}}{dx}) \eta(x)\,dx = 0$$

and this is possible only if both

$$(F_y - y'F_{y'}) \frac{dx_1}{d\alpha} - (F_y - y'F_{y'})_o \frac{dx_o}{d\alpha} + (F_{y'})_1 \frac{dy_1}{d\alpha} - (F_{y'})_o \frac{dy_o}{d\alpha} = 0 \qquad (33)$$

and

$$F_y - \frac{d}{dx} F_{y'} = 0$$

The first of the above equations is known as the Transversility Condition, whereas the second is the familiar Euler equation. For the existence of an extremum both of these two equations must be satisfied. A problem of trajectory optimization for small changes in the orbital elements for electric propulsion devices is described by Edelbaum[47] and a problem of optimum proportioning of two propellants to obtain maximum burnt velocity is dealt with by Holds[48] using this method.

2.34 Problems of Lagrange, Mayer and Bolza

In dealing with calculus of variation problems one encounters three distinct problems which can be laid down as follows:

i. $U = \Big[G\,(x,\ y_i)\Big]_a^b$ $i = 1,2,3 \ldots n$ to be minimized or maximized

subject to the constraints

$$\phi_j(x,\ y_i,\ y_i) = 0 \quad j = 1,2,3 \ldots p \quad n \qquad (34)$$

and satisfying boundary or end conditions, defined by the functions

$$\Big[\bar{\phi}_k\ (x,\ y_i)\Big]_a^b = 0$$

ii. $U = \int_{x_a}^{x_b} H\,(x,\ y_i,\ y_k')\,dx$ to be minimized or maximized

subject to the constraints

$$\phi_j\ (x,\ y_i,\ y_i') = 0 \qquad (35)$$

and the boundary conditions

$$\Big[\bar{\phi}_k\ (x,\ y_i)\Big]_a^b = 0$$

iii. $U = \int_{x_a}^{x_b} H\,(x,\ y_i,\ y_i')\,dx + \Big[G\,(x,\ y_i)\Big]_a^b$

subject to constraints

$$\phi_j\ (x,\ y_i,\ y_i) = 0 \qquad (36)$$

and satisfying boundary conditions

$$\Big[\bar{\phi}_k\ (x,\ y_i)\Big]_a^b = 0$$

First of these three is known as Mayer's problem and is by far the most common in engineering and particularly in design. The second one is the Lagrange's problem and the third one is the Bolza's problem. Bolza's problem is basically the more general problem and is the combination of Mayer and Lagrange's problems.

In each case using Lagrange's multipliers an augmented function

$$F = H + \sum_{j=1}^{n} \lambda_j \, \phi_j$$ can be formulated.

This, then along with boundary conditions, the Euler equation and the transversility conditions allows one to solve for n + m unknowns $\lambda_j(x)$ and $y_i(x)$.

A problem of determining the two dimensional wing having minimum pressure drag in supersonic flow with given profile area or given moment of inertia of the profile area is dealt with by Miele[49].

2.35 Some Recent Developments

On account of the importance of the variational calculus in the field of aerospace and aerodynamics both in U.S.S.R. and U.S.A., extensive researches are under way to find better methods of solving such problems. As a result of this two very important methods have recently been added in this field. One of these is the American mathematician Bellman's Dynamic Programming[50], and the other is the Russian Professor Pontryagin's Maximum Principle[51]. Dynamic programming will be dealt with in section 2.5 under Mathematical Programming where as the Maximum Principle will be briefly dealt with in this section.

The Maximum Principle can be stated as follows:

If $U = [u(t), t_o, t_1, \ldots x_o]$ be an admissible control process, and if X(t) be the corresponding integral curve of the system

$\frac{dx^i}{dt} = f^i\,(x, u)$ $(i = 0,1,2, \ldots n)$ passing through the point $(0, x_o^1, \ldots x_o^n)$ for $t = 0$, and satisfying the conditions

$$x^1(t_1) = x_1^1, \ldots x^n\,(t_1) = x_1^n$$

where x^1 indicates the first derivative

x^n indicates the nth derivative, etc.

for $t = t_1$. Then if the control process U is optimal, there exists a continuous vector function $\phi(t) = \phi_o(t), \phi_1(t), \ldots \phi_n(t)$ such that

1. the function $\phi(t)$ satisfies the system

$$\frac{d\,\phi_i}{d\,t} = \sum_{\alpha=0}^{n} \frac{\partial\, f^{\alpha}(x,u)}{\partial\, x^i}\,\phi_{\alpha} \qquad (i = 0,1,2, \ldots n) \qquad (37)$$

for $x = x(t)$, $u = u(t)$.

2. For all t in (t_o, t_1), the function $\Pi(\phi, x, u) = \sum_{\alpha=0}^{n} \phi_\alpha f^\alpha(x,u)$

attains its maximum for $u = u(t)$, i.e.

$$\Pi\left[\phi(t), \quad x(t), u(t)\right] = K\ \phi(t), x(t) \quad , \tag{38}$$

where $K(\phi, x) = \sup_{u \in \Omega} \Pi(\phi, x, u)$ (39)

where Sup denotes supremum or least upper bound.

3. The relation

$$\phi_o\left[(t_1) \leq 0, K(\phi(t_1), u(t_1)\right] = 0 \tag{40}$$

hold at time t_1.

Pontryagin et.al[41] have described in detail the application of the maximum principle to the solution of some time optimum synthesis problems of control systems.

2.40 Direct or Numerical Methods

The methods of optimization, described in the preceding sections usually end up in differential equations which quite frequently are very difficult to solve. To overcome this difficulty, direct methods have been evolved. A direct method can be defined as one wherein values of a function at two or more points are compared to reach an extremum. All such methods are approximate and do not give precise results. However, simplicity and the ease with which solutions can be obtained by using these methods has made them very popular and explains their wide use.

Direct methods are not new; their existance would be traced to the time of Euler and Ritz and, if solutions of equations are considered, to the time of Algorithm.[1,2,3,4,5,6] However, since the appearance of computers in the last decade, their importance suddenly increased multifold and more attention is being paid to such methods during the present time than ever before.

Many such methods are at present available, such as direct methods of variational calculus, analytical cum numerical methods, deterministic methods of direct search, and stohastic methods of direct search. All these will be dealt with separately in the following sections.

2.41 Direct Methods of Variational Calculus

Quite a few methods are available which can be described under this heading. However, four important ones are Euler's method of finite differences, Ritz method, Kantrovic's[45] method and the method of linear integral[52].

The basic idea behind the so-called Euler's method is that a functional of the type U y(x) can be considered as a sum of a finite set of variables so that for a particular functional

$$U\left[y(x)\right] = F(x, y, y')dx, \qquad y(x_o) = a \qquad \text{and } y(x_1) = b$$

$U\left[y(x)\right]$ turns into a function $\phi\,(y_1, y_2, \ldots y_{n-1})$ of ordinates $y_1, y_2, \ldots y_{n-1}$ for a polygonal curve divided into n line segments $x_o + \Delta x$, $x_o + 2\Delta x$, . . . $x_o + (n-1)\,\Delta x$. Ordinates $y_1, y_2, \ldots y_{n-1}$ are so chosen that the function has an extremum. Referring back to theory of maxima and minima

$$\frac{\partial \phi}{\partial y_1} = 0\ ,\quad \frac{\partial \phi}{\partial y_2} = 0\ ,\ \ldots\ldots,\quad \frac{\partial \phi}{\partial y_{n-1}} = 0$$

The next step to this is to pass the limit with $n \to \infty$. Thus

$$U\left[y(x)\right] = \int_{x_o}^{x_1} F(x, y, y')\,dx \approx \phi\,(y_1, y_2, \ldots y_{n-1})$$

and
$$\phi\,(y_1, y_2 \ldots y_{n-1}) = \sum_{i=o}^{n-1} F\left(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\right)\Delta x \qquad (41)$$

Since only the ith and (i-1)th terms in the above equation depend on y_i

$$\frac{\partial \phi}{\partial y_i} = 0 \text{ for } i = 1,2,3, \ldots n-1$$

can be of the form

$$F_y\left(x_i, y_i, \frac{y_{i+1}-y_i}{\Delta x}\right)\Delta x + F_{y'}\left(x_i, y_i, \frac{y_{i+1}-y_i}{\Delta x}\right)\frac{(-1)}{\Delta x}\,\Delta x$$

$$+ F_{y'}\left(x_{i-1}, y_{i-1}, \frac{y_i - y_{i-1}}{\Delta x}\right)\left(\frac{1}{\Delta x}\right)\Delta x = 0$$

for i = 1,2,3 . . . n-1

i.e.

$$F_y(x_i, y_i, \Delta y_i/\Delta x) - F_{y'}(x_i, y_i, \Delta y_i/\Delta x) - F_{y'}(x_{i-1}, y_{i-1}, \Delta y_{i-1}/\Delta x) = 0$$

or $F(x_i, y_i, \Delta y_i/\Delta x) - \Delta F_{y'}/\Delta x = 0$

where $\frac{\partial F}{\partial y} = F_y$

$\frac{\partial F}{\partial y'} = F_{y'}$

Now when $n \to \infty$ this becomes

$F_y - \frac{d}{dx} F_{y'} = 0$ (which is Euler's equation derived in preceding sections and must be satisfied by the function $y(x)$ for an extremum.

In case approximations are sought these can be determined from equations $\partial\phi/\partial y_i = 0$, $i = 1,2,3, \ldots n-1$ without applying the limitation process.

The other important method is Ritz's Method[45]. In this method a functional is considered as a linear functional combination. Thus a functional $U = U\left[y(x)\right]$.

can be written as $U_n = U_n\left[y_n(x)\right]$

$$\text{where } y_n(x) = \sum_{i=1}^{n} \alpha W_i(x) \qquad (42)$$

coefficients $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are constants which are adjusted to yield the desired extremum. $W_i(x)$ are functions of x satisfying given boundary condition. Thus with these linear combinations, functional $U\left[y(x)\right]$ becomes a function $\phi(\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n)$ of coefficients $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$. For an extremum the coefficients can then be determined by the system of equations

$$\frac{\partial\phi}{\partial\alpha_i} = 0 \text{ for } i=1,2,3, \ldots n.$$

In the end applying the limits, e.g., $n \to \infty$, limit function

$$y = \sum_{i=1}^{\infty} \alpha_i W_i(x)$$

is obtained. If this converges, under certain assumptions about the functional $U\left[y(x)\right]$ and $W_i(x)$ this will give an exact solution. If the limiting process is not applied, an n term approximate solution is obtained.

The so-called method of Kantrovic[45] is a generalization of the Ritz's method. In this method coefficients $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are no longer constants; instead they are functions of the independent variables. These functions are to be so chosen that the functional $U\ y(x)$ has an extremum. The Russian mathematicians claim that by using this method, approximate solutions obtained are usually better than those obtained by Ritz's method.

In the linear integral method[52] certain new variables are introduced, and by integrating by parts, certain formulae are obtained which are commonly known as Green's Formula. Green's formula is then simplified by using certain multipliers so that dependent variables drop out. This simplified integral is then maximized using a theorem which Faulkner[52] calls the maximum principle.

Elsgolc[45] has used the first three methods for solving various design problems and Faulkner[52] has used the last method for determining optimum trajectories on digital computers.

2.42 Numerical cum Analytical Methods

It is mentioned in section 2.4 that the differential equations obtained by the methods of previous sections are not always easy to solve. Similarly the set of simultaneous constraining equations are very often difficult to solve. Therefore, in certain cases it is customary to solve these equations by using various approximate numerical methods such as Algorithm's method, Newton-Raphson's method and the so called method of finite differences.

In Algorithm's Method of Errors (Al-Hisab-ul-Khataayn)[53,54*] if a root $\bar{x}$ of $U(x)=0$ can be isolated between two points x_0 and x_1 in the interval x_0 and x_1, the graph of $Y=U(x)$ would be as shown in Figure 2.42. If the point A_0 and A_1 in Figure 2.42 are joined by a straight line, this line will cut the x-axis at a point say x_2 which is close to the root $\bar{x}$. Therefore using similar triangles $x_0A_0x_2$ and $x_1A_1x_2$ it can be shown that

$$\frac{x_2 - x_0}{U(x_0)} = \frac{x_1 - x_2}{-U(x_1)} \qquad \text{i.e.} \qquad x_2 = \frac{x_0 U(x_1) - x_1 U(x_0)}{U(x_1) - U(x_0)}$$

$$\therefore \quad x_{n+1} = \frac{x_0 U(x_n) - x_n U(x_0)}{U(x_n) - U(x_0)}$$

To get a closer approximation to the root x another straight line closer to the curve can be drawn, and by repeating this process a series of values $x_2, x_3, x_4, \ldots\ldots\ldots, x_n$ can be obtained which in the end converge to the real root $\bar{x}$.

This method, although it gives the root, is slow in convergence, and hence Newton-Raphson's method is usually preferred. However, in case $f'(x)=0$ or close to zero, Newton-Raphson's methods fails. Algorithim's method is found to be of value in such cases.

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* Translated in Latin as Regula Falsi

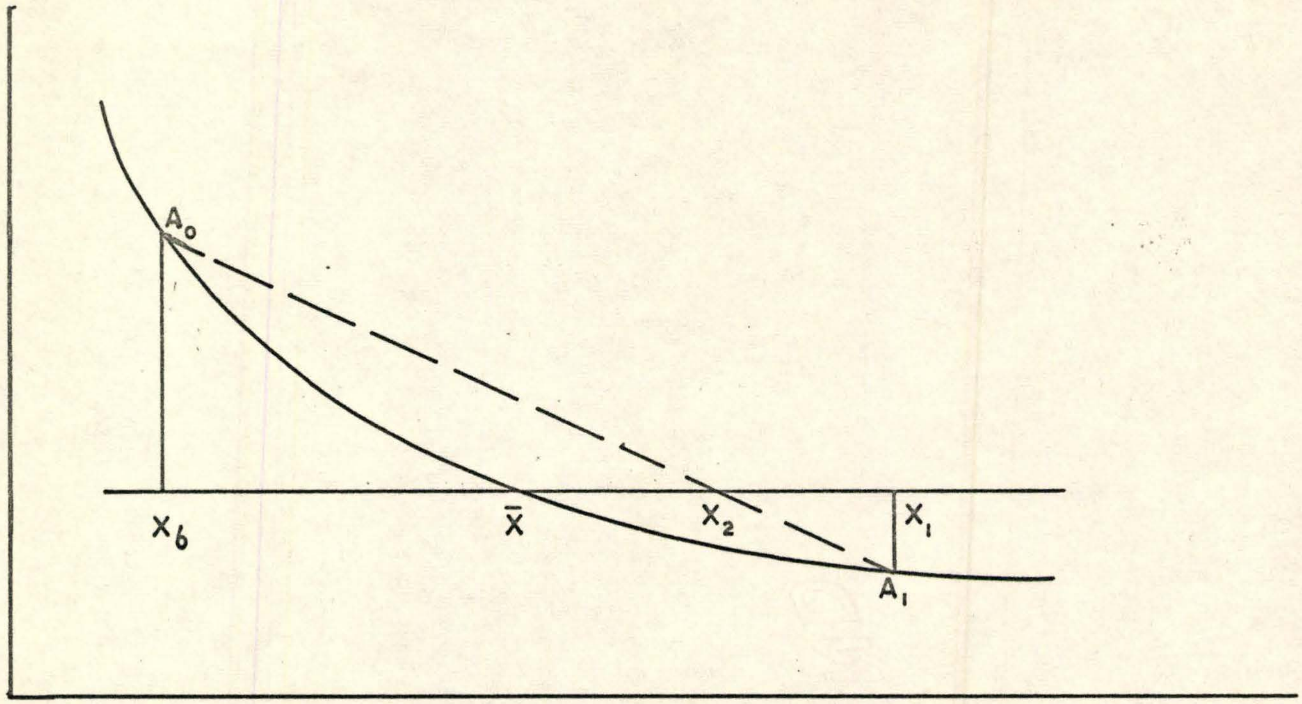

FIG. 2-42

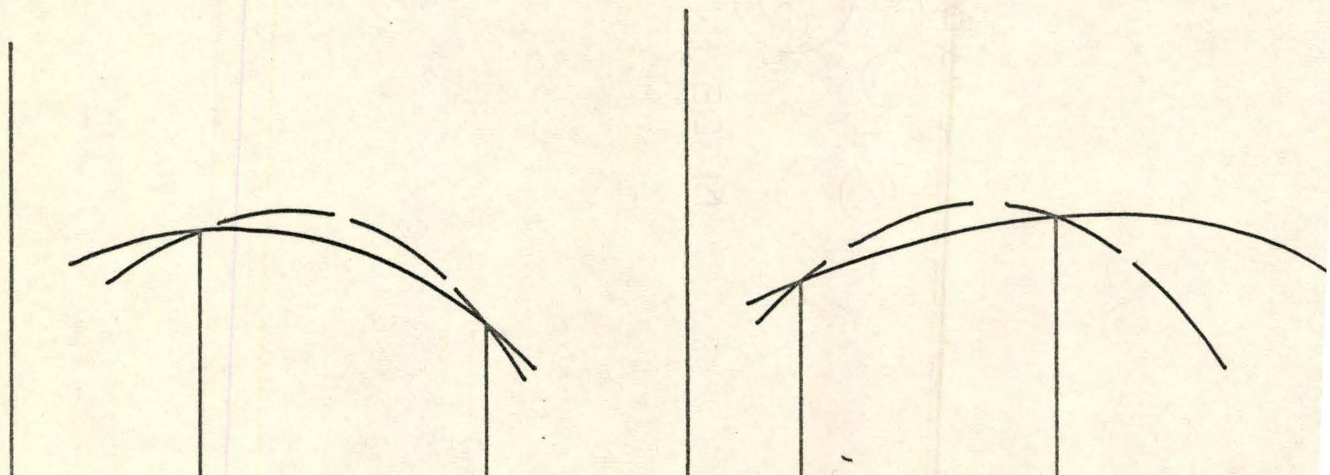

FIG. 2-43 u

In the Newton-Raphson's method [55,56] the root of the equation $U(x)=0$ is found by the formula

$$x_{n+1} = x_n - U(x_n)/U'(x_n) \qquad n = 1,2,\ldots\ldots\ldots, \text{etc.}$$

and convergence is comparatively faster. The formula is derived by the use of Taylor's expansion. The above expression can also be written as

$$x_{n+1} = x_n - U(x_n) \cdot U'^{-1}(x_n)$$

This is similar to matrix form; suggesting that a set of silmultaneous equations can be solved by this method. The possibility of this can be investigated in the following manner: Let us assume that $\emptyset$ is a vector consisting of a set of functions $\emptyset_1, \emptyset_2, \ldots\ldots, \emptyset_n$ and that $\overline{X}_0$ is an initial starting vector. Adding a small increment $\Delta\overline{X}$ to this vector we arrive at a point where

$$\text{vector} \quad \overline{X}_1 = \overline{X}_0 + \Delta\overline{X}$$

Similarly for the rth vector $\overline{X}_r$ and r+1th vector $\overline{X}_{r+1}$ we can write

$$\overline{X}_{r+1} = \overline{X}_r + \Delta\overline{X} \qquad (43)$$

Now considering that somehow we have arrived at rth approximation $\overline{X}_r$ and that at this point $\overline{\emptyset}$ has a value $\overline{\emptyset}_r$. We can now expand $\emptyset$ using Taylor's series expansion so that

$$\overline{\emptyset}_{r+1} = \overline{\emptyset}_r + \Delta\overline{X} \cdot \overline{\emptyset}'_r + \ldots\ldots\ldots\ldots$$

Neglecting higher terms this can be written as

$$\overline{\emptyset}_{r+1} \approx \overline{\emptyset}_r + \Delta\overline{X} \cdot \overline{\emptyset}'_r$$

To reach close to the solution point we equate $\overline{\emptyset}$ to zero so that

$$\Delta\overline{X} \cdot \overline{\emptyset}'_r = -\overline{\emptyset}_r$$

Now multiplying both sides by the inverse matrix $\overline{\emptyset}'^{-1}_r$ we get

$$\Delta\overline{X} \cdot I = -\overline{\emptyset}_r \cdot \overline{\emptyset}'^{-1}_r \qquad (44)$$

where I is the unit vector and hence we can further reduce this to

$$\Delta \bar{X} = - \phi_r \cdot \phi_r'^{-1} \tag{45}$$

Substituting (45) in (43) we get

$$\bar{X}_{r+1} = - \phi_r \cdot \phi_r'^{-1} + \bar{X}_r$$

This equation can very easily be handled by computers[36] and hence even lengthy and difficult sets of nonlinear algebraic equations can be dealt with by this method.

For solution of certain differential equations the so called method of finite differences[57,58] is found to be very useful. This can be explained by using the first order differential equation

$$d\,y/d\,x = U(x,y) = y' \tag{46}$$

with initial conditions $y=y_0$ for $x=x_0$ when $x > x_0$.

For the solution, starting with the known ordinate and calculating ordinates successively, we can write

$$y_1 = y(x_0+h)$$
$$y_2 = y(x_0+2h) = y(x_1+h)$$
$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$
$$y_n = y(x_0+nh) = y(x_{n-1}+h)$$

where h is a finite increment.

Using Taylor's series expansion we can write

$$y_n = y(x_{n-1}+h) = y(x_{n-1}) + hy' + \frac{h^2}{2!}y'' + \cdots\cdots\cdots$$

Using truncated approximation up to y'

$$y_n = y(x_{n-1}) + h\,y' \quad \text{or} \quad y_n = y_{n-1} + h\,y' \tag{47}$$

Starting with n=0 values of ordinates at n=1, 2, 3, ... can be calculated and the process is repeated so that a step-by-step method of finding the solution is obtained.

The above equation can also be written as

$$y' = (y_n - y_{n-1})/h \tag{48}$$

Similarly values of second and third derivatives can also be calculated. This facilitates synthesis of kinematic mechanisms where y, y', y'' etc. are needed to represent velocity, acceleration, etc. Due to truncation this method is not accurate and hence certain modified versions such as Heun's method and the mid point methods[57,58] are commonly used.

For a special case when criterion as well as constraints happen to be homogeneous, Bedford, Willis and Dodson[59] have shown that a method which they call the "Infinitisimal per-unit increment" method can easily be applied and has the property that the designer is kept in touch with the elements of design.

This method can be described as follows

$$U = U(x_1, x_2, x_3, \dots\dots\dots\dots, x_n) \tag{49}$$

and the constraint

$$\phi = \phi\,(x_1, x_2, x_3, \dots\dots, x_n)$$

are both homogeneous. By using the definition of homogeneity[60] we can write $\bar{x} = \lambda x_n$ for $n = 1, 2, \dots\dots$

so that the above equations take the form

$$U = \lambda_u^{\alpha} \cdot U(x_1, x_2, \dots\dots\dots, x_n) \tag{50}$$

and $$\phi = \lambda_\phi^{\beta} \cdot \phi(x_1, x_2, \dots\dots\dots, x_n)$$

where α and β are the degree of homogeneity of U and ϕ respectively. At this stage using Euler's Homogeneity Relation we can write

$$\frac{\partial u}{\partial x_1}\,dx_1 + \frac{\partial u}{\partial x_2}\,dx_2 + \dots\dots + \frac{\partial u}{\partial x_n}\,dx_n = \alpha U \tag{51}$$

$$\frac{\partial \phi}{\partial x_1}\,dx_1 + \frac{\partial \phi}{\partial x_2}\,dx_2 + \dots\dots + \frac{\partial \phi}{\partial x_n}\,dx_n = \beta \phi \tag{52}$$

Multiplying (52) with $\frac{\alpha U}{\beta \phi}$ and subtracting from (51) we get

$$dx_1\left(\frac{\partial U}{\partial x_1} - \frac{\alpha U}{\beta \phi} \cdot \frac{\partial \phi}{\partial x_1}\right) + \cdots\cdots + dx_n\left(\frac{\partial U}{\partial x_n} - \frac{\alpha U}{\beta \phi} \cdot \frac{\partial \phi}{\partial x_n}\right) \quad (53)$$

which imply that

$$\frac{\partial U}{\partial x_n} - \frac{\alpha U}{\beta \phi} \cdot \frac{\partial \phi}{\partial x} = 0 \qquad \text{for all } n=1,\ 2,\ \ldots\ldots\ldots$$

This can now be written as

$$\frac{\partial U}{\partial x_n} = \frac{\alpha U}{\beta \phi} \cdot \frac{\partial \phi}{\partial x_n} \qquad \text{i.e.} \quad \frac{1/U \ \cdot \ \partial U/\partial x_n}{1/\phi \ \cdot \ \partial \phi/\partial x_n} = \frac{\alpha}{\beta} \quad (54)$$

Using the finite increment approximations this can be expressed as

$$\frac{\Delta U}{U} \Big/ \frac{\Delta \phi}{\phi} = \frac{\alpha}{\beta} \quad (55)$$

The procedure of attack for solution of a problem of this type would be to start with any arbitrary values of the design parameter and calculate the ratio as shown in the right side of the last equation. This can then be checked with the ratio of the degrees of homogeneity $\frac{\alpha}{\beta}$, and if it happens to be the same, the optimum is reached; otherwise, one of the design parameters is changed (increased or decreased depending on whether the ratio happens to be smaller or larger than the ratio of degree of homogeneity) and the method is repeated till these two ratios tally. This method can therefore be stated as follows:

> If any device has optimum shape, the ratio between the resulting per-unit increments in optimizing characteristics is equal to the ratio between degrees of homogeneity of these characteristics.

For illustration of this method we will use the example of the design of a cylindrical fuel tank to have maximum volume for a given surface area which has been used in section 2.22.

Using the relations mentioned under section 2.22 we can write

$U = \pi r^2 h$ and $\phi = (\pi r^2 + \pi r h)$

The degree of homogeneity of U is 3 and that of ϕ is 2 so that the ratio $\frac{\alpha}{\beta} = 3/2 = 1.5$

Using the equation derived above we can write

$$\frac{\triangle U \cdot 1/U}{\triangle \phi \cdot 1/\phi} = \frac{\alpha}{\beta} = 1.5$$

Assuming r=2 and h=2 as our first guess, then if $\frac{\Delta h}{h} = 2K$

$$\triangle U/U = h/U \cdot \partial U/\partial h \cdot \triangle h/h = 1/\pi r^2 \cdot \pi r^2/1 \cdot 2K/1 = 2K$$

$$\triangle \phi/\phi = h/\phi \cdot \partial \phi/\partial h \cdot \triangle h/h = 1/2\pi r \cdot h/r+h \cdot 2\pi r(2K)/1 = K$$

This gives a ratio of 2.0, which is too large, so that h, which seems to be too small, should be increased. Using h=3 as our next step, we get the ratio 1.8 which is still too large. Trying h=4 gives the ratio equal to 1.5 which is exactly the same as the required ratio. Hence the optimum shape of the fuel tank would have a relation h=2r, which is the same as the one obtained in section 2.22.

Asimow[36] has solved a bearing problem by using Newton-Raphson's method extended to simultaneous equations along with the method of Lagrange's undetermined multipliers. A problem of transformer design for optimum geometric configuration is solved by Jackson[61] by using the method of finite differences as applied to homogeneous partial differential equations, and a problem of synthesis of kinematic mechanisms by using method of finite differences is dealt with by Shaffer and Krause.[62] Johnson[34] has also used this method for solving various optimum design problems of machine elements.

2.43 Deterministic Methods of Direct Search

Direct search is a method of arriving at an optimum by continuous searching through experimentation. A deterministic problem is one which does not depend on any random factors. A deterministic method of direct search can therefore be defined as a method of search for an optimum of an unknown function having no random factors.[63] Deterministic problems can be divided into two main types, e.g., univariable and multi-variable problems. By univariable problems we mean a problem having not more than one independent variable. Similarly a multivariable problem is one which contains more than one independent variable. The techniques applicable for solution of univariable problems are relatively simple but unfortunately these are not applicable to multivariable problem solutions.

Search methods are usually either sequential or simultaneous type. In sequential methods co-ordinates are examined one after the other, whereas, in simultaneous methods co-ordinates are examined simultaneously. A sequential method takes more time than a simultaneous method; however, sequential methods are more effective since errors involved are usually less than the simultaneous methods.

The experimental search of optimum is a statistical technique and Hotelling[64] may be called the enunciator of efforts in this direction. Later G. E. P. Box[65] applied these methods for solving optimization problems of chemical process systems.

We will discuss these methods briefly in this and the following section.

Univariable Methods:

There are several rigorous mathematical techniques which could be applied for the search of optimum for the so called univariable unimodal problems. However, the best one is a sequential method due to Kiefer.[66] This can be described as follows:

A function $u=f(x)$ is said to be unimodal if there exists a specific value x_0 of x such that u either increases for $x \leq x_0$ and decreases for $x > x_0$ or increases for $x < x_0$ and also increases for $x \geq x_0$. If such a function is prescribed in a given interval $0 \leq x \leq L_n$ and if U_n is the supremum of all L_n such that the maximum of u on a subinterval of unit length can always be located by calculating n values of the function then

$$U_n = U_{n-1} + U_{n-2} \text{ where } n \geq 2^* \tag{56}$$

and $U_0 = U_1 = 1$

This indicates that U_n will assume values

$U_0=1,\ U_1=1,\ U_2=2,\ U_3=3,\ U_4=5,\ U_5=8,\ U_6=13,\ U_7=21,\ U_8=34,$

$U_9=55,\ U_{10}=89,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

so that $U_{20} > 10000$. This simply means that a maximum can always be located to the accuracy of .0001 of the original interval length within 20 calculations.

Since the ratio $R = U_{n+1}/U_n = 1 + U_{n-1}/U_n = 1 + 1/1 + U_{n-2}/U_{n-1}$ which is a continued fraction and its limiting value is

$$R_1 = \sqrt{5}+1\ /2$$

a good procedure for starting the search of this type is to use two

---

* For proof see Ref. 63, 66, 67

values of x such that

$$L - x_1 = L/R_1 = .618\ L$$

and $x_2 = L/R_1 = .618\ L$ where L is the original given interval

The interval of uncertainty would reduce to .618 L after first trial and would lie either between 0 and x_2 or x_1 and L depending on whether

$$u_1 > u_2 \qquad u_2 > u_1 \qquad \text{(see Fig. 2.43)}$$

The new interval L' can then be treated in the same manner and the procedure is repeated till a point very close to optimum (with a negligible interval of uncertainty) is attained.

Multivariable Methods

Multivariable search methods can be described as methods of local exploration for finding the location of the surfaces of higher response and then of making a decision which way to move, so that the criterion may be improved most effectively. The search can be divided into three phases.[63] In the first phase exploration at the base is confined to a suitably selected small region and linear approximations are used for explorations. In the second phase explorations are very infrequent and the progress is usually by jumps and for most of this area linear approximations will do. In the third and the last phase explorations should be more carefully organized since we are in the vicinity of the peak. If large steps are taken there is risk of missing the maximum; hence, quadratic or higher approximations of non-linearities are needed. Usually the multivariable functions are first reduced to unimodal response surfaces by using parametric representations. In search problems of this type generally two basic approaches are used -- the elimination approach and the climb approach.

In the elimination approach, which is sometimes called the contour tangent elimination approach,[63,68] a contour tangent is first determined at some arbitrary point. Using this contour tangent, which divides the given area into two sections, the lower section which is less liable to have an extremum is eliminated. This procedure is then repeated in the other section which is considered as the feasi area. In this way the feasibility area is successively reduced by rep searches, so that in the end this converges to the required peak or what we call an optimum. This is illustrated in Fig. 2.43m which is self-explanatory.

Since this method is based on a decision regarding the probability of the extremum lying in a section of the experimentation area, there is a possibility that it might end in a failure due to an incorrect decision. For the strongly unimodal functions which are defined as functions whose summit 's' could be joined to any point 'p' in the region of experimentation by a line ps which is always a rising path, this method always leads to the summit.

The climb or ascent methods,[63] also known as gradient methods, are the methods of search along the gradient or the direction perpendicular to the contour tangent. Since the direction of the gradient is the direction of the maximum rate of change of criterion for each individual step taken, these methods converge to the peak quicker than any other method and are less liable to error.

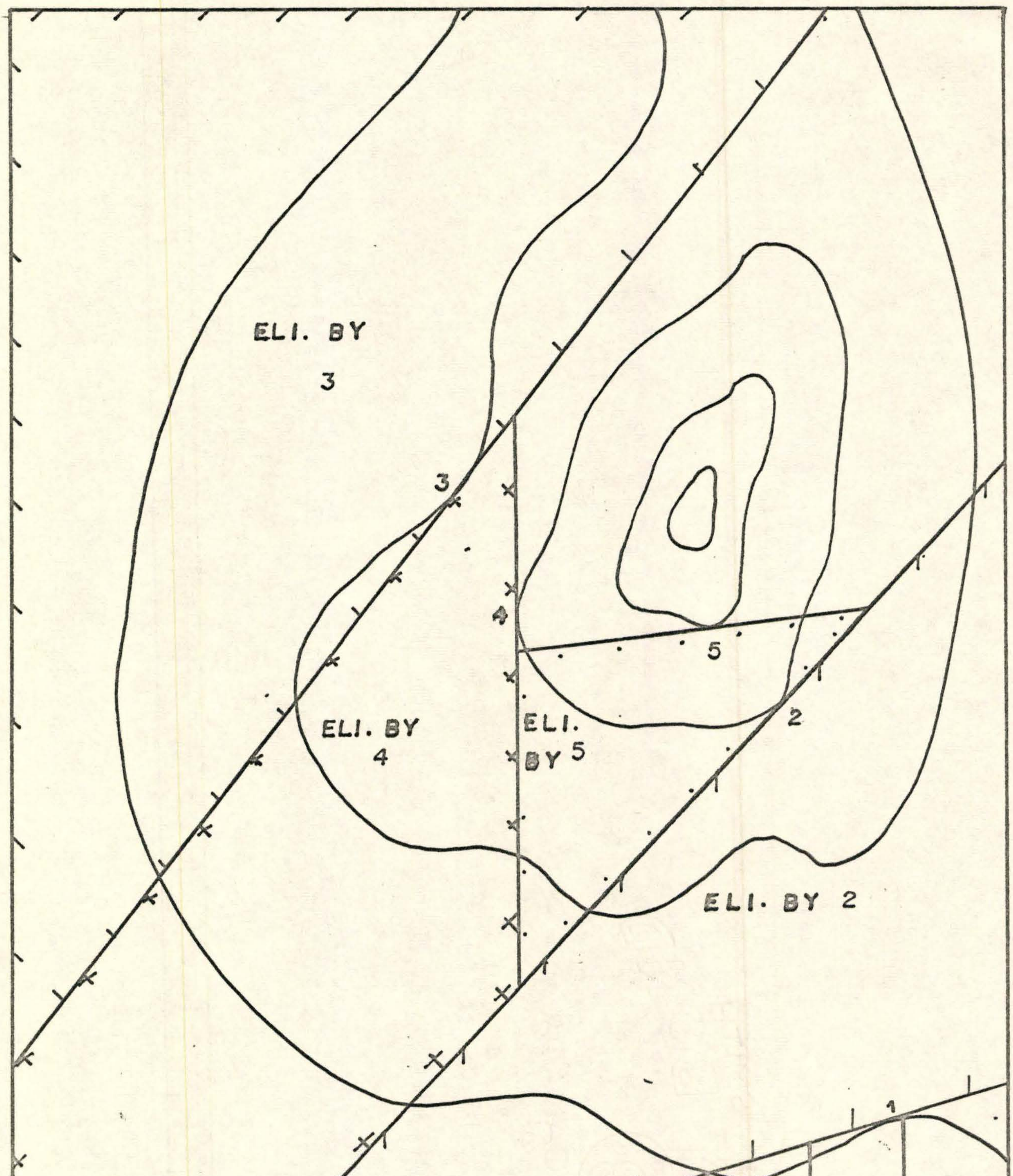

FIG 2·43m

Acceleration Methods

Along with the methods described in the preceding paragraph, certain comparatively new methods are often used since they increase the rate of convergence and hence are generally known as acceleration methods. Some of these are the partan or parallel tangent method of Shah, Buehler and Kemphthorne,[69] pattern search method of Hooke and Jeeves,[70] poor man's optimizer of Mugele,[71] and rotating coordinate method of Rosenbrock.[72] The parallel tangent method, which is often used with the gradient method by the name of the accelerated gradient method, is found to be very useful in design optimization problems and hence will be described here.

The parallel tangent method is based on the fact that the alternate paths in gradient methods are usually approximately parallel to the respective contour tangent. Therefore this property can be used to reduce the number of steps in the final phase of the gradient method, which are usually more than the number of steps near the start. In the accelerated ascent or accelerated gradient method, the first two steps are taken along the gradient and then the final step is taken along the line joining the first (P_0) and third (P_2) point, as shown in Figure 2.43a. In a general case of n independent variables, if the response hyper surface contours happen to be concentric ellipsoids, the peak is attained after exactly n-1 steps.

Other modifications or extensions of the basic gradient method are also discussed in section 2.54.

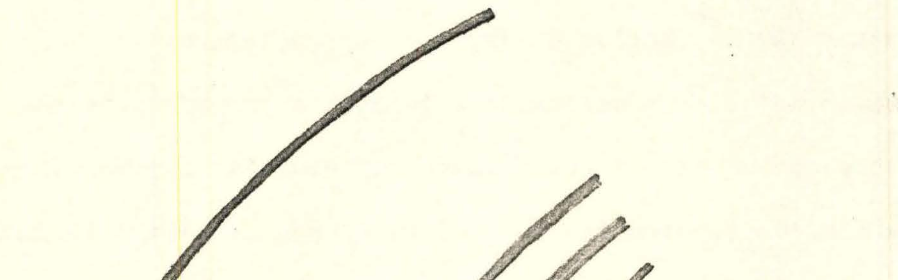

FIG. 2·43 a

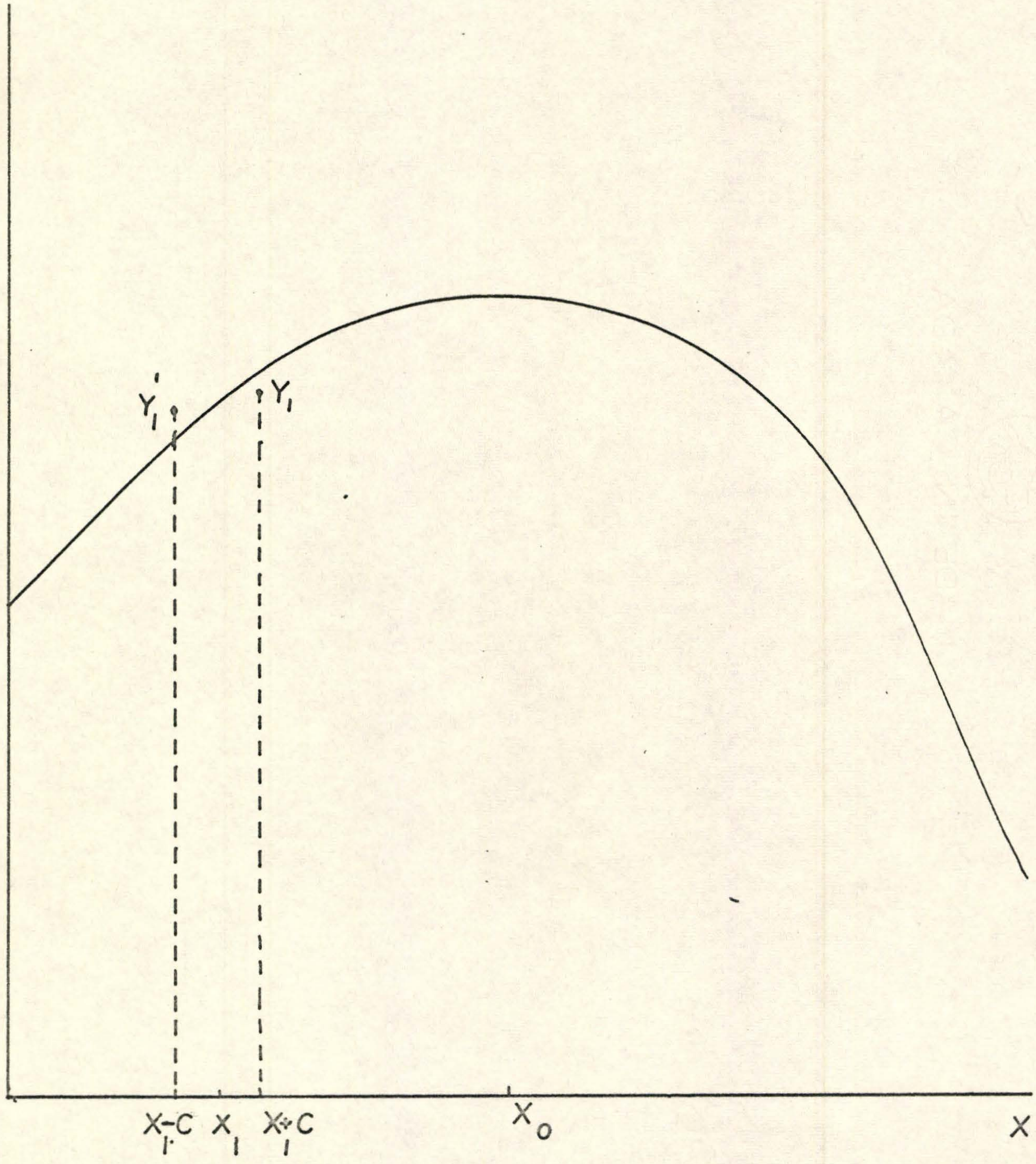

FIG. 2·44

(CHANG)

will be such that in the limit of large n, x_n converges to x. Lapidus[76] et. al. have applied this procedure to process performance optimization and Bertram[77] applied it to control systems. Chang[78] in his book on control systems optimum synthesis has allotted one full chapter to stochastic processes and optimum design of adaptive control.

2.50 Mathematical Programming

Mathematical programming can be defined as the method of planning of various activities in such a manner that the objective of optimization is achieved. These methods are comparatively recent and are said to have initially been evolved during World War II for military purposes. Hitchcock[79] perhaps formulated the first of the various techniques in the field; e.g., the formulation of the transportation problem for minimum cost distribution which he submitted in 1941. In 1947 Dantzig[80] came out with his famous simplex method which solved the general linear programming problem. Bellman[81] evolved the more sophisticated technique, which he called Dynamic Programming, in 1955. In the meantime computers entered in the field of research and proved to be the strongest tool of research man ever had. The ease with which programming problems can be handled by the computers made them very popular among programmers, and soon new and better methods of programming and many refinements of the older programming methods were evolved and are being evolved every day.

Initially such methods were confined to the so called linear problems; i.e., problems dealing with a linear criterion or optimization function and linear constraints. However, very soon it was realized that many practical problems do not fall under this category. Hence new methods for handling non-linear problems were soon introduced.

In the following sections we will therefore consider both linear as well as non-linear programming methods. The first section will deal with comparatively simple problems; e.g., the Transportation

Problem and its refinements. In the next section the simplex and variations in the simplex method will be dealt with. The last two sections will cover dynamic programming and various versions of the so-called Gradient Method which solved several nonlinear many variable problems. Before considering any mathematical programming method we will have to familiarize ourselves with the following terms and theorems which commonly occur in mathematical programming problems.*

1. An n dimensional Euclidean Space is a set of vectors with the property that there exists n linearly independent vectors for every set of n+1 linearly dependent vectors.
2. A convex set is a combination of points such that if any two points lying in the set are joined, the line joining these points will also lie wholly in the set.
3. A vertex, often called an extreme point, is a corner of the convex set.
4. A basis for an n dimensional Euclidean Space is a set of n linearly independent vectors.
5. A basic solution for a linear programming problem is the solution of the constraining equations with n-m variables set equal to zero.
6. A feasible solution is a vector which satisfies both the constraining equations and the non-negativity condition, but not the optimization equation.
7. A basic feasible solution is a feasible solution with not more than m positive independent variables where $m \angle n$. (m is the number of conatraining equations) The remaining variables are set equal to zero.

* For proof see Ref. 82, 83, 84, 85

8. A non-degenerate basic feasible solution is a basic feasible solution with exactly m positive independent variables.

9. An optimum basic feasible solution is a basic feasible solution which optimizes the criterion function.

10. All feasible solutions to the linear programming problem constitute a convex set. (Theorem)

11. If a linear programming problem has a feasible solution it must also have a basic feasible solution. (Theorem)

12. If an optimization function has a finite extremum, then it is a basic feasible solution. (Theorem)

2.51 Transportation Method

As the name implies, the transportation method was originally devised for the solution of distribution problems. It falls under the cases of mathematical programming called linear programming because the criterion as well as the constraints involved in this type of problem have a linear relationship. The transportation method is simpler than many other linear programming methods. However, it is applicable only in special cases. On the other hand the so-called simplex method which will be described in the next section can solve transportation problems as well as many other types of problems which can not be solved by the transportation method. In spite of this, on account of its simplicity and better efficiency compared to the simplex method, the transportation method has been retained and is expected to retain its position in the solution of a certain category of linear programming problems. Over and above the famous distribution problem this method has been successfully utilized for the solution of many multiple assignment problems,[84,85] electrical network design problems[86] and production design problems.[87] It can also be used for the solution of certain assembly design problems and for this purpose a simple example, which is developed during this study, is illustrated at the end of this section.

The Transportation Problem can be stated in the following manner: Suppose there are n sources S_i where certain products are manufactured and there are m destinations D_j where these products are to be shipped for storage prior to their final disposal. At each source

s_i, p_i products can be produced and at each destination D_j, k_j products can be stored. If the number of products shipped from a source S_i to a destination D_j can be called x_{ij} and the shipping cost from a source S_i to a destination D_j can be called C_{ij} the requirement is to minimize total shipping cost. Expressing this mathematically we can write

$$\sum_{j=1}^{n} x_{ij} = p_i \qquad \text{for } i = 1, 2, \ldots\ldots\ldots m$$

$$\sum_{i=1}^{m} x_{ij} = k_j \qquad \text{for } j = 1, 2, \ldots\ldots\ldots n$$

and $U = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij}\, x_{ij}$ is to be minimized subject to the consistency condition $\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} k_j$ and non-negative condition $x_{ij} \geq 0$ for all values of i and j. However, in certain cases the consistency condition may not be satisfied and hence we will be dealing with inequalities instead of equalities. In order to overcome this difficulty a so-called slack variable v is added so that the inequality is finally converted into an equality and hence satisfies the required consistency condition. For illustration, see the example at the end of this section. A theorem of the transportation problem tells us that it has a triangular basis; i.e., the system of linear equations associated with transportation problems can be given such a form that there exists at least one equation that contains only one unknown and when this is evaluated its deletion will evolve a new set which once again will have at least one equation that contains only one unknown. Proceeding in this way unknowns can be evaluated. For example

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= P_1 \\ a_{22}x_2 + a_{23}x_3 &= P_2 \\ a_{33}x_3 &= P_3 \end{aligned}$$

is a set having a triangular basis.

A transportation problem can be solved by an algorithm consisting of five steps:

1. Formulation of the transportation array -- In this the given data can be arranged in the form of an array similar to the one shown in Table 1.
2. Determination of an initial solution -- Many different methods for arriving at an initial solution are presently available; however, the so-called North West Corner method[83,84,85,88] (also called stepping stone method) is the simplest and easiest for computer applications. The other important methods are Inspection method, Mutually Preferred method and Vogal's approximate method.[88] These methods reduce the computation time by reducing the number of iteration for arriving at an optimum but they can not be applied so easily to the computers as the north west corner method.

   In the north west corner method, as its name suggests, the north west corner cell is first selected and the maximum number of products, without violating the capacity restrictions, are shipped to this cell; i.e., from S_1 to D_1. The triangular basis of the problem permits us to delete either a column or a row where the capacity restraints are satisfied thus leaving a new smaller array which can then be dealt with in exactly the same manner. The procedure is repeated till all capacity restraints are fully satisfied. For illustration see Table 2.
3. Evaluation of empty cells -- After determining the initial solution each empty cell is evaluated term by term for possible movement in

closed circuits to bring about a possible reduction in cost. All such evaluations are then compared to choose the one which can bring the maximum reduction.

4. Altering the solution -- The initial solution is now altered by shifting a maximum number of products without violating the capacity restraints according to the circuit of maximum reduction found in Step 4.

5. Evaluation of the altered solution -- The altered solution is evaluated in the same manner as Step 3. There could be three possibilities at this stage; e.g., i. there could be a possibility of further reduction in cost in which case the procedure is to repeat Step 4.

ii. there could be no possibility of further reduction in cost but a circuit exists which is indifferent, indicating that the optimum is reached but an alternate solution is also available.

iii. there is no possibility of further reduction and there is no indifferent circuit available indicating that optimum is reached and that no other alternate optimum solution is available.

Every transportation solution, says a transportation theorem, must have m+n-1 variables. If at any stage of iterations the number of variables happen to be less than this number the solution is said to have degenerated. Further evaluation of a degenerated solution is either not possible or very difficult to handle. Therefore, to overcome this difficulty an infinitisimally small quantity ε is introduced. This quantity can then be handled as any other variable but having a negligible value.

For proof of the authenticity of this procedure, References 82, 83, 84, 85 and 87 can be consulted.

Example

A product consists of 4 components D_1, D_2, D_3 and D_4. In each product 8 components of D_1, 5 components of D_2, and 17 components of D_3 are to be mounted on a frame D_4. For the assembly of component D_1 either five screws or five rivets are needed. Similarly for mounting each D_2 five screws or five rivets are again required. For the assembly of D_3 either two screws or two rivets are needed. The cost of screwing, including the cost of the screw itself, is $.15 for D_1, $.25 for D_2 and $.15 for D_3. Similarly for rivetting this cost is $.10 for D_1, $.15 for D_2, and $.15 for D_3. One hundred products are to be assembled daily and the capacity of both screwing and rivetting sections is 5000 screws or 5000 rivets a day. It is required to find the components where rivets are to be used and the components where screws are to be used so that the total cost be a minimum.

Step 1. Formulation of the array

$$
\begin{aligned}
x_{11} + x_{12} + x_{13} &= p_1 \\
x_{11} + x_{21} &= k_1 \\
x_{12} + x_{22} &= k_2 \\
x_{13} + x_{23} &= k_3 \\
x_{21} + x_{22} + x_{23} &= p_2
\end{aligned}
$$

where x_{11}, x_{12} and x_{13} are number of screws required for fastening D_1, D_2 and D_3 on the frame D_4 respectively. Similarly x_{21}, x_{22} and x_{23} are number of rivets required for fastening D_1, D_2 and D_3 respectively. C_{11}, C_{12} and C_{13} are the costs of screwing x_{11}, x_{12} and x_{13}. Similarly C_{21}, C_{22} and C_{23} are costs of rivetting x_{21}, x_{22} and x_{23}.

The total capacity of screw section is $p_1 = 5000$

The total capacity of rivet section is $p_2 = 5000$

Each component D_1 requires a total of 5 screws or rivets, hence the total requirements of screws or rivets for component D_1 for one hundred products to be manufactured would be

$$k_1 = 100.8.5 = 4000$$

In a similar manner, total number of screws or rivets required for the manufacture of component D_2 for one hundred products would be

$$k_2 = 100.5.5 = 2500$$

Similarly for the component D_3 this would be

$$k_3 = 100.17.2 = 3400$$

Now to convert the above inequalities into equalities we include slack variables v_1 and v_2 corresponding to fictitious product D_s. Here v_1 and v_2 actually represent the idle capacities of the screw and rivet sections. Doing this we can formulate our problem as follows:

$$U = c_{11}x_{11}+c_{12}x_{12}+c_{13}x_{13}+c_{21}x_{21}+c_{22}x_{22}+c_{23}x_{23}+c_1v_1+c_2v_2 \text{ to be minimized.}$$

subject to the condition

$$x_{11}+x_{12}+x_{13}+v_1 = p_1 = 5000$$

$$x_{11}+x_{21} = k_1 = 4000$$

$$x_{12}+x_{22} = k_2 = 2500$$

$$x_{13}+x_{23} \qquad = 3400$$

$$x_{21}+x_{22}+x_{23}+v_2 = p = 5000\!\!/$$

and $x_{ij} = 0$ for $i = 1, 2, 3$ and $j = 1, 2, 3$ where $C_1 = C_2 = 0$.

The data can now be arranged in the tabular form as shown in Table I. The last column is the fastening capacity of each section, whereas the last row represents the number of fasteners required. Column 1 is the component D_1 column whereas row 1 is the screw section row so that x_{11} represents the number of screws required for fastening component D_1 and c_{11} represents the cost of using one screw for fastening a component D_1. Similarly x_{33} represents number of rivets required to fasten component D_3 and c_{33}, the cost of using one rivet for fastening a component D_3.

TABLE I

	D_1	D_2	D_3	D_s	p_i
s_1	x_{11} .15	x_{12} .25	x_{13} .15	v_1 .00	5000
s_2	x_{21} .10	x_{22} .15	x_{23} .15	v_2 .00	5000
k_j	4000	2500	3400	100	10000

Step 2. Initial Solution using North-west Corner Method

TABLE II

	D_1	D_2	D_3	D_s	P_i
s_1	4000 .15	1000 .25	.15	0 .00	5000
s_2	0 .10	1500 .15	3400 .15	100 .00	5000
k_j	4000	2500	3400	100	10000

Step 3. Evaluation of Empty Cells

Using circuit 1-○ as shown in Table II possible recuction

in cost $-.25 - .15 + .15 + .15 = -.10$

Using circuit 2-□ reduction in cost $+.25 + .10 - .15 - .15 = +.05$

Using circuit 3-△ reduction in cost $-.25 - .00 + .15 + .00 = -.10$

Thus the largest possible reduction can be obtained by using circuit 1

Step 4. Alteration of the Initial Solution

TABLE III

	D_1	D_2	D_3	D_s	P_i
S	4000 .15	0 .25	1000 .15	0 .00	5000
S	0 .10	2500 .15	2400 .15	100 .00	5000
k_j	4000	2500	3400	100	10000

Step 5. Evaluation of the Altered Solution

Possible reduction by using circuit 1-○ -.15-.15 + .10+.15 = -.05

Possible reduction by using circuit 2-□ -.15-.00 + .15+.00 = -.00

Possible reduction using circuit 3-△ +.25+.15 - .15-.15 = +.05

Largest reduction is possible by using circuit 1.

Step 6. Repetition of Step 4.

TABLE IV

	D_1	D_2	D_3	D_s	p_i
s_1	⊞ 1600 △ ⊖ .15	0 ⊕ .25	⊟3400 .15	0 △ .00	5000
s_2	⊟ 2400 △ ⊕ .10	2500 ⊖ .15	⊞ 0 .15	100 △ .00	5000
k_0	4000	2500	3400	100	10000

Step 7. Repetition of Step 5.

Possible reduction using circuit 1-○ +.25+.10 - .15-.15 = +.05

Possible reduction using circuit 2-□ +.15+.15 - .10-.15 = +.05

Possible reduction using circuit -△ -.15-.00 + .10+.00 = -.05

Largest possible reduction is possible through circuit 3.

Step 8. Repetition of Step 4.

TABLE V

	D_1	D_2	D_3	D_s	p_i
s_1	⊞ 1500 △ ⊖ .15	0 ⊕ .25	⊟ 3400 .15	100 △ .00	5000
s_2	⊟ 2500 △ ⊕ .10	2500 ⊖ .15	⊞ 0 .15	0 △ .00	5000
k_j	4000	2500	3400	100	10000

Step 9. Repetition of Step 5.

Possible reduction using circuit 1-○ +.25+.10 - .15-.15 = +.05

Possible reduction using circuit 2-□ +.15+.15 - .15-.10 = +.05

Possible reduction using circuit 3-△ +.15+.00 - .10-.00 = +.05

No reduction in cost is possible; therefore, the above solution is an optimum solution. No indifferent situation is possible since there is no zero reduction and hence no other alternate optimum solution is possible.

By the so-called inspection method this result could be obtained in just two iterations as shown below:

TABLE VI

	D_1		D_2		D_3		D_s		p_i
s_1	+ 0	.15	_ 1500	.25	3400	.15	100	.00	5000
s_2	_ 4000	.10	+ 1000	.15	0	.15	0	.00	5000
k_j	4000		2500		3400		100		10000

On evaluation of this initial solution it is found that the circuit shown in Table VI can bring about a reduction in cost of \$.05 for the movement of a unit to cell SD thus giving the altered solution as shown in Table VII which on evaluation is found to be optimal solution. Comparing with the Table V it can be seen that Table V and Table VII are exactly the same.

TABLE VII

	D_1		D_2		D_3		D_s		p_i
s_1	1500	.15	0	.25	3400	.15	100	.00	5000
s_2	2500	.10	2500	.15	0	.15	0	.00	5000
k_j	4000		2500		3400	.15	100		10000

The optimum programme obtained from Table V or VII is as follows:

3400 screws are to be used for mounting components D_3, i.e.,

$\frac{3400 \cdot 1}{1100 \cdot 2}$ or all 17 components D_3 are to be screwed.

2500 rivets are to be used for mounting components D_2 that is

$\frac{2500 \cdot 1}{100 \cdot 5}$ or all of the 5 components of D_2 are to be rivetted.

1500 screws are to be used for mounting components D_1 that is

$\frac{1500 \cdot 1}{100 \cdot 5}$ or 3 of the 8 components of D_1 are to be screwed.

2500 rivets are to be used for mounting the remaining 5 components on each of the 100 parts to be assembled.

The total cost for this optimum solution would be

$$U = .15x1500+.25x0+.15x3400+.10x2500+.15x2500+.15x0+.00x100$$

$$= \$1360.00$$

Bowman[87] has described various production design problems and Dennis[86] electric net-work design problems that could be solved by this method.

A variation of the transportation problem is the so-called multiple assignment[84,85] problem and yet another is tanker routing problem,[84] but since they do not at present apply to design problems these are not considered here.

Computer program codes for various transportation problems are available for the 704, 709 and many other high speed digital computers. The sources of information are referred to in Gass's paper on "Recent Advances in Linear Programming".[88,89]

2.52 Simplex and its Variations

Simplex is defined as an n-dimensional convex polyhydron having n+1 vertices.[82] As has been mentioned in the preceding section, a problem of minimizing (or maximizing) a function

$$U = \sum_{i=1}^{n} c_i x_i$$

subject to the conditions

$$\phi = \sum_{i=1}^{n} a_{ij} x_i \qquad \text{for } j = 1, 2, \ldots\ldots\ldots\ldots\ldots, m$$

and $x_i \geq 0$ for $i = 1, 2, \ldots\ldots\ldots\ldots n$

constitutes a linear programming problem. However, geometrically these conditions represent an n-dimensional polyhydron defined above as a simplex. Hence each linear programming problem is also a simplex problem. Dantzig was the first to notice this and came out with a new iterative method of solution called the simplex method.[82]

From above it is clear that every simplex problem also has a graphical solution. For $n > 3$ the graphical solution becomes too involved and is very difficult to handle. Simplex on the other hand, though lengthy, can easily be handled; particularly so on computers. But, since the simplex requires quite a few algebraic manipulations and is somewhat difficult to understand, it is customary to start with a simple graphical solution which facilitates the understanding of the algebraic simplex.

For example, let us consider the following two-dimensional case study of a small manufacturing concern producing two different models of iron-clad mains switches for home use. The concern consists of a sheet

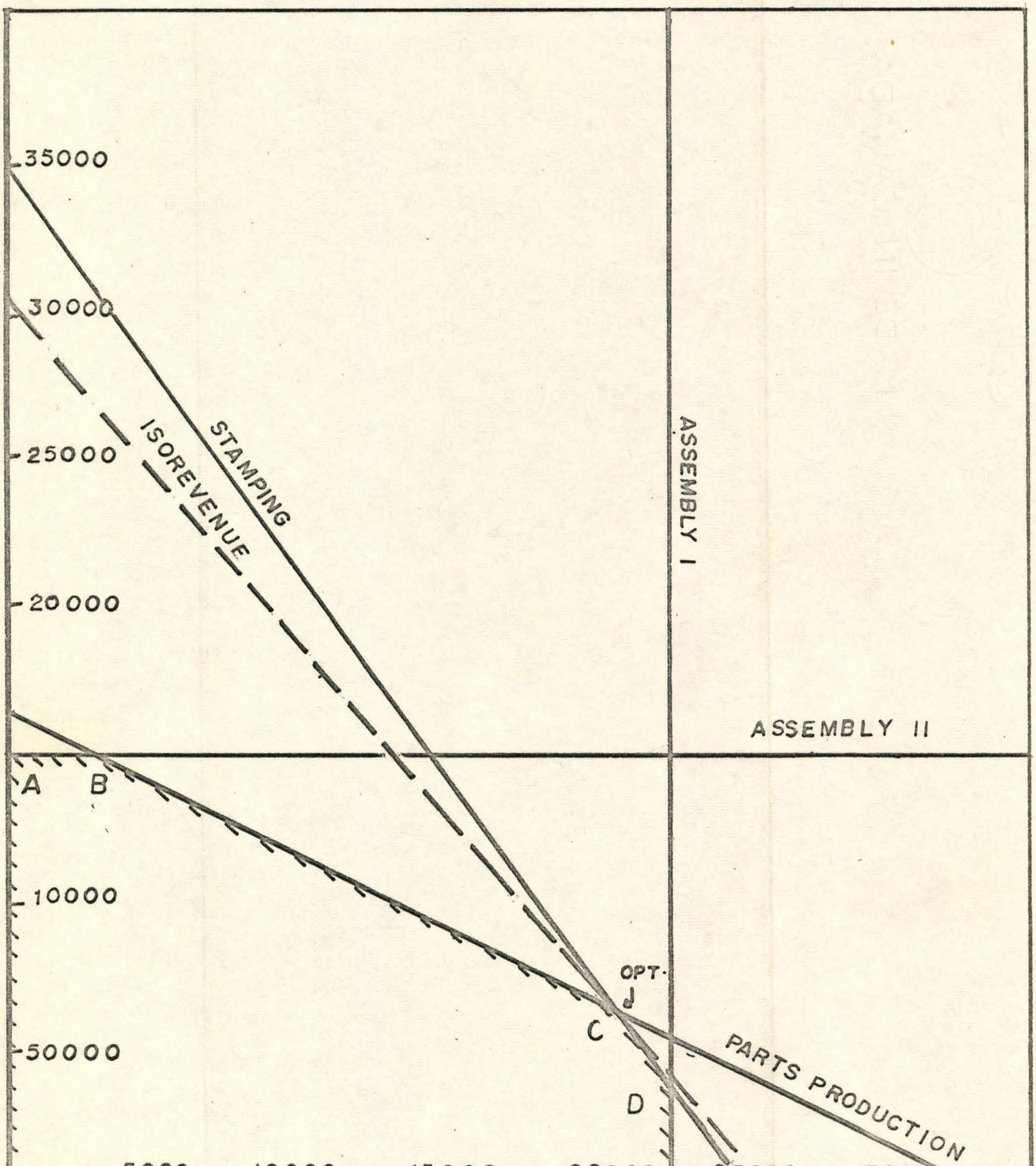
35000
30000
25000
20000
10000
50000
STAMPING
ISOREVENUE
ASSEMBLY I
ASSEMBLY II
A
B
OPT.
J
C
D
E
PARTS PRODUCTION
5000
10000
15000
20000
25000
30000

FIG. 2·52

metal stamping division, a parts production division, assembly division for model I and assembly division for model II. It is assumed that raw materials, labour, and other inputs are available at constant prices within the demand range of the concern. The production capacities of various divisions is as follows:

Metal Stampings	25000 Model I or 35000 Model II per month	
Parts Production	33333 ,,	16667 ,, ,, ,,
Model I Assembly	22500 per month	
Model II Assembly	15000 per month	

The sales value of Model I is greater than the total costs of purchased materials, labour and other direct costs attributable to its manufacture by an amount equal to \$ 3.00. Model II in the same way yields \$ 2.50. An optimum production programme is to be found which will maximize the total contribution towards profit.

According to the above data there are four constraints which affect the economic programming. These are

1. Stamping constraints, combinations of Model I and II in the stamping division are defined by the following equation:

$$35000\ x_1 + 25000\ x_2 \leqq 35000\ .\ 25000$$

2. Parts Production constraints, combinations of Model I and II in the parts production division are defined by the following equation:

$$33333\ x_2 \pm 16667\ x_1 \leqq 33333\ .\ 16667$$

3. Model I Assembly constraints can be expressed by the following relation:

$$x_1 \leqq 22500$$

4. Model II Assembly constraints can be expresse by the relation

$$x_2 \leqq 15000$$

The variables x_1 and x_2 are the quantities of Models I and II respectively The graph of the constraining relations is as shown in fig. 2.52. From the curves as well as from the relations it is clear that the slope of

curve 1 is - 1.4

curve 2 is - 1.5

curve 3 is ∞

curve 4 is - 0.0

Each of these curves is a straight line (hence linear programming problem), first two are drooping, rest of the two are parallel to x_2 - axis and x_1 - axis respectively. These together form a feasibility area A B C D E along with origin O which has 5 vertices.

The criterion

$$U = 3.0\ x_1 + 2.5\ x_2$$

is the total contribution towards profit and hence is to be maximized. The curve representing this relation is also a straight line giving a slope of - 1.2. All points on one such line will bring the same contribution towards profits and hence this line is known as an iso-revenue line. However, as the profit is not fixed this line too is not fixed. Actually it represents a family of lines having a constant slope of - 1.2. The iso-revenue line which will be closest to the origin will bring the minimum profit, where as the one which will be the farthest from the origin will give the maximum profit. Now comparing the slopes of the constraint lines it can be readily seen that maximum profit iso-revenue line will be closer to the - 1.4 slope line and would lie between this line and the -0.5 slope line. This indicates that these two lines e.g. the stamping line and the parts production line are the dominating constraints and hence point C, the point of intersection of these two lines will be the optimum point. Any iso-revenue line passing through this point does not cross the feasibility area except at this point. Any othe iso-revenue line lying towards the origin will cross the feasibility area at two points rather than one. Similarly every iso-revenue line away from the origin does not pass through the feasibility area at all. This confirms that the point C is the optimum point giving maximum contribution towards profits. As such the optimum production programme would be

20400 units of Model I per month

and 6400 units of Model II per month

The total contribution towards profit would therefore be

$$U = 3.0 \;.\; 20400 + 2.5 \;.\; 6500$$

$$= .07745 \text{ million dollars per month}$$

It is worth while to mention at this stage that if returns from either of the two models change, the slope of the iso-revenue line would also change and hence the optimal point would shift to some other vertex of the feasibility area.

In a three dimensional problem each line of fig. 2.52 would be replaced by a plane and the graphic solution, though possible, is difficult to handle. For more than three dimensional problems even physical visualization of the problem is difficult.

After studying the linear programming problem by graphic interpretations we are now ready to consider the algebraic simplex. For this let us consider the following numerical example.

<u>Example</u>: A small manufacturing concern is presently manufacturing

two products 'A' and 'B'. Each product is processed on two machines m_1 and m_2. Product 'A' takes 3 min. on machine m_1 and then 6 min. on machine m_2. Similarly product 'B' takes 5 min. on machine m_1 and 3 min. on machine m_2. By selling each product 'A' the concern gets a profit of \$ 3.0 by a similar measure product 'B' earns a profit of \$ 4.0. The weekly machine capacities are 4200 min. for each machine. It is desired to find the manufacturing policy for which the profit can be expected to attain a maximum value.

The solution procedure in this case would be similar to the one used for transportation problem. This can be described as follows:

1. Formulation of the problem.
2. Determination of an initial solution.
3. Evaluation of an initial solution.
4. Determination of the variable to be replaced.
5. Alteration of the solution according to 3 and 4 to maximize profit.
6. Repetition of steps 3 to 5 till no favourable alternative can be evaluated. (This solution would then be the optimum solution.)

Proceeding in this way we can solve the problem as depicted below.

Step 1. Assuming x_1 and x_2 to be the number of products of 'A' and 'B' respectively which we would produce a_{11} and a_{12} to be the time of manufacture of part 'A' on machine m_1 and machine m_2 respectively, and a_{12} and a_{22} the time of manufacture of part 'B' on machines m_1 and m_2 respectively, we can write

$$a_{11}\, x_1 + a_{12}\, x_2 - b_1 = 4200$$

$$a_{21}\, x_1 + a_{22}\, x_2 - b_2 = 4200$$

Where b_i's are capacity restrictions.

Now assuming c_1 to be the profit earned by the sale of each product 'A' and

c_2 by each product 'B', U, the total profit earned would be

$$U = c_1\, x_1 + c_2\, x_2$$

Since no negative manufacture is possible we must impose the non-negativity conditions

$$x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0$$

In the above formulation the capacity restrictions have resulted in constraining inequalities rather than equalities. In order to convert these into equalities we will have to introduce slack variables x_3 and x_4, the number of hypothetical products M_1 and M_2. These hypothetical products will be such that each M_1 can be produce in one minute on machine m_1 and each M_2 can be produced on machine m_2 in one minute. Further, we will also assume that no M_1 can be produced on machine m_2 and similarly M_2 can not be produced on machine m_1. With these assumptions we are able to construct the set

$$a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 + a_{14}\, x_4 = b_1$$

$$a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 + a_{24}\, x_4 = b_2$$

The value of a_{13} and a_{24} is one and that of a_{14} and a_{23} is zero. With these co-efficients the form of the set is known as canonical form. The total profit U can now be written as

$$U = c_3\, x_3 + c_4\, x_4 + c_1\, x_1 + c_2\, x_2 \qquad \text{to be maximized.}$$

This new system is known as augmented system. It should be noted here that our interpretation of the hypothetical product is idle time of machines. The above results can now be set in the form of a table as shown.

TABLE

c_j		c_1	c_2	c_3	c_4
Basis	Value	x_1	x_2	x_3	x_4
x_3	b_1	a_{11}	a_{12}	a_{13}	a_{14}
x_4	b_2	a_{21}	a_{22}	a_{23}	a_{24}

The first step of formulation of the problem is now complete.

Step 2. Our assumptions under step 1 that the hypothetical products M_1 and M_2 are idle times gives us an opportunity to start with an initial solution when only M_1 and M_2 are produced i.e. a situation when both machines are kept idle. This is equivalent in the graphical solution to starting at the origin, before pushing the iso-revenue lines as far as possible in the feasibility area. This situation is usually possible and any problem can be started in this manner. However, in case it is not possible, simple transformation techniques can be used to convert the set into canonical form[84]. With this initial solution we can write

$$x_3 = 4200$$

$$x_4 = 4200$$

Our assumptions regarding $a_{13}, a_{14}, a_{23}, a_{24}$ have led us to an artificial initial solution and hence this set is known as as artificial basis.

Step 3. Now we will have to evaluate the initial solution for any probable improvements in profit. This we will do by calculating the variation in profit by introducing first one 'A' and then one 'B' -- one at a time. This means that for each case we will have to calculate profit variation

$$u = c_j' \quad \text{(for minimization problem } u = - c_j\text{)}$$

where $c'_j = \sum_{i=n+1}^{n+m} a_{ij}\, c_i + c_j$

Here n is the number of variables in the actual basis and m in the artificial basis. (in our case n=2 and m=2)

From the above formula each c_j will be

$$c'_1 = -(a_{31}\, c_3 + a_{41}\, c_4) + c_1$$

$$c'_2 = -(a_{32}\, c_3 + a_{42}\, c_4) + c_2$$

$$c'_3 = -(a_{33}\, c_3 + a_{43}\, c_4) + c_3$$

$$c'_4 = -(a_{34}\, c_3 + a_{44}\, c_4) + c_4$$

Substituting numerical values we get

$$c'_1 = 3\ ,\ c'_2 = 4\ ,\ c'_3 = 0\ ,\ c'_4 = 0$$

At this stage we will introduce a row c'_j in the table constructed in step 1 and substituting numerical values obtain a table generally known as tableau No. 1.

TABLEAU No. 1

c_j		3	4	0	0
Basis	Value	x_1	x_2	x_3	x_4
x_3	4200	3	5	1	0
x_4	4200	6	3	0	1
c'_j		3	4	0	0

Among all c'_j , j = 1,2,3,4, c_2 is the largest i.e. if a product corresponding to this is brought into the solution the maximum improvements can be expected. We may encounter three possibilities at this stage; the largest c'_j may be zero, less than zero, or more than zero. In case it happens to be either zero or less than zeroit is clear that no improvement in profit earnings could be brought about and hence the solution under consideration will be the best possible, or what we would prefer to call an optimum solution. On the other hand, if it happens to be more than zero it simply means that further improvements are possible by introducing x_i corresponding to c_j (i.e. for $i = j$). In our case we have seen that $j = 2$; hence $i = 2$ and x_2 will be the variable to be introduced i.e. product 'B' will be produced.

Step 3. After finding out which variable is to be introduced we will have to determine the variable which is to be replaced. This can easily be done by finding the ratio b_i/a_{ij} i.e. by finding maximum number of products or the variables to be introduced. But here we are bounded by the capacity restrictions and therefore can not select any ratio. Instead we actually have to select the smallest one. For our numerical example we have decided that we will introduce product 'B'. For doing this we can either produce 4200/5 i.e. 840 products or 4200/3 i.e. 1400 products. However, each product which is to be manufactured has to be processed on both m_1 and m_2. If we decide to produce 1400 products we will not be able to process them on machine m_1 since it can not produce or process more than 840 parts. The dominent constraint, therefore, is that of machine m_1 and we have no other choice except to produce 840 parts. A general rule that we have deduced from the above is that we should calculate the ratio b_i/a_{ij} and choose the one which is the smallest. By doing this we have used the full capacity of m_1 whereas m_2 is used only partly. Its idle capacity would be

$$4200 - 3 \cdot 840 = 1680$$

The new solution would therefore be

$$x_2 = 840 \quad \text{and} \quad x_4 = 1680$$

Thus we can write

$x_{\underline{i}} = b_i / a_{ij}$,

All other $x_i = b_i - x_{\underline{i}}\ a_{i\underline{j}}$

To facilitate understanding the problem, these values are shown on the right side of each tableau. In this step we have determined that M_1 products, or the idle capacity of machine m_1, is to be replaced.

Step 5. The solution is now altered by introducing x_2 for x_3 and b_1= 840 and b_2= 1680. Since

$$x_2 = 5\ x_3 + 3\ x_4 \quad \text{and} \quad x_1 = 3\ x_3 + 6\ x_4$$

we get

$$x_3 = x_2/5 - 3\ x_1/5 \quad \text{and} \quad x_1 = 3\ x_2/5 + 21\ x_4/5$$

A new table, tableau 2 will now be constructed.

Step 6. The construction of the table will be completed by repeating steps 3 to 5.

TABLEAU No. 2

c_j		3	4	0	0		
Basis	Value	x_1	x_2	x_3	x_4	$b_i/a_{i\underline{j}}$	$b_i - x_{\underline{i}}\ a_{i\underline{j}}$
x_2	840	3/5	1	1/5	1	$\frac{840\ .\ 5}{3}$	840-400 . 1/5
x_4	1680	21/5	0	-3/5	0	$\frac{1680\ .\ 5}{21}$	
c'_j		3/5	0	-4/5	0		

Repeating the same procedure we can construct tableau 3

TABLEAU No. 3

c_j		3	4	0	0		
Basis	Value	x_1	x_2	x_3	x_4	$b_i/a_{i\underline{j}}$	$bi - x_{\underline{i}}\ a_{i\underline{j}}$
x_2	600	0	1	2/7	-1/7		
x_1	400	1	0	-1/7	5/21		
c_j'		0	0	-5/7	-1/7		

In tableau 3 the values of c_j''s are either zeros or negative numbers. This indicates that no further improvement in profit contribution is possible and hence the optimum solution is

$$x_{1_{opt}} = 400 \quad \text{and} \quad x_{2_{opt}} = 600$$

Maximum profit

$$U = x_{1_{opt}}\ c_1 + x_{2_{opt}}\ c_2$$

$$= \$\ 3600\ /\text{week}$$

Checking for any round off error we can calculate capaity restrictions

$$b_1 = a_{11}\ x_1 + a_{12}\ x_2 = 6\ .\ 400 + 3\ .\ 600 = 4200$$

$$b_2 = a_{21}\ x_1 + a_{22}\ x_2 = 3\ .\ 400 + 5\ .\ 600 = 4200$$

This indicates that our solution is exact.

Now we are in aposition to write down a generalized simplex algorithm (also known as Dantzig's Basic Algorithm)

1. Formulate the problem.

2. Start with an artificial basis.

3. Select the largest c_j' and call it $c_{\underline{j}}'$

4. a. If $c_{\underline{j}}^{\prime} \leq 0$ we have the optimum solution.

   b. If $c_{\underline{j}}^{\prime} > 0$ choose those $a_{i\underline{j}}$'s which are more than zero.

   c. If all $a_{i\underline{j}}$'s are less than or equal to zero there would be no finite solution.

   d. If there are certain $a_{i\underline{j}}$'s that are more than zero calculate the ratio $b_i/a_{i\underline{j}}$.

5. Choose the smallest ratio $b_i/a_{i\underline{j}}$ and rename the corresponding x_i as $x_{\underline{i}}$. This is the variable which is to be replaced by $x_{\underline{\underline{i}}}$

6. Substitute $x_{\underline{\underline{i}}}$, new b_i's, new a_{ij}'s so as to construct a new tableau.

7. Repeat all steps from step 2 to step 6 till $c_j^{\prime} \leq 0$.

From the above we come to the conclusion that the simplex is an iterative method of seeking optimum by moving from one basic feasible solution (an extreme point or the vertex of a simplex) to an adjacent basic feasible solution having higher value of criterion (lower value for minimization problem).

Over and above the algorithm described in the preceding paragraphs, three other algorithms are also in common use. To differentiate among them, and to examine their compartive merits, we can name them in the following manner.

1. Dantzig's Basic Simplex Algorithm.
2. Dantzig's [90] Inverse Form Revised Algorithm, commonly called the revised simplex.
3. Dantzig and Hay's [91] Product Form Revised Algorithm.
4. Zoutendijk 's Product Form Revised Algorithm.

For comparatively smaller computers and for smaller linear programming problems basic algorithm is the best one since its code is the simplest although it entails round off errors. For medium size computers the inverse form revised algorithm is to be prefered since it generally requires fewer multiplications and its code is also not very difficult. Its round off error is smaller than the basic algorithm, though not comparable to that of product form algorithm. For large computers Zoutendijk's product form algorithm is preferablesince it needs fewer computations than any other algorithm, requires fewer tape transfers, better restart properties and can work out re-inversions without appreciable increase in computing time[92].

In the numerical example solved in the preceding pages, on the right side of the last table in the column of c_j', net evaluations of -5/7 and -1/7 appear. The significance of these evaluations can be considered as follows.

The variable x_4 takes an evaluation of -5/7 which means that,if a unit of this variable is introduced into the solution at this stage, it would reduce the criterion (profit in this case) by \$ 5/7. This is equivalent to saying that had the capacity of m_1 been 1 unit larger than its present value of 4200, the optimum solution would have a criterion of 5/7 more. In other words, worth or value of the marginal unit of the capacity of m_1 is expressed by these evaluations. Dwelling upon this idea we can say that the total worth of the machine m_1 is 4200 . 5 / 7 or \$3000. Similarly the worth of machine m_2 is 4200 . 1 / 7 or \$ 600. The total worth of the concern would therefore be \$ 3600. This value is exactly same as the maximum profit value obtained by the criterion function. Thus we have come to the conclusion that the values imputed to the variables not in the solution basis equals the value of the criterion. This suggests the existence of a dual in linear programming.

Every simplex problem has associated with it a dual problem. writing the previous equations and interchanging columns and rows, we get

$4200\ y_1 + 4200\ y_2 = \bar{\bar{U}}$ to be minimized

subject to

$$3y_1 \pm 6y_2 = 3$$

$$5y_1 + 3y_2 = 4$$

If this problem would have been solved instead of solving the original problem, the dual criterion would have the same value of 3600 as explained earlier. On account of this, some problems where the number of constraints in the dual is less than the number of constraints in primal (the original problem is called the primal), it is preferable to convert the problem to dual and solve it as dual rather than primal. However, the basic simplex will not evaluate the original x variables.

In the simplex solution, as in the transportation solution, occasionally the total number of variables in the solution fall short of what they should be. This results in difficulty in further evaluation of the solution and some time takes the form of what is known as cycling (cycling is the process of repetition of one of the preceding bases in the new solution having the same value of the criterion) This is known as degeneracy and to resolve this, similar to the transportation method, a very small quantity is introduced into the solution. This and various other techniques of resolving degeneracy are described in detail in references 82, 83 and 85. Since the problem of this type does not occur too often we shall not consider it here.

It is possible to apply linear programming to parametric and stochastic situations and references 83 and 85 deal with these subjects in detail. Similarly sensitivity analysis and variou other topics are also dealt with in these above qouted references.

So far as design problems are concerned the so called light weight limit design has been dealt with by several authors (see bibliography under section 3.7). Hays[93] has given an example of least cost optimum vehicle performance by using linear programming and Evans[94] has solved an electronic miniaturized package design problem by using a technique which is based on linear programming. Orden [95] has given an example of optical filter design, Kuchn and Porter[96] in process control system design and Fanshel and Lynes[97] have given an example of economic power generation --all by using linear programming.

Analogue computations for linear programming problems are discussed by Pyne[98].

2.53 Dynamic Programming

In dealing with certain optimization problems pertaining to system engineering design, control engineering design, aerospace technology and mechanical engineering one often encounters the problem of maximization or minimization of a function

$$G(x_1, x_2, \ldots\ldots\ldots, x_n) = g(x_1) + g(x_2) + \ldots\ldots\ldots + g(x_n)$$

under the constraints

$$x = x_1 + x_2 + \ldots\ldots\ldots\ldots + x_n$$

and subject to the condition

$$x_n \geq 0$$

This type of problem is usually very difficult to handlesince extrema at boundaries can not be located by the present known methods of calculus[99] and the so called direct search methods require too many calculations which are very difficult to handle even on computers.

Such problems can easily be handled by using the so called principle of optimality recently developed by Bellman[50]. Regarding its development Bellman[50] writes "The principle of optimality is actually a particular application of what we have called 'the principle of invariant imbedding.' A special form of the invariance principle was used by Ambarzumian* 'On the Scattering of Light by Diffuse Medium,' C.R. Doklady, Sci. U.R.S.S. 38 (1943), pp. 237 and extensively developed by S. Chanderasekhar$^{\Delta}$, 'Radiative Transfer,' Oxford, 1950."

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* Russian Scientist Δ Indian Scientist

The principle of optimality can be stated as follows "An optimal policy has the property that what ever the initial state and and initial decision are, the remaining decision must constitute an optimal policy with regard to the state resulting from the first decision.[99] "

The type of problems falling under the perview of the above described principle of optimality are the so called multi-stage processes. This is a process composed of sequences of operations in which the out come of the preceding stage may be used for the decision regarding the course of action to be taken for the succeeding operations. Information, data or materials which are fed into an operation as input are usually known as feed, and the out put from the operation as the returns. If the returns can be determined the process will be deterministic; otherwise, on account of the presence of random factors, the process will be stochastic. Each stage of such a process is connected with certain courses of action and hence these stages are known as activities. An n stage process will therefore have n activities associated with it.

For solution of such a problem first of all a quantity (feed) is assigned to the nth activity, then to the n-1 th activity and so on. This gives it the time like property and hence the word dynamic programming is coined to describe and to emphasize this property. This method is therefore ideally suited to the time dependent problems, but its applications are not confined to such problems alone. Several deterministic as well as stochastic problems of ordinary and variational calculus have been successfully solved by this method and many more are expected to be solved. The design problem stated above is just one example of the many problems that can be handled by this method.

We can now formulate our problem in the following manner. Let us call

$$x = x_1 + x_2 + \ldots\ldots\ldots\ldots + x_n$$

as total feed where $x_1, x_2, \ldots\ldots, x_n$ are the feeds for activities 1,2,... ...,n respectively. Assuming G to be returns from this total feed x, and that G can be represented by the sum of g's which are returns from feed corresponding to each activity we can write

$$G(x_1, x_2, \ldots\ldots\ldots\ldots, x_n) = g_1(x_1) + g_2(x_2) + \ldots\ldots\ldots + g_n(x_n)$$

We can impose the restriction $x_i \_\_ 0$ if we do not want any negative feed to be considered.

From the above it is clear that the extremum of returns depends on total feed x and activities n only. Hence we can introduce a sequence of functions $U_n(x)$ such that

$$U_n(x) = \max.\ G(x_1, x_2, \ldots\ldots, x_n)$$

This function $U_n(x)$ would then represent the optimum returns for feed quantities x to activities n. Now, since there could be no returns if there would be no feed, we can write

$$U_n(0) = 0 \text{ for } n = i, 2, 3, \ldots\ldots\ldots, n.$$

provided that

$$g_n(0) = 0 \text{ for each value of } n.$$

For the first activity the feed quantity would be x and hence we can write

$$U_1(x) = \max.\ g(x_1)$$

subject to the condition that

$$0 \leqq x_1 \leqq x$$

If it is assumed that the maximum returns result when feed is x, we can write

$$U_1(x) = g_1(x)$$

Now assuming any quantity x_n connected with the nth activity such that

$$0 \leqq x_n \leqq x$$

the remaining feed quantity $x - x_n$ regardless of the value of x_n, can be used to get the extremum returns from the rest n-1 activities.

Thus the optimum returns $U_{n-1}(x - x_n)$ for n-1 activities, along with initial allocation of x to the nth activity, result in total returns

$$g_n(x) + U_{n-1}(x - x_n)$$

This will be maximum or minimum for an optimal choice of x_n. Hence

$$U_n(x) = \max.\ g_n(x_n) + U_{n-1}(x - x_n) \qquad \text{for } n = 2, 3, \ldots,$$

and $\quad x \geqq x_n \geqq 0$

The case of $n = 1$ is omitted since $U_1(x)$ is already determined. This recursive relationship permits us to determine the optimum return from the nth stage problem if we know it for the n-1 th stage.

This would now be further illustrated by solving a simple example similar to economic models treated by Bowman and Fetter[87].

<u>Example:</u> It is required to design a change speed gear box for a to tool room milling machine by incorporating as many speed changes as possible. Four standard gear systems are available which can be used in combination. The sets can not be fractioned and must be used as whole sets i.e. 1,2,3 etc. . The total funds available for purchase of the sets are limited to $ 1000. The specifications of sets are as follows:

	Cost in $	No. of Speed Changes
Set I	700	9
Set II	500	4
Set III	400	3

Set IV 300 2

This is a simple multi-stage allocation problem in which allocation of funds for purchase of gear systems are feed or resources, available speed changes are returns and selection of sets are the activities. The problem can now be formulated as follows:

$$G = g_1(x_1) + g_2(x_2) + g_3(x_3) + g_4(x_4)$$

subject to

$$x \geqq c_1\, x_1 + c_2\, x_2 + c_3\, x_3 + c_4\, x_4$$

and $\quad x_n \geqq 0$ for $n = 1, 2, \ldots\ldots,$

where $g_n(x_n) = v_n\, x_n \qquad$ for $n = 1, 2, \ldots\ldots\ldots$

Note that the return functions $g_n(x_n)$ are nonlinear step functions.

In the above formulation, x_n's are the number of whole gear sets, x is the total available funds in dollars, v_n's are the speed changes and c_n's are the cost of sets. Substituting numerical values we obtain

$$G = 9\, x_1 + 4\, x_2 + 3\, x_3 + 2\, x_4$$

and $g_1(x_1) = 9\, x_1$, $\; g_2(x_2) = 4\, x_2$, $\; g_3(x_3) = 3\, x_3$, $\; g_4(x_4) = 2\, x_4$

$$1000 \geqq 700\, x_1 + 500\, x_2 + 400\, x_3 \pm 300\, x_4$$

$$U_1(x) = 9\, x_1 \quad \text{for} \quad 1000/700 \geqq x_1 \geqq 0$$

$$U_2(x) = \max.\; 4x_2 - U_1(x - 500x_2) \qquad \text{for} \quad 1000/500 \geqq x_2 \geqq 0$$

$$U_3(x) = \max.\; 3x_3 + U_2(x - 400x_3) \qquad \text{for} \quad 1000/400 \geqq x_3 \geqq 0$$

$$U_4(x) = \max.\; 2x_4 + U_3(x - 300x_4) \qquad \text{for} \quad 1000/300 \geqq x_4 \geqq 0$$

Calculations for each activity are performed and the results are tabulated as shown below.

First Activity Feed 0-1000 with an increment of $ 100

TABLE I

Allocation	Units	Returns	Maximum Returns
x	$1 \geqq x_1 \geqq 0$	$9\ x_1$	$U_1(x)$
0	0	9 . 0 = 0	0
100	0	9 . 0 = 0	0
200	0	9 . 0 = 0	0
300	0	9 . 0 = 0	0
400	0	9 . 0 = 0	0
500	0	9 . 0 = 0	0
600	0	9 . 0 = 0	0
700	0	9 . 0 = 0	
	1	9 . 1 = 9	9
800	0	9 . 0 = 0	
	1	9 . 1 = 9	9
900	0	9 . 0 = 0	
	1	9 . 1 = 9	9
1000	0	9 . 0 = 0	
	1	9 . 1 = 9	9

Maximum Returns - 9 Speed Changes at an optimum feed of $ 700

Second Activity Feed 0-1000 with an increment of \$ 100

TABLE II

Allocation	Units	Returns	Maximum Returns
x	$1 \geq x_2 \geq 0$	$4 \cdot x_2 + U_1(x - 500x_2)$	$U_2(x)$
0	0	$4 \cdot 0 + U_1(0) = 0$	0
100	0	$4 \cdot 0 + U_1(100) = 0$	0
200	0	$4 \cdot 0 + U_1(200) = 0$	0
300	0	$4 \cdot 0 + U_1(300) = 0$	0
400	0	$4 \cdot 0 + U_1(400) = 0$	0
500	0	$4 \cdot 0 + U_1(500) = 0$	
	1	$4 \cdot 1 + U_1(0) = 4$	4
600	0	$4 \cdot 0 + U_1(600) = 0$	
	1	$4 \cdot 1 + U_1(100) = 4$	4
700	0	$4 \cdot 0 + U_1(700) = 9$	9
	1	$4 \cdot 1 + U_1(200) = 4$	
800	0	$4 \cdot 0 + U_1(800) = 9$	9
	1	$4 \cdot 1 + U_1(300) = 4$	4
900	0	$4 \cdot 0 + U_1(900) = 9$	9
	1	$4 \cdot 1 + U_1(400) = 4$	
1000	0	$4 \cdot 0 + U_1(1000) = 9$	9
	1	$4 \cdot 1 + U_1(500) = 4$	
	2	$4 \cdot 2 + U_1(0) = 8$	

Maximum Returns - 9 Speed Changes at an optimum Feed of \$ 700

Third Activity Feed 0-1000 with an increment of \$ 100

TABLE III

Allocation	Units	Returns	Maximum Returns
x	$2 \geq x_3 \geq 0$	$3 \cdot x_3 + U_2(x - 400x_3)$	$U_3(x)$
0	0	$3 \cdot 0 + U_2(0) = 0$	0
100	0	$3 \cdot 0 + U_2(100) = 0$	0
200	0	$3 \cdot 0 + U_2(200) = 0$	0
300	0	$3 \cdot 0 + U_2(300) = 0$	0
400	0	$3 \cdot 0 + U_2(400) = 0$	
	1	$3 \cdot 1 + U_2(0) = 3$	3
500	0	$3 \cdot 0 + U_2(500) = 4$	4
	1	$3 \cdot 1 + U_2(100) = 3$	
600	0	$3 \cdot 0 + U_2(600) = 4$	4
	1	$3 \cdot 1 + U_2(200) = 3$	
700	0	$3 \cdot 0 + U_2(700) = 9$	9
	1	$3 \cdot 1 + U_2(300) = 3$	
800	0	$3 \cdot 0 + U_2(800) = 9$	9
	1	$3 \cdot 1 + U_2(400) = 3$	
	2	$3 \cdot 2 + U_2(0) = 6$	
900	0	$3 \cdot 0 + U_2(900) = 9$	9
1	1	$3 \cdot 1 + U_2(500) = 7$	
	2	$3 \cdot 2 + U_2(100) = 6$	
1000	0	$3 \cdot 0 + U_2(1000) = 9$	9
	1	$3 \cdot 1 + U_2(600) = 7$	
	2	$3 \cdot 2 ++U_2(200) = 6$	

Maximum Returns - 9 Speed Changes at an optimum Feed of \$ 700

Fourth Activity — Feed 0-1000 with an increment of $ 100

TABLE IV

Allocation	Units	Returns	Maximum Returns
x	$3 \geqq x_4 \geqq 0$	$2 \cdot x_4 + U_3(x - 300x_4)$	$U_4(x)$
0	0	$2 \cdot 0 + U_3(0) = 0$	0
100	0	$2 \cdot 0 + U_3(100) = 0$	0
200	0	$2 \cdot 0 + U_3(200) = 0$	0
300	0	$2 \cdot 0 + U_3(300) = 0$	
	1	$2 \cdot 1 + U_3(0) = 2$	2
400	0	$2 \cdot 0 + U_3(400) = 3$	3
	1	$2 \cdot 1 + U_3(100) = 2$	
500	0	$2 \cdot 0 + U_3(500) = 4$	4
	1	$2 \cdot 1 + U_3(200) = 2$	
600	0	$2 \cdot 0 + U_3(600) = 4$	4
	1	$2 \cdot 1 + U_3(300) = 2$	
	2	$2 \cdot 2 + U_3(0) = 4$	
700	0	$2 \cdot 0 + U_3(700) = 9$	9
	1	$2 \cdot 1 + U_3(400) = 5$	
	2	$2 \cdot 2 + U_3(100) = 4$	
800	0	$2 \cdot 0 + U_3(800) = 9$	9
	1	$2 \cdot 1 + U_3(500) = 6$	
	2	$2 \cdot 2 + U_3(200) = 4$	
900	0	$2 \cdot 0 + U_3(900) = 9$	9
	1	$2 \cdot 1 + U_3(600) = 6$	
	2	$2 \cdot 2 + U_3(300) = 4$	
	3	$2 \cdot 3 + U_3(0) = 6$	

1000	0	$2 \cdot 0 + U_3(1000) = 9$	
	1	$2 \cdot 1 + U_3(700) = 11$	11
	2	$2 \cdot 2 + U_3(400) = 7$	
	3	$2 \cdot 3 + U_3(100) = 6$	

Maximum Returns - 11 Speed Changes at an optimum Feed of $ 1000

Optimum Returns from 4th activity are obtained when one set IV and optimum returns from 3rd activity are used.

Optimum returns from 3rd activity are optimum returns from 2nd activity.

Optimum Returns from 2nd activity are optimum returns from 1st activity.

Optimum returns from first activity are obtained when one set I is used.

Optimum combination would therefore be

<u>one set I + one set IV</u>

By solving this example we have seen that the dynamic programming method can be successfully utilised for the solution of such problems. However, we have also noted that the solution is lengthy and involves many computations. On the other hand, in the situation where too many variables are involved, it is almost impossible to solve the problem by any other method except this technique. Further, Since computers can easily be used for simple calculations like this, number of computations is no longer a limitation. This is the reason why, inspite of some of its inherent disadvantages, this method is so popular with designers and many process system designs and mechanical system designs have been successfully dealt with by this method. The problem formulation, however, is different in each case. For applications of this method to design problems see section 3.

2.54 Nonlinearities and Gradient Methods

In the preceding sections under mathematical programming we have examined the transportation method and simplex method, both of them dealing with linear criterion and linear constraints. Unfortunately most of the design problems as well as other optimization problems do not fall under this class of functions. Many linear programming examples are actually approximations and not truly linear. Several attempts have been made in the past and are still under way to find reasonably accurate and yet efficient methods of solving nonlinear problems. However, the results of such efforts have not been too fruitful as yet. Possibly some significant progress has been made in the last year or two. Commenting on this situation Wolfe[121a] writes, "there is no lack of information as to how nonlinear programming might be done, but on the other hand there is almost no information as to how nonlinear programming should be done." Under a situation like this it is very difficult to handle design problems which are nonlinear. In spite of all these difficulties, persons like Brown,[101] Dickinson[102] and a few others have solved some design problems; and there is a likelyhood that many more problems will be tackled and come to light since a reasonably satisfactory number of efficient methods have quite recently become available.

Nonlinear problems can be classified into three distinct categories e.g. [100]

1. Nonlinear criterion and linear constraints
2. Nonlinear constraints and linear criterion.
3. Nonlinear criterion and nonlinear constraints.

All nonlinear programming methods will solve problems having constraints that may be equalities, inequalities or combinations of both. The first two of these methods are comparatively easier than the third: the first one being the simplest. Several methods of solving these problems have been described in the literature and most of them use the gradient of the function as a guide to reach the optimum and hence are classified under the common name of the gradient methods. Such methods when the criterion function is unknown or only partially known have been dealt with under sec. 2.43. In this section we are concerned with the problem of known criterion.

Early attempts of using gradient method for the location of optimum were confined to the classical approach of converting a constrained extremizing problem into an unconstrained extremizing problem. Curry[103], Brown[101], Harris[104], Carrol[13], Dickinson[102], Asimow[36] Bryan et. al.[105 & 106] and several others have used this approach, however the results were not very satisfactory. Hence new and better methods were investigated and some of them successfully utilized for optimization problems of economics. A few important methods of this class will be described in the following pages.

On the basis of the comments made above, it is but natural to start with the simplest case -- case 1 -- of nonlinear criterion and linear constraints. Four methods of solving specifically linear constrained problems are presented here. Separable programming[107] methods can also solve these problems by separating a nonlinear function into a sum of regional or local linear combinations; hence the nonlinear programming problem is converted into a linear programming problem and can easily be handled by the simplex method. However separable programming is not covered since it does not fall into the preview of the so called gradient methods. These four methods are:

1. Wolfe's Reduced Gradient Method[108].
2. Rosen's Gradient Projection Method I[109].
3. Wolfe's Accelerated Gradient Method[110].
4. Beal's Simplex Method for Quadratic Programming[111]. (A slightly different method under the same title is due to Wolfe[112].)

Wolfe's reduced gradient method can be considered as the extension of Dantzig's simplex method, since it can provide solutions other than vertices of the constraint set and its computational basis is the same as that of simplex method. By this method, after the formulation of the problem $U(x)$ a nonlinear criterion subject to the linear constraints

$$\phi_i = \sum_{j=1}^{n} a_{ij} x_j = b_j \text{ for each } i,\ i = 1,2,\ldots\ldots$$

The problem can be solved by using the following algorithm.

Step 1. Assuming that a simplex basis and a feasible point is known, and using the fact that the c_j's of the simplex method are but the gradient of the criterion, the c_j's for this case can be determined. In simplex reduced costs $c'_j = c_j - \sum_{i=1}^{n} a_{ij} c_i$, in the same way the reduced costs c'_j can be expressed by

$$c'_j = [\nabla U(x^k)]_j - \sum_{i=1}^{n} a_{ij} [\nabla U(x^k)]_i$$

for the non-basic variables.

Step 2. Since a gradient vector of a function can be defined as a direction in which the directional derivative of the function is maximum i.e. a vector Δ x maximizing

$$\Delta U(x) = \frac{\partial U}{\partial x_1} \Delta x_1 + \dots\dots\dots\dots + \frac{\partial U}{\partial x_n} \Delta x_n$$

such that $\quad \Delta x = (\Delta x_1, \dots\dots\dots, \Delta x_n)$,

and $\quad \Delta x = \Delta x_1 + \dots\dots\dots\dots + \Delta x_n$

then all of its components, except the jth one, must be zero where partial derivative of U with respect to x_j is the maximum of all partial derivatives and Δx_j is the direction of $(\frac{\partial U}{\partial x_j})$. This can also be stated as $\Delta x_j = c_j$ if $x_j > 0$ or $c_j > 0$ otherwise $\Delta x_j = 0$ for non-basic variables. Since the basic variables Δx can be defined so that $\sum_{j=1}^{n} \Delta_j x_j$ vanishes, therefore if $\Delta x = 0$ the problem is solved, otherwise we have to proceed to the next step.

Step 3. Step length can be determined by calculating π_m where it is defined by $\pi_m = \max. \pi(x + \Delta x\pi > 0)$ where π is a constant determining the step length, the x will then be replaced by $x + \Delta x \cdot \pi_m$

Step 4. Except in the case where all basic variables of new x's are positive, a simple pivot step of interchanging the vanishing basic variable with a non-vanishing non-basic variable is performed, and steps 1 to 4 are repeated. If on the other hand all basic variables happen to be positive a direct return to step 1 is taken. The method is said to converge to a solution if the criterion is bounded and if the constraints are non-degenerate.

The gradient projection method I[109] is also known as the large step method or walking method since for reaching the peak, it allows one to take larger steps, without the possibility of leaving the constraint set. The method will be illustrated by using a two dimensional example of a non-linear function $U(x_1, x_2)$ to be maximized subject to the linear constraints $x_1 \geq 0$, $x_2 \geq 0$, $x_2 \leq a_1 + b_1 x_1$, $x_2 \leq a_2 - b_2 x_1$ and $x_2 \leq a_3 - b_3 x_1$. This results in the situation shown in fig. 2.54. Straight lines are the constraints and the contour lines are the criterion lines depicting the value of the criterion function. Starting with an initial feasible point p in the interior of the constraint set, gradient $(\nabla U(x))$ is calculated; this is obviously perpendicular to the associated contour line. A largest possible step in this direction without leaving the constraint set, unless the criterion reached an optimum for a shorter step, is then taken, so that point p_1, a point on the boundary of the constraint set, is reached. A gradient $[\nabla U(x)]_1$, is now calculated at this point. The projection of this gradient on the constraint line associated with p_1 is calculated. Since this is more than zero, a ray from p_1 is extended in the direction of this projection to the farthest point on the ray but lying within the constraint set. The farthest point thus obtained is p_2. The value of the function at points p_1 and p_2 are calculated. Since U has a larger value at p_2 than at p_1, (from fig. 2.54) this cycle is complete. The gradient at p_2 is now calculated and the procedure is repeated so that we reach at point p_3. Since value of the criterion is improved at this point as well this cycle is also completed. Again calculating the gradient, its projection and the direction of the projection we move along this direction to the farthest point p_4 of the ray lying in the constraint set. Calculating the value of the function at this point we find that the criterion has reduced. This means that the optimum might be some where between points p_3 and p_4. Choosing a point p on this

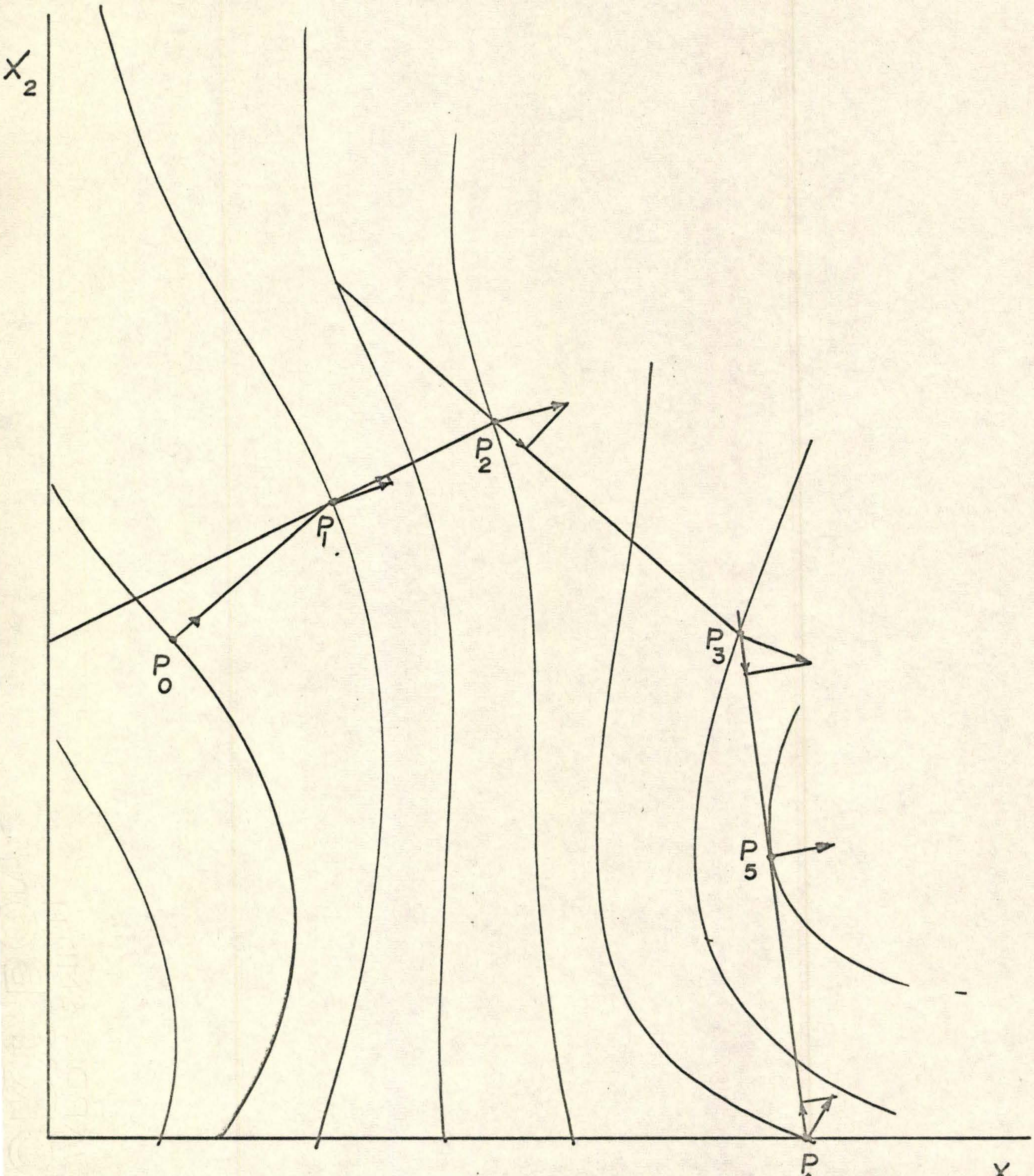

FIG. 2·54

segment such that the criterion is maximized after several trials, we reach the point p_5 where the projection of the gradient at that point vanishes. Hence x maximizes the criterion and no further interpolation is necessary. In a multi-dimensional hyper-space this would not be sufficient and hence we have to add certain other conditions. Except for this, all other steps can be generalized for any number of variables. The recursion formula for this method would be

$$x_i^{k+1} = x_i^{k} - \left(\frac{\partial U}{\partial x_i} + \sum_{j=1}^{n} \lambda_j \frac{\partial \phi}{\partial x_i} \right) \Delta\sigma$$

where $\Delta\sigma$ is step size and λ_j Lagrange's multipliers. A detailed description of the method and the computational algorithm are given by Rosen[108 & 109].

Wolfe's accelerated version of the cutting plane method[110] is a special case of Kelley's Cutting Plane method[113] and can best be explained after dealing with that method. We, therefore, postpone our consideration of this method till that time.

The details of Beal's[111] method of quadratic programming were not available at the time of this study and therefore not included here.

The second case, which is maximizing or minimizing linear criterion under non-linear constraints, can best be explained by using the cutting plane method due to Kelley[113]. The requirement for the method is that the non-linear constraints be convex. The constraints are first linearized by using Taylor's first order approximations. Starting with a known initial solution, the value of the criterion and that of constraints are calculated at this point. If the original constraints are satisfied this value of criterion is the required solution.

If it is not so, by using th Taylors first order approximation obtained earlier, an approximation equation of the constraints at the initial solution point is calculated. This will be the equation of a hyper plane cutting the problem space, hence the method is known as cutting plane method. Now, since the initial solution, as tested above, does not satisfy the original constraints, this solution is altered. This, at the initial stage, can be done by keeping all of the variables except one as constant and altering this one. At each stage, using the previous procedure, the value of the criterion and the original constraints are calculated. If, as has been mentioned before, the constraints are satisfied the value of the criterion will be the extreme value otherwise the preceding steps are repeated. The procedure, in the end, converges to the required solution.

For the illustration of the method we will consider Kelley's two dimensional example with comparatively better trials than his so that we will reach the solution in fewer iterations than he did.

Example. Find a vector $x=(x_1, x_2)$ such that the criterion $U=x_1 - x_2$ is a minimum subject to the condition

$$\phi = -3x_1^2 - 2x_1 \cdot x_2 + x_2^2 - 1 \leq 0$$

The above constraint boundary is an ellipse and hence the constraints are convex. The convexity requirement is therefore fulfilled. Linearizing the constraints by using Taylor's first order approximation we can write the transformed version of the problem as

$$\text{minimize } U = x_1 - x_2$$

subject to $\quad \bar{\phi} = \phi(x^k) + U(x^k) \,.\, (x - x^k) \quad 0$

where $\quad x^k = (x_1, x_2)$ and $U(x^k) = (\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2})$

Suppose we have at hand an initial solution $x_1 = -2$ and $x_2 = 2$. The value of criterion at this point is -4 and that of original constraint is 23. Since 23 is larger than 0, the constraints are not satisfied. Now keeping the value of $x_2 = 2$ same as before we can find the value of x by using the linearized approximation of the constraint $\bar{\phi}$ at $x = (-2, 2)$. This value is found to be -0.56250. With this solution we can repeat the preceding steps; several such steps are tabulated as shown below.

k	$\bar{\phi}$	x_1	x_2	U	ϕ
0	$-16.00000x_1 + 8.00000x_2 - 25.00000$	-2.00000	2.00000	-4.00000	23.00000
1	$-7.37500x_1 + 5.12500x_2 - 8.19922$	-0.56250	2.00000	-2.56250	6.19922
2	$-2.33157x_1 + 3.44386x_2 - 4.11958$	+0.27807	2.00000	-1.72193	2.11978
3	$-0.97970x_1 + 2.09198x_2 - 2.24875$	+0.27807	1.32406	-1.04599	0.24873
4	$-1.88542x_1 + 2.09514x_2 - 2.10291$	+0.05243	1.10000	-1.04757	0.10290
5	$-1.96520x_1 + 2.00552x_2 - 2.00574$	+0.01000	1.01276	-1.00276	0.00572
6		+0.00011	1.00000	-0.99989	-0.00021

The solution lies between step 5 and 6 and is close to 6, i.e. the required vector is $x = (0, 1)$ and the minimum value of $U = -1$. The Cutting plane method is not confined to linear criterion only; any general non-linear problem consisting of both non-linear criterion and non-linear constraints can be handled by this method.

After considering Kelley's cutting plane method we are now in a position to consider Wolfe's accelerated version[108] which is applicable to the linear constraints only. For this, at every other step a point $\bar{x}$ is defined such that $\bar{x} = \sum_{k=1}^{K} u^k \cdot x^k$

where u^k is the sum of the values of the dual variables associated with the constraints generated from x^k.

As regards the general case of non-linear criterion and non-linear constraints, we have already seen that the cutting plane method can successfully be used for this purpose if constraints are convex. Two other methods, the separable programming method and the decomposition method, are also used for nonlinear programming of the general type. However, since these methods do not use gradient technique, they will not be discussed here. Rosen's gradient projection method II[114] is a modified version of method I, discussed under linear constraint programming and can handle general non-linear programming problems successfully. Zoutendijk has proposed several variants of this method; and Witzgall[115] has shown that Frisch's[116] multiplex method, Lemke's[117] constrained gradient method and Rosen's gradient projection methods are all variants of one basic scheme. The only class of methods which is not dealt with as yet is the so called differential gradient methods family, a particular case of this has been studied by many under the name of Gradient Method.

The basic idea behind the differential gradient method is taht the direction of the gradient of the criterion is the direction of the steepest ascent and hence if one would like to reach the peak, one merely has to follow this direction by taking one step at a time so as always to increase one's altitude till one encounters a restraint. From then onward, one has to move along the restraint so long as one is capable of going higher. By using two different approaches for enforcing the constraints, two different methods of solving the problem are proposed e.g. the direct differential gradient method and the Lagrangian differential gradient method[100, 118, 119, 120].

If we follow the above procedure and at any stage find that the constraint $\phi_i(x) \leq 0$ is violated it simply means that the value of x is such that $\phi_i(x)$ is too big. We would, therefore, like to reduce this function. As we have seen in the preceding paragraph, the gradients of a function is the direction of increase of the function, the direction of the negative gradient would be the direction of decrease of a function. Using this idea we can incorporate the negative of the gradient of constraint in our search plan so that if there is any tendency towards violation of the constraint, this negative gradient will kick back the variable inside the constraint set. Mathematically we can express this as follows:

$$\frac{dx}{dt} = \nabla U(x) - \sum_{i=1}^{n} \partial_i(x) \nabla \phi_i(x)$$

where $\partial_i = 0$ if $\phi_i(x) \leq 0$ and $\partial_i = K$ if $\phi_i(x) > 0$

Here K is chosen larger than the maximum of $|\nabla U(x)| / |\nabla \phi_i(x)|$ for any x lying on the boundary of the constraints. This method is known as the direct differential gradient method. Curry, Brown, Harris, Carrol, Dickinson and several others have tried this method, though not in the same form for solutions of certain design problems. However the method is very inefficient and hence acceleration techniques such as that of Shah's (described under section 2.4.3) and Forsythe's have been used to improve the efficiency. Pyne[98], Deland[120a] and Ablow[120b] have tried this method by analogue computation rather than digital computations. All of these attempts were for finding efficient methods of system design and equipment design.

In the Lagrangian differential gradient method, using the classical approach dealt with under section 2.23 the following augmented function

$$\bar{U}(x,\lambda) = U(x) + \sum_{i=1}^{n} \lambda_i \phi_i(x)$$

is first derived. Now a necessary condition that x solves the programming problem, $U(x)$ to be minimized (or maximized) subject to the constraints $\phi(x) \leq 0$ or $\phi(x) = 0$ is that (x, λ) solves the problem $\underset{x \geq 0 \ \lambda \ 0}{\text{Min Max.}} \ \bar{U}(x,\lambda)$

If the criterion $U(x)$ happens to be convex the necessary condition also becomes the sufficient condition. This implies that any non-negative x that minimizes $U(x)$ subject to $\phi = 0$ or $\phi \ 0$ must satisfy the conditions

$$\frac{\partial \bar{U}}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial \bar{U}}{\partial x} = 0$$

Yet another approach due to Courant[120c] is the so called penalty function method. In this method for minimizing (or maximizing) a function $U(x)$ subject to the condition $\phi(x) \geq 0$, an augmented function

$$\bar{U}(x) = U(x) + h\,[\phi(x)]^2$$

is formed. This new unconstrained function is then minimized for successively increasing values of h. Courant has proved that as h goes to infinity the solution of the problem approaches the solution of the original problem.

In the preceding pages we have studied several nonlinear programming methods. Most of these are limited to the solution of convex criterion problems. However, three of these, e.g. reduced gradient method, gradient projection method and the separable programming method are not so restricted. The reduced gradient method seems to be more efficient but no computational evidence to this effect is yet known.

The gradient projection method is said to have been coded for the IBM 704 and 709 by Rosen[114] and Merril[121] and computational experience has been reported. The separable programming is limited to the functions that could be easily separated i.e. could be expressed as the sum of separate functions of the independent variables x_j. This method is said to have been coded for the IBM 709 and is in use by the Standard Oil Company of California[121a]. For general convex programming, computational experience with Kelley's cutting plane method is reported by Dornheim[121b], and by Griffith and Stewart[121c] and for Lagrangian differential gradient method by Manne[122a] and by Marschack[122b]. However, this method is said to be very slow and does not seem to be promising. As regards the direct differential gradient method, it has already been mentioned that considerable experimentation has been undertaken but the method, although it works well with some particular small problems, does not seem to be promising. Some aerospace problems are said to have been dealt with by Kelley[120c] using the penalty function method, but sufficient information regarding its usefulness is not presently available. Quadratic programming methods of Beale[111] and of Wolfe[112] have both been computationaly tried and are said to be reasonably efficient in handling quadratic programming problems.

2.61 Johnson's Method

For the design of mechanical elements Johnson has suggested a method based on the classical approach of converting a constrained extremizing problem into an unconstrained extremizing problem. He divides the optimum design problem into three different cases which he calls

1. Design Under Normal Specifications
2. Design Under Redundant Specifications
3. Design Under Incompatible Specifications.

In case 1, it is possible to combine constraining equations or inequalities into the criterion (Primary Design Equation, according to Johnson) in such a manner that the limiting conditions apply to the independent variables existing in the newly developed criterion. A curve indicating the relation between the criterion and the independent parameter can then be drawn for each feasible material and, applying the limit, the optimum parameters and extreme criterion can be found. Of all these the parameters corresponding to the material which gives the minimum (or maximum) criterion can then be selected.

In case 2 the above procedure cannot be adopted on account of the existince of constraints in excess numbers. The procedure then is to ignore such constraints temporarily and handle the problem exactly in the same manner as case 1 is dealt with. Curves can then be drawn with different independent parameters as abscissa. These steps are taken sequentially so that the results of application of limits for one independent parameter are carried to the next consecutive case. Thus the parameters obtained by the last curve would be the optimum parameters. Usually the transformed criterion is nonlinear and difficult to handle, hence logarithimized criterion rather than the actual criterion is used. This linearization simplifies the problem and mathematical manipulation becomes easier.

Case 3 is a special case of case 2 and can be handled in exactly the same manner. However, though it does not provide any feasible solution, it gives a clue to how an incompatible problem can be converted into a compatible problem by varying certain limits or by using other materials. Hence when all other methods fail this method can provide data for altering the specifications to convert an incompatible design situation into a compatible design situation.

Although Johnson's approach is an ingenious technique for design of mechanical elements, it seems that it is incapable of handling inequality constraints which are functions and it cannot be applied to a general problem of a set of large number of equations and inequalities. Klein's method described in section 2.26 may generally take care of the first difficulty whereas the nonlinear programming methods described in section 2.54 can handle the second one.

For the application of this method to design of mechanical elements Johnson's book Optimum Design of Mechanical Elements (ref. 34) can be referred to.

2.62 Other Methods

After dealing with several mathematical tools of optimization we would consider in this section the tools which involve little or no mathematics. Several attempts have been made to find the best or optimum conditions when it is very difficult or very expensive to formulate a mathematical model by theoretical or experimentation methods. A few important techniques of this type are Simulation, Statistical Control Procedures, Simplification Methods, Standardization Practices and the Method Time Measurement Techniques.

In simulation, a real system is duplicated in some sense so that by using this approximate model sufficient data can be collected for making a decision. A special simulation method which takes into consideration the stochastic or random factors, i.e., a method of simulation under risk or uncertainty, is the so-called Monte Carlo Method. The subject is dealt with in detail by Morgenthaler[123]. Applications to product design problems are dealt with by Starr[124] and production design problems by Bowman and Fetter[86].

Statistical control procedures can be used to optimize product design by determining the optimum or the so-called natural tolerances. A method of determining tolerance by this method is dealt with in detail by Mir[125]. Eary[126] has used a similar method for finding the best coolant for a machining operation.

Simplification methods are attempts to reduce the cost of a product by simplifying design from the point of view of manufacture, use and maintenance. Several examples of this method are described by Maynarad[127] and various other authors[128].

Standardization practices[127-128] are very important in the search of finding new means for reducing the cost of an over-all design of an equipment; these may be considered as sub-optimization attempts which could be utilized later for primary optimization.

The method time measurement technique, though comparatively new, have been widely used for optimizing manufacturing methods[127,129]. Similarly, for scheduling problems the PERT system[129] of evaluation (Program Evaluation Research Task) initially developed for the U.S. navy has been found to be very valuable, and since then has been extended to resource incorporation, performance incorporation and programme balancing.

3

APPLICATIONS

3.1 Design of Mechanical Elements

The dictionary meaning of the word mechanical is 'concerned with machines or pertaining to machines. In common usage the word machine is used to signify machine tools; this is incorrect. From technical point of view, a machine is defined as a device which used power to accomplish a physical effect. According to Johnson[34], a machine can be defined as a mechanical structure which is characterized by mechanical elements having relative motion and generally capable of transmitting or dissipating significant amount of energy. The above definitions include machine tools as well as several other equipments such as engines, turbines, motors, measuring devices, heat transfer equipments etc. Keeping this in view, we have captioned this section "Design of Mechanical Elements" and we intend to cover the applications of optimization techniques to the design of various members of all such types of equipment.

Although the use of optimizing techniques in this field is not older than a decade or two, substantial work has been done and quite a few optimum design problems have been solved by using such techniques. Most of the earlier work was confined to graphical and graphical cum analytical methods. If any analytical approach was applied at all, it was mostly concerned with unconstrained design. It is only during last few years that the problem of design of such under equality and inequality constraints and limit or boundary conditions was extensively studied. Several methods of solving problems falling under the linear programming class have been discussed and attempts have been made to solve nonlinear problems by using linear approximations. A bibliography of applications is given below.

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Also see ref. 11, 27, 30, 36.

3.2 Synthesis of Mechanisms

German kinematicians such as Beyer[131], Rosenaeur[131] etc. have, in the recent past, tried to rationalize the methods of synthesis of mechanisms by using the previous works of German and Russian kinematicians such as Bermeister[131] and Tchebycheff[132]. With the migration of such persons into United States during the fifties, the research in kinematics in this country received a new impetus. Persons like Freudenstein, Hartenberg, Rothbart, Hall, Hirschhorn etc. have, since then, further extended this work by using Tchebycheff's polynomial approximations, the method of finite differences and complex notation. Efforts have been made to synthesize four bar mechanisms for best transmission properties and function generators with minimum errors by using high speed digital computers. Problems of high speed cam design and intermittent motion generators have also been studied. Some of the important works in the connection are referred to below.

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Also see ref. 1, 2, 4, 15, 17, 18, 21, 22 under Bib. Sec. 3.1 and ref. 25, 29, 23 under general references.

3.3 Design of Machines

A problem of design of a machine is a primary optimization problem. Hence it is comparatively more difficult than the sub-optimization problem of design of mechanical elements. The situation further deteriorates when nonlinearities are encountered. On account of these difficulties work in this field has not been extended to general design of machines. However, several examples of design of standard machines are reported to have been practically handled by using optimization techniques. Extensive use of such techniques is reported in the field of rotating machine design only. Examples of machine design problems where only a few constraints and not more than two or three variables occur in fairly good numbers. Some of the references available to the author are given in the bibliography that follows.

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Also see ref. 14, 21, 23, 42, 61, 105 under general References and ref. 18-29 under Sec. 3.8.

3.4 System Design

A system may be defined as a combination of several parts and activities usually with one basic objective. It is this basic objective of singleness of purpose which gives a unique characteristic to a system. Often, a change in one of the many variables of a system results in corresponding changes in many other variables. Thus we can say that systems are usually complex.

From the above discussion it is clear that the problem of system design, like the problem of machine design, is a primary optimization problem. Most of the difficulties encountered in the optimum design of machines are also experienced in the synthesis of optimum systems. Inspite of all these difficulties the problem of optimum design of systems has received the utmost attention of the researchers. It is perhaps due to the fact that the systems are usually complex, no alternative approach is presently known and that mere guess work cannot be tolerated since it may lead to disastrous situations. Statistical methods, dynamic programming, simulation techniques etc. have all been applied at some time or the other for the synthesis of different type of systems. These researches have recently culminated into what is presently known as adaptive or self-optimizing system. A self optimizing or adaptive system is one which "learns about its environments and adjusts itself to expected performance in a continual process of measuring and adjusting [78]". Many of the techniques developed in Operations Research have been applied to system optimization.

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.5 Aerospace Design

The two world wars, the present armament race between United States and Union of Soviet Socialist Republic and the age long desire of man to conquer outer space have all worked for the developments in the aerospace design field. Since a single failure costs millions of dollars, an aerospace engineer must be very careful and instead of relying on cut and try methods he must use more sophisticated techniques. The problem of fuel consumption, problem of long life power supply units, the exact nature of trajectories, transfer of rockets and satellites from one orbit to another, reliability of system performance, attainment of required speeds in a given interval of time and capability of exerting the required thrust are questions which can be handled by optimization techniques in a better way than by any other techniques. Almost all optimization techniques described in the preceding sections have been tried and in several cases more sophisticated methods particularly suited to the individual problems have been devised. The author has not extensively studied aerospace design; however, to be comprehensive, a bibliography of some selected applications are included.

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3.6 Tooling and Processing Design

Tooling and processing design is the most neglected branch of engineering design field. This may be due to the fact that the so called cut and try methods usually work and because a single failure, even if it occurs, does not cost too much. However, with the rapid growth of technologies it is felt that this status should not be allowed to continue. Rationalizing the rules of thumb, either by standard practices or by developing relevent theories is more common these days.

The knowledge of existence of optimum parameters in tooling is not new, however due to lack of theoretical corelation with practical results, it is usually difficult to arrive at any precise value of such parameters. A few examples of such parameters are the half angle of the wire drawing die, the radius of the punch nose in deep drawing dies, the cutting angle in the single point cutting tools etc. Where theoretical corelation exists it is easy to calculate such optimum parameters mathematically by using elementary procedures of sec. 2.2. An example of wire drawing die half angle will be illustrated here.

Using the following relation from Ford [132]

$$P/A_2 = q\,(1 - \mu \cot\alpha) \cdot r/(1-r)$$

where P is drawing force, A_2 the area of the drawn wire, A_1 the area of wire before drawing, q the normal die pressure, μ the coefficient of friction, α the half angle of the die and r, ratio equal to $(1 - A_2/A_1)$, we can proceed as follows:

$$r = (1 - A_2/A_1) = 1 - D_2^2/D_1^2$$

where D_1 and D_2 are the diameter of the wire at the entry and exit respectively.

From the geometry of the die we can write

$D_2 = D_1 - L \tan \alpha$ where 'L' is the effective length of the die.

Substituting this in the basic relation we get

$$\frac{P}{A_2} = \frac{qL}{D_1} \left(2D_1 \tan \alpha - L \tan^2 \alpha - 2 \mu D_1 - \mu L \tan \alpha\right)$$

i.e.

$$= k_1 \left(k_2 \tan \alpha - k_3 \tan^2 \alpha - k_4\right)$$

where $k_1 = q\,L / D_1$, $k_2 = 2D_1 - \mu L$, $k_3 = L$ and $k_4 = 2 \mu D_1$

Differentiating the stress P/A_2 and equating to zero we get

$$k_2 = 2 k_3 \tan \alpha \quad \text{or} \quad \alpha = \tan^{-1} k_2/2k_3$$

$$= \tan^{-1} \frac{2D_1 - \mu L}{2 L}$$

for a '1' inch effective length carbide die ($\mu = .04$) with an entry diameter of 1/8 inch the optimum value of the half angle

$$\alpha = \tan^{-1} \frac{2 \cdot .125 - .04 \cdot 1}{2 \cdot 1}$$

$$= \tan^{-1} .105$$

$$= 6^\circ$$

which is the value commonly used.

Similarly, in processing, the maximum height of draw, the optimum blank for deep drawing, the optimum machining tolerance, the best coolant for maching can be determined. Some relevant examples are referred in the bibliography that follows.

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3.7 Structural Design

In dealing with problems of structural design one often encounters the problem of choice of member parameters to meet a certain criterion. In such cases the problem becomes extremely difficult if the structure is assumed to be perfectly elastic. On the other hand, by assuming the structure as rigid and perfectly plastic, it is possible to handle such cases in a fairly simple manner. The determination of optimum design of structures that are just able to carry a specified load on the above assumptions is usually called the method of limit design. The problem can then be so reduced that the criterion, usually weight, can be minimized subject to certain linear inequality constraints. Often such problems can easily be reduced to linear form and hence linear programming procedures can be applied.

Several structural design problems have been handled by this technique. Some of the important references are given in the Bibliography that follows.

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3.8 Electrical Design

In electrical design the problems of synthesis of networks, design of rotating machines, design of electrical apparatus, transmission and distribution systems design and synthesis of control systems have all been dealt with by using optimizing techniques. Transportation method and its variations have been applied for network synthesis, linear programming for electronic package assemblies and distribution system design, dynamic programming for the control system design and various analytical and analytical cum graphical methods for the electrical rotating machines and non-rotating or stationary apparatus design. Actually the pioneering work regarding the application of optimization techniques in engineering design was done in this sector. Even at present time most of the research regarding application of optimization techniques is being handled by organizations like I.B.M., G.E. and Westinghouse, all of whom are primarily interested in the manufacture of electrical apparatus.

Since control systems have already been dealt with under system design, in the bibliography of the applications of optimization techniques to electrical design which follows, these references are not repeated. They are merely referred back.

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3.9 Production Design

Production design problems such as that of machine scheduling, inventory control, product choice, purchase planning, quality control, standardization, method improvements, work simplifications, investment analysis, etc., are problems where one or the other optimization technique is successfully applied. It would not be an exageration to say that production design is the only branch of engineering where optimization techniques have been extensively applied and almost every time without any failure. Some of the more important applications are given in the following bibliography. They are closely related to Operations Research.

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COMMENTS

Summary

The method of finding extrema of multivariable functions by using ordinary theory of maximum and minimum has been presented. The problem is simple if the function is differentiable with respect to its independent variables and has an extremum in the allowable range of its variables. However, in case the function contains transcendental terms the method is almost impossible to apply.

When the function has to satisfy certain equality constraints the method of undetermined multipliers can usually be applied. If constraints happen to be inequalities rather than equalities, an approach due to Valentine, Klein and others can be used to transform them to equality constraints. The undetermined multipliers method is limited to the functions which allow the value and the location of the extremum to be expressed in terms of the newly introduced variable, the undetermined multiplier. If the function to be handled in this way happens to be a polynomial of its independent variables Zener's method permits one to evaluate the extremum without determining the independent variables. It reduces the computational time by transforming a problem of solving a set of nonlinear equations into the problem of solving a small set of relatively simple linear equations.

In certain cases when the criterion function is convex the non-linear inequality constraints can be reduced to linear inequality constraints by using Charnes and Cooper's technique and convex programming can be used to determine the extremum.

Several practical problems require the determination of an extremum of a functional rather than a function. If the end conditions are fixed and the functional is not constrained the extremum can readily be found by using the simple variational calculus procedures. If the problem happens to be isoperimetric, undetermined multipliers can be introduced and the augmented function can be used in a similar manner to the preceding case, and constrained functional extrema can be determined. If the end conditioned are not fixed the problem cannot be handled so easily. Euler equation as well as transversility conditions must be satisfied. If the problem happens to be Mayer or Lagrange's type, which is generally the case in the design field, it can be handled by an approach similar to the two cases discussed above.

In practical design situations most of the analytical methods are to be applied on a high speed digital computer. For this, a problem must often be solved by using numerical iterative techniques. For the variational calculus problems, Euler's Method of finite differences reduces the nonlinear functional problem into polygonal function problem which can then be applied to computers. If the nonlinear functional can be approximated by the sum of linear functionals Ritz method can be applied. If more accurate results are required Kantrovic's method can be used. For multistage multivariable functional dynamic programming or Pontryagin's Maximum Principle can often be applied successfully.

If the set of constraining equations is difficult to solve, Newton-Raphson's approximations can be used provided the first derivative does not vanish. In case the first derivative vanishes Algorithm's method of errors (regula falsi) can be used.

In several instances the extremizing problems end up in differential equations which are very difficult to solve. For solution of such equations, the finite differences approach can generally be used with success. If the criterion as well as constraints happen to be homogeneous Bedford's finite increment technique can successfully be used. Since parameters are changed one by one for reaching the extremum, the designer is kept aware of the inside picture of the design situation.

If the criterion function is unknown or only partially known, the statistical experimentation technique of search can be used. For the univariable unimodal deterministic model, Kiefer's method very quickly reaches the extremum. For multivariable strongly unimodal functions the contour tangents elimination technique is very valuable. Other cases can generally be handled by various versions of gradient methods. Gradient methods are not very efficient and hence acceleration techniques are usually applied to improve efficiency. If constraints are involved pattern search technique is of considerable value. For stochastic situations Kiefer's approximations permits easy handling of the error problem. Dvoretzky's method gives still better results.

If the extremum happens to be on the boundary rather than in the interior of the prescribed region, programming methods can be of very great value. Linear cases subjected to both equality and inequality constraints can easily be handled by linear programming methods. If the basis of the problem happens to be triangular the simple transportation method is the best to apply, otherwise the simplex method may be tried. Simplex is a strong optimization tool and can be applied to several design situations.

For multistage nonlinear functions, parametric or otherwise, dynamic programming often solves the problem by reducing the multi-variable problem into a series of sequential single variable problems. The calculations are usually lengthly and hence computer memory increases exponentially with the number of variables. The method, however, is reliable and can be applied in both ordinary and variational problems. Pontryagin's maximum principle in its special digitized form can be successfully utilised to reduce the dynamic programming defect of rise in computer memory capacity. However, this gives rise to a cumulative error which in ordinary circumstances is very difficult to determine.

The problem of optimization of quadratic functions subjected to linear equality or inequality constraints can be handled by Beal's extension of the simplex method. For linear equality or inequality constraints and nonlinear criterion Wolfe's reduced gradient method, Rosen's gradient projection method I and Kelley's cutting plane method seem to be promising. Economic models are said to have been successfully handled by these methods but the computations are not available for common use.

Rosen's gradient projection method and Kelley's cutting plane method along with the separable programming method and decomposition method are applicable to problems where the criterion as well as constraints (equalities or inequalities) are nonlinear. Rosen's Method seems more suited since it does not depend on the properties of the criterion other than smoothness and computational experience is available.

Conclusions

Choice of a particular method varies with the type of problem and hence it is very difficult to suggest one particular technique. However, as soon as the problem is formulated, the nature of the criterion and the constraints suggests a method to be preferred. Experience is limited in many of the newer techniques, making it difficult to give firm recommendations on choice of method.

For the sub-optimization problems such as the design of mechanical elements, synthesis of mechanisms, design of structures or machine scheduling problems of production design, if the criterion and the constraints happen to be linear, simplex can be selected. Since it is generally sufficient to know the how's of it rather than the why's of it, one can handle simplex by knowing just elementary mathematical principles. Many sub-optimization problems do not fall under linear programming case. In such cases, sub-optimization problems can usually be solved by Lagrange's method of undetermined multipliers with Klein's extension and Newton-Raphson's equation solution procedure. For fairly large number of variables Zener's Method along with Charnes and Cooper's extenion and convex programming seem to be promising provided it satisfies Zener's method's requirements. An alternative choice is Johnson's approach which is more or less a graphical approximation technique and is particularly useful when the design problem involves a choice of materials. Certain multivariable problems of design can be handled by dynamic programming methods.

For the primary optimization problems such as the problem of design of a machine, synthesis of systems, synthesis of electrical networks and process equipment design problems, programming methods are better. If such problems happen to be non-stochastic and criterion can not easily be determined, which is generally the case with the process system design, particularly with chemical process systems, direct search methods can be used. If the problems are not error free, such as control system design problems, these can be handles by Kiefer or Dvoretzky's method. Certain control system design problems can be solved by dynamic programming or the digitized maximum principle. Some of the electrical apparatus such as transformers or reactors can be handled by Bedfords's finite increment technique where as electrical network problems can generally be solved by the transportation method or the convex programming methods. For a special type of problem which may occur in process design Zener's method of minimizing system cost in terms of sub-system cost can be used to reach the optimum with fewer computations. When reliability is of paramount importance, such as in aerospace design, certain control system designs and some of the process design problems, simulation techniques are the only choice for determining the behaviour of the system before it is actually designed and manufactured.

Some of the nonlinear programming methods such as gradient projection method can handle the problem of finding the extremum whether it lies in the interior or on the boundary of the given region. However, generally such problems can first be tried by simpler method of ordinary calculus. If these methods fail, it is at once known that the extremum does not lie in the interior of the region and hence the programming methods can be applied for the location of the extremum at the boundary.

Certain fields in engineering design are more or less neglected so far and depend mostly on cut and try methods; rationalization of such methods by using standardization procedures and various statistical techniques such as statistical control method and regression methods seems advisable.

Since no experience with nonlinear programming in the design field is so far reported, it would be worth while if further research were done in this field. It would seem to be more appropriate to conduct such research in one of the few promising nonlinear programming techniques with the intension of evolving a simple code that could be handled by less experienced designers engaged in practical fields, rather than in research. This thesis is prepared with the point of view of the designers and though some of the recent techniques such as search methods, simulation, dynamic programming, maximum principle and nonlinear programming are included, simulation and search techniques are just touched, dynamic programming and maximum principle are covered only in part, and nonlinear programming methods are dealt with very briefly. On account of the rapid pace of development these days it is very difficult to call any work comprehensive and up to date. However, it is an attempt to provide an extensive up to date survey of optimization techniques and their applications.

For the guide of the designers a comparative chart of various methods is included at the end of this section.

USE CHART OF METHODS

	Method	Criterion	Constraints	Nature of Const.	Number	Eq. to be solved	Comp. Exp.	Comp. Prog.
1.	Lagrange's Method of Undetermined Multipliers	Nonlinear	Equality	Linear or Nonlinear	Few	Differential	--	--
2.	Klein's Extension of Lagrange's Method	Linear or Nonlinear	Any	Linear or Nonlinear	Few	Simultaneous any	--	--
3.	Zener's Method	Linear or Nonlinear	No	--	--	Simultaneous Lin.	--	--
4.	Charnes and Cooper's Extension of Zener's Method	Nonlinear(Convex)	Any	Linear or Nonlinear	Many	Prog.	Available	Simple
5.	Isoperimetric Problem	Functional	Equality	Functional	Few	Differential	--	--
6.	Mayer's Problem	Functional	Equality	Functional	Few	Differential	--	--
7.	Pontryagin's Maximum Priniciple	Functional	Any	Any	Many	Prog.	Available	Available
8.	Euler's Method of Finite Differences	Functional	No	--	--	Prog.	--	Simple
9.	Ritz Method	Functional	No	--	--	Prog.	Available	Available
10.	Kiefer's Search Method (deterministic)	Unknown (unimodal) (univariable)	No	--	--	Experimental	--	--
11.	Contour Tangent Elimination Method (deterministic)	Unknown (unimodal) (multivariable)	No	--	--	Experimental	--	--
12	Gradient Method (deterministic)	Known or Unknown	Sp. Method-any	Linear or Nonlinear	Few	Differential	Available	Available
13.	Accelerated Gradient Method (deterministic)	Same as above (more effeicient)						
14.	Hooke and Jeeves Method(deterministic)	Unknown	Any	Any	Few	Experimental	--	--
15.	Kiefer and Wolfowitz Method (stochastic)	Unknown(stochas.)	No	--	--	Experimental	--	--
16.	Transportation Method	Linear	Equality(con.ineq)	Linear	Many	Prog.	Available	Available
17.	Simplex	Linear	Any	Linear	Many	Prog.	Available	Available
18.	Dynamic Programming	Nonlinear(function or Functional)	Any	Sum	Many	Prog.	Available	Available
19.	Simplex Method of Quadratic Programming	Quadratic	Any	Linear	Many	Prog.	Available	Available
20.	Reduced Gradient Method	Nonlinear	Any	Linear or Nonlinear	Many	Prog.	Not Available	--
21.	Separable Programming	Nonlinear(sep.)	Any	Linear or Nonlinear	Many	Prog.	Available	Not Available
22.	Cutting Plane Method	Nonlinear(convex)	Any	Linear or Nonlinear	Many	Prog.	Available	Not Available
23.	Gradient Projection Method	Nonlinear	Any	Linear or Nonlinear	Many	Prog	Available	Expectedto to be available.
24.	Differential Gradient Method	Nonlinear	Any	Linear or Nonlinear	Few	Prog.	Available	Available
25.	Johnson's Method	Nonlinear	Any(function inequalities can not be)	Linear or Nonlinear	Few	Graphically		
26.	Simple case of ordinary theory of maxima and minima	Nonlinear	No	--	--	--	--	--

Solution of simultaneous equations--- Newton-Raphson's Method

Solution of differential equations--- Method of finite differences

In the use chart of methods following abreviations are used:

i. Const.	Constraints
ii. Eq.	Equations
iii. Comp. Exp.	Computational Experience with Computers
iv. Comp. Prog.	Computer Programme (nature of)
v. Prog.	Programming
vi. Con. ineq.	Convertable to equality type inequality

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