RM GAMES FOR HOTEL ROOM INVENTORY CONTROL

GAME THEORETIC REVENUE MANAGEMENT MODELS FOR HOTEL ROOM INVENTORY CONTROL

BY

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Abstract

In this thesis, we focus on the rationing polices for the hotel room inventory control problems. Our study begins with a brief overview of revenue management in hotel industry, emphasizing the importance of room inventory control in revenue management problems. Mathematical models for controlling the room inventory in the literature are then reviewed along with recently developed game theoretic applications in revenue management. In game theoretic context, we establish three types of models to solve the hotel room inventory control problem in three different situations: 1) two-player two-fare-class static single-period game with complete information; 2) two-player two-fare-class dynamic multiple-period game with complete information; and 3) two-player two-fare-class single-period game with incomplete information.

In the first situation, we find the existence of unique Nash equilibrium and Stackelberg equilibrium in the non-cooperative case. We provide the exact forms for these equilibria and corresponding conditions. Next, under the dynamic game settings, we provide the sufficient conditions for the unique Nash equilibrium. In the last situation, we consider the static single-period games with incomplete information and discuss the optimal strategies for the uninformed case, secret information case, private information case and public information case. The unique Bayesian Nash equilibrium in each case is found. We then analyze the values of different types of information and study their relations in different situations. Under each game theoretic setting, we present the managerial implications of our solutions along with the numerical examples. The thesis is concluded by a discussion of how game theory can is useful in hotel industry, and its relationship to other topics in revenue management.

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Chapter 1 Introduction

This chapter provides an introduction to the topic of revenue management (RM) and game theoretic applications. We start with an explanation of RM and its history. We then briefly describe the industry profiles of hotels which make the RM practices distinct from the applications in other industries and the objective of hotel RM. Next, we highlight the importance of combining ideas from RM and game theory. Finally, we conclude by giving an outline of the remaining chapters in this thesis.

1.1 Revenue Management and Its History

Revenue management (RM) first appeared in the airline industry in the late 1970s when the deregulated industry attempted to maximize profit by ensuring all seats were occupied before take-off and offering varied prices to the customers. RM marries operations research/management science, statistics, economics, and software development to manage demand for a firm's inventory with the goal of maximizing revenue. Practitioners usually find that it is easier to define the objective of revenue management rather than explain what it actually is—the outcomes are easier to understand than the process. Revenue management is about marketing mix, cost/price relationships and product distribution, which allows a business to "sell the right product, to the right customer, at the right time, at the right price" (Smith et al. [45]). It is a suite of components that, when working in harmony, will present the best opportunity to maximize revenue. In 1992, Weatherford and Bodily [50] proposed to replace the term revenue management with a new, more appropriate term, Perishable-Asset Revenue Management (PARM). The element that links the industries implementing revenue management is that all of their inventories are perishable. Once the plane takes off, there is nothing one can do about trying to sell any of the seats on the plane. Similarly, when a room is empty overnight, the opportunity for revenue is lost forever.

Today, revenue management has also spread out to other industries such as hotels, retailers, car rental agencies, Internet service providers (ISP), railways, cruise lines, electric power supply and restaurants. Basically, these industry sectors all share certain characteristics that make them particularly suited for revenue management. These characteristics have been identified by Kimes [26] as: relatively fixed capacity; ability to segment markets; perishable inventory; product sold in advance; fluctuating demand and low marginal sales cost and high marginal capacity change costs. Although similar in these respects, there are still some explicit differences when different industries are subject to different combinations of duration control and variable pricing. Kimes and Chase [27] demonstrate such differences with a pricing and duration positioning table; see Table 1.

Comparatively, airlines, hotels and car-rental firms are more able to apply variable pricing for a product which has a more predictable duration. They conclude that successful revenue management applications generally occur in these industries (Quadrant 2), because they can manage both price and capacity more effectively.

		Price	
		Fixed	Variable
Duration	Predictable	Quadrant 1: Movies Stadiums/Arenas	Quadrant 2: Hotel Rooms Airline Seats Rental Cars
	Unpredictable	Quadrant 3: Restaurants Golf Courses Internet Service Providers	Quadrant 4: Continuing Care Hospitals

Table 1. Pricing and duration positioning table for various industries.

However, there are still a wide variety of complications when firms implement RM techniques because every firm possesses its own industry-specific characteristics such as technology standards, consumer behavior, pricing policies etc. Carrol and Grimes [10] summarize the impact of these factors on three industries: airline, hotels and car rental firms. Nair and Bapna [35] use Weatherford and Bodily's taxonomy [50] to compare the Internet Service Provider (ISP) problem with hotel and airline revenue management problems. Several review papers describing the theory and applications of revenue management in the airline industry have been published in recent years (see Bitran and Caldentey [5], Kevin and Piersma [25], McGill and van Ryzin [33], and Weatherford and Bodily [50]).

1.2 Hotel Revenue Management

The hotel industry began to apply the concept of revenue management in late 1980s when the industry faced excess capacities, competitive markets, liquidity problems and recession; all of which affected operations and resulted in lower revenue (See Hansen and Eringa [21]). Hotels can be classified as business, resorts, extended-stay, or a mix of business and leisure and also by size and location. Some hotels manage only individual properties, while large hotel chains can own hundreds of properties. A hotel, typically, offers rooms for many day-to-day lodgings of various types of customers. Despite some of the similarity with the airline customer types, the segmentation used in hotel RM are different. For example, advance-purchase discounts, a prominent segmentation mechanism of airlines, are not commonly used by hotels. Since hotels also generate significant revenues from other sources such as food, entertainment, and function space, the value of a customer is hard to determine exactly. However, these additional sources of revenue are usually not considered in hotel RM applications.

There are many different room types, such as standard rooms, deluxe rooms, executive rooms, rooms with a view, single or double bed rooms, smoking and non-smoking rooms, etc. They can be grouped together into three or four categories for capacity control purposes. Hotels typically aggregate both the room rate and the customer types, leading to about 3 to 10 rate bands for RM purposes. The room rates are usually adjusted only once or twice a year. Normally, a hotel room booking is made directly with the hotel (walk-in, through Internet, or by call). However, in a large hotel, approximately 20 to 40 percent of bookings come from Global Distribution Systems (GDS). The *P*lation of a booking happens not only when the customer cancels the booking before the date for accommodation, but also when the customer decides to check out early. Therefore, the future capacity of the hotel is often uncertain and overbooking is widely practiced in the hotel industry.

Hotel RM mainly focuses on selling rooms in a way that maximizes total room revenue, rather than trying to sell all available rooms. For example, hotels sometimes make the customer "walk" (i.e., send elsewhere) a less valuable customer even when a room is available, to avoid walking a more valuable customer who is arriving later. This strategy may be risky since the arrivals of high-revenue customers in the future are not guaranteed. However, it is a systemized occupancy-price strategy for controlling the room rates and occupancies to maximize the total revenue. Some recent studies perceive revenue management as a managerial tool for maximizing profits, rather than revenue. For example, Donaghy et al. [13] and Griffin [20] point out that the total income calculations should include cost considerations and revenue management should move from a revenue- to a profit-generating tool. However, due to the high capital investments but low variable costs of hotel operations, increasing revenue essentially results in an increase in operating profits.

1.3 Game Theory and Revenue Management

Game theory concerns itself with the analysis of competition and cooperation situations. It has found applications in diverse areas such as anthropology, auctions, biology, business, economics, management-labour arbitration, philosophy, politics, sports and warfare. During the 1950s and the 1960s, academic researchers began to apply game theory in operations research/management science area. Several reviews focussing on the application of game theory in economics or management science have appeared in the last five decades. An early survey of game theoretic applications in management science was given by Shubik [43]. Feichtinger and Jørgensen [14] published a review that was restricted to differential game applications in management science and operations research. A review of applications of differential games in advertising was given by Jørgensen [22]. Wang and Parlar [49] presented a survey of the static game theory applications in management science problems. In addition, several books (e.g., Chatterjee and Samuelson [11], Gautschi [17], and Sheth et al. [42]) partially reviewed some specific game-related topics in management science. More recently, Leng and Parlar [29] present a review of the existing supply chain game models, under a topic classification of five areas: (i) Inventory control, (ii) production and pricing competition, (iii) service and product quality competition, (iv) sharing issues in supply chain management, and (v) strategic competition in marketing.

To the best of our knowledge, there are currently no detailed survey papers on game theoretic models in RM problems and there are very few published works directly concerning such problems. Most studies assume that the company handling perishable products (such as airline, hotel, restaurant, etc.) exists as a distinct entity. In reality, there are usually more than one company dealing with "substitutable" products in a specific geographical market. In this situation, one company's decisions on inventory rationing, pricing, or both might be affected by the decisions of other companies. Therefore, more significant and interesting topics arising from revenue management allow us to address the following questions: How do they set the booking limits or protection levels of multiple classes products? Is there an equilibrium in inventory allocations? Is it more beneficial to be the "leader" in a Stackelberg game? How to find the optimal rationing policies when one firm has an incomplete information of the others? How much can RM increase the overall revenue if the firms cooperate? As a result, a prime methodological tool for dealing with these problems is game theory that focuses on the simultaneous or sequential decision-making of multi-players under complete or incomplete information in a competitive or cooperative context.

1.4 Organization of the Thesis

The rest of this thesis is organized as follows.

Chapter 2 presents a comprehensive discussion of the existing mathematical models which can be applied to hotel room inventory control problems. We then look at several game models which can be applied to hotel revenue management.

Chapter 3 addresses a single-period two-player two-fare-class hotel room rationing game. First, we investigate the best response functions of both players and corresponding properties. A unique Nash equilibrium of booking limit decisions is found in the competitive situation. Next, we assume that one hotel acts as the "leader" and the other as the "follower"; under this scenario we examine the Stackelberg equilibrium. For this case, we identify a situation in which the Stackelberg game is equivalent to the Nash game. This result shows that if one player's booking limit is reached, i.e., if he always rejects low-fare customers, neither of the two players prefers to be the "leader" in the game. Finally, we examine the cooperative case where the hotels "cooperate" to maximize a system-wide objective function and find that the profit loss is substantial if there is a lack of cooperation between two players.

In Chapter 4, we study a multi-period hotel room rationing problem. This problem is formulated as a dynamic programming model. First, we find that each player's optimal expected future revenue is a non-decreasing function of its own room inventory level and a non-increasing function of the other hotel's room inventory at any time. Second, we provide the sufficient conditions for the unique Nash equilibrium of dynamic accept/reject decisions and identify the situations in which the game admits multiple Nash equilibria (MNE). Finally, by defining expected marginal values of the hotels' rooms, we simplify the optimal accept/reject decision into sets of critical values. We also provide some numerical examples along with the managerial implications for our solutions of the competitive and cooperative games.

In Chapters 3 and 4, we assume that the games are played under complete information, i.e., each player knows the booking arrival patterns, transfer rates and rejection costs of both players. In Chapter 5, we relax this assumption and examine the static game problem under incomplete information. More specifically, we assume one player's rejection cost and transfer rate of low-fare class customer as the incomplete information. By employing the different types of information (secret, private and public information), we discuss the game theoretic solution for the incomplete information game, which is known as Bayesian Nash equilibrium. Another goal of the study in this chapter is to evaluate the different information types. Accordingly, we first analyze the conditions in which the value of information is positive (or negative). Then, we compare the values of different information to see which type is most valuable for one player, and under what conditions the information value benefits the player most and in what content.

Chapter 6 summarizes the concluding remarks of this thesis and provides some potential research directions for future studies.

Chapter 2 Literature Review

One of the fundamental questions of perishable asset inventory control that must be answered each time a demand arrives is whether to accept it or to reserve the unit of inventory for possible sales later to a potentially higher-paying customer [34]. Before reviewing room inventory control solutions in the literature, some vocabulary should be introduced first. *Booking limit and protection level* are the two important concepts used for the room inventory control problem. Netessine and Shumsky [36] define a booking limit to be the maximum number of rooms that may be sold at the discount price, and protection level is the number of rooms which will be protected for full-price customers. If the optimal values of these two variables can be determined, it will be easy to decide whether to accept the low-revenue booking or to reserve the room to a potentially higher fare class customer later on.

Consider the arrival of a booking request that requires one or more rooms starting on a specified date, at a given price. One of the basic revenue management decisions is whether or not to accept or reject this request in order to maximize the total expected revenue. These types of problems are known as *room inventory control* problems faced by hotel management. The point is that at a certain time it is more profitable to reject a lower-revenue customer in order to be able to accept a higher-revenue customer at a later time. Clearly, if the hotel waits too long for higher-revenue customers to appear, at the end of the selling horizon, there might be some unsold rooms that could have been sold to lower revenue customers at an earlier time.

Revenue and Cost Factors
Cancellation penalties
Denied lodging cost
Lost of goodwill cost
Demand Factors
Demand dependencies between booking classes
Length of customer stay
Demand for future arrival dates
Demand for future room types
Competing hotels' effect on future demand
Room Inventory Control
Booking lead-time
Overbooking up-limit

Table 2. Significant factors which affect hotel RM.

Based on the discussion in McGill and van Ryzin [33] and Upchurch et al. [48], we identify the elements in Table 2 which can be used to model a generic hotel revenue management problem. In the rest of this chapter, we review some RM models that incorporate these elements.

2.1 Static Models for Hotel Room Allocations

Static RM models have been used frequently in recent years. They are typically formulated assuming that demand is segmented in predetermined fare classes. Instead of a distinct control mechanism, they assume a nested booking limit system (See

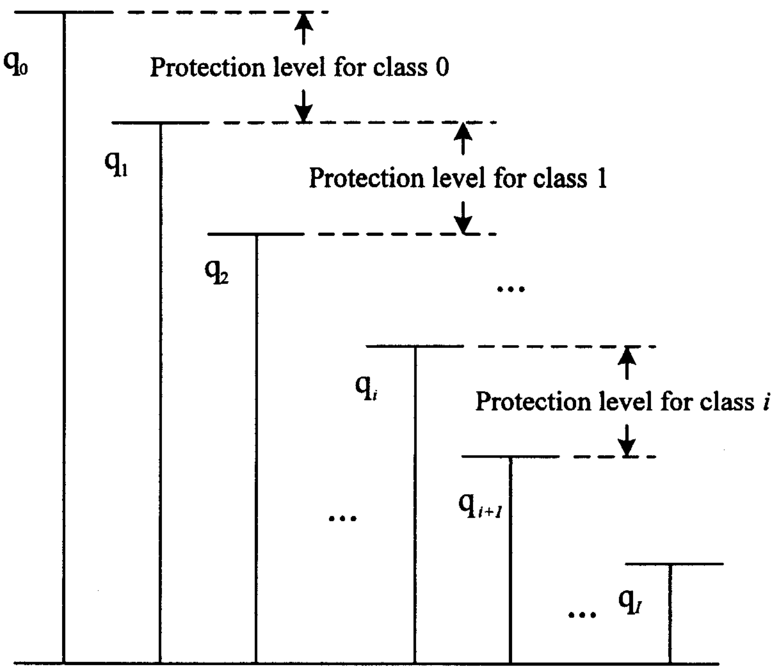


Figure 1. Nested booking limits and protection levels.

Belobaba [4]). This approach sets booking limits that are nested from above, e.g., the full-price class has a booking limit up to the total capacity of the hotel. Most of the earlier works only consider two price classes: full price and discount price. However, the nested booking limit method can be applied to any number of price classes. Bodily and Weatherford [50] define a '*bucket*', q_i, $i = 0, 1, 2, ..., I$, as the booking limit for price π_i class; these buckets are nested as in Figure 1.

The total capacity is equal to q_0, which is only available to those willing to pay the highest price. An amount q_i is available for the ith highest price class, with the difference, $q_i - q_{i+1}$, protected for sale to ith class customer from those in lower price classes. For the two fare classes problem, the earliest model for inventory control problem is due to Littlewood [32]. He assumes that there are x units of capacity remaining and there is a booking request from discount price class. Therefore, if such request is accepted, the company collects a revenue of π_2. If the request is rejected, the company will sell unit x at π_1 if and only if demand for full price class is x units or higher. Demand for class j is denoted by D_j, and its distribution is denoted by F_j.

Littlewood's Rule: *There is an optimal protection level, denoted by y_1^*, such that we accept the discount price class if the remaining capacity exceeds y_1^* and reject it otherwise. Such y_1^* must satisfy*

$$\pi_2 < \pi_1 \Pr\left(D_1 \geq y_1^*\right) \quad \text{and} \quad \pi_2 \geq \pi_1 \Pr\left(D_1 \geq y_1^* + 1\right). \clubsuit$$

If one can use a continuous distribution $F_1(x)$ to model demand, then y_1^* can be given by a simpler expression $y_1^* = F_1^{-1}(1 - \pi_2/\pi_1)$. See Liang [31] for an analysis of similar continuous-time version of a dynamic model. In general, the lower the ratio π_2/π_1, the more capacity the company should reserve for full price class. This makes sense since the company would be always willing to accept a low price only when the chances of selling at a high price are lower.

Brumelle et al. [9] propose another decision rule to obtain the optimal booking limit for the discount class.

Brumelle's Rule 1: *Booking limit of discounted class is*

$$q^* = \max_q \left\{ 0 \leq q \leq Q : \Pr\left[Y > Q - q \mid B \geq q\right] < \frac{P_B}{P_Y} \right\} \quad (1)$$

where Q is the total capacity, Y the number of full-fare customers (random variable), B the number of discount customers (random variable), q the booking limit of discount customers and P the price subscripted for a given fare class. ♣

This stopping rule is developed under the assumption that discount demand occurs earlier than the full-fare demand occurs and a booking class will not be re-opened once it has been closed. In addition, this model assumes that there is no cost involved when rejecting a reservation. Bodily and Weatherford [7] present a similar model:

Weatherford and Bodily's Rule 1: *Reserve an additional space for a discount customer if*

$$p = \Pr[X < q_0 - q_1] > 1 - \frac{R_1}{R_0} \tag{2}$$

where q_0 is total capacity, q_1 the booking limit for discount class (decision variable), X the number of full-fare customers (random variable) and R_0 and R_1 represent the revenue of full-fare class and discount class, respectively. ♣

Brumelle et al. [9] generalize their model by assuming a cost (loss of goodwill) is involved when turning away further requests once current reservations reach capacity. It is a modification of their previous rule (see (1)) where the value of lost of goodwill, P_G, is added to the full price to give the rule that the optimal booking limit,

$$q^* = \max_q \left\{ 0 \le q \le Q : \Pr\left[Y > Q - q \mid B \ge q\right] < \frac{P_B}{P_Y + P_G} \right\}.$$

They also present a decision rule which incorporates the compensation of customers if overbooking occurs.

We note that all the above rules assume that the discount class customers only accept the discount price and full-fare customers only pay full price. In hotel business, the business travellers (full-fare class) may also want to pay the discount price and as the discounts class is closed the leisure travelers (discount class) may accept the full price. This is defined as diversion by Pfeifer [41] who provides the following rule to determine the booking limit of discounted class customers.

Pfeifer's Rule: *Reserve an additional discount customer if*

$$p_1 p_2 < 1 - \frac{P_D}{P_F}$$

where p_1 *is the probability that the* $(q+1)st$ *customer will only accept discounted price and* p_2 *the probability that* $Q - q - 1$ *(*Q *being the total capacity) full price units will satisfy all subsequent demands from those who would pay the full price and those who would prefer the discount but if unavailable would accept the full price.* ♣

Considering the probability γ of an upgrade if a discounted class customer is rejected, Brumelle et al. [9] develop a different model which is similar to Pfeifer's rule.

Brumelle et al.'s Rule 2: *Booking limit of discounted class is*

$$q^* = \max_q \left\{ 0 \le q \le Q : \Pr\left[Y + U(q) > Q - q \mid B \ge q\right] < \frac{P_B - \gamma P_\gamma}{(1-\gamma) P_Y} \right\}$$

where $U(q)$ is the number who will upgrade if discounted class is closed at q. ♣

Another model by Belobaba [4] considers the probability that a discounted customer may upgrade vertically to full-fare class.

Belobaba's Rule: *Booking limit of discount-class is*

$$BL_2 = C - (S_2^1 + V_2^1)$$

where C is the total available capacity and $(S_2^1 + V_2^1)$ the total protection level for discounted class from full-fare class. The quantity $(S_2^1 + V_2^1)$ is determined by

$$EMSR_1 \left(S_2^1 + V_2^1\right) \left[1 - P_2(v)\right] + f_1 P_2(v) = f_2$$

where $EMSR_1$ is the expected marginal seat revenue for full-fare class when the number of seats available to the class is increased by one; $P_2(v)$ the probability that a refused discount-class customer will accept a booking in full-fare class; and f_1 and f_2 the average fare level of full-fare class and discounted class, respectively. ♣

All of these three rules mentioned above solve the diversion problem by introducing probabilities of 'upgrade' of discounted class customer. These probabilities increase as the booking limits increase because the lower the inventory level the more likely a discounted customer will buy up. However, in reality, estimating these probabilities is still not easy.

The decision rules described above just concern two-fare classes problem. In practice, a hotel manager might often face three or more fare classes for a single type of rooms. Based on the solution for two fare classes problem, Weatherford and Bodily [7] extend the decision analysis to a more general case considering any number of fare classes. In order to establish the general decision rules for each of the booking limit, some other definitions and notations are defined first. For each class ($i = 0, 1, 2, ...I$), R_i is defined as the contribution from a unit sold to a class i customer ($R_0 > R_1 > R_2 > ... > R_I$); X_i as a random variable for class i; Y_i as a random variable representing the demand for units in all price classes $\leq i$ subsequent to the arrival of the $(q_{i+1} + 1)$st customer; β_i as the probability that the next customer requesting a reservation is in class i; p_i as probability that $Y_i \leq q_0 - (q_{i+1} + 1)$.

Weatherford and Bodily's Rule 2: *Accept an additional class i ($i = 1, 2, ..., I-1$) customer if*

$$\frac{\beta_i p_{i-1}}{\sum_{k=0}^{i} \beta_k} > \frac{R_{i-1} - R_i}{R_{i-1}}$$

and accept an additional class I customer if

$$\beta_I p_{I-1} > \frac{R_{I-1} - R_I}{R_{I-1}}. \clubsuit$$

For the first time, this decision rule provides an approach to solve the diversion problem with any number of price classes. Bodily and Weatherford [7] also evaluate the revenue improvement by simulation using the actual airline data for demand. The results indicate that the revenue gains are between 3.6% and 5.35% when considering seven fare classes. But its weakness also resides in the difficulty of estimating β_i and p_i. And besides, this model just considers the 'neighboring diversion effect' (upgrade from one fare class to its immediately higher class), it does not consider the diversion from one class to all possible higher fare classes.

In the next section, we will discuss the models concerning the rationing policies in a dynamic situation (multiple periods).

2.2 Dynamic Models for Hotel Room Allocations

Gerchak et al.'s model [18] is one of the first dynamic models dealing with concurrent demand problem in revenue management applications. The authors were motivated from a real situation observed at a delicatessen store where the manager felt that it might be more profitable if he refuses the request from a low-revenue customer in order to offer a food item (i.e., a bagel) to a high-revenue customer later. In the case of hotel business, this policy can be used to decide the optimal booking limit for lower fare class customers. The authors assume that the time horizon is divided into discrete intervals. In each interval, the arrival rates of the full-fare and discounted class customers are assumed known as λ_1 and λ_2, respectively. And the time interval is short enough to make the probability of more than one customer arriving in any interval negligible. The high-revenue customers generate a unit revenue of ρ_1 and the low-revenue customers generate ρ_2 revenue per unit ($\rho_1 > \rho_2$). Each customer requests a single unit each time. In the basic model, the authors also assume that there is no salvage value for unsold units and there is no loss of goodwill when rejecting a low-fare customer.

Gerchak et al.'s Rule: *The maximum expected future total revenue* $V(n,t)$ *is*

$$V(n,t) = (1-\lambda_1-\lambda_2)V(n,t-1)+\lambda_1\left[\rho_1+V(n-1,t-1)\right] + \lambda_2 \max\begin{cases}\rho_2+V(n-1,t-1) & : \text{Accept}\\ V(n,t-1) & : \text{Reject}\end{cases} \tag{3}$$

with $V(n,0)=0$, *for all* n; $V(0,t)=0$, *for all* $t\in\{0,1,...,T\}$; *and* n *denotes units on hand,* t *time intervals remaining until the end of time horizon and* T *the number of intervals in the planning period. If* $n\geq t$, *we shall never reject a customer no matter which class he/she belongs to. If* $n<t$, *and if we reject a discounted customer at* (n,t),*he/she should also be rejected at* $(n-1,t)$ *and* $(n,t+1)$. *And, if we accept a discounted customer at* (n,t),*she/he should also be accepted at* $(n+1,t)$ *and* $(n,t-1)$.

♣

Clearly, the decisions of "reject" and "accept" depend on the values of (n,t) in case $n<t$. The decision should be made based on two state variables (i) available rooms (n) and (ii) number of remaining time intervals (t). According to the policy, at any given t, there must exist an n^* (booking limit) which the decision should be "reject" if the number of available rooms is less than n^*, and "accept" otherwise. On the other hand, for any given n, there also exists a t^* which our decision should be "reject" if the time remaining until the end of the time horizon is before that point and "accept" otherwise. So by linking all such (n,t^*) and (n^*,t) together, a rejection-acceptance curve can be formed.

Next, the authors extend the basic model to a more complicated situation in which loss of goodwill is involved. Denote the loss of goodwill per rejected customer by g, the only change required is to alter $V(n,t-1)$ to $V(n,t-1)-g$. In addition, if the per unit revenues for the K types of customers are $\rho_1 > \rho_2 > ... > \rho_K$, and their arrival rates are $\lambda_1 > \lambda_2 > ... > \lambda_K$ respectively ($\sum_{i=1}^K \lambda_i < 1$), then the problem can be formulated with the similar way just simply substituting $\lambda_1+\lambda_2$ by $\sum_{i=1}^K \lambda_i$, ρ_2 by ρ_i and λ_2 by $\sum_{i=2}^K \lambda_i$ in (3).

Lee and Hersh [28] consider a general case in which there are more than 2 booking classes and each request may be for more than one unit of product. Using the same discrete-time scheme as that in Gerchak et al.'s paper, they denote G_{im}^n as the probability that a booking from class i in decision period n is for m products, $m = 1, 2, ..., M_i$, where M_i is the maximum number of products allowed for each booking.

Lee and Hersh's Rule: *A booking from fare class i ($i = 1, 2, ..., k$) will be accepted only if $mF_i + f_{s-m}^{n-1} \geq f_s^{n-1}$. The recursive function of f_s^n, the optimal expected revenue generated for next n period given booking capacity s, is:*

$$f_s^n = \begin{cases} \left(1-\sum_{i=1}^{k} P_i^n\right) f_s^{n-1} + \sum_{i=1}^{k} P_i^n \sum_{m=1}^{M_i} G_{im}^n \max\left\{mF_i + f_{s-m}^{n-1}, f_s^{n-1}\right\} & \text{for } n>0, s>0 \\ 0, & \text{otherwise,} \end{cases}$$

where F_i denotes the value of accepting a booking request in fare class i and P_i^n the probability that a request in class i will arrive during a decision period n. ♣

This model implies that for a given request size, booking limit, and decision period, there exists a critical discount class; for a given request size, discount class and booking limit, there exist a critical decision period. But for a given request size, decision period and discount class, the critical booking limits may NOT apply. It is because the expected marginal value for reducing the available rooms of size s by m simultaneously (as a group) in period n, $\delta_m(n,s)$, is not necessarily non-increasing (See Lee and Hersh [28]).

You [51] generalizes Gerchak et al.'s model by relaxing the assumption that the rejected customers are lost sales. He assumes that at any given decision period, a discounted class ℓ customer may upgrade to the next higher class, $\ell+n$, if his initial booking request and subsequent upgrade requests to classes $\ell+1, \ell+2, ..., \ell+n-1$ are rejected. You denotes r_n^ℓ as the probability of such event.

You's Rule: *Accept a booking from fare class ℓ ($\ell = 1, 2, \ldots, L$) if and only if $v_{t-1}(i-1) + x^\ell \geq u_t^\ell(i,1)$. The maximum total expected revenue, $v_t(i)$, with i units available stock with t periods to go is given by*

$$v_t(i) = \begin{cases} \lambda_t^0 v_{t-1}(i) + \lambda_t^L \left[x^L + v_{t-1}(i)\right] \\ + \sum\limits_{\ell=1}^{L-1} \lambda_t^\ell \max\left\{u_t^\ell(i,1), x^\ell + v_{t-1}(i-1)\right\}, \; i \geq 1, t \geq 1 \\ 0, \text{ otherwise} \end{cases}$$

where λ_t^ℓ denotes the probability of a booking request from fare class ℓ ($\ell = 1, 2, ...L$) in period t; λ_t^0 the probability of no arrival; x^ℓ the fare charged for class ℓ ($\ell = 1, 2, ...L$). The maximum total expected revenue, $u_t^\ell(i,n)$, with i units available stock with t periods to go under the condition that the fare class ℓ customer's initial request and *subsequent upgrade requests $\ell+1, \ell+2, ..., \ell+n-1$ are rejected, is given by*

$$\begin{cases} r_n^\ell \max\left\{u_t^\ell\left(i, n+1\right), x^{\ell+n} + v_{t-1}(i-1)\right\} + \left(1 - r_n^\ell\right) v_{t-1}(i), & \text{if } n < L - \ell, \\ r_{L-\ell}^\ell \left[x_L + v_{t-1}(i-1)\right] + \left(1 - r_{L-\ell}^\ell\right) v_{t-1}(i), & \text{if } n = L - \ell. \end{cases}$$

♣

The dynamic models discussed above can be characterized as follows: the rationing policy can be controlled using either a set of critical inventory levels or a set of critical decision periods. For a hotel, a booking request from a discounted class in a decision period is accepted if the remaining decision periods is less than or equal to the critical decision period for the current available rooms and rejected otherwise. On the other hand, in a decision period, a booking request from a discounted class is accepted if the number of current available rooms is greater than or equal to the critical booking limit for that period and rejected otherwise. Similar research on this kind of policy has also been done by Banerjee and Viswanathan [1], Bitran and Gilbert [6], You [52], and Zhao and Zheng [53].

One of the shortcomings of these models is the assumption that the arrival rates of customers do not vary in time which can be unrealistic in some real applications. Slyke and Young [44] model the random number of arrivals in $[0, t]$, as a time-dependent Poisson process $N(t)$ with $\lambda_k(t)$ as the Poisson arrival rate of the kth type of requests as a function of time.

Slyke and Young's Rule: *Accept a request from fare class k ($k = 1, 2, \ldots, K$) if and only if $b_k + f(y-1, t) - f(y, t) > 0$. Here, b_k is the positive real valued benefit* *of a type k request and $f(y,t)$ the maximum total expected sum of the benefits of the accepted requests given that y units of inventory $(0 \leq y \leq W)$ and t time $(0 \leq t \leq T)$ remain. This function is given as:*

$$f(y,t) = \int_0^t \lambda(s) e^{-\int_s^t \lambda(\tau)\, d\tau} \sum_{k=1}^{K} p_k(s) \max\{b_k + f(y-1,s) - f(y,s), 0\}\ ds$$

with $f(y,0) = 0$, for all y; $y(0,t) = 0$, for all $t \in [0,T]$, and $\lambda(s) = \sum_{k=1}^{K} \lambda_k(s)$, $p_k(s) = \lambda_k(s)/\lambda(s)$ is the probability that, given one request arrived, it belongs to class k. ♣

Slyke and Young [44] prove that if $f(y,t)$ is absolutely continuous in t and $\lambda_k(t) > 0$ for all k and t, then $f(y,t)$ is strictly monotone increasing in t. This implies that there exist a critical time (threshold point) t^* for a given inventory level and a fare class k, where the optimal policy is to reject that request before t^* and accept otherwise. This solution is very similar to the policies in discrete time we described above. In the two-price model, Feng and Gallego [15] give the exact solution for such threshold point for any given inventory level. Feng and Xiao [16] extend the results obtained by Feng and Gallego to a more general situation which considers multiple fare classes.

Feng and Xiao's Rule: *For a request in fare class p_k $(1 \leq k \leq K)$, with inventory level n $(1 \leq n \leq M)$, we accept the request until the threshold is reached. Such threshold, x_n^k, is determined recursively by*

$$x_n^k = \inf\left\{0 \leq t \leq T : \int_t^T L_k(s,n) e^{-\lambda_k(s-t)}\, ds > 0\right\}$$

where

$$L_k(t,n) = \frac{\partial V_{k+1}(t,n)}{\partial t} + \lambda_k \left[\bar{V}_k(t,n) + p_k\right],$$

$$\bar{V}_k(t,n) = \begin{cases} \int_t^T L_k(s,n)\, e^{-\lambda_k(s-t)} ds, & \text{if } t > x_n^k \\ 0, & \text{Otherwise.} \end{cases}$$

Here, $V_k(t,n)$ represents the maximum expected revenue over the interval $[t,T]$ given the inventory level $n(t)=n$, which is achieved by keeping the current price p_k until x_n^k, and $\bar{V}_k(t,n) = V_k(t,n) - V_{k+1}(t,n)$. ♣

Brumelle and Walczak [8] present a model with a continuous-time, multi-period arrival process with multiple demands following a Markov process, and where decisions are made at the end of each period. They consider the situation that a customer can request more than one unit of product and that request may be splittable, i.e., the requests can be partially satisfied.

For many hotels, booking requests are only recorded when the customer accepts the price hotel offers. No purchase decision can be observed from the available stored data. Thus, it is sometimes difficult to distinguish between periods with no arrival and periods in which there was an arrival and his/her booking request was rejected. Talluri and van Ryzin [47] overcome this incomplete data problem by applying the expectation-maximization (EM) method to solve the discrete-time revenue management problem. They assume in each period that there is at most one arrival with probability of λ. There are n fare classes and $N = \{1, 2, ..., n\}$ denotes the entire set of fare classes where each fare class $j \in N$ has an associated revenue r_j. Hence, in each equal-length period t, a subset $S \subseteq N$ of fare classes must be decided to open. When the fares S are offered, the probability that a customer chooses fare class product $j \in S$ is $P_j(S)$ where $P_0(S)$ denotes the no-purchase probability.

Talluri and Van Ryzin's Rule: *At any given time t $(0 \le t \le T)$, one set of fare classes S should be opened and such a set should satisfy*

$$S = \arg\max_{S \subseteq N} \{\lambda (R(S) - Q(S) \Delta V_{t-1}(x))\}$$

where $Q(S) = \sum_{j \in S} P_j(S) = 1 - P_0(S)$ is the total probability of purchase, $R(S) = \sum_{j \in S} P_j(S) r_j$ is the total expected revenue from offering set S and $\Delta V_{t-1}(x) = V_{t-1}(x) - V_{t-1}(x-1)$ the marginal cost of capacity. The maximum expected revenue, $V_t(x)$, obtained from period $t, t-1, \ldots, 0$, given that there are x inventory units remaining at time t is

$$V_t(x) = \max_{S \subseteq N} \left\{ \sum_{j \in S} \lambda P_j(S) (r_j - \Delta V_{t-1}(x)) \right\} + V_{t-1}(x)$$

$V_t(x) = \max_{S \subseteq N} \left\{ \sum_{j \in S} \lambda P_j(S) (r_j - \Delta V_{t-1}(x)) \right\} + V_{t-1}(x) with\ V_t(0) = 0,$ $t = 1, 2, ..., T$ *and* $V_0(x) = 0, x = 1, 2, ..., C$. ♣

The most significant characteristic of this model is that the optimal sets of fare classes are only those efficient sets. In many cases, this observation reduces the number of sets we have to consider. Moreover, it shows that these efficient sets can be sequenced in a natural way and that the more capacity we have, the higher the set we should select. But the limitation of this model is in its difficulty of estimating the probability of $P_j(S)$. There are many potential strategic behaviors which can affect customer's choice, e.g., a customer's choice may depend on the strategies of other customers; or his or her past choices or past events in the system, etc.

2.3 Game Theoretic Models for Hotel Room Allocations

As we discussed in Section 1.3, game theory should be applied for the simultaneous or sequential decision-making if one hotel's revenue is affected by the rationing policies of another hotel's rooms. To the best of our knowledge, there is no related literature addressing the room inventory control problem using game theoretic tools even though game theory has found frequent use in problems involving competition in supply chains. For example, in one of the earliest papers in this field, Parlar [40] has modelled the substitutable product inventory problem with two newsvendors whose profits are determined as a function of both players' order quantities u and v. The newsvendors attempt to maximize their expected profits $J_1(u,v)$ and $J_2(u,v)$, respectively, where the first retailer's objective is given as

$$\begin{aligned} J_1(u,v) &= (s_1+p_1)\left[\int_0^u xf(x)dx + u\int_u^\infty f(x)dx\right] - p_1E(X) + q_1\int_0^u (u-x)f(x)dx \\ &\quad +(s_1-q_1)\int_0^u\left[\int_v^B b(y-v)g(y)dy + \int_B^\infty (u-x)g(y)dy\right]f(x)dx - c_1u, \end{aligned}$$

with $f(x)$ and $g(y)$ as the demand densities faced by each retailer, a and b $(0 \leqslant a, b \leqslant 1)$ are the substitution rates of the retailer's products when they are sold out; s_1, c_1, q_1 and p_1 are the unit selling price, purchase cost, salvage value and shortage penalty cost for first player's product, and $B = [(u-x)/b] + v$ and $A = [(v-y)/a] + u$. For this model Parlar proved the existence and uniqueness of the Nash equilibrium and showed that cooperation between two players can increase their profits.

The importance of this work is that it establishes the existence of a unique Nash equilibrium. The essential differences between the RM games and newsvendor games reside in the capacity and variation of price: the former accounts for a fixed capacity and offers different prices for the same product (i.e., airline seat, hotel room), which are seldom considered by the latter.

More recently, Netessine and Shumsky [37] presented a seat inventory control problem in which two airlines compete for passengers on the same flight leg. Their model cannot guarantee an equilibrium, because they assume the airline's demand depends on the booking limit, which makes the problem more complicated than any newsvendor game problem presented in the literature.

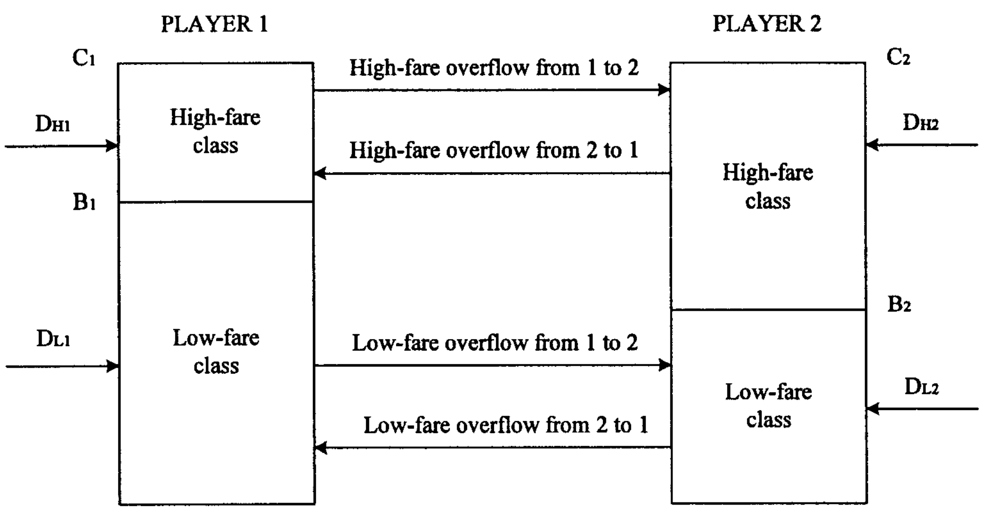


Figure 2. Overflow process of two-player two-fare class game.

In general, such a problem can be summarized as two-player two-fare class non-zero-sum game problem. Each player i has capacity C_i and there are two fare classes available for the customers: low fare and high fare. If either type of customer is denied by one player, the customer will attempt to purchase a substitutable product from another player. Figure 2 shows the overflow processes.

The total revenue for player i is

$$\pi_i = E\left[p_{Li} \min\left(D_{Li}^T, B_i\right) + p_{Hi} \min\left(D_{Hi}^T, R_i\right)\right],$$

where $D_{Li}^T = D_{Li} + (D_{Lj} - B_j)^+$, total demand for low-fare demands for player i, $i, j = 1, 2$ and $i \neq j$; $R_i = C_i - \min\left(D_{Li}^T, B_i\right)$, the number of seats available for high-fare class customers for player i, $i = 1, 2$ and $D_{Hi}^T = D_{Hi} + (D_{Hj} - R_j)^+$ which is the total demand from high-fare class. Netessine and Shumsky [37] provide sufficient conditions for a pure-strategy Nash equilibrium and also present the results for a cooperative situation. Even though they evaluated both the direction and magnitude of revenue losses due to competition, they do not clearly take into account the cost savings in a cooperative situation and their effects on the decisions. In addition, their model assumes that the customers who are rejected by one player will transfer to the other one (transfer rate is equal to 1). In the next chapter, we will also study a single-period static game model. However, our model will relax the assumption on the transfer rate (it can be any number between 0 and 1). Moreover, we will analyze the cost savings in the cooperative situation and discuss the effects of some parameters (e.g., rejection cost) on the optimal rationing policies.

Chapter 3 Static Game Model for Hotel Room Allocations

In this Section, we study a two-player, two-fare class room inventory control problem arising in hotel business. The booking requests for a given date from each fare class in each hotel are assumed to be random and independent. Each hotel is assumed to have complete information of the prices, booking request distributions, costs, and all other parameter values related to the rooms of both hotels. In order to maximize the total expected revenue (objective), each hotel has to decide the maximum level of rooms (*booking limit* [36]) to be sold at a lower price. Such a decision complements the minimum level (*protection level*) which should be reserved for high-fare customers who prefer better quality rooms. If either type of customer is rejected at one hotel, a fraction of these customers will attempt to book a room from the other hotel. We define these customers as *transfer customers.* However, the rest of the rejected customers are totally lost to both hotels. Since the booking limits decided by both hotels affect their respective objectives, the hotel RM problem we are considering should be modeled using a game-theoretic framework.

3.1 The Model

Before we present the objective functions for the two hotels, it would be helpful to summarize the underlying assumptions. Two hotels dealing with substitutable rooms are assumed as the two players which are denoted by Pi, $i = 1, 2$. Two fare classes (L : Low and H : High) have been set a priori. Booking requests for different fare classes are assumed to be independent random variables with continuous probability distribution functions. If one hotel has excess capacity, the excess rooms will be filled partly or fully by the other hotel according to the transfer rate. Furthermore, it is assumed that there are neither no-shows nor cancellations by accepted customers (obviating the need for overbooking). Finally, in order to simplify our model further we also assume that there are no buy-ups to the high-fare class by rejected low-fare customers.

We use the following notations ($i = 1, 2,\ K = H, L$):

- C_i: capacity of Pi,
- b_{iL}: booking limit chosen by Pi (our decision variables),
- b_{iH}: protection level chosen by Pi,
- X_{iK}: random booking request from fare class K customers for a given date accommodation in Pi, with probability density function (p.d.f.) $f_{iK}(x_{iK})$, cumulative distribution function (c.d.f.) $F_{iK}(x_{iK}) = \int_0^{x_{iK}} f_{iK}(t_{iK})\ dt_{iK}$ and complementary c.d.f. $\bar{F}_{iK}(x_{iK}) = 1 - F_{iK}(x_{iK})$,
- r_{iK}: fare paid per night by Pi's K-fare class customer,
- q_{iK}: rejection penalty cost per K-fare class customer incurred on Pi,
- u_{iK}: the fraction of Pi's rejected K-fare class booking requests which switch to the other hotel,
- $\Pi_i(b_{1L}, b_{2L})$: random revenue for Pi, with $J_i = E(\Pi_i)$.

We note that since the hotel's capacity is fixed, *protection level* and *booking* *limit* complement each other, that is, the sum of these levels equals the hotel's capacity. Thus, we choose one of these the decision variable and express the other one in terms of the chosen decision variable and the fixed capacity. In our study, we use *booking limit* as our decision variable, so that *protection level* can be expressed as $C_i - b_{iL}$. However, in some cases, it is more convenient to use both of them to make the expressions more compact and simpler. Therefore, we define *protection level* for Pi to be b_{iH} $(i = 1, 2)$.

Because of the existence of transfer customers, each player's revenue function will depend on not only its own booking limit but also on the other player's *booking limit.* Thus, game theory should be used for analyzing the optimal booking decisions for both players.

3.1.1 Objective Functions

We denote J_{iK} as Pi's expected revenue from K-fare class customers, where $i = 1, 2$, $K = H, L$. We begin by analyzing $P1$'s expected revenue generated by low-fare customers. For any given b_{1L} and b_{2L}, there are four mutually exclusive cases in which transfer happens between $P1$ and $P2$. Therefore, $P1$'s revenue function in each case can be expressed by the following:

(1) $x_{1L} \leq b_{1L}$, $x_{2L} \leq b_{2L} : \pi_{1L}^{1} = r_{1L}x_{1L}$

(2) $x_{1L} \leq b_{1L}$, $x_{2L} \geq b_{2L}$:

$$\begin{aligned} \pi_{1L}^{2} = \quad & r_{1L}x_{1L} + r_{1L}\min\left[u_{2L}\left(x_{2L} - b_{2L}\right), b_{1L} - x_{1L}\right] \\ & -q_{1L}\max\left[0, u_{2L}\left(x_{2L} - b_{2L}\right) - \left(b_{1L} - x_{1L}\right)\right] \end{aligned}$$

In this case, $P1$ has excess rooms and $P2$ is in shortage. $P1$ will then accept any transfer customer from $P2$ until its booking limit is reached. Hence, the revenue and penalty cost of transfer customers from $P2$ are expressed with the second and third terms respectively.

(3) $x_{1L} \geq b_{1L}$, $x_{2L} \leq b_{2L}$: $\pi_{1L}^3 = r_{1L}b_{1L} - q_{1L}\left(x_{1L} - b_{1L}\right)$

Here, there are no transfer customers from $P2$ to $P1$, and $P1$'s low-fare class has been closed. Hence, the excess booking requests, $x_{1L}-b_{1L}$, will be penalized with q_{1L} per room.

(4) $x_{1L} \geq b_{1L}$, $x_{2L} \geq b_{2L}$: $\pi_{1L}^4 = r_{1L}b_{1L} - q_{1L}\left(x_{1L} - b_{1L}\right) - u_{2L}q_{1L}\left(x_{2L} - b_{2L}\right)$ Since the low-fare classes in both hotels have been closed, all of $P1$'s own low-fare customers and transfer customers from $P2$ to $P1$ must be rejected which cost $q_{1L}\left(x_{1L} - b_{1L}\right)$ and $u_{2L}q_{1L}\left(x_{2L} - b_{2L}\right)$, respectively.

The expected revenue from $P1$'s low-fare customers can be obtained by integrating the four revenue expressions above over their respective regions. Using a similar procedure as discussed above, we can also obtain the expected revenue from $P1$'s high-fare customers. Thus, the total expected revenue of $P1$ is given as $J_1 = J_{1L}+J_{1H}$. Analogously, we can obtain $P2$'s objective J_2. After some simplifications, the total expected revenue of Pi, $i = 1, 2$, is found as follows:

$$J_i = \sum_{K=L,H} \left\{ \int_0^{b_{iK}} r_{iK}x_{iK}F_{jK}\left(b_{jK}\right) f_{iK}\, dx_{iK} \right.$$

$$+ \int_{b_{jK}}^{B_{jK}} \int_0^{b_{iK}} r_{iK}\left(x_{iK} + b_{iK} - M_{iK}\right) f_{iK}f_{jK}\, dx_{iK}\, dx_{jK}$$

$$
\begin{aligned}
&+\int_{b_{jK}}^{B_{jK}} \int_{b_{iK}}^{\infty} \left[r_{iK} b_{iK} - q_{iK}\left(x_{iK} - M_{iK}\right)\right] f_{iK} f_{jK}\, dx_{iK}\, dx_{jK} \\
&+\int_{B_{jK}}^{\infty} \int_{0}^{\infty} \left[r_{iK} b_{iK} - q_{iK}\left(x_{iK} - M_{iK}\right)\right] f_{iK} f_{jK}\, dx_{iK}\, dx_{jK} \\
&+\int_{b_{iK}}^{\infty} \left[r_{iK} b_{iK} - q_{iK}\left(x_{iK} - b_{iK}\right)\right] F_{jK}\left(b_{jK}\right) f_{iK}\, dx_{iK} \Bigg\}
\end{aligned} \tag{4}
$$

where $b_{iH} = C_i - b_{iL}$, $M_{iK} = b_{iK} - u_{jK}\left(x_{jK} - b_{jK}\right)$, and $B_{jK} = b_{jK} + \left(b_{iK} - x_{iK}\right)/u_{jK}$ for $i, j = 1, 2$ and $i \neq j$.

3.1.2 Best Response (BR) Functions

With each player's objective function given by (4), we now examine the optimal decision (i.e., best response) of each player in response to an arbitrary decision by the other one. For instance, suppose $P2$ announces her low-fare booking limit b_{2L}. Given this, $P1$ can determine his best response $b_{1L}^{R}(b_{2L})$ to maximize his objective function. These results will be helpful when we consider different solution concepts using Nash, leader-follower Stackelberg, and cooperative strategies.

Let us first examine the properties of J_i, $i = 1, 2$ for further information.

Lemma 1 *Pi's objective function is strictly concave in b_{iL} for $i = 1, 2$.*

Proof. By differentiating $J_1(b_{1L}, b_{2L})$ with respect to b_{1L}, after some simplification we find

$$\frac{\partial J_1}{\partial b_{1L}} = V_1(b_{1L}, b_{2L}) = (r_{1L}+q_{1L})\left[\int_0^{b_{1L}}\int_{N_{2L}}^{\infty} f_{1L}f_{2L}\ dx_{2L}\ dx_{1L} + \bar{F}_{1L}(b_{1L})\right]$$
$$-(r_{1H}+q_{1H})\left[\int_0^{b_{1H}}\int_{N_{2H}}^{\infty} f_{1H}f_{2H}\ dx_{2H}\ dx_{1H} + \bar{F}_{1H}(b_{1H})\right], \quad (5)$$

where $N_{2K} = b_{2K} + (b_{1K} - x_{1K})/u_{2K}$. Next, we obtain the second derivative of J_1 with respect to b_{1L} as

$$\frac{\partial^2 J_1}{\partial b_{1L}^2} = -\sum_{K=L,H}(r_{1K}+q_{1K})\left[\int_0^{b_{1K}} \tfrac{1}{u_{2K}} f_{1K}f_{2K}(N_{2K})\ dx_{1K} + f_{1K}(b_{1K})F_{2K}(b_{2K})\right]. \quad (6)$$

It is not difficult to see that $\partial^2 J_1/\partial b_{1L}^2 < 0$ for any $b_{1L} \in [0, C_1]$. Similarly, we can show that $\partial^2 J_2/\partial b_{2L}^2 < 0$ for any $b_{2L} \in [0, C_2]$. Thus, J_i is strictly concave in Pi's own decision variable b_{iL}, $i = 1, 2$. ■

Let us define S_{iK} ($i = 1, 2$ and $K = L, H$) to be the probability of "spill" (an event that unsatisfied booking request occurs; see, McGill and van Ryzin [33]). It is not difficult to see that Pi's K-fare class customers will spill in two cases: 1) $X_{iK} > b_{iK}$; and 2) $X_{iK} + u_{jK}(X_{jK} - b_{jK}) > b_{iK}$ with $X_{iK} \le b_{iK}$. Thus, the spill rate of Pi's K-fare customers can be expressed as:

$$S_{iK} = \Pr\{X_{iK} > b_{iK}\} + \Pr\{X_{iK} + u_{jK}(X_{jK} - b_{jK}) > b_{iK} \text{ and } X_{iK} \le b_{iK}\}.$$

Integrating over the two respective regions, we obtain

$$S_{iK} = \int_0^{b_{iL}}\int_{N_{jL}}^{\infty} f_{iL}f_{jL}\ dx_{jL}\ dx_{iL} + \bar{F}_{iL}(b_{iL}), \quad (7)$$

where $N_{jK} = b_{jK} + (b_{iK} - x_{iK})/u_{jK}$, $i,j = 1,2$, $K = L,H$ and $i \neq j$. Thus, the first order partial derivative, i.e. V_i can be expressed in terms of S_{iK} as

$$V_i = (r_{iL} + q_{iL}) S_{iL} - (r_{iH} + q_{iH}) S_{iH}, \quad i = 1,2. \tag{8}$$

Lemma 2 *$V_i(b_{1L}, b_{2L}) = 0$, $i = 1,2$, is a strictly decreasing curve in the (b_{1L}, b_{2L}) plane.*

Proof. Note that it is impossible to express b_{2L} as an explicit function of b_{1L}. However, we can use implicit differentiation to obtain the derivative of $V_1 = 0$ with respect to b_{1L}, which we denote by b_1'. We immediately observe, using chain rule, that

$$b_1' = -\left(\frac{\partial V_1}{\partial b_{1L}}\right) / \left(\frac{\partial V_1}{\partial b_{2L}}\right) < 0$$

since

$$\frac{\partial V_1}{\partial b_{2L}} = -\sum_{K=L,H} u_{2K} (r_{1K} + q_{1K}) \int_0^{b_{1K}} f_{1K} f_{2K} (N_{2K}) \, dx_{1K} < 0, \quad \text{and} \quad \frac{\partial V_1}{\partial b_{1L}} < 0$$

from (6). Thus, $V_1 = 0$ is a strictly decreasing curve in the (b_{1L}, b_{2L}) plane. Similarly, defining b_2' as the derivative of $V_2 = 0$ with respect to b_{1L}, implicit differentiation gives

$$b_2' = -\left(\frac{\partial V_2}{\partial b_{1L}}\right) / \left(\frac{\partial V_2}{\partial b_{2L}}\right) < 0$$

since, from symmetry,

$$\frac{\partial V_2}{\partial b_{1L}} = -\sum_{K=L,H} u_{1K} (r_{2K} + q_{2K}) \int_0^{b_{2K}} f_{1K} (N_{1K}) f_{2K} \, dx_{2K} < 0, \quad \text{and} \quad \frac{\partial V_2}{\partial b_{2L}} < 0$$

Thus, $V_2 = 0$ is also a strictly decreasing curve in the (b_{1L}, b_{2L}) plane. This proves the lemma. ■

We now want to determine whether Pi's best response to Pj's decision, i.e., $b_{iL}^R(b_{jL})$ (where $b_{jL} \in [0, C_j]$, $i, j = 1, 2$ and $i \neq j$), can always be obtained by solving $V_i = 0$. By Lemma 1, we note that Pi's objective function is strictly concave in his/her own decision variable for any given b_{jL}. However, mathematically, for a given b_{jL}, J_i may be either strictly increasing concave or decreasing concave in b_{iL}. Therefore, the optimal solution of b_{iL} which maximizes J_i can be found on the boundary if such cases occur. The optimal solution resides in $(0, C_i)$ only when J_i is not monotone in b_{iL}. Then the best response of Pi can be obtained by solving $V_i = 0$.

The following theorem provides the exact form of the best response (BR) functions.

Theorem 1 *Pi's best response $b_{iL}^R(b_{jL})$ $(i, j = 1, 2$ and $i \neq j)$ is given by*

$$b_{iL}^R(b_{jL}) = \begin{cases} 0, & \text{if } \xi_{jL} \leq b_{jL} \leq C_j; \\ b_{iL}^*, & \text{if } 0 \leq b_{jL} \leq \xi_{jL}. \end{cases} \tag{9}$$

where

$$\xi_{jL} = \begin{cases} b_{jL}\text{-axis intercept of } V_i = 0, & \text{if } V_i = 0 \text{ intersects with } b_{jL}\text{-axis;} \\ C_{jL}, & \text{if } V_i = 0 \text{ intersects with } b_{jL} = C_{jL}; \\ 0, & \text{if } V_i < 0 \text{ for any } b_{jL} \in [0, C_j], \end{cases} \tag{10}$$

and b_{iL}^* can be obtained by solving $V_i = 0$.

Proof. From (7), we find that $S_{iH}(C_i, b_{jL}) = 1$. Therefore, it is not difficult to see $V_i(C_i, b_{jL}) = (r_{iL} + q_{iL}) S_{iL}(b_{iL}, C_j) - r_{iH} - q_{iH} < 0$, which means for any b_{jL}, J_i is not increasing in b_{iL} at C_j. However, J_i may be a strictly decreasing concave function ($V_i < 0$) for any b_{iL}. In this case, the best response $b_{iL}^R(b_{jL})$ should always be zero [see Scenario (a) in Figure 3]. On the other hand, if $\xi_{jL} > 0$ [Scenarios (b) and (c) in Figure 3], due to the strictly decreasing property of $V_i = 0$ in the (b_{1L}, b_{2L}) plane, the optimal solution for any $b_{jL} \in [0, \xi_{jL}]$ can be obtained by solving $V_i = 0$. In other words, for any $b_{jL} \in [0, \xi_{jL}]$, we can use the curve of $V_i = 0$ as Pi's BR curve. As for $b_{jL} \in [\xi_{jL}, C_j]$, V_i is always less than zero. Therefore, the best response to any b_{jL} belonging to this region should be zero too. ■

Referring to (8), we recall that V_i $(i = 1, 2)$ can be expressed in terms of the spill rates S_{iL} and S_{iH} of the two fare class customers of Pi. Since Pi's best response to an announcement of Pj is determined in terms of V_i, this implies that the spill rates play an important role in determining the best response of a player.

Remark 1 *We note that the BR curve is non-increasing in the (b_{1L}, b_{2L}) plane. It is optimal for one player to decrease the booking limit if the other one increases the booking limit, and vice versa. We see in Figure 3 that ξ_{iL}^i, the upper bound of best response of Pi, is always less than C_i. (In this figure, δ_{iL}^i denotes the b_{iL}-axis intercept of Pi's BR curve.) Therefore, in practice, the hotel manager should always set a booking limit less than the capacity. This is reasonable since a high-fare customer always generates more revenue if accepted, and incurs more cost if rejected.*

Therefore, the hotel should always reserve some rooms for high-fare customers if there is any possibility of booking requests from them. ◁

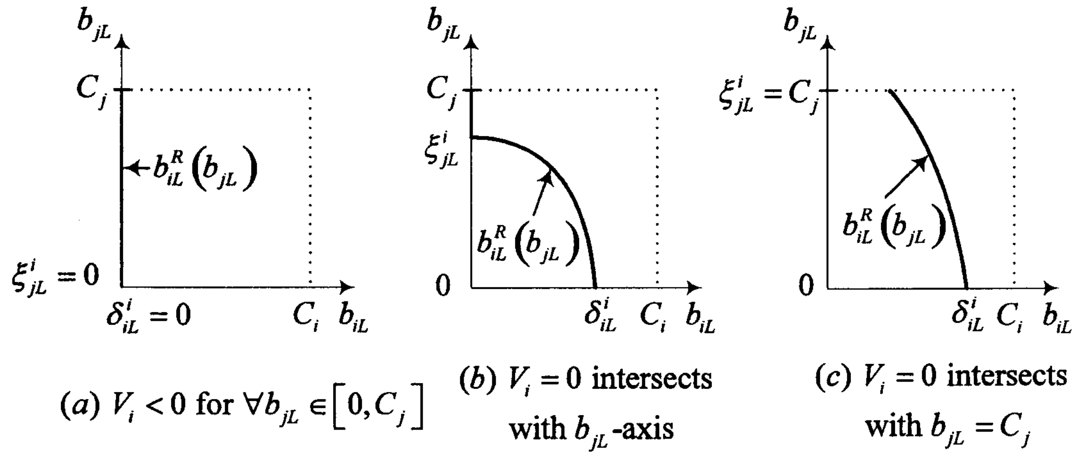


Figure 3. Best response curves in three scenarios where δ^i_{iL} is the b_{iL}-axis intercept of Pi's BR curve, $i=1,2$. Also, ξ^1_{iL} is the smallest value of b_{jL} for which b_{iL} assumes the smallest value.

Example 1 *Our goal in this example is to demonstrate the structure of the BR curve for one of the hotels, say, P1. We assume $C_1=40$ and $C_2=45$, as the capacities of hotels P1 and P2, respectively. The room rates, penalty costs, and transfer rates of K-fare class customers in Pi ($K=L,H$ and $i=1,2$) are given in Table 3.*

	Low-fare ($K=L$)			High-fare ($K=H$)		
	r_{iL}	q_{iL}	u_{iL}	r_{iH}	q_{iH}	u_{iH}
P1	\$99	\$30	0.6	\$159	\$70	0.8
P2	\$105	\$35	0.65	\$165	\$75	0.8

Table 3. Prices, rejection costs, and transfer rates of $P1$ and $P2$

	P1		P2	
	λ_{1L}	λ_{1H}	λ_{2L}	λ_{2H}
Scenario 1	35	50	35	50
Scenario 2	35	20	35	50
Scenario 3	35	5	35	50

Table 4. Booking request expectations in three scenarios

The random booking requests of K-fare class customers for Pi's rooms are represented by the exponential r.v. X_{iK} with density $f_{iK} = \frac{1}{\lambda_{iK}} \exp(-\frac{x_{iK}}{\lambda_{iK}})$, $i = 1, 2$ and $K = L, H$ (and mean λ_{iK}). In order to show the different BR functions as given in Theorem 1, we generate three scenarios where we vary only λ_{1H} and fix all other parameters. Table 4 provides the booking request expectation (λ_{iK}) of each fare class of the two players.

We obtain the BR function of $P1$ in each scenario according to Theorem 1. As depicted in Figure 4(a) of Scenario 1, if $\lambda_{1H} = 50$, then $P1$'s best response is always $b_{1L} = 0$ which implies that $b_{1H} = C_1 = 40$; that is, every room in hotel 1 is protected for high-fare customers. From Figure 4(b) of Scenario 2, we see that when $\lambda_{1H} = 20$, a moderate level, then $P1$'s best response of b_{1L} will be between 0 and 10 as long as P2 decides to choose a booking limit of $b_{2L} \in [0, 40]$. If $b_{2L} > 40$, then $b_{1L} = 0$. Finally, if $\lambda_{1H} = 5$, a low value, Figure 4(c) of Scenario 3 show that regardless of which value b_{2L} is chosen by P2, it is always optimal for $P1$ to reserve some rooms (between 6 and 23) for the low-fare customers. ♦

From the above example, we note that if all other parameters are fixed in our basic model, changing λ_{1H} affects the structure of $P1$'s BR curve. We will present a

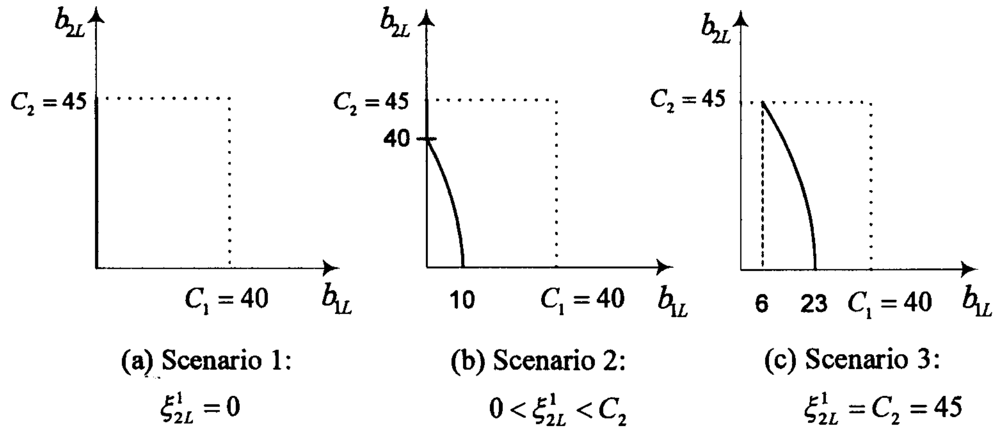


Figure 4. $P1$'s BR curves in three scenarios : (a) $\lambda_{1H} = 50$; (b) $\lambda_{1H} = 20$; (c) $\lambda_{1H} = 5$.

sensitivity analysis to show the major factors affecting the structure of BR functions in Section 3.5. In this chapter, we use Maple 10 to carry out the results for all of the numerical computations.

3.2 Nash Equilibrium

In a non-cooperative environment, two players make decisions simultaneously. In this situation, players are assumed to be "rational", i.e., that one would not lower his/her objective functions for the sole purpose of inflicting damage on the opponent. Thus, in such situations the solution concept that is used is known as the Nash strategy. Mathematically, it is a pair (b^N_{1L}, b^N_{2L}) such that

$$\begin{aligned} J_1(b^N_{1L}, b^N_{2L}) &\geq J_1(b_{1L}, b^N_{2L}), \text{ for all } b_{1L} \\ J_2(b^N_{1L}, b^N_{2L}) &\geq J_2(b^N_{1L}, b_{2L}), \text{ for all } b_{2L}. \end{aligned} \tag{11}$$

This strategy leads to what is known as the Nash equilibrium, as it ensures that Pi $(i = 1, 2)$ receives at least $J_i(b^N_{1L}, b^N_{2L})$ if he uses (b^N_{1L}, b^N_{2L}) and he will not receive more than this amount if he deviates from it unilaterally. The best response curves of both players were found by Theorem 1. Therefore, the Nash equilibrium exists if and only if these two response curves intersect in the (b_{1L}, b_{2L}) plane where $b_{iL} \in [0, C_i]$. According to Nikaido and Isora [39], we know that if each player's objective function is continuous in all decision variables and concave in its own decision variable, the game is convex and admits at least one Nash equilibrium. These conditions obviously hold for our problem (see Lemma 1). Hence, in order to see whether there is only one Nash equilibrium, let us examine the properties of $V_i = 0$ further.

Lemma 3 *The derivative of $V_1 = 0$ with respect to b_{1L} is always less than the derivative of $V_2 = 0$ with respect to b_{1L}.*

Proof. Referring to Lemma 2, the implicit derivative of $V_1 = 0$ with respect to b_{1L} is

$$b_1' = -\frac{\sum_{K=L,H} (r_{1K} + q_{1K}) \left[\int_0^{b_{1K}} \frac{1}{u_{1K}} f_{1K} f_{2K} (N_{2K}) \ dx_{1K} + f_{1K}(b_{1K}) F_{2K}(b_{2K}) \right]}{\sum_{K=L,H} u_{2K} (r_{1K} + q_{1K}) \int_0^{b_{1K}} f_{1K} f_{2K} (N_{2K}) \ dx_{1K},} \tag{12}$$

and the derivative of $V_2 = 0$ with respect to b_{1L} is,

$$b_2' = -\frac{\sum_{K=L,H} u_{1K} (r_{2K} + q_{2K}) \int_0^{b_{2K}} f_{1K} (N_{1K}) f_{2K} \ dx_{2K}}{\sum_{K=L,H} (r_{2K} + q_{2K}) \left[\int_0^{b_{2K}} \frac{1}{u_{2K}} f_{1K} (N_{1K}) f_{2K} \ dx_{2K} + f_{2K}(b_{2K}) F_{1K}(b_{1K}) \right]} \tag{13}$$

where $i, j = 1, 2$, $i \neq j$ and $K = L, H$. It is easy to see that $b_1' < -1$ and $b_2' > -1$.

Therefore, b'_1 is always less than b'_2. ■

Lemma 3 has a useful interpretation: In order to maximize the expected revenue, $P1$ should decrease b_{1L} by more than one unit if $P2$ increases her low-fare booking limit by one unit, and vice versa. Lemma 3 is crucial because it is an important sufficient condition for existence of unique Nash equilibrium.

Theorem 2 *The game admits a unique Nash equilibrium which can be expressed as*

$$(b_{1L}^N, b_{2L}^N) = \begin{cases} (0, \delta_{2L}^2), & \text{if } \xi_{2L}^1 \leq \delta_{2L}^2 \\ (\delta_{1L}^1, 0), & \text{if } \xi_{2L}^1 > \delta_{2L}^2 \text{ and } \delta_{1L}^1 \geq \xi_{1L}^2 \\ (b_{1L}^*, b_{2L}^*), & \text{if } \xi_{2L}^1 > \delta_{2L}^2 \text{ and } \delta_{1L}^1 < \xi_{1L}^2, \end{cases} \tag{14}$$

where (b_{1L}^*, b_{2L}^*) *can be obtained by solving* $V_1(b_{1L}^*, b_{2L}^*) = 0$ *and* $V_2(b_{1L}^*, b_{2L}^*) = 0$ *(see Figure* 5 *which corresponds to this case where* $\xi_{2L}^1 > \delta_{2L}^2$ *and* $\delta_{1L}^1 < \xi_{1L}^2$*).*

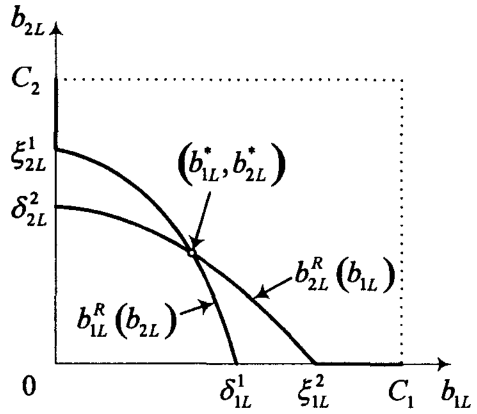


Figure 5. Nash equilibrium in Scenario 3: $\xi_{2L}^1 > \delta_{2L}^2$ and $\delta_{1L}^1 < \xi_{1L}^2$.

Proof. From Lemma 2 and Lemma 3, we find that $V_1 = 0$ and $V_2 = 0$ are monotone-decreasing curves and the implicit derivative b_1' is strictly less than b_2'. We also know that the derivative of $b_{1L}^R = 0$ (a vertical line) with respect to b_{1L} is infinite [from Figure 3(a)] and the derivative of $b_{2L}^R = 0$ (a horizontal line) with respect to b_{1L} is zero [from Figure 3(b)]. Thus, the implicit derivative of $P1$'s best response function with respect to b_{1L} is also strictly smaller than that of $P1$'s best response function. In addition, Pi's best response to any b_{jL} ($i, j = 1, 2$ and $i \neq j$) is unique and continuous over $[0, C_j]$ (see Theorem 1). Accordingly, the best response curves of the two players intersect only once in (b_{1L}, b_{2L}) plane. It indicates that the game admits a unique Nash equilibrium.

As depicted in Figure 3, there are three types of best response curves for each player. Hence, there is a total of nine situations in which we can combine different best response curves of the two players. It is not difficult to show that the two best response curves intersect either on the b_{iL} axis or inside the first quadrant. The b_{1L}- and b_{2L}-coordinates of the intersection of two curves, which is the Nash equilibrium, are determined by the relations between ξ_{iL}^1 and δ_{iL}^2. (Note here that, as we see in Figure 3, ξ_{iL}^1 is the smallest value of b_{jL} for which b_{iL} assumes the smallest value.) For example, if $\xi_{2L}^1 > \delta_{2L}^2$ and $\delta_{1L}^1 < \xi_{1L}^2$, then $V_1 = 0$ and $V_2 = 0$ will admit a unique intersection point, (b_{1L}^*, b_{2L}^*). Similarly, we can obtain the same results of the other cases shown in (14). ■

Example 2 *Here, we use the same values as in Table* 3 *of Example* 1 *for prices, costs, and transfer rates. The goal in this example is to demonstrate the Nash equilibrium in different situations. Since each player's BR function could be one of the three types* *as given by Theorem 1, there is a total of nine situations in which the two BR curves intersect in (b_{1L}, b_{2L}) plane. In this example, we only present three of them and point out that in all cases, the Nash equilibrium must be one of the three types as given by Theorem 2.*

	$(\lambda_{1L}, \lambda_{1H})$	$(\lambda_{2L}, \lambda_{2H})$	(b_{1L}^N, b_{2L}^N)	$(J_1(b_{1L}^N, b_{2L}^N), J_2(b_{1L}^N, b_{2L}^N))$
Scenario 1	(35,20)	(10,80)	(6.73,0)	(1208.61,1296.64)
Scenario 2	(25,80)	(25,25)	(0,7.62)	(422.55,1643.07)
Scenario 3	(80,40)	(85,45)	(6.81,11.94)	(2893.45,3631.97)

Table 5. Nash equilibria and profits in three scenarios

We use three different sets of λ's as in Table 5 and compute the resulting Nash equilibria. The Nash solution and the corresponding expected revenues for each scenario is summarized in Table 5 and displayed in Figure 6.

In Scenario 1 [Figure 6(a)], since the two BR curves only intersect at $(b_{1L}^N, b_{2L}^N) = (6.73, 0)$, P2 only accepts high-fare class customers. Similarly, in Scenario 2 [Figure 6(b)], the two BR curves only intersect at $(b_{1L}^N, b_{2L}^N) = (0, 7.62)$. In this case, since $\lambda_{1L} = 25$ (a low value), and $\lambda_{1H} = 80$ (a high value), P1 does not reserve any low-fare rooms but keeps them all for high fare customers. Finally, the two BR curves intersect at $(b_{1L}^N, b_{2L}^N) = (6.81, 11.94)$ which is obtained by solving $V_1 = 0$ and $V_2 = 0$.

♦

3.3 Stackelberg Equilibrium

When the decision makers choose their strategies simultaneously as was the case in the previous section, then the proper solution concept that should be used

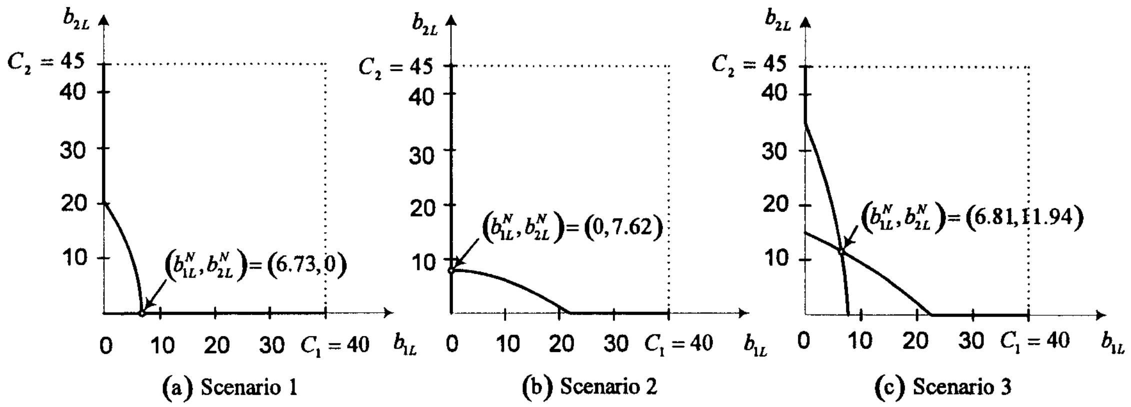


Figure 6. Nash equilibria in three scenarios.

in a non-cooperative game is the Nash equilibrium. However, in some cases, one player may assume the role of the "leader" (perhaps because he/she can act before the other one) and the other is the "follower." Here, the leader announces his strategy first and the follower must make a decision to optimize his objective function after observing the leader's decision. Thus, the leader is able to not only determine the follower's response, but also optimize his objective accordingly. This strategy was first introduced by von Stackelberg [46] in 1934. For a rigorous treatment of the Stackelberg strategy, see Başar and Olsder [2].

Without loss of generality, we assume $P1$ as the leader and $P2$ as the follower in our game theoretical framework. Thus, $P1$ announces his booking limit b_{1L}, and $P2$ chooses an optimal booking limit b_{2L}^R as a function of b_{1L} that maximizes her expected revenue $J_2(b_{1L}, b_{2L})$. Since $P1$ can identify $P2$'s best response b_{2L}^R for each b_{1L}, the Stackelberg equilibrium $\left(b_{1L}^S, b_{2L}^S\right)$ can always be obtained by solving the following nonlinear programming problem (NLP):

$$\begin{aligned} &\max\ J_1\left(b_{1L}, b_{2L}\right), \\ \text{s.t.}\ \ & b_{2L} = b_{2L}^R(b_{1L}), \\ & b_{1L} \in \left[0, C_1\right]. \end{aligned} \tag{15}$$

Unfortunately, the objective function $J_1(b_{1L}, b_{2L}^R(b_{1L}))$ may not be a concave function in b_{1L} after the substitution $b_{2L} = b_{2L}^R(b_{1L})$. Let us examine the optimal solution (Stackelberg equilibrium) of the above NLP according to the different types of b_{2L}^R as given by (9).

The following proposition determines the Stackelberg equilibrium when $b_{2L}^R = 0$ for $\forall b_{1L} \in [0, C_1]$.

Proposition 1 *If $b_{2L}^R\left(b_{1L}\right) = 0$ for $\forall b_{1L} \in [0, C_{1L}]$, then the Stackelberg equilibrium is identical to the Nash equilibrium, i.e.,*

$$(b_{1L}^S, b_{2L}^S) = (b_{1L}^N, b_{2L}^N).$$

Proof. If $P1$ always sets his booking limit as 0, i.e., $b_{1L} = 0$, then $P2$'s best response to $P1$'s decision is $b_{2L}^R\left(0\right) = \delta_{2L}^2$. [This can be seen in parts (b) and (c) in Figure 3 with $j = 1$ and $i = 2$.] This implies that the Stackelberg equilibrium solution which maximizes J_1 is $\left(0, \delta_{2L}^2\right)$. Clearly, it is identical to the Nash equilibrium as given by Theorem 2 with $\xi_{2L}^1 \leq \delta_{2L}^2$. ■

Remark 2 *From Proposition 1, we find that the "Stackelberg game" is equiva-*

lent to "Nash game" if $\xi_{2L}^1 = 0$ $(\Leftrightarrow b_{1L} = 0)$. *In fact, the Stackelberg equilibrium is identical to the Nash equilibrium as long as* J_1 *is only maximized at* (b_{1L}^N, b_{2L}^N). *This could happen in many situations. For example, if* $J_1(0, b_{2L})$ *is strictly concave in* b_{2L} *and* $b_{1L} = 0$ *for* $\forall b_{2L} \in [0, C_{2L}]$, *then the Stackelberg equilibrium is also identical to the Nash equilibrium since for any point* (b_{1L}, b_{2L}) *on P2's BR curve* $J_1(0, \delta_{2L}^2) \geq J_1(0, b_{2L}) \geq J_1(b_{1L}, b_{2L})$ *always holds. Therefore, in this situation* J_1 *is always maximized at* $(0, \delta_{2L}^2)$ *which is the Nash equilibrium. However, in general,* $J_1(b_{1L}^S, b_{2L}^S)$ *should be no less than* $J_1(b_{1L}^N, b_{2L}^N)$ *because* $P1$ *can, at worst, play the strategy corresponding to the most favorable (from the leader's point of view) Nash equilibrium.* $\lhd$

We note that if $\xi_{1L}^2 = C_{1L}$, then $P2$'s BR function is $V_2 = 0$ for $\forall b_{1L} \in [0, C_{1L}]$. Therefore, in this situation, the objective function $J_1(b_{1L}, b_{2L}^R(b_{1L}))$ may not be a concave function. It can be seen that the second order derivatives of J_1 with respect to b_{1L}

$$\frac{\partial^2 J_1}{\partial b_{1L}^2} + 2b_2' \frac{\partial^2 J_1}{\partial b_{1L} \partial b_{2L}} + (b_2')^2 \frac{\partial^2 J_1}{\partial b_{2L}^2} + b_2'' \frac{\partial J_1}{\partial b_{2L}}$$

where $b_2'' = db_2'/db_{1L}$, involves the probability density functions of the booking requests whose monotonicities and concavities are unknown. In fact, we find that the concavity of $J_1(b_{1L}, b_{2L}^R(b_{1L}))$ with respect to b_{1L} still can not be guaranteed even if assuming the probability densities as functions with the monotone and concave properties. This fact makes the solution of the Stackelberg game more complicated than that of the Nash game.

Example 3 *We again use the same data as in Example* 2 *and the same* λ *values as in Table* 5 *for each of the three scenarios to examine the Stackelberg equilibrium. In order to compare the results in different leadership structures, we also calculate the Stackelberg equilibrium solution by assigning P*2 *as the leader. These results are presented in Table* 6.

	P_1 leader, P_2 follower		P_2 leader, P_1 as follower	
	(b_{1L}^S, b_{2L}^S)	(J_1^S, J_2^S)	(b_{1L}^S, b_{2L}^S)	(J_1^S, J_2^S)
Scenario 1	(6.73,0)	(1208.61,1296.64)	(6.73,0)	(1208.61,1296.64)
Scenario 2	(0,7.17)	(422.55,1643.07)	(0,7.17)	(422.55,1643.07)
Scenario 3	(1.77,13.92)	(2910.28,3555.63)	(8.64,7.258)	(2819.67,3646.64)

Table 6. Stackelberg equilibria and profits in three scenarios

As expected, Stackelberg equilibria are identical to Nash equilibria in Scenarios 1 and 2. However, in Scenario 3 where neither of the two players' BR is 0 at all the time, the Stackelberg solution is not identical to the Nash equilibrium. We also observe that the follower's Stackelberg revenue is less than her Nash revenue in this scenario. Therefore, the follower could prefer to play "Nash game" instead of "Stackelberg game", if such an option is open. ♦

3.4 Cooperative Solution

We shall now discuss the case of cooperation between the two players. When cooperating with each other, one player does not incur a rejection cost if a booking request is satisfied by its cooperative player. Hence the cooperative player whose booking limit or capacity has been reached should switch its unsatisfied bookings, if any, to the other player with excess inventory so that the previous player can save in rejection penalty costs. In addition, we also assume that there are no transfer customers between $P1$ and $P2$ when both players' booking limits or capacities have been reached. Thus, they save rejection costs incurred by transfer customers. This is reasonable since when two players act as one player, a rejected customer will be noticed that both players are fully filled and therefore avoid the transfer from happening. Let us denote J_c to be the joint expected revenue of $P1$ and $P2$ when they cooperate. Intuitively, J_c is expected to be higher than the sum of two expected revenue under any other strategy. We will prove it with the following theorem.

Theorem 3 $J_c \geq J_1 + J_2$.

Proof. Let us consider the rejection cost savings generated by low-fare transfer customers first. There are three mutually exclusive cases in which these can take place.

(1) $x_{1L} \leq b_{1L}$, $x_{2L} \geq b_{2L}$:

For $P1$, the cost savings are

$$q_{1L} \max[0, u_{2L}(x_{2L} - b_{2L}) - (b_{1L} - x_{1L})]$$

since there will be no transfer customers from $P2$ to $P1$ when $P1$'s booking limit is reached. And for $P2$, the transfer customers who are satisfied by $P1$ will not incur penalty costs, therefore $P2$ saves

$$q_{2L}\min\left[u_{2L}\left(x_{2L}-b_{2L}\right),b_{1L}-x_{1L}\right].$$

(2) $x_{1L}\geq b_{1L}$, $x_{2L}\leq b_{2L}$:

The cost savings for $P1$ and $P2$ are

$$q_{1L}\min\left[u_{1L}\left(x_{1L}-b_{1L}\right),b_{2L}-x_{2L}\right]$$

and

$$q_{2L}\max[0,u_{1L}\left(x_{1L}-b_{1L}\right)-\left(b_{2L}-x_{2L}\right)],$$

respectively.

(3) $x_{1L}\geq b_{1L},x_{2L}\geq b_{2L}$:

In this case, both players's booking limits have been reached. There are no transfer customers between them. Therefore, the penalty cost savings for $P1$ and $P2$ are $u_{2L}q_{1L}\left(x_{2L}-b_{2L}\right)$ and $u_{1L}q_{2L}\left(x_{1L}-b_{1L}\right)$ respectively.

Integrating these three cost savings over the respective regions, we can obtain the total expected cost savings by low-fare customers for both players. Similarly, we can also obtain the expected cost savings by high-fare customers. After some simplifications, the expected joint revenue is found as

$$J_c = J_1 + J_2 + \sum_{i=1,2} \sum_{K=L,H} \left\{ \int_{M_{iK}}^{b_{iK}} \int_{b_{jK}}^{\infty} q_{iK} \left(x_{iK} - M_{iK} \right) f_{iK} f_{jK} \; dx_{iK} \; dx_{jK} \right.$$

$$+ \int_{b_{iK}}^{\infty} \int_{0}^{M_{jK}} q_{iK} u_{iK} \left(x_{iK} - b_{iK} \right) f_{iK} f_{jK} \; dx_{iK} \; dx_{jK}$$

$$+ \int_{b_{iK}}^{\infty} \int_{M_{jK}}^{b_{jK}} q_{iK} \left(b_{jK} - x_{jK} \right) f_{iK} f_{jK} \; dx_{iK} \; dx_{jK}$$

$$\left. + \int_{b_{iK}}^{\infty} \int_{b_{jK}}^{\infty} q_{iK} u_{jK} \left(x_{jK} - b_{jK} \right) f_{iK} f_{jK} dx_{iK} dx_{jK} \right\}.$$

Clearly, we have $J_c \geq J_1 + J_2$ which means that the expected revenue under cooperation would be higher than the sum of two expected revenues under any other strategy.

■

The optimal solution for the cooperative game can be obtained by solving the following nonlinear programming problem:

$$\begin{aligned} & \max \;\; J_c \left(b_{1L}, b_{2L} \right), \\ \text{Subject to} \;\; & b_{1L} \leq C_1, \; b_{2L} \leq C_2 \\ & b_{1L}, \; b_{2L} \geq 0. \end{aligned}$$

For the existence of unique optimal solution of above problem, one must show that J_c is a strictly concave function of b_{1L} and b_{2L}. This is not pursued in our study since it deviates from the general game theoretic theme.

Example 4 *We still use the same values for all parameters in Example 2 and the* *same λ values as in Table 5 for each of the three scenarios to examine the optimal booking limits of both players and corresponding joint revenue in cooperative situation. The results for three scenarios are shown in Table 7 where J_{ci}, $i = 1, 2$ is Pi's expected revenue when the two players cooperate.*

	(b_{1L}^*, b_{2L}^*)	(J_{c1}^*, J_{c2}^*)	J_c^*
Scenario 1	(11.77,0)	(3454.82,2357.14)	5811.96
Scenario 2	(0,12.87)	(1342.02,4433.45)	5775.47
Scenario 3	(20.81,25.83)	(3026.63,3916.40)	6943.03

Table 7. Cooperative solutions in three scenarios

Comparing to the results with Nash strategy and Stackelberg strategy (see Table 5 and Table 6), each player's expected revenue has increased in the cooperative situation. We note that such improvement is more than 100 percent for each player in scenario 1 and 2. This indicates that cooperation becomes useful when one player has a high booking rate of high-fare customers. ♦

3.5 Sensitivity Analysis

From our previous discussion, we see that solutions using different strategies depend very much on the position of the BR function of each player. Referring to Example 1, we note that the position of one player's best response varies with the booking request expectations. On the other hand, the position of each player's BR function is also sensitive to the values of transfer rates and rejection costs of each fare class. Thus, in this section we discuss sensitivity analyses according to these important parameters and present their effects on optimal decisions and corresponding objectives when adopting different strategies.

In our analyses, we use all parameters in Example 2 as the base parameters, and unless otherwise indicated, the solutions are computed with these parameters. Then, we vary parameters λ_{1L}, λ_{1H}, u_{1L}, u_{1H}, q_{1L}, and q_{1H} one-at-a-time and re-solve the problem to find the solutions with different strategies. In order to make each of these parameters cover a large range, we (i) vary the values of two demand-related parameters λ_{1L} and λ_{1H} by progressively halving or doubling the base value; (ii) vary the values of transfer rates as $u_{1K} = 0.2(0.2)1$, $K = L, H$, and (iii) vary the values of rejection costs as $q_{1L} = 10(10)50$ and $q_{1H} = 30(20)110$. The sensitivity analysis results with Nash, Stackelberg and cooperative strategy are presented in Table 8, Table 9 and Table 10 respectively, where the base values and the corresponding solutions are indicated in **bold.**

First, we examine the effect of changing parameter values on the Nash solution and expected Nash revenue of each player.

Changes in λ_{1L}. Referring to the results in Table 8, we observe that for increased values of λ_{1L}, the Nash equilibrium moves in the northeast direction in the (b_{1L}, b_{2L}) plane. We also find that when $\lambda_{1L} \to \infty$, the best responses of both players b_{1L}^R and b_{2L}^R approach 27 and 33, respectively, which indicates that there exists a minimum protection level for high-fare customers in each hotel. This should be expected since in lower traffic season hotels should raise booking limits if the amount of booking requests of low-fare customers increases. However, hotels still have to keep some rooms for "more valuable" customers even though the booking rate of

Varying parameters						Nash strategy	
λ_{1L}	λ_{1H}	u_{1L}	u_{1H}	q_{1L}	q_{1H}	(b_{1L}^N, b_{2L}^N)	(J_1^N, J_2^N)
6.25						(13.00, 22.93)	(2600.80, 2940.40)
12.5						(16.37, 23.14)	(2669.56, 2988.26)
25						**(19.86, 24.17)**	**(2591.72, 3013.19)**
50						(22.75, 26.05)	(2125.11, 2843.89)
100						(24.70, 28.21)	(845.47, 2136.26)
	3.75					(30.67, 25.97)	(2283.48, 2918.21)
	7.5					(26.20, 25.58)	(2456.04, 2965.32)
	15					**(19.86, 24.17)**	**(2591.72, 3013.19)**
	30					(11.66, 21.28)	(2446.96, 2913.55)
	60					(1.67, 17.03)	(1436.20, 2233.90)
		0.2				(20.28, 22.55)	(2582.46, 3038.76)
		0.4				(20.04, 23.48)	(2587.49, 3041.10)
		0.6				**(19.86, 24.17)**	**(2591.72, 3013.19)**
		0.8				(19.73, 24.68)	(2595.12, 2968.72)
		1.0				(19.62, 25.07)	(2597.85, 2914.22)
			0.2			(19.25, 26.40)	(2608.35, 2989.95)
			0.4			(19.48, 25.59)	(2601.74, 3018.77)
			0.6			(19.69, 24.83)	(2596.14, 3023.98)
			0.8			**(19.86, 24.17)**	**(2591.72, 3013.19)**
			1.0			(20.01, 23.61)	(2588.26, 2991.62)
				10		(17.76, 24.73)	(2966.98, 3004.66)
				20		(18.86, 24.44)	(2776.44, 3008.54)
				30		**(19.86, 24.17)**	**(2591.72, 3013.19)**
				40		(20.79, 23.91)	(2412.13, 3018.49)
				50		(21.66, 23.66)	(2237.15, 3024.30)
					30	(22.26, 23.48)	(2825.54, 3028.86)
					50	(21.00, 23.85)	(2704.93, 3019.82)
					70	**(19.86, 24.17)**	**(2591.72, 3013.19)**
					90	(18.82, 24.45)	(2484.89, 3008.40)
					110	(17.86, 24.71)	(2383.57, 3004.95)

Table 8. Sensitivity analysis for Nash equilibrium and profits

Varying parameters	Stackelberg strategy			
	$P1$ leader, and $P2$ follower		$P2$ leader, and $P1$ follower	
λ_{1L}	(b_{1L}^S, b_{2L}^S)	(J_1^S, J_2^S)	(b_{1L}^S, b_{2L}^S)	(J_1^S, J_2^S)
6.25	(12.75, 22.97)	(2600.97, 2939.56)	(13.06, 22.74)	(2599.02, 2940.47)
12.5	(16.09, 23.20)	(2669.83, 2987.35)	(16.42, 22.96)	(2668.31, 2988.34)
25	**(19.55, 24.25)**	**(2592.04, 3011.64)**	**(19.92, 23.93)**	**(2590.20, 3013.35)**
50	(22.42, 26.15)	(2125.38, 2841.38)	(22.85, 25.69)	(2123.07, 2844.23)
100	(24.39, 28.31)	(845.706, 2133.13)	(24.83, 27.73)	(843.157, 2136.90)
λ_{1H}				
3.75	(30.72, 25.97)	(2283.48, 2918.35)	(30.74, 25.78)	(2284.02, 2918.32)
7.5	(26.13, 25.59)	(2456.05, 2964.98)	(26.28, 25.29)	(2455.45, 2965.56)
15	**(19.55, 24.25)**	**(2592.04, 3011.64)**	**(19.92, 23.93)**	**(2590.20, 3013.35)**
30	(10.91, 21.56)	(2448.27, 2912.02)	(11.68, 21.18)	(2446.00, 2913.55)
60	(0.13, 17.80)	(1440.91, 2235.980)	(1.66, 17.04)	(1436.36, 2233.90)
u_{1L}				
0.2	(20.09, 22.59)	(2582.57, 3037.41)	(20.36, 22.24)	(2580.91, 3039.06)
0.4	(19.79, 23.54)	(2587.72, 3039.76)	(20.10, 23.23)	(2586.12, 3041.28)
0.6	**(19.55, 24.25)**	**(2592.04, 3011.64)**	**(19.92, 23.93)**	**(2590.20, 3013.35)**
0.8	(19.36, 24.79)	(2595.47, 2966.63)	(19.80, 24.40)	(2593.25, 2968.92)
1.0	(19.21, 25.20)	(2598.32, 2911.38)	(19.72, 24.71)	(2595.36, 2914.57)
u_{1H}				
0.2	(19.07, 26.43)	(2608.43, 2989.89)	(19.26, 26.39)	(2608.24, 2989.97)
0.4	(19.24, 25.63)	(2601.94, 3018.03)	(19.52, 25.45)	(2600.65, 3018.85)
0.6	(19.41, 24.89)	(2596.33, 3022.68)	(19.74, 24.61)	(2594.70, 3024.09)
0.8	**(19.55, 24.25)**	**(2592.04, 3011.64)**	**(19.92, 23.93)**	**(2590.20, 3013.35)**
1.0	(19.67, 23.71)	(2588.60, 2990.05)	(20.07, 23.37)	(2586.91, 2991.77)
q_{1L}				
10	(17.47, 24.81)	(2967.16, 3003.80)	(17.80, 24.59)	(2966.12, 3004.70)
20	(18.56, 24.52)	(2776.70, 3007.35)	(18.91, 24.25)	(2775.31, 3008.66)
30	**(19.55, 24.25)**	**(2592.04, 3011.64)**	**(19.92, 23.93)**	**(2590.20, 3013.35)**
40	(20.46, 24.00)	(2412.45, 3016.51)	(20.87, 23.62)	(2410.20, 3018.75)
50	(21.31, 23.76)	(2237.51, 3021.88)	(21.74, 23.33)	(2234.70, 3024.64)
q_{1H}				
30	(21.88, 23.59)	(2825.91, 3025.94)	(22.35, 23.12)	(2823.22, 3029.24)
50	(20.66, 23.94)	(2705.26, 3017.71)	(21.08, 23.55)	(2703.03, 3020.09)
70	**(19.55, 24.25)**	**(2592.04, 3011.64)**	**(19.92, 23.93)**	**(2590.20, 3013.35)**
90	(18.53, 24.53)	(2485.12, 3007.27)	(18.87, 24.26)	(2483.71, 3008.49)
110	(17.60, 24.78)	(2383.77, 3004.13)	(17.90, 24.56)	(2382.70, 3005.00)

Table 9. Sensitivity analysis for Stackelberg equilibrium and profits

Varying parameters	Cooperative strategy			Cooperation vs. Nash	
λ_{1L}	(b^*_{1L}, b^*_{2L})	(J^*_{c1}, J^*_{c2})	J^*_c	ΔJ^*_1	ΔJ^*_2
6.25	(14.28, 24.75)	(2906.29, 3188.32)	6094.61	11.75%	8.43%
12.5	(17.38, 24.77)	(3022.29, 3268.31)	6290.60	13.21%	9.37%
25	**(20.81, 25.83)**	**(3026.63, 3416.40)**	**6443.03**	16.78%	13.38%
50	(23.83, 27.91)	(2674.35, 3612.25)	6286.60	25.85%	27.02%
100	(25.97, 30.25)	(1521.02, 3805.90)	5326.92	79.90%	78.16%
λ_{1H}					
3.75	(32.00, 25.38)	(2629.33, 3187.59)	5816.91	15.15%	9.23%
7.5	(27.80, 25.76)	(2822.39, 3274.63)	6097.02	14.92%	10.43%
15	**(20.81, 25.83)**	**(3026.63, 3416.40)**	**6443.03**	16.78%	13.38%
30	(11.80, 24.35)	(3003.80, 3681.60)	6685.40	22.76%	26.36%
60	(0.00, 22.56)	(2159.15, 4087.55)	6246.71	50.34%	82.98%
u_{1L}					
0.2	(24.10, 22.07)	(2939.23, 3392.86)	6332.10	13.82%	11.65%
0.4	(22.52, 23.91)	(2985.86, 3415.66)	6401.52	15.40%	12.32%
0.6	**(20.81, 25.83)**	**(3026.63, 3416.40)**	**6443.03**	16.78%	13.38%
0.8	(19.94, 26.82)	(3051.79, 3437.74)	6489.53	17.60%	15.80%
1.0	(18.83, 28.01)	(3077.40, 3443.49)	6520.90	18.46%	18.16%
u_{1H}					
0.2	(17.14, 29.70)	(3066.30, 3261.73)	6328.03	17.56%	9.09%
0.4	(18.51, 28.29)	(3053.79, 3324.02)	6377.81	17.37%	10.11%
0.6	(19.82, 26.89)	(3039.12, 3378.17)	6417.28	17.06%	11.71%
0.8	**(20.81, 25.83)**	**(3026.63, 3416.40)**	**6443.03**	16.78%	13.38%
1.0	(22.61, 23.94)	(3000.09, 3482.30)	6482.39	15.91%	16.40%
q_{1L}					
10	(18.30, 27.07)	(3211.29, 3395.35)	6606.64	8.23%	13.00%
20	(19.74, 26.30)	(3114.90, 3410.95)	6525.84	12.19%	13.38%
30	**(20.81, 25.83)**	**(3026.63, 3416.40)**	**6443.03**	16.78%	13.38%
40	(22.57, 24.59)	(2932.51, 3450.05)	6382.55	21.57%	14.30%
50	(24.02, 23.63)	(2846.05, 3474.17)	6320.22	27.22%	14.88%
q_{1H}					
30	(26.14, 22.27)	(3135.50, 3514.88)	6650.38	10.97%	16.05%
50	(23.27, 24.13)	(3081.25, 3461.35)	6542.60	13.91%	14.62%
70	**(20.81, 25.83)**	**(3026.63, 3416.40)**	**6443.03**	16.78%	13.38%
90	(19.46, 26.52)	(2963.66, 3406.88)	6370.55	19.27%	13.25%
110	(18.04, 27.38)	(2907.52, 3390.32)	6297.84	21.98%	12.82%

Table 10. Sensitivity analysis for cooperative solution

"less valuable" customers is very high. This actually follows the principle of Revenue Management.

Changes in λ_{1H}. Increasing the value of λ_{1H} has an opposite effect on Nash equilibrium comparing to the situation for λ_{1L}. This is due to the shifting of each player's BR curve to a lower position, thus resulting in decreased values of b_{iL}^N.

Changes in u_{1L}. Note that as soon as the value of u_{1L} increases, the Nash equilibrium moves in the northwest direction, resulting in a decrease in b_{1L} and an increase in b_{2L}.

Changes in u_{1H}. In this case, the direction of movement of the Nash equilibrium is exactly opposite to the situation for u_{1L}: increasing the value of u_{1H} raises b_{1L} and reduces b_{2L}.

Changes in q_{1L}. As the value of q_{1L} increases, the Nash equilibrium moves in the southeast direction, resulting in an increase in b_{1L} and a decrease in b_{2L}.

Changes in q_{1H}. Referring to Table 8, we note that the direction of movement of the Nash equilibrium is exactly opposite to the situation for u_{1L}: increasing the value of q_{1H} reduces b_{1L} and raises b_{2L}.

The sensitivity analysis of Stackelberg equilibrium is summarized in Table 9 which reveals that the movements of the Stackelberg solution pair $\left(b_{1L}^S, b_{2L}^S\right)$ parallel those of Nash equilibrium as presented in Table 8.

However, we observe that in the cooperative situation the booking limit b_{2L} does not monotonically decrease when λ_{1H} is increasing (see Table 10). In fact, the optimal booking limit of each player might not monotonically decrease or increase at all times. This is expected since when the two players act as one player and try to maximize the joint revenue, it is always optimal for the two players to keep the "most valuable" customers in the system. After then, they should satisfy the low-fare customers as many as they can. Therefore, when the booking rates of high-fare customers in both players are very low, a small increase of high-fare booking request expectation of one player will not generate high-fare transfer customers if the player decreases the booking limit. However, the low-fare transfer customers will increase due to the decrease of booking limit of the hotel with increased high-fare class booking rate. Hence, it might be optimal for the other hotel to increase its booking limit based on the condition that all high-fare customers are still in the system.

Chapter 4
Dynamic Game Model for Hotel Room Allocations

In this Chapter, we consider a situation in which the booking requests from different fare classes arrive concurrently and two hotels compete with each other. One hotel's accept/reject decision of a booking request depends on the time at which the request arrives, as well as on the available rooms of both hotels at that point in time. One hotel's available room(s) at a specific time might affect another hotel's decision because of the existence of transferred customers. A discrete-time dynamic game is presented to obtain an optimal policy for making accept/reject decisions. This model differs from Chen et al.'s [12] model in that our model is used in the context of hotel business where the capacity is fixed and each player has his own booking requests from two fare classes. Moreover, our model assumes that the probability of a transferred customer who is rejected by one hotel (transfer rate) can be between zero and one. In addition, the rejection costs are incurred when a customer is rejected by a hotel. These assumptions make our model more general in practice.

4.1 The Dynamic Model of Best Response Policies

We assume that there are only two hotels serving a specific geographical market. These two hotels are assumed to be two players ($P1$ and $P2$), each with certain units of rooms to sell within a specified time period $[0, T]$. The customers are classified as low-fare class (L) customers and high-fare class (H) customers, who are charged discounted price and full price, respectively. The booking period is divided into equal time intervals which is short enough to make the probability of more than one customer arriving in each interval negligible. In order to simplify the analysis, we make the following assumptions:

(1) The customer arrival patterns are known to both players.

(2) The room rates of the two fare classes in both hotels are constant and known.

(3) There is no buy-up when a low-fare customer is rejected.

(4) Each customer asks for a single unit of room.

(5) There are no cancellations allowed for any customer and no overbookings in both hotels.

(6) A rejection cost is only incurred by one hotel's own customers and it is only incurred when one player still has available rooms.

The objective of each hotel is to maximize the expected future revenue by finding an optimal accept/reject policy for any combination of the rooms and time remaining. We use the following notation with $i, j = 1, 2$, $K = L, H$ and $i \neq j$:

- λ_{iK} : probability of arrival (in any given interval) of a Pi's own K-fare class customer,
- r_{iK} : revenue per room from a Pi's K-fare class customer,
- c_{iK} : rejection cost per K-fare class customer of Pi,
- μ_{iK} : probability of transfer if a K-fare class customer is rejected by Pi
- n_i : available rooms of Pi at time t.

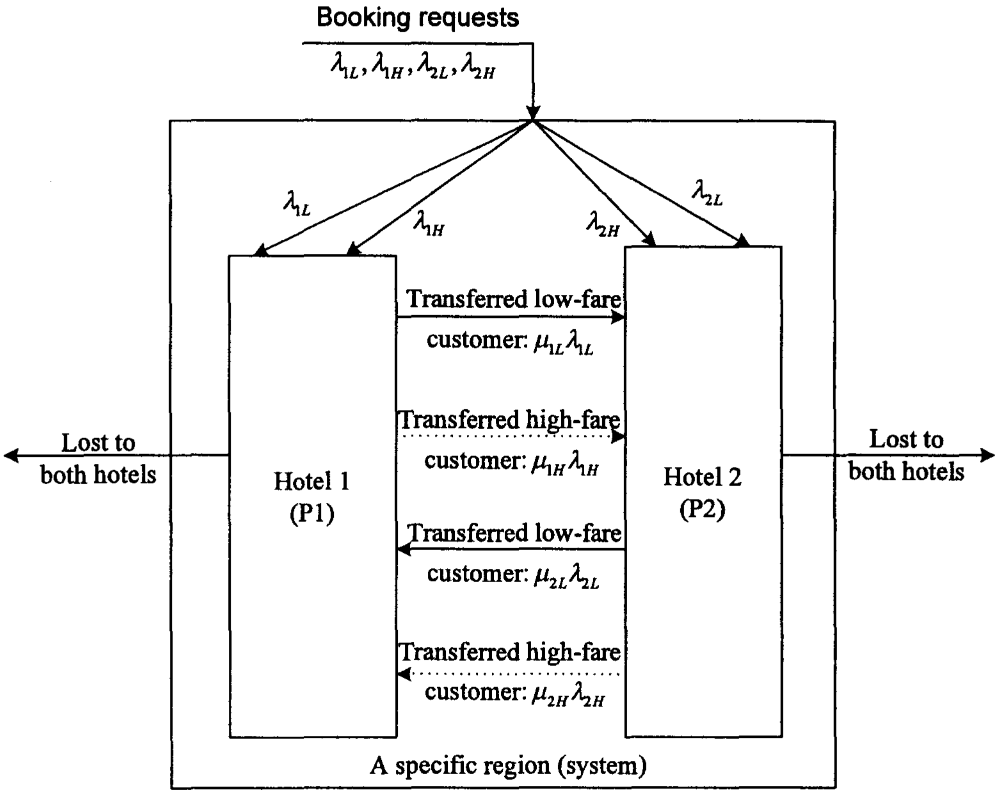


Figure 7. Customer flows in the model.

Referring to Figure 7, we note that in each time interval, only one player can receive a booking request. If he rejects it, the rejected customer may become a transferred customer who will seek accommodation with another hotel, or choose to leave the system. Therefore, in each period, the players have to make decisions on whether to accept or to reject the booking request upon its arrival. Obviously, one player will not reject any high-fare customer unless he does not have available rooms. However, he may reject a low-fare customer in case his room can be sold to a more revenueable customer in later periods. Thus, the decisions of each player at period t

are

$$x_{iL} = \begin{cases} 1, & Pi \text{ accepts his own low-fare customer, if any,} \\ 0, & Pi \text{ rejects his own low-fare customer, if any,} \end{cases}$$

and

$$y_{iL} = \begin{cases} 1, & Pi \text{ accepts a transferred low-fare customer, if any,} \\ 0, & Pi \text{ rejects a transferred low-fare customer, if any,} \end{cases}$$

for $i = 1, 2$. Now defining $V_i(t, n_i, n_j)$ as the maximum expected future total revenue of Pi when Pi and Pj start with n_i and n_j room(s), respectively, and there are t time intervals remaining until the end of the booking period, our problem can be formulated under two situations: (i) only one player has available rooms in period t; and (ii) both players have available rooms in period t.

4.1.1 Case 1: Only One Player Has Available Rooms in Period t

Without loss of generality, let us assume that $P1$ has $n_1 > 0$ rooms at time t while $P2$ has sold out all of his rooms (i.e., $n_2 = 0$). In this case, $P2$ has to reject all his booking requests due to empty inventory, and $P1$'s decisions will have no effects on $P2$'s expected revenue. In this case, $P2$'s expected future total revenue is $V_2(t, n_1, 0) = 0$ for all n_1 and $t \in \{0, 1, 2, ..., T\}$.

Due to the possibilities of transferred high-fare class customers from $P2$ to $P1$, there are five different situations in which $P1$ has to make decisions on x_{1L} and y_{1L}. Figure 8 presents $P1$'s expected revenues under different sequential situations. Thus, $P1$'s expected future total revenue will be

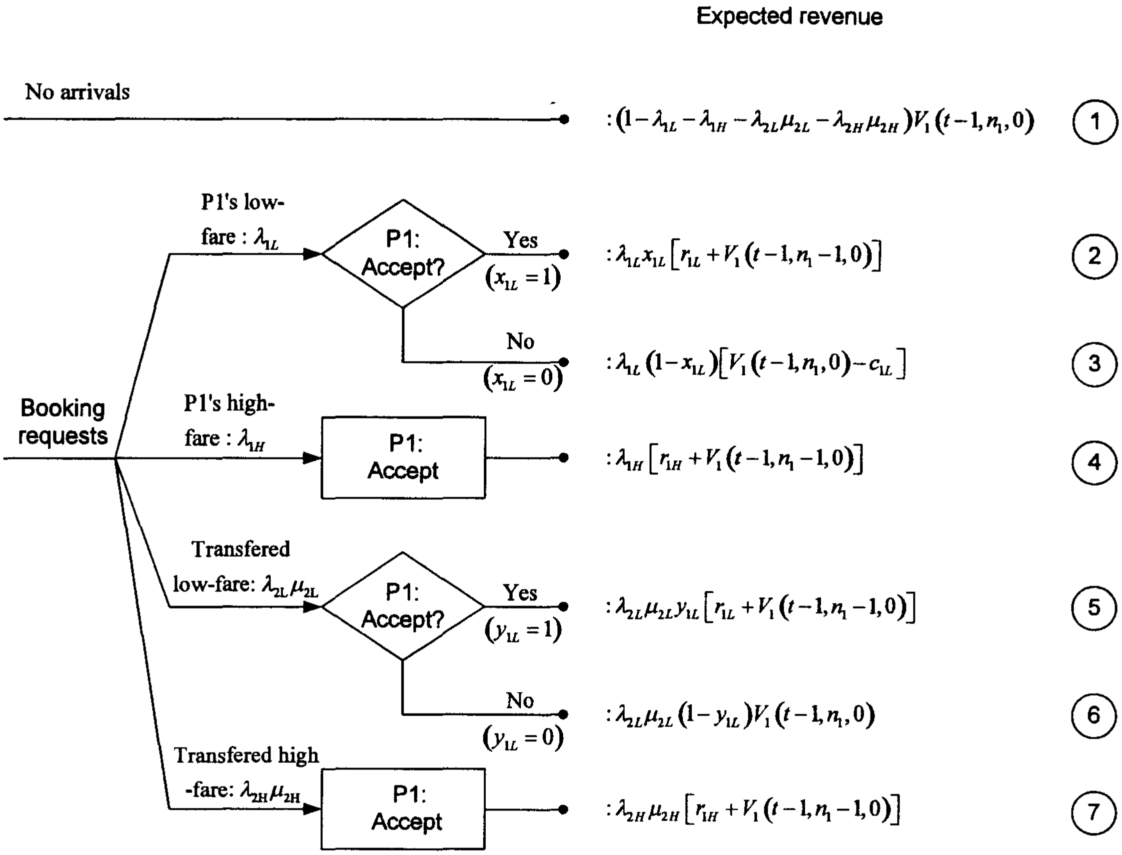


Figure 8. $P1$'s expected revenue when $P2$'s rooms are all sold out.

$$
\begin{aligned}
V_1(t,n_1,0) = \max_{x_{1L},y_{1L}} \quad & \{(1-\lambda_{1L}-\lambda_{1H}-\lambda_{2L}\mu_{2L}-\lambda_{2H}\mu_{2H})V_1(t-1,n_1,0) \\
& +(\lambda_{1H}+\lambda_{2H}\mu_{2H})[r_{1H}+V_1(t-1,n_1-1,0)] \\
& +\lambda_{1L}x_{1L}[r_{1L}+V_1(t-1,n_1-1,0)] \\
& +\lambda_{1L}(1-x_{1L})[V_1(t-1,n_1,0)-c_{1L}] \\
& +\lambda_{2L}\mu_{2L}y_{1L}[r_{1L}+V_1(t-1,n_1-1,0)] \\
& +\lambda_{2L}\mu_{2L}(1-y_{1L})[V_1(t-1,n_1,0)]\}
\end{aligned} \tag{16}
$$

with $V_1(0, n_1, 0) = 0$ for all n_1. After some simplifications, we generalize Pi's expected future total revenue when $n_1 > 0$ and $n_2 = 0$ as follows ($i, j = 1, 2$ and $i \neq j$):

$$\begin{aligned} V_i(t, n_i, 0) = \max_{x_{iL}, y_{iL}} \quad & \{ (1 - \lambda_{iH} - \lambda_{jL}\mu_{jL}y_{iL} - \lambda_{jH}\mu_{jH} - \lambda_{iL}x_{iL}) V_i(t-1, n_i, 0) \\ & + (\lambda_{iH} + \lambda_{jH}\mu_{jH} + \lambda_{jL}\mu_{jL}y_{iL} + \lambda_{iL}x_{iL}) V_i(t-1, n_i - 1, 0) \\ & + (\lambda_{iL}x_{iL} + \lambda_{jL}\mu_{jL}y_{iL}) r_{iL} \\ & + (\lambda_{iH} + \lambda_{jH}\mu_{jH}) r_{iH} - \lambda_{iL}(1 - x_{iL}) c_{iL} \} \end{aligned} \tag{17}$$

with $V_i(0, n_i, 0) = 0$ for all n_i. In order to investigate the decision rule of Pi and the properties of $V_i(t, n_i, 0)$, we now introduce three important concepts.

Definition 1 *We define*

$$\delta_i(t, n_i, n_j) = V_i(t, n_i, n_j) - V_i(t, n_i - 1, n_j),$$

$n_1, n_2 = 1, 2, ..,$ *as Pi's expected marginal value of having an extra room in period* t *given that Pi and Pj have booking capacities of* n_i *and* n_j*, respectively.*

Definition 2 *We define*

$$\xi_i(t, n_i, n_j) = V_i(t, n_i, n_j) - V_i(t, n_i, n_j - 1),$$

$n_1, n_2 = 1, 2, ..,$ *as Pi's expected marginal value arising from Pj having an extra room in period* t *given Pi and Pj have booking capacities of* n_i *and* n_j *respectively.*

Definition 3 *We define*

$$\theta_i(t, n_i, n_2) = V_i(t, n_i, n_j) - V_i(t - 1, n_i, n_j),$$

$n_1, n_2 = 1, 2, ..,$ *as Pi's expected opportunity cost of holding* n_i *rooms from period* t *to* $t-1$.

Referring to the terms involving the decision variables in (17), the decision rules of Pi can be found as follows.

Proposition 2 *When* Pj *has no rooms available* ($n_j = 0$) *but* Pi *has* $n_i > 0$ *rooms at the beginning of period* t, Pi *should,*

(1) *accept any low-fare class booking request if* $\delta_i(t-1, n_i, 0) < r_{iL}$,

(2) *accept his own low-fare class booking request and reject the transferred low-fare booking request if* $r_{iL} \leq \delta_i(t-1, n_i, 0) < r_{iL} + c_{iL}$, *and*

(3) *reject any low-fare class booking request if* $r_{iL} + c_{iL} \leq \delta_i(t-1, n_i, 0)$.

Proof. Referring to Figure 8, the results claimed in Proposition 2 are not difficult to obtain. For example, when $P1$'s own low-fare customer arrives, comparing the expected revenues of accepting and rejecting λ_{1L} which are provided by the second and third expression in Figure 8, we note that it is optimal for $P1$ to accept his own low-fare booking request if $r_{1L} + V_1(t, n_1 - 1, 0) > V_1(t, n_1, 0) - c_{1L}$ (or, equivalently, $\delta_1(t-1, n_1, 0) < r_{1L} + c_{1L}$). Similarly, we obtain the results in other situations in which different booking requests arrive to $P1$. Thus, after some simplifications, the conclusions can be shown in three cases as described above. ■

From Definition 1, we note that $\delta_i(t, n_i, 0)$ is a function of the decision period

(t), and booking capacity (n_i). By fixing the value of t, Pi's expected marginal value in n_i can be expressed as

$$\begin{aligned}\delta_i(t,n_i,0) = &\ V_i(t,n_i,0) - V_i(t,n_i-1,0)\\ = &\ (1-\lambda_{iH}-\lambda_{jH}\mu_{jH}-\lambda_{iL}-\lambda_{jL}\mu_{jL})\delta_i(t-1,n_i,0)\\ &+(\lambda_{iH}+\lambda_{jH}\mu_{jH})\delta_i(t-1,n_i-1,0)\\ &-\lambda_{iL}\{\max[r_{iL}+c_{iL}-\delta_i(t-1,n_i,0),0]\\ &+\max[r_{iL}+c_{iL},\delta_i(t-1,n_i-1,0)]\}\\ &-\lambda_{jL}\mu_{jL}\{\max[r_{iL}-\delta_i(t-1,n_i,0),0]+\max[r_{iL},\delta_i(t-1,n_i-1,0)]\}\end{aligned} \tag{18}$$

for $n_i > 1$, $t > 1$. Using the relations between $\delta_i(t-1,n_i,0)$ and the values of r_{iL} and $r_{iL}+c_{iL}$ in (18), we will show that $\delta_i(t,n_i,0)$ is non-increasing in n_i for a fixed t, and non-decreasing in t for a fixed n_i. By these monotonic properties of $\delta_i(t,n_i,0)$, we are able to simplify the optimal accept/reject policy to sets of critical values, which can be used to control the booking process.

Theorem 4 *For a given t, $\delta_i(t,n_i,0)$ is non-increasing in n_i; and for a given n_i, $\delta_i(t,n_i,0)$ is non-decreasing in t.*

Proof. We will prove this theorem by induction. In last period, i.e., $t=1$, $\delta_i(1,n_i,0) = (\lambda_{iH}+\lambda_{jH}\mu_{jH})r_{iH} + (\lambda_{iL}+\lambda_{jL}\mu_{jL})r_{iL}$. Clearly, $\delta_i(1,n_i,0)$ is non-increasing in n_i. Now, we assume that $\delta_i(t-1,n_i,0)$ is non-increasing in n_i. Referring to (18), we find that $\delta_i(t,n_i,0)$ is a non-negative combination of $\delta_i(t-1,n_i-1,0)$ and $\delta_i(t-1,n_i,0)$. This indicates that $\delta_i(t,n_i,0)$ can always be expressed as a non-negative and linear combination of items which are non-increasing in n_i. Therefore, by induction, $\delta_i(t, n_i, 0)$ is non-increasing in n_i for a given t.

Rearranging (18), we obtain

$$\begin{aligned}&\delta_i(t, n_i, 0) - \delta_i(t, n_i - 1, 0) = \\ &\left(\lambda_{iH} + \lambda_{jH}\mu_{jH}\right)\left[\delta_i(t-1, n_i - 1, 0) - \delta_i(t-1, n_i - 1, 0)\right] \\ &+\lambda_{iL}\left\{\max\left[r_{iL} + c_{iL}, \delta_i(t-1, n_i - 1, 0)\right] - \max\left[r_{iL} + c_{iL}, \delta_i(t-1, n_i, 0)\right]\right\} \\ &+\lambda_{jL}\mu_{jL}\left\{\max\left[r_{iL}, \delta_i(t-1, n_i - 1, 0)\right] - \max\left[r_{iL}, \delta_i(t-1, n_i, 0)\right]\right\}.\end{aligned} \tag{19}$$

Since we know that $\delta_i(t, n_i, 0)$ is non-increasing in n_i for a given t, the RHS of (19) is non-negative. Thus, $\delta_i(t, n_i, 0)$ is non-decreasing in t for a given n_i. This completes the proof. ■

The monotonicity of $\delta_i(t, n_i, 0)$ has the following managerial implications:

- In a booking period t, there exists two *critical booking capacities,* $\hat{n}_i^1(t)$ and $\hat{n}_i^2(t)$, for Pi, (with $\hat{n}_i^1(t) \leq \hat{n}_i^2(t)$), such that (i) any low-fare booking request is accepted for $\hat{n}_i^2(t) < n_i$; (ii) a booking request from Pi's own low-fare class is accepted while a transferred low-fare booking request is rejected for $\hat{n}_i^1(t) \leq n_i < \hat{n}_i^2(t)$; and (iii) any low-fare booking request is rejected for $n_i \leq \hat{n}_i^1(t)$.
- Given the booking capacity n_i for Pi, there exists two *critical booking periods,* $\hat{t}_i^1(n_i)$ and $\hat{t}_i^2(n_i)$, $\left(\text{with } \hat{t}_i^1(n_i) \leq \hat{t}_i^2(n_i)\right)$, such that (i) any low-fare booking request is accepted for $t < \hat{t}_i^1(n_i)$; (ii) a booking request from Pi's own low-fare class is accepted while a transferred low-fare booking request is rejected for $\hat{t}_i^1(n_i) \leq t < \hat{t}_i^2(n_i)$; and (iii) any low-fare booking request is rejected for $\hat{t}_i^2(n_i) \leq t$.

Example 5 *We now present a numerical example for the case where only one player has available rooms in period t. We assume that each hotel has a capacity of 30 rooms. The room rate, penalty cost, and transfer rate of K-fare class customers in Pi ($K = L, H$ and $i = 1, 2$) are given in Table* 11

	Low-fare ($K = L$)				High-fare ($K = H$)			
	r_{iL}	c_{iL}	λ_{iL}	μ_{iL}	r_{iH}	c_{iH}	λ_{iH}	μ_{iH}
P1	\$99	\$10	0.35	0.8	\$159	\$20	0.15	0.6
P2	\$105	\$12	0.25	0.75	\$165	\$25	0.10	0.65

Table 11. Prices, rejection costs, arrival and transfer rates of $P1$ and $P2$.

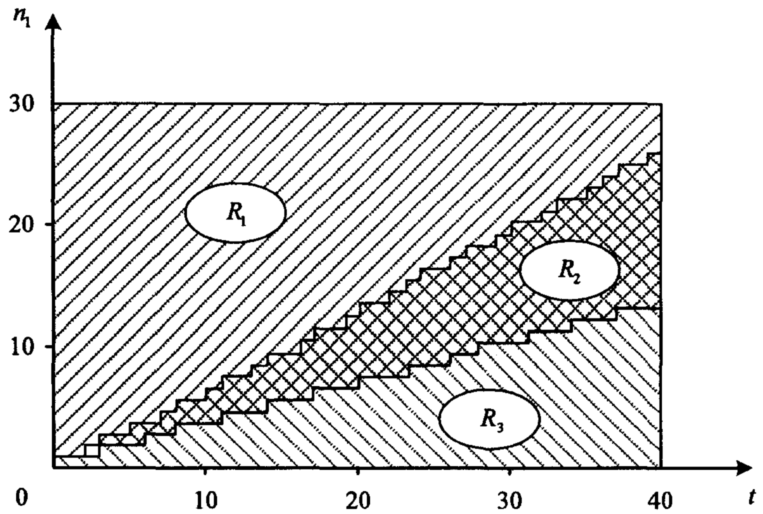


Figure 9. Dynamic optimal decisions of $P1$ when $P2$'s rooms are sold out. In region R_1, $P1$ accepts both his low-fare customer and $P2$'s transfer customer; in region R_2, he accepts only his low fare customer and in region R_3 he rejects any low-fare customer.

Let us assume that P2 has sold out all his rooms, and P1 is attempting to determine his optimal accept/reject decisions. For the data given in Table 11*, Figure*

9 *has three regions: In region* R_1, $P1$ *accepts both his low-fare customer and* $P2$*'s transfer customer. In region* R_2, $P1$ *accepts only his low fare customer and in region* R_3 *rejects any low-fare customer. The figure shows that the cutoff levels for the acceptance/rejection regions are non-decreasing in time* t. *When there are only a few time periods left, and there is sufficient inventory,* $P1$ *accepts almost all low fare and transfer customers. On the other hand, when the time remaining is quite long,* $P1$ *rejects low fare and transfer customers if his inventory is low.* ♦

For all of the numerical examples in the chapter, we use Excel VBA to calculate the optimal solutions.

4.1.2 Case 2: Both Players Have Available Rooms in Period t

In this case, $n_1, n_2 > 0$ at the beginning of period t, there will not be any transfer of high-fare customers between the two players. Hence, each player will have to decide whether to accept his own low-fare customer if such an arrival occurs. On the other hand, if a low-fare customer is rejected by one player, such a customer will probably seek accommodation with another player. Hence, each player also faces the choice of accepting or rejecting the booking request of a transferred customer.

Again, let us consider the situation faced by $P1$ whose expected revenues under different decisions in period t are presented in Figure 10. Here, $P1$'s total expected revenue can be obtained by adding all these expressions on the right in Figure 10 and $P2$'s total expected revenue can be obtained similarly. Therefore, after some

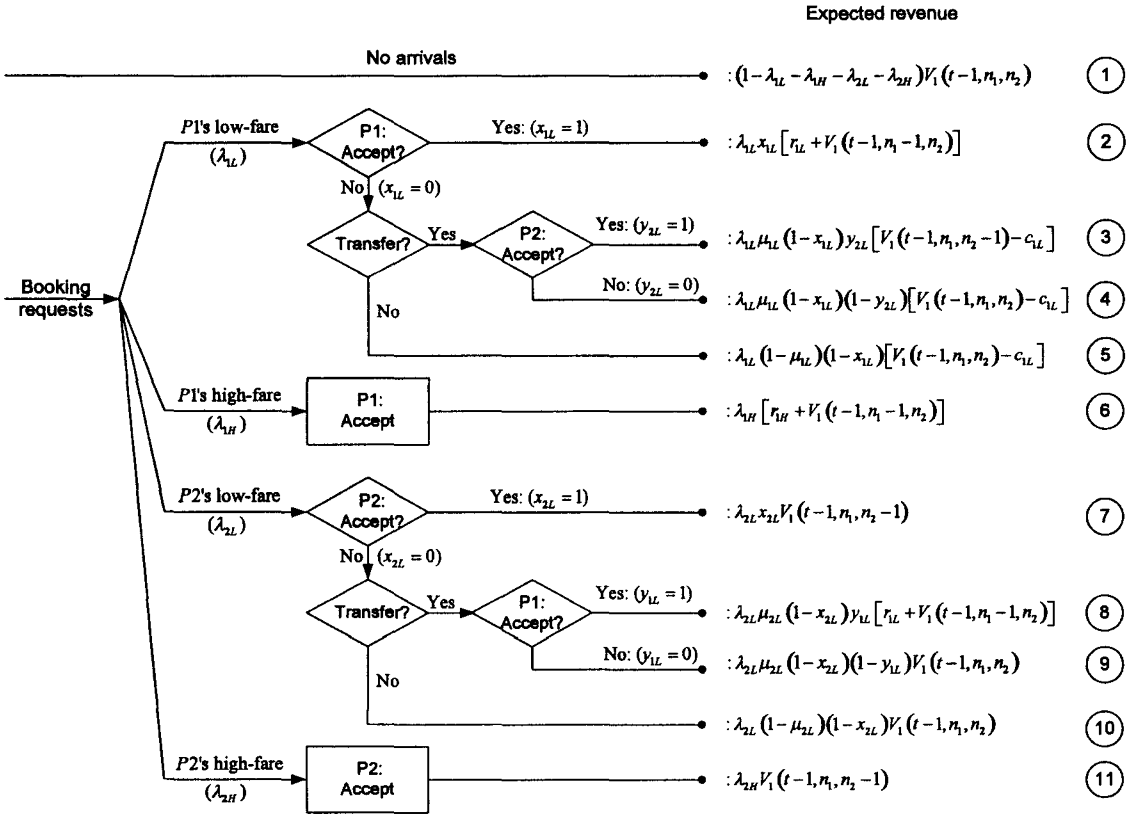


Figure 10. $P1$'s expected revenue when both players have available room(s) at the beginning of period t.

simplifications, Pi's total expected revenue is obtained as follows.

$$
\begin{aligned}
V_i(t,n_i,n_j) = &\max_{x_{iL},y_{iL}} \\
&\{V_i(t-1,n_i,n_j)+\lambda_{iH}r_{iH}+r_{iL}\left[\lambda_{jL}\mu_{jL}(1-x_{jL})y_{iL}+\lambda_{iL}x_{iL}\right]-\lambda_{iL}(1-x_{iL})c_{iL} \\
&+\left[\lambda_{iL}x_{iL}+\lambda_{iH}+\lambda_{jL}\mu_{jL}(1-x_{jL})y_{iL}\right][V_i(t-1,n_i-1,n_j)-V_i(t-1,n_i,n_j)] \\
&+[\lambda_{jL}x_{jL}+\lambda_{jH}+\lambda_{iL}\mu_{iL}(1-x_{iL})y_{jL}][V_i(t-1,n_i,n_j-1)-V_i(t-1,n_i,n_j)]\}
\end{aligned}
\tag{20}
$$

with $V_i(0, n_i, n_j) = 0$ for all $n_i, n_j > 0$ $(i, j = 1, 2$ while $i \neq j)$.

According to Figure 10, when a transferred low-fare booking request occurs with $P1$, his expected revenue is $\lambda_{2L}\mu_{2L}[r_{1L} + V_1(t-1, n_1-1, n_2)]$ if he accepts such a request. On the other hand, if $P1$ rejects such a request, his expected revenue is $\lambda_{2L}\mu_{2L}V_1(t-1, n_1, n_2)$. We can now express $P1$'s optimal accept/reject decision for a transferred low-fare customer from $P2$ (i.e., $y_{1L} = 1$ or 0) in terms of the expected marginal value, $\delta_1(t-1, n_1, n_2)$:

$$\begin{cases} y_{1L} = 1, & \text{if } \delta_1(t-1, n_1, n_2) < r_{1L}; \\ y_{1L} = 0, & \text{if } \delta_1(t-1, n_1, n_2) \geq r_{1L}. \end{cases} \tag{21}$$

Next, we consider the situation in which $P1$'s own low-fare booking request occurs. This is more complicated since $P1$'s accept/reject decisions of his own low-fare customers, (i.e., $x_{1L} = 1$ or 0) have to be made upon $P2$'s decisions on transferred customers. If $P2$ decides to *reject* a transferred customer from $P1$, the optimal accept/reject decision for $P1$'s own low-fare customer can be presented as:

$$\begin{cases} x_{1L} = 1, & \text{if } \delta_1(t-1, n_1, n_2) < r_{1L} + c_{1L} \text{ (and } P2 \text{ rejects)} \\ x_{1L} = 0, & \text{if } \delta_1(t-1, n_1, n_2) \geq r_{1L} + c_{1L} \text{ (and } P2 \text{ rejects).} \end{cases} \tag{22}$$

On the other hand, if $P2$ decides to *accept* a transferred customer, the optimal accept/reject decision for $P1$'s own low-fare customer will be

$$\begin{cases} x_{1L} = 1, & \text{if } \delta_1(t-1, n_1, n_2) < \alpha_1(t-1, n_1, n_2) \text{ (and } P2 \text{ accepts)} \\ x_{1L} = 0, & \text{if } \delta_1(t-1, n_1, n_2) \geq \alpha_1(t-1, n_1, n_2) \text{ (and } P2 \text{ accepts),} \end{cases} \tag{23}$$

where $\alpha_1(t-1, n_1, n_2) = \mu_{1L}\xi_1(t-1, n_1, n_2) + r_{1L} + c_{1L}$. Similarly, we can obtain the optimal decisions for $P2$.

We note that, for Pi $(i = 1, 2)$, there are four possible combinations of strategy mix (x_{iL}, y_{iL}). To simplify the expression, we denote M_i^1 as $(1, 1)$, M_i^2 $(0, 1)$, M_i^3 $(1, 0)$, and M_i^4 $(0, 0)$ respectively. Referring to (21), (22) and (23), we find that the optimal solution pair is determined in terms of $\delta_i(t-1, n_i, n_j)$ and the relations among the three critical values of r_{iL}, $\alpha_1(t-1, n_1, n_2)$, and $r_{iL} + c_{iL}$. In order to identify the optimal strategy mix in the different situations, we now examine the properties of $V_i(t, n_i, n_j)$.

Theorem 5 *For any $t \in [1, T]$, $V_i(t, n_i, n_j)$ $(i, j = 1, 2$ and $i \neq j)$ has the following properties:*

(1) *$V_i(t, n_i, n_j)$ is non-decreasing in n_i, and non-increasing in n_j;*

(2) *$\delta_i(t, n_i, n_j)$ is non-increasing in n_i and n_j;*

(3) *$\delta_i(t, n_i, n_j) - \gamma\xi_i(t, n_i, n_j)$, with $0 \leq \gamma \leq 1$, is non-increasing in n_i and n_j.* (24)

Proof. Again, we shall use induction to prove this theorem. First, let us verify these properties for the last period $(t = 1)$. We see that the player has to accept any low-fare booking request in last period as long as he has unsold rooms on hand. Then, from (20), we obtain $V_i(1, n_i, n_j) = \lambda_{iH}r_{iH} + \lambda_{iL}r_{iL}$ for any $n_i, n_j > 0$. Thus, all properties in (24) hold.

Next, assuming that for any $n_i, n_j > 0$, these properties hold in period $t - 1$,

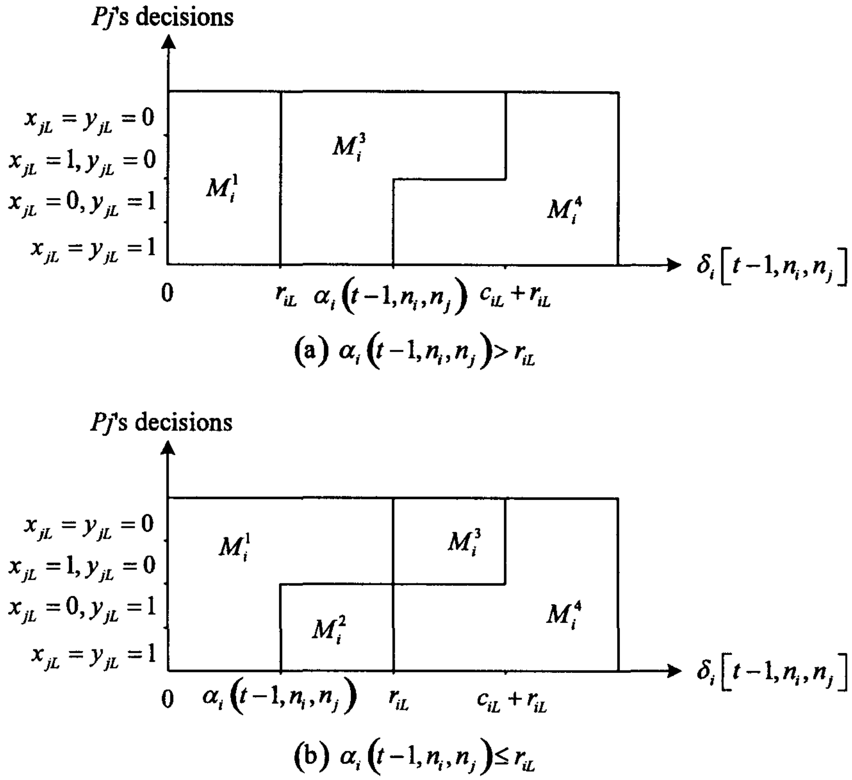


Figure 11. Pi's complete optimal strategy mixes corresponding to Pj's decisions in period t.

we wish to prove that the properties hold for period t. Actually, properties (1) and (2) indicate that $V_i(t, n_i, n_j)$ is non-decreasing quasi-concave in n_i, and non-increasing quasi-concave in n_j. We also note that $\xi_i(t-1, n_i, n_j) \le 0$ since $V_i(t-1, n_i, n_j)$ is non-increasing in n_j. Thus, $\alpha_i(t-1, n_i, n_j) \le r_{iL} + c_{iL}$. According to (21), (22) and (23), we describe Pi's optimal strategy corresponding to Pj's decisions in Figure 11. In any situation, Pi should choose the corresponding strategy mix in order to maximize his expected revenue. For instance, if Pj's decisions in period t are $x_{jL} = 0$ and $y_{jL} = 0$, Pi's expected revenue will be

$$\begin{aligned} V_i\left(t,n_i,n_j\right) = & \ (1-\lambda_{iH}-\lambda_{iL}-\lambda_{jL})V_i\left(t-1,n_i,n_j\right)+\lambda_{jL}\left(1-\mu_{jL}\right)V_i\left(t-1,n_i,n_j\right) \\ & +\lambda_{iL}\max\left[V_i\left(t-1,n_i-1,n_j\right)+r_{iL},V_i\left(t-1,n_i,n_j\right)-c_{iL}\right] \\ & +\lambda_{jL}\mu_{jL}\max\left[V_i\left(t-1,n_i-1,n_j\right)+r_{iL},V_i\left(t-1,n_i,n_j\right)\right] \\ & +\lambda_{iH}[V_i\left(t-1,n_i-1,n_j\right)+r_{iH}]. \end{aligned} \tag{25}$$

By assumption, we find that the RHS of (25) is a combination of terms that is non-decreasing quasi-concave in n_i, and non-increasing quasi-concave in n_j. Thus, in this situation, properties (1) and (2) hold. It is not difficult to validate that in other situations, $V_i\left(t,n_i,n_j\right)$ can also be expressed as a combination of terms which satisfy properties (1) and (2). analogous to the above procedure, we can also prove that $\delta_i\left(t,n_i,n_j\right)-\gamma\xi_i\left(t,n_i,n_j\right)$ $(0\le\gamma\le 1)$ is non-increasing in n_i and n_j, Hence, by induction, all of the properties can be propagated to the other t values. ■

The properties shown in Theorem 5 are intuitive with the following managerial implications:

- In a booking period t, there exists a *critical booking capacity*, $\hat{n}_i(t)$, for Pi, such that any transferred low-fare booking request should be accepted for $n_i > \hat{n}_i(t)$, and any transferred low-fare booking request should be rejected for $n_i \le \hat{n}_i(t)$.
- The non-decreasing property of $V_i(t,n_i,n_j)$ shows that Pi will be better off if Pj has few rooms unsold in period t. This is so, since each player is more likely to reject a low-fare customer when the number of his unsold rooms is lower and

there are still many periods left.

- The non-increasing property of $\delta_i(t, n_i, n_j)$ well fits the classical "marginal revenue decreasing" law in economics.

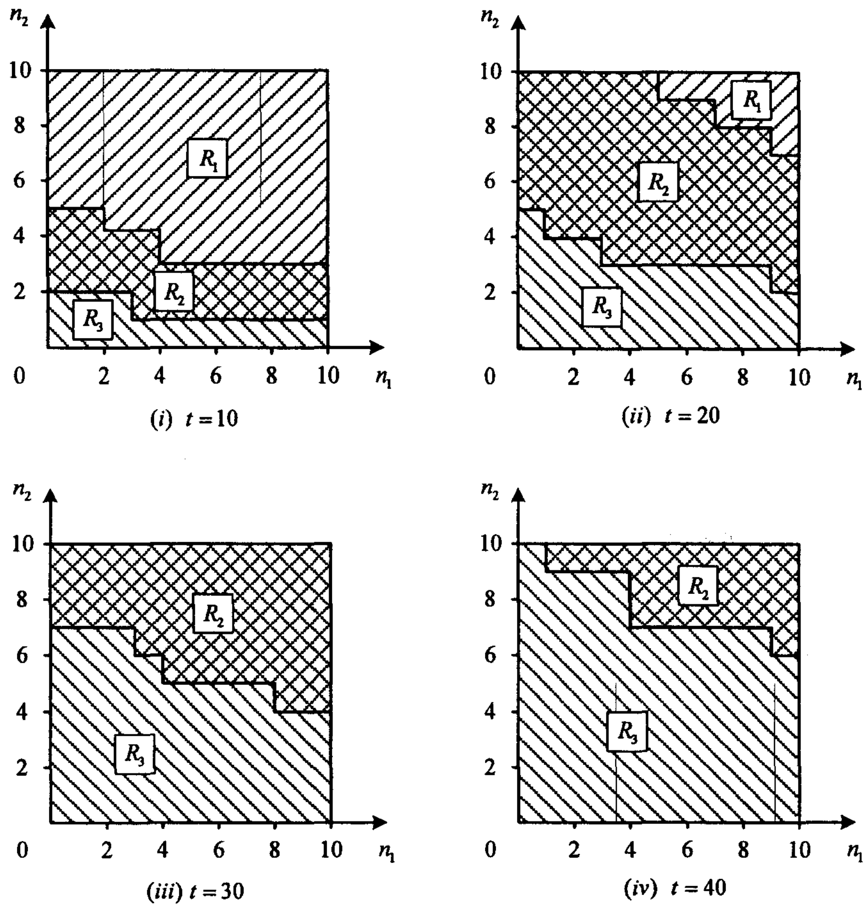


Figure 12. Dynamic best responses of $P1$ when both hotels have rooms and $P2$ follows a non-optimal policy. In region R_1, $P1$ accepts both his low-fare customer and $P2$'s transfer customer; in region R_2, he accepts only his low fare customer and in region R_3 he rejects any low-fare customer.

Example 6 *Now let us examine the optimal decisions of one player, say, $P1$, in* *case both players have available rooms. We use the same data as in Table* 11. *In this case, P*2 *chooses an arbitrary (non-optimal) dynamic policy and P*1 *determines his best response after observing P*2*'s decision in each period. We assume that P*2 *always adopts a first-come-first-served (FCFS) policy where* $x_{2L} = y_{2L} = 1$*; that is, P*2 *accepts both her own low-fare customer and the transferred customer from P*1. *Faced with this policy, P*1 *determines his best response from* (20) *for each period. In Figure* 12, *we present the acceptance/rejection regions for P*1 *at four different time points;* $t = 10, 20, 30$ *and* 40 *(periods-to-go) with 10 rooms remaining in each hotel. As in Example* 5, *in region* R_1, *P*1 *accepts both his low-fare customer and P*2*'s transfer customer. In region* R_2, *P*1 *accepts only his low fare customer and in region* R_3, *he rejects any low-fare customer. It is worth noting that when* $t = 40$, *i.e., when 40 time periods are left, P*1 *almost always rejects any low fare customer, but when* $t = 10$, *he accepts both his low-fare customers and P*2*'s transfers provided that both hotels have sufficient number of rooms left.* ⧫

4.2 Non-cooperative Solution

In this section we shall examine Nash and Stackelberg equilibria in a non-cooperative framework by using the results obtained in Section 4.1. When the players make their decisions simultaneously in each period, the Nash equilibrium applies. On the other hand, when one of the players can act before the other one, we obtains the Stackelberg equilibrium.

4.2.1 Nash Equilibrium

We assume in this section that the two players make their accept/reject decisions simultaneously in each period. We also make the standard assumption that the two players are "rational", i.e., one would not lower his objective function for the sole purpose of inflicting damage on the opponent. Thus, the best strategy they should adopt will give rise to the Nash equilibrium. Since we assume that both players are rational, some optimal strategy mixes shown in Figure 11 may not apply. For example, if $P1$ always accepts his own low-fare customer ($x_{1L} = 1$) in period t, there will be no transferred low-fare customer from $P1$ to $P2$. Thus, $V_1(t, n_1, n_2)$ $(n_1, n_2 > 0)$ is the same for $x_{1L} = 1$, $y_{1L} = 1$ and $x_{1L} = 1$, $y_{1L} = 0$ if $x_{2L} = 1$. For the game problem in this section, we define the possible strategies of Pi as follows:

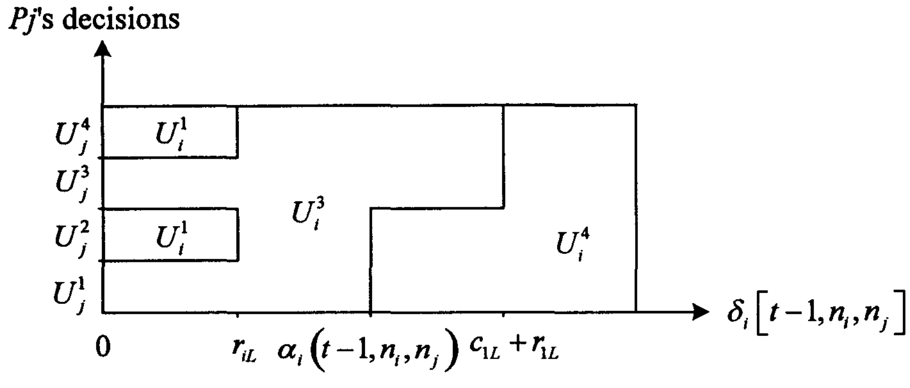
(1) U_i^1 : accept any low-fare customer;

(2) U_i^2 : accept only transferred low-fare customer,

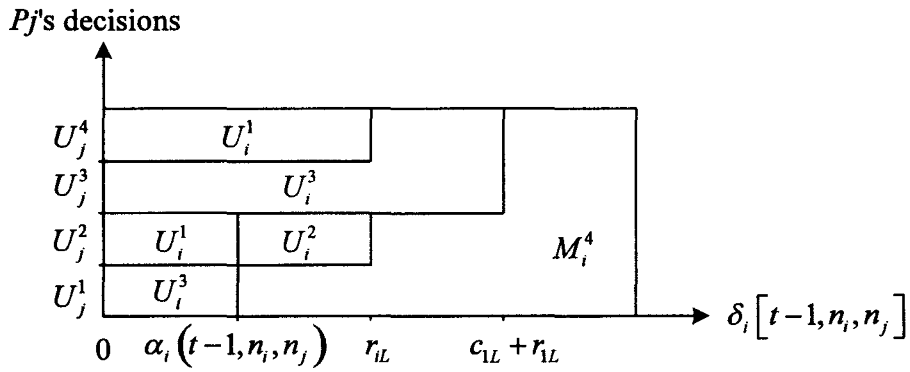
(3) U_i^3 : accept only own low-fare customer, and

(4) U_i^4 : reject any low-fare customer.

Note that the third strategy "accept only own low-fare customer" means that either the transferred low-fare customers from other player should be rejected, or there are no such customers. Thus, we describe Pi's best response corresponding to Pj's decisions in Figure 13. In any situation, Pi should choose the corresponding strategy mix in order to maximize his expected revenue.



(a) $\alpha_i(t-1,n_i,n_j) > r_{iL}$



(b) $\alpha_i(t-1,n_i,n_j) \leq r_{iL}$

Figure 13. Pi's best response corresponding to Pj's decisions in period t.

Mathematically, the Nash strategy is a pair (U_1^N, U_2^N) where $U_i^N \in \{U_i^1, U_i^2, U_i^3, U_i^4\}$, $i = 1, 2$, such that each player's total expected revenue with this mix is always better than those with other strategy mixes. This strategy results in a equilibrium as it ensures that Pi $(i = 1, 2)$ will not receive more than $V_i(t, n_i, n_j)$ with (U_1^N, U_2^N)

if he deviates from it unilaterally. Before examining the Nash strategy, let us first investigate the optimal strategies of one player in response to the chosen strategies of the other player which are the best responses. According to the optimal strategy mixes shown in Figure 13, we find that there are five mutually exclusive cases in which Pi's best response exhibits a unique form:

(1) $\delta_i < \min(r_{iL}, \alpha_i) : \; U_i^R(U_j) = \begin{cases} U_i^1, & \text{if } U_j = U_j^2, U_j^4; \\ U_i^3, & \text{if } U_j = U_j^1, U_j^3. \end{cases}$;

(2) $r_{iL} \le \delta_i < \alpha_i \; (r_{iL} < \alpha_i) : \; U_i^R(U_j) = U_i^3$, for $\forall U_j$;

(3) $\alpha_i \le \delta_i < r_{iL} \; (\alpha_i \le r_{iL}) : \; U_i^R(U_j) = \begin{cases} U_i^1, & \text{if } U_j = U_j^4; \\ U_i^2, & \text{if } U_j = U_j^2; \\ U_i^3, & \text{if } U_j = U_j^3; \\ U_i^4, & \text{if } U_j = U_j^1. \end{cases}$;

(4) $\max(r_{iL}, \alpha_i) \le \delta_i < r_{iL} + c_{iL} : \; U_i^R(U_j) = \begin{cases} U_i^3, & \text{if } U_j = U_j^3, U_j^4; \\ U_i^4, & \text{if } U_j = U_j^1, U_j^2. \end{cases}$; and

(5) $r_{iL} + c_{iL} \le \delta_i : \; U_i^R(U_j) = U_i^4$, where δ_i and α_i are for state $(t-1, n_i, n_j)$, $i, j = 1, 2$, and $i \ne j$.

Therefore, there are twenty-five different combinations of the best responses of two players in (U_1, U_2) plane. Due to the symmetrical property of these combinations, we will show only fifteen of them in the following figures. In practice, we use the cells filled by vertical lines to present $P1$'s best responses corresponding to $P2$'s strategies. On the other hand, the cells filled by horizontal lines are presented as $P2$'s best responses corresponding to $P1$'s strategies. Thus, the cells which are filled by both vertical and horizontal lines are the Nash equilibria.

Theorem 6 *In each period, the optimal rational behavior of the two players has a unique Nash equilibrium* $\left(U_i^N, U_j^N\right)$ *for* $\forall t \in [1,T]$ *and* $\forall n_i, n_j > 0$, $i,j = 1,2$ *for* $i \neq j$ *as follows.*

$$
(U_i^N, U_j^N) = \begin{cases}
(U_i^1, U_j^4), & \text{if } \delta_i < r_{iL} \text{ and } r_{jL} + c_{jL} \leq \delta_j; \\
(U_i^4, U_j^1), & \text{if } r_{iL} + c_{iL} \leq \delta_i \text{ and } \delta_j < r_{jL}; \\
(U_i^3, U_j^4), & \text{if } r_{iL} \leq \delta_i < r_{iL} + c_{iL} \text{ and } r_{jL} + c_{jL} \leq \delta_j; \\
(U_i^4, U_j^3), & \text{if } r_{iL} + c_{iL} \leq \delta_i \text{ and } r_{jL} \leq \delta_j < r_{jL} + c_{jL}; \\
(U_i^3, U_j^3), & \text{if } \begin{cases} \delta_i < \alpha_i \text{ and } \delta_j < \alpha_j, \text{ or} \\ r_{iL} \leq \delta_i < r_{iL} + c_{iL} \text{ and } r_{jL} \leq \delta_j < r_{jL} + c_{jL}, \text{ or} \\ \min(r_{iL}, \alpha_i) < \delta_i < \alpha_i \text{ and } \min(r_{jL}, \alpha_j) < \delta_j < r_{jL}; \end{cases} \\
(U_i^4, U_j^4), & \text{if } r_{iL} + c_{iL} \leq \delta_i \text{ and } r_{jL} + c_{jL} \leq \delta_j.
\end{cases} \tag{26}
$$

In addition, the game admits multiple Nash equilibria (MNE) in other situations. The multiple Nash equilibria and corresponding conditions can be expressed as:

$$
\begin{cases}
(U_i^1, U_j^4) \text{ or } (U_i^3, U_j^3), & \text{if } \begin{cases} \delta_i < r_{iL} \text{ and } \max(r_{jL}, \alpha_j) \leq \delta_j \in r_{jL} + c_{jL}, \text{ or} \\ \delta_i < \alpha_i \text{ and } \alpha_j \leq \delta_j < \max(r_{jL}, \alpha_j); \end{cases} \\
(U_i^4, U_j^1) \text{ or } (U_i^3, U_j^3), & \text{if } \begin{cases} \max(r_{iL}, \alpha_i) \leq \delta_i < r_{iL} + c_{iL} \text{ and } \delta_j <, r_{jL}, \text{ or} \\ \alpha_i \leq \delta_i < \max(r_{iL}, \alpha_i) \text{ and } \delta_j < \alpha_j; \end{cases} \\
\begin{array}{l}(U_i^1, U_j^4),\ (U_i^4, U_j^1) \\ (U_i^2, U_j^2) \text{ or } (U_i^3, U_j^3),\end{array} & \text{if } \alpha_i \leq \delta_i < \max(r_{iL}, \alpha_i) \text{ and } \alpha_j \leq \delta_j < \max(r_{jL}, \alpha_j),
\end{cases} \tag{27}
$$

in which δ_i and α_i are for state $(t-1, n_i, n_j)$.

Proof. Referring to Figure 14 – 17, we notice that if both players make the decisions optimally, the game admits either a unique Nash equilibrium or multiple Nash equilibria. Accordingly, we build three matrices to show the Nash equilibria in different situations (see Figure 18). Simply categorizing areas with same Nash equilibrium, we obtain (U_i^N, U_j^N) for each situation.

We see that there are two Nash equilibria in (c), (d) of Figure 14 and (k) of Figure 17, which are (U_1^1, U_2^4) and (U_1^3, U_2^3) for all three cases. Comparing the two players' expected revenue with them, we obtain

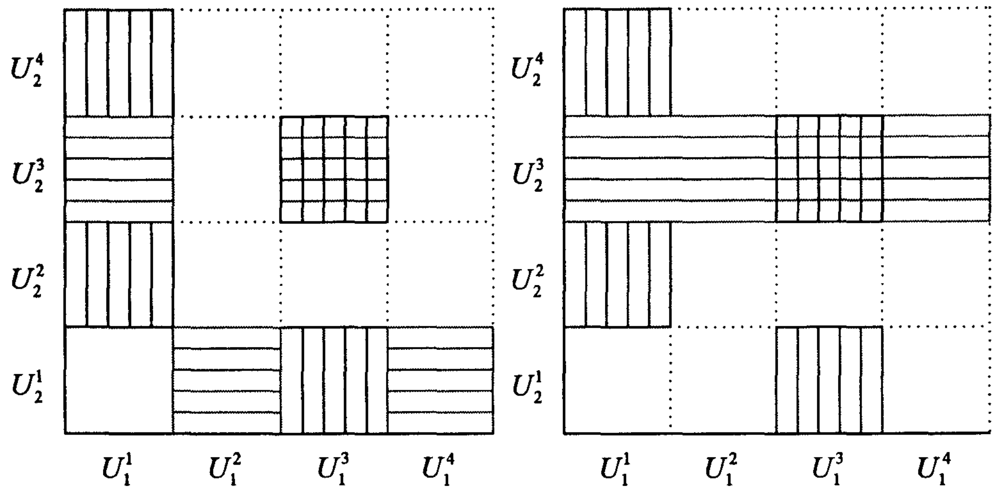
$$V_1 \mid_{(U_1^1,U_2^4)} - V_1 \mid_{(U_1^3,U_2^3)} = \lambda_{2L}\mu_{2L}[r_{1L} - \delta_1(t-1,n_1,n_2)] + \lambda_{2L}\xi_1(t-1,n_1,n_2) \quad (28)$$

and

$$V_2 \mid_{(U_1^1,U_2^4)} - V_2 \mid_{(U_1^3,U_2^3)} = \lambda_{2L}[\delta_2(t-1,n_2,n_1) - \alpha_2(t-1,n_2,n_1)]. \quad (29)$$

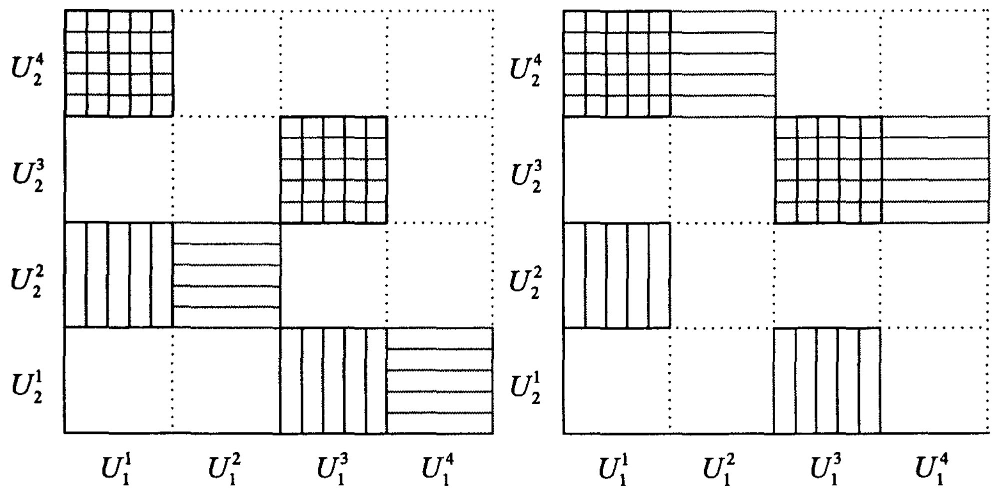
It is not difficult to find that $V_2 \mid_{(U_1^1,U_2^4)} - V_2 \mid_{(U_1^3,U_2^3)} \geq 0$ since $\delta_2(t-1,n_2,n_1) \geq \alpha_2(t-1,n_2,n_1)$ for any case. ∎

Thus, the equilibrium (U_1^1, U_2^4) is better than (U_1^3, U_2^3) for $P2$. He would more likely choose U_2^4 hoping $P1$ choose U_1^1. Unfortunately, $V_1 \mid_{(U_1^1,U_2^4)} - V_1 \mid_{(U_1^3,U_2^3)} \geq 0$ can not be guaranteed. It implies $V_1 \mid_{(U_1^1,U_2^4)}$might be less than $V_1 \mid_{(U_1^3,U_2^3)}$in some situations by which the possible strategy combination of the two players would be (U_1^3, U_2^4). Such deviation from the Nash equilibrium is dangerous since both players' revenue could be badly decreased. However, if both players have noticed this "danger", they might not choose the strategies which lead to (U_1^3, U_2^4). Thus, the two players are more



(a) $P1$'s Case 1 vs $P2$'s Case 1: $(U_1^N, U_2^N) = (U_1^3, U_2^3)$

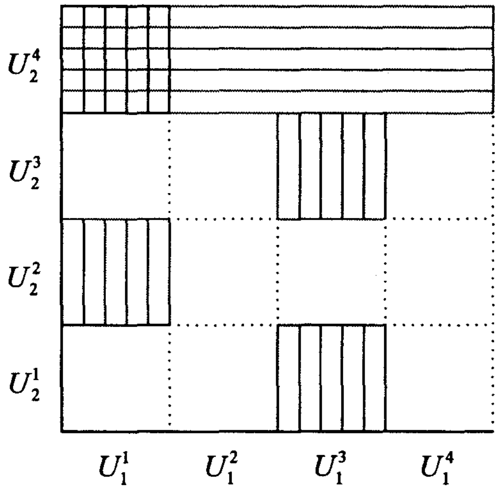
(b) $P1$'s Case 1 vs $P2$'s Case 2: $(U_1^N, U_2^N) = (U_1^3, U_2^3)$



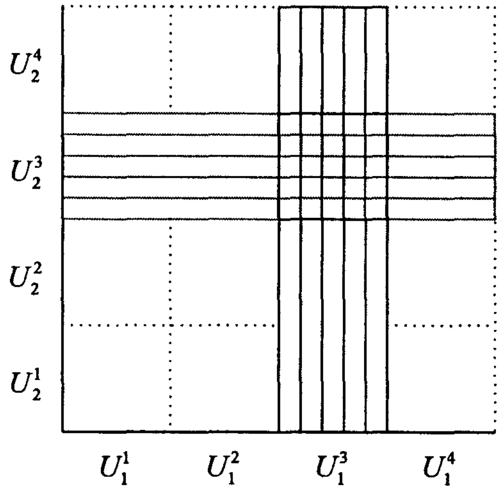
(c) $P1$'s Case 1 vs $P2$'s Case 3: $(U_1^N, U_2^N) = (U_1^1, U_2^4)$ or (U_1^3, U_2^3)

(d) $P1$'s Case 1 vs $P2$'s Case 4: $(U_1^N, U_2^N) = (U_1^1, U_2^4)$ or (U_1^3, U_2^3)

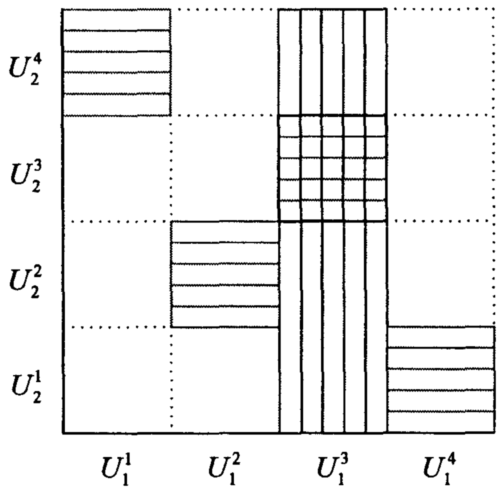
Figure 14. Nash equilibria in the situations where $P1$'s case 1 vs $P2$'s cases 1–4.



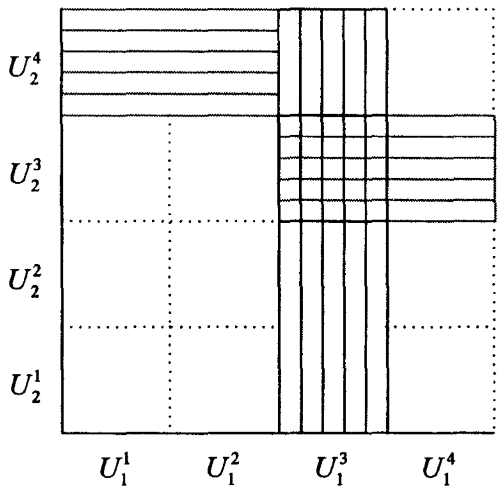
(*e*) *P*1's Case 1 vs *P*2's Case 5:
$(U_1^N, U_2^N) = (U_1^1, U_2^4)$



(*f*) *P*1's Case 2 vs *P*2's Case 2:
$(U_1^N, U_2^N) = (U_1^3, U_2^3)$

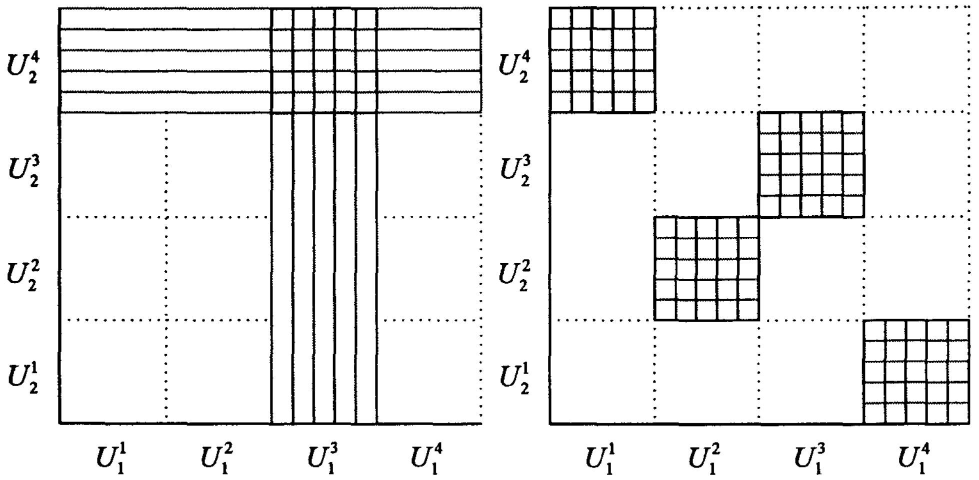


(*g*) *P*1's Case 2 vs *P*2's Case 3:
$(U_1^N, U_2^N) = (U_1^3, U_2^3)$



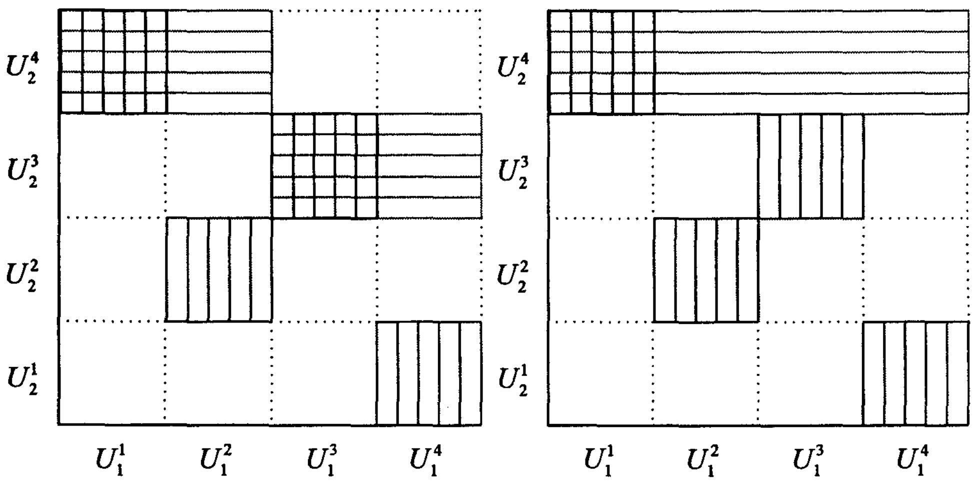
(*h*) *P*1's Case 2 vs *P*2's Case 4:
$(U_1^N, U_2^N) = (U_1^3, U_2^3)$

Figure 15. Nash equilibria in the situations where $P1$'s case 1 vs $P2$'s case 5, and $P1$'s case 2 vs $P2$'s cases 2–4.



(i) P1's Case 2 vs P2's Case 5: $(U_1^N, U_2^N) = (U_1^3, U_2^4)$

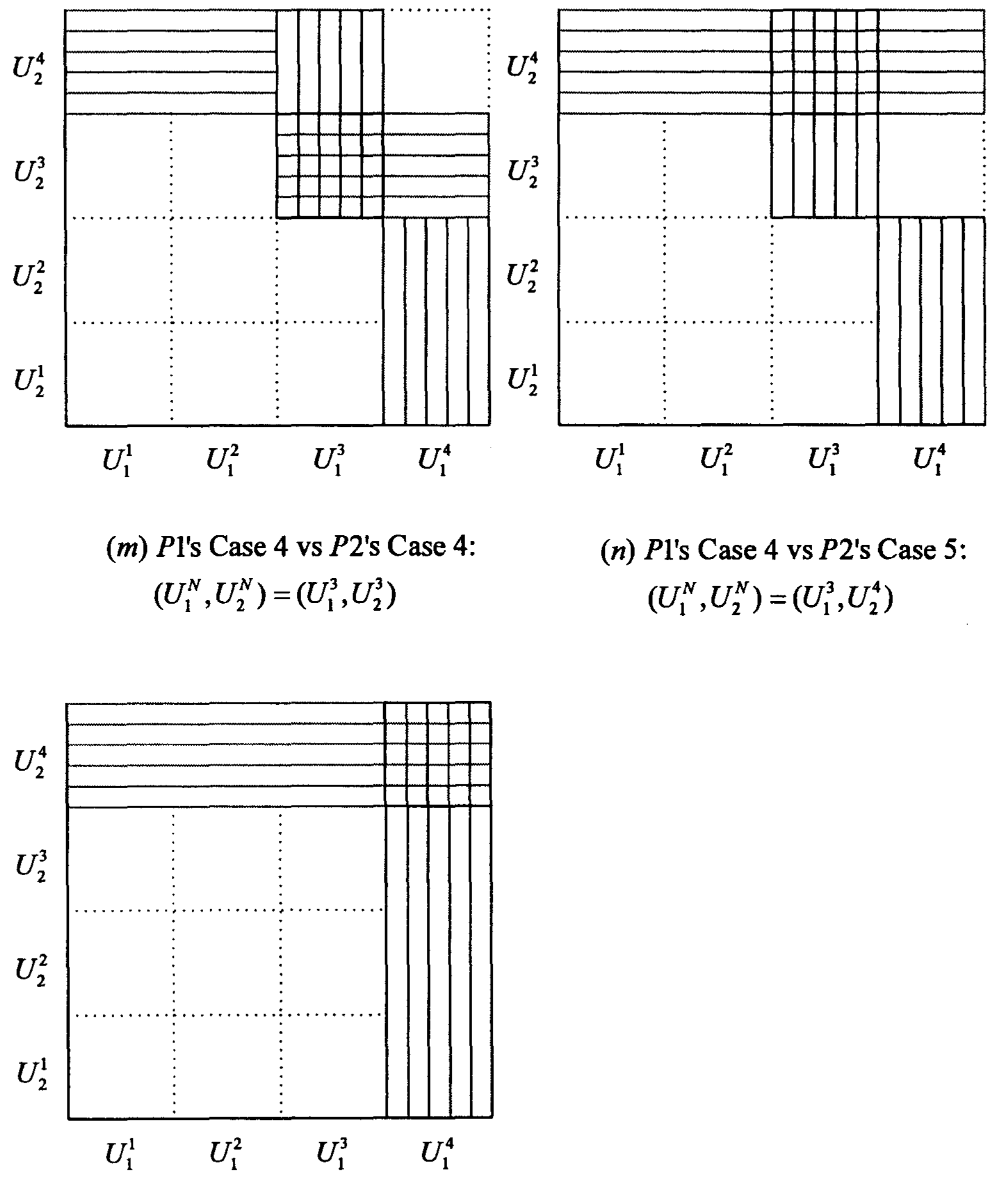
(j) P1's Case 3 vs P2's Case 3: $(U_1^N, U_2^N) = (U_1^1, U_2^4)$, (U_1^2, U_2^2) (U_1^3, U_2^3), or (U_1^4, U_2^1)



(k) P1's Case 3 vs P2's Case 4: $(U_1^N, U_2^N) = (U_1^1, U_2^4)$ or (U_1^3, U_2^3)

(l) P1's Case 3 vs P2's Case 5: $(U_1^N, U_2^N) = (U_1^1, U_2^4)$

Figure 16. Nash equilibria in the situations where $P1$'s case 2 vs $P2$'s case 5, and $P1$'s case 3 vs $P2$'s cases 3–5.



(*m*) *P*1's Case 4 vs *P*2's Case 4: $(U_1^N, U_2^N) = (U_1^3, U_2^3)$

(*n*) *P*1's Case 4 vs *P*2's Case 5: $(U_1^N, U_2^N) = (U_1^3, U_2^4)$

(*o*) *P*1's Case 5 vs *P*2's Case 5: $(U_1^N, U_2^N) = (U_1^4, U_2^4)$

Figure 17. Nash equilibria in the situations where $P1$'s case 4 vs $P2$'s cases 4–5, and $P1$'s case 5 vs $P2$'s case 5.

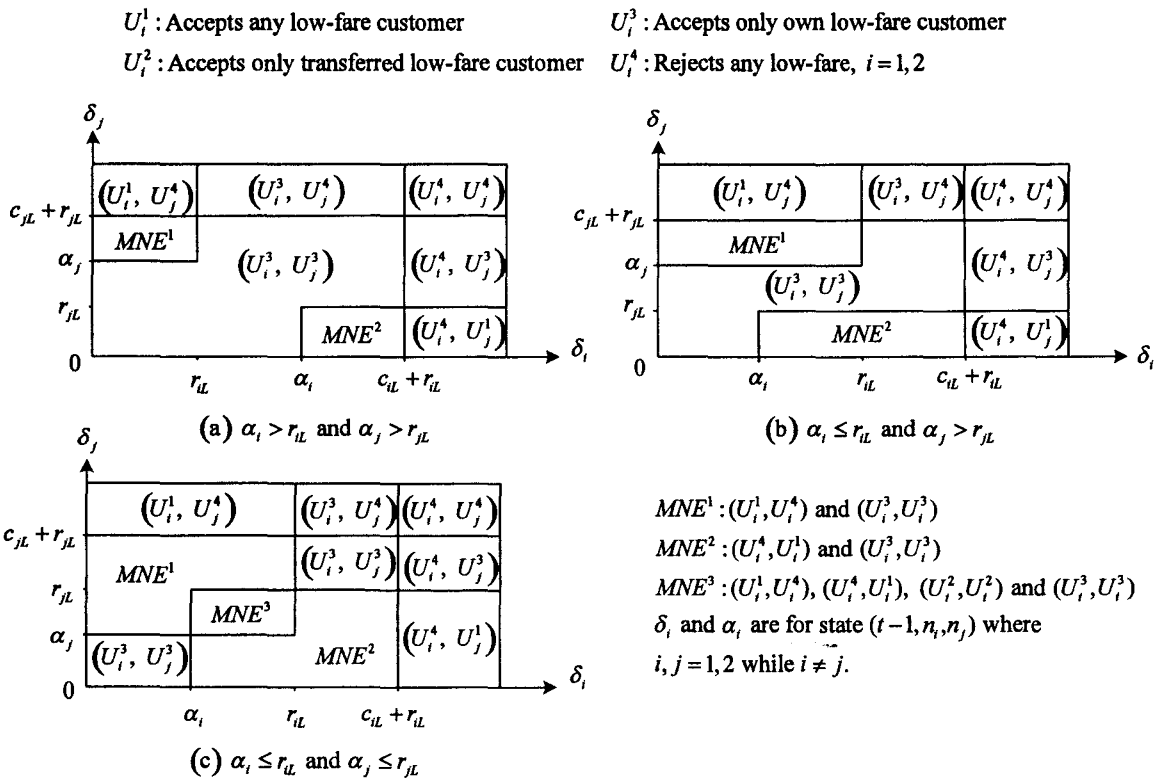


Figure 18. Nash Equilibrium in period t, for $\forall n_1, n_2 > 0$.

likely to make their decisions by negotiation (e.g., they may choose the equilibrium which maximizes their joint revenue). However, this might cause another problem: how do they share those 'extra' revenue by such negotiation?

In addition, there are four Nash equilibria in case (j) of Figure 16: (U_1^1, U_2^4), (U_1^2, U_2^2), (U_1^3, U_2^3) and (U_1^4, U_2^1). Referring to Pi's objective function, we have

$$\begin{aligned} V_i \mid_{(U_i^1, U_j^4)} = \; & V_i(t-1, n_i, n_j) + \lambda_{iH} r_{iH} + (\lambda_{jL}\mu_{jL} + \lambda_{iL}) r_{iL} \\ & -(\lambda_{iL} + \lambda_{iH} + \lambda_{jL}\mu_{jL})\delta_i(t-1, n_i, n_j) - \lambda_{jH}\xi_i(t-1, n_i, n_j), \end{aligned}$$

$$V_i \mid_{(U_i^1,U_j^4)} = V_i(t-1,n_i,n_j) + \lambda_{iH}r_{iH} + \lambda_{jL}\mu_{jL}r_{iL} - \lambda_{iL}c_{iL}$$
$$-(\lambda_{iH} + \lambda_{jL}\mu_{jL})\delta_i(t-1,n_i,n_j) - (\lambda_{jH} + \lambda_{iL}\mu_{iL})\xi_i(t-1,n_i,n_j),$$

$$V_i \mid_{(U_i^3,U_j^3)} = V_i(t-1,n_i,n_j) + \lambda_{iH}r_{iH} + \lambda_{iL}r_{iL}$$
$$-(\lambda_{iL} + \lambda_{iH})\delta_i(t-1,n_i,n_j) - (\lambda_{jL} + \lambda_{jH})\xi_i(t-1,n_i,n_j),$$

and

$$V_i \mid_{(U_i^4,U_j^1)} = V_i(t-1,n_i,n_j) + \lambda_{iH}r_{iH} - \lambda_{iL}c_{iL}$$
$$-\lambda_{iH}\delta_i(t-1,n_i,n_j) - (\lambda_{jL} + \lambda_{jH} + \lambda_{iL}\mu_{iL})\xi_i(t-1,n_i,n_j).$$

Comparing Pi's expected revenue when using these equilibria, we obtain

$$V_i \mid_{(U_i^4,U_j^1)} - V_i \mid_{(U_i^3,U_j^3)} = \lambda_{iL}(\delta_i - \alpha_i) \geq 0,$$

$$V_i \mid_{(U_i^2,U_j^2)} - V_i \mid_{(U_i^1,U_j^4)} = \lambda_{iL}(\delta_i - \alpha_i) \geq 0,$$

and

$$\begin{aligned} V_i \mid_{(U_i^4,U_j^1)} - V_i \mid_{(U_i^2,U_j^2)} &= \lambda_{iL}\mu_{iL}(\delta_i - r_{iL}) - \lambda_{iL}\xi_i \\ &\geq (\tfrac{1}{\mu_{iL}} - \mu_{jL})(r_{iL} - \delta_i) + \tfrac{c_{iL}}{\mu_{iL}} \\ &> 0. \end{aligned}$$

Thus, (U_1^4, U_2^1) is superior to the other three equilibria $P1$; on the other hand, (U_1^1, U_2^4) is the best equilibrium for $P2$. Again, in this case, we can not choose an equilibrium for the two players. The following example will show this four Nash equilibria case.

Example 7 *In order to show the four Nash equilibria situation, we use the following values (see Table 12) for the room rate, penalty cost, and transfer rate of K-fare class* *customers in Pi* $(K = L, H$ *and* $i = 1, 2)$.

	Low-fare $(K = L)$				High-fare $(K = H)$			
	r_{iL}	c_{iL}	λ_{iL}	μ_{iL}	r_{iH}	c_{iH}	λ_{iH}	μ_{iH}
P1	\$99	\$2	0.10	1	\$159	\$5	0.45	1
P2	\$105	\$3	0.35	1	\$165	\$6	0.05	1

Table 12. Prices, rejection costs, arrival and transfer rates of $P1$ and $P2$ for the four Nash equilibrium case.

We assume there is only 2 periods left and each player has only one room unsold. We then calculate the total expected revenue of the two player with each possible strategy mixes combination. The results are shown in Table 13.

P2 \ P1	U_1^1	U_1^2	U_1^3	U_1^4
U_2^4	**(126.38,121.03)**	(124.42,112.88)	(120.23,92.52)	(118.28,84.37)
U_2^3	N/A	N/A	**(135.14,112.96)**	(133.19,104.81)
U_2^2	N/A	**(128.68,118.52)**	N/A	(122.54,90.01)
U_2^1	N/A	N/A	N/A	**(137.45,110.45)**

Table 13. The total expected revenue of the two players with different strategy mixes.

We see that there are four Nash equilibria, which are (U_1^4, U_2^4), (U_1^2, U_2^2), (U_1^3, U_2^3), *and* (U_1^4, U_2^4). *Comparing the revenue obtained by these equilibria, none of them can be the best for both players. Specifically, the equilibrium* (U_1^4, U_2^1) *is the best for* $P1$, *and* (U_1^1, U_2^4) *is the best for* $P2$. ♦

From Example 7, we verify our statement, which says, in the MNE case there is not an optimal decision rule for the two players. Then, we can not calculate one player's total expected revenue in that period. As a result, the optimal strategies for the all former periods are also uncertain. However, our numerical experiments show that in most cases, the game admits only one equilibrium. We will use the following example to illustrate the unique Nash equilibrium cases.

Example 8 *Here, we again use the same values as in Table* 11 *of Example* 5 *for the room rates, penalty costs, and transfer rates for both hotels. The goal in this example is to demonstrate the unique Nash equilibrium arising from equation (*26*) in Theorem* 6. *As in Example* 6, *we consider four time periods (to-go), i.e., 10, 20, 30 and 40. Referring to Figure* 19, *we have up to six different regions which are defined as follows: Region* R_1 *corresponds to* (U_1^2, U_2^2), *that is, both players accept only their low-fare customers. In region* R_2, *we have* (U_1^4, U_2^2) *which corresponds to* $P1$ *reject any low-fare customer and* $P2$*accepting only her own low-fare customer. Similarly, in region* R_3 *we have* (U_1^2, U_2^4), *in* R_4, *we have* (U_1^4, U_2^1). *Finally, in* R_5, *the policy is* (U_1^1, U_2^4) *and in* R_6, *we have* (U_1^4, U_2^4).

According to Figure 19, *we note that when* $t = 40$ *(periods-to-go), both hotels reject any low fare customer—hoping that high-fare customers will arrive in later periods. However, at* $t = 10$, *for a large combination of high* (n_1, n_2) *values, both players accept any low-fare customer, which is intuitive. At* $t = 10$, *when only a few rooms remain, the hotels can be more "choosy" and can reject low-fare customers.*

We have also compared the total expected revenues for both players under two scenarios: Scenario 1: $P2$ *uses an arbitrary dynamic policy (i.e., FCFS) and* $P1$ *responds optimally to* $P2$*'s decisions (as presented in Section* 4.1.2, *and Scenario 2:*

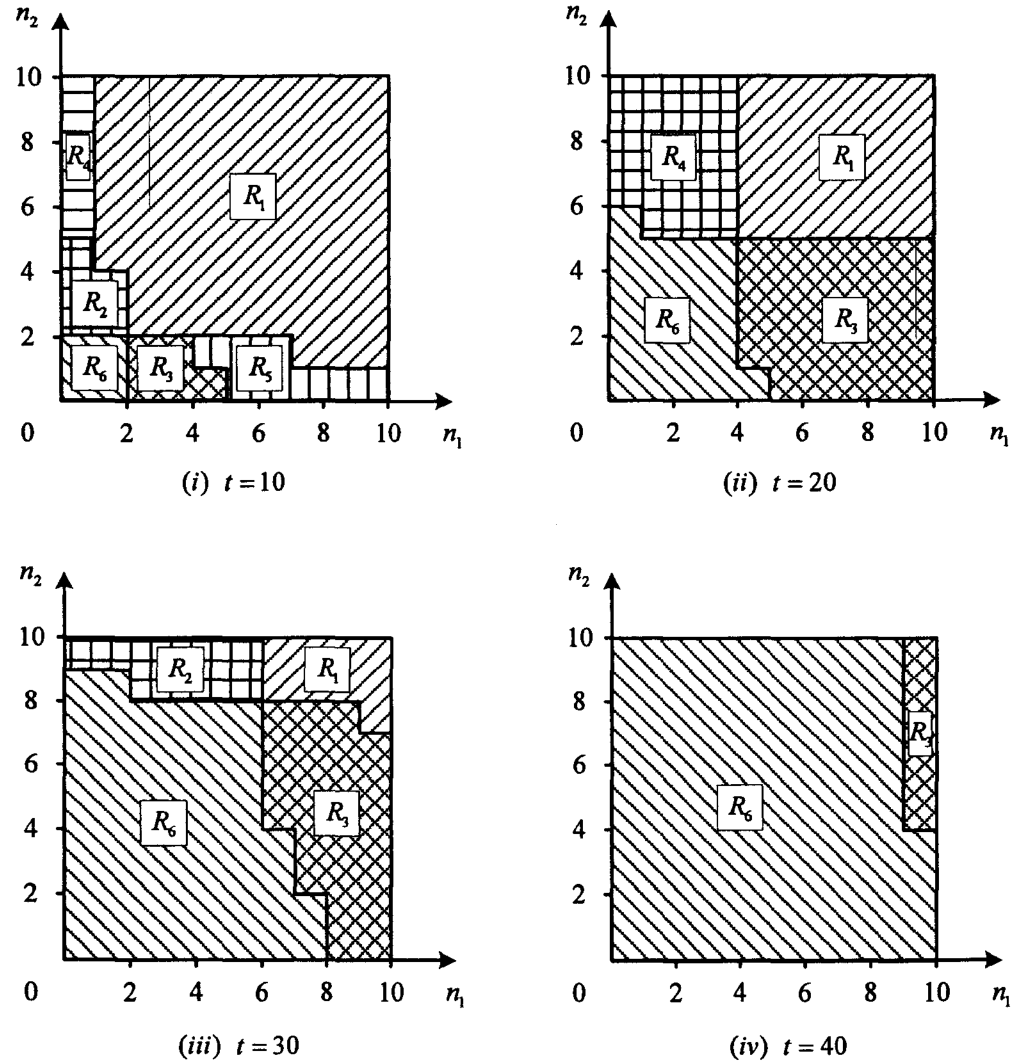


Figure 19. Nash equilibrium for both players. Here, region R_1 corresponds to (U_1^2, U_2^2), that is, both players accept only their low-fare customers. In region R_2, we have (U_1^4, U_2^2) which correponds to $P1$ rejecting any low-fare customer and $P2$ accepting only her own low-fare customer. Similarly, in region R_3 we have (U_1^2, U_2^4), in R_4, we have (U_1^4, U_2^1). Finally, in R_5, the policy is (U_1^1, U_2^4) and in R_6, we have (U_1^4, U_2^4).

	t: Time-to-go							
	15	20	25	30	35	40	45	50
Scenario 1: $P1$ acts optimally and $P2$ uses FCFS rule with $x_{2L} = y_{2L} = 1$								
V_1^*:	826.17	1050.06	1119.59	1141.57	1163.36	1185.40	1211.93	1239.25
V_2:	801.89	1086.06	1169.73	1163.70	1157.29	1151.32	1142.94	1133.54
Scenario 2: Nash strategies								
V_1^N:	826.17	1050.06	1111.81	1120.00	1125.88	1131.11	1136.14	1141.16
V_2^N:	801.89	1086.06	1175.37	1184.52	1188.75	1191.92	1194.96	1198.37
revenue variations (%) as players move from Scenario 1 to Scenario 2:								
$P1$:	0	0	-0.70	-1.93	-3.33	-4.80	-6.67	-8.60
$P2$:	0	0	0.48	1.76	2.65	3.41	4.35	5.41

Table 14. The expected profits of the two players in two Scenarios: $P1$ acts optimally; $P2$ use FCFS VS. Nash game.

Both $P1$ and $P2$ implement the Nash strategies as discussed in this section. Referring to Table 14, we observe that as players move from Scenario 1 (where $P2$ acts in a non-optimal fashion) to Scenario 2 (where both adopt the Nash strategy), $P1$'s expected revenues decreases and $P2$' expected revenue increases. ♦

4.2.2 Stackelberg Equilibrium

In the previous section, the players used the Nash strategy under the assumption that they make their decisions simultaneously. Now we consider another non-cooperative situation which is leader-follower Stackelberg game. Without loss of generality, in this section we assume $P1$ as the leader and $P2$ as the follower in our game theoretical framework. In each period t, $P1$ announces his strategy mix, U_1, first, and $P2$ chooses an optimal accept/reject decision as a function of U_1 to maximize his expected revenue. Recall the best response from Section 4.2.1, we obtain $P2$'s best response for acceptance/rejection of transferred low-fare customer in period t $(t \geq 1)$ for $\forall n_i, n_j > 0$, $i, j = 1, 2$ and $i \neq j$ as,

$$y_{2L}^b(U_1) = \begin{cases} 0, & \text{if } \begin{cases} \delta_2(t-1, n_2, n_1) \geq r_{2L} \text{ and } x_{1L} = 0, \text{ or} \\ x_{1L} = 1; \end{cases} \\ 1, & \text{if } \delta_2(t-1, n_2, n_1) < r_{2L} \text{ and } x_{1L} = 0; \end{cases} \tag{30}$$

Similarly, $P2$'s best response for acceptance/rejection of his own low-fare customer in period t, $x_{2L}^b(U_1)$ can be derived as

$$x_{2L}^b(U_1) = \begin{cases} 1, & \text{if } \begin{cases} \delta_2(t-1, n_2, n_1) < r_{2L} + c_{2L} \text{ and } y_{1L} = 0, \text{ or} \\ \delta_2(t-1, n_2, n_1) < \alpha_2(t-1, n_2, n_1) \text{ and } y_{1L} = 1; \end{cases} \\ 0, & \text{if } \begin{cases} \delta_2(t-1, n_2, n_1) \geq r_{2L} + c_{2L} \text{ and } y_{1L} = 0, \text{ or} \\ \delta_2(t-1, n_2, n_1) \geq \alpha_2(t-1, n_2, n_1) \text{ and } y_{1L} = 1. \end{cases} \end{cases} \tag{31}$$

Proposition 3 *When both players have available rooms $(n_1, n_2 > 0)$ at the beginning of period t, the Stackelberg equilibria and corresponding conditions can be expressed as follows.*

$$(U_1^S, U_2^S) = \begin{cases} (U_1^1, U_2^4), & \text{if } \begin{cases} \delta_1 < r_{1L} \text{ and } c_{2L} + r_{2L} \leq \delta_2, \text{ or} \\ \delta_1 < r_{1L} + \xi_1/\mu_{2L} \text{ and } \alpha_2 \leq \delta_2 < c_{2L} + r_{2L}; \end{cases} \\ (U_1^3, U_2^4), & \text{if } r_{1L} \leq \delta_1 < c_{1L} + r_{1L} \text{ and } c_{2L} + r_{2L} \leq \delta_2; \\ (U_1^4, U_2^1), & \text{if } \alpha_1 \leq \delta_1 \text{ and } \delta_2 < r_{2L}; \\ (U_1^4, U_2^3), & \text{if } c_{1L} + r_{1L} \leq \delta_1 \text{ and } r_{2L} \leq \delta_2 < c_{2L} + r_{2L}; \\ (U_1^4, U_2^4), & \text{if } c_{1L} + r_{1L} \leq \delta_1 \text{ and } c_{2L} + r_{2L} \leq \delta_2; \\ (U_1^3, U_2^3), & \text{otherwise.} \end{cases} \tag{32}$$

Proof. We use the exhaustive enumeration method to prove this proposition by examining each case listed in Figure 14 – 17. For example, if $\alpha_2 \leq r_{2L}$ and $r_2 > \delta_2(t-1, n_2, n_1) \geq \alpha_{2L}$, then

$$U_2^b = \left(x_{2L}^b(U_1), y_{2L}^b(U_1)\right) = \begin{cases} (0,1), & \text{if } U_1 = (1,1) \text{ or } U_1 = (0,1); \\ (1,1), & \text{if } U_1 = (1,0) \text{ or } U_1 = (0,0). \end{cases}$$

Checking each combination of $(U_1, U_2^b(U_1))$, we find that $P1$ could obtain maximum expected revenue if he announce U_1^3 to $P2$ who will choose U_2^3 as the strategy to response U_1^3. Then the optimal strategy combination of the two players, which is the Stackelberg equilibrium, is (U_1^3, U_2^3). Similarly, we can also analyze the optimal strategies of $P1$ in all other cases of Figure 14 – 17. Finally, we obtain (U_1^S, U_2^S) for each situation as described above (see (32)). ■

It can be seen when $(U_1^S, U_2^S) = (U_1^N, U_2^N)$, the corresponding condition in (26) is tighter than that in (32). It implies that the if the game admits a unique Nash equilibrium in state (t, n_1, n_2) and the all lower states, the leader-follower stackelberg game is identical to the Nash game. Based on the data given in Example 8, we calculate the Stackelberg equilibrium and corresponding revenue for both players ($P1$ is assumed as leader). We obtain the same results as shown in Table 14. Actually, comparing the Stackelberg equilibria in (32) with the MNE in (27), we also find that the Stackelberg equilibrium is still one of the multiple Nash equilibria. For example, in four Nash equilibria case the Stackelberg equilibrium is (U_1^4, U_2^1) which is still a Nash equilibrium. We can verify this results by Example 7. Thus, $P1$ can only benefit by his leadership in state (t, n_1, n_2) if and only if there are multiple Nash equilibria in that state or lower states.

4.3 Cooperative Solution

We shall now study the situation of cooperation between the two players. Here, we assume that under cooperation, a player does not incur a rejection cost if a booking request is satisfied by its cooperative partner. In addition, we also assume that the transfer rate between two players is one when one player's rejected customer is acceptable by another player. This is reasonable since when two players cooperate, each player should would encourage an unsatisfied customer to transfer to the other (cooperative) hotel. Under this situation, both players make decisions jointly on which hotel should accept/reject the arriving customer dynamically according to their inventory levels and time-to-go. Since the total revenue of two players is possibly increased at the cost of losing some revenue on one player, the optimal strategy mixes may be different from those under the competitive situation.

We denote $V(t, n_1, n_2)$ as the maximum total expected revenue of two players in state (t, n_1, n_2). In the last period, i.e., $t = 1$, two players should accept any booking request as long as there are some unsold rooms. However, the booking request should be given to the player whose unit revenue per room is higher. Hence, the maximum total expected revenue in state $(1, n_1, n_2)$ can be expressed as

$$V(1, n_1, n_2) = \begin{cases} 0, & \text{if } n_1 = n_2 = 0; \\ \sum_{K=L,H} (\lambda_{1K} + \lambda_{2K}) r_{1K}, & \text{if } n_1 > 0 \text{ and } n_2 = 0; \\ \sum_{K=L,H} (\lambda_{1K} + \lambda_{2K}) r_{2K}, & \text{if } n_1 = 0 \text{ and } n_2 > 0; \\ \sum_{K=L,H} (\lambda_{1K} + \lambda_{2K}) \max(r_{1K}, r_{2K}), & \text{if } n_1 > 0 \text{ and } n_2 > 0. \end{cases} \tag{33}$$

If one player (i.e., Pi, $i = 1, 2$) has sold out all of his rooms before the end of booking period, the maximum total expected revenue can be obtained from (17) and the optimal decisions would be the same as those shown in Proposition 2.

Next, we will focus on the situation in which both players have unsold rooms. It is possible that a player whose unit revenue of a high-fare customer is lower than that of another player may reject his own high-fare customer in order to maximize the total revenue of two players. Thus, we introduce an additional decision variable for such a player; without loss of generality, we that assume that $P1$ is this player. Let us now denote x_{1H} as the decision of $P1$ on his high-fare booking request, i.e.,

$$x_{1H} = \begin{cases} 1, & P1 \text{ accepts his own high-fare customer, if any} \\ 0, & P1 \text{ rejects his own high-fare customer, if any.} \end{cases}$$

Taking into account all other decisions, we present the expected revenue with any possible combination of decisions in period t (See Figure 20).

Then, we have,

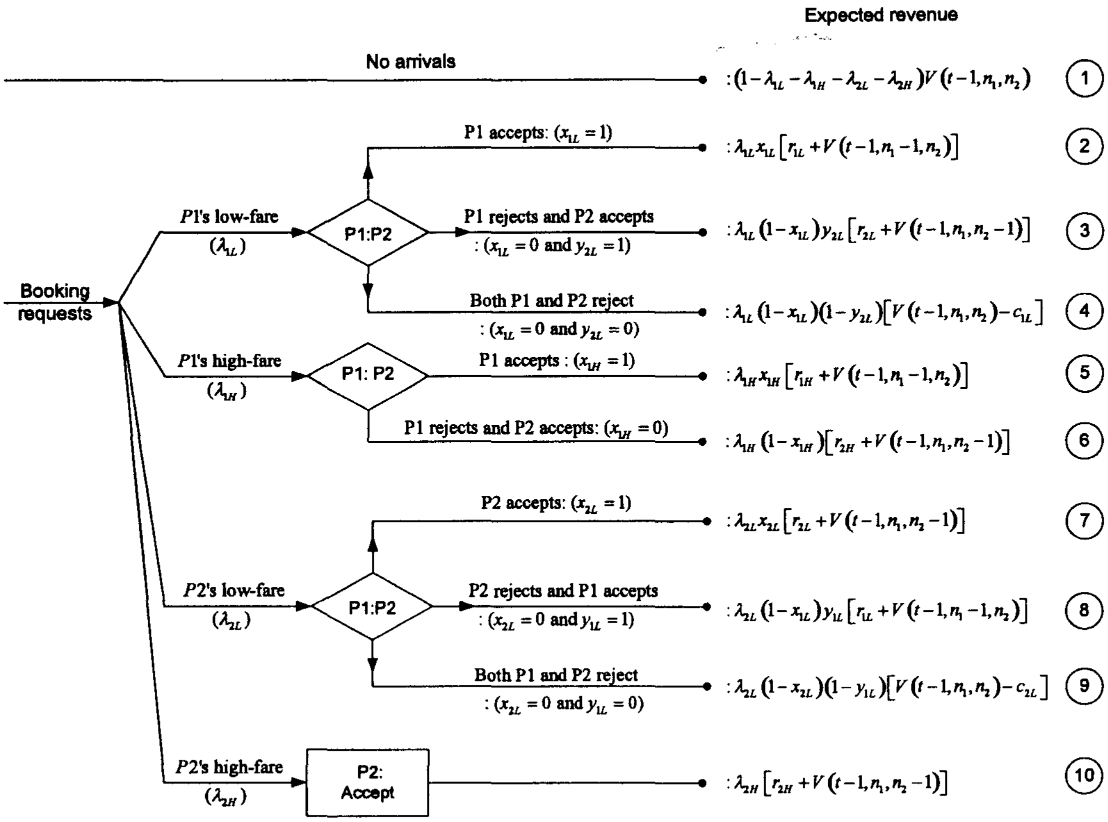


Figure 20. The expected revenue of two players under cooperation $(n_1, n_2 > 0)$.

$$\begin{aligned} V(t,n_1,n_2) = & \max_{x_{1L},x_{1H},y_{1L},x_{2L},y_{2L}} \{V(t-1,n_1,n_2) \\ & +\lambda_{1H}(1-x_{1H})[V(t-1,n_1,n_2-1)-V(t-1,n_1-1,n_2)+r_{2H}-r_{1H}] \\ & +\sum\nolimits_{i=1,2}\{r_{iL}[\lambda_{jL}(1-x_{jL})y_{iL}+\lambda_{iL}x_{iL}]-\lambda_{iL}(1-x_{iL})(1-y_{jL})c_{iL}+\lambda_{iH}r_{iH}\} \\ & +[\lambda_{1L}x_{1L}+\lambda_{1H}+\lambda_{2L}(1-x_{2L})y_{1L}][V(t-1,n_1-1,n_2)-V(t-1,n_1,n_2)] \\ & +[\lambda_{2L}x_{2L}+\lambda_{2H}+\lambda_{1L}(1-x_{1L})y_{2L}][V(t-1,n_1,n_2-1)-V(t-1,n_1,n_2)]\} \end{aligned} \tag{34}$$

with $V(0, n_1, n_2) = 0$, $n_1, n_2 > 0$ and $i, j = 1, 2$ $(i \neq j)$.

According to the cases shown in Figure 20, the optimal accept/reject decisions of the two players in under cooperation is given in the following Proposition.

Proposition 4 *Under cooperative situation, if the two players have available rooms* $(n_i, n_j > 0,\ i, j = 1, 2$ *while* $i \neq j)$ *at the beginning of period* t *and* $r_{1H} < r_{2H}$*, then the accept/reject decision rules are as follows:*

(1) *P1 accepts his own high-fare customer, if* $\beta_1 > r_{2H} - r_{1H}$*, and rejects, otherwise;*

(2) *Pi accepts his own low-fare customer, if* $\beta_i > r_{jL} - r_{iL}$ *and* $\delta_i < r_{iL} + c_{iL}$*;*

(3) *Pi rejects his own low-fare customer, but Pj accepts his own low-fare customer, if* $\beta_i \leq r_{jL} - r_{iL}$ *and* $\delta_j < r_{jL} + c_{iL}$*;*

(4) *both Pi and Pj reject their own low-fare customer, if* $\delta_i \geq r_{iL} + c_{iL}$ *and* $\delta_j \geq r_{jL} + c_{iL}$*,*

where $\beta_1 = -\beta_2 = V(t-1, n_1 - 1, n_2) - V(t-1, n_1, n_2 - 1)$*, and* δ_i *denotes to the marginal revenue of a Pi's room, i.e.,* $\delta_1 = V(t-1, n_1, n_2) - V(t-1, n_1 - 1, n_2)$ *(*β_i *and* δ_i *are for state* $(t-1, n_1, n_2)$*).*

Proof. Referring to Figure 20, we find the accept/reject decision on $P1$'s high-fare booking request by comparing the fifth and sixth expressions, i.e., $P1$ should accept his own high-fare customer if $\beta_1(t-1, n_1, n_2) > r_{2H} - r_{1H}$. Similarly, we can obtain other accept/reject decisions when different booking requests occur. This completes the proof. ■

From Proposition 4, we note that in cooperative situations it might be optimal for $P1$ to reject his own high-fare class customer if such action can improve the total expected revenue of the two players.

Example 9 *From Example 7, we note in the multiple Nash equilibria situation, the optimal solutions for the two players are uncertain and they most likely choose cooperation in order to avoid the danger of the deviation from the Nash equilibria. In this example, we use the same parameter values as those in Example 7. According to the decision rules provided by Proposition 4, we find the optimal solutions are* $x_{1H}^* = x_{1L}^* = y_{1L}^* = 0$ *and* $x_{2L}^* = y_{2L}^* = 1$. *With this strategy, we obtain the expected revenue of the two players for the state* $(2,1,1)$ *as:* $V_1^* = 104.25$ *and* $V_2^* = 194.18$. *Comparing these solutions with the two players' expected revenue in Table 13, we see that even though* $P1$*'s expected revenue is decreased, the expected revenue of* $P2$ *is dramatically increased. As a result, the total expected revenue of the two players is increased more than $50 on average, which account for about 17% of the total revenue in the non-cooperative situation. Thus, if there is an appropriate agreement on the compensation between the two players, such improvement indicates that the cooperation is strongly recommended.* ♦

We will next investigate the structural properties of $V(t, n_1, n_2)$.

Theorem 7 *Under cooperation, for any state* (t, n_1, n_2) *(*$t \in [1, T]$ *and* $n_1, n_2 \geq 0$*), the objective function* $V(t, n_1, n_2)$ *exhibits the following structural properties:*

(i) $V(t, n_1, n_2)$ is non-decreasing quasi-concave in n_1 and n_2;

(ii) $V(t, n_1, n_2) - V(t-1, n_1, n_2)$ is non-decreasing in t and non-decreasing in n_i.

Proof. We will prove this Theorem by induction. First, let us prove the property (i). From (33), it is easy to verify that property (i) is satisfied by for any state $(1, n_1, n_2)$. Now, let us assume that property (i) is also valid for any n_1 and n_2 in period $t-1$ with the hope that such properties can be extended to period t. Referring to expressions shown in Figure 20, the total expected revenue can also be written as

$$
\begin{aligned}
V(t, n_1, n_2) = \; & (1 - \lambda_{1L} - \lambda_{2L} - \lambda_{1H} - \lambda_{2H}) V(t-1, n_1, n_2) \\
& + \lambda_{1L} \max \begin{cases} r_{1L} + V(t-1, n_1 - 1, n_2) \\ r_{2L} + V(t-1, n_1, n_2 - 1) \\ V(t-1, n_1, n_2) - c_{1L} \end{cases} \\
& + \lambda_{2L} \max \begin{cases} r_{2L} + V(t-1, n_1, n_2 - 1) \\ r_{1L} + V(t-1, n_1 - 1, n_2) \\ V(t-1, n_1, n_2) - c_{2L} \end{cases} \\
& + \lambda_{1H} \max \begin{cases} r_{1H} + V(t-1, n_1 - 1, n_2) \\ r_{2H} + V(t-1, n_1, n_2 - 1) \end{cases} \\
& + \lambda_{2L} \left[r_{2H} + V(t-1, n_1, n_2 - 1) \right]
\end{aligned} \tag{35}
$$

It is not difficult to find that $V(t, n_1, n_2)$ is always a positive and linear combination of five non-decreasing quasi-concave functions (by assumption). Hence, we can conclude that $V(t, n_1, n_2)$ for $\forall n_1, n_2 > 0$ is also non-decreasing quasi-concave in n_1 and n_2. By induction, the proof of property (i) is finished.

Next, we shall prove the property (ii) as stated above. In terms of the marginal expected revenue of Pi's one room, we can transform equation (35) as

$$\begin{aligned} & V(t, n_1, n_2) - V(t-1, n_1, n_2) = \\ & +\lambda_{2H}\left(r_{2H} - \delta_2(t-1, n_1, n_2)\right) + \lambda_{1H} \max \begin{cases} r_{1H} - \delta_1(t-1, n_1, n_2) \\ r_{2H} - \delta_2(t-1, n_1, n_2) \end{cases} \\ & +\lambda_{1L} \max \begin{cases} r_{1L} - \delta_1(t-1, n_1, n_2) \\ r_{2L} - \delta_2(t-1, n_1, n_2) \\ -c_{1L} \end{cases} + \lambda_{2L} \max \begin{cases} r_{2L} - \delta_2(t-1, n_1, n_2) \\ r_{1L} - \delta_1(t-1, n_1, n_2) \\ -c_{2L} \end{cases} \end{aligned} \tag{36}$$

We note that each item on the RHS of (36) is non-decreasing in n_1 and n_2 since δ_i for state $(t-1, n_1, n_2)$ is always non-increasing in n_1 and n_2. Hence, the LHS, $V(t, n_1, n_2) - V(t-1, n_1, n_2)$ is also non-decreasing in n_1 and n_2. In other words, $V(t, n_1, n_2)$ is sub-modular in (t, n_1) and (t, n_2). Meanwhile, according to such sub-modularity of $V(t, n_1, n_2)$, we can see that $\delta_i(t-1, n_1, n_2)$ is also non-increasing in t which implies that the LHS of (36) is non-decreasing in t. Therefore, $V(t, n_1, n_2) - V(t-1, n_1, n_2)$ is non-decreasing in t and n_i, $i = 1, 2$. This completes the proof. ■

Similar to the Theorem 5, the properties shown in the Theorem 7 implies the existences of some critical booking capacities and booking periods.

- For an given t and n_2, there always exist a critical booking capacity, $\hat{n}_{1K}(t, n_2)$ $(K = L, H)$, by which it is always optimal to 'assign' the K-fare class customer to $P2$ for $n_1 < \hat{n}_{1K}$; and $P1$ should accept it for $n_1 \geq \hat{n}_{1K}$.
- For an given n_1 and n_2, there always exist a critical booking period, $\hat{t}_{1K}(n_1, n_2)$ $(K = L, H)$, by which it is always optimal to 'assign' the K-fare class customer to $P2$ for $t < \hat{t}_{1K}$; and $P1$ should accept it for $t \geq \hat{t}_{1K}$.
- There exists a critical state $(\hat{t}, \hat{n}_1, \hat{n}_2)$ by which for any "upper" state $((t, n_1, n_2) > (\hat{t}, \hat{n}_1, \hat{n}_2))$ the low-fare booking request should be rejected by both players.

Referring to the optimal decision rules described in Proposition 4, there are sixteen different combinations of the five decision variables in the cooperative situation. The optimal solution can be any of them. However, we can use the critical values in a specific state to summarize each of these five decisions, e.g., for the give state variables, t and n_2, the decision on $P1$'s high-fare class booking request can be expressed as

$$X_{1H}(t, n_1, n_2) = \begin{cases} 1 \text{ (accept)}, & \text{if } n_1 < \hat{n}_{1K}(t, n_2); \\ 0 \text{ (reject)}, & \text{otherwise}, \end{cases}$$

where $\hat{n}_{1H}(t, n_2) = \min\{n_1 : \beta_1(t-1, n_1, n_2) > r_{2H} - r_{1H}\}$. Comparing with the condition for $X_{1H}(t, n_1, n_2)$ in Proposition 4, this expression looks nicer and more understandable. In addition, our numerical experiments show that using these critical values significantly decreases the computing time when calculating the optimal solutions on the computer.

Chapter 5
Static Game Model with Incomplete Information

Note that a very important assumption in the models established in Chapter 3 and Chapter 4 is complete information. One hotel knows all the necessary information (e.g., transfer rate, rejection cost, etc.) of both hotels except for the other hotel's decision on the booking limit. In other words, the two players' objective functions are common knowledge. Obviously, this is not always applicable in practice. Hence, we are going to relax this assumption and study single-period games with incomplete information. Under these game theoretic settings, the expected revenues of the two players are determined by a "chance move", about which the players are partially informed. In this context, we investigate the consequences for the players' expected revenues by varying the states of information on the outcome of the chance move. In general, the value of information for the player refers to the difference of his optimal payoffs with and without the information. In this chapter, we are going to study the value of different information. Specifically, we debate the following questions: 1) Is the value of information always positive in our games? and 2) what type of information is more valuable?

Information value theory is a rather well known subject in classical decision theory. The basic result for a zero-sum Bayesian game indicates that the information is always valuable (see Ponssard [23]). Gilboa and Lehrer [19] characterized the functions that measure the value of information in optimization problems. However, for the non-zero-sum games, some studies indicate that the value of information may become negative in some special cases. Kamien et. al. [24] showed an instance where players might prefer dropping some payoff-relevant information in order to improve their equilibrium payoff. Neyman [38] investigated the reasons why a player might prefer a 'no information' game instead of a game with private information. Bassan et. al. [3] present the conditions under which having more information always improves all players' payoffs. According to the definitions of different types of information, we will examine their values when the chance move is incurred by the incomplete information of a specific parameter. In practice, we identify two parameters for our study. They are the rejection cost and transfer rate respectively. From our problem in Chapter 3, we see that the existence of transferred customers leads to competition between the two players. Therefore, transfer rates significantly affects one player's revenue especially when the booking requests in one hotel are 'rich' and the booking requests in the other one are 'poor'. In addition, we know that the transfer rate is a customer side parameter. Hotels can not totally control it by all means. Then, it is reasonable to assume there is incomplete information about the transfer rate. On the other hand, we note the rejection cost is another suitable parameter since it is normally incurred by loss of goodwill in hotel business. Therefore, it is also on the customers' side and it is important for the player's expected revenue. Our study, in this chapter, will assume these two parameters as the incomplete information to generate the chance move for the game of the two players.

Before analyzing our game model and optimal rationing policies, it is necessary for us to introduce the preliminary considerations of information types. According to Levine and Ponssard [30], there are three distinct types of information that one player may acquire in the incomplete information game.

Type 1: Secret information *One player acquires the information, but the other players are ignorant of this fact and will not modify their strategies. This assumption is actually unreasonable if it is a dynamic game and there exists a unique Nash equilibrium in the 'no information' game. It is because the uninformed players might soon realize that they are playing a different game.*

Type 2: Private information *One player acquires the information and, though he is the only one informed, this fact is known to the other players. This type of information may have several effects on all player's optimal decisions. First, the acquisition of information may give the opportunity to the informed player to use it against the uninformed players. Second, the uninformed players might also modify their own decisions and it may or may not benefit the informed player.*

Type 3: Public information *All players acquire the information and it is known to all players.*

At first sight, we might expect that secret information would be more valuable than private information, which in turn, would be more valuable than public information. Next, we will examine the Bayesian Nash equilibrium in each type of games by assuming the rejection cost q_{1L} and transfer rate u_{1L} as the incomplete information respectively.

5.1 The Value of Information when q_{1L} is unknown

As discussed in Chapter 3, in a static game of complete information, a strategy for Pi is his low-fare booking limit, b_{iL}. We shall use the same assumptions and notations presented in Chapter 3 for our static incomplete information games in this chapter. The normal-form representation of this two-player game of complete information can be written as $G = \{b_{1L}, b_{2L}; J_1, J_2\}$. We now want to develop the normal-form representation of the static incomplete information (Bayesian) game. Let us assume that Pi's objective function is $J_i(b_{1L}, b_{2L}; t_i)$, where t_i is Pi's type and belongs to the type space T_i. To simplify the problem, we assume that $P1$'s rejection cost of his low-fare class customer is the only incomplete information which can be q_{1L}^1 with probability θ_q^1 and q_{1L}^2 with probability of θ_q^2 for $\theta_q^1 + \theta_q^2 = 1$ and $q_{1L}^1 < q_{1L}^2$. Let us first discuss the case when both players are uninformed.

5.1.1 Uninformed Game: Both Players are Uninformed of q_{1L}

We note that if both players are uninformed on the rejection cost of $P1$, then each player has only one type which is $T_1 = \{t_{q1}\}$ and $T_2 = \{t_{q2}\}$. Referring to (4), $P1$'s expected revenue obtained from the low-fare class customers is

$$J_{1L}(b_{1L}, b_{2L}; t_{q1}) = \int_0^{b_{1L}} r_{1L} x_{1L} F_{2L}(b_{2L}) f_{1L}\, dx_{1L} + \int_{b_{2L}}^{B_{2L}} \int_0^{b_{1L}} r_{1L}(x_{1L} + b_{1L} - M_{1L}) f_{1L} f_{2L}\, dx_{1L}\, dx_{2L}$$

$$+\int_{b_{2L}}^{B_{2L}}\int_{b_{1L}}^{\infty}\left[r_{1L}b_{1L}-E(q_{1L})\ \left(x_{1L}-M_{1L}\right)\right]f_{1L}f_{2L}\ dx_{1L}\ dx_{2L}$$
$$+\int_{B_{2L}}^{\infty}\int_{0}^{\infty}\left[r_{1L}b_{1L}-E(q_{1L})\ \left(x_{1L}-M_{1L}\right)\right]f_{1L}f_{2L}\ dx_{1L}\ dx_{2L}$$
$$+\int_{b_{1L}}^{\infty}\left[r_{1L}b_{1L}-E(q_{1L})\ \left(x_{1L}-b_{1L}\right)\right]F_{2L}\left(b_{2L}\right)f_{1L}\ dx_{1L} \tag{37}$$

where $M_{1L}=b_{1L}-u_{2L}\left(x_{2L}-b_{2L}\right)$, $E(q_{1L})=\theta_q^1 q_{1L}^1+\theta_q^2 q_{1L}^2$, and $B_{2L}=b_{2L}+(b_{1L}-x_{1L})/u_{2L}$. We find that $P1$'s expected revenue obtained from the high-fare class customers is same as that in the complete information game, which is

$$J_{1H}(b_{1H},b_{2H};t_{q1})=\int_{0}^{b_{1H}}r_{1H}x_{1H}F_{2H}\left(b_{2H}\right)f_{1H}\ dx_{1H}$$
$$+\int_{b_{2H}}^{B_{2H}}\int_{0}^{b_{1H}}r_{1H}\left(x_{1H}+b_{1H}-M_{1H}\right)f_{1H}f_{2H}\ dx_{1H}\ dx_{2H}$$
$$+\int_{b_{2H}}^{B_{2H}}\int_{b_{1H}}^{\infty}\left[r_{1H}b_{1H}+q_{1H}\left(x_{1H}-M_{1H}\right)\right]f_{1H}f_{2H}\ dx_{1H}\ dx_{2H}$$
$$+\int_{B_{2H}}^{\infty}\int_{0}^{\infty}\left[r_{1H}b_{1H}+q_{1H}\left(x_{1H}-M_{1H}\right)\right]f_{1H}f_{2H}\ dx_{1H}\ dx_{2H}$$
$$+\int_{b_{1H}}^{\infty}\left[r_{1H}b_{1H}-q_{1H}\left(x_{1H}-b_{1H}\right)\right]F_{2H}\left(b_{2H}\right)f_{1H}\ dx_{1H}, \tag{38}$$

where $M_{1H}=b_{1H}-u_{2H}\left(x_{2H}-b_{2H}\right)$, and $B_{2H}=b_{2H}+(b_{1H}-x_{1H})/u_{2H}$. Note that $P2$'s objective function does not involve q_{1L}, we then obtain $P2$'s expected revenue

$$
\begin{aligned}
J_2(b_{1L}, b_{2L}; t_{q2}) = \sum_{K=L,H} \Bigg\{ & \int_0^{b_{2K}} r_{2K} x_{2K} F_{1K}(b_{1K}) f_{2K}\, dx_{2K} \\
& + \int_{b_{1K}}^{B_{1K}} \int_0^{b_{2K}} r_{2K} (x_{2K} + b_{2K} - M_{2K}) f_{2K} f_{1K}\, dx_{2K}\, dx_{1K} \\
& + \int_{b_{1K}}^{B_{1K}} \int_{b_{2K}}^{\infty} [r_{2K} b_{2K} - q_{2K} (x_{2K} - M_{2K})] f_{2K} f_{1K}\, dx_{2K}\, dx_{1K} \\
& + \int_{B_{1K}}^{\infty} \int_0^{\infty} [r_{2K} b_{2K} - q_{2K} (x_{2K} - M_{2K})] f_{2K} f_{1K}\, dx_{2K}\, dx_{1K} \\
& + \int_{b_{2K}}^{\infty} [r_{2K} b_{2K} - q_{2K} (x_{2K} - b_{2K})] F_{1K}(b_{1K}) f_{2K}\, dx_{2K} \Bigg\}
\end{aligned} \tag{39}
$$

where $i = 1, 2$, $b_{2H} = C_2 - b_{2L}$, $M_{2K} = b_{2K} - u_{1K}(x_{1K} - b_{1K})$, and $B_{1K} = b_{1K} + (b_{2K} - x_{2K})/u_{1K}$. Then, the total expected revenue of $P1$ in this situation is

$$
J_1 = J_{1L}(b_{1L}, b_{2L}; t_{q1}) + J_{1H}(b_{1H}, b_{2H}; t_{q1}). \tag{40}
$$

Meanwhile, the total expected revenue of $P2$ is

$$
J_2 = J_2(b_{1L}, b_{2L}; t_{q2}). \tag{41}
$$

Naturally, we will turn to find the Nash equilibrium in this Bayesian game (Bayesian Nash equilibrium). Referring to (5), we obtain

$$
\begin{aligned}
\frac{\partial J_1}{\partial b_{1L}} = V_1 = \; & (r_{1L} + E(q_{1L})) \left[\int_0^{b_{1L}} \int_{N_{2L}}^{\infty} f_{1L} f_{2L}\, dx_{2L}\, dx_{1L} + \bar{F}_{1L}(b_{1L}) \right] \\
& - (r_{1H} + q_{1H}) \left[\int_0^{b_{1H}} \int_{N_{2H}}^{\infty} f_{1H} f_{2H}\, dx_{2H}\, dx_{1H} + \bar{F}_{1H}(b_{1H}) \right],
\end{aligned} \tag{42}
$$

where $N_{2K} = b_{2K} + (b_{1K} - x_{1K})/u_{2K}$ $(K = L, H)$ and $E(q_{1L}) = \theta_q^1 q_{1L}^1 + \theta_q^2 q_{1L}^2$. As for

$P2$, the first order partial derivative of J_2 with respect to b_{2L} is exactly same as that in the complete information game, which is

$$\begin{aligned}\frac{\partial J_2}{\partial b_{2L}} = V_2 = & \ (r_{2L} + q_{2L}) \left[\int_0^{b_{2L}} \int_{N_{1L}}^{\infty} f_{1L} f_{2L} \, dx_{2L} \, dx_{1L} + \bar{F}_{2L} (b_{2L}) \right] \\ & - (r_{2H} + q_{2H}) \left[\int_0^{b_{2H}} \int_{N_{1H}}^{\infty} f_{1H} f_{2H} \, dx_{2H} \, dx_{1H} + \bar{F}_{2H} (b_{2H}) \right],\end{aligned} \tag{43}$$

where $N_{1K} = b_{1K} + (b_{2K} - x_{2K}) / u_{1K}$ $(K = L, H)$.

We see that V_1 is also exactly same as that in the complete information game by substituting $E(q_{1L})$ with q_{1L}. The uninformed game in this situation is not equivalent to the complete information game by assuming q_{1L} in (4) as $E(q_{1L})$. According to Lemmas 1, 2 and 3, we know each player's objective function has the following structural properties:

(1) Pi's objective function is strictly concave in b_{iL} for $i = 1, 2$. (by Lemma 1)

(2) $V_i = 0$, $i = 1, 2$, is a strictly decreasing curve in the (b_{1L}, b_{2L}) plane. (by Lemma 2)

(3) The implicit derivative of $V_1 = 0$ with respect to b_{1L} is always less than the implicit derivative of $V_2 = 0$ with respect to b_{1L}. (by Lemma 3)

All of these properties of the objective functions will lead to the existence of unique Bayesian Nash equilibrium in the uninformed game.

Theorem 8 *When each of the two players has an incomplete information of q_{1L}, the game admits a unique Nash equilibrium.*

Proof. Referring to the properties of J_i, we find that one player's best response function has the same structural properties as that in the complete information game. It implies that, in (b_{1L}, b_{2L}) plane, the two best response curves of the two players intersects and only intersects once. In other words, the Bayesian game admits a unique Nash equilibrium. ■

From Theorem 8, we know that if the two players are both uninformed, they will play a Nash game in order to maximize their total expected revenue. We denote (b_{1L}^*, b_{2L}^*) as the Nash equilibrium and J_i^* the maximum expected revenue in this "uninformed" game. We see that the Nash solution pair is the intersection of $V_1 = 0$ and $V_2 = 0$ in the (b_{1L}, b_{2L}) plane. We now attempt to find the moving direction of (b_{1L}^*, b_{2L}^*) in the (b_{1L}, b_{2L}) plane as $E(q_{1L})$ varies. Also, we will examine the variation of the optimal expected revenue, J_i^*, on $E(q_{1L})$. These findings will help us analyze the conditions by which the value of information is positive (or negative).

Proposition 5 *If the two players are both uninformed of q_{1L} and the Nash solution of each player is greater than 0 ($b_{iL}^* > 0$, $i = 1, 2$), then b_{1L}^* decreases and b_{2L}^* increases as $E(q_{1L})$ decreases; and vice versa.*

Proof. If $b_{1L}^*, b_{2L}^* > 0$, then they must satisfy (42) and (43). The derivative of V_1 with respect to $E(q_{1L})$ is

$$S_{1L} - db_{2L}/dE(q_{1L}) \left[\sum_{K=L,H} u_{2K} \left(r_{1K} + q_{1K}\right) \int_0^{b_{1K}} f_{1K} f_{2K} \left(N_{2K}\right) \, dx_{1K} \right]$$

$$- db_{1L}/dE(q_{1L}) \sum_{K=L,H} (r_{1K} + q_{1K}) \int_0^{b_{1K}} \tfrac{1}{u_{1K}} f_{1K} f_{2K} (N_{2K}) \; dx_{1K}$$
$$- db_{1L}/dE(q_{1L}) \sum_{K=L,H} (r_{1K} + q_{1K}) \; f_{1K} (b_{1K}) F_{2K} (b_{2K}) \qquad = 0,$$

and the derivative of V_2 with respect to $E(q_{1L})$ is

$$-db_{1L}/dE(q_{1L}) \sum_{K=L,H} u_{1K} (r_{2K} + q_{2K}) \int_0^{b_{2K}} f_{1K} (N_{1K}) f_{2K}$$
$$-db_{2L}/dE(q_{1L}) \sum_{K=L,H} (r_{2K} + q_{2K}) \int_0^{b_{2K}} \tfrac{1}{u_{2K}} f_{1K} (N_{1K}) f_{2K} \; dx_{2K}$$
$$-db_{2L}/dE(q_{1L}) \sum_{K=L,H} (r_{2K} + q_{2K}) \; f_{2K} (b_{2K}) F_{1K} (b_{1K}) \qquad = 0.$$

We then obtain $db^*_{1L}/dE(q_{1L})$ and $db^*_{2L}/dE(q_{1L})$ by solving the two equations above. It is not difficult to find that $db^*_{1L}/dE(q_{1L}) > 0$ and $db^*_{2L}/dE(q_{1L}) < 0$. These imply that as $E(q_{1L})$ increases, b^*_{1L} increases and b^*_{2L} decrease; and vice versa. ■

Remark 3 *From Proposition 5, we note as q_{1L} increases, the Nash solution pair move in the southeast direction in the (b_{1L}, b_{2L}) plane, which is identical to the result in the sensitivity analysis in Chapter 3. However, we find the total derivative of J_1 with respect to $E(q_{1L})$, which is (after some simplifications)*

$$\begin{aligned} dJ_1/dq_{1L} = \; & u_{2L} \left[(r_{1L} + E(q_{1L})) \; (S_{1L} - \bar{F}_{1L} (b_{1L}) F_{2L} (b_{2L})) \right] db_{2L}/dE(q_{1L}) \\ & - u_{2H} \left[(r_{1H} + q_{1H}) \; (S_{1H} - \bar{F}_{1H} (b_{1H}) F_{2H} (b_{2H})) \right] db_{2L}/dE(q_{1L}) \\ & - \int_{B^*_{2L}}^{\infty} \int_0^{\infty} (x_{1L} - M_{1L}) f_{1L} f_{2L} \; dx_{1L} \; dx_{2L} \end{aligned}$$

$$-\int_{b_{1L}^*}^{\infty} (x_{1L} - b_{1L}) F_{2L}(b_{2L}) f_{1L}\, dx_{1L}$$

$$-\int_{b_{2L}}^{B_{2L}} \int_{b_{1L}}^{\infty} (x_{1L} - M_{1L}) f_{1L} f_{2L}\, dx_{1L}\, dx_{2L}$$

$$-u_{2L}\; r_{1L} \bar{F}_{2L}(b_{2L})\; db_{2L}/dE(q_{1L})$$

$$+u_{2H}\; r_{1H} \bar{F}_{2H}(b_{2H})\; db_{2L}/dE(q_{1L}),$$

is too complicated to analyze the structural properties of J_1 *with respect to* $E(q_{1L})$. *The monotonicity and concavity (or convexity) of optimal* J_1 *with respect to* q_{1L} *is ambiguous. Similarly, we can also find that the structural properties of optimal* J_2 *on* q_{1L} *is uncertain.* $\lhd$

$P1$'s information on q_{1L}	Objective functions: Type t_{q1}^{1i}	Objective functions: Type t_{q1}^{2i}	Value of information
Secret information ($i = 1$)	$J_1^1(b_{1L}^{11}, b_{2L}^*; t_{q1}^{11})$	$J_1^1(b_{1L}^{21}, b_{2L}^*; t_{q1}^{21})$	ω_{q1}^1
Private information ($i = 2$)	$J_1^2(b_{1L}^{12}, b_{2L}^2; t_{q1}^{12})$	$J_1^2(b_{1L}^{12}, b_{2L}^2; t_{q1}^{22})$	ω_{q1}^2
Public information ($i = 3$)	$J_1^3(b_{1L}^{13}, b_{2L}^{13}; t_{q1}^{13})$	$J_1^3(b_{1L}^{23}, b_{2L}^{23}; t_{q1}^{23})$	ω_{q1}^3

Table 15. $P1$'s objective function and corresponding strategies when $P1$ receives different information.

We will next examine the optimal decisions of one player when he receives the information of q_{1L}. We then analyze the value of information of different types. We denote $P1$'s types, relevant objective functions, and value of information when acquiring different information with the notations in Table 15. analogous to these notations, one might obtain $P2$'s types, relevant objective functions, and value of information. First, let us assume $P1$ received the information secretly. In this case, $P2$'s strategy remains b_{2L}^* since he assume the game is still a Nash game. However,

$P1$ might choose different booking limit in the two types against b_{2L}^{*}. We denote $P1$'s booking limit in type t_{q1}^{j1} as b_{1L}^{j1}, $j = 1, 2$. Second, let assume $P1$ received private information. According to the definition, $P2$ might choose a strategy which is different with b_{2L}^{*}, however, he will use the same strategy for both type of games. We denote the booking limit of $P2$ in this situation as b_{2L}^{2}. On the other hand, $P1$ knows the chance move of the game, then he should adopt different strategies for the two types of games. We use b_{1L}^{j2} to present the booking limit of $P1$ in type t_{q1}^{j2}, $j = 1, 2$. At last, when both players knows the chance move of the game (public information case), we use b_{iL}^{j3} to present the booking limit of Pi ($i = 1, 2$) in type t_{qi}^{j3} ($j = 1, 2$).

5.1.2 Secret Information Game : One Player Acquires q_{1L} Secretly

Now, we will examine value of secret information by examining the one player's objective functions and corresponding strategies.

Proposition 6 *The value of secret information of q_{1L} is always positive for $P1$ if $b_{1L}^{*} > 0$, and it is always zero for $P2$.*

Proof. When $P1$ receives the secret information of q_{1L}, he will make a decision on the booking limit against b_{2L}^{*} according to his best response function in both types of games. Then, the total expected revenue is

$$J_1^1 = \theta_q^1 J_1^1(b_{1L}^{11}, b_{2L}^{*}; t_{q1}^{1}) + \theta_q^2 J_1^1(b_{1L}^{21}, b_{2L}^{*}; t_{q1}^{2}). \tag{44}$$

We see that the optimal booking limit for $P1$ in type t_{q1}^{j1} ($j = 1, 2$), b_{1L}^{j1*}, can be obtained according to the best response function (in this case the BR function is $V_1 = 0$ since $b_{1L}^* > 0$). Referring to Proposition 5, we also know that $b_{1L}^{11*} > b_{1L}^*$ and $b_{1L}^{21*} < b_{1L}^*$, since $q_{1L}^1 > E(q_{1L}) > q_{1L}^2$. Thus, it turns out

$$J_1^1(b_{1L}^{j1*}, b_{2L}^*; t_{q1}^j) > J_1(b_{1L}^*, b_{2L}^*; t_{q1}),$$

for $j = 1, 2$, which indicates the value of secret information of q_{1L} for $P1$

$$\omega_{q1}^1 = J_1^{1*} - J_1^* > 0.$$

Then the value of the secret information of q_{1L} for $P1$ is always positive.

As for $P2$, we know that q_{1L} does not play a role on expected revenue. When receiving secret information, his objective functions in the two types exhibit exactly the same form. It implies that the game ($P2$ acquires secret information of q_{1L}) is equivalent to the uninformed game in each type. Thus,

$$b_{2L}^{11*} = b_{2L}^{21*} = b_{2L}^*,$$

which indicates the value of secret information of q_{1L} for $P2$,

$$\omega_{q2}^1 = J_2^{1*} - J_2^*,$$

is always zero. ■

5.1.3 Private Information Game: One Player Receives Private Information of q_{1L}

Next, we consider the case when one player acquires the private information on q_{1L}. Let us first assume $P1$ acquires the private information of q_{1L}. In this situation,

$P1$'s type space is $T_1 = \{t_{q1}^{12}, t_{q1}^{22}\}$ and $P2$'s type space is $T_2 = \{t_{q2}^{2}\}$. $P1$'s expected revenue from low-fare customers in type t_{q1}^{j2} ($j = 1, 2$) is

$$\begin{aligned} J_{1L}^2(b_{1L}^{j2}, b_{2L}^2; t_{q1}^{j2}) = & \int_0^{b_{1L}^{j2}} r_{1L} x_{1L} F_{2L}\left(b_{2L}^j\right) f_{1L}\, dx_{1L} \\ & + \int_{b_{2L}}^{B_{2L}^{j2}} \int_0^{b_{1L}^{j2}} r_{1L}\left(x_{1L} + b_{1L}^{j2} - M_{1L}^{j2}\right) f_{1L} f_{2L}\, dx_{1L}\, dx_{2L} \\ & + \int_{b_{2L}}^{B_{2L}^{j2}} \int_{b_{1L}^2}^{\infty} \left[r_{1L} b_{1L}^{j2} - q_{1L}^j\left(x_{1L} - M_{1L}^{j2}\right)\right] f_{1L} f_{2L}\, dx_{1L}\, dx_{2L} \\ & + \int_{B_{2L}^{j2}}^{\infty} \int_0^{\infty} \left[r_{1L} b_{1L}^{j2} - q_{1L}^j\left(x_{1L} - M_{1L}^{j2}\right)\right] f_{1L} f_{2L}\, dx_{1L}\, dx_{2L} \\ & + \int_{b_{1L}^2}^{\infty} \left[r_{1L} b_{1L}^{j2} - q_{1L}^j\left(x_{1L} - b_{1L}^{j2}\right)\right] F_{2L}\left(b_{2L}^{j2}\right) f_{1L}\, dx_{1L} \end{aligned} \tag{45}$$

where $M_{1L}^{j2} = b_{1L}^{j2} - u_{2L}(x_{2L} - b_{2L}^2)$, and $B_{2L}^{j2} = b_{2L}^2 + (b_{1L}^{j2} - x_{1L})/u_{2L}$. Referring to (38), we can obtain $P1$'s expected revenue from high-fare customers in type t_{q1}^{j2} ($j = 1, 2$):

$$\begin{aligned} J_{1H}^2(b_{1H}^{j2}, b_{2H}^2; t_{q1}^{j2}) = & \int_0^{b_{1H}^{j2}} r_{1H} x_{1H} F_{2H}\left(b_{2H}^j\right) f_{1H}\, dx_{1H} \\ & + \int_{b_{2H}}^{B_{2H}^{j2}} \int_0^{b_{1H}^{j2}} r_{1H}\left(x_{1H} + b_{1H}^{j2} - M_{1H}^{j2}\right) f_{1H} f_{2H}\, dx_{1H}\, dx_{2H} \\ & + \int_{b_{2H}}^{B_{2H}^{j2}} \int_{b_{1H}^2}^{\infty} \left[r_{1H} b_{1H}^{j2} - q_{1H}\left(x_{1H} - M_{1H}^{j2}\right)\right] f_{1H} f_{2H}\, dx_{1H}\, dx_{2H} \\ & + \int_{B_{2H}^{j2}}^{\infty} \int_0^{\infty} \left[r_{1H} b_{1H}^{j2} - q_{1H}\left(x_{1H} - M_{1H}^{j2}\right)\right] f_{1H} f_{2H}\, dx_{1H}\, dx_{2H} \\ & + \int_{b_{1H}^2}^{\infty} \left[r_{1H} b_{1H}^{j2} - q_{1H}\left(x_{1H} - b_{1H}^{j2}\right)\right] F_{2H}\left(b_{2H}^{j2}\right) f_{1H}\, dx_{1H} \end{aligned} \tag{46}$$

where $M_{1H}^{j2} = b_{1H}^{j2} - u_{2H}\left(x_{2H} - b_{2H}^{j2}\right)$, $b_{1H}^{j2} = C_1 - b_{1L}^{j2}$, and $B_{2H}^{j2} = b_{2H}^2 + (b_{1H}^{j2} - x_{1H})/u_{2H}$. To sum up the two expression of (45) and (46), we obtain $P1$'s objective function in type t_{q1}^{j2} $(j = 1, 2)$:

$$J_1^2(b_{1L}^{j2}, b_{2L}^2; t_{q1}^{j2}) = J_{1L}^2(b_{1L}^{j2}, b_{2L}^2; t_{q1}^{j2}) + J_{1H}^2(b_{1H}^{j2}, b_{2H}^2; t_{q1}^{j2}). \tag{47}$$

We see that $P1$'s objective function in each type is exactly identical with that in the complete information game. Thus, the properties described in Lemmas 1, 2 hold. However, $P2$ has only one possible objective function:

$$\begin{aligned} J_2(b_{1L}^{12}, b_{1L}^{22}, b_{2L}^2; t_{q2}^2) = & \sum_{j=1,2} \theta_q^j \sum_{K=L,H} \\ & \left\{ \int_0^{b_{2K}^2} r_{2K} x_{2K} F_{1K}\left(b_{1K}^{j2}\right) f_{2K}\, dx_{2K} \right. \\ & + \int_{b_{1K}^{j2}}^{B_{1K}^{j2}} \int_0^{b_{2K}^2} r_{2K}\left(x_{2K} + b_{2K}^2 - M_{2K}^{j2}\right) f_{2K} f_{1K}\, dx_{2K}\, dx_{1K} \\ & + \int_{b_{1K}^{j2}}^{B_{1K}^{j2}} \int_{b_{2K}^2}^{\infty} \left[r_{2K} b_{2K}^2 - q_{2K}\left(x_{2K} - M_{2K}^{j2}\right)\right] f_{2K} f_{1K}\, dx_{2K}\, dx_{1K} \\ & + \int_{B_{1K}^{j2}}^{\infty} \int_0^{\infty} \left[r_{2K} b_{2K}^2 - q_{2K}\left(x_{2K} - M_{2K}^{j2}\right)\right] f_{2K} f_{1K}\, dx_{2K}\, dx_{1K} \\ & \left. + \int_{b_{2K}^2}^{\infty} \left[r_{2K} b_{2K}^2 - q_{2K}\left(x_{2K} - b_{2K}^2\right)\right] F_{1K}\left(b_{1K}^{j2}\right) f_{2K}\, dx_{2K} \right\} \end{aligned} \tag{48}$$

where $b_{2H}^2 = C_2 - b_{2L}^2$, $M_{2K}^{j2} = b_{2K}^2 - u_{1K}\left(x_{1K} - b_{1K}^{j2}\right)$, and $B_{1K}^{j2} = b_{1K}^{j2} + (b_{2K}^2 - x_{2K})/u_{1K}$. Differentiating J_2 with respect to b_{2L}^2, we obtain

$$V_2(b_{1L}^{12}, b_{1L}^{22}, b_{2L}^2; t_{q2}^2) = (r_{2L} + q_{2L}) \sum_{j=1,2} \theta_q^j [\int_0^{b_{2L}^2} \int_{N_{1L}^{j2}}^{\infty} f_{1L} f_{2L} dx_{2L} dx_{1L} + \bar{F}_{2L}(b_{2L})]$$
$$- (r_{2H} + q_{2H}) \sum_{j=1,2} [\theta_q^j \int_0^{b_{2H}^2} \int_{N_{1H}^{j2}}^{\infty} f_{1H} f_{2H} dx_{2H} dx_{1H} + \bar{F}_{2H}(b_{2H})],$$

where $N_{1K}^{j2} = b_{1K}^{j2} + (b_{2K}^2 - x_{2K})/u_{1K}$ $(K = L, H)$ and $b_{2H}^2 = C_2 - b_{2L}^2$. Differentiating $V_2(b_{1L}^{12}, b_{1L}^{22}, b_{2L}^2; t_{q2}^2)$ with respect to b_{2L}^2, we obtain the second order derivative of J_2 with respect to b_{2L}^2, which is

$$-\sum_{j=1,2} \theta_q^j \sum_{K=L,H} (r_{2K} + q_{2K}) [\int_0^{b_{2K}^2} \tfrac{1}{u_2} f_{2K} f_{1K}(N_{2K}^{j2})\, dx_{1K} + f_{2K}(b_{2K}^2) F_{1K}(b_{1K}^{j2})] < 0. \tag{49}$$

Therefore, $J_2(b_{1L}^{12}, b_{1L}^{22}, b_{2L}^2; t_{q2}^2)$ is strictly concave in b_{2L}^2. Furthermore, we find the implicit derivative of $V_2 = 0$ with respect to b_{1L}^{j2} is

$$b_2' = \frac{-\theta_q^j \sum_{K=L,H} u_{1K}(r_{2K} + q_{2K}) \int_0^{b_{2K}^2} f_{1K}(N_{1K}^{j2}) f_{2K}\, dx_{2K}}{\sum_{K=L,H} (r_{2K} + q_{2K}) \left[\int_0^{b_{2K}} \frac{1}{u_{2K}} f_{1K}(N_{1K}^{j2}) f_{2K} dx_{2K} + f_{2K}(b_{2K}^2) F_{1K}(b_{1K}^{j2})\right]}, \tag{50}$$

and $-1 < b_2' < 0$. (49) and (50) imply that the properties described in Lemmas 1, 2 also hold for $P2$ in this situation. Comparing b_2' with the implicit derivative of $V_1 = 0$ in each type, we find that b_2' is always greater than the derivative of $V_1 = 0$ with respect to b_{1L}^{j2} in type t_{q1}^{j2} $(j = 1, 2)$.

All of the results obtained above lead to the existence of a unique Bayesian Nash equilibrium.

Theorem 9 *When P1 has the complete information of P2 and himself, and P2 knows only that P1's rejection cost of low-fare class customer is* q_{1L}^1 *with probability* θ_q^1 *and* q_{1L}^2 *with probability of* θ_q^2 *for* $\theta_q^1 + \theta_q^2 = 1$ *and* $q_{1L}^2 > q_{1L}^1$*, the game admits a unique Bayesian Nash equilibrium* $(b_{1L}^{12*}, b_{1L}^{22*}, b_{2L}^{2*})$*. In addition,* $b_{1L}^{22*} \geq b_{1L}^{12*}$ *and* $J_1^{2*}(b_{1L}^{12*}, b_{2L}^{2*}; t_{q1}^{12}) > J_1^{2*}(b_{1L}^{22*}, b_{2L}^{2*}; t_{q1}^{22})$

Proof. Referring to the results we discussed above, we find that $P1$'s best response function in each type has the same structural properties as those in complete information game. On the other hand, $P2$'s best response function to $P1$'s strategy in one type given the strategy in the other type is known also has the same properties as those in the complete information game. Therefore, in (b_{1L}^{j2}, b_{2L}^2) $(j = 1, 2)$ plane, the two best response curves of the two players intersects and only intersects once. It implies that the Bayesian game admits a unique Nash equilibrium $(b_{1L}^{12*}, b_{1L}^{22*}, b_{2L}^{2*})$.

According to (8), we see that if b_{2L} is given, i.e., $b_{2L} = b_{2L}^{2*}$, and $b_{1L} = b_{1L}^{12*} > 0$, then b_{1L}^{12*} and b_{2L}^{2*} must satisfy

$$\left(r_{1L} + q_{1L}^1\right) S_{1L} - \left(r_{1H} + q_{1H}\right) S_{1H} = 0.$$

Moreover, we know that the spill rate of a low-fare customer, S_{1L}, decreases as b_{1L} increases, and vice versa. Thus, as q_{1L} increases from q_{1L}^1 to q_{1L}^2, S_{1L} should decrease while S_{1H} should increase. It leads to the increase of the optimal booking limit: $b_{1L}^{22*} > b_{1L}^{12*}$. However, we know that the optimal booking limit, b_{1L}^{22*}, is zero if

$$V_1(0, b_{2L}^*; q_{1L}^2) = \left(r_{1L} + q_{1L}^2\right) - \left(r_{1H} + q_{1H}\right) S_{1H}(0, b_{2L}^*) \leq 0.$$

In this case, as q_{1L} decreases from q_{1L}^2 to q_{1L}^1, $V_1(0, b_{2L}^*; t_{q1}^{12})$ is still less than zero. It implies that $b_{1L}^{12*} = b_{1L}^{22*} = 0$. Thus, in general, $b_{1L}^{22*} \geq b_{1L}^{12*}$.

Differentiating J_1 with respect to q_{1L}, we obtain

$$\begin{aligned} dJ_1/dq_{1L} = \; & [(r_{1L} + q_{1L})S_{1L} - (r_{1H} + q_{1H})S_{1H}]\; db_{1L}/dq_{1L} \\ & - \int_{B_{2L}}^{\infty} \int_{0}^{\infty} (x_{1L} - M_{1L})\, f_{1L} f_{2L}\; dx_{1L}\; dx_{2L} \\ & - \int_{b_{2L}}^{B_{2L}} \int_{b_{1L}}^{\infty} (x_{1L} - M_{1L})\, f_{1L} f_{2L}\; dx_{1L}\; dx_{2L} \\ & - \int_{b_{1L}}^{\infty} (x_{1L} - b_{1L})\, F_{2L}\,(b_{2L})\, f_{1L}\; dx_{1L}, \end{aligned} \tag{51}$$

where b_{2L} is assumed to be constant. Referring to Proposition 5, we know $db_{1L}/dq_{1L} > 0$. Then the first item on the RHS of (51) is less than or equal to zero since $V_1 \leq 0$. Thus, $dJ_1/dq_{1L} < 0$, which indicates that the maximum expected revenue decreases as q_{1L} increases: $J_1^{2*}(b_{1L}^{12*}, b_{2L}^{2*}; t_{q1}^{12}) > J_1^{2*}(b_{1L}^{22*}, b_{2L}^{2*}; t_{q1}^{22})$. ■

We note that the Bayesian Nash equilibrium $(b_{1L}^{12*}, b_{1L}^{22*}, b_{2L}^{2*})$ can be obtained by solving the BR functions of $P1$ in the two types and $P2$'s BR function. Thus, the value of private information of q_{1L} for $P1$ can be expressed as

$$\omega_{q1}^2 = \theta_q^1 J_1^{2*}(b_{1L}^{12*}, b_{2L}^{2*}; t_{q1}^{12}) + \theta_q^2 J_1^{2*}(b_{1L}^{22*}, b_{2L}^{2*}; t_{q1}^{22}) - J_1^*(b_{1L}^*, b_{2L}^*; t_{q1}^2). \tag{52}$$

Figure 21 depicts the respective positions of the total expected revenue in the (q_{1L}, J_1) plane. Consequently, the value of private information is always positive if the two points, $(q_{1L}^1, J_1^{2*}(b_{1L}^{12*}, b_{2L}^{2*}; t_{q1}^{12}))$ and $(q_{1L}^2, J_1^{2*}(b_{1L}^{22*}, b_{2L}^{2*}; t_{q1}^{22}))$, are both above

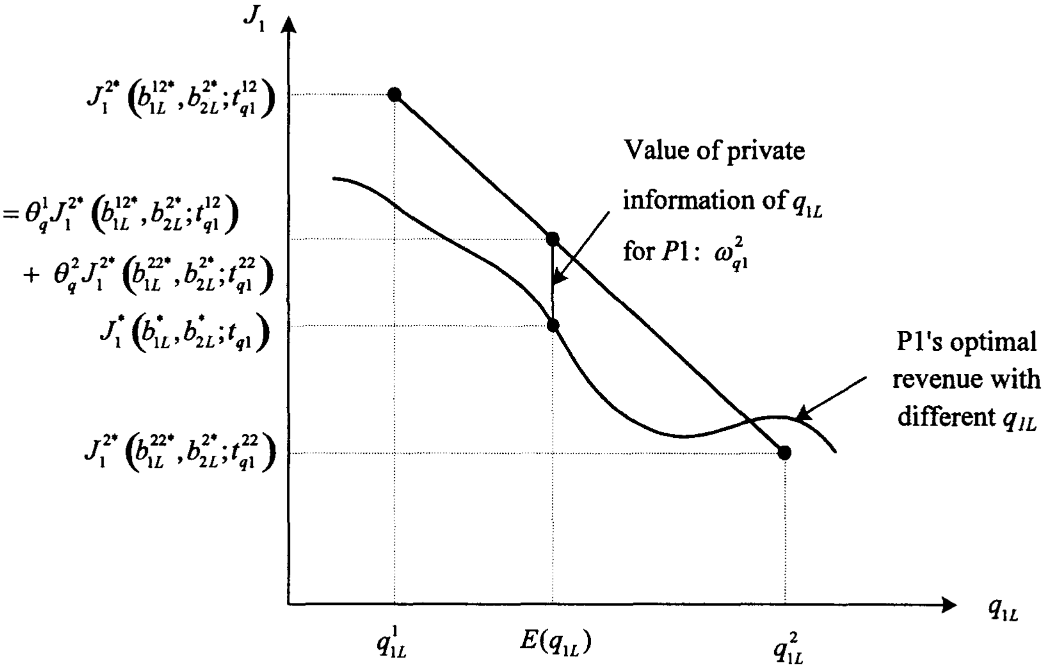


Figure 21. Value of private information of q_{1L} for $P1$.

the curve $J_1(b_{1L}^*(q_{1L}), b_{2L}^*(q_{1L}))$. However, the value of private information is always negative if $(q_{1L}^1, J_1^{2*}(b_{1L}^{12*}, b_{2L}^{2*}; t_{q1}^{12}))$ and $(q_{1L}^2, J_1^{2*}(b_{1L}^{22*}, b_{2L}^{2*}; t_{q1}^{22}))$ are both below the curve $J_1(b_{1L}^*(q_{1L}), b_{2L}^*(q_{1L}))$. In addition, it is also possible that $(q_{1L}^1, J_1^{2*}(b_{1L}^{12*}, b_{2L}^{2*}; t_{q1}^{12}))$ is above J_1 and $(q_{1L}^2, J_1^{2*}(b_{1L}^{22*}, b_{2L}^{2*}; t_{q1}^{22}))$ is below J_1. In this situation, θ_q^1 (or θ_q^2) plays a role on the value of private information. It is not difficult to find that the value of private information is positive if $(E(q_{1L}), J_1^{2*})$ is above $J_1(b_{1L}^*(q_{1L}), b_{2L}^*(q_{1L}))$; and it is negative if $(E(q_{1L}), J_1^{2*})$ is below $J_1(b_{1L}^*(q_{1L}), b_{2L}^*(q_{1L}))$.

Next, we will in turn discuss the case when $P2$ receives the private information. In this situation, $P2$'s type space is $T_2 = \{t_{q2}^{12}, t_{q2}^{22}\}$ and $P1$'s type space is $T_1 = \{t_{q1}^2\}$. $P2$'s objective function in type t_{q2}^{j2} $(j = 1, 2)$ is

$$J_2^2(b_{1L}^2, b_{2L}^{j2}; t_{q2}^{j2}) = \sum_{K=L,H} \left\{ \int_0^{b_{2K}^{j2}} r_{2K} x_{2K} F_{1K}\left(b_{1K}^2\right) f_{2K}\, dx_{2K} \right.$$
$$+ \int_{b_{1K}^2}^{B_{1K}^{j2}} \int_0^{b_{2K}^{j2}} r_{2K}\left(x_{2K} + b_{2K}^{j2} - M_{2K}^{j2}\right) f_{2K} f_{1K} dx_{2K} dx_{1K}$$
$$+ \int_{b_{1K}^2}^{B_{1K}^{j2}} \int_{b_{2K}^{j2}}^{\infty} \left[r_{2K} b_{2K}^{j2} - q_{2K}\left(x_{2K} - M_{2K}^{j2}\right)\right] f_{2K} f_{1K} dx_{2K} dx_{1K}$$
$$+ \int_{B_{1K}^2}^{\infty} \int_0^{\infty} \left[r_{2K} b_{2K}^{j2} - q_{2K}\left(x_{2K} - M_{2K}^{j2}\right)\right] f_{2K} f_{1K} dx_{2K} dx_{1K}$$
$$\left. + \int_{b_{2K}^{j2}}^{\infty} \left[r_{2K} b_{2K}^{j2} - q_{2K}\left(x_{2K} - b_{2K}^{j2}\right)\right] F_{1K}\left(b_{1K}^2\right) f_{2K} dx_{2K} \right\}.$$

Note that $J_2^2(b_{1L}^2, b_{2L}^{j2}; t_{q2}^{j2})$ is identical to the objective function in the uninformed game. It is because q_{1L} does not play a role on $P2$'s objective function, he then will use the same strategy in the two types to against b_{1L}^2. It indicates that

$$b_{2L}^{12*} = b_{2L}^{22*}.$$

Thus, $P1$'s objective function is also identical to that in the uninformed case (see (40)). Accordingly, we conclude that the game when $P2$ acquires the private information of q_{1L} is identical to the uninformed game and the value of private information of q_{1L} for $P2$, ω_{q2}^2, is always zero.

5.1.4 Public Information Game: q_{1L} is Known to Both Players

At last, we are going to discuss the case in which the two players receive public information of q_{1L}. Each player has two types. The type spaces of $P1$ and $P2$ are $T_1^3 = \{t_{q1}^{13}, t_{q2}^{23}\}$ and $T_2^3 = \{t_{q2}^{13}, t_{q2}^{23}\}$ respectively. In this situation, both players have two objective functions. Note that in each type, one player has complete information of both players. Thus, one player's objective function in each type is identical to (4). Then, we have the following conclusions.

Theorem 10 *When the two player receive the public information of q_{1L}, there exist a unique Nash equilibrium in each type of games, which is $(b_{1L}^{13*}, b_{2L}^{13*})$ when $q_{1L} = q_{1L}^1$; and $(b_{1L}^{23*}, b_{2L}^{23*})$ when $q_{1L} = q_{1L}^2$. In addition, if $q_{1L}^1 < q_{1L}^2$, then $b_{1L}^{13*} \leq b_{1L}^{23*}$ and $b_{2L}^{23*} \leq b_{2L}^{13*}$.*

Proof. When the two players receive public information, in each chance move, the two players play a complete information game. Referring to Theorem 2, the game admits a unique Nash equilibrium. Furthermore, we have known that the optimal booking limit of b_{1L} is non-decreasing and b_{2L} is non-increasing as q_{1L} increases (see Proposition 5). Accordingly, we have $b_{1L}^{13*} \leq b_{1L}^{23*}$ and $b_{2L}^{23*} \leq b_{2L}^{13*}$ if $q_{1L}^1 < q_{1L}^2$. ■

Due to the existence of the Nash equilibrium, we can calculate the optimal booking limits of the two players in each chance move. Thus, the value of public information of q_{1L} for $P1$ can be expressed as

$$\omega_{q1}^3 = \theta_q^1 J_1^{3*}(b_{1L}^{13*}, b_{2L}^{13*}; t_{q1}^{13}) + \theta_q^2 J_1^{3*}(b_{1L}^{23*}, b_{2L}^{23*}; t_{q1}^{23}) - J_1^*(b_{1L}^*, b_{2L}^*; t_{q1}), \quad (53)$$

and the value of public information of q_{1L} for $P1$ can be expressed as

$$\omega_{q1}^3 = \theta_q^1 J_2^{3*}(b_{1L}^{13*}, b_{2L}^{13*}; t_{q2}^{13}) + \theta_q^2 J_2^{3*}(b_{1L}^{23*}, b_{2L}^{23*}; t_{q2}^{23}) - J_2^*(b_{1L}^*, b_{2L}^*; t_{q2}), \quad (54)$$

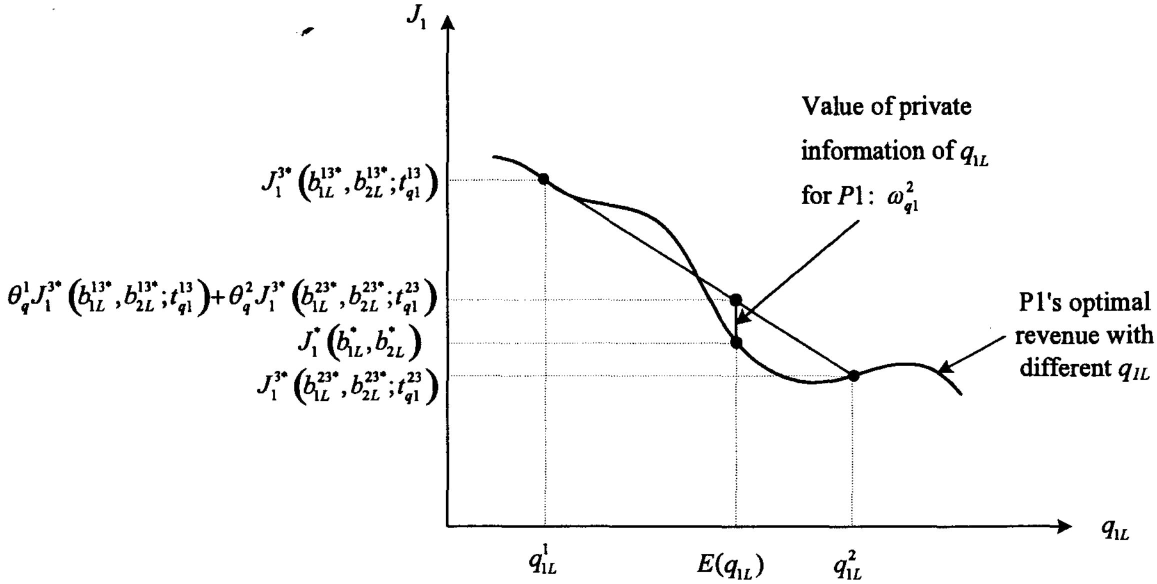


Figure 22. Value of public information of q_{1L} for $P1$.

Remark 4 *From Figure 22, we see that the sign of the value of public information for $P1$ depends on the position of the point $(E(q_{1L}), J_1^{3*})$. The value of public information for $P1$ is positive if the point is above the curve J_1^*; and it is negative if the point is below J_1^*. Note that θ_q^1 (or θ_q^2) also affects the value of public information. As θ_q^1 varies from 0 to 1, $E(q_{1L})$ increases from q_{1L}^1 to q_{1L}^2. Then, the corresponding total expected revenue of $P1$ will decreases from $J_1^{3*}(b_{1L}^{13*}, b_{2L}^{13*}; t_{q1}^{12})$ to $J_1^{3*}(b_{1L}^{23*}, b_{2L}^{23*}; t_{q1}^{23})$ along the line. Consequently, it is possible that the value of public information is negative for some ranges of θ_q^1 and it is positive for other ranges of θ_q^1.* $\triangleleft$

We have seen the secret and private information of q_{1L} are not valuable for $P2$. However, it is interesting to see that the public information might benefit $P2$

in some situations. Even though the relationship between J_2^* and q_{1L} is uncertain (see Remark 3), we know that the value of public information for $P2$ is positive if $\theta_q^1 J_2^{3*}(b_{1L}^{13*}, b_{2L}^{13*}; t_{q2}^{13}) + \theta_q^2 J_2^{3*}(b_{1L}^{23*}, b_{2L}^{23*}; t_{q2}^{23})$ is greater than $J_2^*(b_{1L}^*, b_{2L}^*; t_{q2})$. In addition, the role of θ_q^1 on the sign of ω_{q2}^3 is similar to the case for $P1$ (see Remark 4).

Next, we will provide a numerical example to demonstrate the important results obtained in this section.

Example 10 *In this example, we use the same values as in Table* 3 *of Example* 1 *in Chapter* 3 *for prices and transfer rates. In this case, we assume* $q_{1L}^1 = 10$ *with probability of 0.5 and* $q_{1L}^2 = 60$ *with probability of 0.5. First, we attempt to examine the values of the different information and their relations. We set the booking request expectation of* K*-fare class in each hotel to be a lower value number (10 for low-fare class, and 5 for high-fare class) and a high number (60 for low-fare class, and 35 for high-fare class). We then generate 16 scenarios, by which we calculate the optimal solutions when the two players acquires different information on* q_{1L}*. The relationships among the values of different information for* $P1$ *is shown in Figure* 23*. It can be seen that the values of secret and private information dominate the value of public information in most cases (region* R_1 *and* R_2*). However, we find that when* λ_{1L} *and* λ_{2H} *are high while* λ_{2L} *and* λ_{1H} *are low (region* R_3*), the value of public information dominates the value of private information. In addition, when* λ_{1L} *is high while* λ_{2L}*,* λ_{1H} *and* λ_{2H} *are low (region* R_4*), the value of public information dominates the values of both secret and private information. Next, we examine the value of different information of* q_{1L} *for* $P2$*. As expected, the values of secret and*

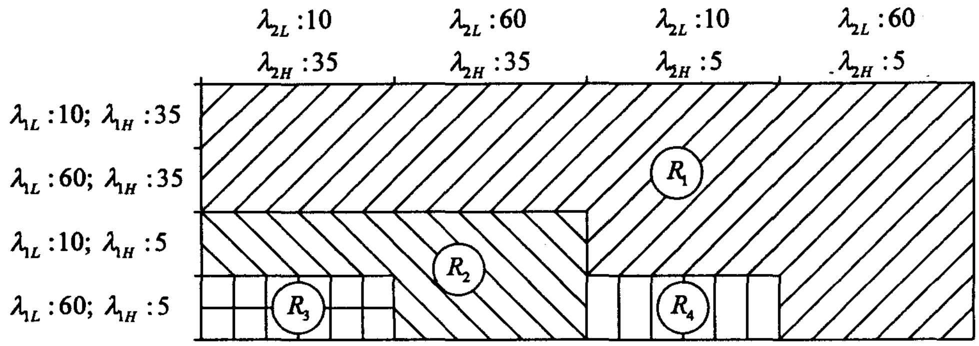


Figure 23. Relations of value of different information of q_{1L} for $P1$. In region R_1, $\omega_{q1}^2 > \omega_{q1}^1 > \omega_{q1}^3$; in region R_2, $\omega_{q1}^1 > \omega_{q1}^2 > \omega_{q1}^3$; in region R_3, $\omega_{q1}^1 > \omega_{q1}^3 > \omega_{q1}^2$ and in region R_4, $\omega_{q1}^3 > \omega_{q1}^2 > \omega_{q1}^1$.

private information for P2 are always zero in any scenarios. We then calculate the values of public information in each scenario and the findings are summarized in Figure 24. Note that when P2's booking request expectation of high-fare class customer is lower (region D_2), P2 is more likely benefit from the public information of q_{1L}. On the other hand, when his booking request expectations of the two fare class customers are both high (region D_1), the value of public information of q_{1L} is negative. We also find that in some scenarios, the sign of ω_{q2}^3 change if we vary θ_q^1. For example, when we set $\lambda_{1H} = \lambda_{2H} = 10$ and $\lambda_{1H} = \lambda_{2H} = 35$, if θ_q^1 is changed from 0.5 to 0.8, ω_{q2}^3 will change from $\omega_{q2}^3 < 0$ to $\omega_{q2}^3 > 0$. This verifies our conclusion presented in Remark 4. ♦

5.2 The Value of Information when u_{1L} is unknown

Note that we have analyzed the value of information of q_{1L} for the two players in pervious section. Now, we will assume another important parameter as the

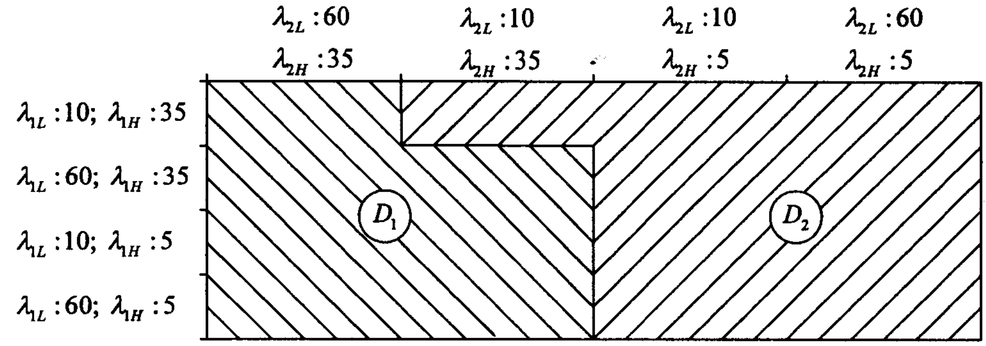


Figure 24. Value of public information of q_{1L} for $P2$. In region D_1, $\omega_{q2}^3 < 0$ and in region D_2, $\omega_{q2}^3 > 0$.

incomplete information, i.e., the transfer rate. Clearly, in each information case, the game model will be different since one player's transfer rate only plays a role in the other player's objective function. Again, we assume $P1$'s transfer rate of his low-fare class customer is the only incomplete information which can be u_{1L}^1 with probability θ_u^1 and u_{1L}^2 with probability of θ_u^2 for $\theta_u^1 + \theta_u^2 = 1$. Let us begin from the uninformed case.

5.2.1 Uninformed Game: Both Players are Uninformed of u_{1L}

When both players are uninformed of u_{1L}, each player has only one type which is $T_1 = \{t_{u1}\}$ and $T_2 = \{t_{u2}\}$. We see that u_{1L} does not affect the expected revenue of $P1$ in any fare class (see (4)). Thus, the objective function in this case is exactly identical to (4) of the complete information game. However, u_{1L} plays a role on the expected revenue of $P2$. Thus, we have $P2$'s objective function as

$$
\begin{aligned}
J_2(b_{1L}, b_{2L}; t_{u2}) = \quad & \sum_{j=1,2} \theta_u^j \left\{ \int_0^{b_{2L}} r_{2L} x_{2L} F_{1L}(b_{1L}) f_{2L} \, dx_{2L} \right. \\
& + \int_{b_{1L}}^{B_{1L}^j} \int_0^{b_{2L}} r_{2L} (x_{2L} + b_{2L} - M_{2L}) f_{2L} f_{1L} \, dx_{2L} \, dx_{1L} \\
& + \int_{b_{1L}}^{B_{1L}} \int_{b_{2L}}^{\infty} \left[r_{2L} b_{2L} - q_{2L} \left(x_{2L} - M_{2L}^j \right) \right] f_{2L} f_{1L} \, dx_{2L} \, dx_{1L} \\
& + \int_{B_{1L}}^{\infty} \int_0^{\infty} \left[r_{2L} b_{2L} - q_{2L} \left(x_{2L} - M_{2L}^j \right) \right] f_{2L} f_{1L} \, dx_{2L} \, dx_{1L} \\
& \left. + \int_{b_{2L}}^{\infty} \left[r_{2L} b_{2L} - q_{2L} (x_{2L} - b_{2L}) \right] F_{1L}(b_{1L}) f_{2L} \, dx_{2L} \right\} \\
& + \int_0^{b_{2L}} r_{2L} x_{2L} F_{1L}(b_{1L}) f_{2L} \, dx_{2L} \\
& + \int_{B_{1H}}^{\infty} \int_0^{\infty} \left[r_{2H} b_{2H} - q_{2H} (x_{2H} - M_{2H}) \right] f_{2H} f_{1H} \, dx_{2H} \, dx_{1H} \\
& + \int_{b_{1H}}^{B_{1H}} \int_{b_{2H}}^{\infty} \left[r_{2H} b_{2H} - q_{2H} (x_{2H} - M_{2H}) \right] f_{2H} f_{1H} \, dx_{2H} \, dx_{1H} \\
& + \int_{B_{1H}}^{\infty} \int_0^{\infty} \left[r_{2H} b_{2H} - q_{2H} (x_{2H} - M_{2H}) \right] f_{2H} f_{1H} \, dx_{2H} \, dx_{1H} \\
& + \int_{b_{2H}}^{\infty} \left[r_{2H} b_{2H} - q_{2H} (x_{2H} - b_{2H}) \right] F_{1H}(b_{1H}) f_{2H} \, dx_{2H}
\end{aligned} \tag{55}
$$

where $b_{2H} = C_2 - b_{2L}$, $M_{2L}^j = b_{2L} - u_{1L}^j (x_{1L} - b_{1L})$, $M_{2H} = b_{2H} - u_{1H} (x_{1H} - b_{1H})$, $B_{1L}^j = b_{1L} + (b_{2L} - x_{2L})/u_{1L}^j$ and $B_{1H} = b_{1H} + (b_{2H} - x_{2H})/u_{1H}$ for $j = 1, 2$. Similar to the uninformed case in previous section, we will attempt to find whether this game has a Nash strategy. Since $P1$'s objective function is identical to that of complete information game. Therefore, his objective function has the same structural properties as described 3. We then focus on the $P2$'s objective function. Partially differentiating J_2 with respect to b_{2L}, we obtain

$$\begin{aligned} V_2 = \ & (r_{2L} + q_{2L}) \left[\int_0^{b_{2L}} (\theta_u^1 \int_{N_{1L}^1}^{\infty} f_{1L} dx_{1L} + \theta_u^2 \int_{N_{1L}^2}^{\infty} f_{1L} dx_{1L}) f_{2L} dx_{2L} + \bar{F}_{2L}(b_{2L}) \right] \\ & - (r_{2H} + q_{2H}) \left[\int_0^{b_{2H}} \int_{N_{1H}}^{\infty} f_{1H} f_{2H} \ dx_{2H} \ dx_{1H} + \bar{F}_{2H}(b_{2H}) \right], \end{aligned} \tag{56}$$

where $N_{1L}^j = b_{1L} + (b_{2L} - x_{2L})/u_{1L}^j$ $(j = 1, 2)$ and $N_{1H} = b_{1H} + (b_{2H} - x_{2H})/u_{1H}$. We find the second order derivative of J_2 with respect to b_{2L},

$$\begin{aligned} & - (r_{2L} + q_{2L}) \left[\int_0^{b_{2L}} f_{2L} (\tfrac{\theta_u^1}{u_{1L}^1} f_{1L}(N_{1L}^1) + \tfrac{\theta_u^1}{u_{1L}^2} f_{1L}(N_{1L}^2)) dx_{1L} + f_{1L}(b_{1L}) F_{2L}(b_{2L}) \right] \\ & - (r_{2H} + q_{2H}) \left[\int_0^{b_{2H}} f_{2H} f_{1H}(N_{1H}) \ dx_{1H} + f_{1H}(b_{1H}) F_{2H}(b_{2H}) \right], \end{aligned}$$

is always less than zero, which implies J_2 is strictly concave in b_{2L}. With further investigation, we find that $V_2(b_{1L}, b_{2L}; t_{u2}) = 0$ is a strictly decreasing curve in the (b_{1L}, b_{2L}) plane, and the implicit derivative of $V_1(b_{1L}, b_{2L}; t_{u1}) = 0$ with respect to b_{1L} is always less than the implicit derivative of $V_2(b_{1L}, b_{2L}; t_{u2}) = 0$ with respect to b_{1L}. All of these results obtained above lead to the existence of a unique Nash equilibrium in the uninformed game.

Theorem 11 *When each of the two players has an incomplete information of u_{1L}, the game admits a unique Nash equilibrium.*

Proof. Analogous to the proof of Theorem 8. ■

Due to the existence of the unique Nash equilibrium in the uninformed game, the two players will more likely play the Nash strategy in this case. Note that the uninformed game in this case is not equivalent to the complete information game by assuming u_{1L} in (4) as $E(u_{1L})$, which is applicable when q_{1L} is uninformed. This will make the analysis of the value of information even more difficult. We will next discuss the optimal strategy and the corresponding expected revenue when one player acquires the information of u_{1L}.

5.2.2 Secret Information Game : One Player Acquires u_{1L} Secretly

We start our analysis of the value of information by assuming one player receives the secret information of u_{1L}.

Proposition 7 The value of secret information of u_{1L} is always zero for $P1$; and it is always non-negative for $P2$. Specially, the value of secret information of u_{1L} is always positive for $P2$ if $b_{2L}^* > 0$.

Proof. When $P1$ acquires secret information of u_{1L}, his objective function in each type is totally identical to that of uninformed game since u_{1L} has no effect on his expected revenue in any fare class. And, we also know that $P2$'s booking limit remains b_{2L}^*. Thus, it is optimal for $P1$ to adopt b_{1L}^* in each chance move to against b_{2L}^*. It indicates that the value of secret information of u_{1L} is meaningless for $P1$. On the other hand, if $P2$ receives the secret information of u_{1L}, he will choose b_{2L}^{j1} in type t_{u2}^{j1} $(j = 1, 2)$, which is probably different to b_{2L}^*. It is because the first order derivative of J_2 in each type is different with that of uninformed game (see 56). Meanwhile, we note that if there exist a feasible solution of b_{2L}^* $(b_{2L}^* > 0)$ which satisfied $V_2 = 0$, the optimal booking limit in each type, b_{2L}^{j1*}, must be different with b_{2L}^*. It incurs an higher expected revenue than J_2^*. Thus, the total expected revenue of $P2$ with secret information is

$$\theta_u^1 J_2^{1*}(b_{1L}^*, b_{2L}^{11*}; t_{u2}^1) + \theta_u^1 J_2^{1*}(b_{1L}^*, b_{2L}^{21*}; t_{u2}^2),$$

which is greater than $J_2(b_{1L}^*, b_{2L}^*; t_{u2})$. It indicates that the value of secret information of u_{1L} is positive for $P2$ in this situation. In general, we have $\omega_{u2}^1 \geq 0$. ■

Similar to the secret information case, the objective functions of $P1$ with private information of u_{1L} are same as (4) in each type. Then, the games in the two types are played as if both players are uninformed. Thus, $P1$ also can not benefit from the private information of u_{1L}.

5.2.3 Private Information Game: One Player Receives Private Information of u_{1L}

In this situation, $P2$'s objective functions in the two types are different when acquiring private information of u_{1L}. We can obtain $P2$'s objective function in each type by simply substituting u_{1L} with u_{1L}^j $(j = 1, 2)$ and b_{2L} with b_{2L}^{j2} in (4). As for

$P1$, his objective function in this case can be expressed as

$$
\begin{aligned}
J_1 = \sum_{j=1,2} \theta_u^j \sum_{K=L,H} & \left\{ \int_0^{b_{1K}^2} r_{1K} x_{1K} F_{2K}\left(b_{2K}^{j2}\right) f_{1K}\, dx_{1K} \right. \\
& + \int_{b_{2K}^{j2}}^{B_{2K}^{j2}} \int_0^{b_{1K}^2} r_{1K}\left(x_{1K} + b_{1K}^2 - M_{1K}^{j2}\right) f_{1K} f_{2K}\, dx_{2K}\, dx_{1K} \\
& + \int_{b_{2K}^{j2}}^{B_{2K}^{j2}} \int_{b_{1K}^2}^{\infty} \left[r_{1K} b_{1K}^2 - q_{1K}\left(x_{1K} - M_{1K}^{j2}\right)\right] f_{2K} f_{1K}\, dx_{2K}\, dx_{1K} \\
& + \int_{B_{2K}^{j2}}^{\infty} \int_0^{\infty} \left[r_{1K} b_{1K}^2 - q_{1K}\left(x_{1K} - M_{1K}^{j2}\right)\right] f_{2K} f_{1K}\, dx_{2K}\, dx_{1K} \\
& \left. + \int_{b_{1K}^2}^{\infty} \left[r_{1K} b_{1K}^2 - q_{1K}\left(x_{1K} - b_{1K}^2\right)\right] F_{2K}\left(b_{2K}^{j2}\right) f_{1K}\, dx_{1K} \right\},
\end{aligned} \tag{57}
$$

where $b_{2H}^{j2} = C_2 - b_{2L}^{j2}$, $b_{1H}^2 = C_1 - b_{1L}^2$, $M_{1K}^{j2} = b_{1K}^2 - u_{2K}\left(x_{1K} - b_{2K}^{j2}\right)$, and $B_{2K}^{j2} = b_{2K} + (b_{1K}^2 - x_{1K})/u_{2K}$ for $j = 1, 2$ and $K = L, H$. Similarly, we in turn investigate the first and second order derivatives of J_1 with respect to b_{1L}^2, the monotonic property of $V_1 = 0$ in the (b_{1L}, b_{2L}) plane. We then find the structural properties of the objective function described in Lemmas 1, 2 also hold for $P1$ in this case. Furthermore, Comparing the implicit derivative of $V_1 = 0$ with respect to b_{1L} to that of of $V_2 = 0$ in each type, we find that in each type of game, the property shown in Lemma 3 holds.

Theorem 12 *When $P2$ has the complete information of $P1$ and himself, and $P1$ knows that his transfer rate of low-fare class customer is u_{1L}^1 with probability θ_u^1 and u_{1L}^2 with probability of θ_u^2 for $\theta_u^1 + \theta_u^2 = 1$ and $u_{1L}^1 < u_{1L}^2$, the game admits a unique*

Bayesian Nash equilibrium $(b_{1L}^{2*}, b_{2L}^{12*}, b_{2L}^{22*})$. *In addition,* $b_{2L}^{12*} \leq b_{2L}^{22*}$.

Proof. The proof of the existence of the unique Bayesian Nash equilibrium is analogous to the proof of Theorem 8, and is thereby omitted here. Now, let us prove $b_{2L}^{12*} \leq b_{2L}^{22*}$. We know that if

$$V_2(b_{1L}^{2*}, 0; t_{u2}^{j2}) = (r_{2L} + q_{2L}) - (r_{2H} + q_{2H}) S_{2H}(b_{2L}^{2*}, 0; t_{u2}^{j2}) < 0, \; j = 1, 2,$$

then it is always optimal for $P2$ to set his booking limit as zero for any u_{1L}. It implies that

$$b_{2L}^{12*} = b_{2L}^{22*} = 0.$$

However, in type t_{u2}^{12}, if $b_{2L} = b_{2L}^{12*} > 0$, then it must satisfies

$$(r_{2L} + q_{2L}) S_{2L}(b_{1L}, b_{2L}) - (r_{2H} + q_{2H}) S_{2H}(b_{1L}, b_{2L}) = 0.$$

Differentiating both sides of the above equation with respect to u_{1L} (b_{1L} is constant), we obtain

$$\frac{1}{(u_{1L})^2}(r_{2L} + q_{2L}) \int_0^{b_{2L}} f_{2L} f_{1L}(N_{1L}) \; dx_{2L} + \frac{\partial^2 J_2}{\partial (b_{2L})^2} \frac{db_{2L}}{du_{1L}} = 0.$$

Since $\frac{\partial^2 J_2}{\partial (b_{2L})^2} < 0$, we know that $\frac{db_{2L}}{du_{1L}}$ must be positive, which indicates b_{2L} increases as u_{1L} increases. On the other words, b_{2L}^{12*} is always less than b_{2L}^{22*} if $b_{2L}^{12*} > 0$. Thus, in general, $b_{2L}^{12*} \leq b_{2L}^{22*}$. ■

We might also compare the optimal expected revenue of $P2$ in each type. Differentiating J_2 with respect to u_{1L} (b_{1L} is constant), we obtain

$$
\begin{aligned}
dJ_2/du_{1L} = & \int_0^{b_{2L}} \int_{b_{1L}}^{B_{1L}} r_{2L} b_{2L} f_{1L} f_{2L} \, dx_{1L} \, dx_{2L} \\
& - \int_0^{b_{2L}} \int_{B_{1L}}^{\infty} q_{2L} \left[(x_{2L} - b_{2L}) \right] f_{1L} f_{2L} \, dx_{1L} \, dx_{2L} \\
& - \int_{b_{2L}}^{\infty} \int_{b_{1L}}^{\infty} q_{2L} \left[(x_{2L} - b_{2L}) \right] f_{1L} f_{2L} \, dx_{1L} \, dx_{2L} \\
& + \left(\partial J_2^2 / \partial b_{2L} \right) \; \left(db_{2L} / du_{1L} \right) .
\end{aligned} \tag{58}
$$

Unfortunately, the relationship between $J_2^*(b_{1L}^{2*}, b_{2L}^{12*}; t_{u2}^{12})$ and $J_2^*(b_{1L}^{2*}, b_{2L}^{22*}; t_{u2}^{22})$ is uncertain since the sign of dJ_2/du_{1L} in (58) can not be determined. However, the value of private information of u_{1L} for $P2$ can be obtain by the following expression:

$$
\omega_{u2}^2 = \theta_u^1 J_2^{2*}(b_{1L}^{2*}, b_{2L}^{12*}; t_{u2}^{12}) + \theta_u^2 J_2^{2*}(b_{1L}^{2*}, b_{2L}^{22*}; t_{u2}^{22}) - J_2^*(b_{1L}^*, b_{2L}^*; t_{u2}).
$$

5.2.4 Public Information Game: u_{1L} is Known to Both Players

When the two players receive public information of u_{1L}, the type spaces of $P1$ and $P2$ are $T_1 = \{t_{u1}^{13}, t_{u2}^{23}\}$ and $T_2 = \{t_{u2}^{13}, t_{u2}^{23}\}$ respectively. Similar to the public information case in Section 5.1, one player' objective function in each type is identical to (4).

Theorem 13 *When the two players receive the public information of u_{1L}, there exists a unique Nash equilibrium in each type of game, which is $(b_{1L}^{13*}, b_{2L}^{13*})$ when $u_{1L} = u_{1L}^1$; and $(b_{1L}^{23*}, b_{2L}^{23*})$ when $u_{1L} = u_{1L}^2$. In addition, $b_{1L}^{13*} \leq b_{1L}^{23*}$ and $b_{2L}^{23*} \leq b_{2L}^{13*}$ if $u_{1L}^1 < u_{1L}^2$.*

Proof. The proof of the existence of the unique Nash equilibrium in each type is analogous to the proof of Theorem 10, and the proof the relations between b_{iL}^{13*} and b_{iL}^{23*} $(i = 1, 2)$ is analogous to the proof of Proposition 5. Then, we thereby omit the proof of Theorem 13. ■

Even though we know that as u_{1L} increases the Nash equilibrium moves in the northwest direction in (b_{1L}, b_{2L}) plane, which leads to a decrease in b_{1L} and an increase in b_{2L}, we could not generalize the specific situation(s) in which the value of public information of u_{1L} for one player is positive or negative. This is due to the uncertainty of the objective function with respect to u_{1L} in both uninformed and public information games. In general, the value of public information of u_{1L} for $P1$ can be expressed as

$$\omega_{u1}^3 = \theta_u^1 J_1^{3*}(b_{1L}^{13*}, b_{2L}^{13*}; t_{u1}^{13}) + \theta_u^2 J_1^{3*}(b_{1L}^{23*}, b_{2L}^{23*}; t_{u1}^{23}) - J_1^*(b_{1L}^*, b_{2L}^*; t_{u1}),$$

and the value of public information of u_{1L} for $P2$ is

$$\omega_{u2}^3 = \theta_u^1 J_2^{3*}(b_{1L}^{13*}, b_{2L}^{13*}; t_{u2}^{13}) + \theta_u^2 J_2^{3*}(b_{1L}^{23*}, b_{2L}^{23*}; t_{u2}^{23}) - J_2^*(b_{1L}^*, b_{2L}^*; t_{u2}).$$

We are going to use the following numerical example to demonstrate the value of different information of u_{1L} and some important results shown in this section.

Example 11 *In this example, we examine the value of different information of u_{1L} for each player and their relationships in different situation. The procedures to attain this goal are very similar to Example* 10*. Specifically, we fix q_{1L} as 30 and $u_{1L}^1 = 0.2$ with probability of 0.5 and $u_{1L}^2 = 0.9$ with probability of 0.5. probability* *of 0.5. Again, we vary the booking request expectations to generate 16 scenarios, by which we examine the relationships among the values of different information for the two players. As expected, the value of public information of u_{1L} is dominated by the values of secret and private information in most cases (see Figure* 25*). Comparing to Example* 10*, the relationships among the values of different information of u_{1L} exhibit an additional form, which is $\omega_{u2}^3 > \omega_{u2}^1 > \omega_{u2}^2$. In addition, we find that $\omega_{u2}^3 < 0$ in any scenario in which the booking request expectations of the two fare classes of P2 is high. As for P1, the situations in which the value of public information of u_{1L}*

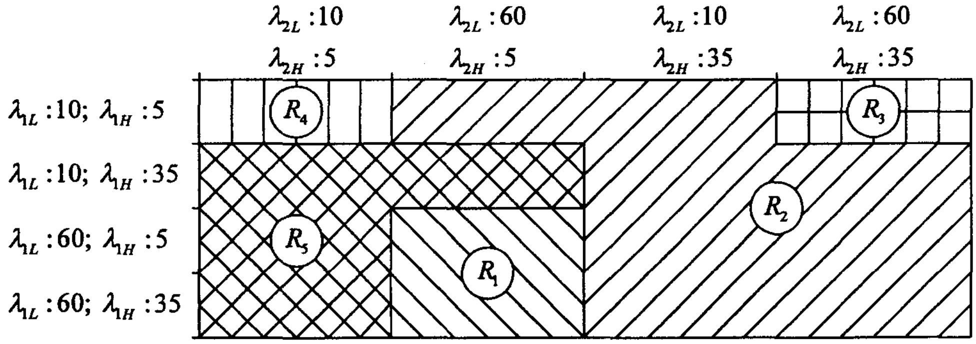


Figure 25. Relations of value of different information of u_{1L} for $P2$. In region R_1, $\omega_{u2}^2 > \omega_{u2}^1 > \omega_{u2}^3$; in region R_2, $\omega_{u2}^1 > \omega_{u2}^2 > \omega_{u2}^3$; in region R_3, $\omega_{u2}^1 > \omega_{u2}^3 > \omega_{u2}^2$; in region R_4, $\omega_{u2}^3 > \omega_{u2}^2 > \omega_{u2}^1$ and in region R_5, $\omega_{u2}^3 > \omega_{u2}^1 > \omega_{u2}^2$.

is positive or negative can be seen from Figure 26*. In most cases, P1 can benefit from the public information of u_{1L}. Similarly, we also find in some scenarios the relationships between the values of different information of u_{1L} for P2 and the sign of the value of public information for P1 change as we vary θ_u^2. In addition, we find that in the scenario when the booking requests expectations of P1 are high and the booking requests expectations of P2 are low, the value of any type of information for*

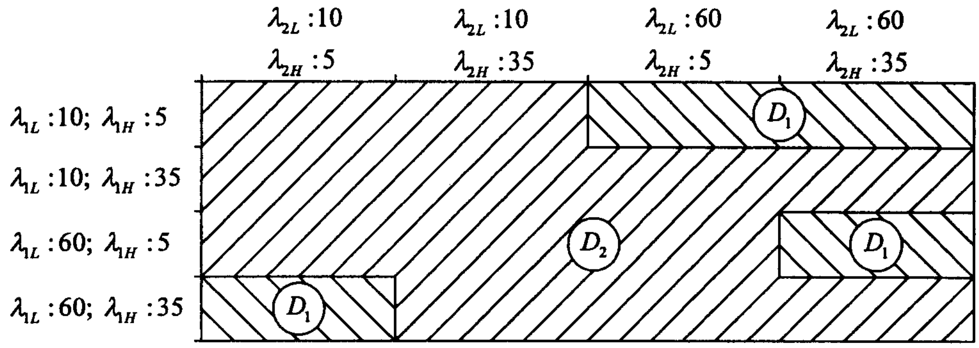


Figure 26. Value of public information of u_{1L} for $P2$. In region D_1, $\omega_{u1}^3 < 0$ and in region D_2, $\omega_{u1}^3 > 0$.

P2 is \$55+ which account for 3% of his total expected revenue. ♦

Chapter 6
Thesis Summary and Concluding Remarks

As discussed in Chapter 1, game theory is a quantitative approach used in the study of two or more decision makers' interactive behaviors in competitive or cooperative situations. In Section 1.3, we indicated that game theory can be applied in revenue management to deal with single-period multiple-class games with complete information, and with multiple-period multiple-class games with complete information, etc. In Chapter 3, we established a two-player two-fare-class (high-fare and low-fare) static game model to solve the hotel room inventory control problem. Under this game theoretic setting, we obtained the optimal rationing policies for the two hotels under competitive and cooperative situations. We also characterized the structural properties of the corresponding objective functions and analyzed the equilibria of competitive and cooperative games, respectively. Our study indicates some important managerial implications on this revenue management problem: First, our game model indicates that as a best response, one hotel should always decrease its booking limit for low-fare customer by more than one unit if another hotel increases the low-fare booking limit by one unit, and vice versa. Secondly, when the hotels compete, we have proved the existence and uniqueness of Nash equilibrium and have presented the structural properties of these equilibria in different situations. Also, we identify the situation in which Stackelberg game is equivalent to Nash game. This result shows that if one player's booking limit is reached, i.e., he always rejects low-fare customers, neither of the two players would like to "lead" the game. Finally, we find that the revenue loss is substantial if there is a lack of cooperation between two players. Our numerical experiments suggest that such loss can be more than 10% in most cases.

In Chapter 4, we formulated a two-player two-fare-class dynamic game model. In this situation, the problem becomes more complicated since one hotel's accept/reject decisions in each period are not only affected by the decisions of the other hotel, but also affected by room inventory levels of both players at the beginning of the period. Analogous to the analysis in Chapter 3, we examine our model using Nash, Stackelberg and cooperative strategies. The main contributions can be presented as follows. First, each hotel's optimal future revenue is a non-decreasing function of its own room inventory and a non-increasing function of the other hotel's room inventory at any time. Secondly, we establish the unique Nash equilibrium of dynamic accept/reject decisions for the two hotels under competitive situation. Finally, by defining expected marginal value of hotels' room, we simplify the optimal accept/reject decision into sets of critical values.

In Chapter 5, we studied the two-player two-fare-class static game with incomplete information. We first clarified the various definitions of information value in our games. Then we assume the rejection cost and transfer rate of one player as part of incomplete information and examine the optimal booking policies of each player with different information structures. We find that in the uninformed game, there exist a unique Nash equilibrium for the two players. Specially, as the rejection cost of one player is uninformed, the game is equivalent to the complete information game by using the rejection cost expectation as the real rejection cost. We also proved the existence of a unique Bayesian Nash equilibrium in the game when one player receives the private information of rejection cost or transfer rate. Furthermore, we provided the formulations of the value of different information. We see that the value of secret information is always non-negative for both players. We also evaluated the values of private and public information for one player and we provided the conditions by which one player might use or drop the information of rejection cost and transfer rate. Finally, we presented experimental results corroborating our theoretical analysis of the value of different information.

The research discussed in this paper could be extended in several directions One natural extension should be to consider the incomplete information game in the dynamic context. Again, we can examine the existence of Nash equilibrium (perfect Bayesian Nash equilibrium) and the uniqueness. Second, under each game-theoretic setting, static or dynamic, complete or incomplete information, we can consider three or more players. For example, under the static game with complete information, one possibility is to analyze a competitive/cooperative game in which at least two or more players cooperate to increase their total expected revenue at the expense of the other players. Another possibility is to assume that they all compete in which case Nash strategy can be used to by all players as before. However, we suspect that it will be more difficult to prove the existence and/or uniqueness of Nash equilibrium since the two-way transfers of low-fare bookings would greatly complicate the substitution structure of the model. A third possible extension would be to consider a game with the consideration of the overbooking by relaxing the assumption of no cancellations of booked customer. This situation is fairly common in practice and many RM researchers have worked on different aspects of this problem in recent years. Generally, within game-theoretic context, we might progressively relax the one or more assumptions in our problems to make the games 'richer' and rule out the booking policies and plausible equilibrium for them.

References

[1] P. K. Banerjee and B. Viswanathan. On optimal rationing policies. *Canadian Journal of Administrative Sciences*, 6(2):1–6, 1989.

[2] T. Basar and G.J. Olsder. *Dynamic Noncooperative Game Theory*. Academic Press, London, UK, 1982.

[3] B. Bassan, O. Gossner, M. Scarsini, and S. Zamir. A class of games with positive value of information. *International Journal of Game Theory*, 32:17–31, 2003.

[4] P. P. Belobaba. Application of a probabilistic decision model to airline seat inventory control. *Operations Research*, 37:183–197, 1989.

[5] G. R. Bitran and R. Caldentey. An overview of pricing models for revenue management. *Manufacturing & Service Operations Management*, 5(3):203–229, 2003.

[6] G. R. Bitran and S. M. Gilbert. Managing hotel reservations with uncertain arrivals. *Operations Research*, 44:35–49, 1996.

[7] S. Bodily and L. Weatherford. Perishable-asset revenue management: Generic and multiple-price yield management with diversion. *Omega*, 23(2):173–185, 1995.

[8] S. Brumelle and D. Walczak. Dynamic airline revenue management with multiple semi-markov demand. *Operations Research*, 51(1):137–148, 2003.

[9] S. L. Brumelle, J. I. McGill, T. H. Oum, K. Sawaki, and M. W. Tretheway.

Allocation of airline seats between stochastically dependent demands. *Transportation Science*, 24:183–192, 1990.

[10] W. J. Carrol and R. C. Grimes. Evolutionary change in product management: Experiences in the car rental industry. *Interfaces*, 25(5):84–104, 1995.

[11] K. Chatterjee and W. F. Samuelson, editors. *Game Theory and Business Applications.* Kluwer, Boston, 2001.

[12] F. Y. Chen, J. Chen, and Y. Xiao. Competitive and cooperative inventory models for firms serving both store and online third-party's demand. Technical report, Chinese University of Hong Kong, 2006. Working Paper.

[13] K. Donaghy, U. McMahone, and D. McDowell. Yield management: An overiew. *International Journal of Hospitality Management*, 14(2):139–151, 1995.

[14] G. Feichtinger and S. Jørgensen. Differential game models in management science. *European Journal of Operational Research*, 14:137–155, 1983.

[15] Y. Feng and G. Gallego. Optimal starting times for end-of-season sales and optimal stopping times for promotional fares. *Management Science*, 41:1371–1391, 1995.

[16] Y. Feng and B. Xiao. Optimal policies of yield management with multiple predetermined prices. *Operations Research*, 48(2):332–343, March-April 2000.

[17] D. A. Gautschi, editor. *Productivity and Efficiency in Distribution Systems.* Elsevier Science Publishing Co., Inc., New York, 1983.

[18] Y. Gerchak, M. Parlar, and T. K. M. Yee. Optimal rationing policies and production quantities for products with serveral demand classes. *Canadian*

Journal of Administrative Sciences, 2(1):161–176, June 1985.

[19] I. Gilboa and E. Lehrer. The value of information - an axiomatic approach. *Journal of Mathematical Economics*, 20:443–459, 1991.

[20] R. K. Griffin. A categorization scheme for critical factors of lodging yield management systems. *International Journal of Hospitality Management*, 14(3):325–338, 1995.

[21] C. N. Hansen and K. Eringa. Critical success factors in yield management: A development and analysis. *Progress In Tourism and Hospitality Research*, 4:229–244, 1998.

[22] S. Jørgensen. A survey of some differential games in advertising. *Journal of Economic Dynamics and Control*, 4:341–369, 1982.

[23] J.P J.P. Ponssard. On the concept of the value of information in competitive situations. *Management Science 22 (7), 1976.*, 22(7):739–747, 1976.

[24] M. Kamien, Y. Tauman, and S. Zamir. The value of information in a strategic conflict. *Games and Economic Behavior*, 2:129–153, 1990.

[25] P. Kevin and N. Piersma. Airline revenue management: An overview of OR techniques 1982-2001. *Econometric Institute Report EI 2002-2003*, 2002.

[26] S. E. Kimes. Yield management: A tool for capacity-constrained service firms. *Journal of Operations Management*, 8(4):348–363, 1989.

[27] S. E. Kimes and R. B. Chase. The strategic levers of yield management. *Journal of Service Research*, 1(2):156–166, 1998.

[28] T. C. Lee and M. Hersh. A model for dynamic airline seat inventory control with multiple seat bookings. *Transportation Science*, 27(3):252–265, August 1993.

[29] M. Leng and M. Parlar. Game theoretic applications in supply chain management: A review. *Infor*, 43(3):187–220, 2005.

[30] P. Levine and J. P. Ponssard. Tile values of information in some nonzero sum games. *International Journal of Game Theory,*, 6(4):221–229, 1977.

[31] Y. Liang. Solution to the continuous time dynamic yield management model. *Transportation Science*, 13:117–123, 1999.

[32] K. Littlewood. Forecasting and control of passenger bookings. *AGIFORS Symposium Proceeding*, 12, 1972.

[33] J. I. McGill and G. J. van Ryzin. Revenue management: Research overview and prospects. *Transportation Science*, 33(2):233–256, May 1999.

[34] G. E. Monahan, N. C. Petruzzi, and W. Zhao. The dynamic pricing problem from a newsvendorąŕs perspective. *Manufacturing & Service Operations Management*, 6(1):73–91, 2004.

[35] S. K. Nair and R. Bapna. An application of yield management for internet service providers. *Naval Research Logistics*, 48:348–362, 2001.

[36] S. Netessine and R. Shumsky. Introduction to the theory and practice of yield management. *INFORMS Transcations on Education*, 3(1):34–44, 2002.

[37] S. Netessine and R. A. Shumsky. Revenue management games: Horizontal and vertical competition. *Management Science*, 51(5):813–831, 2005.

[38] A. Neyman. The positive value of information. *Games and Economic Behavior*, 3:350–355., 1991.

[39] H. Nikaido and K. Isora. Notes on noncooperative convex games. *Pacific Journal of Mathematics*, 5:807–815, 1955.

[40] M. Parlar. Game theoretic analysis of the substitutable product inventory problem with random demands. *Naval Research Logistics*, 35:397–409, 1988.

[41] P. E. Pfeifer. The airline discount fare allocation problem. *Decision Sciences*, 20:149–157, 1989.

[42] J. Sheth, L. P. Bucklin, and J. M. Carman, editors. *Research in Marketing: Distribution Channels and Institutions*, volume 8. JAI press Inc., London, England, 1986.

[43] M. Shubik. The uses of game theory in management science. *Management Science*, 2(1):40–54, October 1955.

[44] R. V. Slyke and Y. Young. Finite horizon stochastic knapsacks with applications to yield management. *Operations Research*, 48(1):155–172, January-Feburary 2000.

[45] B. C. Smith, J. F. Leimkuhler, and R. M. Darrow. Yield management at american airlines. *Interfaces*, 22(1):8–31, 1992.

[46] H. Von Stackelberg, editor. *Marktform und Gleichgewicht.* Springer, Vienna, 1934. an English translation of this book published by Oxford University Press entitled by The Theory of Market Economy, appeared in 1952.

[47] K. Talluri and G. Van Ryzin. Revenue management under a general discrete

choice model of consumer behavior. *Management Science*, 50(1):15–33, 2004.

[48] R. S. Upchurch, T. Ellis, and J. Seo. Revenue management underpinnings: an exploratory review. *International Journal of Hospitality Management*, 21:67–83, 2002.

[49] Q. Wang and M. Parlar. Static game theory models and their applications in management science. *European Journal of Operational Research*, 42:1–21, 1989.

[50] L. Weatherford and S. Bodily. A taxonomy and research overview of perishable-asset revenue management: Yield management, overbooking, and pricing. *Operations Research*, 40:831–844, 1992.

[51] P.-S. You. A dynamic model for a two-cabin yield management with free upgrading decision. *Journal of Operations Research Society of Japan*, 44(4):313–325, December 2001.

[52] P.-S. You. Dynamic rationing policies for product with incremental upgrading demands. *European Journal of Operational Research*, 144:128–137, 2003.

[53] W. Zhao and Y.-S. Zheng. A dynamic model for airline seat allocation with passenger diversion and no-shows. *Transportation Science*, 35(1):80–98, February 2001.