RM GAMES FOR HOTEL ROOM INVENTORY CONTROL
GAME THEORETIC REVENUE MANAGEMENT MODELS FOR HOTEL ROOM INVENTORY CONTROL

BY

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In this thesis, we focus on the rationing policies for the hotel room inventory control problems. Our study begins with a brief overview of revenue management in hotel industry, emphasizing the importance of room inventory control in revenue management problems. Mathematical models for controlling the room inventory in the literature are then reviewed along with recently developed game theoretic applications in revenue management. In game theoretic context, we establish three types of models to solve the hotel room inventory control problem in three different situations: 1) two-player two-fare-class static single-period game with complete information; 2) two-player two-fare-class dynamic multiple-period game with complete information; and 3) two-player two-fare-class single-period game with incomplete information.

In the first situation, we find the existence of unique Nash equilibrium and Stackelberg equilibrium in the non-cooperative case. We provide the exact forms for these equilibria and corresponding conditions. Next, under the dynamic game settings, we provide the sufficient conditions for the unique Nash equilibrium. In the last situation, we consider the static single-period games with incomplete information and discuss the optimal strategies for the uninformed case, secret information case, private information case and public information case. The unique Bayesian Nash equilibrium in each case is found. We then analyze the values of different types of information and study their relations in different situations. Under each game theoretic setting, we present the managerial implications of our solutions along with the numerical examples. The thesis is concluded by a discussion of how game theory can is useful in hotel industry, and its relationship to other topics in revenue management.
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Last, but not least, I thank my family, my parents, elder brother, and my wife for their unconditional support and encouragement to pursue my interests, even when the interests went beyond boundaries of language, field and geography.
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Chapter 1
Introduction

This chapter provides an introduction to the topic of revenue management (RM) and game theoretic applications. We start with an explanation of RM and its history. We then briefly describe the industry profiles of hotels which make the RM practices distinct from the applications in other industries and the objective of hotel RM. Next, we highlight the importance of combining ideas from RM and game theory. Finally, we conclude by giving an outline of the remaining chapters in this thesis.

1.1 Revenue Management and Its History

Revenue management (RM) first appeared in the airline industry in the late 1970s when the deregulated industry attempted to maximize profit by ensuring all seats were occupied before take-off and offering varied prices to the customers. RM marries operations research/management science, statistics, economics, and software development to manage demand for a firm’s inventory with the goal of maximizing revenue. Practitioners usually find that it is easier to define the objective of revenue management rather than explain what it actually is—the outcomes are easier to understand than the process. Revenue management is about marketing mix, cost/price relationships and product distribution, which allows a business to “sell the right product, to the right customer, at the right time, at the right price” (Smith et al. [45]). It is a suite of components that, when working in harmony, will present the best oppor-
tunity to maximize revenue. In 1992, Weatherford and Bodily [50] proposed to replace the term revenue management with a new, more appropriate term, Perishable-Asset Revenue Management (PARM). The element that links the industries implementing revenue management is that all of their inventories are perishable. Once the plane takes off, there is nothing one can do about trying to sell any of the seats on the plane. Similarly, when a room is empty overnight, the opportunity for revenue is lost forever.

Today, revenue management has also spread out to other industries such as hotels, retailers, car rental agencies, Internet service providers (ISP), railways, cruise lines, electric power supply and restaurants. Basically, these industry sectors all share certain characteristics that make them particularly suited for revenue management. These characteristics have been identified by Kimes [26] as: relatively fixed capacity; ability to segment markets; perishable inventory; product sold in advance; fluctuating demand and low marginal sales cost and high marginal capacity change costs. Although similar in these respects, there are still some explicit differences when different industries are subject to different combinations of duration control and variable pricing. Kimes and Chase [27] demonstrate such differences with a pricing and duration positioning table; see Table 1.

Comparatively, airlines, hotels and car-rental firms are more able to apply variable pricing for a product which has a more predictable duration. They conclude that successful revenue management applications generally occur in these industries (Quadrant 2), because they can manage both price and capacity more effectively.
Table 1. Pricing and duration positioning table for various industries.

However, there are still a wide variety of complications when firms implement RM techniques because every firm possesses its own industry-specific characteristics such as technology standards, consumer behavior, pricing policies etc. Carrol and Grimes [10] summarize the impact of these factors on three industries: airline, hotels and car rental firms. Nair and Bapna [35] use Weatherford and Bodily's taxonomy [50] to compare the Internet Service Provider (ISP) problem with hotel and airline revenue management problems. Several review papers describing the theory and applications of revenue management in the airline industry have been published in recent years (see Bitran and Caldentey [5], Kevin and Piersma [25], McGill and van Ryzin [33], and Weatherford and Bodily [50]).
1.2 Hotel Revenue Management

The hotel industry began to apply the concept of revenue management in late 1980s when the industry faced excess capacities, competitive markets, liquidity problems and recession; all of which affected operations and resulted in lower revenue (See Hansen and Eringa [21]). Hotels can be classified as business, resorts, extended-stay, or a mix of business and leisure and also by size and location. Some hotels manage only individual properties, while large hotel chains can own hundreds of properties. A hotel, typically, offers rooms for many day-to-day lodgings of various types of customers. Despite some of the similarity with the airline customer types, the segmentation used in hotel RM are different. For example, advance-purchase discounts, a prominent segmentation mechanism of airlines, are not commonly used by hotels. Since hotels also generate significant revenues from other sources such as food, entertainment, and function space, the value of a customer is hard to determine exactly. However, these additional sources of revenue are usually not considered in hotel RM applications.

There are many different room types, such as standard rooms, deluxe rooms, executive rooms, rooms with a view, single or double bed rooms, smoking and non-smoking rooms, etc. They can be grouped together into three or four categories for capacity control purposes. Hotels typically aggregate both the room rate and the customer types, leading to about 3 to 10 rate bands for RM purposes. The room rates are usually adjusted only once or twice a year. Normally, a hotel room booking is made directly with the hotel (walk-in, through Internet, or by call). However, in a large
hotel, approximately 20 to 40 percent of bookings come from Global Distribution Systems (GDS). The Plation of a booking happens not only when the customer cancels the booking before the date for accommodation, but also when the customer decides to check out early. Therefore, the future capacity of the hotel is often uncertain and overbooking is widely practiced in the hotel industry.

Hotel RM mainly focuses on selling rooms in a way that maximizes total room revenue, rather than trying to sell all available rooms. For example, hotels sometimes make the customer “walk” (i.e., send elsewhere) a less valuable customer even when a room is available, to avoid walking a more valuable customer who is arriving later. This strategy may be risky since the arrivals of high-revenue customers in the future are not guaranteed. However, it is a systemized occupancy-price strategy for controlling the room rates and occupancies to maximize the total revenue. Some recent studies perceive revenue management as a managerial tool for maximizing profits, rather than revenue. For example, Donaghy et al. [13] and Griffin [20] point out that the total income calculations should include cost considerations and revenue management should move from a revenue- to a profit-generating tool. However, due to the high capital investments but low variable costs of hotel operations, increasing revenue essentially results in an increase in operating profits.

1.3 Game Theory and Revenue Management

Game theory concerns itself with the analysis of competition and cooperation situations. It has found applications in diverse areas such as anthropology, auctions, biology, business, economics, management-labour arbitration, philosophy, politics,
sports and warfare. During the 1950s and the 1960s, academic researchers began to apply game theory in operations research/management science area. Several reviews focusing on the application of game theory in economics or management science have appeared in the last five decades. An early survey of game theoretic applications in management science was given by Shubik [43]. Feichtinger and Jørgensen [14] published a review that was restricted to differential game applications in management science and operations research. A review of applications of differential games in advertising was given by Jørgensen [22]. Wang and Parlar [49] presented a survey of the static game theory applications in management science problems. In addition, several books (e.g., Chatterjee and Samuelson [11], Gautschi [17], and Sheth et al. [42]) partially reviewed some specific game-related topics in management science. More recently, Leng and Parlar [29] present a review of the existing supply chain game models, under a topic classification of five areas: (i) Inventory control, (ii) production and pricing competition, (iii) service and product quality competition, (iv) sharing issues in supply chain management, and (v) strategic competition in marketing.

To the best of our knowledge, there are currently no detailed survey papers on game theoretic models in RM problems and there are very few published works directly concerning such problems. Most studies assume that the company handling perishable products (such as airline, hotel, restaurant, etc.) exists as a distinct entity. In reality, there are usually more than one company dealing with “substitutable” products in a specific geographical market. In this situation, one company’s decisions on inventory rationing, pricing, or both might be affected by the decisions of other companies. Therefore, more significant and interesting topics arising from revenue
management allow us to address the following questions: How do they set the booking limits or protection levels of multiple classes products? Is there an equilibrium in inventory allocations? Is it more beneficial to be the “leader” in a Stackelberg game? How to find the optimal rationing policies when one firm has an incomplete information of the others? How much can RM increase the overall revenue if the firms cooperate? As a result, a prime methodological tool for dealing with these problems is game theory that focuses on the simultaneous or sequential decision-making of multi-players under complete or incomplete information in a competitive or cooperative context.

1.4 Organization of the Thesis

The rest of this thesis is organized as follows.

Chapter 2 presents a comprehensive discussion of the existing mathematical models which can be applied to hotel room inventory control problems. We then look at several game models which can be applied to hotel revenue management.

Chapter 3 addresses a single-period two-player two-fare-class hotel room rationing game. First, we investigate the best response functions of both players and corresponding properties. A unique Nash equilibrium of booking limit decisions is found in the competitive situation. Next, we assume that one hotel acts as the “leader” and the other as the “follower”; under this scenario we examine the Stackelberg equilibrium. For this case, we identify a situation in which the Stackelberg game is equivalent to the Nash game. This result shows that if one player’s booking limit is reached, i.e., if he always rejects low-fare customers, neither of the two play-
ers prefers to be the “leader” in the game. Finally, we examine the cooperative case where the hotels “cooperate” to maximize a system-wide objective function and find that the profit loss is substantial if there is a lack of cooperation between two players.

In Chapter 4, we study a multi-period hotel room rationing problem. This problem is formulated as a dynamic programming model. First, we find that each player’s optimal expected future revenue is a non-decreasing function of its own room inventory level and a non-increasing function of the other hotel’s room inventory at any time. Second, we provide the sufficient conditions for the unique Nash equilibrium of dynamic accept/reject decisions and identify the situations in which the game admits multiple Nash equilibria (MNE). Finally, by defining expected marginal values of the hotels’ rooms, we simplify the optimal accept/reject decision into sets of critical values. We also provide some numerical examples along with the managerial implications for our solutions of the competitive and cooperative games.

In Chapters 3 and 4, we assume that the games are played under complete information, i.e., each player knows the booking arrival patterns, transfer rates and rejection costs of both players. In Chapter 5, we relax this assumption and examine the static game problem under incomplete information. More specifically, we assume one player’s rejection cost and transfer rate of low-fare class customer as the incomplete information. By employing the different types of information (secret, private and public information), we discuss the game theoretic solution for the incomplete information game, which is known as Bayesian Nash equilibrium. Another goal of the study in this chapter is to evaluate the different information types. Accordingly, we first analyze the conditions in which the value of information is positive (or nega-
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tive). Then, we compare the values of different information to see which type is most valuable for one player, and under what conditions the information value benefits the player most and in what content.

Chapter 6 summarizes the concluding remarks of this thesis and provides some potential research directions for future studies.
Chapter 2
Literature Review

One of the fundamental questions of perishable asset inventory control that must be answered each time a demand arrives is whether to accept it or to reserve the unit of inventory for possible sales later to a potentially higher-paying customer [34]. Before reviewing room inventory control solutions in the literature, some vocabulary should be introduced first. *Booking limit and protection level* are the two important concepts used for the room inventory control problem. Netessine and Shumsky [36] define a booking limit to be the maximum number of rooms that may be sold at the discount price, and protection level is the number of rooms which will be protected for full-price customers. If the optimal values of these two variables can be determined, it will be easy to decide whether to accept the low-revenue booking or to reserve the room to a potentially higher fare class customer later on.

Consider the arrival of a booking request that requires one or more rooms starting on a specified date, at a given price. One of the basic revenue management decisions is whether or not to accept or reject this request in order to maximize the total expected revenue. These types of problems are known as *room inventory control* problems faced by hotel management. The point is that at a certain time it is more profitable to reject a lower-revenue customer in order to be able to accept a higher-revenue customer at a later time. Clearly, if the hotel waits too long for higher-revenue customers to appear, at the end of the selling horizon, there might be
some unsold rooms that could have been sold to lower revenue customers at an earlier time.

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<td>Demand for future arrival dates</td>
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<td>Booking lead-time</td>
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<td>Overbooking up-limit</td>
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Table 2. Significant factors which affect hotel RM.

Based on the discussion in McGill and van Ryzin [33] and Upchurch et al. [48], we identify the elements in Table 2 which can be used to model a generic hotel revenue management problem. In the rest of this chapter, we review some RM models that incorporate these elements.

### 2.1 Static Models for Hotel Room Allocations

Static RM models have been used frequently in recent years. They are typically formulated assuming that demand is segmented in predetermined fare classes. Instead of a distinct control mechanism, they assume a nested booking limit system (See
Belobaba [4]). This approach sets booking limits that are nested from above, e.g., the full-price class has a booking limit up to the total capacity of the hotel. Most of the earlier works only consider two price classes: full price and discount price. However, the nested booking limit method can be applied to any number of price classes. Bodily and Weatherford [50] define a ‘bucket’, $q_i$, $i = 0, 1, 2, ..., I$, as the booking limit for price $\pi_i$ class; these buckets are nested as in Figure 1.

The total capacity is equal to $q_0$, which is only available to those willing to pay the highest price. An amount $q_i$ is available for the $i$th highest price class, with the difference, $q_i - q_{i+1}$, protected for sale to $i$th class customer from those in lower price classes. For the two fare classes problem, the earliest model for inventory control
problem is due to Littlewood [32]. He assumes that there are $x$ units of capacity remaining and there is a booking request from discount price class. Therefore, if such request is accepted, the company collects a revenue of $\pi_2$. If the request is rejected, the company will sell unit $x$ at $\pi_1$ if and only if demand for full price class is $x$ units or higher. Demand for class $j$ is denoted by $D_j$, and its distribution is denoted by $F_j$.

Littlewood's Rule: There is an optimal protection level, denoted by $y_i^*$, such that we accept the discount price class if the remaining capacity exceeds $y_i^*$ and reject it otherwise. Such $y_i^*$ must satisfy

$$\pi_2 < \pi_1 Pr(D_1 \geq y_1^*) \quad \text{and} \quad \pi_2 \geq \pi_1 Pr(D_1 \geq y_1^* + 1).$$

If one can use a continuous distribution $F_1(x)$ to model demand, then $y_1^*$ can be given by a simpler expression $y_1^* = F_1^{-1}(1 - \pi_2/\pi_1)$. See Liang [31] for an analysis of similar continuous-time version of a dynamic model. In general, the lower the ratio $\pi_2/\pi_1$, the more capacity the company should reserve for full price class. This makes sense since the company would be always willing to accept a low price only when the chances of selling at a high price are lower.

Brumelle et al. [9] propose another decision rule to obtain the optimal booking limit for the discount class.

Brumelle's Rule 1: Booking limit of discounted class is

$$q^* = \max_q \left\{ 0 \leq q \leq Q : Pr[Y > Q - q | B \geq q] < \frac{P_B}{R'} \right\} \quad (1)$$

13
where \( Q \) is the total capacity, \( Y \) the number of full-fare customers (random variable), \( B \) the number of discount customers (random variable), \( q \) the booking limit of discount customers and \( P \) the price subscripted for a given fare class.

This stopping rule is developed under the assumption that discount demand occurs earlier than the full-fare demand occurs and a booking class will not be re-opened once it has been closed. In addition, this model assumes that there is no cost involved when rejecting a reservation. Bodily and Weatherford [7] present a similar model:

**Weatherford and Bodily’s Rule 1:** *Reserve an additional space for a discount customer if*

\[
p = \Pr [X < q_0 - q_1] > 1 - \frac{R_1}{R_0}
\]

where \( q_0 \) is total capacity, \( q_1 \) the booking limit for discount class (decision variable), \( X \) the number of full-fare customers (random variable) and \( R_0 \) and \( R_1 \) represent the revenue of full-fare class and discount class, respectively.

Brumelle et al. [9] generalize their model by assuming a cost (loss of goodwill) is involved when turning away further requests once current reservations reach capacity. It is a modification of their previous rule (see (1)) where the value of lost of goodwill, \( P_G \), is added to the full price to give the rule that the optimal booking
They also present a decision rule which incorporates the compensation of customers if overbooking occurs.

We note that all the above rules assume that the discount class customers only accept the discount price and full-fare customers only pay full price. In hotel business, the business travellers (full-fare class) may also want to pay the discount price and as the discounts class is closed the leisure travelers (discount class) may accept the full price. This is defined as diversion by Pfeifer [41] who provides the following rule to determine the booking limit of discounted class customers.

**Pfeifer’s Rule:**  *Reserve an additional discount customer if*

\[ p_1 p_2 < 1 - \frac{P_D}{P_F} \]

*where \( p_1 \) is the probability that the \((q + 1)\)st customer will only accept discounted price and \( p_2 \) the probability that \( Q - q - 1 \) \((Q\text{ being the total capacity})\) full price units will satisfy all subsequent demands from those who would pay the full price and those who would prefer the discount but if unavailable would accept the full price.*  ♦

Considering the probability \( \gamma \) of an upgrade if a discounted class customer is rejected, Brumelle et al. [9] develop a different model which is similar to Pfeifer’s rule.
Brumelle et al.'s Rule 2: Booking limit of discounted class is

\[ q^* = \max_q \left\{ 0 \leq q \leq Q : \Pr \left[ Y + U(q) > Q - q \mid B \geq q \right] < \frac{P_B - \gamma P_Y}{(1 - \gamma) P_Y} \right\} \]

where \( U(q) \) is the number who will upgrade if discounted class is closed at \( q \). ♦

Another model by Belobaba [4] considers the probability that a discounted customer may upgrade vertically to full-fare class.

Belobaba's Rule: Booking limit of discount-class is

\[ BL_2 = C - (S_2^1 + V_2^1) \]

where \( C \) is the total available capacity and \( (S_2^1 + V_2^1) \) the total protection level for discounted class from full-fare class. The quantity \( (S_2^1 + V_2^1) \) is determined by

\[ EMSR_1 (S_2^1 + V_2^1) [1 - P_2(v)] + f_1 P_2(v) = f_2 \]

where \( EMSR_1 \) is the expected marginal seat revenue for full-fare class when the number of seats available to the class is increased by one; \( P_2(v) \) the probability that a refused discount-class customer will accept a booking in full-fare class; and \( f_1 \) and \( f_2 \) the average fare level of full-fare class and discounted class, respectively. ♦

All of these three rules mentioned above solve the diversion problem by introducing probabilities of 'upgrade' of discounted class customer. These probabilities increase as the booking limits increase because the lower the inventory level the more likely a discounted customer will buy up. However, in reality, estimating these prob-
abilities is still not easy.

The decision rules described above just concern two-fare classes problem. In practice, a hotel manager might often face three or more fare classes for a single type of rooms. Based on the solution for two fare classes problem, Weatherford and Bodily [7] extend the decision analysis to a more general case considering any number of fare classes. In order to establish the general decision rules for each of the booking limit, some other definitions and notations are defined first. For each class \(i = 0, 1, 2, \ldots, I\), \(R_i\) is defined as the contribution from a unit sold to a class \(i\) customer \((R_0 > R_1 > R_2 > \ldots > R_I)\); \(X_i\) as a random variable for class \(i\); \(Y_i\) as a random variable representing the demand for units in all price classes \(\leq i\) subsequent to the arrival of the \((q_{i+1} + 1)st\) customer; \(\beta_i\) as the probability that the next customer requesting a reservation is in class \(i\); \(p_i\) as probability that \(Y_i \leq q_0 - (q_{i+1} + 1)\).

**Weatherford and Bodily's Rule 2:** Accept an additional class \(i\) \((i = 1, 2, \ldots, I-1)\) customer if

\[
\beta_i p_{i-1} > \frac{R_{i-1} - R_i}{\sum_{k=0}^{i} \beta_k}
\]

and accept an additional class 1 customer if

\[
\beta_I p_{I-1} > \frac{R_{I-1} - R_I}{R_{I-1}}.
\]

For the first time, this decision rule provides an approach to solve the diversion problem with any number of price classes. Bodily and Weatherford [7] also evaluate the revenue improvement by simulation using the actual airline data for demand. The
results indicate that the revenue gains are between 3.6% and 5.35% when considering seven fare classes. But its weakness also resides in the difficulty of estimating $\beta_i$ and $p_i$. And besides, this model just considers the ‘neighboring diversion effect’ (upgrade from one fare class to its immediately higher class), it does not consider the diversion from one class to all possible higher fare classes.

In the next section, we will discuss the models concerning the rationing policies in a dynamic situation (multiple periods).

### 2.2 Dynamic Models for Hotel Room Allocations

Gerchak et al.'s model [18] is one of the first dynamic models dealing with concurrent demand problem in revenue management applications. The authors were motivated from a real situation observed at a delicatessen store where the manager felt that it might be more profitable if he refuses the request from a low-revenue customer in order to offer a food item (i.e., a bagel) to a high-revenue customer later. In the case of hotel business, this policy can be used to decide the optimal booking limit for lower fare class customers. The authors assume that the time horizon is divided into discrete intervals. In each interval, the arrival rates of the full-fare and discounted class customers are assumed known as $\lambda_1$ and $\lambda_2$, respectively. And the time interval is short enough to make the probability of more than one customer arriving in any interval negligible. The high-revenue customers generate a unit revenue of $p_1$ and the low-revenue customers generate $p_2$ revenue per unit ($p_1 > p_2$). Each customer requests a single unit each time. In the basic model, the authors also assume that there is no salvage value for unsold units and there is no loss of goodwill when rejecting
Gerchak et al.'s Rule: The maximum expected future total revenue $V(n,t)$ is

$$V(n,t) = (1 - \lambda_1 - \lambda_2)V(n,t-1) + \lambda_1 [\rho_1 + V(n-1,t-1)] + \lambda_2 \max \left\{ \begin{array}{ll}
\rho_2 + V(n-1,t-1) & : \text{Accept} \\
V(n,t-1) & : \text{Reject}
\end{array} \right. $$

with $V(n,0) = 0$, for all $n$; $V(0,t) = 0$, for all $t \in \{0, 1, \ldots, T\}$; and $n$ denotes units on hand, $t$ time intervals remaining until the end of time horizon and $T$ the number of intervals in the planning period. If $n \geq t$, we shall never reject a customer no matter which class he/she belongs to. If $n < t$, and if we reject a discounted customer at $(n,t)$, he/she should also be rejected at $(n-1,t)$ and $(n,t+1)$. And, if we accept a discounted customer at $(n,t)$, she/he should also be accepted at $(n+1,t)$ and $(n,t-1)$.

Clearly, the decisions of “reject” and “accept” depend on the values of $(n,t)$ in case $n < t$. The decision should be made based on two state variables (i) available rooms $(n)$ and (ii) number of remaining time intervals $(t)$. According to the policy, at any given $t$, there must exist an $n^*$ (booking limit) which the decision should be “reject” if the number of available rooms is less than $n^*$, and “accept” otherwise. On the other hand, for any given $n$, there also exists a $t^*$ which our decision should be “reject” if the time remaining until the end of the time horizon is before that point and “accept” otherwise. So by linking all such $(n,t^*)$ and $(n^*,t)$ together, a rejection-acceptance curve can be formed.
Next, the authors extend the basic model to a more complicated situation in which loss of goodwill is involved. Denote the loss of goodwill per rejected customer by $g$, the only change required is to alter $V(n, t - 1)$ to $V(n, t - 1) - g$. In addition, if the per unit revenues for the $K$ types of customers are $\rho_1 > \rho_2 > \ldots > \rho_K$, and their arrival rates are $\lambda_1 > \lambda_2 > \ldots > \lambda_K$ respectively ($\sum_{i=1}^{K} \lambda_i < 1$), then the problem can be formulated with the similar way just simply substituting $\lambda_1 + \lambda_2$ by $\sum_{i=1}^{K} \lambda_i$, $\rho_2$ by $\rho_i$ and $\lambda_2$ by $\sum_{i=2}^{K} \lambda_i$ in (3).

Lee and Hersh [28] consider a general case in which there are more than 2 booking classes and each request may be for more than one unit of product. Using the same discrete-time scheme as that in Gerchak et al.’s paper, they denote $G_{im}^n$ as the probability that a booking from class $i$ in decision period $n$ is for $m$ products, $m = 1, 2, \ldots, M_i$, where $M_i$ is the maximum number of products allowed for each booking.

**Lee and Hersh’s Rule:** A booking from fare class $i$ ($i = 1, 2, \ldots, k$) will be accepted only if $mF_i + f_{s-m}^{n-1} \geq f_s^{n-1}$. The recursive function of $f_s^n$, the optimal expected revenue generated for next $n$ period given booking capacity $s$, is:

$$f_s^n = \begin{cases} \left(1 - \sum_{i=1}^{k} P_i^n\right) f_s^{n-1} + \sum_{i=1}^{k} P_i^n \sum_{m=1}^{M_i} G_{im}^n \max \{mF_i + f_{s-m}^{n-1}, f_s^{n-1}\} & \text{for } n > 0, s > 0 \\
0, & \text{otherwise,} \end{cases}$$

where $F_i$ denotes the value of accepting a booking request in fare class $i$ and $P_i^n$ the probability that a request in class $i$ will arrive during a decision period $n$. 

This model implies that for a given request size, booking limit, and decision
period, there exists a critical discount class; for a given request size, discount class
and booking limit, there exist a critical decision period. But for a given request size,
decision period and discount class, the critical booking limits may NOT apply. It is
because the expected marginal value for reducing the available rooms of size \( s \) by \( m \)
simultaneously (as a group) in period \( n \), \( \delta_m(n,s) \), is not necessarily non-increasing
(See Lee and Hersh [28]).

You [51] generalizes Gerchak et al.'s model by relaxing the assumption that
the rejected customers are lost sales. He assumes that at any given decision period, a
discounted class \( \ell \) customer may upgrade to the next higher class, \( \ell + n \), if his initial
booking request and subsequent upgrade requests to classes \( \ell + 1, \ell + 2, \ldots, \ell + n - 1 \)
are rejected. You denotes \( r^\ell_n \) as the probability of such event.

You's Rule: Accept a booking from fare class \( \ell \) (\( \ell = 1, 2, \ldots, L \)) if and only if
\( v_{t-1}(i - 1) + x^\ell \geq u^\ell_t(i, 1) \). The maximum total expected revenue, \( v_t(i) \), with \( i \) units
available stock with \( t \) periods to go is given by

\[
v_t(i) = \begin{cases} 
\lambda^0_t v_{t-1}(i) + \lambda^L_t [x^L + v_{t-1}(i)] \\
+ \sum_{\ell=1}^{L-1} \lambda^\ell_t \max \{ u^\ell_t(i, 1), x^\ell + v_{t-1}(i - 1) \}, & i \geq 1, t \geq 1 \\
0, & \text{otherwise}
\end{cases}
\]

where \( \lambda^\ell_t \) denotes the probability of a booking request from fare class \( \ell \) (\( \ell = 1, 2, \ldots, L \)) in
period \( t \); \( \lambda^0_t \) the probability of no arrival; \( x^\ell \) the fare charged for class \( \ell \) (\( \ell = 1, 2, \ldots, L \)).
The maximum total expected revenue, \( u^\ell_t(i,n) \), with \( i \) units available stock with \( t \)
periods to go under the condition that the fare class \( \ell \) customer's initial request and
subsequent upgrade requests \(\ell + 1, \ell + 2, \ldots, \ell + n - 1\) are rejected, is given by

\[
\begin{cases}
    r_n^\ell \max \left\{ u_i^\ell (i, n + 1), x^\ell + n + v_{t-1}(i-1) \right\} + (1 - r_n^\ell) v_{t-1}(i), & \text{if } n < L - \ell, \\
    r_{L-\ell}^\ell [x_L + v_{t-1}(i-1)] + (1 - r_{L-\ell}^\ell) v_{t-1}(i), & \text{if } n = L - \ell.
\end{cases}
\]

The dynamic models discussed above can be characterized as follows: the rationing policy can be controlled using either a set of critical inventory levels or a set of critical decision periods. For a hotel, a booking request from a discounted class in a decision period is accepted if the remaining decision periods is less than or equal to the critical decision period for the current available rooms and rejected otherwise. On the other hand, in a decision period, a booking request from a discounted class is accepted if the number of current available rooms is greater than or equal to the critical booking limit for that period and rejected otherwise. Similar research on this kind of policy has also been done by Banerjee and Viswanathan [1], Bitran and Gilbert [6], You [52], and Zhao and Zheng [53].

One of the shortcomings of these models is the assumption that the arrival rates of customers do not vary in time which can be unrealistic in some real applications. Slyke and Young [44] model the random number of arrivals in \([0, t]\), as a time-dependent Poisson process \(N(t)\) with \(\lambda_k(t)\) as the Poisson arrival rate of the \(k\)th type of requests as a function of time.

**Slyke and Young’s Rule:** Accept a request from fare class \(k\) \((k = 1, 2, \ldots, K)\) if and only if \(b_k + f(y-1,t) - f(y,t) > 0\). Here, \(b_k\) is the positive real valued benefit
of a type $k$ request and $f(y, t)$ the maximum total expected sum of the benefits of the accepted requests given that $y$ units of inventory ($0 \leq y \leq W$) and $t$ time ($0 \leq t \leq T$) remain. This function is given as:

$$f(y, t) = \int_0^t \lambda(s) e^{-\int_r^t \lambda(r) \, dr} \sum_{k=1}^K p_k(s) \max \{b_k + f(y-1, s) - f(y, s), 0\} \, ds$$

with $f(y, 0) = 0$, for all $y$; $y(0, t) = 0$, for all $t \in [0, T]$, and $\lambda(s) = \sum_{k=1}^K \lambda_k(s)$, $p_k(s) = \lambda_k(s) / \lambda(s)$ is the probability that, given one request arrived, it belongs to class $k$. ⬠

Slyke and Young [44] prove that if $f(y, t)$ is absolutely continuous in $t$ and $\lambda_k(t) > 0$ for all $k$ and $t$, then $f(y, t)$ is strictly monotone increasing in $t$. This implies that there exist a critical time (threshold point) $t^*$ for a given inventory level and a fare class $k$, where the optimal policy is to reject that request before $t^*$ and accept otherwise. This solution is very similar to the policies in discrete time we described above. In the two-price model, Feng and Gallego [15] give the exact solution for such threshold point for any given inventory level. Feng and Xiao [16] extend the results obtained by Feng and Gallego to a more general situation which considers multiple fare classes.

**Feng and Xiao's Rule:** For a request in fare class $p_k$ ($1 \leq k \leq K$), with inventory level $n$ ($1 \leq n \leq M$), we accept the request until the threshold is reached. Such threshold, $x_n^k$, is determined recursively by

$$x_n^k = \inf \left\{ 0 \leq t \leq T : \int_t^T L_k(s, n) e^{-\lambda_k(s-t)} \, ds > 0 \right\}$$
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where

\[
L_k(t, n) = \frac{\partial V_{k+1}(t, n)}{\partial t} + \lambda_k \left[ \tilde{V}_k(t, n) + p_k \right],
\]

\[
\tilde{V}_k(t, n) = \begin{cases} 
\int_t^T L_k(s, n) e^{-\lambda_k(s-t)} ds, & \text{if } t > x^k_n \\
0, & \text{Otherwise.}
\end{cases}
\]

Here, \( V_k(t, n) \) represents the maximum expected revenue over the interval \( [t, T] \) given the inventory level \( n(t) = n \), which is achieved by keeping the current price \( p_k \) until \( x^k_n \), and \( \tilde{V}_k(t, n) = V_k(t, n) - V_{k+1}(t, n) \).

Brumelle and Walczak [8] present a model with a continuous-time, multi-period arrival process with multiple demands following a Markov process, and where decisions are made at the end of each period. They consider the situation that a customer can request more than one unit of product and that request may be splittable, i.e., the requests can be partially satisfied.

For many hotels, booking requests are only recorded when the customer accepts the price hotel offers. No purchase decision can be observed from the available stored data. Thus, it is sometimes difficult to distinguish between periods with no arrival and periods in which there was an arrival and his/her booking request was rejected. Talluri and van Ryzin [47] overcome this incomplete data problem by applying the expectation-maximization (EM) method to solve the discrete-time revenue management problem. They assume in each period that there is at most one arrival with probability of \( \lambda \). There are \( n \) fare classes and \( N = \{1, 2, ..., n\} \) denotes the entire set of fare classes where each fare class \( j \in N \) has an associated revenue \( r_j \). Hence, in each equal-length period \( t \), a subset \( S \subseteq N \) of fare classes must be decided.
to open. When the fares $S$ are offered, the probability that a customer chooses fare class product $j \in S$ is $P_j(S)$ where $P_0(S)$ denotes the no-purchase probability.

**Talluri and Van Ryzin's Rule:** At any given time $t$ ($0 \leq t \leq T$), one set of fare classes $S$ should be opened and such a set should satisfy

\[
S = \arg \max_{S \subseteq N} \{ \lambda (R(S) - Q(S) \Delta V_{t-1}(x)) \}
\]

where $Q(S) = \sum_{j \in S} P_j(S) = 1 - P_0(S)$ is the total probability of purchase, $R(S) = \sum_{j \in S} P_j(S) r_j$ is the total expected revenue from offering set $S$ and $\Delta V_{t-1}(x) = V_{t-1}(x) - V_{t-1}(x - 1)$ the marginal cost of capacity. The maximum expected revenue, $V_t(x)$, obtained from period $t, t-1, \ldots, 0$, given that there are $x$ inventory units remaining at time $t$ is

\[
V_t(x) = \max_{S \subseteq N} \left\{ \sum_{j \in S} \lambda P_j(S) (r_j - \Delta V_{t-1}(x)) \right\} + V_{t-1}(x)
\]

\[
V_t(x) = \max_{S \subseteq N} \left\{ \sum_{j \in S} \lambda P_j(S) (r_j - \Delta V_{t-1}(x)) \right\} + V_{t-1}(x) \text{ with } V_t(0) = 0,
\]

$t = 1, 2, \ldots, T$ and $V_0(x) = 0, x = 1, 2, \ldots, C$. ♦

The most significant characteristic of this model is that the optimal sets of fare classes are only those efficient sets. In many cases, this observation reduces the number of sets we have to consider. Moreover, it shows that these efficient sets can be sequenced in a natural way and that the more capacity we have, the higher the set we should select. But the limitation of this model is in its difficulty of estimating the probability of $P_j(S)$. There are many potential strategic behaviors which can affect
customer's choice, e.g., a customer's choice may depend on the strategies of other customers; or his or her past choices or past events in the system, etc.

2.3 Game Theoretic Models for Hotel Room Allocations

As we discussed in Section 1.3, game theory should be applied for the simultaneous or sequential decision-making if one hotel's revenue is affected by the rationing policies of another hotel's rooms. To the best of our knowledge, there is no related literature addressing the room inventory control problem using game theoretic tools even though game theory has found frequent use in problems involving competition in supply chains. For example, in one of the earliest papers in this field, Parlar [40] has modelled the substitutable product inventory problem with two newsvendors whose profits are determined as a function of both players' order quantities \( u \) and \( v \). The newsvendors attempt to maximize their expected profits \( J_1(u, v) \) and \( J_2(u, v) \), respectively, where the first retailer's objective is given as

\[
J_1(u, v) = (s_1 + p_1) \left[ \int_0^u x f(x)dx + u \int_0^\infty f(x)dx \right] - p_1 E(X) + q_1 \int_0^u (u - x)f(x)dx \\
+ (s_1 - q_1) \int_0^u \left[ \int_B b(y - v)g(y)dy + \int_B^\infty (u - x)g(y)dy \right] f(x)dx - c_1 u,
\]

with \( f(x) \) and \( g(y) \) as the demand densities faced by each retailer, \( a \) and \( b \) (\( 0 \leq a, b \leq 1 \)) are the substitution rates of the retailer's products when they are sold out; \( s_1, c_1, q_1 \) and \( p_1 \) are the unit selling price, purchase cost, salvage value and shortage penalty cost for first player's product, and \( B = [(u - x)/b] + v \) and \( A = [(v - y)/a] + u \). For this model Parlar proved the existence and uniqueness of the Nash equilibrium and showed that cooperation between two players can increase their profits.
The importance of this work is that it establishes the existence of a unique Nash equilibrium. The essential differences between the RM games and newsvendor games reside in the capacity and variation of price: the former accounts for a fixed capacity and offers different prices for the same product (i.e., airline seat, hotel room), which are seldom considered by the latter.

More recently, Netessine and Shumsky [37] presented a seat inventory control problem in which two airlines compete for passengers on the same flight leg. Their model cannot guarantee an equilibrium, because they assume the airline's demand depends on the booking limit, which makes the problem more complicated than any newsvendor game problem presented in the literature.

![Diagram](attachment:overflow_diagram.png)

**Figure 2. Overflow process of two-player two-fare class game.**

In general, such a problem can be summarized as two-player two-fare class non-zero-sum game problem. Each player $i$ has capacity $C_i$ and there are two fare
classes available for the customers: low fare and high fare. If either type of customer is denied by one player, the customer will attempt to purchase a substitutable product from another player. Figure 2 shows the overflow processes.

The total revenue for player $i$ is

$$\pi_i = E \left[ p_{Li} \min \left( D^T_{Li}, B_i \right) + p_{Hi} \min \left( D^T_{Hi}, R_i \right) \right],$$

where $D^T_{Li} = D_{Li} + (D_{Li} - B_j)^+$, total demand for low-fare demands for player $i$, $i, j = 1, 2$ and $i \neq j$; $R_i = C_i - \min (D^T_{Li}, B_i)$, the number of seats available for high-fare class customers for player $i$, $i = 1, 2$ and $D^T_{Hi} = D_{Hi} + (D_{Hi} - R_j)^+$ which is the total demand from high-fare class. Netessine and Shumsky [37] provide sufficient conditions for a pure-strategy Nash equilibrium and also present the results for a cooperative situation. Even though they evaluated both the direction and magnitude of revenue losses due to competition, they do not clearly take into account the cost savings in a cooperative situation and their effects on the decisions. In addition, their model assumes that the customers who are rejected by one player will transfer to the other one (transfer rate is equal to 1). In the next chapter, we will also study a single-period static game model. However, our model will relax the assumption on the transfer rate (it can be any number between 0 and 1). Moreover, we will analyze the cost savings in the cooperative situation and discuss the effects of some parameters (e.g., rejection cost) on the optimal rationing policies.
Chapter 3
Static Game Model for Hotel Room Allocations

In this Section, we study a two-player, two-fare class room inventory control problem arising in hotel business. The booking requests for a given date from each fare class in each hotel are assumed to be random and independent. Each hotel is assumed to have complete information of the prices, booking request distributions, costs, and all other parameter values related to the rooms of both hotels. In order to maximize the total expected revenue (objective), each hotel has to decide the maximum level of rooms (booking limit [36]) to be sold at a lower price. Such a decision complements the minimum level (protection level) which should be reserved for high-fare customers who prefer better quality rooms. If either type of customer is rejected at one hotel, a fraction of these customers will attempt to book a room from the other hotel. We define these customers as transfer customers. However, the rest of the rejected customers are totally lost to both hotels. Since the booking limits decided by both hotels affect their respective objectives, the hotel RM problem we are considering should be modeled using a game-theoretic framework.

3.1 The Model

Before we present the objective functions for the two hotels, it would be helpful to summarize the underlying assumptions. Two hotels dealing with substitutable rooms are assumed as the two players which are denoted by \( P_i, i = 1, 2 \). Two fare
classes \((L: \text{Low} \text{ and } H: \text{High})\) have been set a priori. Booking requests for different fare classes are assumed to be independent random variables with continuous probability distribution functions. If one hotel has excess capacity, the excess rooms will be filled partly or fully by the other hotel according to the transfer rate. Furthermore, it is assumed that there are neither no-shows nor cancellations by accepted customers (obviating the need for overbooking). Finally, in order to simplify our model further we also assume that there are no buy-ups to the high-fare class by rejected low-fare customers.

We use the following notations \((i = 1, 2, K = H, L)\):

- \(C_i\): capacity of \(P_i\),
- \(b_{iL}\): booking limit chosen by \(P_i\) (our decision variables),
- \(b_{iH}\): protection level chosen by \(P_i\),
- \(X_{iK}\): random booking request from fare class \(K\) customers for a given date accommodation in \(P_i\), with probability density function (p.d.f.) \(f_{iK}(x_{iK})\), cumulative distribution function (c.d.f.) \(F_{iK}(x_{iK}) = \int_0^{x_{iK}} f_{iK}(t_{iK}) \, dt_{iK}\) and complementary c.d.f. \(\bar{F}_{iK}(x_{iK}) = 1 - F_{iK}(x_{iK})\),
- \(r_{iK}\): fare paid per night by \(P_i\)'s \(K\)-fare class customer,
- \(q_{iK}\): rejection penalty cost per \(K\)-fare class customer incurred on \(P_i\),
- \(u_{iK}\): the fraction of \(P_i\)'s rejected \(K\)-fare class booking requests which switch to the other hotel,
- \(\Pi_i(b_{1L}, b_{2L})\): random revenue for \(P_i\), with \(J_i = E(\Pi_i)\).

We note that since the hotel's capacity is fixed, protection level and booking
limit complement each other, that is, the sum of these levels equals the hotel’s capacity. Thus, we choose one of these the decision variable and express the other one in terms of the chosen decision variable and the fixed capacity. In our study, we use booking limit as our decision variable, so that protection level can be expressed as $C_i - b_{iL}$. However, in some cases, it is more convenient to use both of them to make the expressions more compact and simpler. Therefore, we define protection level for $P_i$ to be $b_{iH}$ ($i = 1, 2$).

Because of the existence of transfer customers, each player’s revenue function will depend on not only its own booking limit but also on the other player’s booking limit. Thus, game theory should be used for analyzing the optimal booking decisions for both players.

3.1.1 Objective Functions

We denote $J_{iK}$ as $P_i$’s expected revenue from $K$-fare class customers, where $i = 1, 2$, $K = H, L$. We begin by analyzing $P_1$’s expected revenue generated by low-fare customers. For any given $b_{1L}$ and $b_{2L}$, there are four mutually exclusive cases in which transfer happens between $P_1$ and $P_2$. Therefore, $P_1$’s revenue function in each case can be expressed by the following:

1. $x_{1L} \leq b_{1L}$, $x_{2L} \leq b_{2L}$: $\pi_{1L}^1 = r_{1L}x_{1L}$

2. $x_{1L} \leq b_{1L}$, $x_{2L} \geq b_{2L}$:

$$\pi_{1L}^2 = r_{1L}x_{1L} + r_{1L} \min \left[ u_{2L} (x_{2L} - b_{2L}), b_{1L} - x_{1L} \right] - q_{1L} \max \left[ 0, u_{2L} (x_{2L} - b_{2L}) - (b_{1L} - x_{1L}) \right]$$
In this case, Pl has excess rooms and P2 is in shortage. Pl will then accept any transfer customer from P2 until its booking limit is reached. Hence, the revenue and penalty cost of transfer customers from P2 are expressed with the second and third terms respectively.

(3) \( x_{1L} \geq b_{1L}, \ x_{2L} \leq b_{2L} : \pi_{1L}^3 = r_{1L}b_{1L} - q_{1L}(x_{1L} - b_{1L}) \)

Here, there are no transfer customers from P2 to P1, and P1’s low-fare class has been closed. Hence, the excess booking requests, \( x_{1L} - b_{1L} \), will be penalized with \( q_{1L} \) per room.

(4) \( x_{1L} \geq b_{1L}, \ x_{2L} \geq b_{2L} : \pi_{1L}^4 = r_{1L}b_{1L} - q_{1L}(x_{1L} - b_{1L}) - u_{2L}q_{1L}(x_{2L} - b_{2L}) \) Since the low-fare classes in both hotels have been closed, all of P1’s own low-fare customers and transfer customers from P2 to P1 must be rejected which cost \( q_{1L}(x_{1L} - b_{1L}) \) and \( u_{2L}q_{1L}(x_{2L} - b_{2L}) \), respectively.

The expected revenue from P1’s low-fare customers can be obtained by integrating the four revenue expressions above over their respective regions. Using a similar procedure as discussed above, we can also obtain the expected revenue from P1’s high-fare customers. Thus, the total expected revenue of P1 is given as \( J_1 = J_{1L} + J_{1H} \). Analogously, we can obtain P2’s objective \( J_2 \). After some simplifications, the total expected revenue of \( Pi, i = 1, 2 \), is found as follows:

\[
J_i = \sum_{K=L,H} \left\{ \int_0^{b_{iK}} r_{iK}x_{iK}F_{jK}(b_{jK})f_{iK} \ dx_{iK} + \int_{b_{jK}}^{B_{jK}} \int_0^{b_{iK}} r_{iK}(x_{iK} + b_{iK} - M_{iK})f_{iK}f_{jK} \ dx_{iK} \ dx_{jK} \right\}
\]
3.1.2 Best Response (BR) Functions

With each player's objective function given by (4), we now examine the optimal decision (i.e., best response) of each player in response to an arbitrary decision by the other one. For instance, suppose $P_2$ announces her low-fare booking limit $b_{2L}$. Given this, $P_1$ can determine his best response $b_{1L}(b_{2L})$ to maximize his objective function. These results will be helpful when we consider different solution concepts using Nash, leader-follower Stackelberg, and cooperative strategies.

Let us first examine the properties of $J_i, i = 1, 2$ for further information.

**Lemma 1** $P_i$'s objective function is strictly concave in $b_{iL}$ for $i = 1, 2$.

**Proof.** By differentiating $J_1(b_{1L}, b_{2L})$ with respect to $b_{1L}$, after some simplification we find...
\[
\frac{\partial J_1}{\partial b_{1L}} = V_1(b_{1L}, b_{2L}) = (r_{1L} + q_{1L}) \left[ \int_0^{b_{1L}} \int_{N_{2L}} \infty f_{1L} f_{2L} \, dx_{2L} \, dx_{1L} + \bar{F}_{1L}(b_{1L}) \right] \\
- (r_{1H} + q_{1H}) \left[ \int_0^{b_{1H}} \int_{N_{2H}} \infty f_{1H} f_{2H} \, dx_{2H} \, dx_{1H} + \bar{F}_{1H}(b_{1H}) \right],
\]

where \( N_{2K} = b_{2K} + (b_{1K} - x_{1K}) / u_{2K} \). Next, we obtain the second derivative of \( J_1 \) with respect to \( b_{1L} \) as

\[
\frac{\partial^2 J_1}{\partial b_{1L}^2} = - \sum_{K=L,H} (r_{1K} + q_{1K}) \left[ \int_0^{b_{1K}} \frac{1}{u_{2K}} f_{1K} f_{2K} (N_{2K}) \, dx_{1K} + f_{1K} (b_{1K}) F_{2K}(b_{2K}) \right].
\]

It is not difficult to see that \( \partial^2 J_1 / \partial b_{1L}^2 < 0 \) for any \( b_{1L} \in [0, C_1] \). Similarly, we can show that \( \partial^2 J_2 / \partial b_{2L}^2 < 0 \) for any \( b_{2L} \in [0, C_2] \). Thus, \( J_i \) is strictly concave in \( P_i \)'s own decision variable \( b_{iL}, i = 1, 2 \).

Let us define \( S_{iK} \) \((i = 1, 2 \text{ and } K = L, H)\) to be the probability of "spill" (an event that unsatisfied booking request occurs; see, McGill and van Ryzin [33]). It is not difficult to see that \( P_i \)'s \( K \)-fare class customers will spill in two cases: 1) \( X_{iK} > b_{iK} \); and 2) \( X_{iK} + u_{jK} (X_{jK} - b_{jK}) > b_{iK} \) with \( X_{iK} \leq b_{iK} \). Thus, the spill rate of \( P_i \)'s \( K \)-fare customers can be expressed as:

\[
S_{iK} = \Pr\{X_{iK} > b_{iK}\} + \Pr\{X_{iK} + u_{jK} (X_{jK} - b_{jK}) > b_{iK} \text{ and } X_{iK} \leq b_{iK}\}.
\]

Integrating over the two respective regions, we obtain

\[
S_{iK} = \int_0^{b_{1L}} \int_{N_{2L}} \infty f_{iL} f_{jL} \, dx_{jL} \, dx_{iL} + \bar{F}_{iL}(b_{iL}),
\]
where \( N_{jK} = b_{jK} + (b_{iK} - x_{iK})/u_{jK}, \ i, j = 1, 2, K = L, H \) and \( i \neq j \). Thus, the first order partial derivative, i.e. \( V_i \) can be expressed in terms of \( S_{iK} \) as

\[
V_i = (r_{iL} + q_{iL}) S_{iL} - (r_{iH} + q_{iH}) S_{iH}, \ i = 1, 2.
\]

**Lemma 2** \( V_i(b_{1L}, b_{2L}) = 0, \ i = 1, 2, \) is a strictly decreasing curve in the \( (b_{1L}, b_{2L}) \) plane.

**Proof.** Note that it is impossible to express \( b_{2L} \) as an explicit function of \( b_{1L} \). However, we can use implicit differentiation to obtain the derivative of \( V_1 = 0 \) with respect to \( b_{1L} \), which we denote by \( b'_1 \). We immediately observe, using chain rule, that

\[
b'_1 = - \frac{\partial V_1}{\partial b_{1L}} / \left( \frac{\partial V_1}{\partial b_{2L}} \right) < 0
\]

since

\[
\frac{\partial V_1}{\partial b_{2L}} = - \sum_{K=L,H} u_{2K} (r_{1K} + q_{1K}) \int_0^{b_{1K}} f_{1K} f_{2K} (N_{2K}) \ dx_{1K} < 0, \ \text{and} \ \frac{\partial V_1}{\partial b_{1L}} < 0
\]

from (6). Thus, \( V_1 = 0 \) is a strictly decreasing curve in the \( (b_{1L}, b_{2L}) \) plane. Similarly, defining \( b'_2 \) as the derivative of \( V_2 = 0 \) with respect to \( b_{1L} \), implicit differentiation gives

\[
b'_2 = - \frac{\partial V_2}{\partial b_{1L}} / \left( \frac{\partial V_2}{\partial b_{2L}} \right) < 0
\]

since, from symmetry,

\[
\frac{\partial V_2}{\partial b_{1L}} = - \sum_{K=L,H} u_{1K} (r_{2K} + q_{2K}) \int_0^{b_{2K}} f_{1K} (N_{1K}) f_{2K} \ dx_{2K} < 0, \ \text{and} \ \frac{\partial V_2}{\partial b_{2L}} < 0
\]
Thus, \( V_2 = 0 \) is also a strictly decreasing curve in the \((b_1L, b_2L)\) plane. This proves the lemma. ■

We now want to determine whether \( P_i \)'s best response to \( P_j \)'s decision, i.e.,
\( b_{iL}^R(b_{jL}) \) (where \( b_{jL} \in [0, C_j] \), \( i, j = 1, 2 \) and \( i \neq j \)), can always be obtained by solving \( V_i = 0 \). By Lemma 1, we note that \( P_i \)'s objective function is strictly concave in his/her own decision variable for any given \( b_{jL} \). However, mathematically, for a given \( b_{jL} \), \( J_i \) may be either strictly increasing concave or decreasing concave in \( b_{iL} \). Therefore, the optimal solution of \( b_{iL} \) which maximizes \( J_i \) can be found on the boundary if such cases occur. The optimal solution resides in \((0, C_i)\) only when \( J_i \) is not monotone in \( b_{iL} \). Then the best response of \( P_i \) can be obtained by solving \( V_i = 0 \).

The following theorem provides the exact form of the best response (BR) functions.

**Theorem 1** \( P_i \)'s best response \( b_{iL}^R(b_{jL}) \) \((i, j = 1, 2 \) and \( i \neq j \)) is given by

\[
b_{iL}^R(b_{jL}) = \begin{cases} 
0, & \text{if } \xi_{jL} \leq b_{jL} \leq C_j; \\
\ast_{iL}, & \text{if } 0 \leq b_{jL} \leq \xi_{jL}.
\end{cases}
\]  

where

\[
\xi_{jL} = \begin{cases} 
b_{jL}-axis \text{ intercept of } V_i = 0, & \text{if } V_i = 0 \text{ intersects with } b_{jL}-axis; \\
C_{jL}, & \text{if } V_i = 0 \text{ intersects with } b_{jL} = C_{jL}; \\
0, & \text{if } V_i < 0 \text{ for any } b_{jL} \in [0, C_j],
\end{cases}
\]

and \( \ast_{iL} \) can be obtained by solving \( V_i = 0 \).
Proof. From (7), we find that $S_{iH} (C_i, b_{jL}) = 1$. Therefore, it is not difficult to see $V_i (C_i, b_{jL}) = (r_{iL} + q_{iL}) S_{iL} (b_{iL}, C_j) - r_{iH} - q_{iH} < 0$, which means for any $b_{jL}$, $J_i$ is not increasing in $b_{iL}$ at $C_j$. However, $J_i$ may be a strictly decreasing concave function ($V_i < 0$) for any $b_{iL}$. In this case, the best response $b_{iL}^R (b_{jL})$ should always be zero [see Scenario (a) in Figure 3]. On the other hand, if $\xi_{jL} > 0$ [Scenarios (b) and (c) in Figure 3], due to the strictly decreasing property of $V_i = 0$ in the $(b_{1L}, b_{2L})$ plane, the optimal solution for any $b_{jL} \in [0, \xi_{jL}]$ can be obtained by solving $V_i = 0$. In other words, for any $b_{jL} \in [0, \xi_{jL}]$, we can use the curve of $V_i = 0$ as $Pi$’s BR curve. As for $b_{jL} \in [\xi_{jL}, C_j]$, $V_i$ is always less than zero. Therefore, the best response to any $b_{jL}$ belonging to this region should be zero too. ■

Referring to (8), we recall that $V_i (i = 1, 2)$ can be expressed in terms of the spill rates $S_{iL}$ and $S_{iH}$ of the two fare class customers of $Pi$. Since $Pi$’s best response to an announcement of $Pj$ is determined in terms of $V_i$, this implies that the spill rates play an important role in determining the best response of a player.

Remark 1 We note that the BR curve is non-increasing in the $(b_{1L}, b_{2L})$ plane. It is optimal for one player to decrease the booking limit if the other one increases the booking limit, and vice versa. We see in Figure 3 that $\xi_{iL}^i$, the upper bound of best response of $Pi$, is always less than $C_i$. (In this figure, $\delta_{iL}^i$ denotes the $b_{iL}$-axis intercept of $Pi$’s BR curve.) Therefore, in practice, the hotel manager should always set a booking limit less than the capacity. This is reasonable since a high-fare customer always generates more revenue if accepted, and incurs more cost if rejected.
Therefore, the hotel should always reserve some rooms for high-fare customers if there is any possibility of booking requests from them.

(a) \( V_i < 0 \) for \( \forall b_{jL} \in [0, C_j] \)  
(b) \( V_i = 0 \) intersects with \( b_{jL} \)-axis  
(c) \( V_i = 0 \) intersects with \( b_{jL} = C_j \)

Figure 3. Best response curves in three scenarios where \( \delta_{iL}^i \) is the \( b_{iL} \)-axis intercept of \( P_i \)'s BR curve, \( i = 1, 2 \). Also, \( \xi_{iL}^1 \) is the smallest value of \( b_{jL} \) for which \( b_{iL} \) assumes the smallest value.

**Example 1** Our goal in this example is to demonstrate the structure of the BR curve for one of the hotels, say, \( P_1 \). We assume \( C_1 = 40 \) and \( C_2 = 45 \), as the capacities of hotels \( P_1 \) and \( P_2 \), respectively. The room rates, penalty costs, and transfer rates of \( K \)-fare class customers in \( P_i \) (\( K = L, H \) and \( i = 1, 2 \)) are given in Table 3.

<table>
<thead>
<tr>
<th></th>
<th>Low-fare ((K = L))</th>
<th>High-fare ((K = H))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( r_{iL} )</td>
<td>( q_{iL} )</td>
</tr>
<tr>
<td>P1</td>
<td>$99</td>
<td>$30</td>
</tr>
<tr>
<td>P2</td>
<td>$105</td>
<td>$35</td>
</tr>
</tbody>
</table>

Table 3. Prices, rejection costs, and transfer rates of \( P_1 \) and \( P_2 \)
Table 4. Booking request expectations in three scenarios

<table>
<thead>
<tr>
<th>Scenario</th>
<th>P1</th>
<th>P2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda_{1L}$</td>
<td>$\lambda_{1H}$</td>
</tr>
<tr>
<td>Scenario 1</td>
<td>35</td>
<td>50</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>35</td>
<td>20</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>35</td>
<td>5</td>
</tr>
</tbody>
</table>

The random booking requests of $K$-fare class customers for Pi’s rooms are represented by the exponential r.v. $X_{iK}$ with density $f_{ik} = \frac{1}{\lambda_{ik}} \exp(-\frac{x_{ik}}{\lambda_{ik}})$, $i = 1, 2$ and $K = L, H$ (and mean $\lambda_{ik}$). In order to show the different BR functions as given in Theorem 1, we generate three scenarios where we vary only $\lambda_{1H}$ and fix all other parameters. Table 4 provides the booking request expectation ($\lambda_{ik}$) of each fare class of the two players.

We obtain the BR function of $P1$ in each scenario according to Theorem 1. As depicted in Figure 4(a) of Scenario 1, if $\lambda_{1H} = 50$, then $P1$’s best response is always $b_{1L} = 0$ which implies that $b_{1H} = C_1 = 40$; that is, every room in hotel 1 is protected for high-fare customers. From Figure 4(b) of Scenario 2, we see that when $\lambda_{1H} = 20$, a moderate level, then $P1$’s best response of $b_{1L}$ will be between 0 and 10 as long as $P2$ decides to choose a booking limit of $b_{2L} \in [0, 40]$. If $b_{2L} > 40$, then $b_{1L} = 0$. Finally, if $\lambda_{1H} = 5$, a low value, Figure 4(c) of Scenario 3 show that regardless of which value $b_{2L}$ is chosen by $P2$, it is always optimal for $P1$ to reserve some rooms (between 6 and 23) for the low-fare customers.

From the above example, we note that if all other parameters are fixed in our basic model, changing $\lambda_{1H}$ affects the structure of $P1$’s BR curve. We will present a
sensitivity analysis to show the major factors affecting the structure of BR functions in Section 3.5. In this chapter, we use Maple 10 to carry out the results for all of the numerical computations.

### 3.2 Nash Equilibrium

In a non-cooperative environment, two players make decisions simultaneously. In this situation, players are assumed to be “rational”, i.e., that one would not lower his/her objective functions for the sole purpose of inflicting damage on the opponent. Thus, in such situations the solution concept that is used is known as the Nash strategy. Mathematically, it is a pair \((b^N_1, b^N_2)\) such that

\[
J_1(b^N_1, b^N_2) \geq J_1(b_1, b^N_2), \quad \text{for all } b_1
\]

\[
J_2(b^N_1, b^N_2) \geq J_2(b^N_1, b_2), \quad \text{for all } b_2.
\]

This strategy leads to what is known as the Nash equilibrium, as it ensures that \(P_i\) \((i = 1, 2)\) receives at least \(J_i(b^N_1, b^N_2)\) if he uses \((b^N_1, b^N_2)\) and he will not receive more...
than this amount if he deviates from it unilaterally. The best response curves of both players were found by Theorem 1. Therefore, the Nash equilibrium exists if and only if these two response curves intersect in the \((b_1L, b_2L)\) plane where \(b_iL \in [0, Ci]\). According to Nikaido and Isora [39], we know that if each player’s objective function is continuous in all decision variables and concave in its own decision variable, the game is convex and admits at least one Nash equilibrium. These conditions obviously hold for our problem (see Lemma 1). Hence, in order to see whether there is only one Nash equilibrium, let us examine the properties of \(V_i = 0\) further.

**Lemma 3** The derivative of \(V_1 = 0\) with respect to \(b_1L\) is always less than the derivative of \(V_2 = 0\) with respect to \(b_1L\).

**Proof.** Referring to Lemma 2, the implicit derivative of \(V_1 = 0\) with respect to \(b_1L\) is

\[
b'_1 = -\frac{\sum_{K=L,H} (r_{1K} + q_{1K}) \left[ \int_0^{b_{1K}} \frac{1}{u_{1K}} f_{1K} f_{2K} (N_{2K}) \, dx_{1K} + f_{1K} (b_{1K}) F_{2K} (b_{2K}) \right]}{\sum_{K=L,H} u_{2K} (r_{1K} + q_{1K}) \int_0^{b_{1K}} f_{1K} f_{2K} (N_{2K}) \, dx_{1K}},
\]

and the derivative of \(V_2 = 0\) with respect to \(b_1L\) is,

\[
b'_2 = -\frac{\sum_{K=L,H} u_{1K} (r_{2K} + q_{2K}) \int_0^{b_{2K}} f_{1K} (N_{1K}) f_{2K} \, dx_{2K}}{\sum_{K=L,H} (r_{2K} + q_{2K}) \left[ \int_0^{b_{2K}} \frac{1}{u_{2K}} f_{1K} (N_{1K}) f_{2K} \, dx_{2K} + f_{2K} (b_{2K}) F_{1K} (b_{1K}) \right]}.
\]

where \(i, j = 1, 2, i \neq j\) and \(K = L, H\). It is easy to see that \(b'_1 < -1\) and \(b'_2 > -1\).
Therefore, $b_1'$ is always less than $b_2'$. ■

Lemma 3 has a useful interpretation: In order to maximize the expected revenue, $P1$ should decrease $b_{1L}$ by more than one unit if $P2$ increases her low-fare booking limit by one unit, and vice versa. Lemma 3 is crucial because it is an important sufficient condition for existence of unique Nash equilibrium.

**Theorem 2** The game admits a unique Nash equilibrium which can be expressed as

$$
(b_{1L}^*, b_{2L}^*) = \begin{cases} 
(0, \delta_{2L}^2), & \text{if } \xi_{2L}^1 \leq \delta_{2L}^2 \\
(\delta_{1L}^1, 0), & \text{if } \xi_{2L}^1 > \delta_{2L}^2 \text{ and } \delta_{1L}^1 \geq \xi_{1L}^2 \\
(b_{1L}^*, b_{2L}^*), & \text{if } \xi_{2L}^1 > \delta_{2L}^2 \text{ and } \delta_{1L}^1 < \xi_{1L}^2,
\end{cases}
$$

(14)

where $(b_{1L}^*, b_{2L}^*)$ can be obtained by solving $V_1(b_{1L}^*, b_{2L}^*) = 0$ and $V_2(b_{1L}^*, b_{2L}^*) = 0$ (see Figure 5 which corresponds to this case where $\xi_{2L}^1 > \delta_{2L}^2$ and $\delta_{1L}^1 < \xi_{1L}^2$).

**Proof.** From Lemma 2 and Lemma 3, we find that $V_1 = 0$ and $V_2 = 0$ are monotone-
decreasing curves and the implicit derivative $b'_1$ is strictly less than $b'_2$. We also know that the derivative of $b^R_1 = 0$ (a vertical line) with respect to $b_{1L}$ is infinite [from Figure 3(a)] and the derivative of $b^R_2 = 0$ (a horizontal line) with respect to $b_{1L}$ is zero [from Figure 3(b)]. Thus, the implicit derivative of $P1$'s best response function with respect to $b_{1L}$ is also strictly smaller than that of $P1$'s best response function. In addition, $P_i$'s best response to any $b_{jL}$ ($i, j = 1, 2$ and $i \neq j$) is unique and continuous over $[0, C_j]$ (see Theorem 1). Accordingly, the best response curves of the two players intersect only once in $(b_{1L}, b_{2L})$ plane. It indicates that the game admits a unique Nash equilibrium.

As depicted in Figure 3, there are three types of best response curves for each player. Hence, there is a total of nine situations in which we can combine different best response curves of the two players. It is not difficult to show that the two best response curves intersect either on the $b_{iL}$ axis or inside the first quadrant. The $b_{1L}$- and $b_{2L}$-coordinates of the intersection of two curves, which is the Nash equilibrium, are determined by the relations between $\xi_{iL}$ and $\delta_{iL}$. (Note here that, as we see in Figure 3, $\xi_{iL}$ is the smallest value of $b_{jL}$ for which $b_{iL}$ assumes the smallest value.) For example, if $\xi_{1L} > \delta_{2L}$ and $\delta_{1L} < \xi_{2L}$, then $V_1 = 0$ and $V_2 = 0$ will admit a unique intersection point, $(b^*_1, b^*_2)$. Similarly, we can obtain the same results of the other cases shown in (14). □

**Example 2**  Here, we use the same values as in Table 3 of Example 1 for prices, costs, and transfer rates. The goal in this example is to demonstrate the Nash equilibrium in different situations. Since each player's BR function could be one of the three types
as given by Theorem 1, there is a total of nine situations in which the two BR curves intersect in \((b_{1L}, b_{2L})\) plane. In this example, we only present three of them and point out that in all cases, the Nash equilibrium must be one of the three types as given by Theorem 2.

\[
\begin{array}{|c|c|c|c|}
\hline
& (\lambda_{1L}, \lambda_{1H}) & (\lambda_{2L}, \lambda_{2H}) & (b_{1L}^N, b_{2L}^N) & (J_1(b_{1L}^N, b_{2L}^N), J_2(b_{1L}^N, b_{2L}^N)) \\
\hline
\text{Scenario 1} & (35,20) & (10,80) & (6.73,0) & (1208.61,1296.64) \\
\text{Scenario 2} & (25,80) & (25,25) & (0,7.62) & (422.55,1643.07) \\
\text{Scenario 3} & (80,40) & (85,45) & (6.81,11.94) & (2893.45,3631.97) \\
\hline
\end{array}
\]

Table 5. Nash equilibria and profits in three scenarios

We use three different sets of \(\lambda\)'s as in Table 5 and compute the resulting Nash equilibria. The Nash solution and the corresponding expected revenues for each scenario is summarized in Table 5 and displayed in Figure 6.

In Scenario 1 [Figure 6(a)], since the two BR curves only intersect at \((b_{1L}^N, b_{2L}^N) = (6.73,0)\), \(P2\) only accepts high-fare class customers. Similarly, in Scenario 2 [Figure 6(b)], the two BR curves only intersect at \((b_{1L}^N, b_{2L}^N) = (0,7.62)\). In this case, since \(\lambda_{1L} = 25\) (a low value), and \(\lambda_{1H} = 80\) (a high value), \(P1\) does not reserve any low-fare rooms but keeps them all for high fare customers. Finally, the two BR curves intersect at \((b_{1L}^N, b_{2L}^N) = (6.81,11.94)\) which is obtained by solving \(V_1 = 0\) and \(V_2 = 0\).

3.3 Stackelberg Equilibrium

When the decision makers choose their strategies simultaneously as was the case in the previous section, then the proper solution concept that should be used
in a non-cooperative game is the Nash equilibrium. However, in some cases, one player may assume the role of the “leader” (perhaps because he/she can act before the other one) and the other is the “follower.” Here, the leader announces his strategy first and the follower must make a decision to optimize his objective function after observing the leader’s decision. Thus, the leader is able to not only determine the follower’s response, but also optimize his objective accordingly. This strategy was first introduced by von Stackelberg [46] in 1934. For a rigorous treatment of the Stackelberg strategy, see Başar and Olsder [2].

Without loss of generality, we assume $P_1$ as the leader and $P_2$ as the follower in our game theoretical framework. Thus, $P_1$ announces his booking limit $b_{1L}$, and $P_2$ chooses an optimal booking limit $b_{2L}^R$ as a function of $b_{1L}$ that maximizes her expected revenue $J_2(b_{1L}, b_{2L})$. Since $P_1$ can identify $P_2$’s best response $b_{2L}^R$ for each $b_{1L}$, the Stackelberg equilibrium $(b_{1L}^S, b_{2L}^S)$ can always be obtained by solving the following
Jingpu Song

nonlinear programming problem (NLP):

\[
\begin{align*}
\text{max } & \ J_1(b_{1L}, b_{2L}), \\
\text{s.t. } & \ b_{2L} = b_{2L}^R(b_{1L}), \\
& \ b_{1L} \in [0, C_1].
\end{align*}
\]

Unfortunately, the objective function \( J_1(b_{1L}, b_{2L}(b_{1L})) \) may not be a concave function in \( b_{1L} \) after the substitution \( b_{2L} = b_{2L}^R(b_{1L}) \). Let us examine the optimal solution (Stackelberg equilibrium) of the above NLP according to the different types of \( b_{2L}^R \) as given by (9).

The following proposition determines the Stackelberg equilibrium when \( b_{2L}^R = 0 \) for \( \forall b_{1L} \in [0, C_1] \).

**Proposition 1** If \( b_{2L}^R(b_{1L}) = 0 \) for \( \forall b_{1L} \in [0, C_1] \), then the Stackelberg equilibrium is identical to the Nash equilibrium, i.e.,

\[
(b_{1L}^S, b_{2L}^S) = (b_{1L}^N, b_{2L}^N).
\]

**Proof.** If P1 always sets his booking limit as 0, i.e., \( b_{1L} = 0 \), then P2's best response to P1's decision is \( b_{2L}^R(0) = \delta_{2L}^2 \). [This can be seen in parts (b) and (c) in Figure 3 with \( j = 1 \) and \( i = 2 \).] This implies that the Stackelberg equilibrium solution which maximizes \( J_1 \) is \( (0, \delta_{2L}^2) \). Clearly, it is identical to the Nash equilibrium as given by Theorem 2 with \( \xi_{2L}^1 \leq \delta_{2L}^2 \). ■

**Remark 2** From Proposition 1, we find that the "Stackelberg game" is equiva-
lent to “Nash game” if \( \xi_{2L}^1 = 0 \) (\( \iff b_{1L} = 0 \)). In fact, the Stackelberg equilibrium is identical to the Nash equilibrium as long as \( J_1 \) is only maximized at \((b_{1L}^N, b_{2L}^N)\). This could happen in many situations. For example, if \( J_1(0, b_{2L}) \) is strictly concave in \( b_{2L} \) and \( b_{1L} = 0 \) for \( \forall b_{2L} \in [0, C_{2L}] \), then the Stackelberg equilibrium is also identical to the Nash equilibrium since for any point \((b_{1L}, b_{2L})\) on \( P2 \)'s BR curve \( J_1(0, \delta_{2L}^2) \geq J_1(0, b_{2L}) \geq J_1(b_{1L}, b_{2L}) \) always holds. Therefore, in this situation \( J_1 \) is always maximized at \((0, \delta_{2L}^2)\) which is the Nash equilibrium. However, in general, \( J_1(b_{1L}^S, b_{2L}^S) \) should be no less than \( J_1(b_{1L}^N, b_{2L}^N) \) because \( P1 \) can, at worst, play the strategy corresponding to the most favorable (from the leader's point of view) Nash equilibrium. \( \triangleq \)

We note that if \( \xi_{1L}^2 = C_{1L} \), then \( P2 \)'s BR function is \( V_2 = 0 \) for \( \forall b_{1L} \in [0, C_{1L}] \). Therefore, in this situation, the objective function \( J_1(b_{1L}, b_{2L}^R(b_{1L})) \) may not be a concave function. It can be seen that the second order derivatives of \( J_1 \) with respect to \( b_{1L} \)

\[
\frac{\partial^2 J_1}{\partial b_{1L}^2} + 2b_2^2 \frac{\partial^2 J_1}{\partial b_{1L} \partial b_{2L}} + (b_2')^2 \frac{\partial^2 J_1}{\partial b_{2L}^2} + b_2' \frac{\partial J_1}{\partial b_{2L}}
\]

where \( b_2' = db_2/db_{1L} \), involves the probability density functions of the booking requests whose monotonicities and concavities are unknown. In fact, we find that the concavity of \( J_1(b_{1L}, b_{2L}^R(b_{1L})) \) with respect to \( b_{1L} \) still cannot be guaranteed even if assuming the probability densities as functions with the monotone and concave properties. This fact makes the solution of the Stackelberg game more complicated than that of the Nash game.
Example 3  We again use the same data as in Example 2 and the same $\lambda$ values as in Table 5 for each of the three scenarios to examine the Stackelberg equilibrium. In order to compare the results in different leadership structures, we also calculate the Stackelberg equilibrium solution by assigning $P_2$ as the leader. These results are presented in Table 6.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$P_1$ leader, $P_2$ follower</th>
<th>$P_2$ leader, $P_1$ as follower</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 1</td>
<td>$(b_{1L}^S, b_{2L}^S)$, $(J_f^S, J_f^S)$</td>
<td>$(b_{1L}^S, b_{2L}^S)$, $(J_f^S, J_f^S)$</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>$(0.7, 1.17)$, $(422.55, 1643.07)$</td>
<td>$(0.7, 1.17)$, $(422.55, 1643.07)$</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>$(1.77, 13.92)$, $(2910.28, 3555.63)$</td>
<td>$(8.64, 7.258)$, $(2819.67, 3646.64)$</td>
</tr>
</tbody>
</table>

Table 6. Stackelberg equilibria and profits in three scenarios

As expected, Stackelberg equilibria are identical to Nash equilibria in Scenarios 1 and 2. However, in Scenario 3 where neither of the two players' BR is 0 at all the time, the Stackelberg solution is not identical to the Nash equilibrium. We also observe that the follower's Stackelberg revenue is less than her Nash revenue in this scenario. Therefore, the follower could prefer to play "Nash game" instead of "Stackelberg game", if such an option is open. ♦

3.4 Cooperative Solution

We shall now discuss the case of cooperation between the two players. When cooperating with each other, one player does not incur a rejection cost if a booking request is satisfied by its cooperative player. Hence the cooperative player whose booking limit or capacity has been reached should switch its unsatisfied bookings, if
any, to the other player with excess inventory so that the previous player can save in rejection penalty costs. In addition, we also assume that there are no transfer customers between $P_1$ and $P_2$ when both players' booking limits or capacities have been reached. Thus, they save rejection costs incurred by transfer customers. This is reasonable since when two players act as one player, a rejected customer will be noticed that both players are fully filled and therefore avoid the transfer from happening. Let us denote $J_c$ to be the joint expected revenue of $P_1$ and $P_2$ when they cooperate. Intuitively, $J_c$ is expected to be higher than the sum of two expected revenue under any other strategy. We will prove it with the following theorem.

**Theorem 3** $J_c \geq J_1 + J_2$.

**Proof.** Let us consider the rejection cost savings generated by low-fare transfer customers first. There are three mutually exclusive cases in which these can take place.

1. $x_{1L} \leq b_{1L}, x_{2L} \geq b_{2L}$:

   For $P_1$, the cost savings are
   
   $$q_{1L} \max[0, u_{2L}(x_{2L} - b_{2L}) - (b_{1L} - x_{1L})]$$

   since there will be no transfer customers from $P_2$ to $P_1$ when $P_1$'s booking limit is reached. And for $P_2$, the transfer customers who are satisfied by $P_1$ will not incur
penalty costs, therefore $P2$ saves

$$q_{2L} \min \left[u_{2L} \left(x_{2L} - b_{2L}\right), b_{1L} - x_{1L}\right].$$

(2) $x_{1L} \geq b_{1L}, \quad x_{2L} \leq b_{2L}$:

The cost savings for $P1$ and $P2$ are

$$q_{1L} \min \left[u_{1L} \left(x_{1L} - b_{1L}\right), b_{2L} - x_{2L}\right]$$

and

$$q_{2L} \max[0, u_{1L} \left(x_{1L} - b_{1L}\right) - (b_{2L} - x_{2L})],$$

respectively.

(3) $x_{1L} \geq b_{1L}, \quad x_{2L} \geq b_{2L}$:

In this case, both players's booking limits have been reached. There are no transfer customers between them. Therefore, the penalty cost savings for $P1$ and $P2$ are $u_{2L}q_{1L} \left(x_{2L} - b_{2L}\right)$ and $u_{1L}q_{2L} \left(x_{1L} - b_{1L}\right)$ respectively.

Integrating these three cost savings over the respective regions, we can obtain the total expected cost savings by low-fare customers for both players. Similarly, we can also obtain the expected cost savings by high-fare customers. After some
simplifications, the expected joint revenue is found as

\[ J_c = J_1 + J_2 + \sum_{i=1,2} \sum_{K=L,H} \left\{ \int_{M_iK}^{b_iK} \int_{b_{jK}}^{\infty} q_{iK} (x_{iK} - M_{iK}) f_{iK} f_{jK} \, dx_{iK} \, dx_{jK} 
+ \int_{b_{iK}}^{\infty} \int_{0}^{M_{jK}} q_{iK} u_{iK} (x_{iK} - b_{iK}) f_{iK} f_{jK} \, dx_{iK} \, dx_{jK} 
+ \int_{b_{iK}}^{\infty} \int_{M_{jK}}^{b_{jK}} q_{iK} (b_{jK} - x_{jK}) f_{iK} f_{jK} \, dx_{iK} \, dx_{jK} 
+ \int_{b_{iK}}^{\infty} \int_{b_{jK}}^{\infty} q_{iK} u_{jK} (x_{jK} - b_{jK}) f_{iK} f_{jK} \, dx_{iK} \, dx_{jK} \right\} \].

Clearly, we have \( J_c \geq J_1 + J_2 \) which means that the expected revenue under cooperation would be higher than the sum of two expected revenues under any other strategy.

The optimal solution for the cooperative game can be obtained by solving the following nonlinear programming problem:

\[
\max J_c (b_{1L}, b_{2L}), \\
\text{Subject to } b_{1L} \leq C_1, b_{2L} \leq C_2, \\
b_{1L}, b_{2L} \geq 0.
\]

For the existence of unique optimal solution of above problem, one must show that \( J_c \) is a strictly concave function of \( b_{1L} \) and \( b_{2L} \). This is not pursued in our study since it deviates from the general game theoretic theme.

Example 4 We still use the same values for all parameters in Example 2 and the
same \( \lambda \) values as in Table 5 for each of the three scenarios to examine the optimal booking limits of both players and corresponding joint revenue in cooperative situation. The results for three scenarios are shown in Table 7 where \( J_{ci} \), \( i = 1, 2 \) is \( P_i \)'s expected revenue when the two players cooperate.

<table>
<thead>
<tr>
<th>Scenario 1</th>
<th>((b_{1L}^<em>, b_{2L}^</em>))</th>
<th>((J_{c1}^<em>, J_{c2}^</em>))</th>
<th>(J_c^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 1</td>
<td>(11.77,0)</td>
<td>(3454.82,2357.14)</td>
<td>5811.96</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>(0,12.87)</td>
<td>(1342.02,4433.45)</td>
<td>5775.47</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>(20.81,25.83)</td>
<td>(3026.63,3916.40)</td>
<td>6943.03</td>
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</tbody>
</table>

Table 7. Cooperative solutions in three scenarios

Comparing to the results with Nash strategy and Stackelberg strategy (see Table 5 and Table 6), each player's expected revenue has increased in the cooperative situation. We note that such improvement is more than 100 percent for each player in scenario 1 and 2. This indicates that cooperation becomes useful when one player has a high booking rate of high-fare customers.

3.5 Sensitivity Analysis

From our previous discussion, we see that solutions using different strategies depend very much on the position of the BR function of each player. Referring to Example 1, we note that the position of one player's best response varies with the booking request expectations. On the other hand, the position of each player's BR function is also sensitive to the values of transfer rates and rejection costs of each fare class. Thus, in this section we discuss sensitivity analyses according to these
important parameters and present their effects on optimal decisions and corresponding objectives when adopting different strategies.

In our analyses, we use all parameters in Example 2 as the base parameters, and unless otherwise indicated, the solutions are computed with these parameters. Then, we vary parameters $\lambda_{1L}$, $\lambda_{1H}$, $u_{1L}$, $u_{1H}$, $q_{1L}$, and $q_{1H}$ one-at-a-time and re-solve the problem to find the solutions with different strategies. In order to make each of these parameters cover a large range, we (i) vary the values of two demand-related parameters $\lambda_{1L}$ and $\lambda_{1H}$ by progressively halving or doubling the base value; (ii) vary the values of transfer rates as $u_{1K} = 0.2(0.2)1$, $K = L, H$, and (iii) vary the values of rejection costs as $q_{1L} = 10(10)50$ and $q_{1H} = 30(20)110$. The sensitivity analysis results with Nash, Stackelberg and cooperative strategy are presented in Table 8, Table 9 and Table 10 respectively, where the base values and the corresponding solutions are indicated in **bold**.

First, we examine the effect of changing parameter values on the Nash solution and expected Nash revenue of each player.

**Changes in $\lambda_{1L}$**. Referring to the results in Table 8, we observe that for increased values of $\lambda_{1L}$, the Nash equilibrium moves in the northeast direction in the $(b_{1L}, b_{2L})$ plane. We also find that when $\lambda_{1L} \rightarrow \infty$, the best responses of both players $b_{1L}^R$ and $b_{2L}^R$ approach 27 and 33, respectively, which indicates that there exists a minimum protection level for high-fare customers in each hotel. This should be expected since in lower traffic season hotels should raise booking limits if the amount of booking requests of low-fare customers increases. However, hotels still have to keep some rooms for “more valuable” customers even though the booking rate of
<table>
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<th>Varying parameters</th>
<th>Nash strategy</th>
</tr>
</thead>
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<td>$\lambda_L$</td>
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<td>25</td>
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</tr>
<tr>
<td>50</td>
<td>(22.75, 26.05)</td>
</tr>
<tr>
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<td>(24.70, 28.21)</td>
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<td>3.75</td>
<td>(30.67, 25.97)</td>
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<td>7.5</td>
<td>(26.20, 25.58)</td>
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<td>(19.86, 24.17)</td>
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<tr>
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<td>(20.04, 23.48)</td>
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<td>(20.79, 23.91)</td>
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<td>(21.66, 23.66)</td>
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<td>70</td>
<td>(19.86, 24.17)</td>
</tr>
<tr>
<td>90</td>
<td>(18.82, 24.45)</td>
</tr>
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<td>110</td>
<td>(17.86, 24.71)</td>
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Table 8. Sensitivity analysis for Nash equilibrium and profits
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<th>Varying parameters</th>
<th>Stackelberg strategy</th>
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<tr>
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<td>$P1$ leader, and $P2$ follower</td>
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<td>$\lambda_{IL}$</td>
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<tr>
<td>6.25</td>
<td>(12.75, 22.97), (2600.97, 2939.56)</td>
</tr>
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<td>12.5</td>
<td>(16.09, 23.20), (2609.83, 2987.35)</td>
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<tr>
<td>25</td>
<td>(19.55, 24.25), (2592.04, 3011.64)</td>
</tr>
<tr>
<td>50</td>
<td>(22.42, 26.15), (2125.38, 2941.38)</td>
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<td>100</td>
<td>(24.39, 28.31), (845.706, 2841.38)</td>
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<tr>
<td>$\lambda_{IH}$</td>
<td></td>
</tr>
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<td>(30.72, 25.97), (2283.48, 2918.35)</td>
</tr>
<tr>
<td>7.5</td>
<td>(26.13, 25.59), (2456.05, 2964.98)</td>
</tr>
<tr>
<td>15</td>
<td>(19.55, 24.25), (2592.04, 3011.64)</td>
</tr>
<tr>
<td>30</td>
<td>(10.91, 21.56), (2448.27, 2912.02)</td>
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<tr>
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<tr>
<td>$u_{IL}$</td>
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<td>$u_{IH}$</td>
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Table 9. Sensitivity analysis for Stackelberg equilibrium and profits
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<thead>
<tr>
<th>Varying parameters</th>
<th>Cooperative strategy</th>
<th>Cooperation vs. Nash</th>
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<td>40</td>
<td>(22.57, 24.59)</td>
<td>(2932.51, 3450.05)</td>
</tr>
<tr>
<td>50</td>
<td>(24.02, 23.63)</td>
<td>(2846.05, 3474.17)</td>
</tr>
<tr>
<td>( q_{1H} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>(26.14, 22.27)</td>
<td>(3135.50, 3514.88)</td>
</tr>
<tr>
<td>50</td>
<td>(23.27, 24.13)</td>
<td>(3081.25, 3461.35)</td>
</tr>
<tr>
<td>70</td>
<td>(20.81, 25.83)</td>
<td>(3026.63, 3416.40)</td>
</tr>
<tr>
<td>90</td>
<td>(19.46, 26.52)</td>
<td>(2963.66, 3406.88)</td>
</tr>
<tr>
<td>110</td>
<td>(18.04, 27.38)</td>
<td>(2907.52, 3390.32)</td>
</tr>
</tbody>
</table>

Table 10. Sensitivity analysis for cooperative solution
“less valuable” customers is very high. This actually follows the principle of Revenue Management.

**Changes in** $\lambda_{1H}$. Increasing the value of $\lambda_{1H}$ has an opposite effect on Nash equilibrium comparing to the situation for $\lambda_{1L}$. This is due to the shifting of each player’s BR curve to a lower position, thus resulting in decreased values of $b_{1L}^N$.

**Changes in** $u_{1L}$. Note that as soon as the value of $u_{1L}$ increases, the Nash equilibrium moves in the northwest direction, resulting in a decrease in $b_{1L}$ and an increase in $b_{2L}$.

**Changes in** $u_{1H}$. In this case, the direction of movement of the Nash equilibrium is exactly opposite to the situation for $u_{1L}$: increasing the value of $u_{1H}$ raises $b_{1L}$ and reduces $b_{2L}$.

**Changes in** $q_{1L}$. As the value of $q_{1L}$ increases, the Nash equilibrium moves in the southeast direction, resulting in an increase in $b_{1L}$ and a decrease in $b_{2L}$.

**Changes in** $q_{1H}$. Referring to Table 8, we note that the direction of movement of the Nash equilibrium is exactly opposite to the situation for $u_{1L}$: increasing the value of $q_{1H}$ reduces $b_{1L}$ and raises $b_{2L}$.

The sensitivity analysis of Stackelberg equilibrium is summarized in Table 9 which reveals that the movements of the Stackelberg solution pair $(b_{1L}^S, b_{2L}^S)$ parallel those of Nash equilibrium as presented in Table 8.

However, we observe that in the cooperative situation the booking limit $b_{2L}$ does not monotonically decrease when $\lambda_{1H}$ is increasing (see Table 10). In fact, the optimal booking limit of each player might not monotonically decrease or increase at all times. This is expected since when the two players act as one player and try
to maximize the joint revenue, it is always optimal for the two players to keep the “most valuable” customers in the system. After then, they should satisfy the low-fare customers as many as they can. Therefore, when the booking rates of high-fare customers in both players are very low, a small increase of high-fare booking request expectation of one player will not generate high-fare transfer customers if the player decreases the booking limit. However, the low-fare transfer customers will increase due to the decrease of booking limit of the hotel with increased high-fare class booking rate. Hence, it might be optimal for the other hotel to increase its booking limit based on the condition that all high-fare customers are still in the system.
Chapter 4
Dynamic Game Model for Hotel Room Allocations

In this Chapter, we consider a situation in which the booking requests from different fare classes arrive concurrently and two hotels compete with each other. One hotel’s accept/reject decision of a booking request depends on the time at which the request arrives, as well as on the available rooms of both hotels at that point in time. One hotel’s available room(s) at a specific time might affect another hotel’s decision because of the existence of transferred customers. A discrete-time dynamic game is presented to obtain an optimal policy for making accept/reject decisions. This model differs from Chen et al.’s [12] model in that our model is used in the context of hotel business where the capacity is fixed and each player has his own booking requests from two fare classes. Moreover, our model assumes that the probability of a transferred customer who is rejected by one hotel (transfer rate) can be between zero and one. In addition, the rejection costs are incurred when a customer is rejected by a hotel. These assumptions make our model more general in practice.

4.1 The Dynamic Model of Best Response Policies

We assume that there are only two hotels serving a specific geographical market. These two hotels are assumed to be two players (P1 and P2), each with certain units of rooms to sell within a specified time period $[0, T]$. The customers are classified as low-fare class ($L$) customers and high-fare class ($H$) customers, who are
charged discounted price and full price, respectively. The booking period is divided into equal time intervals which is short enough to make the probability of more than one customer arriving in each interval negligible. In order to simplify the analysis, we make the following assumptions:

(1) The customer arrival patterns are known to both players.
(2) The room rates of the two fare classes in both hotels are constant and known.
(3) There is no buy-up when a low-fare customer is rejected.
(4) Each customer asks for a single unit of room.
(5) There are no cancellations allowed for any customer and no overbookings in both hotels.
(6) A rejection cost is only incurred by one hotel’s own customers and it is only incurred when one player still has available rooms.

The objective of each hotel is to maximize the expected future revenue by finding an optimal accept/reject policy for any combination of the rooms and time remaining. We use the following notation with $i, j = 1, 2$, $K = L, H$ and $i \neq j$:

- $\lambda_{iK}$: probability of arrival (in any given interval) of a $P_i$’s own $K$-fare class customer,
- $r_{iK}$: revenue per room from a $P_i$’s $K$-fare class customer,
- $c_{iK}$: rejection cost per $K$-fare class customer of $P_i$,
- $\mu_{iK}$: probability of transfer if a $K$-fare class customer is rejected by $P_i$
- $n_i$ : available rooms of $P_i$ at time $t$. 
Figure 7. Customer flows in the model.

Referring to Figure 7, we note that in each time interval, only one player can receive a booking request. If he rejects it, the rejected customer may become a transferred customer who will seek accommodation with another hotel, or choose to leave the system. Therefore, in each period, the players have to make decisions on whether to accept or to reject the booking request upon its arrival. Obviously, one player will not reject any high-fare customer unless he does not have available rooms. However, he may reject a low-fare customer in case his room can be sold to a more revenueable customer in later periods. Thus, the decisions of each player at period $t$
are
\[ x_{iL} = \begin{cases} 
1, & P_i \text{ accepts his own low-fare customer, if any,} \\
0, & P_i \text{ rejects his own low-fare customer, if any,}
\end{cases} \]
and
\[ y_{iL} = \begin{cases} 
1, & P_i \text{ accepts a transferred low-fare customer, if any,} \\
0, & P_i \text{ rejects a transferred low-fare customer, if any,}
\end{cases} \]
for \( i = 1, 2 \). Now defining \( V_i(t, n_i, n_j) \) as the maximum expected future total revenue of \( P_i \) when \( P_i \) and \( P_j \) start with \( n_i \) and \( n_j \) room(s), respectively, and there are \( t \) time intervals remaining until the end of the booking period, our problem can be formulated under two situations: (i) only one player has available rooms in period \( t \); and (ii) both players have available rooms in period \( t \).

4.1.1 Case 1: Only One Player Has Available Rooms in Period \( t \)

Without loss of generality, let us assume that \( P_1 \) has \( n_1 > 0 \) rooms at time \( t \) while \( P_2 \) has sold out all of his rooms (i.e., \( n_2 = 0 \)). In this case, \( P_2 \) has to reject all his booking requests due to empty inventory, and \( P_1 \)'s decisions will have no effects on \( P_2 \)'s expected revenue. In this case, \( P_2 \)'s expected future total revenue is \( V_2(t, n_1, 0) = 0 \) for all \( n_1 \) and \( t \in \{0, 1, 2, \ldots, T\} \).

Due to the possibilities of transferred high-fare class customers from \( P_2 \) to \( P_1 \), there are five different situations in which \( P_1 \) has to make decisions on \( x_{1L} \) and \( y_{1L} \). Figure 8 presents \( P_1 \)'s expected revenues under different sequential situations. Thus, \( P_1 \)'s expected future total revenue will be

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Figure 8. P1’s expected revenue when P2’s rooms are all sold out.

\[
V_1(t, n_1, 0) = \max_{x_1L, y_1L} \{(1 - \lambda_{1L} - \lambda_{1H} - \lambda_{2L}\mu_{2L} - \lambda_{2H}\mu_{2H}) V_1(t - 1, n_1, 0) \\
+ (\lambda_{1H} + \lambda_{2H}\mu_{2H}) [r_{1H} + V_1(t - 1, n_1 - 1, 0)] \\
+ \lambda_{1L} x_{1L} [r_{1L} + V_1(t - 1, n_1 - 1, 0)] \\
+ \lambda_{1L} (1 - x_{1L}) [V_1(t - 1, n_1, 0) - c_{1L}] \\
+ \lambda_{2L}\mu_{2L} y_{1L} [r_{1L} + V_1(t - 1, n_1 - 1, 0)] \\
+ \lambda_{2L}\mu_{2L} (1 - y_{1L}) [V_1(t - 1, n_1, 0)]\} 
\]  \(16\)
with $V_i(0, n_i, 0) = 0$ for all $n_i$. After some simplifications, we generalize $P_i$’s expected future total revenue when $n_1 > 0$ and $n_2 = 0$ as follows ($i, j = 1, 2$ and $i \neq j$):

$$V_i(t, n_i, 0) = \max_{x_{iL}, y_{iL}} \left\{ (1 - \lambda_i H - \lambda_j M_{jL} y_{iL} - \lambda_j H M_{jL} - \lambda_i L x_{iL}) V_i(t - 1, n_i, 0) 
+ (\lambda_i H + \lambda_j H M_{jL} + \lambda_j L M_{jL} y_{iL} + \lambda_i L x_{iL}) V_i(t - 1, n_i - 1, 0) 
+ (\lambda_i L x_{iL} + \lambda_j L M_{jL}) r_{iL}
+ (\lambda_i H + \lambda_j H M_{jL}) r_{iH} - \lambda_i L (1 - x_{iL}) c_{iL} \right\}$$

(17)

with $V_i(0, n_i, 0) = 0$ for all $n_i$. In order to investigate the decision rule of $P_i$ and the properties of $V_i(t, n_i, 0)$, we now introduce three important concepts.

**Definition 1** We define

$$\delta_i(t, n_i, n_j) = V_i(t, n_i, n_j) - V_i(t, n_i - 1, n_j),$$

$n_1, n_2 = 1, 2, \ldots$, as $P_i$’s expected marginal value of having an extra room in period $t$ given that $P_i$ and $P_j$ have booking capacities of $n_i$ and $n_j$, respectively.

**Definition 2** We define

$$\xi_i(t, n_i, n_j) = V_i(t, n_i, n_j) - V_i(t, n_i, n_j - 1),$$

$n_1, n_2 = 1, 2, \ldots$, as $P_i$’s expected marginal value arising from $P_j$ having an extra room in period $t$ given $P_i$ and $P_j$ have booking capacities of $n_i$ and $n_j$ respectively.

**Definition 3** We define

$$\theta_i(t, n_i, n_2) = V_i(t, n_i, n_j) - V_i(t - 1, n_i, n_j),$$
Proposition 2  When $P_j$ has no rooms available ($n_j = 0$) but $P_i$ has $n_i > 0$ rooms at the beginning of period $t$, $P_i$ should,

1. accept any low-fare class booking request if $\delta_i(t-1,n_i,0) < r_{iL}$,
2. accept his own low-fare class booking request and reject the transferred low-fare booking request if $r_{iL} \leq \delta_i(t-1,n_i,0) < r_{iL} + c_{iL}$, and
3. reject any low-fare class booking request if $r_{iL} + c_{iL} \leq \delta_i(t-1,n_i,0)$.

Proof. Referring to Figure 8, the results claimed in Proposition 2 are not difficult to obtain. For example, when $P_1$'s own low-fare customer arrives, comparing the expected revenues of accepting and rejecting $\lambda_{1L}$ which are provided by the second and third expression in Figure 8, we note that it is optimal for $P_1$ to accept his own low-fare booking request if $r_{1L} + V_1(t,n_1 - 1,0) > V_1(t,n_1,0) - c_{1L}$ (or, equivalently, $\delta_1(t-1,n_1,0) < r_{1L} + c_{1L}$). Similarly, we obtain the results in other situations in which different booking requests arrive to $P_1$. Thus, after some simplifications, the conclusions can be shown in three cases as described above. ■

From Definition 1, we note that $\delta_i(t,n_i,0)$ is a function of the decision period
(t), and booking capacity (n_i). By fixing the value of t, Pi's expected marginal value in n_i can be expressed as

\[ \delta_i(t, n_i, 0) = V_i(t, n_i, 0) - V_i(t, n_i - 1, 0) \]

\[ = (1 - \lambda_iH - \lambda_{jH}\mu_{jH} - \lambda_{iL} - \lambda_{jL}\mu_{jL})\delta_i(t - 1, n_i, 0) \]

\[ + (\lambda_iH + \lambda_{jH}\mu_{jH})\delta_i(t - 1, n_i - 1, 0) \]

\[ - \lambda_iL\{\max[r_{iL} + c_{iL} - \delta_i(t - 1, n_i, 0), 0] \]

\[ + \max[r_{iL} + c_{iL}, \delta_i(t - 1, n_i - 1, 0)]\} \]

\[ - \lambda_{jL}\mu_{jL}\{\max[r_{iL} - \delta_i(t - 1, n_i, 0), 0] + \max[r_{iL}, \delta_i(t - 1, n_i - 1, 0)]\} \]

(18)

for n_i > 1, t > 1. Using the relations between \(\delta_i(t - 1, n_i, 0)\) and the values of \(r_{iL}\) and \(r_{iL} + c_{iL}\) in (18), we will show that \(\delta_i(t, n_i, 0)\) is non-increasing in n_i for a fixed t, and non-decreasing in t for a fixed n_i. By these monotonic properties of \(\delta_i(t, n_i, 0)\), we are able to simplify the optimal accept/reject policy to sets of critical values, which can be used to control the booking process.

**Theorem 4** For a given t, \(\delta_i(t, n_i, 0)\) is non-increasing in n_i; and for a given n_i, \(\delta_i(t, n_i, 0)\) is non-decreasing in t.

**Proof.** We will prove this theorem by induction. In last period, i.e., t = 1, \(\delta_i(1, n_i, 0) = (\lambda_{iH} + \lambda_{jH}\mu_{jH})r_{iH} + (\lambda_{iL} + \lambda_{jL}\mu_{jL})r_{iL}\). Clearly, \(\delta_i(1, n_i, 0)\) is non-increasing in n_i. Now, we assume that \(\delta_i(t - 1, n_i, 0)\) is non-increasing in n_i. Referring to (18), we find that \(\delta_i(t, n_i, 0)\) is a non-negative combination of \(\delta_i(t - 1, n_i - 1, 0)\) and \(\delta_i(t - 1, n_i, 0)\). This indicates that \(\delta_i(t, n_i, 0)\) can always be expressed as a non-negative and linear combination of items which are non-increasing in n_i. Therefore,
by induction, \( \delta_i(t, n_i, 0) \) is non-increasing in \( n_i \) for a given \( t \).

Rearranging (18), we obtain

\[
\delta_i(t, n_i, 0) - \delta_i(t, n_i - 1, 0) = \\
(\lambda_{iH} + \lambda_{jH} \mu_{jH}) [\delta_i(t - 1, n_i - 1, 0) - \delta_i(t - 1, n_i - 1, 0)] \\
+ \lambda_{iL} \{\max [r_{iL} + c_{iL}, \delta_i(t - 1, n_i - 1, 0)] - \max [r_{iL} + c_{iL}, \delta_i(t - 1, n_i, 0)]\} \\
+ \lambda_{jL} \mu_{jL} \{\max [r_{iL}, \delta_i(t - 1, n_i - 1, 0)] - \max [r_{iL}, \delta_i(t - 1, n_i, 0)]\}. \tag{19}
\]

Since we know that \( \delta_i(t, n_i, 0) \) is non-increasing in \( n_i \) for a given \( t \), the RHS of (19) is non-negative. Thus, \( \delta_i(t, n_i, 0) \) is non-decreasing in \( t \) for a given \( n_i \). This completes the proof. ■

The monotonicity of \( \delta_i(t, n_i, 0) \) has the following managerial implications:

- In a booking period \( t \), there exists two critical booking capacities, \( \hat{n}_i^1(t) \) and \( \hat{n}_i^2(t) \), for \( Pi \), (with \( \hat{n}_i^1(t) \leq \hat{n}_i^2(t) \)), such that (i) any low-fare booking request is accepted for \( \hat{n}_i^2(t) < n_i \); (ii) a booking request from \( Pi \)'s own low-fare class is accepted while a transferred low-fare booking request is rejected for \( \hat{n}_i^1(t) \leq n_i < \hat{n}_i^2(t) \); and (iii) any low-fare booking request is rejected for \( n_i \leq \hat{n}_i^1(t) \).

- Given the booking capacity \( n_i \) for \( Pi \), there exists two critical booking periods, \( \hat{t}_i^1(n_i) \) and \( \hat{t}_i^2(n_i) \), (with \( \hat{t}_i^1(n_i) \leq \hat{t}_i^2(n_i) \)), such that (i) any low-fare booking request is accepted for \( t < \hat{t}_i^1(n_i) \); (ii) a booking request from \( Pi \)'s own low-fare class is accepted while a transferred low-fare booking request is rejected for \( \hat{t}_i^1(n_i) \leq t < \hat{t}_i^2(n_i) \); and (iii) any low-fare booking request is rejected for \( \hat{t}_i^2(n_i) \leq t \).
Example 5  We now present a numerical example for the case where only one player has available rooms in period $t$. We assume that each hotel has a capacity of 30 rooms. The room rate, penalty cost, and transfer rate of $K$-fare class customers in $P_i$ ($K = L, H$ and $i = 1, 2$) are given in Table 11.

<table>
<thead>
<tr>
<th></th>
<th>Low-fare ($K = L$)</th>
<th>High-fare ($K = H$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_{iL}$ $c_{iL}$ $\lambda_{iL}$ $\mu_{iL}$</td>
<td>$r_{iH}$ $c_{iH}$ $\lambda_{iH}$ $\mu_{iH}$</td>
</tr>
<tr>
<td>P1</td>
<td>$99$ $10$ $0.35$ $0.8$</td>
<td>$159$ $20$ $0.15$ $0.6$</td>
</tr>
<tr>
<td>P2</td>
<td>$105$ $12$ $0.25$ $0.75$</td>
<td>$165$ $25$ $0.10$ $0.65$</td>
</tr>
</tbody>
</table>

Table 11. Prices, rejection costs, arrival and transfer rates of $P1$ and $P2$.

Figure 9. Dynamic optimal decisions of $P1$ when $P2$'s rooms are sold out. In region $R_1$, $P1$ accepts both his low-fare customer and $P2$'s transfer customer; in region $R_2$, he accepts only his low fare customer and in region $R_3$ he rejects any low-fare customer.

Let us assume that $P2$ has sold out all his rooms, and $P1$ is attempting to determine his optimal accept/reject decisions. For the data given in Table 11, Figure
9 has three regions: In region $R_1$, $P_1$ accepts both his low-fare customer and $P_2$'s transfer customer. In region $R_2$, $P_1$ accepts only his low-fare customer and in region $R_3$ rejects any low-fare customer. The figure shows that the cutoff levels for the acceptance/rejection regions are non-decreasing in time $t$. When there are only a few time periods left, and there is sufficient inventory, $P_1$ accepts almost all low-fare and transfer customers. On the other hand, when the time remaining is quite long, $P_1$ rejects low-fare and transfer customers if his inventory is low.

For all of the numerical examples in the chapter, we use Excel VBA to calculate the optimal solutions.

4.1.2 Case 2: Both Players Have Available Rooms in Period $t$

In this case, $n_1, n_2 > 0$ at the beginning of period $t$, there will not be any transfer of high-fare customers between the two players. Hence, each player will have to decide whether to accept his own low-fare customer if such an arrival occurs. On the other hand, if a low-fare customer is rejected by one player, such a customer will probably seek accommodation with another player. Hence, each player also faces the choice of accepting or rejecting the booking request of a transferred customer.

Again, let us consider the situation faced by $P_1$ whose expected revenues under different decisions in period $t$ are presented in Figure 10. Here, $P_1$'s total expected revenue can be obtained by adding all these expressions on the right in Figure 10 and $P_2$'s total expected revenue can be obtained similarly. Therefore, after some
Expected revenue

\[ V_i(t, n_i, n_j) = \max_{x_{iL}, y_{iL}} \]
\[ \{ V_i(t - 1, n_i, n_j) + \lambda_{iH} r_{iH} + r_{iL} \left[ \lambda_{jL} \mu_{jL} (1 - x_{jL}) y_{iL} + \lambda_{iL} x_{iL} \right] - \lambda_{iL} (1 - x_{iL}) c_{iL} \\
+ \left[ \lambda_{iL} x_{iL} + \lambda_{iH} + \lambda_{jL} \mu_{jL} (1 - x_{jL}) y_{iL} \right] [V_i(t - 1, n_i - 1, n_j) - V_i(t - 1, n_i, n_j)] \\
+ \left[ \lambda_{jL} x_{jL} + \lambda_{jH} + \lambda_{iL} \mu_{iL} (1 - x_{iL}) y_{jL} \right] [V_i(t - 1, n_i, n_j - 1) - V_i(t - 1, n_i, n_j)] \} \]

Figure 10. P1’s expected revenue when both players have available room(s) at the beginning of period t.

simplifications, Pi’s total expected revenue is obtained as follows.

\[ V_i(t, n_i, n_j) = \max_{x_{iL}, y_{iL}}\]
with \( V_i(0, n_i, n_j) = 0 \) for all \( n_i, n_j > 0 \) \((i, j = 1, 2 \text{ while } i \neq j)\).

According to Figure 10, when a transferred low-fare booking request occurs with \( P1 \), his expected revenue is \( \lambda_2 \mu_2 [r_{1L} + V_1(t - 1, n_1 - 1, n_2)] \) if he accepts such a request. On the other hand, if \( P1 \) rejects such a request, his expected revenue is \( \lambda_2 \mu_2 V_1(t - 1, n_1, n_2) \). We can now express \( P1 \)'s optimal accept/reject decision for a transferred low-fare customer from \( P2 \) (i.e., \( y_{1L} = 1 \) or \( 0 \)) in terms of the expected marginal value, \( \delta_1(t - 1, n_1, n_2) \):

\[
\begin{align*}
y_{1L} = 1, & \quad \text{if } \delta_1(t - 1, n_1, n_2) < r_{1L}; \\
y_{1L} = 0, & \quad \text{if } \delta_1(t - 1, n_1, n_2) \geq r_{1L}.
\end{align*}
\] (21)

Next, we consider the situation in which \( P1 \)'s own low-fare booking request occurs. This is more complicated since \( P1 \)'s accept/reject decisions of his own low-fare customers, (i.e., \( x_{1L} = 1 \) or \( 0 \)) have to be made upon \( P2 \)'s decisions on transferred customers. If \( P2 \) decides to reject a transferred customer from \( P1 \), the optimal accept/reject decision for \( P1 \)'s own low-fare customer can be presented as:

\[
\begin{align*}
x_{1L} = 1, & \quad \text{if } \delta_1(t - 1, n_1, n_2) < r_{1L} + c_{1L} \text{ (and } P2 \text{ rejects)}; \\
x_{1L} = 0, & \quad \text{if } \delta_1(t - 1, n_1, n_2) \geq r_{1L} + c_{1L} \text{ (and } P2 \text{ rejects)}.
\end{align*}
\] (22)

On the other hand, if \( P2 \) decides to accept a transferred customer, the optimal accept/reject decision for \( P1 \)'s own low-fare customer will be

\[
\begin{align*}
x_{1L} = 1, & \quad \text{if } \delta_1(t - 1, n_1, n_2) < \alpha_1(t - 1, n_1, n_2) \text{ (and } P2 \text{ accepts)}; \\
x_{1L} = 0, & \quad \text{if } \delta_1(t - 1, n_1, n_2) \geq \alpha_1(t - 1, n_1, n_2) \text{ (and } P2 \text{ accepts)},
\end{align*}
\] (23)

where \( \alpha_1(t - 1, n_1, n_2) = \mu_{1L} \xi_1(t - 1, n_1, n_2) + r_{1L} + c_{1L} \). Similarly, we can obtain the optimal decisions for \( P2 \).
We note that, for \( Pi \,(i = 1, 2) \), there are four possible combinations of strategy mix \((x_{iL}, y_{iL})\). To simplify the expression, we denote \( M^1_i \) as \((1, 1)\), \( M^2_i \) \((0, 1)\), \( M^3_i \) \((1, 0)\), and \( M^4_i \) \((0, 0)\) respectively. Referring to (21), (22) and (23), we find that the optimal solution pair is determined in terms of \( \delta_i (t - 1, n_i, n_j) \) and the relations among the three critical values of \( r_{iL} \), \( \alpha_i (t - 1, n_1, n_2) \), and \( r_{iL} + c_{iL} \). In order to identify the optimal strategy mix in the different situations, we now examine the properties of \( V_i (t, n_i, n_j) \).

**Theorem 5**  For any \( t \in [1, T] \), \( V_i (t, n_i, n_j) \,(i, j = 1, 2 \text{ and } i \neq j) \) has the following properties:

1. \( V_i (t, n_i, n_j) \) is non-decreasing in \( n_i \), and non-increasing in \( n_j \);
2. \( \delta_i (t, n_i, n_j) \) is non-increasing in \( n_i \) and \( n_j \);
3. \( \delta_i (t, n_i, n_j) - \gamma \xi_i (t, n_i, n_j) \), with \( 0 \leq \gamma \leq 1 \), is non-increasing in \( n_i \) and \( n_j \).

(24)

**Proof.** Again, we shall use induction to prove this theorem. First, let us verify these properties for the last period \( (t = 1) \). We see that the player has to accept any low-fare booking request in last period as long as he has unsold rooms on hand. Then, from (20), we obtain \( V_i (1, n_i, n_j) = \lambda_{iH} r_{iH} + \lambda_{iL} r_{iL} \) for any \( n_i, n_j > 0 \). Thus, all properties in (24) hold.

Next, assuming that for any \( n_i, n_j > 0 \), these properties hold in period \( t - 1 \),
we wish to prove that the properties hold for period $t$. Actually, properties (1) and (2) indicate that $V_i(t, n_i, n_j)$ is non-decreasing quasi-concave in $n_i$, and non-increasing quasi-concave in $n_j$. We also note that $\xi_i(t-1, n_i, n_j) \leq 0$ since $V_i(t-1, n_i, n_j)$ is non-increasing in $n_j$. Thus, $\alpha_i(t-1, n_i, n_j) \leq r_i + c_i$. According to (21), (22) and (23), we describe $P_i$'s optimal strategy corresponding to $P_j$'s decisions in Figure 11. In any situation, $P_i$ should choose the corresponding strategy mix in order to maximize his expected revenue. For instance, if $P_j$'s decisions in period $t$ are $x_{jL} = 0$ and $y_{jL} = 0$, $P_i$'s expected revenue will be
\[ V_i(t, n_i, n_j) = (1 - \lambda_iH - \lambda_iL - \lambda_jL)V_i(t-1, n_i, n_j) + \lambda_jL (1 - \mu_jL) V_i(t-1, n_i, n_j) \]
\[ + \lambda_iL \max [V_i(t-1, n_i-1, n_j) + r_iL, V_i(t-1, n_i, n_j) - c_iL] \]
\[ + \lambda_jL \mu_jL \max [V_i(t-1, n_i-1, n_j) + r_iL, V_i(t-1, n_i, n_j)] \]
\[ + \lambda_iH [V_i(t-1, n_i-1, n_j) + r_iH]. \]

By assumption, we find that the RHS of (25) is a combination of terms that is non-decreasing quasi-concave in \( n_i \), and non-increasing quasi-concave in \( n_j \). Thus, in this situation, properties (1) and (2) hold. It is not difficult to validate that in other situations, \( V_i(t, n_i, n_j) \) can also be expressed as a combination of terms which satisfy properties (1) and (2). analogous to the above procedure, we can also prove that \( \delta_i(t, n_i, n_j) - \gamma \xi_i(t, n_i, n_j) (0 \leq \gamma \leq 1) \) is non-increasing in \( n_i \) and \( n_j \), Hence, by induction, all of the properties can be propagated to the other \( t \) values.

The properties shown in Theorem 5 are intuitive with the following managerial implications:

- In a booking period \( t \), there exists a critical booking capacity, \( \hat{n}_i(t) \), for \( P_i \), such that any transferred low-fare booking request should be accepted for \( n_i > \hat{n}_i(t) \), and any transferred low-fare booking request should be rejected for \( n_i \leq \hat{n}_i(t) \).

- The non-decreasing property of \( V_i(t, n_i, n_j) \) shows that \( P_i \) will be better off if \( P_j \) has few rooms unsold in period \( t \). This is so, since each player is more likely to reject a low-fare customer when the number of his unsold rooms is lower and
there are still many periods left.

- The non-increasing property of $\delta_i(t, n_i, n_j)$ well fits the classical “marginal revenue decreasing” law in economics.

![Diagram](image)

Figure 12. Dynamic best responses of $P_1$ when both hotels have rooms and $P_2$ follows a non-optimal policy. In region $R_1$, $P_1$ accepts both his low-fare customer and $P_2$'s transfer customer; in region $R_2$, he accepts only his low-fare customer and in region $R_3$ he rejects any low-fare customer.

**Example 6** Now let us examine the optimal decisions of one player, say, $P_1$, in
case both players have available rooms. We use the same data as in Table 11. In this case, P2 chooses an arbitrary (non-optimal) dynamic policy and P1 determines his best response after observing P2's decision in each period. We assume that P2 always adopts a first-come-first-served (FCFS) policy where \(x_{2L} = y_{2L} = 1\); that is, P2 accepts both her own low-fare customer and the transferred customer from P1. Faced with this policy, P1 determines his best response from (20) for each period. In Figure 12, we present the acceptance/rejection regions for P1 at four different time points; \(t = 10, 20, 30\) and \(40\) (periods-to-go) with 10 rooms remaining in each hotel. As in Example 5, in region \(R_1\), P1 accepts both his low-fare customer and P2's transfer customer. In region \(R_2\), P1 accepts only his low fare customer and in region \(R_3\), he rejects any low-fare customer. It is worth noting that when \(t = 40\), i.e., when 40 time periods are left, P1 almost always rejects any low-fare customer, but when \(t = 10\), he accepts both his low-fare customers and P2's transfers provided that both hotels have sufficient number of rooms left.

4.2 Non-cooperative Solution

In this section we shall examine Nash and Stackelberg equilibria in a non-cooperative framework by using the results obtained in Section 4.1. When the players make their decisions simultaneously in each period, the Nash equilibrium applies. On the other hand, when one of the players can act before the other one, we obtains the Stackelberg equilibrium.
4.2.1 Nash Equilibrium

We assume in this section that the two players make their accept/reject decisions simultaneously in each period. We also make the standard assumption that the two players are “rational”, i.e., one would not lower his objective function for the sole purpose of inflicting damage on the opponent. Thus, the best strategy they should adopt will give rise to the Nash equilibrium. Since we assume that both players are rational, some optimal strategy mixes shown in Figure 11 may not apply. For example, if $P1$ always accepts his own low-fare customer ($x_{1L} = 1$) in period $t$, there will be no transferred low-fare customer from $P1$ to $P2$. Thus, $V_1(t, n_1, n_2) (n_1, n_2 > 0)$ is the same for $x_{1L} = 1$, $y_{1L} = 1$ and $x_{1L} = 1$, $y_{1L} = 0$ if $x_{2L} = 1$. For the game problem in this section, we define the possible strategies of $P_i$ as follows:

(1) $U_i^1$: accept any low-fare customer;
(2) $U_i^2$: accept only transferred low-fare customer,
(3) $U_i^3$: accept only own low-fare customer, and
(4) $U_i^4$: reject any low-fare customer.

Note that the third strategy “accept only own low-fare customer” means that either the transferred low-fare customers from other player should be rejected, or there are no such customers. Thus, we describe $P_i$’s best response corresponding to $P_j$’s decisions in Figure 13. In any situation, $P_i$ should choose the corresponding strategy mix in order to maximize his expected revenue.
Figure 13. Pi’s best response corresponding to Pj’s decisions in period t.

Mathematically, the Nash strategy is a pair \((U_i^N, U_j^N)\) where \(U_i^N \in \{U_i^1, U_i^2, U_i^3, U_i^4\}\), \(i = 1, 2\), such that each player’s total expected revenue with this mix is always better than those with other strategy mixes. This strategy results in an equilibrium as it ensures that \(Pi (i = 1, 2)\) will not receive more than \(V_i (t, n_i, n_j)\) with \((U_i^N, U_j^N)\).
if he deviates from it unilaterally. Before examining the Nash strategy, let us first investigate the optimal strategies of one player in response to the chosen strategies of the other player which are the best responses. According to the optimal strategy mixes shown in Figure 13, we find that there are five mutually exclusive cases in which $P_i$’s best response exhibits a unique form:

1. $\delta_i < \min(r_{iL}, \alpha_i): U_i^R(U_j) = \begin{cases} U_i^1, & \text{if } U_j = U_j^2, U_j^4; \\ U_i^3, & \text{if } U_j = U_j^1, U_j^3. \end{cases}$

2. $r_{iL} \leq \delta_i < \alpha_i (r_{iL} < \alpha_i): U_i^R(U_j) = U_i^3$, for $\forall U_j$;

3. $\alpha_i \leq \delta_i < r_{iL} (\alpha_i \leq r_{iL}): U_i^R(U_j) = \begin{cases} U_i^1, & \text{if } U_j = U_j^2; \\ U_i^2, & \text{if } U_j = U_j^3; \\ U_i^3, & \text{if } U_j = U_j^3; \\ U_i^4, & \text{if } U_j = U_j^1. \end{cases}$

4. $\max(r_{iL}, \alpha_i) \leq \delta_i < r_{iL} + c_{iL}: U_i^R(U_j) = \begin{cases} U_i^3, & \text{if } U_j = U_j^3, U_j^4; \\ U_i^4, & \text{if } U_j = U_j^1, U_j^2. \end{cases}$

5. $r_{iL} + c_{iL} \leq \delta_i: U_i^R(U_j) = U_i^4$, where $\delta_i$ and $\alpha_i$ are for state $(t-1, n_i, n_j)$, $i, j = 1, 2$, and $i \neq j$.

Therefore, there are twenty-five different combinations of the best responses of two players in $(U_1, U_2)$ plane. Due to the symmetrical property of these combinations, we will show only fifteen of them in the following figures. In practice, we use the cells filled by vertical lines to present $P_1$’s best responses corresponding to $P_2$’s strategies. On the other hand, the cells filled by horizontal lines are presented as $P_2$’s best responses corresponding to $P_1$’s strategies. Thus, the cells which are filled by both
vertical and horizontal lines are the Nash equilibria.

**Theorem 6**  In each period, the optimal rational behavior of the two players has a unique Nash equilibrium \((U^N_i, U^N_j)\) for \(\forall t \in [1, T] \) and \(\forall n_i, n_j > 0, i, j = 1, 2\) for \(i \neq j\) as follows.

\[
(U^N_i, U^N_j) = \begin{cases} 
(U^1_i, U^4_j), & \text{if } \delta_i < r_i L \text{ and } r_j L + c_j L \leq \delta_j; \\
(U^4_i, U^1_j), & \text{if } r_i L + c_i L \leq \delta_i \text{ and } \delta_j < r_j L; \\
(U^2_i, U^3_j), & \text{if } r_i L \leq \delta_i < r_i L + c_i L \text{ and } r_j L + c_j L \leq \delta_j; \\
(U^3_i, U^2_j), & \text{if } r_i L + c_i L \leq \delta_i \text{ and } r_j L \leq \delta_j < r_j L + c_j L; \\
(U^4_i, U^3_j), & \text{if } r_i L + c_i L \leq \delta_i \text{ and } r_j L + c_j L \leq \delta_j.
\end{cases}
\]

In addition, the game admits multiple Nash equilibria (MNE) in other situations. The multiple Nash equilibria and corresponding conditions can be expressed as:

\[
\begin{align*}
(U^1_i, U^4_j) \text{ or } (U^3_i, U^3_j), \quad & \text{if } \delta_i < r_i L \text{ and } \max (r_j L, \alpha_j) \leq \delta_j \leq r_j L + c_j L, \text{ or } \delta_i < \alpha_i \text{ and } \alpha_j \leq \delta_j < \max (r_j L, \alpha_j); \\
(U^4_i, U^1_j) \text{ or } (U^3_i, U^3_j), \quad & \text{if } \max (r_i L, \alpha_i) \leq \delta_i < r_i L + c_i L \text{ and } \delta_j < r_j L, \text{ or } \alpha_i \leq \delta_i \leq \max (r_i L, \alpha_i) \text{ and } \delta_j < \alpha_j; \\
(U^1_i, U^4_j), (U^4_i, U^1_j) \text{ or } (U^3_i, U^3_j), \quad & \text{if } \alpha_i \leq \delta_i < \max (r_i L, \alpha_i) \text{ and } \alpha_j \leq \delta_j < \max (r_j L, \alpha_j),
\end{align*}
\]
in which $\delta_i$ and $\alpha_i$ are for state $(t - 1, n_i, n_j)$.

**Proof.** Referring to Figure 14 - 17, we notice that if both players make the decisions optimally, the game admits either a unique Nash equilibrium or multiple Nash equilibria. Accordingly, we build three matrices to show the Nash equilibria in different situations (see Figure 18). Simply categorizing areas with same Nash equilibrium, we obtain $(U_i^N, U_j^N)$ for each situation.

We see that there are two Nash equilibria in (c), (d) of Figure 14 and (k) of Figure 17, which are $(U_1^1, U_2^4)$ and $(U_1^3, U_2^2)$ for all three cases. Comparing the two players' expected revenue with them, we obtain

\[
V_1 |_{(U_1^1, U_2^2)} - V_1 |_{(U_1^3, U_2^2)} = \lambda_2 L \mu_2 L [r_1 L - \delta_1(t - 1, n_1, n_2)] + \lambda_2 L \xi_1(t - 1, n_1, n_2)
\]  
(28)

and

\[
V_2 |_{(U_1^1, U_2^2)} - V_2 |_{(U_1^3, U_2^2)} = \lambda_2 L [\delta_2(t - 1, n_2, n_1) - \alpha_2(t - 1, n_2, n_1)].
\]  
(29)

It is not difficult to find that $V_2 |_{(U_1^1, U_2^2)} - V_2 |_{(U_1^3, U_2^2)} \geq 0$ since $\delta_2(t - 1, n_2, n_1) \geq \alpha_2(t - 1, n_2, n_1)$ for any case. ■

Thus, the equilibrium $(U_1^1, U_2^4)$ is better than $(U_1^3, U_2^2)$ for $P2$. He would more likely choose $U_2^4$ hoping $P1$ choose $U_1^1$. Unfortunately, $V_1 |_{(U_1^1, U_2^4)} - V_1 |_{(U_1^3, U_2^2)} \geq 0$ can not be guaranteed. It implies $V_1 |_{(U_1^1, U_2^4)}$ might be less than $V_1 |_{(U_1^3, U_2^2)}$ in some situations by which the possible strategy combination of the two players would be $(U_1^3, U_2^4)$. Such deviation from the Nash equilibrium is dangerous since both players' revenue could be badly decreased. However, if both players have noticed this "danger", they might not choose the strategies which lead to $(U_1^3, U_2^4)$. Thus, the two players are more
Figure 14. Nash equilibria in the situations where P1's case 1 vs P2's cases 1–4.
Figure 15. Nash equilibria in the situations where $P1$'s case 1 vs $P2$'s case 5, and $P1$'s case 2 vs $P2$'s cases 2–4.
(i) $P_1$'s Case 2 vs $P_2$'s Case 5:
$(U_1^N, U_2^N) = (U_1^3, U_2^4)$

(j) $P_1$'s Case 3 vs $P_2$'s Case 3:
$(U_1^N, U_2^N) = (U_1^1, U_2^4), (U_1^2, U_2^2), (U_1^3, U_2^3), \text{ or } (U_1^4, U_2^1)$

(k) $P_1$'s Case 3 vs $P_2$'s Case 4:
$(U_1^N, U_2^N) = (U_1^1, U_2^4) \text{ or } (U_1^3, U_2^3)$

(l) $P_1$'s Case 3 vs $P_2$'s Case 5:
$(U_1^N, U_2^N) = (U_1^1, U_2^1)$

Figure 16. Nash equilibria in the situations where $P_1$'s case 2 vs $P_2$'s case 5, and $P_1$'s case 3 vs $P_2$'s cases 3-5.
(m) PI's Case 4 vs P2's Case 4:
\((U_1^N, U_2^N) = (U_1^3, U_2^2)\)

(n) PI's Case 4 vs P2's Case 5:
\((U_1^N, U_2^N) = (U_1^3, U_2^4)\)

(o) PI's Case 5 vs P2's Case 5:
\((U_1^N, U_2^N) = (U_1^4, U_2^4)\)

Figure 17. Nash equilibria in the situations where PI's case 4 vs P2's cases 4-5, and PI's case 5 vs P2's case 5.
likely to make their decisions by negotiation (e.g., they may choose the equilibrium which maximizes their joint revenue). However, this might cause another problem: how do they share those ‘extra’ revenue by such negotiation?

In addition, there are four Nash equilibria in case (j) of Figure 16: $(U_1^1, U_2^4)$, $(U_1^2, U_2^2)$, $(U_1^3, U_2^3)$ and $(U_1^4, U_2^1)$. Referring to $P_i$’s objective function, we have

$$V_i |_{(U_1^i, U_2^j)} = V_i(t-1, n_i, n_j) + \lambda_i H r_{iH} + (\lambda_j L \mu_j L + \lambda_i L) r_{iL}$$

$$- (\lambda_i L + \lambda_i H + \lambda_j L \mu_j L) \delta_i(t-1, n_i, n_j) - \lambda_j H \xi_i(t-1, n_i, n_j),$$
Comparing $P_i$'s expected revenue when using these equilibria, we obtain

$$V_i \mid_{(U_i^1, U_j^1)} - V_i \mid_{(U_i^2, U_j^2)} = \lambda_{iL}(\delta_i - \alpha_i) \geq 0,$$

$$V_i \mid_{(U_i^3, U_j^3)} - V_i \mid_{(U_i^1, U_j^1)} = \lambda_{iL}(\delta_i - \alpha_i) \geq 0,$$

and

$$V_i \mid_{(U_i^4, U_j^4)} - V_i \mid_{(U_i^3, U_j^3)} = \lambda_{iL}\mu_{iL}(\delta_i - r_{iL}) - \lambda_{iL}\xi_i$$

$$\geq \left( \frac{1}{\mu_{iL}} - \mu_{jL} \right)(r_{iL} - \delta_i) + \frac{\xi_{iL}}{\mu_{iL}}$$

$$> 0.$$  

Thus, $(U_i^4, U_j^4)$ is superior to the other three equilibria $P1$; on the other hand, $(U_i^1, U_j^2)$ is the best equilibrium for $P2$. Again, in this case, we can not choose an equilibrium for the two players. The following example will show this four Nash equilibria case.

**Example 7**  In order to show the four Nash equilibria situation, we use the following values (see Table 12) for the room rate, penalty cost, and transfer rate of $K$-fare class.
customers in \( P_i \) (\( K = L, H \) and \( i = 1,2 \)).

<table>
<thead>
<tr>
<th>( K = L )</th>
<th>( K = H )</th>
</tr>
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<tbody>
<tr>
<td>( r_{iL} )</td>
<td>( r_{iH} )</td>
</tr>
<tr>
<td>( c_{iL} )</td>
<td>( c_{iH} )</td>
</tr>
<tr>
<td>( \lambda_{iL} )</td>
<td>( \lambda_{iH} )</td>
</tr>
<tr>
<td>( \mu_{iL} )</td>
<td>( \mu_{iH} )</td>
</tr>
<tr>
<td>P1</td>
<td>$99</td>
</tr>
<tr>
<td>P2</td>
<td>$105</td>
</tr>
</tbody>
</table>

Table 12. Prices, rejection costs, arrival and transfer rates of \( P1 \) and \( P2 \) for the four Nash equilibrium case.

We assume there is only 2 periods left and each player has only one room unsold. We then calculate the total expected revenue of the two player with each possible strategy mixes combination. The results are shown in Table 13.

| \( U_2^A \) | \( (126.38,121.03) \) | \( (124.42,112.88) \) | \( (120.23,92.52) \) | \( (118.28,84.37) \) |
|---|---|---|---|
| \( U_2^3 \) | N/A | N/A | \( (135.14,112.96) \) | \( (133.19,104.81) \) |
| P2 : \( U_2^2 \) | \( (128.68,118.52) \) | N/A | \( (122.54,90.01) \) |
| \( U_2^1 \) | N/A | N/A | N/A | \( (137.45,110.45) \) |
| \( U_1^1 \) | \( U_1^2 \) | \( U_1^3 \) | \( U_1^4 \) |

Table 13. The total expected revenue of the two players with different strategy mixes.

We see that there are four Nash equilibria, which are \((U_1^A, U_2^A)\), \((U_1^2, U_2^2)\), \((U_1^3, U_2^3)\), and \((U_1^4, U_2^4)\). Comparing the revenue obtained by these equilibria, none of them can be the best for both players. Specifically, the equilibrium \((U_1^4, U_2^4)\) is the best for \( P1 \), and \((U_1^4, U_2^4)\) is the best for \( P2 \).

From Example 7, we verify our statement, which says, in the MNE case there is not an optimal decision rule for the two players. Then, we can not calculate one player’s total expected revenue in that period. As a result, the optimal strategies for
the all former periods are also uncertain. However, our numerical experiments show that in most cases, the game admits only one equilibrium. We will use the following example to illustrate the unique Nash equilibrium cases.

**Example 8** Here, we again use the same values as in Table 11 of Example 5 for the room rates, penalty costs, and transfer rates for both hotels. The goal in this example is to demonstrate the unique Nash equilibrium arising from equation (26) in Theorem 6. As in Example 6, we consider four time periods (to-go), i.e., 10, 20, 30 and 40. Referring to Figure 19, we have up to six different regions which are defined as follows: Region $R_1$ corresponds to $(U_1^2, U_2^2)$, that is, both players accept only their low-fare customers. In region $R_2$, we have $(U_1^4, U_2^2)$ which corresponds to $P_1$ reject any low-fare customer and $P_2$ accepting only her own low-fare customer. Similarly, in region $R_3$ we have $(U_1^2, U_2^4)$, in $R_4$, we have $(U_1^4, U_2^1)$. Finally, in $R_5$, the policy is $(U_1^1, U_2^4)$ and in $R_6$, we have $(U_1^4, U_2^4)$.

According to Figure 19, we note that when $t = 40$ (periods-to-go), both hotels reject any low fare customer—hoping that high-fare customers will arrive in later periods. However, at $t = 10$, for a large combination of high $(n_1, n_2)$ values, both players accept any low-fare customer, which is intuitive. At $t = 10$, when only a few rooms remain, the hotels can be more "choosy" and can reject low-fare customers.

We have also compared the total expected revenues for both players under two scenarios: Scenario 1: $P_2$ uses an arbitrary dynamic policy (i.e., FCFS) and $P_1$ responds optimally to $P_2$'s decisions (as presented in Section 4.1.2, and Scenario 2:
Figure 19. Nash equilibrium for both players. Here, region $R_1$ corresponds to $(U_1^2, U_2^2)$, that is, both players accept only their low-fare customers. In region $R_2$, we have $(U_1^4, U_2^3)$ which corresponds to $P_1$ rejecting any low-fare customer and $P_2$ accepting only her own low-fare customer. Similarly, in region $R_3$ we have $(U_1^3, U_2^3)$, in $R_4$, we have $(U_1^4, U_2^1)$. Finally, in $R_5$, the policy is $(U_1^1, U_2^4)$ and in $R_6$, we have $(U_1^4, U_2^4)$. 

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Scenario 1: P1 acts optimally and P2 uses FCFS rule with $x_{2L} = y_{2L} = 1$

<table>
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<tr>
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<tr>
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Scenario 2: Nash strategies

<table>
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</tr>
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<tbody>
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<td>50</td>
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</table>

V1: 801.89 1086.06 1169.73 1163.70 1157.29 1151.32 1142.94 1133.54

V2: 801.89 1086.06 1175.37 1184.52 1188.75 1191.92 1194.96 1198.37

Table 14. The expected profits of the two players in two Scenarios: P1 acts optimally; P2 use FCFS VS. Nash game.

Both P1 and P2 implement the Nash strategies as discussed in this section. Referring to Table 14, we observe that as players move from Scenario 1 (where P2 acts in a non-optimal fashion) to Scenario 2 (where both adopt the Nash strategy), P1’s expected revenues decreases and P2’s expected revenue increases. ♦

4.2.2 Stackelberg Equilibrium

In the previous section, the players used the Nash strategy under the assumption that they make their decisions simultaneously. Now we consider another non-cooperative situation which is leader-follower Stackelberg game. Without loss of generality, in this section we assume P1 as the leader and P2 as the follower in our game theoretical framework. In each period $t$, P1 announces his strategy mix, $U_1$, first, and P2 chooses an optimal accept/reject decision as a function of $U_1$ to max-
imize his expected revenue. Recall the best response from Section 4.2.1, we obtain $P_2$'s best response for acceptance/rejection of transferred low-fare customer in period $t \ (t \geq 1)$ for $\forall n_i, n_j > 0, \ i, j = 1, 2$ and $i \neq j$ as,

$$y_{2L}^b(U_1) = \begin{cases} 
0, & \text{if} \quad \delta_2(t-1, n_2, n_1) \geq r_{2L} \text{ and } x_{1L} = 0; \\
1, & \text{if} \quad \delta_2(t-1, n_2, n_1) < r_{2L} \text{ and } x_{1L} = 0; 
\end{cases} \quad (30)$$

Similarly, $P_2$'s best response for acceptance/rejection of his own low-fare customer in period $t$, $x_{2L}^b(u_1)$ can be derived as

$$x_{2L}^b(U_1) = \begin{cases} 
1, & \text{if} \quad \delta_2(t-1, n_2, n_1) < r_{2L} + c_{2L} \text{ and } y_{1L} = 0; \\
0, & \text{if} \quad \delta_2(t-1, n_2, n_1) \geq r_{2L} + c_{2L} \text{ and } y_{1L} = 0; 
\end{cases} \quad (31)$$

**Proposition 3** When both players have available rooms $(n_1, n_2 > 0)$ at the beginning of period $t$, the Stackelberg equilibria and corresponding conditions can be expressed as follows.

$$(U_1^S, U_2^S) = \begin{cases} 
(U_1^1, U_2^4), & \text{if} \quad \delta_1 < r_{1L} \text{ and } c_{2L} + r_{2L} \leq \delta_2; \\
(U_1^3, U_2^4), & \text{if} \quad r_{1L} \leq \delta_1 < c_{1L} + r_{1L} \text{ and } c_{2L} + r_{2L} \leq \delta_2; \\
(U_1^4, U_2^1), & \text{if} \quad \alpha_1 \leq \delta_1 \text{ and } \delta_2 < r_{2L}; \\
(U_1^4, U_2^3), & \text{if} \quad c_{1L} + r_{1L} \leq \delta_1 \text{ and } r_{2L} \leq \delta_2 < c_{2L} + r_{2L}; \\
(U_1^4, U_2^4), & \text{if} \quad c_{1L} + r_{1L} \leq \delta_1 \text{ and } c_{2L} + r_{2L} \leq \delta_2; \\
(U_1^3, U_2^3), & \text{otherwise.} 
\end{cases} \quad (32)$$
Proof. We use the exhaustive enumeration method to prove this proposition by examining each case listed in Figure 14 – 17. For example, if $\alpha_2 \leq r_{2L}$ and $r_2 > \delta_2 (t - 1, n_2, n_1) \geq \alpha_{2L}$, then

$$U_2^b = (x_{2L}^b(U_1), y_{2L}^b(U_1)) = \begin{cases} (0, 1), & \text{if } U_1 = (1, 1) \text{ or } U_1 = (0, 1); \\
(1, 1), & \text{if } U_1 = (1, 0) \text{ or } U_1 = (0, 0). \end{cases}$$

Checking each combination of $(U_1, U_2^b(U_1))$, we find that $P1$ could obtain maximum expected revenue if he announce $U_1^3$ to $P2$ who will choose $U_2^3$ as the strategy to response $U_1^3$. Then the optimal strategy combination of the two players, which is the Stackelberg equilibrium, is $(U_1^3, U_2^3)$. Similarly, we can also analyze the optimal strategies of $P1$ in all other cases of Figure 14 – 17. Finally, we obtain $(U_1^S, U_2^S)$ for each situation as described above (see (32)).

It can be seen when $(U_1^S, U_2^S) = (U_1^N, U_2^N)$, the corresponding condition in (26) is tighter than that in (32). It implies that the if the game admits a unique Nash equilibrium in state $(t, n_1, n_2)$ and the all lower states, the leader-follower stackelberg game is identical to the Nash game. Based on the data given in Example 8, we calculate the Stackelberg equilibrium and corresponding revenue for both players ($P1$ is assumed as leader). We obtain the same results as shown in Table 14. Actually, comparing the Stackelberg equilibria in (32) with the MNE in (27), we also find that the Stackelberg equilibrium is still one of the multiple Nash equilibria. For example,
in four Nash equilibria case the Stackelberg equilibrium is \((U_4, U_2^1)\) which is still a Nash equilibrium. We can verify this results by Example 7. Thus, \(P1\) can only benefit by his leadership in state \((t, n_1, n_2)\) if and only if there are multiple Nash equilibria in that state or lower states.

4.3 Cooperative Solution

We shall now study the situation of cooperation between the two players. Here, we assume that under cooperation, a player does not incur a rejection cost if a booking request is satisfied by its cooperative partner. In addition, we also assume that the transfer rate between two players is one when one player's rejected customer is acceptable by another player. This is reasonable since when two players cooperate, each player should encourage an unsatisfied customer to transfer to the other (cooperative) hotel. Under this situation, both players make decisions jointly on which hotel should accept/reject the arriving customer dynamically according to their inventory levels and time-to-go. Since the total revenue of two players is possibly increased at the cost of losing some revenue on one player, the optimal strategy mixes may be different from those under the competitive situation.

We denote \(V(t, n_1, n_2)\) as the maximum total expected revenue of two players in state \((t, n_1, n_2)\). In the last period, i.e., \(t = 1\), two players should accept any booking request as long as there are some unsold rooms. However, the booking request should be given to the player whose unit revenue per room is higher. Hence, the maximum total expected revenue in state \((1, n_1, n_2)\) can be expressed as
If one player (i.e., $P_i$, $i = 1, 2$) has sold out all of his rooms before the end of booking period, the maximum total expected revenue can be obtained from (17) and the optimal decisions would be the same as those shown in Proposition 2.

Next, we will focus on the situation in which both players have unsold rooms. It is possible that a player whose unit revenue of a high-fare customer is lower than that of another player may reject his own high-fare customer in order to maximize the total revenue of two players. Thus, we introduce an additional decision variable for such a player; without loss of generality, we that assume that $P_1$ is this player. Let us now denote $x_{1H}$ as the decision of $P_1$ on his high-fare booking request, i.e.,

$$x_{1H} = \begin{cases} 
1, & P_1 \text{ accepts his own high-fare customer, if any} \\
0, & P_1 \text{ rejects his own high-fare customer, if any.}
\end{cases}$$

Taking into account all other decisions, we present the expected revenue with any possible combination of decisions in period $t$ (See Figure 20).

Then, we have,

$$V(1, n_1, n_2) = \begin{cases} 
0, & \text{if } n_1 = n_2 = 0; \\
\sum_{K=L,H} (\lambda_{1K} + \lambda_{2K}) r_{1K}, & \text{if } n_1 > 0 \text{ and } n_2 = 0; \\
\sum_{K=L,H} (\lambda_{1K} + \lambda_{2K}) r_{2K}, & \text{if } n_1 = 0 \text{ and } n_2 > 0; \\
\sum_{K=L,H} (\lambda_{1K} + \lambda_{2K}) \max(r_{1K}, r_{2K}), & \text{if } n_1 > 0 \text{ and } n_2 > 0.
\end{cases}$$

(33)
Figure 20. The expected revenue of two players under cooperation \((n_1, n_2 > 0)\).

\[
V(t, n_1, n_2) = \max_{x_1L, x_1H, y_1L, y_1H, x_2L, y_2L} \{ V(t-1, n_1, n_2) \\
+ \lambda_1H (1 - x_1H) [V(t-1, n_1, n_2 - 1) - V(t-1, n_1 - 1, n_2)] + r_{2H} - r_{1H} \\
+ \sum_{i=1,2} \{ r_{iL} [\lambda_{jL} (1 - x_{jL}) y_{iL} + \lambda_{iL} x_{iL}] - \lambda_{iL} (1 - x_{iL}) (1 - y_{jL}) c_{iL} + \lambda_{iH} r_{iH} \} \\
+ [\lambda_1L x_{1L} + \lambda_1H + \lambda_2L (1 - x_{2L}) y_{1L}] [V(t-1, n_1 - 1, n_2) - V(t-1, n_1, n_2)] \\
+ [\lambda_2L x_{2L} + \lambda_2H + \lambda_1L (1 - x_{1L}) y_{2L}] [V(t-1, n_1, n_2 - 1) - V(t-1, n_1, n_2)] \}
\]

(34)
with \( V(0, n_1, n_2) = 0, n_1, n_2 > 0 \) and \( i, j = 1, 2 \) \((i \neq j)\).

According to the cases shown in Figure 20, the optimal accept/reject decisions of the two players in under cooperation is given in the following Proposition.

**Proposition 4** Under cooperative situation, if the two players have available rooms
\((n_i, n_j > 0, i, j = 1, 2 \text{ while } i \neq j)\) at the beginning of period \(t\) and \(r_{1H} < r_{2H}\), then the accept/reject decision rules are as follows:

1. \(P_1\) accepts his own high-fare customer, if \(\beta_1 > r_{2H} - r_{1H}\), and rejects, otherwise;

2. \(P_i\) accepts his own low-fare customer, if \(\beta_i > r_{jL} - r_{iL}\) and \(\delta_i < r_{iL} + c_{iL}\);

3. \(P_i\) rejects his own low-fare customer, but \(P_j\) accepts his own low-fare customer,
   if \(\beta_i \leq r_{jL} - r_{iL}\) and \(\delta_j < r_{jL} + c_{iL}\);

4. both \(P_i\) and \(P_j\) reject their own low-fare customer, if \(\delta_i \geq r_{iL} + c_{iL}\) and \(\delta_j \geq r_{jL} + c_{iL}\),

where \(\beta_1 = -\beta_2 = V(t - 1, n_1 - 1, n_2) - V(t - 1, n_1, n_2 - 1)\), and \(\delta_i\) denotes to the marginal revenue of a \(P_i\)'s room, i.e., \(\delta_1 = V(t - 1, n_1, n_2) - V(t - 1, n_1 - 1, n_2)\) \((\beta_i\) and \(\delta_i\) are for state \((t - 1, n_1, n_2))\).

**Proof.** Referring to Figure 20, we find the accept/reject decision on \(P_1\)'s high-fare booking request by comparing the fifth and sixth expressions, i.e., \(P_1\) should accept his own high-fare customer if \(\beta_1(t - 1, n_1, n_2) > r_{2H} - r_{1H}\). Similarly, we can obtain other accept/reject decisions when different booking requests occur. This completes
the proof. ■

From Proposition 4, we note that in cooperative situations it might be optimal for $P_1$ to reject his own high-fare class customer if such action can improve the total expected revenue of the two players.

**Example 9** From Example 7, we note in the multiple Nash equilibria situation, the optimal solutions for the two players are uncertain and they most likely choose cooperation in order to avoid the danger of the deviation from the Nash equilibria. In this example, we use the same parameter values as those in Example 7. According to the decision rules provided by Proposition 4, we find the optimal solutions are $x_{1H}^* = x_{1L}^* = y_{1L}^* = 0$ and $x_{2L}^* = y_{2L}^* = 1$. With this strategy, we obtain the expected revenue of the two players for the state $(2,1,1)$ as: $V_1^* = 104.25$ and $V_2^* = 194.18$. Comparing these solutions with the two players’ expected revenue in Table 13, we see that even though $P_1$’s expected revenue is decreased, the expected revenue of $P_2$ is dramatically increased. As a result, the total expected revenue of the two players is increased more than $50$ on average, which account for about 17% of the total revenue in the non-cooperative situation. Thus, if there is an appropriate agreement on the compensation between the two players, such improvement indicates that the cooperation is strongly recommended. ♦

We will next investigate the structural properties of $V(t, n_1, n_2)$. 

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Theorem 7 Under cooperation, for any state \((t, n_1, n_2)\) \((t \in [1, T] \text{ and } n_1, n_2 > 0)\), the objective function \(V(t, n_1, n_2)\) exhibits the following structural properties:

(i) \(V(t, n_1, n_2)\) is non-decreasing quasi-concave in \(n_1\) and \(n_2\);
(ii) \(V(t, n_1, n_2) - V(t - 1, n_1, n_2)\) is non-decreasing in \(t\) and non-decreasing in \(n_i\).

Proof. We will prove this Theorem by induction. First, let us prove the property (i). From (33), it is easy to verify that property (i) is satisfied by for any state \((1, n_1, n_2)\). Now, let us assume that property (i) is also valid for any \(n_1\) and \(n_2\) in period \(t - 1\) with the hope that such properties can be extended to period \(t\). Referring to expressions shown in Figure 20, the total expected revenue can also be written as

\[
V(t, n_1, n_2) = (1 - \lambda_{1L} - \lambda_{2L} - \lambda_{1H} - \lambda_{2H}) V(t - 1, n_1, n_2)
\]

\[
+ \lambda_{1L} \max \begin{cases} 
  r_{1L} + V(t - 1, n_1 - 1, n_2) \\
  r_{2L} + V(t - 1, n_1, n_2 - 1) \\
  V(t - 1, n_1, n_2) - c_{1L}
\end{cases}
\]

\[
+ \lambda_{2L} \max \begin{cases} 
  r_{2L} + V(t - 1, n_1, n_2 - 1) \\
  r_{1L} + V(t - 1, n_1 - 1, n_2) \\
  V(t - 1, n_1, n_2) - c_{2L}
\end{cases}
\]

\[
+ \lambda_{1H} \max \begin{cases} 
  r_{1H} + V(t - 1, n_1 - 1, n_2) \\
  r_{2H} + V(t - 1, n_1, n_2 - 1)
\end{cases}
\]

\[
+ \lambda_{2L} [r_{2H} + V(t - 1, n_1, n_2 - 1)]
\]

It is not difficult to find that \(V(t, n_1, n_2)\) is always a positive and linear combination of
five non-decreasing quasi-concave functions (by assumption). Hence, we can conclude that $V(t, n_1, n_2)$ for $\forall n_1, n_2 > 0$ is also non-decreasing quasi-concave in $n_1$ and $n_2$. By induction, the proof of property (i) is finished.

Next, we shall prove the property (ii) as stated above. In terms of the marginal expected revenue of $P_i$'s one room, we can transform equation (35) as

$$V(t, n_1, n_2) - V(t - 1, n_1, n_2) =$$

$$+ \lambda_{2H} (r_{2H} - \delta_2(t - 1, n_1, n_2)) + \lambda_{1H} \max \left\{ \begin{array}{l}
r_{1H} - \delta_1(t - 1, n_1, n_2) \\
r_2H - \delta_2(t - 1, n_1, n_2)
\end{array} \right\}$$

$$+ \lambda_{1L} \max \left\{ \begin{array}{l}
r_1L - \delta_1(t - 1, n_1, n_2) \\
r_2L - \delta_2(t - 1, n_1, n_2) \\
-c_{1L} + \lambda_{2L} \max \left\{ \begin{array}{l}
r_1L - \delta_1(t - 1, n_1, n_2) \\
r_2L - \delta_2(t - 1, n_1, n_2) \\
-c_{2L}
\end{array} \right\} \right\}$$

(36)

We note that each item on the RHS of (36) is non-decreasing in $n_1$ and $n_2$ since $\delta_i$ for state $(t - 1, n_1, n_2)$ is always non-increasing in $n_1$ and $n_2$. Hence, the LHS, $V(t, n_1, n_2) - V(t - 1, n_1, n_2)$ is also non-decreasing in $n_1$ and $n_2$. In other words, $V(t, n_1, n_2)$ is sub-modular in $(t, n_1)$ and $(t, n_2)$. Meanwhile, according to such sub-modularity of $V(t, n_1, n_2)$, we can see that $\delta_i(t - 1, n_1, n_2)$ is also non-increasing in $t$ which implies that the LHS of (36) is non-decreasing in $t$. Therefore, $V(t, n_1, n_2) - V(t - 1, n_1, n_2)$ is non-decreasing in $t$ and $n_i$, $i = 1, 2$. This completes the proof. □

Similar to the Theorem 5, the properties shown in the Theorem 7 implies the existences of some critical booking capacities and booking periods.
• For an given $t$ and $n_2$, there always exist a critical booking capacity, $\hat{n}_{1K}(t,n_2)$ ($K = L, H$), by which it is always optimal to 'assign' the $K$-fare class customer to $P2$ for $n_1 < \hat{n}_{1K}$; and $P1$ should accept it for $n_1 \geq \hat{n}_{1K}$.

• For an given $n_1$ and $n_2$, there always exist a critical booking period, $\hat{t}_{1K}(n_1,n_2)$ ($K = L, H$), by which it is always optimal to 'assign' the $K$-fare class customer to $P2$ for $t < \hat{t}_{1K}$; and $P1$ should accept it for $t \geq \hat{t}_{1K}$.

• There exists a critical state $(\hat{t}, \hat{n}_1, \hat{n}_2)$ by which for any "upper" state $((t, n_1, n_2) > (\hat{t}, \hat{n}_1, \hat{n}_2))$ the low-fare booking request should be rejected by both players.

Referring to the optimal decision rules described in Proposition 4, there are sixteen different combinations of the five decision variables in the cooperative situation. The optimal solution can be any of them. However, we can use the critical values in a specific state to summarize each of these five decisions, e.g., for the given state variables, $t$ and $n_2$, the decision on $P1$'s high-fare class booking request can be expressed as

$$X_{1H}(t,n_1,n_2) = \begin{cases} 1 \text{ (accept), if } n_1 < \hat{n}_{1K}(t,n_2); \\ 0 \text{ (reject), otherwise,} \end{cases}$$

where $\hat{n}_{1H}(t,n_2) = \min\{n_1 : \beta_1(t-1,n_1,n_2) > r_{2H} - r_{1H}\}$. Comparing with the condition for $X_{1H}(t,n_1,n_2)$ in Proposition 4, this expression looks nicer and more understandable. In addition, our numerical experiments show that using these critical values significantly decreases the computing time when calculating the optimal solutions on the computer.
Chapter 5
Static Game Model with Incomplete Information

Note that a very important assumption in the models established in Chapter 3 and Chapter 4 is complete information. One hotel knows all the necessary information (e.g., transfer rate, rejection cost, etc.) of both hotels except for the other hotel's decision on the booking limit. In other words, the two players' objective functions are common knowledge. Obviously, this is not always applicable in practice. Hence, we are going to relax this assumption and study single-period games with incomplete information. Under these game theoretic settings, the expected revenues of the two players are determined by a "chance move", about which the players are partially informed. In this context, we investigate the consequences for the players' expected revenues by varying the states of information on the outcome of the chance move. In general, the value of information for the player refers to the difference of his optimal payoffs with and without the information. In this chapter, we are going to study the value of different information. Specifically, we debate the following questions: 1) Is the value of information always positive in our games? and 2) what type of information is more valuable?

Information value theory is a rather well known subject in classical decision theory. The basic result for a zero-sum Bayesian game indicates that the information is always valuable (see Ponssard [23]). Gilboa and Lehrer [19] characterized the
functions that measure the value of information in optimization problems. However, for the non-zero-sum games, some studies indicate that the value of information may become negative in some special cases. Kamien et. al. [24] showed an instance where players might prefer dropping some payoff-relevant information in order to improve their equilibrium payoff. Neyman [38] investigated the reasons why a player might prefer a 'no information' game instead of a game with private information. Bassan et. al. [3] present the conditions under which having more information always improves all players' payoffs. According to the definitions of different types of information, we will examine their values when the chance move is incurred by the incomplete information of a specific parameter. In practice, we identify two parameters for our study. They are the rejection cost and transfer rate respectively. From our problem in Chapter 3, we see that the existence of transferred customers leads to competition between the two players. Therefore, transfer rates significantly affects one player's revenue especially when the booking requests in one hotel are 'rich' and the booking requests in the other one are 'poor'. In addition, we know that the transfer rate is a customer side parameter. Hotels can not totally control it by all means. Then, it is reasonable to assume there is incomplete information about the transfer rate. On the other hand, we note the rejection cost is another suitable parameter since it is normally incurred by loss of goodwill in hotel business. Therefore, it is also on the customers' side and it is important for the player's expected revenue. Our study, in this chapter, will assume these two parameters as the incomplete information to generate the chance move for the game of the two players.

Before analyzing our game model and optimal rationing policies, it is necessary
for us to introduce the preliminary considerations of information types. According to Levine and Ponssard [30], there are three distinct types of information that one player may acquire in the incomplete information game.

**Type 1: Secret information** *One player acquires the information, but the other players are ignorant of this fact and will not modify their strategies. This assumption is actually unreasonable if it is a dynamic game and there exists a unique Nash equilibrium in the 'no information' game. It is because the uninformed players might soon realize that they are playing a different game.*

**Type 2: Private information** *One player acquires the information and, though he is the only one informed, this fact is known to the other players. This type of information may have several effects on all player’s optimal decisions. First, the acquisition of information may give the opportunity to the informed player to use it against the uninformed players. Second, the uninformed players might also modify their own decisions and it may or may not benefit the informed player.*

**Type 3: Public information** *All players acquire the information and it is known to all players.*

At first sight, we might expect that secret information would be more valuable than private information, which in turn, would be more valuable than public information. Next, we will examine the Bayesian Nash equilibrium in each type of games by assuming the rejection cost $q_{1L}$ and transfer rate $u_{1L}$ as the incomplete information respectively.
5.1 The Value of Information when $q_{1L}$ is unknown

As discussed in Chapter 3, in a static game of complete information, a strategy for $P_i$ is his low-fare booking limit, $b_i\,L$. We shall use the same assumptions and notations presented in Chapter 3 for our static incomplete information games in this chapter. The normal-form representation of this two-player game of complete information can be written as $G = \{b_{1L}, b_{2L}; J_1, J_2\}$. We now want to develop the normal-form representation of the static incomplete information (Bayesian) game.

Let us assume that $P_i$'s objective function is $J_i(b_{1L}, b_{2L}; t_i)$, where $t_i$ is $P_i$'s type and belongs to the type space $T_i$. To simplify the problem, we assume that $P_1$'s rejection cost of his low-fare class customer is the only incomplete information which can be $q_{1L}$ with probability $\theta_q^1$ and $q_{2L}$ with probability of $\theta_q^2$ for $\theta_q^1 + \theta_q^2 = 1$ and $q_{1L} < q_{2L}$.

Let us first discuss the case when both players are uninformed.

5.1.1 Uninformed Game: Both Players are Uninformed of $q_{1L}$

We note that if both players are uninformed on the rejection cost of $P_1$, then each player has only one type which is $T_1 = \{t_{q1}\}$ and $T_2 = \{t_{q2}\}$. Referring to (4), $P_1$'s expected revenue obtained from the low-fare class customers is

$$J_{1L}(b_{1L}, b_{2L}; t_{q1}) = \int_0^{b_{1L}} r_{1L} x_{1L} f_{2L}(b_{2L}) f_{1L} \, dx_{1L}$$

$$+ \int_{b_{2L}}^{B_{2L}} \int_0^{b_{1L}} r_{1L} (x_{1L} + b_{1L} - M_{1L}) f_{1L} f_{2L} \, dx_{1L} \, dx_{2L}$$
where \( M_{1L} = b_{1L} - u_{2L} (x_{2L} - b_{2L}) \), \( E(q_{1L}) = \theta_1^1 q_{1L}^1 + \theta_2^2 q_{1L}^2 \), and \( B_{2L} = b_{2L} + (b_{1L} - x_{1L})/u_2L \). We find that \( P1 \)'s expected revenue obtained from the high-fare class customers is same as that in the complete information game, which is

\[
J_{1H}(b_{1H}, b_{2H}; t_{q1}) = \int_0^{b_{1H}} r_{1H} x_{1H} F_{2H}(b_{2H}) f_{1H} \, dx_{1H} \\
+ \int_{b_{2H}}^{B_{2H}} \int_0^{b_{1H}} r_{1H} (x_{1H} + b_{1H} - M_{1H}) f_{1H} f_{2H} \, dx_{1H} \, dx_{2H} \\
+ \int_{b_{2H}}^{B_{2H}} \int_{b_{1H}}^{\infty} [r_{1H} b_{1H} + q_{1H} (x_{1H} - M_{1H})] f_{1H} f_{2H} \, dx_{1H} \, dx_{2H} \\
+ \int_{B_{2H}}^{\infty} \int_0^{b_{1H}} [r_{1H} b_{1H} + q_{1H} (x_{1H} - M_{1H})] f_{1H} f_{2H} \, dx_{1H} \, dx_{2H} \\
+ \int_{b_{1H}}^{\infty} [r_{1H} b_{1H} - q_{1H} (x_{1H} - b_{1H})] F_{2H}(b_{2H}) f_{1H} \, dx_{1H},
\]

(38)

where \( M_{1H} = b_{1H} - u_{2H} (x_{2H} - b_{2H}) \), and \( B_{2H} = b_{2H} + (b_{1H} - x_{1H})/u_2H \). Note that \( P2 \)'s objective function does not involve \( q_{1L} \), we then obtain \( P2 \)'s expected revenue
\[ J_2(b_{1L}, b_{2L}; t_{q2}) = \sum_{K=L,H} \left\{ \int_0^{b_{2K}} r_{2K} x_{2K} F_{1K} (b_{1K}) f_{2K} \, dx_{2K} \\
+ \int_{b_{1K}}^{b_{1K}} \int_0^{b_{2K}} r_{2K} (x_{2K} + b_{2K} - M_{2K}) f_{2K} f_{1K} \, dx_{2K} \, dx_{1K} \\
+ \int_{b_{1K}}^{b_{1K}} \int_{b_{2K}}^{\infty} [r_{2K} b_{2K} - q_{2K} (x_{2K} - M_{2K})] f_{2K} f_{1K} \, dx_{2K} \, dx_{1K} \\
+ \int_{b_{1K}}^{\infty} \int_{b_{2K}}^{\infty} [r_{2K} b_{2K} - q_{2K} (x_{2K} - M_{2K})] f_{2K} f_{1K} \, dx_{2K} \, dx_{1K} \\
+ \int_{b_{2K}}^{\infty} [r_{2K} b_{2K} - q_{2K} (x_{2K} - b_{2K})] F_{1K} (b_{1K}) f_{2K} \, dx_{2K} \right\} \tag{39} \]

where \( i = 1, 2 \), \( b_{2H} = C_2 - b_{2L} \), \( M_{2K} = b_{2K} - u_{1K} (x_{1K} - b_{1K}) \), and \( B_{1K} = b_{1K} + (b_{2K} - x_{2K})/u_{1K} \). Then, the total expected revenue of \( P1 \) in this situation is

\[ J_1 = J_{1L}(b_{1L}, b_{2L}; t_{q1}) + J_{1H}(b_{1H}, b_{2H}; t_{q1}) \tag{40} \]

Meanwhile, the total expected revenue of \( P2 \) is

\[ J_2 = J_2(b_{1L}, b_{2L}; t_{q2}) \tag{41} \]

Naturally, we will turn to find the Nash equilibrium in this Bayesian game (Bayesian Nash equilibrium). Referring to (5), we obtain

\[ \frac{\partial J_1}{\partial b_{1L}} = V_1 = (r_{1L} + E(q_{1L})) \left[ \int_0^{b_{1L}} \int_{N_{2L}}^{\infty} f_{1L} f_{2L} \, dx_{2L} \, dx_{1L} + \tilde{F}_{1L} (b_{1L}) \right] \\
- (r_{1H} + q_{1H}) \left[ \int_0^{b_{1H}} \int_{N_{2H}}^{\infty} f_{1H} f_{2H} \, dx_{2H} \, dx_{1H} + \tilde{F}_{1H} (b_{1H}) \right], \tag{42} \]

where \( N_{2K} = b_{2K} + (b_{1K} - x_{1K})/u_{2K} \) (\( K = L, H \)) and \( E(q_{1L}) = \theta_1 q_{1L} + \theta_2 q_{1L}^2 \). As for
P2, the first order partial derivative of $J_2$ with respect to $b_{2L}$ is exactly same as that in the complete information game, which is

$$\frac{\partial J_2}{\partial b_{2L}} = V_2 = (r_{2L} + q_{2L}) \left[ \int_0^{b_{2L}} \int_{N_{1L}}^\infty f_{1L} f_{2L} \, dx_{2L} \, dx_{1L} + \tilde{F}_{2L}(b_{2L}) \right]$$

$$- (r_{2H} + q_{2H}) \left[ \int_0^{b_{2H}} \int_{N_{1H}}^\infty f_{1H} f_{2H} \, dx_{2H} \, dx_{1H} + \tilde{F}_{2H}(b_{2H}) \right],$$

(43)

where $N_{1K} = b_{1K} + (b_{2K} - x_{2K}) / u_{1K}$ ($K = L, H$).

We see that $V_1$ is also exactly same as that in the complete information game by substituting $E(q_1L)$ with $q_1L$. The uninformed game in this situation is not equivalent to the complete information game by assuming $q_1L$ in (4) as $E(q_1L)$. According to Lemmas 1, 2 and 3, we know each player’s objective function has the following structural properties:

1. $\Pi$’s objective function is strictly concave in $b_{iL}$ for $i = 1, 2$. (by Lemma 1)
2. $V_i = 0, i = 1, 2$, is a strictly decreasing curve in the $(b_{1L}, b_{2L})$ plane. (by Lemma 2)
3. The implicit derivative of $V_1 = 0$ with respect to $b_{1L}$ is always less than the implicit derivative of $V_2 = 0$ with respect to $b_{1L}$. (by Lemma 3)

All of these properties of the objective functions will lead to the existence of unique Bayesian Nash equilibrium in the uninformed game.

**Theorem 8** When each of the two players has an incomplete information of $q_{1L}$, the game admits a unique Nash equilibrium.
Proof. Referring to the properties of $J_i$, we find that one player’s best response function has the same structural properties as that in the complete information game. It implies that, in $(b_{1L}, b_{2L})$ plane, the two best response curves of the two players intersects and only intersects once. In other words, the Bayesian game admits a unique Nash equilibrium.

From Theorem 8, we know that if the two players are both uninformed, they will play a Nash game in order to maximize their total expected revenue. We denote $(b^*_{1L}, b^*_{2L})$ as the Nash equilibrium and $J^*_i$ the maximum expected revenue in this “uninformed” game. We see that the Nash solution pair is the intersection of $V_1 = 0$ and $V_2 = 0$ in the $(b_{1L}, b_{2L})$ plane. We now attempt to find the moving direction of $(b^*_{1L}, b^*_{2L})$ in the $(b_{1L}, b_{2L})$ plane as $E(q_{1L})$ varies. Also, we will examine the variation of the optimal expected revenue, $J^*_i$, on $E(q_{1L})$. These findings will help us analyze the conditions by which the value of information is positive (or negative).

Proposition 5 If the two players are both uninformed of $q_{1L}$ and the Nash solution of each player is greater than 0 ($b^*_{iL} > 0$, $i = 1, 2$), then $b^*_{1L}$ decreases and $b^*_{2L}$ increases as $E(q_{1L})$ decreases; and vice versa.

Proof. If $b^*_{1L}, b^*_{2L} > 0$, then they must satisfy (42) and (43). The derivative of $V_1$ with respect to $E(q_{1L})$ is

$$S_{1L} - db_{2L}/dE(q_{1L}) \left[ \sum_{K=L,H} u_{2K} (r_{1K} + q_{1K}) \int_0^{b_{1K}} f_{1K} f_{2K} (N_{2K}) \, dx_{1K} \right]$$
and the derivative of $V_2$ with respect to $E(q_{1L})$ is

\[-db_{1L}/dE(q_{1L}) \sum_{K=L,H} (r_{1K} + q_{1K}) \int_0^{b_{1K}} \frac{1}{u_{1K}} f_{1K} f_{2K} (N_{2K}) \, dx_{1K} \]

\[-db_{2L}/dE(q_{1L}) \sum_{K=L,H} (r_{1K} + q_{1K}) f_{1K} (b_{1K}) F_{2K} (b_{2K}) = 0,\]

We then obtain $db_{1L}^*/dE(q_{1L})$ and $db_{2L}^*/dE(q_{1L})$ by solving the two equations above. It is not difficult to find that $db_{1L}^*/dE(q_{1L}) > 0$ and $db_{2L}^*/dE(q_{1L}) < 0$. These imply that as $E(q_{1L})$ increases, $b_{1L}^*$ increases and $b_{2L}^*$ decrease; and vice versa.

**Remark 3** From Proposition 5, we note as $q_{1L}$ increases, the Nash solution pair move in the southeast direction in the $(b_{1L}, b_{2L})$ plane, which is identical to the result in the sensitivity analysis in Chapter 3. However, we find the total derivative of $J_1$ with respect to $E(q_{1L})$, which is (after some simplifications)

\[dJ_1/dq_{1L} = u_{2L} [(r_{1L} + E(q_{1L})) (S_{1L} - \bar{F}_{1L} (b_{1L}) F_{2L} (b_{2L}))] \, db_{2L}/dE(q_{1L})\]

\[-u_{2H} [(r_{1H} + q_{1H}) (S_{1H} - \bar{F}_{1H} (b_{1H}) F_{2H} (b_{2H}))] \, db_{2L}/dE(q_{1L})\]

\[-\int_{B_{2L}^*}^{\infty} \int_0^{\infty} (x_{1L} - M_{1L}) f_{1L} f_{2L} \, dx_{1L} \, dx_{2L}\]
is too complicated to analyze the structural properties of $J_1$ with respect to $E(q_{1L})$.
The monotonicity and concavity (or convexity) of optimal $J_1$ with respect to $q_{1L}$ is ambiguous. Similarly, we can also find that the structural properties of optimal $J_2$ on $q_{1L}$ is uncertain.

<table>
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<tr>
<th>$P_1$'s information on $q_{1L}$</th>
<th>Objective functions</th>
<th>Value of information</th>
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<tr>
<td>Secret information ($i = 1$)</td>
<td>$J_1(b_{11}^{1L}, b_{21}^{1L}; t_{q_{11}})$</td>
<td>$J_1(b_{12}^{1L}, b_{22}^{1L}; t_{q_{12}})$</td>
</tr>
<tr>
<td>Private information ($i = 2$)</td>
<td>$J_2(b_{11}^{2L}, b_{21}^{2L}; t_{q_{11}})$</td>
<td>$J_2(b_{12}^{2L}, b_{22}^{2L}; t_{q_{12}})$</td>
</tr>
<tr>
<td>Public information ($i = 3$)</td>
<td>$J_3(b_{11}^{3L}, b_{21}^{3L}; t_{q_{11}})$</td>
<td>$J_3(b_{12}^{3L}, b_{22}^{3L}; t_{q_{12}})$</td>
</tr>
</tbody>
</table>

Table 15. $P_1$’s objective function and corresponding strategies when $P_1$ receives different information.

We will next examine the optimal decisions of one player when he receives the information of $q_{1L}$. We then analyze the value of information of different types. We denote $P_1$’s types, relevant objective functions, and value of information when acquiring different information with the notations in Table 15. analogous to these notations, one might obtain $P_2$’s types, relevant objective functions, and value of information. First, let us assume $P_1$ received the information secretly. In this case, $P_2$’s strategy remains $b_{2L}^*$ since he assume the game is still a Nash game. However,
P1 might choose different booking limit in the two types against $b_{2L}^*$. We denote P1’s booking limit in type $t_{q1}^j$, $j = 1, 2$ as $b_{1L}^j$, $j = 1, 2$. Second, let assume P1 received private information. According to the definition, P2 might choose a strategy which is different with $b_{2L}^*$, however, he will use the same strategy for both type of games. We denote the booking limit of P2 in this situation as $b_{2L}^2$. On the other hand, P1 knows the chance move of the game, then he should adopt different strategies for the two types of games. We use $b_{1L}^j$ to present the booking limit of P1 in type $t_{q1}^j$, $j = 1, 2$. At last, when both players knows the chance move of the game (public information case), we use $b_{1L}^{j3}$ to present the booking limit of $P_i (i = 1, 2)$ in type $t_{q1}^j$, $(j = 1, 2)$.

5.1.2 Secret Information Game : One Player Acquires $q_{1L}$ Secretly

Now, we will examine value of secret information by examining the one player’s objective functions and corresponding strategies.

**Proposition 6** The value of secret information of $q_{1L}$ is always positive for P1 if $b_{1L}^* > 0$, and it is always zero for P2.

**Proof.** When P1 receives the secret information of $q_{1L}$, he will make a decision on the booking limit against $b_{2L}^*$ according to his best response function in both types of games. Then, the total expected revenue is

$$J_1^1 = \theta_q^1 J_1^1 (b_{1L}^1, b_{2L}^*, t_{q1}^1) + \theta_q^2 J_1^1 (b_{1L}^2, b_{2L}^*, t_{q1}^2).$$

(44)

We see that the optimal booking limit for P1 in type $t_{q1}^j$ $(j = 1, 2)$, $b_{1L}^{j1*}$, can be
obtained according to the best response function (in this case the BR function is $V_i = 0$ since $b^*_i > 0$). Referring to Proposition 5, we also know that $b^{1*}_{1L} > b^*_{1L}$ and $b^{2*}_{1L} < b^*_{1L}$, since $q^1_{1L} > E(q_{1L}) > q^2_{1L}$. Thus, it turns out

$$J^1_J(b^{1*}_{1L}, b^{*}_{2L}; t^{*}_{q1}) > J^*_J(b^*_{1L}, b^*_{2L}; t_{q1}),$$

for $j = 1, 2$, which indicates the value of secret information of $q_{1L}$ for $P1$

$$\omega^1_{q1} = J^1_J - J^*_J > 0.$$

Then the value of the secret information of $q_{1L}$ for $P1$ is always positive.

As for $P2$, we know that $q_{1L}$ does not play a role on expected revenue. When receiving secret information, his objective functions in the two types exhibit exactly the same form. It implies that the game ($P2$ acquires secret information of $q_{1L}$) is equivalent to the uninformed game in each type. Thus,

$$b^{1*}_{2L} = b^{2*}_{2L} = b^*_{2L},$$

which indicates the value of secret information of $q_{1L}$ for $P2$,

$$\omega^1_{q2} = J^1_J - J^*_J,$$

is always zero. ■

5.1.3 Private Information Game: One Player Receives Private Information of $q_{1L}$

Next, we consider the case when one player acquires the private information on $q_{1L}$. Let us first assume $P1$ acquires the private information of $q_{1L}$. In this situation,
PI's type space is $T_1 = \{ t_{q_1}^{i_1}, t_{q_1}^{i_2} \}$ and $P2$'s type space is $T_2 = \{ t_{q_2}^{j_2} \}$. PI's expected revenue from low-fare customers in type $t_{q_1}^{i_2}$ ($j = 1, 2$) is

$$J_{1L}^2(b_{1L}^i, b_{2L}^j; t_{q_1}^{i_2}) = \int_0^{b_{1L}^i} r_{1L} x_{1L} F_{2L} (b_{2L}^j) f_{1L} dx_{1L}$$

$$+ \int_{b_{1L}^i}^{B_{1L}^j} \int_0^{b_{1L}^i} r_{1L} (x_{1L} + b_{1L}^i - M_{1L}^j) f_{1L} f_{2L} dx_{1L} dx_{2L}$$

$$+ \int_{b_{1L}^i}^{B_{1L}^j} \int_b^{b_{1L}^i} \int_0^{b_{1L}^i} [r_{1L} b_{1L}^j - q_{1L} (x_{1L} - M_{1L}^j)] f_{1L} f_{2L} dx_{1L} dx_{2L}$$

$$+ \int_{b_{1L}^i}^{b_{1L}^j} \int_0^{b_{1L}^i} \int_0^{b_{1L}^i} [r_{1L} b_{1L}^j - q_{1L} (x_{1L} - M_{1L}^j)] f_{1L} f_{2L} dx_{1L} dx_{2L}$$

$$+ \int_{b_{1L}^i}^{\infty} \int_0^{b_{1L}^i} \int_0^{b_{1L}^i} [r_{1L} b_{1L}^j - q_{1L} (x_{1L} - M_{1L}^j)] f_{1L} f_{2L} dx_{1L} dx_{2L}$$

$$+ \int_{b_{1L}^i}^{\infty} \int_0^{b_{1L}^i} \int_0^{b_{1L}^i} [r_{1L} b_{1L}^j - q_{1L} (x_{1L} - M_{1L}^j)] f_{1L} f_{2L} dx_{1L} dx_{2L}$$

$$+ \int_{b_{1L}^i}^{\infty} \int_0^{b_{1L}^i} \int_0^{b_{1L}^i} [r_{1L} b_{1L}^j - q_{1L} (x_{1L} - M_{1L}^j)] f_{1L} f_{2L} dx_{1L} dx_{2L}$$

$$+ \int_{b_{1L}^i}^{\infty} \int_0^{b_{1L}^i} \int_0^{b_{1L}^i} [r_{1L} b_{1L}^j - q_{1L} (x_{1L} - M_{1L}^j)] f_{1L} f_{2L} dx_{1L} dx_{2L}$$

(45)

where $M_{1L}^j = b_{1L}^j - u_{2L} (x_{2L} - b_{2L}^j)$, and $B_{2L}^j = b_{2L}^j + (b_{1L}^j - x_{1L})/u_{2L}$. Referring to (38), we can obtain PI's expected revenue from high-fare customers in type $t_{q_1}^{i_2}$ ($j = 1, 2$):

$$J_{1H}^2(b_{1H}^i, b_{2H}^j; t_{q_1}^{i_2}) = \int_0^{b_{1H}^i} r_{1H} x_{1H} F_{2H} (b_{2H}^j) f_{1H} dx_{1H}$$

$$+ \int_{b_{1H}^i}^{B_{1H}^j} \int_0^{b_{1H}^i} r_{1H} (x_{1H} + b_{1H}^i - M_{1H}^j) f_{1H} f_{2H} dx_{1H} dx_{2H}$$

$$+ \int_{b_{1H}^i}^{B_{1H}^j} \int_b^{b_{1H}^i} \int_0^{b_{1H}^i} [r_{1H} b_{1H}^j - q_{1H} (x_{1H} - M_{1H}^j)] f_{1H} f_{2H} dx_{1H} dx_{2H}$$

$$+ \int_{b_{1H}^i}^{\infty} \int_0^{b_{1H}^i} \int_0^{b_{1H}^i} [r_{1H} b_{1H}^j - q_{1H} (x_{1H} - M_{1H}^j)] f_{1H} f_{2H} dx_{1H} dx_{2H}$$

$$+ \int_{b_{1H}^i}^{\infty} \int_0^{b_{1H}^i} \int_0^{b_{1H}^i} [r_{1H} b_{1H}^j - q_{1H} (x_{1H} - M_{1H}^j)] f_{1H} f_{2H} dx_{1H} dx_{2H}$$

$$+ \int_{b_{1H}^i}^{\infty} \int_0^{b_{1H}^i} \int_0^{b_{1H}^i} [r_{1H} b_{1H}^j - q_{1H} (x_{1H} - M_{1H}^j)] f_{1H} f_{2H} dx_{1H} dx_{2H}$$

$$+ \int_{b_{1H}^i}^{\infty} \int_0^{b_{1H}^i} \int_0^{b_{1H}^i} [r_{1H} b_{1H}^j - q_{1H} (x_{1H} - M_{1H}^j)] f_{1H} f_{2H} dx_{1H} dx_{2H}$$

(46)
where $M_{1H}^j = b_{1H}^j - u_{2H} (x_{2H} - b_{2H}^j)$, $b_{1H}^j = C_1 - b_{1L}^j$, and $B_{2H}^j = b_{2H}^j + (b_{1H}^j - x_{1H})/u_{2H}$.

To sum up the two expression of (45) and (46), we obtain $P_1$'s objective function in type $t_{q1}^j$ ($j = 1, 2$):

$$J_1^j (b_{1L}^j, b_{2L}^j; t_{q1}^j) = J_1^j (b_{1L}, b_{2L}; t_{q1}) + J_1^j (b_{1H}, b_{2H}; t_{q1}).$$  \hspace{1cm} (47)

We see that $P_1$'s objective function in each type is exactly identical with that in the complete information game. Thus, the properties described in Lemmas 1, 2 hold.

However, $P_2$ has only one possible objective function:

$$J_2 (b_{1L}^2, b_{2L}^2, b_{2L}^2; t_{q2}) = \sum_{j=1,2} \theta^j \sum_{K=L,H} \int_0^{b_{2K}} \int_0^{b_{2K}} r_{2K} f_{1K} (b_{1K}^j) f_{2K} dx_{2K}$$

$$+ \int_{b_{1K}^2}^{b_{2K}} \int_0^{b_{2K}} r_{2K} (x_{2K} + b_{2K}^j - M_{2K}^j) f_{2K} f_{1K} dx_{2K} dx_{1K}$$

$$+ \int_{b_{1K}^2}^{b_{2K}} \int_{b_{2K}^2}^\infty [r_{2K} b_{2K}^2 - q_{2K} (x_{2K} - M_{2K}^j)] f_{2K} f_{1K} dx_{2K} dx_{1K}$$

$$+ \int_{b_{1K}^2}^{b_{2K}} \int_{b_{2K}^2}^\infty [r_{2K} b_{2K}^2 - q_{2K} (x_{2K} - b_{2K}^j)] f_{2K} f_{1K} dx_{2K} dx_{1K}$$

$$+ \int_{b_{2K}^2}^\infty [r_{2K} b_{2K}^2 - q_{2K} (x_{2K} - b_{2K}^j)] F_{1K} (b_{1K}^j) f_{2K} dx_{2K} \right \} \hspace{1cm} (48)$$

where $b_{2H}^2 = C_2 - b_{2L}^2$, $M_{2K}^2 = b_{2K}^2 - u_{1K} (x_{1K} - b_{1K}^2)$, and $B_{1K}^2 = b_{1K}^2 + (b_{2K}^2 - x_{2K})/u_{1K}$. Differentiating $J_2$ with respect to $b_{2L}^2$, we obtain
\[ V_2(b_{1L}^2, b_{1L}^2, b_{2L}^2; t_{q2}^2) = (r_{2L} + q_{2L}) \sum_{j=1,2} \theta_j^2 \int_0^{b_{2L}^2} f_{1L} f_{2L} dx_{2L} dx_{1L} + F_{2L} (b_{2L}^2) \]
\[ - (r_{2H} + q_{2H}) \sum_{j=1,2} \theta_j^2 \int_0^{b_{2L}^2} f_{1H} f_{2H} dx_{2H} dx_{1H} + F_{2H} (b_{2L}^2), \]

where \( N_{1K}^{j2} = b_{1K}^2 + (b_{2K}^2 - x_{2K}) / u_{1K} \) \((K = L, H)\) and \( b_{2H}^2 = C_2 - b_{2L}^2 \). Differentiating \( V_2(b_{1L}^2, b_{1L}^2, b_{2L}^2; t_{q2}^2) \) with respect to \( b_{2L}^2 \), we obtain the second order derivative of \( J_2 \) with respect to \( b_{2L}^2 \), which is

\[ - \sum_{j=1,2} \theta_j^2 \sum_{K=L,H} (r_{2K} + q_{2K}) \left[ \int_0^{b_{2K}^2} \frac{1}{u_{2K}} f_{2K} f_{1K} \left( N_{1K}^{j2} \right) dx_{1K} + f_{2K} (b_{2K}^2) F_{1K} (b_{1K}^{j2}) \right] < 0. \] (49)

Therefore, \( J_2(b_{1L}^2, b_{1L}^2, b_{2L}^2; t_{q2}^2) \) is strictly concave in \( b_{2L}^2 \). Furthermore, we find the implicit derivative of \( V_2 = 0 \) with respect to \( b_{1L}^{j2} \) is

\[ b_2^2 = \frac{-\theta_j^2 \sum_{K=L,H} u_{1K} (r_{2K} + q_{2K}) \int_0^{b_{2K}^2} f_{1K} \left( N_{1K}^{j2} \right) f_{2K} dx_{2K}} {\sum_{K=L,H} (r_{2K} + q_{2K}) \left[ \int_0^{b_{2K}^2} \frac{1}{u_{2K}} f_{1K} \left( N_{1K}^{j2} \right) f_{2K} dx_{2K} + f_{2K} (b_{2K}^2) F_{1K} (b_{1K}^{j2}) \right]}, \] (50)

and \(-1 < b_2^2 < 0. \) (49) and (50) imply that the properties described in Lemmas 1, 2 also hold for \( P_2 \) in this situation. Comparing \( b_2^2 \) with the implicit derivative of \( V_1 = 0 \) in each type, we find that \( b_2^2 \) is always greater than the derivative of \( V_1 = 0 \) with respect to \( b_{1L}^{j2} \) in type \( t_{q1}^{j2} \) \((j = 1, 2)\).

All of the results obtained above lead to the existence of a unique Bayesian Nash equilibrium.
Theorem 9  When $P_1$ has the complete information of $P_2$ and himself, and $P_2$ knows only that $P_1$'s rejection cost of low-fare class customer is $q_{1L}^1$ with probability $\theta_q^1$ and $q_{1L}^2$ with probability of $\theta_q^2$ for $\theta_q^1 + \theta_q^2 = 1$ and $q_{1L}^2 > q_{1L}^1$, the game admits a unique Bayesian Nash equilibrium $(b_{1L}^{1*}, b_{1L}^{2*}, b_{2L}^{2*})$. In addition, $b_{1L}^{1*} \geq b_{1L}^{12*}$ and $J_1^2(b_{1L}^{12*}, b_{2L}^{2*}, r_{1L}) > J_1^2(b_{1L}^{12*}, b_{2L}^{2*}, r_{1L})$

Proof. Referring to the results we discussed above, we find that $P_1$'s best response function in each type has the same structural properties as those in complete information game. On the other hand, $P_2$'s best response function to $P_1$'s strategy in one type given the strategy in the other type is known also has the same properties as those in the complete information game. Therefore, in $(b_{1L}^j, b_{2L}^j)$ $(j = 1, 2)$ plane, the two best response curves of the two players intersects and only intersects once. It implies that the Bayesian game admits a unique Nash equilibrium $(b_{1L}^{1*}, b_{1L}^{2*}, b_{2L}^{2*})$.

According to (8), we see that if $b_{2L}$ is given, i.e., $b_{2L} = b_{2L}^{2*}$, and $b_{1L} = b_{1L}^{12*} > 0$, then $b_{1L}^{12*}$ and $b_{2L}^{2*}$ must satisfy

$$ (r_{1L} + q_{1L}^1) S_{1L} - (r_{1H} + q_{1H}) S_{1H} = 0. $$

Moreover, we know that the spill rate of a low-fare customer, $S_{1L}$, decreases as $b_{1L}$ increases, and vice versa. Thus, as $q_{1L}$ increases from $q_{1L}^1$ to $q_{1L}^2$, $S_{1L}$ should decrease while $S_{1H}$ should increase. It leads to the increase of the optimal booking limit: $b_{1L}^{22*} > b_{1L}^{12*}$. However, we know that the optimal booking limit, $b_{1L}^{22*}$, is zero if

$$ V_1(0, b_{2L}^{2*}; q_{1L}^2) = (r_{1L} + q_{1L}^2) - (r_{1H} + q_{1H}) S_{1H}(0, b_{2L}^{2*}) \leq 0. $$
In this case, as \( q_{1L} \) decreases from \( q_{2L}^1 \) to \( q_{1L}^1, \) \( V_1(0, b_{1L}^*; t_{q1}^1) \) is still less than zero. It implies that \( b_{1L}^{12*} = b_{1L}^{22*} = 0. \) Thus, in general, \( b_{1L}^{2*} \geq b_{1L}^{1*}. \)

Differentiating \( J_1 \) with respect to \( q_{1L}, \) we obtain

\[
d J_1/dq_{1L} = [(r_{1L} + q_{1L}) S_{1L} - (r_{1H} + q_{1H}) S_{1H}] \ db_{1L}/dq_{1L} \\
- \int_{b_{2L}}^{b_{1L}} \int_0^\infty (x_{1L} - M_{1L}) f_{1L} f_{2L} \ dx_{1L} \ dx_{2L} \\
- \int_{b_{2L}}^{b_{1L}} \int_0^\infty (x_{1L} - M_{1L}) f_{1L} f_{2L} \ dx_{1L} \ dx_{2L} \\
- \int_{b_{1L}}^\infty (x_{1L} - b_{1L}) F_{2L} (b_{2L}^*) \ f_{1L} \ dx_{1L},
\]

where \( b_{2L} \) is assumed to be constant. Referring to Proposition 5, we know \( db_{1L}/dq_{1L} > 0. \) Then the first item on the RHS of (51) is less than or equal to zero since \( V_1 \leq 0. \) Thus, \( d J_1/dq_{1L} < 0, \) which indicates that the maximum expected revenue decreases as \( q_{1L} \) increases: \( J_1^2(b_{1L}^{12*}, b_{2L}^{2*}; t_{q1}^2) > J_1^1(b_{1L}^{22*}, b_{2L}^{2*}; t_{q1}^2). \)

We note that the Bayesian Nash equilibrium \((b_{1L}^{12*}, b_{2L}^{2*}, b_{2L}^{2*})\) can be obtained by solving the BR functions of \( P1 \) in the two types and \( P2 \)’s BR function. Thus, the value of private information of \( q_{1L} \) for \( P1 \) can be expressed as

\[
\omega_{q1}^2 = \theta_{q1}^1 J_1^2(b_{1L}^{12*}, b_{2L}^{2*}; t_{q1}^2) + \theta_{q1}^2 J_1^2(b_{1L}^{22*}, b_{2L}^{2*}; t_{q1}^2) - J_1^1(b_{1L}^*, b_{2L}^*; q_{1L}^1).
\]

Figure 21 depicts the respective positions of the total expected revenue in the \((q_{1L}, J_1)\) plane. Consequently, the value of private information is always positive if the two points, \((q_{1L}^1, J_1^2(b_{1L}^{12*}, b_{2L}^{2*}; t_{q1}^2))\) and \((q_{1L}^2, J_1^2(b_{1L}^{22*}, b_{2L}^{2*}; t_{q1}^2))\), are both above
Figure 21. Value of private information of \( q_{1L} \) for \( P1 \).

the curve \( J_1(b_{1L}^*(q_{1L}), b_{2L}^*(q_{1L})) \). However, the value of private information is always negative if \((q_{1L}^1, J_1^*(b_{1L}^{12*}, b_{2L}^{12*}, t_{q_{1L}}))\) and \((q_{1L}^2, J_1^*(b_{1L}^{22*}, b_{2L}^{22*}, t_{q_{1L}}))\) are both below the curve \( J_1(b_{1L}^*(q_{1L}), b_{2L}^*(q_{1L})) \). In addition, it is also possible that \((q_{1L}^1, J_1^*(b_{1L}^{12*}, b_{2L}^{12*}, t_{q_{1L}}))\) is above \( J_1 \) and \((q_{1L}^2, J_1^*(b_{1L}^{22*}, b_{2L}^{22*}, t_{q_{1L}}))\) is below \( J_1 \). In this situation, \( \theta_q^1 \) (or \( \theta_q^2 \)) plays a role on the value of private information. It is not difficult to find that the value of private information is positive if \((E(q_{1L}), J_1^*)\) is above \( J_1(b_{1L}^*(q_{1L}), b_{2L}^*(q_{1L})) \); and it is negative if \((E(q_{1L}), J_1^*)\) is below \( J_1(b_{1L}^*(q_{1L}), b_{2L}^*(q_{1L})) \).

Next, we will in turn discuss the case when \( P2 \) receives the private information. In this situation, \( P2 \)'s type space is \( T_2 = \{t_{q_{21}}^{12}, t_{q_{21}}^{22}\} \) and \( P1 \)'s type space is \( T_1 = \{t_{q_{11}}^{2}\} \). P2's objective function in type \( t_{q_{2j}}^j \) \( (j = 1, 2) \) is
\[ J_2^2(b_{1L}, b_{2L}; t_{q2}) = \sum_{K=L,H} \left\{ \int_0^{B_{1K}} \int_0^{B_{2K}} r_{2K} x_{2K} F_{1K}(b_{1K}^2) f_{2K} \, dx_{2K} \right. \\
+ \int_{B_{1K}}^{B_{1K}} \int_0^{B_{2K}} r_{2K} (x_{2K} + b_{2K}^2 - M_{2K}^{j2}) f_{2K} f_{1K} \, dx_{2K} \, dx_{1K} \\
+ \int_{B_{1K}}^{\infty} \int_0^{B_{2K}} [r_{2K} b_{2K}^2 - q_{2K} (x_{2K} - M_{2K}^{j2})] f_{2K} f_{1K} \, dx_{2K} \, dx_{1K} \\
+ \int_{B_{1K}}^{\infty} \int_0^{\infty} [r_{2K} b_{2K}^2 - q_{2K} (x_{2K} - b_{2K}^2)] f_{2K} f_{1K} \, dx_{2K} \, dx_{1K} \\
+ \int_{B_{2K}}^{\infty} [r_{2K} b_{2K}^2 - q_{2K} (x_{2K} - b_{2K}^2)] F_{1K}(b_{1K}^2) f_{2K} \, dx_{2K} \right\} . \]

Note that \( J_2^2(b_{1L}, b_{2L}; t_{q2}) \) is identical to the objective function in the uninformed game. It is because \( q_{1L} \) does not play a role on \( P2 \)'s objective function, he then will use the same strategy in the two types to against \( b_{1L}^2 \). It indicates that

\[ b_{2L}^{12} = b_{2L}^{22} . \]

Thus, \( P1 \)'s objective function is also identical to that in the uninformed case (see (40)). Accordingly, we conclude that the game when \( P2 \) acquires the private information of \( q_{1L} \) is identical to the uninformed game and the value of private information of \( q_{1L} \) for \( P2, \omega_{q2}^2 \), is always zero.

5.1.4 Public Information Game: \( q_{1L} \) is Known to Both Players

At last, we are going to discuss the case in which the two players receive public
information of $q_{1L}$. Each player has two types. The type spaces of $P1$ and $P2$ are $T_1^3 = \{t_{q1}^{13}, t_{q2}^{23}\}$ and $T_2^3 = \{t_{q2}^{13}, t_{q2}^{23}\}$ respectively. In this situation, both players have two objective functions. Note that in each type, one player has complete information of both players. Thus, one player’s objective function in each type is identical to (4).

Then, we have the following conclusions.

**Theorem 10** When the two players receive the public information of $q_{1L}$, there exist a unique Nash equilibrium in each type of games, which is $(b_{1L}^{13*}, b_{2L}^{13*})$ when $q_{1L} = q_{1L}^1$; and $(b_{1L}^{23*}, b_{2L}^{23*})$ when $q_{1L} = q_{1L}^2$. In addition, if $q_{1L}^1 < q_{1L}^2$, then $b_{1L}^{13*} \leq b_{1L}^{23*}$ and $b_{2L}^{23*} \leq b_{2L}^{13*}$.

**Proof.** When the two players receive public information, in each chance move, the two players play a complete information game. Referring to Theorem 2, the game admits a unique Nash equilibrium. Furthermore, we have known that the optimal booking limit of $b_{1L}$ is non-decreasing and $b_{2L}$ is non-increasing as $q_{1L}$ increases (see Proposition 5). Accordingly, we have $b_{1L}^{13*} \leq b_{1L}^{23*}$ and $b_{2L}^{23*} \leq b_{2L}^{13*}$ if $q_{1L}^1 < q_{1L}^2$. 

Due to the existence of the Nash equilibrium, we can calculate the optimal booking limits of the two players in each chance move. Thus, the value of public information of $q_{1L}$ for $P1$ can be expressed as

$$\omega_{q1}^3 = \theta_q^1 J_1^3 (b_{1L}^{13*}, b_{1L}^{23*}, t_{q1}^1) + \theta_q^2 J_1^3 (b_{1L}^{23*}, b_{2L}^{23*}, t_{q1}^3) - J_1^3 (b_{1L}^{*}, b_{2L}^{*}, t_{q1})$$

and the value of public information of $q_{1L}$ for $P1$ can be expressed as

$$\omega_{q1}^3 = \theta_q^1 J_2^3 (b_{1L}^{13*}, b_{2L}^{13*}, t_{q2}^1) + \theta_q^2 J_2^3 (b_{1L}^{23*}, b_{2L}^{23*}, t_{q2}^3) - J_2^3 (b_{1L}^{*}, b_{2L}^{*}, t_{q2})$$
Remark 4 From Figure 22, we see that the sign of the value of public information for P1 depends on the position of the point \((E(q_{1L}), J^*_1)\). The value of public information for P1 is positive if the point is above the curve \(J^*_1\); and it is negative if the point is below \(J^*_1\). Note that \(\theta^1_q\) (or \(\theta^2_q\)) also affects the value of public information. As \(\theta^1_q\) varies from 0 to 1, \(E(q_{1L})\) increases from \(q^1_{1L}\) to \(q^2_{1L}\). Then, the corresponding total expected revenue of P1 will decreases from \(J^3_1(b_{11L}^{13*}, b_{12L}^{13*}, t_{q1})\) to \(J^3_1(b_{21L}^{23*}, b_{22L}^{23*}, t_{q1})\) along the line. Consequently, it is possible that the value of public information is negative for some ranges of \(\theta^1_q\) and it is positive for other ranges of \(\theta^1_q\). \(\triangleleft\)

We have seen the secret and private information of \(q_{1L}\) are not valuable for P2. However, it is interesting to see that the public information might benefit P2.
in some situations. Even though the relationship between $J_2^*$ and $q_{1L}$ is uncertain (see Remark 3); we know that the value of public information for $P_2$ is positive if $\theta_q^1 J_2^* (b_{1L}, b_{2L}; q_2) + \theta_q^2 J_2^* (b_{1L}, b_{2L}; t_{q2})$ is greater than $J_2^* (b_1^*, b_2^*; t_q)$. In addition, the role of $\theta_q^1$ on the sign of $\omega_{q2}^3$ is similar to the case for $P_1$ (see Remark 4).

Next, we will provide a numerical example to demonstrate the important results obtained in this section.

**Example 10** In this example, we use the same values as in Table 3 of Example 1 in Chapter 3 for prices and transfer rates. In this case, we assume $q_{1L}^1 = 10$ with probability of 0.5 and $q_{1L}^2 = 60$ with probability of 0.5. First, we attempt to examine the values of the different information and their relations. We set the booking request expectation of K-fare class in each hotel to be a lower value number (10 for low-fare class, and 5 for high-fare class) and a high number (60 for low-fare class, and 35 for high-fare class). We then generate 16 scenarios, by which we calculate the optimal solutions when the two players acquire different information on $q_{1L}$. The relationships among the values of different information for $P_1$ is shown in Figure 23.

It can be seen that the values of secret and private information dominate the value of public information in most cases (region $R_1$ and $R_2$). However, we find that when $\lambda_{1L}$ and $\lambda_{2H}$ are high while $\lambda_{2L}$ and $\lambda_{1H}$ are low (region $R_3$), the value of public information dominates the value of private information. In addition, when $\lambda_{1L}$ is high while $\lambda_{2L}$, $\lambda_{1H}$ and $\lambda_{2H}$ are low (region $R_4$), the value of public information dominates the values of both secret and private information. Next, we examine the value of different information of $q_{1L}$ for $P_2$. As expected, the values of secret and
Figure 23. Relations of value of different information of $q_{1L}$ for $P1$. In region $R_1$, $\omega_{q1}^2 > \omega_{q1}^1 > \omega_{q1}^3$; in region $R_2$, $\omega_{q1}^1 > \omega_{q1}^2 > \omega_{q1}^3$; in region $R_3$, $\omega_{q1}^1 > \omega_{q1}^3 > \omega_{q1}^2$ and in region $R_4$, $\omega_{q1}^3 > \omega_{q1}^2 > \omega_{q1}^1$.

private information for $P2$ are always zero in any scenarios. We then calculate the values of public information in each scenario and the findings are summarized in Figure 24. Note that when $P2$’s booking request expectation of high-fare class customer is lower (region $D_2$), $P2$ is more likely benefit from the public information of $q_{1L}$. On the other hand, when his booking request expectations of the two fare class customers are both high (region $D_1$), the value of public information of $q_{1L}$ is negative. We also find that in some scenarios, the sign of $\omega_{q2}^3$ change if we vary $\theta_q^1$. For example, when we set $\lambda_{1H} = \lambda_{2H} = 10$ and $\lambda_{1H} = \lambda_{2H} = 35$, if $\theta_q^1$ is changed from 0.5 to 0.8, $\omega_{q2}^3$ will change from $\omega_{q2}^3 < 0$ to $\omega_{q2}^3 > 0$. This verifies our conclusion presented in Remark 4.

5.2 The Value of Information when $u_{1L}$ is unknown

Note that we have analyzed the value of information of $q_{1L}$ for the two players in previous section. Now, we will assume another important parameter as the
incomplete information, i.e., the transfer rate. Clearly, in each information case, the game model will be different since one player’s transfer rate only plays a role in the other player’s objective function. Again, we assume $P1$’s transfer rate of his low-fare class customer is the only incomplete information which can be $u^1_{1L}$ with probability $\theta^1_u$ and $u^2_{1L}$ with probability of $\theta^2_u$ for $\theta^1_u + \theta^2_u = 1$. Let us begin from the uninformed case.

5.2.1 Uninformed Game: Both Players are Uninformed of $u_{1L}$

When both players are uninformed of $u_{1L}$, each player has only one type which is $T_1 = \{t_{u1}\}$ and $T_2 = \{t_{u2}\}$. We see that $u_{1L}$ does not affect the expected revenue of $P1$ in any fare class (see (4)). Thus, the objective function in this case is exactly identical to (4) of the complete information game. However, $u_{1L}$ plays a role on the
expected revenue of $P2$. Thus, we have $P2$'s objective function as

$$J_2(b_{1L}, b_{2L}; t_{u2}) = \sum_{j=1,2} \theta_u^j \left\{ \int_0^{b_{2L}} r_{2L} x_{2L} F_{1L}(b_{1L}) f_{2L} \, dx_{2L} ight\}$$

$$+ \int_{b_{1L}}^{B_{1L}} \int_0^{b_{2L}} r_{2L} (x_{2L} + b_{2L} - M_{2L}) f_{2L} f_{1L} \, dx_{2L} \, dx_{1L}$$

$$+ \int_{b_{1L}}^{B_{1L}} \int_{b_{2L}}^{\infty} \left[ r_{2L} b_{2L} - q_{2L} (x_{2L} - M_{2L}^j) \right] f_{2L} f_{1L} \, dx_{2L} \, dx_{1L}$$

$$+ \int_{B_{1L}}^{\infty} \int_0^{\infty} \left[ r_{2L} b_{2L} - q_{2L} (x_{2L} - M_{2L}^j) \right] f_{2L} f_{1L} \, dx_{2L} \, dx_{1L}$$

$$+ \int_{b_{2L}}^{B_{2L}} \int_0^{\infty} \left[ r_{2L} b_{2L} - q_{2L} (x_{2L} - b_{2L}) \right] F_{1L}(b_{1L}) f_{2L} \, dx_{2L}$$

$$+ \int_{0}^{B_{2L}} r_{2L} x_{2L} F_{1L}(b_{1L}) f_{2L} \, dx_{2L}$$

$$+ \int_{B_{1H}}^{\infty} \int_0^{\infty} \left[ r_{2H} b_{2H} - q_{2H} (x_{2H} - M_{2H}) \right] f_{2H} f_{1H} \, dx_{2H} \, dx_{1H}$$

$$+ \int_{b_{1H}}^{B_{1H}} \int_{b_{2H}}^{\infty} \left[ r_{2H} b_{2H} - q_{2H} (x_{2H} - M_{2H}) \right] f_{2H} f_{1H} \, dx_{2H} \, dx_{1H}$$

$$+ \int_{B_{1H}}^{\infty} \int_0^{\infty} \left[ r_{2H} b_{2H} - q_{2H} (x_{2H} - M_{2H}) \right] f_{2H} f_{1H} \, dx_{2H} \, dx_{1H}$$

$$+ \int_{b_{2H}}^{B_{2H}} \int_0^{\infty} \left[ r_{2H} b_{2H} - q_{2H} (x_{2H} - b_{2H}) \right] F_{1H}(b_{1H}) f_{2H} \, dx_{2H}$$

(55)

where $b_{2H} = C_2 - b_{2L}$, $M_{2L}^j = b_{2L} - u_{1L}^j (x_{1L} - b_{1L})$, $M_{2H} = b_{2H} - u_{1H} (x_{1H} - b_{1H})$, $B_{1L} = b_{1L} + (b_{2L} - x_{2L})/u_{1L}^j$ and $B_{1H} = b_{1H} + (b_{2H} - x_{2H})/u_{1H}$ for $j = 1, 2$. Similar to the uninformed case in previous section, we will attempt to find whether this game has a Nash strategy. Since $P1$'s objective function is identical to that of com-
plete information game. Therefore, his objective function has the same structural properties as described 3. We then focus on the P2’s objective function. Partially differentiating $J_2$ with respect to $b_2L$, we obtain

$$V_2 = (r_{2L} + q_{2L}) \left[ \int_0^{b_{2L}} (\theta_u^1 \int_{N_{1L}^1}^{\infty} f_{1L} dx_{1L} + \theta_u^2 \int_{N_{1L}^2}^{\infty} f_{1L} dx_{1L}) f_{2L} dx_{2L} + \bar{F}_{2L}(b_{2L}) \right]$$

$$- (r_{2H} + q_{2H}) \left[ \int_0^{b_{2H}} \int_{N_{1H}}^{\infty} f_{1H} f_{2H} dx_{2H} dx_{1H} + \bar{F}_{2H}(b_{2H}) \right],$$

where $N_{1L}^j = b_{1L} + (b_{2L} - x_{2L})/u_{1L}^j (j = 1, 2)$ and $N_{1H} = b_{1H} + (b_{2H} - x_{2H})/u_{1H}$. We find the second order derivative of $J_2$ with respect to $b_2L$,

$$- (r_{2L} + q_{2L}) \left[ \int_0^{b_{2L}} f_{2L} \left( \frac{\partial^2}{\partial u_{1L}^1} f_{1L}(N_{1L}^1) + \frac{\partial^2}{\partial u_{1L}^2} f_{1L}(N_{1L}^2) \right) dx_{1L} + f_{1L}(b_{1L}) F_{2L}(b_{2L}) \right]$$

$$- (r_{2H} + q_{2H}) \left[ \int_0^{b_{2H}} f_{2H} f_{1H} (N_{1H}) dx_{1H} + f_{1H}(b_{1H}) F_{2H}(b_{2H}) \right],$$

is always less than zero, which implies $J_2$ is strictly concave in $b_2L$. With further investigation, we find that $V_2(b_{1L}, b_{2L}; t_{u2}) = 0$ is a strictly decreasing curve in the $(b_{1L}, b_{2L})$ plane, and the implicit derivative of $V_1(b_{1L}, b_{2L}; t_{u1}) = 0$ with respect to $b_{1L}$ is always less than the implicit derivative of $V_2(b_{1L}, b_{2L}; t_{u2}) = 0$ with respect to $b_{1L}$.

All of these results obtained above lead to the existence of a unique Nash equilibrium in the uninformed game.

**Theorem 11** When each of the two players has an incomplete information of $u_{1L}$, the game admits a unique Nash equilibrium.
Due to the existence of the unique Nash equilibrium in the uninformed game, the two players will more likely play the Nash strategy in this case. Note that the uninformed game in this case is not equivalent to the complete information game by assuming $u_{1L}$ in (4) as $E(u_{1L})$, which is applicable when $q_{1L}$ is uninformed. This will make the analysis of the value of information even more difficult. We will next discuss the optimal strategy and the corresponding expected revenue when one player acquires the information of $u_{1L}$.

5.2.2 Secret Information Game: One Player Acquires $u_{1L}$ Secretly

We start our analysis of the value of information by assuming one player receives the secret information of $u_{1L}$.

**Proposition 7** The value of secret information of $u_{1L}$ is always zero for $P1$; and it is always non-negative for $P2$. Specially, the value of secret information of $u_{1L}$ is always positive for $P2$ if $b_{2L}^* > 0$.

**Proof.** When $P1$ acquires secret information of $u_{1L}$, his objective function in each type is totally identical to that of uninformed game since $u_{1L}$ has no effect on his expected revenue in any fare class. And, we also know that $P2$’s booking limit remains $b_{2L}^*$. Thus, it is optimal for $P1$ to adopt $b_{1L}^*$ in each chance move to against $b_{2L}^*$. It indicates that the value of secret information of $u_{1L}$ is meaningless for $P1$. On
the other hand, if $P2$ receives the secret information of $u_{1L}$, he will choose $b_{2L}^{j1}$ in type $t_{u2}^j$ ($j = 1, 2$), which is probably different to $b_{2L}^{*}$. It is because the first order derivative of $J_2$ in each type is different with that of uninformed game (see 56). Meanwhile, we note that if there exist a feasible solution of $b_{2L}^{*}$ ($b_{2L}^{*} > 0$) which satisfied $V_2 = 0$, the optimal booking limit in each type, $b_{2L}^{1*}$, must be different with $b_{2L}^{*}$. It incurs an higher expected revenue than $J_2^{*}$. Thus, the total expected revenue of $P2$ with secret information is

$$\theta^1_uJ_2^{1*}(b_{1L}^{*}, b_{2L}^{1*}, t_{u2}^1) + \theta^1_uJ_2^{1*}(b_{1L}^{*}, b_{2L}^{*}, t_{u2}^2),$$

which is greater than $J_2(b_{1L}^{*}, b_{2L}^{*}, t_{u2})$. It indicates that the value of secret information of $u_{1L}$ is positive for $P2$ in this situation. In general, we have $\omega_{u2}^1 \geq 0$.

Similar to the secret information case, the objective functions of $P1$ with private information of $u_{1L}$ are same as (4) in each type. Then, the games in the two types are played as if both players are uninformed. Thus, $P1$ also can not benefit from the private information of $u_{1L}$.

5.2.3 Private Information Game: One Player Receives Private Information of $u_{1L}$

In this situation, $P2$'s objective functions in the two types are different when acquiring private information of $u_{1L}$. We can obtain $P2$'s objective function in each type by simply substituting $u_{1L}$ with $u_{1L}^j$ ($j = 1, 2$) and $b_{2L}$ with $b_{2L}^{j2}$ in (4). As for
P1, his objective function in this case can be expressed as

\[
J_1 = \sum_{j=1,2} \theta_u^j \sum_{K=L,H} \left\{ \int_0^{b_{2K}^j} r_{1K} x_{1K} F_{2K} \left( b_{2K}^{j2} \right) f_{1K} dx_{1K} \\
+ \int_{b_{2K}^j}^{B_{2K}^{j2}} \int_0^{b_{1K}^j} r_{1K} (x_{1K} + b_{1K}^j - M_{1K}^{j2}) f_{1K} f_{2K} dx_{2K} dx_{1K} \\
+ \int_{b_{2K}^j}^{B_{2K}^{j2}} \int_{b_{1K}^j}^{\infty} \left[ r_{1K} b_{1K}^j - q_{1K} (x_{1K} - M_{1K}^{j2}) \right] f_{2K} f_{1K} dx_{2K} dx_{1K} \\
+ \int_{B_{2K}^{j2}}^{\infty} \int_0^{\infty} \left[ r_{1K} b_{1K}^j - q_{1K} (x_{1K} - b_{1K}^j) \right] f_{2K} f_{1K} dx_{2K} dx_{1K} \\
+ \int_{b_{1K}^j}^{\infty} \left[ r_{1K} b_{1K}^j - q_{1K} (x_{1K} - b_{1K}^j) \right] F_{2K} \left( b_{2K}^{j2} \right) f_{1K} dx_{1K} \right\},
\]

(57)

where \( b_{2H} = C_2 - b_{2L}^2, b_{1H} = C_1 - b_{1L}^2, M_{1K}^{j2} = b_{1K}^j - u_{2K} (x_{1K} - b_{2K}^{j2}), \) and \( B_{2K}^{j2} = b_{2K} + (b_{1K}^j - x_{1K})/u_{2K} \) for \( j = 1,2 \) and \( K = L, H. \) Similarly, we in turn investigate the first and second order derivatives of \( J_1 \) with respect to \( b_{2L}^2, \) the monotonic property of \( V_1 = 0 \) in the \((b_{1L}, b_{2L})\) plane. We then find the structural properties of the objective function described in Lemmas 1, 2 also hold for \( P1 \) in this case. Furthermore, Comparing the implicit derivative of \( V_1 = 0 \) with respect to \( b_{1L} \) to that of \( V_2 = 0 \) in each type, we find that in each type of game, the property shown in Lemma 3 holds.

**Theorem 12** When \( P2 \) has the complete information of \( P1 \) and himself, and \( P1 \) knows that his transfer rate of low-fare class customer is \( u_{1L}^1 \) with probability \( \theta_u^1 \) and \( u_{1L}^2 \) with probability of \( \theta_u^2 \) for \( \theta_u^1 + \theta_u^2 = 1 \) and \( u_{1L}^1 < u_{1L}^2, \) the game admits a unique
Bayesian Nash equilibrium \((b_{1L}^*, b_{2L}^{12*}, b_{2L}^{22*})\). In addition, \(b_{2L}^{12*} \leq b_{2L}^{22*}\).

**Proof.** The proof of the existence of the unique Bayesian Nash equilibrium is analogous to the proof of Theorem 8, and is thereby omitted here. Now, let us prove \(b_{2L}^{12*} \leq b_{2L}^{22*}\). We know that if

\[
V_2(b_{1L}^*, 0; t_{2j}^j) = (r_{2L} + q_{2L}) - (r_{2H} + q_{2H}) S_2(b_{2L}^*, 0; t_{2j}^j) < 0, \ j = 1, 2,
\]

then it is always optimal for \(P2\) to set his booking limit as zero for any \(u_{1L}\). It implies that

\[
b_{2L}^{12*} = b_{2L}^{22*} = 0.
\]

However, in type \(t_{2L}^{12}\), if \(b_{2L} = b_{2L}^{12*} > 0\), then it must satisfies

\[
(r_{2L} + q_{2L}) S_2(b_{1L}, b_{2L}) - (r_{2H} + q_{2H}) S_2(b_{1L}, b_{2L}) = 0.
\]

Differentiating both sides of the above equation with respect to \(u_{1L}\) \((b_{1L}\) is constant), we obtain

\[
\frac{1}{(u_{1L})^2} (r_{2L} + q_{2L}) \int_0^{b_{2L}} f_{2L} f_{1L} (N_{1L}) \ dx_{2L} + \frac{\partial^2 J_2}{\partial (b_{2L})^2} \frac{db_{2L}}{du_{1L}} = 0.
\]

Since \(\frac{\partial^2 J_2}{\partial (b_{2L})^2} < 0\), we know that \(\frac{db_{2L}}{du_{1L}}\) must be positive, which indicates \(b_{2L}\) increases as \(u_{1L}\) increases. On the other words, \(b_{2L}^{12*}\) is always less than \(b_{2L}^{22*}\) if \(b_{2L}^{12*} > 0\). Thus, in general, \(b_{2L}^{12*} \leq b_{2L}^{22*}\).

We might also compare the optimal expected revenue of \(P2\) in each type. Differentiating \(J_2\) with respect to \(u_{1L}\) \((b_{1L}\) is constant), we obtain
\begin{align*}
dJ_2/du_1L &= \int_0^{b_{2L}} \int_{B_{1L}} r_{2L}b_{2L}f_{1L}f_{2L} \, dx_1L \, dx_2L \\
&\quad - \int_0^{b_{2L}} \int_{B_{1L}} q_{2L}[(x_{2L} - b_{2L})] f_{1L}f_{2L} \, dx_1L \, dx_2L \\
&\quad - \int_{b_{2L}}^{\infty} \int_{b_{1L}}^{\infty} q_{2L}[(x_{2L} - b_{2L})] f_{1L}f_{2L} \, dx_1L \, dx_2L \\
&\quad + (\partial J_2^2/\partial b_{2L}) \, (db_{2L}/du_{1L}).
\end{align*}

Unfortunately, the relationship between $J_2^*(b_{1L}^2, b_{2L}^{2*}; t_{u_2})$ and $J_2^*(b_{1L}^2, b_{2L}^{2*}; t_{u_2})$ is uncertain since the sign of $dJ_2/du_{1L}$ in (58) can not be determined. However, the value of private information of $u_{1L}$ for $P2$ can be obtain by the following expression:

$$\omega_{u_2}^2 = \theta_u^1 J_2^*(b_{1L}^{2*}, b_{2L}^{2*}; t_{u_2}) + \theta_u^2 J_2^*(b_{1L}^{2*}, b_{2L}^{2*}; t_{u_2}) - J_2^*(b_{1L}^{1*}, b_{2L}^{2*}; t_{u_2}).$$

### 5.2.4 Public Information Game: $u_{1L}$ is Known to Both Players

When the two players receive public information of $u_{1L}$, the type spaces of $P1$ and $P2$ are $T_1 = \{t_{u_1}^{13}, t_{u_2}^{23}\}$ and $T_2 = \{t_{u_2}^{13}, t_{u_2}^{23}\}$ respectively. Similar to the public information case in Section 5.1, one player’ objective function in each type is identical to (4).

**Theorem 13** When the two players receive the public information of $u_{1L}$, there exists a unique Nash equilibrium in each type of game, which is $(b_{1L}^{13*}, b_{2L}^{13*})$ when $u_{1L} = u_{1L}^1$; and $(b_{1L}^{23*}, b_{2L}^{23*})$ when $u_{1L} = u_{1L}^2$. In addition, $b_{1L}^{13*} \leq b_{1L}^{23*}$ and $b_{2L}^{23*} \leq b_{2L}^{13*}$ if $u_{1L}^1 < u_{1L}^2$.  

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Proof. The proof of the existence of the unique Nash equilibrium in each type is analogous to the proof of Theorem 10, and the proof the relations between $b_{1L}^{13*}$ and $b_{2L}^{23*}$ ($i = 1, 2$) is analogous to the proof of Proposition 5. Then, we thereby omit the proof of Theorem 13.

Even though we know that as $u_{1L}$ increases the Nash equilibrium moves in the northwest direction in $(b_{1L}, b_{2L})$ plane, which leads to a decrease in $b_{1L}$ and an increase in $b_{2L}$, we could not generalize the specific situation(s) in which the value of public information of $u_{1L}$ for one player is positive or negative. This is due to the uncertainty of the objective function with respect to $u_{1L}$ in both uninformed and public information games. In general, the value of public information of $u_{1L}$ for $P1$ can be expressed as

$$
\omega_{u1}^3 = \theta_u^1 J_1^3*(b_{1L}^{13*}, b_{2L}^{13*}; t_{u1}) + \theta_u^2 J_2^3*(b_{1L}^{23*}, b_{2L}^{23*}; t_{u2}) - J_1^*(b_{1L}^{*}, b_{2L}^{*}; t_{u1}),
$$

and the value of public information of $u_{1L}$ for $P2$ is

$$
\omega_{u2}^3 = \theta_u^1 J_2^3*(b_{1L}^{13*}, b_{2L}^{13*}; t_{u2}) + \theta_u^2 J_2^3*(b_{1L}^{23*}, b_{2L}^{23*}; t_{u2}) - J_2^*(b_{1L}^{*}, b_{2L}^{*}; t_{u2}).
$$

We are going to use the following numerical example to demonstrate the value of different information of $u_{1L}$ and some important results shown in this section.

Example 11  In this example, we examine the value of different information of $u_{1L}$ for each player and their relationships in different situation. The procedures to attain this goal are very similar to Example 10. Specifically, we fix $q_{1L}$ as 30 and $u_{1L} = 0.2$ with probability of 0.5 and $u_{1L}^2 = 0.9$ with probability of 0.5. probability
of 0.5. Again, we vary the booking request expectations to generate 16 scenarios, by which we examine the relationships among the values of different information for the two players. As expected, the value of public information of $u_{1L}$ is dominated by the values of secret and private information in most cases (see Figure 25). Comparing to Example 10, the relationships among the values of different information of $u_{1L}$ exhibit an additional form, which is $\omega^3_{u2} > \omega^1_{u2} > \omega^2_{u2}$. In addition, we find that $\omega^3_{u2} < 0$ in any scenario in which the booking request expectations of the two fare classes of $P_2$ is high. As for $P_1$, the situations in which the value of public information of $u_{1L}$ is positive or negative can be seen from Figure 26. In most cases, $P_1$ can benefit from the public information of $u_{1L}$. Similarly, we also find in some scenarios the relationships between the values of different information of $u_{1L}$ for $P_2$ and the sign of the value of public information for $P_1$ change as we vary $\theta^2_u$. In addition, we find that in the scenario when the booking requests expectations of $P_1$ are high and the booking requests expectations of $P_2$ are low, the value of any type of information for
Figure 26. Value of public information of $u_{1L}$ for $P2$. In region $D_1$, $\omega_{u1}^3 < 0$ and in region $D_2$, $\omega_{u1}^3 > 0$.

$P2$ is $55+$ which account for 3% of his total expected revenue. ♦
Chapter 6
Thesis Summary and Concluding Remarks

As discussed in Chapter 1, game theory is a quantitative approach used in the study of two or more decision makers' interactive behaviors in competitive or cooperative situations. In Section 1.3, we indicated that game theory can be applied in revenue management to deal with single-period multiple-class games with complete information, and with multiple-period multiple-class games with complete information, etc. In Chapter 3, we established a two-player two-fare-class (high-fare and low-fare) static game model to solve the hotel room inventory control problem. Under this game theoretic setting, we obtained the optimal rationing policies for the two hotels under competitive and cooperative situations. We also characterized the structural properties of the corresponding objective functions and analyzed the equilibria of competitive and cooperative games, respectively. Our study indicates some important managerial implications on this revenue management problem: First, our game model indicates that as a best response, one hotel should always decrease its booking limit for low-fare customer by more than one unit if another hotel increases the low-fare booking limit by one unit, and vice versa. Secondly, when the hotels compete, we have proved the existence and uniqueness of Nash equilibrium and have presented the structural properties of these equilibria in different situations. Also, we identify the situation in which Stackelberg game is equivalent to Nash game. This result shows that if one player's booking limit is reached, i.e., he always rejects low-fare
customers, neither of the two players would like to “lead” the game. Finally, we find that the revenue loss is substantial if there is a lack of cooperation between two players. Our numerical experiments suggest that such loss can be more than 10% in most cases.

In Chapter 4, we formulated a two-player two-fare-class dynamic game model. In this situation, the problem becomes more complicated since one hotel’s accept/reject decisions in each period are not only affected by the decisions of the other hotel, but also affected by room inventory levels of both players at the beginning of the period. Analogous to the analysis in Chapter 3, we examine our model using Nash, Stackelberg and cooperative strategies. The main contributions can be presented as follows. First, each hotel’s optimal future revenue is a non-decreasing function of its own room inventory and a non-increasing function of the other hotel’s room inventory at any time. Secondly, we establish the unique Nash equilibrium of dynamic accept/reject decisions for the two hotels under competitive situation. Finally, by defining expected marginal value of hotels’ room, we simplify the optimal accept/reject decision into sets of critical values.

In Chapter 5, we studied the two-player two-fare-class static game with incomplete information. We first clarified the various definitions of information value in our games. Then we assume the rejection cost and transfer rate of one player as part of incomplete information and examine the optimal booking policies of each player with different information structures. We find that in the uninformed game, there exist a unique Nash equilibrium for the two players. Specially, as the rejection cost of one player is uninformed, the game is equivalent to the complete information game by
using the rejection cost expectation as the real rejection cost. We also proved the existence of a unique Bayesian Nash equilibrium in the game when one player receives the private information of rejection cost or transfer rate. Furthermore, we provided the formulations of the value of different information. We see that the value of secret information is always non-negative for both players. We also evaluated the values of private and public information for one player and we provided the conditions by which one player might use or drop the information of rejection cost and transfer rate. Finally, we presented experimental results corroborating our theoretical analysis of the value of different information.

The research discussed in this paper could be extended in several directions. One natural extension should be to consider the incomplete information game in the dynamic context. Again, we can examine the existence of Nash equilibrium (perfect Bayesian Nash equilibrium) and the uniqueness. Second, under each game-theoretic setting, static or dynamic, complete or incomplete information, we can consider three or more players. For example, under the static game with complete information, one possibility is to analyze a competitive/cooperative game in which at least two or more players cooperate to increase their total expected revenue at the expense of the other players. Another possibility is to assume that they all compete in which case Nash strategy can be used to by all players as before. However, we suspect that it will be more difficult to prove the existence and/or uniqueness of Nash equilibrium since the two-way transfers of low-fare bookings would greatly complicate the substitution structure of the model. A third possible extension would be to consider a game with the consideration of the overbooking by relaxing the assumption.
of no cancellations of booked customer. This situation is fairly common in practice and many RM researchers have worked on different aspects of this problem in recent years. Generally, within game-theoretic context, we might progressively relax the one or more assumptions in our problems to make the games ‘richer’ and rule out the booking policies and plausible equilibrium for them.
References


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