PRODUCTS AND FACTORIZATIONS OF GRAPHS

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SCOPE AND CONTENTS: It is shown that the cardinal product of graphs does not satisfy unique prime factorization even for a very restrictive class of graphs. It is also proved that every connected graph has a decomposition as a weak cartesian product into indecomposable factors and that this decomposition is unique to within isomorphisms. This latter result is established by considering a certain class of equivalence relations on the edge set of a graph and proving that this collection is a principal filter in the lattice of all equivalences. The least element of this filter is then used to decompose the graph into a weak cartesian product of prime graphs that is unique to within isomorphisms.

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INTRODUCTION

We consider four products for a family of graphs, where the vertex set of the product is the product of the vertex sets of the factors. They are the cartesian, weak, strong and cardinal products (1.2). The cartesian product is due to Shapiro [7] and Sabidussi [5], [6], the weak and strong products to Sabidussi [5], [6], and the cardinal product to Čulik [1]. The cardinal product is called the Kronecker product by Weichsel [9], and the conjunction by Hedetniemi [2]. Szamkolowicz [8] poses the question (due to Mycielski) of unique prime factorization for the cartesian and cardinal product. (It is always of general mathematical interest to know if a product defined in any algebraic system satisfies unique prime factorization.) Sabidussi [6] had already established that unique prime factorization holds for connected graphs containing a vertex of finite degree and connected graphs of finite type. He imposed the finiteness conditions on the graphs to ensure that the number of factors in the decomposition was finite, since he was interested in applying the decomposition theorem to prove a result on the automorphism group of a graph. In this same paper he also shows that the strong product does not satisfy unique prime factorization. In fact he shows that there exist complete graphs that do not have a prime factorization and complete graphs that have infinitely many essentially distinct prime factorizations.

In the discussion of unique prime factorization of graphs with respect to any of the above products, it is essential to exclude graphs with isolated vertices, as is seen in the following example: let

X = path of length 3 together with two isolated vertices,

Y = complete 2-graph,

X' = complete 2-graph together with a single isolated vertex,

Y' = path of length 3 .

Then X, Y, X', Y' are indecomposable with respect to cartesian (weak, strong, cardinal) multiplication, and are pair wise non-isomorphic. However

$$XY \cong X'Y' ,$$

where juxtaposition denotes cartesian (weak, strong, cardinal) multiplication.

For the class of graphs without isolated vertices, we again have non-unique prime factorization. Let

A = disjoint union of two complete 3-graphs,

B = complete 5-graph,

A' = complete 3-graph,

B' = disjoint union of two complete 5-graphs.

Then

$$AB \cong A'B'$$

where juxtaposition again denotes cartesian (weak, strong, cardinal) multiplication. Moreover, A, B, A', B' are indecomposable, in the class of graphs without isolated vertices, with respect to cartesian (weak, strong, cardinal) multiplication.

Hence, in considering the question of unique prime factorization, we restrict ourselves to the class of connected graphs and are then led to investigate the question of the connectedness of products.

Since the cartesian product of connected graphs is connected if and only if the number of factors is finite, we introduce the weak cartesian product of a family of rooted graphs (1.26) which ensures that the product of an arbitrary family of connected graphs is always connected. The weak cartesian product of a family of connected graphs is also due to Sabidussi [6]. He introduced this product to show the existence of connected graphs that are idempotent with respect to cartesian multiplication. Our main result is an extension of Sabidussi's theorem ([6], 2.15). We show (2.41) that every connected graph is decomposable as a weak cartesian product and that the decomposition is unique to withinisomorphisms. Roughly speaking, to prove his decomposition theorem, Sabidussi constructs an equivalence relation on the edge set of a graph such that two edges are equivalent if and only if they project to the same factor. Here we consider a particular collection of equivalence relations (the acyclic equivalences (2.6) that contain certain binary relations α and β (2.8)) in the complete lattice of all equivalences on the edge set of a graph and prove that this collection is a principal filter (2.23). We show that each equivalence in this filter gives rise to a weak cartesian decomposition of the graph such that two edges are equivalent if and only if they project to the same factor, moreover if the equivalence is least then the factor are indecomposable. This correspondence between equivalences and decompositions also enables us to prove the conjecture of Sabidussi's ([6], p.449) that an idempotent graph (i.e., $X \times X \cong X$) with respect to cartesian multiplication does not have a cartesian decomposition into indecomposable factors.

Since the weak product of a family of connected graphs is also connected if and only if the number of factors is finite, we introduce the weak product of a family of rooted graphs (1.28) to get connectedness of the product for an arbitrary family of connected graphs. This product does not however satisfy unique prime factorization.

The cardinal product has the unpleasant property that the product of two connected graphs need not be connected. A necessary and sufficient condition that the cardinal product of two connected graphs be connected is that at least one factor be non-bipartite (1.11). (While preparing this dissertation we discovered that Weichsel [9] had already established this result. Our proof is essentially the same as his; however, we make use of the fact that the cardinal product is categorical whereas he does not.) It is then natural to ask if the cardinal product of a family of non-bipartite graphs is itself non-bipartite. This question is fully answered in 1.13. In particular we have that the product of two connected non-bipartite graphs is again connected and non-bipartite.

Our decomposition theorem (1.20) shows that even by restricting to the class of finite connected non-bipartite graphs unique prime factorization does not hold for the cardinal product. Marica and Bryant [3] prove that finite unary algebras (i.e., functional directed graphs) have unique square roots. It would be of interest to know if a similar result holds for the cardinal product of finite graphs. We have not however attempted this problem.

CHAPTER I

SECTION I: Preliminaries.

1.1. DEFINITIONS: By a <u>graph</u> X we mean an ordered pair $(V(X), E(X))$, where $V(X)$ is a set and $E(X)$ is a set of unordered pairs of distinct elements of $V(X)$. (We can consider a graph to be a set together with a symmetric, irreflexive relation on the set.) We shall denote an unordered pair by brackets. The elements of $V(X)$ will be called the <u>vertices</u> of X and the elements of $E(X)$ the <u>edges</u> of X. We denote the cardinal of the set $V(X)$ by $|X|$. The empty graph, i.e., the graph with empty vertex set, will be denoted by $\emptyset$.

A <u>subgraph</u> Y of a graph X is a graph whose vertex and edge sets are respectively subsets of the vertex and edge sets of X. A subgraph Y of X is called <u>saturated</u> if and only if $x, y \in V(Y)$, $[x,y] \in E(X)$ imply $[x,y] \in E(Y)$. Y is called a <u>spanning</u> subgraph of X if $V(Y) = V(X)$. An edge e is said to be <u>incident</u> with a vertex x if and only if $e = [x,y]$ for some vertex y. Two edges $e = [x,y]$ and $e' = [x',y']$ are said to be <u>adjacent</u> if and only if exactly two of the vertices x,y,x',y' are equal, i.e., two edges are adjacent if and only if they are distinct and incident with a common vertex. A subset V of the vertex set $V(X)$ is called <u>independent</u> if and only if $x,y \in V$ implies $[x,y] \notin E(X)$.

Let X and Y be graphs. By $X \cup Y$ and $X \cap Y$ we mean the graphs defined by

$$V(X \cup Y) = V(X) \cup V(Y) \ ,$$

$$E(X \cup Y) = E(X) \cup E(Y) \ ,$$

and

$$V(X \cap Y) = V(X) \cap V(Y) \ ,$$

$$E(X \cap Y) = E(X) \cap E(Y) \ .$$

If $x \in V(X)$ we let (x) denote the subgraph of X for which

$$V((x)) = \{x\} \text{ and } E((x)) = \emptyset \ .$$

If $e = [x,y] \in E(X)$, (e) denotes the subgraph of X for which

$$V((e)) = \{x,y\} \text{ and } E((e)) = \{e\} \ .$$

Whenever there is no likelihood of confusion we shall write x for (x) and e for (e).

If Y is a subgraph of X we define the <u>relative complement</u> $X\backslash Y$ of Y in X to be the smallest subgraph with

$$E(X\backslash Y) = E(X) - E(Y) \ ,$$

Let X and Y be graphs. By a <u>homomorphism</u> of X into Y we mean a function $\phi: V(X) \longrightarrow V(Y)$ such that $[\phi x, \phi y] \in E(Y)$ whenever $[x,y] \in E(X)$. For a homomorphism $\phi : V(X) \rightarrow V(Y)$ we shall write $\phi : X \longrightarrow Y$. A <u>monomorphism</u> of X into Y is a one-one homomorphism. If $\phi : X \longrightarrow Y$ is a homomorphism then ϕ induces a function $\phi^{\#} : E(X) \longrightarrow E(Y)$ as follows: for $[x,y] \in E(X)$ define

$$\phi^{\#}[x,y] = [\phi x, \phi y] \ .$$

A homomorphism $\phi : X \longrightarrow Y$ is called an <u>epimorphism</u> if and only if ϕ and $\phi^{\#}$ are both onto. By an <u>isomorphism</u> of X onto Y we mean a monomorphism $\phi : X \longrightarrow Y$ such that ϕ and $\phi^{\#}$ are both onto. We shall frequently write

ϕe for $\phi^{\#} e$.

Given graphs X and Y let ϕ be a function from $V(X)$ to $V(Y)$. If A is a subgraph of X , we let ϕA denote that subgraph of Y defined by

$$V(\phi A) = \phi(V(A)) ,$$

$$E(\phi A) = \{[\phi x, \phi x'] \in E(Y) \mid [x,x'] \in E(X)\} .$$

If ϕ is a homomorphism then $E(\phi A) = \phi^{\#}(E(A))$. The only functions we consider that are not homomorphisms are projections (1.3) .

Let $x, y \in V(X)$. A <u>path</u> of X joining x and y is a subgraph P of X such that $V(P)$ is the set of elements of a finite sequence $(x_o, x_1, \ldots, x_n)$ of distinct vertices of X with $x_o = x$ and $x_n = y$, and

$$E(P) = \{[x_i, x_{i+1}] \mid 0 \leqq i \leqq n-1\} .$$

We shall denote the path P by $[x_o, x_1, \ldots, x_n]$. n is called the <u>length</u> of P . A path P is called <u>proper</u> if the length of P is $\geqq 1$. A graph X is called <u>connected</u> if any two vertices of X are joined by a path in X , otherwise it is called <u>disconnected</u>. A path P joining x and y is called a shortest path if and only if for any path Q joining x and y the length of P does not exceed the length of Q . Let X be connected $x, y \in V(X)$. By the <u>distance</u> $d_X(x,y)$ of x and y in X we mean the length of a shortest path joining x and y in X . When no confusion is likely we shall write $d(x,y)$ for $d_X(x,y)$. By the <u>diameter</u> diam X is meant

$$\operatorname{diam} X = \sup_{x,y \in V(X)} d_X(x,y) .$$

A maximal connected subgraph is called a <u>component</u>.

Let $P_n = [x_o, x_1, \ldots, x_n]$ and $P_{n-2} = [y_o, y_1, \ldots, y_{n-2}]$ be paths of length n and $n-2$ respectively, $n \geq 3$. Then $\phi : P_n \longrightarrow P_{n-2}$ defined by

$$\phi x_i = \begin{cases} y_i & , i = 1,2,\ldots,n-2 , \\ y_{n-3} & , i = n-1 , \\ y_{n-2} & , i = n , \end{cases}$$

is an epimorphism with

$$\phi x_o = y_o \quad \text{and} \quad \phi x_n = y_{n-2} \quad .$$

By a <u>circuit</u> of a graph X we mean a subgraph C of X such that $V(C)$ is the set of elements of a sequence $(x_1, x_2, \ldots, x_n)$ of distinct vertices of X, and

$$E(C) = \{[x_i, x_{i+1}] \mid i = 1,\ldots,n-1\} \cup \{[x_o, x_n]\}, \; n \geq 3 \quad .$$

We shall denote the circuit C by $[x_1, \ldots, x_n]$. n is called the <u>order</u> of C and we shall frequently call C an n-circuit. A circuit will be called <u>even</u> or <u>odd</u> according as n is even or odd.

Let $C_n = [x_1, \ldots, x_n]$ and $C_{n-2} = [y_1, \ldots, y_{n-2}]$ be circuits of order n and $n-2$ respectively, $n \geq 5$. The mapping $\phi : C_n \longrightarrow C_{n-2}$ defined by

$$\phi x_i = \begin{cases} y_i & , i = 1,\ldots,n-2 , \\ y_1 & , i = n-1 , \\ y_{n-2} & , i = n , \end{cases}$$

is an epimorphism from C_n onto C_{n-2}.

A graph X is called <u>acyclic</u> if X contains no circuits. A <u>tree</u> is a connected acyclic graph.

A graph X is called <u>complete</u> if and only if $x, y \in V(X)$, $x \neq y$, implies $[x,y] \varepsilon E(X)$. Let n be any cardinal; a complete n-graph is a complete graph on n vertices. We shall frequently denote a complete n-graph by $C(n)$.

We call a graph X bipartite if and only if $E(X) \neq \emptyset$ and every circuit in X is even. It is well-known (c.f. [4] 7.1.1) that X is bipartite if and only if there exists a epimorphism $\phi : X \longrightarrow C(2)$, i.e., $V(X)$ is the disjoint union of two non-empty independent sets of vertices. If X is a bipartite graph with $V(X) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, V_i independent, $i = 1, 2$, then X is called a complete bipartite graph if and only if $x_1 \varepsilon V_1$, $x_2 \varepsilon V_2$ implies $[x_1,x_2] \varepsilon E(X)$. If $|V_1| = n$, $|V_2| = m$ then we denote the complete bipartite graph X by $K_{n,m}$.

If a graph X is non-bipartite we define the odd mesh of X to be $\min |C|$, the minimum taken over all circuits C of odd order.

Let X be a graph. For $x \varepsilon V(X)$ we let

$$V(X;x) = \{y \mid [x,y] \varepsilon E(X)\} .$$

$|V(X;x)|$ is called the degree of x in X and is denoted by $d(x;X)$ or simply d_x when no confusion is likely. X is said to have bounded degree if and only if

$$\sup_{x \varepsilon V(X)} d(x;X) < \infty .$$

Let R be an equivalence relation on the vertex set of a graph X and let $R[x] = \{y \varepsilon V(X) \mid xRy \}$. We define the quotient graph X/R as follows:

$$V(X/R) = \{R[x] \mid x \varepsilon V(X)\} ;$$

For $R[x], R[y] \in V(X/R)$,

$[R[x], R[y]] \in E(X/R)$ if and only if $R[x] \neq R[y]$ and there exist

$$x' \in R[x] , \; y' \in R[y] \text{ with } [x',y'] \in E(X) .$$

By a <u>cover</u> of a graph X we mean a collection $\mathfrak{A}$ of subgraphs such that

(i) $\bigcup_{a \in \mathfrak{A}} A = X$, and

(ii) $E(A) \cap E(A') = \emptyset$ for $A, A' \in \mathfrak{A}$ with $A \neq A'$.

SECTION II: Products and connectedness

1.2. DEFINITIONS: Let $(X_a)_{a\varepsilon A}$ be a family of graphs. $\prod_{a\varepsilon A} V(X_a)$ will denote the usual cartesian product of the sets $\{V(X_a):a\varepsilon A\}$, and $pr_b:\prod_{a\varepsilon A} V(X_a) \longrightarrow V(X_b)$ the projection into the b^{th} factor. We define

1.2 (I) the <u>cartesian product</u> $X = \prod_{a\varepsilon A} X_a$ of the graphs X_a by:

$V(X) = \prod_{a\varepsilon A} V(X_a)$,

$E(X) = \{[x,y] : x,y \ \varepsilon \ V(X), \ [pr_a x, \ pr_a y] \ \varepsilon \ E(X_a)$ for exactly one

$a\varepsilon A, \ pr_b x = pr_b y$ for all $b\varepsilon A - \{a\}\}$,

1.2(II) the <u>weak product</u> $X^w = \prod_{a\varepsilon A} {}^w X_a$ of the graphs X_a by:

$V(X^w) = \prod_{a\varepsilon A} V(X_a)$;

For $x,y \ \varepsilon \ V(X^w)$,

$[x,y] \ \varepsilon \ E(X^w)$ if and only if there exists a non-empty finite subset $B \subset A$ such that

$$[pr_b x, \ pr_b y] \ \varepsilon \ E(X_b) \quad , \text{ for } b\varepsilon B ,$$

and

$$pr_a x = pr_a y \quad , \ a\varepsilon A-B,$$

1.2 (III) the <u>strong product</u> $X^* = \prod_{a\varepsilon A} {}^* X_a$ of the graphs X_a by:

$V(X^*) = \prod_{a\varepsilon A} V(X_a)$;

For $x,y \ \varepsilon \ V(X^*)$,

$[x,y] \ \varepsilon \ E(X^*)$ if and only if there exists a non-empty subset $B \subset A$

such that

$$[pr_b x,\ pr_b y] \in E(X_b) \qquad , \quad b \in B ,$$

and

$$pr_a x = pr_a y \qquad , \quad a \in A-B .$$

1.2 (IV) the <u>cardinal product</u> $X^o = \prod_{a\in A}{}^o X_a$ of the graphs X_a by:

$V(X^o) = \prod_{a\in A} V(X_a)$,

$E(X^o) = \{[x,y] : x,y \in V(X^o) , [pr_a x, pr_a y] \in E(X_a) \text{ for all } a\in A\}$,

If the index set A is finite, it is clear that weak and strong product coincide. For a finite family of graphs $(X_i)_{i=1,\ldots,n}$ we shall frequently denote the cartesian product by $X_1 \times X_2 \times \ldots \times X_n$, the strong product by $X_1 * X_2 * \ldots * X_n$, and the cardinal product by $X_1 o X_2 o \ldots o X_n$.

1.3. REMARK: Observe that $pr_b : \prod_{a\in A}{}^o X_a \longrightarrow X_b$ is a homomorphism; whereas $pr_b : \prod_{a\in A} X_a \longrightarrow X_b$ ($pr_b : \prod_{a\in A}{}^* X_a \longrightarrow X_b$; $pr_b : \prod_{a\in A}{}^w X_a \longrightarrow X_b$) is not a homomorphism provided that one of the factors X_a, $a \neq b$, has an edge.

We note here the fact that the <u>cardinal product is categorical</u>, i.e., given a graph Y and a family of homomorphisms $\phi_b : Y \longrightarrow X_b$, $b\in A$, then the mapping $\phi : Y \longrightarrow \prod_{a\in A}{}^o X_a$ defined by

$$pr_b(\phi(y)) = \phi_b(y) \qquad , \quad y\in Y,\ b\in A ,$$

is a homomorphism (Fig. 1.1) . We will denote ϕ by $\prod_{a\in A}{}^o \phi_a$.

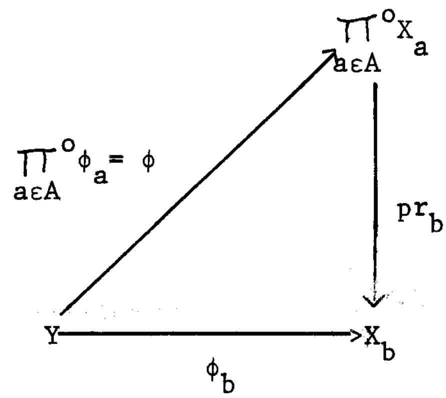


FIGURE 1.1

Moreover if there exists an $a_o \varepsilon A$ such that $\phi_{a_o} : Y \longrightarrow X_{a_o}$ is a monomorphism then $\phi : Y \longrightarrow \prod_{a\varepsilon A}{}^{o} X_a$ is also a monomorphism.

If $\phi_i : X_i \longrightarrow Y_i$ is a homomorphism, $i = 0, 1$, then $\phi : X_o \circ X_1 \longrightarrow Y_o \circ Y_1$ defined by

$$\phi(x_o, x_1) = (\phi_o x_o, \phi_1 x_1)$$

is also a homomorphism. Moreover if ϕ_i, $i = 0, 1$, is an epimorphism (monomorphism) then ϕ is an epimorphism (monomorphism).

1.4. DEFINITION: For each $b \varepsilon A$ and each $x \varepsilon \prod_{a\varepsilon A} V(X_a)$ we define the <u>injection</u> mapping $i_b^x : V(X_b) \longrightarrow \prod_{a\varepsilon A} V(X_a)$ as follows: For each $x_b \varepsilon V(X_b)$,

$$pr_a i_b^x x_b = \begin{cases} x_b & \text{if } b = a \\ pr_a x & \text{if } b \neq a \end{cases} , \quad a\varepsilon A .$$

Clearly $i_b^x : X_b \longrightarrow \prod_{a\varepsilon A} X_a$ ($i_b^x : X_b \longrightarrow \prod_{a\varepsilon A}{}^{*} X_a$; $i_b^x : X_b \longrightarrow \prod_{a\varepsilon A}{}^{w} X_a$) is a monomorphism; however $i_b^x : X_b \longrightarrow \prod_{a\varepsilon A}{}^{o} X_a$ is not even a homomorphism.

Under the identification of isomorphic graphs cartesian, strong, and cardinal multiplication are commutative and associative. For cartesian and strong multiplication the trivial graph (i.e., the graph consisting of a single vertex and hence an empty set of edges) acts as a unit. Cardinal multiplication does not have a unit. A graph will be called non-trivial if its vertex set is non-empty and it is not the trivial graph.

1.5. DEFINITION: A graph X is called prime (or indecomposable) with respect to cartesian multiplication if and only if X is non-trivial and $X \cong Y \times Z$ implies either Y or Z is trivial. Analogous definitions of indecomposable graphs can be introduced for the cardinal and strong product.

1.6. PROPOSITION: The cartesian (strong, cardinal) product of any graph with a disconnected graph is disconnected.

PROOF. Trivial.

1.7. PROPOSITION: The cartesian product of finitely many connected graphs is connected.

PROOF. Let $X = X_1 \times X_2 \times \ldots \times X_n$, where X_i is a connected graph for $i = 1, 2, \ldots, n$, and let $x,y \in V(X)$. Since X_i is connected there exists a path, P_i say, joining $pr_i x$ and $pr_i y$ in X_i, $i = 1, \ldots, n$. Let $x_1 = x$, $x_2 = i_1^{x_1}\, pr_1 y$, $x_3 = i_2^{x_2}\, pr_2 y$, ..., $x_n = i_n^{x_{n-1}}\, pr_{n-1} y$. Then

$$P = i_1^{x_1} P_1 \cup i_2^{x_2} P_2 \cup \ldots \cup i_n^{x_n} P_n .$$

is a path joining x and y in X .

1.8. PROPOSITION: The weak product of infinitely many non-trivial graphs is disconnected.

PROOF. Let $(X_a)_{a \varepsilon A}$ be an infinite family of non-trivial graphs. For each $a \varepsilon A$, let x_a and y_a be distinct vertices of X_a . Define $x, y \varepsilon \prod_{a \varepsilon A}^{w} X_a = X^w$ by $pr_a x = x_a$, and $pr_a y = y_a$ for $a \varepsilon A$. Suppose x and y are joined in X^w by a path $P = [x_o, \ldots, x_n]$. Since $[x_i, x_{i+1}] \varepsilon E(X^w)$, $pr_a x_i$ and $pr_a x_{i+1}$ differ for at most a finite number of $a \varepsilon A$. Hence x and y differ for at most a finite number of projections, contradicting $pr_a x \neq pr_a y$ for all $a \varepsilon A$.

1.9. COROLLARY: The cartesian product of infinitely many non-trivial graphs is disconnected.

PROOF. The proof follows immediately from the fact that $\prod_{a \varepsilon A} X_a$ is a spanning subgraph of $\prod_{a \varepsilon A}^{w} X_a$.

1.10. PROPOSITION: The strong product of a family $(X_a)_{a \varepsilon A}$ of connected graphs is connected if and only if

$$B = \{b \varepsilon A \mid \operatorname{diam} X_b = \infty \}$$

is finite and

$$D = \{\operatorname{diam} X_a \mid a \varepsilon A - B \}$$

is bounded.

PROOF. Let $X = \prod_{a \varepsilon A}^{*} X_a$ and assume that X is connected. If B is infinite or D is unbounded, then for $a \varepsilon A$ there exist $x_a, y_a \varepsilon V(X_a)$ such that

1.10 (1) $$\sup_{a\in A} d_{X_a}(x_a,y_a) = \infty .$$

Define $x,y \in V(X)$ by $pr_a x = x_a$, $pr_a y = y_a$, $a\in A$. X connected implies there exists a path P joining x and y in X. $pr_a P$ is a connected subgraph of X_a containing x_a and y_a and hence contains a path joining x_a and y_a. Therefore

$$d_{X_a}(x_a,y_a) \leqq |pr_a P| \leqq |P| \qquad , a\in A ,$$

contradicting 1.10 (1)

Suppose that B is finite and D is bounded. Take any $x,y \in V(X)$. Since X_a is connected for each $a\in A$, $pr_a x$ and $pr_a y$ can be joined in X_a by a shortest path

$$P_a = [pr_a x = x_o^a, x_1^a, \ldots, x_{n(a)}^a = pr_a y].$$

Since B is finite, $k_1 = \max_{b\in B} n(b)$ exists and since D is bounded $k_2 = \max_{a\in A-B} n(a)$ exists. Let $k = \max\{k_1,k_2\}$.

For $0 \leqq i \leqq k$, define $x_i \in V(X)$ as follows:

$$pr_a x_i = \begin{cases} x_i^a & , 0 \leqq i \leqq n(a) \\ x_{n(a)}^a & , n(a) < i \leqq k \end{cases} , a\in A.$$

To show that $[x_i, x_{i+1}] \in E(X)$, $0 \leqq i \leqq k-1$, we first note that for $a\in A$ either

$$[pr_a x_i, pr_a x_{i+1}] \in E(X_a)$$

or

$$pr_a x_i = pr_a x_{i+1} .$$

Since $k = \max \{k_1, k_2\}$ there exists an $a_o \varepsilon A$ such that $n(a_o) = k$, i.e.,

$$[pr_{a_o} x_i, pr_{a_o} x_{i+1}] \varepsilon E(X_{a_o}) .$$

Hence $[x_i, x_{i+1}] \varepsilon E(X)$ and

$$P = [x_o, \ldots, x_n]$$

is a path joining x and y in X .

SECTION III: The cardinal product.

Let X_1 and X_2 be connected graphs. Fix $x_1 \varepsilon V(X_1)$ and $x_2 \varepsilon V(X_2)$ and for each $y \varepsilon V(X_1 \circ X_2)$ define

$$n(y) = d_{X_1}(x_1, pr_1 y) + d_{X_2}(x_2, pr_2 y) .$$

For $i = 1, 2$ let Y_i be the saturated subgraph of $X_1 \circ X_2$ with

$$V(Y_i) = \{y \varepsilon X_1 \circ X_2 \mid n(y) \equiv i (\mathrm{mod}\ 2)\} .$$

Note that $Y_1 \cup Y_2$ is a spanning subgraph of $X_1 \circ X_2$ and that $Y_1 \cap Y_2 = \emptyset$. If moreover we have that X_1 and X_2 are non-trivial then it is easily seen that Y_1 and Y_2 are connected subgraphs of $X_1 \circ X_2$.

1.11. PROPOSITION: ([9], p.49) Let X_1 and X_2 be connected non-trivial graphs. Then the following statements are equivalent:

(i) $X_1 \circ X_2$ is disconnected (consisting of exactly two components),

(ii) $X_1 \circ X_2 = Y_1 \cup Y_2$,

(iii) X_1 and X_2 are both bipartite.

PROOF. From the remarks preceding the proposition we immediately have that (i) is equivalent to (ii).

To establish that (ii) implies (iii) we assume that (iii) does not hold, i.e., at least one of X_1, X_2 is non-bipartite. Without loss of generality suppose that X_1 contains an odd circuit, say C , and moreover we may take C to have least odd order and let $x_1 \varepsilon V(C)$. C having least odd order implies there exist $x', x'' \varepsilon V(C)$ such that

(a) $$d_{X_1}(x_1, x') = d_{X_1}(x_1, x'')$$

and

(b) $$[x', x''] \varepsilon E(C) .$$

Since X_2 is non-trivial and connected there exists $y \varepsilon V(X_2)$ with

$$[x_2, y] \varepsilon E(X_2).$$

Then

$$[(x', x_2), (x'',y)] \varepsilon E(X_1 \circ X_2) .$$

If $(x', x_2) \varepsilon V(Y_1)$ ($\varepsilon V(Y_2)$) then $(x'', y) \varepsilon V(Y_2)$ ($\varepsilon V(Y_1)$) and hence $Y_1 \cup Y_2 \subsetneqq X_1 \circ X_2$. Therefore (ii) implies (iii).

Now we show (iii) implies (i). For $i = 1, 2$, X_i bipartite implies there exists an epimorphism

$$\phi_i : X_i \longrightarrow C(2) .$$

Hence $\phi : X_1 \circ X_2 \longrightarrow C(2) \circ C(2)$ defined by

$$\phi(x_1,x_2) = (\phi_1 x_1, \phi_2 x_2)$$

is also an epimorphism. But $C(2) \circ C(2)$ is disconnected and therefore $X_1 \circ X_2$ is disconnected. Hence (iii) => (i) . This completes the proof.

1.12. PROPOSITION: Let X_1 be a bipartite graph and X_2 any graph with $E(X_2) \neq \emptyset$, then $X_1 \circ X_2$ is bipartite.

PROOF. X_1 bipartite implies there exists an epimorphism $\phi : X_1 \longrightarrow C(2)$. Hence $\phi \circ pr_1 : X_1 \circ X_2 \longrightarrow C(2)$ is an epimorphism and therefore $X_1 \circ X_2$ is bipartite.

1.13. PROPOSITION: For each $a \varepsilon A$, let X_a be a non-bipartite graph with odd mesh $= n_a$. Then the cardinal product $\prod_{a \varepsilon A}{}^{o} X_a$ is non-bipartite (with odd mesh $= \sup_{a \varepsilon A} n_a$) if and only if $\sup_{a \varepsilon A} n_a < \infty$.

PROOF. Let $X = \prod_{a \varepsilon A}{}^{o} X_a$. First assume $\sup_{a \varepsilon A} n_a = n < \infty$. For each $a \varepsilon A$ let C_a be a circuit of odd order n_a in X_a . Then there exists an $a_o \varepsilon A$ with $|C_{a_o}| = n$. For each $a \varepsilon A$, n_a, n odd and $n_a \leqq n$ imply that there exists an epimorphism

$$\phi_a : C_{a_o} \longrightarrow C_a$$

Since ϕ_{a_o} is a monomorphism, $\phi = \prod_{a \varepsilon A}{}^{o} \phi_a$ is a monomorphism from C_{a_o} to $\prod_{a \varepsilon A}{}^{o} X_a$. Hence $\phi\, C_{a_o} \subset X$ is an odd circuit of order n , i.e., X is non-bipartite.

Now let $C \subset X$ be an odd circuit. For $a \varepsilon A$, $pr_a : X \longrightarrow X_a$ a homomorphism implies $pr_a C$ is a non-bipartite subgraph of X_a and has odd mesh $\leqq |pr_a C|$. Hence

$$n_a \leqq |pr_a C| \leqq |C| , \qquad \text{for all } a \varepsilon A .$$

This proves the necessity part of the theorem, as well as, in combination with the first part of the proof, that $n =$ odd mesh of $\prod_{a \varepsilon A}{}^{o} X_a$.

Let A be an index set and each $a \varepsilon A$ let X_a be a graph with chromatic number $\chi(X_a) = n_a$, i.e., n_a is the least cardinal for which there exists a homomorphism $\phi_a : X_a \longrightarrow C(n_a)$. Since $pr_b : \prod_{a\epsilon A}{}^{o} X_a \longrightarrow X_b$ is a homomorphism for each $b \epsilon A$, we have that $\phi_b \circ pr_b : \prod_{a\epsilon A}{}^{o} X_a \longrightarrow C(n_b)$ is also a homomorphism, i.e.,

$$\chi(\prod_{a\epsilon A}{}^{o} X_a) \leqq \min_{a\epsilon A} \chi(X_a) .$$

It has been conjectured ([2], Conj.,1.2) that equality holds for A finite. By 1.13 we have that $\prod_{n\geqq 1}{}^{o} C_{2n+1}$ is bipartite, i.e., $\chi(\prod_{n\geqq 1}{}^{o} C_{2n+1}) = 2$, where C_{2n+1} is a circuit of order $2n+1$, whereas $\chi(C_{2n+1}) = 3$, for $n \geqq 1$. Hence the above conjecture can not be extended to A countable.

We will describe $\prod_{n\geqq 1}{}^{o} C_{2n+1}$ in greater detail after 1.15.

1.14. LEMMA: <u>Let</u> X <u>be a connected non-bipartite graph of finite diameter</u> d, $x,y \in V(X)$ <u>not necessarily distinct, and</u> $P = [p_o,\ldots,p_s]$ <u>a path of even length</u> $\geq 4d$. <u>Then there exists a homomorphism</u> $\phi : P \longrightarrow X$ <u>such that</u> $\phi p_o = x$ <u>and</u> $\phi p_s = y$.

PROOF. Let $C = [z_o, z_1, \ldots, z_n]$ be a circuit of least odd order. Note that $n \leqq 2d$. Let Q_1 be a shortest path joining x and z_o in X of length r_1 , and Q_3 a shortest path joining z_n and y in X of length r_3 . Let

$$Q_2 = \begin{cases} C\setminus[z_o,z_n] & , \text{ if } r_1+r_3 \text{ is even,} \\ [z_o, z_n] & , \text{ if } r_1+r_3 \text{ is odd.} \end{cases}$$

Then

$$r = r_1 + r_2 + r_3 ,$$

where r_2 is the length of Q_2, is even and $r \leq 4d$. Let $P' = [p_o, \ldots, p_r]$. Clearly there exists a homomorphism $\psi : P' \longrightarrow Q_1 \cup Q_2 \cup Q_3$ such that $\psi p_o = x$ and $\psi p_r = y$. But r, s even and $r \leq s$ implies there exists a homomorphism $\nu : P \longrightarrow P'$ such that $\nu p_o = p_o$ and $\nu p_s = p_r$. Then $\psi \circ \nu : P \longrightarrow X$ is the desired homomorphism.

1.15. PROPOSITION: <u>The cardinal product of a family $(X_a)_{a\varepsilon A}$ of connected non-bipartite graphs is connected if and only if</u>

$$B = \{b\varepsilon A \mid \text{diam } X_b = \infty\}$$

<u>is finite, and</u>

$$D = \{\text{diam } X_a \mid a\varepsilon A\text{-}B\}$$

<u>is bounded.</u>

PROOF. Let $X = \prod_{a\varepsilon A}{}^{o} X_a$ and assume that B is finite and D is bounded. Let $X_1 = \prod_{b\varepsilon B}{}^{o} X_b$ and $X_2 = \prod_{a\varepsilon A-B}{}^{o} X_a$; then $X \cong X_1 \circ X_2$. B finite implies by 1.11 that X_1 is connected and by 1.13 that X_1 is bipartite, and hence to show that X is connected it suffices by 1.11 to show that X_2 is connected.

Let $x,y \varepsilon V(X_2)$ and let

$$P = [p_o, \ldots, p_{4s}]$$

be a path of length $4s$, where

$$s = \sup_{a\varepsilon A-B} \text{diam } X_a .$$

By the lemma preceding the proposition there exists a homomorphism

$$\phi_a : P \longrightarrow X_a$$

such that

$$\phi_a p_o = pr_a x \quad \text{and} \quad \phi_a p_{4s} = pr_a y \qquad , a \varepsilon A-B .$$

Let $\phi = \prod^o_{a \varepsilon A-B} \phi_a : P \longrightarrow X_2$. Then

$$\phi p_o = x \quad \text{and} \quad \phi p_{4s} = y .$$

Since ϕP is a connected subgraph of X_2 and $x, y \varepsilon \phi P$ we have that X_2 is connected and therefore X is connected.

The proof that X connected implies B is finite and D is bounded is the same as that in 1.10.

1.16. COROLLARY: <u>Let</u> $(X_a)_{a \varepsilon A}$ <u>be a family of connected non-bipartite graphs</u>. <u>If</u> $X = \prod^o_{a \varepsilon A} X_a$ <u>is connected then</u> X <u>is non-bipartite</u>.

PROOF. X connected implies B is finite and D is bounded. Let n_a be the odd mesh of X_a . Then $n_a \leqq 2 \text{ diam } X_a + 1$ for all $a \varepsilon A-B$. Hence

$$\sup_{a \varepsilon A} n_a < \infty$$

and therefore X is non-bipartite by 1.13. This completes the proof.

It is obvious that the converse of the corollary is not true.

We now investigate $X = \prod^o_{n \geqq 1} C_{2n+1}$. The reason for doing so is that this graph is the simplest of the pathological cardinal products that exist by 1.13 and 1.15 and hence its structure is of general interest. It will be convenient to consider the vertex set of C_{2n+1} as the additive group of integers mod $2n+1$, i.e.,

$$C_{2n+1} = [-n, -(n-1), \ldots, -1, 0, 1, \ldots, n] .$$

pr_n will denote the projection to C_{2n+1} and $|pr_n x|$ will denote the distance of 0 and $pr_n x$ in C_{2n+1} .

Since the automorphism group of a circuit acts transitively on the vertices we have that the automorphism group of X acts transitively on $V(X)$. This is easily seen as follows: let $x, y \in V(X)$ and let ϕ_n be an automorphism of C_{2n+1} such that $\phi_n(pr_n x) = pr_n y$. Then ϕ defined by

$$pr_n \phi(z) = \phi_n(pr_n z) \qquad , z \in V(X), n \geqq 1 ,$$

is an automorphism of X such that $\phi(x) = y$. Hence the automorphisms of X act transitively and therefore the components of X are all isomorphic.

Next we show that the number of components of X is $2^{\aleph_0}$. For any subset A of the positive integers N define $x_A \in V(X)$ by

$$pr_n x_A = \begin{cases} n & , n \in A , \\ 0 & , n \notin A . \end{cases}$$

Let $\mathfrak{A}$ be an uncountable subset of the power set of N such that

(i) $A \in \mathfrak{A}$ implies that A and $N-A$ are infinite,

(ii) $A, A' \in \mathfrak{A}$, $A \neq A'$, implies $A \cap A'$ is finite.

For $A, B \in \mathfrak{A}$, $A \neq B$ we have x_A and x_B belong to different components of X since

$$\sup_{n \geqq 1} d_{C_{2n+1}}(pr_n x_A, pr_n x_B) = \infty .$$

Hence the number of components of X is $2^{\aleph_o}$.

Since all components of X are isomorphic we need only consider that component X_o that contains x_o, where $x_o \in V(X)$ is defined by $pr_n x_o = 0$, $n \geqq 1$. To see that there are uncountably many vertices in X_o, let $A \subset N$ and define $y_A \in V(X)$ by

$$pr_n y_A = \begin{cases} 1 & , \ n \in A , \\ -1 & , \ n \notin A . \end{cases}$$

Then $y_A \in V(X_o)$ in fact $[x_o, y_A] \in E(X_o)$, and hence $|X_o| = 2^{\aleph_o}$.

For $i = 0, 1$, let

$$V_i = \{x \in V(X) \mid d(x_o, x) \equiv i (\mathrm{mod}\ 2)\} .$$

For $i = 0, 1$, we have that $x \in V_i$ if and only if there exists some integer $j \geqq 1$ such that

(i) $|pr_n x| \leqq j$ for all $n \geqq 1$, and

(ii) $pr_n x \equiv i \pmod 2$ for all $n > j$.

We only consider the case $i = 0$. First suppose $x \in V(X)$ satisfies (i) and (ii). Let $P = [p_o, \ldots, p_{2j}]$ be a path of length $2j$. For each $n \geqq 1$, it is obvious from (i) and (ii) that there exists a homomorphism $\phi_n : P \longrightarrow C_{2n+1}$ such that $\phi_n p_o = 0$, $\phi_n p_{2j} = pr_n x$. Hence $\phi = \prod^o_{n \geqq 1} \phi_n : P \longrightarrow X$ is a homomorphism with $\phi p_o = x_o$ and $\phi p_{2j} = x$. Since X_o is bipartite, we have by 1.19 that $d(x_o, x)$ is even. Now suppose $j = d(x_o, x)$ is even, i.e., $x \in V_o$, and let P be a path joining x_o and x in X of length j. Since $pr_n P$ is a connected subgraph of C_{2n+1} containing $pr_n x_o = 0$ and $pr_n x$ we have

$$|pr_n x| \leq |pr_n P| - 1 \leq j \quad , \quad \text{for} \quad n \geq 1 .$$

For $n > j$ we also have that $pr_n P$ is a path and since P has even length the distance of $pr_n x_o = 0$ and $pr_n x$ in $pr_n P$ is even . Hence the distance of $pr_n x_o = 0$ and $pr_n x$ in C_{2n+1} is even since $n > j$.

1.17 DEFINITION: Let X, X_o be graphs. X will be called X_o-<u>admissible</u> if and only if there exists a graph X_1 such that

(i) $X_o \circ X_1$ is a spanning subgraph of X ;

(ii) $[(x_o,x_1), (x_o', x_1')] \in E(X)$ implies $[x_o,x_o'] \in E(X_o)$, and $[x_1,x_1'] \in E(X_1)$ or $x_1 = x_1'$;

(iii) if $[(x_o,x_1), (x_o',x_1)] \in E(X)$ for some $[x_o,x_o'] \in E(X_o)$ then $[(y_o,x_1), (y_o',x_1)] \in E(X)$ for all $[y_o,y_o'] \in E(X_o)$.

In view of (iii) we can introduce, for convenience, the following subset $V \subset V(X_1)$:

$x_1 \in V$ if and only if $[(x_o,x_1), (x_o',x_1)] \in E(X)$ for some $[x_o,x_o'] \in E(X_o)$. Condition (iii) can then be restated as: for each $[x_o,x_o'] \in E(X_o)$ and each $x_1 \in V$, $[(x_o,x_1), (x_o',x_1)] \in E(X)$. We shall also apply the term X_o-admissible to any graph Y isomorphic to a graph X which is X_o-admissible in the sense just defined. X will be called <u>properly X_o-admissible</u> if it is X_o-admissible and does not have X_o as a factor with respect to cardinal multiplication.

Note that condition (ii) implies that if X is X_o-admissible then $pr_o : X \longrightarrow X_o$ is a homomorphism.

1.18. REMARK: For $V = V(X_1)$ the definition of admissibility can be phrased in terms of still another graph multiplication as follows. Let X_o, X_1 be graphs. Define $X_o \otimes X_1$ by

$$V(X_o \otimes X_1) = V(X_o) \times V(X_1) ,$$

$E(X_o \otimes X_1) = \{[x,y] : [pr_o x, pr_o y] \in E(X_o),$ and $[pr_1 x, pr_1 y] \in E(X_1)$ or $pr_1 x = pr_1 y\}$. Then a graph X is X_o-admissible if there exists a graph X_1 such that

$$X \cong X_o \otimes X_1 .$$

1.19. EXAMPLE: For any non-zero cardinals $m,n,r,$ the complete bipartite graph $K_{mr,nr}$ is properly $K_{m,n}$-admissible. This follows from

$$K_{m,n} \otimes C(r) \cong K_{mr,nr}$$

and the fact that every complete bipartite graph is indecomposable with respect to cardinal multiplication. This can be seen as follows. If

$$K_{m,n} \cong X_1 \circ X_2$$

then each factor is a homomorphic image of $K_{m,n}$. But trivially any homomorphic image of $K_{m,n}$ is of the form $K_{r,s}$, with $r \leq m$, $s \leq n$, and hence bipartite. By 1.11 this would imply $K_{m,n}$ is disconnected, a contradiction. Hence $K_{m,n}$ is indecomposable.

We will investigate the existence of further properly X_o-admissible graphs after proving the following theorem.

1.20. THEOREM: If X is X_o-admissible and Y is any graph, then there exists a graph Y_o such that

$$X \circ Y \cong X_o \circ Y_o \ .$$

Moreover the graph Y_o is Y-admissible.

PROOF. Since X is X_o-admissible there exists a graph X_1 and a subset $V \subset V(X_1)$ such that 1.17 (i)—(iii) hold. Put $Z = X \circ Y$ Then

$$V(Z) = V(X_o) \times V(X_1) \times V(Y)$$

For each $x_1 \in V(X_1)$ and each $y \in V(Y)$ let

$$W_{x_1,y} = \{z \in V(Z) \mid pr_1 z = x_1 \text{ and } pr_2 z = y\} ,$$

where pr_2 denotes projection of $V(Z)$ onto $V(Y)$.

1.20 (1) $$W_{x_1,y} \cap W_{x_1',y'} = \emptyset \text{ whenever } x_1 \neq x_1' \text{ or } y \neq y' ,$$

and

1.20 (2) $$\bigcup_{\substack{x_1 \in V(X_1) \\ y \in V(Y)}} W_{x_1,y} = V(Z)$$

This says that the sets $W_{x_1,y}$ are equivalence classes on $V(Z)$ with respect to some equivalence relation R . Put $Y_o = Z/R$. The vertex set of Y_o is the set of all equivalence classes $W_{x_1,y}$.

Define $\phi : Z \longrightarrow X_o \circ Y_o$ by

$$\phi(x_o,x_1,y) = (x_o,W_{x_1,y}) \ .$$

In view of 1.20 (1), 1.20 (2), ϕ is clearly one-one and onto.

To show that ϕ is a homomorphism take any $z = (x_o,x_1,y)$,

$z' = (x_o', x_1', y') \in V(Z)$ with $[z, z'] \in E(Z)$. Then $[(x_o,x_1), (x_o',x_1')] \in E(X)$ and $[y,y'] \in E(Y)$. Since $pr_o : X \longrightarrow X_o$ is a homomorphism this implies $[x_o,x_o'] \in E(X_o)$. It remains to show $[W_{x_1,y}, W_{x_1',y'}] \in E(Y_o)$. Since $y \neq y'$, $W_{x_1,y} \neq W_{x_1',y'}$. $z \in W_{x_1,y}$, $z' \in W_{x_1',y'}$, $[z,z'] \in E(Z)$ then imply $[W_{x_1,y}, W_{x_1',y'}] \in E(Y_o)$.

To prove that ϕ is an epimorphism let

$$[(x_o, W_{x_1,y}), (x_o', W_{x_1',y'})] \in E(X_o \circ Y_o) .$$

Then $[x_o,x_o'] \in E(X_o)$, $[W_{x_1,y}, W_{x_1',y'}] \in E(Y_o)$. This means there exist s_o, $s_o' \in V(X_o)$ such that

$$[(s_o, x_1', y), (s_o', x_1', y')] \in E(Z) .$$

Hence $[(s_o, x_1), (s_o',x_1')] \in E(X)$ and $[y,y'] \in E(Y)$. Now either $[x_1,x_1'] \in E(X_1)$ and then

$$[(x_o,x_1,y), (x_o',x_1',y')] \in E(X_o \circ X_1 \circ Y) \subset E(Z)$$

or $x_1 = x_1' \in V$ and then by 1.17 (iii) $[(x_o,x_1), (x_o',x_1')] \in E(X)$, so that again

$$[(x_o,x_1,y), (x_o',x_1',y')] \in E(Z) .$$

This completes the proof that ϕ is an isomorphism.

In order to show that Y_o is Y-admissible define an equivalence relation R_o on $V(Y_o)$ by

$$W_{x_1,y} \; R_o \; W_{x_1',y'} \quad \text{if and only if} \quad x_1 = x_1' .$$

Denote Y_o/R_o by Y_1. It is clear that the equivalence classes mod R_o are in one-one correspondence with the vertices of X_1. We shall therefore denote the equivalence class $R_o[W_{x_1,y}]$ by $\bar{x}_1$. Put

$$\bar{V} = \{\bar{x}_1 \in Y_1 \mid x_1 \in V\} ,$$

and define Z' as follows:

$$V(Z') = V(Y) \times V(Y_1) ,$$

$$E(Z') = \{[(y,\bar{x}_1),\ (y',\bar{x}_1')] \mid [y,y'] \in E(Y), \text{ and } [\bar{x}_1,\bar{x}_1'] \in E(Y_1) \text{ or } \bar{x}_1 = \bar{x}_1' \in \bar{V}\} .$$

Clearly Z' is Y-admissible. We will show $Z' \cong Y_o$. Define $\eta : Y_o \longrightarrow Z'$ by

$$\eta W_{x_1,y} = (y,\bar{x}_1) .$$

Clearly η is one-one and onto.

To prove that η is a homomorphism take $[W_{x_1,y}, W_{x_1',y'}] \in E(Y_o)$. Then there exist $x_o,x_o' \in V(X_o)$ such that $[(x_o,x_1,y), (x_o',x_1',y')] \in E(Z)$. Hence $[y,y'] \in E(Y)$, and $[x_1,x_1'] \in E(X_1)$ or $x_1 = x_1' \in V$. If $x_1 = x_1' \in V$ then $\bar{x}_1 = \bar{x}_1' \in \bar{V}$ and hence $[(y,\bar{x}_1), (y',\bar{x}_1')] \in E(Z')$. If $[x_1,x_1'] \in E(X_1)$ then $\bar{x}_1 \neq \bar{x}_1'$; hence $[\bar{x}_1,\bar{x}_1'] \in E(Y_1)$ since $[W_{x_1,y}, W_{x_1',y_1'}] \in E(Y_o)$, and therefore $[(y,\bar{x}_1), (y',\bar{x}_1')] \in E(Y_1)$. This shows that η is a homomorphism.

To show that η is an epimorphism take $[(y,\bar{x}_1'), (y',\bar{x}_1')] \in E(Z')$.

Then $[y,y'] \varepsilon E(Y)$; and $[\overline{x}_1,\overline{x}_1'] \varepsilon E(Y_1)$ or $\overline{x}_1 = \overline{x}_1' \varepsilon \overline{V}$. In the first case $[W_{x_1,z}, W_{x_1',z'}] \varepsilon E(Y_o)$ for some $z, z' \varepsilon V(Y)$. Hence there exist $x_o,x_o' \varepsilon V(X_o)$ such that $[(x_o,x_1,z), (x_o',x_1',z')] \varepsilon E(Z)$. This implies $[(x_o,x_1,y), (x_o',x_1',y')] \varepsilon E(Z)$, i.e., $[W_{x_1,y}, W_{x_1',y'}] \varepsilon E(Y_o)$. In the second case where $\overline{x}_1 = \overline{x}_1' \varepsilon \overline{V}$, we argue as follows: if $E(X_o) = \emptyset$ then X being X_o-admissible implies $E(X) = \emptyset$ and hence we could take $V = \emptyset$. Then $\overline{V} = \emptyset$, and hence $\overline{x}_1 = \overline{x}_1' \varepsilon \overline{V}$ could not ar- rise. If $E(X_o) \neq \emptyset$, take any $[x_o,x_o'] \varepsilon E(X_o)$. Hence again $[W_{x_1,y}, W_{x_1,y'}] \varepsilon E(Y_o)$.

Although this completes the proof we will finally show that if $E(Y) \neq \emptyset$ and $E(X_o) \neq \emptyset$ then $X_1 \cong Y_1$. Let $\psi : X_1 \longrightarrow Y_1$ be defined by

$$\psi\, x_1 = \overline{x}_1 \ .$$

As remarked earlier ψ is one-one and onto. To show that ψ is a homomor- phism let $[x_1,x_1'] \varepsilon E(X_1)$. Since $E(X_o) \neq \emptyset$, there exists $[s_o,s_o'] \varepsilon E(X_o)$ and hence

$$[(s_o,x_1), (s_o',x_1')] \varepsilon E(X) \ .$$

Since $E(Y) \neq \emptyset$, there exist $y,y' \varepsilon V(Y)$ with

$$[y,y'] \varepsilon E(Y) \ .$$

Therefore

$$[(s_o,x_1,y), (s_o',x_1',y')] \varepsilon E(Z) \ ,$$

i.e., $[W_{x_1,y}, W_{x_1',y'}] \in E(Y_o)$. Now $x_1 \neq x_1'$ implies $\overline{x}_1 \neq \overline{x}_1'$ and therefore $[\overline{x}_1,\overline{x}_1'] \in E(Y_1)$.

To show that ψ is an epimorphism is trivial and hence we have that ψ is an isomorphism. In particular if $X \cong X_o \otimes X_1$ then $Y_o \cong Y \otimes X_1$.

We now return to the question of the existence of properly X_o-admissible graphs. Let X_o be a finite graph with $E(X_o) \neq \emptyset$, X_1 a graph of odd order. Then $X = X_o \otimes X_1$ is properly X_o-admissible. This follows from

$$|E(X_o \otimes X_1)| = m_o(2m_1+n_1) \ ,$$

$$|E(X_o \circ Z)| = 2m_o k \ ,$$

where $m_i = |E(X_i)|$, $i = 0, 1,$ $k = |E(Z)|$, $n_1 = |X_1|$. Hence if $X_o \otimes X_1 \cong X_o \circ Z$, then

$$2m_1+ n_1 = 2k \ ,$$

contrary to n_1 being odd.

Now take Y to be any finite graph with $E(Y) \neq \emptyset$. By 1.20 there exists a Y-admissible graph Y_o such that

$$X \circ Y \cong X_o \circ Y_o \ .$$

From the proof of 1.20 $Y_o \cong Y \otimes X_1$ and hence we have that Y_o is properly Y-admissible.

This shows that the decomposition of connected graphs into a cardinal product of indecomposable factors is non-unique in a very strong sense. For if we take Y and X_o to be indecomposable and non-isomorphic as well then Y does not occur as a factor in either X_o or Y_o since Y_o is properly Y-admissible, and X_o does not appear as a factor in either X or Y , since X is properly X_o-admissible.

SECTION IV: Decomposability of products with respect to other multiplications

As a consequence of the following proposition we have that the cardinal product of two non-trivial graphs is in general not a prime graph with respect to cartesian multiplication.

1.21. PROPOSITION: Let X_1 and X_2 be connected graphs of bounded degree. Then $X_1 \circ X_2 \cong X_1 \times X_2$ if and only if $X_1 \cong X_2 \cong C_n$, where C_n is an n-circuit of odd order.

PROOF. If $X_1 \circ X_2 \cong X_1 \times X_2$, X_i connected $i = 1, 2,$ we have by 1.7 and 1.11 that at least one of the X_i's is non-bipartite, say X_1. If X_2 is bipartite then $X_1 \circ X_2$ is also bipartite by 1.11, contrary to $X_1 \times X_2$ being non-bipartite. Hence both X_1 and X_2 are non-bipartite. Let the odd mesh of X_1 and X_2 be k_1 and k_2 respectively. Clearly $X_1 \times X_2$ has odd mesh = $\min\{k_1, k_2\}$ and by 1.13 the odd mesh of $X_1 \circ X_2 = \max\{k_1, k_2\}$. Therefore $k_1 = k_2$.

We now use the fact that X_1 and X_2 are of bounded degree. For $i = 1, 2$ let

$$d_i = \sup_{x \in X_i} d(x;X_i) .$$

By hypothesis $d_i < \infty$, $i = 1, 2$. Then

$$\sup_{x \in X_1 \circ X_2} d(x; X_1 \circ X_2) = d_1 d_2$$

and

$$\sup_{x \varepsilon X_1 \times X_2} d(x; X_1 \times X_2) = d_1 + d_2 .$$

Since $X_1 \circ X_2 \cong X_1 \times X_2$ we have $d_1 d_2 = d_1 + d_2$, i.e., $d_1 = 2 = d_2$. This together with X_1 and X_2 being non-bipartite graphs of the same odd mesh implies $X_1 \cong X_2 \cong C_n$, where C_n is an odd circuit.

To prove the converse let $C_n = [x_o, x_1, \ldots, x_{n-1}]$ and define

$$\phi : C_n \times C_n \longrightarrow C_n \circ C_n$$

as follows: for $0 \leqq i \leqq n-1$, $0 \leqq j \leqq n-1$, define

$$\phi(x_i, x_j) = (x_{j+i}, x_{j-i})$$

where the subscripts are taken mod n .

Since n is odd we have that $\phi : V(C_n \times C_n) \longrightarrow V(C_n \circ C_n)$ is one-one and onto. Moreover it is easily verified that $\phi : C_n \times C_n \longrightarrow C_n \circ C_n$ is an isomorphism.

1.22. PROPOSITION: <u>In the class of graphs without isolated vertices, the cartesian (strong) product of two non-trivial graphs is indecomposable with respect to strong (cartesian) multiplication</u>.

PROOF. Assume the contrary, i.e., there exists an isomorphism $\phi : X_o \times X_1 \longrightarrow Y_o * Y_1$, where X_i, Y_i, $i = 0, 1$, are non-trivial graphs without isolated vertices. Let

$$E_i = \{[x,y] \ \varepsilon\ E(X_o \times X_1) \mid [pr_i x, pr_i y] \ \varepsilon\ E(X_i)\}, \quad i = 0, 1,$$

$$F_i = \{[x,y] \ \varepsilon\ E(Y_o * Y_1) \mid [pr_i x, pr_i y] \ \varepsilon\ E(Y_i),\ pr_{1-i} x = pr_{1-i} y\}, \quad i = 0, 1,$$

and

$$G = E(Y_o * Y_1) - (F_o \cup F_1) .$$

Since X_i has an edge, $i = 0, 1$, there exists a saturated 4-circuit $C = [x_o, x_1, x_2, x_3] \subset X_o \times X_1$ with $e_o, e_2 \varepsilon E_o$, $e_1, e_3 \varepsilon E_1$, where $e_i = [x_i, x_{i+1}]$, $i = 0, 1, 2$ and $e_3 = [x_3, x_o]$. Denote ϕx_i by y_i and ϕe_i by e'_i, $i = 0, \ldots, 3$. There are three cases to consider;

(1^o) $e'_o, e'_1 \varepsilon G$,

(2^o) exactly one of e'_o, e'_1 is in G,

and

(3^o) $e'_o, e'_1 \notin G$.

Case (1^o): $e'_j \varepsilon G$ implies the subgraph A_j generated by $y_j, y_{j+1}, i_o^{y_j} pr_o y_{j+1}, i_o^{y_{j+1}} pr_o y_j$ is a complete 4-graph in $Y_o * Y_1$, $j = 0, 1$. Clearly $\phi^{-1} A_j \subset E_j$, $j = 0, 1$. Let $z_o = i_o^{y_1} pr_o y_o$ and $z_1 = i_o^{y_2} pr_o y_1$. $[z_o, z_1] \varepsilon G$ since $[pr_j z_o, pr_j z_1] \varepsilon E(Y_j)$, $j = 0, 1$. However $[\phi^{-1} z_j, x_1] \varepsilon E_j$ since $\phi^{-1} A_j \subset E_j$, $j = 0, 1$, and hence $[\phi^{-1} z_o, \phi^{-1} z_1] \notin E(X_o \times X_1)$, a contradiction to ϕ being an isomorphism.

Case (2^o): Without loss of generality take $e'_o \varepsilon F_o$ and $e'_1 \varepsilon G$. $e'_1 \varepsilon G$ implies the subgraph A generated by y_1, y_2, $i_o^{y_1} pr_o y_2, i_o^{y_2} pr_o y_1$ is a complete 4-graph in $Y_o * Y_1$. Let $z = i_o^{y_2} pr_o y_1$ and $e'' = [y_1, z]$. $[y_o, z] \varepsilon G$. However $\phi^{-1} A \subset E_1$ implies $\phi^{-1} e'' \varepsilon E_1$; $\phi^{-1} e'' \varepsilon E_1$ and $e_o \varepsilon E_o$ imply $[x_o, \phi^{-1} z] \notin E(X_1 \times X_2)$, contrary to ϕ being an isomorphism.

Case (3^o): Without loss of generality we need only consider the two cases $e'_o, e'_1 \varepsilon F_o$, or $e'_o \varepsilon F_o$ and $e'_1 \varepsilon F_1$. Suppose e'_o, $e'_1 \varepsilon F_o$. Since Y_1 contains no isolated vertices there exists $z \varepsilon Y_o * Y_1$ with $e' = [y_1,z] \varepsilon F_1$ and $[z,y_o]$, $[z,y_2] \varepsilon G$. Without loss of generality let $e = \phi^{-1}e' \varepsilon E_o$. $e \varepsilon E_o$, $e_1 \varepsilon E_1$ and e, e_1 adjacent imply $[x_2, \phi^{-1}z] \notin E(X_o \times X_1)$. This is a contradiction since $[\phi x_2, z] \varepsilon E(Y_o * Y_1)$. The case $e'_o \varepsilon F_o$, $e'_1 \varepsilon F_1$ immediately yields a contradiciton.

1.23. COROLLARY: <u>The cartesian (strong) product of two non-trivial connected graphs is indecomposable with respect to strong (cartesian) multiplication</u>.

1.24. PROPOSITION: <u>In the class of graphs with at least one edge, the strong (cardinal) product is indecomposable with respect to cardinal (strong) multiplication</u>.

PROOF. Assume instead that there exists an isomorphism $\phi : Y_o * Y_1 \longrightarrow X_o \circ X_1$, where X_i, Y_i, $i = 0, 1$ are graphs with at least one edge. There exists a complete 4-graph $A \subset Y_o * Y_1$ with vertex set $\{y_o,y_1,y_2,y_3\}$ such that $e_o, e_2 \varepsilon F_o$, $e_1, e_3 \varepsilon F_1$ and $[y_o,y_2]$, $[y_1,y_3] \varepsilon G$, where $e_i = [y_i, y_{i+1}]$, $i = 0, 1, 2$, $e_3 = [y_3,y_o]$ (F_o, F_1 and G as in 1.22). It is easily verified that $pr_j\phi A$ is a complete 4-graph in X_j, $j = 0, 1$. Set $\phi y_i = x_i$, $i = 0, \ldots, 3$ and $pr_j x_i = x_i^{(j)}$, $j = 0, 1$, $i = 0, \ldots, 3$. Let $x'_o = (x_2^{(o)}, x_o^{(1)})$, $x'_1 = x_3^{(o)}, x_1^{(1)})$, $x'_2 = (x_o^{(o)}, x_2^{(1)})$, $x'_3 = (x_1^{(o)}, x_3^{(1)})$ (Fig. 1.2).

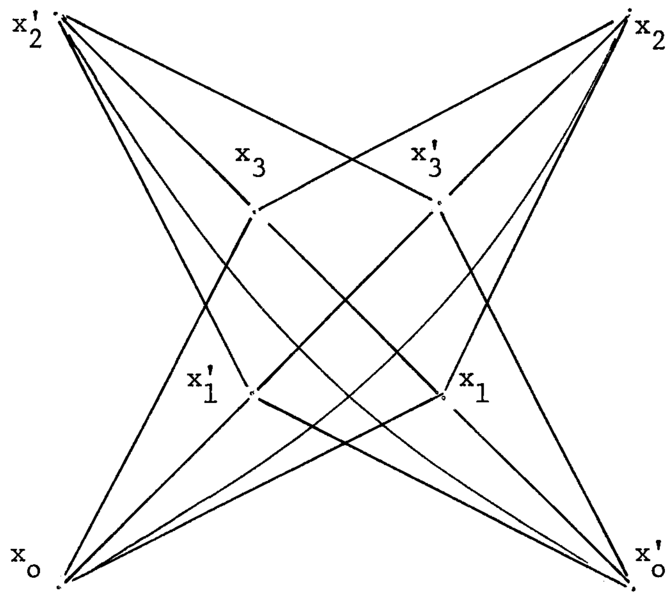


FIGURE 1.2

Now $[x_1, x_3'] \notin X_o \times X_1$ implies $[y_1, \phi^{-1}x_3'] \notin Y_o * Y_1$ and hence $[y_2, \phi^{-1}x_3'] \varepsilon F_1$, otherwise $[y_1, y_2, \phi^{-1}x_3']$ would form a triangle contradicting $[y_1, \phi^{-1}x_3'] \notin Y_o * Y_1$. But $[y_2, y_3] \varepsilon F_o$, $[y_2, \phi^{-1}x_3'] \varepsilon F_1$ imply $[y_3, \phi^{-1}x_3'] \varepsilon Y_o * Y_1$, contrary to $[x_3, x_3'] \notin X_o \circ X_1$.

1.25. COROLLARY: The strong (cardinal) product of two non-trivial connected graphs is indecomposable with respect to cardinal (strong) multiplication.

SECTION V: Products of rooted graphs.

We now turn our attention to the concept of products of rooted graphs. By 1.8 and 1.9 we have that the weak and cartesian product of infinitely many non-trivial connected graphs is disconnected. Since connectedness is essential to the question of unique prime factorization of graphs we introduce the following definitions:

1.26. DEFINITION: Let $(X_a)_{a\varepsilon A}$ be a family of graphs and let $r_a \varepsilon V(X_a)$, $a\varepsilon A$. By the <u>weak cartesian product</u> $\prod_{a\varepsilon A}(X_a,r_a)$ of the rooted graphs (X_a,r_a) we mean the graph X defined by:

$V(X) = \{x \varepsilon \prod_{a\varepsilon A} V(X_a) \mid pr_a x \neq r_a$ for at most finitely many $a\varepsilon A\}$

$E(X) = \{[x,y] \mid x,y\varepsilon V(X),\ [pr_a x,\ pr_a y] \varepsilon E(X_a)$ for exactly one $a\varepsilon A$,

$pr_b x = pr_b y$ for $b\varepsilon A - \{a\}\}$.

For each $b\varepsilon A$ and for each $x\varepsilon V(\prod_{a\varepsilon A}(X_a,r_a)$ we define the <u>injection</u> $i_b^x : V(X_b) \longrightarrow V(\prod_{a\varepsilon A}(X_a,r_a))$ as in 1.4. Here $i_b^x : X_b \longrightarrow \prod_{a\varepsilon A}(X_a,r_a)$ is also a monomorphism.

If the index set A is finite then the weak cartesian product of the rooted graphs (X_a,r_a) does not depend on the roots and is equal to $\prod_{a\varepsilon A} X_a$.

1.27. PROPOSITION: <u>Let</u> $((X_a,r_a))_{a\varepsilon A}$ <u>be a family of rooted graphs</u>. <u>If</u> X_a <u>is connected for each</u> $a\varepsilon A$, <u>then</u> $X = \prod_{a\varepsilon A}(X_a,r_a)$ <u>is connected</u>.

PROOF. Define $r \varepsilon V(X)$ by $pr_a r = r_a$, $a \varepsilon A$. We will show that X is the connected component of $\prod_{a \varepsilon A} X_a$ containing r . For $x \varepsilon V(X)$, $pr_a x \neq r_a$ for at most finitely many $a \varepsilon A$.

Let

$$X'_a = \begin{cases} X_a , & \text{if } pr_a x \neq r_a \\ (r_a) , & \text{if } pr_a x = r \end{cases} , \quad a \varepsilon A .$$

Then $X' = \prod_{a \varepsilon A} (X'_a, r_a) \subset \prod_{a \varepsilon A} (X_a, r_a)$ is connected (since X'_a is connected and non-trivial for only finitely many $a \varepsilon A$) and contains r and x . It is easily verified that for $y \varepsilon V(\prod_{a \varepsilon A} X_a) - V(X)$, there does not exist a path joining y and r (see proof of 1.8).

1.28. DEFINITION: Let $((X_a, r_a))_{a \varepsilon A}$ be a family of rooted graphs. By the <u>weak product</u> $\prod^{w}_{a \varepsilon A} (X_a, r_a)$ <u>of the rooted graphs</u> (X_a, r_a) we mean the graph X defined by:

$V(X) = \{x \varepsilon \prod_{a \varepsilon A} V(X_a) \mid pr_a x \neq r_a$ for at most finitely many $a \varepsilon A \}$.

For $x,y \varepsilon V(X)$,

$[x,y] \varepsilon E(X)$ if and only if there exists a non-empty finite subset $B \subset A$ such that

$$[pr_b x, pr_b y] \varepsilon E(X_b) \quad , b \varepsilon B ,$$

and

$$pr_a x = pr_a y \quad , a \varepsilon A-B$$

Since $pr_b x \neq pr_b y$ for only finitely many b's the condition that the subset B be finite can be dropped. Hence if we introduce a similar definition of the strong product of the family of rooted graphs this will be identical to the weak product of the family of rooted graphs.

If the index set A is finite then the weak product of the rooted graphs (X_a, r_a) does not depend on the roots and is equal to $\prod_{a\varepsilon A}^{w} X_a$.

1.29. PROPOSITION: Let $((X_a, r_a))_{a\varepsilon A}$ be a family of rooted graphs. If X_a is connected for each $a\varepsilon A$, then $\prod_{a\varepsilon A}^{w}(X_a, r_a)$ is connected.

PROOF. Similar to 1.27.

1.30. PROPOSITION: Let $(n_a)_{a\varepsilon A}$ be a family of cardinals, $C(n_a)$ a complete n_a-graph, $r_a \varepsilon V(C(n_a))$, and let $C = \prod_{a\varepsilon A}^{w}(C(n_a), r_a)$. Then

$$C \cong C(n)$$

where

$$n = \begin{cases} \prod_{a\varepsilon A} n_a & , \quad A \text{ finite,} \\ \sum_{a\varepsilon A} n_a & , \quad A \text{ infinite.} \end{cases}$$

PROOF. If A is finite then clearly $|C| = \prod_{a\varepsilon A} n_a$, since here the weak product of the rooted graphs is independent of the roots and equal to the weak product. Suppose A is infinite.
Take $x \varepsilon V(C)$ and define

$$B_x = \{a\varepsilon A \mid pr_a x \neq r_a\} \quad .$$

This set is finite and the mapping

$$f(x) = \{(pr_a x, a) : a\varepsilon B_x\}$$

is obviously a one-one function from $V(C)$ into the set F of all finite subsets of $\bigcup_{a\varepsilon A} N_a$, where $N_a = V(C(n_a)) \times \{a\}$. Hence

$$|C| \leqq |F| = \left|\bigcup_{a\varepsilon A} N_a\right| = \sum_{a\varepsilon A} n_a ,$$

since A is infinite.

Now define $g : \bigcup_{a\varepsilon A} N_a \longrightarrow V(C)$ by

$$pr_b g(\alpha, a) = \begin{cases} \alpha \text{ if } b = a \\ r_b \quad \text{if} \quad b \neq a . \end{cases}$$

Clearly, g is one-one, and hence $\sum_{a\varepsilon A} n_a \leqq |C|$. Therefore if A is infinite,

$$|C| = \sum_{a\varepsilon A} n_a .$$

To show that C is complete we argue as follows: for $x,y \ \varepsilon \ V(C)$, $x \neq y$ let

$$B = \{a\varepsilon A \mid pr_a x \neq pr_a y\} .$$

Since $pr_a x = pr_a y = r_a$ for almost all a, B is finite, and since $x \neq y$, $B \neq \emptyset$. Since $C(n_a)$ is complete, $pr_a x \neq pr_a y$ implies $[pr_a x, pr_a y] \ \varepsilon \ E(C(n_a))$, and therefore $[x,y] \ \varepsilon \ V(C)$.

We will now use the preceding proposition to show that <u>the weak product of rooted graphs does not satisfy unique prime factorization</u>. First we note that for any integer n , C(n) is indecomposable if and only if n is a prime. $C(\aleph_\alpha)$ can be decomposed in infinite many distinct ways into prime factors as follows: let A be any index set with $|A| = \aleph_\alpha$, p and q distinct primes. For each $a \varepsilon A$, let $n_a = p$, $n'_a = q$, $r_a \varepsilon V(C(n_a))$, $r'_a \varepsilon V(C(n'_a))$. Since $\aleph_\alpha = \sum_{a \varepsilon A} n_a = \sum_{a \varepsilon A} n'_a$ we have by 1.22 that

$$C(\aleph_\alpha) \cong \prod_{a \varepsilon A}{}^{w}(C(n_a), r_a) \cong \prod_{a \varepsilon A}{}^{w}(C(n'_a), r'_a) .$$

CHAPTER II

DECOMPOSITION OF GRAPHS INTO WEAK CARTESIAN PRODUCTS

This chapter is primarily devoted to showing that every connected graph X has a weak cartesian decomposition into indecomposable factors that is unique to within isomorphisms. Roughly speaking we will exhibit an invariant equivalence relation on E(X) such that two edges are equivalent if and only if they project to the same factor. To be more explicit we will investigate a particular set of equivalence relations (the acyclic equivalences (2.6) which contain $\alpha \cup \beta$ (2.8)) in the lattice of all equivalence relations on E(X) and show that this is a principal filter with the following property: each equivalence in this filter gives rise to a weak cartesian decomposition of X such that two edges are equivalent if and only if they project to the same factor and the least element of the filter decomposes the graph X into prime factors. We will moreover show that to each decomposition of X as a weak cartesain product there corresponds an equivalence relation in this filter with the property that two edges are equivalent if and only if they project to the same factor. The least element will correspond to a prime decomposition.

Unless otherwise stated X, Y, ... will denote arbitrary graphs.

SECTION I: ρ- compatible graphs and acyclic equivalence relations.

2.1. DEFINITIONS: Let ρ be an equivalence relation on $E(X)$. A subgraph Y of X will be called ρ-compatible if and only if Y has a cover $\mathcal{B}$ such that

(i) every $B \in \mathcal{B}$ is a proper path, and

(ii) for $B, B' \in \mathcal{B}$, $E(B) \times E(B') \subset \rho$ or $\bar{\rho}$ according as $B = B'$ or $B \neq B'$.

It will be convenient to apply the term ρ-compatible to the cover $\mathcal{B}$ as well.

2.2. CONVENTION: Let ρ be an equivalence relation on $E(X)$. When a ρ-compatible path P is written in the form $P = P_1 \cup \ldots \cup P_n$ it is automatically understood that

(i) P_i is a proper path, $i = 1, \ldots, n$,

(ii) $P_i \cap P_j \neq \emptyset$ if and only if $|i-j| \leq 1$, and

(iii) $E(P_i) \times E(P_j) \subset \rho$ or $\bar{\rho}$ according as $i = j$ or $i \neq j$.

Similarly, if a ρ-compatible circuit C is written in the form $C = P_o \cup \ldots \cup P_n$ it is understood that

(i) P_i is a proper path, $i = 0, i, \ldots, n$,

(ii) $P_i \cap P_j \neq \emptyset$ if and only if either $|i-j| \leq 1$ or $|i-j| = n$, and

(iii) $E(P_i) \times E(P_j) \subset \rho$ or $\bar{\rho}$ according as $i = j$ or $i \neq j$.

2.3. REMARK: Let C be a circuit such that $E(C) \times E(C) \not\subset \rho$, where ρ is an equivalence on $E(X)$. Then clearly C can be uniquely expressed as the union of proper paths, each path being maximal with respect to its edges belonging to one equivalence class mod ρ, i.e., $C = P_o \cup \ldots \cup P_n$, where P_i is a maximal proper path such that

2.3 (1) $$E(P_i) \times E(P_i) \subset \rho \ , \ i = 0, 1, \ldots, n \ .$$

This decomposition will be called the ρ-<u>decomposition</u> of C or the <u>decomposition</u> of C <u>determined</u> by ρ, and $n + 1$ will be called the ρ-<u>degree of</u> C. We will denote the ρ-degree of C by $\deg_\rho C$. Whenever the ρ-decomposition of a circuit C is written in the form

$$C = P_o \cup \ldots \cup P_n$$

it will automatically be understood

$$P_i \cap P_j \neq \emptyset \text{ if and only if either } |i-j| \leq 1 \text{ or } |i-j| = n \ .$$

By the maximality of the P_i we have

2.3 (2) $$E(P_i) \times E(P_j) \subset \bar{\rho} \text{ if } |i-j| = 1 \text{ or } |i-j| = n \ .$$

If $C = P_o \cup \ldots \cup P_n$ is not ρ-compatible there exist integers i_o and j_o with $i_o < j_o$ such that $E(P_{i_o}) \times E(P_{j_o}) \subset \rho$. If the path $P_{i_o+1} \cup P_{i_o+2} \cup \ldots \cup P_{j_o}$ is not ρ-compatible, there exist i_1 and j_1 with $i_o < i_1 < j_1 \leq j_o$ such that $E(P_{i_1}) \times E(P_{j_1}) \subset \rho$. If $P_{i_1+1} \cup P_{i_1+2} \cup \ldots \cup P_{j_1}$ is not ρ-compatible we can repeat the above process. Since this can only be done a finite number of times there exist integers $i < j$ such that

2.3 (3) $$E(P_i) \times E(P_j) \subset \rho$$

and

2.3 (4) $$P_{i+1} \cup P_{i+2} \cup \dots \cup P_j \text{ is a } \rho\text{-compatible path.}$$

2.4 PROPOSITION: Let Y be a connected ρ-compatible subgraph of X. Then given any two distinct vertices $x,y \in Y$ there exists a ρ-compatible path joining x and y in Y.

PROOF: Since Y is connected there is a path $[x_o, \dots, x_m] \subset Y$ such that $x_o = x$, $x_m = y$. Let $\mathfrak{L}$ be a ρ-compatible cover of Y and let W_i be that path belonging to $\mathfrak{L}$ which contains the edge $[x_{i-1}, x_i]$, $i = 1, \dots, m$. This means that $W_1, \dots, W_m$ are paths in $\mathfrak{L}$ such that $x \in W_1$, $y \in W_m$, and $W_i \cap W_{i+1} \neq \emptyset$, $i = 1, \dots, m-1$. Now let n be the smallest number for which $\mathfrak{L}$ contains n paths $P^{(1)}, \dots, P^{(n)}$ such that

(i) $x \in P^{(1)}$, $y \in P^{(n)}$, and

(ii) $P^{(j)} \cap P^{(j+1)} \neq \emptyset$, $j = 1, \dots, n-1$.

Then $P^{(k)} \cap P^{(j+1)} = \emptyset$ for all $k < j < n$. For if there exists a $k < j$ with $P^{(k)} \cap P^{(j+1)} \neq \emptyset$, then

$$P^{(1)}, \dots, P^{(k)}, P^{(j+1)}, \dots, P^{(n)}$$

is set of fewer than n paths in $\mathfrak{L}$ with properties (i) and (ii). Now put $y_o = x$, $y_n = y$, and for $i = 1, \dots, n-1$ define y_i inductively to be a vertex in $P^{(i)} \cap P^{(i+1)}$ such that no other vertex of $Q_i = P^{(i)}_{y_{i-1}y_i}$

also belongs to $P^{(i+1)}$. Then

$$P = \bigcup_{i=1}^{n} Q_i$$

is a ρ-compatible path joining x and y in Y .

2.5 PROPOSITION: Let Y be a ρ-compatible subgraph of X which is not acyclic. Then Y contains a ρ-compatible finite circuit.

PROOF. Since Y is not acylic there exists a finite circuit $C = [x_o, \ldots, x_n] \subset Y$. Let $\mathcal{L}$ be a ρ-compatible cover of Y . Let $W^{(o)}$ be that path belonging to $\mathcal{L}$ which contains the edge $[x_o, x_n]$ and $W^{(i)}$ that path belonging to $\mathcal{L}$ which contains the edge $[x_{i-1}, x_i]$, $i = 1, \ldots, n$. $W^{(o)} \neq W^{(i)}$ for at least one i , $1 \leq i \leq n$. Otherwise $C \subset W^{(o)}$, a contradiction to $W^{(o)}$ being a path. Hence there exist $x_h, x_k \in W^{(o)} \cap C$, $o \leq h < k \leq n$, such that $x_t \notin V(W^{(o)})$ for $h < t < k$.

$$Z = \bigcup_{i=h+1}^{k} W^{(i)}$$

is a connected ρ-compatible subgraph containing x_h and x_k and hence there exists a ρ-compatible path P joining x_h and x_k in Z . Either $W^{(o)}_{x_h x_k} \cup P$ is the desired ρ-compatible circuit or there exists an $x \in W^{(o)}_{x_h x_k} \cap P$ such that $W^{(o)}_{x_h x} \cup P_{x x_h}$ is the required circuit.

2.6. DEFINITION: An equivalence relation ρ on $E(X)$ is called acyclic if and only if every ρ-compatible subgraph of X is acyclic.

2.7. PROPOSITION: A necessary and sufficient condition that ρ be acyclic is that X contain no ρ-compatible finite circuit.

PROOF. Necessity: Assume that ρ is not acyclic. By definition there exists a ρ-compatible subgraph of X which is not acyclic and hence by 2.5, X contains a ρ-compatible finite circuit.

Sufficiency: Trivial.

SECTION II: The binary relations α and β

The following two binary relations α and β on $E(X)$ are of considerable importance in our subsequent considerations.

2.8. DEFINITION: Let X be a graph, $e, e' \varepsilon E(X)$.

$e \, \alpha \, e'$ if and only if

(i) e and e' are adjacent, and

(ii) among the saturated subgraphs of X which contain e and e' there is no 4-circuit.

$e \, \beta \, e'$ if and only if

(i) e and e' are not adjacent, and

(ii) among the saturated subgraphs of X which contain e and e' there is a 4-circuit.

In general, neither α nor β is an equivalence relation. By ρ_o we shall denote the smallest equivalence on $E(X)$ which contains $\alpha \cup \beta$.

Note that if X is connected and contains no 4-circuit, or if X is connected and every 4-circuit of X has a diagonal, then $\rho_o = E(X) \times E(X)$.

2.9. PROPOSITION: Let X be a connected graph and let ρ be an equivalence on $E(X)$ containing $\alpha \cup \beta$. Then given any vertex $x \varepsilon X$ and any equivalence class E mod ρ, there is an $e \varepsilon E$ which is incident with x. Hence $E(X)$ consists of at most $\min_{x \varepsilon X} d_x$ equivalence classes mod ρ.

PROOF. Suppose there is no edge in E which is incident with x. Let x_o be a vertex of X such that

(i) x_o is incident with some $e \varepsilon E$, and

(ii) among all vertices having property (i), x_o has minimal distance from x.

Let P be a shortest path joining x and x_o and let $e_o = [x_o, x_1]$ be that edge of P which is incident with x_o. Note that $e_o \notin E$, for otherwise x_1 would be incident with an edge of E, contrary to (ii). $e_o \notin E$ implies $e_o \overline{\rho} e$, hence $e_o \overline{\alpha} e$. Since e_o and e are adjacent, there is a saturated 4-circuit C which contains both e_o and e. Let e' be the edge of C opposite e. Then $e' \beta e$, hence $e' \varepsilon E$. But e' is incident with x_1, a contradiction against (ii).

SECTION III: Construction of ladders.

2.10. CONSTRUCTION: By a ladder is meant a graph which is isomorphic to the cartesian product of an edge with a proper path.

Let ρ be an equivalence on $E(X)$ which contains $\alpha \cup \beta$. Let x, y, x' be distinct vertices of X, $e = [x, x'] \in E(X)$, and let P be a path joining x and y with $E(P) \times \{e\} \subset \overline{\rho}$. We will now give a method for constructing a ladder in X from e and P provided that one of the following conditions holds:

(i) P is a shortest path joining x and y,

(ii) ρ is acyclic and $P = P_1 \cup \ldots \cup P_n$ is a ρ-compatible path joining x and y (here P need not be a shortest path joining x and y).

Denote the consecutive vertices of P by $x_o = x, x_1, \ldots, x_{s-1}, x_s = y$ and let $e_i = [x_{i-1}, x_i]$, $i = 1, \ldots, s$. $e \overline{\rho} e_1$ implies $e \overline{\alpha} e_1$. Since e and e_1 are adjacent X contains a saturated 4-circuit

$$C_1 = [x_o, x_1, x_1', x_o' = x']$$

such that $e, e_1 \in E(C_1)$. Let

$$e^{(o)} = e,\ e^{(1)} = [x_1, x_1'] \text{ and } e_1' = [x', x_1'].$$

Then $e_1 \beta e_1'$, and $e \beta e^{(1)}$, so that $e^{(1)} \overline{\rho} e_2$ (otherwise $e \beta e^{(1)} \rho e_2$, contrary to $e \overline{\rho} e_2$). This implies $e^{(1)} \overline{\alpha} e_2'$, hence again X contains a saturated 4-circuit

$$C_2 = [x_1,x_2,x_2',x_1']$$

such that $e^{(1)}$, $e_2 \varepsilon E(C_2)$. Thus we obtain two new edges

$$e^{(2)} = [x_2,x_2'] \quad \text{and} \quad e_2' = [x_1',x_2']$$

such that $e_2 \ \beta \ e_2'$ and $e^{(1)} \ \beta \ e^{(2)}$. Proceeding in this manner we produce two new sequences,

$$e_1', \ldots, e_s' \quad \text{and} \quad e^{(o)}, \ldots, e^{(s)}$$

of edges of X such that

2.10 (1) e_i' and e_{i+1}' are either equal or adjacent, $i = 1,\ldots, s - 1$,

2.10 (2) $e_i \ \beta \ e_i'$, $i = 1, \ldots, s$,

2.10 (3) $e^{(o)} \ \beta \ e^{(1)} \ \beta \ldots \beta \ e^{(s)}$.

Let $Q = e_1' \cup e_2' \cup \ldots \cup e_s'$.

Now if we assume that P is a shortest path joining x and y we can show the above construction yields a ladder. We first prove that $P \cap Q = \emptyset$. Assume instead that there exist

$$x_i \varepsilon P \ , \quad o \leqq i \leqq s \ ,$$

$$x_j' \varepsilon Q \ , \quad o \leqq j \leqq s \ ,$$

with

$$x_i = x_j' \ ,$$

and without loss of generality we may take $i < j$. $j \neq i + 1$ since

$C_i = [x_i, x_{i+1}, x_{i+1}', x_i']$ was a saturated 4-circuit and hence $x_i \neq x_{i+1}'$.
Therefore $j - i \geqq 2$ and hence

$$[x = x_o, x_1, \ldots, x_{i-1}, x_i = x_j', x_j, \ldots, x_s = y]$$

is a path joining x and y of length less than s . This is a contradiction to the minimality of the length of P . Hence $P \cap Q = \emptyset$. Next we show Q is a path of length s . Assume the contrary, i.e., there exist

$$x_i', x_j' \in Q, o \leqq i \leqq s, o \leqq j \leqq s \quad \text{with} \quad x_i' = x_j',$$

and take $i < j$. If $i + 2 < j$ then

$$[x = x_o, x_1, \ldots, x_i, x_i' = x_j', \ldots, x_s = y]$$

is a path of length less than s joining x and y , contradicting the minimality of the length of P . If $i + 2 = j$ then

$$C = [x_i, x_{i+1}, x_{i+2}, x_{i+2}' = x_i']$$

is a 4-circuit containing $e^{(i)}$ and e_{i+1} (Fig. 2.1) . $e^{(1)} \bar{\rho} e_{i+1}$ implies $e^{(i)} \bar{\alpha} e_{i+1}$ and hence C has at least one diagonal.

$[x_i', x_{i+1}] \notin E(X)$ since $C_i = [x_i, x_{i+1}, x_{i+1}', x_i']$ was a saturated 4-circuit. Therefore $[x_i, x_{i+2}] \in E(X)$. Hence

$$[x = x_o, x_1, \ldots, x_i, x_{i+2}, \ldots, x_s = y]$$

is a path of length $s - 1$ joining x and y , contradicting the minimality of the length of P .

Therefore Q is a path of length s and $P \cup Q \cup e^{(o)} \cup \ldots \cup e^{(s)}$ is isomorphic to $P \times \{e\}$.

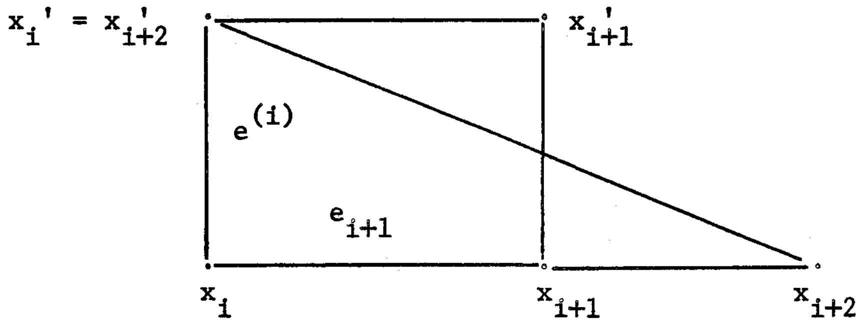


FIGURE 2.1

Now assume instead that ρ is acyclic and that $P = P_o \cup \ldots \cup P_n$ is a ρ-compatible path joining x and y . (Here P need not be a shortest path.) We shall only consider the case $n = 1$. The reader will have no difficulty in extending the argument to $n \geqq 2$. Suppose there exist vertices $x_i', x_j' \varepsilon Q$ with $x_i' = x_j'$ and $i < j$. Since $\alpha \cup \beta \subset \rho$, 2.10 (3) implies $e^{(i)} \; \rho \; e^{(j)}$. Hence

$$e_{i+1} \cup \ldots \cup e_j , \; e^{(j)} \cup e^{(i)}$$

would form a ρ-compatible circuit contradicting the acyclicity of ρ (Fig. 2.2). Therefore Q is again a path length s .

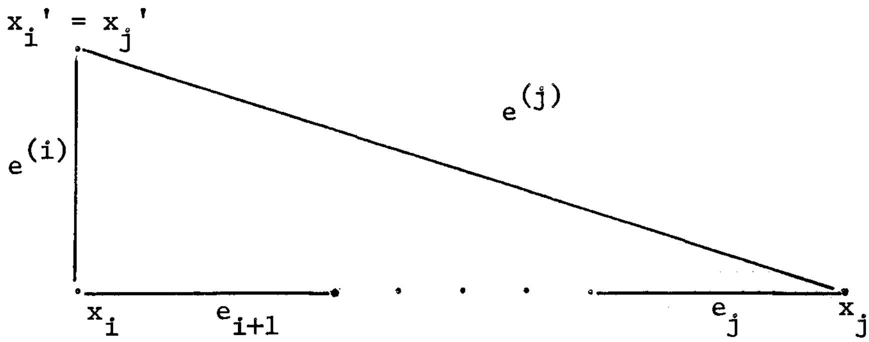


FIGURE 2.2

If $P \cap Q \neq \emptyset$, then there exist $x_i \varepsilon P$, $x_j' \varepsilon Q$, $o \leqq i \leqq s$, $o \leqq j \leqq s$ with $x_i = x_j'$. Again we may assume without loss of generality that $i < j$. $P \times \{e\} \subset \bar{\rho}$ and $e \, \rho \, e^{(j)}$ imply $P \times \{e^{(j)}\} \subset \bar{\rho}$. Hence

$$e^{(i)}, \; e_{i+1} \cup \ldots \cup e_j, \; e^{(j)}$$

form a ρ-compatible circuit contradicting the acyclicity of ρ. Thus condition (ii) also insures that the construction yields a ladder.

In the above construction we will refer to Q as the path *opposite* P and to $e^{(i)}$ as the ith *rung* of the ladder.

2.11. PROPOSITION: *Let* ρ *be an equivalence on* $E(X)$ *containing* $\alpha \cup \beta$. *Let* x, y, x' *be distinct vertices of* X, $e = [x, x'] \; \varepsilon \; E(X)$, *and let* P *be a path joining* x *and* y *such that* $E(P) \times \{e\} \subset \bar{\rho}$. *If*

(i) $P \cup e$ *is a shortest path joining* x' *and* y

(ii) ρ *is acyclic and* P *is a shortest path joining* x *and* y

(iii) ρ *is acyclic and* P *is a saturated* ρ-*compatible path joining* x *and* y

then a saturated ladder can be constructed in X from P and e.

REMARK: If ρ is an acyclic equivalence on $E(X)$ containing $\alpha \cup \beta$ with the further condition that every circuit has a ρ-decomposition then one can show that every ρ-compatible path is saturated.

PROOF. Condition (ii) or (iii) immediately implies that a ladder can be constructed. If $P \cup e$ is a shortest path joining x' and y then P must be a shortest path joining x and y. Therefore in any of the three cases a ladder can be constructed. Again denote the consecutive vertices of P by $x = x_o, x_1, \ldots, x_s = y$, let $e_i = [x_{i-1}, x_i]$, $i = 1, \ldots, s$ and let

$$P \cup Q \cup e \cup e^{(1)} \cup \ldots \cup e^{(s)}$$

be the ladder constructed in 2.10 where

$$Q = [x' = x_o', x_1', \ldots, x_s'] ,$$

and

$$e_i' = [x_{i-1}', x_i'] , \ i = 1, \ldots, s.$$

We first show if (i) holds the ladder is saturated. P being a shortest path joining x and y immediately implies $[x_i, x_j] \notin E(X)$ for $x_i, x_j \in P$ with $|i-j| > 1$. Suppose $[x_i, x_j'] \in E(X)$, $x_i \in P$, $x_j' \in Q$, $(i \neq j)$ and without loss of generality we may take $j-i \geq 1$. Then

$$[x = x_o, x_1, \ldots, x_i, x_j', x_j, \ldots, x_s = y]$$

is a path joining x and y of length $s + 2 - (j-i)$, and by the minimality of the length of P this implies either $j = i + 1$ or

$j = i + 2$. But by the construction of the ladder $C_i = [x_i, x_{i+1}, x_{i+1}', x_i']$ is a saturated 4-circuit and therefore $[x_i, x_{i+1}'] \notin E(X)$. Hence $j \neq i + 1$. If $j = i + 2$ then

$$C = [x_i, x_{i+1}, x_{i+2}, x_{i+2}']$$

is a 4-circuit containing e_{i+1} and $e^{(i+2)}$. $e_{i+1} \bar{\rho} e^{(i+2)}$ implies $e_{i+1} \bar{\beta} e^{(i+2)}$, hence C must contain a diagonal. This is a contradiction since we have already shown $[x_{i+1}, x_{i+2}'] \notin E(X)$ and $[x_i, x_{i+2}] \notin E(X)$. (Note - we have not used the fact that $P \cup e$ is a shortest path joining x' and y so far, only that P is a shortest path.) The minimality of the length of $P \cup e$ immediately implies $[x_i', x_j'] \notin E(X)$ for $x_i', x_j' \varepsilon Q$ with $|j-i| > 1$. Hence if (i) holds the ladder is saturated.

Now assume ρ is acyclic and P is a shortest path joining x and y (here $P \cup e$ need not be a shortest path). From the paragraph above we already have $[x_i, x_j] \notin E(X)$ for $x_i, x_j \varepsilon P$ with $|j-i| > 1$ and $[x_i, x_j'] \notin E(X)$ for $x_i \varepsilon P$ and $x_j' \varepsilon Q$, $i \neq j$. Suppose there exist $x_i', x_j' \varepsilon Q$ with

$$e_o = [x_i', x_j'] \varepsilon E(X)$$

and take $j - i > 1$. Then

$$[x = x_o, x_1, \ldots, x_i, x_i', x_j', x_j, \ldots, x_s = y]$$

is a path joining x and y of length $s + 3 - (j-i)$, and by the minimality of the length of P this implies either $j = i + 2$ or $j = i + 3$.

Suppose $j = i + 2$. Since e_o, e_{i+1}', e_{i+2}' are the edges of a triangle the acyclicity of ρ implies $e_o \rho e_{i+1}' \rho e_{i+2}'$. Recall that $e_i \beta e_i'$. Hence $e_o \rho e_{i+1} \rho e_{i+2}$ so that $e_o \bar{\rho} e^{(i)}$. This implies X contains a saturated 4-circuit

$$C = [x_i, x_i', x_{i+2}', z]$$

(Fig. 2.3). Clearly $z \neq x_{i+1}, x_{i+2}$. $e^{(i)} \beta [x_{i+2}', z]$, $e_o \beta [x_i, z]$, respectively, imply $e^{(i)} \rho [x_{i+2}', z]$ and $e_o \rho [x_i, z]$. $e^{(i+2)} \rho [x_{i+2}', z]$ and $e_{i+1} \rho e_{i+2} \rho [x_i, z]$ imply $[x_{i+2}, x_{i+2}', z, x_i, x_{i+1}]$ is a ρ-compatible circuit contradicting the acyclicity of ρ.

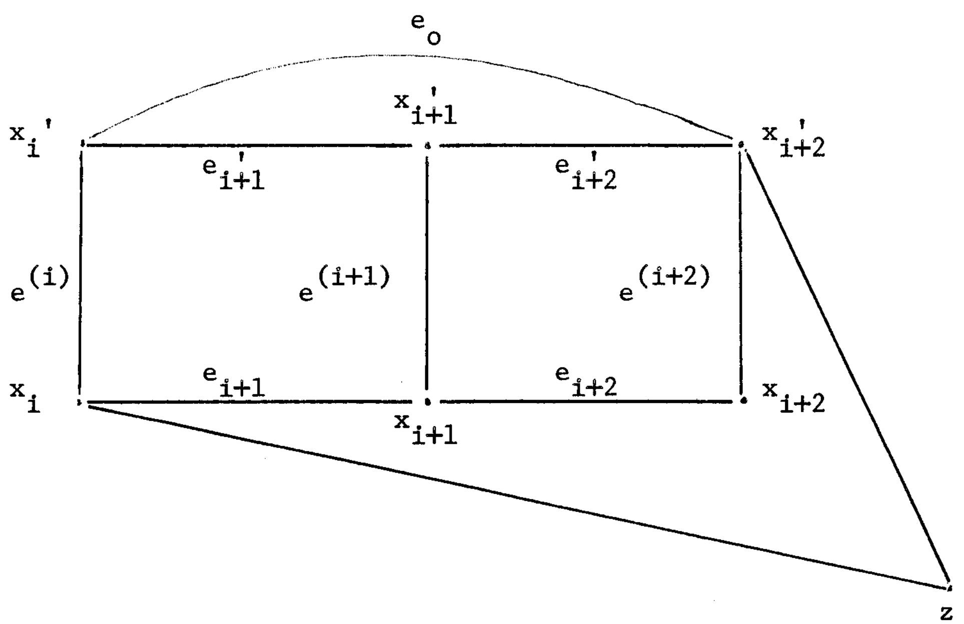


FIGURE 2.3

If $j = i + 3$, then

$$C = [x_i', x_{i+1}', x_{i+2}', x_{i+3}']$$

is a 4-circuit. Since we have already shown that $[x_i', x_{i+2}'] \notin E(X)$ and $[x_{i+1}', x_{i+3}'] \notin E(X)$, C is a saturated 4-circuit. Therefore $e_o \beta e_{i+2}'$ and $e_{i+1}' \beta e_{i+3}'$ and hence $e_o \rho e_{i+2}'$ and $e_{i+1}' \rho e_{i+3}'$. $e_o \overline{\rho} e^{(i)}$ implies that there exists a saturated 4-circuit

$$C' = [x_i, x_i', x_{i+3}', z] .$$

It is easy to verify that $z \neq x_{i+k}$, $k = 1,2,3$. $e^{(i)} \rho e^{(i+3)}$ and $e^{(i)} \rho [x_{i+3}', z]$ implies $e^{(i+3)} \rho [x_{i+3}', z]$. If $e_o \rho e_{i+1}'$ then

$$e_{i+3} \rho e_{i+2} \rho e_{i+1} \rho [x_i, z] .$$

Hence

$$[x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+3}', z]$$

is a ρ-compatible circuit, contradicting the acyclicity of ρ. If $e_o \overline{\rho} e_{i+1}'$ then $e_{i+1} \overline{\rho} e_{i+2}$ (since $e_{i+1}' \rho e_{i+1}$ and $e_o \rho e_{i+2}' \rho e_{i+2}$), and hence X contains a saturated 4-circuit

$$[x_i, x_{i+1}, x_{i+2}, z']$$

(Fig. 2.4). Clearly $z' \neq x_{i+3}$. Hence $[z, x_i, z', x_{i+2}, x_{i+3}, x_{i+3}']$ or $[z, x_{i+2}, x_{i+3}, x_{i+3}']$ is a ρ-compatible circuit, according as $z \neq z'$ or $z = z'$, a contradiction to the acyclicity of ρ. Hence if (ii) holds the ladder is saturated.

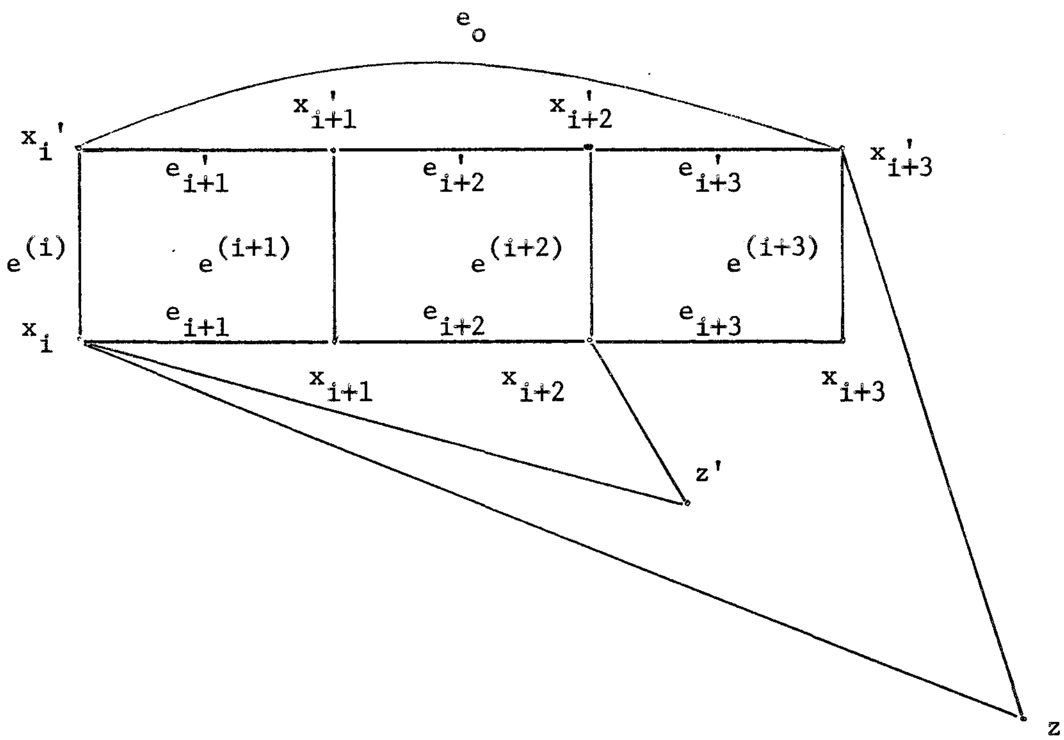


FIGURE 2.4

Finally we show that if ρ is acyclic and $P = P_o \cup \ldots \cup P_n$ is a saturated ρ-compatible path joining x and y, the ladder is saturated. Suppose

$$e_o = [x_i, x_j'] \; \varepsilon \; E(X)$$

for $x_i \varepsilon P$, $x_j' \varepsilon Q$, $i \neq j$. Without loss of generality take $i < j$. Since P is a ρ-compatible saturated path with $E(P) \times \{e\} \subset \bar{\rho}$, $P_{x_i x_j}$ is also a ρ-compatible, saturated path with $E(P_{x_i x_j}) \times \{e^{(i)}\} \subset \bar{\rho}$, and the ladder which can be constructed from $P_{x_i x_j}$ and $e^{(i)}$ can be taken as a subgraph of the ladder constructed from P and e. Therefore we need only consider the case $i = 0$, $j = s$. If

$$E(P) \times \{e_o\} \subset \bar{\rho}$$

then either

$$P_o,\ldots,P_n,e^{(s)},e_o \quad \text{or} \quad P_o,\ldots,P_n,e^{(s)} \cup e_o$$

form a ρ-compatible circuit according as

$$e_o \bar{\rho} e^{(s)} \quad \text{or} \quad e_o \rho e^{(s)}$$

a contradiction to the acyclicity of ρ . Suppose

$$E(P_k) \times \{e_o\} \subset \rho \text{ for some } k,\ 0 \leqq k \leqq n .$$

We may assume that $0 < k \leqq n$, for if $k = 0$, then

$$e_o \cup P_o,P_1,\ldots,P_n,e^{(s)}$$

form a ρ-compatible circuit contradicting the acyclicity of ρ . We may, moreover, assume that k is the smallest subscript with $E(P_k) \times \{e_o\} \subset \rho$. Let

$$P' = P_o \cup \ldots \cup P_{k-1} .$$

Then $E(P') \times \{e_o\} \subset \bar{\rho}$ and hence a ladder can be constructed from P' and e_o . Let

$$Q' = Q_o' \cup \ldots \cup Q_{k-1}'$$

be the path opposite P' and $e_o^{(m)}$ the final rung of the ladder. Let z be the common vertex of Q' and $e_o^{(m)}$, and let z' be the end-vertex of P_k not incident with $e_o^{(m)}$. $\{e_o\} \times E(P_k) \subset \rho$ and $e_o \rho e_o^{(m)}$

imply $\{e_o^{(m)}\} \times E(P_k) \subset \rho$. z and z' can be joined by a path P_k'' in $e_o^{(m)} \cup P_k$ with $E(P_k'') \times E(P_k) \subset \rho$. Then

$$Q_o', \ldots Q_{k-1}', P_k'', P_{k+1}, \ldots P_n, e^{(s)}$$

form a ρ-compatible subgraph which is not acyclic, contradicting the acyclicity of ρ . Hence $[x_i, x_j'] \notin E(X)$ for $i \neq j$. Now we show that Q is saturated. Suppose $e_o = [x_i', x_j'] \in E(X)$ for $x_i' \in Q$, $x_j' \in Q$ and take $i < j$. Without loss of generality we need only consider the case $i = o$, $j = s$. If

$$E(Q) \times \{e_o\} \subset \bar{\rho} \quad \text{then}$$

$$e_o, Q_o, \ldots, Q_n$$

form a ρ-compatible circuit contradicting the acyclicity of ρ . Suppose

$$E(Q_k) \times \{e_o\} \subset \rho \text{ , for some } k,\ o \leqq k \leqq n \text{ .}$$

If $n \geqq 1$ an argument similar to the one above showing that

$$[x_i, x_j'] \notin E(X) \quad \text{for} \quad i \neq j$$

will yield a contradiction here. Suppose $n = o$. Then $e_o \bar{\rho} e$ (otherwise $e \rho e_o \rho e_1' \beta e_1$, contradicting $e \bar{\rho} e_1$) implies that X contains a saturated 4-circuit $[x_s', x_o', x_o, z]$ (Fig. 2.5). Clearly $z \notin V(P)$. Hence $e^{(s)} \cup [x_s', z]$, $[z, x_o] \cup P_o$ form a ρ-compatible circuit contradicting the acyclicity of ρ . Thus we again have a saturated ladder.

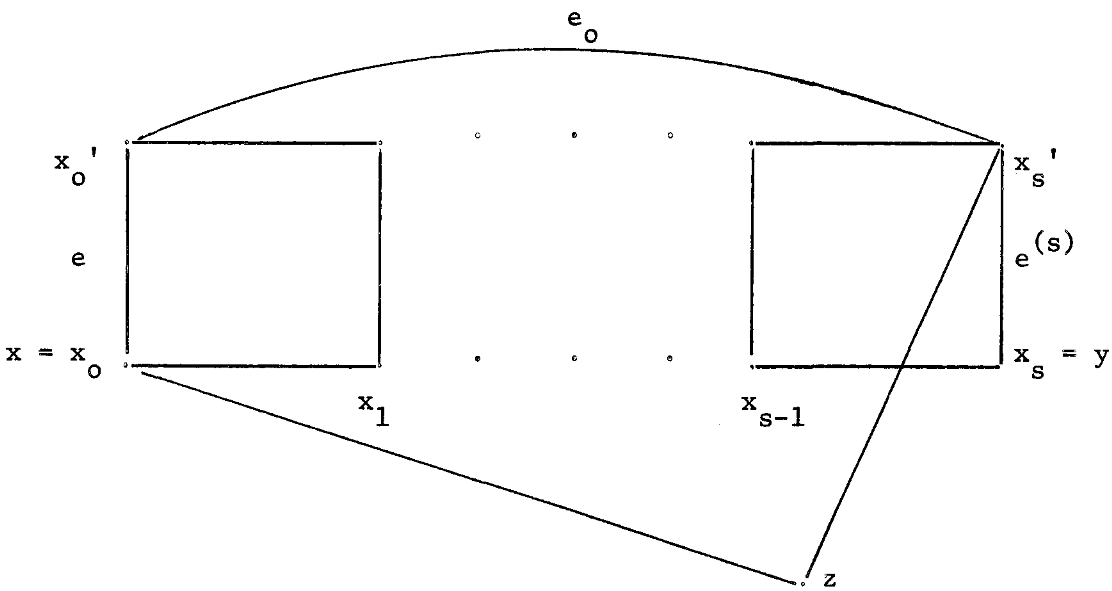


FIGURE 2.5

SECTION IV: Application of ladders.

Proposition 2.12 below will be proved by a straight forward application of 2.10. This proposition will be used later to show among other things that the collection of all acyclic equivalences on $E(X)$ which contain $\alpha \cup \beta$ is a filter.

2.12. PROPOSITION: *Let* ρ *be an equivalence on* $E(X)$ *containing* $\alpha \cup \beta$, $P = P_1 \cup \ldots \cup P_n$ *a* ρ-*compatible path joining* x *and* y . *If* ρ *is acyclic, or if* P *is a shortest path joining* x *and* y *(here* ρ *need not be acyclic) then there exists a* ρ-*compatible path* $Q = Q_1 \cup \ldots \cup Q_n$ *joining* x *and* y *such that*

(i) $|P_i| = |Q_{i+1}|$, $i = 1, \ldots, n-1$

(ii) $|P_n| = |Q_1|$

(iii) $E(P_i) \times E(Q_{i+1}) \subset \rho$, $i = 1, \ldots, n-1$

(iv) $E(P_n) \times E(Q_1) \subset \rho$.

REMARK: (i) and (ii) imply $|P| = |Q|$.

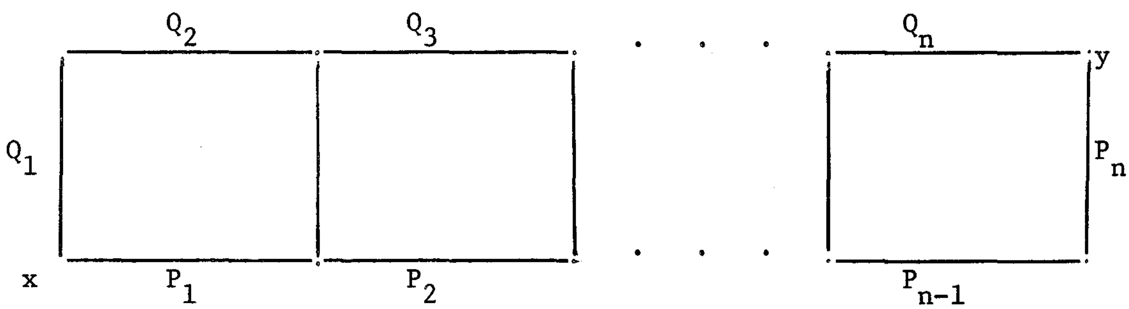


FIGURE 2.6

PROOF. We shall only consider the case $n = 2$, the reader will have no difficulty in extending the argument to $n \geq 3$. Denote the consecutive vertices of P_2 by

$$x_o, x_1, \ldots, x_r = y$$

and let

$$e_i = [x_{i-1}, x_i] , \; i = 1, \ldots, r .$$

If P is a shortest path joining x and y then $P_1 \cup e_1$ is a shortest path joining x and x_1. Hence if ρ is acyclic or P is a shortest path joining x and y, 2.10 implies that a ladder can be constructed from e_1 and P_1. Let

$$e_1' = [x, x_1']$$

be the final rung of the ladder and

$$P_1^{(1)} \text{ the path opposite } P_1 .$$

Again it is clear that a ladder can be constructed from e_2 and $P_1^{(1)}$. Let

$$e_2' = [x_1', x_2']$$

be the final rung of the ladder and

$$P_1^{(2)} \text{ the path opposite } P_1^{(1)} .$$

Continuing in this manner we get a path $P_1^{(r)}$, which we shall denote Q_2, such that

$$E(Q_2) \times E(P_1) \subset \rho ,$$

and

$$|Q_2| = |P_1| ,$$

and a sequence

$$e_1', \dots, e_r'$$

of edges of X such that

(a) e_i' and e_{i+1}' are either equal or adjacent for $i = 1, \dots, r-1$,

and

(b) $e_i \rho e_i'$, $i = 1, \dots, r$.

Let

$$Q_1 = e_1' \cup e_2' \cup \dots \cup e_r' .$$

Then (b) implies

$$E(Q_1) \times E(P_2) \subset \rho .$$

If ρ is acyclic and $x_i' = x_j'$, $i \neq j$, then $P_1^{(i)} \cup P_1^{(j)}$ and the segment of P_2 determined by x_i and x_j form a ρ-compatible circuit contradicting the acyclicity of ρ . Hence Q_1 is a path with

$$|Q_1| = |P_2| .$$

Also it is clear that $Q = Q_1 \cup Q_2$ is a path joining x and y (otherwise we again get a contradiction to the acyclicity of ρ).

Now we assume that P is a shortest path joining x and y. By (a), $Q = Q_1 \cup Q_2$ is a connected subgraph of X joining x and y ; hence Q is a path and

$$|Q_1| = |P_2| ,$$

otherwise P is not a shortest path joining x and y .

2.13 DEFINITION: Let ρ be any equivalence on $E(X)$ and let C be a circuit with ρ-decomposition $C = P_o \cup \ldots \cup P_n$. C is called weakly ρ-compatible if and only if there exists an i , $0 \leq i \leq n$, such that $E(P_i) \times E(P_j) \subset \bar{\rho}$ for all $j, i \neq j, 0 \leq j \leq n$.

2.14 PROPOSITION: Let ρ be an acyclic equivalence on $E(X)$ containing $\alpha \cup \beta$. Then X does not contain any weakly ρ-compatible circuits.

PROOF. Assume the contrary. Among all weakly ρ-compatible circuits choose one, $C = P_o \cup \ldots \cup P_n$ say, whose ρ-degree is minimal, and let the notation be so chosen that $E(P_o) \times E(P_j) \subset \bar{\rho}$, $1 \leq j \leq n$. Since ρ is acyclic, 2.3 implies that there exist P_i, P_j, $i < j$, such that

2.14 (1) $$E(P_i) \times E(P_j) \subset \rho$$

and

2.14 (2) $$Y = P_{i+1} \cup P_{i+2} \cup \ldots \cup P_j$$

is a ρ-compatible path. Let the end vertices of Y be x and y . 2.12 implies that there exists a ρ-compatible path

$$Y' = P'_{i+1} \cup \ldots \cup P'_j$$

such that

2.14 (3) $$E(P_{i+k}) \times E(P'_{i+k+1}) \subset \rho, \; k = 1, \ldots, j-i-1$$

and

2.14 (4) $$E(P_j) \times E(P'_{i+1}) \subset \rho .$$

Let

$$C' = P_o \cup \ldots \cup P_i \cup P'_{i+1} \cup \ldots \cup P'_j \cup P_{j+1} \cup \ldots \cup P_n .$$

If C' is a circuit then it is weakly ρ-compatible (since $E(P_o) \times E(P_k) \subset \overline{\rho}$, $k = 1, \ldots, i, j+1, \ldots, n$, and $E(P_o) \times E(P_k') \subset \overline{\rho}$, $k = i+1, \ldots, j$) and $\deg_\rho C' = n - 1$ (since $E(P_i) \times E(P_{i+1}') \subset \rho$ by 2.14 (1) and 2.14 (4)), a contradiction to the minimality of n. Suppose C' is not a circuit. Set $Y'' = C \setminus Y'$, then $C' = Y'' \cup Y'$. Choose $z, w \in V(Y'')$ such that

(i) $E(Y''_{zw} \cap P_o) \neq \emptyset$,

and

(ii) $V(Y''_{zw}) \cap V(Y') = \{z,w\}$.

Then $C'' = Y''_{zw} \cup Y'_{wz}$ is weakly ρ-compatible circuit with $\deg_\rho C'' \leqq n-1$, again a contradiction to the minimality of n.

2.15. PROPOSITION: Let ρ be an equivalence on $E(X)$ containing $\alpha \cup \beta$ and let $x, y \in V(X)$. Let $P = P_1 \cup \ldots \cup P_n$ and $Q = Q_1 \cup \ldots \cup Q_m$ be two ρ-compatible paths joining x and y. Then $n = m$ and for each P_i there exists a $Q_{j(i)}$ such that $E(P_i) \times E(Q_{j(i)}) \subset \rho$, $i = 1, \ldots, n$. Moreover if P and Q are shortest paths joining x and y then $|P_i| = |Q_{j(i)}|$, $i = 1, \ldots, n$.

PROOF. Without loss of generality we may assume that $P \cup Q$ is a circuit. Since P and Q are both ρ-compatible, we have that $m = n$ (otherwise $P \cup Q$ is weakly ρ-compatible) and that for each P_i there exists a $Q_{j(i)}$ such that

$$E(P_i) \times E(Q_{j(i)}) \subset \rho .$$

Now let us also assume that P and Q are shortest paths joining x and y and suppose that $|P_i| \neq |Q_{j(i)}|$ for some i,

$0 \leqq i \leqq n$. 2.12 implies that we may, without loss of generality, take $i = 1$ and $j(i) = 1$. However we can not assume here that $P \cup Q$ is a circuit. Let $z_1 = P_1 \cap P_2$ and $z_2 = Q_1 \cap Q_2$. $|P_1| \neq |Q_1|$ implies $z_1 \neq z_2$, otherwise we get a contradiction to either P or Q being a shortest path joining x and y . Let W be a path contained in $P_1 \cup Q_1$ joining z_1 and z_2 . $E(W) \times E(W) \subset \rho$ since $E(P_1) \times E(Q_1) \subset \rho$. $E(W) \times E(P_k) \subset \overline{\rho}$, $2 \leqq k \leqq n$, and $E(W) \times E(Q_k) \subset \overline{\rho}$, $2 \leqq k \leqq n$. Hence

$$C = W \cup P_2 \cup \ldots \cup P_n \cup Q_n \cup \ldots \cup Q_2$$

is a weakly ρ-compatible circuit or contains a weakly ρ-compatible circuit, contradicting 2.14.

SECTION V: ρ-saturated subgraphs.

2.16. DEFINITION: Let ρ be any equivalence on $E(X)$. A subgraph Y of X will be called ρ-saturated if and only if

(i) Y is connected,

and

(ii) $e \rho e'$ for $e \in E(X)$, $e' \in E(Y)$ implies $Y \cup (e)$ is disconnected or $e \in E(Y)$.

This is equivalent to saying that there exists a set $\mathfrak{A}$ of equivalence classes mod ρ such that Y is a maximal connected subgraph of X with $E(Y) \subset \bigcup \mathfrak{A}$.

2.17. PROPOSITION: Let ρ be any equivalence on $E(X)$ containing $\alpha \cup \beta$, Y a ρ-saturated subgraph of X . Then any two distinct vertices of Y can be joined by a shortest path in Y which is ρ-compatible.

PROOF. We fix $x \in Y$ and use induction of $d(x,y)$, the distance of x and y in Y . For $k = 1, 2, \ldots$ put

$$A_k = \{y \in Y: d(x,y) = k\} .$$

If $y \in A_1$ then $e = [x,y] \in Y$, hence $P = (e)$ trivially is a ρ-compatible path joining x and y . Assume the proposition true for all $z \in A_k$, and let $y \in A_{k+1}$. Then there is a $z \in A_k$ with $e = [y,z] \in Y$. By the induction hypothesis there is a ρ-compatible path

$P = P_1 \cup \ldots \cup P_n$ such that P is a shortest path joining z and x in Y. If

$$\{e\} \times E(P_i) \subset \overline{\rho} \quad \text{for} \quad i = 1, \ldots, n ,$$

we are finished, because then

$$P_1, \ldots, P_n \quad \text{and} \quad P_{n+1} = e$$

form the required ρ-compatible path from x to y. We may therefore assume that

$$\{e\} \times E(P_m) \subset \rho \text{ for some } m, 1 \leqq m \leqq n .$$

We may then assume that $2 \leqq m \leqq n$, for if $m = 1$, then $e \cup P_1, P_2, \ldots, P_n$ form a ρ-compatible shortest path from x to y. By 2.10 there exist paths $Q_1, Q_2, \ldots, Q_{m-1}$ and an edge e' such that

$$Q_1, \ldots, Q_{m-1}, e' \cup P_m, P_{m+1}, \ldots, P_n$$

form a ρ-compatible shortest path from x to y. (By the maximality of Y, $Q_1, \ldots, Q_{m-1}$ and e' belong to Y).

2.18. COROLLARY: _If_ ρ _is acyclic and contains_ $\alpha \cup \beta$, _then any_ ρ_-saturated subgraph of_ X _is saturated_.

PROOF. Let Y be a ρ-saturated subgraph of X. Suppose there exist two distinct vertices $x, y \in Y$ such that $e = [x,y] \in E(X) - E(Y)$. By the maximality of Y, e is not equivalent to any edge of Y. By 2.17 there is a ρ-compatible path P joining x and y in Y. Hence $P \cup e$ is a ρ-compatible circuit, contrary to the acyclicity of ρ.

SECTION VI: The principal filter of all acyclic equivalence relations containing $\alpha \cup \beta$.

We shall denote by $\mathcal{E}_{\alpha \cup \beta}$ (more precisely $\mathcal{E}_{\alpha \cup \beta}(X)$) <u>the collection of all acyclic equivalence relations</u> on $E(X)$ <u>which contain</u> $\alpha \cup \beta$. $\mathcal{E}_{\alpha \cup \beta}(X)$ is non-empty, since $E(X) \times E(X) \varepsilon \mathcal{E}_{\alpha \cup \beta}(X)$.

If $\phi : X \longrightarrow Y$ is an isomorphism and $\rho \varepsilon \mathcal{E}_{\alpha \cup \beta}(X)$ then $\rho_{\phi} \varepsilon \mathcal{E}_{\alpha \cup \beta}(Y)$, where $e \rho_{\phi} e' (e,e' \varepsilon E(Y))$ if and only if $(\phi^{-1}e)\rho(\phi^{-1}e')$. This follows from the fact that both acyclicity and the relations α , β are defined in invariant terms.

2.19. PROPOSITION: $\mathcal{E}_{\alpha \cup \beta}$ <u>is closed under intersection of chains, and hence contains a minimal element</u>.

PROOF. Let Σ be a chain in $\mathcal{E}_{\alpha \cup \beta}$, $\rho = \bigcap_{\sigma \varepsilon \Sigma} \sigma$. Clearly ρ contains $\alpha \cup \beta$. It remains to show that ρ is acyclic. Suppose there exists a ρ-compatible circuit $C = P_o \cup \ldots \cup P_n$. Let $E_{ij} = E(P_i) \times E(P_j)$. $E_{ii} \subset \rho$ implies $E_{ii} \subset \sigma$ for every $\sigma \varepsilon \Sigma$. $E_{ij} \subset \overline{\rho}$ (for $i \neq j$) implies that there is a $\sigma_{ij} \varepsilon \Sigma$ with $E_{ij} \subset \overline{\sigma}_{ij}$. Let

$$\sigma_o = \bigcap_{o \leq i \leq j \leq n} \sigma_{ij}$$

Then $\sigma_o \varepsilon \Sigma$, and $E_{ij} \subset \overline{\sigma}_o$ whenever $i \neq j$. Also $E_{ii} \subset \sigma_o$, $i = 0, \ldots,$ n so that σ_o is not acyclic, a contradiction.

2.20. PROPOSITION: <u>Let</u> $\rho \varepsilon \mathcal{E}_{\alpha \cup \beta}(X)$ <u>and let</u> σ <u>be any equivalence with</u> $\alpha \cup \beta \subset \sigma \subset \rho$. <u>If</u> C <u>is a</u> σ-<u>compatible circuit then</u> $E(C) \times E(C) \subset \rho$.

PROOF. Assume the contrary, i.e., there exist σ-compatible circuits that have a ρ-decomposition. Among all σ-compatible circuits of minimal σ-degree, choose one, $C = P_o \cup \ldots \cup P_n$ say, whose ρ-degree is minimal. Let $C = Q_o \cup \ldots \cup Q_r$ be the decomposition of C determined by ρ and let the notation be so chosen that $P_o \subset Q_o$. Note that $\sigma \subset \rho$ implies

$$Q_i = \bigcup \{P_j : E(P_j) \cap E(Q_i) \neq \emptyset\} , \; i = o, \ldots, r .$$

ρ is acyclic. Hence 2.3 implies without loss of generality that there exists an integer s, $0 < s < r$, such that

2.20 (2) $$E(Q_o) \times E(Q_s) \subset \rho$$

and

2.20 (3) $$Y = Q_1 \cup Q_2 \cup \ldots \cup Q_s \text{ is a } \rho\text{-compatible path .}$$

Let the end vertices of Y be x and y. By 2.12, there exists a ρ-compatible path

$$Y' = Q_1' \cup \ldots \cup Q_s'$$

joining x and y such that

$$E(Q_i) \times E(Q_{i+1}') \subset \rho , \; i = 1, \ldots, s - 1$$

and

$$E(Q_s) \times E(Q_1') \subset \rho .$$

Let

$$C' = Q_o \cup Q_1' \cup Q_2' \cup \ldots \cup Q_s' \cup Q_{s+1} \cup \ldots \cup Q_r .$$

We now show that C' is a σ-compatible circuit. Since the notation was chosen so that $P_o \subset Q_o$, 2.20 (1) implies that

$$Y = P_k \cup P_{k+1} \cup \ldots \cup P_m \ , \quad 0 < k < m < n \ .$$

$C = P_o \cup \ldots \cup P_n$ is a σ-compatible circuit and hence $Y = P_k \cup P_{k+1} \cup \ldots \cup P_m$ is a σ-compatible path. By the construction of Y' we have that Y' is also a σ-compatible path with $E(Y')$ belonging to the same set of equivalence classes modulo σ as $E(Y)$. Hence C' is a σ-compatible subgraph with a σ-compatible cover of cardinality $n + 1$. Again by 2.20 (1) and the fact that $C = P_o \cup \ldots \cup P_n$ is a σ-compatible circuit we have

$$E(Q_o) \times E(Q_s) \subset \bar{\sigma} \ .$$

By the construction of Q_1' we have $E(Q_1')$ is contained in the same set of equivalence classes mod σ as $E(Q_s)$. Hence

$$E(Q_o) \times E(Q_1') \subset \bar{\sigma} \ .$$

Therefore $E(Q_o) \cap E(Q_1') = \emptyset$ and C' is not acyclic. If C' is not a circuit then by 2.5, C' contains a σ-compatible circuit which can be covered by less than $n + 1$ paths. A contradiction to the minimality of n. Hence C' is a σ-compatible circuit with a σ-compatible cover of cardinality $n + 1$. But $E(Q_o) \times E(Q_1') \subset \rho$. Hence the ρ-decomposition of C' has less than $r + 1$ paths. This is a contradiction to our choice of r.

2.21. PROPOSITION: $\mathcal{E}_{\alpha \cup \beta}$ <u>is closed under finite intersections.</u>

PROOF. Assume that there exist $\rho_1, \rho_2 \in \mathcal{E}_{\alpha \cup \beta}$ such that $\rho = \rho_1 \cap \rho_2 \notin \mathcal{E}_{\alpha \cup \beta}$. Since $\alpha \cup \beta \subset \rho$ this implies that ρ is not acyclic.

Let $C = P_o \cup P_1 \cup \ldots \cup P_n$ be a ρ-compatible circuit. By the previous proposition $E(C) \times E(C) \subset \rho_i$ for $i = 1, 2$ and hence $E(C) \times E(C) \subset \rho_1 \cap \rho_2 = \rho$, a contradiction.

2.22. PROPOSITION: Let $\rho \in \mathcal{E}_{\alpha \cup \beta}$ and let σ be any equivalence containing ρ. Then $\sigma \in \mathcal{E}_{\alpha \cup \beta}$.

PROOF. Suppose $\sigma \notin \mathcal{E}_{\alpha \cup \beta}$. Then $\alpha \cup \beta \subset \rho \subset \sigma$ implies σ is not acyclic. Let C be any σ-compatible circuit. $E(C) \times E(C) \not\subset \sigma$ and $\rho \subset \sigma$ imply $E(C) \times E(C) \not\subset \rho$. Hence every σ-compatible circuit has a ρ-decomposition. Among all σ-compatible circuits of minimal order choose one, say C, whose ρ-decomposition is minimal. Let $C = P_o \cup \ldots \cup P_m$ be the σ-decomposition of C and let $C = Q_o \cup \ldots \cup Q_r$ be the ρ-decomposition of C. Note that $\rho \subset \sigma$ implies that

$$P_i = \bigcup \{Q_j : E(P_i) \cap E(Q_j) \neq \emptyset\}, \quad i = 1, \ldots, m.$$

Since ρ is acyclic, 2.3 implies there exist Q_i, Q_j, $i < j$, such that

2.22 (1) $$E(Q_i) \times E(Q_j) \subset \rho$$

and

2.22 (2) $$Y = Q_{i+1} \cup Q_{i+2} \cup \ldots \cup Q_j \text{ is a } \rho\text{-compatible path}.$$

$\rho \subset \sigma$ and 2.22 (1) imply $Y \subset P_k$ for some k, $0 \leq k \leq m$. Let the end vertices of Y be x and y. By 2.12, there exists a ρ-compatible path

$$Y' = Q'_{i+1} \cup Q'_{i+2} \cup \ldots \cup Q_j'$$

such that

$$E(Q_{i+k}) \times E(Q'_{i+k+1}) \subset \rho, \ k = 1, \ldots, j-i-1,$$

and

$$E(Q_j) \times E(Q'_{i+1}) \subset \rho.$$

Let

$$C' = Q_o \cup \ldots \cup Q_i \cup Q_{i+1}' \cup \ldots \cup Q_j' \cup Q_{j+1} \cup \ldots \cup Q_r .$$

By the construction of Y' we have $E(Y') \times E(P_k) \subset \sigma$. Hence it is clear that C' is a σ-compatible circuit of minimal order. Since $E(Q_i) \times E(Q_{i+1}') \subset \rho$ the ρ-decomposition of C' has less than $r + 1$ paths - a contradiction to the minimality of r.

2.23. THEOREM: $\mathcal{E}_{\alpha \cup \beta}$ <u>is a principal filter in the lattice of all equivalence relations on</u> $E(X)$.

PROOF. 2.21 and 2.22 imply that $\mathcal{E}_{\alpha \cup \beta}$ is a filter. 2.19 implies that it is a principal filter.

2.24. PROPOSITION: <u>Let</u> ρ <u>be the least element of</u> $\mathcal{E}_{\alpha \cup \beta}(X)$ <u>and let</u> E_a, $a \in A$, <u>denote the equivalence classes of</u> $E(X)$ <u>mod</u> ρ. <u>Then</u> $\mathcal{E}_{\alpha \cup \beta}(X)$ <u>is isomorphic to the lattice of all equivalence relations on</u> A.

PROOF. The proposition is readily established even if we replace $\mathcal{E}_{\alpha \cup \beta}(X)$ by an arbitrary principal filter.

2.25. COROLLARY: <u>Let</u> ρ <u>be the least element of</u> $\mathcal{E}_{\alpha \cup \beta}(X)$. $\mathcal{E}_{\alpha \cup \beta}(X)$ <u>is finite if and only if</u> $E(X)$ <u>consists of a finite number of equivalence classes mod</u> ρ. <u>If</u> $E(X)$ <u>has</u> $n \geq \aleph_o$ <u>equivalence classes mod</u> ρ <u>then</u> $|\mathcal{E}_{\alpha \cup \beta}(X)| = 2^n$.

2.26. PROPOSITION: <u>Let</u> ρ <u>be the least element of</u> $\mathcal{E}_{\alpha \cup \beta}$. <u>Then</u> $e \rho e'$ <u>implies that</u> e <u>and</u> e' <u>belong to the same component of</u> X.

PROOF. Define σ by $e\sigma e'$ if and only if e and e' belong to the same component of X and $e\rho e'$. Since circuits are connected, σ is acyclic. If $e\alpha e'$, then e and e' are adjacent, and hence belong to the same component of X; similarly, if $e\beta e'$, the two edges belong to a 4-circuit, and hence again to the same component. That is, $\sigma \supset \alpha \cup \beta$. By the minimality of ρ, $\sigma = \rho$.

SECTION VII: Weak cartesian products and acyclic equivalences.

We now turn our investigations to the relationships between weak cartesian products on the one hand and acyclic equivalences on the other.

Let $X = \prod_{a \in A} (X_a, x_a)$ be the weak cartesian product of the rooted graphs (X_a, x_a). Then for each $e = [x,y] \in E(X)$ there exists exactly one $a \in A$ such that $[pr_a x,\ pr_a y] \in E(X_a)$. We will denote this unique member of A by $a(e)$.

2.27. PROPOSITION: Let $X = \prod_{a \in A} (X_a, x_a)$. Then $e \rho_o e'$ implies $a(e) = a(e')$, where ρ_o is the smallest equivalence on $E(X)$ containing $\alpha \cup \beta$.

PROOF. Assume first that $e \alpha e'$. Let $e = [x,y]$, and let $e' = [x,y']$. Abbreviate $a(e)$ by a, and $a(e')$ by a', and suppose $a \neq a'$. Define $z \in V(X)$ by $pr_a z = pr_a y$, $pr_b z = pr_b y'$, $b \neq a$. $[z,y'] \in E(X)$ since $[pr_a z,\ pr_a y'] = [pr_a y,\ pr_a x] \in E(X_a)$ and $pr_b z = pr_b y'$, $b \neq a$. $[y,z] \in E(X)$ since $[pr_{a'} z,\ pr_{a'} y] = [pr_{a'} y',\ pr_{a'} x] \in E(X_a)$ and $pr_b z = pr_b y' = pr_b x = pr_b y$, $b \neq a$, a', $pr_a z \neq pr_a y$. Hence x,y,z,y' form a 4-circuit $pr_a y' = pr_a x \neq pr_a y$, $pr_{a'} y' \neq pr_{a'} x = pr_{a'} y$, i.e., y differs from y' in more than one coordinate, hence $[y,y'] \notin E(X)$. Similarly $[x,z] \notin E(X)$, a contradiction.

Now assume that $e \beta e'$. Let $e = [x,y]$ and let $e' = [x',y']$. Then x,y,y',x' form a 4-circuit of X. Let $a(e) = a$, $a(e') = a'$, $a([x,x']) = b$, and $a([y,y']) = b'$. It follows that

2.27 (1) $pr_c x = pr_c y \iff c \neq a$

2.27 (2) $pr_c y' = pr_c x' \iff c \neq a'$

2.27 (3) $pr_c x' = pr_c x \iff c \neq b$

2.27 (4) $pr_c y = pr_c y' \iff c \neq b'$

By 2.27 (1), 2.27 (3) $pr_c x' = pr_c y \iff c \neq a,b$; by 2.27 (2) and 2.27 (4) $pr_c x' = pr_c y \iff c \neq a',b'$. Hence there are two possiblities: either (i) $a = a'$, $b = b'$ or (ii) $a = b'$, $a' = b$. We show that (ii) can not occur. Suppose $a = b'$ and $a' = b$. By 2.27 (1) and 2.27 (4) $pr_{a'} x = pr_{a'} y'$; by 2.27 (2) and 2.27 (3) $pr_c x = pr_c y'$, $c \neq a'$. Hence $x = y'$, a contradiction.

2.28. DEFINITION: Let $X = \prod_{a \varepsilon A} (X_a, x_a)$. For each $a \varepsilon A$ define $E_a = \{e \varepsilon E(X) : a(e) = a\}$. Let ρ_a be the equivalence determined by the partition $\{E_a, E(X) - E_a\}$, i.e., $e \rho_a e'$ if and only if either (i) $a(e) = a = a(e')$ or (ii) $a(e) \neq a \neq a(e')$.

2.29. PROPOSITION: <u>Let</u> $X = \prod_{a \varepsilon A} (X_a, x_a)$. <u>For each</u> $a \varepsilon A$, $\rho_a \varepsilon \mathcal{E}_{\alpha \cup \beta}(X)$.

PROOF. We first show that ρ_a is acyclic. Assume the contrary. Since $E(X)$ consists of at most two equivalence classes modulo ρ there exists a ρ_a-compatible circuit $C = P_o \cup P_1$; without loss of generality $E(P_o) \subset E_a$. Let $\{x,y\} = P_o \cap P_1$. $x,y \varepsilon P_o$ and $E(P_o) \subset E_a$ imply $pr_a x \neq pr_a y$. But $x,y \varepsilon P_1$ and $E(P_1) \subset E(X) - E_a$ imply $pr_a x = pr_a y$ – a contradiction, hence ρ_a is acyclic.

From 2.27 we immediately have that $\alpha \cup \beta \subset \rho_a$.

2.30. REMARK: Let $\rho = \bigcap_{a\in A} \rho_a$. Then $\rho \in \mathcal{E}_{\alpha\cup\beta}(X)$ and $e\rho e'$ if and only if $a(e) = a(e')$. Hence $E(X)$ consists of exactly $|A|$ equivalence classes mod ρ .

2.31. THEOREM: <u>Let</u> $X = \prod_{a\in A}(X_a, x_a)$ <u>and let</u> ρ <u>be the least element in</u> $\mathcal{E}_{\alpha\cup\beta}(X)$. <u>Then</u> $e\rho e'$ <u>implies</u> $a(e) = a(e')$.

PROOF. ρ being the least element in $\mathcal{E}_{\alpha\cup\beta}(X)$ and $\rho_a \in \mathcal{E}_{\alpha\cup\beta}(X)$ for each $a\varepsilon A$ imply $\rho \subset \rho_a$ for each $a\varepsilon A$. In particular $\rho \subset \rho_{a(e)}$, hence $a(e) = a(e')$.

2.32. PROPOSITION: <u>Let</u> $x\varepsilon X$, $e = [x_o, y_o] \varepsilon E(X)$, $a(e) = a$. <u>If</u> x <u>and</u> x_o <u>belong to the same component of</u> X <u>then</u> $(i_a^x pr_a e)\rho_o e$.

PROOF. Since x and x_o belong to the same component of X it suffices to assume x and x_o are adjacent, i.e., $e_o = [x_o, x] \varepsilon E(X)$. Let $a(e_o) = a_o$. If $a_o = a$, then $i_a^x pr_a e = e$. If $a_o \neq a$ let y be given by $pr_a y = pr_a y_o$, $pr_{a_o} y = pr_{a_o} x$, $pr_b y = pr_b x_o$ for $b \neq a, a_o$. Then $C = [x_o, y_o, y, x]$ is a 4-circuit without diagonals, and $[x,y] = i_a^x pr_a e$. Thus e and $i_a^x pr_a e$ are opposite edges of C , so that $(i_a^x pr_a e)\beta e$.

2.33. DEFINITION: Let $X = \prod_{a\in A}(X_a, x_a)$ and let σ be an equivalence relation on $E(X_a)$ for some $a\varepsilon A$. Then σ can be extended to an equivalence relation $\tilde{\sigma}$ on $E(X)$ as follows: for $e, e' \varepsilon E(X)$ define $e\tilde{\sigma}e'$ if and only if either

(i) $a(e) = a = a(e')$ and $pr_a e \; \sigma \; pr_a e'$

or

(ii) $a(e) \neq a \neq a(e')$.

Note that by taking $\sigma = E(X_a) \times E(X_a)$ we get 2.28 as a special case of this definition. Similarly 2.29 is a special case of the following proposition.

2.34. PROPOSITION: <u>Let</u> $X = \prod_{a\varepsilon A}(X_a,x_a)$ <u>and let</u> $\sigma \varepsilon \mathcal{E}_{\alpha\cup\beta}(X_a)$ <u>for some</u> $a\varepsilon A$. <u>Then</u> $\tilde{\sigma} \varepsilon \mathcal{E}_{\alpha\cup\beta}(X)$.

PROOF. To show that $\tilde{\sigma}$ is acyclic assume the contrary, i.e., there exists a $\tilde{\sigma}$-compatible circuit $C = P_o \cup \ldots \cup P_n$ in X . Without loss of generality we may assume that $e \varepsilon E(P_o)$ implies $a(e) \neq a$ (otherwise $pr_a C = pr_a P_o \cup \ldots \cup pr_a P_n$ is a σ-compatible circuit). Let $\{x,y\} = V(P_o) \cap V(P_1 \cup \ldots \cup P_n)$. $x,y \varepsilon V(P_o)$ implies $pr_a x = pr_a y$. But $x,y \varepsilon V(P_1 \cup \ldots \cup P_n)$ implies $pr_a x \neq pr_a y$, a contradiction. Hence $\tilde{\sigma}$ is acyclic.

To show that $\alpha \cup \beta \subset \tilde{\sigma}$ we first show that $\alpha \subset \tilde{\sigma}$. Let $e, e' \varepsilon E(X)$ and $e\alpha e'$. By 2.27 we have $a(e) = a(e')$. If $a(e) \neq a$ then $e\tilde{\sigma}e'$. If $a(e) = a(e') = a$, then $pr_a e \,\alpha\, pr_a e'$ on X_a and therefore $pr_a e \sigma pr_a e'$. Hence in either case we have $e\alpha e'$ implies $e\tilde{\sigma}e'$. A similar argument shows that $\beta \subset \tilde{\sigma}$.

2.35. PROPOSITION: <u>Let</u> $X = \prod_{a\varepsilon A}(X_a,x_a)$. <u>If</u> $\rho \varepsilon \mathcal{E}_{\alpha\cup\beta}(X)$ <u>then</u> <u>for each</u> $x\varepsilon X$, $\rho|i_a^x X_a \varepsilon \mathcal{E}_{\alpha\cup\beta}(X_a)$, <u>and if moreover</u> ρ <u>is least then</u> $\rho|i_a^x X_a$ <u>is least</u>.

PROOF. If $\rho \varepsilon \mathcal{E}_{\alpha\cup\beta}(X)$ then $\rho|i_a^x X_a$ is acyclic (in fact ρ restricted to any subgraph is acyclic if ρ is acyclic). Hence we need to show that $\rho|i_a^x X_a$ contains $\alpha\cup\beta$ on $i_a^x X_a$ to establish that $\rho|i_a^x X_a \varepsilon \mathcal{E}_{\alpha\cup\beta}(X_a)$.

Let $e, e' \in E(i_a^x X_a)$ with $e\alpha e'$ on $i_a^x X_a$, i.e., e and e' are adjacent and among the saturated subgraphs of $i_a^x X_a$ there does not exist a 4-circuit. If $e\overline{\rho|i_a^x X_a}e'$ then $e\bar{\rho}e'$ and hence $e\bar{\alpha}e'$ on X. But e, e' adjacent then implies that e, e' are contained in a saturated 4-circuit say C. Since $e, e' \in E(i_a^x X_a)$ it is easily verified that $C \subset i_a^x X_a$, a contradiction. Hence $e\rho|i_a^x X_a e'$, i.e., $\rho|i_a^x X_a$ contains α on $i_a^x X_a$. If $e\beta e'$ on $i_a^x X_a$ then $e\beta e'$ on X and hence $e\rho e'$. But $e, e' \in E(i_a^x X_a)$ implies $e\rho|i_a^x X_a e'$. Hence we have that $\rho|i_a^x X_a$ contains $\alpha \cup \beta$ on $i_a^x X_a$.

Finally we assume that ρ is the least element of $\mathcal{E}_{\alpha \cup \beta}(X)$ and suppose there exists $\sigma \in \mathcal{E}_{\alpha \cup \beta}(i_a^x X_a)$ with $\sigma \subsetneq \rho|i_a^x X_a$. By 2.34, $\tilde{\sigma} \in \mathcal{E}_{\alpha \cup \beta}(X)$ and hence $\tilde{\sigma} \cap \rho \in \mathcal{E}_{\alpha \cup \beta}(X)$. $\tilde{\sigma} \cap \rho \subsetneq \rho$ contradicts the minimality of ρ. Hence $\rho|i_a^x X_a$ is the least element of $\mathcal{E}_{\alpha \cup \beta}(i_a^x X_a)$.

2.36. COROLLARY: <u>Let</u> $X = \prod_{a \in A} (X_a, x_a)$ <u>be connected, and for each</u> $a \in A$ <u>let</u> ρ_a <u>be the least element of</u> X_a. <u>Then</u> $\bigcap_{a \in A} \tilde{\rho}_a$ <u>is the least element of</u> $\mathcal{E}_{\alpha \cup \beta}(X)$.

PROOF. $\tilde{\rho}_a \in \mathcal{E}_{\alpha \cup \beta}(X)$ and $\mathcal{E}_{\alpha \cup \beta}(X)$ a principal filter implies $\bigcap_{a \in A} \tilde{\rho}_a \in \mathcal{E}_{\alpha \cup \beta}(X)$. Let $\rho = \bigcap_{a \in A} \tilde{\rho}_a$ and let σ be the least element of $\mathcal{E}_{\alpha \cup \beta}(X)$. Let $e, e' \in E(X)$ with $e\rho e'$. Now $e\rho e'$ implies $a(e) = a(e') = a$ and $pr_a e \rho_a pr_a e'$. By 2.35 $\sigma|i_a^x X_a$ is the least element of $\mathcal{E}_{\alpha \cup \beta}(i_a^x X_a)$. This together with ρ_a the least element of X_a and $i_a^x : X_a \longrightarrow i_a^x X_a$ an isomorphism imply $i_a^x pr_a e \sigma i_a^x pr_a e'$. By 2.32 $e\rho_o(i_a^x pr_a e)$ and $e'\rho_o(i_a^x pr_a e')$ and hence $e\sigma(i_a^x pr_a e)$ and $e'\sigma(i_a^x pr_a e')$.

Therefore $e\sigma e'$, i.e., $\rho \subset \sigma$, but σ least implies $\rho = \sigma$.

2.37. EXAMPLE: The connectedness of X is actually needed in the previous corollary as is seen in the following example:

Take $X_1 = C(2)$, $X_2 = C(2)$ together with an isolated vertex, then $X_1 \times X_2$ is as in figure 2.7.

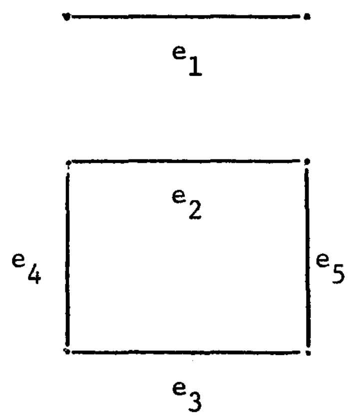


FIGURE 2.7

Now ρ_i the least element of $\mathcal{E}_{\alpha \cup \beta}(X_i)$, $i = 1,2$, implies $\tilde{\rho}_1 \cap \tilde{\rho}_2$ partitions $E(X_1 \times X_2)$ into the two classes $\{e_1,e_2,e_3\}$ and $\{e_4,e_5\}$. However the least element of $\mathcal{E}_{\alpha \cup \beta}(X_1 \times X_2)$ partitions $E(X_1 \times X_2)$ into the three classes $\{e_1\}$, $\{e_2,e_3\}$ and $\{e_4,e_5\}$.

We conclude this section with a summary of $\mathcal{E}_{\alpha \cup \beta}(X)$.

Let $\mathcal{E}(X)$ denote the complete lattice of all equivalence relations on $E(X)$. Then $\mathcal{E}_{\alpha \cup \beta}(X)$ is a principal filter in $\mathcal{E}(X)$. If

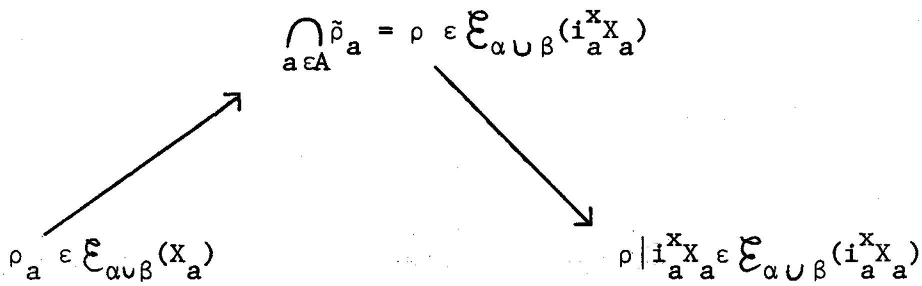
$$\phi : X \longrightarrow Y$$

is a graph isomorphism, then ϕ induces a lattice isomorphism from

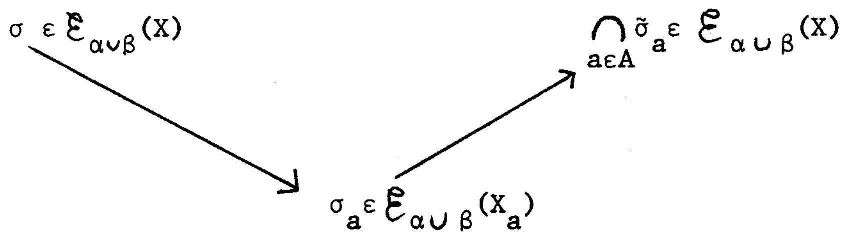
$\mathcal{E}(X) \longrightarrow \mathcal{E}(Y)$ by $\rho \longrightarrow \rho_\phi$ (where $e \rho_\phi e'$ $(e,e' \in E(Y))$ if and only if $(\phi^{-1}e)\ \rho\ (\phi^{-1}e')$) such that the restriction of this function to $\mathcal{E}_{\alpha \cup \beta}(X)$ is an isomorphism onto $\mathcal{E}_{\alpha \cup \beta}(Y)$.

Now let $X = \prod_{a \in A} (X_a, x_a)$, $x \in V(X)$. If $\rho_a \in \mathcal{E}_{\alpha \cup \beta}(X_a)$ $a \in A$, then ρ_a extends to an equivalence $\tilde{\rho}_a \in \mathcal{E}_{\alpha \cup \beta}(X)$ and if $\sigma \in \mathcal{E}_{\alpha \cup \beta}(X)$ then σ restricts to an equivalence $\sigma | i_a^x X_a \in \mathcal{E}_{\alpha \cup \beta}(i_a^x X_a)$ such that the following statements hold:

(1) If we denote $\bigcap_{a \in A} \tilde{\rho}_a$ by ρ then $\rho | i_a^x X_a = (\rho_a)_{i_a^x}$.



(2) If we let σ_a denote that equivalence in $\mathcal{E}_{\alpha \cup \beta}(X_a)$ such that $(\sigma_a) i_a^x = \sigma | i_a^x X_a$ then $\bigcap_{a \in A} \tilde{\sigma}_a \subset \sigma$ if X is connected.



(3) If ρ_a is the least element of $\mathcal{E}_{\alpha \cup \beta}(X_a)$, for each $a \in A$, then $\bigcap_{a \in A} \tilde{\rho}_a$ is least.

(4) If X is connected and σ is the least element of $\mathcal{E}_{\alpha \cup \beta}(X)$ then $\sigma | i_a^X X_a$ is least.

(5) By taking $\rho_a = E(X_a) \times E(X_a)$ we have that the weak cartesian decomposition of X determines an equivalence in $\mathcal{E}_{\alpha \cup \beta}(X)$, namely $\rho = \bigcap_{a \varepsilon A} \tilde{\rho}_a$, such that $e \rho e'$ if and if $a(e) = a(e')$

In the next section we will show that for X connected every equivalence $\rho \ \varepsilon \ \mathcal{E}_{\alpha \cup \beta}(X)$ gives rise to a weak cartesian decomposition and if ρ is least the factors are indecomposable.

SECTION VIII: Unique Prime Factorization Theorem.

Let X be a connected graph and let $\rho \in \mathcal{E}_{\alpha \cup \beta}$. Denote the collection of equivalence classes of $E(X)$ mod ρ by E_ν , $0 \leqq \nu < \nu_o$, ν_o an ordinal. For any vertex $z \in X$ and any ordinal ν, $0 < \nu < \nu_o$, let Y_ν^z be the largest connected subgraph of X such that $z \in Y_\nu^z$ and $E(Y_\nu^z) \subset E_\nu$. For any vertex $z \in X$ and any ordinal ν, $0 < \nu \leqq \nu_o$, let X_ν^z be the largest connected subgraph of X such that $z \in X_\nu^z$ and $E(X_\nu^z) \subset \bigcup_{\mu<\nu} E_\mu$.

Note that $Y_\mu^z \subset X_\nu^z$ and $X_\mu^z \subset X_\nu^z$ for $0 \leqq \mu < \nu \leqq \nu_o$. By 2.9, $E(X_\nu^z) \cap E_\mu \neq \emptyset$, for $0 \leqq \mu < \nu \leqq \nu_o$.

In our succeeding considerations we will let r be an arbitrary but fixed vertex of X , and for convenience we will denote Y_ν^r by Y_ν and X_ν^r by X_ν .

2.38. PROPOSITION: Let x and y be distinct vertices of $X_{\nu+1}$, $0 \leqq \nu < \nu_o$. Then X_ν^y and Y_ν^x have exactly one vertex in common.

PROOF. We first show $X_\nu^y \cap Y_\nu^x \neq \emptyset$. Assume the contrary, and let x_o and y_o be chosen such that

(i) $x_o \in X_\nu^y$, $y_o \in Y_\nu^x$, and

(ii) among all parts of vertices having property (i), $d(x_o,y_o)$ is minimal, where $d(x_o,y_o)$ is the distance of x_o and y_o in $X_{\nu+1}$.

Let $P = P_o \cup \ldots \cup P_n$ be a shortest path joining x_o and y_o in $X_{\nu+1}$ which is ρ-compatible (By 2.17 such a path exists since $X_{\nu+1}$ is a ρ-saturated subgraph of X) . $E(P_o) \subset E_\nu$ and $n \geqq 1$; otherwise we get a contradiction to (ii). Denote the consecutive vertices of P_o by $x_o, x_1, \ldots, x_m$ and let $e = [x_{m-1}, x_m]$.

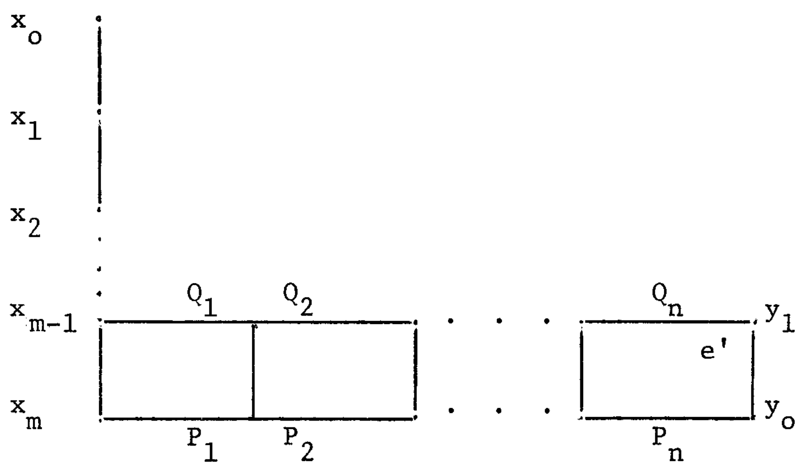


FIGURE 2.8

By 2.10 we can construct a ladder in $X_{\nu+1}$ from e and $P' = P_1 \cup \ldots \cup P_n$. Let $Q = Q_1 \cup \ldots \cup Q_n$ denote the path of the ladder opposite P' and let $e' = [y_o, y_1]$ denote the path of the ladder opposite e . Clearly $y_1 \in Y_\nu^x$, and $d(x_o, y_1) < d(x_o, y_o)$. A contradiction against (ii).

Now suppose there exists at least two distinct vertices z_1 and z_2 in $X_\nu^y \cap Y_\nu^x$. By 2.17 there exist a ρ-compatible path P' joining z_1 and z_2 in X_ν^y and a ρ-compatible path Q' joining z_1 and z_2 in Y_ν^x . Clearly $P' \cup Q'$ is a connected, ρ-compatible subgraph of X which is not acyclic, and hence by 2.5 contains a finite ρ-compatible circuit. This contradicts the acyclicity of ρ .

2.39. THEOREM: Let X be a connected graph, $r \in X$, $\rho \in \mathcal{E}_{\alpha \cup \beta}$. Then $X \cong \prod_{\nu<\nu_0} (Y_\nu, r_\nu)$ where $r_\nu = r$ for $0 \leqq \nu < \nu_0$. Moreover if ρ is the least element of $\mathcal{E}_{\alpha \cup \beta}$ then each Y_ν, $0 \leqq \nu < \nu_0$ is indecomposable.

PROOF. The proof is by transfinite induction. First we show that for non-limit ordinals $\nu+1$, $0 < \nu \leqq \nu_0$,

$$X_{\nu+1} \cong X_\nu \times Y_\nu$$

Let $x \in X_\nu$, $y \in Y_\nu$. By 2.38 there exists a unique vertex in $X_\nu^y \cap Y_\nu^x$ which we denote by z_{xy}. Define $\phi_\nu : X_\nu \times Y_\nu \longrightarrow X_{\nu+1}$ by

$$\phi_\nu(x,y) = z_{xy} .$$

Let z be an arbitrary vertex in $X_{\nu+1}$. By 2.38 Y_ν^z and X_ν have exactly one vertex in common, say x, and X_ν^z and Y_ν have exactly one vertex in common, say y, clearly $Y_\nu^z = Y_\nu^x$ and $X_\nu^z = X_\nu^y$. Hence $z = z_{xy}$. Thus every vertex of $X_{\nu+1}$ has a unique pre image with respect to ϕ_ν and hence ϕ_ν is one-one and onto.

To prove that ϕ_ν is a homomorphism take $[(x,y), (x',y')] \in X_\nu \times Y_\nu$. Then either $[x,x'] \in X_\nu$ and $y = y'$, or $[y,y'] \in Y_\nu$ and $x = x'$. Suppose $[x,x'] \in X_\nu$ and $y = y'$. Let P be a shortest path joining x and z_{xy} in Y_ν^x. Construct a ladder from $[x,x']$ and P, and let e be the final edge opposite $[x,x']$. Then e is incident with z_{xy}, $e \in X_\nu^y$ and hence clearly $e = [z_{xy}, z_{x'y'}] \in X_{\nu+1}$. Similarly, if $[y,y'] \in Y_\nu$ and $x = x'$, then $[z_{xy}, z_{x'y'}] \in X_{\nu+1}$. Hence $[(x,y), (x',y')] \in X_\nu \times Y_\nu$ implies $[z_{xy}, z_{x'y'}] \in X_{\nu+1}$.

To prove that ϕ_ν is an epimorphism let $e = [z_{xy}, z_{x'y'}] \in X_{\nu+1}$. Then $e \in \bigcup_{\mu \leq \nu} E_\mu$ and hence either $e \in E_\nu$ or $e \in \bigcup_{\mu<\nu} E_\mu$. If $e \in E_\nu$ then $e \in Y_\nu^x$ and $e \in Y_\nu^{x'}$ by the maximality of Y_ν^x, $Y_\nu^{x'}$. Hence $x = x'$. Moreover it is easy to verify that $[y,y'] \in Y_\nu$. If $e \in \bigcup_{\mu<\nu} E_\mu$ then $e \in X_\nu^y$ and $e \in X_\nu^{y'}$, and hence $y = y'$. Again it is easy to show that $[x,x'] \in X_\nu$. Hence ϕ_ν is an isomorphism.

Let $Z = \prod_{\nu<\nu_o} (Y_\nu, r_\nu)$ and for each ordinal $\nu \leq \nu_o$ let $Z_\nu = \prod_{\mu<\nu_o} (Y'_\mu, r'_\mu)$ where $(Y'_\mu, r'_\mu) = (Y_\mu, r_\mu)$ for $\mu<\nu$ and $(Y'_\mu, r'_\mu) = ((r), r)$ for $\nu \leq \mu < \nu_o$. Note that $Z = \bigcup_{\nu\leq\nu_o} Z_\nu$.

Suppose there exists a monomorphism $\psi_\nu : X_\nu \longrightarrow Z$ with $\psi_\nu(X_\nu) = Z_\nu$. Then we can construct a monomorphism $\psi_{\nu+1} : X_{\nu+1} \longrightarrow Z$ with $\psi_{\nu+1}(X_{\nu+1}) = Z_{\nu+1}$ and such that $\psi_{\nu+1}|X_\nu = \psi_\nu$. We already have an isomorphism

$$\phi_\nu : X_\nu \times Y_\nu \longrightarrow X_{\nu+1} .$$

Define $\eta_\nu : X_\nu \times Y_\nu \longrightarrow Z$ by

$$pr_\lambda \eta_\nu(x,y) = \begin{cases} pr_\lambda \psi_\nu(x) & , \lambda < \nu \\ y & , \lambda = \nu \\ r & , \nu < \lambda < \nu_o \end{cases}$$

Set $\psi_{\nu+1} = \eta_\nu \circ \phi_\nu^{-1}$

Clearly $\psi_{\nu+1} : X_{\nu+1} \longrightarrow Z$ is a monomorphism with

$$\psi_{\nu+1}(X_{\nu+1}) = Z_{\nu+1} \quad \text{and} \quad \psi_{\nu+1}|X_\nu = \psi_\nu .$$

Next let ν be a limit ordinal and assume that for each ordinal $\mu < \nu$ there exists a monomorphism $\psi_\mu : X_\mu \longrightarrow Z$ with $\psi_\mu(X_\mu) = Z_\mu$ and such that $\psi_\mu|X_\lambda = \psi_\lambda$ for $\lambda < \mu$. $X_\nu = \bigcup_{\mu<\nu} X_\mu$ and hence $x \in X_\nu$ implies $x \in X_\mu$ for some $\mu < \nu$. Set $\psi_\nu(x) = \psi_\mu(x)$. Then clearly $\psi_\nu : X_\nu \longrightarrow Z$ is a monomorphism and $\psi_\nu(X_\nu) = Z_\nu$. Hence $X_\nu \cong \prod_{\mu<\nu}(Y_\mu, r_\mu)$. Since $X = X_{\nu_o}$ we have $X \cong \prod_{\mu<\nu_o}(Y_\mu, r_\mu)$.

Finally if we assume that ρ is the least element of $\mathcal{E}_{\alpha\cup\beta}(X)$ then by 2.35 we have that $\rho|Y_\nu$ is the least element of $\mathcal{E}_{\alpha\cup\beta}(Y_\nu)$. Since $E(Y_\nu)$ consists of exactly one equivalence class modulo $\rho|Y_\nu$ we have that Y_ν is indecomposable.

2.40. PROPOSITION: Let $X = \prod_{a\in A}(X_a, x_a)$ be connected, and for each $a \in A$ let X_a be indecomposable. Let ρ be the least element of $\mathcal{E}_{\alpha\cup\beta}(X)$. Then for $e, e' \in E(X)$ $e\rho e'$ if and only if $a(e) = a(e')$.

PROOF. Since X_a is indecomposable for each $a\in A$, we have that $\rho_a = E(X_a) \times E(X_a)$ is the least element of $\mathcal{E}_{\alpha\cup\beta}(X_a)$. Hence by 2.30 and 2.35 we have the desired result.

We are now in a position to prove the following theorem which is our main result.

2.41. THEOREM: If X is a connected graph then X has a weak cartesian decomposition into indecomposable factors which is unique to within isomorphisms.

PROOF. From 2.39 we have that the least element ρ of $\mathcal{E}_{\alpha \cup \beta}(X)$ determines a weak cartesian decomposition of X into indecomposable factors, where the factors are taken to be ρ-saturated subgraphs with respect to the individual equivalence classes of E(X) mod ρ. If we take any other decomposition of X into prime factors, we have by 2.40 that the number of factors in each decomposition is the same. Since the injection mappings are monomorphisms, and the injections of these latter prime factors are ρ-saturated subgraphs with respect to the individual equivalence classes of E(X) mod ρ we have that the decomposition is unique to within isomorphisms.

2.42 PROPOSITION: <u>Let</u> X <u>be connected</u>. $\mathcal{E}_{\alpha \cup \beta}(X)$ <u>is finite if and only if</u> X <u>has a cartesian decomposition into indecomposable factors.</u>

PROOF. Follows from 2.25 and 2.39.

2.43. COROLLARY: <u>If</u> X <u>is a connected indempotent graph then</u> X <u>does not have a cartesian decomposition into indecomposable factors</u>.

PROOF. Let $f_n : X \longrightarrow \prod_{i=1}^{n} X_i$ be an isomorphism, where $X_i = X$, $i = 1, \ldots, n$. For $e = [x,y]$, $e' = [x',y'] \in E(X)$ define $e\rho^n e'$ if and only if $[pr_i f_n(x), pr_i f_n(y)] = [pr_i f_n(x'), pr_i f_n(y')] \in E(X_i)$ for some i. $\rho^n \in \mathcal{E}_{\alpha \cup \beta}(X)$ and by 2.30, ρ^n has exactly n equivalences classes. Hence $n \neq m$ implies $\rho^n \neq \rho^m$. Therefore $\mathcal{E}_{\alpha \cup \beta}(X)$ is infinite and hence by 2.42 X does not have a cartesian decomposition.

SECTION IX: Acyclic completion.

2.44. DEFINITION: Let ρ be any equivalence on $E(X)$. We define the acyclic completion of ρ, which we denote by ρ^*, as follows: Put $\rho^{(o)} = \rho$. Assume that $\rho^{(n)}$, $n \geq o$ has already been defined. For $e, e' \in E(X)$, we define a binary relation $\tau^{(n)}$ by: $e\, \tau^{(n)} e'$ if and only if there exists a $\rho^{(n)}$-compatible circuit C with $e, e' \in E(C)$. Let $\rho^{(n+1)}$ be the smallest equivalence on $E(X)$ containing $\rho^{(n)} \cup \tau^{(n)}$. Finally we take $\rho^* = \bigcup_{n=o}^{\infty} \rho^{(n)}$.

2.45. PROPOSITION: Let ρ be any equivalence on $E(X)$ and ρ^* the acyclic completion of ρ. Then

(i) $\rho \subset \rho^*$

(ii) ρ^* is acyclic

(iii) $\rho = \rho^*$ if ρ is acyclic

PROOF. (i) and (iii) are trivial. To prove (ii) assume the contrary, i.e., that there is a ρ^*-compatible circuit $C = P_o \cup \ldots \cup P_m$. Since C is finite and the $\rho^{(n)}$'s form an increasing sequence there exists an n such that $C = P_o \cup \ldots \cup P_m$ is a $\rho^{(n)}$-compatible circuit. Let $E_{ij} = E(P_i) \times E(P_j)$, $o \leqq i \leqq m$, $o \leqq j \leqq m$. $E_{ij} \subset \overline{\rho^{(n)}}$ for $i \neq j$ implies $E_{ij} \subset \tau^{(n)}$. Hence $E_{ij} \subset \rho^*$ for $i \neq j$, a contradiction to $C = P_o \cup \ldots \cup P_m$ being a ρ^*-compatible circuit.

2.46. PROPOSITION: Let ρ be any equivalence on $E(X)$ containing $\alpha \cup \beta$. Then $\rho^* = \rho^{(1)}$ is the smallest acyclic equivalence containing ρ .

PROOF. To show that $\rho^* = \rho^{(1)}$ it suffices to show that $\rho^{(1)}$ is acyclic. Assume the contrary. The proof is almost identical to that in 2.22 by replacing σ by $\rho^{(1)}$. There is only one change required. To establish that $C = Q_o \cup \ldots \cup Q_r$ is not ρ-compatible in 2.22 we used the acyclicity of ρ. Here we argue as follows: $C = Q_o \cup \ldots \cup Q_r$ is not ρ-compatible since otherwise $E(C) \times E(C) \subset \rho^{(1)}$, a contradiction to $C = Q_o \cup \ldots \cup Q_r$ being $\rho^{(1)}$-compatible.

Now let σ be any acyclic equivalence relation on $E(X)$ containing ρ . We will show $\rho^* \subset \sigma$. Let $e, e' \in E(X)$ with $e\rho^* e'$. Then there exists a sequence $e_1, \ldots, e_r$ of edges of X with $e_1 = e$, $e_r = e'$ and for each k, $1 \leqq k \leqq r-1$, either $e_k \rho e_{k+1}$ or $e_k \tau^{(o)} e_{k+1}$. $e_k \rho e_{k+1}$ implies $e_k \sigma e_{k+1}$. $e_k \tau^{(o)} e_{k+1}$ implies there exists a ρ-compatible circuit C with $e_k, e_{k+1} \in E(C)$. $\alpha \cup \beta \subset \rho \subset \sigma$, and σ acyclic imply $E(C) \times E(C) = \sigma$ (otherwise there exist ρ-compatible circuits with a σ-decomposition; by choosing one with minimal σ-degree and applying 2.12 we get a contradiction to the acyclicity of σ) and hence $e_k \sigma e_{k+1}$. Therefore $e\sigma e'$, i.e., $\rho^* \subset \sigma$. Hence if $\alpha \cup \beta \subset \rho$, then ρ^* is the smallest equivalence relation on $E(X)$ containing ρ .

For a given equivalence ρ let $\mathcal{E}_\rho$ denote the collection of all acyclic equivalence relations on $E(X)$ which contain ρ . Then 2.46 implies that if $\rho \supset \alpha \cup \beta$, $\mathcal{E}_\rho$ is a principal filter in the lattice of all equivalences on $E(X)$ with ρ^* as its least element. In general $\mathcal{E}_\rho$ does not have a least element as is seen in the following example:

2.47. EXAMPLE: Let X be a circuit of order $n \geq 4$, $E(X) = \{e_1, \ldots, e_n\}$ and δ the identity relation on $E(X)$. Let e_i, e_j be two distinct non-adjacent edges of X

Put

$$\rho_{ij} = \delta \cup \{(e_i, e_j),\ (e_j, e_i)\}$$

ρ_{ij} is a minimal acyclic equivalence on $E(X)$. Hence there are $\frac{1}{2}n(n-3)$ distinct minimal acyclic equivalences on $E(X)$. (This shows that for a given equivalence relation ρ there need not exist a smallest acylic equivalence containing ρ. Here $\rho = \delta$). X is an δ-compatible circuit. Hence $\delta^* = \delta^{(1)} = E(X) \times E(X)$, i.e., δ^* is not a minimal acyclic equivalence.

Let $\mathcal{S}$ denote the set of all equivalence relations ρ on $E(X)$ for which $\mathcal{E}_\rho$ has a least element. $\rho_o \in \mathcal{S}$ and $\sigma \in \mathcal{S}$ for all $\sigma \supset \rho_o$. In general $\mathcal{S}$ is not a filter in the lattice of all equivalence relations. Consider X and ρ_{ij} as in the previous example. Here $\rho_{ij} \in \mathcal{S}$ since $\mathcal{E}_{\rho_{ij}}$ has a least element for each ρ_{ij}. $\rho_{ij} \cap \rho_{kh} = \delta$ for $\rho_{ij} \neq \rho_{kh}$; however $\delta \notin \mathcal{S}$. Hence $\mathcal{S}$ is not closed under intersections, i.e., is not a filter. The following example shows that even if $\mathcal{S}$ is a principal filter the least element of $\mathcal{S}$ need not be ρ_o.

2.48. EXAMPLE: Let X be a tree. Here every equivalence is acyclic and hence $\mathcal{S}$ is the set of all equivalence relations on $E(X)$. The least element of $\mathcal{S}$ is the identity relation but $\rho_o = E(X) \times E(X)$.

SECTION X: Construction of a non-acyclic equivalence containing $\alpha \cup \beta$.

We conclude this chapter with an example of a connected graph Y and an equivalence σ on $E(Y)$, containing $\alpha \cup \beta$, with the property that for every integer $n \geq 2$ there exists a σ-compatible circuit C in Y with $\deg_\sigma C = n$. We proceed by first proving a lemma based on the following definition.

2.49. DEFINITION: Let X be a graph, E a subset of $E(X)$. Let X_1 be defined by $V(X_1) = V(X)$, $E(X_1) = E(X) - E$. Set $Y = X_1 \times X_2$, where X_2 is a complete graph on two vertices say o and 1. We define the <u>interchange</u> X_E of X <u>relative to</u> E by:

$$V(X_E) = V(Y)$$

$$E(X_E) = E(Y) \cup D$$

where

$$D = \{ [x,y] : x,y \in V(Y),\ [pr_1 x,\ pr_1 y] \in E,$$
$$[pr_2 x,\ pr_2 y] \in E(X_2)\}$$

For each $e = [x,y] \in E$, $[(x,o),\ (y,o),\ (y,1),\ (y,o)]$ is a saturated 4-circuit in $X \times X_2$. X_E is obtained from $X \times X_2$ by deleting the edges $[(x,o),\ (y,o)]$, $[(x,1),\ (y,1)]$ and adjoining the diagonals $[(x,o),\ (y,1)]$, $[(x,1),\ (y,o)]$.

If ρ is an equivalence on $E(X)$ then ρ induces an equivalence ρ_E on $E(X_E)$ as follows:

For $e = [x,y]$, $e' = [x',y'] \in E(X_E)$

$e \rho_E e'$ if and only if either

(i) $pr_1 x = pr_1 y$ and $pr_1 x' = pr_1 y'$, or

(ii) $[pr_1 x, pr_1 y]$, $[pr_1 x', pr_1 y'] \in E(X)$ and

$[pr_1 x, pr_1 y] \rho [pr_1 x', pr_1 y']$

REMARK: $\rho_E | X_1 \times X_2 = (\rho | X_1)^\sim$

2.50. PROPOSITION: Let ρ be an equivalence on $E(X)$ containing $\alpha \cup \beta$, E an equivalence class of $E(X)$ mod ρ . Then the induced equivalence ρ_E on the interchange graph X_E of X relative to E contains $\alpha \cup \beta$.

PROOF. $\rho \supset \alpha \cup \beta$ and E an equivalence class of $E(X)$ mod ρ imply $\rho | X_1$ contains $\alpha \cup \beta$. (X_1, X_2, D as in 2.49. Hence $(\rho | X_1)^\sim$ contains $\alpha \cup \beta$ on $X_1 \times X_2$. Since $\rho_E | X_1 \times X_2 = (\rho | X_1)^\sim$ we have that $\rho_E | X_1 \times X_2 \supset \alpha \cup \beta$. $E \times E \subset \rho$ implies $D \times D \subset \rho_E$. Therefore to show that ρ_E contains $\alpha \cup \beta$ we need only show that $e \in E(X_1 \times X_2)$, $e' \in E(X_E)$, $e \rho_o e'$ imply $e' \in E(X_1 \times X_2)$. Assume the contrary, i.e., $e' \in D$. Let $e = [x,y]$, $e' = [x',y']$ and without loss of generality take $pr_2 x' = pr_2 x$.

We first assume that $e \beta e'$. Then without loss of generality $C = [x,y,x',y']$ is a saturated 4-circuit in X_E, i.e. $[y,x']$, $[y',x] \in E(X_E)$, $[x,x']$, $[y,y'] \notin E(X_E)$. Let $pr_1 x = x_1$, $pr_2 x = x_2$, $pr_1 x' = x_1'$, etc. There are two cases to consider (i) $[x_1, y_1] \in E(X_1)$, $x_2 = y_2$ or (ii) $[x_2, y_2] \in E(X_2)$ and $x_1 = y_1$. If (i) holds then $x_2' = x_2 = y_2$ and $x_2 \neq y_2'$. It is easy to verify that $y_1' \neq y_1, x_1'$.

To show that $y_1' \neq x_1$ assume the contrary. Then $[y_1' = x_1, x_1', y_1]$ is a triangle in X. Let $e_1' = [x_1', y_1']$ and $e_1' = [x_1, y_1]$. Since $\rho \supset \alpha \cup \beta$ we then have that $e_1' \rho e_1$ (otherwise $e_1' \bar{\alpha} e_1$ contradicting $[x_1', y_1] \varepsilon E(X))$. $e_1' \varepsilon E$ and E an equivalence class of $E(X)$ mod ρ then imply $e_1 \varepsilon E$, a contradiction to $[x,y] \varepsilon E(X_1 \times X_2)$. Therefore $y_1' \neq x_1$. Hence $C_1 = [x_1, y_1, x_1', y_1']$ is a 4-circuit in X. It is easily verified that C_1 is saturated, and therefore $e_1' \bar{\rho} [y_1', x_1]$ This is a contradiction since $[x',y'] \varepsilon D$, $[x,y] \varepsilon D$ (since $x_1 \neq y_1'$, $x_2 \neq y_2'$) respectively imply $[x_1', y_1']$, $[x_1,y_1'] \varepsilon E$ and hence $[x_1', y_1'] \rho [y_1', x_1]$. Therefore $e\beta e'$ implies $e' \varepsilon E(X_1 \times X_2)$.

Now assume that $e\alpha e'$. Then e and e' are adjacent and $x = x'$ (since we assumed $x_2' = x_2$). Again there are two cases to consider (i) $[x_1,y_1] \varepsilon E(X_1)$, $x_2 = y_2$, or (ii) $[x_2,y_2] \varepsilon E(X_2)$ and $x_1 = y_1$. Suppose (i) holds let $e_1 = [x_1, y_1]$ and $e_1' = [x_1', y_1']$. $e_1 \varepsilon E(X) - E$, $e_1' \varepsilon E$, and E an equivalence class of $E(X)$ mod ρ imply $e_1 \bar{\rho} e_1'$. Hence $e_1 \alpha e_1'$ and therefore X contains a saturated 4-circuit $[y_1', x_1 = x_1', y_1, w]$. Define $z \varepsilon V(X_E)$ by

$$pr_1 z = w \quad , \; pr_2 z = y_2'$$

Then clearly $[y', x = x', y, z]$ is a saturated 4-circuit in X_E contradicting $e\alpha e'$. If (ii) holds define $v \varepsilon V(X_E)$ by

$$pr_1 v = y_1' \; , \; pr_2 v = x_2 (=x_2') \; .$$

Then $[y', x = x', y, v]$ is clearly a saturated 4-circuit in X_E, contradicting $e\alpha e'$. Hence $e\alpha e'$ implies $e' \varepsilon E(X_1 \times X_2)$.

2.51. EXAMPLE: Let X be a 4-circuit, $e \varepsilon E(X)$, $\rho = E(X) \times E(X)$. Let X_2 be the interchange of X relative to e and ρ_2 be the equivalence on $E(X_2)$ induced by ρ (Fig. 2.9).

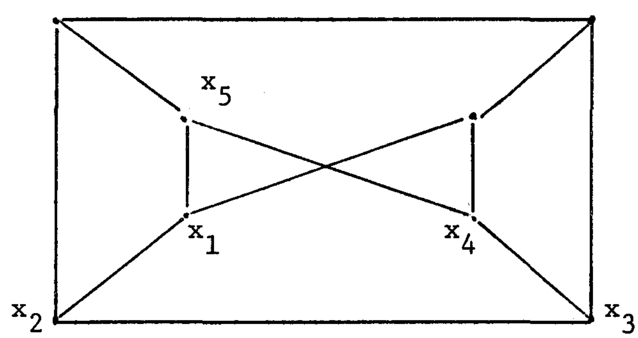


FIGURE 2.9

It is easily verified that ρ_2 contains $\alpha \cup \beta$ on X_2 and that $C_2 = [x_1, x_2, x_3, x_4, x_5]$ is a ρ_2-compatible circuit. Take $E_2 = \{[x,y] \varepsilon X_2 : pr_1 x = pr_1 y\}$.

Let X_3 be the interchange of X_2 relative to E_2 and ρ_3 the equivalence induced on $E(X_3)$ by ρ_2 (Fig. 2.10) . Since E_2 is an equivalence class of $E(X_2)$ mod ρ_2 2.44 implies that $\rho_3 \supset \alpha \cup \beta$.

$C_3 = [(x_1,o), (x_2,o), (x_3,o), (x_4,o), (x_5,o), (x_1,1)]$ is a ρ_3-compatible circuit with $\deg_{\rho_3} C_3 = 3$.

Continuing this process we can construct for each integer n , $n \geqq 2$, a connected graph X_n , an equivalence ρ_n on $E(X_n)$, and a ρ_n-compatible circuit C_n with $\deg_{\rho_n} C_n = n$.

Take $Y = \prod_{n=2}^{\infty} (X_n, x_n)$, $\sigma = \bigcap_{n=2}^{\infty} \rho_n$, where $x_n \in V(X_n)$. Then Y is connected, $\sigma \supset \alpha \cup \beta$, and for each integer $n \geq 2$, there exists a σ-compatible circuit C in Y with $\deg_{\rho} C = n$.

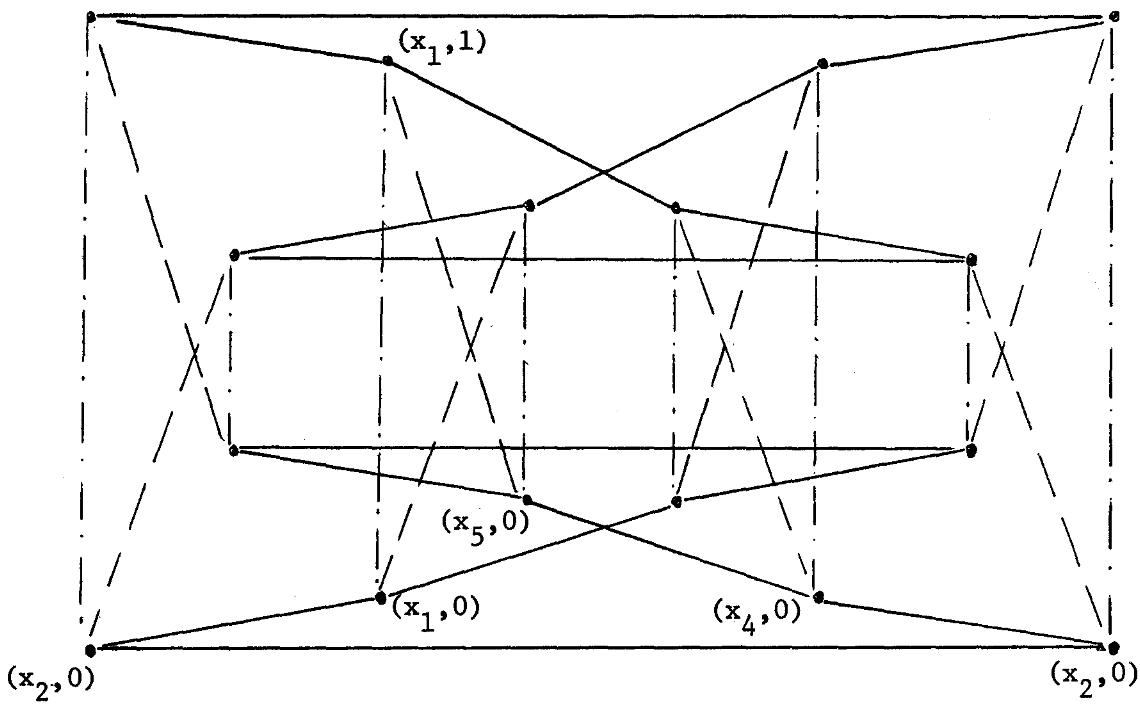


FIGURE 2.10

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