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SCOPE AND CONTENTS: In recent years, a number of qualitative calculations of various nuclear properties have been undertaken based on a "pairing-plus-quadrupole" force model. The present work involves an extension of the techniques associated with this model and is an investigation of the extent to which it can quantitatively predict properties of the low-lying levels of spherical even-even nuclei.

As a particular application, the energies of several of the low-lying levels and the reduced transition probabilities for the first 2^+ levels are calculated for the even isotopes of lead.

These calculations show in a natural way how collective levels arise, and help to clarify the nature both of these and of the non-collective levels.

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INTRODUCTION

The presence of collective effects in the spectra of certain nuclei has been recognized for many years. Most non-spherical, heavy nuclei have low-energy spectra which can be attributed to a rotation of part of their nuclear matter, while many nuclei near closed shells, and possessing a spherical equilibrium shape, exhibit spectra which can be attributed to a vibration of the surface. Volume excitations are possible, but because of the high degree of incompressibility of nuclear matter, such modes will occur at energies that are much higher than the range observed for surface vibrations.

The present work is concerned chiefly with a study of the spectra of even-even spherical nuclei. Such nuclei show significant regularities (S-55). The first excited level nearly always has spin and parity 2^+ and possesses an E2 transition rate to the ground state that is enhanced over the single particle value. The energy of this level varies more or less regularly from isotope to isotope, and increases as a closed shell is approached. In most cases the next excited level lies at roughly twice the energy of the first, and has spin and parity 2^+, 4^+ or, occasionally, 0^+.

In particular, this work treats the even isotopes of lead. These isotopes are appropriate for a quantitative study since the single particle energies needed in the calculation are well known from studies on the one-hole nucleus Pb^{207}.

The present work is a microscopic approach to the problem in which an attempt is made to fit the experimentally observed energies and E2 transition rates by considering shell model particles and allowing them to interact by means of a residual two-body force.

The treatment is reasonably exact insofar as the force chosen for the residual interaction is a good approximation to the actual residual interaction. Exchange terms in the interaction, which have been ignored in all "pairing-model" calculations, as well as in many other treatments, are here treated exactly. Their effect is in no way negligible but their inclusion does complicate the calculations to a considerable degree.

The main purpose of this work is to find out whether or not the pairing force, along with a quadrupole force, is capable of yielding quantitative results in the calculation of nuclear spectra, and at the same time, to investigate the nature of the states which combine coherently to produce the observed collective excitations.

CHAPTER I

METHODS OF TREATING COLLECTIVE VIBRATIONS OF SPHERICAL NUCLEI

The rapidly growing body of experimental data being accumulated on both collective and non-collective excitations has motivated the introduction of many nuclear models.

The success of the shell model in predicting many of the ground state spins and magnetic moments as well as the discontinuities associated with the magic numbers has shown that the main part of the two-nucleon interaction can be assumed to produce a spherically symmetric, static potential (E-57).

The unified model developed by Bohr, Mottelson, and others, is based on the assumption that from the remaining part of the interaction it is possible to extract an additional self-consistent field which is non-spherical and time dependent (B-55).

However, the real two-nucleon interaction can not be completely reduced to a self-consistent field, and some residual interaction will always remain. While this latter

interaction can be neglected in the calculation of many nuclear properties, especially those that depend on gross features, there are other phenomena in which the residual interaction plays a crucial role (B-55), (K-60), (B-59). It is responsible, for example, for the shift of the intrinsic observed levels from the independent particle prediction; for the stability of the spherical shape near closed shell nuclei, and the sharp transition to the deformed shape away from the closed shells; for the reduction of the moment of inertia from the rigid body value; and for the energy gap observed in the spectra of some even-even nuclei.

Let us now discuss, in more detail, some of the above concepts as they apply to collective phenomena.

1. Liquid Drop Model

One of the first models that was introduced in an attempt to discuss specifically collective effects was the liquid drop model (M-57). In this model, the wavefunction contains no explicit reference to individual nucleons but is defined in terms of the shape of the nuclear surface. The equation for the surfaces of constant density can be written as (B-52)

$$R(\theta,\varphi) = R_0\left[1 + \Sigma_{\lambda\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}(\theta\varphi)\right]$$

where the $\alpha_{\lambda\mu}$ are a set of coordinates which describe the deformation of the nuclear surface. For small deformations, the potential energy, corresponding to a given equilibrium shape, can be written in the form

$$E(\alpha) = E(0) + \frac{1}{2} \sum_{\lambda,\mu} C_\lambda |\alpha_{\lambda\mu}|^2 \tag{1.1}$$

where the coefficients C_λ give a measure of the resistance of the nucleus against deformation. Collective motions are introduced by allowing the deformation parameters to vary with time. If these parameters change slowly, with respect to intrinsic nucleon motions, the kinetic energy can be written in the form

$$T = \frac{1}{2} \sum_\lambda B_\lambda \sum_\mu |\dot{\alpha}_{\lambda\mu}|^2 \tag{1.2}$$

where the coefficients B_λ give a measure of the inertia of the nucleus with respect to changes in deformation.

The total Hamiltonian which describes the motion of the nucleus can be written

$$H = E(0) + \frac{1}{2} \sum_\lambda C_\lambda \sum_\mu |\alpha_{\lambda\mu}|^2 + \frac{1}{2} \sum_\lambda B_\lambda \sum_\mu |\dot{\alpha}_{\lambda\mu}|^2 \tag{1.3}$$

from which the classical frequencies of oscillation are given by

$$\omega_\lambda = \left(\frac{C_\lambda}{B_\lambda} \right)^{1/2} \tag{1.4}$$

and the energy levels by

$$E = E(0) + \Sigma_{\lambda\mu} (n_{\lambda\mu} + \tfrac{1}{2}) \hbar w \qquad (1.5)$$

where the $n_{\lambda\mu}$ give the number of phonons in the mode $\lambda\mu$. We note that the formulae given above are a consequence only of the assumed spherical equilibrium shape, of the smallness of the shape oscillations, and of the adiabatic hypothesis. The parameters C_λ and B_λ contain all the dependence on the detailed properties of nuclear matter.

Most studies of the vibrations of spherical nuclei have been concerned with quadrupole vibrations, i.e., the mode $\lambda = 2$. We are thus led to consider the spectrum predicted by the Hamiltonian

$$H = \frac{1}{2} C_2 \Sigma_\mu |\alpha_{2\mu}|^2 + \frac{1}{2} B_2 \Sigma_\mu |\dot{\alpha}_{2\mu}|^2 \qquad (1.6)$$

Only positive parity states can arise, since the parity, given by $(-1)^\lambda$, is +1 for quadrupole vibrations.

The Hamiltonian, (1.6), predicts a ground state spin of $I = 0^+$, a single phonon state with $I = 2^+$ at an energy $\hbar w_2$ above the ground state, and a triplet of states with $I = 0^+, 2^+, 4^+$ at an energy $2\hbar w_2$ above the ground state. This triplet of states results from the coupling of two one-phonon states, each with $I = 2^+$.

These features are seen to be in partial agreement with the experimental facts as outlined earlier. The

liquid drop model, however, fails to predict either the correct magnitude for $\hbar\omega_2$, or the correct dependence of this energy on A. Using the hydrodynamic estimates for B_2 and C_2 (P-62), $\hbar\omega_2$ has a value slightly over 2 Mev for $A \approx 100$, whereas the observed values lie in the range 0.5 to 1 Mev. The agreement is better for $A \approx 200$, where the calculated value drops to 1 Mev, but the dependence on A is still not correct.

The liquid drop model can also be used to calculate the γ-ray transition rates between the low-lying levels. The following features, all in qualitative agreement with experiment, are found (E-58):

a) The cross-over transition from the second 2^+ level to the 0^+ ground state is strongly inhibited.

b) The M_1 component of the $2^+ \rightarrow 2^+$ decay is very weak.

c) The E2 rates between adjacent levels are considerably higher than the single particle estimates.

Again, however, the estimates of B_2 and C_2 obtained from these studies are markedly different from the values required by the experimental data.

Thus, to summarize the contribution of the liquid

drop model, it may be said that the actual nuclear oscillations resemble those of a liquid drop in some respects, but an attempt at detailed calculations on the basis of this model are rather less than successful.

There might thus appear to be some justification for assuming that the form of the equations given by the liquid drop model may be quite adequate. In that case, there would be some interest in attempting to calculate the parameter C_λ and B_λ within the framework of a more sophisticated model. One method of determining the inertial parameters B_λ involves using the "cranking" model formula of Inglis (I-54),

$$B(\alpha) = 2\hbar^2 \sum_{i \neq o} \frac{|\langle i | \partial/\partial\alpha | o \rangle|^2}{W_i - W_o} \qquad (1.7)$$

where the states $|i\rangle$ refer to the intrinsic single particle states, and Wi is their total energy. This formula, for an harmonic oscillator potential, leads to a value of $B(\alpha)$, so small that the adiabatic condition is violated.

2. Shell Model

Shell model calculations in which two-body interactions are taken into account are restricted, because of the complexities involved in computation, to light nuclei

or to nuclei in the vicinity of closed shells. The nucleus Pb^{206}, falling in the latter category, has been particularly well studied (T-58), (G-61).

The first extensive calculation was carried out by True and Ford. These authors obtained reasonably good agreement for the energies of the levels of Pb^{206} up to 3.2 Mev, using a singlet two-body force with the same effective range and strength as for the low energy two-body system. Their calculations indicated that better agreement could be obtained if some coupling to collective surface vibration was introduced, and further, that such coupling was essential to yield the correct E2 transition rates.

A more recent calculation, more in the spirit of the unified model, in which surface coupling is introduced at the start, was carried out by Guman et al (G-61). Let us briefly outline the method used in this calculation. The nucleus Pb^{206} is regarded as a system consisting of two neutron holes and the surface of the core. The Hamiltonian is written as

$$H = H_s + H_p(1) + H_p(2) + V_s(1) + V_s(2) + V_p(12)$$

where $H_s = \hbar w \left(\Sigma_\mu b^+_\mu b_\mu + 5/2\right)$

$$H_p = -\frac{h^2}{2m} \nabla^2 + V_c$$

$$V_c = V(r) - \lambda \left(\frac{\hbar}{2mc}\right)^2 \frac{\vec{l}\cdot\vec{s}}{r} \frac{\partial V(r)}{\partial r} \qquad 1.8)$$

$$V(r) = \frac{-V_0}{(1 + e^{\alpha(r-r_0)})}$$

$$V_p = - \left[v_t \Pi_t + v_s \Pi_s\right] e^{- \frac{|r_1 - r_2|^2}{\rho^2}}$$

$$V_s(r_k) = -x(r_k) \Sigma_\mu \alpha_{2\mu} Y_{2\mu}(\theta_k \varphi_k)$$

The parameters of the central potential, V_c, are assumed to be known from previous work (S-59).

In the expression for V_p, v_t and v_s are the triplet and singlet interactions, and Π_t and Π_s are the corresponding projection operators. ρ is the effective range of the pair interaction.

$V_s(r_k)$ describes the interaction between one external nucleon and the field generated by all the others, and leads to a deformation of the potential surface of the core. For small deformation, the matrix element of the radial part of this quadrupole force is taken to be

$$\langle n\ell \mid x(r_k) \mid n'\ell' \rangle \approx (-1)^{n+n'} \cdot 40 \text{ Mev} \qquad (1.9)$$

The parameter $\alpha_{2\mu}$ is represented by

$$\alpha_{2\mu} = \sqrt{\frac{\hbar w}{2C}} \left(b_{2\mu} + (-1)^\mu b^\dagger_{2\mu}\right)$$

where C is the effective surface tension, and $b^\dagger_{2\mu}$ and $b_{2\mu}$

are the annihilation and creation operators for a phonon with angular momentum 2 and z - component, μ . The frequency of the core surface oscillations is w.

The basic set of eigenfunctions are chosen to be of the form

$$|j_1 j_2\ J;\ NR;\ IM\rangle$$

which corresponds to a state with total angular momentum I and z - component M formed from a state of two nucleons coupled to angular momentum J and N phonons with angular momentum R.

These functions form the basis for a calculation of the eigenfunctions of the complete Hamiltonian H.

There are four constants to be determined, namely hw, C_0, ρ , and v_s. The authors show that, with suitable choices for these parameters, all experimentally known excited levels up to an energy of 3 Mev can be reproduced. The authors note one exception, namely, the 4^+ level at 1.66 Mev, and cite this exception as evidence that this level should be reinvestigated. Good agreement with experiment is also achieved for the calculated transition probabilities between the various levels.

This calculation shows that collective states are admixed in all the excited levels, and that the short range

pair interaction leads to a mixing of the single particle states.

3. Pairing-Plus-Quadrupole Force Method

One of the most difficult problems associated with the residual interaction has been that of suitably accounting for the short range part. A major breakthrough in this direction occurred when Bardeen, Cooper and Schrieffer introduced their theory of superconductivity (B-57). The ideas associated with this theory have been taken over to the case of nuclear matter and have formed the basis for the introduction of the pairing force. It was observed that nuclear structure exhibits certain features which are similar to those of electron structure in metals, the most important of these being an energy gap observed both in the spectra of certain nuclei and in the band structure of superconducting metals. Bohr, Mottelson and Pines (BM-58) suggested a possible analogy between the correlation in the two systems responsible for the gap. The physical basis for this analogy is the similarity between the pairing energy of two nucleons with equal and opposite projections of the angular momentum, and quasi-bound states of electron pairs with equal and opposite linear momentum.

The pairing force which has emerged from the above considerations can be defined (K-60) as one which has constant matrix elements in a $|jm\rangle\ |j-m\rangle$ configuration. This is equivalent to saying that the matrix elements of the interaction between two particles in a j-shell and two-particles in a j'-shell vanishes unless the total angular momentum in each state is zero, in which case it is proportional to $\frac{1}{2}\sqrt{(2j+1)(2j'+1)}$

Assuming then, that the pairing force adequately describes the short range part of the residual interaction, it remains to describe the long range part. The presence of quadrupole vibrations of even-even nuclei, and the appearance of several regions of permanently deformed nuclei possessing large quadrupole moments, suggests that a quadrupole force is required as at least part of the field-producing portion of the residual interaction. It appears that higher multipoles are not required.

Let us now discuss, in more detail, some of the main features of the pairing model.

a) Pairing Model (B-59)

Let us introduce the Fermi operators a_ν^+ and a_ν which create and annihilate a particle in the state ν The index ν can refer to any appropriate set of quantum

numbers, in particular, for the case of a spherical nucleus, $\nu = [n, l, j, m]$. We will define such single particle states as

$$|\nu\rangle = a^{+}_{\nu}|0\rangle$$

where $|0\rangle$ denotes the vacuum state with respect to the operators a^{+}, a, and is defined by the set of all equations

$$a_{\nu}|0\rangle = 0$$

If we impose the condition of time reversal invariance on the system then, the states $|\nu\rangle$ and $|-\nu\rangle$ are degenerate. Here, $|-\nu\rangle$ is the state with the same set of quantum numbers as $|\nu\rangle$ but with the opposite sign for the projection of angular momentum. The relation between the two states , apart from a phase factor, is

$$|-\nu\rangle = \tau|\nu\rangle \qquad (1.10)$$

where τ is the time reversal operator.

In terms of the usual angular momentum states used, for example, by Condon and Shortley (C-57), a state with angular momentum j and component m can be written

$$\psi_{jm} = \sum_{m_s + m_l = m} C^{l m_l ; s m_s}_{j m} \; i^{l} Y_{l m_l} \chi_{s m_s} \qquad (1.11)$$

where $C^{l m_l ; s m_s}_{jm}$ is a Clebsch-Gordon coefficient, $\chi_{s m_s}$ is the usual spin function, and the spherical harmonics have been defined as $i^{l}Y_{lm}$. To obtain the time-reversed

state, we make use of the relations (W-59)

$$\tau\, Y_{\ell m_\ell} = (-1)^{m_\ell} Y_{\ell -m_\ell} \qquad \tau\, \chi_{s m_s} = i^{2m_s} \chi_{s -m_s}$$

and write

$$\tau\, \psi_{jm} = \sum C_{jm}^{\ell m_\ell ; s m_s} (-1)^{\ell} Y_{\ell -m_\ell} (i)^{2(m_\ell + m_s)} \chi_{s -m_s}$$

$$= (-1)^{j-m} \sum C_{j-m}^{\ell m_\ell ; s m_s} (-i)^{\ell} Y_{\ell m_\ell} \chi_{s m_s}$$

$$= (-1)^{j-m} \psi_{j-m}$$

Thus, the states $|-\vartheta\rangle = |j\text{-}m\rangle$ have the phases $(-1)^{j-m}$ times those in the usual Condon and Shortley notation (allowing for the extra phase factor introduced in the spherical harmonics).

The Hamiltonian for a system of particles moving in a spherically-symmetric, self-consistent well, and interacting through a residual two-body force, can be written

$$H' = \sum_{\vartheta} \varepsilon_j a^{+}_{\vartheta} a_{\vartheta} - \frac{1}{2} \sum \langle \vartheta_1 \vartheta_2 | V | \vartheta_2' \vartheta_1' \rangle a^{+}_{\vartheta_1} a^{+}_{\vartheta_2} a_{\vartheta_2'} a_{\vartheta_1'} \qquad (1.12)$$

where ε_j is the energy of a particle in the shell j.

The pairing interaction is one in which each state is assumed to be correlated with its time-reversed conjugate. A convenient way of introducing these correlations is to define new Fermi operators as follows:

$$\alpha_{\vartheta} = u_{\vartheta} a_{\vartheta} - v_{\vartheta} a^{+}_{-\vartheta} \qquad (1.13)$$

$$\beta_{\vartheta} = u_{\vartheta} a_{-\vartheta} + v_{\vartheta} a^{+}_{\vartheta}$$

The canonical nature of the transformation is insured by choosing the u_ν and v_ν to be real and to satisfy the relation

$$u_\nu^2 + v_\nu^2 = 1 \tag{1.14}$$

This canonical transformation does not conserve the number of particles, and instead of considering the Hamiltonian H', it is necessary to introduce the Hamiltonian

$$H = H' - \lambda N \tag{1.15}$$

where λ is chosen in such a manner that the average number of particles in the N-body system is N.

The inverse transformation to (1.13) is

$$\begin{aligned} a_\nu &= u_\nu \alpha_\nu + v_\nu \beta_\nu^+ \\ a_{-\nu} &= u_\nu \beta_\nu - v_\nu \alpha_\nu^+ \end{aligned} \tag{1.16}$$

Substitution of this transformation into (1.15) yields a Hamiltonian with the following structure:

$$H = U + (H_{20} + H_{02}) + H_{11} + H_{int} \tag{1.17}$$

U is a constant term

$$\begin{aligned} U = {} & \Sigma'_\nu (\bar{\varepsilon}_\nu - \lambda) 2 v_\nu^2 - \Sigma'_\nu \Delta_\nu u_\nu v_\nu \\ & + \Sigma'_{\nu\nu'} \langle \nu \nu' | \bar{G} | \nu' \nu \rangle v_{\nu'}^2 v_\nu^2 \end{aligned} \tag{1.17a}$$

$(H_{02} + H_{20})$ and H_{11} are quadratic in the new operators.

$$\begin{aligned} (H_{02} + H_{20}) = {} & \Sigma_\nu \Big[(\bar{\varepsilon}_\nu - \lambda) 2 u_\nu v_\nu - \Delta_\nu (u_\nu^2 - v_\nu^2) \Big] \\ & \times (\alpha_\nu^+ \beta_\nu^+ + \beta_\nu \alpha_\nu) \end{aligned} \tag{1.17b}$$

$$H_{11} = \Sigma_\nu \left[(\bar{\varepsilon}_\nu - \lambda)(u_\nu^2 - v_\nu^2) + \Delta_\nu 2 u_\nu v_\nu \right] \times (\alpha_\nu^+ \alpha_\nu + \beta_\nu^+ \beta_\nu) \tag{1.17c}$$

where $\bar{\varepsilon}_\nu$ and Δ_ν are defined by

$$\bar{\varepsilon}_\nu = \varepsilon_\nu - \Sigma'_{\nu_1} \langle \nu \nu_1 | \bar{V} | \nu_1 \nu \rangle v_{\nu_1}^2 \tag{1.18}$$

$$\Delta_\nu = \Sigma'_{\nu_1} \langle \nu \bar{\nu} | V | \nu_1 \bar{\nu}_1 \rangle u_{\nu_1} v_{\nu_1} \tag{1.19}$$

The matrix elements in (1.18) and (1.19) are defined by

$$\langle \nu_1 \nu_2 | V | \nu_2' \nu_1' \rangle = \langle j_1 m_1 j_2 -m_2 | V | j_2' -m_2' j_1' m_1' \rangle - \langle j_1 m_1 j_2 -m_2 | V | j_1' m_1' j_2' -m_2' \rangle$$

$$\langle \nu_1 \nu_2 | \bar{V} | \nu_2' \nu_1' \rangle = \langle \nu_1 \nu_2 | V | \nu_2' \nu_1' \rangle + \langle j_1 m_1 j_2' m_2' | V | j_2 m_2 j_1' m_1' \rangle - \langle j_1 m_1 j_2' m_2' | V | j_1' m_1' j_2 m_2 \rangle$$

Σ' means that the sum is restricted to positive values of the angular momentum components. The remaining term in the Hamiltonian, H_{int}, describes the interaction between the new particles, and can be written in the form

$$H_{int} = (H_{40} + H_{04}) + (H_{31} + H_{13}) +$$

Explicit expressions for the ter

given in the appendix to reference (B-59).

Let us, for the moment, neglect the terms in H_{int}. Then the Hamiltonian (1.17) will describe a set of independent particle-like entities, or 'quasiparticles', if the term $(H_{20} + H_{02})$ is set equal to zero. This condition yields the relation

$$(\bar{\varepsilon}_\nu - \lambda)\, 2 u_\nu v_\nu - \Delta_\nu (u_\nu^2 - v_\nu^2) = 0 \tag{1.20}$$

which, along with (1.14) yields the familiar relations for u_ν and v_ν.

$$u_\nu^2 = \frac{1}{2}\left[1 + \frac{\bar{\varepsilon}_\nu - \lambda}{E_\nu}\right] \qquad v_\nu^2 = \frac{1}{2}\left[1 - \frac{\bar{\varepsilon}_\nu - \lambda}{E_\nu}\right] \tag{1.21}$$

where $E_\nu = \left[(\bar{\varepsilon}_\nu - \lambda)^2 + \Delta_\nu^2\right]^{1/2}$

is the energy of a quasi-particle in the state ν.

Using (1.21) and (1.20), the equation for Δ_ν can be written in the form

$$\Delta_\nu = \frac{1}{2} \sum'_{\nu'} \langle \nu\bar{\nu} | V | \nu'\bar{\nu}' \rangle \frac{\Delta_{\nu'}}{E_{\nu'}} \tag{1.22}$$

Specializing now to the case where $\nu = [j,m]$ (we can suppress the dependence on n and l for what follows), the pairing matrix element in (1.22) can be written in the following form:

$$\langle jj|V|j'j'\rangle = C_{JM}^{j'-m';j'm'} C_{JM}^{j-m;jm} \langle JM|V|JM\rangle$$

$$- C_{JM}^{j'm';j'-m'} C_{JM}^{j-m;jm} \langle JM|V|JM\rangle$$

$$= C_{JM}^{j'-m';j'm'} C_{JM}^{j-m;jm} \left(1 + (-1)^{J}\right) \langle JM|V|JM\rangle \qquad (1.23)$$

Since the pairing force we are considering vanishes unless J = 0, we set J = 0 in (1.23). Then, using the relation

$$C_{00}^{jm;j-m} = (-1)^{j-m} (2j+1)^{-1/2}$$

and defining $\langle 00 | V | 00 \rangle$ as $\frac{G}{2} \sqrt{(2j+1)(2j'+1)}$, we can rewrite (1.22) in the form

$$(-1)^{j-m} \Delta jm = \frac{1}{2} G \sum'_{j'm'} \frac{(-1)^{j'-m'} \Delta j'm'}{E j'} \qquad (1.24)$$

This expression shows that $(-1)^{j-m} \Delta jm$ is a constant. We then define

$$(-1)^{j-m} \Delta jm = \Delta \qquad (1.25)$$

and this reduces (1.24) to the form

$$1 = \frac{G}{2} \sum j' \frac{\Omega j'}{E j'} = \frac{G}{2} \sum j' \frac{\Omega j'}{\sqrt{(\bar{E}_j - \lambda)^2 + \Delta^2}} \qquad (1.26)$$

where $\Omega_j = j + 1/2$ is the pair degeneracy of the j-shell.

As mentioned earlier, the constant λ is found by

requiring that the average number of particles in the system be N. This requirement can be written

$$\langle o | \hat{N} | o \rangle = N = \Sigma_j \Omega_j \left[1 - \frac{\varepsilon_j - \lambda}{E_j} \right] \qquad (1.27)$$

Thus, if the shell model energies, $\bar{\varepsilon}_j$, are known, and a value for G, the strength of the pairing force is estimated, then (1.26) and (1.27) can be solved simultaneously for λ and Δ . It should be noted that, while in equations (1.26) and (1.27) the sum includes all distinct neutron, or proton, energies, in practice only states in the partially filled shell under consideration are considered. This is not a serious approximation, since states that are distant from the Fermi energy, λ , will make little contribution to the sum, since, for these states, $v^2 \approx 1$ or 0.

The main results of the pairing model can be summarized as follows:

1). The Hamiltonian describes a system of independent quasiparticles

$$H = U + \Sigma'_j E_j \left(\alpha^+_{jm} \alpha_{jm} + \beta^+_{jm} \beta_{jm} \right)$$

2). The single quasiparticle excitation energy is given by

$$E_j = \sqrt{(\bar{\varepsilon}_j - \lambda)^2 + \Delta^2}$$

3). The quasiparticle vacuum is defined by

$$\alpha_j |0\rangle = 0 \qquad \text{for all } j.$$

The approximations involved in achieving this simplified picture are as follows:

1) The number of particles is no longer fixed. This means that solutions of a Hamiltonian which has quasiparticles as eigenstates, will describe only average properties of nuclei, that is, in calculating the properties of a nucleus with N particles, an average over the properties of nuclei with N, $N \pm 2$, $N \pm 4$, etc. is involved. In practice, this does not appear to be too serious, but it is important that the property being studied varies smoothly from one nucleus to another.

2) The matrix elements of the pairing force are set equal to a constant, for states coupled to total angular momentum zero. Such a force does seem to reproduce the important characteristics of a short range interaction. In particular, it gives rise to a two-particle spectrum in which one state, that with $J = 0$, is split off from all the others (M-58). It is hoped that the differences between the pairing force and the short range part of the actual two-body interaction will be relatively unimportant.

b) Work of Kisslinger and Sorensen (K-60)

A nuclear model in which the residual interaction is represented by just these two components, the pairing force and a quadrupole force, was first studied in some detail in a fundamental paper by Belyaev(B-59). The first realistic quantitative calculations based on this model were performed by Kisslinger and Sorensen (K-60). In order to avoid the difficult problem of treating the short range interaction between neutrons and protons, the authors considered only the case of nuclei in which either the neutron shell or proton shell was closed. They followed the procedure of Belyaev, in which the pairing force correlations are taken into account by means of a canonical transformation from the original interacting shell-model nucleons to new independent quasiparticles. The ground state of the system in terms of the new quasiparticles is the "vacuum" state. The pairing correlations enter into the "vacuum" energy, and into the intrinsic structure of the quasiparticles. Thus, if the interaction between quasiparticles is ignored, one is left with a system of new independent "particles" which implicitly contain the effect of the pairing force. The effect of the quadrupole force is then taken into account by introducing the total quadrupole moment of the nucleus as a

parameter, and allowing the particles defined above to interact with the deformed field generated by this parameter. The adiabatic hypothesis is then invoked, and the energy is calculated for a fixed value of the quadrupole moment, $W(Q)$. The Hamiltonian of the collective motion, for small Q, can be written

$$H = \frac{1}{2} C(Q)\, Q^2 + \frac{1}{2} B(Q)\, \dot{Q}^2 \qquad (1.28)$$

where the inertial parameter $B(Q)$ is found from the usual cranking model formula of Inglis (1.7).

This Hamiltonian describes harmonic quadrupole surface vibrations and yields, for the energy of the first 2^+ level,

$$\hbar w = \sqrt{\frac{C(Q)}{B(Q)}} \qquad (1.29)$$

As mentioned previously, the potential energy term, $C(Q)$, and the kinetic energy term, $B(Q)$, depend upon the details of the intrinsic nucleon motion. The authors calculate these quantities for the quasiparticle model, and for the case of a single j-shell, eq. (1.29) can be written in the form (B-61)

$$\hbar w = 2E \sqrt{1 - \Theta} \qquad (1.30)$$

where E is the energy of a quasiparticle in the j-shell being considered, and Θ is a function of the number of particles that varies from nearly zero at closed shells, to nearly unity as the shell becomes half-filled. This is in qualitative agreement with the observed trend, where the first 2^+ excitation is smaller than the lowest "single-particle" excitation, $2E$, and decreases as particles are added to a closed shell nucleus. The authors also calculate the reduced matrix elements for excitations from the ground state to the 2^+ level

$$B(E2) = \frac{5}{2} \frac{e_{eff}^2}{\sqrt{BC}} \qquad (1.31)$$

where e_{eff} is the effective charge of the external nucleons. The derivation of this formula involves the assumption that this transition exhausts the sum rule for quadrupole matrix elements to the ground state. For the isotopes of lead, the authors obtain agreement with experiment by using the experimentally measured effective charge in Pb^{207},

$e_{eff} = 1.1$ (T-58).

The results of Kisslinger and Sorensen do seem to indicate that a quite adequate description of those properties of nuclei which depend strongly on the residual interaction can be achieved by the use of a pairing force and a quadrupole force.

The calculations of Kisslinger and Sorensen, in which a parameter is introduced to describe a specific type of collective motion can be considered as a macroscopic description of the problem. From such a viewpoint, the 2^+ excited levels occur as a result of oscillations of the quadrupole moment of the external nucleons. The static polarizability of the core then gives rise to surface oscillations (B-61).

A more fundamental approach to the problem would be a microscopic description, in which one starts from the basic single particle states and an appropriate residual interaction, calculates the "elementary excitations" of the system, and from these builds up the collective levels. The only parameters appearing in such a treatment would be the strengths of the various components of the residual interaction, for example, the strengths of the pairing and quadrupole forces. Besides being more fundamental, a microscopic description becomes mandatory if extensions of the theory to include higher excitations than the 2^+ levels are attempted for here, the adiabatic hypothesis breaks down. A further advantage lies in the fact that collective levels, weakly-collective levels, and non-collective two-particle

excitations can be described by the same set of equations.

4. Method of Approximate Second Quantization

There are two general methods which have been used to investigate, from a microscopic viewpoint, the collective excitations. The first method, discussed in this section, is the "method of approximate second quantization," although it is also known under various other names (B-60). The second method, and the one to be used in this work, is the method of Green's function (B-61) (G-58). An introductory note on this latter method is contained in the next section.

The Hamiltonian for the residual interaction is written in the form

$$H = H_p + H_Q \tag{1.32}$$

where H_p is the pairing Hamiltonian, and

$$H_Q = -\frac{1}{2} \times \sum \langle \nu_1 \nu_2 | r_1^2 r_2^2 \sum_\mu Y_{2\mu}(1)\, Y^*_{2\mu}(2) | \nu_2' \nu_1' \rangle \times a^+_{\nu_1} a^+_{\nu_2} a_{\nu_2'} a_{\nu_1'} \tag{1.33}$$

The usual Bogoliubov transformation converts these Hamiltonians to the following forms.

$$H_p = \sum'_{jm} E_j \left(\alpha^+_{jm} \alpha_{jm} + \beta^+_{jm} \beta_{jm} \right) \tag{1.32a}$$

$$H_Q = H'_Q + H''_Q \tag{1.32b}$$

Here,

$$H'_Q = -1/2 \times \Sigma (q_2)_{jj'} (q_2)_{j_1 j_1'} (u_j v_{j'} + u_{j'} v_j)$$
$$\times (u_{j_1} v_{j_1'} + u_{j_1'} v_{j_1}) \left(A^+_{jj';\kappa\mu} + (-1)^{\mu} A_{j'j;\kappa-\mu} \right)$$
$$\times \left(A^+_{j_1 j_1';\kappa\mu} + (-1)^{\mu} A_{j_1' j_1;\kappa-\mu} \right) \qquad (1.34)$$

and

$$(q_2)_{jj'} = \langle j \| r^2 Y_2 \| j' \rangle \left(5(1+\delta_{jj'})\right)^{-1/2} \qquad (1.35)$$

$$A^+_{jj';2\mu} = \Sigma_{m,m'} C^{jm;j'm'}_{2\mu} \alpha^+_{jm} \alpha^+_{j'm'} \qquad (1.36)$$

The operator A^+ thus creates a quasiparticle pair in a state with a fixed angular momentum and z- component.

H''_Q contains the remaining terms which contribute to H_{int} defined by equation (1.17d). The error made in neglecting H''_Q is $\sim (\rho_o \Delta)^{-1} \sim A^{-1/3}$. Here, ρ_o is the density of levels near the Fermi surface, and Δ is the energy gap. To the same order of approximation, the operators A and A^+ can be regarded as Boson operators obeying the commutation relations

$$\left[A^+_{jj';\kappa\mu}, A_{j_1 j_1';\kappa\mu} \right] \simeq \delta_{jj_1} \delta_{j'j_1'} - \delta_{jj_1'} \delta_{j'j_1}$$
$$\left[A^+_{jj';\kappa\mu}, A^+_{j_1 j_1';\kappa\mu} \right] = \left[A_{jj';\kappa\mu}, A_{j_1 j_1';\kappa\mu} \right] = 0 \qquad (1.37)$$

The excitation energy w of an excited level with angular momentum 2 and z-component μ can then be obtained by taking matrix elements of the equations of motion

for A^+ and A between the ground state and the excited level.

Let us denote the exact ground state by $|\Phi_0\rangle$ and the excited level by $|2\mu\rangle$. The matrix element of the equation of motion for A^+ can then be written

$$\langle\Phi_0|\left[A^+_{jj';2\mu}, H\right]|2\mu\rangle = w\langle\Phi_0|A^+_{jj';2\mu}|2\mu\rangle \tag{1.38}$$

where $H = H_p + H_Q'$

A similar equation holds for the operator A. A system of coupled equations for the amplitudes

$$\langle\Phi_0|A^+|2\mu\rangle \quad and \quad \langle\Phi_0|A|2\mu\rangle$$

is thus obtained. The condition that these equations be consistent yields the secular equation

$$\frac{1}{\chi} = \Sigma_{jj'}\frac{2(E_j+E_{j'})(g_2)^2_{jj'}(u_jv_{j'}+v_ju_{j'})^2}{(E_j+E_{j'})^2-w^2} = F_{jj'}(w) \tag{1.39}$$

A graph of this function is shown below in Figure 1. Only positive values of w should be considered since, by definition, the ground state has zero energy and negative values of w are thus unphysical.

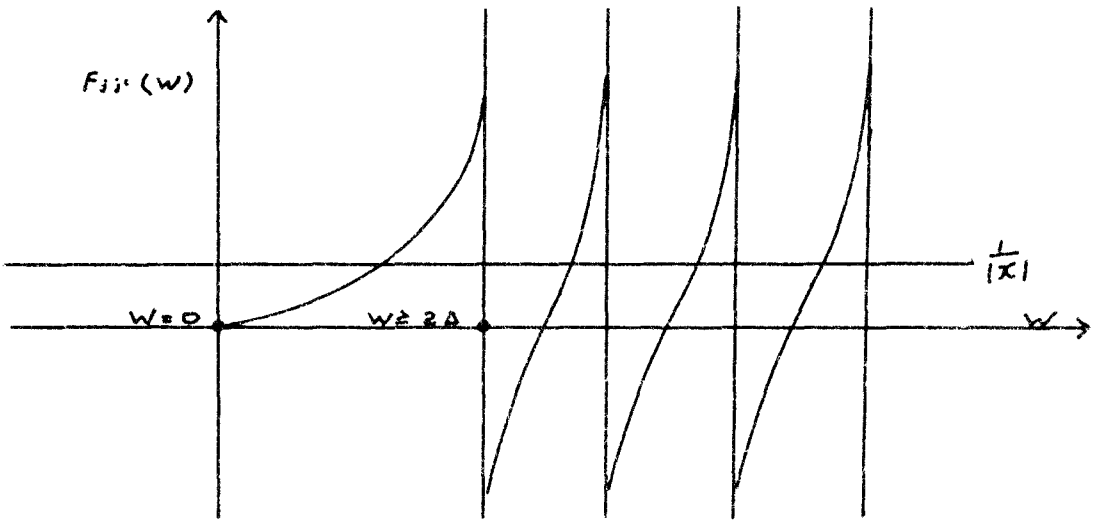


Figure 1

The function $F_{jj}i\ (w)$ has poles at the quasiparticle energies $(E_j + E_{ji})$. The first such pole occurs at an energy $\geq 2\Delta$ since Δ is the smallest value that E can assume. As can be seen from Figure 1, if χ is positive, and not too small, a collective level is split off from the otherwise regular two-quasiparticle spectrum. The level is collective in nature since it is only through a coherent interaction among several single particle states that an energy can be obtained which is lower than that of the states taken individually.

The same equation of motion approach can be used to find the state vector for the excited collective level.

If we define an operator $B^{+}_{2\mu}$ such that

$$|2\mu\rangle = B^{+}_{2\mu}|\Phi_0\rangle \tag{1.40}$$

and assume that $B^{+}_{2\mu}$ can be expanded in terms of the operators A^{+} and A, then (1.38) can be used to find the coefficients of expansion. It turns out that these coefficients are just proportional to the amplitudes $\langle\Phi_0|A^{+}|2\mu\rangle$ and $\langle\Phi_0|A|2\mu\rangle$. Thus, once the energy of the 2^{+} level is found, the amplitudes, and hence the state vector, can also be found.

The assumption that the operators A^{+} and A can be regarded as Boson operators, has led to the above approach being called the 'Quasi-Boson' method.

An equivalent approach, called the "method of linearized equations of motion," has also been used to derive the same secular equation and state vector (B-60). In this method, the equation of motion technique is again used, but the exact commutators of A and A^{+} are found with the complete Hamiltonian $H'_Q + H''_Q$. The approximation consists in dropping all terms containing four single quasiparticle operators, and keeping only those with two such operators. This procedure leads to a set of equations which are linear in A and A^{+}, the same set, in fact, which arise from the

Quasi-Boson approximation. The justification for dropping terms with four operators, according to Baranger, is that these terms involve more energetic excitation. Beyond this statement, however, an investigation of the validity of the approximation is not carried out. Presumably, there is a close connection between this approximation and the quasi-boson treatment where only that part of the Hamiltonian which will lead to linear equations is used.

Besides these approximations, which are rather difficult to assess, it should be mentioned that the simplicity in form of the secular equation for w, (1.39), results from a neglect of certain exchange terms. This point will be taken up again when the equations to be used in this work have been derived.

5. Method of Green's Functions

The method used in the present work to study the low-lying collective excitations in spherical, even-even nuclei involves the use of Green's functions (B-61). Let us briefly outline the method to be followed.

Single particle excitations can be described by single particle Green's functions, whose poles in the complex energy plane determine the energy and damping of these excitations (GM-58). Similarly, the poles of the two particle

Green's function yield the energies and damping rates of the two-particle excitations. Thus, to describe the collective states in the presence of a pairing force and a quadrupole force, the following procedure is carried out. Considering first only the pairing force, the single-particle Green's functions are calculated and the single-particle excitations are found. These excitations will be called "quasiparticles." We note that this definition of quasiparticles, as will be discussed in the next chapter, is not the same as that introduced in connection with the canonical transformation technique discussed earlier. The effect of the quadrupole force is then introduced as an interaction between the quasiparticles, and the Green's function for the two quasiparticle excitations is calculated, yielding the two quasiparticle energy spectrum. Collective levels can also arise, depending on the nature of the interaction. Since only a quadrupole force is being considered, the appearance of collective levels will depend upon the strength and the sign of this force.

Because of the presence of pairing, which leads to a condensate of Cooper pairs being formed in the ground state, the usual methods of quantum field theory, in particular the analysis of the problem in terms of Feynman

diagrams, break down (B-58). This difficulty is associated with the non-vanishing ground state expectation values associated with uncontracted pairs of operators. This problem can be circumvented by considering particles outside the condensate, and then treating the condensate as an external field. The system will no longer be closed with regard to the number of particles, since allowance must be made for the absorption of a pair of particles from the condensate, or the loss of a pair of particles which may combine to form a bound state with zero momentum and drop into the condensate.

The presence of the condensate necessitates the introduction of three single-particle Green's functions: the normal one, and two more which allow for propagation into and out of the condensate. Similarly, we must introduce three two-particle Green's functions. A system of coupled equations for these functions is then derived. Each of the two-particle Green's functions can be reduced to an amplitude for a specific kind of pair, particle-particle, particle-hole or hole-hole. By pair is meant just a product of two real particle creation or annihilation operators. These amplitudes are transition matrices between the exact ground

state and an excited level characterized by angular momentum K, and energy w. They can be interpreted physically as the proportions of the various two-quasiparticle states from which the collective level is built. The system of coupled equations forms a homogeneous set in terms of these amplitudes, with coefficients depending upon the energy w. A secular determinant is then constructed from which the energy of the collective level is determined. Having found the energy, the amplitudes are then calculated and the state vector found. Once the state vector is found, of course, it is possible to calculate electromagnetic transition rates.

The next chapter describes in detail how the above calculations are carried out.

CHAPTER II

FORMULATION OF THE GREEN'S FUNCTION METHOD

1. One-Particle Green's Function

We introduce the single particle Green's function, defined in the coordinate representation,

$$G(\vec{r}_1 t_1; \vec{r}_2 t_2) = -i \langle \Phi_0^N | T\left\{ \Psi(\vec{r}_1 t_1)\, \Psi^+(\vec{r}_2 t_2) \right\} | \Phi_0^N \rangle \tag{2.1}$$

where $|\Phi_0^N\rangle$ denotes the exact ground state for a system of N particles, Ψ^+ and Ψ are creation and annihilation operators in the Heisenberg picture, and T is the Wick Chronological operator.

Let us write the Hamiltonian of the system in the form

$$H_0 = \int \Psi^+(\vec{r}t) \frac{\vec{p}^2}{2M} \Psi(\vec{r}t)\, d\vec{r}$$

$$+ \int\int \Psi^+(\vec{r}t)\, \Sigma(\vec{r}t; \vec{r}'t')\, \Psi(\vec{r}'t')\, d\vec{r}\, d\vec{r}'\, dt' \tag{2.2}$$

The potential Σ, which in general is non-local, comprises the set of all proper self-energy diagrams.

Corresponding to this Hamiltonian, the equation of motion for the Green's function, (2.1), takes the form (KK-60)

$$\left(i\frac{\partial}{\partial t_1} - \frac{\vec{p}^{\,2}}{2M} \right) G(\vec{r}_1 t_1 ; \vec{r}_2 t_2) = \delta(\vec{r}_1 - \vec{r}_2)\,\delta(t_1 - t_2)$$

$$+ \int \Sigma(\vec{r}_1 t_1 ; \vec{r}' t')\; G(\vec{r}' t' ; \vec{r}_2 t_2)\, d\vec{r}'\, dt' \qquad (2.3)$$

We first assume that the Hamiltonian (2.2) does not depend explicitly on time, in which case both G and Σ are functions of time only through the difference $t_1 - t_2 = \tau$ Taking the Fourier transform of (2.3) with respect to this time difference, we get

$$\left(\varepsilon - \frac{\vec{p}^{\,2}}{2M} \right) G(\vec{r}_1, \vec{r}_2 ; \varepsilon) = \delta(\vec{r}_1 - \vec{r}_2)$$

$$+ \int \Sigma(\vec{r}_1, \vec{r}' ; \varepsilon)\; G(\vec{r}', \vec{r}_2 ; \varepsilon)\, d\vec{r}' \qquad (2.4)$$

It is well-known that the poles of G in the complex energy plane determine the energy and damping of the elementary excitations of the system (GM-58). Also, the single particle states with which we will be dealing are those with energies close to the Fermi energy ε_0 .

It is thus necessary to investigate the expansion of $\Sigma(\varepsilon)$ in the neighborhood of the Fermi energy. This has been carried out in some detail by Migdal (M-62). He emphasizes the difference in character of $\Sigma(\varepsilon)$, depending on whether or not Cooper pairs may be formed near the Fermi surface. If Σ_R denotes the regular part of Σ containing no Cooper pairing, and if Σ_K denotes the part

of the self energy corresponding to the presence of Cooper pairs, then equation (2.4) reduces, as shown by Migdal, to the form

$$\left(\varepsilon - \frac{\vec{p}^{\,2}}{2M_{eff}} - U(\vec{r}_1)\right)\left(1 - \frac{\partial \Sigma_R(p_0,\varepsilon_0)}{\partial \varepsilon_0}\right) G(\vec{r}_1,\vec{r}_2;\varepsilon)$$

$$= \delta(\vec{r}_1-\vec{r}_2) + \int \Sigma_K(\vec{r}_1,\vec{r}\,';\varepsilon)\, G(\vec{r}\,',\vec{r}_2;\varepsilon)\, d\vec{r}\,' \tag{2.5}$$

where

$$M_{eff} = M \frac{1 - \frac{\partial \Sigma_R}{\partial \varepsilon_0}}{1 + \left(\frac{\partial \Sigma_R}{\partial p_0}\right)\frac{M}{p_0}} \tag{2.6}$$

and

$$U(\vec{r}) = \frac{\Sigma_R(\vec{r}, p_0 \varepsilon_0) - \varepsilon_0 \frac{\partial \Sigma_R}{\partial \varepsilon_0} - \frac{p_0}{2}\frac{\partial \Sigma_R}{\partial p_0}}{1 - \frac{\partial \Sigma_R}{\partial \varepsilon_0}} \tag{2.7}$$

In these equations, p_0 and ε_0 denote the momentum and energy associated with the Fermi surface. The above equations are accurate to order $r_0/R \approx A^{-1/3}$ where r_0 is the interparticle distance at the Fermi surface, and R is the size of the system.

One of the basic assumptions of the present work, as well as of those outlined earlier, is that the same set of intrinsic single particle states can be used throughout an entire major oscillator shell. Another way of stating this assumption is to assert that the self-energy term, Σ_R,

does not change as particles are added to, or subtracted from, the system, provided that such changes do not occur across a magic nucleus. The mathematical expression of this assumption is contained in the statement

$$\frac{\partial \Sigma_R}{\partial \varepsilon_0} = 0 \tag{2.8}$$

since the only quantity affecting the position of the Fermi energy is now assumed to be the number of particles, and Σ_R is independent of this number. Similarly, we can set

$$\frac{\partial \Sigma_R}{\partial \rho_0} = 0 \tag{2.9}$$

The equation for the Green's function can now be written

$$\left(\varepsilon - \frac{\vec{p}^2}{2M} - U(\vec{r}_1)\right) G(\vec{r}_1, \vec{r}_2 ; \varepsilon)$$

$$= \delta(\vec{r}_1 - \vec{r}_2) + \int \Sigma_K(\vec{r}_1, \vec{r}' ; \varepsilon)\, G(\vec{r}', \vec{r}_2 ; \varepsilon)\, d\vec{r}' \tag{2.10}$$

where $U(r) = \Sigma_R(\vec{r}, \rho_0, \varepsilon_0)$ (2.11)

In the absence of pairing, $\Sigma_K = 0$, equation (2.10) describes the motion of independent "quasiparticles" in a potential well U(r) according to the Hamiltonian

$$i\frac{\partial \psi}{\partial t} = H\psi \qquad H = \frac{\vec{p}^2}{2M} + U(\vec{r}) \tag{2.12}$$

We note here, that the term "quasiparticle" does not have the same meaning as in the context of the pairing model described earlier. The term "quasiparticle," in this and in subsequent sections, will be used to describe real particles modified by the self energy, Σ_R , and interacting with other such particles through the pairing interaction, Σ_K.

The energies of the single particle excitations of the system can be found from the eigenvalues of H as given by 2.12.

$$H \varphi_\lambda = \varepsilon_\lambda \varphi_\lambda \tag{2.13}$$

In actual practice, it is assumed that the ε_j , given by (2.13), are just the single particle energies given by experiment for the case of one particle outside a double closed-shell. In the present work, these energies for the lead isotopes are known from work on Pb^{207}. As mentioned earlier, the variation with A of the single particle energies is ignored. This assumption is shown by the work of (K-60) to be reasonably valid, and certainly for the case of the lead isotopes, where A changes only from 200 to 206, it should be quite valid.

For the case of spherical nuclei, the index λ in (2.13) denotes the set of quantum numbers [n,l,j,m].

Let us now turn to the solution of (2.10) for the case where $\Sigma_\kappa \neq 0$. We proceed by expanding G in terms of the eigenfunctions φ_λ of (2.13).

$$G(\vec{r}_1, \vec{r}_2; \varepsilon) = \Sigma_{\lambda_1 \lambda_2} G_{\lambda_1 \lambda_2}(\varepsilon) \varphi_{\lambda_1}(\vec{r}_1) \varphi^*_{\lambda_2}(\vec{r}_2) \tag{2.14}$$

Substituting this expansion in (2.10) yields the equation

$$(\varepsilon - \varepsilon_{\lambda_1}) G_{\lambda_1 \lambda_2}(\varepsilon) = \delta_{\lambda_1 \lambda_2} + \Sigma_{\lambda'} (\Sigma_\kappa(\varepsilon))_{\lambda_1 \lambda'} G_{\lambda' \lambda_2}(\varepsilon) \tag{2.15}$$

Introducing the single particle Green's function in the absence of pairing,

$$G^\circ_{\lambda_1 \lambda_2}(\varepsilon) = \frac{\delta_{\lambda_1 \lambda_2}}{\varepsilon - \varepsilon_{\lambda_1} + i\delta_{\lambda_0}} \tag{2.16}$$

where $\delta_{\lambda_0} \longrightarrow +0$ for $\varepsilon - \varepsilon_{\lambda_0} > 0$

$\delta_{\lambda_0} \longrightarrow -0$ for $\varepsilon - \varepsilon_{\lambda_0} < 0$

and ε_{λ_0} is the energy of the highest filled state, we can rewrite (2.15) in the form

$$G_{\lambda_1 \lambda_2}(\varepsilon) = G^\circ_{\lambda_1 \lambda_2}(\varepsilon) + \Sigma_{\lambda' \lambda''} G^\circ_{\lambda_1 \lambda'}(\varepsilon) (\Sigma_\kappa(\varepsilon))_{\lambda' \lambda''} G_{\lambda'' \lambda_2}(\varepsilon) \tag{2.17}$$

Comparing (2.17) with the corresponding equation that would result from (2.4), namely

$$G_{\lambda_1 \lambda_2}(\varepsilon) + \Sigma_{\lambda' \lambda''} G^\circ_{\lambda_1 \lambda'}(\varepsilon) (\Sigma_R(\varepsilon))_{\lambda' \lambda''} G_{\lambda'' \lambda_2}(\varepsilon) + \Sigma_{\lambda' \lambda''} G^\circ_{\lambda_1 \lambda'}(\varepsilon) (\Sigma_\kappa(\varepsilon))_{\lambda' \lambda''} G_{\lambda'' \lambda_2}(\varepsilon)$$

where

$$G^\circ_{\lambda_1 \lambda_2}(\varepsilon) = \delta_{\lambda_1 \lambda_2} (\varepsilon - \varepsilon^*_{\lambda_1} + i\delta_{\lambda_0})^{-1}$$

and

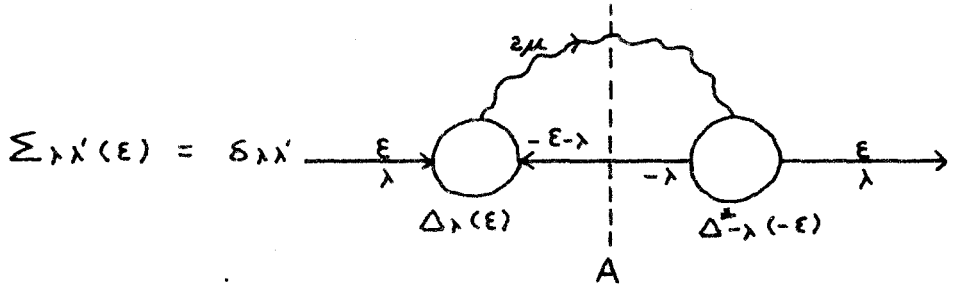
$$H'\varphi_\lambda = \varepsilon_\lambda \varphi_\lambda \qquad H' = \frac{\vec{p}^2}{2M}$$

we note that the simplicity of form of (2.17) results from assuming that the effect of $\Sigma_R(\varepsilon)$ can be absorbed into the function $G^0_{\lambda_1,\lambda_2}(\varepsilon)$.

a) Structure of Σ_k

Let us consider the structure of Σ_k, the self-energy term giving rise to the pairing correlations. Since this is discussed in detail by Migdal (M-62), we will just sketch the main ideas.

The interaction Σ_k which we will hereafter denote simply by Σ, can be represented graphically



(2.18)

The wavy line denotes a Cooper pair, that is, a bound pair with total angular momentum zero; $-i\,\Delta_\lambda(\varepsilon)$ is the amplitude for a pair of particles to form a bound intermediate state with total angular momentum zero, and $i\,\Delta^*_{-\lambda}(-\varepsilon)$ is the amplitude for a pair of particles

to come out of this bound state; $-\lambda$ denotes the state of the hole that is left when the Cooper pair is formed. The initial energy of the system is $E_o(N) + \varepsilon$ and the energy of the state A is $E_o(N+2) - \varepsilon_{-\lambda}$. If the above process obeys the conservation laws, then we can write

$$\Sigma_{\lambda\lambda'}(\varepsilon) = \delta_{\lambda\lambda'} \frac{\Delta_\lambda(\varepsilon)\ \Delta^*_{-\lambda}(-\varepsilon)}{(\varepsilon + \varepsilon_{-\lambda} - 2\mu)} \tag{2.19}$$

Equation (2.18) can be interpreted in the following fashion. The initial state of the system consists of a condensate of N/2 bound Cooper pairs and an external particle in the state λ with energy ε. This particle interacts with another external particle in the state $-\lambda$ to form a bound Cooper pair, leaving behind a hole in the state $-\lambda$. The initial state is again restored when the bound pair $(\lambda, -\lambda)$ is broken up allowing the particle in state $-\lambda$ to combine with the hole in that same state, and leaving the particle in state λ to propagate freely once more.

In analogy with the definition of the Green's function for a particle without pairing, (2.16), let us introduce the Green's function corresponding to a hole in the absence of pairing,

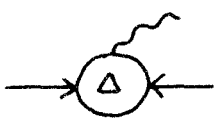
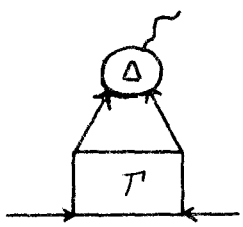
$$\bar{G}^{\circ}_{\lambda_1\lambda_2}(\varepsilon) = \frac{\delta_{\lambda_1\lambda_2}}{\varepsilon + \varepsilon_{\lambda_1} - 2\mu + i\delta_{\lambda_1}} = -G^{\circ}_{\lambda_1\lambda_2}(-\varepsilon + 2\mu) \tag{2.20}$$

Using this definition, equation (2.19) can be written in the form

$$\Sigma_{\lambda\lambda'}(\varepsilon) = \delta_{\lambda\lambda'}\,\Delta_{\lambda}(\varepsilon)\,\bar{G}^{\circ}_{-\lambda}(\varepsilon)\,\Delta^{*}_{-\lambda}(-\varepsilon) \tag{2.21}$$

b) Structure of Δ

Let us now investigate the structure of the vertex part Δ . This quantity must be determined from the self-consistent equation

$\Delta \equiv$  $=$  (2.22)

where $-i\Gamma$ represents the sum of all interactions which scatter a pair of particles but which allow for the fact that in the intermediate state, one line denotes the complete Green's function G, while the other denotes the Green's function defined by (2.20). This asymmetry in structure follows from equation (2.18), where it can be seen that in the state A, the single hole propagator must not include the pairing effect, for otherwise the diagram is not irreducible.

Thus Δ can be expressed in the form

$$\Delta_\lambda(\varepsilon) = -\frac{i}{4\pi} \Sigma_{\lambda'} \int d\varepsilon' \; T(\lambda'\varepsilon', -\lambda'-\varepsilon'; -\lambda-\varepsilon, \lambda\varepsilon) \times G_{\lambda'}(\varepsilon') \Delta_{\lambda'}(\varepsilon') \bar{G}^{\circ}_{-\lambda'}(\varepsilon') \tag{2.23}$$

Introducing the new function

$$F^{-}_{\lambda}(\varepsilon) = - G_\lambda(\varepsilon) \Delta_\lambda(\varepsilon) \bar{G}^{\circ}_{-\lambda}(\varepsilon) \tag{2.24}$$

allows us to write (2.23) in the form

$$\Delta_\lambda(\varepsilon) = \frac{i}{2\pi} \Sigma_{\lambda'} \int d\varepsilon' \; T(\lambda'\varepsilon', -\lambda'-\varepsilon'; -\lambda-\varepsilon, \lambda\varepsilon) \; F^{-}_{\lambda'}(\varepsilon') \tag{2.25}$$

In a similar fashion, the equations for Δ^* can be written

$$\Delta^{*}_{-\lambda}(-\varepsilon) = \frac{i}{2\pi} \Sigma_{\lambda'} \int d\varepsilon' \; T(\lambda'\varepsilon', -\lambda'-\varepsilon'; -\lambda-\varepsilon, \lambda\varepsilon) \; F^{+}_{-\lambda'}(-\varepsilon') \tag{2.26}$$

where

$$F^{+}_{-\lambda}(-\varepsilon) = \left(F^{-}_{-\lambda}(\varepsilon)\right)^* = - G^{\circ}_{-\lambda}(\varepsilon) \Delta^{*}_{-\lambda}(-\varepsilon) G_\lambda(\varepsilon) \tag{2.27}$$

Using (2.21) and (2.27), and dropping the factors $\delta_{\lambda\lambda'}$, equation (2.17) can be written in the form

$$G_\lambda(\varepsilon) = G^{\circ}_{\lambda}(\varepsilon) + G^{\circ}_{\lambda}(\varepsilon) \Delta_\lambda(\varepsilon) \bar{G}^{\circ}_{-\lambda}(\varepsilon) \Delta^{*}_{-\lambda}(-\varepsilon) G_\lambda(\varepsilon)$$

$$= G^{\circ}_{\lambda}(\varepsilon) - G^{\circ}_{\lambda}(\varepsilon) \Delta_\lambda(\varepsilon) F^{+}_{-\lambda}(-\varepsilon) \tag{2.28}$$

Thus, using (2.16) and (2.20) for G° and $\bar{G}^{\circ}$ respectively, equation (2.28) reduces to

$$G_\lambda(\varepsilon) = \frac{G^0_\lambda(\varepsilon)}{1 - G^0_\lambda(\varepsilon)\bar{G}^0_{-\lambda}(\varepsilon)|\Delta_\lambda(\varepsilon)|^2}$$

$$= \frac{\varepsilon + \varepsilon_{-\lambda} - 2\mu}{(\varepsilon - \varepsilon_\lambda)(\varepsilon + \varepsilon_{-\lambda} - 2\mu) - |\Delta_\lambda(\varepsilon)|^2} \qquad (2.29)$$

The functions F^+ and F^- can now be written

$$F^+_{-\lambda}(-\varepsilon) = \frac{-\Delta^*_{-\lambda}(-\varepsilon)}{(\varepsilon - \varepsilon_\lambda)(\varepsilon + \varepsilon_{-\lambda} - 2\mu) - |\Delta_\lambda(\varepsilon)|^2} \qquad (2.30)$$

$$F^-_\lambda(\varepsilon) = \frac{-\Delta_\lambda(\varepsilon)}{(\varepsilon - \varepsilon_\lambda)(\varepsilon + \varepsilon_{-\lambda} - 2\mu) - |\Delta_\lambda(\varepsilon)|^2} \qquad (2.31)$$

In order to reduce the system of equations (2.29)-(2.31) to a more managable form, we introduce two approximations.

1) T' is restricted to the class of instantaneous interactions.

2) $\Delta(\vec{r})$ does not depend on $\vec{r}$

Statement 1) implies that T' does not depend on energy, and from (2.25) we see that this means that $\Delta_\lambda(\varepsilon)$ is independent of energy.

The assumption in 2), which leads to an error of order $A^{-1/3}$ (M-62), means that Δ_λ is a constant, apart from a phase factor. We write, for the [j,m] representation

$$\Delta_{jm} = (-1)^{j-m} \Delta_j = (-1)^{j-m} \Delta$$

We also note that for constant Δ, equations (2.30) and (2.31) are invariant under the transformation

$$\Delta \longrightarrow \Delta' e^{i\alpha} \qquad F \longrightarrow F' e^{-i\alpha}$$

so without loss in generality we can choose Δ real, that is, $\alpha = 0$.

To simplify the form of the equations, let us measure all energies with respect to μ. We thus set

$$\varepsilon - \mu \longrightarrow \varepsilon \qquad \varepsilon_\lambda - \mu \longrightarrow \varepsilon_\lambda \tag{2.32}$$

The system of equations for the three functions G, F^+ and F^- can now be written

$$G_\lambda(\varepsilon) = \frac{\varepsilon + \varepsilon_\lambda}{\varepsilon^2 - \varepsilon_\lambda^2 - \Delta^2} = \frac{\varepsilon + \varepsilon_\lambda}{\varepsilon^2 - E_\lambda^2} \tag{2.33a}$$

$$F_\lambda^{\pm}(\varepsilon) = \frac{\pm \Delta (-1)^{j-m}}{\varepsilon^2 - E_\lambda^2} = \frac{\pm \Delta_\lambda}{\varepsilon^2 - E_\lambda^2} \tag{2.33b}$$

where

$$E_\lambda = \sqrt{\varepsilon_\lambda^2 + \Delta_\lambda^2} \tag{2.34}$$

In order to calculate physical quantities of interest from the functions defined in (2.33) it is usually necessary to integrate over the energy variable. To do this, the singularities of these functions in the complex ε - plane

must be known. The procedure for specifying the contour for this integration can be formulated as follows:

$\mathcal{E}_\lambda$ is replaced by $\mathcal{E}_\lambda - i\delta_1 \frac{\mathcal{E}_\lambda}{|\mathcal{E}_\lambda|}$ (where $\delta_1 \longrightarrow +0$) in the denominators of (2.33), (2.34) and (2.35). This is equivalent to the introduction of a vanishingly small damping of the states describing the particle and hole (M-59). Thus

$$E_\lambda \longrightarrow \sqrt{\mathcal{E}_\lambda^2 + \Delta^2 - 2i\delta_1 \mathcal{E}_\lambda^2/|\mathcal{E}_\lambda|^2} = E_\lambda \sqrt{1 - \frac{2i\delta_1 \mathcal{E}_\lambda^2}{E_\lambda^2 |\mathcal{E}_\lambda|^2}}$$
$$\simeq E_\lambda \left(1 - i\delta_1/E_\lambda\right) = E_\lambda - i\delta_1$$

The contour of integration in the $\mathcal{E}$ - plane now consists of the real axis plus a semi-circle enclosing the upper half-plane.

For purposes of calculation, it is thus convenient to replace the set of equations (2.33) by the following:

$$G_\lambda(\mathcal{E}) = \frac{u_\lambda^2}{\mathcal{E} - E_\lambda + i\delta} + \frac{v_\lambda^2}{\mathcal{E} + E_\lambda - i\delta} \qquad (2.35a)$$

$$F_\lambda^{\pm}(\mathcal{E}) = \pm u_\lambda v_\lambda \left[\frac{1}{\mathcal{E} - E_\lambda + i\delta} - \frac{1}{\mathcal{E} + E_\lambda - i\delta}\right] \qquad (2.35b)$$

where

$$u_\lambda^2 = \frac{1}{2}\left(1 + \frac{\mathcal{E}_\lambda}{E_\lambda}\right) \qquad v_\lambda^2 = \frac{1}{2}\left(1 - \frac{\mathcal{E}_\lambda}{E_\lambda}\right)$$

$$u_\lambda v_\lambda = \frac{\Delta_\lambda}{2E_\lambda} = (-1)^{j-m} \frac{\Delta}{2E_\lambda} \qquad (2.36)$$

The equation for Δ_λ , (2.25) now takes the form

$$\Delta_\lambda = \frac{-1}{2\pi i} \Sigma_{\lambda'} \Gamma(\lambda'-\lambda';-\lambda\lambda) \int F^-_{\lambda'}(\varepsilon)\, d\varepsilon$$
$$= -\Sigma_{\lambda'} \Gamma(\lambda'-\lambda';-\lambda\lambda) \frac{\Delta_{\lambda'}}{2\varepsilon_{\lambda'}} \qquad (2.37)$$

In its first approximation, Γ has the form

$$\Gamma(12;34) = \langle 12|V|34\rangle - \langle 12|V|43\rangle \qquad (2.38)$$

Thus, equations (2.34), (2.36) and (2.37), when the approximation (2.38) is used, are the same as those derived by the canonical transformation technique. See, for example, equations (1.21) and (1.22) of the introduction. In principle, these results could be improved by relaxing approximations 1) and 2) and including a larger set of graphs in the calculation of Γ . In practice, however, since Γ depends on G, and G of course depends on Γ , solutions for G, F^+ and F^- involve the solution of a set of complicated coupled equations and it is difficult to advance much beyond the approximations used in this section.

We now come to the question concerning the number of particles in the system. The single particle density matrix can be written

$$\rho(\vec{r}_1,\vec{r}_2) = \langle \Phi_o^N | \psi^+(\vec{r}_1)\psi(\vec{r}_2) | \Phi_o^N\rangle$$
$$= -i\, G(\vec{r}_1,\vec{r}_2;\tau)\Big|_{\tau\to -o}$$
$$= \frac{1}{2\pi i}\int G(\vec{r}_1,\vec{r}_2;\varepsilon)\, d\varepsilon \qquad (2.39)$$

where the contour again consists of the real axis and a semi-circle of infinite radius in the upper half plane. In the φ_λ representation, this becomes

$$\rho_{\lambda\lambda'} = \frac{1}{2\pi i} \int G_{\lambda\lambda'}(\varepsilon)\, d\varepsilon \tag{2.40}$$

Using (2.35a) for G, the number of particles in the state λ is

$$\rho_\lambda = n_\lambda = \frac{1}{2\pi i} \int \frac{V_\lambda^2}{\varepsilon + E_\lambda - i\delta}\, d\varepsilon = V_\lambda^2 \tag{2.41}$$

and the total number of particles in the system is

$$n = \Sigma_\lambda\, n_\lambda = \Sigma_\lambda\, V_\lambda^2 = \Sigma_j\, V_j^2\, \Sigma_m\, 1$$

$$= 2\, \Sigma_j\, \Omega_j\, V_j^2 \tag{2.42}$$

where $\Omega_j = j + 1/2$ Equation (2.42) shows that v_j^2 can be interpreted as the probability that any one of the (2j + 1)-degenerate states in the j-shell is occupied. From (2.36) we see that u_j^2 is then the probability that this same state is empty. These quantities thus have the usual interpretation.

For the case of the one-particle Green's functions G, F^+ and F^-, the essential problem can be summarized as follows:

1) The single particle energies, $\varepsilon_\lambda \equiv \varepsilon_j$ are taken from experiment.

2) An appropriate value of the strength of the

pairing force, G, is chosen.

3) The parameters μ , and Δ are found, for fixed n, from the simultaneous equations

$$\frac{2}{G} = \Sigma_j \frac{\Omega_j}{E_j} = \Sigma_j \frac{\Omega_j}{\sqrt{(\varepsilon_j-\mu)^2+\Delta^2}}$$

and

$$n = \Sigma_j \Omega_j \left[1 - \frac{\varepsilon_j - \mu}{\sqrt{(\varepsilon_j-\mu)^2+\Delta^2}} \right]$$

The first equation results from (2.37) by setting

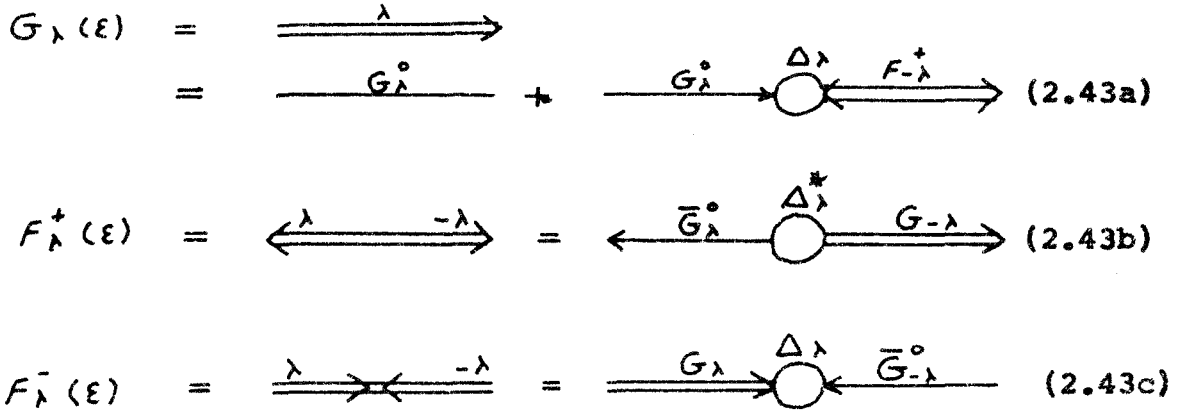
$T' = -G/2$, $\Delta_{jm} = (-1)^{j-m}\Delta$ and the second is just (2.42).

c) Diagram Convention

The structure of the functions F^+, F^- and G suggests the following physical interpretation of their roles.

F^- can be considered as a propagator that effectively replaces a particle by a hole, while F^+ can be considered as a propagator that changes a hole into a particle. The function G is just the usual propagator for either a hole or a particle.

The following diagram convention can be introduced as an aid to the understanding of the equations discussed in this section.



2. Two-Particle Green's Function

We define the normal two-particle Green's function as

$$K_{22}(x_1x_2;\, x_3x_4) =$$

$$= \langle \Phi^N_0 \,|\, T\{\, \psi(x_1)\, \psi^+(x_2)\, \psi(x_3)\, \psi^+(x_4)\,\} \,|\, \Phi^N_0 \rangle \qquad (2.44)$$

where $x = [\, \vec{r},\, t]$.

This function obeys the Dyson equation (G-51)

$$K_{22}(x_1x_2;\, x_3x_4) =$$

$$G(x_1x_4)\, G(x_3x_2) - G(x_1x_2)\, G(x_3x_4)$$

$$+ i \int d^4x_5\, d^4x_6\, d^4x_7\, d^4x_8\; G(x_1x_5)\, G(x_6x_2)\; \times$$

$$\Gamma(x_5x_6;\, x_7x_8)\, K_{22}(x_7x_8;\, x_3x_4) \qquad (2.45)$$

where all the symbols have the same meaning as in the previous section, and G is the exact single particle Green's function defined by equation (2.1), not the pairing approximation Green's function.

Let us consider a particular ordering of the times in (2.44)

$$t_1, t_2 > t_3, t_4$$

In this case we can write

$$K_{22}(x_1x_2; x_3x_4)$$

$$= \Sigma_s \langle \Phi_0^N | T\{\psi(x_1)\psi^+(x_2)\} | s\rangle\langle s | T\{\psi(x_3)\psi^+(x_4)\} | \Phi_0^N\rangle$$

$$= \Sigma_s \varphi_s(x_1 x_2)\,\bar{\varphi}_s(x_3 x_4) \qquad (2.46)$$

φ_s can be interpreted as an amplitude describing the systematic motion of a particle and a hole (KK-60). In particular, for $t_1 = t_2 = t$, $\varphi_s(r_1\ r_1; t)$ can be interpreted as an amplitude describing the behavior of a particle and a hole in the state s. It is possible to show that when bound states are being considered, the inhomogeneous part of (2.45) does not contribute, and upon substituting (2.46) into (2.45), an equation for the amplitude φ_s can be extracted (G-51) (KK-60).

We can thus write an equation for φ_s in the form

$$\varphi_s(x_1x_2)$$

$$= i\int d^4x_5 d^4x_6 d^4x_7 d^4x_8\ \ G(x_1x_5)\ \ G(x_6x_2)\ \ \times$$

$$\Gamma(x_5x_6; x_7x_8)\ \ \varphi_s(x_7x_8) \qquad (2.47)$$

We now introduce again the assumption that Γ describes an instantaneous interaction, and that the propagators

G depend on their time coordinates only through the difference $t_1 - t_2 = \tau$. This assumption means that the time coordinates in φ_s are equal.

We thus define, from

$$\varphi_s(x_1 x_2) = \varphi_s(\vec{r}_1 t_1, \vec{r}_2 t_2)$$

$$= \langle \Phi_0^N | T\{\psi(\vec{r}_1 t_1)\, \psi^+(\vec{r}_2 t_2)\} | s \rangle$$

the function

$$\varphi_s(\vec{r}_1\, \vec{r}_2\, ; t) = -\langle \Phi_0^N | \psi^+(\vec{r}_2 t)\, \psi(\vec{r}_1 t) | s \rangle \qquad \tau = t_1 - t_2 \to -0 \tag{2.48}$$

Equation (2.47) can now be written

$$\varphi_s(r_1 r_2;\ t)$$

$$= i \int dr_5\, dr_6\, dr_7\, dr_8\, d\tau\ G(r_1 r_5;\ -\tau)\ G(r_6 r_2;\ \tau)$$

$$\times\ \Gamma(r_5 r_6;\ r_7 r_8)\ \varphi_s(r_7 r_8;\ t + \tau) \tag{2.49}$$

The opposite sign for τ in the two Green's functions in (2.49) arises because one function describes the propagation of a particle while the other describes the propagation of a hole.

Equation (2.49) can be interpreted in the following sense.

The left hand side describes a bound state of a

particle and a hole at time t. In the right hand side, $G(\tau)$, for $\tau > 0$ describes the propagation of a particle from time t to time $t' = t + \tau$, while $G(-\tau)$ describes the propagation of a hole from time t to time t'. At time t', the particle and hole interact to form a bound state.

If the single particle Green's function is known, and if the interaction Γ is known, then (2.49) gives a prescription for calculating the two-particle amplitude. In practice, these functions are usually not known, and approximations for them must be introduced. We are interested in the case where the system contains pairing correlations, but as we have seen in the previous section, this involves the introduction of two new single particle propagators. The modification which this requires in the structure of the two-particle amplitude* will be discussed in the next chapter.

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*We will see that two new amplitudes besides φ must be introduced.

CHAPTER III

DERIVATION OF THE EIGENVALUE EQUATION FOR THE EXCITED LEVELS

1. Derivation of the General Equations for the Amplitudes

The equation for the amplitude of a particle-hole pair in a state $|\Phi_s^N\rangle$ was derived in the previous chapter, and can be written

$$\varphi_s(\vec{r}_1, \vec{r}_2; t) = i\int d\vec{r}_3\, d\vec{r}_4\, d\vec{r}_5\, d\vec{r}_6\; G(\vec{r}_1 \vec{r}_5; -\tau)\, G(\vec{r}_6 \vec{r}_2; \tau) \times \Gamma(\vec{r}_5 \vec{r}_6; \vec{r}_3 \vec{r}_4)\, \varphi_s(\vec{r}_3 \vec{r}_4; t+\tau) \tag{3.1}$$

However, as was shown in section 1 of Chapter II, the single particle (or hole) propagation, in the presence of pairing, is described not only by G, but also by the new propagators F^+ and F^-. As was indicated there, F^+ describes the transition of a hole into a particle, while F^- describes the transition of a particle into a hole. The modification to the two-particle case required by the presence of these new propagators can be introduced as follows.

In equation (3.1), we make the following changes.

$$G(\vec{r}_1 \vec{r}_5 ; -\tau) \longrightarrow G(\vec{r}_1 \vec{r}_5 ; -\tau) + F^{+}(\vec{r}_1 \vec{r}_5 ; -\tau) \tag{3.2a}$$

$$G(\vec{r}_6 \vec{r}_2 ; \tau) \longrightarrow G(\vec{r}_6 \vec{r}_2 ; \tau) + F^{-}(\vec{r}_6 \vec{r}_2 ; \tau) \tag{3.2b}$$

$$\Gamma(\vec{r}_5 \vec{r}_6 ; \vec{r}_3 \vec{r}_4)$$

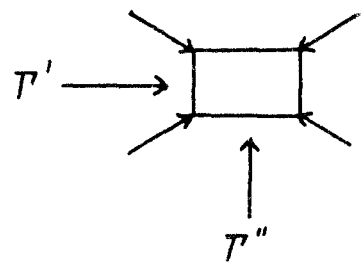
$$\longrightarrow \Gamma'(\vec{r}_5 \vec{r}_6 ; \vec{r}_3 \vec{r}_4) + \Gamma''(\vec{r}_5 \vec{r}_6 ; \vec{r}_3 \vec{r}_4) \tag{3.3}$$

and we introduce

$$\bar{\chi}(\vec{r}_1 \vec{r}_2 ; t) = -\langle \Phi_0^N | \psi^{+}(\vec{r}_2 t)\, \psi^{+}(\vec{r}_1 t) | \Phi_s^{N-2} \rangle \tag{3.4}$$

$$\chi(\vec{r}_1 \vec{r}_2 ; t) = -\langle \Phi_0^N | \psi(\vec{r}_2 t)\, \psi(\vec{r}_1 t) | \Phi_s^{N+2} \rangle \tag{3.5}$$

Here, Γ' is defined as the set of diagrams that is irreducible in the particle-hole direction, and Γ'' is the set that is ireducible in the particle-particle or hole-hole direction. This is illustrated in the accompanying figure.



The function $\bar{\chi}$ is an amplitude describing the behavior of two particles in the state s , while χ_s is the

corresponding function for two holes.

These functions enter in the following manner. The left hand side of (3.1) still describes a particle-hole pair at time t, so the right hand side must still describe a particle and a hole propagating from time t. If the particle and hole each propagate by a G-function, then at time t', the interaction Γ' leads to the formation of a bound particle-hole pair, described by φ. Consider what now happens when the propagation of the particle is described by F^-, and the propagation of the hole is described by G. The particle at t becomes a hole at t' and the interaction, which is now Γ'', leads to the formation of a bound hole-hole pair, described by χ. Similarly, if the hole propagates by F^+, and the particle by G, then the interaction Γ'' leads to the formation of a particle-particle pair described by $\bar{\chi}$. Finally, if the hole propagates by F^+ and the particle propagates by F^-, then the interaction is again Γ', and a bound particle-hole pair is again formed.

With the aid of this rather "picturesque" description, it is easy to see that equation (3.1) should be replaced by

$$\varphi_s(\vec{r}_1\vec{r}_2;t) = i\int d\vec{r}_5\, d\vec{r}_6\, d\vec{r}_3\, d\vec{r}_4 \times$$

$$\Big\{ \Big[G(\vec{r}_1\vec{r}_5;-\tau)\; G(r_6 r_2;\tau) + F^+(\vec{r}_1\vec{r}_5;-\tau)\; F^-(\vec{r}_6\vec{r}_2;\tau) \Big] \times \Gamma'(\vec{r}_5\vec{r}_6;\vec{r}_3\vec{r}_4)\; \varphi_s(r_3 r_4;t+\tau)$$

$$+ G(\vec{r}_1\vec{r}_5:-\tau)\; F^-(\vec{r}_6\vec{r}_2;\tau)\; \Gamma''(\vec{r}_5\vec{r}_6;\vec{r}_3\vec{r}_4)\; \chi_s(\vec{r}_3\vec{r}_4;t+\tau)$$

$$+ G(\vec{r}_6\vec{r}_2;\tau)\; F^+(\vec{r}_1\vec{r}_5;-\tau)\; \Gamma''(\vec{r}_5\vec{r}_6;\vec{r}_3\vec{r}_4)\; \bar{\chi}_s(r_3 r_4;t+\tau) \Big\} \qquad (3.6)$$

Passing to the φ_λ representation, this equation becomes

$$\varphi^s_{\lambda_1\lambda_2}(t)$$

$$= i\,\Sigma_{\lambda_3\lambda_4}\int d\tau \Big\{ G_{\lambda_1}(-\tau)\; G_{\lambda_2}(\tau)\; \Gamma'(\lambda_1\lambda_4;\lambda_2\lambda_3)\; \varphi^s_{\lambda_3\lambda_4}(t+\tau)$$

$$+ F^+_{\lambda_1}(-\tau)\; F^-_{\lambda_2}(\tau)\; \Gamma'(-\lambda_2\lambda_4;-\lambda_1\lambda_3)\; \varphi^s_{\lambda_3\lambda_4}(t+\tau)$$

$$+ G_{\lambda_1}(-\tau)\; F^-_{\lambda_2}(\tau)\; \Gamma''(\lambda_1-\lambda_2;\lambda_4\lambda_3)\; \chi^s_{\lambda_3\lambda_4}(t+\tau)$$

$$+ G_{\lambda_2}(\tau)\; F^+_{\lambda_1}(-\tau)\; \Gamma''(\lambda_3\lambda_4;-\lambda_1\lambda_2)\; \bar{\chi}_{\lambda_3\lambda_4}(t+\tau) \Big\} \qquad (3.7)$$

where

$$G(\vec{r}_1\vec{r}_2;\pm\tau) = \Sigma_{\lambda_1\lambda_2} G_{\lambda_1\lambda_2}(\pm\tau)\; \varphi_{\lambda_1}(\vec{r}_1)\; \varphi^*_{\lambda_2}(\vec{r}_2)$$

$$F^+(\vec{r}_1\vec{r}_2;\pm\tau) = \Sigma_{\lambda_1\lambda_2} F^+_{\lambda_1\lambda_2}(\pm\tau)\; \varphi_{\lambda_1}(\vec{r}_1)\; \varphi_{\lambda_2}(\vec{r}_2)$$

$$F^-(\vec{r}_1\vec{r}_2;\pm\tau) = \Sigma_{\lambda_1\lambda_2} F^-_{\lambda_1\lambda_2}(\pm\tau)\; \varphi^*_{\lambda_1}(\vec{r}_1)\; \varphi^*_{\lambda_2}(\vec{r}_2) \qquad (3.8)$$

$$\varphi_s(r_1 r_2;\tau) = \Sigma_{\lambda_1\lambda_2}\; \varphi^s_{\lambda_1\lambda_2}(\tau)\; \varphi_{\lambda_1}(\vec{r}_1)\; \varphi_{\lambda_2}(\vec{r}_2)$$

$$\bar{\chi}_s(\vec{r}_1\vec{r}_2;\tau) = \Sigma_{\lambda_1\lambda_2}\; \bar{\chi}^s_{\lambda_1\lambda_2}(\tau)\; \varphi^*_{\lambda_1}(\vec{r}_1)\; \varphi^*_{\lambda_2}(\vec{r}_2)$$

$$\chi_s(\vec{r}_1\vec{r}_2;\tau) = \Sigma_{\lambda_1\lambda_2}\; \chi^s_{\lambda_1\lambda_2}(\tau)\; \varphi_{\lambda_1}(\vec{r}_1)\; \varphi_{\lambda_2}(\vec{r}_2)$$

and

$$T(\lambda_1\lambda_2;\lambda_3\lambda_4)$$

$$= \int d\vec{r}_1\, d\vec{r}_2\, d\vec{r}_3\, d\vec{r}_4\ \varphi^*_{\lambda_1}(\vec{r}_1)\ \varphi^*_{\lambda_3}(\vec{r}_3)\ \varphi_{\lambda_4}(\vec{r}_4)\ \varphi_{\lambda_2}(\vec{r}_2) \times T(\vec{r}_1\vec{r}_3;\vec{r}_4\vec{r}_2) \tag{3.9}$$

The orthonormality of the complete set of functions φ_λ

$$\int d\vec{r}\ \varphi^*_\lambda(\vec{r})\ \varphi_{\lambda'}(\vec{r}) = \delta_{\lambda\lambda'} \tag{3.10}$$

along with the δ - functions appearing in the definitions of G, F^+ and F^-, have been invoked to reduce the number of independent summation indices in (3.7) from four to two.

We note that T'' and T' obey the symmetry condition

$$T'^{(\prime\prime)}(\lambda_1\lambda_2;\lambda_3\lambda_4) = -T'^{(\prime\prime)}(\lambda_2\lambda_1;\lambda_3\lambda_4) = T'^{(\prime\prime)}(\lambda_2\lambda_1;\lambda_4\lambda_3) \tag{3.11a}$$

while $\bar{\chi}$ and χ satisfy

$$\bar{\chi}_{\lambda_1\lambda_2}(\tau) = -\bar{\chi}_{\lambda_2\lambda_1}(\tau)$$
$$\chi_{\lambda_1\lambda_2}(\tau) = -\chi_{\lambda_2\lambda_1}(\tau) \tag{3.11b}$$

The relation (3.11a) follows immediately from equation (2.38) while (3.11b) follows from the definition of $\bar{\chi}$ and χ

Let us now consider the structure of the equation satisfied by $\bar{\chi}$. We recall that this function describes the bound state of two particles. The propagation of two

particles, initially at time t, can be described by

$$F^{-}(\vec{r}_6\,\vec{r}_2;\tau) \quad + \quad G(\vec{r}_6\,\vec{r}_2;\tau)$$

and

$$F^{-}(\vec{r}_5\,\vec{r}_1;\tau) \quad + \quad G(\vec{r}_5\,\vec{r}_1;\tau) \tag{3.12}$$

The equation for $\bar{\chi}$ in terms of $\bar{\chi}$, χ and φ follows immediately, except for the relative signs of the terms, as

$$\bar{\chi}_s(\vec{r}_1\,\vec{r}_2;t) = i\int d\vec{r}_5\, d\vec{r}_6\, d\vec{r}_3\, d\vec{r}_4 \times$$

$$\Big\{ \Big[G(\vec{r}_6\,\vec{r}_2;\tau)\, F^{-}(\vec{r}_5\,\vec{r}_1;\tau) - G(\vec{r}_5\,\vec{r}_1;\tau)\, F^{-}(r_6\,r_2;\tau) \Big] \times T'(\vec{r}_5\,\vec{r}_6;\vec{r}_3\,\vec{r}_4)\ \varphi_s(\vec{r}_3\,\vec{r}_4;t+\tau)$$

$$+ G(\vec{r}_6\,\vec{r}_2;\tau)\, G(\vec{r}_5\,\vec{r}_1;\tau)\, T''(\vec{r}_5\,\vec{r}_6;\vec{r}_3\,\vec{r}_4)\ \bar{\chi}_s(\vec{r}_3\,\vec{r}_4;t+\tau)$$

$$+ F^{-}(\vec{r}_6\,\vec{r}_2;\tau)\, F^{-}(r_5\,r_1;\tau)\, T''(\vec{r}_5\,\vec{r}_6;\vec{r}_3\,\vec{r}_4)\ \chi_s(r_3\,r_4;t+\tau) \Big\} \tag{3.13}$$

In the φ_λ representation, this becomes

$$\bar{\chi}^{s}_{\lambda_1\lambda_2}(t)$$

$$= i\sum_{\lambda_3\lambda_4}\int d\tau \Big\{ G_{\lambda_2}(\tau)\, F_{\bar{\lambda}_1}(\tau)\, T'(\lambda_4-\lambda_1;\lambda_3\lambda_2)\ \varphi^{s}_{\lambda_3\lambda_4}(t+\tau)$$

$$- G_{\lambda_1}(\tau)\, F_{\bar{\lambda}_2}(\tau)\, T'(\lambda_4-\lambda_2;\lambda_3\lambda_1)\ \varphi^{s}_{\lambda_3\lambda_4}(t+\tau)$$

$$+ G_{\lambda_2}(\tau)\, G_{\lambda_1}(\tau)\, T''(\lambda_4\lambda_3;\lambda_1\lambda_2)\ \bar{\chi}^{s}_{\lambda_3\lambda_4}(t+\tau)$$

$$+ F_{\bar{\lambda}_2}(\tau)\, F_{\bar{\lambda}_1}(\tau)\, T''(-\lambda_1-\lambda_2;\lambda_4\lambda_3)\ \chi^{s}_{\lambda_3\lambda_4}(t+\tau) \Big\} \tag{3.14}$$

Finally, for the function χ_s, propagation by the

combination

$$F^{+}(\vec{r}_1 \vec{r}_5 ; -\tau) + G(\vec{r}_1 \vec{r}_5 ; -\tau)$$
$$F^{+}(\vec{r}_2 \vec{r}_6 ; -\tau) + G(\vec{r}_2 \vec{r}_6 : -\tau) \qquad (3.15)$$

leads to the equation

$$\chi^{s}_{\lambda_1 \lambda_2}(t)$$
$$= i \Sigma_{\lambda_3 \lambda_4} \int d\tau \Big\{ - G_{\lambda_1}(-\tau)\, F^{+}_{\lambda_2}(-\tau)\, \Gamma'(\lambda_4 \lambda_1 ; \lambda_3 - \lambda_2)\, \varphi^{s}_{\lambda_3 \lambda_4}(t+\tau)$$
$$+ G_{\lambda_2}(-\tau)\, F^{+}_{\lambda_1}(-\tau)\, \Gamma'(\lambda_4 \lambda_2 ; \lambda_3 - \lambda_1)\, \varphi^{s}_{\lambda_3 \lambda_4}(t+\tau)$$
$$+ G_{\lambda_2}(-\tau)\, G_{\lambda_1}(-\tau)\, \Gamma''(\lambda_1 \lambda_2 ; \lambda_4 \lambda_3)\, \chi^{s}_{\lambda_3 \lambda_4}(t+\tau)$$
$$+ F^{+}_{\lambda_2}(-\tau)\, F^{+}_{\lambda_1}(-\tau)\, \Gamma''(\lambda_3 \lambda_4 ; -\lambda_2 - \lambda_1)\, \bar{\chi}^{s}_{\lambda_3 \lambda_4}(t+\tau) \Big\} \qquad (3.16)$$

The form of (3.12) and of (3.15) with respect to the signature is a consequence of the definitions of $\bar{\chi}$ and χ that are implied by the form of equation (3.6) or (3.7). A convenient method of verifying this self-consistency is to adopt the diagram convention outlined in section 1 of the preceding chapter, along with the labelling convention for Γ' discussed in the next section, and then to substitute (3.14) and (3.16) into (3.7).

We now define

$$\varphi^{s}_{\lambda_1 \lambda_2}(t) \equiv - \langle \Phi^{N}_{0} |\, a^{+}_{\lambda_2}(t)\, a_{\lambda_1}(t) \,| \Phi^{N}_{s} \rangle$$
$$= - e^{i(E^{N}_{0} - E^{N}_{s})t} \langle \Phi^{N}_{0} |\, a^{+}_{\lambda_2} a_{\lambda_1} \,| \Phi^{N}_{s} \rangle \qquad (3.17)$$
$$= e^{-i\omega t}\, \varphi_{\lambda_1 \lambda_2}(\omega, \alpha)$$

where $\omega = E_s^N - E_o^N$ is the energy of the N-particle excited state with respect to the ground state of N particles. The label α denotes the remaining quantum numbers that are needed to completely specify the state s.

In a similar fashion we define

$$\begin{aligned}\bar{\chi}^s_{\lambda_1\lambda_2}(t) &\equiv -\langle \Phi_o^N | \, a^+_{\lambda_2}(t)\, a^+_{\lambda_1}(t) \, | \Phi_s^{N-2}\rangle \\ &= -e^{i(E_o^N - E_s^{N-2})} \langle \Phi | \, a^+_{\lambda_2} a^+_{\lambda_1} | \Phi_s^{N-2}\rangle \\ &= e^{-i\omega t}\, \bar{\chi}_{\lambda_1\lambda_2}(\omega,\alpha) \end{aligned} \tag{3.18}$$

We note that

$$\begin{aligned} -E_o^N + E_s^{N-2} &= (E_s^{N-2} - E_s^N) + (E_s^N - E_o^N) \\ &= \omega - 2\mu \longrightarrow \omega \end{aligned}$$

since all energies are being measured with respect to μ.

Finally, we have

$$\chi^s_{\lambda_1\lambda_2}(t) = e^{-i\omega t}\, \chi_{\lambda_1\lambda_2}(\omega,\alpha) \tag{3.19}$$

Substituting (3.17), (3.18) and (3.19) in the equations for $\bar{\chi}$, χ and φ, and making use of the relations

$$\int d\tau\; G_{\lambda_1}(-\tau) G_{\lambda_2}(\tau) e^{-i\omega\tau} = \frac{1}{2\pi}\int d\varepsilon\; G_{\lambda_2}(\varepsilon)\, G_{\lambda_1}(\varepsilon-\omega) \tag{3.20a}$$

$$\int d\tau\; G_{\lambda_1}(-\tau)\, G_{\lambda_2}(-\tau)\, e^{-i\omega\tau} = \frac{1}{2\pi}\int d\varepsilon\; G_{\lambda_2}(\varepsilon)\, G_{\lambda_1}(-\varepsilon-\omega) \tag{3.20b}$$

$$\int d\tau\; G_{\lambda_1}(\tau)\, G_{\lambda_2}(\tau)\, e^{-i\omega\tau} = \frac{1}{2\pi}\int d\varepsilon\; G_{\lambda_2}(\varepsilon) G_{\lambda_1}(-\varepsilon+\omega) \tag{3.20c}$$

the equations for the Fourier components $\varphi(\omega)$, $\bar{\chi}(\omega)$ and $\chi(\omega)$ can be written.

$$\varphi_{\lambda_1\lambda_2}(\omega,\alpha)$$

$$= \frac{i}{2\pi}\sum_{\lambda_3\lambda_4}\int d\varepsilon \Big\{ G_{\lambda_2}(\varepsilon)\, G_{\lambda_1}(\varepsilon-\omega)\, T'(\lambda_4\lambda_1;\lambda_3\lambda_2)\,\varphi_{\lambda_3\lambda_4}(\omega,\alpha)$$
$$+ F^{+}_{\lambda_1}(\varepsilon-\omega)\, F^{-}_{\lambda_2}(\varepsilon)\, T'(\lambda_4-\lambda_2;\lambda_3-\lambda_1)\,\varphi_{\lambda_3\lambda_4}(\omega,\alpha)$$
$$+ G_{\lambda_1}(\varepsilon-\omega)\, F^{-}_{\lambda_2}(\varepsilon)\, T''(\lambda_1-\lambda_2;\lambda_4\lambda_3)\,\chi_{\lambda_3\lambda_4}(\omega,\alpha)$$
$$+ F^{+}_{\lambda_1}(\varepsilon-\omega)\, G_{\lambda_2}(\varepsilon)\, T''(\lambda_3\ \lambda_4;-\lambda_1\lambda_2)\,\bar{\chi}_{\lambda_3\lambda_4}(\omega,\alpha)\Big\}$$

$$\bar{\chi}_{\lambda_1\lambda_2}(\omega,\alpha) \qquad (3.21)$$

$$= \frac{i}{2\pi}\sum_{\lambda_3\lambda_4}\int d\varepsilon \Big\{ G_{\lambda_2}(\varepsilon)\, F^{-}_{\lambda_1}(-\varepsilon+\omega)\, T'(\lambda_4-\lambda_1;\lambda_3\lambda_2)\,\varphi_{\lambda_3\lambda_4}(\omega,\alpha)$$
$$- G_{\lambda_1}(\varepsilon)\, F^{-}_{\lambda_2}(-\varepsilon+\omega)\, T'(\lambda_4-\lambda_2;\lambda_3\lambda_1)\,\varphi_{\lambda_3\lambda_4}(\omega,\alpha)$$
$$- G_{\lambda_1}(\varepsilon)\, G_{\lambda_2}(-\varepsilon+\omega)\, T''(\lambda_3\lambda_4;\lambda_1\lambda_2)\,\bar{\chi}_{\lambda_3\lambda_4}(\omega,\alpha)$$
$$+ F^{-}_{\lambda_2}(\varepsilon)\, F^{-}_{\lambda_1}(-\varepsilon+\omega)\, T''(-\lambda_1-\lambda_2;\lambda_4\lambda_3)\,\chi_{\lambda_3\lambda_4}(\omega,\alpha)\Big\}$$

(3.22)

$$\chi_{\lambda_1\lambda_2}(\omega,\alpha)$$

$$= \frac{i}{2\pi}\sum_{\lambda_3\lambda_4}\int d\varepsilon \Big\{ G_{\lambda_2}(\varepsilon)\, F^{+}_{\lambda_1}(-\varepsilon-\omega)\, T'(\lambda_4\lambda_2;\lambda_3-\lambda_1)\,\varphi_{\lambda_3\lambda_4}(\omega,\alpha)$$
$$- G_{\lambda_1}(\varepsilon)\, F^{+}_{\lambda_2}(-\varepsilon-\omega)\, T'(\lambda_4\lambda_1;\lambda_3-\lambda_2)\,\varphi_{\lambda_3\lambda_4}(\omega,\alpha)$$
$$+ G_{\lambda_2}(\varepsilon)\, G_{\lambda_1}(-\varepsilon-\omega)\, T''(\lambda_1\lambda_2;\lambda_4\lambda_3)\ \chi_{\lambda_3\lambda_4}(\omega,\alpha)$$
$$- F^{+}_{\lambda_2}(\varepsilon)\, F^{+}_{\lambda_1}(-\varepsilon-\omega)\ T''(\lambda_3\lambda_4;-\lambda_1-\lambda_2)\,\bar{\chi}_{\lambda_3\lambda_4}(\omega,\alpha)\Big\}$$

(3.23)

The integrations over ε can now be carried out, using the relations for $G_\lambda(\varepsilon)$, $F^+_\lambda(\varepsilon)$ and $F^-_\lambda(\varepsilon)$ derived in Chapter II (equations (2.35)).

To simplify the form of the resulting equations, let us define the following quantities.

$$E_{\lambda_1} + E_{\lambda_2} = A_{\lambda_1\lambda_2} \qquad u^2_{\lambda_1} v^2_{\lambda_2} + v^2_{\lambda_1} u^2_{\lambda_2} = B_{\lambda_1\lambda_2}$$

$$u^2_{\lambda_1} - u^2_{\lambda_2} = C_{\lambda_1\lambda_2} \qquad u^2_{\lambda_1} - v^2_{\lambda_2} = D_{\lambda_1\lambda_2} \qquad (3.24)$$

$$u_{\lambda_1} \times v_{\lambda_1} = M_{\lambda_1} \qquad u^2_{\lambda_1} - v^2_{\lambda_1} = P_{\lambda_1} = D_{\lambda_1\lambda_1}$$

The systems of equations for φ, $\bar{\chi}$ and χ can now be written in the form

$$(A^2_{\lambda_1\lambda_2} - \omega^2)\,\varphi_{\lambda_1\lambda_2}(\omega,\alpha)$$

$$= (A_{\lambda_1\lambda_2} B_{\lambda_1\lambda_2} - \omega C_{\lambda_1\lambda_2}) \Sigma_{\lambda_3\lambda_4} T'(\lambda_4\lambda_1;\lambda_3\lambda_2)\,\varphi_{\lambda_3\lambda_4}(\omega,\alpha)$$

$$+ 2 M_{\lambda_1} M_{\lambda_2} A_{\lambda_1\lambda_2} \Sigma_{\lambda_3\lambda_4} T'(\lambda_4-\lambda_2;\lambda_3-\lambda_1)\,\varphi_{\lambda_3\lambda_4}(\omega,\alpha)$$

$$+ M_{\lambda_2}(A_{\lambda_1\lambda_2} P_{\lambda_1} - \omega) \Sigma_{\lambda_3\lambda_4} T''(\lambda_1-\lambda_2;\lambda_4\lambda_3)\,\chi_{\lambda_3\lambda_4}(\omega,\alpha)$$

$$- M_{\lambda_1}(A_{\lambda_1\lambda_2} P_{\lambda_2} + \omega) \Sigma_{\lambda_3\lambda_4} T''(\lambda_3\lambda_4;-\lambda_1\lambda_2)\,\bar{\chi}_{\lambda_3\lambda_4}(\omega,\alpha) \qquad (3.25)$$

$$(A^2_{\lambda_1\lambda_2} - \omega)\,\bar{\chi}_{\lambda_1\lambda_2}(\omega,\alpha)$$

$$= M_{\lambda_1}(A_{\lambda_1\lambda_2} P_{\lambda_2} + \omega) \Sigma_{\lambda_3\lambda_4} T'(\lambda_4-\lambda_1;\lambda_3\lambda_2)\,\varphi_{\lambda_3\lambda_4}(\omega,\alpha)$$

$$- M_{\lambda_2}(A_{\lambda_1\lambda_2} P_{\lambda_1} + \omega) \Sigma_{\lambda_3\lambda_4} T'(\lambda_4-\lambda_2;\lambda_3\lambda_2)\,\varphi_{\lambda_3\lambda_4}(\omega,\alpha)$$

$$+ \left[A_{\lambda_1\lambda_2}(1 - B_{\lambda_1\lambda_2}) + \omega D_{\lambda_1\lambda_2}\right] \Sigma_{\lambda_3\lambda_4} T''(\lambda_3\lambda_4;\lambda_1\lambda_2)\,\bar{\chi}_{\lambda_3\lambda_4}(\omega,\alpha)$$

$$- 2 M_{\lambda_1} M_{\lambda_2} A_{\lambda_1\lambda_2} \Sigma_{\lambda_3\lambda_4} T''(-\lambda_1-\lambda_2;\lambda_4\lambda_3)\,\chi_{\lambda_3\lambda_4}(\omega,\alpha) \qquad (3.26)$$

$$(A^2_{\lambda_1\lambda_2} - \omega^2)\ \chi_{\lambda_1\lambda_2}(\omega,\alpha)$$

$$= M_{\lambda_2}(A_{\lambda_1\lambda_2}P_{\lambda_1} - \omega)\sum_{\lambda_3\lambda_4} T'(\lambda_4\lambda_1;\lambda_3-\lambda_2)\ \varphi_{\lambda_3\lambda_4}(\omega,\alpha)$$

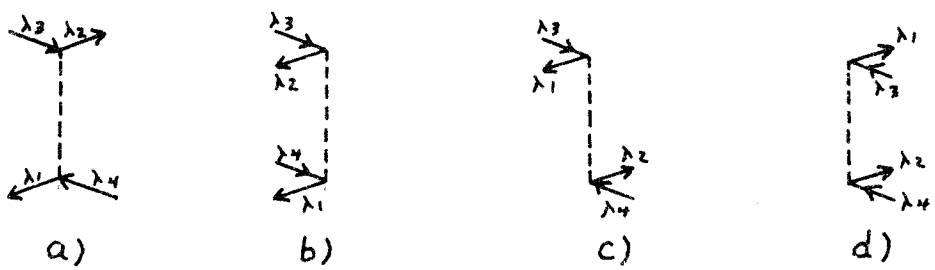
$$- M_{\lambda_1}(A_{\lambda_1\lambda_2}P_{\lambda_2} - \omega)\sum_{\lambda_3\lambda_4} T'(\lambda_4\lambda_2;\lambda_3-\lambda_1)\ \varphi_{\lambda_3\lambda_4}(\omega,\alpha)$$

$$- \left[A_{\lambda_1\lambda_2}(1-\beta_{\lambda_1\lambda_2}) - \omega D_{\lambda_1\lambda_2}\right]\sum_{\lambda_3\lambda_4} T''(\lambda_1\lambda_2:\lambda_4\lambda_3)\ \chi_{\lambda_3\lambda_4}(\omega,\alpha)$$

$$+ 2M_{\lambda_1}M_{\lambda_2}A_{\lambda_1\lambda_2}\sum_{\lambda_3\lambda_4} T''(\lambda_3\lambda_4;-\lambda_1-\lambda_2)\ \bar{\chi}_{\lambda_3\lambda_4}(\omega,\alpha) \qquad (3.27)$$

2. The Interaction

Let us consider the structure of T . We first consider T' , which has been defined as the sum of all irreducible diagrams in which a particle-hole pair is scattered. In first order, the following diagrams contribute to $T'(\lambda_1\lambda_2;\lambda_3\lambda_4)$ (T-61)



(3.28a)

The convention adopted in these diagrams is as follows:

1) Arrows pointing away from the interaction denote the creation of particles or of holes, while arrows pointing toward the interaction denote the annihilation of particles

or holes.

2) Arrows directed to the right, which is taken as the direction of increasing time, refer to particles, while those directed to the left, refer to holes.

The contribution from each of a) and b) is

$$(1/2)\langle \lambda_1 \lambda_2 | V | \lambda_3 \lambda_4 \rangle$$

and from each of c) and d) is

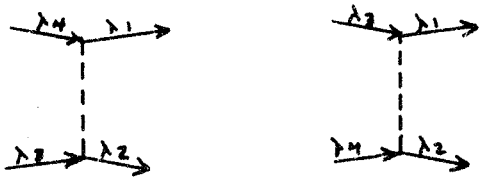
$$-(1/2)\langle \lambda_1 \lambda_2 | V | \lambda_4 \lambda_3 \rangle$$

Thus, in first order,

$$\Gamma'(\lambda_1 \lambda_2 ; \lambda_3 \lambda_4)$$

$$= \langle \lambda_1 \lambda_2 | V | \lambda_3 \lambda_4 \rangle - \langle \lambda_1 \lambda_2 | V | \lambda_4 \lambda_3 \rangle \qquad (3.29)$$

is the sum of all irreducible interactions that scatter particle-particle or hole-hole pairs. In first order, the following diagrams contribute to $\Gamma''(\lambda_1 \lambda_2 ; \lambda_3 \lambda_4)$



(3.28b)

Thus in first order,

$$2\,\Gamma''(\lambda_1 \lambda_2 ; \lambda_3 \lambda_4)$$

$$= \langle \lambda_1 \lambda_2 | V | \lambda_3 \lambda_4 \rangle - \langle \lambda_1 \lambda_2 | V | \lambda_4 \lambda_3 \rangle \qquad (3.30)$$

Since we are interested in the case of spherical systems, let us now proceed to write the matrix elements of the potential V in a form that exhibits its invariance under rotations and reflections.

We write the two-body potential in the form

$$V(r_1 r_2) = \Sigma_k v_k(r_1 r_2)\, T^k(1) \cdot T^k(2) \qquad (3.31)$$

where T^k is a tensor operator of degree k which transforms under rotations like the spherical harmonics Y^k.

An arbitrary matrix element of this potential between the single particle states $|\alpha\rangle \equiv |jm\rangle$ can be written

$$\langle \alpha_1 \alpha_2 | V | \alpha_3 \alpha_4 \rangle = \Sigma_K \langle \alpha_1 \alpha_2 | v(r_1 r_2)\, T^K(1) \cdot T^K(2) | \alpha_3 \alpha_4 \rangle$$

$$= \Sigma_{K\mu} (-1)^{\mu} \langle 12 | v(r_1 r_2) | 34 \rangle \langle j_1 m_1 | T^K_{\mu}(1) | j_4 m_4 \rangle \times$$

$$\langle j_2 m_2 | T^K_{-\mu}(2) | j_3 m_3 \rangle \qquad (3.32)$$

We define

$$R(12;34) = \langle 12 | v(r_1 r_2) | 34 \rangle$$

Using the Wigner-Eckart theorem (E-58), we write

$$\langle j_1 m_1 | T^K_{\mu} | j_4 m_4 \rangle = C^{j_4 m_4 ; K\mu}_{j_1 m_1} \langle 1 \| T^K \| 4 \rangle (2j_1+1)^{-1/2} \qquad (3.33)$$

$$= (-1)^{j_4 - m_4} C^{j_1 m_1 ; j_4 - m_4}_{K\mu} \langle 1 \| T^K \| 4 \rangle (2K+1)^{-1/2}$$

and similarly

$$\langle j_2 m_2 | T^k_{-\mu} | j_3 m_3 \rangle = (-1)^{j_3 - m_3} C^{j_2 m_2 ; j_3 - m_3}_{K\,-\mu} \frac{\langle 2 \| T^K \| 3 \rangle}{\sqrt{2K+1}}$$

We define

$$F'_k(12;34) = -\frac{R(12;34)}{2k+1} \langle 1 \| T^k \| 4 \rangle \langle 2 \| T^k \| 3 \rangle$$

and we finally get

$$\langle \alpha_1 \alpha_2 | V | \alpha_3 \alpha_4 \rangle$$

$$= -\Sigma_{k\mu} (-1)^{\mu} F'_k(12;34)\, (-1)^{j_4 - m_4} C^{j_1 m_1 ; j_4 - m_4}_{K\mu} (-1)^{j_3 - m_3}$$

$$\times\; C^{j_2 m_2 ; j_3 - m_3}_{K\,-\mu} \qquad (3.34)$$

F_k is a rotationally invariant particle-hole matrix element. However, the interaction V can also be expressed in terms of a rotationally invariant particle-particle or hole-hole matrix element, G_k, defined by

$$\langle \vartheta_1 \vartheta_2 | V | \vartheta_3 \vartheta_4 \rangle = - \sum_{K\mu} G'_k(12;34)\, C^{j_1 m_1 ; j_2 m_2}_{K\mu}\, C^{j_3 m_3 ; j_4 m_4}_{K\mu} \tag{3.35}$$

The functions F'_k and G'_k are related by

$$F'_K(12;34) = \sum_\ell (-1)^{j_1+j_3+\ell} (2\ell+1)\, G'_\ell(12;34)\, W(1423;K\ell)$$

$$G'_K(12;34) = (-1)^{j_1+j_2+K} \sum_\ell (2\ell+1)\, F'_\ell(12;34)\, W(1243;K\ell) \tag{3.36}$$

(See Appendix A, equations (A-3).) Here, W (1234; kl) is the usual Racah coefficient as defined by Rose (R-57).

Let us consider the case where the interaction consists of a quadrupole force and a pairing force. For the former we have

$$V = - \chi\, r_1^2 r_1^2 \sum_\mu (-1)^\mu\, Y_{2\mu}\, Y_{2-\mu} \tag{3.37}$$

Then we get

$$F'_2(12;34)$$

$$= (\chi/5) \langle 1 \| r_1^2 \| 4 \rangle \langle 2 \| r_2^2 \| 3 \rangle \langle 1 \| Y_2 \| 4 \rangle \langle 2 \| Y_2 \| 3 \rangle$$

Now the reduced matrix element is given by (E-58)

$$\langle 1 \| Y_2 \| 4 \rangle = C^{j_4 1/2 ; 20}_{j_1 1/2} \left[\frac{5}{4\pi} (2j_4 + 1) \right]^{1/2}$$

so we get the particle-hole matrix element F in the form

$$F_2'(12;34)$$

$$= \frac{\chi}{4\pi} \langle 1 \| r_1^2 \| 4 \rangle \langle 2 \| r_2^2 \| 3 \rangle \left[(2j_4+1)(2j_3+1)\right]^{1/2} \times C^{j_4\,1/2\,;\,20}_{j_1\,1/2} \; C^{j_3\,1/2\,;\,20}_{j_2\,1/2} \qquad (3.38)$$

Evaluating the radial part of the matrix element yields (K-60)

$$\langle j \| r^2 \| j^1 \rangle = \left(\frac{h}{M\omega_o}\right)^2 (n + 3/2)^2 \qquad (3.39)$$

where M is the nucleon mass, $\hbar\omega_o = 41A^{-1/3}$ Mev, and n is the number of oscillator quanta associated with the single-particle states in the region being studied. For the lead isotopes, n = 5. We follow Kisslinger and Sorensen and define

$$\frac{5}{4\pi} \chi \langle j \| r^2 \| j' \rangle^2 = X \qquad (3.40)$$

We can thus write (3.38) as

$$F_2'(12;34) = \frac{X}{5}\left[(2j_4+1)(2j_3+1)\right]^{\frac{1}{2}2} C^{j_4\,1/2\,;\,20}_{j_1\,1/2} \; C^{j_3\,1/2\,;\,20}_{j_2\,1/2} \qquad (3.41)$$

It follows from (3.41) that F_2' obeys the following symmetry properties

$$F_2'(12;34) = (-1)^{j_1 - j_4} F_2'(42;31)$$
$$F_2'(12;34) = (-1)^{j_1 + j_2 + j_3 + j_4} F_2'(43;21) \qquad (3.42)$$

It further follows from (3.36) that G_2' obeys the symmetry properties

$$G_2'(12;34) = G_2'(43;21)$$
$$G_2'(12;34) = (-1)^{j_1 + j_2 + j_3 + j_4} G_2'(21;34) \qquad (3.43)$$

The pairing force result has been mentioned previously and can be written

$$G(12;34) = G\ \delta_{12}\ \delta_{34} \left[\Omega_1 \Omega_3\right]^{1/2} \qquad (3.44)$$

As mentioned earlier, the particle-hole interaction consists of two types of terms, shown below

direct exchange (3.45)

We can see that for a given multipole moment of the potential, say the k^{th}, the direct term is the dominating one. This is readily seen from the above diagram, where a particle and hole coupled to an angular momentum K can annihilate only through the kth term in the multipole expansion. The exchange term, on the other hand, depends on all the multi-

pole moments, these moments being recoupled to k by means of a Racah coefficient (3.37). The ratio of the contribution of the exchange term to the contribution of the direct term for a single multipole will, in general, be small, but by no means negligible.

We now want to express the interactions T' and T'' in terms of the matrix elements F and G. From Appendix A we obtain the following relations

$$T'_{\mu}(j_4 m_4 j_1 m_1;\ j_3 m_3 j_2 m_2)$$

$$= \sum_k F_k(41;32)\left[(-1)^{j_2-m_2} C^{j_1 m_1;\ j_2 -m_2}_{k\mu}\right]\left[(-1)^{j_4-m_4} C^{j_3 m_3;\ j_4-m_4}_{k\mu}\right] \tag{3.46}$$

where $F_k(41;32) = \left[F'_k(13;42) + f'_k(41;32)\right]$ (3.47)

and $f'_k(41;32) = -\sum_l (2l+1)\, F'_l(41;32)(-1)^{j_1+j_2+k+l} W(j_1 j_2 j_3 j_4;\ k l)$ (3.48)

$$T'_{\mu}(j_4 m_4\ j_2 -m_2;\ j_3 m_3\ j_1 -m_1)$$

$$= \sum_k \bar{F}_k(42;31)(-1)^{j_1-j_2+\mu}\left[(-1)^{j_2-m_2} C^{j_1 m_1;\ j_2-m_2}_{k\mu}\right]$$

$$\times\left[(-1)^{j_4-m_4} C^{j_3 m_3;\ j_4-m_4}_{k\mu}\right] \tag{3.49}$$

where

$$\bar{F}_k(42;31) = \left[F'_k(13;42)(-1)^k + (-1)^{j_1 - j_2} \bar{f}'_k(42;31) \right] \quad (3.50)$$

and

$$\bar{f}'_k(42;31) = -\Sigma_1 (21+1)\, F'_1(42;31)(-1)^{j_1 + j_2 + \ell}\, W(j_1 j_2 j_4 j_3; k1) \quad (3.51)$$

The extra phase factor in the definition of (3.49) will prove to be convenient later.

$$2\, T''_\mu (j_1 m_1\ j_2 m_2; j_3 m_3\ j_4 m_4)$$

$$= \Sigma_k\, G_k(43;21)\ C^{j_1 m_1\ ;\ j_2 - m_2}_{k\mu}\ C^{j_3 m_3\ ;\ j_4 m_4}_{k\mu} \quad (3.52)$$

where

$$G_k(43;21)$$

$$= \Sigma_1 (-1)^{j_1 + j_4}\ (21+1) \left[F'_1(12;34)\, W(1243;k1) + (-1)^k \times \right.$$

$$\left. F'_\ell\ (21;34)\ W\ (1234;k1) \right] \quad (3.53)$$

$$2\, T''_\mu (j_4 m_4\ j_3 m_3;\ j_2 m_2\ j_1 - m_1)$$

$$= -\Sigma_\kappa\ G_\kappa (43;21)(-1)^{j_1 - m_1} \left[(-1)^{\kappa + \mu}\ C^{j_4 - m_4\ ;\ j_3 - m_3}_{\kappa\mu} \right]$$

$$\times \quad \left[(-1)^{j_2 - m_2}\ C^{j_1 m_1\ ;\ j_2 - m_2}_{\kappa\mu} \right] \quad (3.54)$$

It will also prove convenient to extract from the equations for the amplitudes, the remaining dependence on the quantum number m.

From the relations (1.25) and (2.36),

$$\Delta_\lambda = \Delta_{jm} = (-1)^{j-m} \Delta \quad \text{and} \quad u_\lambda v_\lambda = \frac{\Delta_\lambda}{2E_\lambda}$$

and, using the fact that $E_\lambda = E_j$ is independent of m, we write

$$u_{jm}\, v_{jm} = (-1)^{j-m}\, u_j\, v_j$$

or

$$u_{jm} = u_j \quad , \quad v_{jm} = (-1)^{j-m}\, v_j$$

This definition gives u_{jm} and v_{jm} the symmetry properties adopted in the usual pairing model treatments

$$u_{jm} = u_{j-m} \qquad v_{jm} = -v_{j-m}$$

Using these relations, we can thus write, from (3.25)

$$M_\lambda = M_{jm} = (-1)^{j-m} M_j \tag{3.55}$$

Thus, all the m- dependence in equations (3.26) (3.28) can be separated out.

We thus see that for an interaction F, there corresponds a value of G. This value will, in general, be smaller than F because the recoupling of the angular momentum spreads the strength of G over many angular momenta. This is the same situation occurring with the exchange term to F, namely f. However, an accurate treatment of the problem would require taking both of these effects into

account. Since G is associated with the particle-particle, or hole-hole wave functions, a finite value of this quantity means that the system of equations can not be uncoupled. Thus, even if there is no (p-p) or (h-h) interaction as such, we can not take G to be zero, at least not in an accurate treatment.

A first approximation would thus involve neglecting both f_k' and G_k', in which case the equation for φ_K is closed, and a solution can be obtained.

A second approximation which might be valid would be to consider f_k' but not G_k' even though these quantities will contribute roughly equal amounts. The advantage would be, of course, that we could still consider only the equation for φ .

The exact treatment would have to include the full set of three coupled equations.

Let us consider the case where the interaction consists of a quadrupole force and a pairing force.

$$F_k(41;32)$$

$$= F_k'(13;42)\delta_{k2} - 5\, F_2'(41;32)(-1)^{j_1+j_2+K} W(1234;k2) \qquad (3.56)$$

$$\bar{F}_k(42;31)$$

$$= F_k'(13;42)\delta_{k,2} + 5\, F_2'(42;31)\, W(1243;k2) \qquad (3.57)$$

$$G_k(42;31)$$

$$= 2G(\Omega_1\Omega_3)^{1/2}\delta_{43}\delta_{12}\delta_{k0}$$

$$+ 5\left[(-1)^{j_1+j_2+K} F_2'(41;32)\, W(1234;k2)\right.$$

$$\left. - F_2'(42;31)\, W(1243;k2)\right] \qquad (3.58)$$

3. Equations for Amplitudes of States with Definite Angular Momentum

Let us now define

$$\hat{\varphi}_{12;k\mu} = \sum_{m_1,m_2} (-1)^{j_2-m_2} C^{j_1 m_1;\, j_2 -m_2}_{K\mu}\, a^+_{j_2 m_2}\, a_{j_1 m_1} \qquad (3.59a)$$

$$\hat{\bar{\chi}}_{12;k\mu} = \sum_{m_1,m_2} (-1)^{K+\mu} C^{j_1 m_1;\, j_2 m_2}_{K\,-\mu}\, a^+_{j_2 m_2}\, a^+_{j_1 m_1} \qquad (3.59b)$$

$$\hat{\chi}_{12;k} = \sum_{m_1,m_2} C^{j_1 m_1;\, j_2 m_2}_{K\mu}\, a_{j_2 m_2}\, a_{j_1 m_1} \qquad (3.59c)$$

The phase factors arise because of the way in which we have defined the conjugate states. These equations define pair operators for states with angular momentum k and z- component μ .

Finally, to simplify the equations for the amplitudes we introduce two new operators

$$\hat{\chi}^{\pm}_{12;k\mu} = \hat{\bar{\chi}}_{12;k\mu} \pm \hat{\chi}_{12;k\mu} \qquad (3.60)$$

and the corresponding amplitudes

$$\langle \Phi_0 |\ \hat{\chi}^{\pm}_{12;K\mu}\ | K\mu \rangle \qquad (3.61)$$

We note from (3.59b) and (3.59c) that $\hat{\chi}^{\pm}$ obeys the symmetry relation

$$\hat{\chi}^{\pm}_{12;k\mu} = (-1)^{j_1 - j_2 + \kappa} \hat{\chi}^{\pm}_{21;k\mu} \tag{3.62}$$

Substituting (3.46), (3.49), (3.52), (3.54) and (3.61) in equations (3.25) to (3.27), we obtain the following set of equations for the amplitudes corresponding to the functions (3.59),

$$\begin{aligned}
&(A_{12}^2 - \omega^2)\, \varphi_{12;k}(\omega) \\
&= (A_{12}B_{12} - \omega C_{12}) \Sigma_{34} F_k(41;32)\, \varphi_{34;k}(\omega) \\
&\quad + 2M_1M_2A_{12} \Sigma_{34} \bar{F}_k(42;31)\, \varphi_{34;k}(\omega) \\
&+ 1/4 \left[(M_1A_{12}P_2 + M_2A_{12}P_1) + (M_1 - M_2)\omega \right] \Sigma_{34} G_k(43;21)\, \chi^{-}_{34;k} \\
&+ 1/4 \left[(M_1A_{12}P_2 - M_2A_{12}P_1) + (M_1 + M_2)\omega \right] \Sigma_{34} G_k(43;21)\, \chi^{+}_{34;k}
\end{aligned} \tag{3.63a}$$

$$\begin{aligned}
&(A_{12}^2 - \omega^2)\, \chi^{-}_{12;k}(\omega) \\
&= \left[(M_1A_{12}P_2 + M_2A_{12}P_1) + (M_1 - M_2)\omega \right] \Sigma_{34} F_k(41;32)\, \varphi_{34;k}(\omega) \\
&+ \left[(M_2A_{12}P_1 + M_1A_{12}P_2) + (M_2 - M_1)\omega \right] \Sigma_{34} \bar{F}_k(42;31)\, \varphi_{34;k}(\omega) \\
&+ 1/2 \left[A_{12}(1 - B_{12}) - 2M_1M_2A_{12} \right] \Sigma_{34} G_k(43;21)\, \chi^{-}_{34;k}(\omega) \\
&+ 1/2\, \omega D_{12} \Sigma_{34} G_k(43;21)\, \chi^{+}_{34;k}(\omega)
\end{aligned} \tag{3.63b}$$

$$(A_{12}^2 - \omega^2)\ \chi^+_{12;k}\ (\omega)$$

$$= \left[(M_1 A_{12} P_2 - M_2 A_{12} P_1) + (M_1+M_2)\omega\right] \Sigma_{34} F_k(41;32)\ \varphi_{34;k}\ (\omega)$$

$$+ \left[(M_2 A_{12} P_1 - M_1 A_{12} P_2) + (M_1+M_2)\omega\right] \Sigma_{34} \bar{F}_k(42;31)\ \varphi_{34;k}\ (\omega)$$

$$+ 1/2 \left[A_{12}(1-B_{12}) + 2M_1 M_2 A_{12}\right] \Sigma_{34} G_k(43;21)\ \chi^+_{34;k}\ (\omega)$$

$$+ (1/2)\omega\, D_{12} \Sigma_{34} G_k(43;21)\ \chi^-_{34;k}\ (\omega) \qquad (3.63c)$$

This is the final form for the equations relating the set of amplitudes which describe the excited state $|K\mu\omega\rangle$ when the paired states are interacting by means of the functions F_k and G_k.

Let us consider the case where there is only a direct quadrupole interaction. In this case, the φ's are uncoupled from χ^+ and χ^- and we get

$$(A^2 - \omega^2)\ \varphi_{12;2}$$

$$= \left[A B - \omega C + 2 M_1 M_2 A\right] \Sigma_{34}\ F_2'(13;42)\ \varphi_{34;2}$$

where

$$F_2'(13;42) = \frac{X}{5} \sqrt{(2j_4+1)(2j_2+1)}\ C^{j_4 1/2;20}_{j_3 1/2}\ C^{j_2 1/2;20}_{j_1 1/2}$$

$$= \frac{X}{5}\ F''_{12}\ F''_{34} \qquad (3.64)$$

Thus we can write

$$\frac{(A^2-\omega^2)\varphi_{12;2}}{[AB-\omega C+2M_1M_2A]F''_{12}} = \frac{X}{5}\Sigma_{34}F''_{34}\varphi_{34;2} = \text{constant}$$

$$\frac{(A^2-\omega^2)\varphi_{34;2}}{[AB-\omega C+2M_3M_4A]F''_{34}} = \frac{X}{5}\Sigma_{12}\frac{F''_{12}\times\text{constant}\times[AB-\omega C+2M_1M_2A]F''_{12}}{(A^2-\omega^2)}$$

$$= \text{constant}$$

$$\therefore \Sigma_{12}\frac{X}{5}(2j_2+1)C^{j_2\frac{1}{2};20}_{j_1\frac{1}{2}}\left[\frac{(E_1+E_2)(u_1v_2+u_2v_1)^2-(u_1^2-u_2^2)\omega}{(E_1+E_2)^2-\omega^2}\right] = 1$$

This expression is symmetric in j_1 and j_2 except for the second term in brackets, which is antisymmetric. This latter term will vanish when j_1 and j_2 are summed over, so it can be set equal to zero in the above expression.

We are left with

$$5/X =$$

$$\Sigma_{12}(2j_2+1)C^{j_2\frac{1}{2};20}_{j_1\frac{1}{2}}\frac{(E_1+E_2)(u_1v_2+u_2v_1)^2}{(E_1+E_2)^2-\omega^2} \qquad (3.65)$$

This is the same expression as that found by Baranger and by Tamura and Udagawa (see equation (1.39)).

We see that the situation is much more complex when both direct and exchange interactions are considered. It is not possible to write down a simple expression corresponding to (3.65) and some other method must be used to

find the eigenvalues ω of the set of equations (3.62). When we are only interested in the lowest 2^+ level, that is, the collective level, approximate numerical techniques can be adopted. These are outlined in Appendix B.

CHAPTER IV

CONSTRUCTION OF THE STATE VECTOR FOR THE EXCITED LEVELS

1. State Vector

We assume that the state vector for the level characterized by angular momentum k, z- component μ , and energy E , can be expanded in terms of the operators $\hat{\varphi}$, $\hat{x}^+$ and $\hat{x}^-$. The state vector for this level can then be defined by the relation

$$\hat{B}^{\dagger}_{K\mu E} \, | \Phi_0 \rangle = | K\mu E \rangle \tag{4.1}$$

where

$$\hat{B}^{\dagger}_{K\mu E} = \frac{1}{N} \Sigma_{12} \Big[X^{*}_{12;KE} \, \hat{\varphi}^{\dagger}_{12;K\mu} + Y^{*}_{12;KE} \, (\hat{x}^{+}_{12;K\mu})^{\dagger} + Z^{*}_{12;KE} \, (\hat{x}^{-}_{12;K\mu})^{\dagger} \Big] \tag{4.2}$$

and

$$\hat{B}_{K-\mu E} = \frac{1}{N} \Sigma_{12} \Big[X_{12;KE} \, \hat{\varphi}_{12;K-\mu} + Y_{12;KE} \, \hat{x}^{+}_{12;K-\mu} + Z_{12;KE} \, \hat{x}^{-}_{12;K-\mu} \Big] \tag{4.3}$$

The equation that results when μ is replaced by $-\mu$ in (4.2) or (4.3) must be the same as that obtained by the application of the time-reversal operator to these same equations. Using relations (c.2) of Appendix C, we thus obtain the following condition which must be satisfied by the coefficients X, Y and Z.

$$\begin{aligned} X^{*}_{12;K\mathcal{E}} &= -(-1)^{j_1+j_2+K} X_{12;K\mathcal{E}} \\ Y^{*}_{12;K\mathcal{E}} &= (-1)^{j_1+j_2+K} Y_{12;K\mathcal{E}} \\ Z^{*}_{12;K\mathcal{E}} &= (-1)^{j_1+j_2+K} Z_{12;K\mathcal{E}} \end{aligned} \tag{4.4}$$

Using these equations along with equations (c.3) of Appendix C, we can rewrite equation (4.2) in the form

$$\begin{aligned} &\hat{B}^{\dagger}_{K\mu\mathcal{E}} \\ &= \frac{1}{N} \Sigma_{12} \Big[X_{12;K\mathcal{E}}\, \hat{\varphi}_{21;K-\mu} - Y_{12;K\mu}\, \hat{x}^{+}_{21;K-\mu} \\ &\qquad + Z_{12;K\mathcal{E}}\, \hat{x}^{-}_{21;K-\mu} \Big] (-1)^{j_1-j_2+K} \end{aligned} \tag{4.5}$$

2. System of Equations for the Coefficients X, Y and Z

One method of obtaining the system of equations satisfied by the coefficients X, Y and Z is to proceed as follows:

We take the matrix element between the ground state and the state $|K\mu\mathcal{E}\rangle$ of the operator defined by equation (4.2). The left hand side becomes

$$\langle K\mu\mathcal{E} | \hat{B}^{\dagger}_{K\mu\mathcal{E}} | \Phi_0 \rangle = \langle K\mu\mathcal{E} | K\mu\mathcal{E} \rangle = 1$$

since we require the state to be normalized. The right hand side of (4.2) can be written as

$$\frac{1}{N}\sum_{12}\Big[\; X^{*}_{12;K\mathcal{E}}\langle K\mu E|\,\hat{\varphi}^{\dagger}_{12;K\mu}\,|\Phi_0\rangle + Y^{*}_{12;K\mathcal{E}}\langle K\mu E|\,(\hat{x}^{+}_{12;K\mu})^{\dagger}\,|\Phi_0\rangle + Z^{*}_{12;K\mathcal{E}}\langle K\mu E|\,(\hat{x}^{-}_{12;K\mu})^{\dagger}\,|\Phi_0\rangle\;\Big]$$

$$=\frac{1}{N}\sum_{12}\Big[\; X^{*}_{12;K\mathcal{E}}\,\varphi^{*}_{12;K\mathcal{E}} + Y^{*}_{12;K\mathcal{E}}\,(x^{+}_{12;K\mathcal{E}})^{*} + Z^{*}_{12;K\mathcal{E}}\,(x^{-}_{12;K\mu})^{*}\;\Big]$$

$$=\frac{1}{N}\sum_{12}\Big[\; X_{12;K\mathcal{E}}\,\varphi_{12;K\mathcal{E}} + Y_{12;K\mathcal{E}}\,x^{+}_{12;K\mathcal{E}} + Z_{12;K\mathcal{E}}\,x^{-}_{12;K\mathcal{E}}\;\Big]$$

The normalization condition is thus

$$N = \sum_{12}\Big[\; X_{12;K\mathcal{E}}\,\varphi_{12;K\mathcal{E}} + Y_{12;K\mathcal{E}}\,x^{+}_{12;K\mathcal{E}} + Z_{12;K\mathcal{E}}\,x^{-}_{12;K\mathcal{E}}\;\Big] \tag{4.6}$$

We have seen that the amplitudes φ, x^{+} and x^{-} are the components of the eigenvector, for the state $|K\mu E\rangle$, of the energy matrix $A(\omega = E)$. It thus follows from (4.6) that the coefficients X, Y and Z are the corresponding components of the eigenvector, for this same state of the matrix transpose to A, that is $A^{T}(\omega = E)$.

If A is symmetric, then of course the two eigenvectors are identical. In the present case, however, A is not symmetric, so both eigenvectors must be obtained.

A second, and more direct method of obtaining the equations for these coefficients is to substitute (4.3) directly into the equations of motion for φ, χ^+ and χ^- making use of the double commutation relation

$$\langle \Phi_0 | \Big[H, \big[H, \hat{B}_{K\mu E} \big] \Big] | K\mu E \rangle$$
$$= \omega^2 \langle \Phi_0 | \hat{B}_{K\mu E} | K\mu E \rangle \qquad (4.7)$$

The secular matrix resulting from this procedure can be readily verified to be just the transpose of the matrix derived previously.

3. Normalization

Let us now investigate the form of the normalization condition for the state vectors $|K\mu E\rangle$ From (4.2) and (4.3) we get

$$\frac{1}{N^2} \langle \Phi_0 | \hat{B}_{K'\mu'E'} \hat{B}^\dagger_{K\mu E} | \Phi_0 \rangle = I$$

$$= \frac{1}{N^2} \Sigma_{1234} \Big[X_{12;K'E'} \Big\{ X^*_{34;KE} \langle \Phi_0 | \hat{\varphi}_{12;K'\mu'} \hat{\varphi}^\dagger_{34;K\mu} | \Phi_0 \rangle$$
$$+ Y^*_{34;K\mu} \langle \Phi_0 | \hat{\varphi}_{12;K'\mu'} (\hat{\chi}^+_{34;K\mu})^\dagger | \Phi_0 \rangle$$
$$+ Z^*_{34;K\mu} \langle \Phi_0 | \hat{\varphi}_{12;K'\mu'} (\hat{\chi}^-_{34;K\mu})^\dagger | \Phi_0 \rangle \Big\}$$

$$+ Y_{12;K'E'} \Big\{ X^*_{34;KE} \langle \Phi_0 | \hat{\chi}^+_{12;K'\mu'} \hat{\varphi}^\dagger_{34;K\mu} | \Phi_0 \rangle +$$

$$+ Y^{*}_{34;K\varepsilon} \langle \Phi_0 | \hat{X}^{+}_{12;K'\mu'} (\hat{X}^{+}_{34;K\mu})^{\dagger} | \Phi_0 \rangle$$

$$+ Z^{*}_{34;K\varepsilon} \langle \Phi_0 | \hat{X}^{+}_{12;K'\mu'} (\hat{X}^{-}_{34;K\mu})^{\dagger} | \Phi_0 \rangle \Big\}$$

$$+ Z_{12;K'\varepsilon'} \Big\{ X^{*}_{34;K\varepsilon} \langle \Phi_0 | \hat{X}^{-}_{12;K'\mu'} \hat{\varphi}^{\dagger}_{34;K\mu} | \Phi_0 \rangle$$

$$+ Y^{*}_{34;K\varepsilon} \langle \Phi_0 | \hat{X}^{-}_{12;K'\mu'} (\hat{X}^{+}_{34;K\mu})^{\dagger} | \Phi_0 \rangle$$

$$+ Z^{*}_{34;K\varepsilon} \langle \Phi_0 | \hat{X}^{-}_{12;K'\mu'} (\hat{X}^{-}_{34;K\mu})^{\dagger} | \Phi_0 \rangle \Big\} \Big] \quad (4.8)$$

Inserting a complete set of states between the pairs of operators appearing in the ground state expectation values of (4.8), and making use of the relation

$$\langle \Phi_0 | \hat{\varphi}_{12;K'\mu'} | K''\mu'' \varepsilon'' \rangle = \varphi_{12;K'\varepsilon''} \delta_{K'K''} \quad (4.9)$$

allow us to rewrite equation (4.8) in the form

$$N^2 = \delta_{KK'} \delta_{\mu\mu'} \sum_{\varepsilon''} \times$$

$$\Big[\sum_{12} \Big\{ X_{12;K\varepsilon'} \varphi_{12;K\varepsilon''} + Y_{12;K\varepsilon'} X^{+}_{12;K\varepsilon''} + Z_{12;K\varepsilon'} X^{-}_{12;K\varepsilon''} \Big\}$$

$$\sum_{34} \Big\{ X_{34;K\varepsilon} \varphi_{34;K\varepsilon''} + Y_{34;K\varepsilon} X^{+}_{34;K\varepsilon''} + Z_{34;K\varepsilon} X^{-}_{34;K\varepsilon''} \Big\} \Big] \quad (4.10)$$

However, it is shown in Appendix B, that each of the summations in (4.10) is zero unless the energies appearing in that sum are equal. The normalization condition thus becomes

$$N^2 = \delta_{KK'} \delta_{\mu\mu'} \delta_{\varepsilon\varepsilon'} \times$$

$$\Big[\sum_{12} \Big\{ X_{12;K\varepsilon} \varphi_{12;K\varepsilon} + Y_{12;K\varepsilon} X^{+}_{12;K\varepsilon} + Z_{12;K\varepsilon} X^{-}_{12;K\varepsilon} \Big\} \Big]^2 \quad (4.11)$$

As we shall see in the next chapter, the matrix element, between the ground state and the excited level $|K\mu E\rangle$ can be expressed in terms of the suitably normalized functions

$$\frac{1}{N'}\ \varphi_{ij;KE} \tag{4.12}$$

If we operate on both sides of equation (4.2) with $\varphi_{12;K\mu}$ and then take the matrix element between $|\Phi_0\rangle$ and $|K\mu E\rangle$ of the resulting operator, we obtain

$$\varphi_{12;KE} = \langle \Phi_0 |\ \hat{\varphi}_{12;K\mu}\ |K\mu E\rangle$$

$$= \Sigma_{34}\Big[\ X^{*}_{34;KE}\ \langle \Phi_0|\ \hat{\varphi}_{12;K\mu}\ \hat{\varphi}^{\dagger}_{34;K\mu}\ |\Phi_0\rangle$$

$$+\ Y^{*}_{34;KE}\ \langle \Phi_0|\ \hat{\varphi}_{12;K\mu}\ (\hat{\chi}^{+}_{34;K\mu})^{\dagger}\ |\Phi_0\rangle$$

$$+\ Z^{*}_{34;KE}\ \langle \Phi_0|\ \hat{\varphi}_{12;K\mu}\ (\hat{\chi}^{-}_{34;K\mu})^{\dagger}\ |\Phi_0\rangle\ \Big] \tag{4.13}$$

Now, since the functions φ, χ^{+}, χ^{-} and X, Y, Z each form a complete set, it is not difficult to see that in order to obtain the required set of normalized functions (4.12), it is not enough to write N^2 in the form (4.10) and $\varphi_{12;KE}$ in the corresponding form from (4.13), but rather, the expectation values in (4.8) and in (4.13) must be explicitly evaluated. Let us now consider these two equations which we rewrite as follows.

$$\varphi_{12;\kappa\varepsilon} = (-1)^{\kappa+\mu} \Sigma_{34} (-1)^{j_3 - j_4 + \kappa} \times$$

$$\Big[X_{34;\kappa\varepsilon} \langle \Phi_0 | \hat{\varphi}_{12;\kappa\mu} \hat{\varphi}_{43;\kappa-\mu} | \Phi_0 \rangle - Y_{34;\kappa\varepsilon} \langle \Phi_0 | \hat{\varphi}_{12;\kappa\mu} \hat{x}^{+}_{43;\kappa-\mu} | \Phi_0 \rangle + Z_{34;\kappa\varepsilon} \langle \Phi_0 | \hat{\varphi}_{12;\kappa\mu} \hat{x}^{-}_{43;\kappa-\mu} | \Phi_0 \rangle \Big] \qquad (4.14)$$

$$N^2 = (-1)^{\kappa+\mu} \Sigma_{1234} (-1)^{j_3 - j_4 + \kappa} \times$$

$$\Big[X_{12;\kappa\varepsilon} \Big\{ X_{34;\kappa\varepsilon} \langle \Phi_0 | \hat{\varphi}_{12;\kappa\mu} \hat{\varphi}_{43;\kappa-\mu} | \Phi_0 \rangle - Y_{34;\kappa\varepsilon} \langle \Phi_0 | \hat{\varphi}_{12;\kappa\mu} \hat{x}^{+}_{43;\kappa-\mu} | \Phi_0 \rangle + Z_{34;\kappa\varepsilon} \langle \Phi_0 | \hat{\varphi}_{12;\kappa\mu} \hat{x}^{-}_{43;\kappa-\mu} | \Phi_0 \rangle \Big\}$$

$$+ Y_{12;\kappa\varepsilon} \Big\{ X_{34;\kappa\varepsilon} \langle \Phi_0 | \hat{x}^{+}_{12;\kappa\mu} \hat{\varphi}_{43;\kappa-\mu} | \Phi_0 \rangle - Y_{34;\kappa\varepsilon} \langle \Phi_0 | \hat{x}^{+}_{12;\kappa\mu} \hat{x}^{+}_{43;\kappa-\mu} | \Phi_0 \rangle + Z_{34;\kappa\varepsilon} \langle \Phi_0 | \hat{x}^{+}_{12;\kappa\mu} \hat{x}^{-}_{43;\kappa-\mu} | \Phi_0 \rangle \Big\}$$

$$+ Z_{12;\kappa\varepsilon} \Big\{ X_{34;\kappa\varepsilon} \langle \Phi_0 | \hat{x}^{-}_{12;\kappa\mu} \hat{\varphi}_{43;\kappa-\mu} | \Phi_0 \rangle - Y_{34;\kappa\varepsilon} \langle \Phi_0 | \hat{x}^{-}_{12;\kappa\mu} x^{+}_{43;\kappa-\mu} | \Phi_0 \rangle + Z_{34;\kappa\varepsilon} \langle \Phi_0 | \hat{x}^{-}_{12;\kappa\mu} \hat{x}^{-}_{43;\kappa-\mu} | \Phi_0 \rangle \Big\} \Big] \qquad (4.15)$$

Let us consider the form of the expectation values appearing in these expressions. We have, for example,

$$\langle \Phi_0 | \hat{\varphi}_{12;\kappa\mu} \hat{\varphi}_{43;\kappa-\mu} | \Phi_0 \rangle$$

$$= \Sigma_{m_1 m_2 m_3 m_4} (-1)^{j_2 - m_2} C^{j_1 m_1 ; j_2 - m_2}_{\kappa\mu} (-1)^{j_3 - m_3} C^{j_4 m_4 ; j_3 - m_3}_{\kappa-\mu} \times$$

$$\langle \Phi_0 | a^{+}_{j_2 m_2} a_{j_1 m_1} a^{+}_{j_3 m_3} a_{j_4 m_4} | \Phi_0 \rangle \qquad (4.16)$$

The last factor in (4.16) can be regarded as an equal-times, two-particle Green's function. A first approximation to this expectation value can thus be obtained by keeping only the inhomogeneous part of the Dyson expansion (2.45) and allowing also for pairings of two creation or two annihilation operators. Provided the interaction is not too strong, this approximation should be quite valid.

We thus have

$$\begin{aligned} &\langle \Phi_0 | \, a^+_{\nu_2} a_{\nu_1} a^+_{\nu_3} a_{\nu_4} \, | \Phi_0 \rangle \\ \simeq \;& \langle \Phi_0 | \, a^+_{\nu_2} a_{\nu_1} \, | \Phi_0 \rangle \langle \Phi_0 | \, a^+_{\nu_3} a_{\nu_4} \, | \Phi_0 \rangle \\ &- \langle \Phi_0 | \, a^+_{\nu_2} a^+_{\nu_3} \, | \Phi_0 \rangle \langle \Phi_0 | \, a_{\nu_1} a_{\nu_4} \, | \Phi_0 \rangle \\ &+ \langle \Phi_0 | \, a^+_{\nu_2} a_{\nu_4} \, | \Phi_0 \rangle \langle \Phi_0 | \, a_{\nu_1} a^+_{\nu_3} \, | \Phi_0 \rangle \end{aligned} \qquad (4.17)$$

The ground state expectation values of products of two creation or annihilation operators are known (see Appendix C, equations (C.4)) and so the expression (4.17) can be evaluated.

A special case occurs in the expansion of a product of four operators, such as (4.17), arising from two "pair" operators, such as the two φ'_s in (4.16). Since these "pair" operators are coupled to angular momentum K, that term in the expansion (4.17) which involves the ground state expectation value of these operators must vanish unless $k=0$.

To be more precise, consider the first term in (4.17). Substituting this in the right hand side of (4.16), and using equation (C.4a), we obtain

$$\sum_{m_1 m_2 m_3 m_4} (-1)^{j_2 - m_2} C^{j_1 m_1 ; j_2 - m_2}_{K \mu} (-1)^{j_3 - m_3} C^{j_4 m_4 ; j_3 - m_3}_{K - \mu} \times$$

$$\langle \Phi_0 | a^+_{\nu_2} a_{\nu_1} | \Phi_0 \rangle \langle \Phi_0 | a^+_{\nu_3} a_{\nu_4} | \Phi_0 \rangle$$

$$= \sum_{m_1 m_2 m_3 m_4} (-1)^{j_2 - m_2} C^{j_1 m_1 ; j_2 - m_2}_{K \mu} (-1)^{j_3 - m_3} C^{j_4 m_4 ; j_3 - m_3}_{K - \mu} \times$$

$$\delta_{K0} \, \delta_{\nu_1 \nu_2} \, \delta_{\nu_3 \nu_4} \, v^2_{j_1} \, v^2_{j_3}$$

Making use of the relation

$$\sum_m (-1)^{j-m} \delta_{K0} C^{jm ; j-m}_{K0} = \delta_{K,0} \sum_m \frac{1}{\sqrt{2j+1}} = \delta_{K,0} \sqrt{2j+1}$$

the above expression reduces to

$$2 \, \delta_{\nu_1 \nu_2} \, \delta_{\nu_3 \nu_4} \, \delta_{K0} \, v^2_{j_1} \, v^2_{j_3} \sqrt{\Omega_{j_1} \, \Omega_{j_3}} \qquad (4.18)$$

Expressions similar to (4.18) arise from the other ground state expectation values appearing in (4.14) and (4.15). We will omit these terms until we come to discuss the k=0 state.

Thus, in this approximation, and for $k \neq 0$, the normalization condition can be written in the form (see Appendix C)

$$N^2$$

$$= \delta_{KK'}\,\delta_{\mu\mu'}\,\delta_{EE'}\,\Sigma_{1\le 2}\,(1+\delta_{12})^{-1}\,\times$$

$$\Big[\, X_{12;KE}\,v_2 u_1 + X_{21;KE}\,v_1 u_2 (-1)^{j_1 - j_2 + K}$$

$$+ 2\,(u_1 u_2 + v_1 v_2)\,Y_{12;KE}$$

$$- 2\,(u_1 u_2 - v_1 v_2)\,Z_{12;KE}\,\Big]^2 \qquad (4.19)$$

4. Spurious States

It has been demonstrated (M-58)(K-60)(B-60) that treatments of excited levels which do not conserve the number of particles, contain spurious solutions. In the present work, there is one spurious state present with k=0. It can be found as follows. Since the ground state we are using is not an eigenstate of the number of particles, the state $\hat{N}\,|\Phi_0\rangle$ is different from $|\Phi_0\rangle$ where

$$\hat{N} = \Sigma_{jm}\, a^{+}_{jm}\, a_{jm} \qquad (4.20)$$

is the operator for the number of particles. This extra state is spurious, and should be eliminated from the discussion. The number operator can be rewritten as

$$\hat{N} = \Sigma_{jm}\, a^{+}_{jm}\, a_{jm}$$

$$= \Sigma_{jm}\, C_{00}^{jm;\,j-m}\,(-1)^{j-m}\sqrt{2j+1}\;\; a^{+}_{jm}\, a_{jm}$$

$$= \Sigma_{jm}\,\sqrt{2\Omega_j}\;\;\hat{\varphi}_{jj;00} \qquad (4.21)$$

Here, we have made use of the relation

$$C_{00}^{jm\,;\,j-m} = (-1)^{j-m}\,(2j+1)^{-1/2}$$

We thus have

$$\hat{B}^{\dagger}_{00\omega_0} = \Sigma_j \sqrt{2\Omega_j}\ \hat{\varphi}_{jj;00} \tag{4.22}$$

as the operator that creates the spurious k=0 level. Thus, in order to eliminate this level, we must set the amplitude

$$\langle \Phi_0 |\, B_{00\omega_0} \,|\, 00\omega_0 \rangle = \Sigma_j \sqrt{2\Omega_j}\ \varphi_{jj;0\omega_0} \tag{4.23}$$

equal to zero. This condition yields the set of relations

$$\varphi_{jj;0\omega_0} = 0 \tag{4.24}$$

For the 0^+ levels, equations (3.63) become

$$(4E_1^2 - \omega^2)\,\varphi_{11;0}(\omega)$$

$$= 8E_1 u_1^2 v_1^2 \Sigma_2\ F_0(21;21)\,\varphi_{22;0}(\omega)$$

$$+ E_1 u_1 v_1 (u_1^2 - v_1^2)\,\Sigma_2\ G_0(22;11)\,X^{-}_{22;0}(\omega)$$

$$+ \frac{u_1 v_1}{2}\,\omega\ \Sigma_2\ G_0(22;11)\,X^{+}_{22;0}(\omega) \tag{4.25a}$$

$$(4E_1^2 - \omega^2)\,X^{-}_{11;0}(\omega)$$

$$= 8E_1 u_1 v_1 (u_1^2 - v_1^2)\,\Sigma_2\ F_0(21;21)\,\varphi_{22;0}(\omega)$$

$$+ E_1 (1 - 4u_1^2 v_1^2)\,\Sigma_2 G_0(22;11)\,X^{-}_{22;0}(\omega)$$

$$+ \frac{1}{2}\,\omega\,(u_1^2 - v_1^2)\,\Sigma_2\ G_0(22;11)\,X^{+}_{22;0}(\omega) \tag{4.25b}$$

$$(4E_1^2 - \omega^2)\, X^{+}_{11;0}(\omega)$$

$$= 4u_1 v_1 \omega\, \Sigma_2\, F_0(21;21)\, \varphi_{22;0}(\omega)$$

$$+ E_1\, \Sigma_2\, G_0(22;11)\, X^{+}_{22;0}(\omega)$$

$$+ \tfrac{1}{2}\omega(u_1^2 - v_1^2)\, \Sigma_2\, G_0(22;11)\, X^{-}_{22;0}(\omega) \qquad (4.25c)$$

where

$$F_0(21;21) = -X\left(C^{j_2 1/2;\,20}_{j_1 1/2}\right)^2 \sqrt{\Omega_{j_2}/\Omega_{j_1}} \qquad (4.26)$$

$$G_0(22;11) = 2G\sqrt{\Omega_{j_1}\Omega_{j_2}} + 2X\left(C^{j_2 1/2;\,20}_{j_1 1/2}\right)^2 \sqrt{\frac{\Omega_{j_2}}{\Omega_{j_1}}} \qquad (4.27)$$

To obtain these equations we have used the relation

$$W(jj\,j'j';02) = (-1)^{j+j'}\left(4\Omega_j\,\Omega_{j'}\right)^{-1/2}$$

For the spurious level, we set $\varphi_{ii;0\omega_0} = 0$ and obtain

$$(4E_1^2 - \omega_0^2)\, X^{-}_{11;0\omega_0}$$

$$= E_1(1 - 4u_1^2 v_1^2)\, \Sigma_2\, G_0(22;11)\, X^{-}_{22;0\omega_0}$$

$$+ \tfrac{1}{2}\omega_0(u_1^2 - v_1^2)\, \Sigma_2\, G_0(22;11)\, X^{+}_{22;0\omega_0} \qquad (4.28a)$$

$$(4E_1^2 - \omega_0^2)\, X^{+}_{11;0\omega_0}$$

$$= E_1\, \Sigma_2\, G_0(22;11)\, X^{+}_{22;0\omega_0}$$

$$+ \tfrac{1}{2}\omega_0(u_1^2 - v_1^2)\, \Sigma_2\, G_0(22;11)\, X^{-}_{22;0\omega_0} \qquad (4.28b)$$

If we consider the case where the exchange terms are neglected, then

$$F_0(21;21) = 0 \qquad G_0(22;11) = 2G\sqrt{\Omega_1\Omega_2}$$

and it can be readily verified that a solution exists with $\omega_0 = 0$, and $X^- = 0$. In this case we get

$$4E_1 X^+_{11;00} = 2\Sigma_2 G\sqrt{\Omega_1\Omega_1} X^+_{22;00}$$

which leads to the identity

$$G \equiv \Sigma_j \frac{2E_j}{\Omega_j}$$

When the exchange contribution is included, the energy of the spurious level will not necessarily be zero. It is still hoped, however, that this level can be easily picked out from among the solutions which appear. It will probably still be the lowest solution. The important point, however, is that the spurious level will be orthogonal to the other 0^+ levels, so the latter will contain no components from the spurious solution.

CHAPTER V

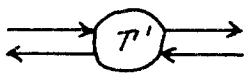
CORE POLARIZATION EFFECTS

Up to this point we have treated the nuclear core as inert, however, there are some effects in which the core plays a crucial role, two of which are especially important in this work. It should first be noted that the term "nuclear core" is intended to include both proton and neutron filled shells, as well as higher unoccupied shells which are important when hole states are being considered. The first effect arises because of the ability of the few particles or holes outside a closed shell to polarize that core. This can lead to an enhancement of electromagnetic moments and transition rates. Secondly, we note that interactions between particles or holes outside a core can take place not only directly, but also indirectly by way of particles or holes in the closed shells. This can lead to a renormalization of the interaction between external nucleons. Let us consider these two effects in more detail.

1. Renormalization of the interaction (B-61)

We recall that the interaction between particle-hole

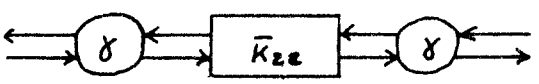
states was represented in diagram form as

$\Gamma'(41;32) \longrightarrow$ 

Let us denote by $\bar{\Gamma}'$ that part of Γ' corresponding to an interaction, via the core, between an external particle and hole. We can represent $\bar{\Gamma}'$ by the equation

$$\bar{\Gamma}'(41;32) = -i \sum_{5,6,7,8} \gamma'(61;52)\, \bar{K}_{22}(56;78)\, \gamma'(47;38)$$

or in diagram form by



Here, γ describes the interaction of an outer particle or hole with the core. If this interaction is assumed to be the same as that between external particles and holes, then γ' is just Γ'. Even if this assumption is not true, these two functions will still have the same form. In the above equation, $\bar{K}_{22}$ denotes the Green's function of the core, defined as

$$\bar{K}_{22}(56;78)$$
$$= \langle \Phi_0 |\, T \left\{ a^{+}_{\vartheta_5}(t)\, a_{\vartheta_6}(t)\, a^{+}_{\vartheta_7}(t')\, a_{\vartheta_8}(t') \right\} | \Phi_0 \rangle$$

We can rewrite the equation for $\bar{\Gamma}'$ in a representation in which the angular momentum of the particle-hole pairs

is fixed. This is the same procedure as that used in Chapter III. We thus have

$$\bar{F}'_K(41;32)$$

$$= -i \sum_{5,6,7,8} f_K(61;52)\, \bar{K}_{22}(56,K;78,K)\, f'_K(47;38) \tag{5.1}$$

where f bears the same relation to γ, as F' does to Γ'. Here,

$$\bar{K}_{22}(56,K;78,K)$$

$$= \langle \Phi_0 | T\left\{ \left(a^+_{\nu_5}(t)\, a_{\nu_6}(t)\right)_K \left(a^+_{\nu_7}(t')\, a_{\nu_8}(t')\right)_K \right\} | \Phi_0 \rangle$$

Let us restrict ourselves to the situation where the external nucleons are of one kind only, say neutrons. It is thus necessary to distinguish two cases, an interaction of the external neutrons with the proton core, and an interaction with the neutron core. Let us consider the former case first.

The only first order term contributing to $\bar{\Gamma}'$ will be of the form

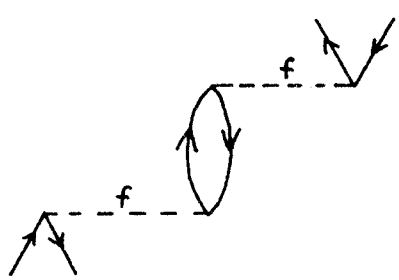


Figure 1

In this case we can write

$$f_k(12;34) = f'_k(21;34) = f''_k(24)\ f''_k(13)$$

(Note the analogous equation for $F_k(12;34)$ in Chapter III) Equation (5.1) thus simplifies to

$$\bar{F}'_K(41;32)$$

$$= -i\, f''_K(12)\left[\Sigma_{5,6,7,8}\, f''_K(6,5)\, \bar{K}_{22}(56,K;78,K)\, f''_K(7,8)\right] f''_K(4,3)$$

$$= \eta'_p\, f''_K(12)\, f''(43) = \eta'_p\, f_K(41;32) \qquad (5.2)$$

The total interaction between external holes and particles can thus be written

$$F_k(41;32) \longrightarrow F_k(41;32) + \bar{F}'_k(41;32)$$

$$= F_k(41;32) + \eta'_p\, f_k(41;32)$$

Since $f_k = F_k$ except possibly for a change in the coupling constant, we can write

$$F_k(41;32) \longrightarrow F_k(41;32) + \eta_p\, F_k(41;32)$$

$$= (1 + \eta_p)\ F_k(41;32) \qquad (5.3)$$

The presence of the proton core thus leads to a renormalization of the interaction between external nucleons.

The treatment of the neutron core is slightly more complicated because of the effect of exchange graphs. Thus, along with Figure 1, where the intermediate states now refer to neutron core particle-hole pairs, we must consider the

following processes.

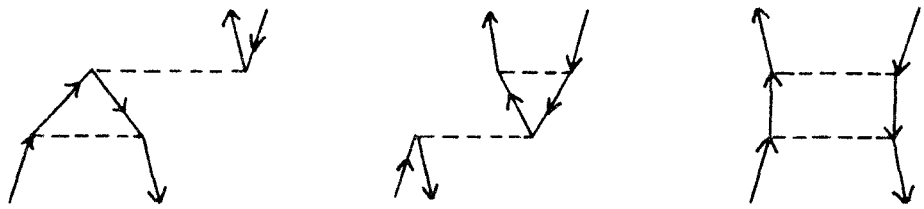


Figure 2

These allow for the scattering of external neutrons into neutron core states.

The complication arises because the functions f are no longer factorable. However, we can still write

$$\bar{F}'_K(41;32) = -i\Sigma_{5,6,7,8} f_K(61;52)\bar{K}_{22}(56,K;78,K) f'_K(47;38)$$

$$= \eta'_n f_K(41;32)$$

where

$$\eta'_n = \frac{\Sigma_{5,6,7,8} f_K(61;52)\ \bar{K}_{22}(56,K;78,K) f'_K(47;38)}{f_K(41;32)} \tag{5.4}$$

is a function of the indices 1, 2, 3, 4.

We can again write

$$F_K(41;32) \longrightarrow F_K(41;32) + \eta'_n f_K(41;32)$$

$$= (1+\eta_n)\ F_K(41;32) \tag{5.5}$$

The total renormalized interaction can then be written

$$F_K(41;32) \longrightarrow (1+\eta_n+\eta_p)\, F_K(41;32)$$

$$= (1+\eta_T)\, F_K(41;32) \qquad (5.6)$$

for the direct terms, and

$$F_K(41;32) \longrightarrow (1+\eta_n)\, F_K(41;32) \qquad (5.7)$$

for the exchange terms. We emphasize again that η_p involves a sum over proton core states, while η_n involves a sum over neutron core states.

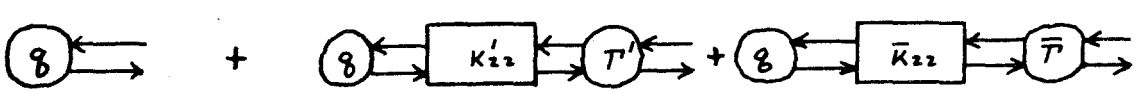
2. Effective Charge

The matrix element for the transition of an external nucleon from a state ϑ_1 to a state ϑ_2 under the action of a multipole operator $\hat{Q}_{\lambda\mu}$ can be represented by the algebraic equation

$$\langle \vartheta_1 | \hat{Q}_{\lambda\mu} | \vartheta_2 \rangle$$

$$= q(12) + \Sigma_{3,4,5,6}\, q(34)\, K_{22}(34;56)\, T'(51;62) \qquad (5.8)$$

or by the equivalent equation in diagram form



where  denotes an interaction with an external multipole field. K'_{22} and T' refer to external nucleons, while $\bar{K}_{22}$ and $\bar{T}$ refer to core nucleons.

The first and second order diagrams that contribute to the transition matrix element are shown below.

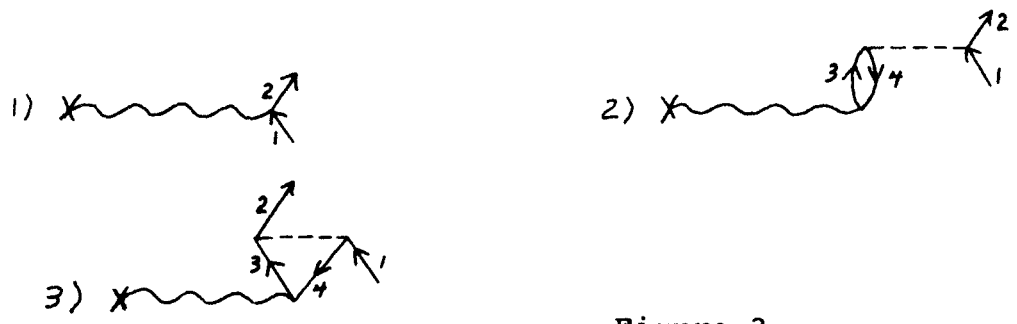


Figure 3

In general, the intermediate states formed can be composed either of external nucleons or of core nucleons. Since we are interested in neutron transitions, however, it is only when core protons are excited that there will be any contribution to the electric multipole moments. For reasons identical to those put forth in a) only processes such as 2) in Figure 3 should be included. This again means that $\bar{\bar{T}}$ is factorable. We thus have

$$\langle \vartheta_1 | \hat{Q}_{\lambda\mu} | \vartheta_2 \rangle$$

$$= \Sigma_{3,4,5,6} \; g(34) \; \bar{K}_{22}(34;56) \; \bar{\bar{T}}(51;62) \qquad (5.9)$$

Using the Wigner-Eckart theorem we can rewrite this as

$$\langle j_1 \| \hat{Q}_\lambda \| j_2 \rangle$$

$$\rightarrow \Sigma_{3,4,5,6} \; \bar{\bar{g}}_\lambda(j_3 j_4) \; K_{22}(34,K;56,K) \; \bar{F}_\lambda(51;62)$$

where the double bar over q denotes a reduced matrix element.

For the quadrupole transition rate we have

$$\bar{F}_2(51;62) = \bar{F}_2'(15;62) = \frac{\chi}{5}\langle j_5 \| r^2 Y_2 \| j_6\rangle\langle j_1 \| r^2 Y_2 \| j_2\rangle$$

$$= \frac{4\pi\chi}{25\langle r^2\rangle^2 e^2}\langle j_5 \| \hat{Q}_2 \| j_6\rangle\langle j_1 \| \hat{Q}_2 \| j_2\rangle$$

$$\therefore \langle j_1 \| \hat{Q}_2 \| j_2\rangle \longrightarrow \left(4\pi\chi / 25e^2\langle r^2\rangle^2\right)\langle j_1 \| \hat{Q}_2 \| j_2\rangle \times$$

$$\Sigma_{3,4,5,6}\langle j_3 \| \hat{Q}_2 \| j_4\rangle \bar{K}_{22}(34,2;56,2)\langle j_5 \| \hat{Q}_2 \| j_6\rangle \quad (5.10)$$

We can thus replace $\langle j_1 \| \hat{Q}_2 \| j_2\rangle$ by

$$\frac{e_{eff}}{e}\langle j_1 \| \hat{Q}_2 \| j_2\rangle$$

where

$$\frac{e_{eff}}{e} = \frac{4\pi\chi}{25e^2\langle r^2\rangle^2}\Sigma_{3,4,5,6} \times$$

$$\langle j_3 \| \hat{Q}_2 \| j_4\rangle \bar{K}_{22}(34,2;56,2)\langle j_5 \| \hat{Q}_2 \| j_6\rangle \quad (5.11)$$

where the sum is taken over all proton states.

We note that we can rewrite the expression for the effective charge in the form

$$e_{eff} = e\,\Sigma_{3,4,5,6} f(3,4)\bar{K}_{22}(34,2;56,2) f(56)$$

$$= \eta_p e \approx \frac{Z}{A}\eta_T e \quad (5.12)$$

Thus, the ratio e_{eff}/e is the same quantity as that which leads to a renormalization of the interaction by the proton core.

3. Electromagnetic Transition Rates

The electric multipole operator is defined as

$$\hat{Q}_{\lambda\mu} = \int d\tau \; r^{\lambda} Y_{\lambda\mu}(\theta, \varphi) \hat{\rho}(\vec{r}) \tag{5.13}$$

The polar coordinates are referred to a coordinate system with its origin in the nuclear mass centre and with a fixed direction of the polar axis.

We assume that the nuclear charge density is made up of point charges, in which case the density operator in the second quantized representation can be written

$$\begin{aligned} \hat{\rho}(\vec{r}) &= e \; \psi^{+}(\vec{r}) \; \psi(\vec{r}) \\ &= e \sum_{\vartheta_1 \vartheta_2} \varphi^{*}_{\vartheta_1}(\vec{r}) \varphi_{\vartheta_2}(\vec{r}) \, a^{+}_{\vartheta_1} a_{\vartheta_2} \end{aligned} \tag{5.14}$$

We can thus write $\hat{Q}_{\lambda\mu}$ in the form

$$\hat{Q}_{\lambda\mu} = e \sum_{\vartheta_1 \vartheta_2} \langle \vartheta_1 | r^{\lambda} Y_{\lambda\mu} | \vartheta_2 \rangle \, a^{+}_{\vartheta_1} a_{\vartheta_2} \tag{5.15}$$

where

$$\langle \vartheta_1 | r^{\lambda} Y_{\lambda\mu} | \vartheta_2 \rangle = \int d\tau \; \varphi^{*}_{\vartheta_1}(\vec{r}) \, r^{\lambda} Y_{\lambda\mu} \, \varphi_{\vartheta_2}(\vec{r}) \tag{5.16}$$

and the sum is taken over all the proton states in the nucleus (e is zero for neutron states).

We are interested in the operator for electric quadrupole transitions. This can be rewritten in the following way.

$$\hat{Q}_{2\mu} = e \sum_{\nu_1 \nu_2} \langle \nu_1 | r^2 Y_{2\mu} | \nu_2 \rangle a^+_{\nu_1} a_{\nu_2}$$

$$= e \sum_{l_1 l_2} \frac{\langle l_1 \| r^2 Y_2 \| l_2 \rangle}{\sqrt{2j_1+1}} \sum_{m_1 m_2} C^{j_2 m_2 ; 2\mu}_{j_1 m_1} a^+_{j_1 m_1} a_{j_2 m_2}$$

$$= e \sum_{l_1 l_2} \frac{\langle l_1 \| r^2 Y_2 \| l_2 \rangle}{\sqrt{5}} \sum_{m_1 m_2} (-1)^{j_2 - m_2} C^{j_1 m_1 ; j_2 - m_2}_{2\mu} a^+_{j_1 m_1} a_{j_2 m_2}$$

$$= (e/\sqrt{5}) \sum_{l_1 l_2} (-1)^{j_1 - j_2 + \mu} \langle l_1 \| r^2 Y_2 \| l_2 \rangle \hat{\varphi}_{2l;\kappa-\mu} \qquad (5.17)$$

We again use harmonic oscillator wave functions to evaluate the radial integral. The reduced matrix element of Y_2 is given by

$$\langle l \| Y_2 \| l_2 \rangle = \sqrt{\frac{5}{4\pi}} \sqrt{2j_2+1} \; C^{j_2 1/2 ; 20}_{j_1 1/2}$$

We thus get

$$\hat{Q}_{2\mu} = (e/\sqrt{4\pi}) \langle r^2 \rangle^{\frac{k}{2}} \times$$

$$\sum_{l_2} (-1)^{j_2 - j_1 + \mu} \sqrt{2j_2+1} \; C^{j_2 1/2 ; 20}_{j_1 1/2} \; \hat{\varphi}_{2l;2-\mu} \qquad (5.18)$$

The reduced matrix element for an electric multipole transition is defined as (P-62)

$$B(E\lambda; I_i \to I_f)$$

$$= \frac{1}{2I_i+1} \sum_{M_i M_f} \left| \langle I_f M_f | \hat{Q}_{\lambda\mu} | I_i M_i \rangle \right|^2 \qquad (5.19)$$

where a sum is taken over the spin orientations of the final

state, and an average is taken over the spin orientations of the initial state.

For the case of an electric quadrupole transition from the ground state to the 2^+ excited level, the reduced matrix element becomes

$$B(E2; 0^+ \rightarrow 2^+) = \Sigma_\mu \left| \langle 2\mu E | \hat{Q}_{2\mu} | \Phi_0 \rangle \right|^2$$

$$= \frac{e^2 \langle r^2 \rangle^2}{4\pi} \Sigma_\mu \times$$

$$\left[\Sigma_{12} \sqrt{2j_2+1} \; C^{j_2 1/2; 20}_{j_1 1/2} (-1)^{j_2 - j_1 + \mu} \langle 2\mu E | \hat{\varphi}_{21;2\mu} | \Phi_0 \rangle \right]^2$$

Since the matrix element of $\hat{\varphi}$ is independent of μ, the sum over μ just contributes a factor of 5. We thus get

$$B(E2; 0^+ \rightarrow 2^+)$$

$$= \frac{5 e^2 \langle r^2 \rangle^2}{4\pi} \times$$

$$\left[\Sigma_{12} \sqrt{2\Omega_2} \; C^{j_2 1/2; 20}_{j_1 1/2} (-1)^{j_2 - j_1} \langle 2\mu E | \hat{\varphi}_{21;2\mu} | \Phi_0 \rangle \right]^2 \quad (5.20)$$

In the case of the isotopes of lead, the collective 2^+ level is made up of neutron quasiparticles, which would normally mean that the reduced transition probability would vanish. However, as we saw in the last chapter, when core polarization effects are included, an effective charge should be used in equation (5.20) and this leads to a non-zero value for B(E2).

In Chapter IV, the problem of normalizing the amplitudes, $\varphi_{ij;\kappa\varepsilon}$, was discussed. This normalization is given in Appendix C. Using equation (C.12) we finally obtain

$$N^2 B(E2; 0^+ \rightarrow 2^+)$$

$$= \frac{5 e_{eff}^2 \langle r^2 \rangle^2}{4\pi} \Bigg[\sum_{1 \le 2} \frac{\sqrt{2\Omega_1}}{1+\delta_{12}} (-1)^{j_2 - j_1} C^{j_1 1/2; 20}_{j_2 1/2} (v_1 u_2 + v_2 u_1) \times$$

$$\Big\{ \left(u_2 v_1 X_{21;2\varepsilon} (-1)^{j_2 - j_1} + v_2 u_1 X_{12;2\varepsilon} \right)$$

$$+ 2 (u_1 u_2 + v_1 v_2) Y_{12;2\varepsilon} - 2 (u_1 u_2 - v_1 v_2) Z_{12;2\varepsilon} \Big\} \Bigg]^2 \quad (5.21)$$

where N^2 is given by (4.19).

CHAPTER VI

APPLICATION TO ISOTOPES OF LEAD

1. Choice of Parameters

In order to apply the formulas derived in the preceding chapters to the case of specific nuclei, it is necessary to know the single particle energy levels in the absence of residual interactions, ε_j, the strength of the pairing force, G, and the strength of the quadrupole force, X.

The energy levels for the single hole in Pb^{207} are known quite accurately (A-55). The strength of the pairing force is known approximately from experimental odd-even mass differences and has a value between 23/A to 30/A MeV (KK-60). The strength of the quadrupole force can be assigned roughly by choosing a value such that, in conjunction with a value of G from the above range, the energies of the 2^+ levels are given correctly. On this basis, a value of $X \approx 110/A$ MeV has been previously used to give reasonable results (KK-60), (T-61). Once these rough values have been given, both parameters can be varied within their respective ranges until the best fit to the energy levels is found.

As mentioned previously, one way in which the present work improves on earlier treatments is in the inclusion of exchange terms. In these earlier treatments the strength of the quadrupole force, found by fitting calculations to experimental results, is really the renormalized interaction as discussed in the preceding chapter. However, as was noted there, the renormalization of the direct and exchange parts of the interaction are not the same. Thus a third parameter, the ratio of the direct to exchange parts of the interaction, is introduced. We call this new parameter Y. Using the notation of the last chapter, we have

$$Y = \frac{1 + \eta_n + \eta_p}{1 + \eta_n} = \frac{1 + \eta_T}{1 + \eta_n} \qquad (6.1)$$

However, $\eta_p e$ is just the value of the effective charge. For Pb^{207} the value has been obtained from a measurement of E2 transition rates and is $\sim$ 1.15 e (T-58). If we assume that

$$\eta_T \approx (A/Z)\,\eta_p \qquad (6.2)$$

then we also have $Y \simeq 1.4$. This can be used as a rough value, with the final value again being decided upon by the agreement with experiment of the energy levels.

Once the eigenvectors for the 2^+ levels are known,

the calculation of the E2 transition rates is then a straightforward matter. Although an estimate of the effective charge could be made from the defining formulas in Chapter V (see in this regard (T-62)), it was decided to assume that the value given by Pb^{207} remains constant over the range of A values being considered. This value is probably more correct than any approximate calculations could claim to be. The agreement with experiment of the transition rates can also be used, in conjunction with the energies of the 2^+ levels, to fix the values of the parameters G, X and Y. The values of the transition rates in Pb^{204} and in Pb^{206} are known, and because of the approximations involved in deriving the theoretical estimates, the ratio of the values in these two isotopes provides a more meaningful quantity for comparison than do the values taken individually.

2. Experimental Data

It will be convenient to collect here all the data on energy levels (up to about 2 Mev) and transition rates with which we will be dealing. All the data reproduced here can be found in the Nuclear Data Sheets (N-61).

Table 1

SINGLE-PARTICLE LEVELS IN Pb^{207} (Mev)

Level	$p_{1/2}$	$f_{5/2}$	$p_{3/2}$	$i_{13/2}$	$f_{7/2}$
Energy	0	0.570	0.894	1.634	2.338

Table 2

ENERGY LEVELS OF EVEN LEAD ISOTOPES (Mev)

Level	2^+	(0^+)	3^+	(2^+)	4^+	(1^+)	(2^+)	$(4)^+$	$(7)^-$
Energy	0.803	1.15	1.341	1.47	1.684	1.78	1.82	1.998	2.200

Pb^{206}

Level	2+	4^+	4^+	?	?	?	9^-
Energy	0.899	1.274	1.563	1.816	1.945	2.065	2.186

Pb^{204}

Level	2^+	4^+	$(4)^+$	5^-	9^-
Energy	0.961	1.384	1.624	2.041	2.171

Pb^{202}

Level	2^+	?
Energy	1.027	1.489

Pb^{200}

Table 3

REDUCED TRANSITION PROBABILITIES (N-62)

$(e^2 \times 10^{-48}\ cm^4)$

Isotope	206	204
$B(E2;0^+ \longrightarrow 2^+)$	0.13	0.22

It should be noted that in Table 1, the energies of the hole states are taken to be positive with respect to the $p_{1/2}$ level. In Table 2, the spins and parities in parentheses denote values which have not been measured directly and are just 'probable.'

3. Preliminary Results

When this work was undertaken, it was intended that the calculations would be carried out on the Bendix G-15 computer in the Computing Center of this university. It was realized that the storage capacity of this unit would restrict the size of the energy matrix which could be treated and so it was decided to carry out the calculations, neglecting the effects of the $f_{7/2}$ single-particle level. As shown in Table 1, this level is well separated from the others (the $i_{13/2}$ level has opposite parity and thus does not enter) and is far enough from the Fermi surface so that its probability

of being occupied (by a hole) is close to zero (see Table4§). The effect of neglecting this level was investigated by treating different parts of the full energy matrix (including the $f_{7/2}$ level). It was found that in the case of Pb^{206}, there appeared to be no change in the energy of the 2^+ level when this extra level was included, and also, that the amplitudes involving the $f_{7/2}$ components were essentially zero. The effect is not negligible for the other isotopes, however, and for Pb^{200}, the lowest A value treated, it was estimated that the calculated value for the energy of the 2^+ level, neglecting the $f_{7/2}$ components, would be about 10% too high. An error of this magnitude will not affect the proposed test of the formalism outlined in this work, although the actual values of the parameters that will be required to give agreement with experiment will not be the same as those which would come from a treatment of the full energy matrix. However, as will be seen, the difference in the two sets of parameters should not be great.

At the time this work was begun, no satisfactory general method was known for finding the eigenvalues and eigenvectors of large non-symmetric matrices. For this reason, an approximation technique (discussed in Appendix B) was used which could extract from such a matrix, the lowest

eigenvalue and its corresponding eigenvector. This procedure, which depends on the lowest eigenvalue being well separated from the second lowest, was quite satisfactory for the study of the first 2^+ level, since this collective level is split off from the rest of the spectrum of 2^+ levels. For the other excited levels, however, this technique was found to be less satisfactory.

While these calculations were in progress, a paper was published (E-62) containing a new method for finding all the eigenvalues and eigenvectors of a non-symmetric matrix. It was then decided, on the basis of the success of the present calculations, to take advantage of the greater accuracy and completeness of this method by obtaining the use of a larger and faster computer. At the same time, this would permit the full energy matrix to be treated. Plans are now in progress for using the IBM 7090 computer at the Computing Center of the University of Toronto to carry out a thorough study of all the low-lying excited levels in the isotopes of lead. The results of this set of computations will be published at a later date.

Let us now discuss the results of the present calculations.

Figure 1 shows the variations in the energies of the first 2^+ levels as a function of $X = Y$ for two values of G. It will be noted that the separation in energy, for fixed X, of any two isotopes is roughly independent of G, but for fixed G, this separation decreases as X is increased. It is thus apparent that the correct sequence of energy separations for the isotopes being considered will be mainly dependent on X and the correct placing of the individual levels on the energy scale will then mainly be dependent upon G.

Figure 2 shows a similar variation for one value of G, but this time as a function of Y with X fixed at 0.4.

A better estimate of G itself can be obtained from a study of the levels with spin different from 2. In these cases, only the exchange contribution to the quadrupole force enters and the variation in energy of these levels as a function of the strength of this force is much less than in the case of the 2^+ levels.

The 9^- level is convenient in this regard because the high spin and odd parity reduces the number of single particle levels which can contribute. Thus, if we neglect the $f_{7/2}$ level, only the $f_{5/2}$ and $i_{13/2}$ levels need to

Energy of the first 2^+ level as a function of the strength of the quadrupole forces.

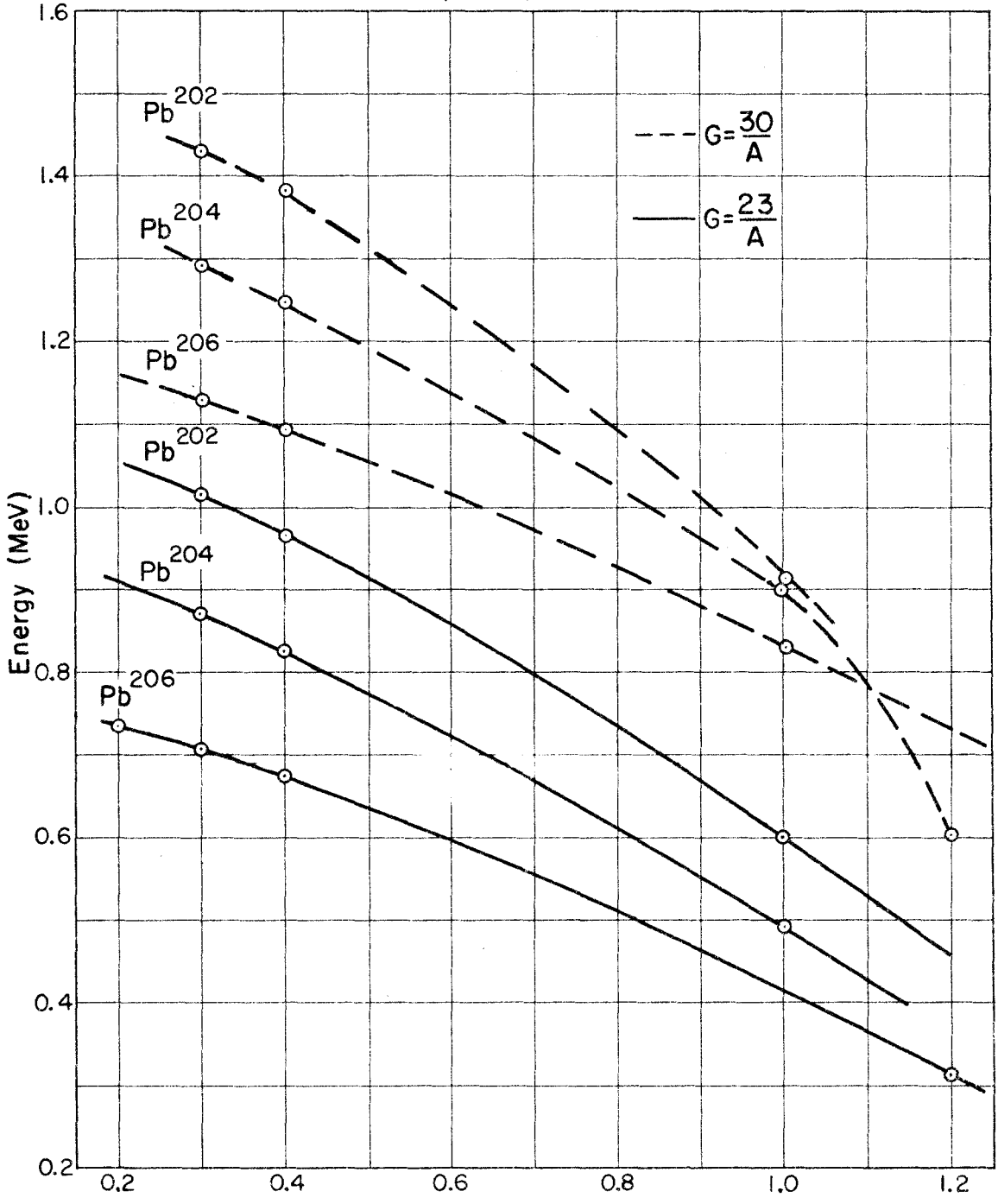


Figure 1

Energy of the first 2^+ level as a function of the relative strength of direct and exchange quadrupole forces.

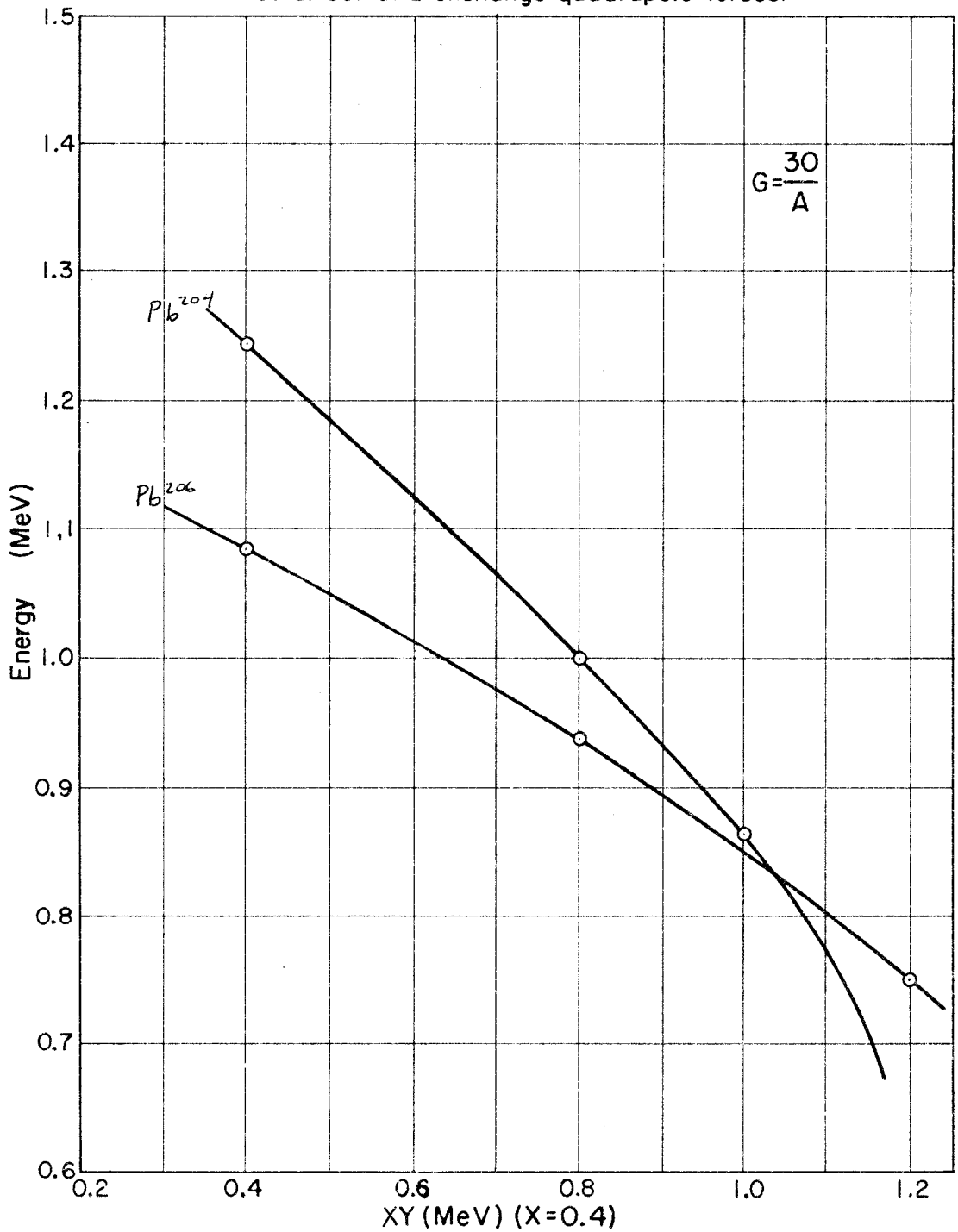


Figure 2

be considered. The experimental value for the energy of this level in Pb^{204} is 2.19 Mev. A value of $G = 28/A$ along with $X = 0.4$ yields the result 2.21 Mev and increasing X to 0.8 lowers this value by only 0.02 Mev. On the other hand, a value of $G = 27/A$ and $X = 0.4$ yields the result 2.14 Mev.

Table 4 gives the values of μ, Δ, E_j and u_j^2, corresponding to $G = 28/A$, for each of the isotopes being treated. The units for μ, Δ and E_j are Mev.

Table 4

PARAMETERS μ, Δ, E_j AND u_j^2 FOR $G = 28/A$

	Pb^{206}	Pb^{204}	Pb^{202}	Pb^{200}
μ	0.000	0.220	0.415	0.606
Δ	0.408	0.590	0.725	0.826
$E_{1/2}$	0.4080	0.6297	0.8353	1.0247
$E_{5/2}$	0.7010	0.6860	0.7414	0.8268
$E_{3/2}$	0.9827	0.8958	0.8689	0.8746
$E_{13/2}$	1.6842	1.5323	1.4183	1.3184
$E_{7/2}$	2.3732	2.1987	2.0551	1.9185
$u^2_{1/2}$	0.5000	0.3253	0.2516	0.2041
$u^2_{5/2}$	0.9066	0.7551	0.6045	0.4770
$u^2_{3/2}$	0.9549	0.8762	0.7756	0.6643
$u^2_{13/2}$	0.9851	0.9614	0.9297	0.8896
$u^2_{7/2}$	0.9926	0.9816	0.9679	0.9513

4. The First 2^+ Levels

The energies of the first 2^+ levels are shown in Figures 3 and 4 for the case $G = 28/A$. Figure 3 shows the variation in energy of these levels as the strength of the quadrupole force is varied for the case $X = Y$. Figure 4 shows a similar variation, but this time as a function of Y, with X held fixed at 0.4. The short solid horizontal lines denote the experimentally observed values. In both these cases, reasonable agreement with experiment for all the isotopes except Pb^{200} is achieved with a value of slightly greater than 0.8 for the abscissa.

Table 5 shows the amplitudes of the "two-particle" states that contribute to the 2^+ level for each of the isotopes treated and corresponding to the parameters $G = 28/A$, $X = Y = 0.8$. The most noticeable feature is that the 2^+ level becomes less collective as the closed shell of neutrons is approached. In Pb^{206}, the main contributions come from the (1/2 5/2) components, with an appreciable part also being contributed by the (5/2 5/2) components. However, for the case of Pb^{200} there are only a few "small" components, and the 2^+ level is highly collective.

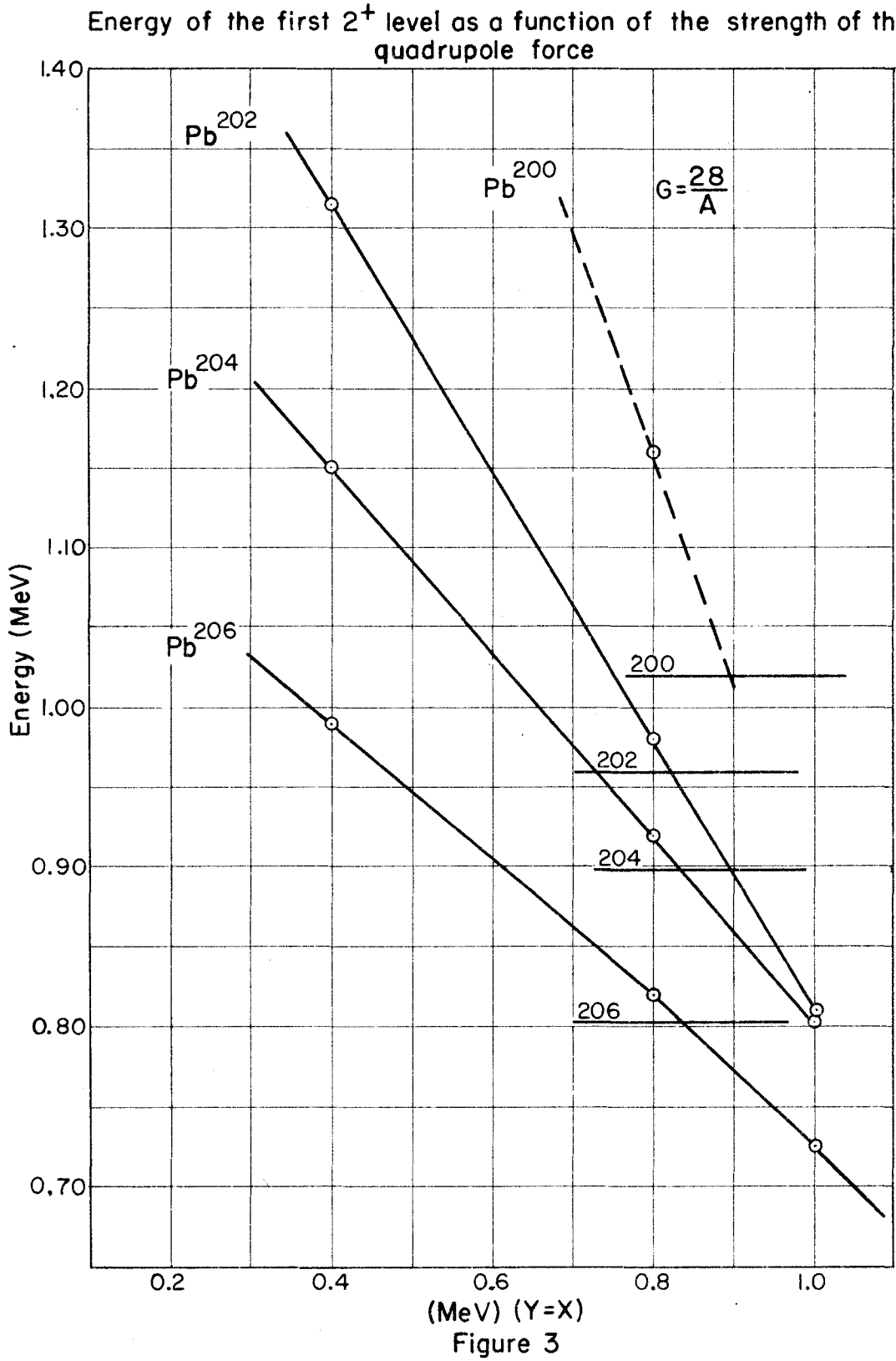

Energy of the first 2+ level as a function of the strength of the quadrupole force
1.40
1.30
1.20
1.10
1.00
0.90
0.80
0.70
Energy (MeV)
Pb202
Pb200
G=28/A
Pb204
Pb206
200
202
204
206
0.2
0.4
0.6
0.8
1.0
(MeV) (Y=X)

Figure 3

Energy of the first 2^+ levels as a function of the strength of the direct part of the quadrupole forces.

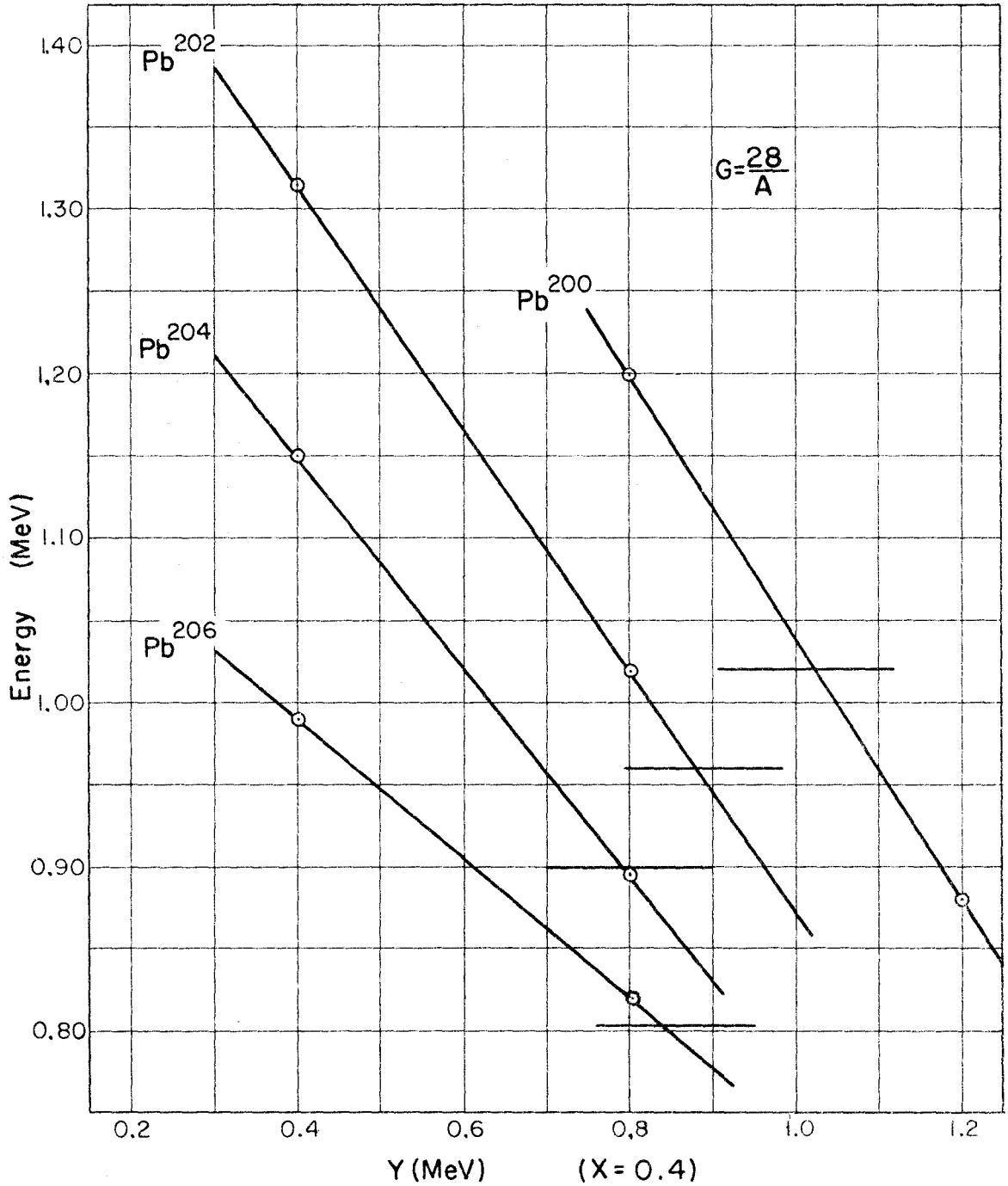


Figure 4

Table 5

AMPLITUDES OF COMPONENTS CONTRIBUTING TO THE COLLECTIVE LEVELS (G = 28/A X = Y = 0.8)

	i, j	Pb^{206}	Pb^{204}	Pb202	Pb^{200}
$\varphi_{ij;2\varepsilon}$	1/2 5/2	1.000	1.000	1.000	1.000
	1/2 3/2	- .423	-.544	-.759	-.914
	5/2 3/2	.059	.166	.404	.659
	3/2 3/2	- .047	-.128	-.365	-.736
	5/2 5/2	- .189	-.492	-.959	-1.307
	5/2 1/2	.390	.472	.563	.603
	3/2 1/2	.122	.188	.334	.423
	3/2 5/2	-.042	-.120	-.305	-.499
$x^{+}_{ij;2\varepsilon}$	1/2 5/2	.672	.127	-.232	-.558
	1/2 3/2	-.319	-.174	-.039	.200
	5/2 3/2	.166	.236	.291	.167
	3/2 3/2	-.208	-.292	-.482	-.513
	5/2 5/2	-.494	-.584	-.412	.113
$x^{-}_{ij;2\varepsilon}$	1/2 5/2	1.075	1.027	1.097	1.269
	1/2 3/2	-.384	-.457	-.672	-.926
	5/2 3/2	.163	.273	.516	.868
	3/2 3/2	-.202	-.316	-.626	-1.183
	5/2 5/2	-.576	-.968	-1.496	-2.035

Table 5[a] shows the values of the reduced transition probabilities for the 2^{+} levels in Pb^{206} and Pb^{204}. The units are $e^2 \times 10^{-48}$ cm^4. The symbol R denotes the ratio of the value for Pb^{204} to that of Pb^{206}.

Table 5[a]

E2 TRANSITION RATES ($e^2 \times 10^{-48}$ cm^4)

	Pb^{206}	Pb^{204}	R
X = Y = 0.8	.107	.253	2.36
X = 0.4 Y= 0.8	.116	.210	1.81

These results were obtained by using oscillator wave functions to evaluate $\langle r^2 \rangle^2$ and by taking $e_{eff} = 1.15e$. The B(E2) values shown in Table 5[a] agree reasonably well with the observed values, but the case $X = 0.4$ $Y = 0.8$ yields a value for the ratio R which is much closer to the observed value of 1.73.

5. The 9^- Levels

Because of the high spin and odd parity of these levels, the only contribution will come from the $f_{5/2}$ and $i_{13/2}$ single-particle levels. We are again neglecting the $f_{7/2}$ level. The justification for doing so here, is that the smallness of the off diagonal elements of the energy matrix introduces a negligible amount of coupling between the (5/2 13/2) and (7/2 13/2) components. Thus including the latter terms would not affect the energy given by the former terms alone, but of course would introduce another solution with a value nearly equal to the sum of the $f_{7/2}$ and $i_{13/2}$ quasiparticles. This energy lies above the region in which we are interested.

The following values were obtained for the energies of these levels.

Table 6

ENERGIES OF THE 9^- LEVELS

Nuclide	X(Mev)	Energy (Mev)
Pb^{204}	0.4	2.21
	0.8	2.19
Pb^{202}	0.4	2.16
	0.8	2.14

From the equations for the amplitudes of the 9^- levels, it will be observed that there are 4 positive and 4 negative solutions. The latter are unphysical, and can be dropped. The energies quoted above are, in each case, averages of the four calculated energies (none of which varied by more than ± 0.01 from the average). The multiplicity in the solutions is a result of the fact that our treatment does not conserve the number of particles, and should not be taken to indicate that four 9^- levels should be observed. It is well known that the pairing model leads to solutions for the energy which are averages over the energies of the nucleus being studied and the neighboring nuclei differing by an even number of particles.

Because the energies of the four solutions were so close in value, it was not possible, without using an unwarranted amount of computer time, to obtain the corresponding eigenvectors.

6. The 4^+ Levels

For the 4^+ levels, contributions can come from the (5/2 3/2) and (5/2 5/2) components. It seems reasonable to postulate that the two 4^+ levels in each of Pb^{204} and Pb^{202} are mainly composed of (5/2 5/2) and (5/2 3/2) components respectively. Under this assumption, the equationsof motion yield the following values for the energies.

Table 7

ENERGIES OF THE 4^+ LEVELS (Mev)

Nuclide	X	First 4^+ Level	Second 4^+ Level
Pb^{204}	0.8	1.33	1.56
Pb^{202}	0.8	1.47	1.60

The values quoted here are the smallest and largest eigenvalues of the energy matrix for each of the two nuclei treated. Because of the unfeasibility of obtaining accurate eigenfunctions, attempts made to calculate the other eigenfunctions were not successful beyond indicating that they

were close to one or the other of the values in Table 7. This clustering of solutions around two values is, of course, expected from the stucture of the energy matrix for the 4^+ levels, and is similar to the situation that occurred in the case of the 9^- levels. Table 8 shows, for Pb^{204}, the approximate values of the amplitudes of the two-particle components for each of the two solutions.

Table 8

AMPLITUDES OF THE 4^+ STATE VECTORS FOR Pb^{204}

Level	$\varphi_{5/2\,3/2}$	$\varphi_{5/2\,5/2}$	$\varphi_{5/2\,5/2}$	$x^+_{5/2\,3/2}$	$x^+_{5/2\,5/2}$	$x^-_{5/2\,3/2}$	$x^-_{5/2\,5/2}$
1.33	0.00	0.00	0.92	0.00	0.93	0.00	1.00
1.56	1.00	0.01	0.04	0.42	0.01	0.70	0.02

These values indicate that the two levels are indeed mainly (5/2 5/2) and (5/2 3/2) components respectively.

7. Concluding Remarks

Because of the slowness of convergence of the approximation technique being used in this work, attempts to calculate some of the other low-lying levels in Pb^{206} were not very successful. As mentioned earlier, further work is now in progress which will allow these levels to be treated. The success of the present calculations indicates that the formalism

presented here is indeed capable of yielding quantitative results and suggests that further calculations would be quite fruitful.

There are several interesting features which such an extended program should help to explain. For example, the energy matrix for the 4^+ levels in the case of Pb^{206} indicates that levels are expected at energies of about 1.30 and 1.61 Mev corresponding to the levels observed in Pb^{204} and Pb^{202}. However, the lowest observed 4^+ level is at an energy of 1.68 Mev. This probably corresponds to the higher of the two calculated values and the question is raised as to why the lower level has not been observed. A similar question can also be asked about the lack of a 9^- level in Pb^{206}. A possible explanation is offered by Kisslinger and Sorensen, who point out that their calculations indicate the presence of several other negative parity levels with spins less than 9^- which lie above the 9^- level in Pb^{204} and Pb^{202}, but just below it in Pb^{206}. The transition from the 9^- level in Pb^{206} could thus easily be missed.

From the energy matrix for the 4^+ levels in Pb^{200}, a solution with energy about 1.55 Mev is expected. This probably corresponds to the unlabelled level observed at 1.49 Mev.

As a final note, it might also be possible, from a careful observation of the variation in energy of the 2^+ levels as the parameters of the quadrupole force are changed, to estimate the relative strengths of the direct and exchange contributions and thus to estimate the relative polarizing effects of the closed cores of neutrons and protons.

Although we have restricted this work to a study of the isotopes of lead, the method is of course much more general in scope. We will conclude by expressing the hope that this method will prove to be helpful in the study of the low-lying levels of many other even-even spherical nuclei, and that it will help to clarify the nature of these levels.

APPENDIX A

DERIVATION OF FORMULAS INVOLVING THE INTERACTIONS

Along with equation (3.34) which expresses the interaction in terms of a rotationally-invariant particle-hole pair, there corresponds an equation expressing the interaction in terms of a rotationally invariant particle-particle or hole-hole matrix element. We thus write

$$\langle \nu_1 \nu_2 | V | \nu_3 \nu_4 \rangle$$

$$= -\sum_{K\mu} G_K(12;34)\, C^{j_1 m_1 ; j_2 m_2}_{K\mu}\, C^{j_4 m_4 ; j_3 m_3}_{K\mu} \qquad \text{(A.1)}$$

This expansion must be the same as (3.34)

$$\langle \nu_1 \nu_2 | V | \nu_3 \nu_4 \rangle$$

$$= -\sum_{K\mu} (-1)^{\mu} F'_K(12;34)(-1)^{j_4 - m_4}\, C^{j_1 m_1 ; j_4 - m_4}_{K\mu}\, (-1)^{j_3 - m_3}\, C^{j_2 m_2 ; j_3 - m_3}_{K-\mu}$$

Equating the right hand sides, and using the following relation between pairs of Clebsch-Gordan coefficients and the Racah coefficient, W (R-57)

$$C_{e\,\alpha+\beta}^{a\alpha\,;\,b\beta}\; C_{e\,\alpha+\beta}^{c\,\alpha+\beta+\delta\,;\,d-\delta}$$

$$= \Sigma_f\,(2e+1)(-1)^{a+c+e-\alpha-\delta}\,W(abcd;ef)\,C_{f\,\beta+\delta}^{b\beta\,;\,d\delta}\; C_{f\,\beta+\delta}^{c\,\alpha+\beta+\delta\,;\,a-\alpha} \qquad (A.2)$$

we get

$$F_k'(12;34) = \Sigma_l\,(-1)^{j_1+j_3+l}\,(2l+1)\,G_l(12;34)\,W(4132;kl) \qquad (A.3)$$

$$G_k'(12;34) = (-1)^{j_1+j_3+K}\,\Sigma_l\,(2l+1)\,F_l(12;34)\,W(4321;kl)$$

Let us now express the interactions T' and T'' in terms of the matrix elements F and G.

$$T'(\nu_4\nu_1\,;\,\nu_3\nu_2)$$

$$= \langle j_4 m_4\, j_1 m_1 \mid V \mid j_3 m_3\, j_2 m_2\rangle \;-\; \langle j_1 m_1\, j_4 m_4 \mid V \mid j_3 m_3\, j_2 m_2\rangle$$

$$= -\Sigma_{K\mu}\,(-1)^{\mu}\,F_K'(41;32)\,(-1)^{j_2-m_2}\,C_{K\mu}^{j_4 m_4\,;\,j_2-m_2}\,(-1)^{j_3-m_3}\,C_{K-\mu}^{j_1 m_1\,;\,j_3-m_3}$$

$$+\,\Sigma_{K\mu}\,(-1)^{\mu}\,F_K'(14;32)\,(-1)^{j_2-m_2}\,C_{K\mu}^{j_1 m_1\,;\,j_2-m_2}\,(-1)^{j_3-m_3}\,C_{K-\mu}^{j_4 m_4\,;\,j_3-m_3}$$

The second term can be rewritten as

$$\Sigma_{K\mu}\;F_K'(14;32)\,(-1)^{j_4-j_3}\;\times$$

$$\left[(-1)^{j_2-m_2}\,C_{K\mu}^{j_1 m_1\,;\,j_2-m_2}\right]\left[(-1)^{j_4-m_4}\,\overline{C}_{K\mu}^{j_3 m_3\,;\,j_4-m_4}\right]$$

and the first term can be written in the same form to give

$$\Sigma_{K\mu}\, f'_K(41;32)\, x$$

$$\left[(-1)^{j_2-m_2}\, C^{j_1 m_1;\, j_2-m_2}_{K\mu}\right]\left[(-1)^{j_4-m_4}\, C^{j_3 m_3;\, j_4-m_4}_{K\mu}\right]$$

where

$$f'_k(41;32)$$

$$= -\ \Sigma_\ell\ (2l+1)(-1)^{j_1+j_2+K+\ell}\ F'_k(41;32)\ W(1234;kl) \qquad (A.4)$$

This last relation can be derived using the same procedure as that which led to equation (A.3).

We now define

$$F_k(41;32)$$

$$= F'_k(13;42) + f'_k(41;32) \qquad (A.5)$$

We thus have

$$T'_\mu(\vartheta_4\vartheta_1;\vartheta_3\vartheta_2) = \Sigma_K\, F_K(41;32)\, x$$

$$\left[(-1)^{j_2-m_2}\, C^{j_1 m_1;\, j_2-m_2}_{K\mu}\right]\left[(-1)^{j_4-m_4}\, C^{j_3 m_3;\, j_4-m_4}_{K\mu}\right] \qquad (A.6)$$

In a similar manner, we write

$$T'_\mu(\vartheta_4-\vartheta_2;\vartheta_3-\vartheta_1) = \Sigma_K\, \bar{F}_K(42;31)(-1)^{j_1-j_2+\mu}\, x$$

$$\left[(-1)^{j_2-m_2}\, C^{j_1 m_1;\, j_2-m_2}_{K\mu}\right]\left[(-1)^{j_4-m_4}\, C^{j_3 m_3;\, j_4-m_4}_{K\mu}\right]$$

where

$$\bar{F}_k(42;31)$$

$$= (-1)^K\, F'_k(13;42) + (-1)^{j_1-j_2}\, f'_k(42;31) \qquad (A.7)$$

and

$$f'_k(42;31)$$

$$= - \Sigma_\ell \; (21 + 1)(-1)^{j_1 + j_2 + \ell} \; F'_1(42;31) \; W\,(1243;k1) \qquad \text{(A.8)}$$

Now let us consider the terms in T''

$$2\,T''(\nu_4 \nu_3 ; \nu_2 - \nu_1)$$

$$= - \Sigma_{K\mu} \; G'_K(43;21) \; C^{j_4 m_4 ; j_3 m_3}_{K-\mu} \; C^{j_1 - m_1 ; j_2 m_2}_{K-\mu}$$

$$+ \Sigma_{K\mu} \; G'_K(34;21) \; C^{j_3 m_3 ; j_4 m_4}_{K-\mu} \; C^{j_1 - m_1 ; j_2 m_2}_{K-\mu}$$

$$= \Sigma_{K\mu} \left[G'_K(34;21) - (-1)^{j_3 + j_4 + K} \, G'_K(43;21) \right]$$

$$\left[(-1)^{K+\mu} \, C^{j_4 - m_4 ; j_3 - m_3}_{K\mu} \right] \left[(-1)^{j_2 - m_2} \, C^{j_1 m_1 ; j_2 - m_2}_{K\mu} \right]$$

Let us define

$$G_k(43;21)$$

$$= G'_k(34;21) \; - \; (-1)^{j_3 + j_4 + K} \; G'_k(43;21) \qquad \text{(A.9)}$$

We can then write

$$T''_\mu(\nu_4 \nu_3 ; \nu_2 - \nu_1) \; = \; \frac{1}{2} \, \Sigma_K \; G_K(43;21)(-1)^{j_1 + m_1} \; X$$

$$\left[(-1)^{K+\mu} \; C^{j_4 - m_4 ; j_3 - m_3}_{K\mu} \right] \left[(-1)^{j_2 - m_2} \; C^{j_1 m_1 ; j_2 - m_2}_{K\mu} \right]$$

(A.10)

The value of G which corresponds to the particle-hole quadrupole force is given by

$$G'_k(43;21)$$

$$= (-1)^{j_4 + j_2 + K} \Sigma_\ell (21 + 1) F'_1(43;21) W(1243;k1) \quad (A.11)$$

We thus get

$$G_k(43;21)$$

$$= \Sigma_\ell (21 + 1)(-1)^{j_1 + j_4} \left[F'_1(12;34) W(1243;k1) + (-1)^K F'_1(21;34) W(1234;k1) \right] \quad (A.12)$$

In a similar fashion, and making use of the symmetry properties (3.43), we get

$$T''_\mu(\nu_1 - \nu_2 ; \nu_3 \nu_4) = \frac{1}{2} \Sigma_K G_K(43;21) \times \left[C^{j_3 m_3 ; j_4 m_4}_{K\mu} \right] \left[C^{j_1 m_1 ; j_2 - m_2}_{K\mu} \right] \quad (A.13)$$

APPENDIX B

THE ENERGY MATRIX

The energy matrix, formed from the set of three homogeneous equations for φ , χ^+ and χ^- can be expressed in the form

$$\left(\tilde{I}\,\omega^2 - \tilde{A}_1\,\omega - \tilde{A}_2 \right) \tilde{X} = 0 \tag{B.1}$$

where $\tilde{A}_2$ and $\tilde{A}_1$ are square matrices of order n , and $\tilde{I}$ is the unit matrix of the same order The components of the vector X are just the functions φ, χ^+ and χ^-. Let us define a new vector

$$\tilde{Y} = \omega \tilde{X}$$

We can then write (B.1) in the usual form of an eigenvalue equation

$$\begin{pmatrix} \tilde{A}_1 & -\tilde{I}\,\omega & \tilde{A}_2 \\ \tilde{I} & & -\omega\,\tilde{I} \end{pmatrix} \begin{pmatrix} \tilde{Y} \\ \tilde{X} \end{pmatrix} = 0 \tag{B.2}$$

Besides (B.1) we also need to consider the equation

$$\left(\tilde{I}\omega^2 - \tilde{A}_1^T \omega - \tilde{A}_2^T \right) \tilde{X}' = 0 \tag{B.3}$$

where $\tilde{A}^T$ denotes the transpose of $\tilde{A}$.

The equation corresponding to (B.2) is

$$\begin{pmatrix} \tilde{A}_1^T - \tilde{I}\omega & \tilde{A}_2^T \\ \tilde{I} & -\omega \tilde{I} \end{pmatrix} \begin{pmatrix} \tilde{Y}' \\ \tilde{X}' \end{pmatrix} = 0 \tag{B.4}$$

Thus to obtain the complete solution to the eigenvalue problem, we are required to find the eigenvalues of a square matrix of order $2n$ and the eigenvectors of this matrix and also of its transpose.

Let us now show that the eigenvector $\begin{pmatrix} \tilde{Y}_r' \\ \tilde{X}_r' \end{pmatrix}$ belonging to the eigenvalue w_r, is orthogonal to the eigenvector $\begin{pmatrix} \tilde{Y}_s \\ \tilde{X}_s \end{pmatrix}$ belonging to the eigenvalue ω_s

We write (B.2) as

$$\begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{I} & \tilde{O} \end{pmatrix} \begin{pmatrix} \tilde{Y}_s \\ \tilde{X}_s \end{pmatrix} = \omega_s \begin{pmatrix} \tilde{Y}_s \\ \tilde{X}_s \end{pmatrix} \tag{B.5}$$

and (B.4) as

$$(\tilde{Y}_r' \ \tilde{X}_r') \begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{I} & \tilde{O} \end{pmatrix} = \omega_r (\tilde{Y}_r' \ \tilde{X}_r') \tag{B.6}$$

Taking the scalar product of (B.5) with $(\tilde{Y}_r' \ \tilde{X}_r')$ and of (B.6) with $\begin{pmatrix} \tilde{Y}_s \\ \tilde{X}_s \end{pmatrix}$ we get

$$(\omega_r - \omega_s)(\tilde{Y}_r' \ \tilde{X}_r') \begin{pmatrix} \tilde{Y}_s \\ X_s \end{pmatrix} = 0$$

Thus if $\omega_r \neq \omega_s$, we have

$$\tilde{Y}_r' \tilde{Y}_s + \tilde{X}_r' \tilde{X}_s = (1 + \omega_r \omega_s)(\tilde{X}_r' \tilde{X}_s) = 0 \tag{B.7}$$

The orthogonality relation is thus

$$\tilde{X}_r' \tilde{X}_s = 0 \quad \text{if} \quad r \neq s \tag{B.8}$$

We note from (B.7) that if $\omega_r \omega_s = -1$ then the corresponding vectors are not necessarily orthogonal. However, since we are only interested in positive eigenvalues, this point is of no further importance.

Let us define a new matrix

$$\tilde{A} = \begin{pmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{I} & \tilde{O} \end{pmatrix}$$

Then

$$\tilde{A}\,\tilde{X}_m = \omega_m\,\tilde{X}_m$$

Let $\tilde{X}_1$, $\tilde{X}_2$ . . . . be the eigenvectors of $\tilde{A}$ associated with the eigenvalues ω_1 , ω_2 , ... respectively.

Now an arbitrary vector $\tilde{Y}_o$ can be expanded in terms of the $\tilde{X}_m$.

$$\tilde{Y}_o = a_1\,\tilde{X}_1 + a_2\,\tilde{X}_2 + \ldots + a_n\,\tilde{X}_n$$

where the a_n are constants.

Multiplying by $\tilde{A}$ we get

$$A\,\tilde{Y}_o = a_1\,\omega_1\,\tilde{X}_1 + a_2\,\omega_2\,\tilde{X}_2 + \ldots + a_n\,\omega_n\,\tilde{X}_n$$

and after k iterations we get

$$\tilde{A}^k\,\tilde{Y}_o = a_1\omega_1^k\,\tilde{X}_1 + a_2\,\omega_2^k\,\tilde{X}_2 + \ldots + a_n\,\omega_n^k\,\tilde{X}_n$$

Define $\tilde{A}^k\,\tilde{Y}_o = \tilde{Y}_k = (y_{1k}, y_{2k} \ldots y_{nk})$

$$\tilde{X}_1 = (x_{11}, x_{21}, x_{31}, \ldots x_{n1})$$

$$\tilde{X}_2 = (x_{12}, x_{22}, x_{32}, \ldots x_{n2})$$

$$\vdots \qquad\qquad \vdots$$

$$\tilde{X}_n = (x_{1n}, x_{2n}, \ldots \qquad x_{nn})$$

We can then write, for an arbitrary element of the k^{th} iterate

$$y_{ik} = a_1 x_{i1} \omega_1^k + a_2 x_{i2} \omega_2^k + \ldots a_n x_{in} \omega_n^k$$

The coefficient of ω_1^k reduces to zero only if $a_1 = 0$ or $x_{i1} = 0$, and since all x_{i1} are not zero (that is, $\tilde{X} \neq 0$) the coefficients of ω_1^k in all components can be zero only if $a_1 = 0$.

We thus see that any component of the vector $\tilde{Y}_k$ depends linearly on $\omega_1^k, \omega_2^k, \ldots \omega_n^k$.

Let us write, for such a component

$$y_k = c_1 \omega_1^k + c_2 \omega_2^k + \ldots + c_n \omega_n^k$$

Let us assume that $\tilde{Y}_o$ has been chosen so that $a_1 \neq 0$ (that is, $\tilde{Y}_o$ is not orthogonal to $\tilde{X}_1$).

If $|\omega_1| > |\omega_2| > |\omega_3| \cdots$, then

$$\frac{y_{k+1}}{y_k} = \frac{c_1 \omega_1^{k+1} + c_2 \omega_2^{k+1} + \cdots c_n \omega_n^{k+1}}{c_1 \omega_1^k + c_2 \omega_2^k + \cdots c_n \omega_n^k}$$

$$= \omega_1 \left[1 - \frac{c_2}{c_1} \frac{\omega_2^k}{\omega_1^k} \left(1 - \frac{\omega_2}{\omega_1}\right) - \frac{c_3}{c_1} \frac{\omega_3^k}{\omega_1^k} \left(1 - \frac{\omega_3}{\omega_1}\right) \cdots \right]$$

Thus, if ω_1 is sufficiently larger than ω_2

and if k is large enough, then

$$\omega_1 \simeq \frac{Y_{K+1}}{Y_K}$$

We thus see that the largest eigenvalue is approximately equal to the ratio of any two corresponding elements of two sufficiently high successive iterates of an arbitrary vector $\tilde{Y}_o$ by the matrix $\tilde{A}$.

Since we are interested in the smallest eigenvalue rather than the largest one, we can perform the following transformations:

$$\tilde{A}\,\tilde{X}_n = \omega_n\,\tilde{X}_n$$

Multiply both sides on the left by $\tilde{A}^{-1}$. We have

$$\tilde{A}^{-1}\,\tilde{A}\,\tilde{X}_n = \omega_n\,\tilde{A}^{-1}\,\tilde{X}_n$$

or $$\tilde{A}^{-1}\,\tilde{X}_n = \lambda_n X_n \quad \text{where} \quad \lambda = \frac{1}{\omega}$$

Thus, by forming iterates with $\tilde{A}^{-1}$ rather than with $\tilde{A}$, we will find the largest value of λ, which corresponds to the smallest value of ω.

This procedure also yields the components of the eigenvector corresponding to the largest eigenvalue, since the components of the k^{th} iterate, Y_k are equal, apart from a normalization factor, to the components of the

eigenvector in question. This can be seen as follows:

$$\tilde{Y}_k = \tilde{A}^k \tilde{Y}_o = \omega_1^k a_1 \tilde{X}_1 + a_2 \omega_2^k \tilde{X}_2 + \ldots.$$

$$= \omega_1^k \left[a_1 \tilde{X}_1 + \left(\frac{\omega_2}{\omega_1}\right)^k a_2 \tilde{X}_2 + \ldots. \right]$$

$$\approx a_1 \omega_1^k \tilde{X}_1$$

APPENDIX C

PROPERTIES OF THE AMPLITUDES AND NORMALIZATION OF THE STATE VECTOR

1. Time Reversal and Hermiticity Properties of the Amplitudes

We have defined the following operators:

$$\hat{\varphi}_{12;K\mu} = \sum_{m_1 m_2} (-1)^{j_2 - m_2} C^{j_1 m_1 ; j_2 - m_2}_{K\mu} a^{+}_{j_2 m_2} a_{j_1 m_1}$$

$$\hat{\chi}_{12;K\mu} = \sum_{m_1 m_2} C^{j_1 m_1 ; j_2 - m_2}_{K\mu} a_{j_2 m_2} a_{j_1 m_1}$$

$$\hat{\bar{\chi}}_{12;K\mu} = (-1)^{K+\mu} \sum_{m_1 m_2} C^{j_1 m_1 ; j_2 m_2}_{K-\mu} a^{+}_{j_2 m_2} a^{+}_{j_1 m_1} \qquad \text{(C.1)}$$

$$\hat{\chi}^{\pm}_{12;K\mu} = \hat{\bar{\chi}}_{12;K-\mu} \pm \hat{\chi}_{12;K\mu}$$

Operating on $\hat{\varphi}$ with the time-reversal operator yields the result

$$\tau \hat{\varphi}_{12;K\mu} = \sum_{m_1 m_2} (-1)^{j_2 - m_2} C^{j_1 m_1 ; j_2 - m_2}_{K\mu} a^{+}_{j_2 - m_2} a_{j_1 - m_1}$$

$$= \sum_{m_1 m_2} (-1)^{j_2 + m_2} C^{j_1 - m_1 ; j_2 m_2}_{K\mu} a^{+}_{j_2 m_2} a_{j_1 m_1}$$

$$= (-1)^{j_1 - j_2 + K} \sum_{m_1 m_2} (-1)^{j_2 - m_2} C^{j_1 m_1 ; j_2 m_2}_{K-\mu} a^{+}_{j_2 m_2} a_{j_1 m_1}$$

$$\therefore \quad \tau \hat{\varphi}_{12;K\mu} = -(-1)^{j_1 + j_2 + K} \hat{\varphi}_{12;K-\mu} \qquad \text{(C.2a)}$$

Similarly we obtain

$$\tau\ \hat{\chi}_{12;K\mu} = (-1)^{j_1+j_2+K}\ \hat{\chi}_{12;K-\mu} \tag{C.2b}$$

$$\tau\ \hat{\bar{\chi}}_{12;K\mu} = (-1)^{j_1+j_2+K}\ \hat{\bar{\chi}}_{12;K-\mu} \tag{C.2c}$$

from which we get

$$\tau\ \hat{\chi}^{\pm}_{12;K\mu} = (-1)^{j_1+j_2+K}\ \hat{\chi}^{\pm}_{12;K-\mu} \tag{C.2d}$$

The Hermitian adjoint of $\hat{\varphi}$ is given as follows:

$$\hat{\varphi}^{\dagger}_{12;K-\mu} = \sum_{m_1 m_2} (-1)^{j_2-m_2}\ C^{j_1 m_1;\, j_2-m_2}_{K\mu}\ a^{\dagger}_{j_1-m_1}\ a_{j_2-m_2}$$

$$= (-1)^{K+\mu}\ \hat{\varphi}_{21;K\mu} \tag{C.3a}$$

Similarly we obtain

$$\hat{\chi}^{\dagger}_{12;K-\mu} = (-1)^{K+\mu}\ \hat{\bar{\chi}}_{21;K\mu}$$

$$\hat{\bar{\chi}}^{\dagger}_{12;K-\mu} = (-1)^{K+\mu}\ \hat{\chi}_{21;K\mu}$$

from which we obtain

$$(\hat{\chi}^{\pm})^{\dagger}_{12;K-\mu} = \pm (-1)^{K+\mu}\ \hat{\chi}^{\pm}_{21;K\mu} \tag{C.3d}$$

2. The Evaluation of the Equal-times, Single-particle Green's Functions

$$\langle \Phi_0 | a^+_{\nu_1} a_{\nu_2} | \Phi_0 \rangle$$

$$= -i \lim_{\tau \to 0^-} -i \langle \Phi_0 | T \left\{ a_{\nu_2}(\tau)\, a^+_{\nu_1}(0) \right\} | \Phi_0 \rangle$$

$$= -i\, G_{\nu_1}(\tau)\, \delta_{\nu_1 \nu_2} \Big|_{\tau \to 0^-} = \delta_{\nu_1 \nu_2} \lim_{\tau \to 0^-} \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{i\varepsilon\tau} G_{\nu_1}(\varepsilon)\, d\varepsilon$$

$$= \delta_{\nu_1 \nu_2} \frac{1}{2\pi i} \int G_{\nu_1}(\varepsilon)\, d\varepsilon = \delta_{\nu_1 \nu_2} \int_C \frac{\varepsilon + \varepsilon_{\nu_1}}{\varepsilon^2 - E^2_{\nu_1}}\, d\varepsilon$$

$$= \delta_{\nu_1 \nu_2} \int \frac{(\varepsilon + \varepsilon_{\nu_1})\, d\varepsilon}{(\varepsilon - E_{\nu_1} + i\delta)(\varepsilon + E_{\nu_1} - i\delta)} = \frac{\varepsilon_{\nu_1} - E_{\nu_1}}{-2E_{\nu_1}}\, \delta_{\nu_1 \nu_2}$$

$$= \delta_{\nu_1 \nu_2} \frac{1}{2} \left[1 - \frac{\varepsilon_{\nu_1}}{E_{\nu_1}} \right] = v^2_{\nu_1}\, \delta_{\nu_1 \nu_2}$$

$$= v_1^2\, \delta_{j_1 j_2}\, \delta_{m_1 m_2}$$

(C.4a)

$$\langle \Phi_0 | a^+_{\nu_1} a^+_{\nu_2} | \Phi_0 \rangle$$

$$= -i \lim_{\tau \to 0^-} -i \langle \Phi_0 | T \left\{ a^+_{\nu_2}(\tau)\, a^+_{\nu_1}(0) \right\} | \Phi_0 \rangle$$

$$= -i\, \delta_{\nu_1 - \nu_2}\, F^+_{\nu_2}(\tau) \Big|_{\tau \to 0^-} = -\delta_{\nu_1 - \nu_2} \frac{\Delta_{\nu_2}}{2E_{\nu_2}}$$

$$= (-1)^{j_2 + m_2}\, u_2 v_2\, \delta_{j_1 j_2}\, \delta_{m_1 - m_2}$$

(C.4b)

Similarly

$$\langle \Phi_0 | a_{\nu_1} a_{\nu_2} | \Phi_0 \rangle = -(-1)^{j_2+m_2} u_2 v_2 \delta_{j_1 j_2} \delta_{m_1,-m_2} \quad \text{(C.4c)}$$

3. Evaluation of the Ground State Expectation Values of Four Operators

Let us evaluate, in the first approximation, the ground state expectation values of the bilinear combinations of φ, χ^+ and χ^- which appear in the expressions for the normalization condition (4.15) and for the amplitudes φ, (4.14).

Using equation (C.4) we can write

$$\langle \Phi_0 | a^+_{\nu_1} a_{\nu_2} a^+_{\nu_3} a_{\nu_4} | \Phi_0 \rangle \approx \langle \Phi_0 | a^+_{\nu_1} a_{\nu_2} | \Phi_0 \rangle \langle \Phi_0 | a^+_{\nu_3} a_{\nu_4} | \Phi_0 \rangle$$

$$- \langle \Phi_0 | a^+_{\nu_1} a^+_{\nu_3} | \Phi_0 \rangle \langle \Phi_0 | a_{\nu_2} a_{\nu_4} | \Phi_0 \rangle + \langle \Phi_0 | a^+_{\nu_1} a_{\nu_4} | \Phi_0 \rangle \times$$

$$\langle \Phi_0 | a_{\nu_2} a^+_{\nu_3} | \Phi_0 \rangle = \delta_{\nu_1 \nu_2} \delta_{\nu_3 \nu_4} v_1^2 v_3^2$$

$$+ (-1)^{j_3+m_3} (-1)^{j_4+m_4} \delta_{\nu_1,-\nu_3} \delta_{\nu_2,-\nu_4} u_3 v_3 u_4 v_4 + \delta_{\nu_1 \nu_4} \delta_{\nu_2 \nu_3} v_1^2 u_3^2$$

(C.5a)

Similarly

$$\langle \Phi_0 | a^+_{\nu_1} a_{\nu_2} a^+_{\nu_3} a^+_{\nu_4} | \Phi_0 \rangle$$

$$\approx \delta_{\nu_1 \nu_2} \delta_{\nu_3,-\nu_4} (-1)^{j_4+m_4} v_1^2 u_4 v_4 - \delta_{\nu_1,-\nu_3} \delta_{\nu_2 \nu_4} (-1)^{j_3+m_4} u_3 v_3 u_4^2$$

$$+ \delta_{\nu_1,-\nu_4} \delta_{\nu_2 \nu_3} (-1)^{j_4+m_4} u_4 v_4 u_3^2 \quad \text{(C.5b)}$$

$$\langle \Phi_0 | a^+_{\nu_1} a_{\nu_2} a_{\nu_3} a_{\nu_4} | \Phi_0 \rangle$$

$$\approx -\delta_{\nu_1\nu_2}\delta_{\nu_3-\nu_4}(-1)^{j_4+m_4} v_1^2 u_4 v_4 + \delta_{\nu_1\nu_3}\delta_{\nu_2-\nu_4}(-1)^{j_4+m_4} v_1^2 u_4 v_4$$

$$- \delta_{\nu_1\nu_4}\delta_{\nu_2-\nu_3}(-1)^{j_3+m_4} v_1^2 u_3 v_3 \qquad \text{(C.5c)}$$

Using (4.15) and (C.5a) we get

$$(-1)^{K+\mu}\Sigma_{34}(-1)^{j_3-j_4+K} X_{34;KE} \langle \Phi_0 | \hat{\varphi}_{21;K'\mu'} \hat{\varphi}_{43;K-\mu} | \Phi_0 \rangle$$

$$= (-1)^{K+\mu}\Sigma_{34}\Sigma_{m_1m_2m_3m_4}(-1)^{j_1-m_1} C^{j_2m_2;j_1-m_1}_{K'\mu'} (-1)^{j_3-m_3} C^{j_4m_4;j_3-m_3}_{K-\mu}$$

$$\times (-1)^{j_3-j_4+K} X_{34;KE} \langle \Phi_0 | a^+_{\nu_1} a_{\nu_2} a^+_{\nu_3} a_{\nu_4} | \Phi_0 \rangle$$

$$\approx 2\delta_{12}\delta_{K0}\delta_{K'0} v_1^2 \sqrt{\Omega_1}\,\Sigma_3 X_{33;KE} v_3^2 \sqrt{\Omega_3}$$

$$+ \delta_{KK'}\delta_{\mu\mu'}\left[X_{12;KE} u_1 v_1 u_2 v_2 (-1)^{j_1-j_2+K} + X_{21;KE} v_1^2 u_2^2 \right] \qquad \text{(C.6a)}$$

Similarly we get

$$(-1)^{K+\mu}\Sigma_{34}(-1)^{j_3-j_4+K} Y_{34;KE} \langle \Phi_0 | \hat{\varphi}_{21;K'\mu'} \hat{\bar{\chi}}_{43;K+\mu} | \Phi_0 \rangle$$

$$\approx 2\delta_{12}\delta_{K0}\delta_{K'0} v_1^2 \sqrt{\Omega_1}\,\Sigma_3 Y_{33;KE} u_3 v_3 \sqrt{\Omega_3}$$

$$- \delta_{KK'}\delta_{\mu\mu'} u_1 v_1 u_2^2 \left(Y_{12;KE}(-1)^{j_1-j_2+K} + Y_{21;KE} \right) \qquad \text{(C.6b)}$$

$$\left[(-1)^{K+\mu} \sum_{34} (-1)^{j_3 - j_4 + K} Y_{34;KE} \langle \Phi_0 | \hat{\Phi}_{21;K'\mu'} \hat{X}_{43;K-\mu} | \Phi_0 \rangle\right]$$

$$\approx 2\delta_{12}\delta_{K0}\delta_{K'0} v_1^2 \sqrt{\Omega_1} \sum_3 Y_{33;KE} u_3 v_3 \sqrt{\Omega_3}$$

$$- \delta_{KK'}\delta_{\mu\mu'} u_2 v_2 v_1^2 \left(Y_{12;KE}(-1)^{j_1 - j_2 + K} + Y_{21;KE}\right) \quad \text{(C.6c)}$$

$$\left[(-1)^{K+\mu} \sum_{34} (-1)^{j_3 - j_4 + K} X_{34;KE} \langle \Phi_0 | \hat{\bar{X}}_{21;K'-\mu'} \hat{\Phi}_{43;K-\mu} | \Phi_0 \rangle\right]$$

$$\approx -2\delta_{12}\delta_{K0}\delta_{K'0} u_1 v_1 \sqrt{\Omega_1} \sum_3 X_{33;KE} v_3^2 \sqrt{\Omega_3}$$

$$+ \delta_{KK'}\delta_{\mu\mu'} \left(v_2^2 u_1 v_1 (-1)^{j_1 - j_2 + K} + v_1^2 u_2 v_2 X_{21;KE}\right) \quad \text{(C.6d)}$$

$$\left[(-1)^{K+\mu} \sum_{34} (-1)^{j_3 - j_4 + K} X_{34;KE} \langle \Phi_0 | \hat{X}_{21;K'\mu'} \hat{\Phi}_{43;K-\mu} | \Phi_0 \rangle\right]$$

$$\approx 2\delta_{12}\delta_{K0}\delta_{K'0} u_1 v_1 \sqrt{\Omega_1} \sum_3 X_{33;KE} v_3^2 \sqrt{\Omega_3}$$

$$+ \delta_{KK'}\delta_{\mu\mu'} \left(u_1^2 u_2 v_2 X_{12;KE}(-1)^{j_1 - j_2 + K} + u_2^2 u_1 v_1 X_{21;KE}\right) \quad \text{(C.6e)}$$

$$\left[(-1)^{K+\mu} \sum_{34} (-1)^{j_3 - j_4 + K} Z_{34;KE} \langle \Phi_0 | \hat{\bar{X}}_{21;K'-\mu'} \hat{\bar{X}}_{43;K\mu} | \Phi_0 \rangle\right]$$

$$\approx 2\delta_{12}\delta_{K0}\delta_{K'0} u_1 v_1 \sqrt{\Omega_1} \sum_3 Z_{33;KE} u_3 v_3 \sqrt{\Omega_3}$$

$$- \delta_{KK'}\delta_{\mu\mu'} u_1 u_2 v_1 v_2 \left(Z_{12;KE}(-1)^{j_1 - j_2 + K} + Z_{21;KE}\right) \quad \text{(C.6f)}$$

$$\left[(-1)^{K+\mu} \Sigma_{34} (-1)^{j_3 - j_4 + K} Z_{34;KE} \; \langle \Phi_0 | \; \hat{\bar{\chi}}_{21;K'-\mu'} \; \hat{\chi}_{43;K-\mu} | \Phi_0 \rangle \right]$$

$$\approx -2 \delta_{12} \delta_{K0} \delta_{K'0} u_1 v_1 \sqrt{\Omega_1} \; \Sigma_3 \; Z_{33;KE} \; u_3 v_3 \sqrt{\Omega_3}$$

$$- \delta_{KK'} \delta_{\mu\mu'} v_1^2 v_2^2 \left(Z_{12;KE} (-1)^{j_1 - j_2 + K} + Z_{21;KE} \right) \tag{C.6g}$$

$$\left[(-1)^{K+\mu} \Sigma_{34} (-1)^{j_3 - j_4 + K} Z_{34;KE} \; \langle \Phi_0 | \; \hat{\chi}_{21;K'\mu'} \; \hat{\bar{\chi}}_{43;K\mu} | \Phi_0 \rangle \right]$$

$$\approx -2 \delta_{12} \delta_{K0} \delta_{K'0} u_1 v_1 \sqrt{\Omega_1} \; \Sigma_3 \; Z_{33;KE} \; u_3 v_3 \sqrt{\Omega_3}$$

$$- \delta_{KK'} \delta_{\mu\mu'} u_1^2 u_2^2 \left(Z_{12;KE} (-1)^{j_1 - j_2 + K} + Z_{21;KE} \right) \tag{C.6h}$$

Equations (C.6) can be recombined to yield the following expressions:

$$(-1)^{K+\mu} \Sigma_{34} (-1)^{j_3 - j_4 + K} X_{34;KE} \langle \Phi_0 | \; \hat{\varphi}_{12;K'\mu'} \; \hat{\varphi}_{43;K-\mu} | \Phi_0 \rangle$$

$$\approx 2 \delta_{12} \delta_{KK'} \delta_{K0} v_1^2 \sqrt{\Omega_1} \; \Sigma_3 \; X_{33;KE} \; v_3^2 \sqrt{\Omega_3}$$

$$+ \delta_{KK'} \delta_{\mu\mu'} v_2 u_1 \left[X_{21;KE} \; u_2 v_1 (-1)^{j_1 - j_2 + K} + X_{12;KE} \; v_2 u_1 \right] \tag{C.7a}$$

$$(-1)^{K+\mu} \Sigma_{34} (-1)^{j_3 - j_4 + K} Y_{34;KE} \langle \Phi_0 | \; \hat{\varphi}_{12;K'\mu'} \; \hat{\chi}^{+}_{43;K-\mu} | \Phi_0 \rangle$$

$$\approx 4 \delta_{12} \delta_{KK'} \delta_{K0} v_1^2 \sqrt{\Omega_1} \; \Sigma_3 \; Y_{33;KE} \; u_3 v_3 \sqrt{\Omega_3}$$

$$- 2 \delta_{KK'} \delta_{\mu\mu'} v_2 u_1 \left(u_1 u_2 + v_1 v_2 \right) Y_{12;KE} \tag{C.7b}$$

$$(-1)^{K+\mu} \Sigma_{34} (-1)^{j_3 - j_4 + K} Y_{34;KE} \langle \Phi_0 | \hat{\varphi}_{12;K'\mu'} \hat{\chi}^{-}_{43;K-\mu} | \Phi_0 \rangle$$

$$\approx -2 \delta_{KK'} \delta_{\mu\mu'} v_2 u_1 \left(u_1 u_2 - v_1 v_2 \right) Y_{12;KE} \qquad \text{(C.7c)}$$

$$(-1)^{K+\mu} \Sigma_{34} (-1)^{j_3 - j_4 + K} X_{34;KE} \langle \Phi_0 | \hat{\chi}^{+}_{12;K'\mu'} \hat{\varphi}_{43;K-\mu} | \Phi_0 \rangle$$

$$\approx \delta_{KK} \delta_{\mu\mu'} \left(u_1 u_2 + v_1 v_2 \right) \left[v_1 u_2 X_{21;KE} (-1)^{j_1 - j_2 + K} + u_1 v_2 X_{12;KE} \right] \qquad \text{(C.7d)}$$

$$(-1)^{K+\mu} \Sigma_{34} (-1)^{j_3 - j_4 + K} X_{34;KE} \langle \Phi_0 | \hat{\chi}^{-}_{12;K'\mu'} \hat{\varphi}_{43;K-\mu} | \Phi_0 \rangle$$

$$\approx -4 \delta_{12} \delta_{KK'} \delta_{K0} u_1 v_1 \sqrt{\Omega_1} \Sigma_3 X_{33;KE} v_3^2 \sqrt{\Omega_3}$$

$$- \delta_{KK'} \delta_{\mu\mu'} \left(u_1 u_2 - v_1 v_2 \right) \left[v_1 u_2 X_{21;KE} (-1)^{j_1 - j_2 + K} + u_1 v_2 X_{12;KE} \right] \qquad \text{(C.7e)}$$

$$(-1)^{K+\mu} \Sigma_{34} (-1)^{j_3 - j_4 + K} Z_{34;KE} \langle \Phi_0 | \hat{\chi}^{+}_{12;K'\mu'} \hat{\chi}^{+}_{43;K-\mu} | \Phi_0 \rangle$$

$$\approx -2 \delta_{KK'} \delta_{\mu\mu'} \left(u_1 u_2 + v_1 v_2 \right)^2 Z_{12;KE} \qquad \text{(C.7f)}$$

$$(-1)^{K+\mu} \Sigma_{34} (-1)^{j_3 - j_4 + K} Z_{34;KE} \langle \Phi_0 | \hat{\chi}^{-}_{12;K'\mu'} \hat{\chi}^{-}_{43;K-\mu} | \Phi_0 \rangle$$

$$\approx 8 \delta_{12} \delta_{KK'} \delta_{K0} u_1 v_1 \sqrt{\Omega_1} \Sigma_3 Z_{33;KE} u_3 v_3 \sqrt{\Omega_3}$$

$$+ 2 \delta_{KK'} \delta_{\mu\mu'} \left(u_1 u_2 - v_1 v_2 \right)^2 Z_{12;KE} \qquad \text{(C.7g)}$$

$$(-1)^{K+\mu} \Sigma_{34} (-1)^{j_3 - j_4 + K} Z_{34;KE} \langle \Phi_0 | \hat{X}^{+}_{12;K'\mu'} \hat{X}^{-}_{43;K\mu} | \Phi_0 \rangle$$

$$\approx -2 \delta_{KK'} \delta_{\mu\mu'} \left(u_1^2 u_2^2 - v_1^2 v_2^2 \right) Z_{12;KE} \qquad \text{(C.7h)}$$

$$(-1)^{K+\mu} \Sigma_{34} (-1)^{j_3 - j_4 + K} Z_{34;KE} \langle \Phi_0 | \hat{X}^{-}_{12;K'\mu'} \hat{X}^{+}_{43;K-\mu} | \Phi_0 \rangle$$

$$\approx 2 \delta_{KK'} \delta_{\mu\mu'} \left(u_1^2 u_2^2 - v_1^2 v_2^2 \right) Z_{12;KE} \qquad \text{(C.7i)}$$

Equation (4.14) can now be written

$$\varphi_{12;KE}$$

$$= 2 \delta_{12} \delta_{K0} v_1^2 \sqrt{\Omega_1} \Sigma_3 \left[X_{33;KE} v_3^2 + 2 Y_{33;KE} u_3 v_3 \right] \sqrt{\Omega_3}$$

$$+ v_2 u_1 \left[X_{21;KE} u_2 v_1 (-1)^{j_1 - j_2 + K} + X_{12;KE} v_2 u_1 \right]$$

$$+ 2 v_2 u_1 \left[Y_{12;KE} (u_1 u_2 + v_1 v_2) - Z_{12;KE} (u_1 u_2 - v_1 v_2) \right] \qquad \text{(C.8)}$$

However, we are more interested in the quantity

$$\Sigma_{12} \sqrt{2\Omega_1} \; C^{j_1 \, 1/2 \, ; \, 20}_{j_2 \, 1/2} \; (-1)^{j_1 - j_2} \varphi_{12;2E} \qquad \text{(C.9)}$$

which appears in the expression for the quadrupole transition rate

We write (C.9) in the form

$$\Sigma_{1 \le 2} \frac{\sqrt{2\Omega_1}}{1 + \delta_{12}} \; C^{j_1 \, 1/2 \, ; \, 20}_{j_2 \, 1/2} \left[(-1)^{j_1 - j_2} \varphi_{12;2E} + \varphi_{21;2E} \right] \qquad \text{(C.10)}$$

Substituting (C.8) in this expression we get

$$\Sigma_{1\le 2}\frac{\sqrt{2\Omega_1}}{1+\delta_{12}}(-1)^{j_1-j_2}C^{j_1\,1/2;\,20}_{j_2\,1/2}\left(v_1u_2+v_2u_1\right)\times$$
$$\Big[\left(u_2v_1X_{21;2E}(-1)^{j_1-j_2}+v_2u_1X_{12;2E}\right)$$
$$+2\left(u_1u_2+v_1v_2\right)Y_{12;2E}-2\left(u_1u_2-v_1v_2\right)Z_{12;2E}\Big] \qquad \text{(C.11)}$$

The normalization condition can be written

$$N^2=\delta_{KK'}\delta_{\mu\mu'}\Sigma_{12}\times$$
$$\Big[X_{12;K'E}\Big\{2\delta_{12}\delta_{K0}v_1^2\sqrt{\Omega_1}\,\Sigma_3X_{33;KE}v_3^2\sqrt{\Omega_3}$$
$$+v_2u_1\left(X_{21;KE}u_2v_1(-1)^{j_1-j_2+K}+X_{12;KE}v_2u_1\right)$$
$$-+\delta_{12}\delta_{K0}v_1^2\sqrt{\Omega_1}\,\Sigma_3Y_{33;KE}u_3v_3\sqrt{\Omega_3}$$
$$+2v_2u_1\left(u_1u_2+v_1v_2\right)Y_{12;KE}$$
$$-2v_2u_1\left(u_1u_2-v_1v_2\right)Z_{12;KE}\Big\}$$
$$+Y_{12;K'E}\Big\{\left(u_1u_2+v_1v_2\right)\left(v_1u_2X_{21;KE}(-1)^{j_1-j_2+K}+u_1v_2X_{12;KE}\right)$$
$$+2\left(u_1u_2+v_1v_2\right)^2Y_{12;KE}$$
$$-2\left(u_1^2u_2^2-v_1^2v_2^2\right)Z_{12;KE}\Big\}$$
$$+Z_{12;K'E}\Big\{-4\delta_{12}\delta_{K0}u_1v_1\sqrt{\Omega_1}\,\Sigma_3X_{33;KE}v_3^2\sqrt{\Omega_3}$$
$$-\left(u_1u_2-v_1v_2\right)\left(v_1u_2X_{21;KE}(-1)^{j_1-j_2+K}+u_1v_2X_{12;KE}\right)$$
$$-2\left(u_1^2u_2^2-v_1^2v_2^2\right)Y_{12;KE}$$
$$+8\delta_{12}\delta_{K0}u_1v_1\sqrt{\Omega_1}\,\Sigma_3Z_{33;KE}u_3v_3\sqrt{\Omega_3} \qquad \text{(C.13)}$$
$$+2\left(u_1u_2-v_1v_2\right)^2Z_{12;KE}\Big\}\Big]$$

Collecting terms we obtain

$$N^2 = \delta_{KK'}\delta_{\mu\mu'} \sum_{1\le 2} \frac{1}{1+\delta_{12}} \times$$

$$\Big[4\delta_{12}\delta_{K0} X_{12;KE} v_1^2 \sqrt{\Omega_1} \sum_3 (X_{33;KE} v_3^2 - 2 Y_{33;KE} u_3 v_3)\sqrt{\Omega_3}$$

$$- 8\delta_{12}\delta_{K0} Z_{12;KE} u_1 v_1 \sqrt{\Omega_1} \sum_3 (X_{33;KE} v_3^2 + 2 Z_{33;KE} u_3 v_3)\sqrt{\Omega_3}$$

$$+ 2\Big\{ X_{12;KE} v_2 u_1 + X_{21;KE} v_1 u_2 (-1)^{j_1 - j_2 + K}$$

$$+ 2(u_1 u_2 + v_1 v_2) Y_{12;KE} - 2(u_1 u_2 - v_1 v_2) Z_{12;KE} \Big\}^2 \Big] \qquad \text{(C.14)}$$

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