AN EXTENSION PROBLEM IN O-MINIMALITY

AN ALMOST EVERYWHERE EXTENSION THEOREM FOR CONTINUOUS DEFINABLE FUNCTIONS IN AN O-MINIMAL STRUCTURE

By

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Abstract

Let $\mathcal{R} = (R, <, \mathcal{S})$ be an o-minimal expansion of an ordered group. In this thesis, we define the class $\mathcal{C}$ of *asymptotically monotone* cells and we show they have the property that, for any cell $C \in \mathcal{C}$ and for any definable, continuous, bounded function $f : C \to R$, it is always possible to continuously extend f "almost everywhere" to the frontier of C. We make this notion precise using a theory of dimension for sets definable in an o-minimal structure. This result is a generalization of a known fact about continuous extensions of definable, continuous, bounded functions on open cells; we show by way of counterexample that the original result does not generalize to the class of all cells and hence that the assumption that our cells are asymptotically monotone is required. Background on o-minimality and the theory of dimension for definable sets is provided.

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Chapter 1

Introduction and preliminaries

O-minimality is the study of ordered structures, in the sense of first-order logic, whose definable subsets of one variable have finitely many connected components. Remarkably, this property turns out to imply that *all* definable subsets have finitely many connected components. This fact allows for the development of a "tame" theory of topology within the framework of such ordered structures. Naturally, one would expect that these tameness results stemming from o-minimality have numerous applications to other areas of mathematics; indeed, o-minimality has been successfully applied, for instance, to Tarski's problem of the decidability of the real field with the exponential function (as in [1]), as well as to long-standing questions arising from Diophantine geometry (see, e.g., [6]). The study of o-minimality and related conditions emerged from model theory via the study of the theory of dense linear orders and the definable subsets of its models. One of the initial motivations for studying o-minimal structures, however, was to find a suitable generalization of semialgebraic and subanalytic geometry and hence it is possible to view o-minimality as a subject in its own right, without explicit reference to its place within the vast sea of model theory – this is the approach we take in our development of the theory of o-minimal structures.

We begin by recalling the basic definitions and main results of o-minimality. Almost nothing in this chapter will be proven; we refer the reader to [2, Chapters 1-4] for the relevant details.

1.1 O-minimal structures

Throughout this paper, $\mathbb{N}$ will denote the set of non-negative integers. Given a function $f : X \to Y$ between two sets X and Y, we denote by $\mathrm{gr}(f) := \{(x, f(x)) : x \in X\}$ the *graph* of f, viewed as a subset of the cartesian product $X \times Y$.

Definition 1.1. Let R be a non-empty set. A *structure* on R is a sequence $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ such that for each $n \geq 0$:

(S1) $\mathcal{S}_n$ is a boolean algebra of subsets of R^n, i.e. $\mathcal{S}_n$ is a collection of subsets of R^n which is closed under the operations of taking finite unions and complementation.

(S2) If $A \in \mathcal{S}_n$, then $R \times A$ and $A \times R$ belong to $\mathcal{S}_{n+1}$.

(S3) The set $\{(x_1, \dots, x_n) \in R^n : x_1 = x_n\}$ belongs to $\mathcal{S}_n$.

(S4) If $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$ where $\pi : R^{n+1} \to R^n$ is the projection

$$(x_1, \dots, x_n, x_{n+1}) \mapsto (x_1, \dots, x_n)$$

onto the first n coordinates.

In this case we also say that $(R, \mathcal{S})$ is a structure. Given a structure $\mathcal{S}$ on R and a set $A \subseteq R^n$, we say that A is *definable in* $\mathcal{S}$ (or just *definable* if $\mathcal{S}$ is clear from context) if $A \in \mathcal{S}$. We say a function $f : R^n \to R$ is *definable in* $\mathcal{S}$ if $\mathrm{gr}(f) \subseteq R^{n+1}$ is definable.

Axioms (S1), (S2) and (S4) in the definition provide a correspondence between the above set-theoretic operations and the standard logical operations of disjunction, negation, existential quantification and the operation of adding free variables, while axiom (S3) provides us with a notion of equality. We shall be interested exclusively in structures defined on sets which are equipped with a linear order.

Definition 1.2. Let R be a non-empty set with a linear order $<$ such that R has no endpoints (that is, R has no maximal or minimal element with respect to $<$). We say the linearly ordered set $(R, <)$ is *dense* if, whenever $a, b \in R$ satisfy $a < b$, there exists $c \in R$ such that $a < c < b$. Add two endpoints $-\infty, +\infty$ such that $-\infty < a < +\infty$ for all $a \in R$. An *interval* is a subset of R of the form

$$(a, b) := \{x \in R : a < x < b\}$$

where $-\infty \leq a < b \leq +\infty$. For us, an interval will always refer to a non-empty "open" interval as above; that is, sets of the form $[a, b)$ with $-\infty < a < b \leq +\infty$, $(a, b]$ with $-\infty \leq a < b < +\infty$, or $[a, b]$ with $-\infty < a \leq b < +\infty$ are not considered intervals.

Given a non-empty, densely linearly ordered set $(R, <)$, we can equip R with the interval topology by declaring that the intervals form a base. We then equip each cartesian product R^n with the corresponding product topology. The base for the product topology is given by the collection of all boxes in R^n, where a *box* in R^n is a cartesian product of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$ where (a_i, b_i) is an interval for each $i = 1, \ldots, n$. One can check that R^n is Hausdorff under this topology. Given a set $A \subseteq R^n$, its topological closure is denoted by $\mathrm{cl}(A)$ and its topological interior in R^n is denoted by $\mathrm{int}(A)$.

Definition 1.3. Let $(R, <)$ be a non-empty, densely linearly ordered set without endpoints. An *o-minimal structure* on $(R, <)$ is a structure $\mathcal{S}$ on R such that:

(O1) The set $\{(x, y) \in R^2 : x < y\}$ belongs to $\mathcal{S}_2$.

(O2) The sets in S_1 are exactly the finite unions of points and intervals.

In this case, we also say that $(R, <, \mathcal{S})$ is an o-minimal structure.

Axiom (O1) allows us to define subsets of R^n using the dense linear order $<$, while axiom (O2) – often referred to as the *o-minimality axiom* – says that the definable subsets of R are precisely the ones obtained from quantifier-free first-order formulas involving $=$ and $<$ (that is, from quantifier-free formulas in the language of dense linear orders). We will see later that the o-minimality axiom has many significant consequences which, for instance, eventually allow us to decompose definable sets into finitely many definable subsets of a "nice" form.

Example 1.4. (i) (See [3, Example 1.2].) Let **alg** be the structure on the ordered set of real numbers $(\mathbb{R}, <)$ containing the set $\{(x, y) \in \mathbb{R}^2 : x < y\}$, each set of the form $\{r\}$ for $r \in \mathbb{R}$, and the graphs of $+ : \mathbb{R}^2 \to \mathbb{R}$ and $\cdot : \mathbb{R}^2 \to \mathbb{R}$, viewed as subsets of $\mathbb{R}^3$. Then one easily verifies that the subsets of the form

$$\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \ldots, g_m(x) > 0\}$$

are definable, where $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ are real polynomials in n variables. Finite unions of sets of this form are called *semialgebraic sets*; hence the semialgebraic sets are definable in **alg**. It is a deep theorem, however, that the semialgebraic sets are *exactly* the sets definable in **alg**. So any subset A of $\mathbb{R}^1$ definable in **alg** is a semialgebraic set and so A must be a finite union of points and intervals in $\mathbb{R}$, which shows that the structure $\mathbb{R}_{\mathbf{alg}} := (\mathbb{R}, <, \mathbf{alg})$ is o-minimal.

(ii) Let **sine** be the structure on $(\mathbb{R}, <)$ given by **alg**, together with the graph of the sine function $\sin : \mathbb{R} \to \mathbb{R}$. Then $(\mathbb{R}, <, \mathbf{sine})$ is not o-minimal, because the set

$$\{x \in \mathbb{R} : \sin(\pi x) = 0\}$$

is definable and is an infinite union of points in $\mathbb{R}$. In general, any structure in which the integers are definable cannot be o-minimal.

(iii) (See [3, Example 1.6].) Denote by $I := [-1, 1]$ the closed unit interval in $\mathbb{R}$. Let **an** be the structure on $(\mathbb{R}, <)$ given by **alg**, together with the graphs of all the *restricted analytic functions*, that is, the graphs of all the functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $f \restriction_{I^n}$ is analytic and f is identically zero outside of I^n. Then the structure $\mathbb{R}_{\mathbf{an}} := (\mathbb{R}, <, \mathbf{an})$ turns out to be o-minimal. This fact is highly non-trivial and uses results from subanalytic geometry; see, for instance, [5] for a detailed treatment of the theory of subanalytic sets.

For the rest of this chapter, we fix an o-minimal structure $\mathcal{S}$ on $(R, <)$. Let

$$R_\infty := R \cup \{-\infty, +\infty\}.$$

We begin with some basic facts about o-minimal structures which we will use repeatedly, without mention.

Lemma 1.5. (i) *If $A \subseteq R$ is definable, then* $\inf(A)$ *and* $\sup(A)$ *exist in R_∞ (where* $\inf(A)$ *and* $\sup(A)$ *denote the greatest lower bound of A and the least upper bound of A, respectively.)*

(ii) *If $A \subseteq R^n$ is definable, then so are* $\mathrm{cl}(A)$ *and* $\mathrm{int}(A)$.

(iii) *If $A \subseteq B \subseteq R^n$ are definable and A is open in B, then there is a definable open set $U \subseteq R^n$ such that $U \cap B = A$.*

The usual notion of topological connectedness has a definable analogue:

Definition 1.6. A definable set $A \subseteq R^n$ is *definably connected* if A is not the union of two disjoint, non-empty, definable open subsets of A. A *definably connected component* of a non-empty definable set $X \subseteq R^n$ is a maximal definably connected subset of X.

Note that *a priori* is it not evident that the definably connected components of a definable set are definable; this issue will be addressed in the next section.

Lemma 1.7. (i) *The definably connected subsets of R are precisely the empty set, the intervals, and the sets of the form $(a, b]$, $[a, b)$ and $[a, b]$ described in Definition 1.2.*

(ii) *The image $f(A)$ of a definably connected set $A \subseteq R^n$ under a definable continuous function $f : A \to R^m$ is definably connected; in particular, definable continuous functions have the intermediate value property.*

(iii) *If A and B are definably connected subsets of R^n and $A \cap B \neq \varnothing$, then $A \cup B$ is definably connected.*

(iv) *If A is definably connected, then so is* $\mathrm{cl}(A)$.

For the next result, recall that an *ordered group* $(G, <, +)$ is a group $(G, +)$ equipped with a linear order $<$ such that, for all $x, y, z \in G$,

$$x < y \implies z + x < z + y \text{ and } x + z < y + z.$$

The following result heavily determines the nature of groups definable in an o-minimal structure:

Proposition 1.8. *Let $(R, <, \mathcal{S})$ be an o-minimal structure and suppose $\mathcal{S}$ contains group operations $0 : R^0 \to R$ and $+ : R^2 \to R$ such that $(R, <, 0, +)$ is an ordered group. Then the group $(R, +)$ is abelian, divisible and torsion-free.*

We also recall that an *ordered ring* $(G, <, 0, 1, +, \cdot)$ is an associative ring with unity equipped with a linear order $<$ such that, for all $x, y, z \in G$,

(i) $0 < 1$,

(ii) $x < y$ implies $x + z < y + z$ (and hence $z + x < z + y$ by the above proposition), and

(iii) $x < y$ and $z > 0$ implies $x \cdot z < y \cdot z$.

An *ordered field* is then an ordered ring G as above with commutative multiplication, with the additional property that for each $0 \neq x \in G$ there is a $y \in G$ such that $x \cdot y = 1$.

Definition 1.9. Let $(R, <, \mathcal{S})$ be an o-minimal structure. We say that $(R, <, \mathcal{S})$ *expands an ordered group* if there are group operations $0 : R^0 \to R, + : R^2 \to R$ and $- : R \to R$ definable in $\mathcal{S}$ such that $(R, <, 0, +, -)$ is an ordered group. In this case we also say $(R, <, \mathcal{S})$ *expands the ordered group* $(R, <, 0, +, -)$. Similarly, we say that $(R, <, \mathcal{S})$ *expands an ordered ring (resp. field)* if there are definable ring operations $0, 1 : R^0 \to R$, $+, \cdot : R^2 \to R$ and $- : R \to R$ such that $(R, <, 0, 1, +, -, \cdot)$ is an ordered ring (resp. field). In this case we say $(R, <, \mathcal{S})$ *expands the ordered ring (resp. field)* $(R, <, 0, 1, +, -, \cdot)$.

1.2 The cell decomposition theorem

Recall that, given two linearly ordered sets $(R_1, <)$ and $(R_2, \prec)$ and a function $f : R_1 \to R_2$, we say that f is *strictly increasing* if $x < y$ in R_1 implies $f(x) \prec f(y)$ in R_2, and we say that f is *strictly decreasing* if $x < y$ in R_1 implies $f(y) \prec f(x)$ in R_2; f is *strictly monotone* if f is either strictly increasing or strictly decreasing.

Theorem 1.10 (Monotonicity)**.** *Let $f : (a, b) \to R$ be a definable function on the interval $(a, b) \subseteq R$, where $a < b$. Then there is a finite set of points $\{a_1, \ldots, a_k\}$ in R such that*

$$a =: a_0 < a_1 \cdots < a_k < a_{k+1} := b$$

and, for each $j = 0, \ldots, k$, the restriction of f to the subinterval (a_j, a_{j+1}) is either constant, or strictly monotone and continuous.

The monotonicity theorem says that functions definable in an o-minimal structure are piecewise "well-behaved," and easily implies the following result, which yields a quick proof that if a structure is o-minimal, then the usual trigonometric functions are not definable. (More generally, any periodic function cannot be definable.) First, given a function $f : X \to H$ from a set X into a Hausdorff space H together with a point $p \in \mathrm{cl}(X \setminus \{p\})$, we define $\lim_{x \to p} f(x)$ to be the (necessarily unique) point $q \in H$ such that, for every neighbourhood V of q, there is a neighbourhood U of p such that

$$f(U \cap (X \setminus \{p\})) \subseteq V.$$

The point q is the *limit* of $f(x)$ as x approaches p. Similarly, we can define the one-sided limits $\lim_{x\to p^-} f(x)$ and $\lim_{x\to p^+} f(x)$ when f is a function of one variable.

Corollary 1.11. *Let $f : (a, b) \to R$ be definable. Then the limits $\lim_{x\to b^-} f(x)$ and $\lim_{x\to a^+}$ exist in R_∞.*

Our goal in this section is to state the fundamental theorem of o-minimality – the cell decomposition theorem – which says that it is possible to decompose definable sets into finitely many "cells," each of which is a definable set of a particularly nice form. This theorem forms the basis of much of the machinery of o-minimality; the reader is encouraged to peruse through the latter chapters of [2] to witness some of its uses.

Let $X \subseteq R^n$ be a definable set and denote by $C(X)$ the set of all continuous definable functions $f : X \to R$. Given $f, g \in C(X)$, we write $f < g$ if $f(x) < g(x)$ for all $x \in X$. We also allow for the possibility that $f = -\infty$ or $g = +\infty$, where we regard $-\infty$ and $+\infty$ as constant functions on X, and so we set

$$C_\infty(X) := C(X) \cup \{-\infty, +\infty\}.$$

Given two functions $f, g \in C_\infty(X)$, we define the *interval between f and g above X* to be the (definable) set

$$(f, g)_X := \{(x, r) \in X \times R : f(x) < r < g(x)\}.$$

We also refer to such a set as an *interval of functions* when f, g and X are clear from context. It is a straightforward exercise to check that if X is definably connected then $(f, g)_X$ is also definably connected, provided that f and g are continuous and definable.

Definition 1.12. Let $(i_1, \ldots, i_n)$ be a sequence of n zeros and ones. An $(i_1, \ldots, i_n)$-*cell* is a definable subset of R^n obtained by induction on $n \geq 1$, as follows:

(i) A (0)-cell is a point $\{r\}$ in R; a (1)-cell is an interval $(a, b) \subseteq R$.

(ii) Suppose the class of all $(i_1, \ldots, i_n)$-cells has been defined. An $(i_1, \ldots, i_n, 0)$-cell is the graph $\mathrm{gr}(f) \subseteq R^{n+1}$ of a function $f \in C(X)$ defined on an $(i_1, \ldots, i_n)$-cell

$X \subseteq R^n$. An $(i_1, \ldots, i_n, 1)$-cell is an interval of functions $(f, g)_X$ where $X \subseteq R^n$ is an $(i_1, \ldots, i_n)$-cell, $f, g \in C_\infty(X)$ and $f < g$.

A *cell* in R^n is an $(i_1, \ldots, i_n)$-cell for some sequence $(i_1, \ldots, i_n) \in \{0, 1\}^n$. If C is $(i_1, \ldots, i_n)$-cell with $i_j = 1$ for all $j = 1, \ldots, n$, then we call C an *open cell* in R^n.

For instance, the cells in R^2 consist of: the graphs of continuous definable functions $f : I \to R$ where I is a definable interval in R; the intervals $(f, g)_I$ where $f, g \in C_\infty(I)$ and $f < g$; the points $\{(r, s)\} \subseteq R^2$; and the "vertical intervals" of the form $\{a\} \times R$, where $a \in R$. Also, note that a box in R^n is an $(i_1, \ldots, i_n)$-cell with $i_j = 1$ for all $j = 1, \ldots, n$.

Proposition 1.13. (i) *The union of finitely many non-open cells in R^n has empty interior.*

(ii) *Let $C \subseteq R^n$ be a cell. Then C is open as a subset of its closure* $\mathrm{cl}(C)$.

(iii) *Let $C \subseteq R^n$ be an $(i_1, \ldots, i_n)$-cell, and let $i := i_1 + \cdots + i_n$. Then there exists a coordinate projection $\pi : R^n \to R^i$ such that $\pi(C)$ is an open cell in R^i.*

(iv) *If C is a cell in R^n then $\Pi_{n-1}(C)$ is a cell in R^{n-1}, where $\Pi_{n-1} : R^n \to R^{n-1}$ is the projection map onto the first $n - 1$ coordinates.*

(v) *Each cell is definably connected.*

The coordinate projection described in part (iii) of the proposition is a definable homeomorphism when restricted to C, and we refer to it as the *canonical projection* associated to C. We also note that, from now on, we will always denote the projection map

$$R^n \to R^{n-1} : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$$

by Π_{n-1}, as in part (iv) of the above proposition.

In addition to part (v) of Proposition 1.13 we also need another "connectedness" property of cells which is not found in [2], and so we state and prove it here for convenience. First, we say that a definable set $A \subseteq R^n$ is *locally connected* at $x \in R^n$ if, for every open neighbourhood U of x, there is an open neighbourhood V of x such that $V \subseteq U$ and $V \cap A$ is definably connected.

Lemma 1.14. *Let $C \subseteq R^n$ be a cell. Then C is locally connected at each $x \in C$.*

Proof. By induction on n: If $n = 1$ then C is either a point or an open interval in R, in which case the result is immediate. So let $n > 1$ and suppose the result holds for lower values of n. Let $(x, y) \in R^{n-1} \times R$ be a point in C, let $U \times I \subseteq R^{n-1} \times R$ be an open box about (x, y), and set $D := \Pi_{n-1}(C)$.

Suppose first that $C = \mathrm{gr}(f)$ for some continuous definable function $f : D \to R$. Since f is continuous, the pre-image $f^{-1}(I)$ of I under f is an open subset of D such that $x \in f^{-1}(I)$; write $f^{-1}(I) = V \cap D$ for some open set $V \subseteq R^{n-1}$. Then $W := U \cap V$ is an open neighbourhood of x in R^{n-1} and so by the inductive hypothesis we may shrink W, if necessary, so that $W \cap D = W \cap D \cap f^{-1}(I)$ is definably connected. Then

$$\begin{aligned}(W \times I) \cap C &= \{(\xi, f(\xi)) : \xi \in W \cap D, f(\xi) \in I\} \\ &= \{(\xi, f(\xi)) : \xi \in W \cap D \cap f^{-1}(I)\} \\ &= \mathrm{gr}\left(f \restriction_{W \cap D \cap f^{-1}(I)}\right)\end{aligned}$$

and $\mathrm{gr}\left(f \restriction_{W \cap D \cap f^{-1}(I)}\right)$ is definably connected since f is definable and continuous. Hence $(W \times I) \cap C$ is definably connected.

Now suppose $C = (f, g)_D$ for continuous definable functions $f, g \in C_\infty(D)$ such that $f < g$. Then, by continuity of f, there exists an open box $V_1 \times J_1$ about (x, y) such that $(V_1 \times J_1) \cap \mathrm{gr}(f)$ is empty, since otherwise we would have $(x, y) \in \mathrm{cl}(\mathrm{gr}(f))$. But $(x, y) \in D \times R$ and $\mathrm{gr}(f)$ is a closed subset of $D \times R$ by continuity (and since our topology is Hausdorff) and hence $(x, y) \in \mathrm{gr}(f)$, a contradiction. Similarly we can find an open box $V_2 \times J_2$ about (x, y) such that $(V_2 \times J_2) \cap \mathrm{gr}(g)$ is empty. Let $W := U \cap V_1 \cap V_2$ and $J = I \cap J_1 \cap J_2$, so that $W \times J$ is an open box about (x, y) which is disjoint from both $\mathrm{gr}(f)$ and $\mathrm{gr}(g)$; write $J = (a, b)$ for $a, b \in R$ such that $a < b$ and note that $f(\xi) < a < b < g(\xi)$ for each $\xi \in W$. Then

$$\begin{aligned}(W \times J) \cap C &= \{(\xi, \eta) : \xi \in W \cap D, \eta \in J, f(\xi) < \eta < g(\xi)\} \\ &= \{(\xi, \eta) : \xi \in W \cap D, a < \eta < b\} \\ &= (a, b)_W\end{aligned}$$

where $a, b : W \to R$ are viewed as constant functions on W taking the values $a \in R$ and $b \in R$, respectively. Hence $(W \times J) \cap C$ can be written as an interval of continuous definable functions defined on a definably connected set, and so $(W \times J) \cap C$ is definably connected. $\square$

Before we can state the cell decomposition theorem, we need more terminology:

Definition 1.15. Let $n \geq 1$. A *decomposition* of R^n is a partition of R^n into finitely many cells, obtained by induction on n:

(i) A decomposition of R^1 is a finite collection

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \{a_2\}, \ldots, \{a_k\}\}$$

where each a_i is a point in R.

(ii) Let $n > 1$. A decomposition of R^n is a finite partition $\mathcal{C}$ of R^n into cells such that the collection

$$\Pi_{n-1}(\mathcal{D}) := \{\Pi_{n-1}(D) : D \in \mathcal{D}\}$$

is a decomposition of R^{n-1}.

Furthermore, a decomposition $\mathcal{D}$ of R^n is said to *partition* a set $S \subseteq R^n$ if S can be written as a finite union of cells in $\mathcal{D}$, i.e. if, for every cell $D \in \mathcal{D}$, either $D \subseteq S$ or $D \cap S = \varnothing$.

Theorem 1.16 (Cell decomposition)**.** *Let $n \geq 1$.*

(I_n). *Given any finite collection of definable sets $A_1, \ldots, A_k \subseteq R^n$, there is a decomposition of R^n partitioning each of $A_1, \ldots, A_k$.*

(II_n). *For any definable function $f : A \to R$, $A \subseteq R^n$, there is a decomposition $\mathcal{D}$ of R^n partitioning A such that the restriction $f \restriction_B : B \to R$ of f to each cell $B \in \mathcal{D}$ with $B \subseteq A$ is continuous.*

We only mention that the proof is by induction on n; the base case $n = 1$ follows immediately from the o-minimality axiom together with the monotonicity theorem. We also record the following consequences of the cell decomposition theorem.

Theorem 1.17. *Let $X \subseteq R^n$ be a non-empty definable set. Then X has only finitely many definably connected components, each of which is definable. They are open and closed in X and form a finite partition of X.*

Given a definable set $S \subseteq R^n \times R^m$ and $a \in R^n$, we let

$$S_a := \{x \in R^m : (a, x) \in S\}$$

denote the *fibre of S above a*; in this way we can view S as describing a *definable family* $(S_a)_{a\in R^n}$ of subsets of R^m, with parameter space R^n.

Theorem 1.18. *Let $S \subseteq R^n \times R^m$ be definable. Then there is a number $M_S \in \mathbb{N}$ such that, for each $a \in R^n$, the set $S_a \subseteq R^m$ has a partition into at most M_S cells. In particular, each fibre S_a has at most M_S definably connected components.*

Corollary 1.19. *Let $X \subseteq R^n$ be a definable set and let $x \in R^n$. There exists an open box $B \subseteq R^n$ about x such that the number M of definably connected components of $B \cap X$ is maximal as B ranges over all possible open boxes about x.*

Proof. Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ be points in R^n. Define a subset $\mathcal{B}$ of R^{3n} by setting

$$\mathcal{B} := \{(\alpha, \beta, \xi) \in R^n \times R^n \times X : \alpha_i < x_i < \beta_i \text{ and } \alpha_i < \xi_i < \beta_i \text{ for all } i\}.$$

Then for $(a, b) \in R^n \times R^n$, the fibre $\mathcal{B}_{(a,b)}$ is simply the intersection of the open box $\Pi_{i=1}^n(a_i, b_i)$ about x with X, where $\Pi_{i=1}^n(a_i, b_i)$ denotes the n-fold cartesian product $(a_1, b_1) \times \cdots \times (a_n, b_n)$ of the intervals $(a_i, b_i), i = 1, \dots, n$. By Theorem 1.18 there is an $M \in \mathbb{N}$ such that the number of definably connected components of each fibre $\mathcal{B}_{(a,b)}$, as (a, b) ranges over $R^n \times R^n$, is at most M; choose a tuple $(a, b) \in R^n \times R^n$ witnessing the maximality of M and set B to be $\Pi_{i=1}^n(a_i, b_i)$. □

1.3 Dimension and curve selection

The cell decomposition theorem allows us to introduce a well-defined notion of dimension; the first goal of this section is to provide a list of properties of dimension to be used in the next chapter (see [2, Chapter 4] for details).

Definition 1.20. The *dimension* of a non-empty definable set $X \subseteq R^n$ is given by

$$\dim(X) := \max\{i_1 + \cdots + i_n : X \text{ contains an } (i_1, \dots, i_n)\text{-cell.}\}.$$

We set $\dim(\varnothing) := -\infty$.

By definition, $\dim(X) \in \{-\infty, 0, 1, \dots, n\}$, and $\dim(X) = n$ if and only if X contains an open cell.

Proposition 1.21. (i) *If $X \subseteq Y \subseteq R^n$ and X, Y are definable, then* $\dim(X) \leq \dim(Y) \leq n$.

(ii) *If $X \subseteq R^n$ and $Y \subseteq R^m$ are definable and there is a definable bijection between X and Y, then* $\dim(X) = \dim(Y)$.

(iii) *If $X, Y \subseteq R^n$ and X, Y are definable, then* $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.

(iv) *Let $X \subseteq R^n$ be definable and let $f : X \to R^m$ be a definable function. Then* $\dim(X) \geq \dim(f(X))$.

(v) *For any pair of definable sets X and Y,* $\dim(X \times Y) = \dim(X) + \dim(Y)$.

We will make use of the following proposition in a few places, often without mention; the proof is by induction on n.

Proposition 1.22. *Let C, D be cells in R^n, where C is an $(i_1, \ldots, i_n)$-cell and $D \subseteq C$. Then the following are equivalent:*

(i) *D is an $(i_1, \ldots, i_n)$-cell.*

(ii) $\dim(C) = \dim(D)$.

(iii) *D is open in C.*

Given a definable set $S \subseteq R^n$, we define the *frontier* of S to be the set $\mathrm{fr}(S) := \mathrm{cl}(S) \setminus S$. Here we point out that, for subsets S, T of a topological space, $\mathrm{fr}(S \cup T) \subseteq \mathrm{fr}(S) \cup \mathrm{fr}(T)$ – this fact will be used repeatedly in the next chapter. We also have the following result, which will become crucial in the next chapter.

Theorem 1.23. *Let $S \subseteq R^n$ be a non-empty definable set. Then* $\dim(\mathrm{fr}(S)) < \dim(S)$. *In particular,* $\dim(\mathrm{cl}(S)) = \dim(S)$.

Note that one cannot expect such a result to hold in a general setting: Consider, for instance, any structure on $(\mathbb{R}, <)$ which contains the graph X of the *topologist's sine curve*, i.e. the graph of the function $x \mapsto \sin(1/x)$ for $x > 0$, $x \in \mathbb{R}$. The closure $\mathrm{cl}(X)$ of this curve is the union of X together with the closed straight line segment L connecting the points $(0, 1)$ and $(0, -1)$ in the plane $\mathbb{R}^2$. Hence $L \subseteq \mathrm{fr}(X)$ and so $\dim(\mathrm{fr}(X)) = \dim(X) = 1$.

The second goal of this section is to introduce *curve selection*, which allows us to formalize a notion of sequential limit which makes sense in an o-minimal setting; we are unable to make use of the usual notion of a sequential limit, since the domain of a sequence (e.g. the natural numbers) is a set with infinitely many definably connected components and hence cannot be definable in an o-minimal structure. If $\mathcal{R}$ is an o-minimal expansion of an ordered group, then curve selection provides us with a useful alternative to sequences.

For the rest of this section, fix an o-minimal expansion $\mathcal{R} = (R, <, \mathcal{S})$ of an ordered group $(R, <, 0, +, -)$. First, a preliminary result:

Theorem 1.24 (Definable choice)**.** *If $S \subseteq R^{n+m}$ is a definable set and $\pi : R^{n+m} \to R^n$ denotes the projection map onto the first n coordinates, then there is a definable map $f : \pi(S) \to R^m$ such that $\mathrm{gr}(f) \subseteq S$.*

Definable choice can be used, for instance, to show that for any definable equivalence relation $E \subseteq R^{2n}$ on R^n, there is a definable function $f : R^n \to R^k$ for some $k \in \mathbb{N}$ such that, for all $x, y \in R^n$, xEy if and only if $f(x) = f(y)$. In model-theoretic terms, this says that $\mathcal{R}$ has elimination of imaginaries and so $\mathcal{R}^{\mathrm{eq}} = \mathcal{R}$.

We will make use of the following corollary of definable choice:

Corollary 1.25 (Curve selection)**.** *Let X be a definable set and let $x \in \mathrm{fr}(X)$. There is a definable continuous injective map $\gamma : (0, \epsilon) \to X$, for some $\epsilon > 0$, such that $\lim_{t\to 0} \gamma(t) = x$.*

In this situation, we call γ a *definable curve*. Curve selection does not hold for arbitrary o-minimal structures, and so the assumption that $\mathcal{R}$ expands an ordered group is necessary (see [2, Chapter 6] for a counterexample).

Chapter 2

An extension theorem for continuous bounded functions on cells

Given an arbitrary structure $\mathcal{R} = (R, <, \mathcal{S})$, a definable set $A \subseteq R^n$ and a definable continuous function $f : A \to R$, one may ask if there exists a definable continuous function $\widetilde{f} : B \to R$ on a definable set $B \supseteq A$ such that $\widetilde{f} \upharpoonright_A = f$. In this case we say that f *extends continuously* to B, or that $\widetilde{f}$ is a *continuous extension* of f to B. When $\mathcal{R}$ is an o-minimal expansion of an ordered field, we have the following result (see [2, Chapter 8]).

Theorem 2.1. *Let $\mathcal{R}$ be an o-minimal expansion of an ordered field $(R, <, 0, 1, +, -, \cdot)$ and let $A \subseteq B$ be definable subsets of R^n such that A is closed in B. Then every definable continuous function $f : A \to R$ can be extended to a definable continuous function $\widetilde{f} : B \to R$. Furthermore, if f is bounded then so is $\widetilde{f}$.*

This theorem and its proof, however, tell us nothing about definable functions defined on non-closed subsets of R^n. Furthermore, the proof relies on a definable triangulation result which only makes sense when dealing with expansions of ordered fields. The aim of this chapter is to consider extension theorems in o-minimality which do not rely on either of these hypotheses; the bulk of the chapter consists of a proof of an extension theorem for definable, continuous, bounded functions on bounded cells which are asymptotically "well-behaved," a notion which we make precise later. The proof of the result makes use of definable curve selection and hence we must assume that $\mathcal{R}$ is an o-minimal expansion of an ordered group. Prior to the statement and proof of this result, we consider an extension theorem found in [4] for definable, continuous, bounded functions on bounded open cells, and we show by way of counterexample that this result cannot be generalized to arbitrary cells without requiring additional hypotheses.

2.1 The open cell case

From now on we work in an arbitrary o-minimal structure $\mathcal{R} = (R, <, \mathcal{S})$. In light of Theorem 2.1, one may ask if it is possible to extend continuous, definable functions on arbitrary subsets of R^n. The first result in this direction is proved in [4], in which the authors show that it is possible to extend continuous, bounded functions defined on a bounded open cell "almost everywhere" to the frontier of the cell.

Theorem 2.2. *Let $\mathcal{R} = (R, <, \mathcal{S})$ be an o-minimal structure, $C \subseteq R^n$ be a bounded open cell and $F : C \to R$ be a definable, continuous, bounded function. Then there is a definable set $X \subseteq \operatorname{fr}(C)$ such that*

$$\dim(\operatorname{fr}(C) \setminus X) \leq \dim(C) - 2$$

and F extends continuously to $C \cup X$.

(Note that Theorem 2.1 cannot be applied to obtain a result of this kind since each cell C is open in its closure $\operatorname{cl}(C)$.) The resulting extension $\widetilde{F}$ of F to $C \cup X$ is given by $\widetilde{F}(y) := \lim_{x \to y} F(x)$ and so $\widetilde{F}$ is always definable and unique.

Theorem 2.2 is obtained by proving the following two claims together by induction on $n \geq 1$ under the same assumptions stated in the theorem.

(I_n). There is a definable $X \subseteq \operatorname{fr}(C)$ such that $\dim(\operatorname{fr}(C) \setminus X) \leq \dim(C) - 2$ and C is locally connected at every $x \in X$.

(II_n). There is a definable $Y \subseteq \operatorname{fr}(C)$ such that $\dim(\operatorname{fr}(C) \setminus Y) \leq \dim(C) - 2$ and F extends continuously to $C \cup Y$.

Hence, as a corollary to the proof of Theorem 2.2, we also obtain (I_n) as an independent result whenever C is a bounded open cell in R^n.

The following examples show that the upper bounds on $\dim(\operatorname{fr}(C) \setminus X)$ and $\dim(\operatorname{fr}(C) \setminus Y)$ above are optimal in the sense that equality can occur.

Example 2.3. Let $\mathcal{R}$ be an o-minimal expansion of the real field $(\mathbb{R}, <, 0, 1, +, -, \cdot)$.

(i) Let C be the open cell $\{(x, y) \in \mathbb{R}^2 : 0 < y < x < 1\}$ and define a function $f : C \to \mathbb{R}$ by $f(x, y) = \frac{y}{x}$. Then f is a definable, continuous, bounded function which does not extend continuously to the origin $(0, 0) \in \mathbb{R}^2$, but does extend continuously to every other point of $\mathrm{fr}(C)$. Hence we can take Y to be $\mathrm{fr}(C) \setminus \{(0, 0)\}$ and so $\dim(\mathrm{fr}(C) \setminus Y) = \dim(\{(0, 0)\}) = 0 = \dim(C) - 2$.

(ii) Let C be the open cell

$$\left\{(x, y, z) \in \mathbb{R}^3 : |x| < 1,\ 0 < y < 1,\ \text{and } -1 < z < \frac{\sqrt{|x|}}{y}\right\},$$

i.e. C is the cell $\left(-1, \frac{\sqrt{|x|}}{y}\right)_D$ where D is the open box $(-1, 1) \times (0, 1)$. Then C is bounded and is not locally connected at any point of $Z := \{(0, 0, z) : 0 < z < 1\}$, and so we can take X to be $\mathrm{fr}(C) \setminus Z$. Then $\dim(\mathrm{fr}(C) \setminus X) = \dim(Z) = 1 = \dim(C) - 2$.

Naturally, the following question arises: Does Theorem 2.2 hold when C is not assumed to be open? It turns out that, without too much effort, one can construct an example which yields a negative answer to this question; we do so as follows. Fix an o-minimal expansion $\mathcal{R} = (R, <, \mathcal{S})$ of an ordered group $(R, <, 0, +, -)$ and fix an arbitrary positive element $1 \in R$. Then, by Proposition 1.8, $\mathcal{R}$ is a torsion-free, divisible abelian group and hence we can regard $\mathcal{R}$ as a vector space over $\mathbb{Q}$ (see [2] for details). Since $\mathcal{R}$ is an ordered group, we can define the absolute value function $|\cdot| : R \to R^{\geq 0}$ by setting

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

where $R^{\geq 0}$ is the set of non-negative elements of R. We will use the absolute value function to construct a cell which fails to satisfy the conclusion of Theorem 2.2.

Example 2.4. Let $C = \mathrm{gr}(f) \subseteq R^3$ where $f : (-1, 1)^2 \to R$ is given by

$$f(x, y) = \begin{cases} 1 & \text{if } y \geq \frac{1}{2}(1 - x) \text{ or } y \leq \frac{1}{2}(x - 1) \\ 2\frac{|y|}{(1-x)} & \text{otherwise} \end{cases}.$$

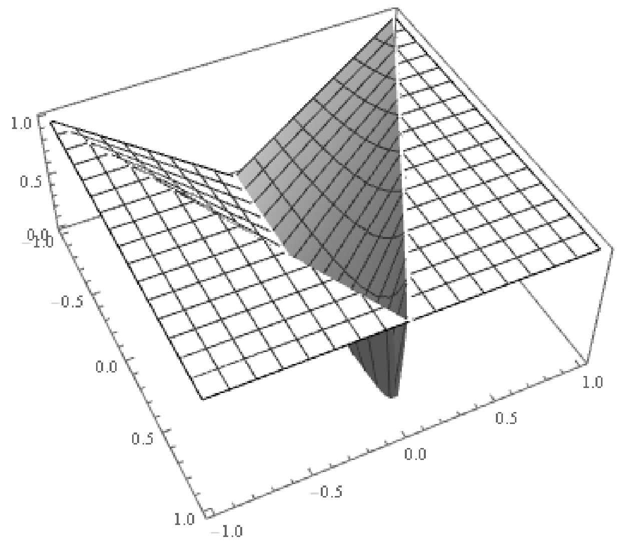

Figure 2.1: A plot of the function f given in Example 2.4.

Then C is a bounded cell definable in $\mathcal{R}$. Define $F : C \to R$ by

$$F(x, y, z) = \begin{cases} -1 & \text{if } y \leq \frac{1}{4}(x-1) \\ -2z & \text{if } \frac{1}{4}(x-1) \leq y \leq 0 \\ 2z & \text{if } 0 \leq y \leq \frac{1}{4}(1-x) \\ 1 & \text{if } y \geq \frac{1}{4}(1-x) \end{cases}.$$

Then F is continuous, bounded and definable. Note that the line segment $Z := \{(1, 0, t) : 0 < t < 1\}$ is contained in the frontier $\mathrm{fr}(C)$ of C. Furthermore, F does not extend continuously to Z: If $z \in Z$ then every neighbourhood of z intersects the domain of F restricted to

$$\{(x, y) : y \leq \frac{1}{4}(x-1)\} \cup \{(x, y) : y \geq \frac{1}{4}(1-x)\}$$

and so z is approached by points taking on the values 1 and -1. Thus $\lim_{x \to z} F(x)$ does not exist and hence F does not extend continuously to Z, a set of dimension 1. In particular, this means that if there was a definable $X \subseteq \mathrm{fr}(C)$ such that

$$\dim(\mathrm{fr}(C) \setminus X) \leq \dim(C) - 2 = 0$$

and F extends continuously to X, we must have $Z \subseteq \mathrm{fr}(C) \setminus X$ and so $\dim(Z) \leq 0$

must hold, which contradicts $\dim(Z) = 1$.

By studying the proof of Theorem 2.2 given in [4], one will note that the proof of (II_n) does not rely at all on the fact that our cells are open and merely depends on the existence of the set guaranteed by (I_n). So we can reduce the problem of generalizing Theorem 2.2 to that of determining sufficient conditions for a bounded cell to be "almost everywhere locally connected" at its frontier. The guiding question is thus:

Question 2.5. *What conditions must one impose on a bounded cell C in order to guarantee the existence of a set $X \subseteq \mathrm{fr}(C)$ as in (I_n) above? In other words, when is a bounded cell "almost everywhere locally connected" at its frontier?*

2.2 Asymptotically monotone cells

We wish to determine a class of cells (preferably the largest such class) which are almost everywhere locally connected at their frontier, and hence for which Theorem 2.2 holds unconditionally. The problem with the cell C given in Example 2.4 is that the frontier of C contains an asymptote which comes from the frontiers of two disjoint subcells of C. Indeed, using the same set-up given in Example 2.4, let

$$C_1 := \mathrm{gr}\left(f \restriction_{(-1,1)\times(\frac{1-x}{4},1)}\right)$$

and

$$C_2 := \mathrm{gr}\left(f \restriction_{(-1,1)\times(-1,\frac{x-1}{4})}\right)$$

so that C_1 is the portion of C lying above $(-1,1)\times(\frac{1-x}{4},1)$ and C_2 is the portion of C lying above $(-1,1)\times(-1,\frac{x-1}{4})$. Then the line segment Z intersects both $\mathrm{fr}(C_1)$ and $\mathrm{fr}(C_2)$ in a set of maximal dimension in $\mathrm{fr}(C)$. So C contains disjoint subcells C_1, C_2 of maximal dimension such that C_1 and C_2 are, in some sense, "far apart" in C, yet

$$\dim(\mathrm{fr}(C_1) \cap \mathrm{fr}(C_2) \cap Z) = \dim(C) - 1.$$

If we can rule such phenomena out then we obtain a class of cells for which the theorem holds; the main definition of this section attempts to make this precise.

From now on, we work in an arbitrary o-minimal structure $\mathcal{R} = (R, <, \mathcal{S})$ unless otherwise specified.

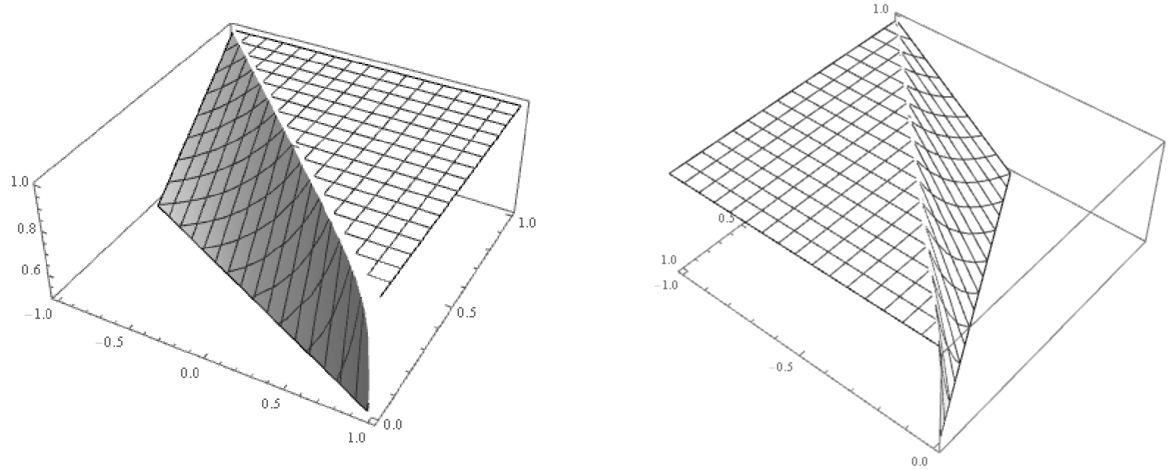

Figure 2.2: The two subcells C_1 and C_2 constructed above.

Definition 2.6. Let $C = \mathrm{gr}(f : D \to R)$ be a cell and let

$$X := \{x \in \mathrm{fr}(C) : \left|\Pi_{n-1}^{-1}(\Pi_{n-1}(x)) \cap \mathrm{fr}(C)\right| > 1\}.$$

Then C is *asymptotically monotone at* $x \in X$ if, for every open box $U \times I \subseteq R^n$ about x and for every pair D_i, D_j of distinct definably connected components of $U \cap f^{-1}(I)$, we have

$$\mathrm{fr}(\mathrm{gr}(f \restriction_{D_i})) \cap \mathrm{fr}(\mathrm{gr}(f \restriction_{D_j})) \cap X = \varnothing,$$

and if the implication

$$\Pi_{n-1}(x) \in \mathrm{cl}(D_i) \implies D_i \text{ is locally connected at } \Pi_{n-1}(x)$$

holds for each such component D_i.

Definition 2.7. We define the class of *asymptotically monotone* cells by induction on the dimension $n \geq 1$ of the ambient space R^n, as follows:

(i) If $C \subseteq R$ is an open interval or a point, then C asymptotically monotone.

(ii) Let $D \subseteq R^{n-1}$ be an asymptotically monotone cell. Let $f, g : D \to R$ be continuous definable functions such that $f < g$. If $C = (f, g)_D$, $C = (-\infty, f)_D$, or $C = (f, +\infty)_D$, then C is asymptotically monotone.

(iii) Let $D \subseteq R^{n-1}$ be an asymptotically monotone cell. Suppose $C = \mathrm{gr}(f)$ where $f : D \to R$ is definable and continuous, and let X be as in Definition 2.6. Then

C is asymptotically monotone if there is a definable subset X_0 of X such that $\dim(X \setminus X_0) \leq \dim(C) - 2$ and C is asymptotically monotone at x for every $x \in X_0$.

We point out that X is a definable set, and if X is empty then C is vacuously asymptotically monotone. Notice that each open cell is asymptotically monotone. Furthermore, the definition of asymptotic monotonicity in the final case above allows for exceptions at sets of small dimension (compared to that of the frontier of the cell), rather than requiring the cell to be asymptotically monotone at every point of its frontier. This loosening allows us to capture as many cells as possible when attempting to determine the largest class of cells for which the conclusion to Theorem 2.2 holds.

2.3 The general case

Before we can prove a general result in the direction of Theorem 2.2, we need a preliminary result about the frontier of cells with a "top" and a "bottom."

Definition 2.8. Let $C \subseteq R^n$ be a bounded cell of the form $(f, g)_{\Pi_{n-1}(C)}$ for continuous functions $f, g : \Pi_{n-1}(C) \to R$ such that $f < g$. The *side* of C is defined as

$$\mathrm{side}(C) := (\mathrm{fr}(\Pi_{n-1}(C)) \times R) \cap \mathrm{cl}(C).$$

Furthermore, we denote by $T(C) := \mathrm{gr}(g)$ the *top* of C and by $B(C) := \mathrm{gr}(f)$ the *bottom* of C. Note that each of $\mathrm{side}(C), T(C)$ and $B(C)$ are subsets of $\mathrm{fr}(C)$.

Lemma 2.9. *Suppose $C = (f, g)_D \subseteq R^n$ is a bounded cell where $D = \Pi_{n-1}(C)$ and $n > 1$. Then $\mathrm{side}(C), T(C)$ and $B(C)$ form a partition of $\mathrm{fr}(C)$.*

Proof. Let $(x, y) \in \mathrm{fr}(C)$ and suppose $(x, y) \notin T(C) \cup B(C)$. Then $(x, y) \in \mathrm{cl}(C)$ and so it suffices to check that $x \in \mathrm{fr}(D)$. By continuity of Π_{n-1}, $x \in \mathrm{cl}(D)$. Now suppose $x \in D$. Then, by the fact that $(x, y) \in \mathrm{cl}(C)$, either $y = f(x)$ or $y = g(x)$, since if $y < f(x)$ or $y > g(x)$ then there must be a neighbourhood of (x, y) which is disjoint from C. So one of $y = f(x)$ or $y = g(x)$ must hold, which contradicts the assumption that $(x, y) \notin T(C) \cup B(C)$. Thus $x \notin D$ and hence $x \in \mathrm{side}(C)$. Furthermore, $T(C)$ and $B(C)$ are disjoint by definition, while $\mathrm{side}(C)$ and $B(C)$ (resp. $T(C)$) are disjoint since their projections $\Pi_{n-1}(\mathrm{side}(C)) \subseteq \mathrm{fr}(D)$ and $\Pi_{n-1}(B(C)) = D$ are disjoint. Thus $\mathrm{side}(C), T(C)$ and $B(C)$ are pairwise disjoint and cover $\mathrm{fr}(C)$. $\square$

We also need the following easy fact about the frontier of cells without a top and bottom; we will use this occasionally, and always without mention.

Lemma 2.10. *Let $C = \mathrm{gr}(f : D \to R) \subseteq R^n$ be a cell and let $x = (x_1, \ldots, x_n) \in \mathrm{fr}(C)$. Then $x' := \Pi_{n-1}(x) \in \mathrm{fr}(D)$.*

Proof. The fact that $x' \in \mathrm{cl}(D)$ is immediate by continuity of Π_{n-1}. Suppose $x' \in D$; then $(x', x_n) = x \in D \times R$. Since f is continuous, its graph is a closed subset of $D \times R$ and hence $x \in \mathrm{cl}(C) = C$, which contradicts the assumption that $x \in \mathrm{fr}(C)$. $\square$

We are now in a position to state and prove our main result, which generalizes Theorem 2.2 to non-open cells in the case where our cells are all assumed to be asymptotically monotone, assuming $\mathcal{R}$ expands an ordered group. The proof of (I_n), Case 1 below is inspired by that of the corresponding result given in the proof of Theorem 2.2 in [4].

Theorem 2.11. *Let $\mathcal{R} = (R, <, \mathcal{S})$ be an o-minimal expansion of an ordered group $(R, <, 0, +, -)$, let $C \subseteq R^n$ be a bounded asymptotically monotone cell and let $F : C \to R$ be a definable, continuous, bounded function. Then there is a definable set $X \subseteq \mathrm{fr}(C)$ such that*

$$\dim(\mathrm{fr}(C) \setminus X) \leq \dim(C) - 2$$

and F extends continuously to $C \cup X$.

Proof. We will prove the following claims together by induction on $n \geq 1$.

(I_n). There is a definable $X \subseteq \mathrm{fr}(C)$ such that $\dim(\mathrm{fr}(C) \setminus X) \leq \dim(C) - 2$ and C is locally connected at every $x \in X$.

(II_n). There is a definable $Y \subseteq \mathrm{fr}(C)$ such that $\dim(\mathrm{fr}(C) \setminus Y) \leq \dim(C) - 2$ and F extends continuously to $C \cup Y$.

If $n = 1$ then C is either a point or an open interval, so (I_1) is clear and (II_1) follows immediately from the monotonicity theorem. Now let $n > 1$ and suppose (I_n) and (II_n) hold for lower values of n. Throughout, we set $D := \Pi_{n-1}(C)$.

(I_n). **Case 1**: $C = (f, g)_D$ for definable, continuous, bounded functions $f, g : D \to R$ such that $f < g$.

First note that C is locally connected at every $x \in T(C) \cup B(C)$. Indeed, let $x \in B(C) = \mathrm{gr}(f)$ (the case where $x \in T(C)$ is similar) and let $U \times I \subseteq R^n$ be an open box about x where I is an open interval $(a,b) \subseteq R$ for $a, b \in R$ such that $a < b$. By continuity of g, we may shrink $U \times I$ so that $b < g(\xi)$ for all $\xi \in D \cap U$, since otherwise $x \in \mathrm{cl}(\mathrm{gr}(g)) = \mathrm{gr}(g)$ which contradicts $\mathrm{gr}(f) \cap \mathrm{gr}(g) = \varnothing$. Let $x' := (x_1, \ldots, x_{n-1}) \in D$ be the projection of x onto the first $n-1$ coordinates, so that $(x', f(x')) = x$. Consider the pre-image $f^{-1}(I)$ of I under f. By continuity, $f^{-1}(I)$ is open in D; let D_0 be the definably connected component of $f^{-1}(I)$ which contains x' and note that D_0 is open in D. Hence $D_0 = W \cap D$ for some open box $W \subseteq R^{n-1}$ and so $x' \in W \cap U$. Since, by Lemma 1.14, cells are locally connected at each of their points, there is an open box $V \subseteq W \cap U$ in R^{n-1} such that $V \cap D$ is definably connected and $x' \in V$. Then

$$\begin{aligned}(V \times I) \cap C &= \{(z,r) \in R^n : f(z) < r < g(z), z \in V \cap D, r \in I\} \\ &= \{(z,r) \in R^n : f(z) < r < b, z \in V \cap D\} \\ &= (f, b)_{V \cap D}\end{aligned}$$

where $b : V \cap D \to R$ is the constant function taking the value $b \in R$ everywhere. But $(f,b)_{V\cap D}$ is an interval of definable continuous functions defined on a definably connected set, and so $(V \times I) \cap C$ is definably connected.

So by Lemma 2.9, it suffices to show the existence of a set $X \subseteq \mathrm{side}(C)$ such that $\dim(\mathrm{side}(C) \setminus X) \leq \dim(C) - 2$ and C is locally connected at every $x \in X$, since then $T(C) \cup B(C) \cup X \subseteq \mathrm{fr}(C)$ and

$$\dim(\mathrm{fr}(C) \setminus (T(C) \cup B(C) \cup X)) = \dim(\mathrm{side}(C) \setminus X) \leq \dim(C) - 2$$

and C is locally connected at every $x \in T(C) \cup B(C) \cup X$. By (I_{n-1}) and (II_{n-1}), there is a definable $Z \subseteq \mathrm{fr}(D)$ such that $\dim(\mathrm{fr}(D) \setminus Z) \leq \dim(D) - 2$, D is locally connected at each $z \in Z$ and f, g extend continuously to functions $\tilde{f}, \tilde{g} : D \cup Z \to R$. Let

$$X := \{(z,r) : z \in Z, \tilde{f}(z) < r < \tilde{g}(z)\}.$$

Then $X \subseteq \mathrm{side}(C)$; we claim that

$$\mathrm{side}(C) \setminus X \subseteq \mathrm{fr}(\mathrm{gr}(f)) \cup \mathrm{fr}(\mathrm{gr}(g)) \cup ((\mathrm{fr}(D) \setminus Z) \times R).$$

Indeed, if $(x,y) \in \mathrm{side}(C) \setminus X$ then either $x \notin Z$, or $x \in Z$ and $y \notin (\tilde{f}(x), \tilde{g}(x))$. If the former holds then $(x, y) \in (\mathrm{fr}(D) \setminus Z) \times R$ by definition of side(C), so assume the latter holds. Since $y \notin (\tilde{f}(x), \tilde{g}(x))$, we must have one of $y = \tilde{f}(x)$ or $y = \tilde{g}(x)$; if not, then by continuity of f and g we obtain an open box $B \subseteq R^n$ about (x, y) such that $B \cap C = \varnothing$, contradicting $(x, y) \in \mathrm{cl}(C)$. So one of $y = \tilde{f}(x)$ or $y = \tilde{g}(x)$ must hold, which implies $(x, y) \in \mathrm{fr}(\mathrm{gr}(f)) \cup \mathrm{fr}(\mathrm{gr}(g))$ since $\tilde{f}$ and $\tilde{g}$ are continuous extensions of f and g, respectively. Therefore the above inclusion holds and so $\dim(\mathrm{side}(C) \setminus X) \leq \dim(C) - 2$.

Now let $(z, r) \in X$ and suppose U is an open neighbourhood of (z, r) in R^n. By continuity of $\tilde{f}$ and $\tilde{g}$ there is an open box $B \times I \subseteq U$ containing (z, r) such that $B \times I$ is disjoint from $\mathrm{gr}(\tilde{f})$ and from $\mathrm{gr}(\tilde{g})$: Otherwise we would have, say, $(z, r) \in \mathrm{cl}(\mathrm{gr}(\tilde{f})) = \mathrm{gr}(\tilde{f})$, which contradicts the assumption that $\tilde{f}(z) < r$. By the inductive hypothesis, D is locally connected at $z \in Z$ and so we may shrink B so that $B \cap D$ is definably connected. Then

$$(B \times I) \cap C = (B \cap D) \times I$$

is definably connected, and so C is locally connected at every $y \in X$, thus proving (I_n) in the case where C is of the form $(f, g)_D$.

Case 2: $C = \mathrm{gr}(f)$ for a definable, continuous, bounded function $f : D \to R$.

By (I_{n-1}) and (II_{n-1}), there is a definable $Z \subseteq \mathrm{fr}(D)$ such that $\dim(\mathrm{fr}(D) \setminus Z) \leq \dim(D) - 2$, D is locally connected at each $z \in Z$ and f extends continuously to $\tilde{f} : D \cup Z \to R$. We consider two possible subcases, depending on the asymptotic behaviour of $\mathrm{fr}(C)$.

Subcase 2.1: $\{x \in \mathrm{fr}(C) : \left|\Pi_{n-1}^{-1}(\Pi_{n-1}(x)) \cap \mathrm{fr}(C)\right| > 1\} = \varnothing$.

Let $\psi := \Pi_{n-1} \restriction_{\mathrm{cl}(C)}$; ψ is a bijection since the above assumption implies injectivity. Let

$$X := \mathrm{gr}(\tilde{f} \restriction_Z) = \{(z, \tilde{f}(z)) : z \in Z\}.$$

Then C is locally connected at each $x \in X$: Given an open box $U \times I$ in R^n about a point $(z, \tilde{f}(z)) \in X$, the continuity of $\tilde{f}$ implies

$$\tilde{f}^{-1}(I) = U' \cap (D \cup Z)$$

for some open box $U' \subseteq R^{n-1}$ such that $z \in U'$. By local connectedness of D at z there exists an open box $V \subseteq U'$ in R^{n-1} about z such that $V \cap D$ is definably connected. Note that $V \cap D$ is contained in $f^{-1}(I)$, since $x \in V \cap D$ implies $x \in U'$ and $x \in D \cup Z$, i.e. $x \in \tilde{f}^{-1}(I)$, so $\tilde{f}(x) \in I$. But f and $\tilde{f}$ agree on D and so $\tilde{f}(x) = f(x)$ and hence $x \in f^{-1}(I)$. Then, after replacing V with $V \cap U$ if necessary to ensure $V \times I \subseteq U \times I$, we have

$$\begin{aligned}(V \times I) \cap C &= \{(x, f(x)) : x \in V \cap D, f(x) \in I\} \\ &= \mathrm{gr}(f \restriction_{V \cap D}) \cap (R^{n-1} \times I) \\ &= \mathrm{gr}(f \restriction_{V \cap D})\end{aligned}$$

where the last equality follows from the fact that $\mathrm{im}(f \restriction_{V \cap D}) \subseteq I$. But $V \cap D$ is definably connected and f is definable and continuous, and so the graph of $f \restriction_{V \cap D}$ is definably connected.

Furthermore, $\dim(\mathrm{fr}(C) \setminus X) \leq \dim(C) - 2$: If not, then there is a cell E such that $\dim(E) = \dim(C) - 1$ and $E \subseteq \mathrm{fr}(C) \setminus X$. By injectivity of ψ,

$$\psi(E) \subseteq \psi(\mathrm{fr}(C) \setminus X) = \psi(\mathrm{fr}(C)) \setminus \psi(X) \subseteq \mathrm{fr}(D) \setminus Z,$$

where we also use the fact that ψ is continuous and so $\psi(\mathrm{cl}(C)) \subseteq \mathrm{cl}(\psi(C)) = \mathrm{cl}(D)$ which, together with injectivity, yields

$$\psi(\mathrm{fr}(C)) = \psi(\mathrm{cl}(C) \setminus C) = \psi(\mathrm{cl}(C)) \setminus \psi(C) \subseteq \mathrm{cl}(D) \setminus D = \mathrm{fr}(D).$$

But ψ is a bijection and hence

$$\dim(E) = \dim(\psi(E)) \leq \dim(D) - 2 = \dim(C) - 2$$

which contradicts the fact that $\dim(E) = \dim(C) - 1$. Thus C is locally connected at every point in X and $\dim(\mathrm{fr}(C) \setminus X) \leq \dim(C) - 2$.

Subcase 2.2: $X' := \{x \in \mathrm{fr}(C) : \left|\Pi_{n-1}^{-1}(\Pi_{n-1}(x)) \cap \mathrm{fr}(C)\right| > 1\} \neq \varnothing$.

To begin, we claim that

$$\dim(\mathrm{fr}(C) \setminus (\mathrm{gr}(\tilde{f} \restriction_Z) \cup X')) \leq \dim(C) - 2.$$

Otherwise there is a cell $E \subseteq \mathrm{fr}(C)$ with $\dim(E) = \dim(C) - 1$, $E \subseteq \mathrm{fr}(C) \setminus \mathrm{gr}(\tilde{f} \restriction_Z)$ and $E \subseteq \mathrm{fr}(C) \setminus X'$. Since E is disjoint from X', the restriction φ of Π_{n-1} to the union $E \cup \mathrm{gr}(\tilde{f} \restriction_Z)$ must be injective, since otherwise there are points

$$(\xi, \zeta_1) \in E, (\xi, \zeta_2) \in E \cup \mathrm{gr}(\tilde{f} \restriction_Z)$$

such that $\zeta_1 \neq \zeta_2$. But then $|\Pi_{n-1}^{-1}(\xi) \cap \mathrm{fr}(C)| \geq 2$ and so $(\xi, \zeta_1) \in X'$, which contradicts the disjointness of E from X'. Hence by injectivity E must be of the form $\mathrm{gr}(\gamma : \Pi_{n-1}(E) \to R)$ for some definable continuous function γ, and so

$$\dim(\Pi_{n-1}(E)) = \dim(C) - 1 = \dim(D) - 1.$$

Then by injectivity of φ

$$\varnothing = \varphi(E \cap \mathrm{gr}(\tilde{f} \restriction_Z)) = \varphi(E) \cap \varphi(\mathrm{gr}(\tilde{f} \restriction_Z)) = \Pi_{n-1}(E) \cap Z$$

which implies $\Pi_{n-1}(E) \subseteq \mathrm{fr}(D) \setminus Z$. But now

$$\dim(\Pi_{n-1}(E)) \leq \dim(\mathrm{fr}(D) \setminus Z) \leq \dim(D) - 2$$

contradicting $\dim(\Pi_{n-1}(E)) = \dim(D) - 1$. Now, since C is asymptotically monotone, there is a definable subset X_0 of X' such that $\dim(X' \setminus X_0) \leq \dim(C) - 2$ and C is asymptotically monotone at every $x \in X_0$. Then

$$\mathrm{fr}(C) \setminus (\mathrm{gr}(\tilde{f} \restriction_Z) \cup X_0) \subseteq \Big(\mathrm{fr}(C) \setminus (\mathrm{gr}(\tilde{f} \restriction_Z) \cup X')\Big) \cup X' \setminus X_0$$

and thus

$$\dim\Big(\mathrm{fr}(C) \setminus (\mathrm{gr}(\tilde{f} \restriction_Z) \cup X_0)\Big) \leq \dim(C) - 2.$$

So we set $X := \mathrm{gr}(\tilde{f} \restriction_Z) \cup X_0$ and we aim to show that C is locally connected at each $x \in X$.

By the same argument as in Subcase 2.1, C is locally connected at every $x \in \mathrm{gr}(\tilde{f} \restriction_Z)$ and so it remains to check that C is locally connected at every $x \in X_0$. Let $U \times I$ be an open box about a point $x \in X_0$ and let $x' := (x_1, \ldots, x_{n-1})$. By Corollary 1.19 there exists an open box $V \subseteq U$ about x' such that the number M of definably connected components of $V \cap f^{-1}(I)$ is maximal as V ranges over all possible open boxes in R^{n-1}

about x' contained in U. We write

$$V \cap f^{-1}(I) = D_1 \cup \cdots \cup D_M$$

as a disjoint union of its definably connected components $D_i, i \in \{1, \ldots, M\}$. Now let

$$\widetilde{D}_i := \operatorname{gr}(f \restriction_{D_i}),\ i \in \{1, \ldots, M\}$$

and notice that each set

$$\operatorname{fr}(\widetilde{D}_i) \cap (\{x'\} \times R)$$

is a closed subset of $\{x'\} \times R$ with $\dim(\operatorname{fr}(\widetilde{D}_i) \cap (\{x'\} \times R)) = 1$. By maximality of M there are exactly M such sets, each of which is definably connected; hence each such set forms a "closed interval" lying above $\{x'\}$.

Our first goal is to show that $x \in \operatorname{fr}(\widetilde{D}_i)$ for precisely one $i \in \{1, \ldots, M\}$. Note

$$x \in \operatorname{fr}(C) \subseteq \operatorname{fr}\left(\operatorname{gr}(f \restriction_{V \cap f^{-1}(I)})\right) \cup \operatorname{fr}\left(\operatorname{gr}(f \restriction_{D \setminus V})\right) \cup \operatorname{fr}\left(\operatorname{gr}(f \restriction_{D \setminus f^{-1}(I)})\right).$$

If $x \in \operatorname{fr}\left(\operatorname{gr}(f \restriction_{D \setminus V})\right)$ then in particular $V \times I$ must intersect $\operatorname{gr}(f \restriction_{D \setminus V})$ and so there is a point $(\xi, f(\xi))$ such that $\xi \in V \cap (D \setminus V)$, and so it must be that $x \notin \operatorname{fr}\left(\operatorname{gr}(f \restriction_{D \setminus V})\right)$. Similarly, $x \in \operatorname{fr}\left(\operatorname{gr}(f \restriction_{D \setminus f^{-1}(I)})\right)$ implies there is a point $(\xi, f(\xi))$ such that $f(\xi) \in I \cap (R \setminus I)$, and so we must have $x \notin \operatorname{fr}\left(\operatorname{gr}(f \restriction_{D \setminus f^{-1}(I)})\right)$. Thus $x \in \operatorname{fr}\left(\operatorname{gr}(f \restriction_{V \cap f^{-1}(I)})\right)$ must hold. But

$$\operatorname{gr}(f \restriction_{V \cap f^{-1}(I)}) = \bigcup_{i=1}^{M} \operatorname{gr}(f \restriction_{D_i}) = \bigcup_{i=1}^{M} \widetilde{D}_i$$

and so $x \in \operatorname{fr}(\widetilde{D}_i)$ for at least one $i \in \{1, \ldots, M\}$. But C is asymptotically monotone at x and so the sets $\operatorname{fr}(\widetilde{D}_i)$ are pairwise disjoint in X_0. Hence $x \in \operatorname{fr}(\widetilde{D}_i)$ for exactly one $i \in \{1, \ldots, M\}$; let η be this unique index. Now, since the $\operatorname{fr}(\widetilde{D}_i)$ are pairwise disjoint in X_0 and form disjoint, closed intervals above $\{x'\}$, there is an open interval J about x_n, the last coordinate of x, such that

$$(V \times J) \cap (\{x'\} \times R) = \operatorname{fr}(\widetilde{D}_\eta) \cap (\{x'\} \times R).$$

We claim that there is an open box $W \subseteq V$ about x' such that

$$\operatorname{gr}(f \restriction_{W \cap D_\eta}) \subseteq W \times J.$$

If not, then for every open box $W \subseteq V$ such that $x' \in W$, the set $W \cap D_\eta \cap f^{-1}(I \setminus J)$ is non-empty, and so $x' \in \mathrm{cl}(D_\eta \cap f^{-1}(I \setminus J))$. By curve selection, we obtain a definable curve

$$\gamma : (0, \epsilon) \to D_\eta \cap f^{-1}(I \setminus J)$$

such that $\lim_{t\to 0} \gamma(t) = x'$. We lift γ to a definable curve

$$\Gamma : (0, \epsilon) \to \mathrm{gr}(f \restriction_{D_\eta \cap f^{-1}(I\setminus J)}) = \widetilde{D}_\eta \cap (R^{n-1} \times (I \setminus J))$$

by setting $\Gamma(t) := (\gamma(t), f(\gamma(t)))$. Then

$$\lim_{t\to 0} \Gamma(t) = \left(\lim_{t\to 0} \gamma(t), \lim_{t\to 0} f(\gamma(t))\right) = \left(x', \lim_{t\to 0} f(\gamma(t))\right) \in \{x'\} \times R.$$

Furthermore, $\lim_{t\to 0} \Gamma(t)$ belongs to $\mathrm{fr}(\widetilde{D}_\eta)$ and so by definition of J we must have

$$\lim_{t\to 0} \Gamma(t) \in (V \times J) \cap (\{x'\} \times R)$$

which contradicts $\lim_{t\to 0} f(\gamma(t)) \notin J$; indeed, $I \setminus J$ is a closed set and so

$$\lim_{t\to 0} f(\gamma(t)) \in \mathrm{cl}(I \setminus J) = I \setminus J$$

must hold. Thus we can take an open box $W \subseteq V$ about x' with the desired property.

Next, we also claim that there is an open box $W_0 \subseteq W$ about x' such that $W_0 \cap f^{-1}(J) \subseteq W_0 \cap D_\eta$. If not then $x' \in \mathrm{cl}(f^{-1}(J) \cap (D \setminus D_\eta))$ and so, by curve selection, there is a definable curve

$$\gamma : (0, \epsilon) \to f^{-1}(J) \cap (D \setminus D_\eta)$$

such that $\lim_{t\to 0} \gamma(t) = x'$. Lift γ to a definable curve

$$\Gamma : (0, \epsilon) \to \left(\bigcup_{i\neq\eta} \widetilde{D}_i\right) \cap (R^{n-1} \times J)$$

as before, so that

$$\lim_{t\to 0} \Gamma(t) \in (V \times J) \cap (\{x'\} \times R)$$

and so $\lim_{t\to 0} \Gamma(t) \in \mathrm{fr}(\widetilde{D}_\eta)$ by definition of J. But the image of Γ is definably connected and so

$$\operatorname{im}(\Gamma) \subseteq \widetilde{D}_i \cap (R^{n-1} \times J)$$

for a unique $i \in \{1, \dots, M\} \setminus \{\eta\}$. Then $\lim_{t \to 0} \Gamma(t) \in \operatorname{fr}(\widetilde{D}_i)$ for some $i \neq \eta$, which contradicts the fact that

$$\operatorname{fr}(\widetilde{D}_\eta) \cap \operatorname{fr}(\widetilde{D}_i) \cap X = \varnothing$$

when $i \neq \eta$.

Thus we can find an open box $W_0 \subseteq W$ about x' such that $W_0 \cap f^{-1}(J) \subseteq W_0 \cap D_\eta$. By asymptotic monotonicity of C at x, D_η is locally connected at x' and so we may shrink W_0 if necessary so that $W_0 \cap D_\eta$ is definably connected. To finish the proof of (I_n) we note that by definition of W we have

$$\operatorname{gr}(f \restriction_{W_0 \cap D_\eta}) \subseteq W_0 \times J$$

and so $W_0 \cap D_\eta \subseteq W_0 \cap f^{-1}(J)$. Then, combined with the fact that $W_0 \cap f^{-1}(J) \subseteq W_0 \cap D_\eta$, we obtain $W_0 \cap f^{-1}(J) = W_0 \cap D_\eta$. So we have

$$W_0 \times J \subseteq W \times J \subseteq V \times I \subseteq U \times I,$$

and the set

$$\begin{aligned}
(W_0 \times J) \cap C &= \{(\xi, f(\xi)) : \xi \in W_0 \cap D, f(\xi) \in J\} \\
&= \{(\xi, f(\xi)) : \xi \in W_0 \cap f^{-1}(J)\} \\
&= \operatorname{gr}\left(f \restriction_{W_0 \cap f^{-1}(J)}\right) \\
&= \operatorname{gr}\left(f \restriction_{W_0 \cap D_\eta}\right)
\end{aligned}$$

is definably connected, since f is definable and continuous and $W_0 \cap D_\eta$ is definably connected.

(II_n). We now use (I_n) and the inductive hypothesis to prove (II_n) for all asymptotically monotone cells. (The proof of (II_n) given here is based on that of the corresponding result given in [4].)

Let $C \subseteq R^n$ be a bounded asymptotically monotone cell and let $F : C \to R$ be a definable, continuous, bounded function. Let Z be the set of all points $z \in \operatorname{fr}(C)$ such that $\lim_{x \to z} F(x)$ exists. We first claim that $\dim(\operatorname{fr}(C) \setminus Z) \leq \dim(C) - 2$: If not, then there is a cell $E \subseteq \mathrm{fr}(C)$ such that $\dim(E) = \dim(C) - 1$ and

$$\liminf_{x \to y} F(x) < \limsup_{x \to y} F(x)$$

for every $y \in E$. By (I_n) there is a definable $X \subseteq \mathrm{fr}(C)$ such that $\dim(\mathrm{fr}(C) \setminus X) \leq \dim(C) - 2$ and C is locally connected at every $x \in X$. Let E_0 be the set of $x \in E$ such that C is locally connected at x. Since

$$\dim(C) - 1 = \dim(E) = \max\{\dim(E_0), \dim(E \setminus E_0)\}$$

and

$$\dim(E \setminus E_0) \leq \dim(\mathrm{fr}(C) \setminus X) \leq \dim(C) - 2,$$

we must have $\dim(E_0) = \dim(C) - 1$ and so we may replace E with E_0 so that C is locally connected at each $y \in E$. But then

$$\mathrm{fr}(\mathrm{gr}(F)) \supseteq \{(y, r) \in R^{n+1} : y \in E \text{ and } \liminf_{x \to y} F(x) < r < \limsup_{x \to y} F(x)\}$$

since, given any such $(y, r) \in R^{n+1}$ and any open neighbourhood $U \times I$ about (y, r), by local connectedness we may assume $U \cap C$ is definably connected after shrinking U, if necessary. Furthermore, by definition of the $\liminf$ and the $\limsup$, there are points $\xi_1, \xi_2 \in U \cap C$ such that $F(\xi_1) < r$ and $F(\xi_2) > r$. Since $U \cap C$ is definably connected and F is definable and continuous, the image $F(U \cap C)$ is definably connected and so there exists $\zeta \in U \cap C$ such that $F(\zeta) = r$. Thus the intersection $(U \times I) \cap \mathrm{gr}(F)$ is non-empty and so $(y, r) \in \mathrm{cl}(\mathrm{gr}(F))$. The above inclusion then yields

$$\dim(\mathrm{fr}(\mathrm{gr}(F))) \geq \dim(E) + 1 = \dim(C) = \dim(\mathrm{gr}(F)) > \dim(\mathrm{fr}(\mathrm{gr}(F)))$$

where the last inequality is by Theorem 1.23 and hence we obtain a contradiction.

Now define $G : Z \to R$ by

$$G(z) := \lim_{x \to z} F(x)$$

and let Y be the set of points of continuity of G. By the cell decomposition theorem,

$\dim(Z \setminus Y) < \dim(Z)$ and so

$$\dim(\mathrm{fr}(C) \setminus Y) = \dim((\mathrm{fr}(C) \setminus Z) \cup (Z \setminus Y)) = \max\{\dim(\mathrm{fr}(C) \setminus Z), \dim(Z \setminus Y)\}$$

which implies $\dim(\mathrm{fr}(C) \setminus Y) \leq \dim(C) - 2$. Finally, define $\widetilde{F} : C \cup Y \to R$ by setting $\widetilde{F} \restriction_C := F$ and $\widetilde{F} \restriction_Y := G \restriction_Y$. Then $\widetilde{F}$ is a continuous extension of F. This completes the proof of (II_n) and hence the theorem is proven. □

Given a bounded cell $C \subseteq R^n$, we say that C has the *almost everywhere extension property* if C satisfies the conclusion of Theorem 2.11 for any definable, continuous, bounded function $F : C \to R$. Theorem 2.11 then says that, whenever $\mathcal{R}$ expands an ordered group, all asymptotically monotone cells have the almost everywhere extension property.

Corollary 2.12. *Let $\mathcal{R}$ be an o-minimal expansion of an ordered group and let C be a bounded cell. Then* (1) $\implies$ (2) $\implies$ (3)*, where:*

(1) *C is asymptotically monotone.*

(2) *There is a definable $X \subseteq \mathrm{fr}(C)$ such that $\dim(\mathrm{fr}(C) \setminus X) \leq \dim(C) - 2$ and C is locally connected at every $x \in X$.*

(3) *C has the almost everywhere extension property.*

Question 2.13. *Let C be a bounded cell. If C has the almost everywhere extension property, must C be asymptotically monotone? Must C be "almost everywhere locally connected" at its frontier?*

Notice that the assumption that $\mathcal{R}$ expands an ordered group is only used in the proof of (I_n), Subcase 2.2; in fact, definable curve selection is only used in two places in the proof, and it is plausible that the proof still goes through when $\mathcal{R}$ is an arbitrary o-minimal structure:

Conjecture 2.14. *Let $\mathcal{R}$ be an arbitrary o-minimal structure and let C be a bounded asymptotically monotone cell. Then C has the almost everywhere extension property.*

We have at the very least the following result for an arbitrary o-minimal structure. First we define a class of cells which is strictly contained in the class of asymptotically monotone cells.

Definition 2.15. We define the class of *non-asymptotic* cells by induction on n:

(i) If $C \subseteq R$ is an open interval or a point, then C non-asymptotic.

(ii) Let $D \subseteq R^{n-1}$ be a non-asymptotic cell. Let $f, g : D \to R$ be continuous definable functions such that $f < g$. If $C = (f, g)_D$, $C = (-\infty, f)_D$, or $C = (f, +\infty)_D$, then C is non-asymptotic.

(iii) Let $D \subseteq R^{n-1}$ be a non-asymptotic cell. Suppose $C = \mathrm{gr}(f)$ where $f : D \to R$ is definable and continuous, and let X be as in Definition 2.6. Then C is non-asymptotic if X is empty.

The proof of Theorem 2.11 then goes through, since Subcase 2.2 becomes an empty case. Hence, as a corollary to the proof, we obtain the following:

Corollary 2.16. *Let $\mathcal{R}$ be an o-minimal structure and let C be a bounded cell. Then* (1) $\implies$ (2) $\implies$ (3), *where:*

(1) *C is non-asymptotic.*

(2) *There is a definable $X \subseteq \mathrm{fr}(C)$ such that $\dim(\mathrm{fr}(C) \setminus X) \leq \dim(C) - 2$ and C is locally connected at every $x \in X$.*

(3) *C has the almost everywhere extension property.*

If we also assume that $\mathcal{R}$ expands an ordered field then we have the following result, which could potentially be used to answer Question 2.13 in the affirmative. First let us say that a cell C is *asymptotically non-injective* if there exist disjoint, open subcells $C_1, C_2 \subseteq C$ such that

$$\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2) \cap C = \varnothing \text{ and } \dim(\mathrm{fr}(C_1) \cap \mathrm{fr}(C_2) \cap X) = \dim(C) - 1$$

where X is as in Definition 2.6.

Theorem 2.17. *Let $\mathcal{R}$ be an o-minimal expansion of an ordered field $(R, <, 0, 1, +, -, \cdot)$ and let $C \subseteq R^n$ be a bounded cell which is asymptotically non-injective. Then there exists a definable, continuous, bounded function $F : C \to R$ such that F does not extend continuously almost everywhere to $\mathrm{fr}(C)$, i.e. there is a definable $Z \subseteq \mathrm{fr}(C)$ such that $\dim(Z) = \dim(C) - 1$ and F does not extend continuously to x for all $x \in Z$.*

Proof. Let X be as in Definition 2.6 and let $C_1, C_2 \subseteq C$ be disjoint open subcells of C such that $\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2) \cap C = \varnothing$ and $\dim(\mathrm{fr}(C_1) \cap \mathrm{fr}(C_2) \cap X) = \dim(C) - 1$. Define a function

$$F : (\mathrm{cl}(C_1) \cup \mathrm{cl}(C_2)) \cap C \to R$$

by setting

$$F \restriction_{\mathrm{cl}(C_1) \cap C} := c_1 \text{ and } F \restriction_{\mathrm{cl}(C_2) \cap C} := c_2$$

where c_1, c_2 are arbitrary elements of R such that $c_1 \neq c_2$. Note that F is well-defined since $\mathrm{cl}(C_1) \cap \mathrm{cl}(C_2) \cap C$ is empty. Furthermore, F is continuous since the pre-image of an open set in R under F is either empty, or all of $(\mathrm{cl}(C_1) \cup \mathrm{cl}(C_2)) \cap C$, or $\mathrm{cl}(C_1) \cap C$, or $\mathrm{cl}(C_2) \cap C$; the latter two sets are open since they are definably connected components of $(\mathrm{cl}(C_1) \cup \mathrm{cl}(C_2)) \cap C$. Hence F is a definable, continuous, bounded function on $(\mathrm{cl}(C_1) \cup \mathrm{cl}(C_2)) \cap C$. Note that $(\mathrm{cl}(C_1) \cup \mathrm{cl}(C_2)) \cap C$ is closed in C and so by Theorem 2.1 we can continuously extend F to a definable, bounded function $\widetilde{F} : C \to R$. But $\widetilde{F}$ does not extend continuously almost everywhere to $\mathrm{fr}(C)$ since $\dim(\mathrm{fr}(C_1) \cap \mathrm{fr}(C_2) \cap X) = \dim(C) - 1$ and, for any $x \in \mathrm{fr}(C_1) \cap \mathrm{fr}(C_2) \cap X$, x belongs to the closure of the domain of $\widetilde{F}$ restricted to $\mathrm{cl}(C_i) \cap C$, for each $i \in \{1, 2\}$, and so the limit of $\widetilde{F}(z)$ as z approaches x does not exist. $\square$

Hence, if one could show that the negation of asymptotic monotonicity implies asymptotic non-injectivity, then we would obtain (in the case where $\mathcal{R}$ expands an ordered field) the implication (3) $\implies$ (1), using the shorthand of Corollary 2.16.

Question 2.18. *Let C be a bounded cell which is not asymptotically monotone. Is C asymptotically non-injective?*

If not, then it is not unreasonable to suggest that there is a property which satisfies each of our requirements:

Conjecture 2.19. *Let $\mathcal{R}$ be an o-minimal expansion of an ordered field. There is a property P of cells such that the following are equivalent for a bounded cell C:*

(1) *C has property P.*

(2) *There is a definable $X \subseteq \mathrm{fr}(C)$ such that $\dim(\mathrm{fr}(C) \setminus X) \leq \dim(C) - 2$ and C is locally connected at every $x \in X$.*

(3) *C has the almost everywhere extension property.*

Furthermore, it is not known if asymptotic monotonicity at a point is a definable condition, nor is it known if one can partition an arbitrary definable set into asymptotically monotone cells. Hence we conclude with the following:

Question 2.20. *Let C and X be as in Definition 2.6. Is the set of all $x \in X$ such that C is asymptotically monotone at x a definable set? Is there a cell decomposition theorem for asymptotically monotone cells?*

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