# BLOCK-COLOURINGS OF STEINER 2-DESIGNS

# SPECIAL BLOCK-COLOURINGS OF STEINER 2-DESIGNS

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## Abstract

Let $t, k, v$ be three positive integers such that $2 \leq t < k \leq v$. A *Steiner system* $S(t, k, v)$ is a pair $(\mathcal{V}, \mathcal{B})$ where $|\mathcal{V}| = v$ and $\mathcal{B}$ is a collection of $k$–subsets of $\mathcal{V}$, called blocks, such that every $t$–subset of $\mathcal{V}$ occurs in exactly one block in $\mathcal{B}$. When $t = 2$, the Steiner system $S(2, k, v)$ is sometimes called a Steiner 2-design.

Given a Steiner 2-design, $S = (\mathcal{V}, \mathcal{B})$, with general block size $k$, a block-colouring of $S$ is a mapping $\phi : \mathcal{B} \to C$, where $C$ is a set of colours. If $|C| = n$, then $\phi$ is an $n$-block-colouring. In this thesis we focus on block-colourings for Steiner 2-designs with $k = 4$ with some results for general block size $k$.

In particular, we present known results for $S(2, 4, v)s$ and the classical chromatic index. A *classical* block-colouring is a block-colouring in which any two blocks containing a common element have different colours. The smallest number of colours needed in a classical block-colouring of a design $S = (\mathcal{V}, \mathcal{B})$, denoted by $\chi'(S)$, is the *classical chromatic index.*

We also discuss $n$-block-colourings of type $\pi$, where $\pi = (\pi_1, \pi_2, \ldots, \pi_s)$ is a partition of the replication number $r = \frac{v-1}{k-1}$ for a Steiner system $S(2, k, v)$. In particular, we focus on $S(2, 4, v)s$ and the partitions $(2, 1, 1, \ldots, 1)$, $(3, 1, 1 \ldots, 1)$, and partitions of the form $\pi = (\pi_1, \pi_2, \ldots, \pi_s)$, where $|\pi_j - \pi_i| \leq 1$ for all $1 \leq i < j \leq s$. These latter partitions are called *equitable partitions* and the corresponding block-colourings are called *equitable block-colourings.*

Finally, we present results on the $T$-chromatic index for $S(2, 4, v)s$ for various configurations $T$. The $T$-chromatic index for a Steiner system $S(2, k, v)$, $S$, is the minimum number of colours needed to colour the blocks of $S$ such that there are no monochromatic copies of $T$. In particular, we focus on configurations containing 2 lines and configurations containing 3 lines for both $S(2, 4, v)s$ and general $S(2, k, v)s$.

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# Contents

# Chapter 1

# Introduction

A $t-(v,k,\lambda)$ design is a pair $(\mathcal{V},\mathcal{B})$ where $|\mathcal{V}|=v$ and $\mathcal{B}$ is a collection of $k$–subsets of $\mathcal{V}$, called blocks, such that every $t$–subset of $\mathcal{V}$ occurs in exactly $\lambda$ blocks in $\mathcal{B}$. When $\lambda=1$ we call the $t$-design a Steiner system, denoted $S(t,k,v)$. When $t=2$, the Steiner system $S(2,k,v)$ is sometimes called a Steiner 2-design. In general, for an $S(2,k,v)$, every element is contained in $r=\frac{v-k}{(k-1)}$ blocks; $r$ is called the *replication number.*

When $k=3$ and $t=2$ the Steiner system $S(2,3,v)$ is called a *Steiner triple system* of order $v$ and is denoted by $STS(v)$. For example, the blocks

$$\begin{array}{llll}\{0,1,3,\} & \{1,2,4\} & \{2,3,5\} & \{3,4,6\} \\ \{4,5,0\} & \{5,6,1\} & \{6,0,2\} & \end{array}$$

are the blocks of the unique (up to isomorphism) $STS(7)$. This design is known as the *Fano plane.* It is well known that an $STS(v)$ exists if and only if $v\equiv 1,3 \pmod 6$ [14].

Although much research and literature has been devoted to Steiner triple systems, there has been little done with respect to the case where $t=2$ and $k=4$, a Steiner system $S(2,4,v)$. An example is the unique (up to isomorphism) $S(2,4,13)$ which has as its blocks:

$$\begin{array}{lllll}\{0,1,3,9\} & \{0,2,8,c\} & \{0,4,5,7\} & \{0,6,a,b\} & \\ \{1,2,4,a\} & \{1,5,6,8\} & \{1,7,b,c\} & \{2,3,5,b\} & \\ \{2,6,7,9\} & \{3,4,6,c\} & \{3,7,8,a\} & \{4,8,9,b\} & \{5,9,a,c\},\end{array}$$

where $a=10, b=11, c=12$. It is well known that an $S(2,4,v)$ exists if and only if $v\equiv 1,4 \pmod{12}$ [22].

Both the unique $STS(7)$ and the unique $S(2,4,13)$ are examples of projective planes of orders 2 and 3, respectively. *A finite projective plane of order* $n$ can be viewed as a Steiner system $S(2,n+1,n^2+n+1)$; every two blocks intersect in one element and $n^2+n+1=|\mathcal{B}|$, the number of blocks in the design. Projective planes are examples of *symmetric designs,* a 2-design with $|\mathcal{B}| = v$ (or equivalently, $r = k$).

The orders of $v$ for which the necessary conditions for the existence of an $S(2,k,v)$ are satisfied are called *admissible.* However, if a $t-(v,k,\lambda)$ design does not exist for given $t,k,v,\lambda$, then it is an interesting problem to determine how close we can come to a $t$-design. A $t-(v,k,\lambda)$ *covering design* consists of a $v$-element set $\mathcal{V}$ and a collection of $k$-element subsets (blocks) of $\mathcal{V}$ such that every $t$-element subset of $\mathcal{V}$ is contained in at least $\lambda$ blocks. The *covering number* $C_\lambda(v,k,t)$ is the *minimum* number of blocks in a covering design. When $\lambda = 1$ we denote the covering number by $C(v,k,t)$.

A $t-(v,k,\lambda)$ *packing design* consists of a $v$-element set $\mathcal{V}$ and a collection of $k$-element subsets (blocks) of $\mathcal{V}$ such that every $t$-element subset of $\mathcal{V}$ is contained in at most $\lambda$ blocks. The *packing number* $D_\lambda(v,k,t)$ is the *maximum* number of blocks in a packing design. When $\lambda = 1$ we denote the packing number by $D(v,k,t)$. A $2-(v,k,1)$ packing design is sometimes called a *partial Steiner system.*

A *partial parallel class* in a Steiner design $S(t,k,v), S = (\mathcal{V},\mathcal{B})$, is set of blocks in $\mathcal{B}$ which are mutually disjoint. A *parallel class* is a set of blocks in $\mathcal{B}$ which partitions $\mathcal{V}$. A Steiner system is *resolvable* if the blocks of $\mathcal{B}$ can be partitioned into parallel classes.

A resolvable $STS(v)$ is called a *Kirkman triple system.* It is well known that a resolvable $STS(v)$ exists if and only if $v \equiv 3 \pmod 6$ [14]. Hanani et al showed in [22] that a resolvable $S(2,4,v)$ exists if and only if $v \equiv 4 \pmod{12}$. There exists no special name for resolvable $S(2,4,v)s$.

An *almost parallel class (APC)* is a partial parallel class that partitions $\mathcal{V}\backslash\{\,x\,\}$, for some $x \in \mathcal{V}$. A *Hanani triple system, HATS*, is a Steiner triple system with a partition of its blocks into $\frac{v-1}{2}$ almost parallel classes and a single partial parallel class with $\frac{v-1}{6}$ blocks. Hanani triple systems are known to exist if and only if $v \equiv 1 \pmod 6$ and $v \notin \{\,7,13\,\}$ [38]. It is not known whether there exists an analogous system $S(2,4,v)$ for $v \equiv 1 \pmod{12}$.

An *automorphism* of an $S(t,k,v)$ $S = (\mathcal{V},\mathcal{B})$, is a bijection $\rho : \mathcal{V} \to \mathcal{V}$ such that $\rho(\mathcal{B}) = \mathcal{B}$, where $\rho(\mathcal{B})$ is the set of blocks obtained by applying $\rho$ to the elements in each block of $\mathcal{B}$. In other words, the blocks of the design are preserved under the mapping $\rho$. An $S(t,k,v)$ $, S = (\mathcal{V},\mathcal{B})$, is *cyclic* if its automorphism group contains an automorphism $\alpha$ consisting of a single cycle of length $v$.

It is well known that a cyclic $STS(v)$ exists for all $v \equiv 1,3 \pmod 6, v \neq 9$, [32]. Although the existence of cyclic $STS(v)s$ has been solved completely, the existence question for cyclic $S(2,4,v)s$ remains open. There are, however, several partial results. In 1939, Bose [3] gave a construction of an infinite class of cyclic $S(2,4,v)s$ where $v$ is a prime number. It was shown in [5], [8] that a cyclic $S(2,4,v)$ and a cyclic $S(2,4,4v)$ exist whenever $v$ is a product of primes which are congruent to 1 (mod 12). In [4] Buratti proved that a cyclic $S(2,4,4v)$ exists whenever $v$ has a prime factor $p$ congruent to 1 (mod 6) and $gcd(\frac{p-1}{6}, 20!) \neq 1$. In [6] Chang has shown that there exists a cyclic $S(2,4,4^n u)$ where $n \geq 3$ is a positive integer and $u$ is a product of primes, each congruent to 1 (mod 6), or $n = 2$ and $u$ is a product of primes each congruent to 1 (mod 6) such that $gcd(u, 7, 13, 19) \neq 1$. For small orders, it was shown in [1], [7], that a cyclic $S(2,4,v)$ exists for all $v \equiv 1, 4 \pmod{12}$ and $v \leq 613$, except for the cases $v = 16, 25, 28$ where it is known that they do not exist. The existence of cyclic $S(2,4,v)s$ for general $v \equiv 1, 4 \pmod{12}$ remains an open problem, but it is widely believed that such a system exists for all $v \geq 37$.

A *colouring* of a Steiner system, $S = (\mathcal{V}, \mathcal{B})$, is a surjective mapping $\phi : \mathcal{V} \to C$, where $C$ is the set of colours. A proper element-colouring of $S$ has the property that $|\phi(B)| > 1$ for all $B \in \mathcal{B}$, where $\phi(B) = \cup_{x \in B} \phi(x)$. If $|C| = m$ then we say $S$ has a proper $m$-colouring. The *chromatic number* of $S$, denoted by $\chi(S)$, is the smallest $m$ for which there exists a proper $m$-colouring of $S$.

In other words, in a proper element-colouring of $S$, no block is monochromatic. This type of colouring has also been called *weak* (or *classical*), in accordance with Berge's definition of weak and strong colouring of hypergraphs, respectively. However, strong colourings of hypergraphs require all elements of a block (hyperedge) to be coloured with a different colour. This is not very interesting when colouring Steiner systems as this requirement would imply that each element must get a different colour. There exists extensive literature on the subject of colourings; see [12] and [14] for surveys. See [37] for a survey of colourings of $S(2,4,v)s$.

Although the majority of the literature on colourings has focused on weak colourings, recently there have been investigations into specialized types of colourings of designs in which the elements of each block must have a prescribed colour pattern; for example, see [11], [20], [29], [30], and [31]. Voloshin's concept of mixed hypergraph colourings (cf. [39]), where each block is considered as a $C$-edge and a $D$-edge simultaneously, requires the absence of not only monochromatic but also of polychromatic blocks. The latter papers deal with a refinement of this concept which leads to several types of colourings with specified block patterns. If the block is of size $k$, there are $P_k$ possible colour patterns on a block where $P_k$ is the number of partitions of $k$.

We will consider a dual situation as it pertains to Steiner systems: we colour the blocks in such a way that the collection of blocks containing a given element are coloured according to a prescribed colour partition.

Given an $S(2,k,v)$, $S = (\mathcal{V},\mathcal{B})$, a *block-colouring* of $S$ is a mapping $\phi : \mathcal{B} \to C$ where $C$ is a set of colours. If the set of colours $|C| = n$ then $\phi$ is an $n$-block-colouring. For each $c \in C, \phi^{-1}(c)$ is a *block colour class.* A (*classical*) or *proper* block-colouring is a block-colouring in which any two blocks containing a common element have different colours. The smallest number of colours needed in a proper block-colouring of $S$, denoted by $\chi'(S)$, is the *(classical) chromatic index.* We also define $\chi'(v) = \min\{\, \chi'(S) \mid S \text{ is a Steiner system}\,\}$. It is well known that when $k = 3$ and $v \equiv 3 \pmod 6$ or $v \equiv 1 \pmod 6$, then $\chi'(S) \geq \frac{v-1}{2}$ or $\chi'(S) \geq \frac{v+1}{2}$ , respectively [14]. Similarly, for $k = 4$, $\chi'(S) \geq \frac{v-1}{3}$ or $\chi'(S) \geq \frac{v+2}{3}$ for $v \equiv 4 \pmod{12}$ and $v \equiv 1 \pmod{12}$, respectively. In fact, if $v \equiv 3 \pmod 6$ and $S = (\mathcal{V},\mathcal{B})$ is a resolvable $STS(v)$, then $\chi'(S) = \frac{v-1}{2}$. To obtain such a block-colouring we colour the $\frac{v-1}{2}$ parallel classes with $\frac{v-1}{2}$ distinct colours, where all blocks in each parallel class are coloured with the same colour. Similarly, if $v \equiv 4 \pmod{12}$ and $S = (\mathcal{V},\mathcal{B})$ is a resolvable $S(2,4,v)$, then $\chi'(S) = \frac{v-1}{3}$. If $v \equiv 1 \pmod 6$ and $S = (\mathcal{V},\mathcal{B})$ is a Hanani triple system then $\chi'(S) = \frac{v+1}{2}$. To obtain such a block-colouring we colour the $\frac{v-1}{2}$ almost parallel classes with $\frac{v-1}{2}$ distinct colours and we colour the single partial parallel class with another distinct colour, therefore giving us a $\frac{v+1}{2}$-block-colouring.

It is easy to see that if there exists a proper $m$-block-colouring of an $S(t,k,v)\,,S$, then there exists a proper $(m+1)$-block colouring. Therefore, if we define $\Omega(S) = \{\, m \mid \text{ there exists a proper } m\text{-block-colouring of } S \,\}$ and $\Omega(v) = \bigcup \Omega(S)$ where the union is taken over the set of all $S(2,k,v)$, then $\Omega(S) = \left\{ \chi'(S), \chi'(S)+1, \ldots, \frac{v(v-1)}{k(k-1)} \right\}$ and $\Omega(v) = \left\{ \chi'(v), \ldots, \frac{v(v-1)}{k(k-1)} \right\}$. Thus we have the following corollary.

**Corollary 1.1.** *Let $v \equiv 1,3 \pmod 6$ and consider $STS(v)$s. Then*

$$\Omega(v) = \begin{cases} \{\,7\,\} & \text{if } v = 7; \\ \{\,8, 9, \ldots, 26\,\} & \text{if } v = 13; \\ \{\,\frac{v+1}{2}, \ldots, \frac{v(v-1)}{6}\,\} & \text{if } v \equiv 1 \pmod 6 \text{ and } v > 7; \\ \{\,\frac{v-1}{2}, \ldots, \frac{v(v-1)}{6}\,\} & \text{if } v \equiv 3 \pmod 6 \text{ and } v > 13. \end{cases}$$

As far as we can tell, there has not been much work done on the chromatic index of $S(2,4,v)$s. We will discuss the chromatic index of $S(2,4,v)$s in Chapter 3. In this chapter we will present some basic results for the minimum chromatic index of $S(2,4,v)$s, we determine $\Omega(v)$ for all $v \equiv 4 \pmod{12}$, and in Theorem 3.7 we show that

$$\{\, 2r-1, 2r, \ldots, \frac{v(v-1)}{12}\,\} \subseteq \Omega(v)$$

for all $v \equiv 1 \pmod{12}$.

We now discuss colouring notions inspired by Voloshin's concept of mixed hypergraph colouring ([39]). A *valid* block-colouring of a design $(\mathcal{V}, \mathcal{B})$ is a block-colouring where the blocks are coloured in such a way that every element in $\mathcal{V}$ is contained in at least two blocks of one colour and in at least two blocks of a different colour. A valid $k$-block-colouring is a valid block-colouring which uses $k$ colours; it is important to note that all $k$ colours must be used. We note that valid block-colourings gives rise to the question of the *largest* number of colours possible in such a block-colouring. The *lower chromatic index* $\underline{\chi}'(S)$ of an $S(2,k,v)$ $S = (\mathcal{V}, \mathcal{B})$ is defined as the smallest $k$ for which there exists a valid $k$-block-colouring of $S$. The *upper chromatic index* $\overline{\chi}'(S)$ of an $S(2,k,v)$, $S = (\mathcal{V}, \mathcal{B})$, is defined as the largest $k$ for which there exists a valid $k$-block-colouring of $S$. In other words,

$$\underline{\chi}'(S) = \min\{\, k \mid \text{there exists a valid } k\text{-block-colouring of } S\,\}$$

$$\overline{\chi}'(S) = \max\{\, k \mid \exists \text{ a valid } k\text{-block-colouring of } S\,\}.$$

The following theorems regarding the upper and lower chromatic index for Steiner triple systems were proved in [13].

**Theorem 1.2.** *([13]) Let $S$ be a nontrivial Steiner triple system of order $v$. Then the lower chromatic index $\underline{\chi}'(S) = 3$ if $v = 7$ and $\underline{\chi}'(S) = 2$ otherwise.*

**Theorem 1.3.** *([13]) For any Steiner triple system $S$ of order $v$, the upper chromatic index $\overline{\chi}'(S) \leq \frac{(v-1)(v-3)}{6} - 1$, and for every $v \equiv 1, 3 \pmod 6), v \geq 19$, there exists an $STS(v)$ with $\overline{\chi}'(S) = \frac{(v-1)(v-3)}{6} - 1$.*

We will present analogous theorems in Chapter 3.

The definition of valid block-colourings was generalized in [13]. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_s)$ be a partition of the replication number $r$. For a Steiner triple system $STS(v)$ $S = (\mathcal{V}, \mathcal{B})$, a $k$-block-colouring of type $\pi$ is a block-colouring of $S$ with $k$ colours such that for all elements $x \in \mathcal{V}$, the $r = \frac{v-1}{2}$ blocks containing $x$ are coloured according to the partition $\pi$. For example, given an $STS(v)$ $S = (\mathcal{V}, \mathcal{B})$, take an arbitrary element $x \in \mathcal{V}$ and colour two blocks containing $x$ with colour red and the other blocks containing $x$ with colour blue, then colour all of the remaining blocks of the design with colour green. This gives us a 3-block-colouring of type $\pi = (2,1)$. This definition can easily be extended to any Steiner system $S(2, k, v)$. We will focus on the case for $k = 4$.

The smallest number of colours needed to colour a Steiner system $S$ according to the partition $\pi$ is denoted by $\underline{\chi}'_\pi(S)$ and the maximum number of colours is denoted by $\overline{\chi}'_\pi(S)$. We also define

$$\underline{\chi}'_\pi(v) = \min\{\, \underline{\chi}'_\pi(S) \mid S \text{ is an } S(2,4,v)\,\},$$

$$\overline{\chi}'_\pi(v) = \max\{\, \overline{\chi}'_\pi(S) \mid S \text{ is an } S(2,4,v)\,\},$$

$$\Omega_\pi(v) = \{\, k \mid \exists \text{ an } S(2,4,v) \text{ which admits a } k\text{-block-colouring of type } \pi\,\}.$$

In Chapter 3 we present basic results for general partitions $\pi$. In particular we will consider block-colourings of type $\pi = (2, 1, 1, \ldots, 1)$ and $\pi = (3, 1, 1, \ldots 1)$ for Steiner systems $S(2, 4, v)$. Our new results on these two types of partitions include upper bounds on $\overline{\chi}'(v)$ and partial results on $\Omega_\pi(v)$. We will also include partial results for the general partition $\pi = (s, 1, 1, \ldots, 1)$.

In Chapter 3 we also discuss equitable colourings. An *equitable partition* of the replication number $r$ is a partition $\pi = (\pi_1, \pi_2, \ldots .\pi_s)$ such that $|\,\pi_i - \pi_j\,| \leq 1$ for all $i, j$. A block-colouring of type $\pi$ is *equitable* if $\pi$ is an equitable partition. Equitable colourings for Steiner triple systems are considered in [18]. Chapter 3 presents basic results for equitable bicolourings and equitable tricolourings for general $S(2, k, v)s$ and $S(2, 4, v)s$, and, most importantly, we show that $\chi'_\pi(v) = \overline{\chi}'_\pi = 2$ for any equitable bicolouring of an $S(2, 4, v)$.

In Chapter 2 we discuss the block-colourings obtained when we colour Steiner triple systems $STS(v)s$ and Steiner systems $S(2, 4, v)s$ with two colours.

In Chapter 4 we consider a generalization of the chromatic index of Steiner systems. The $T$-chromatic index for a Steiner system $S(2, k, v)$, $S$, is the minimum number of colours needed to colour the blocks of $S$ such that there are no monochromatic copies of $T$. We first consider the configurations in $S(2, k, v)s$ containing two blocks. There are two possibilities for such a configuration. If $T$ consists of only two intersecting blocks then the $T$-chromatic index is the classical chromatic index. If $T$ consists of only two parallel blocks then the $T$-chromatic index is the minimum number of colours required to colour the blocks of a Steiner system such that any two parallel blocks receive different colours. This is also called the *2-parallel chromatic index, denoted* $\chi''$. Theorem 4.7 shows that $\chi''(v) \leq \frac{v-1}{k-1}$ for a general $S(2,k,v)$ and when $v \geq k^3 + 2k$. We were then able to use this result to show that, for $S(2,4,v)s$, if $v \geq 73$ and $v \equiv 4, 13 \pmod{36}$ then $\underline{\chi}''(v) = (v-1)/3$.

We also consider the case when $T$ is a configuration of $S(2,k,v)$ containing three blocks and present results for all possible such configurations, except for the case where $T$ is a triangle. We were able to prove some results, both partial and complete, on the upper and lower bounds for the chromatic index of these 3-line configurations.

Finally, in the last chapter we give our concluding remarks and present some open problems on block-colourings designs.

# Chapter 2

# Block-colourings of $S(2,4,v)$s with two colours

In this chapter we consider block-colourings of $S(2,4,v)$s using only two colours. In particular we will consider block-colourings of type $P \subseteq K_r^2$, defined below.

**Definition 1.** Let $K_r^2$ be the set of all partitions of $r$ with at most two parts, and let $P \subseteq K_r^2$. Then an $S(2,k,v)$, $S = (\mathcal{V},\mathcal{B})$, has a *$t$-block-colouring (a colouring with $t$ colours where each colour must be used) of profile $P$* if every element in $\mathcal{V}$ is incident with blocks coloured according to a partition in $P$ and for every partition $\pi$ in $P$ there exists an $x \in \mathcal{V}$ which is incident with blocks coloured according to $\pi$.

We first give a result for the whole set $K_r^2$ and general $S(2,k,v)$.

**Theorem 2.1.** *Let $S = (\mathcal{V},\mathcal{B})$ be an $S(2,k,v)$. Then there exists a 2-block-colouring of profile $K_r^2$.*

*Proof.* Let $S = (\mathcal{V},\mathcal{B})$ be an $S(2,k,v)$. In step 1, colour $r-1$ blocks containing $x_1 \in \mathcal{V}$ with colour 1. In step $i$, let $x_i \in \mathcal{V}, x_i \neq x_1, x_2, \ldots, x_{i-1}$. Now let $c_i$ be the number of blocks containing $x_i$ which are coloured with colour 1 in steps $1, 2, \ldots, i-1$. We will colour $r - c_i - i$ blocks containing $x_i$ with colour 1 in step $i$ for all $2 \leq i \leq \lceil \frac{r}{2} \rceil$. Colour all remaining blocks with colour 2. We now have a 2-block-colouring of profile $K_r^2$.

$\square$

We will now only focus on the cases when $|P| = 2, 3$.

Let us first consider the case when $|P| = 2$.

**Theorem 2.2.** *If $v \equiv 1, 4 \pmod{12}, v \geq 16$, then there exists an $S(2,4,v)$ which admits a 2-block-colouring of profile $P = \{(m, r-m), (m+1, r-$*

$m-1)\,\}$ *for all partitions* $(m, r-m)$ *of* $r = \frac{v-1}{3}$, *including the case when* $m = r$.

*Proof.* Let $v \geq 16$. Assume $v \equiv 4 \pmod{12}$. Let $S = (\mathcal{V}, \mathcal{B})$ be a resolvable $S(2,4,v)$ with parallel classes $R_1, R_2, \ldots, R_r$ $(r = \frac{v-1}{3})$. Let $x \in \mathcal{V}$ and let $B_1, B_2, \ldots, B_r$ be the blocks containing $x$. Colour $m$ of these blocks with colour 1 and colour the remaining $r-m$ blocks with colour 2. We let $R_i' = R_i \setminus B_i$. Using colours 1 and 2, colour the $R_i'$s, for all $1 \leq i \leq r$, according to the partition $(m, r-m-1)$ such that all blocks in each $R_i'$ are coloured with the same colour. The result follows.

Assume $v \equiv 1 \pmod{12}$. Let $S = (\mathcal{V}, \mathcal{B})$ be a 1-near-resolvable $S(2,4,v)$ with almost parallel classes $R_1, R_2, \ldots, R_r$. We now proceed as in the case for $v \equiv 4 \pmod{12}$. □

**Theorem 2.3.** *Let* $v \equiv 4, 13 \pmod{36}$. *Then there exists an* $S(2,4,v)$ *which admits a 2-block-colouring of profile* $P = \{\, (\frac{v-4}{9}, \frac{2v+1}{9})\,, (m, r-m)\,\}$ *for all partitions* $(m, r-m)$ *of* $r = \frac{v-1}{3}$, *including the case when* $m = r$.

*Proof.* We will make use of the "$v \to 3v+1$ rule". Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,4,v)$, $v \equiv 1, 4 \pmod{12}$, and let $\mathcal{X}$ be such that $|\mathcal{X}| = 2v+1$ and $\mathcal{X} \bigcap \mathcal{V} = \emptyset$. Let $T = (\mathcal{X}, \mathcal{C})$ be a resolvable $STS(2v+1)$, and let $\mathcal{R} = \{\, R_1, R_2, \ldots, R_v \,\}$ be a resolution of $(\mathcal{X}, \mathcal{C})$. Form the set of quadruples $D_i = \{\, \{\, v_i, x, y, z \,\} \mid v_i \in \mathcal{V}, \{\, x, y, z \,\} \in R_i \,\}$, and put $\mathcal{D} = \bigcup_i D_i$. Then $Y = (\mathcal{V} \bigcup \mathcal{X}, \mathcal{B} \bigcup \mathcal{D})$ is an $S(2,4,3v+1)$.

Now colour the blocks of $\mathcal{B}$ with colour 1. Then colour the blocks of $\mathcal{D}$ with colours 1 and 2 according to the partition $(m, r-m)$ such that every block in $D_i$ get coloured the same colour, for all $1 \leq i \leq v$. The result follows.

□

For the next Theorem we will consider $S(2,4,v)s$ with resolvable subsystems. A resolvable $S(2,4,w)$ is a subsystem of a $S(2,4,v)$ only if the parallel classes of the $S(2,4,w)$ are induced by the parallel classes of the $S(2,4,v)$. It was shown in [16] that

**Theorem 2.4** ([16])**.** *A resolvable* $S(2,4,v)$ *contains a resolvable subsystem* $S(2,4,w)$ *if and only if* $v \geq 4w$ *and* $v \equiv w \equiv 4 \pmod{12}$, *with possible exceptions when* $w < 1852$.

**Theorem 2.5.** *Let* $v \equiv 4 \pmod{12}$ *and let* $S = (\mathcal{V}, \mathcal{B})$ *be a resolvable* $S(2,4,v)$ *which contains a resolvable* $S(2,4,w)$ *subsystem* $T = (\mathcal{W}, \mathcal{C})$. *If* $(s, t-s)$ *and* $(p, t-p)$ *are partitions of* $t = \frac{w-1}{3}$ *then there exists a 2-block-colouring of profile* $P = \{\, (m, r-m), (m-s+p, r-m+s-p)\,\}$.

*Proof.* Let $S = (\mathcal{V}, \mathcal{B})$ be a resolvable $S(2,4,v)$ which contains a resolvable $S(2,4,w)$ subsystem $T = (\mathcal{W}, \mathcal{C})$. Let $\mathcal{R} = R_1, R_2, \dots, R_r$ be the parallel classes of $S$ and let $\mathcal{P} = P_1, P_2, \dots, P_t$ be the parallel classes of $T$. We can assume that $P_i \subset R_i$ for all $1 \leq i \leq t$. We first colour the blocks of $\mathcal{P}$ according to the partition $(p, t-p)$, using colours 1 and 2, such that the blocks in each parallel class are coloured the same colour. Now colour the blocks of $R_i \setminus P_i$ according to the partition $(s, t-s)$, using the colours 1 and 2. Finally, colour the remaining parallel classes of $\mathcal{R}$ according to the partition $(m-s, r-m-t+s)$. the result follows.

□

We now look at the case where $|P| = 3$.

**Theorem 2.6.** *Let* $P = \{(m, r-m), (s, r-s), (s-1, r-s+1)\}$. *If* $v \equiv 4 \pmod{12}, v \geq 16$, *then there exists an* $S(2,4,v)$ *which admits a 2-block-colouring of profile* $M$, *where* $P \subseteq M$. *If* $v \equiv 1 \pmod{12}, v \geq 25$, *then there exists an* $S(2,4,v)$ *which admits a 2-block-colouring of profile* $P$.

*Proof.* Let $v \geq 16$. Assume $v \equiv 4 \pmod 4$. Let $S = (\mathcal{V}, \mathcal{B})$ be a resolvable $S(2,4,v)$. We first colour the parallel classes of $S$ according to the partition $(s, r-s)$ using the colours 1 and 2. Let $x \in \mathcal{V}$. We now recolour all blocks incident with $x$ according to the partition $(m, r-m)$, using the same two colours. We now have a 2-block-colouring of profile $M$, where $P \subseteq M$.

We now assume that $v \equiv 1 \pmod{12}$. Let $S = (\mathcal{V}, \mathcal{B})$ be a 1-near-resolvable $S(2,4,v)$. We first colour the almost parallel classes of $S$ according to the partition $(s-1, r-s)$ using the colours 1 and 2. Let $x \in \mathcal{V}$. We now recolour all blocks incident with $x$ according to the partition $(m, r-m)$, using the same two colours. We now have a 2-block-colouring of profile $P$ and our result follows.

□

We also present some results for sets $P \geq 2$ in general.

**Theorem 2.7.** *Let* $v \equiv 4, 13 \pmod{36}$ *and let* $P$ *be a set of partitions of* $r = \frac{v-1}{3}$ *such that* $|P| \leq r$ *and each partition is of the type* $(m, r-m)$ *and* $m \geq \frac{v-4}{9}$. *Then there exists an* $S(2,4,v)$ *with a 2-block-colouring of profile* $M$, *where* $P \subseteq M$.

*Proof.* We again make use of the "$v \to 3v+1$ rule" to create an $S(2,4,3v'+1) = (\mathcal{V}, \mathcal{B})$ with a subsystem $S(2,4,v') = (\mathcal{W}, \mathcal{C})$. We first colour the blocks of $\mathcal{C}$ with colour 1. Take an arbitrary partition $(m, r-m)$ from $P$ and let $w \in \mathcal{W}$. Colour the blocks in $\mathcal{B} \backslash \mathcal{C}$ incident with $w$ with colours 1 and 2 according to the partition $(m - \frac{v-4}{9}, r-m)$. Continue this process for all

$w \in \mathcal{W}$ and for all $(m, r-m) \in P$. We now have a 2-block-colouring of profile $M$, where $P \subseteq M$.

$\square$

**Theorem 2.8.** *Let $v \equiv 4, 13 \pmod{36}$ and let $P$ be a set of partitions of $r = \frac{v-1}{3}$ such that $|P| \leq r$ and each partition is of the type $(m, r-m)$ and $m \geq \frac{v-4}{9}$. If there exists an $S(2,4,\frac{v-1}{3})$ with a 2-block-colouring of type $(s, \frac{v-4}{9} - s)$ then there exists an $S(2,4,v)$ with a 2-block-colouring of profile $M$, where $P \subseteq M$.*

*Proof.* We again make use of the "$v \longrightarrow 3v+1$ rule" to create an $S(2,4,3v'+1) = (\mathcal{V}, \mathcal{B})$ with an $S(2,4,v') = (\mathcal{W}, \mathcal{C})$ subsystem. Assume that $(\mathcal{W}, \mathcal{C})$ admits a 2-block-colouring of type $(s, \frac{v-4}{9} - s)$. We give $(\mathcal{W}, \mathcal{C})$ such a colouring by colouring $s$ blocks with colour 1 and the remaining blocks with colour 2. Take an arbitrary partition $(m, r-m)$ from $P$ and let $w \in \mathcal{W}$. Colour the blocks in $\mathcal{B} \backslash \mathcal{C}$ incident with $w$ with colours 1 and 2 according to the partition $(m-s, r - \frac{v-4}{9} + s)$. Continue this process for all $w \in \mathcal{W}$ and for all $(m, r-m) \in P$. We now have a 2-block-colouring of profile $M$, where $P \subseteq M$.

$\square$

# Chapter 3

# Block-colourings of $S(2,4,v)$s

In this chapter we explore the various types of block-colourings of $S(2,4,v)$s. Section 3.2.1 discusses colourings of type $\pi = (2,1,1,\ldots,1)$, section 3.2.2 deals with colourings of type $\pi = (3,1,1,\ldots,1)$, section 3.3 deals with equitable colourings, section 3.4 lists the known results for block-colourings of $S(2,4,v)s$ with small $v$, and section 3.5 gives further miscellaneous results on block-colourings.

We first define group divisible designs.

Let $\mathcal{V}$ be a $v$-set and $K$ a set of positive integers $\geq 2$. A *group divisible design* $K-GDD$ of type $g_1^{u_1} g_2^{u_2} \ldots g_n^{u_n}$ is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ with the following properties.

1. $\mathcal{G}$ is a partition of the set $\mathcal{V}$ into $g_i$-subsets (groups), for all $1 \leq i \leq n$, such that $u_1 g_1 + u_2 g_2 + \ldots + u_n g_n = v$,
2. $\mathcal{B}$ is a collection of $k$-subsets (blocks) of $\mathcal{V}$, where $k \in K$,
3. each pair of elements in $\mathcal{V}$ occurs in exactly one block in $\mathcal{B}$ or in exactly one group in $\mathcal{G}$.

The *GDD* is *resolvable* if the blocks can be partitioned into parallel classes.

A *transversal design* $TD(k,v)$ is a $k-GDD$ of type $v^k$.

Throughout the remainder of the thesis we will be using $[a,b]$ to denote the set $\{\, c \mid a \leq c \leq b \,\}$.

## 3.1 Chromatic index for $S(2,4,v)s$

In this section we first discuss the classical chromatic index of $S(2,4,v)s$.

As was previously discussed in the Introduction, the minimum chromatic index $\chi'(v)$ for $STS(v)s$ was completely determined. But there has been little research done on the chromatic index for $S(2,4,v)s$, and in particular, on the minimum chromatic index for $S(2,4,v)s$. The chromatic index of a Steiner system $S(2,4,v)$ cannot exceed $\frac{4v}{3}$. This follows by considering the block-intersection graph of an $S(2,4,v)$ and utilizing Brooks' Theorem ([40]) which gives an upper bound on the chromatic number of the graph. Given a Steiner system, $S = (\mathcal{V}, \mathcal{B})$, its *block-intersection graph* $G_S$ is the graph having vertex set $\mathcal{B}$ such that two vertices $B_1$ and $B_2$ are adjacent if and only if $B_1$ and $B_2$ have non-empty intersection. The *chromatic number* of a graph $G$ is the minimum number of colours needed to colour the vertices of a graph so that there is no monochromatic edge. Therefore, the chromatic number of a block-intersection graph is equal to the chromatic index of the underlying Steiner system. If we denote the chromatic number of a graph $G$ by $\chi(G)$ and its maximum degree by $d$, Brooks' Theorem states the following:

**Theorem 3.1** (Brooks' Theorem,[40]). *If $G$ is not an odd length circuit or a complete graph, then $\chi(G) \leq d$.*

The block-intersection graph of an $S(2,4,v)$ is regular of degree $4(r-1) = \frac{4v-16}{3}$, thus by Brooks' Theorem its chromatic number cannot exceed $\frac{4v-16}{3}$. This trivial upper bound was improved for cyclic Steiner 2-designs by Colbourn and Colbourn ([9]), who reduced this upper bound to $v$. This is obtained by observing that a cyclic $S(2,4,v)$ for $v \equiv 1 \pmod{12}, v = 12t+1$, contains $t = \frac{v-1}{12}$ full length block orbits; the block-intersection graph of a single such orbit is regular of degree 12, hence there exists a proper block-colouring of such a cyclic $S(2,4,v)$ with $12t = v-1$ colours: just colour each orbit with 12 colours. Proceeding similarly when $v \equiv 4 \pmod{12}, v = 12t+4$, where a cyclic $S(2,4,v)$ contains $t$ full-length block orbits plus a single short orbit–assigning one extra colour to the short orbit which is in fact a parallel class–yields the upper bound of $v$ on the chromatic index. However, apart from $PG(2,3)$, the unique $S(2,4,13)$, which indeed requires $v = 13$ colours, the "real" chromatic index is apparently much smaller. By how much smaller remains an open question.

Let us consider the case $v \equiv 4 \pmod{12}$. It is easy to see that for such a $v$, $\chi'(v) = r = \frac{v-1}{3}$ and $\Omega(v) = \{\frac{v-1}{3}, \frac{v-1}{3}+1, \ldots, \frac{v(v-1)}{12}\}$. In fact, for $v \equiv 4 \pmod{12}$ take a resolvable $S(2,4,v)$, $S$, which exists by [22], with $r$ parallel classes $R_1, \ldots, R_r$. We colour the blocks of $S$ with $r$ distinct colours such that all blocks in each parallel class get coloured with the same colour. This gives us a proper $r$-block-colouring. We can now obtain a proper $(r+i)$-block-colouring by colouring $i$ arbitrary blocks with $i$ distinct colours not already used, where $1 \le i \le b - r$.

The case when $v \equiv 1 \pmod{12}$ is more interesting and much more difficult. When $v \equiv 1 \pmod 6$ we obtain the proper $k$-block colouring spectrum $\Omega(v)$ for $STS(v)s$ by utilizing Hanani triple systems. However, there is no known result for an analogous design for $S(2,4,v)s$. By this we mean that, since $\chi'(v) \ge \frac{v+2}{3}$ for all $v \equiv 1 \pmod{12}$, an $S(2,4,v), S$, with minimum chromatic index $\chi'(S) = \frac{v+2}{3}$ is not yet known for any $v \equiv 1 \pmod{12}$. But we do know the chromatic index of the unique $S(2,4,13)$ and the chromatic indices for the 18 non-isomorphic $S(2,4,25)s$. Since the unique $S(2,4,13)$ is a projective plane of order 3, any two blocks of the design intersect in exactly one point. Thus it is easy to see that $\chi'(13) = 13$ and $\Omega(13) = \{\,13\,\}$.

The chromatic index for the 18 non-isomorphic $S(2,4,25)s$ was determined by Meszka ([27]). Using the numbering scheme from [10] it was shown that the $S(2,4,25)s$ No. 12, 13, 15, and 16, with automorphism groups of order 9,9,21, and 63, respectively, have chromatic index 12, the $S(2,4,25)$ No. 17 has chromatic index 10, while the remaining 13 of the 18 $S(2,4,25)s$ have chromatic index 11. Therefore, $\chi'(25) = 10$ and $\Omega(25) = \{\,10, 11, \ldots, 50\,\}$. So for $v \in \{\,13, 25\,\}$ there is no $S(2,4,v), S$, with minimum chromatic index $\chi'(S) = \frac{v+2}{3}$. Thus, if an $S(2,4,v)$, $v \equiv 1 \pmod{12}$, with minimum chromatic index $\frac{v+2}{3}$ exists, then necessarily $v \ge 37$. However, the existence of such an $S(2,4,v)$ is open at present.

Now consider the case $v = 37$. There are exactly two non-isomorphic cyclic $S(2,4,37)$. The base blocks for these systems are

$$\begin{array}{llll} 1) & \{\,0,1,2,24\,\} \ \{\,0,4,9,15\,\} & \{\,0,7,17,25\,\} & \text{and} \\ 2) & \{\,0,1,3,24\,\} \ \{\,0,4,26,32\,\} & \{\,0,7,17,25\,\} & , \end{array}$$

respectively. The first of these systems has no almost-parallel class but contains 555 distinct partial parallel classes with 8 blocks. The maximum number of such pairwise disjoint classes is 11 which implies that the classical chromatic index of system 1) is at least 15. On the other hand, the following block-colouring with 15 colours establishes that the classical chromatic index of the cyclic system 1) equals 15;

$C1: \{0,1,3,24\}$ $\{26,4,1,7\}$ $\{4,26,30,35\}$ $\{5,9,14,20\}$
$\{7,8,10,31\}$ $\{12,13,15,36\}$ $\{18,22,27,33\}$ $\{19,23,28,34\}$

$C2: \{0,2,23,36\}$ $\{1,5,10,16\}$ $\{3,25,29,34\}$ $\{4,12,24,31\}$
$\{6,7,9,30\}$ $\{11,15,20,26\}$ $\{13,17,22,28\}$ $\{19,32,33,35\}$

$C3: \{0,4,9,15\}$ $\{1,12,16,34\}$ $\{3,10,20,28\}$ $\{5,13,25,32\}$
$\{7,19,26,36\}$ $\{8,21,22,24\}$ $\{14,18,23,29\}$ $\{17,30,31,33\}$

$C4: \{0,5,11,33\}$ $\{1,8,18,26\}$ $\{2,14,21,31\}$ $\{3,16,17,19\}$
$\{4,10,32,36\}$ $\{7,15,27,34\}$ $\{9,22,23,25\}$ $\{20,24,29,35\}$

$C5: \{0,6,28,32\}$ $\{1,2,4,25\}$ $\{3,9,31,35\}$ $\{5,18,19,21\}$
$\{7,11,16,22\}$ $\{8,12,17,23\}$ $\{13,26,27,29\}$ $\{20,33,34,36\}$

$C6: \{0,7,17,25\}$ $\{1,9,21,28\}$ $\{4,8,13,19\}$ $\{10,23,24,26\}$
$\{11,12,14,35\}$ $\{16,29,30,32\}$

$C7: \{0,8,20,27\}$ $\{3,13,21,33\}$ $\{5,12,22,30\}$ $\{6,18,25,35\}$
$\{7,14,24,32\}$ $\{9,16,26,34\}$ $\{15,22,29,31\}$

$C8: \{0,10,18,30\}$ $\{1,23,27,32\}$ $\{2,7,13,35\}$ $\{3,8,14,36\}$
$\{4,11,21,29\}$ $\{5,17,24,34\}$ $\{6,19,20,22\}$ $\{12,25,26,28\}$

$C9: \{0,12,19,29\}$ $\{2,8,30,34\}$ $\{3,15,22,32\}$ $\{6,14,26,33\}$
$\{10,17,27,35\}$ $\{11,18,28,36\}$ $\{16,20,25,31\}$

$C10: \{0,13,14,16\}$ $\{2,10,22,29\}$ $\{3,4,6,27\}$ $\{7,20,21,23\}$
$\{8,9,11,32\}$ $\{15,19,24,30\}$

$C11: \{0,21,34,35\}$ $\{1,11,19,31\}$ $\{2,3,5,26\}$ $\{4,17,18,20\}$
$\{6,16,24,36\}$ $\{8,15,25,33\}$ $\{14,27,28,30\}$

$C12: \{0,22,26,31\}$ $\{1,14,15,17\}$ $\{2,9,19,27\}$ $\{3,7,12,18\}$
$\{4,16,23,33\}$ $\{5,6,8,29\}$ $\{10,11,13,34\}$ $\{21,25,30,36\}$

$C13 : \{1,7,29,33\}$ $\{2,12,20,32\}$ $\{3,11,23,30\}$ $\{4,14,22,34\}$
$\{5,27,31,36\}$ $\{10,14,19,25\}$ $\{12,16,21,27\}$ $\{18,31,32,34\}$

$C14 : \{1,13,20,30\}$ $\{2,24,28,33\}$ $\{5,15,23,35\}$ $\{9,17,29,36\}$
$\{10,14,19,25\}$ $\{12,16,21,27\}$ $\{18,31,32,34\}$

$C15 : \{1,22,35,36\}$ $\{2,15,16,18\}$ $\{4,5,7,28\}$ $\{6,13,23,31\}$
$\{9,10,12,33\}$ $\{11,24,25,27\}$ $\{17,21,26,32\}$.

This means that the classical chromatic index of system 1) exceeds the minimum possible chromatic index (13) by 2.

Cyclic system 2) has 37 almost-parallel classes and 1443 partial parallel classes with 8 blocks. The minimum number of such pairwise disjoint parallel classes is 12, thus the classical chromatic index of system 2) is at least 14. It is conceivable, but unlikely, that the chromatic index of system 2) equals 14. Therefore, the chromatic index of system 2) is equal to 14 or 15 since a 15-block-colouring for system 2) also exists.

We now consider projective spaces $PG(d,3)$, an $S(2,4,v)$ with $v = \frac{3^{d+1}-1}{2}$. We are able to show that the design of points and lines of the projective space $PG(d,3)$ of even dimension $d$ cannot have chromatic index equal to $\frac{v+2}{3}$. We follow an argument analogous to the one employed by Wilson for the case of $STS(v)s$ (cf. [14], Theorem 19.6). We will abuse the language slightly by speaking simply of $PG(d,3)$ instead of "the design of points and lines of $PG(d,3)$."

**Theorem 3.2.** *If $d$ is even then $PG(d,3)$ contains no almost parallel class.*

*Proof.* It is well known ([23]) that a projective space $PG(d,3)$ of dimension $d$ has $3^d+3^{d-1}+\ldots+3+1 = \frac{3^{d+1}-1}{2}$ points and $(3^{d+1}-1)(3^{d+1}-3)/48$ lines. The number of elements $v = \frac{3^{d+1}-1}{2}$ satisfies $v \equiv 1 \pmod{12}$ only when $d$ is even. In any $S(2,4,12n+1)$ with a (maximal) subsystem $S(2,4,4n)$, an $APC$ cannot miss an element not in the subsystem. Indeed, every block contains either zero or three elements not in the subsystem. Since the number of elements in an $S(2,4,12n+1)$ not in a subsystem $S(2,4,4n)$ is also a multiple of three, the number of elements not in the subsystem that are missed by any partial parallel class is a multiple of three. But an $APC$ misses exactly one element so it must contain all elements not in the subsystem. On the other hand, in a $PG(d,3)$ for every element $x$ there is a sub$-PG(d-1,3)$ that misses $x$. Since no element can be missed by an $APC$, we conclude that $PG(d,3)$ with $d$ even contains no $APC$.

□

**Lemma 3.3.** *Let $S$ be an $S(2,4,v)$ without an APC. Then $\chi'(S) \geq (v+5)/3$.*

*Proof.* The maximum number of blocks in a block colour class is $\frac{v-5}{4}$. If we had only $\frac{v+2}{3}$ colour classes, we would have at most $\left(\frac{v-5}{4}\right)\left(\frac{v+2}{3}\right)$ blocks which is less than the total number $\frac{v(v-1)}{12}$ of blocks. □

**Corollary 3.4.** *The chromatic index of $PG(d,3)$, $d$ even, is $\geq (3^d+3)/2$.*

For $v \equiv 4 \pmod{12}$, a lemma analogous to Lemma 3.3 gives us a slightly stronger result.

**Lemma 3.5.** *Let $S$ be an $S(2,4,12n+4)$ without a parallel class. Then $\chi'(S) \geq (v+5)/3$.*

*Proof.* The number of blocks in a colour class in a block-colouring of such an $S(2,4,v)$ is at most $\frac{v}{4}-1$. If we had only $\frac{v-1}{3}+1 = \frac{v+2}{3}$ colour classes, we would have at most $\left(\frac{v}{4}-1\right)\left(\frac{v+2}{3}\right)$ blocks which is less than $\frac{v(v-1)}{12}$. □

Thus, when $v \equiv 4 \pmod{12}$, the chromatic index of an $S(2,4,v)$ without a parallel class exceeds the minimum possible chromatic index by at least 2. In [25], Krčadinac constructed $S(2,4,28)s$ which have no parallel class. Therefore there exists an $S(2,4,28), S$, such that $\chi'(S) \geq \frac{v-1}{3}+2 = 11$.

For the remaining cases when $v \equiv 1 \pmod{12}$ we were able to obtain a partial result with respect to $\Omega(v)$ in Theorem 3.7. But we first define some terms needed in the proof of the theorem.

Let $S = (\mathcal{V},\mathcal{B})$ be an $S(2,4,v)$. Let $N \subseteq V$ be a set of $n$ points. A *parallel class with respect* to $N$ is set of blocks in $\mathcal{B}$ which partitions the point set $V \backslash N$. If we delete all blocks of $S$ containing elements of $N$ and are able to partition the remaining blocks into parallel classes with respect to $N$, then we say that $S$ is *$n$-near-resolvable* (this design was first defined by Zeitler in [41]). For example, consider the $S(2,4,v)$ No.1 in [24]: for $N = \{25\}$ this design is 1-near resolvable. The near resolvable classes are shown below:

$$
\begin{array}{llll}
N1: & \{1,2,3,19\} & \{4,5,6,20\} & \{7,8,9,21\} \\
 & \{10,13,16,23\} & \{11,14,17,24\} & \{12,15,18,22\} \\[1em]
N2: & \{2,24,10,9\} & \{14,6,22,1\} & \{23,18,7,5\} \\
 & \{20,3,15,17\} & \{16,21,4,12\} & \{8,11,19,13\} \\[1em]
N3: & \{24,12,20,7\} & \{21,22,13,2\} & \{17,19,23,6\} \\
 & \{1,10,11,4\} & \{15,5,14,8\} & \{18,16,9,3\} \\[1em]
N4: & \{12,8,1,23\} & \{5,13,3,24\} & \{4,9,17,22\} \\
 & \{11,6,21,18\} & \{2,20,14,16\} & \{19,15,7,10\} \\[1em]
N5: & \{8,18,2,17\} & \{6,3,10,12\} & \{14,7,4,13\} \\
 & \{24,1,15,21\} & \{16,22,5,19\} & \{9,11,23,20\} \\[1em]
N6: & \{18,19,24,4\} & \{22,10,20,8\} & \{21,23,14,3\} \\
 & \{12,2,11,5\} & \{15,13,6,9\} & \{7,16,17,1\} \\[1em]
N7: & \{19,9,12,14\} & \{13,20,1,18\} & \{5,17,21,10\} \\
 & \{8,24,16,6\} & \{11,3,22,7\} & \{23,15,4,2\}
\end{array}
$$

A parallel class with respect to $N$ is called an *almost parallel class* when $|N| = 1$. It is easy to see that a 1-near resolvable $S(2,4,v)$ exists if and only if a $4-GDD$ of type $3^r$ exists. If $v \equiv 1 \pmod{12}$, a $4-GDD$ of type $3^r$ exists for all $r \equiv 0 \pmod 4$ [10], which implies a 1-near resolvable $S(2,4,v)$ exists for all admissible $v$. Therefore, we have

**Theorem 3.6** ([41]). *If $v \equiv 1 \pmod{12}$ then there exists a 1-near resolvable $S(2,4,v)$.*

An alternate proof can be found in [41], where the author also proves results for $|N| > 1$.

**Theorem 3.7.** *Let $v \equiv 1,4 \pmod{12}$ and $\pi = (r_1, r_2, \ldots, r_m, 1)$, where $r_1 + r_2 + \cdots + r_m = r - 1$. Then there exists a $(2m+1)$-colouring of type $\pi$ for some $S(2,4,v)$, where $m + 1 \le \frac{v}{4}$ when $v \equiv 4 \pmod{12}$. Furthermore, if $r_1 = r_2 = \ldots = r_j = 1$ for some $1 \le j \le m$, then $\Omega_\pi(v) \supset \{2m+1, 2m+2, \ldots, 2m+1+j\frac{v-1}{4}\}$.*

*Proof.* If $v \equiv 4 \pmod{12}$ and $m + 1 \le \frac{v}{4}$, let $S = (\mathcal{V}, \mathcal{B})$ be a resolvable $S(2,4,v)$. Colour $r_1$ parallel classes with colour 1, another $r_2$ parallel classes with colour 2, and continue in this fashion, colouring another $r_i$ classes with colour $i$ for all $1 \leq i \leq m$. Finally, colour the remaining parallel class with $m+1$ colours. This proves the result for $v \equiv 4 \pmod{12}$.

Now let $S = (\mathcal{V}, \mathcal{B})$ be a 1-near-resolvable $S(2,4,v)$. Without loss of generality, assume the almost parallel classes partition the set $\mathcal{V} \backslash \{ v_0 \}$. Colour $r_1$ almost parallel classes with colour 1, $r_2$ almost parallel classes with colour 2, and continue, colouring $r_i$ classes with colour $i$ for all $1 \leq i \leq m$. Now colour $r_i$ blocks containing $v_0$ with colour $m+i$ for all $1 \leq i \leq m$. Finally, colour the remaining block with colour $2m+1$. Now for each $r_k = 1$ we can recolour an almost parallel class with $j$ additional colours, where $j \in \{ 1, 2, \ldots, \frac{v-1}{4} \}$. □

This gives the following corollary regarding the classical chromatic index.

**Corollary 3.8.** *Let $v \geq 16$ and let $\pi_r = \overbrace{(1,1,\ldots,1)}^{r}$, where $r = \frac{v-1}{3}$. If $v \equiv 4 \pmod{12}$, then $\Omega_{\pi_r}(v) = \{ r, r+1, \ldots, \frac{v(v-1)}{12} \}$. If $v \equiv 1 \pmod{12}$, then $\Omega_{\pi_r}(v) \supseteq \{ 2r-1, 2r, \ldots, \frac{v(v-1)}{12} \}$.*

*Proof.* When $v \equiv 1 \pmod{12}$ we use Theorem 3.7.

When $v \equiv 4 \pmod{12}$ we simply colour $r$ parallel classes of a resolvable $S(2,4,v)$ with $r$ colours such that each block in the same parallel class gets coloured with the same colour. We then recolour each block consecutively until each block gets coloured with a separate colour. The result follows. □

## 3.2 Block-colourings of type $\pi$

Let $\pi = (\pi_1, \pi_2, \ldots, \pi_s)$ be a partition of the replication number $r = \frac{v-1}{3}$. For a Steiner system $S(2,4,v)$, $S = (\mathcal{V}, \mathcal{B})$, a $k$-block-colouring of type $\pi$ is a block-colouring of $S$ such that for all elements $x \in \mathcal{V}$, the $r$ blocks containing $x$ are coloured according to the partition $\pi$.

The smallest number of colours needed to colour a Steiner system $S$ according to the partition $\pi$ is denoted $\underline{\chi}'_\pi(S)$ and the maximum number of colours is denoted $\overline{\chi}'_\pi(S)$. We also define

$$\underline{\chi}'_\pi(v) = \min\{ \underline{\chi}'_\pi(S) \mid S \text{ is an } S(2,4,v) \},$$

$$\overline{\chi}'_\pi(v) = \max\{ \overline{\chi}'_\pi(S) \mid S \text{ is an } S(2,4,v) \},$$

$$\Omega_\pi(v) = \{ k \mid \exists \text{ an } S(2,4,v) \text{ which admits a } k\text{-block-colouring of type } \pi \}.$$

In this section we investigate block-colourings of type $\pi$ for various partitions $\pi$. Our main result will be for the partitions $\pi = (2, 1, 1, \dots, 1)$ and $\pi = (3, 1, 1, \dots, 1)$.

For a general $S(2, k, v)$ and $\pi = (r-1, 1)$, where $r = \frac{v-1}{k-1}$, we have the following result.

**Lemma 3.9.** *For any $S(2, k, v)$ $S$, and the partition $\pi = (r-1, 1)$, we have $\Omega_\pi(S) = \{2, 3, \dots, \frac{v}{k} + 1\}$ if $S$ has a parallel class and $= \{3\}$ otherwise.*

*Proof.* The following proof is analogous to the proof of Theorem 3 in [13]. Let $x$ be an arbitrary element of an $S(2, k, v)S$. Colour $r-1$ blocks containing $x$ colour 1, the remaining block containing $x$ colour 2, and all the remaining blocks in the design colour 3. This gives us a 3-colouring of type $\pi$. When $S$ has a parallel class, we obtain an $(i+1)$-colouring by colouring the blocks of the parallel class with $i \in \{1, 2, \dots, \frac{v}{k}\}$ colours and the remaining blocks one colour. It is easily seen that in any block-colouring of $S$ of type $\pi$, each colour class is either (a), obtained by deleting the blocks of a parallel class from the set of all blocks of $S$, (b) obtained by deleting an element and all blocks containing it, or (c) a set of $r-1$ blocks containing one given element, or (d) a partial parallel class. Since there can be at most one colour class of type (a), (b), and (c) in any colouring of type $\pi$, our proof is complete. □

Thus, for the case $k = 4$ we have

**Corollary 3.10.** *For any $S(2, 4, v)$ $S$, and the partition $\pi = (r-1, 1)$, we have $\Omega_\pi(S) = \{2, 3, \dots, \frac{v}{4} + 1\}$ if $S$ has a parallel class and $= \{3\}$ otherwise.*

We now consider the partitions of the type $\pi = (t, t, \dots, t)$. If $D$ is a non-empty set of integers then a $D$-configuration is a configuration where each element has degree $d \in D$. If $D = \{d\}$ then we refer to a (regular) $d$-configuration. It is easy to see that when a Steiner system admits a block-colouring of type $\pi$ that the colour classes are $D$-configurations, where $D$ is the set of integers in the partition $\pi$.

**Lemma 3.11.** *If an $S(2, 4, v)$ admits a block-colouring of type $\pi$ for some partition $\pi$ of $r = \frac{v-1}{3}$ then each colour class is a $D$-configuration, where $D$ is a set of integers from $\pi$.*

The converse of this lemma is not true for general $D$ but it is true when $D = \{d\}$.

**Lemma 3.12.** *A $k$-block-colouring of type $\pi = (d, d, \dots, d)$ of an $S(2, 4, v)$, $S = (\mathcal{V}, \mathcal{B})$, exists if and only if $\mathcal{B}$ can be decomposed into $k$ $d$-configurations.*

*Proof.* Assume that $S$ admits a $k$-block-colouring of type $\pi$. Then each colour class is a $d$-configuration by Lemma 3.11. Now assume that $\mathcal{B}$ can be decomposed into $k$ $d$-configurations. Colour the configurations with $k$ colours such that all blocks in the same configuration get coloured with one colour. This gives us a $k$-block-colouring of type $\pi$.

□

Let us now consider the case for small $d$. The case $d = 1$ corresponds to the classical chromatic index. See Section 3.1 for results.

In order for an $S(2,4,v)$ to admit a block-colouring of type $\pi = (2,\ldots,2)$, we must have $v \equiv 1 \pmod{12}$. The existence of block-colourings of this type remains an open question.

In order for an $S(2,4,v)$ to admit a block-colouring of type $\pi = (3,\ldots,3)$ we must have $v \equiv 1,28 \pmod{36}$. When $v \equiv 28 \pmod{36}$ we can colour the parallel classes of a resolvable $S(2,4,v)$ according to the partition $\pi$, giving us a colouring using $\frac{v-1}{9}$ colours and $\underline{\chi}'_\pi(v) = \frac{v-1}{9}$. The case for $v \equiv 1 \pmod{36}$ seems more difficult and remains an open question. The spectrum for both cases is also open.

When $d = 4$, the necessary condition for the existence of a block-colouring of type $\pi = (4,4,\ldots,4)$ is $v \equiv 1 \pmod{12}$. Indeed, a cyclic $S(2,4,v)$ for $v \equiv 1 \pmod{12}$ admits such a colouring. Simply colour the orbits of the cyclic design such that each block in the same orbit are assigned the same colour. This gives us a $\frac{v-1}{12}$-block-colouring of type $\pi$ and $\underline{\chi}'_\pi(v) = \frac{v-1}{12}$. Now cyclic $S(2,4,v)s$ are known to exist for all $v \leq 613$. Therefore, such a colouring exists for all $v \leq 613$ and $v \equiv 1 \pmod{12}$. Determining the spectrum for such a colouring seems more difficult, but the following lemma is helpful.

**Lemma 3.13.** *If there exists a $BIBD(v,b,r,k,1)$ with $k \equiv 1,4 \pmod{12}$ then there exists a $b$-colouring of type $\pi = \left(\frac{k-1}{3},\frac{k-1}{3},\ldots,\frac{k-1}{3}\right)$ for some $S(2,4,v)$.*

*Proof.* For such a $BIBD(v,b,r,k,1)$ we take each block and replace it with $S(2,4,k)$, each of which can be viewed as a $\frac{k-1}{3}$-configuration in the $S(2,4,v)$. Lemma 3.12 gives us our result.

□

For example, the existence of a resolvable $BIBD(v,13,1)$ implies $\Omega_\pi(v) = \left[\frac{v-1}{12},\frac{v(v-1)}{156}\right]$. Therefore, the existence of an affine space over $GF(13)$ implies that this equation holds for all $v = 13^n$, where $n \geq 2$.

### 3.2.1 Block-colourings of type $\pi = (2, 1, 1, \ldots, 1)$

In this subsection we consider block-colourings of $S(2,4,v)$s of the type $\pi = (2, 1, 1, \ldots, 1)$. We note that is easy to see that $\underline{\chi}'_\pi(v) \geq \frac{v-4}{3}$.

**Theorem 3.14.** *For $v \equiv 4 \pmod{12}$ we have $\left[\frac{v-4}{3}, \frac{v(v-7)}{12} + 1\right] \subseteq \Omega_\pi(v)$.*

*Proof.* Let $\pi = (2, 1, 1, \ldots, 1)$. Let $v \equiv 4 \pmod{12}, v \geq 16$. Let $R_1, \ldots, R_r$ be the parallel classes of an $S(2,4,v), (\mathcal{V}, \mathcal{B})$. Colour the blocks of $R_1, R_2$ with $C_1$, those of $R_3$ with $C_2$, and those of $R_i$ with $C_{i-1}$ for all $4 \leq i \leq r$. This gives us a $\frac{v-4}{3}$-block-colouring. We can now recolour consecutively each block in the colour classes $C_2, \ldots, C_{r-1}$ until each block has its own colour. Therefore, we have $\left[\frac{v-4}{3}, \frac{v(v-7)}{12} + 1\right] \subseteq \Omega_\pi(v)$.

□

We now have the following corollary.

**Corollary 3.15.** *If $v \equiv 4 \pmod{12}$ then $\underline{\chi}'_\pi(v) = \frac{v-4}{3}$.*

*Proof.* Since $\underline{\chi}'_\pi(v) \geq \frac{v-4}{3}$, Theorem 3.14 gives us $\underline{\chi}'_\pi(v) = \frac{v-4}{3}$ when $v \equiv 4 \pmod{12}$.

□

**Theorem 3.16.** *For $v \equiv 1 \pmod{12}$ we have $\left[\frac{2v-11}{3}, \frac{(v-4)(v-3)}{12}\right] \subseteq \Omega_\pi(v)$.*

*Proof.* Let $v \equiv 1 \pmod{12}$ and let $S = (\mathcal{V}, \mathcal{B})$ be a 1-near-resolvable $S(2,4,v)$ with almost parallel classes $A_1, A_2, \ldots, A_{r-1}$. Assume that the almost parallel classes have $v_0$ as the missing element. Colour all blocks of $A_1, A_2$ with the colour $C_1$, and colour all blocks of $A_i$ with the colour $C_{i-1}$ for all $3 \leq i \leq r-1$. Now colour all blocks incident with $v_0$ with $r-1$ different colours according to the partition $\pi = (2, 1, 1, \ldots, 1)$. This gives us a $(2r-3)$-block-colouring of type $\pi$. Now recolour the blocks of $A_i$, $i \neq 1, 2$, consecutively until each block gets a different colour. This gives us $\left[\frac{2v-11}{3}, \frac{(v-4)(v-3)}{12}\right] \subseteq \Omega_\pi(v)$.

□

**Theorem 3.17.** *For any $S(2,4,v), S = (\mathcal{V}, \mathcal{B})$, of order $v$, the upper chromatic index $\overline{\chi'}(S) \leq \frac{(v-1)(v-4)}{12} - 1$.*

*Proof.* Let $\phi : \mathcal{B} \to C$ be a valid block-colouring of an $S(2,4,v), S$, where $|C| = \frac{(v-1)(v-4)}{12} - 1 + \alpha$ colours with $\alpha > 0$. Then the number of extra blocks in the colour classes is

$$\frac{v(v-1)}{12} - \frac{(v-1)(v-4)}{12} + 1 - \alpha = \frac{v-1}{3} - \alpha + 1.$$

Let $x_i$ denote the number of colour classes containing $i+1$ blocks. Then we have

$$\sum_{i=0}^{b} ix_i = \frac{v-1}{3} - \alpha + 1. \tag{3.1}$$

Now every element occurs in at least two blocks of a non-singleton colour class. Considering the maximum number of such elements that can occur in a colour class with $i+1$ blocks, we have 0 when $i=0$; 1 when $i=1$; 3 when $i=2$; 6 when $i=3$. Now when $i \geq 4$ we will denote $\beta_i$ as the maximum number of elements of degree 2 in a colour class with $i+1$ blocks. So we have

$$x_i + 3x_2 + 6x_3 + \sum_{i=4}^{b} \beta_i x_i \geq v.$$

Subtracting three times equation (3.1) from the latter inequality we find that

$$-2x_1 - 3x_2 - 3x_3 + \sum_{i=4}^{b} (\beta_i - 3i)x_i \geq 3\alpha - 2 > 0.$$

Now $\beta_i \leq \frac{4(i+1)}{2} = 2i+2$. Thus, $\beta_i - 3i < 0$ for all $i \geq 3$. Therefore at least one of the $x_i$'s is negative, which is a contradiction.

□

Extrapolating from the case for $k=3$ and $k=4$ we believe that the following holds for any $k \geq 3$:

**Conjecture 1.** *For any Steiner system $S(2,k,v)$ $S$ of order $v$, the upper chromatic index $\overline{\chi'}(S) \leq \frac{(v-1)(v-k)}{k(k-1)} - 1$, and there exists a $v_0$ such that there is an $S(2,k,v), S'$, with $\overline{\chi'}(S') = \frac{(v-1)(v-k)}{k(k-1)} - 1$ for all admissible $v \geq v_0$.*

We note that Conjecture 1 has been shown to be true for the case $k=3$ (Theorem 1.3).

### 3.2.2 Block-colourings of type $\pi = (3, 1, 1, \ldots, 1)$

In this subsection we consider block-colourings of $S(2,4,v)$s of the type $\pi = (3,1,1,\ldots,1)$. We note that is easy to see that $\underline{\chi}'_\pi(v) \geq \frac{v-7}{3}$.

**Theorem 3.18.** *For $v \equiv 4 \pmod{12}$ we have $\left[\frac{v-7}{3}, \frac{v(v-10)}{12} + 1\right] \subseteq \Omega_\pi(v)$. For $v \geq 39$ and $v \equiv 1 \pmod{12}$ we have*

$$\left[\frac{2v-17}{3}, \frac{v(v-10)}{12} + 1\right] \bigcup \left[\frac{(v-9)(v-1)}{12} - 1, \frac{(v-8)(v-1)}{12} - 2\right] \subseteq \Omega_\pi(v).$$

*Proof.* Let $\pi = (3, 1, 1, \ldots, 1)$. Let $v \equiv 4 \pmod{12}, v \geq 16$. Let $R_1, \ldots, R_r$ be the parallel classes of a resolvable $S(2,4,v), (\mathcal{V}, \mathcal{B})$. Colour the blocks of $R_1, R_2, R_3$ with $C_1$, those of $R_4$ with $C_2$, and those of $R_i$ with $C_{i-2}$ for all $5 \leq i \leq r$. This gives us a $\frac{v-7}{3}$-colouring. We can now recolour consecutively each block in the colour classes $C_2, \ldots, C_{r-2}$ until each block has its own colour. Therefore, we have $\left[\frac{v-7}{3}, \frac{v(v-10)}{12} + 1\right] \subseteq \Omega_\pi(v)$.

Let $v \equiv 1 \pmod{12}, v \geq 49$. We want to construct an $S(2,4,v)$. Take a $4 - GDD, (\mathcal{X}, \mathcal{G}, \mathcal{C})$, of type $12^{\frac{v-1}{12}}$. Such a $GDD$ exists for all $v \geq 49$ [10]. Form an $S(2,4,13) = (\mathcal{V}_G, \mathcal{B}_G)$ on $G \bigcup \{\infty\}$ for all $G \in \mathcal{G}$. This gives us $\frac{v-1}{12}$ disjoint configurations isomorphic to the unique $S(2,4,13)$. Then $S = (\mathcal{V}, \mathcal{C} \bigcup \mathcal{B})$ is an $S(2,4,v)$, where $\mathcal{B} = \bigcup_{G \in \mathcal{G}} \mathcal{B}_G$ and $|\mathcal{V}| = v$. Now take three arbitrary blocks $B_1, B_2, B_3$ which contain $\infty$ and colour them with one colour. Colour the blocks which do not contain $\infty$ in the $\frac{v-1}{12}$ configurations with $i$ colours, where $i \in \{1, 2, \ldots, \frac{v-1}{12}\}$, so that all blocks in each configuration get coloured the same colour. Finally, colour the remaining $c = \frac{v(v-1)}{12} - \frac{3(v-1)}{4} - 3$ blocks with $c$ colours. This gives us $\left[\frac{(v-9)(v-1)}{12} - 1, \frac{(v-8)(v-1)}{12} - 2\right] \subseteq \Omega_\pi(v)$.

Again, let $v \equiv 1 \pmod{12}$ and let $S = (\mathcal{V}, \mathcal{B})$ be a 1-near-resolvable $S(2,4,v)$ with almost parallel classes $A_1, A_2, \ldots, A_{r-1}$. Assume that the almost parallel classes have $v_0$ as the missing element. Colour all blocks of $A_1, A_2, A_3$ with the colour $C_1$, and colour all blocks of $A_i$ with the colour $C_{i-2}$ for all $4 \leq i \leq r-1$. Now colour all blocks incident with $v_0$ with $r-2$ different colours according to the partition $\pi = (3, 1, 1, \ldots, 1)$. This gives us a $(2r-5)$-block-colouring of type $\pi$. Now recolour the blocks of $A_i$, $i \neq 1, 2, 3$, consecutively until each block gets a different colour. This gives us $\left[\frac{2v-17}{3}, \frac{v(v-10)}{12} + 1\right] \subseteq \Omega_\pi(v)$.

□

We now have the following corollary.

**Corollary 3.19.** *If* $v \equiv 4 \pmod{12}$ *then* $\underline{\chi}'_\pi(v) = \frac{v-7}{3}$.

*Proof.* Since $\underline{\chi}'_\pi(v) \geq \frac{v-7}{3}$, Theorem 3.18 gives us $\underline{\chi}'_\pi(v) = \frac{v-7}{3}$ when $v \equiv 4 \pmod{12}$.

□

We define a configuration $P$ in an $S(2,4,v)$, $S = (\mathcal{V}, \mathcal{B})$, as the configuration of nine blocks on twelve elements such that each element has degree three. We can obtain this configuration by taking a copy of the $S(2,4,13)$ and deleting an element and all blocks containing it. It is easy to see that this configuration is equivalent to a $4 - GDD$ of type $3^4$.

We say that an $S(2,4,v)$ $S = (\mathcal{V}, \mathcal{B}), v \equiv 4 \pmod{12}$, has property $M$ if it contains $\frac{v-4}{12}$ element-disjoint configurations, $\frac{v-16}{12}$ of which are the $P$ configuration and one of which is isomorphic to the unique $S(2,4,16)$.

We note that the unique $S(2,4,16)$ trivially has this property, while it is easily seen that no $S(2,4,v)$ with property $M$ exists for $v = 28, 40$.

**Theorem 3.20.** *Let $v \equiv 4 \pmod{12}$. If an $S(2,4,v)$ exists with property $M$ for $v \in \{52, 64, 76, 88, 100, 112, 124, 136, 148, 172, 184\}$ then an $S(2,4,v)$ with property $M$ exists for all $v \geq 16, v \neq 28, 40$.*

*Proof.* Assume that there exists an $S(2,4,v)$ with property $M$ for $v \in \{52, 64, 76, 88, 100, 112, 124, 136, 148, 172, 184\}$. Consider $4 - GDDs$ of type $36^n, n \geq 4$, which exist by [10]. Let $\mathcal{G}$ be the set of groups of the $GDD$ and let $X$ be a set of 16 elements disjoint from the elements of $\mathcal{G}$. Then we can construct an $S(2,4,36n+16), (\mathcal{V}, \mathcal{B})$ as follows: the blocks in $\mathcal{B}$ consist of all of the blocks in the $4 - GDD$, and the blocks from each of the $S(2,4,52)$ formed on $G \bigcup X$ for all $G \in \mathcal{G}$. Similarly, we can construct an $S(2,4,36n+64)$ and $S(2,4,36n+76)$ using a $4 - GDD$ of type $36^n48^1$ and $36^n60^1$, respectively (both exist: see [10]), with the only difference that we form an $S(2,4,64)$ on $G' \bigcup X$ and an $S(2,4,76)$ on $G'' \bigcup X$, where $G'$ and $G''$ are the groups of size 48 and 60, respectively. Then each of the $S(2,4,v)s$ constructed above have property $M$.

□

We conjecture that there exists an $S(2,4,v)$ with property $M$ for all $v \in \{52, 64, 76, 88, 100, 112, 124, 136, 148, 172, 184\}$ and hence an $S(2,4,v)$ with property $M$ exists for all $v \geq 16, v \neq 28, 40$.

We define a configuration $G$ of an $S(2,4,v)$ as a configuration of twelve blocks on sixteen elements. We note that $G$ can be obtained by taking three parallel classes of an $S(2,4,16)$. An $S(2,4,v)$ has property $M'$ if it contains $\frac{v-4}{12}$ element-disjoint configurations, $\frac{v-16}{12}$ of which are the configuration $P$ and one of which is the configuration $G$. We note that if an $S(2,4,v)$ has property $M$ then it trivially has property $M'$.

**Theorem 3.21.** *Let $v \equiv 4 \pmod{12}$. If there exists an $S(2,4,v)$ with property $M$ then $\left[\frac{v(v-10)}{12} + 1, \frac{(v-1)(v-8)}{12} - 1\right] \subseteq \Omega_\pi(v)$.*

*Proof.* Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,4,v)$ with property $M$. Then $S$ also has property $M'$. We colour the $\frac{v-4}{12}$ configurations with $i$ colours, where $i \in \{1, 2, \ldots, \frac{v-4}{12}\}$, such that each block in the same configuration gets coloured with the same colour. There are exactly $\frac{v(v-10)}{12}$ blocks not contained in the $\frac{v-4}{12}$ element disjoint configurations. Colour each of these blocks with $\frac{v(v-10)}{12}$ different colours. Our result follows.

□

**Theorem 3.22.** *Let $\pi = (3, 1, 1, \dots, 1)$ and let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2, 4, v)$. Then $\overline{\chi'_\pi}(S) \leq \frac{(v-1)(v-8)}{12} - 1$.*

*Proof.* Let $\phi : \mathcal{B} \to C$ be a block-colouring of $S$ of type $\pi$ where $|C| = \frac{(v-1)(v-8)}{12} - 1 + \alpha$ colours with $\alpha > 0$. Then the number of extra blocks in the colour classes is

$$\frac{v(v-1)}{12} - \frac{(v-1)(v-8)}{12} + 1 - \alpha = \frac{2(v-1)}{3} - \alpha + 1.$$

Let $x_i$ denote the number of colour classes containing $i + 1$ blocks. Then we have

$$\sum_{i=0}^{b} i x_i = \frac{2(v-1)}{3} - \alpha + 1. \tag{3.2}$$

Now every element is contained in at least three blocks of a non-singleton class. Consider the maximum number of such elements that can occur in a colour class with $i$ +1 blocks: 0 when $i = 0, 1$; 1 when $i = 2, 3$; 2 when $i = 4$; 4 when $i = 5$; 7 when $i = 6$;8 when $i = 7$. Now when $i \geq 8$ let $\beta_i$ denote the number of elements of degree three in a colour class with $i + 1$ blocks. So we have

$$x_2 + x_3 + 2x_4 + 4x_5 + 7x_6 + 8x_7 + \sum_{i=8}^{b} \beta_i x_i \geq v.$$

Subtracting $\frac{3}{2}$ times equation (3.2) from the latter we have

$$-2x_2 - \frac{7}{2}x_3 - 4x_4 - \frac{7}{2}x_5 - 2x_6 - \frac{5}{2}x_7 + \sum_{i=8}^{b} \left(\beta_i - \frac{3}{2}i\right) x_i \geq \frac{3\alpha}{2} > 0.$$

Again, this is a contradiction.

□

**Corollary 3.23.** *Let $v \equiv 4 \pmod{12}$. If there exists an $S(2, 4, v)$ with property $M$ then $\left[\frac{v-7}{3}, \frac{(v-1)(v-8)}{12} - 1\right] = \Omega_\pi(v)$.*

*Proof.* Use Theorems 3.18, 3.21, and 3.22.

□

Now consider the case when $\pi = (s, 1, 1, \dots, 1)$ for a general $s, s \geq 1$.

**Theorem 3.24.** *Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,4,v)$ and let $\pi = (s,1,1,\dots,1)$. For $v \equiv 4 \pmod{12}$ we have $\left[\frac{v-3s+2}{3}, \frac{v(v-1-3s)}{12} + 1\right] \subseteq \Omega_\pi(v)$. For $v \equiv 1 \pmod{12}$ we have $\left[\frac{2v-6s+1}{3}, \frac{v(v-1-3s)}{12} + 1\right] \subseteq \Omega_\pi(v)$.*

*Proof.* Let $R_1, \dots, R_r$ be the parallel classes of a resolvable $S(2,4,v)$, $(\mathcal{V}, \mathcal{B})$. Colour the blocks of $R_1, R_2, \dots, R_s$ with $C_1$ and those of $R_i$ with $C_{i-s+1}$ for all $s+1 \leq i \leq r$. This gives us a $\frac{v-3s+2}{3}$-colouring. We can now recolour consecutively each block in the colour classes $C_2, \dots, C_{r-s+1}$ until each block has its own colour. Therefore, we have $\left[\frac{v-3s+2}{3}, \frac{v(v-1-3s)}{12} + 1\right] \subseteq \Omega_\pi(v)$.

Let $v \equiv 1 \pmod{12}$ and let $S = (\mathcal{V}, \mathcal{B})$ be a 1-near-resolvable $S(2,4,v)$ with almost parallel classes $A_1, A_2, \dots, A_{r-1}$. Assume that the almost parallel classes have $v_0$ as the missing element. Colour all blocks of $A_1, A_2, \dots, A_s$ with the colour $C_1$, and colour all blocks of $A_i$ with the colour $C_{i-s+1}$ for all $s+1 \leq i \leq r-1$. Now colour all blocks incident with $v_0$ with $r-s+1$ different colours according to the partition $\pi = (s,1,1,\dots,1)$. This gives us a $(2r-2s+1)$-block-colouring of type $\pi$. Now recolour the blocks of $A_i$, $i \neq 1,2,\dots,s$, consecutively until each block gets a different colour. This gives us $\left[\frac{2v-6s+1}{3}, \frac{v(v-1-3s)}{12} + 1\right] \subseteq \Omega_\pi(v)$.

□

We can extend the colour spectrum if $s$ is odd and $v \equiv 1 \pmod{12}$, however.

**Theorem 3.25.** *Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,4,v)$ and let $\pi = (s,1,1,\dots,1)$ where $s = 4l+3$ for some $l \geq 0$ and $v \equiv 1 \pmod{12}, v \geq 48(l+1)+1$. Then $\left[c+2, c+3, \dots, c+1+\frac{v-1}{12(l+1)}\right] \subseteq \Omega_\pi(v)$ where*

$$c = \frac{v(v-1)}{12} - s - \left(\frac{v-1}{12(l+1)}\right)((12l+13)(l+1) - s - 1)$$

*Proof.* Let $\pi = (s,1,1,\dots,1)$ where $s = 4l+3$ for some $l \geq 0$. Let $v \equiv 1 \pmod{12}, v \geq 48(l+1)+1$. We want to construct an $S(2,4,v)$. Take a $4-GDD, (\mathcal{X}, \mathcal{G}, \mathcal{C})$, of type $12(l+1)^{\frac{v-1}{12(l+1)}}$. Such a $GDD$ exists for all $v \geq 48(l+1)+1$ [10]. Form an $S(2,4,12(l+1)+1) = (\mathcal{V}_G, \mathcal{B}_G)$ on $G \bigcup \{\infty\}$ for all $G \in \mathcal{G}$. This gives us $\frac{v-1}{12(l+1)}$ disjoint configurations. Then $S = (\mathcal{V}, \mathcal{C} \bigcup \mathcal{B})$ is an $S(2,4,v)$, where $\mathcal{B} = \bigcup_{G \in \mathcal{G}} \mathcal{B}_G$ and $|\mathcal{V}| = v$. Now take $s$ arbitrary blocks $B_1, B_2, \dots, B_s$ which contain $\infty$ and colour them with one colour. Colour the blocks which do not contain $\infty$ in the $\frac{v-1}{12(l+1)}$ configurations with $i$ colours, where $i \in \{1, 2, \dots, \frac{v-1}{12(l+1)}\}$, so that all blocks in each configuration get coloured the same colour. Finally, colour the remaining

$$c = \frac{v(v-1)}{12} - s - \left(\frac{v-1}{12(l+1)}\right)((12l+13)(l+1) - s - 1)$$

blocks with $c$ colours. This gives us $\left[c+2, c+3, \ldots, c+1+\frac{v-1}{12(l+1)}\right] \subseteq \Omega_\pi(v)$.

□

## 3.3 Equitable colourings

In this section we will discuss equitable block-colourings. Partitions of the form $\pi = (\pi_1, \pi_2, \ldots, \pi_s)$, where $|\pi_j - \pi_i| \leq 1$ for all $1 \leq i < j \leq s$, are called *equitable partitions* and the corresponding block-colourings are called *equitable block-colourings.*

**Theorem 3.26.** *If an $S(2,k,v)$ admits a block colouring of type $\pi = (t, t, \ldots, t)$, where $st = \frac{v-1}{k-1}$, then any colour class contains at least $\frac{(v+s-1)(v-1)}{k(k-1)s^2}$ blocks and $\overline{\chi}'_\pi(v) \leq s^2 - 1$.*

*Proof.* Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,k,v)$, $c \in C$ be a colour, and $x \in \mathcal{V}$ be an element incident with a block of colour $c$. Since there are $\frac{r}{s} = \frac{v-1}{s(k-1)}$ blocks of colour $c$ incident with $x$, we have $|V(c)| \geq 1 + (k-1)\frac{v-1}{s(k-1)} = \frac{v+s-1}{s}$, where $V(c)$ is the set of elements in $\mathcal{V}$ which are incident with blocks of colour $c$. Therefore, there are at least $\frac{1}{k}\frac{v+s-1}{s}\frac{v-1}{s(k-1)}$ blocks of colour $c$.

Since $\sum_{c \in C} |V(c)| = sv$, in an $h$-colouring of type $\pi$ we have $h\frac{v+s-1}{s} \leq sv \Rightarrow h \leq \left\lfloor \frac{s^2 v}{v+s-1} \right\rfloor$. If we let $h$ be the maximum number of colours in a colouring of type $\pi$, we get $\overline{\chi}'_\pi(S) \leq \left\lfloor \frac{s^2 v}{v+s-1} \right\rfloor \Rightarrow \overline{\chi}'_\pi(S) \leq s^2 - 1$. □

**Theorem 3.27.** *Let $\pi = \left(\frac{v-k}{2(k-1)}, \frac{v+k-2}{2(k-1)}\right)$. If $S = (\mathcal{V}, \mathcal{B})$ is an $S(2,k,v)$ with a colouring of type $\pi$, then $\overline{\chi}'_\pi(S) \leq 3$.*

*Proof.* Assume that there exists a 4-colouring of type $\pi$. For each colour $c$, we have $|V(c)| \geq 1 + (k-1)\left(\frac{v-k}{2(k-1)}\right) = \frac{v-k+2}{2}$. We call $c$ a *rich* colour if there is an $x \in \mathcal{V}$ which is incident with $\frac{v+k-2}{2(k-1)}$ blocks of colour $c$. If $c$ is rich, we have $|V(c)| \geq 1+(k-1)\frac{v+k-2}{2(k-1)} = \frac{v+k}{2}$. Assume that there is exactly one rich colour $c'$. Then we have

$$2v = |V(c')| + \sum_{c \in C, c \neq c'} |V(c)| \geq v + 3\frac{v-k+2}{2} \Rightarrow 3k-6 \geq v.$$

This is a contradiction for $v > 3k-6$. So assume we have at least two rich colours. Then we have

$$2v \geq 2\frac{v-k+2}{2} + 2\frac{v+k-2}{2} = 2v+2,$$

which is clearly a contradiction.

Therefore, we must have $\overline{\chi}'_{\pi}(S) \leq 3$. □

There are exactly two equitable partitions which give an equitable bi-colouring of an $S(2,4,v)$. The partitions are $\pi_1 = (\frac{v-1}{6}, \frac{v-1}{6})$ and $\pi_2 = (\frac{v-4}{6}, \frac{v+2}{6})$. It is easy to see that colourings of type $\pi_1$ and $\pi_2$ exist only when $v \equiv 1 \pmod{12}$ and $v \equiv 4 \pmod{12}$, respectively. Therefore theorems 3.26 and 3.27 give us the following corollary.

**Corollary 3.28.** *Let $\pi_1 = (\frac{v-1}{6}, \frac{v-1}{6})$ and $\pi_2 = (\frac{v-4}{6}, \frac{v+2}{6})$. If $S = (\mathcal{V}, \mathcal{B})$ is an $S(2,4,v)$ which has a colouring of type $\pi_i$, then $\overline{\chi}'_{\pi_i}(S) \leq 3$.*

**Theorem 3.29.** *Let $S = (\mathcal{V}, \mathcal{B}), v > 13$, be an $S(2,4,v)$ which is $\pi_1$-colourable. Then $\underline{\chi}'_{\pi_1}(S) = \overline{\chi}'_{\pi_1}(S) = 2$.*

*Proof.* By Theorems 3.26 and 3.27 we know that $\overline{\chi}'_{\pi_i} \leq 3$. Therefore we only need to show that there is no 3-colouring of $S$ of type $\pi_1$. So assume that $S$ has a 3-colouring of type $\pi_1$. Let $\mathcal{P}$ be a partition of the set $V$ into three sets $A$, $B$, and $C$ where each element in $A$ is incident with blocks of colour 1 and 2, each element in $B$ is incident with blocks of colour 1 and 3, and each element in $C$ is incident with blocks of colour 2 and 3. Let $|A| = a$, $|B| = b$, and $|C| = c$. We can assume without loss of generality that $a \leq b \leq c$.

We now prove a series of claims which will be used throughout the remainder of the proof.

*Claim 1.* There is no block $T \in \mathcal{B}$ such that $A \bigcap T \neq \emptyset$, $B \bigcap T \neq \emptyset$, and $C \bigcap T \neq \emptyset$. Furthermore, if $A \bigcap T \neq \emptyset$ and $B \bigcap T \neq \emptyset$, then $T$ is coloured with colour 1; if $A \bigcap T \neq \emptyset$ and $C \bigcap T \neq \emptyset$, then $T$ is coloured with colour 2; and if $B \bigcap T \neq \emptyset$ and $C \bigcap T \neq \emptyset$, then $T$ is coloured with colour 3.

*Proof.* Let $T \in \mathcal{B}$. Assume $T \bigcap D \neq \emptyset$ where $D \in \{A, B, C\}$. Each element of $D$ is incident with blocks coloured by the same two colours. Therefore, $T$ must be coloured with one of two colours.

□

*Claim 2.* For the partition $\pi_1$, $a+b \geq \frac{v+1}{2}$, while for the partition $\pi_2$, $a+b \geq \frac{v-2}{2}$.

*Proof.* Let $x \in \mathcal{V}$ be an element incident with a block of colour 1. Then $x$ is incident with $\frac{v-1}{6}$ blocks for the partition $\pi_1$ and with at least $\frac{v-4}{6}$ blocks for the partition $\pi_2$. The blocks of colour 1 incident with $x$ contain $3\frac{v-1}{6}+1 = \frac{v+1}{2}$ elements for $\pi_1$ or at least $3\frac{v-4}{6}+1 = \frac{v-2}{2}$ elements for $\pi_2$. □

*Claim 3.* $\binom{v}{2} \geq \frac{3}{2}(ab+ac+bc)$.

*Proof.* There are $ab+ac+bc$ pairs $Q = \{x,y\}$ of elements in $\mathcal{V}$ where $x$ and $y$ belong to different parts of the partition $\mathcal{P}$. Each pair $Q$ is covered by a block $T$ containing two or three elements from the same part of $\mathcal{P}$. Such a block contains 3 or 4 pairs from different parts of $\mathcal{P}$. Therefore, the number of pairs $\{u,v\}$ where $u,v$ are from the same part of $\mathcal{P}$ is at least $\frac{1}{2}(ab+ac+bc)$. This gives us that $\binom{v}{2}-(ab+ac+bc) \geq \frac{1}{2}(ab+ac+bc)$ and the result follows. □

We now proceed to prove theorem 3.29. Let $d$ be the number of pairs $\{x,y\}$ where either both $x,y \in B$ or both $x,y \in C$, and $\{x,y\}$ is contained in a block of colour 3. By claim 1, all pairs $\{x,y\}$ where $x \in B$ and $y \in C$ are contained in blocks of colour 3. Therefore

$$d \geq \frac{bc}{2}. \tag{3.3}$$

To obtain an upper bound for $d$ we now count the number of pairs $Q = \{x,y\}$ so that either $Q \subset B$ or $Q \subset C$, and $Q$ is contained in a block of colour 1 or 2. Now for the partition $\pi_1$ each element in $A$ is incident with $\frac{v-1}{3}$ of such blocks and each element in $B \bigcup C$ is incident with $\frac{v-1}{6}$. So we have $e = \frac{1}{4}(a\frac{v-1}{3}+(b+c)\frac{v-1}{6}) = \frac{(v-1)(v+a)}{24}$ total blocks of colour 1 and 2 for the partition $\pi_1$. For $\pi_2$ each element in $A$ is incident with $\frac{v-1}{3}$ blocks of colour 1 and 2 and each element in $B \bigcup C$ is incident with at least $\frac{v-4}{6}$ of them. Therefore, for $\pi_2$, we have $e \geq \frac{1}{4}(a\frac{v-1}{3}+(b+c)\frac{v-4}{6}) = \frac{(v-1)(v+a)}{24} - \frac{b+c}{8}$. Let $t$ be the number of blocks $T$ such that $T \subset A$, let $s$ be the number of blocks of the form $AAAB$ or $AAAC$. Then there are $\binom{a}{2}$ $-3s-6t$ blocks of the form $AABB$ or $AACC$. Let $w$ be the number of blocks of the form $ABBB$ or $ACCC$. Then

$$ab + ac = 4\left(\binom{a}{2} - 3s - 6t\right) + 3s + 3w \Rightarrow w = \frac{ab + ac - 4\binom{a}{2} + 9s + 24t}{3} \tag{3.4}$$

There are $e - t - s - w - \binom{a}{2} + 3s + 6t = e + 2s + 5t - w - \binom{a}{2}$ blocks $T$ coloured 1 or 2 and $T \subset B$ or $T \subset C$. Each of these blocks covers 6 pairs. Thus,

$$d = \binom{b}{2} + \binom{c}{2} - 3w - \left(\binom{a}{2} - 3s - 6t\right) - 6\left(e + 2s + 5t - w - \binom{a}{2}\right) \tag{3.5}$$

If we combine the equations (3.3) and (3.5) then we have

$$\binom{a}{2} + \binom{b}{2} + \binom{c}{2} + ab + bc + ac - 6e \geq \frac{3}{2}bc,$$

which simplifies to

$$\binom{v}{2} - 6e \geq \frac{3}{2}bc. \tag{3.6}$$

Substituting for $e$ in the partition $\pi_1$ we get

$$\binom{v}{2} - \frac{a(v-1)}{2} \geq 3bc,$$

which simplifies to

$$(v-1)(v-a) \geq 6bc. \tag{3.7}$$

For the partition $\pi_2$ we get

$$\binom{v}{2} - \frac{a(v-1)}{2} + \frac{3(b+c)}{\cdot 2} \geq 3bc,$$

which simplifies to

$$(v+2)(v-a) \geq 6bc. \tag{3.8}$$

We will first show that, for the partition $\pi_1$, $(v-1)(v-a) < 6bc$ for all $v \geq 13, v \equiv 1 \pmod 4$. We let $a$ be a fixed number and consider two separate cases.

Let $a$ be a fixed number such that $1 \leq a \leq \frac{v-1}{4}$. By Claim 2, we have $b \geq \frac{v+1}{2} - a$, therefore min $6bc$ occurs at $b = \frac{v+1}{2} - a$ and $c = v - (a+b) = \frac{v-1}{2}$.

So it suffices to show that $(v-1)(v-a) < 6(\frac{v+1}{2} - a)(\frac{v-1}{2})$ for all $v$. It is easy to see that this inequality holds for all $a \leq \frac{v-1}{4}$.

We now consider the case where $a \geq \frac{v+3}{4}$. Since $a \leq b \leq c$ we have that the *min* of $6bc$ occurs when $a = b$ and $c = v - 2a$. Therefore it suffices to prove that $(v-1)(v-a) < 6a(v-2a)$ for all $a \geq \frac{v+3}{4}$. That is, we need to show that $f(a) = 6a(v-2a) - (v-1)(v-a) > 0$ for all $\frac{v+3}{4} \leq a \leq \frac{v}{3}$. Now $f\left(\frac{v+3}{4}\right) = \frac{3}{4}(v^2-9) - \frac{3}{4}(v-1)^2 = \frac{3}{2}(v-5) > 0$ And $f\left(\frac{v}{3}\right) = \frac{2v^2}{3} - \frac{2v(v-1)}{3} = \frac{2v}{3} > 0$. This implies that $f(a) > 0$ for all $\frac{v+3}{4} \leq a \leq \frac{v}{3}$ since the graph of $f$ is a parabola opening down. This completes the proof for the partition $\pi_1$.

□

We also offer the following conjecture regarding the partition $\pi_2$.

**Conjecture 2.** *Let $S = (\mathcal{V}, \mathcal{B}), v > 13$, be an $S(2,4,v)$ which is $\pi_2$-colourable. Then $\underline{\chi}'_{\pi_2}(S) = \overline{\chi}'_{\pi_2}(S) = 2$.*

Only three types of equitable tricolourings of $S(2,4,v)$s exist. Those of types $\delta_1 = \left(\frac{v-1}{9}, \frac{v-1}{9}, \frac{v-1}{9}\right), \delta_2 = \left(\frac{v-4}{9}, \frac{v-4}{9}, \frac{v+5}{9}\right)$, and $\delta_3 = \left(\frac{v-7}{9}, \frac{v+2}{9}, \frac{v+2}{9}\right)$. It is easy to see that a colouring of type $\delta_1$, $\delta_2$, and $\delta_3$ exists only when $v \equiv 1, 28 \pmod{36}$, $v \equiv 4, 13 \pmod{36}$, and $v \equiv 16, 25 \pmod{36}$, respectively.

**Lemma 3.30.** *Let $\delta_1 = \left(\frac{v-1}{9}, \frac{v-1}{9}, \frac{v-1}{9}\right), \delta_2 = \left(\frac{v-4}{9}, \frac{v-4}{9}, \frac{v+5}{9}\right)$ and $\delta_3 = \left(\frac{v-7}{9}, \frac{v+2}{9}, \frac{v+2}{9}\right)$. If an $S = (\mathcal{V}, \mathcal{B})$ is an $S(2,4,v)$ which has a colouring of type $\delta_i$, then $\overline{\chi}'_{\delta_i}(S) \leq 9$.*

*Proof.* For $\delta_1$, see Theorem 3.26. Assume $S$ has 10-colouring of type $\delta_2$. For each colour $c$, we have $|V(c)| \geq 1 + 3\frac{v-4}{9} = \frac{v-1}{3}$. For a rich colour $c'$, we have $|V(c')| \geq 1 + 3\frac{v+5}{9} = \frac{v+8}{3}$. Let $c'$ be a rich colour, then we get

$$3v = \sum_{c \in C} |V(c)| = |V(c')| + \sum_{c \in C, c \neq c'} |V(c)| \geq \frac{v+8}{3} + 9\frac{v-1}{3} = \frac{10v-1}{3}$$

This is a contradiction when $v \geq 13$.

Now consider the partition of type $\delta_3$. When $v = 16$ and $S_{16}$ is the unique $S(2,4,16)$ assume that we have a 9−block-colouring of type $(2,2,1)$. Now let $y$ be the number of rich colours. If $y = 5$, with each colour class containing at least 7 elements, then

$$3 \cdot 16 = \sum_{c \in C} |V(c)| \geq 5(7) + 4(4) = 51$$

which is a contradiction. Therefore $y \leq 4$. Now the union of the $y$ colour classes corresponding to the rich colours must contain all 16 elements exactly four times. Therefore this union contains at least 16 blocks, leaving four blocks to distribute amongst $9 - y$ colour classes. Thus, $y \geq 5$. This is a contradiction. Therefore $\overline{\chi}'_{\delta_i}(S_{16}) \leq 8$.

Now assume $v > 16$ and $S$ has a 10-colouring of type $\delta_3$. For each colour $c$, we have $|V(c)| \geq 1 + 3\frac{v-7}{9} = \frac{v-4}{3}$. For a rich colour $c'$, we have $|V(c')| \geq 1 + 3\frac{v+2}{9} = \frac{v+5}{3}$. Let $c'$, $c''$ be a rich colour, then we get

$$\begin{aligned} 3v &= \sum_{c \in C} |V(c)| \\ &= |V(c')| + |V(c'')| + \sum_{c \in C, c \neq c', c''} |V(c)| \geq 2\frac{v+5}{3} + 8\frac{v-4}{3} = \frac{10v-22}{3} \end{aligned}$$

This is a contradiction for $v > 22$.

□

**Lemma 3.31.** *Let $S$ be a cyclic $S(2,k,v)$ with $v \equiv 1 \pmod{3k(k-1)}$. Then there exists a 3-colouring of type $\pi = (kt, kt, kt)$, where $v = 3k(k-1)t + 1$.*

*Proof.* If $v = 3k(k-1)t + 1$, the number of full orbits in a cyclic $S(2,k,v)$ equals $3t$. Colour any $t$ orbits colour 1, any other $t$ orbits colour 2, and the remaining $t$ orbits colour 3. This gives us a 3-colouring of type $\pi$. □

We conclude this section with a result on equitable colourings for an $S(2,k,v)$ and a general partition $\pi^k = \left(\frac{v-k}{k(k-1)}, \ldots, \frac{v-k}{k(k-1)}, \frac{v+k(k-2)}{k(k-1)}\right)$.

**Lemma 3.32.** *Consider an $S(2,k,v)$ with the partition*

$$\pi^k = \left(\frac{v-k}{k(k-1)}, \ldots, \frac{v-k}{k(k-1)}, \frac{v+k(k-2)}{k(k-1)}\right),$$

*a $k$-tuple. If $v \equiv k, k^2 \pmod{k^2(k-1)}$ and there exists a resolvable transversal design $RTD(k,n)$, then $\{k, k+1, \ldots, 2k-1\} \subseteq \Omega_{\pi^k}(v)$.*

*Proof.* Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,k,v)$, let $\mathcal{X} = \mathcal{V} \times \{1, 2, \ldots, k\}$, and let $(\mathcal{X}, \mathcal{G}, \mathcal{C})$ be a resolvable transversal design $RTD(k,v)$ with groups $\mathcal{G} = \{\mathcal{V} \times \{i\} \mid i \in \{1, 2, \ldots, k\}\}$. Construct an $S(2,k,kv)$ by putting an $S(2,k,v), (\mathcal{V} \times \{i\}, B_i)$ on each $\mathcal{V} \times \{i\}$ and adjoining the set of blocks from $\mathcal{C}$. Colour $(k-2)\frac{v-1}{k-1}$ parallel classes with $k-2$ colours, where all blocks in each parallel class get coloured the same colour , and colour $\frac{v+k-2}{k-1}$ parallel classes with colour $k-1$. Finally, colour the blocks $B_i$ with $c$ colours not yet used, where $c \in \{1, 2, \ldots, k\}$ and all blocks in each $B_i$ get coloured the same colour. □

Since there exists a resolvable $RTD(4,v)$ for all $v \equiv 1,4 \pmod{12}$ ([10]) we have the following corollary:

**Corollary 3.33.** *Consider an* $S(2,4,v)$ *with the partition* $\pi = \left(\frac{v-4}{12}, \frac{v-4}{12}, \frac{v-4}{12}, \frac{v+8}{12}\right)$*. If* $v \equiv 4, 16 \pmod{48}$*, then* $\{4,5,6,7\} \subset \Omega_\pi(v)$*.*

Section 3.4 also contains results on equitable colourings for $S(2,4,v)s$ and small $v$.

## 3.4 Block-colouring of type $\pi$ for small $v$

In the following lemma we list known results for special block colourings of $S(2,4,v)s$ for small $v$.

**Lemma 3.34.**

1. $\Omega_{\pi_1}(13) = \{13\}$, *where* $\pi_1 = (1,1,1,1)$.

2. $\Omega_{\pi_2}(13) = \{5,6\}$, *where* $\pi_2 = (2,1,1)$.

3. $\Omega_{\pi_3}(13) = \emptyset$, *where* $\pi_3 = (2,2)$.

4. $\Omega_{\pi_4}(13) = \{3\}$, *where* $\pi_4 = (3,1)$.

5. $\Omega_{\pi_5}(16) = \{5,6,\ldots,20\}.$, *where* $\pi_5 = (1,1,1,1,1)$.

6. $\Omega_{\pi_6}(16) = \{4,5,\ldots,13\}$ *where* $\pi_6 = (2,1,1,1)$.

7. $\Omega_{\pi_7}(16) \supseteq \{3,4,5,6\}$ *where* $\pi_7 = (2,2,1)$.

8. $\Omega_{\pi_8}(16) = \{3,4,\ldots,9\}$ *where* $\pi_8 = (3,1,1)$.

9. $\Omega_{\pi_9}(16) = \{2,3,4,5\}$ *where* $\pi_9 = (4,1)$

*10.* $\Omega_{\pi_{10}}(16) = \{ 2, 3 \}$ *where* $\pi_{10} = (3, 2)$.

*11.* $\Omega_{\pi_{11}}(25) = \{ 10, 11, \ldots, 50 \}$ *where* $\pi_{11} = (1, 1, \ldots, 1)$

*12.* $\Omega_{\pi_{12}}(25) = \{ 3 \}$ *where* $\pi_{12} = (7, 1)$

*13.* $\Omega_{\pi_{13}}(25) \supseteq \{ 5, 6 \ldots, 10 \}$ *where* $\pi_{13} = (6, 1, 1)$

*14.* $\Omega_{\pi_{14}}(25) = \{ 7, 8, \ldots, 16 \}$ *where* $\pi_{14} = (5, 1, 1, 1)$

*15.* $\Omega_{\pi_{15}}(25) = \{ 7, 8, \ldots, 12 \}$ *where* $\pi_{15} = (4, 2, 1, 1)$

*16.* $\Omega_{\pi_{16}}(25) = \{ 7, 8, \ldots, 12 \}$ *where* $\pi_{16} = (3, 3, 1, 1)$

*17.* $\Omega_{\pi_{17}}(25) = \{ 9, 10, \ldots, 26 \}$ *where* $\pi_{17} = (4, 1, 1, 1, 1)$

*18.* $\Omega_{\pi_{18}}(25) = \{ 9, 10, \ldots, 20 \}$ *where* $\pi_{18} = (3, 2, 1, 1, 1)$

*19.* $\Omega_{\pi_{19}}(25) = \{ 11, 12, \ldots, 28 \}$ *where* $\pi_{19} = (2, 2, 1, 1, 1, 1)$

*20.* $\Omega_{\pi_{20}}(25) = \{ 11, 12, \ldots, 34 \}$ *where* $\pi_{20} = (3, 1, 1, 1, 1, 1)$

*21.* $\Omega_{\pi_{21}}(25) = \{ 15, 16, \ldots, 44 \}$ *where* $\pi_{21} = (2, 1, 1, 1, 1, 1, 1)$

*Proof.* We will use results from [31] to prove 1-4. In [31] the authors considered specialized colourings of $STS(v)$s and $S(2,4,v)$s where only specified block-colouring patterns are allowed. For example, they considered colourings of $S(2,4,v)$s where the elements are coloured so that each block has the colour pattern $\{ \times \times \times \Box \}$ (the authors referred to this as a type $B$ colouring). The colourings of type $A, C, D, E$ are colourings where the blocks are coloured according to the patterns $\{ \times \times \times \times \}$, $\{ \times \times \Box \Box \}$, $\{ \times \times \Box \triangle \}$, and $\{ \times \Box \triangle \diamond \}$, respectively, a colouring of type $T$, where $T \subseteq \{ A, B, C, D, E \}$, is a colouring where each block is coloured according to one of the elements of $T$, and $\Omega_T(v) = \{ k \mid \text{ there exists a } k\text{-colouring of type } T \}$. It was shown in [31] that $\Omega_E(13) = \{ 13 \}$, $\Omega_B(13) = \{ 3 \}$, $\Omega_C(13) = \emptyset$, and $\Omega_D(13) = \{ 5, 6 \}$. Recall that the unique $S(2,4,13) = S$ is a projective plane of order 3, hence it is a symmetric $BIBD$. Given a $BIBD$ $P = (\mathcal{V}, \mathcal{B})$, then the *dual* of $P$ is the design $P' = (\mathcal{B}, \mathcal{V})$ where $b \in \mathcal{B}$ is contained in $v \in \mathcal{V}$ if and only if $v$ is contained in $b$ in $P$. Now the dual, $S' = S'(2,4,13)$, of $S = S(2,4,13)$ is isomorphic to $S$, so a colouring of the elements of $S$ is coextensive with a block-colouring of $S'$. For example, a 3-colouring of type $B$ is equivalent to a 3-block-colouring of type $\pi_4 = (3,1)$. Therefore the colourings obtained in [31] allow us to easily obtain the colourings in 1-4.

For 5, let $\mathcal{R} = \{ R_1, R_2, R_3, R_4, R_5 \}$ be the set of parallel classes of the unique $S(2,4,16)$. Colouring each block of $R_i$ with colour $i$ gives us a 5-colouring of type $\pi_5$. Now recolour consecutively the blocks of each of the parallel classes until each block obtains an individual colour. The result follows.

If we colour $R_1 \bigcup R_2$ with colour 1, and each block of $R_i$ with colour $i - 1$ for $3 \leq i \leq 5$, then we obtain a 4-colouring of type $\pi_6$. Now recolour consecutively the blocks of each of the parallel classes $R_2, R_3, R_4$ until each block obtains an individual colour. This gives us $\{ 4, 5, \ldots, 13 \} \subseteq \Omega_{\pi_6}(16)$. Now assume that the unique $S(2,4,16), S$, has a 14-colouring of type $\pi_8 = (2,1,1,1)$. If there exists three rich colours, say $c_1, c_2, c_3$, then $|V(c_i)| \geq 7$ for all $1 \leq i \leq 3$. This gives us the inequality

$$4 \cdot 16 = \sum_{c \in C} |V(c)| \geq 3(7) + 11(4) = 65,$$

which is a contradiction. This implies that there is exactly one or exactly two rich colour classes. If there is exactly one rich colour, say $c_1$, then $|V(c_1)| = v = 16$ which gives us

$$4 \cdot 16 = \sum_{c \in C} |V(c)| \geq 16 + 13(4) = 68,$$

which is a contradiction. So assume that there are exactly two rich colour classes, say $C_1, C_2$. But it is impossible to partition 8 blocks of $S$ amongst two colour classes such that each element of $S$ occurs in exactly two blocks of either $C_1$ or $C_2$. Therefore, there is no 14-colouring of type $\pi_6 = (2,1,1,1)$. This proves 6.

Colouring each block $R_1 \bigcup R_2$ with colour 1, each block of $R_3 \bigcup R_4$ with colour 2, and each block of $R_5$ with colour 3 gives us a 3-colouring of type

$\pi_7$. Recolouring the blocks of $R_5$ consecutively until each block obtains an individual colour gives us 7.

If we colour $R_1 \bigcup R_2 \bigcup R_3$ with colour 1, and each block of $R_4, R_5$ with colours 2 and 3, respectively, then we obtain a 3-colouring of type $\pi_8$. Now recolour consecutively the blocks of each of the parallel classes $R_3, R_4$ until each block obtains an individual colour. This gives us $\{3, 4, \ldots, 9\} \subseteq \Omega_{\pi_8}(16)$. Now assume that the unique $S(2,4,16), S$, has a 10-colouring of type $\pi_8 = (3,1,1)$. Then every rich colour class has at least 10 elements, while every other colour class has at least 4 elements. So we assume first that there are at least two rich colours. This gives us the inequality

$$3 \cdot 16 = \sum_{c \in C} |V(c)| \geq 2(10) + 8(4) = 52,$$

which is a contradiction. Therefore, we must have exactly one rich colour, say $c'$. It is easy to see that $|V(c')| = v = 16$ and

$$3 \cdot 16 = \sum_{c \in C} |V(c)| = 16 + 9(4) = 52,$$

a contradiction. Therefore, there is no 10-colouring of type $\pi_8 = (3,1,1)$. This proves 8.

If we colour each block of $R_1 \bigcup R_2 \bigcup R_3 \bigcup R_4$ with colour 1 and each block of $R_5$ with colour 2 then this gives us a 2-colouring of type $\pi_9$. Now recolour consecutively the blocks of $R_5$ until each block obtains an individual colour. This gives us 9.

If we colour each block of $R_1 \bigcup R_2 \bigcup R_3$ with colour 1, and each block of $R_4 \bigcup R_5$ with colour 2 then we obtain a 2-colouring of type $\pi_{10}$. By Theorem 3.27 we know that if there exists a $k$-colouring of type $\pi_{10}$ then $k \leq 3$. Assume that there exists a 3-colouring of type $\pi_{10}$. If there are exactly two rich colours then $|V(c')| = 16$, where $c'$ is the non-rich colour. . Therefore

$$2 \cdot 16 = \sum_{c \in C} |V(c)| \geq 2(10) + 16 = 36,$$

which is a contradiction. So we have exactly one rich colour, say $c''$, and exactly one rich colour class, say $C''$. Now $|V(c'')| = 16$ so $C''$ contains three parallel classes. However, it is impossible to partition the remaining blocks amongst two parallel classes so that each element in the colour classes have degree two. This proves 10.

Finally, 11 follows from Meszka's computations ([27]), 12 follows from Lemma 3.9, and 13-21 follows from Theorem 3.7.

□

## 3.5 Special block-colourings of $S(2,4,v)$s

**Definition 2.** Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,k,v)$. Let $\Pi^r$ be the set of all partitions of the replication number $r = \frac{v-1}{k-1}$ and let $K \subseteq \Pi^r$. Then $S$ has a *$t$-colouring of profile $K$* if the blocks of $\mathcal{B}$ can be coloured with $t$ colours such that each element $x \in \mathcal{V}$ is contained in blocks coloured according to a partition in $K$ and for every partition $\pi$ in $K$ there exists an $x \in \mathcal{V}$ which is incident with blocks coloured according to $\pi$.

For example, if $K = \{(1,1,\ldots,1),(r)\}$, then we can obtain a $(\frac{(v-1)(v-k)}{k(k-1)}+1)$-colouring of profile $K$ for $S$. Take $x \in \mathcal{V}$ and colour all blocks containing $x$ with one colour. Colour each remaining block with an individual colour. Now each element in $\mathcal{V}$ is obtained in $r$ blocks with $r$ different colours or $r$ blocks with one colour.

In fact, we have

**Lemma 3.35.** *Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,k,v), K_1 = \{\pi_1, \pi^*\}$, and $K_2 = \{\pi_2, \pi^*\}$ where $\pi_1 = (r-1,1), \pi_2 = (1,1,\ldots,1)$, and $\pi^*$ is a partition of $r$ with $m$ parts. Then there exists a $(m+1)$-colouring of profile $K_1$ and a $(\frac{(v-1)(v-k)}{k(k-1)}+m)$-colouring of profile $K_2$. If $S$ is resolvable, then we can obtain a $t$-colouring of profile $K_2$, where $t \in \{r+m, r+m+1, \ldots, \frac{(v-1)(v-k)}{k(k-1)}+m\}$.*

*Proof.* Let $\pi^*$ be a partition of $r$ with $m$ parts. Let $x \in \mathcal{V}$ and colour all blocks containing $x$ with $m$ colours according to the partition $\pi^*$. If we colour the remaining blocks with one colour then we obtain a $(m+1)$-colouring of profile $K_1$. If we colour each of the remaining blocks with an individual colour then we obtain a $(\frac{(v-1)(v-k)}{k(k-1)}+m)$-colouring of profile $K_2$. This proves the first statement of the Lemma.

For the second statement, assume $S$ is resolvable. Take $x \in \mathcal{V}$ and colour all blocks containing $x$ with $m$ colours. Let $R_i'$ denote the partial parallel class obtained from the parallel class $R_i$ by deleting the block containing $x$. Colour $R_i'$ with colour $m+i$ for all $1 \leq i \leq r$ We now have an $(r+m)$-colouring of profile $K_2$. Now recolour the blocks in each $R_i$ consecutively until each block is assigned an individual colour. $\square$

**Corollary 3.36.** *Let $v \equiv 1,3 \pmod 6$. Then there exists an $STS(v)$ which admits an $(m+1)$-block-colouring of profile $K_1$ and a $(\frac{(v-1)(v-3)}{6}+m)$-block-colouring of profile $K_2$. When $v \equiv 3 \pmod 6$ there exits an $STS(v)$ which admits a $t$-block-colouring of profile $K_2$ with $t \in \{\frac{v-1}{2}+m, \ldots, \frac{(v-1)(v-3)}{6}+m\}$.*

*Proof.* Our result follows from Lemma 3.35 and the fact that a resolvable $STS(v)$ exists for all $v \equiv 3 \pmod 6$.

$\square$

**Corollary 3.37.** *Let $v \equiv 1, 4 \pmod{12}$. Then there exists an $S(2,4,v)$ which admits an $(m+1)$-block-colouring of profile $K_1$ and a $(\frac{(v-1)(v-4)}{12}+m)$-block-colouring of profile $K_2$. When $v \equiv 4 \pmod{12}$ there exits a $t$-block-colouring of profile $K_2$ with $t \in \{\frac{v-1}{3}+m, \dots, \frac{(v-1)(v-4)}{12}+m\}$.*

*Proof.* Our result follows from Lemma 3.35 and the fact that a resolvable $S(2,4,v)$ exists for all $v \equiv 4 \pmod{12}$.

$\square$

In this section we will focus on the block-colourings of profile $K$ where $|K| = 2$. We will present results for both $STS(v)s$ and $S(2,4,v)s$ but our focus will be on $S(2,4,v)s$. Given an $S(2,k,v)$, $S$, the minimum number of colours needed in a block-colouring of profile $K$ is denoted by $\underline{\chi}'_K(S)$. We will also use

$$\underline{\chi}'_K(v) = \min\{\,\underline{\chi}'_K(S) \mid S \text{ is an } S(2,4,v)\,\},$$

$$\overline{\chi}'_K(v) = \max\{\,\overline{\chi}'_K(S) \mid S \text{ is an } S(2,4,v)\,\},$$

$$\Omega_K(v) = \{\,p \mid \exists\ S(2,4,v) \text{ which admits a } p\text{-block-colouring of profile } K\,\}.$$

**Theorem 3.38.** *Let $S = (\mathcal{V},\mathcal{B})$ be an $S(2,k,v)$ with an $S(2,k,w)$ subsystem $X = (\mathcal{W},\mathcal{C})$, let $r' = \frac{w-1}{k-1}$, and let $r'' = \frac{v-w}{k-1}$. If $X$ admits a $p$-block-colouring of type $\pi'$ for some partition $\pi'$ of $r'$, then $S$ admits a $t$-block-colouring of profile $K_i$, $1 \le i \le 4$, where*

*(i)* $K_1 = \{\,(\pi',\pi''),(1,1,\dots,1)\,\}$ *and* $t = p + mw + |\mathcal{D}|$

*(ii)* $K_2 = \{\,(\pi',\pi''),(r-w,1,1,\dots,1)\,\}$ *and* $t = p + mw + 1$

*(iii)* $K_3 = \{\,(\pi',\frac{v-w}{3},(r-w,w)\,\}$ *and* $t = p + 2$ *and*

*(iv)* $K_4 = \{\,(\pi',\frac{v-w}{3}),(r)\,\}$ *and* $t = p + 1$

*where $\pi''$ is a partition of $r''$, $m$ is the size the partition $\pi''$, and $\mathcal{D}$ is the set of blocks $\mathcal{D} \subseteq \mathcal{B}$ such that for each $D \in \mathcal{D}, |\mathcal{D} \cap \mathcal{C}| = \emptyset$ for all $C \in \mathcal{C}$.*

*Proof.* Assume that $X$ has a $p$-block-colouring of type $\pi'$. Let $x \in \mathcal{W}$. Colour all blocks in $\mathcal{B}\backslash\mathcal{C}$ which contain $x$ with $m$ colours according to the partition $\pi''$. Continue this process for all $x \in \mathcal{W}$. Colour each remaining block in $\mathcal{B}\backslash\mathcal{C}$ with an individual colour or colour all remaining blocks in $\mathcal{B}\backslash\mathcal{C}$ with one colour. This proves (i) and (ii). To prove (iii), we colour all blocks in $\mathcal{B}\backslash\mathcal{C}$ which contain any $x \in \mathcal{W}$ with one colour and all remaining blocks with one other colour. To prove (iv), we colour all blocks in $\mathcal{B}\backslash\mathcal{C}$ with one colour. □

We will now consider cases where $K_1 = \{ (3,3,\ldots,3,1),(2,2,\ldots,2,1) \}$, $K_2 = \{ (3,3,\ldots,3,1),(2,2,\ldots,2) \}$, $K_3 = \{ (3,3,\ldots,3),(2,2,\ldots,2) \}$, and $K_4 = \{ (3,3,\ldots,3),(2,2,\ldots,1) \}$.

We first consider $STS(v)$s. We note that $v \equiv 3 \pmod{12}$ when we have a block-colouring of type $K_1$.

Throughout this section we will make use of *Kirkman subsystems* of a Kirkman triple system, $KTS(v)$. Formally, $KTS(w)$ is a subsystem of a $KTS(v)$ only if the parallel classes of the $KTS(w)$ are induced by the parallel classes of the $KTS(v)$. It was shown in [34] that

**Theorem 3.39** ([34]). *There is a $KTS(v)$ containing a sub-$KTS(w)$ if and only if $v \equiv w \equiv 3 \pmod 6$ and $v \geq 3w$.*

We will make use of this result to help prove the following:

**Theorem 3.40.** *If $v \equiv 27 \pmod{36}$ and $v \geq 27$, then for $STS(v)$s we have $\left[\frac{5v-3}{12}, \frac{3v-5}{4}\right] \subseteq \Omega_{K_1}(v)$.*

*Proof.* Let $v \equiv 27 \pmod{36}$ and let $(\mathcal{W},\mathcal{C})$ be a sub-$KTS(w)$ of $(\mathcal{V},\mathcal{B})$, a $KTS(v)$, where $v = 3w$.

Let $P_i, 1 \leq i \leq \frac{w-1}{2} = r_w$, be the parallel classes which partition $\mathcal{W}$ and $R_i, 1 \leq i \leq r$, be the parallel classes which partition $\mathcal{V}$. Without loss of generality, we can assume that $P_i \subset R_i$ for all $1 \leq i \leq r_w$.

Colour $P_1 \cup P_2$ with colour 1, $P_3 \cup P_4$ with colour 2, and so on, colouring $P_{r_w-1} \cup P_{r_w}$ with colour $\frac{r_w}{2}$. Consider the partition $\pi = (2,2,\ldots,2,1)$ of $r - r_w = \frac{v-w}{2}$. Colour the parallel classes $R_j, r_w+1 \leq j \leq r$, with $\frac{v-w}{2}$ different colours according to $\pi$. Let $R_{j'} \subset C'$, where $C'$ is the colour class which contains one parallel class. Colour the blocks $R_i/P_i, 1 \leq i \leq r_w$, with $r_w$ colours used in colouring $R_j, r_w+1 \leq j \leq r, j \neq j'$. This gives us a $\frac{5v-3}{12}$-colouring of profile $K_1$. Recolouring the blocks in $C'$ consecutively gives $\left[\frac{5v-3}{12}, \frac{3v-5}{4}\right] \subseteq \Omega_{K_1}(v)$.

□

In order for a block-colouring of profile $K_2$ to exist we need $v \equiv 9 \pmod{12}$.

**Theorem 3.41.** *If $v \equiv 9 \pmod{36}$ and $v \geq 9$, then for $STS(v)$s we have* $\left[\frac{5v-9}{12}, \frac{3v-7}{4}\right] \subseteq \Omega_{K_2}(v)$.

*Proof.* Let $v \equiv 9 \pmod{36}$ and let $(\mathcal{W}, \mathcal{C})$ be a sub-$KTS(w)$ of $(\mathcal{V}, \mathcal{B})$, a $KTS(v)$, where $v = 3w$.

Let $P_i, 1 \leq i \leq \frac{w-1}{2} = r_w$, be the parallel classes which partition $\mathcal{W}$ and $R_i, 1 \leq i \leq r$, be the parallel classes which partition $\mathcal{V}$. Without loss of generality, we can assume that $P_i \subset R_i$ for all $1 \leq i \leq r_w$.

Colour $P_1 \cup P_2$ with colour 1, $P_3 \cup P_4$ with colour 2, and so on, colouring $P_{r_w-2} \cup P_{r_w-1}$ with colour $\frac{r_w-1}{2}$, and then colour $P_{r_w}$ with colour $\frac{r_w+1}{2}$. Consider the partition $\pi' = (2, 2, \ldots, 2, 1)$ of $r - r_w = \frac{v-w}{2}$. Colour $R_{r_w+1}$ with colour $\frac{r_w+1}{2}$. Then colour the parallel classes $R_j, r_w + 2 \leq j \leq r$, with $\frac{v-w-2}{2}$ different colours according to $\pi'$.

Let $R_{j'} \subset C'$, where $C'$ is the colour class which contains one parallel class. Colour the blocks $R_i/P_i, 1 \leq i \leq r_w$, with $r_w$ colours used in colouring $R_j, r_w + 2 \leq j \leq r, j \neq j'$. This gives us a $\frac{5v-9}{12}$-colouring of profile $K_2$. Recolouring the blocks in $C'$ consecutively gives $\left[\frac{5v-9}{12}, \frac{3v-7}{4}\right] \subseteq \Omega_{K_2}(v)$. □

In order for a block-colouring of profile $K_3$ to exist we need $v \equiv 1 \pmod{12}$ and for a block-colouring of profile $K_4$ we need $v \equiv 7 \pmod{12}$. Both of these cases seems a bit more difficult at this point and we hope to investigate this in the future.

Now consider $S(2, 4, v)$s. We note here that it is not possible to have a block-colouring of profile $K_2$ or $K_4$ for $S(2, 4, v)$s. In order for a block-colouring of profile $K_1$ to exist we need $v \equiv 4 \pmod{36}$. In order for a block-colouring of profile $K_3$ to exist we need $v \equiv 1 \pmod{36}$. Both cases seems a bit more difficult at this point and we hope to investigate them in the future.

# Chapter 4

# $T$-chromatic index for Steiner systems $S(2, 4, v)$

Recall that the chromatic index of a Steiner system $S(2,4,v)$ $S = (\mathcal{V}, \mathcal{B})$ is the minimum number of colours needed to colour the blocks of $\mathcal{B}$ such that no element of $\mathcal{V}$ is incident with two blocks of the same colour.

A generalization of this concept was introduced in [17] where the authors considered the problem of colouring the blocks of a Steiner triple system $S$ such that there are no monochromatic copies of a given configuration, $T$. The minimum number of colours required for such a colouring of $S$ is denoted by $\chi(T, S)$.

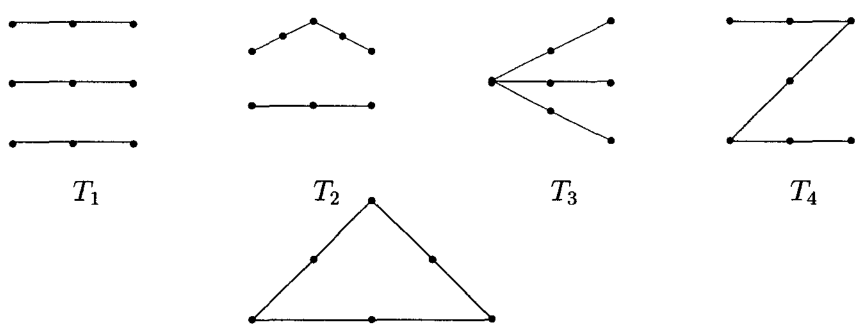
In a general $t - (v, k, \lambda)$ design, an *n-line configuration* is a collection of $n$ $k$-element subsets which collectively have the property that every $t$-element subset is contained in at most $\lambda$ lines (blocks). In an $n$-line configuration the *degree* of a point is the number of lines (blocks) which contain that point. For example, the possible two-line configurations of an $STS(v)$ are two lines intersecting in a point and two parallel (non-intersecting) lines. In the former case $\chi(T, S)$ is just the chromatic index $\chi'(S)$. In [17], the authors consider the latter case where $\chi(T, S)$ was referred to as the 2-parallel chromatic index of $S$, and denoted by $\chi''(S)$. In particular, [17] considers $\underline{\chi}''(v) = \min\{\, \chi''(S) \mid S \text{ is an } STS(v)\,\}$ and $\overline{\chi}''(v) = \max\{\, \chi''(S) \mid S \text{ is an } STS(v)\,\}$. The following results were obtained in [17].

**Theorem 4.1** ([17]). *Let* $\underline{\chi}''(v) = \min\{\, \chi''(S) \mid S \text{ is an } STS(v)\,\}$ *and* $\overline{\chi}''(v) = \max\{\, \chi''(S) \mid S \text{ is an } STS(v)\,\}$. *Then* $\overline{\chi}''(3) = \underline{\chi}''(3) = 1$; $\overline{\chi}''(7) = \underline{\chi}''(7) = 1$; $\overline{\chi}''(9) = \underline{\chi}''(9) = 3$; $\overline{\chi}''(13) = \underline{\chi}''(13) = 6$; $\underline{\chi}''(19) = 8$; $\underline{\chi}''(21) = 9$; $\underline{\chi}''(25) = 12$; $\underline{\chi}''(27) = 13$; $\underline{\chi}''(31) = 15$; $\underline{\chi}''(33) = 17$; $\underline{\chi}''(37) = 19$; *for* $v \geq 39$,

$$\underline{\chi}''(v) = \begin{cases} (v+1)/2, & \text{if } v \equiv 1,9 \pmod{12} \\ (v-1)/2, & \text{if } v \equiv 3,7 \pmod{12} \end{cases}$$

Using a result of Phelps and Rödl in [33], it was also shown that $\overline{\chi}''(v) \leq v - c\sqrt{v \log v}$ for some absolute constant $c$.

In [19] the authors consider the $T$-chromatic index for Steiner triple systems where $T$ is a three-line configuration. The five possible three-line configurations in an $STS(v)$ are a 3-ppc (3-partial parallel class, $T_1$), a hut ($T_2$), a 3-star ($T_3$), a 3-path ($T_4$), and a triangle ($T_5$). These configurations are shown below.



$T_1$ $T_2$ $T_3$ $T_4$

$T_5$

In general, a set of $n$ lines of an $S(2,k,v)$ intersecting in a common point is called an *n-star*, and a set of $n$ parallel lines is called an $n$-partial parallel class, abbreviated *n-ppc*. If $n = \frac{v}{k}$ then an $n$-ppc is called a parallel class. An *n-path* in an $S(2,k,v)$ is a set of $n$ lines, $l_1, l_2, \ldots, l_n$ such that $|\, l_i \bigcap l_j| = 1$ if and only if $|i-j| = 1$ for all $1 \leq i \leq n$. In [19] $\underline{\chi}(T,v) = \min\{\, \chi(T,S) \mid S \text{ is an } STS(v)\,\}$ is obtained for three of the five possible three-line configurations, namely $T_1, T_2, T_3$, and an asymptotic result is obtained for the configuration $T_4$. The results from [19] are as follows.

**Theorem 4.2** ([19]). *$\underline{\chi}(T_1,7) = 1, \underline{\chi}(T_1,9) = 2, \underline{\chi}(T_1,13) = \overline{\chi}(T_1,13) = 3$, and*

$$\underline{\chi}(T_1,v) = \begin{cases} (v+1)/4, & \text{if } v \geq 63 \text{ and } v \equiv 3,7 \pmod{12}; \\ (v+3)/4, & \text{if } v \geq 133 \text{ and } v \equiv 1,9 \pmod{12}. \end{cases}$$

**Theorem 4.3** ([19]). *$\underline{\chi}(T_2, 7) = 1, \underline{\chi}(T_2, 9) = 1, \underline{\chi}(T_2, 13) = 4$ and if $v \geq 49$ then $\underline{\chi}(T_2, v) = (v-1)/2$.*

**Theorem 4.4** ([19]). *If $v = 7$ then $\underline{\chi}(T_3, v) = 3$. If $v \neq 7$ then*

$$\underline{\chi}(T_3, v) = \begin{cases} (v-1)/4 & \text{if } v \equiv 9 \pmod{12}, \\ (v+1)/4 & \text{if } v \equiv 3, 7 \pmod{12}, \\ (v+3)/4 & \text{if } v \equiv 1 \pmod{12}. \end{cases}$$

**Theorem 4.5** ([19]). *$\underline{\chi}(T_4, 7) = 1, \underline{\chi}(T_4, 9) = 3, \underline{\chi}(T_4, 13) = 5$ and as $v \to \infty, \underline{\chi}(T_4, v) = (v-1)/6 + o(v)$.*

The $T$-chromatic index can easily be extended to general Steiner systems $S(2, k, v)$. In this chapter we will discuss the $T$-chromatic index of $S(2, 4, v)$s for various configurations $T$. As far as we can tell, no research has been done in this area.

## 4.1 2-parallel chromatic index

We first prove a result on the upper bound for the minimum 2-parallel chromatic index for $S(2, 4, v)s$. Our proof will utilize independent sets. A subset $I \subset \mathcal{V}$ in a design $\mathcal{D}' = (\mathcal{V}, \mathcal{B})$ is *independent* if there is no $B \in \mathcal{B}$ contained in $I$.

**Theorem 4.6.** *Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2, 4, v)$ where $v \equiv 4, 13 \pmod{36}$. Then $\underline{\chi}''(v) \leq (v-1)/3$.*

*Proof.* We will make use of the "$v \to 3v + 1$ rule". Let $(\mathcal{V}, \mathcal{B})$ be an $S(2, 4, v)$, $v \equiv 1, 4 \pmod{12}$, and let $\mathcal{X}$ be such that $|\mathcal{X}| = 2v + 1$ and $\mathcal{X} \bigcap \mathcal{V} = \emptyset$. Let $(\mathcal{X}, \mathcal{C})$ be a resolvable $STS(2v{+}1)$, and let $\mathcal{R} = \{ R_1, R_2, \ldots, R_v \}$ be a resolution of $(\mathcal{X}, \mathcal{C})$. Form the set of quadruples $D_i = \{ \{ v_i, x, y, z \} \mid v_i \in \mathcal{V}, \{ x, y, z \} \in R_i \}$, and put $\mathcal{D} = \bigcup_i D_i$. Then $(\mathcal{V} \bigcup \mathcal{X}, \mathcal{B} \bigcup \mathcal{D})$ is an $S(2, 4, 3v + 1)$. We note that the elements of $\mathcal{X}$ now form an independent set in the $S(2, 4, 3v+1)$. We have shown that for $v \equiv 4, 13 \pmod{36}$ there exists an $S(2, 4, v)$ with an independent set of size $\frac{2v+1}{3}$. This set is maximum since any independent set in an $S(2, 4, v)$ is of order $\leq \frac{2v+1}{3}$ (For a general $S(2, k, v)$, if $A$ is an independent set, then $|A| \leq \frac{k-2}{k-1}(v-1)$, cf. [37]).

Now consider the case $v \equiv 4, 13 \pmod{36}$. Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2, 4, v)$ which contains an independent set $I$ of order $\frac{2v+1}{3}$. Let $I^* = V/I$. Then $|I^*| = \frac{v-1}{3}$. Colour the elements of $I^*$ with $\frac{v-1}{3}$ distinct colours. Since $I$ is independent $B \bigcap I^* \neq \emptyset$ for all $B \in \mathcal{B}$. If a block $B \in \mathcal{B}$ contains exactly one element in $I^*$, say $x$, then colour $B$ with the same colour as $x$. If a block contains two, three, or four elements of $I^*$ then colour the block with one of the two, three, or four possible colours. If two blocks have the same colour then they intersect, therefore we have a $\frac{v-1}{3}$-block-colouring which contains no monochromatic parallel lines. Thus, $\underline{\chi}''(v) \leq \frac{v-1}{3}$.

□

We now prove a result for general Steiner system $S(2,k,v)$s and $\underline{\chi}''(v)$.

**Theorem 4.7.** *Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,k,v)$. If $v \geq k^3+2k$ then $\underline{\chi}''(v) \geq \frac{v-1}{k-1}$.*

*Proof.* Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,k,v)$ which admits a $p$-block-colouring which contains no monochromatic parallel lines. Assume that $p < \frac{v-1}{k-1}$. Now every colour class is either a star or contains at most $(k-1)^2+(k-1)+1 = k^2-k+1$ blocks.

Let $s$ be the number of star colour classes. We can assume that none of the $(p-s)$ non-star colour classes contains a star center. If it did then we can transfer it to an appropriate star colour class. This may create another colour class but the process will eventually terminate.

We start by assuming that $s < p$. Since no block within a non-star colour class will contain a star center, each star center is incident with $\frac{v-1}{k-1}$ blocks. Counting by star centers gives a total of $s\frac{v-1}{k-1}$ blocks, but some blocks contain anywhere from two to $k$ star centers. If we let $a_i$ denote the number of blocks which contain $i+1$ star centers then

$$\sum_{i=2}^{k} \binom{i}{2} a_{i-1} = \frac{s(s-1)}{2}.$$

Therefore, when we count the total number of blocks within the star colour classes we must account for the multiple counting of the blocks containing more than one star center. Thus, the total number of blocks in the star colour classes is

$$\begin{aligned}
&\frac{s(v-1)}{k-1} - \sum_{i=1}^{k-1} ia_i \\
&= \frac{s(v-1)}{k-1} - \frac{s(s-1)}{2} + \sum_{i=2}^{k-1} \binom{i}{2} a_i \\
&= \frac{s(v-1)}{k-1} - \frac{s(s-1)}{k} + \sum_{i=1}^{k-2} \frac{i(i+1-k)}{k} a_i.
\end{aligned}$$

This gives us the total number of blocks in the non-star colour classes as

$$\begin{aligned} l &= \frac{v(v-1)}{k(k-1)} - \frac{s(v-1)}{k-1} + \frac{s(s-1)}{k} + \sum_{i=1}^{k-2} \frac{i(k-i-1)}{k} a_i. \\ &= \frac{(v-s)(v-1-(k-1)s)}{k(k-1)} + \sum_{i=1}^{k-2} \frac{i(k-i-1)}{k} a_i. \end{aligned}$$

Thus, $l \geq (v-s)(v-1-(k-1)s)/k(k-1)$. However, since each non-star colour class can contain at most $k^2-k+1$ blocks we have $l \leq (k^2-k+1)(p-s)$. Therefore,

$$\frac{(v-s)(v-1-(k-1)s)}{k(k-1)} \leq (k^2-k+1)(p-s).$$

Now define

$$f(x) = \frac{(v-x)(v-1-(k-1)x)}{k(k-1)} - (k^2-k+1)(p-x),$$

which gives us

$$f'(x) = \frac{2(k-1)x - kv + 1}{k(k-1)} + k^2 - k + 1.$$

Since $p < \frac{v-1}{k-1}$ we have $p \leq \frac{v-k}{k-1}$, thus $s \leq \frac{v-2k+1}{k-1}$. For $0 \leq x \leq \frac{v-2k+1}{k-1}$, we have

$$f'(x) \leq f'\left(\frac{v-2k+1}{k-1}\right) \leq \frac{(2-k)v + k^4 - 2k^3 + 2k^2 - 5k + 3}{k(k-1)} < 0$$

for all $v \geq k^3 + 2k$. So $f$ is strictly decreasing on the interval $\left(0, \frac{v-2k+1}{k-1}\right)$ when $v \geq k^3 + 2k$. Hence

$$\begin{aligned} f(s) &\geq f\left(\frac{v-2k+1}{k-1}\right) \\ &= \frac{(k^3-k^2+3k-4)v - (2k^4-3k^3+3k^2-5k+2)}{k(k-1)} - (k^2-k+1)p \\ &\geq \frac{(k^3-k^2+3k-4)v - (2k^4-3k^3+3k^2-5k+2)}{k(k-1)} - (k^2-k+1)\left(\tfrac{v-k}{k-1}\right) \\ &= \frac{(2k-4)v - (k^4-2k^3+2k^2-5k+2)}{k(k-1)} > 0 \text{ if } v \geq k^3 + 2k. \end{aligned}$$

Thus, when $v \geq k^3 + 2k$ the average number of blocks in the non-star colour classes exceeds $k^2 - k + 1$. This is a contradiction and so we must have $s = p$, i.e. every colour class is a star. So the complement of the star centers form an independent set of size $v - p > \frac{(k-2)v+1}{k-1}$. This is also a contradiction, so $p \geq \frac{v-1}{k-1}$ for all $v \geq k^3 + 2k$.

□

This gives us the obvious corollary regarding $S(2,4,v)s$.

**Corollary 4.8.** *If* $v \geq 73$ *and* $v \equiv 1,4 \pmod{12}$ *then* $\underline{\chi}''(v) \geq (v-1)/3$.

By Theorem 4.6 and Corollary 4.8 we have the following result.

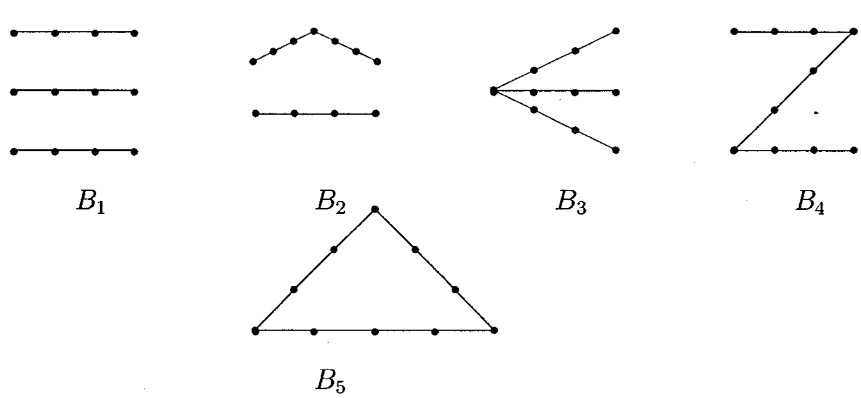
**Corollary 4.9.** *If* $v \geq 73$ *and* $v \equiv 4,13 \pmod{36}$ *then* $\underline{\chi}''(v) = (v-1)/3$

*Proof.* By Theorem 4.6 we have $\underline{\chi}''(v) \leq (v-1)/3$ for all $v \equiv 4,13 \pmod{36}$. By Corollary 4.8 we have $\underline{\chi}''(v) \geq (v-1)/3$ for all $v \equiv 1,4 \pmod{12}$ and $v \geq 73$. The result follows.

□

## 4.2 Three-line chromatic indices

We now consider configurations in $S(2,4,v)s$ which consist of three lines. We let $B_1 = 3 - ppc$, $B_2 = hut$, $B_3 = 3 - star$, $B_4 = 3 - path$, and $B_5 = triangle$. These configurations are shown below.



We first find an upper bound on $\underline{\chi}(B_1, v)$ for all $v \equiv 4, 13 \pmod{36}$.

**Theorem 4.10.** *Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,4,v)$. If $v \equiv 4 \pmod{36}$ then $\underline{\chi}(B_1, v) \leq (v+2)/6$ and if $v \equiv 13 \pmod{36}$ then $\underline{\chi}(B_1, v) \leq (v-1)/6$.*

*Proof.* Let $v \equiv 4, 13 \pmod{36}$. Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,4,v)$ which has a 2-parallel $\frac{v-1}{3}$-colouring. We group the colour classes into pairs and recolour the colour classes with $k$ colours such that all the blocks in the same group are coloured with one colour. If $v \equiv 13 \pmod{36}$ then $\frac{v-1}{3}$ is even and our recolouring gives us $k = \frac{v-1}{6}$ colour classes. If $v \equiv 4 \pmod{36}$ then $\frac{v-1}{3}$ is odd and our recolouring gives us $k = \frac{v-4}{6} + 1 = \frac{v+2}{6}$ colour classes. $\square$

The following lemma will be useful in proving our result for the lower bound on $\underline{\chi}(B_1, v)$.

**Lemma 4.11.** *Suppose $S$ is a set of lines of an $S(2,k,v)$ with no three of these lines being parallel. If $|S| \geq 3k^2 - 2k + 3$, $S$ may be partitioned into two disjoint subsets, $S_1, S_2$, neither of which contains two parallel lines and one of which is a star.*

*Proof.* If $S$ does not contain a pair of parallel lines then we can let $S_1 = S$ and $S_2 = \emptyset$. However, the largest number of blocks in such a design is $k^2 - k + 1$. But $|S_1| \geq 3k^2 - 2k + 3$ so it must be a star. So we can assume that $S$ contains a pair of parallel lines, $l_1, l_2$. Put $A = \{\, l \in S \mid l \bigcap l_1 = \emptyset \,\}$ and $B = \{\, l \in S \mid l \bigcap l_2 = \emptyset \,\}$. If $A$ contains a pair of parallel lines, say $l_3, l_4$, then $\{\, l_1, l_3, l_4 \,\}$ is a set of three parallel lines in $S$, which is a contradiction. Therefore, $A$ contains no parallel lines. Similarly, $B$ contains no parallel lines. Thus, $A$ and $B$ are either stars, or configurations which contain no parallel lines.

Define $C = \{\, l \in S \mid l \bigcap l_1 \neq \emptyset \text{ and } l \bigcap l_2 \neq \emptyset \,\}$. It is clear that $|C| \leq k^2$. Then $A \bigcup B \bigcup C = S$ and $(A \bigcup B) \bigcap C = \emptyset$, therefore $|A \bigcup B| \geq 2k^2 - 2k + 3$ which gives us $|A| \geq k^2 - k + 2$ or $|B| \geq k^2 - k + 2$. Without loss of generality, assume that $|A| \geq k^2 - k + 2$. Then $A$ cannot be a configuration which contains no parallel lines since the largest number of blocks in such a design is $k^2 - k + 1$. Therefore, $A$ must be a star with star center, say $a$, of degree at least $k^2 - k + 2$. Now let $S_1 = \{\, l \in S \mid a \in l \,\}$ and $S_2 = \{\, l \in S \mid a \notin l \,\}$. Since $A \subseteq S_1, |S_1| \geq k^2 - k + 2$. Suppose $S_2$ has two parallel lines, $k_1, k_2$. Then $k_1, k_2$ can intersect with at most $2k$ lines in $S_1$ which pass through $a$. Since the degree of $a$ is $\geq k^2 - k + 2$ there exists a line in $S_1$ which is parallel to both $k_1$ and $k_2$, which is a contradiction. Therefore, $S_2$ does not contain any parallel lines. The result follows. $\square$

We are now ready to prove the result on the lower bound of $\underline{\chi}(B_1, v)$ for general $S(2, k, v)$.

**Theorem 4.12.** *Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2, k, v)$. Then*

$$\underline{\chi}(B_1, v) \geq \begin{cases} \frac{v+k-2}{2(k-1)} & \text{if } v \equiv k \pmod{k(k-1)} \text{ and } v \geq v'; \\ \frac{v-1}{2(k-1)} & \text{if } v \equiv 1 \pmod{k(k-1)} \text{ and } v \geq v'', \end{cases}$$

*where*

$$v' = \frac{1}{2}(3k^3 + k^2 + 6k + 6) + \lceil \frac{7}{k-2} \rceil$$

*and*

$$v'' = \left\lceil \frac{1}{3}(3k^3 + k^2 + 6k + 1) + \frac{8}{3(k-2)} \right\rceil.$$

*Proof.* Assume $S = (\mathcal{V}, \mathcal{B})$ is an $S(2, k, v)$ with $\chi(B_1, S) = p$, where

$$p < \begin{cases} \frac{v+k-2}{2(k-1)} & \text{if } v \equiv k \pmod{k(k-1)} \text{ and } v \geq v'; \\ \frac{v-1}{2(k-1)} & \text{if } v \equiv 1 \pmod{k(k-1)} \text{ and } v \geq v''. \end{cases}$$

Consider the colour classes $C_i$ with $|C_i| \geq q = 3k^2 - 2k + 3$ which are neither a single star nor a union of two stars. By Lemma 4.11 we can partition each $C_i$ into two disjoint subsets $C_i'$, a star, and $C_i''$, a configuration of which contains no parallel lines. We can now group the $C_i'$s, and other colour classes containing single stars, into pairs and the $C_i''$s into pairs so that we obtain a new colouring of $S$ using $p^*$ colours. If the number of $C_i$s in the original colouring was even then we have at most $p^* = p$ colours. If the number of $C_i$s in the original colouring was odd then we have at most $p^* = p + 1$ colours, where exactly one of the colour classes is a star and exactly one is a configuration of which contains no parallel lines. In both cases each colour class is either

**(a)** a union of two stars,

**(b)** a single star (at most one of these), or

**(c)** a class containing at most $q$ blocks and not one of the forms described in $(a)$ or $(b)$.

We can assume that none of the blocks in $(c)$ contain a star center from any of the stars in $(a)$ or $(b)$. Otherwise we could transfer the block to the appropriate star in $(a)$ or $(b)$. This may create a new colour class of type $(a)$ or $(b)$ and we may have to repeat the process, but the process will eventually terminate.

We now let $t$ be the number of colour classes of type $(a)$ and let $s$ be the number of star centers. Then $s = 2t$ or $s = 2t + 1$. Since there are no star centers in $(c)$ we know that each star center is incident with $\frac{v-1}{k-1}$ blocks within the same colour class. Therefore, similar to the proof of Theorem 4.7, we have the number of blocks in the non-star colour classes $(c)$ as

$$l = \frac{(v-s)(v-1-(k-1)s)}{k(k-1)} + \sum_{i=1}^{k-2} \frac{i(k-i-1)}{k} a_i, \tag{4.1}$$

where $a_i$ is the number of blocks containing $i+1$ star centers. The argument now splits into two separate cases.

**Case 1:** If the colouring has $p+1$ colour classes then $s = 2t+1$. The number of type $(c)$ colour classes is $p+1-(t+1) = p-t$. Our construction of our "new" colouring ensures that we have at least one colour class of type $(c)$. Thus, $p - t \geq 1$. We also know that the total number of blocks contained in these $p-t$ colour classes is at most $q(p-t)$. Therefore

$$\begin{aligned} q(p-t) &\geq \frac{(v-s)(v-1-(k-1)s)}{k(k-1)} \\ &= \frac{(v-2t-1)(v-2t(k-1)-k)}{k(k-1)} \end{aligned} \tag{4.2}$$

Define $f(t) = (v-2t-1)(v-2t(k-1)-k) - k(k-1)q(p-t)$. Then

$$f'(t) = 8t(k-1) - 2kv + 4k + k(k-1)q - 2.$$

Since $t \leq p - 1 \leq \frac{v-3k+2}{2(k-1)}$ we have

$$\begin{aligned} f'(t) &\leq 8(k-1)\left(\frac{v-3k+2}{2(k-1)}\right) - 2kv + 4k + k(k-1)q - 2 \\ &= qk(k-1) - 8k - 2v(k-2) + 6. \end{aligned} \tag{4.3}$$

Therefore $f'(t) < 0$ when $v \geq (3k^3 + k^2 + 6k + 4)/2$. So $f$ is strictly decreasing on the interval $\left(0, \frac{v-3k+2}{2(k-1)}\right)$ when $v \geq (3k^3 + k^2 + 6k + 4)/2$.

When $v \equiv k \pmod{k(k-1)}$ we have $t \leq p - 1 \leq \frac{v-3k+2}{2(k-1)}$. Therefore

$$\begin{aligned} f(t) &\geq f\left(\frac{v-3k+2}{2(k-1)}\right) \\ &= \frac{v}{2}(3k^3 - 2k^2 + 6k - 8) - k(k-1)qp - \frac{k(3k-2)q}{2} + 4k - 2 \end{aligned} \tag{4.4}$$

for all $t \leq \frac{(v-3k+2)}{2(k-1)}$. Since $p \leq \frac{v-k}{2(k-1)}$ we have

$$f(t) \geq 2v(k-2) + 4k - k(k-1)q - 2.$$

So $f(t) > 0$ when $v \geq \frac{1}{2}(3k^3 + k^2 + 6k + 6) + \lceil \frac{7}{k-2} \rceil$. In this case the average number of blocks in the non-star colour classes exceeds $q$. This is a contradiction.

When $v \equiv 1 \pmod{k(k-1)}$ we have $t \leq p - 1 \leq \frac{v-4k+3}{2(k-1)}$. Therefore

$$\begin{aligned} f(t) &\geq f\left(\frac{v-4k+3}{2(k-1)}\right) \\ &= \frac{v}{2}(3k^3 - 2k^2 + 8k - 12) - \frac{k(4k-3)q}{2} + 9k - 6 \end{aligned}$$

for all $t \leq \frac{(v-4k+3)}{2(k-1)}$. Since $p \leq \frac{v-2k+1}{2(k-1)}$ we have

$$f(t) \geq 3v(k-2) + 9k - k(k-1)q - 6.$$

So $f(t) > 0$ when $v \geq \left\lceil \frac{1}{3}(3k^3 + k^2 + 6k + 1) + \frac{8}{3(k-2)} \right\rceil$. In this case the average number of blocks in the non-star colour classes exceeds $q$. This is a contradiction.

**Case 2:** Assume that the colouring has $p$ colour classes. Then we either have no classes of type $(c)$ or at least one. So assume that there is at least one colour class of type $(c)$. Then $(i)$ $s = 2t$ and $p \geq t+1$ or $(ii)$ $s = 2t+1$ and $p \geq t+2$. So $s \leq 2t+1$ and $t \leq p-1 \leq \frac{v-3k+2}{2(k-1)}$. Since $s \leq \frac{v-k}{k-1}$ we have $v - k - (k-1)s \geq 0$ and ,thus $v - k - 2t(k-1) > 0$. Hence

$$(v-s)(v-1-(k-1)s) \geq (v-1-2t)(v-k-2t(k-1)).$$

As before, the total number of blocks in a class of type $(c)$ is at most $q(p-t)$ and so

$$q(p-t) \geq \frac{(v-2t-1)(v-k-2t(k-1))}{k(k-1)}.$$

Similar to case 1, this gives us a contradiction and hence our colouring cannot contain any colour classes of type $(c)$.

So it follows that our revised colouring has at most $p$ colours and that every colour class is either a star or a union of two stars. If we take all colour classes consisting of the union of two stars and split them into two constituent stars then we obtain a partition of the blocks of $S$ into at most $2p$ stars. This gives us $\chi''(S) \leq 2p$. Thus

$$\chi''(S) \leq \begin{cases} (v-k)/(k-1) & \text{if } v \equiv k \pmod{k(k-1)}; \\ (v-2k+1)/(k-1) & \text{if } v \equiv 1 \pmod{k(k-1)}. \end{cases}$$

This is a contradiction for all $v \geq k^3 + 2k$ by Theorem 4.7. The result follows.

□

We now have the obvious corollary for $S(2,4,v)$s.

**Corollary 4.13.** *Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,4,v)$. Then*

$$\underline{\chi}(B_1, v) \geq \begin{cases} (v+2)/6 & \text{if } v \equiv 4 \pmod{12} \text{ and } v \geq 148; \\ (v-1)/6 & \text{if } v \equiv 1 \pmod{12} \text{ and } v \geq 80, \end{cases}$$

Theorem 4.10 and Corollary 4.13 give us the following corollary.

**Corollary 4.14.** $\underline{\chi}(B_1, v) = \begin{cases} (v+2)/6 & \text{if } v \equiv 4 \pmod{36} \text{ and } v \geq 148; \\ (v-1)/6 & \text{if } v \equiv 13 \pmod{36} \text{ and } v \geq 80. \end{cases}$

We now give an upper bound on $\underline{\chi}(B_2, v)$.

**Theorem 4.15.** $\underline{\chi}(B_2, v) \leq (v-1)/3$ *for all* $v \equiv 1, 4 \pmod{12}$.

*Proof.* Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,4,v)$. Any block colouring of $S$ which avoids monochromatic intersecting blocks also provides a $B_2$-free colouring. Therefore $\underline{\chi}(B_2, v) \leq \chi'(v)$. Thus, by Corollary 3.8, if $v \equiv 4 \pmod{12}$ we have $\underline{\chi}(B_2, v) \leq \frac{v-1}{3}$.

Now assume that $v \equiv 1 \pmod{12}$. From Theorem 3.6 we know that there exists a 1-near resolvable $S(2,4,v)$, say $S'$. Colour the $r - 1 = \frac{v-4}{3}$ almost parallel classes of $S'$ with $\frac{v-4}{3}$ distinct colours such that all blocks in the same almost parallel class get coloured with one colour. Colour the remaining blocks of $S$ with one colour. This gives us a $B_2$-free colouring. Thus, $\underline{\chi}(B_2, v) \leq \frac{v-1}{3}$. □

The following lemma will help us determine a lower bound on $\underline{\chi}(B_2, v)$. We first recall that $D_\lambda(v,k,t)$ is the *maximum* number of blocks in a packing design.

**Lemma 4.16.** *Suppose that $S$ is a set of lines of an $S(2,k,v) = (\mathcal{V}, \mathcal{B})$ which does not contain a configuration $B_2$. Then either $S$ is a partial parallel class, or a star, or $S$ contains at most $D(4(k-1), 3, 2)$ lines.*

*Proof.* If $S$ is a partial parallel class or a star then we are done. So we assume that $S$ is not a partial parallel class or a star. Consider the points which lie in the intersection of the blocks in $S$. From amongst these choose a point of maximum degree, $d$. We let $a$ be this point.

Assume $d \geq k+2$. Take a line, $l$, in $S$ not passing through $a$. Then $l$ can intersect at most $k$ of the lines which pass through $a$. Therefore, if we take two lines $\lambda_1, \lambda_2$ passing through $a$ which do not intersect $l$ then the set $\{ l, \lambda_1, \lambda_2 \}$ forms a $B_2$ configuration. This is a contradiction. Therefore $d \leq k+1$.

Assume $d = 2$. Let the blocks in $S$ passing through $a$ be $\{ a, a_2, \ldots, a_{k-1} \}$ and $\{ a, b_1, b_2, \ldots, b_{k-1} \}$. Since $d = 2$ we can have at most one block in $S$ passing through each of the points $a_1, \ldots, a_{k-1}, b_1, \ldots, b_{k-1}$. Therefore, $|S| \leq 2k$.

Assume $d = 3$ and let the three blocks in $S$ passing through $a$ be $\{ a, a_1, \ldots, a_{k-1} \}$,$\{ a, b_1, \ldots, b_{k-1} \}$, and $\{ a, c_1, \ldots, c_{k-1} \}$. Since we assume that $S$ does not contain a $B_2$ configuration each line in $S$ must intersect at least two of these three lines. To achieve maximum cardinality in $S$ we will assume that each line in $S$ intersects exactly two of these three lines. Now each point in $S$ can have at most degree 3 so there can be at most $3(k-1)$ additional lines in $S$ with this property. Therefore, $|S| \leq 3k$.

Assume $d = 4$ and let the four blocks in $S$ passing through $a$ be $\{ a, a_1, \ldots, a_{k-1} \}$,$\{ a, b_1, \ldots, b_{k-1} \}$, $\{ a, c_1, \ldots, c_{k-1} \}$, and $\{ a, d_1, \ldots, d_{k-1} \}$. To avoid $B_2$ configurations and achieve maximum cardinality in $S$ we assume each line in $S$ must intersect three of these four lines. So each additional block in $S$ is of the form $\{ x, y, z, w_1, w_2, \ldots, w_{k-1} \}$ where $\{ x, y, z \} \subset T = A \bigcup B \bigcup C \bigcup D$, where $A = \{ a_i \mid 1 \leq i \leq k-1 \}, B = \{ b_i \mid 1 \leq i \leq k-1 \}, C = \{ c_i \mid 1 \leq i \leq k-1 \}$, and $D = \{ d_i \mid 1 \leq i \leq k-1 \}$,and $w_1, w_2, \ldots, w_{k-1} \in \mathcal{V}/T$. Hence the subsets $\{ x, y, z \}$ induce a partial Steiner triple system of order $v^* = 4(k-1)$. It is known that the maximum number of blocks in such a partial triple system is $D(v^*, 3, 2) = \left\lfloor \frac{v^*}{3} \left\lfloor \frac{v^*-1}{2} \right\rfloor \right\rfloor$([14]). It is clear that for each block $\{ x, y, z, w_1, w_2, \ldots, w_{k-1} \}$ in $S$ there is a block in the induced partial triple system. Thus, $|S| \leq D(4(k-1), 3, 2)$.

Assume $d = m, 5 \leq m \leq k+1$. To achieve maximum cardinality each line in $S$ must intersect exactly $m-1$ of the lines containing $a$. Similar to the case for $d = 4$, the additional blocks in $S$ induce a partial Steiner $S(2, m-1, v')$ on the $v' = m(k-1)$ elements, not equal to $a$, which occur in the same blocks as $a$ within $S$. Recall that the maximum number of blocks in such a partial Steiner system is $D(v', m-1, 2) = \left\lfloor \frac{v'}{m-1} \left\lfloor \frac{v'-1}{m-2} \right\rfloor \right\rfloor$ [28]. So for each $m \in \{ 5, 6, \ldots, k+1 \}$ we have $|S| \leq D(v', m-1, 2)$.

Therefore $|S| \leq \max\{ D(d(k-1), d-1, 2) \mid 4 \leq d \leq k+1 \} = D(4(k-1), 3, 2)$. □

Therefore, for $k = 4$ we have:

**Corollary 4.17.** *Suppose that $S$ is a set of lines of an $S(2,4,v) = (\mathcal{V}, \mathcal{B})$ which does not contain a configuration $B_2$. Then either $S$ is a partial parallel class, or a star, or $S$ contains at most 24 lines.*

We can now give a lower bound on $\underline{\chi}(B_2, v)$ for all Steiner systems $S = S(2,k,v)$.

**Theorem 4.18.** *For each $k \geq 3$ there exists a $v^*$ such that if $S = (\mathcal{V}, \mathcal{B})$ is an $S(2,k,v)$ with $v \geq v^*$ then $\underline{\chi}(B_2, v) \geq (v-1)/(k-1)$.*

*Proof.* Suppose that we have a $B_2$-free colouring of $S$ using $m$ colours, where $m < (v-1)/(k-1)$. Let $s$ be the number of colour classes which are stars, let $p$ be the number of colour classes which are partial parallel classes, and let $q$ be the number of other colour classes. Note that the number of blocks in the latter colour classes is $\leq D(4(k-1),3,2)$ by Lemma 4.16. We may assume that none of the latter two colour classes contain a star center. If they did then we could transfer the blocks containing the star center to the appropriate star colour class. This might create a new star or partial parallel class, and the process might have to be repeated, but this process will eventually terminate.

We have $s+p+q = m$ and, as shown before in Theorem 4.7, the number of blocks, $l$, in the non-star colour classes is

$$\frac{(v-s)(v-1-(k-1)s)}{k(k-1)} + \sum_{i=1}^{k-2} \frac{i(k-i-1)}{k} a_i.$$

where $a_i$ is the number of blocks containing $i+1$ star centers for all $1 \leq i \leq k-2$. The maximum number of blocks in the partial parallel classes is $\left\lfloor \frac{v-s}{k} \right\rfloor$. Therefore,

$$p\left(\frac{v-s}{k}\right) + D(4(k-1),3,2)q \geq \frac{(v-s)(v-1-(k-1)s)}{k(k-1)}.$$

Hence

$$\begin{aligned} k(k-1)D(4(k-1),3,2)q &\geq (v-s)(v-1-(k-1)s) - (k-1)p(v-s) \\ &= (v-s)(v-1-(k-1)s-(k-1)p) \\ &= (v-s)(v-1-(k-1)m+(k-1)q). \quad (4.5) \end{aligned}$$

If $m \leq \frac{v-k}{k-1}$ then $v-1-(k-1)m \geq k-1$. Since $s \leq m-q$, equation (4.5) gives us

$$\begin{aligned} k(k-1)D(4(k-1),3,2)q &\geq \left(v - \frac{v-k}{k-1} + q\right)(k-1+(k-1)q) \\ &= ((k-2)v+(k-1)q+k)(1+q) \qquad (4.6) \end{aligned}$$

Now let $v^* = ((k-1)t - k)/(k-2)$ for some $t$. When $v \geq v^*$ we have

$$\begin{aligned} & k(k-1)D(4(k-1),3,2)q \geq ((k-1)t + (k-1)q)(1+q) \\ & \quad \Rightarrow kD(4(k-1),3,2)q \geq (t+q)(1+q) \\ & \qquad \Rightarrow 0 \geq q^2 + (t - kD(4(k-1),3,2)+1)q + t \end{aligned} \tag{4.7}$$

It is easy to see that we can choose $t$ large enough such that the right side of equation (4.7) is greater than 0 for all $q \geq 0$. Therefore, we have a contradiction when $v \geq v^*$. Thus, $m \geq (v-1)/(k-1)$ for all $v \geq v^*$. □

This gives the following when $k = 4$.

**Corollary 4.19.** *If $S = (\mathcal{V}, \mathcal{B})$ is an $S(2,4,v)$ with $v \geq 130$ then $\underline{\chi}(B_2, v) \geq (v-1)/3$.*

Theorem 4.15 and Corollary 4.19 now give us the following result.

**Corollary 4.20.** *$\underline{\chi}(B_2, v) = (v-1)/3$ for all $v \geq 130$ and $v \equiv 1, 4 \pmod{12}$.*

We now give results on the upper and lower bound for $\underline{\chi}(B_3, v)$.

**Theorem 4.21.** $\underline{\chi}(B_3, v) \leq \begin{cases} (v+2)/6 & \text{if } v \equiv 4 \pmod{12}; \\ (v-4)/3 & \text{if } v \equiv 1 \pmod{12}. \end{cases}$

*Proof.* Assume $v \equiv 4 \pmod{12}$. Let $S = (\mathcal{V}, \mathcal{B})$ be a resolvable $S(2,4,v)$ with parallel classes $R_1, R_2, \ldots, R_r$. We first group the $R_i$s into pairs. Since $r = \frac{v-1}{3}$ (the number of $R_i$s) is odd we are left with one unpaired parallel class. We colour all blocks of the unpaired parallel class with one colour, and the remaining blocks of the $R_i$s with $\frac{r-1}{2} = \frac{v-4}{6}$ colours such that all blocks in each pair of $R_i$'s are coloured with the same colour. The result follows.

Assume $v \equiv 1 \pmod{12}$. From Section 3.5 we know that for all such $v$ there exists a 1-near resolvable $S(2,4,v)$. Let $S = (\mathcal{V}, \mathcal{B})$ be such an $S(2,4,v)$ with almost parallel classes $A_1, A_2, \ldots, A_{r-1}$. Let $B_1, B_2, \ldots, B_r$ be the remaining blocks of the design. We first group the $A_i s$ into pairs. Since $r - 1 = \frac{v-4}{3}$ (the number of $A_i s$) is odd we are left with one unpaired parallel class. We colour all blocks of the unpaired parallel class and the blocks $B_1, B_2$ with one colour, and the remaining blocks of the $A_i s$ with $\frac{r-2}{2} = \frac{v-7}{6}$ colours such that all blocks in each pair of $A_i s$ are coloured with the same colour. Finally, colour the remaining $B_j s$, $3 \leq j \leq r$, with $\frac{r-2}{2} = \frac{v-7}{6}$ distinct colours not yet used in the colouring. We have used a total of $1 + \frac{v-7}{6} + \frac{v-7}{6} = \frac{v-4}{3}$ colours. The result follows. □

**Theorem 4.22.** $\underline{\chi}(B_3, v) \geq \begin{cases} (v+2)/6 & \text{if } v \equiv 4 \pmod{12}; \\ (v+5)/6 & \text{if } v \equiv 1 \pmod{12}. \end{cases}$

*Proof.* Since each point in a colour class can have at most degree 2 there can be at most $\lfloor \frac{2v}{4} \rfloor = \lfloor \frac{v}{2} \rfloor$ blocks in a colour class. Therefore,

$$\underline{\chi}(B_3, v) \geq \left\lceil \frac{v(v-1)}{12} / \left\lfloor \frac{v}{2} \right\rfloor \right\rceil.$$

When we consider the cases $v \equiv 4 \pmod{12}$ and $v \equiv 1 \pmod{12}$ separately, we get our result.

□

Theorems 4.21 and 4.22 give us the following corollary.

**Corollary 4.23.** $\underline{\chi}(B_3, v) = (v+2)/6$ *when* $v \equiv 4 \pmod{12}$.

We also believe that the lower bound in Theorem 4.4 is tight for $v \equiv 1 \pmod{12}$, thus we make the following conjecture.

**Conjecture 3.** $\underline{\chi}(B_3, v) = (v+5)/6$ *when* $v \equiv 1 \pmod{12}$.

In fact, if there exists an $S(2,4,v), S$, with $v \equiv 1 \pmod{12}$ such that it admits a classical colouring with $\frac{v+2}{3}$ colours, then we could obtain a $B_3$-free colouring using $\frac{v+5}{6}$ colours. Indeed, we group the colour classes into pairs. Since $\frac{v+2}{3}$ is odd we have $\frac{v-1}{6}$ pairs and one lone colour class. We now recolour the blocks such that all blocks in each group get coloured with the same colour. This gives us a $B_3$-free colouring using $\frac{v+5}{6}$ colours. Combining this fact with the result in Theorem 4.22 we have

**Theorem 4.24.** *Let* $v \equiv 1 \pmod{12}$. *If* $\chi'(v) = \frac{v+2}{3}$ *then* $\underline{\chi}(B_3, v) = \frac{v+5}{6}$.

The contrapositive of the statement in Theorem 4.24 gives us

**Corollary 4.25.** *Let* $v \equiv 1 \pmod{12}$. *If* $\underline{\chi}(B_3, v) > \frac{v+5}{6}$ *then* $\chi'(v) > \frac{v+2}{3}$.

To prove our result on $B_4$ we first prove the following lemma.

**Lemma 4.26.** *If $C$ is a set of lines of an $S(2,k,v)$ and $|C| \geq v+1$ then $C$ contains a configuration $B_4$.*

*Proof.* Assume that $C$ contains at least $v+1$ lines and does not contain a configuration $B_4$. Let $p$ denote the number of points in $C$. Then the average degree in $C$ is $\frac{kv+k}{p} \geq \frac{kv+k}{v} > k$. Thus there is a point in $C$ of degree at least $k+1$. Now take a point in $C$ of maximum degree, $d \geq k+1$, and consider all lines incident with it. If any points on these lines had degree greater than one then $C$ would contain a $B_4$ configuration. Thus all points on these $d$ lines are of degree one and form a $d$-star in $C$. Delete this star from $C$ to form $C'$.

Now $|C'| \geq v + 1 - d$ and $C'$ contains $p - (k-1)d - 1$ points. We note that $p \neq (k-1)d + 1$ because if $d = \frac{p-1}{k-1}$ then we would have $|C'| > 0$. This is impossible since $C'$ cannot contain lines if it does not contain any points. Now the average degree in $C'$ is $k(v+1-d)/(p-(k-1)d-1) > k$, so we have a point in $C'$ of degree at least $k+1$. We iterate the previous process. At each stage of the process we remove more points than lines, eventually arriving at a point where we have a non-empty set of lines spanning no points.

□

We are now ready to give a lower bound for $\underline{\chi}(B_4, v)$.

**Theorem 4.27.** *Given an $S(2,k,v)$, $S = (\mathcal{V}, \mathcal{B})$ we have $\underline{\chi}(B_4, v) \geq (v - 1)/k(k-1)$.*

*Proof.* By the previous lemma each colour class has maximum cardinality $v$, therefore any $B_4$-free colouring must have at least $(v-1)/k(k-1)$ colour classes. □

This now gives us the following corollary.

**Corollary 4.28.** *Given an $S(2,4,v)$, $S = (\mathcal{V}, \mathcal{B})$ we have $\underline{\chi}(B_4, v) \geq (v - 1)/12$.*

The upper bound for $\underline{\chi}(B_4, v)$ is a little more difficult and we hope to investigate it further in the future.

## 4.3 The chromatic indices for small $v$

We give below the values of $\chi''(S)$ and $\chi(B_i, S)$ for $i = 1,2,3,4$ when $S$ is either the unique $S(2,4,13), S_{13}$, or the unique $S(2,4,16), S_{16}$, and we present partial results on these values when $v = 25, 49, 61$.

**Case 1: 2-parallel chromatic index**

Since the unique $S(2,4,13)$ is a projective plane there are no parallel lines. Thus, $\chi''(S_{13}) = 1$.

Now consider the unique $S(2,4,16), S_{16}$. If $\chi''(S_{16}) \leq 3$ then at least one colour class contains six blocks. Therefore, this colour class must contain two parallel blocks. This is a contradiction. Thus, $\chi''(S_{16}) \geq 4$. However, the following block-colouring of $S_{16}$ uses 4 colours and there are no monochromatic parallel blocks.

| | | | | | |
|---|---|---|---|---|---|
| $C_1$ | 0456 | 0789 | $0abc$ | $0def$ | $67cd$ |
| $C_2$ | $147a$ | $15bd$ | $168e$ | $19cf$ | $48bf$ |
| $C_3$ | $24ce$ | $257f$ | $269b$ | $28ad$ | $59ae$ |
| $C_4$ | $358c$ | $36af$ | $37be$ | 3012 | |

Therefore $\chi''(S_{16}) = 4$.

Now consider the case for $v = 25$. Since each element is incident with 8 blocks there can be at most 8 blocks in each colour class, thus we have $\underline{\chi}''(25) \geq 7$.

Consider the case for $v = 49$. Let $(\mathcal{X}, \mathcal{G}, \mathcal{C})$ be a $TD(4, 12)$, where $X = \mathbb{Z}_{12} \times \{1, 2, 3, 4\}$ and $G_i = \mathbb{Z}_{12} \times i$, for all $1 \leq i \leq 4$, and $\mathcal{G} = \bigcup_i G_i$.. Such a $TD$ exists by [10]. Let $S = (\mathcal{V}, \mathcal{B})$ be the $S(2, 4, 49)$ on $\mathcal{X} \bigcup \{\infty\}$ constructed by placing a copy of the unique $S(2, 4, 13)$ on each $G_i \bigcup \{\infty\}$ and combining them with the blocks of the $TD$.

A 15-colouring can be obtained by taking 3 of the above $S(2, 4, 13)s$ and 12 star centers at $(0, 1), (1, 1), \ldots, (11, 1)$. Therefore, $\underline{\chi}''(49) \leq 15$. However, we know that $l \geq \frac{(49-s)(16-s)}{4}$ and $l \leq 13(p - s)$, where $l$ is the number of non-star colour classes in such a colouring using $p$ colours. Now $\frac{(49-s)(16-s)}{4} > 13(14 - s)$ for all $s = 0, 1, 2, \ldots, 14$. Therefore, $\underline{\chi}''(49) = 15$.

Consider the case $v = 61$. Let $(\mathcal{X}, \mathcal{G}, \mathcal{C})$ be a $4 - GDD$ of type $12^5$, where $X = \mathbb{Z}_{12} \times \{1, 2, 3, 4, 5\}$ and $G_i = \mathbb{Z}_{12} \times i$, for all $1 \leq i \leq 5$, and $\mathcal{G} = \bigcup_i G_i$.. Such a $GDD$ exists by [10]. Let $S = (\mathcal{V}, \mathcal{B})$ be the $S(2, 4, 61)$ on $\mathcal{X} \bigcup \{\infty\}$ constructed by placing a copy of the unique $S(2, 4, 13)$ on each $G_i \bigcup \{\infty\}$ and combining them with the blocks of the $GDD$.

We can obtain a 20-colouring by taking 4 of the above $S(2, 4, 13)s$, 12 star centers at $(0, 1), (1, 1), \ldots, (11, 1)$, and 4 star centers at $(0, 2), (1, 2), (2, 2), (3, 2)$. Note that the latter 4 star centers exclude the blocks from the other 16 colour classes and thus, contains 12 blocks. Therefore, $\underline{\chi}''(61) \leq 20$. However, we know that $l \geq \frac{(61-s)(20-s)}{4}$ and $l \leq 13(p - s)$, where $l$ is the number of non-star colour classes in such a colouring using $p$ colours. Now $\frac{(61-s)(20-s)}{4} > 13(19 - s)$ for all $s = 0, 1, 2, \ldots, 19$. Therefore, $\underline{\chi}''(61) = 20$.

**Case 2: the 3-ppc**

Since the unique $S(2, 4, 13)$ is a projective plane there are no parallel lines. Thus, $\chi(B_1, S_{13}) = 1$.

Now consider the design $S_{16}$. We first begin with a 4-block-colouring of $S_{16}$ such that there are no monochromatic parallel blocks. We group the colour classes into pairs and colour the blocks with two colours such that every block in the same group are coloured with the same colour. This gives us a $B_1$-free block-colouring of $S_{16}$ which uses two colours. Therefore $\chi(B_1, S_{16}) = 2$.

Now consider the case $v = 25$. Let $S$ be an $S(2,4,25)$ which admits a $\underline{\chi}''(25)$-colouring such that no there are no monochromatic parallel blocks. From case 1 we know that $\underline{\chi}''(25) \geq 7$. Now group the colour classes into pairs and recolour the blocks such that each block in a group is coloured with the same colour. Then $\underline{\chi}(B_1, 25) \geq 4$.

Similarly, we have $\underline{\chi}(B_1, 49) = 8$ and $\underline{\chi}(B_1, 61) = 10$.

**Case 3: the hut**

Again, since $S_{13}$ contains no parallel lines we have $\chi(B_2, S_{13}) = 1$.

By Theorem 4.15 we have $\chi(B_2, S_{16}) \leq 5$. However, given two parallel lines in the $S_{16}$ a third line would either be disjoint from both or would intersect both, yielding a $3-ppc$ or a $3-path$, respectively. Therefore, the unique $S(2,4,16)$ contains no hut, and thus $\chi(B_2, S_{16}) = 1$.

By Theorem 4.15 we have $\underline{\chi}(B_2, 25) \leq 8$.

**Case 4: the 3-star**

By Theorems 4.21 and 4.22 we have $\chi(B_3, S_{13}) = 3 = \chi(B_3, S_{16})$.

Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,4,25)$ such that $\chi'(S) = 10$. Take the colour classes and group them into pairs. Recolour all the blocks such that all blocks in each pair get coloured with the same colour. Then we have a $B_3$-free block-colouring using 5 colours. Combined with Theorem 4.22 this gives us $\underline{\chi}(B_3, 25) = 5$.

**Case 5: the 3-path**

$S_{13}$ contains no parallel lines, hence no 3-path, so $\chi(B_4, S_{13}) = 1$.

If we colour the blocks of $S_{16}$ with 5 distinct colours so that all blocks in each parallel class get coloured with the same colour then we have a $B_4$-free colouring. So $\chi(B_4, S_{16}) \leq 5$. If we have two parallel lines in $S_{16}$ then a third line gives us a $3 - ppc$ or a $3 - path$. Thus, there can be no $B_4$-free colouring of $S_{16}$ using $k < 5$ colours, since any colouring with less than 5 colour classes contains a configuration $B_4$. So $\chi(B_4, S_{16}) = 5$.

By Theorem 4.27 we have $\underline{\chi}(B_4, 25) \geq 2$.

## 4.4 $T$-chromatic index for configurations consisting of more than three blocks

In this section we will consider the $\underline{\chi}(T, v)$ for $S(2,4,v)s$, where $T$ consists of more than three blocks.

We first consider the case where $T = S_n$ is an $n$-star. Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,4,v)$ which admits a proper block-colouring using $\chi'(v)$ colours. Group the colour classes into groups of cardinality $n-1$. If we recolour the blocks such that all the blocks in each group are coloured with the same colour then this new colouring does not contain any monochromatic copies of $S_n$. Therefore,

$$\underline{\chi}(S_n, v) \leq \underline{\chi}(S_n, S) \leq \left\lceil \frac{\chi'(v)}{n-1} \right\rceil .$$

In particular, when $v \equiv 4 \pmod{12}$ we have $\chi'(v) = r = \frac{v-1}{3}$, thus $\underline{\chi}(S_n, v) \leq \left\lceil \frac{v-1}{3(n-1)} \right\rceil$.

For a lower bound on $\underline{\chi}(S_n, v)$ we notice that any $n$-star-free configuration cannot have a point of degree greater than $n-1$. Therefore the largest possible configuration has at most $\left\lfloor \frac{(n-1)v}{4} \right\rfloor$ blocks. Thus,

$$\underline{\chi}(S_n, v) \geq \left\lceil \frac{v(v-1)}{12} \Big/ \left\lfloor \frac{(n-1)v}{4} \right\rfloor \right\rceil .$$

Now consider the case where $T = P_n$ is an $n$-ppc. Let $S = (\mathcal{V}, \mathcal{B})$ be an $S(2,4,v)$ which admits a block-colouring containing no monochromatic parallel lines and using $\chi''(v)$ colours. Again combine colour classes in groups of up to $n-1$. If we recolour the blocks such that all the blocks in each group are coloured with the same colour then this new colouring does not contain any monochromatic copies of $P_n$. Therefore,

$$\underline{\chi}(P_n, v) \leq \underline{\chi}(P_n, S) \leq \left\lceil \frac{\chi''(v)}{n-1} \right\rceil .$$

In particular, for $v \equiv 4, 13 \pmod{36}$ we have $\underline{\chi}(P_n, v) \leq \left\lceil \frac{v-1}{3(n-1)} \right\rceil$.

# Chapter 5

# Conclusion/Open problems

As demonstrated in previous chapters, many open questions remain, regarding both the classical colourings, and the specialized colourings of $S(2,4,v)s$. Concerning the classical chromatic index, one major open question is whether there exists an $S(2,4,v)$ , $v \equiv 1 \pmod{12}$, with minimum chromatic index equal to $\frac{v+2}{3}$. Somewhat more generally, what is the minimum chromatic index of an $S(2,4,v), v \equiv 1 \pmod{12}, v \geq 37$? We formulate this as open problems 1 and 2.

1. Does there exist an $S(2,4,v)$ $S$ with $\equiv 1 \pmod{12}$ such that $\chi'(S) = (v+2)/3$?

2. Determine the minimum chromatic index $\chi'(v)$ for all $v \equiv 1 \pmod{12}$.

   The spectrum of *minimum* chromatic indices for a given order $v$, over all $S(2,4,v)s$, is also of great interest, although apparently more difficult. By how much can the chromatic index of an $S(2,4,v)$ exceed the minimum possible? This question is appropriate not only for orders $v \equiv 1 \pmod{12}, v > 13$, but also for orders $v \equiv 4 \pmod{12}$. The known spectrum of chromatic indices for $v = 25$ shows that this excess can equal 1,2, or 3. Can it exceed 3? Apart from $PG(2,3)$, no such example is known. A related question is the following.

3. For which orders does there exist an $S(2,4,v)$ without a parallel class? Without an almost parallel class?

   While clearly there is no such system $S(2,4,16)$, several of the $S(2,4,28)s$ constructed by Krčadinac ([25]) have no parallel class. Apparently, this is a difficult question, as even an analogous question for Steiner triple systems remains unsolved.

   Many open questions can also be formulated regarding the lower and upper chromatic index and the spectrum for colourings of type $\pi$, for various partitions $\pi$ of the replication number $r$, or for $T$-colourings of Chapter 4. We list only some that we deem most important and (likely) most feasible.

4. Determine the complete spectrum $\Omega_\pi(v)$ for all $v \equiv 1, 4 \pmod{12}$ where $\pi = (2, 1, 1, \ldots, 1)$.

5. Determine the complete spectrum $\Omega_\pi(v)$ for all $v \equiv 1, 4 \pmod{12}$ where $\pi = (3, 1, 1, \ldots, 1)$.

6. Determine the complete spectrum $\Omega_\pi(v)$ for all $v \equiv 1 \pmod{12}$ where $\pi = (2, 2, 2, \ldots, 2)$.

7. Determine the minimum 2-parallel chromatic index $\underline{\chi}''(v)$ for all $v \equiv 1, 4 \pmod{12}$.

8. Let $B_i, i = 1, 2, 3, 4, 5$, be the three-line configurations of Section 4.2. Determine $\underline{\chi}(B_i, v)$ for all $v \equiv 1, 4 \pmod{12}$, where $1 \leq i \leq 5$.

9. Determine all orders of $v$ for which there exists a cyclic $S(2, 4, v)$.

10. Determine the existence of 2-block-colourings of type $P$ for all $P \subseteq K_r$.

# Chapter 6

# Glossary and Notation

**$\mathbf{K-GDD}$ of type $\mathbf{g_1^{u_1} g_2^{u_2} \ldots g_n^{u_n}}$** Let $\mathcal{V}$ be a $v$-set and $K$ a set of positive integers $\geq 2$. A *group divisible design* $K-GDD$ of type $g_1^{u_1} g_2^{u_2} \ldots g_n^{u_n}$ is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ with the following properties.

1. $\mathcal{G}$ is a partition of the set $\mathcal{V}$ into $g_i$-subsets (groups), for all $1 \leq i \leq n$, such that $u_1 g_1 + u_2 g_2 + \ldots + u_n g_n = v$,
2. $\mathcal{B}$ is a collection of $k$-subsets (blocks) of $\mathcal{V}$, where $k \in K$,
3. each pair of elements in $\mathcal{V}$ occurs in exactly one block in $\mathcal{B}$ or in exactly one group in $\mathcal{G}$.

**$\mathbf{RTD(k, v)}$** A resolvable $TD(k, v)$.

**$\mathbf{TD(k, v)}$** A $k - GDD$ of type $v^k$.

**$\mathbf{S(t, k, v)}$** Let $t, k, v$ be three positive integers such that $2 \leq t < k \leq v$. A *Steiner system* $S(t, k, v)$ is a pair $(\mathcal{V}, \mathcal{B})$ where $|\mathcal{V}| = v$ and $\mathcal{B}$ is a collection of $k$-subsets of $\mathcal{V}$, called blocks, such that every $t$-subset of $\mathcal{V}$ occurs in exactly one block in $\mathcal{B}$. When $t = 2$, the Steiner system $S(2, k, v)$ is sometimes called a Steiner 2-design.

**$\mathbf{STS(v)}$** An $S(t, k, v)$ with $t = 2$ and $k = 3$.

**$\chi'(\mathbf{S})$** Classical chromatic index; the minimum number of colours needed in a classical block-colouring of a Steiner system, $S$.

**$\chi'(\mathbf{v})$** Minimum chromatic index of order $v$;

$$\chi'(v) = \min\{\, \chi'(S) \mid S \text{ is a Steiner system of order } v \,\}$$

**$\underline{\chi}'(\mathbf{S})$**

$$\underline{\chi}'(S) = \min\{\, k \mid \text{there exists a valid } k\text{-block-colouring of } S \,\}$$

$\overline{\chi}'(\mathbf{S})$

$$\overline{\chi}'(S) = \max\{\, k \mid \exists \text{ a valid } k\text{-block-colouring of } S\,\}.$$

$\underline{\chi}'_\pi(\mathbf{S})$ the minimum number of colours needed in a block-colouring of type $\pi$ for a Steiner system, $S$.

$\underline{\chi}'_\pi(\mathbf{v})$

$$\underline{\chi}'_\pi(v) = \min\{\, \underline{\chi}'_\pi(S) \mid S \text{ is a Steiner system of order } v\,\}$$

$\overline{\chi}'_\pi(\mathbf{S})$ the maximum number of colours needed in a block-colouring of type $\pi$ for a Steiner system, $S$.

$\overline{\chi}'_\pi(\mathbf{v})$

$$\overline{\chi}'_\pi(v) = \max\{\, \overline{\chi}'_\pi(S) \mid S \text{ is a Steiner system of order } v\,\}$$

$\chi''(\mathbf{S})$ 2-parallel chromatic index; the minimum number of colours needed in a block-colouring of a Steiner system, $S$, such that there are no monochromatic parallel lines.

$\chi''(\mathbf{v})$

$$\chi''(v) = \min\{\, \chi''(S) \mid S \text{ is a Steiner system of order } v\,\}$$

$\chi(\mathbf{T},\mathbf{S})$ $T$-chromatic index; the minimum number of colours needed in a block-colouring of a Steiner system, $S$, such that there are no monochromatic copies of the configuration $T$.

$\underline{\chi}(\mathbf{T},\mathbf{v})$

$$\underline{\chi}(T,v) = \min\{\, \chi(T,S) \mid S \text{ is a Steiner system of order } v\,\}$$

$\boldsymbol{\Omega}(\mathbf{S})$

$$\Omega(S) = \{\, k \mid \exists \text{ a proper } k\text{-block-colouring of } S\,\}$$

$\boldsymbol{\Omega}(\mathbf{v})$

$$\Omega(v) = \bigcup \Omega(S)$$

$\boldsymbol{\Omega}_\pi(\mathbf{S})$

$$\Omega(S) = \{\, k \mid \exists \text{ a } k\text{-block-colouring of type } \pi \text{ of } S\,\}$$

$\boldsymbol{\Omega}_\pi(\mathbf{v})$

$$\Omega_\pi(v) = \bigcup \Omega_\pi(S)$$

$\boldsymbol{\Omega}_\mathbf{K}(\mathbf{S})$

$$\Omega(S) = \{\, k \mid \exists \text{ a } k\text{-block-colouring of profile } K \text{ of } S\,\}$$

$\boldsymbol{\Omega}_\mathbf{K}(\mathbf{v})$

$$\Omega_K(v) = \bigcup \Omega_K(S)$$

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