DUALITY OVER p-ADIC LIE EXTENSIONS OF GLOBAL FIELDS

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By
MENG FAI LIM, M.SC.

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AUTHOR: Meng Fai Lim, M. Sc. (National University of Singapore)

SUPERVISOR: Dr. Romyar Sharifi

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Abstract

In his monograph [Ne], Nekovář studies cohomological invariants of big Galois representations and looks at the variations of Selmer groups attached to intermediate number fields in a commutative p-adic Lie extension. In view of the formulation of the "main conjecture" for noncommutative extensions, it seems natural to extend the theory to a noncommutative p-adic Lie extension. This thesis will serve as a first step in an extension of this theory, namely, we will develop duality theorems over a noncommutative p-adic Lie extension which are extensions of Tate local duality, Poitou-Tate global duality and Grothendieck duality.

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Contents

Introduction

Iwasawa theory, as does much of number theory, revolves around the study of the relationship between algebraic objects and analytic objects that are naturally attached to number fields, elliptic curves, and even objects as general as motives. In Iwasawa theory, one studies the behavior of algebraic objects, most often Selmer groups, in a tower of number fields, and the analytic objects of comparison are the p-adic L-functions. A precise formulation of this relationship is usually called a "main conjecture", which is known in certain cases but conjectural in general. There are two parts to the conjecture, namely the existence of an appropriate p-adic L-function and the statement of a precise relationship between the algebraic object in question and the p-adic L-function.

It was observed by Iwasawa that a limit up a tower of algebraic objects that are p-torsion groups for a fixed prime p is a module over the completed $\mathbb{Z}_p$-group ring $\mathbb{Z}_p[\![\Gamma]\!]$ of the Galois group Γ of the tower. In the setting of Iwasawa's main conjecture, Γ was isomorphic to $\mathbb{Z}_p$, and the ring $\mathbb{Z}_p[\![\Gamma]\!]$ was then simply isomorphic to a power series ring in one-variable over $\mathbb{Z}_p$ (an observation of Serre). The main conjecture stated that a so-called characteristic power series of an eigenspace of the Galois group $\mathfrak{X}_\infty$ of the maximal abelian pro-p unramified outside p and ∞ extension of the cyclotomic $\mathbb{Z}_p$-extension of an abelian field agrees up to unit with a power series interpolating the values of a Kubota-Leopoldt p-adic L-function.

Of course, it is natural to consider towers with Galois group other than $\mathbb{Z}_p$, and the class of such towers that has come under the greatest consideration is that of the p-adic Lie extensions, i.e., for which the Galois group of the tower is a compact p-adic Lie group, and so isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ for some $n \geq 1$. Such Galois groups arise naturally in number theory: for instance, one obtains a $\mathrm{GL}_2(\mathbb{Z}_p)$-extension by adjoining to $\mathbb{Q}$ the coordinates of all p-power division points of a non-CM elliptic curve E defined over $\mathbb{Q}$. In the past decade, a great deal of activity in the study of Iwasawa theory has been focused on noncommutative generalizations of the main conjecture [CFKSV, FK, RW].

In this thesis, we are interested in exploring duality theorems in Galois cohomology in the context of noncommutative Iwasawa theory. To see why this might be of interest, we observe that in the situation of the cyclotomic $\mathbb{Z}_p$-extension $F_\infty = \cup_n F_n$ of a number field F, the direct limit of cohomology groups

$$\varinjlim_n H^1(\mathrm{Gal}(M_\infty/F_n), \mathbb{Q}_p/\mathbb{Z}_p),$$

where M_∞ is the maximal extension of F_∞ unramified outside p and ∞, is precisely the Pontrjagin dual of the Galois group $\mathfrak{X}_\infty$ appearing in Iwasawa's main conjecture for an abelian F.

Let F be a global field with characteristic not equal to p, and let S be a finite set of primes of F containing all primes above p and all archimedean primes of F. We let $G_{F,S}$ denote the Galois group $\mathrm{Gal}(F_S/F)$ of the maximal unramified outside S extension F_S of F inside a fixed separable closure of F. In its usual formulation, Poitou-Tate duality relates the kernels of the localization maps on the $G_{F,S}$-cohomology of a module and the Tate twist of its Pontrjagin dual. In fact, it can be given a cleaner formulation using compactly supported cohomology groups. For simplicity, we assume that p is odd if F has any real places.

The nth compactly supported $G_{F,S}$-cohomology group $H^n_{c,\mathrm{cts}}(G_{F,S}, M)$ with coefficients in a topological $G_{F,S}$-module M is defined as the nth cohomology group of the complex

$$\mathrm{Cone}\left(C^{\cdot}_{\mathrm{cts}}(G_{F,S}, M) \xrightarrow{\mathrm{res}_S} \bigoplus_{v \in S_f} C^{\cdot}_{\mathrm{cts}}(G_{F_v}, M) \right)[-1],$$

where G_{F_v} is the absolute Galois group of the completion of F at v, and res_S is the sum of restriction maps on the continuous cochain complexes. It therefore fits in a long exact sequence

$$\cdots \to H^n_{c,\mathrm{cts}}(G_{F,S}, M) \to H^n_{\mathrm{cts}}(G_{F,S}, M) \to \bigoplus_{v \in S} H^n_{\mathrm{cts}}(G_{F_v}, M) \to H^{n+1}_{c,\mathrm{cts}}(G_{F,S}, M) \to \cdots.$$

We now let R denote a commutative complete Noetherian local ring with finite residue field of characteristic p. Then we have the following formulation of Poitou-Tate duality due to Nekovář [Ne, Prop. 5.4.3(i)].

Theorem (Poitou-Tate duality). *Let T be a finitely generated R-module with a continuous (R-linear) $G_{F,S}$-action. Then there are isomorphisms*

$$H^n_{\mathrm{cts}}(G_{F,S}, T) \xrightarrow{\sim} H^{3-n}_{c,\mathrm{cts}}(G_{F,S}, T^\vee(1))^\vee$$
$$H^n_{c,\mathrm{cts}}(G_{F,S}, T) \xrightarrow{\sim} H^{3-n}_{\mathrm{cts}}(G_{F,S}, T^\vee(1))^\vee$$

of R-modules for all n, where $T^\vee = \mathrm{Hom}_{\mathrm{cts}}(T, \mathbb{Q}_p/\mathbb{Z}_p)$.

We now recall some notations from the language of derived categories. We denote by $\mathbf{D}(Mod_R)$ the derived category of R-modules which is obtained from the category $\mathrm{Ch}(Mod_R)$ of chain complexes of R-modules by inverting the quasi-isomorphisms, i.e., the maps of complexes that induce isomorphisms on cohomology. We have the derived functors $\mathbf{R}\mathrm{Hom}_R(-,-)$, $\mathbf{R}\Gamma_{\mathrm{cts}}(G_{F,S},-)$ and $\mathbf{R}\Gamma_{c,\mathrm{cts}}(G_{F,S},-)$ that are obtained from $\mathrm{Hom}_R(-,-)$, $C_{\mathrm{cts}}(G_{F,S},-)$ and $C_{c,\mathrm{cts}}(G_{F,S},-)$. Then the Poitou-Tate duality can be reformulated as the following isomorphisms

$$\mathbf{R}\Gamma_{\mathrm{cts}}(G_{F,S},T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\Big(\mathbf{R}\Gamma_{c,\mathrm{cts}}(G_{F,S},T^\vee(1)),\mathbb{Q}_p/\mathbb{Z}_p\Big)[-3]$$
$$\mathbf{R}\Gamma_{c,\mathrm{cts}}(G_{F,S},T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\Big(\mathbf{R}\Gamma_{\mathrm{cts}}(G_{F,S},T^\vee(1)),\mathbb{Q}_p/\mathbb{Z}_p\Big)[-3]$$

in $\mathbf{D}(Mod_R)$.

Nekovář gave a formulation of an analogue of Poitou-Tate duality with a duality of Grothendieck replacing Pontrjagin duality, as we now describe. There exists a bounded complex ω_R of R-modules of finite type, known as a dualizing complex, with the property that for every complex M of modules of finite type, the dual $\mathbf{R}\mathrm{Hom}_R(M,\omega_R) \in \mathbf{D}(Mod_R)$ is quasi-isomorphic to a complex of R-modules of finite type, and moreover, the canonical morphism

$$M \longrightarrow \mathbf{R}\mathrm{Hom}_R\big(\mathbf{R}\mathrm{Hom}_R(M,\omega_R),\omega_R\big)$$

is an isomorphism in $\mathbf{D}(Mod_R)$. We remark that when R is regular (or Gorenstein), the dualizing complex can be taken to be R.

Now suppose T is a bounded complex of $R[G_{F,S}]$-modules that are finitely generated over R. Then there exists a complex T^* of modules of the same form that represents $\mathbf{R}\mathrm{Hom}_R(T,\omega_R)$ in an appropriate derived category of $R[G_{F,S}]$-modules. In this case, we have the following isomorphisms of Nekovář (see [Ne, Prop. 5.4.3(ii)]).

Theorem (Nekovář). *We have the following isomorphisms*

$$\mathbf{R}\Gamma_{\mathrm{cts}}(G_{F,S},T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma_{c,\mathrm{cts}}(G_{F,S},T^*(1)),\omega_R\Big)[-3]$$
$$\mathbf{R}\Gamma_{c,\mathrm{cts}}(G_{F,S},T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma_{c}(G_{F,S},T^*(1)),\omega_R\Big)[-3]$$

in $\mathbf{D}(Mod_R)$.

In this thesis, we study generalizations of the above duality of Poitou-Tate and Grothendieck duality of Nekovář in the context of noncommutative Iwasawa theory. Suppose that F_∞ is a p-adic Lie extension of F contained in F_S. We denote by Γ the Galois group of the extension F_∞/F, and we let $\Lambda = R[\![\Gamma]\!]$ denote the resulting Iwasawa algebra over R. Let T be a finitely generated R-module with a continuous (R-linear) $G_{F,S}$-action, and let A be a cofinitely generated R-module with a continuous (R-linear) $G_{F,S}$-action. The Λ-modules of interest are the following direct and inverse limits of cohomology groups (and their counterparts with compact support)

$$\varinjlim_{F_\alpha} H^n_{\text{cts}}(\mathrm{Gal}(F_S/F_\alpha), A) \quad \text{and} \quad \varprojlim_{F_\alpha} H^n_{\text{cts}}(\mathrm{Gal}(F_S/F_\alpha), T),$$

where the limits are taken over all finite Galois extensions F_α of F which are contained in F_∞. By an application of Shapiro's lemma, one can show that they are respectively isomorphic to

$$H^n_{\text{cts}}(G_{F,S}, F_\Gamma(A)) \quad \text{and} \quad H^n_{\text{cts}}(G_{F,S}, \mathscr{F}_\Gamma(T)),$$

where the Λ-modules $F_\Gamma(A)$ and $\mathscr{F}_\Gamma(T)$ are defined by

$$\varinjlim_{F_\alpha} \mathrm{Hom}_R(R[\mathrm{Gal}(F_\alpha/F)], A) \quad \text{and} \quad \varprojlim_{F_\alpha} R[\mathrm{Gal}(F_\alpha/F)] \otimes_R T$$

respectively. Therefore, we can reduce the question of finding dualities on the Iwasawa modules of interest to that of obtaining dualities over $G_{F,S}$, but with R replaced by Λ.

In his monograph [Ne], Nekovář considers the above situation over a commutative p-adic Lie extension (e.g., a $\mathbb{Z}_p^r$-extension) and develops extensions of Poitou-Tate global duality and the duality of Grothendieck for the above cohomology groups. In view of the noncommutative main conjecture, one would like to extend the work of Nekovář to the noncommutative setting.

In order to prove duality theorems over noncommutative p-adic Lie extensions, we must first understand the structure of the noncommutative Iwasawa algebras and their topological modules. In particular, we shall prove that the Iwasawa algebra Λ is Noetherian (cf. Theorem 3.4.1), generalizing a result of Lazard from the case that $R = \mathbb{Z}_p$.

Theorem. *Let R be a commutative complete Noetherian local ring with finite residue field of characteristic p, and let G be a compact p-adic Lie group. Then $R[\![G]\!]$ is a Noetherian ring.*

Together with the module theory, we carefully develop the theory of continuous group cohomology in our setting. From there, we are able to state and prove our duality theorems (cf. Theorems 5.3.1 and 5.4.1).

Theorem. *Let T be a bounded complex of ind-admissible $R[G_{F,S}]$-modules which are* *finitely generated over R. Then we have the following isomorphisms*

$$\mathbf{R}\Gamma_{\mathrm{cts}}(G_{F,S}, \mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\Big(\mathbf{R}\Gamma_{c,\mathrm{cts}}\big(G_{F,S}, F_\Gamma(T^\vee)(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)[-3]$$
$$\mathbf{R}\Gamma_{\mathrm{cts}}(G_{F,S}, \mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma_{c,\mathrm{cts}}\big(G_{F,S}, \mathscr{F}_\Gamma(T^*)(1)\big), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-3]$$

in the derived category of Λ-modules.

In Nekovář's setting, the group Γ may be taken to be an abelian pro-p p-adic Lie group, and so Λ is a commutative Noetherian complete local ring with finite residue field of characteristic p. Moreover, Nekovář shows that its dualizing complex is isomorphic to $\Lambda \otimes^{\mathbf{L}}_R \omega_R$ in the derived category of Λ-modules. Therefore, the commutative theory described above applies to Λ, and Nekovář is able to deduce his dualities from this. In our thesis, since we are working with noncommutative p-adic Lie extensions, we do not know the existence of a (sufficiently nice) dualizing complex that is compatible with continuous Galois cohomology, and so the proof of the second duality takes another route.

We now give a brief description of the contents of each chapter of the thesis. In Chapter 1, we introduce notations and results from homological algebra required for the thesis. In particular, we will introduce the language of derived categories. We also develop certain derived functors for bimodules over algebras that are central and flat over a commutative ring. These will be applied in the later parts of the thesis. Chapter 2 is about the discussion of adic rings and their topological modules. We also introduce continuous cohomology groups with coefficients in compact modules and discrete modules. In the latter part of Chapter 2, we shall see that the notion of ind-admissible modules (see [Ne, 3.3]) can be carried over to the setting of Noetherian adic rings. We will then describe the category of ind-admissible modules in terms of compact modules and discrete modules.

In Chapter 3, we will investigate the ring-theoretic properties of the completed group algebra of a finitely generated pro-p group. Our study will lead to a generalization of a result of Lazard which essentially says that the Iwasawa algebras in which we are interested are Noetherian. In Chapter 4, we will apply Shapiro's lemma to see that the direct limits and inverse limits of cohomology groups over every intermediate field F_α can be viewed as cohomology groups of certain Λ-modules. We shall also establish certain finiteness results of the cohomology groups in Section 4.2. In that section, we make heavy use of the fact that the Iwasawa algebra of a p-adic Lie extension is Noetherian.

Finally, we will prove the duality theorems in Chapter 5. We will prove an extension of the Tate's local duality and Poitou-Tate duality for a finitely presented module (with a continuous Galois action) over an adic ring, with no restrictions on p. We will then prove the duality theorems of Grothendieck for local fields and global fields over a p-adic Lie extension, with the restriction that p is odd in the number field case if the field has any real places.

Chapter 1

Preliminaries

We begin by reviewing certain objects and notations which will be used in this write-up. Most of the material presented in Section 1.1 and Section 1.2 can be found in [Hart, Ne, Wei]. In Section 1.3, we introduce some derived functors over the derived category of certain bimodules over algebras that are central and flat over a commutative ring. The approach used here is inspired by the paper [Ye] which dealt with algebras that are central and flat over a field. As we shall see in Section 4.1, the Iwasawa algebras we are interested in are central and flat over their coefficient rings.

Throughout the thesis, every ring is associative and has a unit.

1.1 Complexes

Fix an abelian category $\mathfrak{C}$ and denote the category of (cochain) complexes of objects in $\mathfrak{C}$ by $\mathrm{Ch}(\mathfrak{C})$. We also denote the category of bounded below complexes, bounded above complexes and bounded complexes by $\mathrm{Ch}^+(\mathfrak{C})$, $\mathrm{Ch}^-(\mathfrak{C})$ and $\mathrm{Ch}^b(\mathfrak{C})$ respectively. For each $n \in \mathbb{Z}$, the translation by n of a complex X is given by

$$X[n]^i = X^{n+i}, \; d^i_{X[n]} = (-1)^n d^{n+i}_X.$$

If $f : X \longrightarrow Y$ is a morphism of complexes, then $f[n] : X[n] \longrightarrow Y[n]$ is given by $f[n]^i = f^{n+i}$.

A covariant additive functor $F : \mathfrak{C} \longrightarrow \mathfrak{C}'$ induces a functor $F : \mathrm{Ch}(\mathfrak{C}) \longrightarrow \mathrm{Ch}(\mathfrak{C}')$ with $d^i_{FX} = F(d^i_X)$. The identity morphisms in each degree define a canonical isomorphism of complexes

$$F(X[n]) \xrightarrow{\sim} F(X)[n].$$

A contravariant additive functor $F : \mathfrak{C}^o \longrightarrow \mathfrak{C}'$ induces a functor $F : \mathrm{Ch}(\mathfrak{C})^o \longrightarrow \mathrm{Ch}(\mathfrak{C}')$ with $d^i_{FX} = (-1)^{i+1}F(d_X^{-i-1})$. Suppose $G : (\mathfrak{C}')^o \longrightarrow \mathfrak{C}''$ is another contravariant functor. Then, for each i, we have

$$d^i_{G(F(X))} = (-1)^{i+1}G(d_{FX}^{-i-1}) = (-1)^{i+1}G((-1)^i F d^i_X) = -G(F(d^i_X)).$$

On the other hand, we note that $G \circ F$ is a covariant functor. Therefore, taking the sign conventions into consideration, we have an isomorphism

$$G(F(X)) \xrightarrow{\sim} (G \circ F)(X)$$

of complexes which is given by $(-1)^i$ times the identity morphism in degree i.

If X is a complex, we have the following truncations of X:

$$\begin{aligned}
\sigma_{\leq i}X &= [\cdots \longrightarrow X^{i-2} \longrightarrow X^{i-1} \longrightarrow X^i \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots] \\
\tau_{\leq i}X &= [\cdots \longrightarrow X^{i-2} \longrightarrow X^{i-1} \longrightarrow \ker(d^i_X) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots] \\
\sigma_{\geq i}X &= [\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^i \longrightarrow X^{i+1} \longrightarrow X^{i+2} \longrightarrow \cdots] \\
\tau_{\geq i}X &= [\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{coker}(d_X^{i-1}) \longrightarrow X^{i+1} \longrightarrow X^{i+2} \longrightarrow \cdots].
\end{aligned}$$

The cone of a morphism $f : X \longrightarrow Y$ is defined by $\mathrm{Cone}(f) = Y \oplus X[1]$ with differential

$$d^i_{\mathrm{Cone}(f)} = \begin{pmatrix} d^i_Y & f^{i+1} \\ 0 & -d^{i+1}_X \end{pmatrix} : Y^i \oplus X^{i+1} \longrightarrow Y^{i+1} \oplus X^{i+2}.$$

There is an exact sequence of complexes

$$0 \longrightarrow Y \xrightarrow{j} \mathrm{Cone}(f) \xrightarrow{p} X[1] \longrightarrow 0,$$

where j and p are the canonical inclusion and projection respectively. The corresponding boundary map

$$\delta : H^i(X[1]) = H^{i+1}(X) \longrightarrow H^{i+1}(Y)$$

is induced by f^{i+1}.

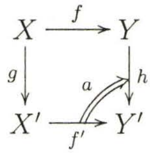
A homotopy a between two morphisms of complexes $f, g : X \longrightarrow Y$ is defined by a collection of maps $a^i : X^{i+1} \longrightarrow Y^i$ such that $g - f = da + ad$. We shall denote this by $a : f \rightsquigarrow g$. If $u : X' \longrightarrow X$ (resp. $v : Y' \longrightarrow Y$) is a morphism of complexes, then $a \star u = (a^i \circ u^{i+1} : (X')^{i+1} \longrightarrow Y^i)$ (resp. $v \star a = (v^i \circ a^i : X^{i+1} \longrightarrow (Y')^i)$) is a homotopy $a \star u : fu \rightsquigarrow gu$ (resp. $v \star a : vf \rightsquigarrow vg$).

A second order homotopy α between two homotopies $a, b : f \rightsquigarrow g$ is defined by a collection of maps $\alpha^i : X^{i+2} \longrightarrow Y^i$ such that $\alpha d - d\alpha = b - a$. We denote this by $\alpha : a \rightsquigarrow b$.

Let $\mathrm{tr}_1(\mathfrak{C})$ be the category defined as follows: the objects are morphisms of complexes $f : X \longrightarrow Y$ in $\mathfrak{C}$. Supposing $f' : X' \longrightarrow Y'$ is another object, a morphism from f to f' is given by

$$(g, h, a) : (f : X \longrightarrow Y) \longrightarrow (f' : X' \longrightarrow Y')$$

where $g : X \longrightarrow X'$ and $h : Y \longrightarrow Y'$ are morphisms of complexes and $a : f'g \rightsquigarrow hf$. We denote this by the following diagram.



The composition

$$(f : X \longrightarrow Y) \stackrel{(g,h,a)}{\longrightarrow} (f' : X' \longrightarrow Y') \stackrel{(g',h',a')}{\longrightarrow} (f'' : X'' \longrightarrow Y'')$$

is defined to be $(g'g, h'h, a' \star g + h' \star a)$.

A morphism $(g, h, a) : (f : X \longrightarrow Y) \longrightarrow (f' : X' \longrightarrow Y')$ induces a morphism of complexes $\mathrm{Cone}(g, h, a) : \mathrm{Cone}(f) \longrightarrow \mathrm{Cone}(f')$ given by

$$\mathrm{Cone}(g, h, a)^i = \begin{pmatrix} h^i & a^i \\ 0 & g^{i+1} \end{pmatrix} : Y^i \oplus X^{i+1} \longrightarrow (Y')^i \oplus (X')^{i+1}.$$

Hence, we have a functor $\mathrm{Cone} : \mathrm{tr}_1(\mathfrak{C}) \longrightarrow \mathrm{Ch}(\mathfrak{C})$.

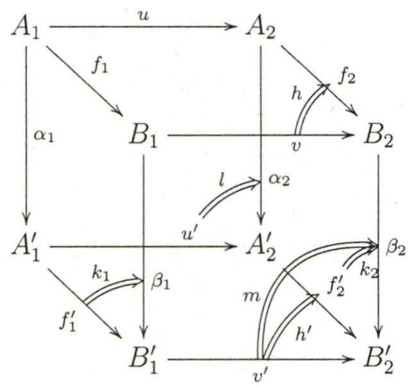
A homotopy $(b, b', \alpha) : (g, h, a) \rightsquigarrow (g', h', a')$ in $\mathrm{tr}_1(\mathfrak{C})$ consists of homotopies $b : g \rightsquigarrow g'$ and $b' : h \rightsquigarrow h'$, and a second order homotopy

$$\alpha : f' \star b + a' \rightsquigarrow b' \star f + a.$$

One can then check that this induces a homotopy (in the usual sense)

$$\begin{pmatrix} b' & \alpha \\ 0 & -b \end{pmatrix} : \mathrm{Cone}(g, h, a) \longrightarrow \mathrm{Cone}(g', h', a').$$

Suppose we have the following cubic diagram of complexes



whose faces are commutative up to homotopy. Suppose further that the boundary of the cube is trivialized by a 2-homotopy $H = (H^i : A_1^{i+2} \to (B'_2)^i)$, i.e.,

$$H : v' \star k_1 + m \star f_1 + \beta_2 \star h \rightsquigarrow k_2 \star u + h' \star \alpha_1 + f'_2 \star l.$$

Then the triple (k_1, k_2, H) defines a homotopy

$$\begin{aligned}(k_1, k_2, H) : (f'_1, f'_2, h') \circ (\alpha_1, \alpha_2, l) = (f'_1\alpha_1, f'_2\alpha_2, h' \star \alpha_1 + f'_2 \star l) \rightsquigarrow \\ (\beta_1 f_1, \beta_2 f_2, m \star f_1 + \beta_2 \star h) = (\beta_1, \beta_2, m) \circ (f_1, f_2, h),\end{aligned}$$

i.e., the following diagram

$$\begin{array}{ccc} \mathrm{Cone}(u) & \xrightarrow{(f_1, f_2, h)} & \mathrm{Cone}(v) \\ {\scriptstyle (\alpha_1,\alpha_2,l)}\downarrow & & \downarrow{\scriptstyle (\beta_1,\beta_2,m)} \\ \mathrm{Cone}(u') & \xrightarrow{(f'_1, f'_2, h')} & \mathrm{Cone}(v') \end{array}$$

is commutative up to homotopy.

Define $\mathbf{K}(\mathfrak{C})$ to be the category of complexes of objects in $\mathrm{Ch}(\mathfrak{C})$ where the morphisms are given by homotopy classes of homomorphisms of complexes. We write $\mathbf{K}^+(\mathfrak{C})$, $\mathbf{K}^-(\mathfrak{C})$ and $\mathbf{K}^b(\mathfrak{C})$ for the subcategories of $\mathbf{K}(\mathfrak{C})$ with objects in $\mathrm{Ch}^+(\mathfrak{C})$, $\mathrm{Ch}^-(\mathfrak{C})$ and $\mathrm{Ch}^b(\mathfrak{C})$ respectively. Given a morphism $f : X \longrightarrow Y$ in $\mathrm{Ch}(\mathfrak{C})$, we say that the following

$$X \xrightarrow{f} Y \xrightarrow{j} \mathrm{Cone}(f) \xrightarrow{-p} X[1]$$

is a *strict exact triangle.* Suppose we are given objects A, B and C in $\mathbf{K}(\mathfrak{C})$ and morphisms $u : A \longrightarrow B, v : B \longrightarrow C$ and $w : C \longrightarrow A[1]$ in $\mathbf{K}(\mathfrak{C})$. We then say that

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

is an *exact triangle* if there exists a strict exact triangle

$$X \xrightarrow{f} Y \xrightarrow{j} \mathrm{Cone}(f) \xrightarrow{-p} X[1]$$

such that we have the following diagram

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha[1]} \\ X & \xrightarrow{f} & Y & \xrightarrow{j} & \mathrm{Cone}(f) & \xrightarrow{-p} & X[1] \end{array}$$

which commutes in $\mathbf{K}(\mathfrak{C})$ (i.e., commutes up to homotopy) with the vertical morphisms being isomorphisms in $\mathbf{K}(\mathfrak{C})$ (i.e., homotopy equivalences). We may sometimes write an exact triangle as $A \longrightarrow B \longrightarrow C$ instead of $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$.

We say that a morphism in $\mathrm{Ch}(\mathfrak{C})$ or $\mathbf{K}(\mathfrak{C})$ is a *quasi-isomorphism* if it induces isomorphisms on the cohomology of the complexes. The derived category $\mathbf{D}(\mathfrak{C})$ is obtained by inverting the quasi-isomorphisms in $\mathbf{K}(\mathfrak{C})$ (see [Wei, Chap. 10]). One has similar definitions for $\mathbf{D}^+(\mathfrak{C})$, $\mathbf{D}^-(\mathfrak{C})$ and $\mathbf{D}^b(\mathfrak{C})$. We remark that we may not always be able to perform such constructions due to certain set-theoretic considerations (loc. cit. 10.3), although when $\mathfrak{C}$ is the category of modules over some ring, the derived category exists (loc. cit. Prop. 10.4.4). However in general, one may have to work with categories other than the category of modules. One way to get around this is to make use of the following proposition (cf. [Wei, Prop. 10.4.8]).

Proposition 1.1.1. *Suppose that $\mathfrak{C}$ has enough injectives. Then $\mathbf{D}^+(\mathfrak{C})$ exists and is equivalent to the full subcategory $\mathbf{K}^+(\mathcal{I})$ of $\mathbf{K}^+(\mathfrak{C})$ whose objects are bounded below complexes of injectives.*

If $\mathfrak{C}$ has enough projectives. Then $\mathbf{D}^-(\mathfrak{C})$ exists and is equivalent to the full subcategory $\mathbf{K}^-(\mathcal{P})$ of $\mathbf{K}^+(\mathfrak{C})$ whose objects are bounded above complexes of projectives. □

Remark. In the case that $\mathbf{D}^-(\mathfrak{C})$ exists, then $\mathbf{D}^b(\mathfrak{C})$ also exists, and is a subcategory of $\mathbf{D}^-(\mathfrak{C})$. A similar statement holds in the case that $\mathbf{D}^+(\mathfrak{C})$ exists.

We shall describe the above equivalence of the categories briefly and refer readers to [Wei] for the details. Suppose $\mathfrak{C}$ has enough injectives. Let A be an object in $\mathbf{K}^+(\mathfrak{C})$. Then one has the Cartan-Eilenberg resolution of A which is a double complex of injectives whose total complex is a bounded below complex and is quasi-isomorphic to A (loc. cit. Ex. 5.7.1). Since any two of such resolutions are homotopic (loc. cit. Ex. 5.7.3), this gives a unique representation of A in $\mathbf{K}^+(\mathcal{I})$. We have a similar construction when $\mathfrak{C}$ has enough projectives.

1.2 Some sign conventions

In this section, we will introduce some sign conventions to which we will adhere throughout the thesis. If X is a complex and $x \in X^i$, we write $\bar{x} = i$ for the degree.

Let Λ, S and T be rings. Let M (resp., N) be a Λ-S-bimodule (resp., a Λ-T-bimodule). Then $\mathrm{Hom}_\Lambda(M, N)$ is taken to be the S-T-bimodule of all left Λ-module homomorphisms from M to N, where the left S-action is given by $(s \cdot f)(m) = f(ms)$ and the right T-action is given by $(f \cdot t)(m) = f(m)t$ for $f \in \mathrm{Hom}_\Lambda(M, N), m \in M, s \in S$ and $t \in T$. If $M^\bullet$ is a complex of Λ-S-bimodules and $N^\bullet$ a complex of Λ-T-bimodules, we define a complex $\mathrm{Hom}^\bullet_\Lambda(M^\bullet, N^\bullet)$ of S-T-bimodules by

$$\mathrm{Hom}^n_\Lambda(M^\bullet, N^\bullet) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_\Lambda(M^i, N^{i+n})$$

with differentials defined as follows: for $f \in \mathrm{Hom}_\Lambda(M^i, N^{i+n})$, we have

$$df = d_N^{i+n} \circ f + (-1)^n f \circ d_M^{i-1}.$$

If $M^\bullet = M$ is a complex concentrated in degree zero, then $\mathrm{Hom}_\Lambda(M, -)$ is a covariant functor and the sign convention for the differentials coincides with that in Section 1.1. Similarly if $N^\bullet = N$ is a complex concentrated in degree zero, then $\mathrm{Hom}_\Lambda(-, N)$ is a contravariant functor with sign convention for the differentials coinciding with that defined in Section 1.1.

In the case when $S = T$, we have a similar definition for the complexes $\mathrm{Hom}^\bullet_{\Lambda-S}(M^\bullet, N^\bullet)$ of abelian groups, where $\mathrm{Hom}_{\Lambda-S}(M, N)$ is the group of all Λ-S-bimodule homomorphisms from M to N. It follows immediately from the definition that for an element $f \in \mathrm{Hom}^0_{\Lambda-S}(M^\bullet, N^\bullet)$, we have $f \in \mathrm{Hom}_{\mathrm{Ch}(\Lambda-S)}(M^\bullet, N^\bullet)$ if and only if $df = 0$. Here $\mathrm{Ch}(\Lambda - S)$ denotes the category of complexes of Λ-S-bimodules.

Suppose that $M^\bullet$ is a complex of Λ-S-bimodules and $L^\bullet$ a complex of S-T-bimodules. We define the complex $M^\bullet \otimes_S L^\bullet$ of Λ-T-bimodules by

$$(M^\bullet \otimes_S L^\bullet)^n = \bigoplus_{i \in \mathbb{Z}} M^i \otimes_S L^{n-i}$$

with differentials

$$d(m \otimes l) = dm \otimes l + (-1)^{\bar{m}} m \otimes dl.$$

Lemma 1.2.1. *The following formulas define isomorphisms of complexes:*

$$\begin{gathered}\mathrm{Hom}^\bullet_\Lambda(M^\bullet, N^\bullet)[n] \cong \mathrm{Hom}^\bullet_\Lambda(M^\bullet, N^\bullet[n])\\ f \mapsto f\\ (M^\bullet[n]) \otimes_S L^\bullet \cong (M^\bullet \otimes_S L^\bullet)[n]\\ m \otimes l \mapsto m \otimes l\\ M^\bullet \otimes_S (L^\bullet[n]) \cong (M^\bullet \otimes_S L^\bullet)[n]\\ m \otimes l \mapsto (-1)^{n\bar{m}} m \otimes l.\end{gathered}$$

Proof: This follows from a straightforward verification of the definition of translation and the sign conventions. □

Lemma 1.2.2. *The adjunction morphisms define morphisms*

$$\begin{gathered}\mathrm{Hom}^\bullet_{\Lambda-T}(M^\bullet \otimes_S L^\bullet, N^\bullet) \longrightarrow \mathrm{Hom}^\bullet_{\Lambda-S}\big(M^\bullet, \mathrm{Hom}^\bullet_{T^o}(L^\bullet, N^\bullet)\big)\\ f \mapsto \big(m \mapsto (l \mapsto f(m \otimes l))\big)\\ \mathrm{Hom}^\bullet_{\Lambda-T}(M^\bullet \otimes_S L^\bullet, N^\bullet) \longrightarrow \mathrm{Hom}^\bullet_{S-T}\big(L^\bullet, \mathrm{Hom}^\bullet_\Lambda(M^\bullet, N^\bullet)\big)\\ f \mapsto \big(l \mapsto (m \mapsto (-1)^{\bar{m}\bar{l}} f(m \otimes l))\big)\end{gathered}$$

of complexes and morphisms

$$\begin{gathered}\mathrm{Hom}_{\mathrm{Ch}(\Lambda-T)}(M^\bullet \otimes_S L^\bullet, N^\bullet) \longrightarrow \mathrm{Hom}_{\mathrm{Ch}(\Lambda-S)}\big(M^\bullet, \mathrm{Hom}^\bullet_{T^o}(L^\bullet, N^\bullet)\big)\\ \mathrm{Hom}_{\mathrm{Ch}(\Lambda-T)}(M^\bullet \otimes_S L^\bullet, N^\bullet) \longrightarrow \mathrm{Hom}_{\mathrm{Ch}(S-T)}\big(L^\bullet, \mathrm{Hom}^\bullet_\Lambda(M^\bullet, N^\bullet)\big)\end{gathered}$$

of abelian groups. All of these maps are monomorphisms; they are isomorphisms if $M^\bullet$ and $L^\bullet$ are bounded above and $N^\bullet$ is bounded below. □

1.3 Some derived functors

Given a ring Λ, we shall denote the category of left Λ-modules by Mod_Λ. Let $\mathbf{K}(Mod_\Lambda)$ denote the category of complexes of left Λ-modules where the morphisms are given by homotopy classes of homomorphisms of complexes. The derived category of Λ-modules is then denoted by $\mathbf{D}(Mod_\Lambda)$.

For a ring Λ, the opposite ring Λ^o is defined to be the ring with underlying additive group Λ and multiplication given by $\lambda_1 \cdot_o \lambda_2 = \lambda_2\lambda_1$ for $\lambda_1, \lambda_2 \in \Lambda$. One can identify the category of right Λ-modules with the category of left Λ^o-modules. From now on, unless otherwise stated, a Λ-module is always taken to be a left Λ-module.

Let R be a fixed commutative ring. For the remainder of this chapter, every ring is taken to be a central R-algebra. In other words, there is a ring homomorphism $R \longrightarrow \Lambda$

whose image is contained in the center of Λ. For two such rings Λ and S, we are interested in a subclass of the class of Λ-S-bimodules, namely, the class of Λ-S-bimodules with the extra property that the left R-action coincides with the right R-action. We can (and shall) identify the category of such Λ-S-bimodules with the category of $\Lambda \otimes_R S^o$-modules, and there are natural exact functors $res_\Lambda : Mod_{\Lambda\otimes_R S^o} \longrightarrow Mod_\Lambda$ and $res_{S^o} : Mod_{\Lambda\otimes_R S^o} \longrightarrow Mod_{S^o}$, which extend to exact functors on the derived categories. By abuse of notation, we also denote the exact restriction from any category of modules over a central R-algebra to the category of R-modules by res_R. One observes that $Mod_{\Lambda\otimes_R R^o} = Mod_\Lambda$ and $Mod_{R\otimes_R S^o} = Mod_{S^o}$. In the case when $\Lambda = S$, we shall write $\Lambda^e = \Lambda \otimes_R \Lambda^o$.

Lemma 1.3.1. (1) *If S is a projective (resp., flat) R-algebra, then res_Λ preserves projective (resp., flat) modules.*

(2) *If Λ is a projective (resp., flat) R-algebra, then res_{S^o} preserves projective (resp., flat) modules.*

(3) *Suppose $\Lambda = S$ is a flat R-algebra. Then res_Λ and res_{Λ^o} preserve injectives.*

Proof: (1) Suppose that S is a projective R-algebra. Since projective modules are exactly the summands of free modules, it suffices to show that $\Lambda \otimes_R S^o$ is a projective Λ-module. Since S is a central R-algebra, we have $S \cong S^o$ as R-modules. Therefore, we have $S^o \oplus P \cong L$ for some projective R-module P and free R-module L. Then $\Lambda \otimes_R S^o$ is a direct summand of $\Lambda \otimes_R L$, which is a free Λ-module. Hence $\Lambda \otimes_R S^o$ is a projective Λ-module.

Now suppose that S is flat over R. Since flat modules are direct limits of finitely generated free modules (see [Lam, Thm. 4.34]) and tensor products preserve direct limits, it suffices to show that $\Lambda \otimes S^o$ is a flat Λ-algebra. Since S is flat over R, we have that S is a direct limit of finitely generated free R-modules, which implies that $\Lambda \otimes S^o$ is a direct limit of finitely generated free Λ-modules.

(2) This follows from a similar argument as in (1).

(3) We shall prove this for res_Λ, the case of res_{Λ^o} being analogous. The functor from Mod_Λ to Mod_{Λ^e} sending M to $\Lambda^e \otimes_\Lambda M$ is exact by our assumption and is left adjoint to the functor res_Λ. The conclusion then follows from [Wei, Prop. 2.3.10]. □

Let Λ, B and S be rings. We now introduce some derived functors which will be used for the rest of this thesis. Let M (resp., N) be a left Λ-module (resp., a Λ-S-bimodule). Recall that $\mathrm{Hom}_\Lambda(M, N)$ is a right S-module, where the right S-action is given by $(f \cdot s)(m) = f(m)s$.

Now let G be a group. If M is a $\Lambda[G]$-B-bimodule, we then define a G-action on $\mathrm{Hom}_\Lambda(M, N)$ by $(g \cdot f)(m) = f(g^{-1}m)$ for $f \in \mathrm{Hom}_\Lambda(M, N), g \in G$ and $m \in M$. We also have a left B-action on $\mathrm{Hom}_\Lambda(M,N)$ given by $(b\cdot f)(m) = f(mb)$ for $f \in \mathrm{Hom}_\Lambda(M,N)$, $b \in B$ and $m \in M$. Therefore, we have that $\mathrm{Hom}_\Lambda(M,N)$ is a $B[G]$-S-bimodule. Thus, we have a bifunctor

$$\mathrm{Hom}_\Lambda(-,-) : (Mod_{\Lambda[G]\otimes_R B^o})^o \times Mod_{\Lambda\otimes_R S^o} \longrightarrow Mod_{B[G]\otimes_R S^o}.$$

Now, if $M^\bullet$ is a complex of $\Lambda[G]$-B-bimodules and $N^\bullet$ a complex of Λ-S-bimodules, we define a complex $\mathrm{Hom}^\bullet_\Lambda(M^\bullet, N^\bullet)$ of $B[G]$-S-bimodules by

$$\mathrm{Hom}^n_\Lambda(M^\bullet, N^\bullet) = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_\Lambda(M^i, N^{i+n})$$

with differentials as in Section 1.2. By abuse of notation, we shall also denote this by $\mathrm{Hom}_\Lambda(M,N)$. By a standard argument (see [Wei, Chap. 10]), we have bifunctors

$$\mathbf{R}\mathrm{Hom}_\Lambda(-,-) : \mathbf{D}^-(Mod_{\Lambda[G]\otimes_R B^o})^o \times \mathbf{D}(Mod_{\Lambda\otimes_R S^o}) \longrightarrow \mathbf{D}(Mod_{B[G]\otimes_R S^o}),$$

where $\mathbf{R}\mathrm{Hom}_\Lambda(M,N)$ can be represented by $\mathrm{Hom}_\Lambda(M,N)$ if M is a bounded above complex of projective $\Lambda[G]\otimes_R B^o$-modules, and

$$\mathbf{R}\mathrm{Hom}_\Lambda(-,-) : \mathbf{D}(Mod_{\Lambda[G]\otimes_R B^o})^o \times \mathbf{D}^+(Mod_{\Lambda\otimes_R S^o}) \longrightarrow \mathbf{D}(Mod_{B[G]\otimes_R S^o}),$$

where $\mathbf{R}\mathrm{Hom}_\Lambda(M,N)$ can be represented by $\mathrm{Hom}_\Lambda(M,N)$ if N is a bounded below complex of injective $\Lambda\otimes_R S^o$-modules. These two bifunctors coincide on $\mathbf{D}^-(Mod_{\Lambda[G]\otimes_R B^o})^o \times \mathbf{D}^+(Mod_{\Lambda\otimes_R S^o})$. We shall write $\mathbf{R}\mathrm{Hom}_\Lambda(M,N) = \mathrm{Hom}_\Lambda(M,N)$ if $\mathbf{R}\mathrm{Hom}_\Lambda(M,N)$ is represented by $\mathrm{Hom}_\Lambda(M,N)$.

Now set $B = R$. Then we have $Mod_{R[G]\otimes_R S^o} = Mod_{S^o[G]}$. The underlying functor $U_\Lambda : Mod_{\Lambda[G]} \longrightarrow Mod_\Lambda$ is exact and induces a functor (which we still denote as U_Λ from $\mathbf{D}(Mod_{\Lambda[G]})$ to $\mathbf{D}(Mod_\Lambda)$). These functors fit into the following commutative diagram.

$$\begin{array}{ccc}
\mathbf{K}(Mod_{\Lambda[G]})^o \times \mathbf{K}(Mod_{\Lambda\otimes_R S^o}) & \xrightarrow{\mathrm{Hom}_\Lambda(-,-)} & \mathbf{K}(Mod_{S^o[G]}) \\
\downarrow{\scriptstyle U_\Lambda\times \mathrm{id}} & & \downarrow{\scriptstyle U_{S^o}} \\
\mathbf{K}(Mod_\Lambda)^o \times \mathbf{K}(Mod_{\Lambda\otimes_R S^o}) & \xrightarrow{\mathrm{Hom}_\Lambda(-,-)} & \mathbf{K}(Mod_{S^o})
\end{array}$$

Since a projective $\Lambda[G]$-module is also projective as a Λ-module, we have the following commutative diagram.

$$\begin{array}{ccc}
\mathbf{D}^-(Mod_{\Lambda[G]})^o \times \mathbf{D}(Mod_{\Lambda\otimes_R S^o}) & \xrightarrow{\mathbf{R}\mathrm{Hom}_\Lambda(-,-)} & \mathbf{D}(Mod_{S^o[G]}) \\
\downarrow{\scriptstyle U_\Lambda\times \mathrm{id}} & & \downarrow{\scriptstyle U_{S^o}} \\
\mathbf{D}^-(Mod_\Lambda)^o \times \mathbf{D}(Mod_{\Lambda\otimes_R S^o}) & \xrightarrow{\mathbf{R}\mathrm{Hom}_\Lambda(-,-)} & \mathbf{D}(Mod_{S^o})
\end{array}$$

Proposition 1.3.2. *Suppose that M is a bounded above complex of $\Lambda[G]$-modules that are projective Λ-modules. Then we have*

$$\mathbf{R}\mathrm{Hom}_\Lambda(M, N) = \mathrm{Hom}_\Lambda(M, N).$$

Proof: Choose a bounded above complex P of projective $\Lambda[G]$-modules such that there is a quasi-isomorphism $f : P \xrightarrow{\sim} M$. Then $\mathrm{Hom}_\Lambda(P, N)$ represents $\mathbf{R}\mathrm{Hom}_\Lambda(M, N)$. Since a projective $\Lambda[G]$-module is also a projective Λ-module, we have $\mathrm{Hom}_\Lambda(U_\Lambda(P), N)$ representing $\mathbf{R}\mathrm{Hom}_\Lambda(U_\Lambda(M), N)$. Since M is a bounded above complex of projective Λ-modules, we also have $\mathrm{Hom}_\Lambda(U_\Lambda(M), N)$ representing $\mathbf{R}\mathrm{Hom}_\Lambda(U_\Lambda(M), N)$. This implies that

$$\mathrm{Hom}_\Lambda(U_\Lambda(M), N) \xrightarrow{f^*} \mathrm{Hom}_\Lambda(U_\Lambda(P), N)$$

is a quasi-isomorphism of complexes of projective S^o-modules. Since f^* is a morphism of complexes of $S^o[G]$-modules, we have a quasi-isomorphism

$$\mathrm{Hom}_\Lambda(M, N) \xrightarrow{f^*} \mathrm{Hom}_\Lambda(P, N)$$

of complexes of $S^o[G]$-modules. This implies that $\mathbf{R}\mathrm{Hom}_\Lambda(M, N) = \mathrm{Hom}_\Lambda(M, N)$, as required. □

Now we set $\Lambda = S$. We have the following commutative diagram.

$$\begin{array}{ccc}
\mathbf{K}(Mod_{\Lambda[G]})^o \times \mathbf{K}(Mod_{\Lambda^e}) & \xrightarrow{\mathrm{Hom}_\Lambda(-,-)} & \mathbf{K}(Mod_{\Lambda^o[G]}) \\
\Big\downarrow{\scriptstyle U_\Lambda \times res_\Lambda} & & \Big\downarrow{\scriptstyle res_R} \\
\mathbf{K}(Mod_\Lambda)^o \times \mathbf{K}(Mod_\Lambda) & \xrightarrow{\mathrm{Hom}_\Lambda(-,-)} & \mathbf{K}(Mod_R)
\end{array}$$

By Lemma 1.3.1, this induces the following commutative diagram.

$$\begin{array}{ccc}
\mathbf{D}^-(Mod_{\Lambda[G]})^o \times \mathbf{D}^+(Mod_{\Lambda^e}) & \xrightarrow{\mathbf{R}\mathrm{Hom}_\Lambda(-,-)} & \mathbf{D}(Mod_{\Lambda^o[G]}) \\
\Big\downarrow{\scriptstyle U_\Lambda \times res_\Lambda} & & \Big\downarrow{\scriptstyle res_R} \\
\mathbf{D}^-(Mod_\Lambda)^o \times \mathbf{D}^+(Mod_\Lambda) & \xrightarrow{\mathbf{R}\mathrm{Hom}_\Lambda(-,-)} & \mathbf{D}(Mod_R)
\end{array}$$

Proposition 1.3.3. *Let M be a complex of $\Lambda[G]$-modules. If Λ is a flat R-algebra, then for a bounded below complex N of Λ^e-modules which are injective Λ-modules, we have* $\mathbf{R}\mathrm{Hom}_\Lambda(M, N) = \mathrm{Hom}_\Lambda(M, N)$.

Proof : Let I be a bounded below complex of injective Λ^e-modules such that there is a quasi-isomorphism $I \xrightarrow{\sim} N$. Then $\mathbf{R}\mathrm{Hom}_\Lambda(M, N)$ is represented by $\mathrm{Hom}_\Lambda(M, I)$. By Lemma 1.3.1(3), it follows that $\mathbf{R}\mathrm{Hom}_\Lambda(M, N)$ is represented by $\mathrm{Hom}_\Lambda(M, res_\Lambda(I))$. It follows from our assumption on N that $\mathbf{R}\mathrm{Hom}_\Lambda(M, res_\Lambda(N))$ is represented by $\mathrm{Hom}_\Lambda(M, res_\Lambda(N))$. This implies that $\mathrm{Hom}_\Lambda(M, res_\Lambda(N))$ is quasi-isomorphic to $\mathrm{Hom}_\Lambda(M, res_\Lambda(I))$, which in turn implies that $\mathrm{Hom}_\Lambda(M, N)$ is quasi-isomorphic to $\mathrm{Hom}_\Lambda(M, I)$. Hence it follows that $\mathbf{R}\mathrm{Hom}_\Lambda(M, N)$ can be represented by $\mathrm{Hom}_\Lambda(M, N)$. □

Now if we set $\Lambda = S = B$ and $G = 1$, we obtain bifunctors

$$\mathrm{Hom}_\Lambda(-,-) : (Mod_{\Lambda^e})^o \times Mod_{\Lambda^e} \longrightarrow Mod_{\Lambda^e}$$

and

$$\mathbf{R}\mathrm{Hom}_\Lambda(-,-) : \mathbf{D}^-(Mod_{\Lambda^e}) \times \mathbf{D}^+(Mod_{\Lambda^e}) \longrightarrow \mathbf{D}(Mod_{\Lambda^e}).$$

By similar arguments as above, we have the following proposition.

Proposition 1.3.4. *If Λ is a projective R-algebra, and if M is a bounded above complex of Λ^e-modules which are projective Λ-modules, then we have* $\mathbf{R}\mathrm{Hom}_\Lambda(M, N) = \mathrm{Hom}_\Lambda(M, N)$.

If Λ is a flat R-algebra, and if N is a bounded below complex of Λ^e-modules which are injective Λ-modules, then we have $\mathbf{R}\mathrm{Hom}_\Lambda(M, N) = \mathrm{Hom}_\Lambda(M, N)$. □

Recall that a complex $N \in \mathrm{Ch}^+(Mod_\Lambda)$ is said to have finite injective dimension over Λ if there exists an integer n_0 such that $\mathrm{Ext}^n_\Lambda(M, N) = 0$ for all $n \geq n_0$ and all Λ-modules M. This is equivalent to N being quasi-isomorphic to a bounded complex of injective Λ-modules (see [Hart, Chap. I, Prop. 7.6]). The following result is a variant of this (see also [Ye, Prop. 2.4]).

Proposition 1.3.5. *Let Λ be a flat R-algebra. Then the following are equivalent for any complex $N \in \mathrm{Ch}^+(Mod_{\Lambda^e})$.*

(1) *N is quasi-isomorphic to a bounded complex of Λ^e-modules which are injective Λ-modules and injective Λ^o-modules.*

(2) *N has finite injective dimension over both Λ and Λ^o.*

Proof: Clearly (1) implies (2). Suppose (2) holds and choose n_0 such that $\mathrm{Ext}^n_\Lambda(M, N) = 0$ and $\mathrm{Ext}^n_{\Lambda^o}(M, N) = 0$ for all $n \geq n_0$ and every Λ-module M and Λ^o-module M'. Let I be a bounded below complex of injective Λ^e-modules that is quasi-isomorphic to N. By Lemma 1.3.1(3), this is also a complex of injective Λ-modules and injective Λ^o-modules. By the hypothesis, the n_0 term of $\tau_{\leq n_0} I$ is an injective Λ-module and an injective Λ^o-module, and the complex $_{\tau \leq n_0}I$ is quasi-isomorphic to I via the natural map $I \longrightarrow {}_{\tau\leq n_0}I$. Thus, we have that N is quasi-isomorphic to $_{\tau\leq n_0}I$ which is a bounded complex of Λ^e-modules that are injective Λ-modules and injective Λ^o-modules. □

For an S-Λ-bimodule L and a $\Lambda[G]$-module M, we endow $L \otimes_\Lambda M$ with the structure of a $S[G]$-module by setting $g(n \otimes m) = n \otimes gm$ for $g \in G, n \in N$ and $m \in M$.

If $M^\bullet$ is a complex of $\Lambda[G]$-modules and $L^\bullet$ is a complex of S-Λ-bimodules, we define the complex $L^\bullet \otimes_\Lambda M^\bullet$ of $S[G]$-modules by

$$(L^\bullet \otimes_\Lambda M^\bullet)^n = \bigoplus_{i\in\mathbb{Z}} L^i \otimes_\Lambda M^{n-i}$$

with differentials defined as in Section 1.2. As in the case of Hom, we shall abuse notation and denote this by $L \otimes_\Lambda M$. By a similar argument to those in Proposition 1.3.2 and Proposition 1.3.3, we have the following result.

Proposition 1.3.6. *The tensor product induces a bifunctor*

$$- \otimes^{\mathbf{L}}_\Lambda - : \mathbf{D}^-(Mod_{S\otimes_R\Lambda^o}) \times \mathbf{D}^-(Mod_{\Lambda[G]}) \longrightarrow \mathbf{D}(Mod_{S[G]}).$$

If M is a bounded above complex of $\Lambda[G]$-modules which are projective Λ-modules, then we have $N \otimes^{\mathbf{L}}_\Lambda M = N \otimes_\Lambda M$.

If S is a flat R-algebra and N is a bounded above complex of $S \otimes_R \Lambda^o$-modules which are flat Λ^o-modules, then $N \otimes^{\mathbf{L}}_\Lambda M = N \otimes_\Lambda M$. □

We detail a relationship between the above defined derived functors in the following proposition.

Proposition 1.3.7. *Let Λ be a central flat R-algebra. For any $M \in \mathbf{D}^-(Mod_{\Lambda[G]}), N \in \mathbf{D}^b(Mod_{\Lambda^e})$ and $I \in \mathbf{D}^+(Mod_{\Lambda^e})$, we have an isomorphism*

$$\mathbf{R}\mathrm{Hom}_\Lambda(N \otimes^{\mathbf{L}}_\Lambda M, I) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_\Lambda\big(M, \mathbf{R}\mathrm{Hom}_\Lambda(N, I)\big)$$

in $\mathbf{D}(Mod_{\Lambda^o[G]})$.

Proof : Replacing M by a bounded above complex of projective $\Lambda[G]$-modules, we may assume that M is itself a bounded above complex of projective $\Lambda[G]$-modules. Similarly, we may assume that I is a bounded below complex of Λ^e-modules which are injective as Λ-modules. Then by Proposition 1.3.6 and Proposition 1.3.3, we have

$$\mathbf{R}\mathrm{Hom}_\Lambda(N \otimes^{\mathbf{L}}_\Lambda M, I) = \mathrm{Hom}_\Lambda(N \otimes_\Lambda M, I)$$

and

$$\mathbf{R}\mathrm{Hom}_\Lambda\big(M, \mathrm{Hom}_\Lambda(N, I)\big) = \mathrm{Hom}_\Lambda\big(M, \mathrm{Hom}_\Lambda(N, I)\big),$$

where one observes that $\mathrm{Hom}_\Lambda(N, I)$ is bounded below. Therefore, we are reduced to showing that there is an isomorphism

$$\mathrm{Hom}_\Lambda(N \otimes_\Lambda M, I) \longrightarrow \mathrm{Hom}_\Lambda\big(M, \mathrm{Hom}_\Lambda(N, I)\big)$$

of complexes, and this follows from Lemma 1.2.2. □

Now if M is a $\Lambda[G]$-module and N is a $\Lambda^o[G]$-module, we define a G-action on $M \otimes_R N$ by $g(m \otimes n) = gm \otimes gn$. This gives $M \otimes_R N$ the structure of a $\Lambda[G]$-Λ-bimodule. Thus, we have the following bifunctor

$$- \otimes_R - : Mod_{\Lambda[G]} \times Mod_{\Lambda^o[G]} \longrightarrow Mod_{\Lambda[G]\otimes_R \Lambda^o}.$$

Lemma 1.3.8. *Given a $\Lambda[G]$-module M, a $\Lambda^o[G]$-module N and a $\Lambda[G]$-Λ-bimodule P with trivial G-action, we have isomorphisms*

$$\begin{aligned} \mathrm{adj} : \mathrm{Hom}_{\Lambda[G]\otimes_R\Lambda^o}(M \otimes_R N, P) &\longrightarrow \mathrm{Hom}_{\Lambda^o[G]}(N, \mathrm{Hom}_\Lambda(M, P)) \\ f &\mapsto \big(n \mapsto (m \mapsto f(m \otimes n))\big) \\ \mathrm{adj}' : \mathrm{Hom}_{\Lambda[G]\otimes_R\Lambda^o}(M \otimes_R N, P) &\longrightarrow \mathrm{Hom}_{\Lambda[G]}(M, \mathrm{Hom}_{\Lambda^o}(N, P)) \\ f &\mapsto \big(m \mapsto (n \mapsto f(m \otimes n))\big) \end{aligned}$$

of abelian groups.

Proof : We shall only prove the first isomorphism, the second being analogous. We first show that $\mathrm{adj}(f)$ lies in $\mathrm{Hom}_{\Lambda^o[G]}(N, \mathrm{Hom}_\Lambda(M, P))$. Let $f \in \mathrm{Hom}_{\Lambda[G]\otimes_R\Lambda^o}(M \otimes_R N, P)$. Then we have the following :

$$\begin{aligned} (\mathrm{adj}(f)(n\lambda))(m) &= f(m \otimes n\lambda) = f(m \otimes n)\lambda \\ &= (\mathrm{adj}(f)(n))(m)\lambda = ((\mathrm{adj}(f)(n)) \cdot \lambda)(m); \\ (\mathrm{adj}(f)(gn))(m) &= f(m \otimes gn) = f(g^{-1}m \otimes n) \\ &= (\mathrm{adj}(f)(n))(g^{-1}m) = (g \cdot (\mathrm{adj}(f)(n)))(m). \end{aligned}$$

The homomorphism adj is clearly injective, so it remains to show that it is surjective. Let $h \in \mathrm{Hom}_{\Lambda^o[G]}(N, \mathrm{Hom}_\Lambda(M, P))$. We define $f : M \otimes_R N \to P$ by $f(m \otimes n) = h(n)(m)$. It suffices to show that $f \in \mathrm{Hom}_{\Lambda[G]\otimes_R\Lambda^o}(M \otimes_{\mathbb{Z}_p} N, P)$. The following records the routine checking:

$$\begin{gathered} f(\lambda m \otimes n) = h(n)(\lambda m) = \lambda(h(n)(m)) = \lambda f(m \otimes n) \\ f(m \otimes n\lambda) = h(n\lambda)(m) = (h(n)\lambda)(m) = (h(n)(m))\lambda = f(m \otimes n)\lambda \\ f(gm \otimes gn) = h(gn)(gm) = (g^{-1} \cdot h(gn))(m) = h(n)(m) = f(m \otimes n). \end{gathered}$$

□

We end this section with a few technical results.

Lemma 1.3.9. *Let M be a finitely generated Λ-module, and let $\{N_\alpha\}$ be a direct system of $\Lambda \otimes_R S^o$-modules. Write $N = \varinjlim_\alpha N_\alpha$. Then we have a canonical monomorphism*

$$\varinjlim_\alpha \mathrm{Hom}_\Lambda(M, N_\alpha) \longrightarrow \mathrm{Hom}_\Lambda(M, N)$$

of S^o-modules. Moreover, if all of the canonical maps $i_\alpha : N_\alpha \to N$ are injective, then the map is an isomorphism.

Proof : Say M is generated by $m_1, ..., m_r$. Suppose $(f_\alpha) \in \varinjlim_\alpha \mathrm{Hom}_\Lambda(M, N_\alpha)$ and $f = \varinjlim_\alpha f_\alpha = 0$. Then for each $j = 1, ..., r$, there exists α_j such that $f_{\alpha_j}(m_j) = 0$ in N_{α_j}. Since there are only finitely many of these, by the directed set property, we can find an α_0 such that $f_{\alpha_0}(m_j) = 0$ for all j. Hence $(f_\alpha) = 0$.

For the second assertion, let $f \in \mathrm{Hom}_\Lambda(M, N)$. Then, for each $j = 1, ..., r$, there exists α_j such that $f(m_j) = i_{\alpha_j}(n_{\alpha_j})$ for some $n_{\alpha_j} \in N_{\alpha_j}$. Since there are only finitely many of these, by the directed set property, we can find an α_0 such that $f(m_j) = i_{\alpha_0}(n_j)$ for some $n_j \in N_{\alpha_0}$ for all j. Thus $i_{\alpha_0}^{-1} f \in \mathrm{Hom}_\Lambda(M, N_{\alpha_0})$, and we have established surjectivity. □

Lemma 1.3.10. *Let R be a commutative ring, and let Λ be an R-algebra. Then for any R-modules M and N, the following map*

$$\begin{aligned} \theta : \Lambda \otimes_R \mathrm{Hom}_R(M, N) &\longrightarrow \mathrm{Hom}_\Lambda(\Lambda \otimes_R M, \Lambda \otimes_R N) \\ \lambda \otimes f &\mapsto (\mu \otimes x \mapsto \mu\lambda \otimes f(x)) \end{aligned}$$

is a homomorphism of Λ^o-modules. Moreover, if M is a finitely presented R-module and Λ is a flat R-algebra, this is an isomorphism.

Proof: Let $\lambda, \mu, \varepsilon \in \Lambda$. Then we have

$$\theta(\lambda\varepsilon \otimes f)(\mu \otimes x) = \mu\lambda\varepsilon \otimes f(x) = (\mu\lambda \otimes f(x))\varepsilon = \theta(\lambda \otimes f)(\mu \otimes x)\varepsilon = (\theta(\lambda \otimes f) \cdot \varepsilon)(\mu \otimes x).$$

This shows that θ preserves the Λ^o-action. Now if M is a finitely presented R-module, we have an exact sequence $R^r \to R^s \to M \to 0$. This in turn induces the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \Lambda \otimes_R \mathrm{Hom}_R(M, N) & \longrightarrow & \Lambda \otimes_R \mathrm{Hom}_R(R^s, N) & \longrightarrow & \Lambda \otimes_R \mathrm{Hom}_R(R^r, N) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Hom}_\Lambda(\Lambda \otimes_R M, \Lambda \otimes_R N) & \longrightarrow & \mathrm{Hom}_\Lambda(\Lambda \otimes_R R^s, \Lambda \otimes_R N) & \longrightarrow & \mathrm{Hom}_\Lambda(\Lambda \otimes_R R^r, \Lambda \otimes_R N) \end{array}$$

since Λ is a flat R-algebra. Since the two maps on the right are clearly isomorphisms, so is the one on the left. □

Lemma 1.3.11. *Let R be a commutative ring, and let Λ be a flat R-algebra. Suppose M is a bounded above complex of finitely presented R-modules and N is a bounded complex of R-modules. Then the following map*

$$\begin{aligned}\theta : \Lambda \otimes_R \mathrm{Hom}_R(M, N) &\longrightarrow \mathrm{Hom}_\Lambda(\Lambda \otimes_R M, \Lambda \otimes_R N)\\ \lambda \otimes f &\mapsto (\mu \otimes x \mapsto \mu\lambda \otimes f(x))\end{aligned}$$

is an isomorphism of chain complexes.

Proof: Since N is bounded, we have

$$\bigoplus_i \mathrm{Hom}_R(M^i, N^{i+n}) = \prod_i \mathrm{Hom}_R(M^i, N^{i+n}),$$

and so the term in degree n for the complex on the left is

$$\bigoplus_i \Lambda \otimes_R \mathrm{Hom}_R(M^i, N^{i+n}).$$

We note that $\Lambda \otimes_R N$ is also bounded, and so the term in degree n for the complex on the right is

$$\bigoplus_i \mathrm{Hom}_\Lambda(\Lambda \otimes_R M^i, \Lambda \otimes_R N^{i+n}).$$

It follows from a direct verification that the map defined in the lemma is a chain map. Since the map is an isomorphism in each degree by the preceding lemma, it follows that the chain map is an isomorphism. □

Lemma 1.3.12. *Given a Λ-module A, a $\Lambda \otimes_R S^o$-module B and an $S \otimes_R (\Lambda')^o$-module C, we have a homomorphism*

$$\begin{aligned}\tau : \mathrm{Hom}_\Lambda(A, B) \otimes_S C &\longrightarrow \mathrm{Hom}_\Lambda(A, B \otimes_S C)\\ f \otimes c &\mapsto \big(a \mapsto f(a) \otimes c\big)\end{aligned}$$

of $(\Lambda')^o$-modules. This is an isomorphism if either of the two following cases holds.

(1) *A is a finitely generated projective Λ-module.*

(2) *A is a finitely presented Λ-module and C is a flat S-module.*

Proof: See [Ish, Lemma 1.1]. □

We extend the above lemma to the derived setting (see also [Ven, Prop. 6.1]).

Lemma 1.3.13. *Let A be a complex of Λ-modules, B be a bounded complex of $\Lambda \otimes_R S^o$-modules and C be a complex of $S \otimes_R (\Lambda')^o$-modules. We assume that Λ' is a flat R-algebra. Suppose A is quasi-isomorphic to a bounded below complex of finitely generated projective Λ-modules, and suppose C is quasi-isomorphic to a bounded complex of $S \otimes_R (\Lambda')^o$-modules which are flat S-modules. Then we have an isomorphism*

$$\mathbf{R}\mathrm{Hom}_\Lambda(A, B) \otimes^{\mathbf{L}}_S C \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_\Lambda(A, B \otimes^{\mathbf{L}}_S C)$$

in $\mathbf{D}(Mod_{(\Lambda')^o})$.

Proof: Without loss of generality, we may assume that A is a bounded below complex of finitely generated projective Λ-modules and C is a bounded complex of $S \otimes_R (\Lambda')^o$-modules which are flat S-modules. Then it suffices to show that there is an isomorphism

$$\mathrm{Hom}_\Lambda(A, B) \otimes_S C \xrightarrow{\sim} \mathrm{Hom}_\Lambda(A, B \otimes_S C)$$

of complexes of $(\Lambda')^o$-modules. Since B and C are bounded, the terms in degree n for both complexes are

$$\bigoplus_{i,j} \mathrm{Hom}(A^i, B^{i+j}) \otimes_S C^{n-j}$$

and

$$\bigoplus_{i,j} \mathrm{Hom}(A^i, B^{i+j} \otimes_S C^{n-j})$$

with differentials given respectively by

$$d(f \otimes c) = d \circ f \otimes c + (-1)^{j-1} f \circ d \otimes c + (-1)^j f \otimes dc$$

for $f \in \mathrm{Hom}(A^i, B^{i+j})$ and $c \in C^{n-j}$, and

$$dg = d \circ g + (-1)^{n-1} g \circ (d \otimes \mathrm{id}) + (-1)^{i+j+n-1} g \circ (\mathrm{id} \otimes d)$$

for $g \in \mathrm{Hom}(A^i, B^{i+j} \otimes_S C^{n-j})$. Let (i, j, n) be a triple of indices with values in $\mathbb{Z}$ such that the following relations hold:

(1) $(-1)^{(0,0,0)} = 1$,
(2) $(-1)^{(i+1,j,n)} = (-1)^{(i,j,n)}$,
(3) $(-1)^{(i,j+1,n)} = (-1)^{n-j}(-1)^{(i,j,n)}$,
(4) $(-1)^{(i,j,n+1)} = (-1)^{i+n-1}(-1)^{(i,j,n)}$.

Then we define a morphism

$$\mathrm{Hom}_\Lambda(A, B) \otimes_S C \longrightarrow \mathrm{Hom}_\Lambda(A, B \otimes_S C)$$

by the following assignment: $f \otimes c \in \mathrm{Hom}_\Lambda(A^i, B^{i+j}) \otimes_S C^{n-j}$ is mapped to $\big(a \mapsto ((-1)^{(i,j,n)} f(a) \otimes c)\big)$. This gives a morphism of complexes by our construction of (i, j, n). By the preceding lemma, each of the individual maps is an isomorphism, and so the chain map is also an isomorphism. □

Chapter 2

Adic rings

Completed group algebras of certain finitely generated profinite groups arise naturally in the study of Iwasawa theory. In particular, an important class of such completed algebras comes in the form of $\mathbb{Z}_p[\![\Gamma]\!]$ where Γ is a compact p-adic Lie group. These rings belong to a class of rings known as adic rings. In this chapter, we shall study the properties of such rings and their (topological) modules. We will also develop a cohomological theory over such rings. This chapter will provide the background knowledge and necessary tools for Chapter 4 and Chapter 5.

Let Λ be an associative (not necessarily commutative) unital ring, and denote by $\mathfrak{M}$ the Jacobson radical of Λ which is the intersection of its left maximal ideals. Then there is a canonical ring homomorphism

$$\Lambda \longrightarrow \varprojlim_n \Lambda/\mathfrak{M}^n$$

with kernel $\bigcap_n \mathfrak{M}^n$. We say that the ring Λ is an *adic ring* if $\Lambda/\mathfrak{M}^n$ is finite for all $n \geq 1$ and the above ring homomorphism is an isomorphism. We remark that this definition mimics that in [FK], where in their definition, $\Lambda/\mathfrak{M}^n$ is taken to be finite of order a power of a prime p. Although, in the context of Iwasawa theory, we usually work with adic rings where each $\Lambda/\mathfrak{M}^n$ is finite of order a power of a prime p, we shall adopt this (slightly) more general definition in the development of the general theory, since there are no extra difficulties involved. In fact, one will see that most of the material presented here parallels that in [NSW, Chap. V §2].

2.1 Finitely presented Λ-modules

From now on, we shall endow an adic ring Λ with the $\mathfrak{M}$-adic topology. It is immediate from the definition of an adic ring that Λ is compact under the $\mathfrak{M}$-adic topology. Since $\mathfrak{M}^n$ is a two-sided ideal, it follows that Λ is an adic ring if and only if Λ^o is. In this section, we will show that there is a natural way to endow finitely presented Λ-modules with the $\mathfrak{M}$-adic topology.

Recall that for a topological abelian group M, the Pontryagin dual $M^\vee$ of M is defined by $\mathrm{Hom}_{\mathrm{cts}}(M, \mathbb{R}/\mathbb{Z})$. When M is profinite (resp., pro-p), we have $M^\vee = \mathrm{Hom}_{\mathrm{cts}}(M, \mathbb{Q}/\mathbb{Z})$ (resp., $\mathrm{Hom}_{\mathrm{cts}}(M, \mathbb{Q}_p/\mathbb{Z}_p)$). If M is discrete (resp., discrete p-torsion), we then have $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ (resp., $\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$). With these descriptions in hand, we are now able to prove the following proposition.

Proposition 2.1.1. *Let M be a finitely presented left Λ-module. Then we have the following:*

(a) $M \cong \varprojlim_n M/\mathfrak{M}^n M$.

(b) $\mathrm{Hom}_\Lambda(M, \Lambda^\vee) \cong M^\vee$ *as Λ^o-modules (where M is endowed with the profinite topology induced by the isomorphism in (a)).*

(c) If M is a Λ-S-bimodule, the isomorphism in (b) is an isomorphism of S-Λ-bimodules.

(d) If M is a left $\Lambda[G]$-module for some group G, the isomorphism in (b) is an isomorphism of $\Lambda^o[G]$-modules.

Proof: Since M is finitely presented, we have an exact sequence $\Lambda^r \to \Lambda^s \to M \to 0$ for some integers r and s. Applying $\Lambda/\mathfrak{M}^n \otimes_\Lambda -$, we obtain an exact sequence

$$(\Lambda/\mathfrak{M}^n)^r \to (\Lambda/\mathfrak{M}^n)^s \to M/\mathfrak{M}^n M \to 0.$$

Since each term in the sequence is finite, taking inverse limits yields an exact sequence

$$(\varprojlim \Lambda/\mathfrak{M}^n)^r \to (\varprojlim \Lambda/\mathfrak{M}^n)^s \to \varprojlim M/\mathfrak{M}^n M \to 0$$

which fits into the following commutative diagram

$$\begin{array}{ccccccc}
\Lambda^r & \longrightarrow & \Lambda^s & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
(\varprojlim \Lambda/\mathfrak{M}^n)^r & \longrightarrow & (\varprojlim \Lambda/\mathfrak{M}^n)^s & \longrightarrow & \varprojlim M/\mathfrak{M}^n M & \longrightarrow & 0
\end{array}$$

with exact rows. Since the two maps on the left are isomorphisms, so is the one on the right, and this proves (a). For (b), we first note that

$$\mathrm{Hom}_{\mathrm{cts}}(M, \mathbb{Q}/\mathbb{Z}) = \varinjlim \mathrm{Hom}_{\mathbb{Z}}(M/\mathfrak{M}^n M, \mathbb{Q}/\mathbb{Z})$$

and observe that

$$\begin{aligned}\mathrm{Hom}_\Lambda(M, \Lambda^\vee) = &\ \mathrm{Hom}_\Lambda(M, \varinjlim_n \mathrm{Hom}_{\mathbb{Z}}(\Lambda/\mathfrak{M}^n, \mathbb{Q}/\mathbb{Z})) \\ \cong &\ \varinjlim_n \mathrm{Hom}_\Lambda(M, \mathrm{Hom}_{\mathbb{Z}}(\Lambda/\mathfrak{M}^n, \mathbb{Q}/\mathbb{Z})) \text{ (by Lemma 1.3.9)} \\ \cong &\ \varinjlim_n \mathrm{Hom}_{\mathbb{Z}}(M/\mathfrak{M}^n M, \mathbb{Q}/\mathbb{Z}) = M^\vee.\end{aligned}$$

It follows from a straightforward calculation that the above isomorphism is given by sending $f \in \mathrm{Hom}_\Lambda(M, \Lambda^\vee)$ to $\big(m \mapsto f(m)(1)\big) \in M^\vee$. Denote this isomorphism by α. Then for $\lambda \in \Lambda$ and $m \in M$,

$$\alpha(f \cdot \lambda)(m) = (f \cdot \lambda)(m)(1) = f(\lambda m)(1) = \alpha(f)(\lambda m) = \big(\alpha(f) \cdot \lambda\big)(m).$$

This shows that the isomorphism preserves the Λ^o-action and we have (b). Part (c) and (d) can be dealt with similarly. □

We have the following corollary. See also [Ne, 2.9.1] for the case when Λ is commutative local adic.

Corollary 2.1.2. *If Λ is left Noetherian, then $\Lambda^\vee$ is an injective left Λ-module.*

Proof: For every left ideal $\mathfrak{A}$ of Λ, we have a map $\mathrm{Hom}_\Lambda(\Lambda, \Lambda^\vee) \longrightarrow \mathrm{Hom}_\Lambda(\mathfrak{A}, \Lambda^\vee)$ induced by the inclusion $\mathfrak{A} \hookrightarrow \Lambda$. By hypothesis, the ideal $\mathfrak{A}$ is Noetherian and hence finitely presented. Thus we may apply the previous proposition and the exactness of Pontryagin dual to obtain the surjectivity of this map. By Baer's Criterion (see [Wei, 2.3.1]), we have the required conclusion. □

Corollary 2.1.3. *Suppose Λ' is another adic ring with Jacobson radical $\mathfrak{M}'$. Let M be a finitely presented Λ-module and N be a $\Lambda' \otimes_R \Lambda^o$-module which is a finitely presented Λ'-module. Then $N \otimes_\Lambda M$ is a finitely presented Λ'-module and*

$$N \otimes_\Lambda M \cong \varprojlim_n (N/\mathfrak{M}'^n N) \otimes_\Lambda M.$$

Proof: We have an exact sequence $\Lambda^r \longrightarrow \Lambda^s \longrightarrow M \longrightarrow 0$. Applying $N \otimes_\Lambda -$, we obtain an exact sequence

$$N^r \longrightarrow N^s \longrightarrow N \otimes_\Lambda M \longrightarrow 0.$$

Therefore, $N \otimes_\Lambda M$ is a finitely presented Λ'-module. By Proposition 2.1.1, we have

$$N \otimes_\Lambda M \cong \varprojlim_n \big((N \otimes_\Lambda M)/\mathfrak{M}'^n(N \otimes_\Lambda M)\big).$$

Now observe that

$$\mathfrak{M}'^n(N \otimes_\Lambda M) = \mathrm{im}\big((\mathfrak{M}'^n N) \otimes_\Lambda M \longrightarrow N \otimes_\Lambda M\big),$$

and so the conclusion follows by the right exactness of the tensor product. □

Proposition 2.1.4. *Let M be a Λ-module such that $M = \varinjlim M_\alpha$, where each M_α is a finite Λ-module. Then $\mathrm{Hom}_\Lambda(M, \Lambda^\vee) \cong \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$. In the case where M is given the discrete topology, we can replace the last term by $M^\vee$. Furthermore, if M and the M_α are Λ-S-bimodules, the isomorphism in (b) is an isomorphism of S-Λ-bimodules. And if M and the M_α are $\Lambda[G]$-modules for some group G, the isomorphism is an isomorphism of $\Lambda^o[G]$-modules.*

Proof : Since each M_α is a finite Λ-module, it follows that the Λ-submodules $\mathfrak{M}^n M_\alpha$ stabilize for big enough n. By Nakayama's lemma [Iss, Thm. 13.11], this implies that $\mathfrak{M}^{n_\alpha} M_\alpha = 0$ for some large enough n_α. Then for each α, we have

$$\begin{aligned}
\mathrm{Hom}_\Lambda(M_\alpha, \Lambda^\vee) &= \mathrm{Hom}_\Lambda\Big(M_\alpha, \varinjlim_n \mathrm{Hom}_\mathbb{Z}(\Lambda/\mathfrak{M}^n, \mathbb{Q}/\mathbb{Z})\Big) \\
&\cong \varinjlim_n \mathrm{Hom}_\Lambda\big(M_\alpha, \mathrm{Hom}_\mathbb{Z}(\Lambda/\mathfrak{M}^n, \mathbb{Q}/\mathbb{Z})\big) \text{ (by Lemma 1.3.9)} \\
&\cong \varinjlim_n \mathrm{Hom}_\mathbb{Z}(M_\alpha/\mathfrak{M}^n M_\alpha, \mathbb{Q}/\mathbb{Z}) \\
&= \mathrm{Hom}_\mathbb{Z}(M_\alpha, \mathbb{Q}/\mathbb{Z})
\end{aligned}$$

The conclusion follows by taking inverse limit over α. □

Corollary 2.1.5. *Let M be a finitely presented Λ-module. Then we have an isomorphism*

$$M \xrightarrow{\sim} \mathrm{Hom}_{\Lambda^o}\big(\mathrm{Hom}_\Lambda(M, \Lambda^\vee), \Lambda^\vee\big)$$

of Λ-modules. Moreover if M is a Λ-Λ-bimodule, then the above isomorphism is of Λ-Λ-bimodules.

Proof : As seen in the proof of Proposition 2.1.1(b), we have $M^\vee \cong \varinjlim_n (M/\mathfrak{M}^n M)^\vee$. Since $(M/\mathfrak{M}^n M)^\vee$ is a finite Λ^o-module for each n, it follows that

$$\begin{aligned}\mathrm{Hom}_{\Lambda^o}(\mathrm{Hom}_\Lambda(M,\Lambda^\vee),\Lambda^\vee) &\cong \mathrm{Hom}_{\Lambda^o}(M^\vee,\Lambda^\vee) \text{ (by Proposition 2.1.1)}\\ &\cong (M^\vee)^\vee \text{ (by Proposition 2.1.4)}\\ &\cong M \text{ (by Pontryagin duality).} \ \square\end{aligned}$$

2.2 Topological Λ-modules

In this section, we will study topological modules over an adic ring. These are Hausdorff topological abelian groups with a continuous Λ-action, where as before, Λ is given the $\mathfrak{M}$-adic topology. In particular, we are interested in the following two classes of topological Λ-modules.

Definition 2.2.1. We say that a topological Λ-module M is a compact (resp., discrete) Λ-module if its underlying topology is compact (resp., discrete). The category of compact Λ-modules (resp., discrete Λ-modules) is denoted by $\mathcal{C}_\Lambda$ (resp., $\mathcal{D}_\Lambda$).

We now describe the structures of compact Λ-modules and discrete Λ-modules.

Proposition 2.2.2. *(i) Every compact Λ-module is a projective limit of finite modules and has a fundamental system of neighborhoods of zero consisting of open submodules. In particular, it is an abelian profinite group.*

(ii) Every discrete Λ-module is the direct limit of finite Λ-modules. In particular, it is an abelian torsion group.

(iii) Pontryagin duality induces a duality between the category $\mathcal{C}_\Lambda$ of compact Λ-modules and the category $\mathcal{D}_{\Lambda^o}$ of discrete Λ^o-modules.

(iv) The category $\mathcal{C}_\Lambda$ is abelian and has enough projectives and exact inverse limits. The category $\mathcal{D}_\Lambda$ is abelian and has enough injectives and exact direct limits.

Proof: Suppose N is a discrete Λ-module, and let $x \in N$. Then $\mathrm{Ann}_\Lambda(x)$ is an open ideal of Λ. Therefore, $\mathrm{Ann}_\Lambda(x)$ contains $\mathfrak{M}^r$ for some r. Thus $\Lambda \cdot x$ is an $\Lambda/\mathfrak{M}^r$-module and so is finite. This shows (ii). Now let M be a compact Λ-module. Then the abstract Λ^o-module $M^\vee$ is a discrete topological group. For this module to be a discrete Λ^o-module, we need the Λ^o-action to be continuous, and this is guaranteed by [F, Prop. 3(b)] (since Λ is compact). By (ii), we have that $M^\vee$ is a direct limit of finite Λ^o-modules. Taking the Pontryagin dual, we have that $M = (M^\vee)^\vee$ is an inverse limit of finite Λ-modules.

Part (iii) is immediate from what we have shown so far. Part (iv) follows from [RZ, Prop. 5.4.2, Prop. 5.4.4]. □

We give another description of discrete Λ-modules in terms of "$\mathfrak{M}$-torsion". If M is a Λ-module, we define

$$M[\mathfrak{M}^n] = \{x \in M \mid \mathfrak{M}^n \subseteq \mathrm{Ann}(x)\}.$$

With this, we have the following lemma.

Lemma 2.2.3. *Let M be an abstract Λ-module. Then M is a discrete Λ-module (i.e., the Λ-action is continuous with respect to the discrete topology on M) if and only if*

$$M = \bigcup_{n=1}^{\infty} M[\mathfrak{M}^n].$$

Proof: Suppose that M is a discrete Λ-module. Let $x \in M$. Then by the continuity of the Λ-action, there exists a positive integer r such that $\mathfrak{M}^r \cdot x = 0$. This implies that $x \in M[\mathfrak{M}^r]$.

Conversely, suppose that

$$M = \bigcup_{n=1}^{\infty} M[\mathfrak{M}^n].$$

We shall show that the action

$$\theta : \Lambda \times M \longrightarrow M$$

is continuous, where M is given the discrete topology. In other words, for each $x \in M$, we need to show that $\theta^{-1}(x)$ is open in $\Lambda \times M$. Let $(\lambda, y) \in \theta^{-1}(x)$. Then $y \in M[\mathfrak{M}^n]$ for some n. Therefore, we have $(\lambda, y) \in (\lambda + \mathfrak{M}^n) \times \{y\}$, and the latter set is an open set contained in $\theta^{-1}(x)$. □

Corollary 2.2.4. *A finite abstract Λ-module is a discrete Λ-module.*

Proof: Let M be a finite abstract module. Then $\mathfrak{M}^n M$ stabilizes, and it follows from Nakayama's lemma [Iss, Thm. 13.11] that we have $\mathfrak{M}^n M = 0$ for big enough n. By Lemma 2.2.3, this implies that M is a discrete Λ-module. □

When working with topological Λ-modules, one will have to consider continuous homomorphisms between the modules. In general, an abstract homomorphism of modules may not be continuous. However, the next lemma will give a few situations where every abstract homomorphism is continuous.

Lemma 2.2.5. *Let M and N be two topological Λ-modules. Suppose one of the following cases holds.*

(1) Both M and N have the $\mathfrak{M}$-adic topology.

(2) Both M and N have the discrete topology.

(3) M has the $\mathfrak{M}$-adic topology and is finitely generated as a Λ-module, and N is a discrete Λ-module.

(4) M has the $\mathfrak{M}$-adic topology and is finitely generated as a Λ-module, and N is a compact Λ-module.

Then every abstract Λ-homomorphism is continuous. In other words, we have

$$\mathrm{Hom}_{\Lambda,\,\mathrm{cts}}(M,N) = \mathrm{Hom}_{\Lambda}(M,N).$$

Proof : (1) This follows from the observation that for every abstract Λ-homomorphism $f : M \to N$, one has $f(\mathfrak{M}^n M) \subseteq \mathfrak{M}^n N$.

(2) This is obvious.

(3) Let $f : M \longrightarrow N$ be an abstract Λ-homomorphism. Since M is finitely generated, there exists a big enough n such that $f(M) \subseteq N[\mathfrak{M}^n]$ by Lemma 2.2.3. This in turn implies that $\mathfrak{M}^n M \subseteq \ker f$.

(4) Let $f : M \longrightarrow N$ be an abstract homomorphism of Λ-modules. By Proposition 2.2.2(i), we may choose a system $\{N_\alpha\}$ of neighborhoods of zero consisting of open submodules of N. Since N/N_α is the quotient of a compact Λ-module by an open Λ-submodule, it follows that N/N_α is a finite discrete abelian group. By the preceding corollary, it is a discrete Λ-module. Therefore, if we denote by $\pi_\alpha : N \longrightarrow N/N_\alpha$ the canonical quotient homomorphism, the following homomorphism

$$M \xrightarrow{f} N \xrightarrow{\pi_\alpha} N/N_\alpha$$

of Λ-modules is continuous by Lemma 2.2.5(1). Therefore, by the universal property of the inverse limits, the induced map

$$\tilde{f} : M \longrightarrow \varprojlim_{\alpha} N/N_\alpha \cong N$$

is continuous and coincides with f. □

Now that we have good descriptions of compact Λ-modules and discrete Λ-modules, we shall give some examples of such modules. It turns out that finitely presented Λ-modules (resp., their Pontryagin duals) give a nice class of compact Λ-modules (resp., discrete Λ^o-modules).

Lemma 2.2.6. *A finitely presented Λ-module endowed with the $\mathfrak{M}$-adic topology is a compact Λ-module.*

Proof: It follows from Proposition 2.1.1 that M is a compact abelian group. It remains to show that the action

$$\theta : \Lambda \times M \longrightarrow M$$

is continuous. Suppose $\theta(\lambda, x) \in y + \mathfrak{M}^n M$. Then one has that $(\lambda, x) \in (\lambda + \mathfrak{M}^n) \times (x + \mathfrak{M}^n M)$ which can be easily verified to be an open set contained in $\theta^{-1}(y + \mathfrak{M}^n M)$. □

Corollary 2.2.7. *If M is a finitely presented Λ-module, then $M^\vee$ is a discrete Λ^o-module.*

Proof: This follows from Proposition 2.2.2 and Lemma 2.2.6. □

Corollary 2.2.8. *Let M be a finitely presented Λ-module, endowed with the $\mathfrak{M}$-adic topology. We have*

$$\mathrm{Hom}_{\Lambda,\,\mathrm{cts}}(M, \Lambda^\vee) = \mathrm{Hom}_\Lambda(M, \Lambda^\vee).$$

Furthermore, if we endow $\mathrm{Hom}_\Lambda(M, \Lambda^\vee)$ with the compact-open topology via the above equality, the isomorphism in Proposition 2.1.1(b) is a homeomorphism of discrete Λ^o-modules.

Proof: The first assertion follows from Lemma 2.2.5(3) and Lemma 2.2.6. The second assertion follows from the general fact that if M is a compact Λ-module and N is a discrete Λ-module, then $\mathrm{Hom}_{\Lambda,\mathrm{cts}}(M, N)$ is discrete under the compact-open topology. □

In view of Lemma 2.2.6, one may ask the following two questions. The first is if one can say anything about the $\mathfrak{M}$-adic topology on an abstract Λ-module M. In general, it is not even clear whether this topology is Hausdorff. The second question that one may ask is if there are other ways to endow a finitely presented Λ-module with a topology such that it becomes a compact Λ-module. In response to these two questions, we have the following proposition. In fact, as we shall see, if M is already a compact Λ-module, the $\mathfrak{M}$-adic topology is Hausdorff, and it is the only one with which one can endow a finitely presented Λ-module in order to make it into a compact Λ-module. One may compare the following proposition with [NSW, Prop. 5.2.17].

Proposition 2.2.9. *Let M be a compact Λ-module. Then the $\mathfrak{M}$-adic topology is finer than the original topology of M, and the canonical homomorphism*

$$\alpha : M \longrightarrow \varprojlim_i M/\mathfrak{M}^i M$$

of Λ-modules is injective. Furthermore, if M is a finitely generated Λ-module, then the topologies coincide, and the above homomorphism is a continuous isomorphism of Λ-modules.

Proof : Let N be an open submodule of M. Then by continuity, for each $x \in M$, there exists a neighborhood V_x of x and a natural number n_x such that $\mathfrak{M}^{n_x} V_x \subseteq N$. Since M is compact, it is covered by finitely many such sets, say $V_{x_1}, V_{x_2}, ..., V_{x_r}$. Setting $n = \max\{n_{x_1}, ..., n_{x_r}\}$, we have $\mathfrak{M}^n M \subseteq N$. This shows the first assertion. Since M is Hausdorff under its original topology, it follows that M is Hausdorff under the $\mathfrak{M}$-adic topology and so

$$\bigcap_{i=1}^{\infty} \mathfrak{M}^i M = 0.$$

Now if M is finitely generated, we have a surjection

$$\Lambda^m \twoheadrightarrow (M \text{ with } \mathfrak{M}\text{−adic topology}),$$

which is continuous by Lemma 2.2.5(1). This implies that M with the $\mathfrak{M}$-adic topology is compact. By the first assertion, the identity map

$$(M \text{ with } \mathfrak{M}\text{−adic topology}) \longrightarrow M$$

is continuous. This in turn gives a continuous bijection between compact spaces and is therefore a homeomorphism. If M is given the $\mathfrak{M}$-adic topology, then the image of α is dense in $\varprojlim_i M/\mathfrak{M}^i M$, and so is surjective since M is compact. □

The following statement is a corollary of 2.2.9.

Corollary 2.2.10. *Let M be a compact Λ-module. Then every finitely generated abstract Λ-submodule of M is a closed subset of M. In particular, every finitely generated (left) ideal of Λ is closed in Λ.*

Proof : Let N be a Λ-submodule of M generated by $x_1, ..., x_r$. Then the following Λ-homomorphism

$$\begin{aligned} \phi : \textstyle\bigoplus_{i=1}^{r} \Lambda \; &\longrightarrow (M \text{ with } \mathfrak{M}\text{−adic topology}) \xrightarrow{\text{id}} M \\ e_i \; &\mapsto x_i \end{aligned}$$

is continuous by Lemma 2.2.5(1) and Proposition 2.2.9. Therefore, we have that N is an image of a compact Λ-module under a continuous map. In particular, this implies that N is closed. □

We have the following version of Nakayama's lemma for compact Λ-modules (see also [NSW, Prop. 5.2.18]).

Proposition 2.2.11. *Let M be a compact Λ-module. Then the following hold.*

(i) If $\mathfrak{M}M = M$, then $M = 0$.

(ii) The Λ-module M is generated by $x_1, ..., x_r$ if and only if $x_1 + \mathfrak{M}M, ..., x_r + \mathfrak{M}M$ generate $M/\mathfrak{M}M$ over $\Lambda/\mathfrak{M}$.

Proof : By Proposition 2.2.9, we have

$$M = \bigcap_{i=0}^{\infty} \mathfrak{M}^i M = 0.$$

This proves (i). For (ii), we shall prove the nontrivial implication. Suppose we have $x_1, ..., x_r \in M$ such that $x_1 + \mathfrak{M}M, ..., x_r + \mathfrak{M}M$ generate $M/\mathfrak{M}M$ over $\Lambda/\mathfrak{M}$. Let N be the Λ-submodule of M generated by $x_1, ..., x_r$. It follows from Corollary 2.2.10 that N is a closed Λ-submodule of M. As a quotient of a compact module by a closed submodule, we have that M/N is a compact Λ-module. By the construction of N, we have that $\mathfrak{M}(M/N) = M/N$. By (i), this implies that $M = N$. □

Proposition 2.2.12. *Suppose M is an abstract Λ-module such that $M = \varprojlim M/M_\alpha$, where $\{M_\alpha\}$ is a direct system of Λ-submodules of finite index. Then M is a compact Λ-module, where the topology on M is given by the inverse limit. Furthermore, for such a module M, it is finitely generated over Λ if and only if $M/\mathfrak{M}M$ is finite.*

Proof : The second assertion follows immediately from the first assertion and Lemma 2.2.11(ii). Thus, it remains to show that the Λ-action

$$\theta : \Lambda \times M \longrightarrow M$$

is continuous with respect to the topology given by the inverse limit. By Corollary 2.2.4, the assertion holds if M is a finite Λ-module. For a general M, let $(\lambda, x) \in \theta^{-1}(y + M_\alpha)$ for $\lambda \in \Lambda$ and $x, y \in M$. This is equivalent to $\lambda(x + M_\alpha) = y + M_\alpha$ in M/M_α. Since M/M_α is finite, it follows from the above that there is an n such that $(\lambda + \mathfrak{M}^n) \cdot (x + M_\alpha) \subseteq y + M_\alpha$. This shows the continuity of θ. □

We now restate what we have done so far in a categorical language.

Proposition 2.2.13. *The underlying functor from the category of topological Λ-modules to the category of abstract Λ-modules induces the following equivalences of categories.*

$$\left\{\begin{matrix}\textit{Discrete } \Lambda\textit{-modules} \\ \textit{with continuous} \\ \Lambda\textit{-homomorphisms}\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\textit{Abstract } \Lambda\textit{-modules such that} \\ M = \bigcup_{n=1}^{\infty} M[\mathfrak{M}^n] \textit{ with abstract} \\ \Lambda\textit{-homomorphisms}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\textit{Finitely presented} \\ \textit{compact } \Lambda\textit{-modules} \\ \textit{with continuous} \\ \Lambda\textit{-homomorphisms}\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\textit{Finitely presented} \\ \textit{abstract } \Lambda\textit{-modules} \\ \textit{with abstract} \\ \Lambda\textit{-homomorphisms}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\textit{Compact and} \\ \textit{discrete } \Lambda\textit{-modules} \\ \textit{with continuous} \\ \Lambda\textit{-homomorphisms}\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\textit{Finite} \\ \textit{abstract } \Lambda\textit{-modules} \\ \textit{with abstract} \\ \Lambda\textit{-homomorphisms}\end{matrix}\right\}$$

Proof : This follows from Lemma 2.2.3, Corollary 2.2.4, Lemma 2.2.5 and Proposition 2.2.9. □

We conclude with a description of projective objects in $\mathcal{C}_\Lambda$ which are finitely generated over Λ.

Proposition 2.2.14. *Let P be a projective object in $\mathcal{C}_\Lambda$ that is finitely generated over Λ. Then P is a projective Λ-module. Conversely, let P be a finitely generated projective Λ-module. Then P, endowed with the $\mathfrak{M}$-adic topology, is a compact Λ-module and is a projective object in $\mathcal{C}_\Lambda$.*

Proof: Let P be a projective object in $\mathcal{C}_\Lambda$ that is finitely generated over Λ. Then there is a surjection $f : \Lambda^r \twoheadrightarrow P$ of Λ-modules. By Proposition 2.2.9, the topology on P is precisely the $\mathfrak{M}$-adic topology, and it follows from Lemma 2.2.5(1) that f is a continuous homomorphism of compact Λ-modules. Now since P is a projective object in $\mathcal{C}_\Lambda$, the map f has a continuous Λ-linear section. In particular, this implies that we have an isomorphism $\Lambda^r \cong P \oplus (\ker f)$ of Λ-modules. Hence P is a projective Λ-module.

Conversely, suppose that P is a finitely generated projective Λ-module. Then there exists a finitely generated projective Λ-module Q such that $P \oplus Q$ is a free Λ-module of finite rank. We then have a surjection $\pi : \Lambda^n \twoheadrightarrow Q$, and this gives a finite presentation

$$\Lambda^n \longrightarrow P \oplus Q \longrightarrow P \longrightarrow 0$$

of P where the first map sends an element x of Λ^n to $(0, \pi(x))$ and the second map is the canonical projection. Hence by Proposition 2.2.6, we have that P is a compact Λ-module under the $\mathfrak{M}$-adic topology. Now suppose we are given the following diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow \alpha \\ M & \overset{\varepsilon}{\twoheadrightarrow} & N \end{array}$$

of compact Λ-modules and continuous Λ-homomorphisms. Since P is a projective Λ-module, there is an abstract Λ-homomorphism $\beta : P \to M$ such that $\varepsilon\beta = \alpha$. On the other hand, it follows from Lemma 2.2.5(3) that β is also continuous. Therefore, we have shown that P is a projective object of $\mathcal{C}_\Lambda$. □

2.3 Noetherian adic rings

Most of the adic rings that we work with in this thesis are Noetherian. This leads us to examine (left) Noetherian adic rings in more detail. We shall see that such rings share certain properties with commutative Noetherian rings.

Throughout this section, unless otherwise stated, all adic rings are assumed to be left Noetherian. As a start, we have the following result, which follows immediately from Proposition 2.2.13, since finitely generated modules over a Noetherian ring are finitely presented.

Proposition 2.3.1. *The forgetful functor*

$$\left\{\begin{array}{c} \textit{Noetherian} \\ \textit{compact } \Lambda\textit{-modules} \\ \textit{with continuous} \\ \Lambda\textit{-homomorphisms} \end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c} \textit{Noetherian} \\ \textit{abstract } \Lambda\textit{-modules} \\ \textit{with abstract} \\ \Lambda\textit{-homomorphisms} \end{array}\right\}$$

is an equivalence of categories. □

Recall from [Wei, Def. 4.1.1] that the projective dimension of an abstract Λ-module M is the minimum integer n (if it exists) such that there is a resolution of M by projective Λ-modules

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

The topological projective dimension (see [NSW, Def. 5.2.10]) of a compact Λ-module is defined similarly, replacing projective Λ-modules by projective objects in $\mathcal{C}_\Lambda$. By Proposition 2.3.1, the two notions coincide. In particular, the global dimension of Λ, which is the supremum of the projective dimensions of all abstract Λ-modules M, coincides with the projective dimension of Λ, which is the supremum of the projective dimensions of all compact Λ-modules M. We denote this common value by pd Λ. Also, for a Noetherian Λ-module M, we denote its projective dimension (there is no ambiguity by the above discussion) by $\mathrm{pd}_\Lambda(M)$.

Denote by $Mod_\Lambda^{\Lambda-ft}$ the category of Noetherian abstract Λ-modules. Since this category has enough projectives, it follows from Proposition 2.2.14 and Proposition 2.3.1 that the category of Noetherian compact Λ-modules (denoted by $\mathcal{C}_\Lambda^{\Lambda-ft}$) also has enough projectives. Therefore, it makes sense to talk about the derived categories $\mathbf{D}^-(Mod_\Lambda^{\Lambda-ft})$ and $\mathbf{D}^-(\mathcal{C}_\Lambda^{\Lambda-ft})$. Denote by $\mathbf{D}^-_{\Lambda-ft}(Mod_\Lambda)$ the full subcategory of $\mathbf{D}^-(Mod_\Lambda)$ where the objects are bounded above complexes X of Λ-modules such that all the cohomology groups $H^i(X)$ are Noetherian Λ-modules.

Lemma 2.3.2. *We have the following equivalences*

$$\mathbf{D}^-(\mathcal{C}_\Lambda^{\Lambda-ft}) \xrightarrow{\sim} \mathbf{D}^-(Mod_\Lambda^{\Lambda-ft}) \xrightarrow{\sim} \mathbf{D}^-_{\Lambda-ft}(Mod_\Lambda)$$

of categories, where the first equivalence is induced by the forgetful functor and the second equivalence is induced by the inclusion $Mod_\Lambda^{\Lambda-ft} \hookrightarrow Mod_\Lambda$.

Proof: The first equivalence is immediate from Proposition 2.3.1 and the above discussion. The second equivalence follows from a similar argument to that in [Ne, 3.2.6-8]. □

We now make the following definition.

Definition 2.3.3. A two sided ideal $\mathfrak{I}$ of a ring S is said to have the (left) Artin-Rees property if for every (left) ideal $\mathfrak{A}$ and every s, there exists $n = n(s)$ such that $\mathfrak{I}^n \cap \mathfrak{A} \subseteq \mathfrak{I}^s\mathfrak{A}$. We shall abbreviate "Artin-Rees property" to "AR property".

When Λ is commutative Noetherian, it follows from the Artin-Rees lemma (see [Mat, Thm. 8.5]) that every ideal satisfies the Artin-Rees property. In the general case of a left Noetherian adic ring (not necessarily commutative), we shall see that the Jacobson radical of the ring has this property. We record the following lemma which extends that in [N]. We do not assume that Λ is Noetherian in the lemma.

Lemma 2.3.4. *Let M be a topological Λ-module whose topology is the $\mathfrak{M}$-adic topology, and let N be a finitely generated Λ-submodule of M such that $\mathfrak{M}^sN$ is closed in M. Then there exists n such that $\mathfrak{M}^nM \cap N \subseteq \mathfrak{M}^sN$.*

Proof: Since N is finitely generated over Λ, it follows that $N/\mathfrak{M}^sN$ is finitely generated over $\Lambda/\mathfrak{M}^s$, and so $N/\mathfrak{M}^sN$ is finite. Thus, the image of $\mathfrak{M}^rM \cap N \longrightarrow N/\mathfrak{M}^sN$ is constant for big enough r. On the other hand, since $\mathfrak{M}^s N$ is closed in M, we have $\bigcap_{r\geq 0}(\mathfrak{M}^r M + \mathfrak{M}^s N) = \mathfrak{M}^s N$. Therefore, we have

$$\bigcap_{r\geq 0}(\mathfrak{M}^r M \cap N + \mathfrak{M}^s N) = N \cap \bigcap_{r\geq 0}(\mathfrak{M}^r M + \mathfrak{M}^s N) = N \cap \mathfrak{M}^s N = \mathfrak{M}^s N,$$

where the first equality follows from the modular law (see the next lemma). Hence there exists n such that $\mathfrak{M}^n M \cap N \subseteq \mathfrak{M}^s N$. □

Lemma 2.3.5. *Let M be a Λ-module with submodules A, B, C, and suppose that $C \subseteq B$. Then*

$$C + (B \cap A) = B \cap (C + A).$$

Proof: This is straightforward. □

Proposition 2.3.6. *If Λ is left Noetherian adic, then $\mathfrak{M}$ has the left AR property.*

Proof: By assumption, every left ideal of Λ is finitely generated and is therefore closed by Corollary 2.2.10. Let $\mathfrak{A}$ be a left ideal of Λ. Then for each s, we have that $\mathfrak{M}^s\mathfrak{A}$ is closed in Λ and the conclusion follows from Lemma 2.3.4. □

In general, the knowledge that a two sided ideal of a Noetherian ring (not necessary adic) satisfies the AR property will yield certain homological relations (see [Bo, N]). In this thesis, we will be interested in the following result.

Proposition 2.3.7. *Suppose Λ is left Noetherian adic. Then the left global dimension of Λ is equal to the projective dimension of $\Lambda/\mathfrak{M}$ as a Λ-module.*

Proof: This follows from Proposition 2.3.6 and the last corollary in [Bo]. □

When Λ has finite global dimension, we can refine the above proposition. In preparation for this, we have the following lemma.

Lemma 2.3.8. *If M is a Noetherian Λ-module with finite projective dimension, then* $\mathrm{pd}_\Lambda(M) = \max\{i \mid \mathrm{Ext}^i_\Lambda(M, \Lambda) \neq 0\}$.

Proof: See [Ven, Rmk. 6.4]. □

Hence, combining the above two results, we obtain the following.

Proposition 2.3.9. *If Λ is left Noetherian adic, then we have*

$$\mathrm{pd}\,\Lambda = \max\{i \mid \mathrm{Ext}^i_\Lambda(\Lambda/\mathfrak{M}, \Lambda) \neq 0\}. \ \square$$

2.4 Continuous cochains

Throughout this section, G denotes a profinite group, and we will be looking at topological Λ-modules with a continuous Λ-linear G-action. In particular, we will be interested in the continuous cochain complex (and its cohomology) of G with coefficients in this class of topological Λ-modules.

Definition 2.4.1. Let M be a topological Λ-module with a continuous Λ-linear G-action. The (inhomogeneous) continuous cochains $C^i_{\text{cts}}(G, M)$ of degree $i \geq 0$ on G with values in M are defined to be the left Λ-module of continuous maps $G^i \to M$ with the usual differential

$$(\delta^i c)(g_1, ..., g_{i+1}) = g_1 c(g_2, ..., g_{i+1}) + \sum_{j=1}^{i} (-1)^j c(g_1, ..., g_j g_{j+1}, ..., g_{i+1}) + (-1)^{i-1} c(g_1, ..., g_i),$$

which maps $C^i_{\text{cts}}(G, M_\alpha)$ to $C^{i+1}_{\text{cts}}(G, M_\alpha)$. It then follows that

$$\cdots \longrightarrow C^i_{\text{cts}}(G, M) \xrightarrow{\delta^i_M} C^{i+1}_{\text{cts}}(G, M) \longrightarrow \cdots$$

is a complex of Λ-modules and its ith cohomology groups are denoted by $H^i_{\text{cts}}(G, M)$. The following lemma is a standard result.

Lemma 2.4.2. *Let*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0$$

be a short exact sequence of topological Λ-modules with a continuous Λ-linear G-action such that the topology of M' is induced by that of M and such that β has a continuous (not necessarily Λ-linear) section. Then

$$0 \to C^\bullet_{\text{cts}}(G, M') \xrightarrow{\alpha_*} C^\bullet_{\text{cts}}(G, M) \xrightarrow{\beta_*} C^\bullet_{\text{cts}}(G, M'') \to 0$$

is an exact sequence of complexes of Λ-modules.

Proof: See [NSW, Lemma 2.3.2]. □

We are particularly interested in the case when M is a compact Λ-module or a discrete Λ-module.

Definition 2.4.3. We define $\mathcal{C}_{\Lambda,G}$ to be the category where the objects are compact Λ-modules with a continuous Λ-linear G-action and the morphisms are continuous $\Lambda[G]$-homomorphisms. Similarly, we define $\mathcal{D}_{\Lambda,G}$ to be the category where the objects are discrete Λ-modules with a continuous Λ-linear G-action and the morphisms are continuous $\Lambda[G]$-homomorphisms.

Proposition 2.4.4. *(i) The category $\mathcal{C}_{\Lambda,G}$ is abelian, has enough projectives and exact inverse limits.*

(ii) The category $\mathcal{D}_{\Lambda,G}$ is abelian, has enough injectives and exact direct limits.

(iii) The Pontryagin duality induces a contravariant equivalence between $\mathcal{C}_{\Lambda,G}$ and $\mathcal{D}_{\Lambda^o,G}$ (resp. $\mathcal{C}_{\Lambda^o,G}$ and $\mathcal{D}_{\Lambda,G}$).

Proof: We shall prove (iii) first. By Proposition 2.2.2, it suffices to show that if M (resp. N) is an object of $\mathcal{C}_{\Lambda,G}$ (resp. $\mathcal{D}_{\Lambda^o,G}$), then $M^\vee$ (resp. $N^\vee$) is an object of $\mathcal{D}_{\Lambda^o,G}$ (resp. $\mathcal{C}_{\Lambda,G}$). We define a G-action on $M^\vee$ by $\sigma \cdot f(m) = f(\sigma^{-1}m)$ for $f \in M^\vee, \sigma \in G$ and $m \in M$. This is clearly Λ^o-linear, and since G is profinite, we may apply [F, Prop. 3] to conclude that the G-action is continuous. The same argument works for N. Hence we have proven (iii). It remains to prove (ii), since (i) will follow from (ii) and (iii).

To prove (ii), we note that it is clear that $\mathcal{D}_{\Lambda,G}$ is abelian and has exact direct limits. It remains to show that it has enough injectives. By the lemma to follow, we see that the functor

$$M \mapsto \bigcup_{n=1}^{\infty} \bigcup_{U} \left(M[\mathfrak{M}^n]\right)^U : Mod_{\Lambda[G]} \longrightarrow \mathcal{D}_{\Lambda,G}$$

is right adjoint to an exact functor, and so preserves injectives by [Wei, Prop. 2.3.10]. Since $Mod_{\Lambda[G]}$ has enough injectives, it follows that $\mathcal{D}_{\Lambda,G}$ also has enough injectives. □

Lemma 2.4.5. *An abstract $\Lambda[G]$-module N is an object in $\mathcal{D}_{\Lambda,G}$ if and only if*

$$N = \bigcup_{n=1}^{\infty} \bigcup_{U} \left(N[\mathfrak{M}^n]\right)^U,$$

where U runs through all the open subgroups of G. Moreover, if M is an abstract $\Lambda[G]$-module, then

$$\bigcup_{n=1}^{\infty} \bigcup_{U} \left(M[\mathfrak{M}^n]\right)^U$$

is an object of $\mathcal{D}_{\Lambda,G}$, and there is a canonical isomorphism

$$\mathrm{Hom}_{\Lambda[G],\mathrm{cts}}\Big(N, \bigcup_{n=1}^{\infty} \bigcup_{U} \left(M[\mathfrak{M}^n]\right)^U\Big) \cong \mathrm{Hom}_{\Lambda[G]}(N, M)$$

for every $N \in \mathcal{D}_{\Lambda,G}$.

Proof: Suppose N is an object in $\mathcal{D}_{\Lambda,G}$. Then, in particular, it is a discrete Λ-module. By Lemma 2.2.3, we have $N = \bigcup_{n=1}^{\infty} N[\mathfrak{M}^n]$. Let $x \in N[\mathfrak{M}^n]$. Then by continuity of the G-action, there exists an open subgroup U of G such that $U \cdot x = x$.

Conversely, suppose that

$$N = \bigcup_{n=1}^{\infty} \bigcup_{U} \left(N[\mathfrak{M}^n]\right)^U.$$

Clearly this implies that $N = \bigcup_{n=1}^{\infty} N[\mathfrak{M}^n]$, and so N is a discrete Λ-module. It remains to show that the G-action

$$\theta : G \times N \longrightarrow N$$

is continuous. Let $x \in N$, and let $(\sigma, y) \in \theta^{-1}(x)$. Then $y \in N[\mathfrak{M}^n]^U$ for some n and open subgroup U. In particular, we have $(\sigma, y) \in \sigma U \times \{y\} \subseteq \theta^{-1}(x)$. Therefore, this proves the first assertion. The second assertion is an immediate consequence of the first. □

Lemma 2.4.6. *Let M be an object of $\mathcal{C}_{\Lambda,G}$. Then M has a fundamental system of neighborhoods of zero consisting of open $\Lambda[G]$-submodules.*

Proof: Let N be an open Λ-submodule of M. Then for each $g \in G$, there exists an open Λ-submodule N_g of M and an open subgroup U_g of G such that $gU_g \cdot N_g \subseteq N$. Since G is compact, it is covered by a finite number of such cosets, say $g_1U_{g_1}, ..., g_rU_{g_r}$. Set $N_0 = \cap_{i=1}^{r} N_{g_i}$. This is an open Λ-submodule of M. Then $\Lambda[G] \cdot N_0$ is a $\Lambda[G]$-submodule of M which contains N_0 and is contained in N. □

For the remainder of this section, we let $\mathfrak{C}$ denote either $\mathcal{C}_{\Lambda,G}$ or $\mathcal{D}_{\Lambda,G}$. Let $M^\bullet$ be a complex of objects in $\mathfrak{C}$ with differentials denoted by d^i_M. We define $C^\bullet_{\text{cts}}(G, M^\bullet)$ by

$$C^n_{\text{cts}}(G, M^\bullet) = \bigoplus_{i+j=n} C^j_{\text{cts}}(G, M^i).$$

Its differential $\delta^{i+j}_{M^\bullet}$ is determined as follows: restriction of $\delta^{i+j}_{M^\bullet}$ to $C^j_{\text{cts}}(G, M^i)$ is the sum of

$$(d^i_M)_* : C^j_{\text{cts}}(G, M^i) \longrightarrow C^j_{\text{cts}}(G, M^{i+1})$$

and

$$(-1)^i \delta^j_{M^i} : C^j_{\text{cts}}(G, M^i) \longrightarrow C^{j+1}_{\text{cts}}(G, M^i).$$

We denote its ith cohomology group by $H^i_{\text{cts}}(G, M^\bullet)$.

Proposition 2.4.7. *Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of objects in $\mathfrak{C}$. Then*

$$0 \to C^\bullet_{\text{cts}}(G, M') \xrightarrow{\alpha_*} C^\bullet_{\text{cts}}(G, M) \xrightarrow{\beta_*} C^\bullet_{\text{cts}}(G, M'') \to 0$$

is an exact sequence of complexes of Λ-modules. The statement also holds true if we replace M', M, M'' by complexes of objects in $\mathfrak{C}$.

Proof: By Lemma 2.4.2, it suffices to show that β has a continuous section. If $\mathfrak{C} = \mathcal{D}_\Lambda$, this is obvious. In the case when $\mathfrak{C} = \mathcal{C}_\Lambda$, since every compact Λ-module is profinite by Proposition 2.2.2, every continuous surjection has a continuous section. □

Let $M^\bullet$ be a complex of objects in $\mathfrak{C}$. The filtration $\tau_{\leq j}M^\bullet$ induces a filtration

$$\tau_{\leq j}C^\bullet_{\text{cts}}(G, M^\bullet) = C^\bullet_{\text{cts}}(G, \tau_{\leq j}M^\bullet)$$

on the cochain groups which fit into the following exact sequence of complexes

$$0 \longrightarrow C^\bullet_{\text{cts}}(G, \tau_{\leq j}M^\bullet) \longrightarrow C^\bullet_{\text{cts}}(G, \tau_{\leq j+1}M^\bullet) \longrightarrow \tau_{\leq j+1}C^\bullet_{\text{cts}}(G, M^\bullet)/\tau_{\leq j}C^\bullet_{\text{cts}}(G, M^\bullet) \longrightarrow 0$$

by Proposition 2.4.7. This filtration gives rise to the following hypercohomology spectral sequence

$$H^i_{\text{cts}}(G, H^j(M^\bullet)) \Longrightarrow H^{i+j}_{\text{cts}}(G, M^\bullet),$$

which is convergent if $M^\bullet$ is cohomologically bounded below.

Lemma 2.4.8. *Let $f : M^\bullet \longrightarrow N^\bullet$ be a quasi-isomorphism of cohomologically bounded below complexes of objects in $\mathfrak{C}$. Then the induced map*

$$f_* : C^\bullet_{\text{cts}}(G, M^\bullet) \longrightarrow C^\bullet_{\text{cts}}(G, N^\bullet)$$

is also a quasi-isomorphism.

Proof: The map f induces isomorphisms

$$H^i_{\text{cts}}(G, H^j(M^\bullet)) \xrightarrow{\sim} H^i_{\text{cts}}(G, H^j(N^\bullet)).$$

By convergence of the above spectral sequence, this implies that the induced maps

$$H^i_{\text{cts}}(G, M^\bullet) \longrightarrow H^i_{\text{cts}}(G, N^\bullet)$$

are isomorphisms. □

Hence we can conclude the following.

Proposition 2.4.9. *The functor*

$$C^\bullet_{\text{cts}}(G, -) : \text{Ch}^+(\mathfrak{C}) \longrightarrow \text{Ch}^+(Mod_\Lambda)$$

preserves homotopy, exact sequences and quasi-isomorphisms, hence induces the following exact derived functors

$$\mathbf{R}\Gamma_{\text{cts}}(G, -) : \mathbf{D}^b(\mathcal{C}_{\Lambda,G}) \longrightarrow \mathbf{D}^+(Mod_\Lambda)$$
$$\mathbf{R}\Gamma_{\text{cts}}(G, -) : \mathbf{D}^+(\mathcal{D}_{\Lambda,G}) \longrightarrow \mathbf{D}^+(Mod_\Lambda).$$

Proof: This proposition follows from what we have done so far. The only subtlety lies in the fact that $\mathcal{C}_{\Lambda,G}$ does not necessarily have enough injectives and therefore we do not know if $\mathbf{D}^+(\mathcal{C}_{\Lambda,G})$ exists. However, we know that $\mathcal{C}_{\Lambda,G}$ has enough projectives. Therefore, $\mathbf{D}^-(\mathcal{C}_{\Lambda,G})$ exists, and we may apply Lemma 2.4.8 to $\mathbf{D}^b(\mathcal{C}_{\Lambda,G})$. □

We now discuss cohomology and limits.

Proposition 2.4.10. *Let* $N = \varinjlim_{\alpha} N_\alpha$ *be an object of* $\mathcal{D}_{\Lambda,G}$*, where* $N_\alpha \in \mathcal{D}_{\Lambda,G}$*. Then we have an isomorphism*

$$C^i_{\mathrm{cts}}(G, N) \cong \varinjlim_{\alpha} C^i_{\mathrm{cts}}(G, N_\alpha)$$

of continuous cochain groups which induces an isomorphism

$$H^i_{\mathrm{cts}}(G, N) \cong \varinjlim_{\alpha} H^i_{\mathrm{cts}}(G, N_\alpha)$$

of cohomology groups.

Proof: The first isomorphism is immediate and the second follows from the first since the direct limit is exact. □

Proposition 2.4.11. *Let* M *be an object in* $\mathcal{C}_{\Lambda,G}$ *which is a finitely generated* Λ*-module. Then we have an exact sequence*

$$0 \longrightarrow \varprojlim_{n}{}^1 H^{i-1}_{\mathrm{cts}}(G, M/\mathfrak{M}^n M) \longrightarrow H^i_{\mathrm{cts}}(G, M) \longrightarrow \varprojlim_{n} H^i_{\mathrm{cts}}(G, M/\mathfrak{M}^n M) \longrightarrow 0.$$

Suppose further that G *has the property that* $H^m_{\mathrm{cts}}(G, N)$ *is finite for all finite discrete* Λ*-modules* N *with a continuous commuting* G*-action and for all* $m \geq 0$*. Then*

$$H^i_{\mathrm{cts}}(G, M) \cong \varprojlim_{n} H^i_{\mathrm{cts}}(G, M/\mathfrak{M}^n M).$$

Proof: By Proposition 2.2.9, the topology on M is precisely the $\mathfrak{M}$-adic topology. Since $\mathfrak{M}^n M$ are also $\Lambda[G]$-submodules, it follows that the isomorphism

$$M \cong \varprojlim_{n} M/\mathfrak{M}^n M$$

in Proposition 2.1.1(a) is an isomorphism of objects in $\mathcal{C}_{\Lambda,G}$. Therefore, we have

$$C^i_{\mathrm{cts}}(G, M) = \varprojlim_{n} C^i_{\mathrm{cts}}(G, M/\mathfrak{M}^n M).$$

The first assertion now follows by a similar argument as in [NSW, Thm. 2.3.4]. The additional assumption implies that the system $\{H^j_{\mathrm{cts}}(G, M/\mathfrak{M}^n M)\}$ satisfies the Mittag-Leffler property (see [Wei, Def. 3.5.6]), and so we have $\varprojlim_n{}^1 H^j_{\mathrm{cts}}(G, M/\mathfrak{M}^n M) = 0$ by [Wei, Prop. 3.5.7]. □

Proposition 2.4.12. *Suppose that G has the property that $H^m_{\mathrm{cts}}(G, N)$ is finite for all finite discrete Λ-modules N with a continuous commuting G-action and for all $m \geq 0$. Let M be an object in $\mathcal{C}_{\Lambda,G}$, and let $\{M_n\}$ be an inverse system of objects in $\mathcal{C}_{\Lambda,G}$ which are also finitely generated Λ-modules. Suppose that $\varprojlim_n M_n \cong M$. Then we have the following isomorphism*

$$H^i_{\mathrm{cts}}(G, M) \cong \varprojlim_n H^i_{\mathrm{cts}}(G, M_n)$$

of cohomology groups for $n \geq 0$.

Proof: Note that we have the following isomorphism

$$C_{\mathrm{cts}}(G, M) = \varprojlim_n C_{\mathrm{cts}}(G, M_n) \cong \varprojlim_{n,k} C_{\mathrm{cts}}(G, M_n/\mathfrak{M}^k M_n)$$

of complexes of Λ-modules. This induces the following spectral sequence

$$\varprojlim_{n,k}{}^i H^j_{\mathrm{cts}}(G, M_n/\mathfrak{M}^k M_n) \Rightarrow H^{i+j}_{\mathrm{cts}}(G, M).$$

Since the inverse is over a countable system, $\varprojlim_{n,k}{}^i H^j_{\mathrm{cts}}(G, M_n/\mathfrak{M}^k M_n) = 0$ for $i > 1$. By the assumption on G, we have $\varprojlim_{n,k}{}^1 H^j_{\mathrm{cts}}(G, M_n/\mathfrak{M}^k M_n) = 0$. Hence, the spectral sequence degenerates and gives the following isomorphism

$$H^i_{\mathrm{cts}}(G, M) \cong \varprojlim_{n,k} H^j_{\mathrm{cts}}(G, M_n/\mathfrak{M}^k M_n).$$

On the other hand, the latter is isomorphic to

$$\varprojlim_n \varprojlim_k H^j_{\mathrm{cts}}(G, M_n/\mathfrak{M}^k M_n) \cong \varprojlim_n H^j_{\mathrm{cts}}(G, M_n)$$

by the preceding proposition. □

Proposition 2.4.13. *Suppose that G has the property that $H^m_{\mathrm{cts}}(G, N)$ is finite for all finite discrete Λ-modules N with a continuous commuting G-action and for all $m \geq 0$. Let $M^\bullet$ be a bounded complex of objects in $\mathcal{C}_{\Lambda,G}$, and let $\{M^\bullet_n\}$ be an inverse system of bounded complexes of objects in $\mathcal{C}_{\Lambda,G}$ which are finitely generated Λ-modules such that $\varprojlim_n M^\bullet_n \cong M^\bullet$ as complexes. Then we have the following isomorphism*

$$H^i_{\mathrm{cts}}(G, M^\bullet) \cong \varprojlim_n H^i_{\mathrm{cts}}(G, M^\bullet_n)$$

of cohomology groups for $n \geq 0$.

Proof : The canonical chain map $M^\bullet \longrightarrow M^\bullet_n \longrightarrow M^\bullet_n/\mathfrak{M}^n M^\bullet_n$ induces the following morphism of (convergent) spectral sequences

$$\begin{array}{ccc} H^i_{\mathrm{cts}}\big(G, H^j(M^\bullet)\big) & \Rightarrow & H^{i+j}_{\mathrm{cts}}(G, M^\bullet) \\ & & \downarrow \\ H^i_{\mathrm{cts}}\big(G, H^j(M^\bullet_n/\mathfrak{M}^n M^\bullet_n)\big) & \Rightarrow & H^{i+j}_{\mathrm{cts}}(G, M^\bullet_n/\mathfrak{M}^n M^\bullet_n) \end{array}$$

which is compatible with n. By hypothesis, the bottom spectral sequence is a spectral sequence of finite Λ-modules. Therefore, the inverse limit is compatible with the inverse system of the spectral sequences, and we have the following morphism

$$\begin{array}{ccc} H^i_{\mathrm{cts}}\big(G, H^j(M^\bullet)\big) & \Rightarrow & H^{i+j}_{\mathrm{cts}}(G, M^\bullet) \\ & & \downarrow \\ \varprojlim_n H^i_{\mathrm{cts}}\big(G, H^j(M^\bullet_n/\mathfrak{M}^n M^\bullet_n)\big) & \Rightarrow & \varprojlim_n H^{i+j}_{\mathrm{cts}}(G, M^\bullet_n/\mathfrak{M}^n M^\bullet_n) \end{array}$$

of (convergent) spectral sequences. By the preceding proposition and the fact that the inverse limit is exact for compact Λ-modules, we have the following isomorphism

$$\varprojlim_n H^i_{\mathrm{cts}}\big(G, H^j(M^\bullet_n/\mathfrak{M}^n M^\bullet_n)\big) \cong H^i_{\mathrm{cts}}\big(G, \varprojlim_n H^j(M^\bullet_n/\mathfrak{M}^n M^\bullet_n)\big) \cong H^i_{\mathrm{cts}}\big(G, H^j(M^\bullet)\big).$$

Hence, by the convergence of the spectral sequences, we obtain the required isomorphism.
□

2.5 Completed tensor products

Let R be a commutative adic ring, and let Λ be a central R-algebra which is also an adic ring (not necessarily commutative). In particular, the ring homomorphism $R \longrightarrow \Lambda$ is continuous. We will introduce certain completed tensor products with which we will work in the thesis.

Let M be a compact Λ^o-module and N be a compact Λ-module. We define the completed tensor product to be the compact R-module

$$M\hat{\otimes}_\Lambda N = \varprojlim_{U,V} M/U \otimes_\Lambda N/V,$$

where U (resp. V) runs through the open Λ^o-submodules of M (resp. open Λ-submodules of N). Note that if we let $B_{U,V}$ denote the R-submodule

$$\mathrm{im}(U \otimes_\Lambda N \longrightarrow M \otimes_\Lambda N) + \mathrm{im}(M \otimes_\Lambda V \longrightarrow M \otimes_\Lambda N)$$

of $M \otimes_\Lambda N$, we have $(M \otimes_\Lambda N)/B_{U,V} = M/U \otimes_R N/V$. In other words, $M\hat{\otimes}_\Lambda N$ is the profinite completion of $M \otimes_\Lambda N$ with respect to the collection of R-submodules $B_{U,V}$.

The completed tensor product satisfies the following universal property (see [Wil, Lemma 7.7.1] and comments before it): For any compact R-module L and any continuous bilinear map $f : M \times N \longrightarrow L$ such that $f(m\lambda, n) = f(m, \lambda n)$ for every $m \in M, n \in N$ and $\lambda \in \Lambda$, there is a unique continuous map $\hat{f} : M\hat{\otimes}_\Lambda N \longrightarrow L$ such that $\hat{f}t = f$, where $t : M \times N \longrightarrow M\hat{\otimes}_\Lambda N$ is the canonical map. It follows from the universal property that in defining the completed tensor product, it suffices to run through a basis of neighborhoods of zero consisting of open Λ^o-submodules of M and a basis of neighborhoods of zero consisting of open Λ-submodules of N.

Lemma 2.5.1. (1) *There are canonical isomorphisms* $M\hat{\otimes}_\Lambda\Lambda \cong M$ *and* $\Lambda\hat{\otimes}_\Lambda N \cong N$.

(2) *Suppose* $M = \varprojlim_i M_i$ *and* $N = \varprojlim_j N_j$, *where each* M_i *(resp.,* N_j*) is a compact* Λ^o*-module (resp., compact* Λ*-module). Then there is an isomorphism*

$$M\hat{\otimes}_\Lambda N \cong \varprojlim_{i,j} M_i\hat{\otimes}_\Lambda N_j.$$

(3) *We have* $M\hat{\otimes}_\Lambda N = M \otimes_\Lambda N$ *if either* M *is a finitely presented* Λ^o*-module or* N *is a finitely presented* Λ*-module.*

(4) *Given a compact* Λ^o*-module* M, *the functor*

$$M\hat{\otimes}_\Lambda - : \mathcal{C}_\Lambda \longrightarrow \mathcal{C}_R$$

is right exact. The analogous assertion holds for a compact Λ*-module* N.

Proof: See [RZ, 5.5] or [Wil, 7.7]. □

Since $\mathcal{C}_{\Lambda^o}$ and $\mathcal{C}_\Lambda$ have enough projectives, we have the following result.

Proposition 2.5.2. *The completed tensor product induces the following derived bifunctor*

$$-\hat{\otimes}^{\mathbf{L}}_{\Lambda}- : \mathbf{D}^-(\mathcal{C}_{\Lambda^o}) \times \mathbf{D}^-(\mathcal{C}_{\Lambda}) \longrightarrow \mathbf{D}^-(\mathcal{C}_R). \ \square$$

Recall that from the discussion before Lemma 2.3.2 that if Λ is Noetherian, then $\mathcal{C}_{\Lambda^o}^{\Lambda^o-ft}$ and $\mathcal{C}_{\Lambda}^{\Lambda-ft}$ have enough projectives. By Lemma 2.5.1(3), we may identify the completed tensor products with the tensor products. Therefore, we have the following result.

Proposition 2.5.3. *If Λ is Noetherian, then we have the following derived bifunctor*

$$-\otimes^{\mathbf{L}}_{\Lambda}- : \mathbf{D}^-(\mathcal{C}_{\Lambda^o}^{\Lambda^o-ft}) \times \mathbf{D}^-(\mathcal{C}_{\Lambda}^{\Lambda-ft}) \longrightarrow \mathbf{D}^-(\mathcal{C}_R). \ \square$$

Now let G be a profinite group. Let M be an object in $\mathcal{C}_{\Lambda,G}$, and let N be an object in $\mathcal{C}_{R,G}$. In this case, the completed tensor product is taken to be

$$M\hat{\otimes}_R N = \varprojlim_{U,V} M/U \otimes_R N/V,$$

where U (resp., V) runs through the open $\Lambda[G]$-submodules of M (resp., open $R[G]$-submodules of N).

Lemma 2.5.4. *The above-defined object is an object of $\mathcal{C}_{\Lambda,G}$.*

Proof : It follows from [Wil, Lemma 7.7.2] that $M\hat{\otimes}_R N$ is a compact Λ-module. By a similar argument to that used in the proof of that lemma, we have that the G-action is continuous. $\square$

As in the case of Lemma 2.5.1, we can show that the completed tensor product defined here is right exact, preserves inverse limits and coincides with the usual tensor product if N is a finitely presented R-module. Recall that by Proposition 2.4.4(i), the categories $\mathcal{C}_{\Lambda,G}$ and $\mathcal{C}_{R,G}$ have enough projective objects. Therefore, we have the following conclusion.

Proposition 2.5.5. *The completed tensor product induces the following derived bifunctor*

$$-\hat{\otimes}^{\mathbf{L}}_{R}- : \mathbf{D}^-(\mathcal{C}_{\Lambda,G}) \times \mathbf{D}^-(\mathcal{C}_{R,G}) \longrightarrow \mathbf{D}^-(\mathcal{C}_{\Lambda,G}). \ \square$$

Finally, let L be an object in $\mathcal{C}_{\Lambda^o}$ and M be an object in $\mathcal{C}_{\Lambda,G}$. In this case, the completed tensor product is taken to be

$$L\hat{\otimes}_\Lambda M = \varprojlim_{W,U} L/W \otimes_\Lambda M/U,$$

where W (resp. U) runs through the open Λ^o-submodules of L (resp. open $\Lambda[G]$-submodules of M). By a similar argument to the above, we have the following.

Lemma 2.5.6. *The above-defined object is in* $\mathcal{C}_{R,G}$. □

Proposition 2.5.7. *There is a derived bifunctor*

$$-\otimes^{\mathbf{L}}_{\Lambda} - : \mathbf{D}^{-}(\mathcal{C}^{\Lambda^{o}-ft}_{\Lambda^{o}}) \times \mathbf{D}^{-}(\mathcal{C}_{\Lambda,G}) \longrightarrow \mathbf{D}^{-}(\mathcal{C}_{R,G}). \ \square$$

2.6 Ind-admissible modules

The notion of ind-admissible modules was introduced in [Ne, 3.3] for commutative Noetherian local rings. In this section, we shall see that the theory can be developed for Noetherian adic rings and that many of the arguments used in [Ne] carry over. We will also describe the category of ind-admissible modules in terms of $\mathcal{C}_{\Lambda,G}$ and $\mathcal{D}_{\Lambda,G}$. Throughout this section, we shall assume that our adic ring Λ is Noetherian.

Definition 2.6.1. Let M be an abstract $\Lambda[G]$-module. Denote by $\mathcal{S}(M)$ the set of $\Lambda[G]$-submodules $M_\alpha \subseteq M$ such that

(a) M_α is a Noetherian Λ-module, and

(b) the action $\lambda_{M_\alpha} : G \times M_\alpha \longrightarrow M_\alpha$ is continuous, where M_α is given the $\mathfrak{M}$-adic topology.

Remark. Note that by Proposition 2.2.9, we have that $M_\alpha \in \mathcal{S}(M)$ is a compact Λ-module under the $\mathfrak{M}$-adic topology.

Lemma 2.6.2. (1) *If* $M_\alpha \in \mathcal{S}(M)$*, then* $N \in \mathcal{S}(M)$ *for every* $\Lambda[G]$*-submodule* N *of* M_α*.*

(2) *If* $M_\alpha \in \mathcal{S}(M)$ *and* N *is a* Λ*-submodule of* M_α*, then* $\Lambda[G] \cdot N \in \mathcal{S}(M)$*.*

(3) *If* $M_\alpha, M_\beta \in \mathcal{S}(M)$*, then* $M_\alpha \oplus M_\beta \in \mathcal{S}(M \oplus M)$*.*

(4) *If* $f : M \to N$ *is a homomorphism of* $\Lambda[G]$*-modules and* $M_\alpha \in \mathcal{S}(M)$*, then* $f(M_\alpha) \in \mathcal{S}(N)$*.*

(5) *If* $M_\alpha, M_\beta \in \mathcal{S}(M)$*, then* $M_\alpha + M_\beta \in \mathcal{S}(M)$*.*

Proof : Part (1) is straightforward. Part (2) follows immediately from (1). For (3), it suffices to check condition (b) of the preceding definition. This follows by observing that the composite

$$G \times M_\alpha \times M_\beta \xrightarrow{\Delta \times \mathrm{id}_{M_\alpha \times M_\beta}} G \times G \times M_\alpha \times M_\beta = G \times M_\alpha \times G \times M_\beta \xrightarrow{\lambda_{M_\alpha} \times \lambda_{M_\beta}} M_\alpha \times M_\beta$$

is continuous.

To see that (4) holds, we note that $f : M_\alpha \to f(M_\alpha)$ is a quotient map with compact kernel. Hence $\mathrm{id} \times f : G \times M_\alpha \to G \times f(M_\alpha)$ is a quotient map and so induces a continuous map $G \times f(M_\alpha) \to f(M_\alpha)$. Hence we have the required conclusion.

Assertion (5) follows from (3) and (4) since $M_\alpha + M_\beta$ is the image of $M_\alpha \oplus M_\beta \in \mathcal{S}(M \oplus M)$ under the sum map $M \oplus M \to M$. □

Corollary 2.6.3. (1) *Let M be a $\Lambda[G]$-module. Then*

$$j(M) := \bigcup_{M_\alpha \in \mathcal{S}(M)} M_\alpha$$

is a $\Lambda[G]$-submodule of M and $j(j(M)) = j(M)$.

(2) *Let $f : M \to N$ be a homomorphism of $\Lambda[G]$-modules. Then we have $f(j(M)) \subseteq j(N)$.* □

Definition 2.6.4. A $\Lambda[G]$-module M is *ind-admissible* if $M = j(M)$.

Proposition 2.6.5. (*a*) *The collection of ind-admissible $\Lambda[G]$-modules forms a full abelian subcategory $(Mod_{\Lambda[G]}^{ind-ad})$ of $(Mod_{\Lambda[G]})$, stable under subobjects, quotients and colimits.*

(*b*) *The embedding functor $i : (Mod_{\Lambda[G]}^{ind-ad}) \hookrightarrow (Mod_{\Lambda[G]})$ is exact and is left adjoint to $j : (Mod_{\Lambda[G]}) \to (Mod_{\Lambda[G]}^{ind-ad})$.*

(*c*) *The functor j is left exact and preserves injectives. Thus $(Mod_{\Lambda[G]}^{ind-ad})$ has enough injectives.*

(*d*) *Let M be an ind-admissible $\Lambda[G]$-module, and let N be a Noetherian Λ-submodule of M. Then $\Lambda[G] \cdot N$ is an ind-admissible $\Lambda[G]$-module which is a Noetherian Λ-module.*

(*e*) *Let M be a $\Lambda[G]$-module. Then $M \in \mathcal{S}(M)$ if and only if M is an ind-admissible $\Lambda[G]$-module which is a Noetherian Λ-module.*

Proof: For (a)-(c), apply similar arguments as in [Ne, Prop. 3.3.5]. The "only if" direction of (e) is obvious. For (d), since N is Noetherian, we can find a finite subcollection of $\mathcal{S}(M)$, say $M_{\alpha_1}, ..., M_{\alpha_n}$, such that

$$N \subseteq M_{\alpha_1} + \cdots + M_{\alpha_n}.$$

The assertion then follows from Lemma 2.6.2(5) and the "only if" direction of (e). It remains to show the "if" direction of (e). But this follows from (d), since $M = \Lambda[G] \cdot M$. □

From now on, for any category $\mathfrak{C}$ whose objects have an underlying Λ-module structure, we denote by $\mathfrak{C}^{\Lambda-ft}$ the category of objects in $\mathfrak{C}$ which are Noetherian Λ-modules. For instance, $Mod_{\Lambda[G]}^{ind-ad,\Lambda-ft}$ will denote the category of ind-admissible $\Lambda[G]$-modules which are Noetherian Λ-modules. For a category $\mathfrak{C}$, the category $\mathrm{Ind}(\mathfrak{C})$ is defined as follows: An object is a functor $F : J \to \mathfrak{C}$, where J is a small filtered category. The morphisms sets are given by

$$\mathrm{Hom}_{\mathrm{Ind}\mathfrak{C}}(F, F') = \varprojlim_{J} \varinjlim_{J'} \mathrm{Hom}_{\mathfrak{C}}(F(j), F'(j')).$$

We are now able to describe the category $(Mod_{\Lambda[G]}^{ind-ad})$ in terms of $\mathcal{C}_{\Lambda,G}^{\Lambda-ft}$.

Proposition 2.6.6. $Mod_{\Lambda[G]}^{ind-ad} = \mathrm{Ind}(\mathcal{C}_{\Lambda,G}^{\Lambda-ft})$.

Proof: By a similar argument to that in [Ne, Prop. 3.3.5(viii)], we have

$$Mod_{\Lambda[G]}^{ind-ad} = \mathrm{Ind}(Mod_{\Lambda[G]}^{ind-ad,\Lambda-ft}).$$

Therefore, it remains to show that $Mod_{\Lambda[G]}^{ind-ad,\Lambda-ft} = \mathcal{C}_{\Lambda,G}^{\Lambda-ft}$. But this follows from Proposition 2.6.5(e) and Proposition 2.3.1.

Definition 2.6.7. The category $(Mod_{\Lambda[G]}^{ind-ad})_{\mathfrak{M}}$ is defined to be the full subcategory of $(Mod_{\Lambda[G]}^{ind-ad})$ consisting of objects M such that $M = \bigcup_{n\geq 1} M[\mathfrak{M}^n]$.

We now give a description of the above category.

Proposition 2.6.8. $(Mod_{\Lambda[G]}^{ind-ad})_{\mathfrak{M}} = \mathcal{D}_{\Lambda,G}$.

Proof: Suppose M is an object of $(Mod_{\Lambda[G]}^{ind-ad})_{\mathfrak{M}}$. We want to show that M is also an object of $\mathcal{D}_{\Lambda,G}$. By Lemma 2.4.5, it suffices to show that for each $x \in M$, we can find some open subgroup U of G and a positive integer n such that $x \in M[\mathfrak{M}^r]^U$. Let $M_\alpha \in \mathcal{S}(M)$. Since M_α is finitely generated, we can find a big enough r such that $M_\alpha = M_\alpha[\mathfrak{M}^r]$. It follows that M_α is a finitely generated $\Lambda/\mathfrak{M}^r$-module and so is finite. Also, it is discrete under the $\mathfrak{M}$-adic topology, since $\mathfrak{M}^s M_\alpha = 0$ for $s \geq r$. Thus, M_α is a finite discrete G-module. Now for each $x \in M$, we have $x \in M_\alpha$ for some $M_\alpha \in \mathcal{S}(M)$. It follows from the above argument that $\ker(G \to \mathrm{Aut}(M_\alpha))$ is an open subgroup of G. Since this is contained in the stabilizer group of x, it follows that the stabilizer group is open and we have $x \in M[\mathfrak{M}^r]^{G_x}$, where G_x denotes the stabilizer subgroup of x.

Conversely, suppose M is an object of $\mathcal{D}_{\Lambda,G}$. Let $x \in M$. By Lemma 2.4.5, we have $x \in M[\mathfrak{M}^r]^U$ for some open subgroup U and positive integer r. Then $\Lambda[G] \cdot x$ is a $\Lambda[G]$-submodule of M. On the other hand, by the choice of x, the $\Lambda[G]$-action on $\Lambda[G] \cdot x$ factors through $\Lambda/\mathfrak{M}^r[G/V]$ for an open normal subgroup V of G, where V is the intersection of all conjugates of U. It then follows that $\Lambda[G] \cdot x$ is a finite $\Lambda[G]$-submodule of M. Note that the $\mathfrak{M}$-adic topology on $\Lambda[G] \cdot x$ is discrete. Thus, we are reduced to showing that the G-action on $\Lambda[G] \cdot x$ is continuous, and this follows from the fact that M is an object of $\mathcal{D}_{\Lambda,G}$. $\square$

Definition 2.6.9. Let M be an ind-admissible $\Lambda[G]$-module. The (inhomogeneous) continuous cochains of degree $i \geq 0$ on G with values in M are defined as

$$C^i_{\text{cts}}(G, M) := \varinjlim_{M_\alpha \in \mathcal{S}(M)} C^i_{\text{cts}}(G, M_\alpha),$$

where $C^i_{\text{cts}}(G, M_\alpha)$ is defined to be the left Λ-module of continuous maps $G^i \to M_\alpha$, where M_α is endowed with the $\mathfrak{M}$-adic topology. In particular, this implies that for each α,

$$C^i_{\text{cts}}(G, M_\alpha) = \varprojlim_n C^i_{\text{cts}}(G, M_\alpha/\mathfrak{M}^n M_\alpha),$$

where the right side is equal to the usual cochain group for profinite groups with finite coefficients. Here, the differential is defined as follows. For each α, we have the usual differential

$$(\delta^i_{M_\alpha} c)(g_1, ..., g_{i+1}) = g_1 c(g_2, ..., g_{i+1}) + \sum_{j=1}^{i} (-1)^j c(g_1, ..., g_j g_{j+1}, ..., g_{i+1}) + (-1)^{i-1} c(g_1, ..., g_i),$$

which maps $C^i_{\text{cts}}(G, M_\alpha)$ to $C^{i+1}_{\text{cts}}(G, M_\alpha)$. Then we define $\delta^i_M = \varinjlim_\alpha \delta^i_{M_\alpha}$, and it follows that

$$\cdots \longrightarrow C^i_{\text{cts}}(G, M) \xrightarrow{\delta^i_M} C^{i+1}_{\text{cts}}(G, M) \longrightarrow \cdots$$

is a complex of Λ-modules.

Lemma 2.6.10. *Let M be an ind-admissible $\Lambda[G]$-module. Suppose $\mathcal{T}$ is a cofinal subset of $\mathcal{S}(M)$ and $M = \bigcup_{M_\beta \in \mathcal{T}} M_\beta$. Then we have a canonical isomorphism*

$$C_{\text{cts}}(G, M) \cong \varinjlim_{M_\beta \in \mathcal{T}} C_{\text{cts}}(G, M_\beta).$$

Proof: This is obvious. □

Proposition 2.6.11. *If M is an object of $(Mod^{ind-ad}_{\Lambda[G]})_{\mathfrak{M}}$, then the continuous cochain groups defined viewing M as an ind-admissible $\Lambda[G]$-module coincide with the continuous cochain groups defined viewing it as an object in $\mathcal{D}_{\Lambda,G}$. Similarly, if M is an object in $(Mod^{ind-ad,\Lambda-ft}_{\Lambda[G]}) = \mathcal{C}^{\Lambda-ft}_{\Lambda,G}$, the continuous cochain groups also agree in both settings.*

Proof: The first assertion follows from Proposition 2.4.10, Proposition 2.6.8 and the above lemma. The second assertion follows from Proposition 2.6.5(e), Proposition 2.6.6 and the above lemma. □

Let $M^\bullet$ be a complex of ind-admissible $\Lambda[G]$-modules with differentials denoted by d_M^i. We define $C^\bullet_{\mathrm{cts}}(G, M^\bullet)$ by

$$C^n_{\mathrm{cts}}(G, M^\bullet) = \bigoplus_{i+j=n} C^j_{\mathrm{cts}}(G, M^i).$$

Its differential $\delta^{i+j}_{M^\bullet}$ is determined as follows: restriction of $\delta^{i+j}_{M^\bullet}$ to $C^j_{\mathrm{cts}}(G, M^i)$ is the sum of

$$(d_M^i)_* : C^j_{\mathrm{cts}}(G, M^i) \longrightarrow C^j_{\mathrm{cts}}(G, M^{i+1})$$

and

$$(-1)^i \delta^j_{M^i} : C^j_{\mathrm{cts}}(G, M^i) \longrightarrow C^{j+1}_{\mathrm{cts}}(G, M^i).$$

Proposition 2.6.12. *Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of $\Lambda[G]$-modules with M ind-admissible. Then M', M'' are also ind-admissible and*

$$0 \to C^\bullet_{\mathrm{cts}}(G, M') \xrightarrow{\alpha_*} C^\bullet_{\mathrm{cts}}(G, M) \xrightarrow{\beta_*} C^\bullet_{\mathrm{cts}}(G, M'') \to 0$$

is an exact sequence of complexes of Λ-modules. The statement also holds true if we replace M', M and M'' by complexes of $\Lambda[G]$-modules.

Proof: The first part follows from Proposition 2.6.5. Since the direct limit is exact, we may assume that M is Noetherian as a Λ-module. Therefore, we are reduced to the case where the exact sequence is an exact sequence in $\mathcal{C}_{\Lambda,G}$. The conclusion now follows from Proposition 2.4.7. □

Definition 2.6.13. The continuous cohomology of G with values in M (resp. $M^\bullet$) is defined as

$$H^i_{\mathrm{cts}}(G, M) = H^i(C^\bullet_{\mathrm{cts}}(G, M))$$
$$\left(\text{resp. } H^i_{\mathrm{cts}}(G, M^\bullet) = H^i(C^\bullet_{\mathrm{cts}}(G, M^\bullet))\right).$$

Proposition 2.6.14. *The functor $C^\bullet_{\mathrm{cts}}(G, -)$ maps bounded below complexes of ind-admissible $\Lambda[G]$-modules to bounded below complexes of Λ-modules and preserves homotopy, exact sequences and quasi-isomorphisms, hence induces an exact derived functor*

$$\mathbf{R}\Gamma_{\mathrm{cts}}(G, -) : \mathbf{D}^+(Mod^{ind-ad}_{\Lambda[G]}) \to \mathbf{D}^+(Mod_\Lambda).$$

Proof: The argument is similar to that in [Ne, 3.5.2-3.5.6]. □

We shall also give the analogous definition for ind-admissible $\Lambda[G]$-Λ-bimodules.

Definition 2.6.15. Let M be an abstract $\Lambda[G]$-Λ-bimodule. Denote by $\mathcal{T}(M)$ the set of $\Lambda[G]$-Λ-submodules $M_\alpha \subseteq M$ such that

(a) M_α is a Noetherian Λ-module and a Noetherian Λ^o-module (by Proposition 2.2.9, the left $\mathfrak{M}$-adic topology coincides with the right $\mathfrak{M}$-adic topology), and

(b) the action $\lambda_{M_\alpha} : G \times M_\alpha \longrightarrow M_\alpha$ is continuous.

We say that M is an ind-admissible $\Lambda[G]$-Λ-bimodule if

$$M = \bigcup_{M_\alpha \in \mathcal{T}(M)} M_\alpha.$$

In this case, we can define the continuous cochain complex

$$C^i_{\text{cts}}(G, M) := \varinjlim_{M_\alpha \in \mathcal{T}(M)} C^i_{\text{cts}}(G, M_\alpha).$$

We also have the analogous definition for complexes of ind-admissible $\Lambda[G]$-Λ-bimodules.

Many of the results shown for an ind-admissible $\Lambda[G]$-module also hold for an ind-admissible $\Lambda[G]$-Λ-bimodule. We shall not dwell on this subject, but instead just mention two of them which we will require.

Lemma 2.6.16. *(1) If $M_\alpha \in \mathcal{T}(M)$, then $N \in \mathcal{T}(M)$ for every $\Lambda[G]$-Λ-submodule N of M_α.*

(2) If $M_\alpha, M_\beta \in \mathcal{T}(M)$, then $M_\alpha + M_\beta \in \mathcal{T}(M)$.

Proof : This follows from a similar argument to that of Lemma 2.6.2. □

For ease of notation, we will drop the '•' for complexes. We also drop the notation 'cts'. Therefore we write $C(G, M)$ as the complex of continuous cochains and $\mathbf{R}\Gamma(G, M)$ for its derived functor. Its ith cohomology group is then written as $H^i(G, M)$.

2.7 Total cup products

We first review the definition of cup-products for topological G-modules (in other words, abelian Hausdorff topological groups with a continuous G-action).

Definition 2.7.1. (Cup products) Let A, B and C be topological G-modules. Suppose

$$\langle\ ,\ \rangle : A \times B \longrightarrow C$$

is a continuous map satisfying $\sigma\langle a, b\rangle = \langle \sigma a, \sigma b\rangle$ for $a \in A, b \in B$ and $\sigma \in G$. Then we define the cup product on the cochain groups

$$C^i(G, A) \times C^j(G, B) \longrightarrow C^{i+j}(G, C)$$

as follows: for $\alpha \in C^i(G, A), \beta \in C^j(G, B)$ and $\sigma_1, ..., \sigma_{i+j} \in G$, we have

$$(\alpha \cup \beta)(\sigma_1, ..., \sigma_{i+j}) = \Big\langle \alpha(\sigma_1, ..., \sigma_i), \sigma_1 \cdots \sigma_i \beta(\sigma_{i+1}, ..., \sigma_{i+j}) \Big\rangle.$$

The cup product satisfies the following relation

$$\delta_C(\alpha \cup \beta) = (\delta_A \alpha) \cup \beta + (-1)^i \alpha \cup (\delta_B \beta)$$

and induces a pairing

$$H^i(G, A) \times H^j(G, B) \longrightarrow H^{i+j}(G, C)$$

on the cohomology groups.

In this thesis, we will mainly work with cup products over an adic ring. For the remainder of the thesis, we shall assume our adic ring Λ has the property that $\Lambda/\mathfrak{M}^n$ is finite of order a power of a prime p for all $n \geq 1$. Let M and N be objects in $\mathcal{C}_{\Lambda,G}$ and $\mathcal{D}_{\Lambda^o,G}$ respectively, and let A be a topological G-module. Suppose there is a continuous pairing

$$\langle\ ,\ \rangle : N \times M \longrightarrow A$$

such that

(1) $\sigma\langle y, x\rangle = \langle \sigma y, \sigma x\rangle$ for $x \in M, y \in N$ and $\sigma \in G$, and

(2) $\langle y\lambda, x\rangle = \langle y, \lambda x\rangle$ for $x \in M, y \in N$ and $\lambda \in \Lambda$.

As before, condition (1) will give rise to the cup product

$$C^i(G, N) \times C^j(G, M) \longrightarrow C^{i+j}(G, A),$$

which is Λ-balanced by condition (2). The cup product induces a group homomorphism

$$C^i(G, N) \otimes_\Lambda C^j(G, M) \longrightarrow C^{i+j}(G, A)$$

which gives rise to the following morphism

$$C(G, N) \otimes_\Lambda C(G, M) \longrightarrow C(G, A)$$

of complexes of abelian groups. Taking the adjoint, we have a morphism

$$C(G, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}\big(C(G, N), C(G, A)\big)$$

of complexes of Λ-modules.

Lemma 2.7.2. *Suppose we are given another continuous pairing*

$$(\ ,\ ) : N' \times M' \longrightarrow A$$

such that (1) $\sigma(y', x') = (\sigma y', \sigma x')$ *for* $x' \in M', y' \in N'$ *and* $\sigma \in G$;

(2) $(y'\lambda, x') = (y', \lambda x')$ *for* $x' \in M', y' \in N'$ *and* $\lambda \in \Lambda$, *and*

(3) *there are morphisms* $f : N' \longrightarrow N$ *in* $\mathcal{D}_{\Lambda^\circ, G}$ *and* $g : M \longrightarrow M'$ *in* $\mathcal{C}_{\Lambda, G}$ *such that the following diagram*

$$\begin{array}{ccc} N' \otimes_\Lambda M & \xrightarrow{\mathrm{id}\otimes g} & N' \otimes_\Lambda M' \\ {\scriptstyle f\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle (\ ,\ )} \\ N \otimes_\Lambda M & \xrightarrow{\langle\ ,\ \rangle} & A \end{array}$$

commutes. Then we have the following commutative diagram

$$\begin{array}{ccc} C(G, M) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}\big(C(G, N), C(G, A)\big) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ C(G, M') & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}\big(C(G, N'), C(G, A)\big) \end{array}$$

of complexes of Λ*-modules.*

Proof: It follows from a direct calculation that the following diagram

$$\begin{array}{ccc} C(G, N') \otimes_\Lambda C(G, M) & \xrightarrow{\mathrm{id}\otimes g} & C(G, N)' \otimes_\Lambda C(G, M') \\ {\scriptstyle f\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \cup_{(\ ,\ )}} \\ C(G, N) \otimes_\Lambda C(G, M) & \xrightarrow{\cup_{\langle\ ,\ \rangle}} & C(G, A) \end{array}$$

is commutative, where $\cup_{(\ ,\ )}$ and $\cup_{\langle\ ,\ \rangle}$ are the cup products induced by the pairings $(\ ,\ )$ and $\langle\ ,\ \rangle$ respectively. By taking the adjoint and another straightforward calculation, we have the commutative diagram in the lemma. □

Now let M and N be bounded complexes of objects in $\mathcal{C}_{\Lambda,G}$ and $\mathcal{D}_{\Lambda^\circ,G}$ respectively, and let A be a bounded complex of topological G-modules. Suppose there is a collection of continuous pairings

$$\langle\ ,\ \rangle_{a,b} : N^a \times M^b \longrightarrow A^{a+b}$$

where each pairing satisfies conditions (1) and (2), and the following hold:

(a) $\langle d_N^a y, x\rangle_{a+1,b} = d_A^{a+b}\big(\langle y, x\rangle_{a,b}\big)$ for $y \in N^a$ and $x \in M^b$, and

(b) $(-1)^a \langle y, d_M^b x\rangle_{a,b+1} = d_A^{a+b}\big(\langle y, x\rangle_{a,b}\big)$ for $y \in N^a$ and $x \in M^b$.

For each pair (a, b), we have a morphism

$$\cup_{ij}^{ab} : C^i(G, N^a) \otimes_\Lambda C^j(G, M^b) \longrightarrow C^{i+j}(G, A^{a+b})$$

of abelian groups induced by the cup product. Then the total cup product

$$\cup : C(G, N) \otimes_\Lambda C(G, M) \longrightarrow C(G, A)$$

is a morphism of complexes of $\mathbb{Z}_p$-modules given by the collection $\cup = \left((-1)^{ib} \cup_{ij}^{ab}\right)$. The definition given for the total cup products follows that in [Ne, 3.4.5.2]. We also have an analogous result to Lemma 2.7.2 for complexes.

We describe another form of (total) cup product with which we will work. Let R be a commutative complete Noetherian local ring with finite residue field of characteristic p, and let Λ be (on top of being an adic ring) a Noetherian central R-algebra. Let M be an object in $\mathcal{C}_{\Lambda,G}^{\Lambda-ft}$ and N be an object in $\mathcal{C}_{\Lambda^o,G}^{\Lambda^o-ft}$. Let W be an ind-admissible $\Lambda[G]$-Λ-bimodule. Suppose there is a continuous pairing

$$\langle\ ,\ \rangle : M \times N \longrightarrow W$$

for which the following hold:

(i) $\langle rx, y\rangle = \langle x, yr\rangle$,

(ii) $\sigma\langle x, y\rangle = \langle \sigma x, \sigma y\rangle$,

(iii) $\lambda\langle x, y\rangle = \langle \lambda x, y\rangle$, and

(iv) $\langle x, y\rangle\lambda = \langle x, y\lambda\rangle$

for $x \in M, y \in N, \sigma \in G, r \in R$ and $\lambda \in \Lambda$.

Then, by a similar argument as above, we have a homomorphism

$$C^i(G, M) \otimes_R C^j(G, N) \longrightarrow C^{i+j}(G, W)$$

of Λ-Λ-bimodules which gives rise to the following morphism

$$C(G, M) \otimes_R C(G, N) \longrightarrow C(G, W)$$

of complexes of Λ-Λ-bimodules.

Suppose M is a bounded complex in $\mathcal{C}_{\Lambda,G}^{\Lambda-ft}$, N is a bounded complex in $\mathcal{C}_{\Lambda^o,G}^{\Lambda^o-ft}$, and W is a bounded complex of ind-admissible $\Lambda[G]$-Λ-bimodules such that there is a collection of continuous pairings

$$\langle\ ,\ \rangle_{a,b} : M^a \times N^b \longrightarrow W^{a+b},$$

where each pairing satisfies the above conditions and the following relations:

(a) $\langle d_M^a x, y\rangle_{a+1,b} = d_W^{a+b}\big(\langle x, y\rangle_{a,b}\big)$, and
(b) $(-1)^a\langle x, d_N^b y\rangle_{a,b+1} = d_W^{a+b}\big(\langle x, y\rangle_{a,b}\big)$
for $x \in M^a$ and $y \in N^b$.

Then, as above, we can construct a total cup product

$$\cup : C(G,M) \otimes_R C(G,N) \longrightarrow C(G,W)$$

under the same sign conventions. Note that this is a morphism of complexes of Λ-Λ-bimodules. There is also a variant of Lemma 2.7.2 in this context.

2.8 Tate cohomology groups

In this section, G is a finite group. We now recall the construction of the Tate cohomology groups of a G-module M from [NSW, Chap. I §2]. The complete cochain groups $\hat{C}^i(G,M)$ are defined by

$$\hat{C}^i(G,M) = \begin{cases} C^i(G,M) & \text{if } i \geq 0, \\ C^{-1-i}(G,M) & \text{if } i \leq -1, \end{cases}$$

where $C^i(G,M)$ is the usual (inhomogeneous) cochain complex, and the differentials are defined for $i \geq 0$ by

$$(\delta^i c)(g_1, \dots, g_{i+1}) = g_1 c(g_2, \dots, g_{i+1}) + \sum_{j=1}^{i}(-1)^j c(g_1, \dots, g_j g_{j+1}, \dots, g_{i+1}) + (-1)^{i-1} c(g_1, \dots, g_i),$$

and for $i > 2$ by

$$\begin{aligned}(\delta^{-1-i}c')(g_1, \dots, g_{i-1}) &= \sum_{\tau\in G}\Big(\tau c'(\tau^{-1}, g_1, \dots, g_{i-1}) - c'(\tau, \tau^{-1}g_1, \dots, g_{i-1}) \\ &\quad + \sum_{r=2}^{i}(-1)^r c'\big(g_1, \dots, (g_1 \cdots g_{r-1})^{-1}\tau, \tau^{-1}g_1 \cdots g_r, \dots, g_{i-1}\big)\Big),\end{aligned}$$

where $c \in \hat{C}^i(G,M), c' \in \hat{C}^{-1-i}(G,M), g_1, \dots, g_{i-1} \in G$, and

$$(\delta^{-2}c'') = \sum_{\tau\in G}\big(\tau c''(\tau^{-1}) - c''(\tau)\big)$$

for $c'' \in \hat{C}^{-2}(G,M)$, and

$$(\delta^{-1}m) = \big(\sum_{\tau\in G}\tau\big)m$$

for $m \in \hat{C}^{-1}(G, M) = M$. The ith cohomology group of $\hat{C}(G, M)$ is denoted by $\hat{H}^i(G, M)$. Clearly, there is a canonical inclusion of complexes

$$C(G, M) \hookrightarrow \hat{C}(G, M)$$

and $\hat{H}^i(G, M) = H^i(G, M)$ for $i \geq 0$.

Following [Ne, 5.7.2], we may extend the above definition to a complex $M^\bullet$ of G-modules by setting

$$\hat{C}^n(G, M^\bullet) = \bigoplus_{i+j=n} \hat{C}^i(G, M^j)$$

with differential defined using the sign conventions of the previous sections. As before, for ease of notation, we will drop the '$\bullet$' for complexes. The usual cup product for Tate cohomology groups [NSW, Prop. 1.4.6] extends to a total cup product with the same sign convention as in the preceding section.

Chapter 3

Completed Group Algebras

In this chapter, we shall investigate the completed group algebra of a finitely generated pro-p group Γ. In particular, we are interested in the case that Γ is uniform (see below for the definition). As we will see soon, the completed group algebra of a uniform group is the completion of the group algebra under a certain norm. This leads us to the study of group algebras of a uniform group. We will show that every element in the group algebra and its completion has a natural series representation which is unique once we fix a minimal set of topological generators for Γ. This will then be applied to prove a generalization of a result of Lazard which says that the completed group algebra $\mathbb{Z}_p[\![\Gamma]\!]$ of a compact p-adic Lie group is a Noetherian ring (see [Laz]). Namely, we will show that the same conclusion holds if one replaces $\mathbb{Z}_p$ by any commutative complete Noetherian local ring with finite residue field of characteristic p. Our argument follows the approach given in [DSMS], aside from some modifications.

We list certain notations to which we will adhere throughout this chapter. We let p denote a fixed prime. We shall then let R be a commutative complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and finite residue field $\mathbb{F}_q$ of order q, where q is a power of p. We also denote $\mathbb{N}$ to be the set of natural numbers including 0 (i.e $\mathbb{N} = \{0, 1, 2, ...\}$).

3.1 Review

We now review some facts, most of which can be found in the book [DSMS]. For a group G, we write $G^p = \langle g^p |\ g \in G\rangle$, that is, the group generated by the p-powers of elements in G. A pro-p group G is said to be *powerful* if $G/\overline{G^p}$ is abelian for odd p, or if $G/\overline{G^4}$ is abelian for $p = 2$. We also recall the lower p-series.

Definition 3.1.1. (Lower p-series) Let G be a pro-p group. Then define $P_1(G) = G$, and for $i \geq 1$

$$P_{i+1}(G) = \overline{P_i(G)^p[P_i(G), G]}.$$

Proposition 3.1.2. *If G is a finitely generated pro-p group, then $P_i(G)$ is open in G for each i, and the set $\{P_i(G) | \ i \geq 1\}$ is a base for the neighborhoods of 1 in G. It follows that*

$$P_{i+1}(G) = P_i(G)^p[P_i(G), G].$$

Moreover, if G is powerful, we have $G^p = \overline{G^p} = P_2(G)$.

Proof: See [DSMS, Prop. 1.16(iii), Cor. 1.20, Lemma 3.4]. □

Theorem 3.1.3. *Let $G = \overline{\langle a_1, ..., a_d \rangle}$ be a finitely generated powerful pro-p group, and put $G_i = P_i(G)$ for each i. We have the following statements.*

(i) $G_i = G^{p^{i-1}} = \{x^{p^{i-1}} | \ x \in G\} = \overline{\langle a_1^{p^{i-1}}, ..., a_d^{p^{i-1}} \rangle}$

(ii) $G = \overline{\langle a_1 \rangle} \cdots \overline{\langle a_d \rangle}$

Proof: See [DSMS, Thm. 3.6]. □

A finitely generated powerful pro-p group G is said to be *uniform* if the p-power map induces isomorphisms

$$P_i(G)/P_{i+1}(G) \xrightarrow{\cdot p} P_{i+1}(G)/P_{i+2}(G), i \geq 1.$$

For any topological group, we denote the minimal cardinality of a generating set of G by $d = d(G)$. If G is a finitely generated pro-p group, we have $d = \dim_{\mathbb{F}_p}(G/P_2(G))$.

Normed rings

As we will be dealing with a certain norm on the (completed) group algebra in our study, we shall review some facts on such norms.

Definition 3.1.4. A (non-Archimedean) norm on a (not necessarily commutative) ring Λ is a function $\| \cdot \| : \Lambda \to \mathbb{R}$ such that for all $a, b \in \Lambda$

(i) $\|a\| \geq 0$; $\|a\| = 0$ if and only if $a = 0$;

(ii) $\|1\| = 1$ and $\|ab\| \leq \|a\| \, \|b\|$ and

(iii) $\|a \pm b\| \leq \max\{\|a\|, \|b\|\}$.

If these hold, then $(\Lambda, \| \cdot \|)$ is said to be a *normed ring*.

A sequence (λ_n) of elements in a normed ring Λ is a *Cauchy sequence* if for every $\varepsilon > 0$, there exists an integer N (depending on ε) such that $\|\lambda_n - \lambda_m\| < \varepsilon$ whenever $n, m \geq N$. We say that the normed ring is *complete* if every Cauchy sequence in Λ converges to an element in Λ.

Definition 3.1.5. A normed ring $\hat{\Lambda}$ is called a completion of Λ if

(a) Λ is a dense subring of $\hat{\Lambda}$ and the norm on $\hat{\Lambda}$ extends the norm on Λ, and

(b) $\hat{\Lambda}$ is complete.

Proposition 3.1.6. *Given a normed ring Λ, there exists a completion $\hat{\Lambda}$ of Λ which is unqiue up to isomorphism.*

Proof: See [DSMS, Prop. 6.3]. □

Lemma 3.1.7. *Let Λ be a ring and*

$$\Lambda = \Lambda_0 \supseteq \Lambda_1 \supseteq \cdots \supseteq \Lambda_i \supseteq \cdots$$

a chain of ideals such that

(i) $\bigcap_{i \geq 0} \Lambda_i = 0$;

(ii) for all i, j, $\Lambda_i \Lambda_j \subseteq \Lambda_{i+j}$.

Fix a real number $c > 1$ and define $\| \cdot \| : \Lambda \to \mathbb{R}$ by

$$\|0\| = 0; \ \|a\| = c^{-k} \text{ if } a \in \Lambda_k \setminus \Lambda_{k+1}.$$

Then $(\Lambda, \| \cdot \|)$ is a normed ring. Furthermore, the completion of Λ under this norm is isomorphic to $\varprojlim_i \Lambda/\Lambda_i$.

Proof: See [DSMS, Lemma 6.5]. □

As we will be dealing with multiple series, we will introduce the following rather general notion of convergence.

Definition 3.1.8. Let Λ be a normed ring. Let T be a countably infinite set, and let $t \mapsto \lambda_t$ be a map of T into Λ. Let $\lambda, s \in \Lambda$.

(a) The family $(\lambda_t)_{t \in T}$ is said to *converge to* λ, written as

$$\lim_{t \in T} \lambda_t = \lambda,$$

if for every $\varepsilon > 0$, there exists a finite subset T' of T such that $\|\lambda - \lambda_t\| < \varepsilon$ for all $t \in T \setminus T'$.

(b) The series $\sum_{t \in T} \lambda_t$ is said to *converge to* s, written as

$$\sum_{t \in T} \lambda_t = s,$$

if for each $\varepsilon > 0$, there exists a finite subset T' of T such that for all finite sets T'' for which $T' \subseteq T'' \subseteq T$, we have $\|s - \sum_{t \in T''} \lambda_t\| < \varepsilon$.

Proposition 3.1.9. *Retaining the notations of the preceding definition, and supposing that $i \mapsto t(i)$ is a bijection from $\mathbb{N}$ to T, the following statements hold.*

(a) $\lim_{t \in T} \lambda_t = \lambda$ *if and only if* $\lim_{i \to \infty} \lambda_{t(i)} = \lambda$.

(b) $\sum_{t \in T} \lambda_t = s$ *if and only if* $\sum_{i=0}^{\infty} \lambda_{t(i)} = s$.

(c) *If* $\sum_{t \in T} \lambda_t$ *converges, then* $\lim_{t \in T} \lambda_t = 0$.

(d) *If* Λ *is complete and* $\lim_{t \in T} \lambda_t = 0$, *then* $\sum_{t \in T} \lambda_t$ *converges.*

Proof: See [DSMS, Prop. 6.9]. □

Associated graded rings

Let $*$ be a commutative and associative binary operation on $\mathbb{N}$ with the properties that

$$i * 0 = i$$
$$i * j = i * k \Rightarrow j = k$$
$$j > k \Rightarrow i * j > i * k$$

for all i, j and k. Let Λ be a ring with a descending chain of ideals $\{\Lambda_i\}$ satisfying the conditions in Lemma 3.1.7 and the following relation

$$\Lambda_i \Lambda_j \subseteq \Lambda_{i*j}$$

for all i and j in $\mathbb{N}$. Note that

$$\Lambda_i \Lambda_{j+1} + \Lambda_{i+1} \Lambda_j \subseteq \Lambda_{i*j+1}$$

for all i and j in $\mathbb{N}$, and one can check easily that $i * j \geq i + j$. Set $E_i = \Lambda_i / \Lambda_{i+1}$ for each $i \geq 0$. The *associated graded ring* (with respect to $*$) is then

$$\Lambda^* = \bigoplus_{i=0}^{\infty} E_i,$$

where the multiplication is induced by the product

$$E_i \times E_j \longrightarrow E_{i*j}.$$

We then have the following proposition.

Proposition 3.1.10. *Let Λ be a ring with a descending chain of ideals $\{\Lambda_i\}$ as above. Suppose Λ is complete under the norm endowed by the chain of ideals as in Lemma 3.1.7.*

(a) If Λ^* is left (or right) Noetherian, so is Λ.

(b) If Λ^* has no zero divisors, then Λ has no zero divisors.

Proof: See [DSMS, Prop. 7.27]. □

3.2 Group algebras

In this section, G will always be a finitely generated pro-p group. Write $G_k = P_k(G)$ and

$$I_k = (G_k - 1)R[G] = \ker(R[G] \to R[G/G_k]).$$

Denote the maximal ideal of $R[G]$ by $J = I_1 + \mathfrak{m}R[G]$. We shall use these notations throughout the section.

Lemma 3.2.1. *We have the following relations.*

(i) $J^k \supseteq I_k + \mathfrak{m}^k R[G]$.

(ii) $I_k + \mathfrak{m}^j R[G] \supseteq J^{j|G/G_k|}$.

Proof: (i) We prove this by induction on k. When $k = 1$, this is true by definition. Let $k > 1$, and suppose that $J^{k-1} \supseteq I_{k-1} + \mathfrak{m}^{k-1}R[G]$. Clearly $\mathfrak{m}^k R[G] \subseteq J^k$. Thus it remains to show that $I_k \subseteq J^k$. By the definition of the lower p-series, G_k is generated by elements of the form $x^p, [x, y]$ for $x \in G_{k-1}, y \in G$. Write $u = x - 1$ and $v = y - 1$. Then

$$x^p - 1 = (u+1)^p - 1 = \begin{cases} u^p + puw & \text{for some } w \in R[G], \text{ if } p \text{ does not divide char } R, \\ u^p & \text{if } p \text{ divides char } R \end{cases}$$

and

$$[x, y] - 1 = (xy - yx)x^{-1}y^{-1} = (uv - vu)x^{-1}y^{-1}.$$

Since $p \in \mathfrak{m}R[G] \subseteq J$, $v \in I_1 \subseteq J$ and $u \in J^{k-1}$ (by induction), it follows that $x^p - 1$ and $[x, y] - 1$ lie in J^k. Therefore $I_k = (G_k - 1)R[G] \subseteq J^k$.

(ii) Write $n = |G/G_k|$. Then G/G_k is a finite p-group acting on the $\mathbb{F}_q$-vector space $\mathbb{F}_q[G/G_k]$ which has dimension n. It follows from [DSMS, 0.8] that $(x_1 - 1)\cdots(x_n - 1) = 0$ in $\mathbb{F}_q[G/G_k]$ for any $x_1, ..., x_n \in G/G_k$. This implies that $(g_1 - 1)\cdots(g_n - 1) \in I_k + \mathfrak{m}R[G]$ for all $g_1, ..., g_n \in G$. It follows that $J^n \subseteq I_k + \mathfrak{m}R[G]$ and hence $J^{jn} \subseteq I_k + \mathfrak{m}^j R[G]$. □

Corollary 3.2.2. *We have*

$$\bigcap_{n=1}^{\infty} J^n = 0.$$

Proof : Let $c = \sum_{i=1}^{m} r_i x_i$, where the x_i's are distinct elements of G and the r_i's are all non-zero. Choose k big enough such that $x_i x_j^{-1} \notin G_k$ for all $i \neq j$ and $r_i \notin \mathfrak{m}^k$ for all i. Consider the canonical map

$$\phi : R[G] \to R/\mathfrak{m}^k[G/G_k].$$

Then by our choice of k, the $\phi(x_i)$'s are distinct elements of G/G_k and $\phi(r_i) \neq 0$ for all i. This implies that $c \notin \ker \phi = I_k + \mathfrak{m}^k R[G]$. By part (ii) of the preceding lemma, we have that $c \notin J^m$ for some m. □

Therefore, the collection $\{J^k\}$ of ideals satisfies the hypotheses of Lemma 3.1.7, and so we can make the following definition.

Definition 3.2.3. The norm on $R[G]$ is defined by

$$\|0\| = 0; \ \ \|c\| = q^{-k} \text{ if } c \in J^k \setminus J^{k+1}.$$

It follows from Lemma 3.2.1 that the topology on $R[G]$ given by the norm induces on G its original topology. For if $x \in G_k$, then $\|x - 1\| \leq q^{-k}$, and conversely, if $x \in G$ and $x - 1 \in I_k + \mathfrak{m}R[G]$, then $x \in G_k$. Also, since $R1_G \cap J^k = \mathfrak{m}^k$, this norm induces a norm on R which coincides with the $\mathfrak{m}$-adic norm. The more important observation that one makes is the following.

Proposition 3.2.4. *The completion of $R[G]$ under the above norm is topologically isomorphic to $R[\![G]\!]$.*

Proof : By Lemma 3.1.7, the completion of $R[G]$ under the given norm is $\varprojlim_k R[G]/J^k$. By Lemma 3.2.1, the two chains of ideals $\{J^k\}$ and $\{I_k + \mathfrak{m}^k R[G]\}$ are cofinal. Therefore, the completion is isomorphic to $\varprojlim_k R[G]/(I_k + \mathfrak{m}^k R[G])$.

On the other hand, since the collection $\{G_k\}$ of subgroups is a system of neighborhoods of 1 in G (see [DSMS, Prop. 1.16]), we have

$$R[\![G]\!] \cong \varprojlim_k R[G/G_k] \cong \varprojlim_{k,j} R/\mathfrak{m}^j[G/G_k].$$

Since R is complete, this last term is isomorphic to $\varprojlim_k R[G]/(I_k + \mathfrak{m}^k R[G])$. □

From now on, unless otherwise stated, G is a finitely generated pro-p group with a minimal set of topological generators $a_1, a_2, ..., a_d$. Write $b_i = a_i - 1$ for each i. For $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ and any d-tuple $\mathbf{v} = (v_1, ..., v_d) \in R[G]^d$, we write

$$\langle \alpha \rangle = \alpha_1 + \cdots + \alpha_d, \ \mathbf{v}^\alpha = v_1^{\alpha_1} \cdots v_d^{\alpha_d}.$$

We are now able to state the following theorem.

Theorem. *Let G be a uniform pro-p group. Then every element of $R[\![G]\!]$ is equal to the sum of a uniquely determined convergent series*

$$\sum_{\alpha \in \mathbb{N}^d} r_\alpha \mathbf{b}^\alpha,$$

where $r_\alpha \in R$ for each $\alpha \in \mathbb{N}^d$. Conversely, every such series converges in $R[\![G]\!]$.

This theorem is a generalization of the well-known case of $\mathbb{Z}_p[\![\Gamma]\!]$, where $\Gamma \cong \mathbb{Z}_p$. In this situation, the theorem is usually expressed as an isomorphism $\mathbb{Z}_p[\![\Gamma]\!] \cong \mathbb{Z}[\![T]\!]$ of topological rings given by the assignment $\gamma - 1 \mapsto T$ for a topological generator γ of Γ. We will prove this theorem in the next section. In this section, we shall state and prove a variant of the above statement for $R[G]$ which will be required for the proof of the theorem.

Theorem 3.2.5. (*i*) *If G is powerful, then each element of $R[G]$ is equal to the sum of a convergent series*

$$\sum_{\alpha \in \mathbb{N}^d} r_\alpha \mathbf{b}^\alpha$$

with $r_\alpha \in R$ for each α.

(*ii*) *If G is uniform, then the series is uniquely determined by its sum.*

When G is uniform, we also have the following result for the norm.

Theorem 3.2.6. *If G is uniform and $c = \sum_{\alpha \in \mathbb{N}^d} r_\alpha \mathbf{b}^\alpha \in R[G]$, where $r_\alpha \in R$ for each α, then*

$$\|c\| = \sup_{\alpha \in \mathbb{N}^d} q^{-\langle \alpha \rangle} \|r_\alpha\|.$$

For the remainder of the section, we will be working towards the proofs of Theorem 3.2.5 and Theorem 3.2.6. As a start, we record a useful lemma.

Lemma 3.2.7. *Let $u_1, ..., u_r \in G$, and write $v_i = u_i - 1$. Then, for $\beta \in \mathbb{N}^r$, we have*

$$\mathbf{u}^\beta = \sum_{\alpha \in \mathbb{N}^r} \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_r}{\alpha_r} \mathbf{v}^\alpha,$$

$$\mathbf{v}^\beta = \sum_{\alpha \in \mathbb{N}^r} (-1)^{\langle\beta\rangle - \langle\alpha\rangle} \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_r}{\alpha_r} \mathbf{u}^\alpha.$$

Proof : See [DSMS, Lemma 7.8]. □

We now examine the structure of $R[G]$ as an R-module. For $k \geq 1$, we define

$$T_k = \{\alpha \in \mathbb{N}^d \mid \alpha_i < p^{k-1} \text{ for } i = 1, ..., d\}.$$

Lemma 3.2.8. *Let $k \geq 1$, and recall that we write $\mathbf{b}^\alpha = b_1^{\alpha_1} \cdots b_d^{\alpha_d}$ for each α.*

(i) If G is powerful, then

$$R[G] = I_k + \sum_{\alpha \in T_k} R\mathbf{b}^\alpha.$$

(ii) If G is uniform, then

$$R[G] = I_k \oplus \bigoplus_{\alpha \in T_k} R\mathbf{b}^\alpha.$$

Proof : (i) Note that I_k is the kernel of the canonical map $\phi : R[G] \to R[G/G_k]$. By Theorem 3.1.3, every element of G/G_k can written as $a_1^{\alpha_1} \cdots a_d^{\alpha_d} G_k$ with $\alpha_i < p^{k-1}$. Hence $\{\phi(\mathbf{a}^\alpha) | \ \alpha \in T_k\}$ generates $R[G/G_k]$ as an R-module. By the preceding lemma, this implies that $\{\phi(\mathbf{b}^\alpha) | \ \alpha \in T_k\}$ also generates $R[G/G_k]$ and so $\phi(R[G]) = \phi(\sum_{\alpha \in T_k} R\mathbf{b}^\alpha)$.

(ii) Since G is uniform, we have $|G/G_k| = p^{(k-1)d}$, and so $\phi(R[G])$ is a free R-module of rank $p^{(k-1)d}$. On the other hand, we also have $|T_k| = p^{(k-1)d}$. Therefore the generating set $\{\phi(\mathbf{b}^\alpha) | \ \alpha \in T_k\}$ is actually a free basis for this module. □

We are in the position to give a proof of Theorem 3.2.5(ii).

Proof of Theorem 3.2.5(ii) : It suffices to show that if $\sum_\alpha r_\alpha \mathbf{b}^\alpha = 0$, then $r_\alpha = 0$ for all α. This is reduced to showing that for every j, we have $r_\alpha \in \mathfrak{m}^j$ for all α. We will prove this by induction on j. The case $j = 0$ is immediate by assumption. Now suppose that $j \geq 1$ and $r_\alpha \in \mathfrak{m}^{j-1}$ for all α. We want to show that $r_\alpha \in \mathfrak{m}^j$ for all α. Let k be an arbitrary positive integer and set $m = |G/G_k|$. By the hypothesis that $\sum_\alpha r_\alpha \mathbf{b}^\alpha = 0$, we have $\| \sum_{\alpha \in S} r_\alpha \mathbf{b}^\alpha \| \leq q^{-jm}$ for some finite set $S \supseteq T_k$. Therefore

$$\sum_{\alpha \in T_k} r_\alpha \mathbf{b}^\alpha = \sum_{\alpha \in S} r_\alpha \mathbf{b}^\alpha - \sum_{\alpha \in S \setminus T_k} r_\alpha \mathbf{b}^\alpha.$$

Now if $\alpha \notin T_k$, then $\alpha_i \geq p^{k-1}$ for some i and so $b_i^{\alpha_i} \in I_k + \mathfrak{m}R[G]$. Together with our induction hypothesis, we have $r_\alpha \mathbf{b}^\alpha \in I_k + \mathfrak{m}^j R[G]$ for $\alpha \in S \setminus T_k$. On the other hand, the first series on the right is in $J^{jm} \subseteq I_k + \mathfrak{m}^j R[G]$. Hence we have

$$\sum_{\alpha \in T_k} r_\alpha \mathbf{b}^\alpha \in I_k + \mathfrak{m}^j R[G].$$

This implies that $\sum_{\alpha \in T_k} r_\alpha \mathbf{b}^\alpha$ lies in the kernel of the composition

$$R[G] \xrightarrow{\phi} R[G/G_k] \xrightarrow{\pi} R/\mathfrak{m}^j[G/G_k]$$

of the two canonical quotient maps. As seen in the proof of Lemma 3.2.8(ii), the set $\{\phi(\mathbf{b}^\alpha) |\ \alpha \in T_k\}$ is actually a free basis for $R[G/G_k]$. Therefore, we have that

$$\sum_{\alpha \in T_k} r_\alpha \phi(\mathbf{b}^\alpha)$$

lies in the kernel of π if and only if $r_\alpha \in \mathfrak{m}^j$ for all $\alpha \in T_k$. Since k is arbitrary large, we have $r_\alpha \in \mathfrak{m}^j$ for all α. □

To prove Theorem 3.2.5(i) and Theorem 3.2.6, we need the two following lemmas which give certain ideal relations in a group algebra of a powerful pro-p group. For $k \geq 1$, we define

$$J_{k,1} = \mathfrak{m}J^{k-1} + J^{k+1}.$$

Lemma 3.2.9. *If G is powerful, then $ux - xu \in J_{k+1,1}$ for every $u \in J^k, x \in G$.*

Proof: Let $x, y \in G$. Then

$$yx - xy = ([y,x] - 1)xy = (z^p - 1)xy$$

for some $z \in G$, since G is powerful (by Proposition 3.1.2). By an argument as in part (i) of Lemma 3.2.1, we have $z^p - 1 \in J^p + \mathfrak{m}J$. If $p \geq 3$, this is contained in $J_{2,1}$. If $p = 2$, G/G_3 is abelian and so $[y,x] - 1 \in I_3 \subseteq J^3$ by Lemma 3.2.1. In either case, we have $yx - xy \in J_{2,1}$. Since every element of $R[G]$ is a R-linear combination of elements in G, we have $ux - xu \in J_{2,1}$ for every $u \in R[G], x \in G$. In particular, this implies the case $k = 1$ of the lemma.

Now suppose $k > 1$ and $vy - yv \in J_{k,1}$ for all $v \in J^{k-1}$ and $y \in G$. As J^k is additively spanned by elements of the form vw with $v \in J^{k-1}$ and $w \in J$, it suffices to show that for any such v and w, we have $vwx - xvw \in J_{k+1,1}$. This follows by observing that

$$vwx - xvw = v(wx - xw) + (vx - xv)w \in J^{k-1}J_{2,1} + J_{k,1}J = J_{k+1,1}$$

by the first paragraph and the inductive hypothesis. □

Lemma 3.2.10. *Assume that G is powerful. Let $k \geq 1$. Then*

$$J^k = J^{k+1} + \sum_{\langle\alpha\rangle \leq\ k} \mathfrak{m}^{k-\langle\alpha\rangle}\mathbf{b}^\alpha.$$

Proof: Write $W_k = \sum_{\langle\alpha\rangle\leq\ k} \mathfrak{m}^{k-\langle\alpha\rangle}\mathbf{b}^\alpha$. Since $\mathfrak{m} \subseteq J$ and $b_i \in J$ for each i, it follows that $\mathfrak{m}^{k-\langle\alpha\rangle}\mathbf{b}^\alpha \subseteq J^k$ and so $W_k + J^{k+1} \subseteq J^k$.

The reverse inclusion is proved by induction on k. By Lemma 3.2.8,

$$R[G] = I_2 + \sum_{\alpha\in T_2} R\mathbf{b}^\alpha = J^2 + \sum_{\langle\alpha\rangle=1} R\mathbf{b}^\alpha + R1_G,$$

where the second equality follows from the facts that $I_2 \subseteq J^2$ and $\mathbf{b}^\alpha \in J^2$ when $\langle\alpha\rangle \geq 2$. It follows by Lemma 2.3.5 that

$$J = J^2 + \sum_{\langle\alpha\rangle=1} R\mathbf{b}^\alpha + (J \cap R1_G) = J^2 + W_1,$$

since $J \cap R1_G = \mathfrak{m}1_G = \mathfrak{m}\mathbf{b}^0$. This establishes the case $k = 1$.

Now suppose $k > 1$ and that $J^l = J^{l+1} + W_l$ for all $l < k$. Then

$$J^k = J^{k-1}J = (J^k + W_{k-1})(J^2 + W_1) \subseteq J^{k+1} + W_{k-1}W_1.$$

Since we already have $W_k + J^{k+1} \subseteq J^k$ from the first part, it remains to show that $W_{k-1}W_1 \subseteq W_k + J^{k+1}$. As $W_1 = \mathfrak{m}1_G + \sum_{i=1}^d Rb_i$ and $\mathfrak{m}W_{k-1} \subseteq W_k$, it suffices to show that for every $\lambda \in \mathfrak{m}^{k-1-\langle\alpha\rangle}$, where $i = 1, ..., d$ and $\langle\alpha\rangle \leq k-1$, the element $\lambda\mathbf{b}^\alpha b_i$ lies in $J^{k+1} + W_k$.

Write $\mathbf{u} = b_1^{\alpha_1}\cdots b_{i-1}^{\alpha_{i-1}}$ and $\mathbf{v} = b_i^{\alpha_i}b_{i+1}^{\alpha_{i+1}}\cdots b_d^{\alpha_d}$. Then

$$\begin{aligned}\mathbf{b}^\alpha b_i = \mathbf{u}\mathbf{v}b_i &= \mathbf{u}b_i\mathbf{v} + \mathbf{u}(\mathbf{v}b_i - b_i\mathbf{v}) \\ &= \mathbf{b}^\beta + \mathbf{u}w,\end{aligned}$$

where $w = \mathbf{v}b_i - b_i\mathbf{v}$ and $\beta_i = 1+\alpha_i$, $\beta_j = \alpha_j$ for $j \neq i$. Now $\mathbf{v} \in J^n$ where $n = \alpha_i + \cdots + \alpha_d$. Thus $w \in J_{n+1,1}$ by Lemma 3.2.9. As $\mathbf{u} \in J^{\langle\alpha\rangle-n}$, it follows that

$$\mathbf{u}w \in J^{\langle\alpha\rangle-n}J_{n+1,1} = J_{\langle\alpha\rangle+1,1} = \mathfrak{m}J^{\langle\alpha\rangle} + J^{\langle\alpha\rangle+2}.$$

Thus

$$\lambda\mathbf{b}^\alpha b_i \in \mathfrak{m}^{k-1-\langle\alpha\rangle}\mathbf{b}^\beta + \mathfrak{m}^{k-\langle\alpha\rangle}J^{\langle\alpha\rangle} + \mathfrak{m}^{k-1-\langle\alpha\rangle}J^{\langle\alpha\rangle+2}.$$

Clearly, the last term on the right is contained in J^{k+1}. Since $\langle\beta\rangle = \langle\alpha\rangle + 1$, the first term lies in W_k. Since $\langle\alpha\rangle \leq k-1$, it follows from the induction hypothesis that the middle term lies in

$$\mathfrak{m}^{k-\langle\alpha\rangle} J^{\langle\alpha\rangle+1} + \mathfrak{m}^{k-\langle\alpha\rangle} W_{\langle\alpha\rangle} \subseteq J^{k+1} + W_k.$$

Hence the conclusion follows. □

We now finish up the section with the promised proofs.

Proof of Theorem 3.2.5(i) and Theorem 3.2.6 : Suppose $\|c\| = q^{-k}$. Then $c \in J^k$ and by Lemma 3.2.10, we can write

$$c = \sum_{\langle\alpha\rangle \leq k} s_{\alpha,k} \mathbf{b}^\alpha + c_{k+1}$$

where $s_{\alpha,k} \in \mathfrak{m}^{k-\langle\alpha\rangle}$ and $c_{k+1} \in J^{k+1}$. Repeating this process, we obtain a sequence $(c_j)_{j\geq k}$ such that $c_j \in J^j$ and

$$c_j - c_{j+1} = \sum_{\langle\alpha\rangle \leq j} s_{\alpha,j} \mathbf{b}^\alpha$$

for some $s_{\alpha,j} \in \mathfrak{m}^{j-\langle\alpha\rangle}$. Set $w_j = \sum_{\langle\alpha\rangle\leq j} s_{\alpha,j}\mathbf{b}^\alpha$. Then we have

$$c - (w_k + \cdots + w_n) \in J^{n+1}.$$

In other words,

$$\|c - \sum_{j=k}^{n} w_j\| \leq q^{-n-1}.$$

Hence

$$c = \sum_{j=k}^{\infty} w_j.$$

Now set $T = \{(\alpha, j) |\ j \geq k, \langle\alpha\rangle \leq j\}$. Since $\mathfrak{m}^{j-\langle\alpha\rangle}\mathbf{b}^\alpha \subseteq J^j$, we have $\|s_{\alpha,j}\mathbf{b}^\alpha\| \leq q^{-j}$ for each j. This implies that

$$\lim_{(\alpha,j)\in T} s_{\alpha,j}\mathbf{b}^\alpha = 0.$$

By Proposition 3.1.9(d), the series $\sum_{(\alpha,j)\in T} s_{\alpha,j}\mathbf{b}^\alpha$ converges in the complete ring $\widehat{R[G]}$. Define

$$s_\alpha = \sum_{j=\max\{k,\langle\alpha\rangle\}}^{\infty} s_{\alpha,j}.$$

We then have

$$\sum_{\alpha\in\mathbb{N}^d} s_\alpha \mathbf{b}^\alpha = \sum_{(\alpha,j)\in T} s_{\alpha,j}\mathbf{b}^\alpha = \sum_{j=k}^{\infty} w_j = c,$$

thus proving Theorem 3.2.5(i).

By Theorem 3.2.5(ii), we have

$$r_\alpha = s_\alpha = \sum_{j=\max\{k,\langle\alpha\rangle\}}^{\infty} s_{\alpha,j}$$

for each α. Note that $s_{\alpha,j} \in \mathfrak{m}^{j-\langle\alpha\rangle} \subseteq \mathfrak{m}^{k-\langle\alpha\rangle}$ since $j \geq \max\{k,\langle\alpha\rangle\} \geq k$. Hence

$$\sup_{\alpha\in\mathbb{N}^d} q^{-\langle\alpha\rangle}\|r_\alpha\| \leq q^{-\langle\alpha\rangle}q^{-k+\langle\alpha\rangle} = q^{-k} = \|c\|.$$

Conversely, we have $\|r_\alpha \mathbf{b}^\alpha\| \leq \|r_\alpha\|\|\mathbf{b}^\alpha\| \leq q^{-\langle\alpha\rangle}\|r_\alpha\|$ for each α. Hence

$$\|c\| \leq \sup_\alpha \|r_\alpha \mathbf{b}^\alpha\| \leq \sup_\alpha q^{-\langle\alpha\rangle}\|r_\alpha\|.$$

This proves Theorem 3.2.6. □

3.3 Completed group algebras

In this section, G is taken to be a uniform pro-p group. As before, R is a commutative complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and a finite residue field $\mathbb{F}_q$, where q is a power of p. In particular, we have that R is a compact ring under the $\mathfrak{m}$-adic topology. We shall now prove the following theorem which was stated in the previous section.

Theorem 3.3.1. *Each element of $R[\![G]\!]$ is equal to the sum of a uniquely determined convergent series*

$$\sum_{\alpha\in\mathbb{N}^d} r_\alpha \mathbf{b}^\alpha,$$

where $r_\alpha \in R$ for each $\alpha \in \mathbb{N}^d$. Conversely, every such series converges in $R[\![G]\!]$.

Proof: Convergence follows from the fact that $\|\mathbf{b}_\alpha\| \leq q^{-\langle\alpha\rangle}$ for each $\alpha \in \mathbb{N}^d$. Uniqueness follows from Theorem 3.2.5(ii).

Now let S denote the subset of $R[\![G]\!]$ consisting of all elements that are equal to the sum of a series as in the theorem. Then S contains $R[G]$ by Theorem 3.2.5(i). Thus S is dense in $R[\![G]\!]$. It remains to prove that S is closed. Write $X = R^{\mathbb{N}^d}$ and define a map $\psi : X \to S$ by

$$\psi\left((r_\alpha)_{\alpha\in X}\right) = \sum_{\alpha\in\mathbb{N}^d} r_\alpha \mathbf{b}^\alpha.$$

Clearly ψ is surjective and X is compact. We shall show that ψ is continuous. Let $\lambda = \psi\left((r_\alpha)_{\alpha \in X}\right)$ and $\varepsilon > 0$. Choose n such that $q^{-n} < \varepsilon$, and set

$$U = \{(s_\alpha) \in X | \ s_\alpha - r_\alpha \in \mathfrak{m}^n \text{ for all } \alpha \text{ with } \langle\alpha\rangle < n\}.$$

Then U is an open set of X containing (r_α) and for each $(s_\alpha) \in U$,

$$\|\psi((s_\alpha)) - r\| = \left\| \sum_{\alpha \in \mathbb{N}^d} (s_\alpha - r_\alpha)\mathbf{b}^\alpha \right\| \leq q^{-n} < \varepsilon.$$

Thus we have shown that S is the image of a compact set under a continuous map. Therefore, S is compact, in particular closed. □

We also have the following statement on the norm in this case.

Theorem 3.3.2. *If* $c = \sum_{\alpha \in \mathbb{N}^d} r_\alpha \mathbf{b}^\alpha \in R[\![G]\!]$, *where* $r_\alpha \in R$, *then*

$$\|c\| = \sup_{\alpha \in \mathbb{N}^d} q^{-\langle\alpha\rangle} \|r_\alpha\|.$$

Proof: Suppose $\|c\| = q^{-k}$. Let $r = \sum_{\langle\alpha\rangle \leq k} r_\alpha \mathbf{b}^\alpha$. Then we have $\|c - r\| \leq q^{-k-1}$. Thus

$$\|c\| = \|r\| = \sup_{\langle\alpha\rangle \leq k} q^{-\langle\alpha\rangle} \|r_\alpha\|$$

by Theorem 3.2.6. For $\langle\alpha\rangle > k$, we have

$$q^{-\langle\alpha\rangle} \|r_\alpha\| < q^{-k} = \|c\|.$$

Hence the conclusion follows. □

Define $A_k = \{c \in R[\![G]\!] \mid \|c\| \leq q^{-k}\}$. One can easily check that $A_0 = R[\![G]\!]$, $A_{k+1} \subseteq A_k$ for each k, and A_k is a two sided ideal of $R[\![G]\!]$. By the norm property, we have $A_i A_j \subseteq A_{i+j}$ for each i and j. Now define

$$A_{k,m} = \mathfrak{m}^m A_{k-m} + A_{k+1} \text{ for } k \geq m.$$

Then one can check that

$$A_k = A_{k,0} \supseteq A_{k,1} \supseteq \cdots \supseteq A_{k,k} \supseteq A_{k,k+1} = A_{k+1};$$

$$A_{i,m} A_{j,n} \subseteq A_{i+j,m+n}.$$

Now for $k \geq 0$ and $0 \leq m \leq k$, we set $E_{k,m} = A_{k,m}/A_{k,m+1}$ and define the associated graded ring of $R[\![G]\!]$ to be

$$R[\![G]\!]^* = \bigoplus_{k=0}^{\infty} \bigoplus_{m=0}^{k} E_{k,m}.$$

with multiplication given by

$$(a + A_{i,m+1})(b + A_{j,n+1}) = ab + A_{i+j,m+n+1}$$

for $a \in A_{i,m}, b \in A_{j,n}$.

Theorem 3.3.3. *Let $\{c_1, ...c_l\}$ be a set of generators for $\mathfrak{m}$. Write $t_i = c_i 1_G + A_{1,2} \in E_{1,1}$ for $i = 1, ..., l$ and $x_j = b_j + A_{1,1} \in E_{1,0}$ for $j = 1, ..., d$. Then we have a surjective $\mathbb{F}_q$-algebra homomorphism*

$$\Phi : \mathbb{F}_q[T_1, ..., T_l, X_1, ..., X_d] \longrightarrow R[\![G]\!]^*$$
$$T_i \mapsto t_i; X_j \mapsto x_j.$$

Proof : Clearly $\mathfrak{m}E_{k,m} = 0$ for all k, m and so $R[\![G]\!]^*$ is an algebra over $R/\mathfrak{m} = \mathbb{F}_q$. Clearly the t_i commute among themselves and with the x_j. Now we shall show that the x_j commute among themselves. By Lemma 3.2.9, we have

$$b_i b_j - b_j b_i \in \mathfrak{m}J + J^3 \subseteq \mathfrak{m}A_1 + A_3 = A_{2,1}.$$

This implies that $x_i x_j - x_j x_i = b_i b_j - b_j b_i + A_{2,1} = 0$. Hence, the assignments $T_i \mapsto t_i$ and $X_j \mapsto x_j$ give a well-defined $\mathbb{F}_q$-algebra homomorphism.

To prove the surjectivity of Φ, it suffices to show that the monomials

$$w_{\alpha,\beta} = t_1^{\alpha_1} \cdots t_l^{\alpha_l} x_1^{\beta_1} \cdots x_d^{\beta_d}$$

with $\langle\alpha\rangle = m$ and $\langle\beta\rangle = k - m$ generate $E_{k,m}$ over $\mathbb{F}_q$. For each $n \geq 0$, set $B_n = \sum_{\langle\beta\rangle=n} R\mathbf{b}^\beta$. Then as seen in the proof of Theorem 3.3.2, we have

$$A_k = \sum_{n=0}^{k-n} \mathfrak{m}^{k-n} B_n + A_{k+1}.$$

It follows that

$$A_{k,m} = \mathfrak{m}^m A_{k-m} + A_{k,m+1} = \sum_{\langle\alpha\rangle=m} R\mathbf{c}^\alpha B_{k-m} + A_{k,m+1} = \sum_{\substack{\langle\alpha\rangle=m \\ \langle\beta\rangle=k-m}} R\mathbf{c}^\alpha \mathbf{b}^\beta + A_{k,m+1}.$$

This implies that

$$E_{k,m} = A_{k,m}/A_{k,m+1} = \sum_{\substack{\langle\alpha\rangle=m \\ \langle\beta\rangle=k-m}} \mathbb{F}_q w_{\alpha,\beta}. \ \square$$

Corollary 3.3.4. *The ring $R[\![G]\!]$ is left and right Noetherian.*

Proof: Set A'_n to be the $(n+1)$th term in the sequence

$$R[\![G]\!] = A_{0,0} \supseteq A_{1,0} \supseteq \cdots \supseteq A_{k,0} \supseteq A_{k,1} \supseteq \cdots \supseteq A_{k,k} \supseteq A_{k+1,0} \supseteq \cdots.$$

Thus, $A'_n = A_{k,m}$, where

$$n = n(k,m) = \frac{1}{2}k(k+1) + m.$$

Note that for a given n, the above equation determines k and m under the constraint $0 \leq m \leq k$. For $n = n(k,m)$ and $n' = n(k',m')$, we define

$$n * n' = n(k+k', m+m').$$

It is straightforward to verify that the descending chain of ideals $\{A'_n\}$ satisfies the conditions of Proposition 3.1.10. Since $\{A'_n\}$ is a refinement of $\{A_n\}$, the norms defined by $\{A'_n\}$ and $\{A_n\}$ are equivalent. Hence $R[\![G]\!]$ is also complete with respect to the norm defined by $\{A'_n\}$, and we can apply Proposition 3.1.10 to obtain the required conclusion. $\square$

In Theorem 3.3.3, we prove that the map Φ is surjective for a general R. In the case when $R = \mathbb{Z}_p$, it is proven in [DSMS, Thm. 7.22] that we can choose generators to make Φ an isomorphism. We shall show that this can be achieved if we impose an extra condition on R and make a careful choice of generators for $\mathfrak{m}$.

Theorem 3.3.5. *Let R be a commutative Noetherian complete local ring with maximal ideal $\mathfrak{m}$ and finite residue field of characteristic p. We also assume that $\bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an integral domain. Choose a set of generators $\{c_1, ...c_l\}$ for $\mathfrak{m}$ such that $c_i \in \mathfrak{m} \setminus \mathfrak{m}^2$ and the images of the c_i in $\mathfrak{m}/\mathfrak{m}^2$ form an $\mathbb{F}_q$-basis for $\mathfrak{m}/\mathfrak{m}^2$. As before, let G be a uniform pro-p group. Write $t_i = c_i 1_G + A_{1,2} \in E_{1,1}$ for $i = 1, ..., l$ and $x_j = b_j + A_{1,1} \in E_{1,0}$ for $j = 1, ..., d$. Then we have an $\mathbb{F}_q$-algebra isomorphism given by*

$$\Phi : \mathbb{F}_q[T_1, ..., T_l, X_1, ..., X_d] \longrightarrow R[\![G]\!]^*$$
$$T_i \mapsto t_i; X_j \mapsto x_j.$$

Proof: By the same argument as in Theorem 3.3.3, the above assignment is a well-defined surjective $\mathbb{F}_q$-algebra homomorphism. It remains to show that the map is injective. To show this, it suffices to show that the monomials

$$w_{\alpha,\beta} = t_1^{\alpha_1} \cdots t_l^{\alpha_l} x_1^{\beta_1} \cdots x_d^{\beta_d}$$

with $\langle\alpha\rangle = m$ and $\langle\beta\rangle = k - m$ are linearly independent over $\mathbb{F}_q$. This is equivalent to showing that if we have $r_{\alpha,\beta} \in R$ such that

$$\sum_{\substack{\langle\alpha\rangle=m \\ \langle\beta\rangle=k-m}} r_{\alpha,\beta}\mathbf{c}^{\langle\alpha\rangle}\mathbf{b}^{\langle\beta\rangle} \in A_{k,m+1},$$

then $r_{\alpha,\beta} \in \mathfrak{m}$. Since $A_{k,m+1} = \mathfrak{m}^{m+1}A_{k-m-1} + A_{k+1} \subseteq \mathfrak{m}^{m+1}R[\![G]\!] + A_{k+1}$, we have

$$\sum_{\substack{\langle\alpha\rangle=m \\ \langle\beta\rangle=k-m}} r_{\alpha,\beta}\mathbf{c}^{\langle\alpha\rangle}\mathbf{b}^{\langle\beta\rangle} = \sum_{\beta} u_\beta \mathbf{b}^{\langle\beta\rangle} + \sum_{\beta} v_\beta \mathbf{b}^{\langle\beta\rangle}$$

where $u_\beta \in \mathfrak{m}^{m+1}$ and

$$\|\sum_{\beta} v_\beta \mathbf{b}^{\langle\beta\rangle}\| \leq q^{-(k+1)}.$$

It follows from Theorem 3.3.2 that we have $\|v_\beta\| \leq q^{-(m+1)}$ for $\langle\beta\rangle = k - m$. This implies that $u_\beta + v_\beta \in \mathfrak{m}^{m+1}$. On the other hand, it follows from Theorem 3.3.1 that we have $r_{\alpha,\beta}\mathbf{c}^{\langle\alpha\rangle} = u_\beta + v_\beta$. This implies that $r_{\alpha,\beta}\mathbf{c}^{\langle\alpha\rangle} \in \mathfrak{m}^{m+1}$. Since the graded ring $\bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an integral domain and each $c_i + \mathfrak{m}^2$ is a nonzero element in the graded ring, it follows that $\mathbf{c}^{\langle\alpha\rangle} + \mathfrak{m}^{m+1}$ is nonzero in the graded ring. By what we have shown, we have

$$(r_{\alpha,\beta} + \mathfrak{m})(\mathbf{c}^{\langle\alpha\rangle} + \mathfrak{m}^{m+1}) = 0.$$

Since $\mathbf{c}^{\langle\alpha\rangle} + \mathfrak{m}^{m+1}$ is nonzero, it follows that $r_{\alpha,\beta} + \mathfrak{m}$ is zero. This implies that $r_{\alpha,\beta} \in \mathfrak{m}$, as required. □

Corollary 3.3.6. *Let R be a commutative Noetherian complete local ring with finite residue field of characteristic p, and assume that the graded ring $\bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is an integral domain. Then for any uniform pro-p group G, the ring $R[\![G]\!]$ has no zero divisors.*

Proof :As seen in Corollary 3.3.4, the ring $R[\![G]\!]$ is complete with respect to the norm induced by the chain of ideals $\{A'_n\}$. Therefore, we may apply Proposition 3.1.10 and Theorem 3.3.5 to obtain the required conclusion. □

3.4 Compact p-adic Lie groups

We shall see that most of the results obtained in the previous section can be carried over to the case of compact p-adic Lie groups. We now recall the following characterization of compact p-adic Lie groups due to Lazard [Laz]. A topological group G is a *compact p-adic Lie group* if and only if G contains a normal open uniform pro-p subgroup of finite index (see [DSMS, Cor. 8.34]). We will use this characterization and refer readers to [DSMS, Def. 8.14] for the definition of a p-adic Lie group. As a start, we shall use this to deduce the main result of this chapter.

Theorem 3.4.1. *Let R be a commutative complete noetherian local ring with finite residue field of characteristic p, and let G be a compact p-adic Lie group. Then $R[\![G]\!]$ is a Noetherian ring.*

Proof : Let U be an open normal uniform pro-p subgroup of G. By Corollary 3.3.4, we have that $R[\![U]\!]$ is left and right Noetherian. Since U is open in G, it is a subgroup with finite index. Therefore, $R[\![G]\!]$ is a finitely generated $R[\![U]\!]$-algebra, and so is also left and right Noetherian. □

The next result is an extension of Corollary 3.3.6.

Proposition 3.4.2. *Suppose R is a commutative complete regular local ring of characteristic $\neq p$ with finite residue field of characteristic p and G is a torsion-free pro-p p-adic Lie group. Then $R[\![G]\!]$ has no zero divisors.*

Proof : The proof follows the ideas in [N], which basically reduces to checking the hypothesis of a theorem of Walker (see loc. cit.). □

Chapter 4

Iwasawa modules

In this chapter, we will introduce certain modules over an Iwasawa algebra. The next paragraph will introduce some notations which will be adhered to throughout this chapter.

Fix a prime p. Let R be a commutative complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, where k is finite of characteristic p. There exists an object $\omega_R \in \mathbf{D}^b_{R-ft}(Mod_R)$ (see [Hart, Ch. V]) with the property that for every $M \in \mathbf{D}(Mod_R^{R-ft})$, we have $\mathbf{R}\mathrm{Hom}_R(M, \omega_R) \in \mathbf{D}_{R-ft}(Mod_R)$, and the canonical morphism

$$M \longrightarrow \mathbf{R}\mathrm{Hom}_R\big(\mathbf{R}\mathrm{Hom}_R(M, \omega_R), \omega_R\big)$$

is an isomorphism in $\mathbf{D}(Mod_R)$. One refers to ω_R as the dualizing complex of R.

Let G and Γ be two profinite groups such that there is a continuous homomorphism $\pi : G \longrightarrow \Gamma$ of profinite groups. Set $\Lambda = R[\![\Gamma]\!]$. We have a map $\iota : \Lambda \to \Lambda$ which is defined by sending γ to γ^{-1} for $\gamma \in \Gamma$. Note that this is only a homomorphism of R-modules. It is a ring homomorphism if and only if Γ is abelian. Denote by

$$\rho = \rho_\Gamma : G \xrightarrow{\pi} \Gamma \subseteq \Lambda^\times$$

the tautological one-dimensional representation of G over Λ.

4.1 Induced modules

We begin with a lemma which tells us that Λ is a central flat R-algebra. Let $\mathscr{U}$ be the collection of open normal subgroups of Γ.

Lemma 4.1.1. *The ring Λ is a central flat R-algebra.*

Proof: By [Wei, Prop. 3.2.4(4)], it suffices to show that for any ideal A of R, the map $\Lambda \otimes_R A \longrightarrow \Lambda$ induced by the inclusion $A \hookrightarrow R$ is injective. For each $U \in \mathscr{U}$, the induced map

$$R[\Gamma/U] \otimes_R A \longrightarrow R[\Gamma/U]$$

is injective, since $R[\Gamma/U]$ is a free R-module. Since the inverse limit is left exact, the following map

$$\varprojlim_U (R[\Gamma/U] \otimes_R A) \longrightarrow \Lambda$$

is injective. Since R is Noetherian, we may view $R[\Gamma/U]$ and A as compact R-modules by Proposition 2.3.1. Therefore, we have

$$\varprojlim_U (R[\Gamma/U] \otimes_R A) = \varprojlim_U (R[\Gamma/U] \hat{\otimes}_R A) \cong \Lambda \hat{\otimes}_R A = \Lambda \otimes_R A,$$

and the conclusion follows. □

Now if M is a Λ-module, we define a Λ^o-module M^ι by the formula $m \cdot_\iota \lambda := \iota(\lambda)m$ for $\lambda \in \Lambda, m \in M$. Similarly, if N is a Λ^o-module, we define a Λ-module, which is also denoted as N^ι, by $\lambda \cdot_\iota m := m\iota(\lambda)$.

We have the following lemma.

Lemma 4.1.2. *(a) If M is a $\Lambda[G]$-module, then M^ι is a $\Lambda^o[G]$-module.*

(b) If M is a $\Lambda[G]$-Λ-module (not necessarily balanced), then M^ι is a $\Lambda^o[G]$-Λ-module (not necessarily balanced).

Proof: (a) Let $g \in G, \lambda \in \Lambda$ and $m \in M^\iota$. Then we have

$$(gm) \cdot_\iota \lambda = \iota(\lambda)gm = g(\iota(\lambda)m) = g(m \cdot_\iota \lambda).$$

(b) Similar argument as above. □

For a given $U \in \mathscr{U}$ and a given $R[G]$-module M, we define two $\Lambda[G]$-Λ-modules as follows:

$$\begin{aligned} {}_U M &= \mathrm{Hom}_R(R[\Gamma/U], M) \\ M_U &= R[\Gamma/U]^\iota \otimes_R M, \end{aligned}$$

where G acts on $R[\Gamma/U]$ via $\rho_{\Gamma/U}$ and Λ acts on $R[\Gamma/U]$ via the canonical projection $\Lambda \twoheadrightarrow R[\Gamma/U]$. Note that the $\Lambda[G]$-Λ-modules defined above are balanced as Λ-Λ-modules. They are balanced as $\Lambda[G]$-Λ-modules if Γ/U is abelian.

Let $V \in \mathscr{U}$ with $U \subseteq V$. Then there is a canonical surjection $\mathrm{pr} : R[\Gamma/U] \twoheadrightarrow R[\Gamma/V]$ and a map $\mathrm{Tr} : R[\Gamma/V] \longrightarrow R[\Gamma/U]$ given by

$$gU \mapsto \sum_{v \in V/U} gvU.$$

These in turn induce the following maps.

$$\begin{aligned} \mathrm{pr}^* &: {}_VM \longrightarrow {}_UM \\ \mathrm{pr}_* &: M_U \longrightarrow M_V \\ \mathrm{Tr}^* &: {}_UM \longrightarrow {}_VM \\ \mathrm{Tr}_* &: M_V \longrightarrow M_U \end{aligned}$$

Denote by $\delta_\beta : G/U \to \mathbb{Z}$ the Kronecker delta-function

$$\delta_\beta(\beta') \cong \begin{cases} 1 & \text{if } \beta = \beta', \\ 0 & \text{if } \beta \neq \beta'. \end{cases}$$

Lemma 4.1.3. *We have the following isomorphism of $R[G]$-modules*

$$\begin{gathered} M_U \xrightarrow{\sim} {}_UM \\ \sum_{\beta \in G/U} \beta \otimes x_\beta \mapsto \sum_{\beta \in G/U} x_\beta \delta_\beta \end{gathered}$$

which is functorial in M. Moreover, if V is another open normal subgroup of G such that $U \subseteq V$, then the isomorphism fits into the following commutative diagrams.

$$\begin{array}{ccc} M_U & \xrightarrow{\sim} & {}_UM \\ \downarrow{\scriptstyle \mathrm{pr}_*} & & \downarrow{\scriptstyle \mathrm{Tr}^*} \\ M_V & \xrightarrow{\sim} & {}_VM \end{array} \qquad \begin{array}{ccc} M_V & \xrightarrow{\sim} & {}_VM \\ \downarrow{\scriptstyle \mathrm{Tr}_*} & & \downarrow{\scriptstyle \mathrm{pr}^*} \\ M_U & \xrightarrow{\sim} & {}_UM \end{array}$$

Proof : This follows from a straightforward calculation. □

Lemma 4.1.4. *We have the following equalities of $\Lambda^o[G]$-modules:*

$$\begin{aligned} ({}_UM)^\iota &= \mathrm{Hom}_R(R[\Gamma/U]^\iota, M), \\ (M_U)^\iota &= R[\Gamma/U] \otimes_R M. \end{aligned}$$

Proof : This is straightforward, noting Lemma 4.1.3. □

Let M be a $R[G]$-module. We define two $\Lambda[G]$-Λ-modules as follows:

$$F_\Gamma(M) = \varinjlim_{U\in\mathscr{U}} {}_U M,$$
$$\mathscr{F}_\Gamma(M) = \varprojlim_{U\in\mathscr{U}} M_U,$$

where the transition maps are induced by the surjections $R[\Gamma/U] \twoheadrightarrow R[\Gamma/V]$ for $U \subseteq V$. Note that the $\Lambda[G]$-Λ-modules defined above are balanced as Λ-Λ-modules. They are balanced as $\Lambda[G]$-Λ-modules if and only if Γ is abelian. One easily sees from Lemma 4.1.3 that

$$F_\Gamma(M)^\iota = \varinjlim_{U\in\mathscr{U}} \mathrm{Hom}_R(R[\Gamma/U]^\iota, M) \text{ and}$$
$$\mathscr{F}_\Gamma(M)^\iota = \varprojlim_{U\in\mathscr{U}} (R[\Gamma/U] \otimes_R M).$$

We now describe another topology on Λ (see [NSW, Chap. V, §2]). Consider the following family of two-sided ideals :

$$\mathfrak{m}^n\Lambda + I(U),\ n \geq 0,\ U \in \mathscr{U}.$$

Here $I(U)$ denotes the kernel of the map $\Lambda \twoheadrightarrow R[\Gamma/U]$. By taking these ideals as a fundamental system of neighborhoods of 0, we call this topology the $(\mathfrak{m}, I)$-topology.

Lemma 4.1.5. *Suppose Γ is a finitely generated profinite group that contains a pro-p subgroup of finite index. Then the $(\mathfrak{m}, I)$-topology concides with the $\mathfrak{M}$-adic topology, and so Λ is an adic ring. Moreover, if M is a finitely presented Λ-module, we can endow M with the $(\mathfrak{m}, I)$-topology by taking the collection $\{\mathfrak{m}^n M + I(U)M\}$ of Λ-submodules as a fundamental system of neighborhoods of 0, and this coincides with the $\mathfrak{M}$-adic topology.*

Proof: This follows from [NSW, Prop. 5.2.16]. □

From now on, Γ will always be a finitely generated profinite group containing a pro-p subgroup of finite index. The next few results will tell us more about these modules under this assumption.

Lemma 4.1.6. *If T is an object of $C_{R,G}$, then $\mathscr{F}_\Gamma(T)$ is isomorphic to $\Lambda^\iota\hat{\otimes}_R T$ and $\mathscr{F}_\Gamma(T)^\iota$ is isomorphic to $\Lambda\hat{\otimes}_R T$. Moreover, if T is a Noetherian R-module, then $\mathscr{F}_\Gamma(T)$ is finitely presented as a left Λ-module (and as a right Λ-module).*

Proof: By the preceding lemma, we have an isomorphism

$$\varprojlim_i \Lambda/\mathfrak{M}^i \cong \varprojlim_{U,n} R/\mathfrak{m}^n[\Gamma/U] \cong \varprojlim_U R[\Gamma/U]$$

of compact rings. Therefore, the results in Section 2.5 will yield

$$\mathscr{F}_\Gamma(T) = \varprojlim_U (R[\Gamma/U]^\iota \otimes_R T) = \varprojlim_U \left(R[\Gamma/U]^\iota \hat{\otimes}_R T\right) \cong \left(\varprojlim_U R[\Gamma/U]^\iota\right)\hat{\otimes}_R T \cong \Lambda^\iota \hat{\otimes}_R T. \ \square$$

Lemma 4.1.7. *If A is an object of $\mathcal{D}_{R,G}$, then $F_\Gamma(A)$ is an object of $\mathcal{D}_{\Lambda,G}$ and*

$$F_\Gamma(A) \cong \varinjlim_{U,n} \mathrm{Hom}_R(R/\mathfrak{m}^n[\Gamma/U], A).$$

Similarly, we have

$$F_\Gamma(A)^\iota \cong \varinjlim_{U,n} \mathrm{Hom}_R(R/\mathfrak{m}^n[\Gamma/U]^\iota, A),$$

a direct limit of finite $\Lambda^o[G]$-modules.

Proof : Let $U \in \mathscr{U}$, and let $\mathfrak{M}_U$ denote the Jacobson radical of $R[\Gamma/U]$. It is clear that $R[\Gamma/U]$ is a compact R-module with the $\mathfrak{M}_U$-adic topology. Since $R[\Gamma/U]$ is finitely geenrated over R, it follows from Proposition 2.2.9 that the $\mathfrak{m}$-adic topology coincides with the $\mathfrak{M}_U$-adic topology. Therefore, we may apply Lemma 2.2.5(4) to conclude that

$$\mathrm{Hom}_R(R[\Gamma/U], A) = \mathrm{Hom}_{R,\mathrm{cts}}(R[\Gamma/U], A) \cong \varinjlim_n \mathrm{Hom}_R(R/\mathfrak{m}^n[\Gamma/U], A).$$

Since A has the discrete topology, it follows that $\mathrm{Hom}_R(R[\Gamma/U], A)$ is a discrete $R[\Gamma/U]$-module under the compact-open topology and hence a discrete Λ-module via the continuous surjection $\Lambda \twoheadrightarrow R[\Gamma/U]$. $\square$

Denote the category of $R[G]$-modules which are Noetherian as R-modules by $Mod_{R[G]}^{R-ft}$. This is an abelian subcategory of Mod_R.

Proposition 4.1.8. *(a) $\Lambda \otimes_R -$ is an exact functor from Mod_R to $Mod_{\Lambda\otimes_R\Lambda^o}$.*

(b) $\mathscr{F}_\Gamma(-)$ is an exact functor from $Mod_{R[G]}^{R-ft}$ to $Mod_{\Lambda[G]}$. In particular, if Λ is Noetherian, then $\mathscr{F}_\Gamma(-)$ is an exact functor from $Mod_{R[G]}^{R-ft}$ to $Mod_{\Lambda[G]}^{\Lambda-ft}$.

(c) $F_\Gamma(-)$ is an exact functor from $Mod_{R[G]}$ to $Mod_{\Lambda[G]}$.

Proof: Assertions (a) and (b) follow from Lemma 4.1.6 and Lemma 4.1.1. Assertion (c) follows from the definition and the facts that $R[\Gamma/U]$ is a free R-module and the direct limit is exact. $\square$

For the remainder of this section, we will try to establish duality relations between the modules defined above. But before we can say something about dualities, we need to have objects serving the roles of the dualizing module and complex as in the case of R. We shall first consider the dualizing module. Motivated by the case of R, we make the following definition.

Definition 4.1.9. We set $I_\Lambda = \mathrm{Hom}_{\mathrm{cts}}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$, where the actions of G on Λ and $\mathbb{Q}_p/\mathbb{Z}_p$ are trivial.

The next result relates I_Λ with I_R, where I_R denotes the Pontryagin dual of R.

Lemma 4.1.10. *We have an isomorphism* $I_\Lambda \cong \varinjlim_{U \in \mathscr{U}} \mathrm{Hom}_R(R[\Gamma/U], I_R)$ *of* Λ-Λ-*bimodules, where* G *acts on* $R[\Gamma/U]$ *and* I_R *trivially.*

Proof: Recall that the $\mathfrak{M}$-adic topology on Λ is equivalent to the $(\mathfrak{m}, I)$-topology. Therefore, we have

$$\begin{aligned} I_\Lambda &\cong \varinjlim_{U,n} \mathrm{Hom}_{\mathbb{Z}_p}(R/\mathfrak{m}^n R[\Gamma/U], \mathbb{Q}_p/\mathbb{Z}_p) \\ &\cong \varinjlim_{U,n} \mathrm{Hom}_R\left(R[\Gamma/U], \mathrm{Hom}_{\mathbb{Z}_p}(R/\mathfrak{m}^n R, \mathbb{Q}_p/\mathbb{Z}_p)\right) \\ &\cong \varinjlim_{U} \mathrm{Hom}_R\left(R[\Gamma/U], \varinjlim_{n} \mathrm{Hom}_{\mathbb{Z}_p}(R/\mathfrak{m}^n R, \mathbb{Q}_p/\mathbb{Z}_p)\right) \\ &\cong \varinjlim_{U} \mathrm{Hom}_R(R[\Gamma/U], I_R). \ \square \end{aligned}$$

Before proving the next proposition, we introduce the following notations:

$$\begin{aligned} D_R(-) &:= \mathrm{Hom}_R(-, I_R), \\ D_\Lambda(-) &:= \mathrm{Hom}_\Lambda(-, I_\Lambda), \\ D^o_\Lambda(-) &:= \mathrm{Hom}_{\Lambda^o}(-, I_\Lambda). \end{aligned}$$

Recall that for $T \in \mathcal{C}_{R,G}^{R-ft}$, we have $D_R(T) \cong T^\vee$ by Proposition 2.1.1.

Proposition 4.1.11. *Let* T *be an object of* $\mathcal{C}_{R,G}^{R-ft}$. *Then we have continuous isomorphisms*

$$D_\Lambda(\mathscr{F}_\Gamma(T)) \cong \mathscr{F}_\Gamma(T)^\vee \cong F_\Gamma(D_R(T))^\iota$$

in $\mathcal{D}_{\Lambda^o,G}$. *Similarly, we have continuous isomorphisms*

$$D^o_\Lambda(\mathscr{F}_\Gamma(T)^\iota) \cong (\mathscr{F}_\Gamma(T)^\iota)^\vee \cong F_\Gamma(D_R(T))$$

in $\mathcal{D}_{\Lambda,G}$.

Proof: The first isomorphism follows from Proposition 2.1.1(b) and Lemma 4.1.6. Since the $(\mathfrak{m}, I)$-topology and $\mathfrak{M}$-adic topology on $\mathscr{F}_\Gamma(T)$ coincide by Lemma 4.1.5, we have a topological isomorphism $\mathscr{F}_\Gamma(T) \cong \varprojlim_{U,n}(R[\Gamma/U]^\iota \otimes_R T/\mathfrak{m}^n T)$ of (compact) Λ-modules. The second isomorphism now follows by the following calculations:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{cts}}(\mathscr{F}_\Gamma(T), \mathbb{Q}_p/\mathbb{Z}_p) &\cong \varinjlim_{U,n} \mathrm{Hom}_{\mathbb{Z}_p}(R[\Gamma/U]^\iota \otimes_R T/\mathfrak{m}^n T, \mathbb{Q}_p/\mathbb{Z}_p) \\
&\cong \varinjlim_{U,n} \mathrm{Hom}_R\left(R[\Gamma/U]^\iota, \mathrm{Hom}_{\mathbb{Z}_p}(T/\mathfrak{m}^n T, \mathbb{Q}_p/\mathbb{Z}_p)\right) \\
&\cong \varinjlim_{U} \mathrm{Hom}_R(R[\Gamma/U]^\iota, T^\vee) \\
&\cong \varinjlim_{U} \mathrm{Hom}_R\left(R[\Gamma/U]^\iota, D_R(T)\right) = F_\Gamma(D_R(T))^\iota. \ \square
\end{aligned}$$

Corollary 4.1.12. *Let T be an object of $\mathcal{C}_{R,G}^{R-ft}$. Then $D_\Lambda(F_\Gamma(D_R(T))) \cong \mathscr{F}_\Gamma(T)^\iota$ as $\Lambda^o[G]$-modules (resp. $D_{\Lambda^o}(F_\Gamma(D_R(T))^\iota) \cong \mathscr{F}_\Gamma(T)$ as $\Lambda[G]$-modules).*

Proof: Applying D_Λ to the second composite isomorphism in Proposition 4.1.11, we see that the conclusion follows from Corollary 2.1.5. □

In the case that Γ is a finitely generated abelian pro-p group, the ring Λ is a commutative complete Noetherian local ring, and its dualizing complex ω_Λ is shown in [Ne, Lemma 8.4.5.6] to be quasi-isomorphic to $\Lambda \otimes_R^{\mathbf{L}} \omega_R$. Inspired by this result, we shall work with the complex $\Lambda \otimes_R^{\mathbf{L}} \omega_R$. Note that this is an object in $\mathbf{D}^b(Mod_{\Lambda \otimes_R \Lambda^o})$.

Lemma 4.1.13. *Let M and N be objects in $\mathcal{C}_{R,G}^{R-ft}$. Then the following map (defined via Lemma 4.1.6)*

$$\begin{gathered}
\phi : \mathscr{F}_\Gamma(M) \otimes_R \mathscr{F}_\Gamma(N)^\iota \longrightarrow \Lambda \otimes_R (M \otimes_R N) \\
\lambda \otimes m \otimes \mu \otimes n \mapsto \iota(\lambda)\mu \otimes m \otimes n
\end{gathered}$$

is a homomorphism of $\Lambda[G]$-Λ-bimodules, where G acts trivially on Λ.

Proof: Let $\lambda, \mu, \gamma \in \Lambda, m \in M, g \in G$ and $n \in N$. We shall check that ϕ preserves the $\Lambda[G]$-Λ-actions:

$$\begin{aligned}
\phi(\gamma \cdot (\lambda \otimes m \otimes \mu \otimes n)) &= \phi(\lambda\iota(\gamma) \otimes m \otimes \mu \otimes n) = \gamma\iota(\lambda)\mu \otimes m \otimes n \\
&= \gamma(\iota(\lambda)\mu \otimes m \otimes n) = \gamma\phi(\lambda \otimes m \otimes \mu \otimes n), \\
\phi((\lambda \otimes m \otimes \mu \otimes n) \cdot \gamma) &= \phi(\lambda \otimes m \otimes \mu\gamma \otimes n) = \iota(\lambda)\mu\gamma \otimes m \otimes n \\
&= (\iota(\lambda)\mu \otimes m \otimes n)\gamma = \phi(\lambda \otimes m \otimes \mu \otimes n)\gamma, \\
\phi(g(\lambda \otimes m \otimes \mu \otimes n)) &= \phi(\rho(g)\lambda \otimes gm \otimes \rho(g)\mu \otimes gn) = \iota(\lambda)\mu \otimes gm \otimes gn \\
&= g \cdot \phi(\lambda \otimes m \otimes \mu \otimes n). \ \square
\end{aligned}$$

Corollary 4.1.14. *Let M and N be bounded complexes of objects in $\mathcal{C}_{R,G}^{R-ft}$. Then the following map (defined via Lemma 4.1.6)*

$$\begin{aligned} \phi : \mathscr{F}_\Gamma(M) \otimes_R \mathscr{F}_\Gamma(N)^\iota &\longrightarrow \Lambda \otimes_R (M \otimes_R N) \\ \lambda \otimes m \otimes \mu \otimes n &\mapsto \iota(\lambda)\mu \otimes m \otimes n \end{aligned}$$

is a morphism of chain complexes, where G acts trivially on Λ.

Proof: This follows from a direct verification. □

Now let X and Y be bounded complexes of $R[G]$-modules which are finitely generated R-modules, and let J be a bounded complex of $R[G]$-modules with trivial G-action. Suppose that $\pi : X \otimes_R Y \longrightarrow J$ is a morphism of complexes of $R[G]$-modules. Then we have a morphism of complexes of $\Lambda[G]$-Λ-bimodules

$$\overline{\pi} : \mathscr{F}_\Gamma(X) \otimes_R \mathscr{F}_\Gamma(Y)^\iota \xrightarrow{\phi} \Lambda \otimes_R (X \otimes_R Y) \xrightarrow{\mathrm{id}\otimes\pi} \Lambda \otimes_R J,$$

which induces the following morphisms of complexes of $\Lambda^o[G]$-modules and $\Lambda[G]$-modules respectively:

$$\begin{aligned} \mathrm{adj}(\overline{\pi}) : \mathscr{F}_\Gamma(Y)^\iota &\longrightarrow \mathrm{Hom}_\Lambda(\mathscr{F}_\Gamma(X), \Lambda \otimes_R J) \\ \mathrm{adj}'(\overline{\pi}) : \mathscr{F}_\Gamma(X) &\longrightarrow \mathrm{Hom}_{\Lambda^o}(\mathscr{F}_\Gamma(Y)^\iota, \Lambda \otimes_R J) \end{aligned}$$

On the other hand, as complexes of Λ^o-modules, we also have the following commutative diagram

$$\begin{array}{ccccc} \mathscr{F}_\Gamma(Y)^\iota & = & \Lambda \otimes_R Y & \xrightarrow{\mathrm{adj}(\overline{\pi})} & \mathrm{Hom}_\Lambda(\Lambda^\iota \otimes_R X, \Lambda \otimes_R J) \\ & & \Big\downarrow {\scriptstyle \mathrm{id}\otimes\mathrm{adj}(\pi)} & & \Big\downarrow {\scriptstyle \iota^*} \\ & & \Lambda \otimes_R \mathrm{Hom}_R(X, J) & \xrightarrow{\theta} & \mathrm{Hom}_\Lambda(\Lambda \otimes_R X, \Lambda \otimes_R J) \end{array}$$

where θ is the morphism defined in Lemma 1.3.11, and this morphism is an isomorphism of complexes of Λ^o-modules since X is a bounded above complex of Noetherian R-modules and Λ is a flat R-algebra. Hence it follows from the above diagram that if $\mathrm{adj}(\pi)$ is a quasi-isomorphism, so is $\mathrm{adj}(\overline{\pi})$. By a similar argument, we have that if $\mathrm{adj}(\pi)$ is a quasi-isomorphism, so is $\mathrm{adj}'(\overline{\pi})$.

We now prove the following result. The point of the result is that even though we do not know the existence of a dualizing complex for Λ in general, the complex $\Lambda \otimes_R^{\mathbf{L}} \omega_R$ still serve as a "dualizing complex" for the type of induced modules we are interested in.

Proposition 4.1.15. *Let T and T^* be two bounded complexes of ind-admissible $R[G]$-modules of finite type over R, and let J be a bounded complex of $R[G]$-modules with trivial G-action. Suppose that there is a morphism $\pi : T \otimes_R T^* \longrightarrow J$ of complexes of $R[G]$-modules such that* $\mathrm{adj}(\pi)$ *induces an isomorphism*

$$T^* \longrightarrow \mathbf{R}\mathrm{Hom}_R(T, J)$$

in $\mathbf{D}(Mod_{R[G]})$. Then we have an isomorphism

$$\mathrm{adj}(\overline{\pi}) : \mathscr{F}_\Gamma(T^*)^\iota \longrightarrow \mathbf{R}\mathrm{Hom}_\Lambda(\mathscr{F}_\Gamma(T), \Lambda \otimes_R^{\mathbf{L}} J)$$

in $\mathbf{D}(Mod_{\Lambda^o[G]})$. We have a similar statement for $\mathrm{adj}'(\pi)$.

Proof: To show this isomorphism, we may disregard the G-action. Let P be a bounded above complex of finitely generated projective R-modules such that $P \xrightarrow{\sim} T$ is a quasi-isomorphism. Since Λ is flat over R, the chain map $\mathscr{F}_\Gamma(P) \longrightarrow \mathscr{F}_\Gamma(T)$ is also a quasi-isomorphism. Consider the map

$$\pi' : P \otimes_R T^* \longrightarrow T \otimes_R T^* \xrightarrow{\pi} J.$$

Then by a similar argument as above, we obtain the following commutative diagram

$$\begin{array}{ccccc} \mathscr{F}_\Gamma(T^*)^\iota & = & \Lambda \otimes_R T^* & \xrightarrow{\mathrm{adj}(\overline{\pi'})} & \mathrm{Hom}_\Lambda(\Lambda^\iota \otimes_R P, \Lambda \otimes_R J) \\ & & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{adj}(\pi')} & & \downarrow{\scriptstyle \iota^*} \\ & & \Lambda \otimes_R \mathrm{Hom}_R(P, J) & \xrightarrow{\theta} & \mathrm{Hom}_\Lambda(\Lambda \otimes_R P, \Lambda \otimes_R J) \end{array}$$

of complexes of Λ^o-modules. Now $\mathrm{adj}(\pi')$ is a quasi-isomorphism by assumption. Hence it follows that $\mathrm{adj}(\overline{\pi'})$ is a quasi-isomorphism. Now since $\Lambda^\iota \otimes_R P$ is a bounded complex of projective Λ-modules, we have $\mathrm{Hom}_\Lambda(\Lambda^\iota \otimes_R P, \Lambda \otimes_R J) = \mathbf{R}\mathrm{Hom}_\Lambda(\mathscr{F}_\Gamma(T), \Lambda \otimes_R^{\mathbf{L}} J)$ and hence the conclusion follows. □

Proposition 4.1.16. *Suppose T is a bounded complex of $R[G]$-modules which are free of finite rank over R. Then we have isomorphisms*

$$\mathscr{F}_\Gamma(T) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^o}(\mathscr{F}_\Gamma(\mathrm{Hom}_R(T, R))^\iota, \Lambda)$$

and

$$\mathscr{F}_\Gamma(\mathrm{Hom}_R(T, R))^\iota \longrightarrow \mathbf{R}\mathrm{Hom}_\Lambda(\mathscr{F}_\Gamma(T), \Lambda)$$

in $\mathbf{D}(\Lambda[G])$ and $\mathbf{D}(\Lambda^o[G])$ respectively.

Proof : The pairing

$$T \otimes_R \mathrm{Hom}_R(T, R) \longrightarrow R$$

satisfies the hypothesis in the preceding proposition. Therefore, we may apply the proposition to obtain our conclusion. □

Though we shall not use it later, we feel it wothwhile to mention the following "local duality" type result. Before that, we have a lemma.

Lemma 4.1.17. *Suppose Γ is a finitely generated profinite group and contains a pro-p subgroup of finite index, and Λ is (left and right) Noetherian. Then we have an isomorphism*

$$\Lambda \otimes_R^{\mathbf{L}} \omega_R \longrightarrow \mathbf{R}\mathrm{Hom}_\Lambda\big(\mathbf{R}\mathrm{Hom}_{\Lambda^o}(\Lambda \otimes_R^{\mathbf{L}} \omega_R, I_\Lambda), I_\Lambda\big)$$

in $\mathbf{D}(Mod_{\Lambda-\Lambda})$.

Proof : Since ω_R is an object of $\mathbf{D}^b(Mod_R^{R-ft})$, we may choose a bounded complex Ω of Noetherian R-modules, which represents ω_R. Since Λ is Noetherian, it follows from Corollary 2.1.2 (and its dual statement) that I_Λ is an injective Λ-module and an injective Λ^o-module. By Proposition 1.3.4 and Proposition 1.3.6, we are reduced to showing that

$$\Lambda \otimes_R \Omega \longrightarrow \mathrm{Hom}_\Lambda\big(\mathrm{Hom}_{\Lambda^o}(\Lambda \otimes_R \Omega, I_\Lambda), I_\Lambda\big)$$

is an isomorphism of complexes. The assumptions on Γ enable us to work with the $\mathfrak{M}$-adic topology, and so it follows from Corollary 2.1.5 that we have the isomorphism in the case when Ω is a single module concentrated at 0. For a general bounded complex of Noetherian R-modules, the term in degree n on the right is

$$\mathrm{Hom}_\Lambda\big(\mathrm{Hom}_{\Lambda^o}(\Omega^n, I_\Lambda), I_\Lambda\big),$$

and the required isomorphism is given by

$$\begin{gathered}\Lambda \otimes_R \Omega \longrightarrow \mathrm{Hom}_\Lambda\big(\mathrm{Hom}_{\Lambda^o}(\Lambda \otimes_R \Omega, I_\Lambda), I_\Lambda\big)\\ \lambda \otimes x \mapsto \big(f \mapsto ((-1)^{\bar{f}\bar{x}} f(\lambda \otimes x))\big),\end{gathered}$$

where the sign conventions make this into a morphism of complexes. □

Theorem 4.1.18. (*Local duality*) *Suppose Γ is a finitely generated profinite group and contains a pro-p subgroup of finite index, and Λ is (left) Noetherian. Let X be a bounded complex of modules in $Mod_{\Lambda[G]}^{\Lambda-ft}$. Then we have the following isomorphism*

$$\mathbf{R}\mathrm{Hom}_\Lambda\big(\mathbf{R}\mathrm{Hom}_{\Lambda^o}(\Lambda \otimes_R^{\mathbf{L}} \omega_R, I_\Lambda) \otimes_\Lambda^{\mathbf{L}} X, I_\Lambda\big) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_\Lambda(X, \Lambda \otimes_R^{\mathbf{L}} \omega_R)$$

in $\mathbf{D}(Mod_{\Lambda^o[G]})$.

Proof : This follows from Proposition 1.3.7 and Lemma 4.1.17. □

4.2 Finite generation of cohomology groups

In this section, we shall show that the cohomology groups of the induced modules are finitely generated under certain finiteness assumptions. These are the groups which we are interested in, and the knowledge that they are finitely generated will help in proving a duality between these groups.

Throughout this section, we assume that Γ is a compact p-adic Lie group. By Theorem 3.4.1, this implies that $\Lambda = R[\![\Gamma]\!]$ is a Noetherian adic ring. We then have the following two lemmas.

Lemma 4.2.1. *Assume that* Γ *is pro-p, and let* M *be a compact* Λ*-module. Then* M *is finitely generated over* Λ *if and only if* M_Γ *is finitely generated over* R.

Proof: Since Γ is pro-p, we have $\mathfrak{M} = \mathfrak{m}\Lambda + I_\Gamma$ and this implies that $M/\mathfrak{M}M = M_\Gamma/\mathfrak{m}M_\Gamma$. Applying Nakayama's lemma (Proposition 2.2.11) for compact Λ-modules, we have that M is finitely generated over Λ if and only if $M/\mathfrak{M}M$ is finite. On the other hand, applying the same proposition for compact R-modules, we have that M_Γ is finitely generated over R if and only if $M/\mathfrak{M}M$ is finite. Thus, the conclusion follows. □

Lemma 4.2.2. *If* M *is a finitely generated* Λ*-module, then* $\mathrm{Tor}_i^\Lambda(R, M)$ *is finitely generated over* R.

Proof : To see this, we first note that since M is finitely generated over Λ and Λ is Noetherian, we can find a resolution P of M consisting of finitely generated projective Λ-modules. Then $R \otimes_\Lambda P$ is a complex of finitely generated R-modules. Therefore, its homology groups $\mathrm{Tor}_i^\Lambda(R, M)$ are finitely generated over R. □

The next lemma will relate two complexes of modules and give a sufficient condition for them to be cohomologically bounded. We will utilize this to derive a relationship between cohomology groups.

Lemma 4.2.3. *Let* T *be a bounded complex in* $\mathcal{C}_{R,G}^{R-ft}$*, and let* $N \in \mathcal{C}_R^{R-ft}$*. Then* N *can viewed as a compact* Λ^o*-module via the augmentation map* $\Lambda \twoheadrightarrow R$*, and we have an isomorphism*

$$N \otimes_\Lambda^{\mathbf{L}} (\Lambda^\iota \otimes_R T) \cong N \otimes_R^{\mathbf{L}} T$$

in $\mathbf{D}^-(\mathcal{C}_{R,G})$*, where the functor* $- \otimes_\Lambda^{\mathbf{L}} -$ *on the left is over* $\mathbf{D}^-(\mathcal{C}_{\Lambda^o}^{\Lambda^o-ft}) \times \mathbf{D}^-(\mathcal{C}_{\Lambda,G})$ *and the one on the right is over* $\mathbf{D}^-(\mathcal{C}_R^{R-ft}) \times \mathbf{D}^-(\mathcal{C}_{R,G})$ *(see Proposition 2.5.7). Moreover, if* $\mathrm{pd}_R(N)$ *is finite, then* $N \otimes_\Lambda^{\mathbf{L}} (\Lambda^\iota \otimes_R T)$ *is an object in* $\mathbf{D}^b(\mathcal{C}_{R,G})$*, and the above isomorphism* *is an an isomorphism in* $\mathbf{D}^b(\mathcal{C}_{R,G})$. *In particular, if* N *is a free* R*-module, we have an isomorphism*

$$N \otimes^{\mathbf{L}}_{\Lambda} (\Lambda^{\iota} \otimes_R T) \cong N \otimes_R T$$

in $\mathbf{D}^b(\mathcal{C}_{R,G})$.

Proof: Let P be a resolution of N consisting of finitely generated projective Λ^o-modules, and let Q be a resolution of N consisting of finitely generated projective R-modules. We may view Q as a resolution of Λ^o-modules via the augmentation map $\Lambda \twoheadrightarrow R$. By comparison (see [Wei, Thm. 2.3.6]), there is a morphism $f : P \longrightarrow Q$ of complexes of Λ^o-modules which is a quasi-isomorphism and lifts the identity map $N \to N$. This induces a quasi-isomorphism

$$f_* : P \otimes_{\Lambda} \Lambda^{\iota} \longrightarrow Q \otimes_{\Lambda} \Lambda^{\iota} \cong Q$$

(since Λ is flat by Lemma 4.1.1) and a morphism

$$f_{**} : P \otimes_{\Lambda} \Lambda^{\iota} \otimes_R T \longrightarrow Q \otimes_{\Lambda} \Lambda^{\iota} \otimes_R T \cong Q \otimes_R T$$

of complexes in $\mathcal{C}_{R,G}$. Since $Q \otimes_R T$ represents $N \otimes^{\mathbf{L}}_R T$ by Proposition 2.2.14, it remains to show that f_{**} is a quasi-isomorphism. Now if $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ is an exact triangle in $\mathbf{D}^b(\mathcal{C}^{R-ft}_{R,G})$, we then have a morphism

$$\begin{array}{ccccccc}
N \otimes^{\mathbf{L}}_{\Lambda} (\Lambda^{\iota} \otimes_R A) & \longrightarrow & N \otimes^{\mathbf{L}}_{\Lambda} (\Lambda^{\iota} \otimes_R B) & \longrightarrow & N \otimes^{\mathbf{L}}_{\Lambda} (\Lambda^{\iota} \otimes_R C) & \longrightarrow & N \otimes^{\mathbf{L}}_{\Lambda} (\Lambda^{\iota} \otimes_R A)[1] \\
\downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_B} & & \downarrow{\scriptstyle f_C} & & \downarrow{\scriptstyle f_A[1]} \\
N \otimes^{\mathbf{L}}_R A & \longrightarrow & N \otimes^{\mathbf{L}}_R B & \longrightarrow & N \otimes^{\mathbf{L}}_R C & \longrightarrow & N \otimes^{\mathbf{L}}_R A[1]
\end{array}$$

of exact triangles. Therefore, if any of the two morphisms f_A, f_B and f_C are isomorphisms, so is the third. For a bounded complex T in $\mathcal{C}^{R-ft}_{R,G}$, we have the following exact triangle

$$\tau_{\leq i-1}T \longrightarrow \tau_{\leq i}T \longrightarrow H^i(T)[-i] \longrightarrow (\tau_{\leq i-1}T)[1].$$

Therefore, by induction, we are reduced to showing that f_{**} is a quasi-isomorphism in the case when T is a single module. To show this, it suffices to show that f_{**} is a quasi-isomorphism of complexes of R-modules. The map f induces the following morphism

$$\begin{array}{c}
\mathrm{Tor}^R_i(H^j(P \otimes_{\Lambda} \Lambda^{\iota}), T) \Rightarrow H^{i+j}(P \otimes_{\Lambda} \Lambda^{\iota} \otimes_R T) \\
\downarrow \\
\mathrm{Tor}^R_i(H^j(Q \otimes_{\Lambda} \Lambda^{\iota}), T) \Rightarrow H^{i+j}(Q \otimes_{\Lambda} \Lambda^{\iota} \otimes_R T)
\end{array}$$

of convergent spectral sequences of R-modules, where the two spectral sequences come from [Wei, Thm. 5.6.4]. Since f_* is a quasi-isomorphism, we have

$$H^j(P \otimes_\Lambda \Lambda^\iota) \cong H^j(Q \otimes_\Lambda \Lambda^\iota),$$

which in turn induces the isomorphisms

$$\mathrm{Tor}_i^R\big(H^j(P \otimes_\Lambda \Lambda^\iota), T\big) \cong \mathrm{Tor}_i^R\big(H^j(Q \otimes_\Lambda \Lambda^\iota), T\big).$$

By convergence of the spectral sequences, this implies that

$$H^j(P \otimes_\Lambda \Lambda^\iota \otimes_R T) \cong H^j(Q \otimes_\Lambda \Lambda^\iota \otimes_R T),$$

as required. The second and third assertions follow immediately from the first. □

Proposition 4.2.4. *Let T be a bounded complex in $\mathcal{C}_{R,G}^{R-ft}$, and let $N \in \mathcal{C}_R^{R-ft}$ with $\mathrm{pd}_R(N) < \infty$. Viewing N as a compact Λ^o-module via the augmentation map $\Lambda \twoheadrightarrow R$, we have an isomorphism*

$$N \otimes_\Lambda^{\mathbf{L}} \mathbf{R}\Gamma(G, \Lambda^\iota \otimes_R T) \cong \mathbf{R}\Gamma(G, N \otimes_R^{\mathbf{L}} T)$$

in $\mathbf{D}(Mod_R)$. In particular, we have the following isomorphism

$$R \otimes_\Lambda^{\mathbf{L}} \mathbf{R}\Gamma(G, \Lambda^\iota \otimes_R T) \cong \mathbf{R}\Gamma(G, T)$$

in $\mathbf{D}(Mod_R)$.

Proof: As before, we let P be a bounded above complex of finitely generated projective Λ^o-modules quasi-isomorphic to N. Then

$$N \otimes_\Lambda^{\mathbf{L}} \mathbf{R}\Gamma(G, \Lambda^\iota \otimes_R T) = P \otimes_\Lambda C(G, \Lambda^\iota \otimes_R T).$$

It is easy to see that there is an isomorphism

$$P \otimes_\Lambda C(G, \Lambda^\iota \otimes_R T) \cong C\big(G, P \otimes_\Lambda (\Lambda^\iota \otimes_R T)\big)$$

of complexes via a similar argument to that of [Ne, Prop. 3.4.4]. As seen in the proof of Lemma 4.2.3, we have that $P \otimes_\Lambda (\Lambda^\iota \otimes_R T)$ is cohomologically bounded and is quasi-isomorphic to $N \hat{\otimes}_R^{\mathbf{L}} T$ in $\mathbf{D}^b(\mathcal{C}_{R,G})$. The conclusion now follows from Lemma 2.4.8. □

Remark. The second assertion of the preceding proposition was proved in [Ne, Prop. 8.4.8.1] for the case $\Gamma \cong \mathbb{Z}_p^r$. We also mention that in the case when T is an object of $\mathcal{C}_{R,G}^{R-ft}$ that is projective as an abstract R-module, and assuming that $\mathrm{cd}_p(G) < \infty$, and that $H^i(G, M)$ is finite for every finite discrete G-module M, the same assertion is a special case of [FK, Prop. 1.6.5(3)].

We recall from [Ne, Prop. 4.2.3] that if G is a profinite group such that $H^i(G, M)$ is finite for every finite G-module M, then for every $T \in \mathcal{C}_{R,G}^{R-ft}$, we have that $H^i(G, T)$ is a Noetherian R-module for every $i \geqslant 0$. In the case when Γ is abelian pro-p, the ring Λ is commutative Noetherian local adic, and so [Ne, Prop. 4.2.3] can be applied. For the case of a noncommutative Γ, we have a weaker result in this direction.

Proposition 4.2.5. *Suppose Γ is a pro-p p-adic Lie group and G is a profinite group satisfying the following properties:*

(1) $\mathrm{cd}_p(G) = n$.

(2) *$H^i(G, M)$ is finite for all finite G-modules M and for all $i \geq 0$.*

Let $T \in \mathcal{C}_{R,G}^{R-ft}$. Then the cohomology groups $H^i(G, \Lambda^\iota \otimes_R T)$ are finitely generated over Λ for all $i \geq 0$.

Proof : Since $\Lambda^\iota \otimes_R T$ is a Noetherian Λ-module, so is $H^0(G, \Lambda^\iota \otimes_R T)$. Also, since $\mathrm{cd}_p(G) = n$, we have $H^i(G, \Lambda^\iota \otimes_R T) = 0$ for $i > n$. Thus, it remains to show that $H^i(G, \Lambda^\iota \otimes_R T)$ is finitely generated over Λ for $1 \leq i \leq n$. We shall prove this by induction downward on i. We apply Proposition 4.2.4 (taking $N = R$) to obtain an isomorphism

$$R \otimes_\Lambda^{\mathbf{L}} \mathbf{R}\Gamma(G, \Lambda^\iota \otimes_R T) \cong \mathbf{R}\Gamma(G, T)$$

in $\mathbf{D}(Mod_R)$ which induces the following bounded convergent spectral sequence

$$E_{r,s}^2 = \mathrm{Tor}_r^\Lambda(R, H^{-s}(G, \Lambda^\iota \otimes_R T)) \Rightarrow H^{-r-s}(G, T).$$

By hypothesis (1), this gives an isomorphism

$$H^n(G, \Lambda^\iota \otimes_R T)_\Gamma \cong H^n(G, T).$$

As seen above, hypothesis (2) allows us to apply [Ne, Prop. 4.2.3] to conclude that $H^n(G, T)$ is a Noetherian R-module. It follows from the above isomorphism that $H^n(G, \Lambda^\iota \otimes_R T)_\Gamma$ is also a Noetherian R-module. By Lemma 4.2.1, this implies that $H^n(G, \Lambda^\iota \otimes_R T)$ is a Noetherian Λ-module. Let $i \leq n$ and suppose $H^j(G, \Lambda^\iota \otimes_R T)$ is a Noetherian Λ-module for $j > i$. Since the spectral sequence is bounded, it follows that

$E^m_{r,s}$ (see [Wei, Def. 5.2.1] for the definition) must stabilize for large enough m, and this stable value is denoted by $E^\infty_{r,s}$. In particular, we have that $E^\infty_{0,-j}$ is a quotient of $E^2_{0,-j}$. By loc. cit. 5.2.5, we have that $E^\infty_{0,-j}$ is a subquotient of $H^j(G,T)$, and so is a Noetherian R-module since $H^j(G,T)$ is a Noetherian R-module by [Ne, Prop. 4.2.3]. On the other hand, it follows from the definition of $E^\infty_{0,-j}$ that the kernel of the map $E^2_{0,-j} \twoheadrightarrow E^\infty_{0,-j}$ is isomorphic to a subquotient of

$$\bigoplus_{j<i\leq n} E^2_{i-j+1,-i}.$$

By our induction hypothesis and Lemma 4.2.2, the above module is a Noetherian R-module. Hence, it now follows that $E^2_{0,-j} = H^j(G, \Lambda^\iota \otimes_R T)_\Gamma$ is a Noetherian R-module. Applying Lemma 4.2.1, we have that $H^j(G, \Lambda^\iota \otimes_R T)$ is a Noetherian Λ-module. □

Corollary 4.2.6. *Suppose* Γ *is a pro-p p-adic Lie group and* G *is a profinite group satisfying the following properties:*

(1) $\mathrm{cd}_p(G) = n$.

(2) $H^i(G,M)$ *is finite for all finite* G*-module* M *for all* $i \geq 0$.

Then for every bounded complex T *of objects in* $\mathcal{C}^{R-ft}_{R,G}$, *the object* $\mathbf{R}\Gamma(G, \Lambda^\iota \otimes_R T)$ *is in* $\mathbf{D}^b_{\Lambda-ft}(Mod_\Lambda)$. □

Proof: Recall from the discussion before Lemma 2.4.8 that for a bounded complex T of objects in $\mathcal{C}^{R-ft}_{R,G}$, we have the following convergent spectral sequence

$$H^i\big(G, H^j(\Lambda^\iota \otimes_R T)\big) \Rightarrow H^{i+j}\big(G, \Lambda^\iota \otimes_R T\big).$$

It follows from Proposition 4.2.5 that $H^i\big(G, H^j(\Lambda^\iota \otimes_R T)\big)$ is a Noetherian Λ-module for all i,j. It follows from [Wei, 5.2.5] that $H^n\big(G, \Lambda^\iota \otimes_R T\big)$ has a finite filtration consisting of subquotients of $H^i\big(G, H^j(\Lambda^\iota \otimes_R T)\big)$ for $i+j=n$. Hence it follows that $H^n\big(G, \Lambda^\iota \otimes_R T\big)$ is also a Noetherian Λ-module. □

4.3 Shapiro's lemma

As before, let R be a commutative complete noetherian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic p. Let G be a profinite group, and let H be a closed subgroup of G such that $\Gamma = G/H$ is a compact p-adic Lie group. We take our continuous homomorphism $\pi : G \longrightarrow \Gamma$ to be the canonical quotient map. It also follows from our assumption on Γ that $\Lambda = R[\![\Gamma]\!]$ is Noetherian. We identify $\mathscr{U}$ as the collection of open normal subgroups of G containing H. Therefore, in this context, for each $U \in \mathscr{U}$, and an ind-admissible $\Lambda[G]$-module M, we have

$$
\begin{aligned}
{}_U M &= \mathrm{Hom}_R(R[G/U], M), \\
M_U &= R[G/U]^\iota \otimes_R M.
\end{aligned}
$$

We will apply Shapiro's lemma to see that the direct limits and inverse limits of cohomology groups over every intermediate field F_α can be viewed as cohomology groups of certain Λ-modules. The results in this section can be found in [Ne, 8.2.2, 8.3.3-5, 8.4.4.2].

Lemma 4.3.1. *Let U be an open normal subgroup of G and N be a bounded below complex of objects of $\mathcal{D}_{R,G}$. Then we have a quasi-isomorphism*

$$C(G, {}_U N) \xrightarrow{\sim} C(U, N)$$

of complexes of Λ-modules.

Proof: We first prove the lemma in the case that N is an object of $\mathcal{D}_{R,G}$. Then we may write $N = \varinjlim_\alpha N_\alpha$, where N_α is a finite $R[G]$-module endowed with the discrete topology. The usual Shapiro's lemma holds for such modules. Also, we note that ${}_U N \cong \varinjlim_\alpha {}_U(N_\alpha)$. Hence, we have

$$C(G, {}_U N) = C\Big(G, \varinjlim_\alpha {}_U(N_\alpha)\Big) \cong \varinjlim_\alpha C\big(G, {}_U(N_\alpha)\big) \xrightarrow{\text{sh}} \varinjlim_\alpha C(U, N_\alpha) = C(U, N)$$

which gives the required conclusion for the case that N is an object of $\mathcal{D}_{R,G}$. For the case that N is a bounded below complex of objects of $\mathcal{D}_{R,G}$, one can prove this by the spectral sequence argument as used in Lemma 2.4.8. □

Recall that if A is a complex in $\mathcal{D}_{R,G}$, then $F_\Gamma(A) = \varinjlim_{U \in \mathscr{U}} {}_U A$ is a complex in $\mathcal{D}_{\Lambda,G}$ by Lemma 4.1.7. We then have the following proposition.

Proposition 4.3.2. *Let A be a bounded below complex of objects of $\mathcal{D}_{R,G}$. Then the composite morphism*

$$C(G, F_\Gamma(A)) \xrightarrow{\sim} \varinjlim_{U \in \mathscr{U}} C(G, {}_U A) \xrightarrow{\text{sh}} \varinjlim_{U \in \mathscr{U}} C(U, A) \xrightarrow{\text{res}} C(H, A)$$

is a quasi-isomorphism of complexes of Λ-modules. In other words, we have an isomorphism

$$\mathbf{R}\Gamma(G, F_\Gamma(A)) \xrightarrow{\sim} \mathbf{R}\Gamma(H, A)$$

in $\mathbf{D}(Mod_\Lambda)$. □

We would also like to have a Shapiro-type relation for cohomology groups of objects in $\mathcal{C}_{R,G}$. But since inverse limits are not necessarily exact, we cannot always do a limit argument on the Shapiro maps as in Lemma 4.3.1. However, we can still say something if we restrict ourselves to objects in $\mathcal{C}_{R,G}^{R-ft}$.

Lemma 4.3.3. *Let U be an open normal subgroup of G. Then for any bounded complex M in $\mathcal{C}_{R,G}^{R-ft}$, we have a quasi-isomorphism*

$$C(G, M_U) \xrightarrow{\sim} C(U, M)$$

of complexes of Λ-modules.

Proof : By the same argument as that in Lemma 4.3.1, it suffices to consider the case when M is an object of $\mathcal{C}_{R,G}$. Note that $M \cong \varprojlim_n M/\mathfrak{m}^n M$ as objects in $\mathcal{C}_{R,G}$ and $M_U \cong \varprojlim_n (M/\mathfrak{m}^n M)_U$. Then we have morphisms

$$C(G, M_U) \cong \varprojlim_n C\big(G, (M/\mathfrak{m}^n M)_U\big) \xrightarrow{\text{sh}} \varprojlim_n C(U, M/\mathfrak{m}^n M) \cong C(U, M)$$

which induces a morphism

$$\begin{array}{ccc} \varprojlim_n{}^i H^j\big(G, (M/\mathfrak{m}^n M)_U\big) & \Rightarrow & H^{i+j}(G, M_U) \\ & & \big\downarrow \\ \varprojlim_n{}^i H^j\big(U, M/\mathfrak{m}^n M\big) & \Rightarrow & H^{i+j}(U, M) \end{array}$$

of convergent spectral sequences. Since $M/\mathfrak{m}^n M$ is finite, the usual Shapiro's lemma implies that

$$H^j\big(G, (M/\mathfrak{m}^n M)_U\big) \cong H^j\big(U, M/\mathfrak{m}^n M\big)$$

is an isomorphism. This in turn implies that

$$\varprojlim_n{}^i H^j\big(G, (M/\mathfrak{m}^n M)_U\big) \cong \varprojlim_n{}^i H^j\big(U, M/\mathfrak{m}^n M\big).$$

By the convergence of the spectral sequences, we have isomorphisms

$$H^i(G, M_U) \cong H^i(U, M),$$

as required. $\square$

To obtain the analogous result to Proposition 4.3.2 for $\mathcal{C}_{R,G}^{R-ft}$, we require more extra assumptions.

Proposition 4.3.4. *Let M be an ind-admissible $R[G]$-module which is Noetherian over R. Then we have the following isomorphism*

$$C(G, \mathscr{F}_\Gamma(M)) \xrightarrow{\sim} \varprojlim_U C(G, M_U)$$

of complexes of Λ-modules. Furthermore, if $H^m(G,N)$ is finite for all finite discrete Λ-modules N with a Λ-linear continuous G-action and all $m \geq 0$, we have

$$H^j(G, \mathscr{F}_\Gamma(M)) \cong \varprojlim_{U \in \mathscr{U}} H^j(U, M).$$

Proof : Since Γ is a compact p-adic Lie group, the $(\mathfrak{m}, I)$-topology and $\mathfrak{M}$-adic topology on $\mathscr{F}_\Gamma(M)$ coincide. This implies that we have a continuous isomorphism

$$\mathscr{F}_\Gamma(M) \cong \varprojlim_U M_U,$$

and thus an identification of the continuous cochain groups. The second assertion now follows from Proposition 2.4.12 (note that Λ is Noetherian and so each M_U is a Noetherian Λ-module) and Lemma 4.3.3. □

4.4 The semilocal case

We now describe Shapiro's lemma in the semilocal case and refer readers to [Ne, Sect. 8.1.7] for the proofs and verifications. We will require the results in this section in the next chapter.

Let $\alpha : \overline{G} \to G$ be a continuous homomorphism of profinite groups, and let U be an open normal subgroup of G. Then $\overline{U} = \alpha^{-1}(U)$ is an open normal subgroup of $\overline{G}$ and α factors through $\overline{G}/\overline{U}$ to give an injection $\overline{G}/\overline{U} \to G/U$, which we also denote by α.

Fix coset representatives $\sigma_i \in G$ of

$$G/U = \bigcup_i \sigma_i \alpha(\overline{G}/\overline{U}) = \bigcup_i \alpha(\overline{G}/\overline{U})\sigma_i^{-1}$$

and set $\alpha_i = \mathrm{Ad}(\sigma_i) \circ \alpha$. Here $\mathrm{Ad}(\sigma_i)$ is the conjugation map on the group G sending g to $\sigma_i g \sigma_i^{-1}$. By abuse of notation, we denote the conjugation map on G/U by $\mathrm{Ad}(\sigma_i)$.

For a G-module X, the $\overline{G}$-module $\alpha^* X$ is defined as follows: as an abelian group, $\alpha^* X = X$, and $\overline{g} \in \overline{G}$ acts on $\alpha^* X$ as $\alpha(\overline{g})$ on X. The $\overline{G}$-module $\alpha_i^* X$ is defined similarly. Note that the action of σ_i defines an isomorphism

$$\sigma_i : \alpha^* X \xrightarrow{\sim} \alpha_i^* X$$
$$x \mapsto \sigma_i x$$

of $\overline{G}$-modules. Now suppose X is a discrete $R[G]$-module. Then we have the following decomposition of $\overline{G}$-modules

$$\alpha^* X_U = \bigoplus_i \alpha^* \left((R[\alpha(\overline{G}/\overline{U})]\sigma_i^{-1})^\iota \otimes_R X \right).$$

Denote the projection on the ith factor by π_i. One then can check that following maps

$$\begin{gathered} \alpha^* \left((R[\alpha(\overline{G}/\overline{U})]\sigma_i^{-1})^\iota \otimes_R X \right) \longrightarrow \alpha_i^* \left(R[\alpha_i(\overline{G}/\overline{U})] \otimes_R X \right) \\ \alpha(\overline{g}U)\sigma_i^{-1} \otimes x \mapsto \sigma_i \alpha(\overline{g}U)\sigma_i^{-1} \otimes \sigma_i x \end{gathered}$$

$$\begin{gathered} \alpha_i^* \left(R[\alpha_i(\overline{G}/\overline{U})] \otimes_R X \right) \longrightarrow (\alpha_i^* X)_{\overline{U}} \\ \alpha_i(\overline{g}\overline{U}) \otimes x \mapsto \overline{g}\overline{U} \otimes x \end{gathered}$$

are isomorphisms of $\overline{G}$-modules. Composing the two isomorphisms with π_i, we obtain a homomorphism

$$w_i : \alpha^* X_U \to (\alpha_i^* X)_{\overline{U}}$$

of $\overline{G}$-modules. Putting all w_i together, we obtain a $\overline{G}$-isomorphism

$$w = (w_i) : \alpha^* X_U \xrightarrow{\sim} \bigoplus_i (\alpha_i^* X)_{\overline{U}}.$$

Then the following diagram of complexes (see [Ne, 8.1.7.2])

$$\begin{array}{ccc} C(G, X_U) & \xrightarrow{w \circ \alpha^*} & \bigoplus_i C(\overline{G}, (\alpha_i^* X)_{\overline{U}}) \\ \downarrow \text{sh} & & \downarrow \text{sh} \\ C(U, X) & \xrightarrow{\alpha_i^*} & \bigoplus_i C(\overline{U}, \alpha_i^* X) \end{array}$$

is commutative up to homotopy and induces a quasi-isomorphism (functorial in X)

$$\mathrm{Cone}(\alpha^*) \longrightarrow \mathrm{Cone}((\alpha_i^*)).$$

Assume $V \subseteq U$ is another open normal subgroup of G. Set $\overline{V} = \alpha^{-1}(V)$. Fix coset representatives $\tau_j \in G$ of

$$G/V = \bigcup_j \tau_j \alpha(\overline{G}/\overline{V}) = \bigcup_j \alpha(\overline{G}/\overline{V})\tau_j^{-1}.$$

Then

$$G = \bigcup_j V\tau_j\alpha(\overline{G}) = \bigcup_i U\sigma_i\alpha(\overline{G}),$$

and for each j, we have $U\tau_j\alpha(\overline{G}) = U\sigma_i\alpha(\overline{G})$ for a unique $i = i(j)$. I.e., $\tau_j = u_{ij}\sigma_i\alpha(\overline{g}_{ij})$ for some $u_{ij} \in U, \overline{g}_{ij} \in \overline{G}$. It is easy to check that the action of τ_i defines an isomorphism

$$\begin{aligned} \tau_i : \alpha^*X &\xrightarrow{\sim} \beta_j^*X \\ x &\mapsto \tau_j x \end{aligned}$$

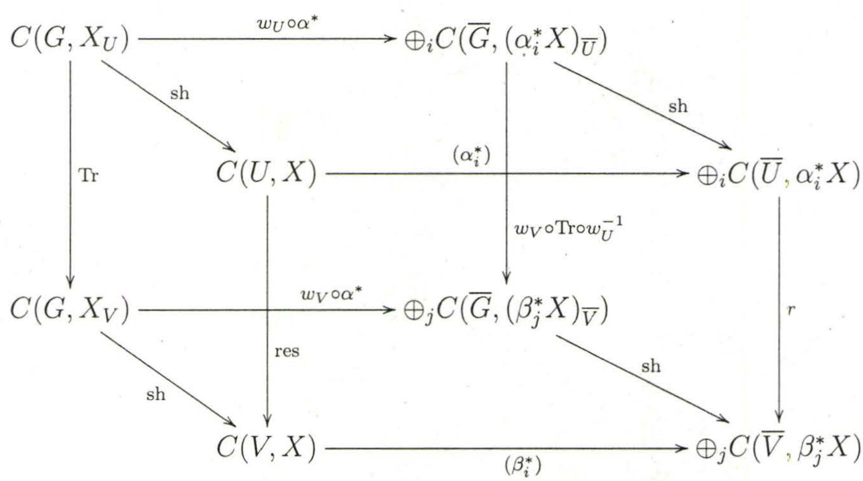
of $\overline{G}$-modules. Set $\beta_j : \overline{G} \xrightarrow{\alpha} G \xrightarrow{\mathrm{Ad}(\tau_j)} G$ and define a morphism of complexes (functorial in X)

$$r = (r_{ij}) : \oplus_i C(\overline{U}, \alpha_i^*X) \longrightarrow \oplus_j C(\overline{V}, \beta_j^*X)$$

by

$$r_{ij} : C(\overline{U}, \alpha_i^*X) \xrightarrow{(\sigma_i^{-1})_*} C(\overline{U}, \alpha^*X) \xrightarrow{\mathrm{Ad}(\overline{g}_{ij}^{-1})} C(\overline{U}, \alpha^*X) \xrightarrow{\mathrm{res}} C(\overline{V}, \alpha^*X) \xrightarrow{(\tau_j)_*} C(\overline{V}, \beta_j^*X).$$

Then we have the following cubic diagram



whose faces commute up to homotopy, and the boundary of the cube is trivialized by a 2-homotopy (see [Ne, 8.1.7.4.2, Lemma 8.1.7.4.3]). By [Ne, Cor. 8.1.7.4.4], the following diagram

$$\begin{array}{ccc} \mathrm{Cone}(w_U \circ \alpha^*) & \xrightarrow{\mathrm{Cone}(\mathrm{sh},\mathrm{sh},h)} & \mathrm{Cone}((\alpha_i^*)) \\ {\scriptstyle \mathrm{Cone}(\mathrm{Tr}, w_V \circ \mathrm{Tr} \circ w_{\overline{U}}^{-1}, 0)} \downarrow & & \downarrow {\scriptstyle \mathrm{Cone}(\mathrm{res}, r, m)} \\ \mathrm{Cone}(w_V \circ \alpha^*) & \xrightarrow{\mathrm{Cone}(\mathrm{sh},\mathrm{sh},h')} & \mathrm{Cone}((\beta_j^*)) \end{array}$$

is commutative up to homotopy, where $h = ((\alpha_i^* \circ \mathrm{sh}) \star h_{\sigma_i})_i$ and $m = ((\beta_j \circ \mathrm{res}) \star h_{u_{ij}})_j$. Here h_σ denotes a fixed homotopy from the identity map to $\mathrm{Ad}(\sigma)$ (see [Ne, 3.6.1.4, 4.5.5]).

There is a similar construction for the corestriction (see [Ne, 8.1.7.5])

$$c : \oplus_j C(\overline{V}, \beta_j^* X) \longrightarrow \oplus_i C(\overline{U}, \alpha_i^* X),$$

which yields the following commutative (up to homotopy) square.

$$\begin{array}{ccc} \mathrm{Cone}(w_V \circ \alpha^*) & \xrightarrow{\mathrm{Cone}(\mathrm{sh},\mathrm{sh},h')} & \mathrm{Cone}((\beta_j^*)) \\ {\scriptstyle \mathrm{Cone}(\mathrm{pr},w_U \circ \mathrm{pr} \circ w_V^{-1},0)} \downarrow & & \downarrow {\scriptstyle \mathrm{Cone}(\mathrm{cor},c,m)} \\ \mathrm{Cone}(w_U \circ \alpha^*) & \xrightarrow{\mathrm{Cone}(\mathrm{sh},\mathrm{sh},h)} & \mathrm{Cone}((\alpha_i^*)) \end{array}$$

Since the above constructions are functorial, and the cochains (as seen in Definition 2.6.9) are compatible with limits, we can extend them to ind-admissible $R[G]$-modules. By functoriality again, we can extend the above constructions to complexes of ind-admissible $R[G]$-modules. We will apply the constructions in this section in the next chapter.

Chapter 5

Duality for Galois Cohomology

We have come to the final chapter of the thesis. In this chapter, we will formulate our duality theorems. We begin by formulating and proving Tate's local duality over adic rings in Section 5.1. We also introduce the cohomology groups with compact support and prove the Poitou-Tate duality over adic rings. In Section 5.2, we will apply the results developed in Chapter 4 to the setting of p-adic Lie extensions. This will allow us to formulate the Grothendieck duality of cohomology groups in Section 5.3 and Section 5.4. In Section 5.5, we shall apply the duality theorems to obtain a generalization of a spectral sequence that first appeared in [Ja]. Finally as a complement, we develop the duality theorems over p-adic Lie extensions of a local field in Section 5.6.

5.1 Duality theorems over adic rings

Let p be a fixed prime. From now on, our adic rings Λ will always have the property that $\Lambda/\mathfrak{M}^n$ is finite of order a power of p for all $n \geq 1$.

Let F be a global field with characteristic not equal to p, and let S be a finite set of primes of F containing all primes above p and all archimedean primes of F (if F is a number field). Let S_f (resp., $S_{\mathbb{R}}$) denote the collection of non-archimedean primes (resp., real primes) of F in S.

Fix a separable closure F^{sep} of F. Set $G_{F,S} = \mathrm{Gal}(F_S/F)$, where F_S is the maximal subextension of F^{sep}/F unramified outside S. For each $v \in S_f$, we fix a separable closure F_v^{sep} of F_v and an embedding $F^{\mathrm{sep}} \hookrightarrow F_v^{\mathrm{sep}}$. This induces a continuous group homomorphism $G_v := \mathrm{Gal}(F_v^{\mathrm{sep}}/F_v) \to G_{F,S}$. If v is a real prime, we also write G_v for $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Lemma 5.1.1. *For each $v \in S_f$, we have*

$$H^j(G_v, \mathbb{Q}_p/\mathbb{Z}_p(1)) \cong \begin{cases} \mathbb{Q}_p/\mathbb{Z}_p & \text{if } j = 2, \\ 0 & \text{if } j > 2. \end{cases}$$

Suppose that Λ is Noetherian. Then for a $\Lambda[G_v]$-module N with trivial G_v-action, we have

$$H^j(G_v, N(1)) \cong \begin{cases} N & \text{if } j = 2, \\ 0 & \text{if } j > 2. \end{cases}$$

In the case where T is a R-module with trivial G_v-action, we have an isomorphism $H^2(G_v, \Lambda \otimes_R T(1)) \cong \Lambda \otimes_R T$ of Λ-Λ-bimodules.

If $v \in S_{\mathbb{R}}$, then we have

$$H^2(G_v, \mathbb{Q}_p/\mathbb{Z}_p(1)) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } p = 2, \\ 0 & \text{if } p \neq 2. \end{cases}$$

Proof: For $j > 2$, the conclusion follows from the fact that G_v has p-cohomological dimension 2 (see [NSW, Thm. 7.1.8(i)]). By [NSW, Thm. 7.1.8(ii)], we have $H^2(G_v, \mathbb{Z}/p^r(1)) \cong \mathbb{Z}/p^r$. Therefore, the first assertion follows from taking direct limits. For a $\Lambda[G_v]$-module N with trivial G_v-action, we have $N(1) = \varinjlim_\alpha N_\alpha(1)$, where N_α is a Noetherian Λ-module. It is easy to verify that the G_v-action on $N_\alpha(1)$ is continuous, where $N_\alpha(1)$ is given the $\mathfrak{M}$-adic topology. Then we have

$$N = \varinjlim_\alpha \varprojlim_n N_\alpha/\mathfrak{M}^n N_\alpha.$$

Recall that cohomology commutes with direct limits by definition. By [NSW, Thm. 7.1.8(iii)] and Proposition 2.4.11, cohomology commutes with inverse limits. Therefore, it suffices to show the assertion for $N_\alpha/\mathfrak{M}^n N_\alpha$, which is a finite discrete abelian p-group. Since cohomology commutes with direct sums, we are reduced to the case of $\mathbb{Z}/p^r$, which follows from the above discussion.

It is easy to see that for any R-module T, the module $\Lambda \otimes_R T$ is an ind-admissible $\Lambda[G_v]$-Λ-bimodule. The conclusion in this case follows by a similar argument as above.

The last assertion follows from [NSW, Thm. 7.2.17] and the fact that G_v is a finite group of order 2. □

Suppose $v \in S_f$. Let N be a complex of ind-admissible $\Lambda[G_v]$-modules. For each $n \in \mathbb{Z}$, we define $\tau^{II}_{\geq n} C(G_v, N)$ to be the total complex of

$$\left(i \mapsto \tau_{\geq n} C(G_v, N^i)\right).$$

We shall also use a similar notation for the above total complex when N is a complex of ind-admissible $\Lambda[G_v]$-Λ-modules. Note that if N is concentrated in degree zero, then $\tau^{II}_{\geq n} C(G_v, N) = \tau_{\geq n} C(G_v, N)$. By the above lemma, the canonical map of complexes of Λ-modules $N[-2] \xrightarrow{i_v(N)} \tau^{II}_{\geq 2} C(G_v, N(1))$ is a quasi-isomorphism. We also have a quasi-isomorphism $\mathbb{Q}_p/\mathbb{Z}_p[-2] \xrightarrow{i_v} \tau_{\geq 2} C(G_v, \mathbb{Q}_p/\mathbb{Z}_p(1))$ of complexes of $\mathbb{Z}_p$-modules. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is an injective $\mathbb{Z}_p$-module, the map i_v has a homotopy inverse. We shall fix one such map

$$r_v : \tau_{\geq 2} C(G_v, \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p[-2].$$

This gives a morphism

$$\theta_v : C(G_v, \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow \tau_{\geqslant 2} C(G_v, \mathbb{Q}_p/\mathbb{Z}_p(1)) \xrightarrow{r_v} \mathbb{Q}_p/\mathbb{Z}_p[-2]$$

of complexes of $\mathbb{Z}_p$-modules.

Let M be a bounded complex of objects in $\mathcal{C}_{\Lambda, G_v}$ which are finitely presented over Λ. Then we have an isomorphism $M = \varprojlim_n M/\mathfrak{M}^n M$ of complexes in $\mathcal{C}_{\Lambda, G_v}$ which induces an isomorphism of complexes

$$C(G_v, M) \cong \varprojlim_n C(G_v, M/\mathfrak{M}^n M)$$

of Λ-modules. Also, we have an isomorphism $M^\vee = \varinjlim_n (M/\mathfrak{M}^n M)^\vee$ of complexes in $\mathcal{D}_{\Lambda^o, G_v}$. For each n, we have the following commutative diagram

$$\begin{array}{ccc} (M/\mathfrak{M}^n M)^\vee(1) \otimes_\Lambda M & \xrightarrow{\mathrm{id} \otimes \pi_n} & (M/\mathfrak{M}^n M)^\vee(1) \otimes_\Lambda M/\mathfrak{M}^n M \\ \downarrow{\scriptstyle \pi_n^\vee \otimes \mathrm{id}} & & \downarrow \\ M^\vee(1) \otimes_\Lambda M & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p(1) \end{array}$$

where the pairings are the obvious ones and $\pi_n : M \twoheadrightarrow M/\mathfrak{M}^n M$ is the canonical quotient. Applying cochains and θ_v, we obtain the following commutative diagram

$$\begin{array}{ccc} C(G_v, (M/\mathfrak{M}^n M)^\vee(1)) \otimes_\Lambda C(G_v, M) & \xrightarrow{\mathrm{id} \otimes \pi_n} & C(G_v, (M/\mathfrak{M}^n M)^\vee(1)) \otimes_\Lambda C(G_v, M/\mathfrak{M}^n M) \\ \downarrow{\scriptstyle \pi_n^\vee \otimes \mathrm{id}} & & \downarrow \\ C(G_v, M^\vee(1)) \otimes_\Lambda C(G_v, M) & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p[-2] \end{array}$$

By Lemma 2.7.2, we obtain the following commutative diagram

$$\begin{array}{ccc} C(G_v, M) & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathbb{Z}_p}\Big(C\big(G_v, M^\vee(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)[-2] \\ \downarrow & & \downarrow \\ C(G_v, M/\mathfrak{M}^n M) & \xrightarrow{\alpha_n} & \mathrm{Hom}_{\mathbb{Z}_p}\Big(C\big(G_v, (M/\mathfrak{M}^n M)^\vee(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)[-2] \end{array}$$

of complexes of Λ-modules. By a similar argument, one can check that this is functorial in n and hence we have the following commutative diagram

$$\begin{array}{ccc} C(G_v, M) & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathbb{Z}_p}\Big(C\big(G_v, M^\vee(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)[-2] \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ \varprojlim_n C(G_v, M/\mathfrak{M}^n M) & \xrightarrow{\varprojlim \alpha_n} & \varprojlim_n \mathrm{Hom}_{\mathbb{Z}_p}\Big(C\big(G_v, (M/\mathfrak{M}^n M)^\vee(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)[-2] \end{array}$$

of complexes of Λ-modules. We are now able to prove the following formulation of Tate's local duality.

Theorem 5.1.2. *Let $v \in S_f$, and let M be a bounded complex of objects in $\mathcal{C}_{\Lambda, G_v}$ which are finitely generated over Λ. Then we have the following isomorphism*

$$\mathbf{R}\Gamma(G_v, M) \longrightarrow \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\Big(\mathbf{R}\Gamma\big(G_v, M^\vee(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)[-2]$$

in $\mathbf{D}(Mod_\Lambda)$.

Proof: We shall show that α (in the above diagram) is a quasi-isomorphism. By considering the exact triangle

$$\sigma_{\leq i-1} M \longrightarrow \sigma_{\leq i} M \longrightarrow M^i[-i] \longrightarrow (\sigma_{\leq i-1} M)[1]$$

and by a similar argument to that of Lemma 4.2.3, we are reduced to the case when M is a single module. By Proposition 2.4.10 and Proposition 2.4.11, we have that u and v in the above diagram are isomorphisms of complexes, and the vertical maps in the following commutative diagram

$$\begin{array}{ccc} H^i(G_v, M) & \xrightarrow{\alpha_*} & \mathrm{Hom}_{\mathbb{Z}_p}\Big(H^{2-i}\big(G_v, M^\vee(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big) \\ {\scriptstyle u_*}\downarrow & & \downarrow{\scriptstyle v_*} \\ \varprojlim_n H^i(G_v, M/\mathfrak{M}^n M) & \xrightarrow{\varprojlim (\alpha_n)_*} & \varprojlim_n \mathrm{Hom}_{\mathbb{Z}_p}\Big(H^{2-i}\big(G_v, (M/\mathfrak{M}^n M)^\vee(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big) \end{array}$$

are isomorphisms. Since each α_n is a quasi-isomorphism by Tate local duality [NSW, Thm. 7.2.6], we have the required conclusion. □

If M is a complex in $(Mod_{\Lambda[G_{F,S}]}^{ind-ad})$, then we can also view M as a complex in $(Mod_{\Lambda[G_v]}^{ind-ad})$ via the continuous homomorphism $G_v \longrightarrow G_{F,S}$. Therefore, the cochain complexes $C(G_{F,S}, M)$ and $C(G_v, M)$ can be defined. Recall that for $v \in S_f$, we have the restriction map

$$\mathrm{res}_v : C(G_{F,S}, M) \xrightarrow{\mathrm{res}} C(G_v, M)$$

induced by the group homomorphism $G_v \longrightarrow G_{F,S}$. For a real prime v, we have the following

$$\mathrm{res}_v : C(G_{F,S}, M) \longrightarrow C(G_v, M) \hookrightarrow C(G_v, M).$$

We now make the following definition.

Definition 5.1.3. Let M be a complex of ind-admissible $\Lambda[G_{F,S}]$-modules. The complex of continuous cochains of M with compact support is defined as

$$C_c(G_{F,S}, M) = \mathrm{Cone}\left(C(G_{F,S}, M) \xrightarrow{\mathrm{res}_S} \bigoplus_{v \in S_f} C(G_v, M) \oplus \bigoplus_{v \in S_{\mathbb{R}}} \hat{C}(G_v, M) \right) [-1] ,$$

where the elements of

$$C_c^i(G_{F,S}, M) = C^i(G_{F,S}, M) \oplus \left(\bigoplus_{v \in S_f} C^{i-1}(G_v, M) \oplus \bigoplus_{v \in S_{\mathbb{R}}} \hat{C}^{i-1}(G_v, M) \right)$$

have the form (a, a_S) with $a \in C^i(G_{F,S}, M), a_S = (a_v)_{v \in S_f \cup S_{\mathbb{R}}}, a_v \in C^{i-1}(G_v, M)$ if $v \in S_f$, and $a_v \in \hat{C}^{i-1}(G_v, M)$ if $v \in S_{\mathbb{R}}$, and the differential is given by

$$d(a, a_S) = (da, -\mathrm{res}_f(a) - da_S).$$

The ith cohomology group of $C_c(G_{F,S}, M)$ is denoted by $H_c^i(G_{F,S}, M)$.

Remark. If F is a function field in one variable over a finite field or F is a totally imaginary number field, then $S_{\mathbb{R}}$ is empty, and the cone is given by

$$\mathrm{Cone}\left(C(G_{F,S}, M) \xrightarrow{\mathrm{res}_S} \bigoplus_{v \in S_f} C(G_v, M) \right) [-1].$$

Now suppose that p is odd and F is a number field with at least one real prime. Let $v \in S_{\mathbb{R}}$. Then $\hat{H}^i(G_v, M) = 0$ for every $M \in (Mod_{\Lambda[G_{F,S}]}^{ind-ad})$ and for all i since G_v is a finite group of order 2 and M is a direct limit of pro-p-groups. Therefore, it follows that the canonical map

$$\mathrm{Cone}\left(C(G_{F,S}, M) \xrightarrow{\mathrm{res}_S} \bigoplus_{v \in S_f} C(G_v, M) \right)[-1] \longrightarrow C_c(G_{F,S}, M)$$

is a quasi-isomorphism. Therefore, we may take the above cone as a definition of the complex of continuous cochains with compact support in this case.

Proposition 5.1.4. *The functor*

$$C_c(G_{F,S}, -) : \mathrm{Ch}^+(Mod^{ind-ad}_{\Lambda[G_{F,S}]}) \longrightarrow \mathrm{Ch}(Mod_\Lambda)$$

preserves homotopy, exact sequences and quasi-isomorphisms, hence induces the following exact derived functors

$$\mathbf{R}\Gamma_c(G_{F,S}, -) : \mathbf{D}^+(Mod^{ind-ad}_{\Lambda[G_{F,S}]}) \longrightarrow \mathbf{D}(Mod_\Lambda)$$

such that we have the following exact triangle for $M \in \mathbf{D}^+(Mod^{ind-ad}_{\Lambda[G_{F,S}]})$

$$\mathbf{R}\Gamma_c(G_{F,S}, M) \longrightarrow \mathbf{R}\Gamma(G_{F,S}, M) \longrightarrow \bigoplus_{v \in S_f} \mathbf{R}\Gamma(G_v, M)$$

in $\mathbf{D}(\Lambda)$ and the following long exact sequence

$$\cdots \longrightarrow \hat{H}^i_c(G_{F,S}, M) \longrightarrow H^i(G_{F,S}, M)$$

$$\longrightarrow \bigoplus_{v \in S_f} H^i(G_v, M) \oplus \bigoplus_{v \in S_\mathbb{R}} \hat{H}^i(G_v, M) \longrightarrow H^{i+1}_c(G, M) \longrightarrow \cdots.$$

Proof: This is immediate from the definition of the cone. □

The next proposition is the analogous statement to Proposition 2.4.11 for cohomology groups with compact support. We note that by [NSW, Thm. 7.1.8(iii), Thm. 8.3.19], Proposition 2.4.11 can be applied to $G_{F,S}$ and G_v, where $v \in S_f$. For $v \in S_\mathbb{R}$, G_v is a finite group of order 2, and so the finiteness hypothesis in Proposition 2.4.11 is satisfied. Therefore, the conclusion also holds in this case.

Proposition 5.1.5. *The functor $C_c(G_{F,S}, -)$ preserves direct limits in $(Mod^{ind-ad}_{\Lambda[G_{F,S}]})$. Moreover, if M is an object in $\mathcal{C}_{\Lambda, G_{F,S}}$ which is a finitely generated Λ-module, we have the following isomorphism*

$$C_c(G_{F,S}, M) \cong \varprojlim_n C_c(G_{F,S}, M/\mathfrak{M}^n M)$$

of complexes and isomorphisms

$$H^i_c(G_{F,S}, M) \cong \varprojlim_n H^i_c(G_{F,S}, M/\mathfrak{M}^n M)$$

of cohomology groups.

Proof : The isomorphism of complexes is immediate from the definition of $C_c(G_{F,S}, -)$ as a cone. The second isomorphism now follows from the long exact sequence of cohomology groups in the preceding proposition and the above discussion. □

We also state the following two propositions which are variants of Proposition 2.4.12 and Proposition 2.4.13. The proofs are similar to those used in the two propositions.

Proposition 5.1.6. *Let M be an object in $\mathcal{C}_{\Lambda,G_{F,S}}$, and let $\{M_n\}$ be an inverse system of objects in $\mathcal{C}_{\Lambda,G_{F,S}}$ which are also finitely geenrated Λ-modules. Suppose that $\varprojlim_n M_n \cong M$. Then we have the following isomorphism*

$$H^i_c(G_{F,S}, M) \cong \varprojlim_n H^i_c(G_{F,S}, M_n)$$

for $n \geq 0$. □

Proposition 5.1.7. *Let $M^\bullet$ be a bounded complex of objects in $\mathcal{C}_{\Lambda,G_{F,S}}$, and let $\{M^\bullet_n\}$ be an inverse system of bounded complexes of objects in $\mathcal{C}_{\Lambda,G_{F,S}}$ which are finitely generated Λ-modules. Suppose that $\varprojlim_n M^\bullet_n \cong M^\bullet$ as complexes. Then we have the following isomorphism*

$$H^i_c(G_{F,S}, M^\bullet) \cong \varprojlim_n H^i_c(G_{F,S}, M^\bullet_n)$$

for $n \geq 0$. □

Lemma 5.1.8. *We have*

$$H^j_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \cong \begin{cases} \mathbb{Q}_p/\mathbb{Z}_p & \text{if } j = 3, \\ 0 & \text{if } j > 3. \end{cases}$$

Suppose Λ is Noetherian. If M is a Λ-module endowed with a trivial $G_{F,S}$-action, then

$$H^j_c(G_{F,S}, M(1)) \cong \begin{cases} M & \text{if } j = 3, \\ 0 & \text{if } j > 3. \end{cases}$$

In the case where T is a R-module with a trivial $G_{F,S}$-action, we have an isomorphism $H^3_c(G_{F,S}, \Lambda \otimes_R T(1)) \cong \Lambda \otimes_R T$ of Λ-Λ-bimodules.

Proof : By the long exact sequence of Poitou-Tate [NSW, 8.6.13], we have the following exact sequence

$$H^2(G_{F,S}, \mathbb{Z}/p^n\mathbb{Z}(1)) \to \bigoplus_{v\in S_f} H^2(G_v, \mathbb{Z}/p^n\mathbb{Z}(1)) \oplus \bigoplus_{v\in S_\mathbb{R}} \hat{H}^2(G_v, \mathbb{Z}/p^n\mathbb{Z}(1)) \to \mathbb{Z}/p^n\mathbb{Z} \to 0$$

and an isomorphism

$$H^3(G_{F,S}, \mathbb{Z}/p^n\mathbb{Z}(1)) \xrightarrow{\text{res}} \bigoplus_{v\in S_\mathbb{R}} \hat{H}^3(G_v, \mathbb{Z}/p^n\mathbb{Z}(1)).$$

By the definition of continuous cochains with compact support and the fact that $\mathrm{cd}_p(G_v) = 2$ for $v \in S_f$, we have $H^3_c(G_{F,S}, \mathbb{Z}/p^n\mathbb{Z}(1)) \cong \mathbb{Z}/p^n\mathbb{Z}$. The remainder of the lemma will then follow from a similar argument to that in Lemma 5.1.1. □

Let M be a bounded complex of objects in $\mathcal{C}_{\Lambda,G_{F,S}}$ which are finitely presented over Λ. We define two morphisms

$$\begin{aligned} {}_c\cup : C_c(G_{F,S}, M^\vee(1)) \otimes_\Lambda C(G_{F,S}, M) &\longrightarrow C_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \\ \cup_c : C(G_{F,S}, M^\vee(1)) \otimes_\Lambda C_c(G_{F,S}, M) &\longrightarrow C_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \end{aligned}$$

of complexes of abelian groups which are given by the following respective formulas (see [Ne, 5.3.3.2, 5.3.3.3])

$$\begin{aligned} (a, a_S)_c \cup b &= (a \cup b, a_S \cup \mathrm{res}_{S_f}(b)) \\ a \cup_c (b, b_S) &= (a \cup b, (-1)^{\bar{a}}\mathrm{res}_{S_f}(a) \cup b_S) \end{aligned}$$

where $\cup$ is the total cup product

$$C(G_{F,S}, M^\vee(1)) \otimes_\Lambda C(G_{F,S}, M) \longrightarrow C(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1))$$

of Section 2.7.

By Lemma 5.1.8, we have a quasi-isomorphism $\mathbb{Q}_p/\mathbb{Z}_p[-3] \xrightarrow{i} \tau_{\geq 3}C_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1))$ of complexes of $\mathbb{Z}_p$-modules. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is an injective $\mathbb{Z}_p$-module, the map i has a homotopy inverse. We shall fix one such map

$$r : \tau_{\geq 3}C_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p[-3],$$

and this induces the following morphism

$$C_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow \tau_{\geq 3}C_c(G_{F,S}, \mathbb{Q}_p/\mathbb{Z}_p(1)) \xrightarrow{r} \mathbb{Q}_p/\mathbb{Z}_p[-3]$$

of complexes of $\mathbb{Z}_p$-modules. Combining this with the total cup products, we obtain the following morphisms

$$\begin{aligned} C_c(G_{F,S}, M^{\vee}(1)) \otimes_{\Lambda} C(G_{F,S}, M) &\longrightarrow \mathbb{Q}_p/\mathbb{Z}_p[-3] \\ C(G_{F,S}, M^{\vee}(1)) \otimes_{\Lambda} C_c(G_{F,S}, M) &\longrightarrow \mathbb{Q}_p/\mathbb{Z}_p[-3] \end{aligned}$$

of complexes of $\mathbb{Z}_p$-modules. We can now state the following theorem.

Theorem 5.1.9. *Let M be a bounded complex of objects in $\mathcal{C}_{\Lambda, G_{F,S}}$ which are finitely generated over Λ. Then we have the following isomorphisms*

$$\begin{aligned} \mathbf{R}\Gamma(G_{F,S}, M) &\longrightarrow \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\Big(\mathbf{R}\Gamma_c\big(G_{F,S}, M^{\vee}(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)[-3] \\ \mathbf{R}\Gamma_c(G_{F,S}, M) &\longrightarrow \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\Big(\mathbf{R}\Gamma\big(G_{F,S}, M^{\vee}(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)[-3] \end{aligned}$$

in $\mathbf{D}(Mod_{\Lambda})$.

Proof : By a similar limiting argument (using Lemma 5.1.5 for the compact support cohomology) to that of Theorem 5.1.2, we can reduce to the case that M is finite. The conclusion then follows from the usual Poitou-Tate duality [NSW, 8.6.13]. □

Remark. Theorem 5.1.2 and Theorem 5.1.9 are stated in [FK] for the case that M is a finitely generated projective Λ-module.

5.2 Iwasawa setting

We retain the notations introduced in the previous section. Assume further that if $p = 2$ and F is a number field, then F has no real primes. By remarks after Definition 5.1.3, we may (and will) take

$$\mathrm{Cone}\left(C(G_{F,S}, -) \xrightarrow{\mathrm{res}_S} \bigoplus_{v \in S_f} C(G_v, -) \right)[-1]$$

to be our complex of continuous cochains with compact support which we denote by $C_c(G_{F,S}, -)$ by abuse of notation.

Let F_{∞} be a p-adic Lie extension of F which is contained in F_S. In other words, F_{∞} is a Galois extension of F whose Galois group Γ is a (compact) p-adic Lie group. Write $H = \mathrm{Gal}(F_S/F_{\infty})$, and let $\mathscr{U}$ denote the collection of open normal subgroups of $G_{F,S}$ containing H. For each $U \in \mathscr{U}$, we let $F_U = (F_S)^U$ and define S_U to be the set of primes in F_U above S. Note that this is a finite Galois extension of F. Let $(S_U)_f$ denote the collection of non-archimedean primes of F_U in S_U. As before, we write $\Lambda = R[\![\Gamma]\!]$, which is a Noetherian ring.

We begin describing how Section 4.4 may be applied here. Let $v \in S_f$, and fix an embedding $F^{\mathrm{sep}} \hookrightarrow F_v^{\mathrm{sep}}$, which induces a continuous group monomorphism

$$\alpha = \alpha_v : G_v \hookrightarrow G_F,$$

where $G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$. Let X be an ind-admissible $R[G_F]$-module. For a finite Galois extension F' of F, write $U = \mathrm{Gal}(F^{\mathrm{sep}}/F')$ and $X_U = R[G_F/U] \otimes_R X$. The embedding $F^{\mathrm{sep}} \hookrightarrow F_v^{\mathrm{sep}}$ determines a prime v' of F' above v such that $F'_{v'}$ is a finite Galois extension of F_v and $G_{v'} := \mathrm{Gal}(F_v^{\mathrm{sep}}/F'_{v'}) = \alpha^{-1}(U)$.

Fix coset representatives $\sigma_i \in G_F$ of

$$G_F/U = \bigcup_i \sigma_i \alpha(G_v/G_{v'}).$$

Then the set of distinct primes in F' above v is given by the (finite) collection $\{\sigma_i(v')\}$, and we may identify $G_{\sigma_i(v')}$ with the subgroup $\alpha_i(G_{v'})$ of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$, where $\alpha_i = \mathrm{Ad}(\sigma_i) \circ \alpha$. We then have the following isomorphisms

$$C(G_{\sigma_i(v')}, X) \xrightarrow{\sim} C(G_{v'}, \alpha^* X) \xrightarrow{\sim} C(G_{v'}, \alpha_i^* X)$$

of complexes, where the first isomorphism is induced by the pair

$$G_{v'} \xrightarrow{\alpha_i} G_{\sigma_i(v')} \qquad \alpha_i^* X \xrightarrow{\sigma_i^{-1}} \alpha^* X\ ,$$

and the second is induced by the pair

$$G_{v'} \xrightarrow{\mathrm{id}} G_{\sigma_i(v')} \qquad \alpha^* X \xrightarrow{\sigma_i} \alpha_i^* X\ .$$

Recall that in Section 4.4, we have the following decomposition

$$w = (w_i) : \alpha^* X_U \xrightarrow{\sim} \bigoplus_i (\alpha_i^* X)_{G_{v'}}$$

of G_v-modules. This induces the following isomorphism

$$C(G_v, X_U) \cong \bigoplus_i C\big(G_v, (\alpha_i^* X)_{G_{v'}}\big).$$

Suppose that X is of finite type over R or cofinite type over R. Then each summand in the last complex is quasi-isomorphic to $C(G_{v'}, \alpha_i^* X)$ by Lemma 4.3.1 and Lemma 4.3.3. Combining this with the above, we obtain a quasi-isomorphism

$$C(G_v, X_U) \xrightarrow{\sim} \bigoplus_i C(G_{\sigma_i(v')}, X)$$

and isomorphisms

$$H^n(G_v, X_U) \cong \bigoplus_i H^n(G_{\sigma_i(v')}, X)$$

of cohomology groups for $n \geq 0$. We shall apply the above discussion to ind-admissible $R[G_{F,S}]$-modules, which we view as $R[G_F]$-modules via the canonical quotient map $G_F \twoheadrightarrow G_{F,S}$.

Lemma 5.2.1. *Let T be an ind-admissible $R[G_{F,S}]$-module of finite type over R, and let A be an ind-admissible $R[G_{F,S}]$-module of cofinite type over R. Then we have the following isomorphisms*

$$\begin{aligned} H^j(G_{F,S}, \mathscr{F}_\Gamma(T)) &\cong \varprojlim_U H^j(G_{F,S}, T_U) \cong \varprojlim_U H^j(G_{F_U,S_U}, T), \\ H^j(G_{F,S}, F_\Gamma(A)) &\cong \varinjlim_U H^j(G_{F,S}, {}_UA) \cong \varinjlim_U H^j(G_{F_U,S_U}, A) \cong H^j(\mathrm{Gal}(F_S/F_\infty), A), \\ H^j(G_v, \mathscr{F}_\Gamma(T)) &\cong \varprojlim_U H^j(G_v, T_U) \cong \varprojlim_U \bigoplus_{w|v} H^j(G_w, T), \\ H^j(G_v, F_\Gamma(A)) &\cong \varinjlim_U H^j(G_v, {}_UA) \cong \varinjlim_U \bigoplus_{w|v} H^j(G_w, A). \end{aligned}$$

Proof : All the isomorphisms follow immediately from Section 4.3, Section 4.4 and the above discussion. □

We would like to derive an analogue of Shapiro's lemma for compactly supported cohomology. Let F' be a finite Galois extension of F which is contained in F_S. Denote the set of primes of F' above S by S'. Let X be an ind-admissible $R[G_{F,S}]$-module. We write $U = \mathrm{Gal}(F_S/F')$ and $X_U = R[G_{F,S}/U] \otimes_R X$. By the discussion in Section 4.4 and above, we have the following diagram

$$\begin{array}{ccccc} C(G_{F,S}, X_U) & \longrightarrow & \bigoplus_{v \in S_f} C(G_v, X_U) & \xrightarrow{\sim} & \bigoplus_{v \in S_f} \bigoplus_{v'|v} C(G_v, R[G_v/G_{v'}] \otimes_R X) \\ \Big\downarrow \mathrm{sh} & & & & \Big\downarrow \mathrm{sh} \\ C(G_{F',S'}, X) & \longrightarrow & & & \bigoplus_{v' \in S'_f} C(G_{v'}, X) \end{array}$$

which commutes up to homotopy. This in turn induces a quasi-isomorphism (functorial in X)

$$\mathrm{sh}_c : C_c(G_{F,S}, X_U) \longrightarrow C_c(G_{F',S'}, X)$$

which fits into the following commutative (up to homotopy) diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{v\in S_f} C(G_v, X_U)[-1] & \longrightarrow & C_c(G_{F,S}, X_U) & \longrightarrow & C(G_{F,S}, X_U) & \longrightarrow & 0 \\
 & & \wr\downarrow & & \| & & \| & & \\
0 & \longrightarrow & \bigoplus_{v\in S_f} \bigoplus_{v'\mid v} C(G_v, R[G_v/G_{v'}] \otimes_R X)[-1] & \longrightarrow & C_c(G_{F,S}, X_U) & \longrightarrow & C(G_{F,S}, X_U) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{sh}[-1]} & & \downarrow{\scriptstyle \mathrm{sh}_c} & & \downarrow{\scriptstyle \mathrm{sh}} & & \\
0 & \longrightarrow & \bigoplus_{v'\in S'_f} C(G_v, X)[-1] & \longrightarrow & C_c(G_{F',S'}, X) & \longrightarrow & C(G_{F',S'}, X) & \longrightarrow & 0
\end{array}$$

Suppose that $F'' \subseteq F_S$ is another finite Galois extension of F containing F', and write S'' for the set of primes of F'' above S and $V = \mathrm{Gal}(F_S/F'')$. Again from Section 4.4, we have the following morphisms

$$\begin{aligned}
\mathrm{res}_c &: C_c(G_{F',S'}, X) \longrightarrow C_c(G_{F'',S''}, X) \\
\mathrm{cor}_c &: C_c(G_{F'',S''}, X) \longrightarrow C_c(G_{F',S'}, X),
\end{aligned}$$

which are functorial in X and fit in the following diagrams, which are commutative up to homotopy:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \oplus_{v'\in S'_f} C(G_{v'}, X)[-1] & \longrightarrow & C_c(G_{F',S'}, X) & \longrightarrow & C(G_{F',S'}, X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{res}[-1]} & & \downarrow{\scriptstyle \mathrm{res}_c} & & \downarrow{\scriptstyle \mathrm{res}} & & \\
0 & \longrightarrow & \oplus_{v''\in S''_f} C(G_{v''}, X)[-1] & \longrightarrow & C_c(G_{F'',S''}, X) & \longrightarrow & C(G_{F'',S''}, X) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \oplus_{v''\in S''_f} C(G_{v''}, X)[-1] & \longrightarrow & C_c(G_{F'',S''}, X) & \longrightarrow & C(G_{F'',S''}, X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{cor}[-1]} & & \downarrow{\scriptstyle \mathrm{cor}_c} & & \downarrow{\scriptstyle \mathrm{cor}} & & \\
0 & \longrightarrow & \oplus_{v'\in S'_f} C(G_{v'}, X)[-1] & \longrightarrow & C_c(G_{F',S'}, X) & \longrightarrow & C(G_{F',S'}, X) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccc}
C_c(G_{F,S}, X_U) & \xrightarrow{\mathrm{sh}_c} & C_c(G_{F',S'}, X) \\
\downarrow{\scriptstyle \mathrm{Tr}_*} & & \downarrow{\scriptstyle \mathrm{res}_c} \\
C_c(G_{F,S}, X_V) & \xrightarrow{\mathrm{sh}_c} & C_c(G_{F'',S''}, X)
\end{array}
\qquad
\begin{array}{ccc}
C_c(G_{F,S}, X_V) & \xrightarrow{\mathrm{sh}_c} & C_c(G_{F'',S''}, X) \\
\downarrow{\scriptstyle \mathrm{pr}_*} & & \downarrow{\scriptstyle \mathrm{cor}_c} \\
C_c(G_{F,S}, X_U) & \xrightarrow{\mathrm{sh}_c} & C_c(G_{F',S'}, X)
\end{array}$$

Since all the morphisms constructed above are functorial, they can be extended to complexes of ind-admissible modules. Hence, we may conclude the following.

Proposition 5.2.2. (*a*) *For a bounded below complex A of ind-admissible $R[G_{F,S}]$-modules which are cofinite type over R, the canonical morphism of complexes*

$$C_c(G_{F,S}, F_\Gamma(A)) \xrightarrow{\sim} \varinjlim_{U,\mathrm{Tr}} C_c(G_{F,S},\, {}_UA)$$

is an isomorphism.

(*b*) *Let T be an object of $\mathcal{C}_{R,G_{F,S}}$. Then the canonical morphism of complexes*

$$C_c(G_{F,S}, \mathscr{F}_\Gamma(T)) \xrightarrow{\sim} \varprojlim_{U} C_c(G_{F,S}, T_U)$$

is an isomorphism and induces isomorphisms

$$H^j_c(G_{F,S}, \mathscr{F}_\Gamma(T)) \cong \varprojlim_{U,\mathrm{pr}} H^j_c(G_{F,S}, T_U) \cong \varprojlim_{U,\mathrm{cor}_c} H^j_c(G_{F_U,S_U}, T)$$

of cohomology groups for $j \geq 0$. □

In the next proposition, we shall show that if T is an object of $\mathcal{C}^{R-ft}_{R,G_{F,S}}$, the cohomology groups $H^j(G_{F,S}, \mathscr{F}_\Gamma(T)), H^j(G_v, \mathscr{F}_\Gamma(T))$ and $H^j_c(G_{F,S}, \mathscr{F}_\Gamma(T))$ are Noetherian Λ-modules.

Proposition 5.2.3. *If T is an object of $\mathcal{C}_{R,G_{F,S}}$ which is a Noetherian R-module, then the cohomology groups $H^j(G_{F,S}, \mathscr{F}_\Gamma(T))$, $H^j(G_v, \mathscr{F}_\Gamma(T))$ and $H^j_c(G_{F,S}, \mathscr{F}_\Gamma(T))$ are Noetherian Λ-modules for $j \geq 0$.*

Proof : We first assume that Γ is pro-p. By Proposition 4.2.5, the cohomology groups $H^j(G_{F,S}, \mathscr{F}_\Gamma(T))$ and $H^j(G_v, \mathscr{F}_\Gamma(T))$ are Noetherian Λ-modules. It then follows from Proposition 5.1.4 that $H^j_c(G_{F,S}, \mathscr{F}_\Gamma(T))$ is a Noetherian Λ-module.

Now suppose that Γ is a general compact p-adic Lie group. Let Γ' be an open normal (uniform) pro-p subgroup of Γ. Write $L = (F_\infty)^{\Gamma'}$ and denote the set of primes in L above S by S'. Denote by $\mathscr{V}$ the collection of open normal subgroups of $G_{L,S}$ containing H. Since this is a cofinal subset of $\mathscr{U}$, we have isomorphisms

$$H^j(G_{F,S}, \mathscr{F}_\Gamma(T)) \cong \varprojlim_{U\in\mathscr{U}} H^j(G_{F_U,S_U}, T) \cong \varprojlim_{V\in\mathscr{V}} H^j(G_{L_V,S_V}, T) \cong H^j(G_{L,S'}, \mathscr{F}_{\Gamma'}(T))$$
$$H^j_c(G_{F,S}, \mathscr{F}_\Gamma(T)) \cong \varprojlim_{U\in\mathscr{U}} H^j_c(G_{F_U,S_U}, T) \cong \varprojlim_{V\in\mathscr{V}} H^j_c(G_{L_V,S_V}, T) \cong H^j_c(G_{L,S'}, \mathscr{F}_{\Gamma'}(T))$$

of $R[\![\Gamma']\!]$-modules. Since Γ' is pro-p, we have that these cohomology groups are finitely generated over $R[\![\Gamma']\!]$, and hence over Λ. By Proposition 5.1.4, we have that $H^j(G_v, \mathscr{F}_\Gamma(T))$ is a Noetherian Λ-module. □

Corollary 5.2.4. *If T is a bounded complex of objects of $\mathcal{C}^{R-ft}_{R,G_{F,S}}$, then $\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T))$, $\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T))$ and $\mathbf{R}\Gamma_c(G_{F,S}, \mathscr{F}_\Gamma(T))$ are objects in $\mathbf{D}^b(Mod^{\Lambda-ft}_\Lambda)$.* □

5.3 Duality over p-adic Lie extensions I

We retain the notations from the previous two sections and state the following variant of Tate local duality and Poitou-Tate duality for p-adic Lie extensions.

Theorem 5.3.1. *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_{F,S}}$ which are finitely generated over R. Then we have the following isomorphism*

$$\begin{aligned}
\mathbf{R}\Gamma(G_v, \mathscr{F}_\Gamma(T)) &\longrightarrow \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\Big(\mathbf{R}\Gamma\big(G_v, F_\Gamma(T^\vee)^\iota(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)[-2] \\
\mathbf{R}\Gamma(G_{F,S}, \mathscr{F}_\Gamma(T)) &\longrightarrow \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\Big(\mathbf{R}\Gamma_c\big(G_{F,S}, F_\Gamma(T^\vee)^\iota(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)[-3] \\
\mathbf{R}\Gamma_c(G_{F,S}, \mathscr{F}_\Gamma(T)) &\longrightarrow \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\Big(\mathbf{R}\Gamma\big(G_{F,S}, F_\Gamma(T^\vee)^\iota(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)[-3]
\end{aligned}$$

in $\mathbf{D}(Mod_\Lambda)$.

Proof : It follows from Lemma 4.1.6 that $\mathscr{F}_\Gamma(T)$ is a bounded complex of Noetherian (and hence finitely presented) Λ-modules. Therefore, we may apply Theorem 5.1.2 and Theorem 5.1.9 to $\mathscr{F}_\Gamma(T)$. The conclusion will now follow from Proposition 4.1.11. □

Let T be a bounded complex of ind-admissible $R[G_v]$-modules which are finitely generated over R. Choose a bounded complex $\widetilde{\Omega}$ of injective R-modules which represents ω_R in $\mathbf{D}(Mod_R)$. Then $\mathrm{Hom}_R(T, \widetilde{\Omega})$ is a bounded complex of ind-admissible $R[G_v]$-modules with cohomology groups which are finitely generated over R (see [Ne, 4.3.2]). By loc. cit. Prop. 3.3.9, there is a subcomplex T^* of $\mathrm{Hom}_R(T, \widetilde{\Omega})$ which is a complex of ind-admissible $R[G_v]$-modules that are finitely generated over R and is quasi-isomorphic to $\mathrm{Hom}_R(T, \widetilde{\Omega})$ via the inclusion map. Therefore, we have the following morphism

$$\pi : T \otimes_R T^* \longrightarrow T \otimes_R \mathrm{Hom}_R(T, \widetilde{\Omega}) \longrightarrow \widetilde{\Omega}$$

of complexes of $R[G_v]$-modules, where the first morphism is induced by the inclusion and the second is the usual evaluation map. Then we have the following morphism

$$\overline{\pi} : \mathscr{F}_\Gamma(T) \otimes_R \mathscr{F}_\Gamma(T^*)^\iota(1) \xrightarrow{\ \phi\ } \Lambda \otimes_R T \otimes_R T^*(1) \xrightarrow{\mathrm{id}\otimes\pi} \Lambda \otimes_R \widetilde{\Omega}(1)$$

of complexes of $\Lambda[G_v]$-Λ-bimodules, where ϕ is defined as in Corollary 4.1.14. By what we have done in Section 2.7, we have a morphism

$$C\big(G_v, \mathscr{F}_\Gamma(T)\big) \otimes_R C\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big) \longrightarrow C\big(G_v, \Lambda \otimes_R \widetilde{\Omega}(1)\big) \longrightarrow \tau_{\geq 2}^{II} C\big(G_v, \Lambda \otimes_R \widetilde{\Omega}(1)\big)$$

of complexes of Λ-Λ-bimodules. Taking the adjoint, we have the following map of complexes of Λ-modules

$$C\big(G_v, \mathscr{F}_\Gamma(T)\big) \longrightarrow \mathrm{Hom}_{\Lambda^\circ}\Big(C\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \tau_{\geq 2}^{II} C\big(G_v, \Lambda \otimes_R \widetilde{\Omega}(1)\big)\Big).$$

By Lemma 5.1.1, we have a chain map

$$\upsilon : \Lambda \otimes_R \widetilde{\Omega} \longrightarrow \tau^{II}_{\geq 2} C\big(G_v, \Lambda \otimes_R \widetilde{\Omega}(1)\big)$$

which is a quasi-isomorphism. Combining everything, we obtain a morphism

$$\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T)\big) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$

in $\mathbf{D}(Mod_\Lambda)$. In fact, we claim that this morphism is an isomorphism.

Theorem 5.3.2. *Let T be a bounded complex in $\mathcal{C}^{R-ft}_{R,G_v}$. Then there is an isomorphism*

$$\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T)\big) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$

in $\mathbf{D}(Mod_\Lambda)$ and an isomorphism

$$\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T)^\iota\big) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)(1)\big), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$

in $\mathbf{D}(Mod_{\Lambda^o})$, where T^* is defined as above.

Remark. The second morphism in Theorem 5.3.2 is constructed in a similar manner as the first. The remainder of the section will be devoted to showing that the above morphisms are isomorphisms. In fact, we shall only show that the first morphism is an isomorphism, the second being analogous.

Let B be a $\Lambda \otimes_R \Lambda^o$-module, and let A be a $\Lambda \otimes_R \Lambda^o$-submodule of B. Suppose that these modules are endowed with topologies making them both compact Λ-modules and compact Λ^o-modules, and that the topology on A coincides with the subspace topology induced from B. Then for any bounded complex T of objects in $\mathcal{C}^{R-ft}_{R,G}$, we define

$$\mathscr{F}_{B/A}(T) = [A \longrightarrow B] \otimes_\Lambda \mathscr{F}_\Gamma(T),$$

where A and B are in degree -1 and 0 respectively.

Lemma 5.3.3. *Let (A, B) and (A', B') be two pairs of $\Lambda \otimes_R \Lambda^o$-submodules with $A \subseteq B$ and $A' \subseteq B'$. Suppose that these modules are endowed with topologies making them both compact Λ-modules and compact Λ^o-modules, and that the topology on A (resp., A') coincides with the subspace topology induced from B (resp., B'). Suppose $f : B \longrightarrow B'$*

is a continuous $\Lambda \otimes_R \Lambda^o$-homomorphism with $f(A) \subseteq A'$. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathbf{R}\Gamma\big(G_v, \mathscr{F}_{B/A}(T)\big) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), [A \longrightarrow B] \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \\ \downarrow & & \downarrow \\ \mathbf{R}\Gamma\big(G_v, \mathscr{F}_{B'/A'}(T)\big) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), [A' \longrightarrow B'] \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \end{array}$$

Proof : It is easy to see that the following diagram

$$\begin{array}{ccc} [A \longrightarrow B] \otimes_\Lambda \mathscr{F}_\Gamma(T) \otimes_R \mathscr{F}_\Gamma(T^*)^\iota(1) & \xrightarrow{\mathrm{id}\otimes\phi} & [A \longrightarrow B] \otimes_R \widetilde{\Omega}(1) \\ \Big\downarrow{\scriptstyle f\otimes\mathrm{id}} & & \Big\downarrow{\scriptstyle f\otimes\mathrm{id}} \\ [A' \longrightarrow B'] \otimes_\Lambda \mathscr{F}_\Gamma(T) \otimes_R \mathscr{F}_\Gamma(T^*)^\iota(1) & \xrightarrow{\mathrm{id}\otimes\phi} & [A' \longrightarrow B'] \otimes_R \widetilde{\Omega}(1) \end{array}$$

is commutative. Applying the total cup products of continuous cochain groups to this diagram and using the results of Section 2.7, we obtain the required conclusion. □

Let $I = I_\Gamma$ denote the augmentation ideal of Λ. Recall that if G is a profinite group, its maximal pro-p quotient is denoted by $G(p)$. We then have the following lemma.

Lemma 5.3.4. *Let n be an arbitrary positive integer. Then we have the following statements.* (1) *The module I^n is a flat R-module.*

(2) *The module I^n/I^{n+1} is finitely generated over R.*

(3) *The module Λ/I^n is finitely generated over R.*

Proof : (1) We first consider the case when $R = \mathbb{Z}_p$ and Γ is finite. Then I^n is a $\mathbb{Z}_p$-submodule of $\mathbb{Z}_p[\Gamma]$. Since $\mathbb{Z}_p[\Gamma]$ is free over $\mathbb{Z}_p$, so is I^n. For a general R, I^n is the tensor product of R with the nth power of the augmentation ideal in $\mathbb{Z}_p[\Gamma]$ and so is free over R. Now if $\Gamma = \varprojlim \Gamma/U$ is profinite, we then have that I^n is the inverse limit of the nth powers of the augmentation ideals of $R[\Gamma/U]$. The conclusion follows from a similar argument to that used in Lemma 4.1.1.

(2) Let J denote the augmentation ideal in $\mathbb{Z}_p[\![\Gamma]\!]$. In this case, we have that J^n/J^{n+1} is a quotient of $\Gamma^{\mathrm{ab}}(p)^{\otimes n}$ which is finitely generated over $\mathbb{Z}_p$. Finally, one observes that

$$I^n/I^{n+1} \cong R \otimes_{\mathbb{Z}_p} (J^n/J^{n+1}),$$

and hence the conclusion follows.

(3) The case $n = 1$ is immediate. The general case follows from (2) and induction using the following exact sequence

$$0 \longrightarrow I^n/I^{n+1} \longrightarrow \Lambda/I^{n+1} \longrightarrow \Lambda/I^n \longrightarrow 0.$$

□

We are now able to state the following proposition which will be an important ingredient in our proof of Theorem 5.3.2.

Proposition 5.3.5. *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_v}^{R-ft}$. Then we have the following morphism of exact triangles.*

$$\begin{array}{ccc}
\mathbf{R}\Gamma\big(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)\big) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), I^n/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma\big(G_v, \mathscr{F}_{\Lambda/I^{n+1}}(T)\big) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda/I^{n+1} \otimes_R^{\mathbf{L}} \omega_R\Big)[-2] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma\big(G_v, \mathscr{F}_{\Lambda/I^n}(T)\big) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R\Big)[-2]
\end{array}$$

Proof: By Proposition 1.3.6, Lemma 4.1.1 and Lemma 5.3.4(1), we see that $I^n/I^{n+1}\otimes_R^{\mathbf{L}}\omega_R$ and $\Lambda/I^n \otimes_R^{\mathbf{L}} \omega_R$ are represented by $[I^{n+1} \longrightarrow I^n] \otimes_R \widetilde{\Omega}$ and $[I^n \longrightarrow \Lambda] \otimes_R \widetilde{\Omega}$ respectively. Therefore, the commutativity of the diagram in the proposition follows from Lemma 5.3.3. Clearly, the column on the right is an exact triangle. It will follow from the next lemma that the column on the left is also an exact triangle. □

Lemma 5.3.6. *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_v}^{R-ft}$. Then we have a quasi-isomorphism*

$$\mathrm{Cone}\Big(\mathscr{F}_{I^n/I^{n+1}}(T) \longrightarrow \mathscr{F}_{\Lambda/I^{n+1}}(T)\Big) \xrightarrow{\sim} \mathscr{F}_{\Lambda/I^n}(T)$$

of objects in $\mathcal{C}_{R,G}$.

Proof: It suffices to show that there is a quasi-isomorphism

$$\mathrm{Cone}\big([I^{n+1} \longrightarrow I^n] \longrightarrow [I^{n+1} \longrightarrow \Lambda]\big) \xrightarrow{\sim} [I^n \longrightarrow \Lambda].$$

Note that $\mathrm{Cone}\big([I^{n+1} \longrightarrow I^n] \longrightarrow [I^{n+1} \longrightarrow \Lambda]\big)$ is precisely the following complex

$$I^{n+1} \xrightarrow{f} I^{n+1} \oplus I^n \xrightarrow{g} \Lambda,$$

where $f(x) = (x, -x)$ and $g(x, y) = x + y$ for $x \in I^{n+1}$ and $y \in I^n$. One can now easily check that the following diagram

$$\begin{array}{ccccc} I^{n+1} & \longrightarrow & I^{n+1} \oplus I^n & \longrightarrow & \Lambda \\ \downarrow & & \downarrow \nu & & \| \\ 0 & \longrightarrow & I^n & \longrightarrow & \Lambda \end{array}$$

commutes, where ν is given by $\nu(x, y) = x + y$ for $x \in I^{n+1}$ and $y \in I^n$, and the vertical maps induce isomorphisms on cohomology. $\square$

We now describe the idea of the proof of Theorem 5.3.2. We shall first prove that the following morphism

$$\mathbf{R}\Gamma\big(G_v, \mathscr{F}_{\Lambda/I^n}(T)\big) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda/I^n \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$

is an isomorphism for all n. Then, Theorem 5.3.2 will follow from this by a limit argument. To show that the above morphism is an isomorphism, we will utilize Proposition 5.3.5. Note that if any two of the morphisms in Proposition 5.3.5 are quasi-isomorphisms, so is the third one. Therefore, by an inductive argument, we are reduced to showing that the following morphism

$$\mathbf{R}\Gamma\big(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)\big) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$

is an isomorphism for all $n \geq 0$. Note that Γ acts trivially on I^n/I^{n+1}. Therefore, one may view I^n/I^{n+1} as a Λ^o-module via the augmentation map $\Lambda \twoheadrightarrow R$. We now have the following lemma.

Lemma 5.3.7. *Let T be a bounded complex of objects in $\mathcal{C}^{R-ft}_{R,G_v}$. Then we have the following isomorphisms*

$$\mathscr{F}_{I^n/I^{n+1}}(T) \xleftarrow{\sim} I^n/I^{n+1} \otimes^{\mathbf{L}}_\Lambda \mathscr{F}_\Gamma(T) \xrightarrow{\sim} I^n/I^{n+1} \otimes^{\mathbf{L}}_R T$$

in $\mathbf{D}^b(\mathcal{C}_{R,G_v})$. Therefore, it follows that we have an isomorphism

$$\mathbf{R}\Gamma\big(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)\big) \xrightarrow{\sim} \mathbf{R}\Gamma(G_v, I^n/I^{n+1} \otimes^{\mathbf{L}}_R T).$$

Proof : Let P be a resolution of I^n/I^{n+1} consisting of finitely generated projective Λ^o-modules. Since $[I^{n+1} \longrightarrow I^n]$ is also a resolution of I^n/I^{n+1} of Λ^o-modules, there is a quasi-isomorphism

$$\alpha : P \longrightarrow [I^n \longrightarrow I^{n+1}]$$

of complexes of compact Λ^o-modules which lifts the identity map on I^n/I^{n+1}. This induces the following quasi-isomorphism

$$\alpha_* : P \otimes_\Lambda \Lambda^\iota \longrightarrow [I^n \longrightarrow I^{n+1}] \otimes_\Lambda \Lambda^\iota$$

and a morphism

$$\alpha_{**} : P \otimes_\Lambda \Lambda^\iota \otimes_R T \longrightarrow [I^n \longrightarrow I^{n+1}] \otimes_\Lambda \Lambda^\iota \otimes_R T.$$

The fact that α_{**} is a quasi-isomorphism now follows from a spectral sequence argument similar to that used in Lemma 4.2.3, thus proving the first isomorphism. The second isomorphism is immediate from the second assertion of Lemma 4.2.3. □

Lemma 5.3.8. *For each n, there is a commutative diagram*

$$\begin{array}{ccc}
I^n/I^{n+1} \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma(G_v, T) & \longrightarrow & I^n/I^{n+1} \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma\big(G_v, T^*(1)\big), \omega_R\Big)[-2] \\
\downarrow \wr & & \downarrow \wr \\
\mathbf{R}\Gamma(G_v, I^n/I^{n+1} \otimes^{\mathbf{L}}_R T) & \longrightarrow & \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma\big(G_v, T^*(1)\big), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \\
\Big\downarrow \wr & & \downarrow \wr \\
 & & \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big) \otimes^{\mathbf{L}}_\Lambda R, I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \\
 & & \uparrow \wr \\
\mathbf{R}\Gamma\big(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)\big) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]
\end{array}$$

where the vertical morphisms are isomorphisms.

Proof : Let Q be a resolution of I^n/I^{n+1} consisting of finitely generated projective R-modules. Note that such a resolution Q exists because of Lemma 5.3.4(2). Then we have an isomorphism

$$\begin{aligned}
\alpha : Q \otimes_R C(G_v, T) &\longrightarrow C(G_v, Q \otimes_R T) \\
x \otimes \sigma &\mapsto \big((g_1, ..., g_j) \mapsto x \otimes \sigma(g_1, ..., g_j)\big)
\end{aligned}$$

of complexes by [Ne, Prop. 3.4.4]. This fits into the following commutative diagram

$$\begin{array}{ccc}
Q \otimes_R C(G_v, T) \otimes_R C(G_v, T^*(1)) & \longrightarrow & Q \otimes_R C(G_v, \widetilde{\Omega}(1)) \\
\Big\downarrow {\scriptstyle \alpha \otimes \mathrm{id}} & & \Big\downarrow {\scriptstyle \alpha'} \\
C(G_v, Q \otimes_R T) \otimes_R C(G_v, T^*(1)) & \longrightarrow & C(G_v, Q \otimes_R \widetilde{\Omega}(1))
\end{array}$$

where α' is defined as above and is an isomorphism by [Ne, Prop. 3.4.4]. On the other hand, we also have the following commutative diagram.

$$\begin{array}{ccccc} Q \otimes_R C(G_v, \widetilde{\Omega}(1)) & \longrightarrow & Q \otimes_R \tau^{II}_{\geq 2} C(G_v, \widetilde{\Omega}(1)) & \longleftarrow & Q \otimes_R \widetilde{\Omega}[-2] \\ \downarrow{\scriptstyle \alpha'} & & \downarrow & & \| \\ C(G_v, Q \otimes_R \widetilde{\Omega}(1)) & \longrightarrow & \tau^{II}_{\geq 2} C(G_v, Q \otimes_R \widetilde{\Omega}(1)) & \longleftarrow & Q \otimes_R \widetilde{\Omega}[-2] \end{array}$$

Combining this with the above diagram and taking adjoints, we see that the top square in the lemma is commutative with the vertical morphism on the left being an isomorphism. The vertical morphism on the right is an isomorphism by Lemma 1.3.13.

Let P be a resolution of I^n/I^{n+1} consisting of finitely generated projective Λ^o-modules. Then as in Lemma 4.2.3, we view Q as a resolution of Λ^o-modules via the augmentation map $\Lambda \twoheadrightarrow R$, and there is a morphism

$$f : P \otimes_\Lambda \mathscr{F}_\Gamma(T) = P \otimes_\Lambda \Lambda^\iota \otimes_R T \longrightarrow Q \otimes_\Lambda \Lambda^\iota \otimes_R T \cong Q \otimes_R T$$

of complexes of objects in $\mathcal{C}_{R,G_v}$ which is a quasi-isomorphism. Let L be a resolution of R consisting of finitely generated projective Λ-modules. Then by Lemma 4.2.3 and a Λ^o version of Proposition 4.2.4, we have a morphism

$$g : C\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big) \otimes_\Lambda L \longrightarrow C(G_v, T^*(1))$$

of complexes of R-modules which is a quasi-isomorphism. Then we have the following commutative diagram

$$\begin{array}{ccc} C\big(G_v, P \otimes_\Lambda \mathscr{F}_\Gamma(T)\big) \otimes_R C\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big) \otimes_\Lambda L & \longrightarrow & C\big(G_v, P \otimes_\Lambda \Lambda \otimes_R \widetilde{\Omega}(1)\big) \otimes_\Lambda L \\ \downarrow{\scriptstyle f_* \otimes g} & & \downarrow{\scriptstyle \varepsilon} \\ C(G_v, Q \otimes_R T) \otimes_R C(G_v, T^*(1)) & \longrightarrow & C\big(G_v, Q \otimes_R \widetilde{\Omega}(1)\big) \end{array}$$

where ε is induced by the augmentation $L \to R$. Taking adjoints, we obtain the following commutative diagram.

$$\begin{array}{ccc} C(G_v, Q \otimes_R T) & \longrightarrow & \mathrm{Hom}_R\Big(C(G_v, T^*(1)), C\big(G_v, Q \otimes_R \widetilde{\Omega}(1)\big)\Big) \\ \uparrow & & \downarrow \\ C(G_v, P \otimes_\Lambda \mathscr{F}_\Gamma(T)) & \longrightarrow & \mathrm{Hom}_R\Big(C\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big) \otimes_\Lambda L, C\big(G_v, Q \otimes_R \widetilde{\Omega}(1)\big)\Big) \\ \| & & \downarrow \\ C(G_v, P \otimes_\Lambda \mathscr{F}_\Gamma(T)) & \longrightarrow & \mathrm{Hom}_{\Lambda^o}\Big(C\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \mathrm{Hom}_R\big(L, C\big(G_v, Q \otimes_R \widetilde{\Omega}(1)\big)\big)\Big) \end{array}$$

Combining the above diagram with the morphism

$$C\big(G_v, Q \otimes_R \widetilde{\Omega}(1)\big) \longrightarrow \tau^{II}_{\geq 2} C\big(G_v, Q \otimes_R \widetilde{\Omega}(1)\big) \longleftarrow Q \otimes_R \widetilde{\Omega}[-2],$$

we obtain the bottom commutative square in the derived category. The vertical morphisms in this part of the diagram are isomorphisms by Proposition 4.2.4, Lemma 5.3.7 and Proposition 1.3.7. □

Lemma 5.3.9. *For $n \geq 0$, the morphism*

$$\mathbf{R}\Gamma\big(G_v, \mathscr{F}_{I^n/I^{n+1}}(T)\big) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), I^n/I^{n+1} \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$

is an isomorphism.

Proof : The morphism

$$\mathbf{R}\Gamma(G_v, T) \longrightarrow \mathbf{R}\mathrm{Hom}_R\Big(\mathbf{R}\Gamma\big(G_v, T^*(1)\big), \omega_R\Big)[-2]$$

is an isomorphism by [Ne, Prop. 5.2.4(ii)], and so the top morphism of the diagram in Lemma 5.3.8 is an isomorphism. Since all the vertical morphisms in the diagram are isomorphisms, it follows that the bottom morphism is also an isomorphism, as required. □

Proposition 5.3.10. *We have an isomorphism*

$$\mathbf{R}\Gamma\big(G_v, \mathscr{F}_{\Lambda/I^n}(T)\big) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda/I^n \otimes^{\mathbf{L}}_R \omega_R\Big)[-2]$$

for all $n \geq 1$.

Proof : As seen in the above discussion, the preceding lemma allows us to perform an inductive argument using the morphism of exact triangles in Proposition 5.3.5 to obtain the required conclusion. □

We now finish up the proof of Theorem 5.3.2.

Proof of Theorem 5.3.2 : Let Q be a bounded above complex of finitely generated projective Λ^o-modules which represents $\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big)$. Such a complex exists by Corollary 4.2.6. Since $\widetilde{\Omega}$ has cohomology groups which are finitely generated over R, we may find (and fix) a subcomplex Ω of $\widetilde{\Omega}$ such that Ω is a complex of finitely generated R-modules and the inclusion $i : \Omega \hookrightarrow \widetilde{\Omega}$ is a quasi-isomorphism. Write $C_n = [I^n \longrightarrow \Lambda]$. Then $\mathrm{Hom}_{\Lambda^o}(Q, C_n \otimes_R \Omega)$ represents

$$\mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda/I^n \otimes^{\mathbf{L}}_R \omega_R\Big),$$

and $\mathrm{Hom}_{\Lambda^o}(Q, \Lambda \otimes_R \Omega)$ represents

$$\mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big),$$

since C_n is a complex of flat R-modules by Lemma 5.3.4(1). Now for each n, we have the following commutative diagram

$$\begin{array}{ccc} \mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T)\big) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \\ \downarrow & & \downarrow \\ \mathbf{R}\Gamma\big(G_v, \mathscr{F}_{\Lambda/I^n}(T)\big) & \longrightarrow & \mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\Gamma\big(G_v, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda/I^n \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \end{array}$$

which induces the following commutative diagram

$$\begin{array}{ccc} H^j\big(G_v, \mathscr{F}_\Gamma(T)\big) & \longrightarrow & H^j\big(\mathrm{Hom}_{\Lambda^o}(Q, \Lambda \otimes_R \Omega)\big) \\ \downarrow & & \downarrow \\ H^j\big(G_v, \mathscr{F}_{\Lambda/I^n}(T)\big) & \longrightarrow & H^j\big(\mathrm{Hom}_{\Lambda^o}(Q, C_n \otimes_R \Omega)\big) \end{array}$$

of cohomology groups. Since this diagram is compatible with n, we obtain the following commutative diagram.

$$\begin{array}{ccc} H^j\big(G_v, \mathscr{F}_\Gamma(T)\big) & \longrightarrow & H^j\big(\mathrm{Hom}_{\Lambda^o}(Q, \Lambda \otimes_R \Omega)\big) \\ \downarrow & & \downarrow \\ \varprojlim_n H^j\big(G_v, \mathscr{F}_{\Lambda/I^n}(T)\big) & \longrightarrow & \varprojlim_n H^j\big(\mathrm{Hom}_{\Lambda^o}(Q, C_n \otimes_R \Omega)\big) \end{array}$$

It remains to show that the top map is an isomorphism. By Proposition 5.3.10, the bottom map is an isomorphism. Since Q is a bounded above complex of finitely generated Λ-modules, we have an isomorphism

$$\varprojlim_n \mathrm{Hom}_{\Lambda^o}(Q, C_n \otimes_R \Omega) \cong \mathrm{Hom}_{\Lambda^o}(Q, \varprojlim_n C_n \otimes_R \Omega) \cong \mathrm{Hom}_{\Lambda^o}(Q, \Lambda \otimes_R \Omega)$$

of complexes of finitely generated Λ-modules. Since inverse limits are exact for finitely generated Λ-modules (for they are finitely generated compact Λ-modules and the inverse limit is exact for compact Λ-modules), we have isomorphisms

$$\varprojlim_n H^i\big(\mathrm{Hom}_{\Lambda^o}(Q, C_n \otimes_R \Omega)\big) \cong H^i\big(\mathrm{Hom}_{\Lambda^o}(Q, \Lambda \otimes_R \Omega)\big)$$

of cohomology groups, thus showing that the vertical map on the right is an isomorphism. On the other hand, we also have an isomorphism

$$\varprojlim_n (C_n \otimes_\Lambda \mathscr{F}_\Gamma(T)) \cong \mathscr{F}_\Gamma(T)$$

of complexes of objects in $\mathcal{C}^{\Lambda-ft}_{\Lambda,G_v}$ and hence an isomorphism

$$\varprojlim_n C\big(G_v, C_n \otimes_\Lambda \mathscr{F}_\Gamma(T)\big) \cong C\big(G_v, \mathscr{F}_\Gamma(T)\big).$$

By Proposition 2.4.13, we have $\varprojlim_n H^j\big(G_v, C_n \otimes_\Lambda \mathscr{F}_\Gamma(T)\big) \cong H^j\big(G_v, \mathscr{F}_\Gamma(T)\big)$. Therefore, the vertical map on the left is also an isomorphism. Hence the top map is an isomorphism, as required. □

5.4 Duality over p-adic Lie extensions II

We now describe the global analog of Theorem 5.3.2. Let T be a bounded complex of objects in $\mathcal{C}^{R-ft}_{R,G_{F,S}}$. In other words, T is a bounded complex of ind-admissible $R[G_{F,S}]$-modules which are finitely generated over R. As before, we fix a bounded complex $\widetilde{\Omega}$ of injective R-modules representing ω_R in $\mathbf{D}(Mod_R)$. Then $\mathrm{Hom}_R(T, \widetilde{\Omega})$ is a bounded complex of ind-admissible $R[G_{F,S}]$-modules with cohomology groups which are finitely generated over R (see [Ne, 4.3.2]). By loc. cit. Prop. 3.3.9, there is a subcomplex T^* of $\mathrm{Hom}_R(T, \widetilde{\Omega})$ which is a complex of ind-admissible $R[G_{F,S}]$-modules that are finitely generated over R, and is quasi-isomorphic to $\mathrm{Hom}_R(T, \widetilde{\Omega})$ via the inclusion map. Therefore, we have the following morphism

$$\pi : T \otimes_R T^* \longrightarrow T \otimes_R \mathrm{Hom}_R(T, \widetilde{\Omega}) \longrightarrow \widetilde{\Omega}$$

of complexes of $R[G_{F,S}]$-modules, where the first morphism is induced by the inclusion and the second is the usual evaluation map. Then we have the following morphism

$$\overline{\pi} : \mathscr{F}_\Gamma(T) \otimes_R \mathscr{F}_\Gamma(T^*)^\iota(1) \xrightarrow{\phi} \Lambda \otimes_R T \otimes_R T^* \xrightarrow{\mathrm{id}\otimes\pi} \Lambda \otimes_R \widetilde{\Omega}(1)$$

of complexes of $\Lambda[G_{F,S}]$-Λ-bimodules, where ϕ is defined as in Corollary 4.1.14.

We define two morphisms of complexes of Λ-Λ-bimodules

$${}_c\cup : C_c\big(G_{F,S}, \mathscr{F}_\Gamma(T)\big) \otimes_R C\big(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)\big) \longrightarrow C_c\big(G_{F,S}, \Lambda \otimes_R \widetilde{\Omega}(1)\big)$$

$$\cup_c : C\big(G_{F,S}, \mathscr{F}_\Gamma(T)\big) \otimes_R C_c\big(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)\big) \longrightarrow C_c\big(G_{F,S}, \Lambda \otimes_R \widetilde{\Omega}(1)\big)$$

which are given by the following respective formulas

$$\begin{aligned}(a, a_S)_c \cup b &= (a \cup b, a_S \cup_S \mathrm{res}_{S_f}(b)) \\ a \cup_c (b, b_S) &= (a \cup b, (-1)^{\bar{a}} \mathrm{res}_{S_f}(a) \cup_S b_S),\end{aligned}$$

where $\cup$ is the total cup product

$$C\big(G_{F,S}, \mathscr{F}_\Gamma(T)\big) \otimes_R C\big(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)\big) \longrightarrow C\big(G_{F,S}, \Lambda \otimes_R \widetilde{\Omega}(1)\big),$$

and $\cup_S$ is the sum of the local cup products.

By Lemma 5.1.8, we have a chain map

$$v : \Lambda \otimes_R \Omega[-3] \longrightarrow \Lambda \otimes_R \widetilde{\Omega}[-3] \longrightarrow \tau^{II}_{\geqslant 3} C_c\big(G_{F,S}, \Lambda \otimes_R \widetilde{\Omega}(1)\big),$$

where the first morphism is induced by the inclusion $i : \Omega \hookrightarrow \widetilde{\Omega}$. Since Λ is flat, the first morphism is a quasi-isomorphism and hence v is a quasi-isomorphism. Combining everything, we obtain the following morphisms

$$\begin{aligned}\mathbf{R}\Gamma\big(G_{F,S}, \mathscr{F}_\Gamma(T)\big) &\longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma_c\big(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-3] \\ \mathbf{R}\Gamma_c\big(G_{F,S}, \mathscr{F}_\Gamma(T)\big) &\longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma\big(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-3]\end{aligned}$$

in $\mathbf{D}(Mod_\Lambda)$. In fact, these two morphisms are isomorphisms.

Theorem 5.4.1. *Let T be a bounded complex of objects in $\mathcal{C}_{R,G_{F,S}}$ which are finitely generated R-modules. Then we have the following isomorphisms*

$$\begin{aligned}\mathbf{R}\Gamma\big(G_{F,S}, \mathscr{F}_\Gamma(T)\big) &\longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma_c\big(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-3] \\ \mathbf{R}\Gamma_c\big(G_{F,S}, \mathscr{F}_\Gamma(T)\big) &\longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma\big(G_{F,S}, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-3]\end{aligned}$$

in $\mathbf{D}(Mod_\Lambda)$.

Proof : The proof follows a similar argument as that used in Theorem 5.3.2. The limit argument for $C_c(G_{F,S}, -)$ follows from Proposition 5.1.7. □

5.5 Some spectral sequences

Let T be a bounded complex of ind-admissible $R[G_{F,S}]$-modules and finitely generated R-modules. Let T^* be a bounded complex of ind-admissible $R[G_{F,S}]$-modules which are finitely generated R-modules that represents $\mathbf{R}\mathrm{Hom}_R(T,\omega_R)$. Write $A = (T^*)^\vee$. Combining Theorem 5.4.1 and Theorem 5.3.1, we have an isomorphism

$$\mathbf{R}\Gamma\big(G_{F,S},\mathscr{F}_\Gamma(T)\big) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\Lambda^o}\Big(\mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\big(\mathbf{R}\Gamma(G_{F,S},F_\Gamma(A)),\mathbb{Q}_p/\mathbb{Z}_p\big),\Lambda\otimes^{\mathbf{L}}_R\omega_R\Big),$$

which gives rise to a cohomological spectral sequence of Λ-modules

$$\mathbb{E}\mathrm{xt}^i_{\Lambda^o}\Big(H^j\big(G_{F,S},F_\Gamma(A)\big)^\vee,\Lambda\otimes_R\Omega\Big) \Rightarrow H^{i+j}\big(G_{F,S},\mathscr{F}_\Gamma(T)\big),$$

where Ω is some complex of R-modules representing ω_R in $\mathbf{D}(Mod_R)$.

Remark. As mentioned in the introduction, when R is regular (or Gorenstein), the dualizing complex ω_R can be represented by R. Therefore, the above spectral sequence can be rewritten as

$$\mathrm{Ext}^i_{\Lambda^o}\Big(H^j\big(G_{F,S},F_\Gamma(A)\big)^\vee,\Lambda\Big) \Rightarrow H^{i+j}\big(G_{F,S},\mathscr{F}_\Gamma(T)\big).$$

This spectral sequence was first constructed in an unpublished note of Jannsen [Ja].

By a similar argument using the appropriate dualites, we can obtain analogous quasi-isomorphisms and spectral sequences for the local and compact support cases. In fact, all of these combine to give the following isomorphism of exact triangles.

Theorem 5.5.1. *Let T and A be defined as above. Then we have an isomorphism of exact triangles.*

$$\begin{array}{ccc}
\bigoplus_{v\in S_f}\mathbf{R}\Gamma\big(G_v,\mathscr{F}_\Gamma(T)\big)[-1] & \xrightarrow{\sim} & \bigoplus_{v\in S_f}\mathbf{R}\mathrm{Hom}_{\Lambda^o}\big(\mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma(G_v,F_\Gamma(A)),\mathbb{Q}_p/\mathbb{Z}_p),\Lambda\otimes^{\mathbf{L}}_R\omega_R\big)[-1] \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma_c\big(G_{F,S},\mathscr{F}_\Gamma(T)\big) & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\Lambda^o}\big(\mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma_c(G_{F,S},F_\Gamma(A)),\mathbb{Q}_p/\mathbb{Z}_p),\Lambda\otimes^{\mathbf{L}}_R\omega_R\big) \\
\downarrow & & \downarrow \\
\mathbf{R}\Gamma\big(G_{F,S},\mathscr{F}_\Gamma(T)\big) & \xrightarrow{\sim} & \mathbf{R}\mathrm{Hom}_{\Lambda^o}\big(\mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathbf{R}\Gamma(G_{F,S},F_\Gamma(A)),\mathbb{Q}_p/\mathbb{Z}_p),\Lambda\otimes^{\mathbf{L}}_R\omega_R\big)
\end{array}$$

□

5.6 Iwasawa theory over local fields

We shall say something about the situation over local fields. Let F be a local field of characteristic not equal to p with finite residue field. Let F_∞/F be a p-adic Lie extension with Galois group Γ. Write $G_E = \mathrm{Gal}(F^{\mathrm{sep}}/E)$ for every Galois extension E/F. Recall that by [NSW, Thm. 7.1.8(i)], we have $\mathrm{cd}_p(G_F) = 2$.

Let T be a bounded complex of ind-admissible $R[G_F]$-modules which are finitely generated R-modules, and let A be a bounded complex of ind-admissible $R[G_F]$-modules which are cofinitely generated R-modules. By Proposition 4.3.2 and Proposition 4.3.4, we have

$$\begin{gathered} C(G_F, F_\Gamma(A)) \xrightarrow{\sim} \varinjlim C(G_{F_\alpha}, A) \\ H^i(G_F, F_\Gamma(A)) \cong \varinjlim H^i(G_{F_\alpha}, A) \cong H^i(G_{F_\infty}, A) \\ C\big(G_F, \mathscr{F}_\Gamma(T)\big) \xrightarrow{\sim} \varprojlim C(G_{F_\alpha}, T) \\ H^i\big(G_F, \mathscr{F}_\Gamma(T)\big) \cong \varprojlim H^i(G_{F_\alpha}, T), \end{gathered}$$

where F_α runs through all finite Galois extension of F_∞/F. By a similar argument to that in Theorem 5.3.1 and Theorem 5.3.2, we have the following.

Theorem 5.6.1. *Let T be a bounded complex of ind-admissible $R[G_F]$-modules which are finitely generated R-modules, and let T^* be a bounded complex, that represents $\mathbf{R}\mathrm{Hom}_R(T, \omega_R)$, of ind-admissible $R[G_F]$-modules which are finitely generated R-modules. Then we have the following isomorphisms*

$$\begin{gathered} \mathbf{R}\Gamma(G_F, \mathscr{F}_\Gamma(T)) \longrightarrow \mathbf{R}\mathrm{Hom}_{\mathbb{Z}_p}\Big(\mathbf{R}\Gamma\big(G_F, F_\Gamma(T^\vee)^\iota(1)\big), \mathbb{Q}_p/\mathbb{Z}_p\Big)[-2] \\ \mathbf{R}\Gamma\big(G_F, \mathscr{F}_\Gamma(T)\big) \longrightarrow \mathbf{R}\mathrm{Hom}_{\Lambda^\circ}\Big(\mathbf{R}\Gamma\big(G_F, \mathscr{F}_\Gamma(T^*)^\iota(1)\big), \Lambda \otimes^{\mathbf{L}}_R \omega_R\Big)[-2] \end{gathered}$$

in $\mathbf{D}(Mod_\Lambda)$. □

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