NEW FORMULATIONS AND APPROACHES TO FACILITY LOCATION PROBLEMS IN THE PRESENCE OF BARRIERS

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for Ph.D.

McMaster University

Ph.D in Management Science (2010)

McMaster University Hamilton Ontario

TITLE: New Formulations and Approaches to Facility Location Problems in the Presence of Barriers

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NUMBER OF PAGES: xii, 139

Abstract

This dissertation examines the facility location problems in the presence of barrier regions and consists basically of four essays exploring new problems. Despite the fact that the facility location problems considering barriers to travel are more realistic than their unrestricted counterparts, research in the area is relatively limited. This is due to the computational complexity associated with them.

The first essay analyzes the problem of locating a facility in a region in the presence of a probabilistic line barrier. The objective is to locate the facility such that the sum of the volume times distances between the facility and demand points is minimized. Some convexity results are presented and a solution algorithm is proposed.

Another interrelated problem is locating a facility in a region where a fixed line barrier such as a borderline divides the region into two. The regions communicate with each other through a number of passage points located on the line barrier. A version of this problem with minisum objective has been studied in the literature where the locations of the passage points are known. The second essay considers a number of extensions to this problem and proposes an efficient solution methodology based on the Outer Approximation algorithm.

The third essay discusses the problem of locating a rectangular barrier facility in an area where interactions among existing facilities are present. The problem has two objectives. The first objective is to minimize the interference of the barrier facility to the interactions among the existing facilities. The second objective is to find a center (minimax) location for the barrier facility. The problem is formulated as a bi-objective problem and a mixed integer program is proposed as a solution methodology. A Simulated Annealing algorithm is presented for an extension of the problem where expropriation of existing facilities is also possible.

Finally, the last essay suggests a practical analog approach for facility location problems in the presence of barriers. The use of the analog for certain problems is justified through some analytical results and a number of problems that appeared in the literature are solved efficiently.

Acknowledgements

I would like to express my gratitude to Dr. George O. Wesolowsky who helped me unremittingly during the course of my studies. His constant guidance, mentorship and extended financial support need appreciation.

I would like to thank Dr. George Steiner, Dr. Elkafi Hassini, and Dr. Jack Brimberg for their valuable comments and suggestions which improved the content and the presentation of my thesis.

A special thank you goes to my friend and colleague Michael von Massow with whom I shared the most productive times of my PhD studies.

Finally, I express my deep gratitude to my family who are always a source of motivation and inspiration to me.

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Chapter 1

Introduction

Suppose that there is demand at some location and there is a need to locate a facility to satisfy this demand. The best location for the server could be at the same place as the demand if reducing distance travelled is the dominant criterion. However, when demand emerges from different geographical locations in a defined region and there are fewer servers than demand points, there comes a question of optimally locating a number of servers throughout the region. Facility location decisions, in the context of operations research, are strategic decisions which aim to determine the optimal number of servers as well as the best location(s) for these servers. Because of the fact that the facility location decision can be present in a real life problem as a subproblem, facility location is interdisciplinary in nature. Therefore, many researchers in variety of disciplines including but not limited to economics, engineering, geography and management have studied location problems. Choices for the best location(s) differ for various types of objectives. For example for a company that wants to build a warehouse for its retailers, it may be important to find a location that minimizes the sum of the distances from the warehouse to the retailers. But for the location of an emergency facility such as a fire station, the most suitable objective could be to minimize the maximum distance from the facility to the demand points in order for the fire station to respond quick enough to the farthest point. Another example might be the location of a waste incinerator for a local municipality. Residents might want that the facility be located as far as possible from residential areas, while the municipality wants it to be close enough to transport the waste. In that case an objective that maximizes the minimum distance of the facility from the residential areas would be more appropriate.

Research in this area has been highly correlated with the development of computer technology as this technology provides means for serving the underlying mathematical models. Advancements in the operations resarch literature opened some avenues that were not accessible before. There are two basic types of models, with different solution methodologies, that have dominated this area. These involve continuous facility location and discrete facility location. Continuous facility location models, which consider locating a facility anywhere on the plane, are highly dependent on the distance functions that are used in the models. On the other hand, discrete facility location models, which are mainly graph theoretic models, in which possible locations of the new facilities are already known, do not use distance functions at all. Solution methodologies for continuous and discrete models are generally different from each other; the former predominantly uses calculus techniques, while the latter is highly dependent on combinatorial optimization methods. The solution methodology that is common to both problem types is mathematical programming where the model considered may appear as a linear program, nonlinear program, integer program or a mixed version of these three.

In this study, we contribute to the continuous side of facility location literature. Therefore, we will first discuss the fundamentals of continuous facility location literature. We begin with the introduction of the various distance functions that are commonly used in continuous facility location problems in Section 1.1. Section 1.2 is an overview of the major facility location objectives used in continuous location.

1.1 Distance Measures

Facility location models require information on how far two objects are away from each other. To measure this proximity, we require a distance function (metric) induced by a norm $\|\bullet\|_l$. For three points X_1, X_2, X_3 in $\Re^n$, the distance function l should satisfy the following fundamental properties:

- Non-negativity : $l(X_1, X_2) \geq 0$
- Symmetry : $l(X_1, X_2) = l(X_2, X_1)$
- Triangle Inequality : $l(X_1, X_3) \leq l(X_1, X_2) + l(X_2, X_3)$

There are many different types of distance measures used in the facility location literature. We will start with the most commonly used one which is the family of p-norm distance measures. We will also discuss basic properties of some other important distance measures. Without loss of generality, our discussion for distance measures will be in R^2.

1.1.1 p-Norm Distances

The p-norm distance (l_p distance function) between points $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ is defined as :

$$l_p(X_1, X_2) = \left(|x_1 - x_2|^p + |y_1 - y_2|^p\right)^{1/p}$$

Figure 1.1 is a representation of the implicit function $l_p(X_1, 0) = 1$ (unit ball) for various values of p. p-norm distance is a convex function for values $1 \leq p \leq \infty$. A large number of distance measures can be represented by this distance function through varying the parameter p. Among the commonly used values for the parameter p are $p = 1$, $p = 2$ and $p = \infty$. Different values of p are also used in the literature (see Love et al. (1988)) to find the best fit for actual road distances in different regions.

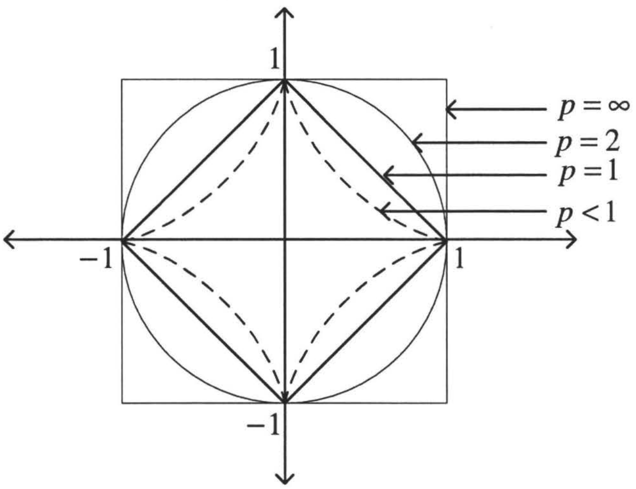


Figure 1.1: Contour lines for p-norm distance for different values of p

For $p = 1$, the p-norm distance is called the rectilinear distance and is also known as rectangular distance, taxicab distance, Manhattan distance and l_1 distance. The rectilinear distance is given as:

$$l_1\left(X_1, X_2\right) = \left|x_1 - x_2\right| + \left|y_1 - y_2\right|,$$

which is the sum of the absolute differences of the coordinates of two points and is a linear convex function (Love et al. (1988)). Because of its linearity, it has been used widely as an approximation to some other distance measures.

For $p = 2$ the p-norm distance is called the Euclidean distance. Euclidean distance is the straight line distance between two points, and the distance that is commonly used in daily life. It is defined as:

$$l_2\left(X_1, X_2\right) = \left(\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2\right)^{1/2}.$$

Another distance measure that is depicted from the p-norm distance is the ∞-norm distance which is also known as the Tchebychev distance or the maximum distance. The reason why it is called the maximum distance is because, when p goes to infinity, the distance between points X_1 and X_2 can be written as:

$$l_\infty (X_1, X_2) = \lim_{p\to\infty} \left(|x_1 - x_2|^p + |y_1 - y_2|^p\right)^{1/p} = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

The maximum distance function can be represented in another convenient form, which makes it easy to use in linear programming models (Ward and Wendell (1985)):

$$\max\{|x_1 - x_2|, |y_1 - y_2|\} = \frac{1}{2}\left(|x_1 + y_1 - x_2 - y_2| + |x_1 - y_1 - x_2 + y_2|\right).$$

The ∞-norm distance is used in cases where simultaneous x direction and y direction movements occur. For example in the context of warehouse automation, the travel time for an automated robotic arm to reach its destination on a shelf, is the maximum of its horizontal and vertical movement times.

The ∞-norm distance has a close relation with the rectilinear distance function. It becomes equivalent to the rectilinear distance in $\Re^2$ when its axes are rotated by an angle of $\pi/4$ and scaled up (See Figure 1.1). This can be done by simple linear transformation. This is an important property because, in general, facility location models in $\Re^2$ which use the ∞-norm distance can be converted into a rectilinear distance model by this axes orientation trick.

1.1.2 Block Norm Distances

A norm is called a block norm, if its corresponding unit ball in $\Re^2$ is a convex symmetric polygon (Klamroth (2002)). A polygon is a finite closed set of a number of linear line segments in a two dimensional plane. Figure 1.2 displays a number of polygons.

Block norms were introduced to the facility location literature by Ward and Wendell, (1980). For a block norm BN, each extreme point (vertex) of the polytope indicates a 'fundamental direction'. Let the set of extreme points be $(b_g : g = \pm1, \pm2, ..., \pm r)$ where $(-b_g = b_{-g})$ since BN is symmetric about the origin. Note that these extreme points also define a unit travel length in their direction. The distance between two points under a block norm measure is the shortest distance that follows combinations

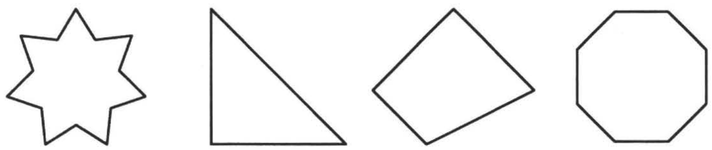


Figure 1.2: Examples of Polygons

of these fundamental directions (See Theorem 1.1.1). It is proven by Hamacher and Klamroth (2000) that for any given block norm, at most two fundamental directions are adequate to determine this distance.

Theorem 1.1.1 (Ward and Wendell, 1980) *A block norm $\|.\|$ has a characterization as:*

$$\|v\| = \min\left\{\sum_{g=1}^{r} |\beta_g| : v = \sum_{g=1}^{r} \beta_g b_g\right\},$$

where $|\beta_g|$ is the distance travelled parallel to b_g.

Note here that, having a symmetric polytope unit ball, the rectilinear distance and the maximum distance also become block norm distances with two fundamental directions ($r = 2$). All block norm distances can be represented as linear functions which makes their underlying models relatively easy to solve. There is an infinite number of possibilities for block norm distances. One of them is the weighted one-infinity norm, suggested by Ward and Wendell (1980), which is a linear combination of the rectilinear distance norm and the maximum distance norm (See Figure 1.3). With the appropriate selection of the weighting parameters, the weighted one-infinity norm provides a good approximation to Euclidean distance as the percentage difference can be reduced to less than 4% (Ward and Wendell (1980)).

1.1.3 Polyhedral Gauges

When the symmetry property is no longer required, a block norm is called a polyhedral gauge. Because of this asymmetry, polyhedral gauges represent a larger class of

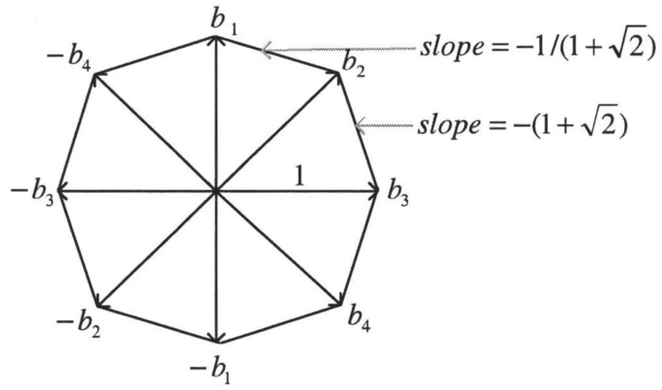


Figure 1.3: One-Infinity Norm (A Block Norm with $r = 4$)

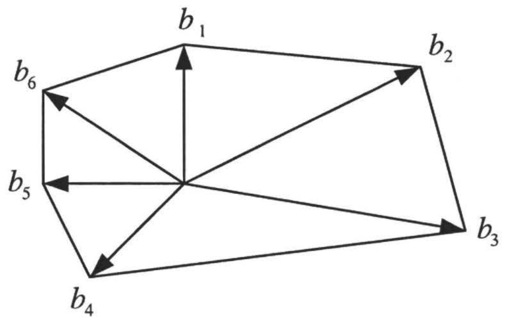


Figure 1.4: A Polyhedral Gauge with $r = 6$

distance functions than block norms, which leads to more general models that are applicable to real life problems. Polyhedral gauges can also be represented as linear functions. A practical example for a polyhedral gauge can be given as the travel time in a big city. For certain time periods, the travel time to some directions might be different. Figure 1.4 is a polyhedral gauge with $r = 6$ fundamental directions.

1.1.4 Closest Distances

To the best of our knowledge, the term 'closest distance' has first been used by Brimberg and Wesolowsky (2000) in the facility location literature to define the distance

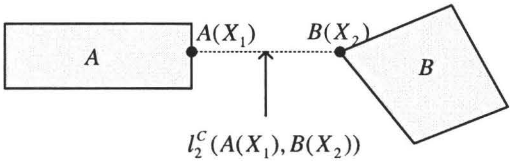


Figure 1.5: Closest Euclidean distance between polygons A and B

between two areas where the distance is measured from the closest point on the boundary of one area to that of the other. The authors also proved that when the areas are convex regions (i.e. polygons), the closest distance between them becomes a convex function. Let $A(X_1), X_1 = (x_1, y_1)$ be the closest point on area A to area B and $B(X_2), X_2 = (x_2, y_2)$ be the closest point on area B to area A. Then we can write the closest p-norm distance between A and B as:

$$l_p^C\left(A(X_1), B(X_2)\right) = \left(|x_1 - x_2|^p + |y_1 - y_2|^p\right)^{1/p}$$

Figure 1.5 shows the closest Euclidean distance between polygons A and B which happened to be the same as the closest rectangular distance.

1.1.5 Spherical Distances

Suppose two points X_1 and X_2 are located on a unit sphere. Location of point X_1 is determined by (x_1, y_1) where x_1 and y_1 are the latitude and longitude coordinates. Because the points are on the sphere, we can draw a circle, known as the 'great circle' through these two points (the largest possible circle on the sphere). The spherical distance then will be the shorter of the two arcs that connects these two points. The spherical distance on a sphere with unit radius is defined as follows:

$$l_S\left(X_1, X_2\right) = \cos^{-1}\left[\cos x_1 \cos x_2 \cos(y_1 - y_2) + \sin x_1 \sin x_2\right],$$

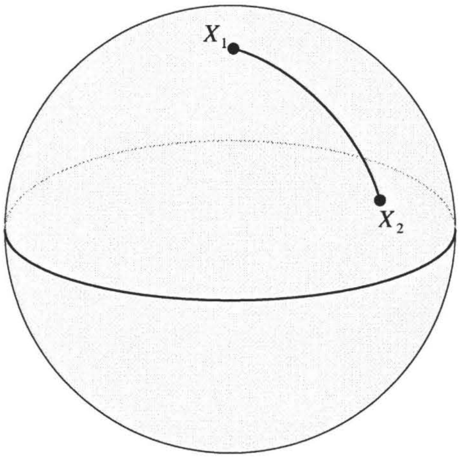


Figure 1.6: Spherical distance between points X_1 and X_2

Drezner and Wesolowsky (1978) proved that the spherical distance is a convex function of X, provided that it is contained in a spherical disk with radius $\pi/2$. Figure 1.6 is an illustration of the spherical distance between points X_1 and X_2.

1.1.6 Weighted Distances

For a given distance, when positive weights are used as a scaling factor that represents the factors that slow down the movement in each direction, then the distance becomes a weighted distance. The weighted p-norm distance between two points X_1 and X_2 can be defined as follows:

$$l_{p,k_1,k_2}(X_1, X_2) = (k_1 |x_1 - x_2|^p + k_2 |y_1 - y_2|^p)^{1/p}$$

A similar structure can be given for other distance norms. Figure 1.7 is an illustration for the weighted Euclidean distance with $k_1 = 1$ and $k_2 = 2$. We can observe that this distance is the same as Euclidean or rectilinear distance if we travel only in its x direction, but it will take more time to travel through the y direction.

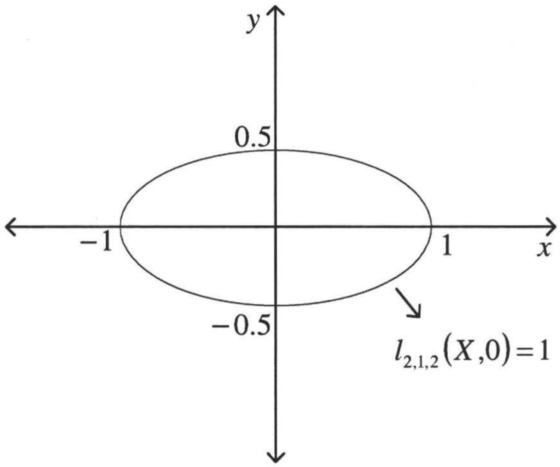


Figure 1.7: Weighted Euclidean distance unit ball with $k_1 = 1$ and $k_2 = 2$

1.1.7 Barrier Distances

Consider a polygonal area on the plane, inside which the distance is a weighted p-norm distance with $k_1 = k_2 = \infty$. If we travel from one point to another on the plane, and if our path requires us to cross this polygonal area to reach our destination, we would avoid crossing through the polygonal area, because if we do, we would never reach to our destination. In facility location literature, we simply call these areas barriers. Barriers are the regions where neither facility location nor crossing through is permitted. Mountains, lakes, military zones, existing facilities with finite sizes, etc., can be given as examples of barriers. For a special type of barriers such as railroads, highways, borders, etc., crossing through may be possible at some passage points. Knowing that we can not cross a barrier region while travelling to our destination, we have to minimize its effect on our travel path. This idea brings another way of looking at the problem which requires the concept of visibility.

Let B be a union of a finite number of barrier regions on the plane. The barrier distance $l_p^B(X_1, X_2)$, between the points X_1 and X_2 is then defined as the infimum of the lengths of all permitted paths (shortest path) between X_1 and X_2.

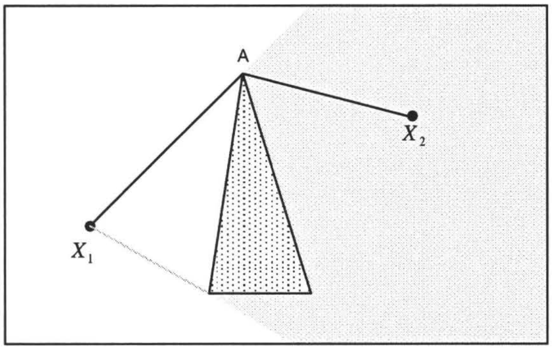


Figure 1.8: The barrier distance between X_1 and X_2 under Euclidean distance norm

For p-norm distances, two points $(x_1, y_1), (x_2, y_2)$ are called p-visible from each other if $l_p^B(X_1, X_2) = l_p(X_1, X_2)$ (there is no barrier region on our path) and p-shadow if $l_p^B(X_1, X_2) > l_p(X_1, X_2)$ (there is at least one barrier region on our path). If the points are not p-visible from each other, then the distance between them becomes a barrier distance. Figure 1.8 illustrates points X_1 and X_2 that are not 2-visible due to the triangular shaped barrier region on the plane. Actually, all the points in the grey region are not visible to point X_1. In our example, the shortest path between these two points will be from vertex A of the barrier region only in this particular case.

No matter what the underlying distance function is, the barrier distances are nonconvex in general, since they require finding the shortest path to reach the destination; in other words, determining the boundary or the vertex of a barrier region to pass through. When the barrier regions have polygonal shapes, there is an important property called 'Barrier Touching Property' proved by Klamroth (2002) and given by the following lemma.

Lemma 1.1.1 (Klamroth, (2002)) *Let $B_1, ..., B_n$ be a finite set of pairwise disjoint, closed, polyhedral barrier sets with a finite set of extreme points $P(B)$. Assume that X_1 is not p-visible from X_2. Then there exists a p-shortest permitted path connecting X_1 and X_2 with the following property: 'The shortest path is a piecewise linear path with breaking points only at extreme points of barriers'.*

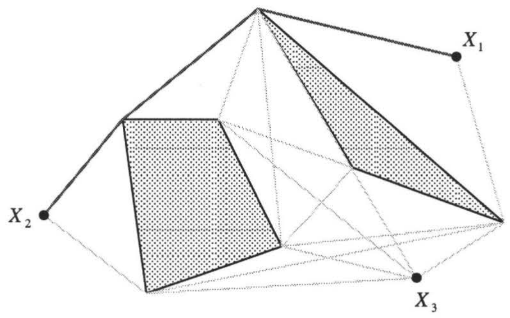


Figure 1.9: A visibility graph for three points and two barriers

The lemma acknowledges that the barriers' effect on the travel from X_1 to X_2 should be minimized. We can think of an example where the distance is represented by a piece of string that connects two points. If a barrier region is present on the path, to obtain the shortest string length, the string has to touch or go along the boundary of the barrier region.

This fundamental property leads to the use of visibility graphs to determine shortest paths between points when barriers are present. For Euclidean distances, visibility graphs can be generated in $O(N^2)$ where N is the number of barrier extreme points (vertices), by drawing lines from the starting point and ending point to all visible vertices of the barriers and drawing lines from every vertex of every barrier to the other vertices that are visible (Klamroth (2002)). The shortest path between any two points can be found in a polynomial time, by using graph theoretical algorithms, such as 'Dijkstra's Algoritm' (Dijkstra, 1959). Figure 1.9 shows a visibility graph for three points X_1, X_2 and X_3 in the presence of two polygonal barrier regions. The shortest path between points X_1 and X_2 is also illustrated.

1.2 Overview of some Facility Location Objectives

1.2.1 The Minisum Objective

Definition 1.2.1 *Given n points ($X_i = (x_i, y_i), i = 1, 2, ..., n$) on the plane, each with a positive weight w_i, the minisum objective finds a point $X = (x, y)$ that minimizes the sum of weighted distances from X to the given points.*

The problem that considers the minisum objective function is called the Weber problem. The Weber problem entails locating a facility on the plane under the minisum objective to serve a finite set of existing demand points with the same or different demand levels. It has been of interest to many researchers since as early as the 17th century, but its practical usage was identified by Weber in 1909. The simplest version of the problem is believed to be originated by Fermat (1601-1665) who issued a challenge by asking *'let he who does not approve of my method attempt the solution of the following problem: Given three points in the plane, find a fourth point such that the sum of its distances to the three given points is a minimum'.*

Mathematically, under the p-norm distances, the problem can be defined as:

$$\min_X \sum_i w_i l_p(X, X_i)$$

Solution methods and difficulty of getting a solution differ with the distance function being used. For rectilinear distances, the objective function is separable into x and y directions, which leads to a simple solution technique that is described by Love et al. (1988). But for Euclidean distances, even though the objective function is convex, there is no closed form solution. Weiszfeld (1937) suggested a numerical analysis based algorithm for the problem which has been a common approach to get a close to optimal solution. There are a number of drawbacks of this algorithm. Firstly, the algorithm will fail if one of the locations generated during the algorithmic process coincides with a fixed point. This is from the fact that the derivatives don't exist at that point. To prevent this from occuring, some researchers suggested using a hyperbolic approximation, which prevents discontinuity in derivatives. Secondly, for some problems, the convergence rate of the Weiszfeld algorithm is reported to be very slow. Another way of solving the problems with a minisum objective is by using off-the-shelf nonlinear solvers. Because the minisum objective function is known to be convex, these solvers will usually provide a local optimal solution which will also be the global optimum.

1.2.2 The Minimax Objective

Definition 1.2.2 *Given n points ($X_i = (x_i, y_i), i = 1, 2, ..., n$) on the plane, each with a positive weight w_i, the minimax objective finds a point $X = (x, y)$ that minimizes the maximum weighted distance from X to the given points.*

Minimax facility location problems have been extensively studied by a large number of researchers because of their importance in locating emergency service facilities. The objective is to locate a new facility to minimize the maximum distance to these existing facilities. The objective function utilizes the fact that even the farthest and/or the weakest facility should get an adequate attention. Network representations of minimax facility location problems are more suitable for practical cases but planar models can also be used for general theoretical models or understanding the situation. The simplest version of the problem on the plane can be solved by finding a circle of the smallest radius (the minimum covering circle (MCC)) which encloses a given set of points that have equal weights. The centre of such a circle will be the minimax point. This follows from the fact that the centre of the circle will have the smallest possible distance (radius) to the farthest point. The MCC problem was initially proposed by Sylvester (1857) in a geometrical context.

Solution methodologies for the problem differ with the underlying model's complexity. For Euclidean distances, Elzinga and Hearn (1972) suggested an algorithm for the unweighted case which finds the minimal covering circle, hence the minimax point, in polynomial number of steps ($O(N^2)$), where N is the number of

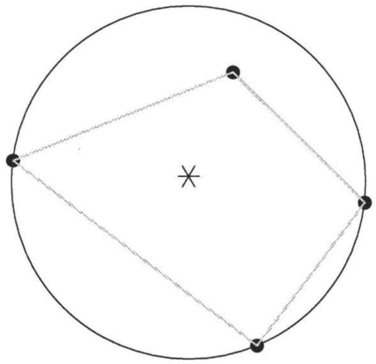


Figure 1.10: The minimum covering circle for four points with equal weights

demand points. The minimal covering circle will sit on a convex hull of the demand points, satisfying the following property.

Property 1.2.1 (Love et al. (1988)) *The minimum covering circle of a convex hull will pass through two or more of its corner points, and all such corner points can not be located on less than half the perimeter of the circle.*

Figure 1.10 shows a minimum covering circle. Observe that the points inside the convex hull do not have any effect on the minimum covering circle.

A practical way of solving the weighted case is to use the compass. Consider a number of demand points on the plane with different weights. If we draw small circles around these demand points with radii inversely proportional to their weights, and slowly enlarge these circles, the first point that is the intersection of all of the circles will be the optimal point. Figure 1.11 illustrates the idea. Similar approaches can be adopted for other distance functions.

1.2.3 The Maximin Objective

Definition 1.2.3 *Given n points ($X_i = (x_i, y_i), i = 1, 2, ..., n$) on the plane, each with a positive weight w_i, the maximin objective finds a point $X = (x, y)$ that maximizes the minimum weighted distance from X to the given points.*

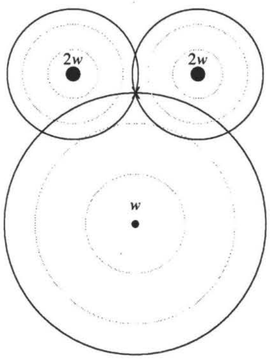


Figure 1.11: Finding the minimax point for three points with different weights

The maximin objective is appropriate for models that deal with locating an 'obnoxious' facility. This type of facility needs to be as far away as possible from existing points, ideally an infinite distance. But if there are mandatory closeness constraints which identify a region that the facility has to stay in, then the maximin objective creates a non-trivial problem. A graphical approach similar to the one used in the minimax objective can be adopted by drawing circles around demand points with radii inversely proportional to their weights, and then slowly enlarging these circles. The last possible remaining point in the region that is outside the coverage of the circles will satisfy the maximin objective.

1.2.4 Covering Objectives

Definition 1.2.4 *Given n points ($X_i = (x_i, y_i), i = 1, 2, ..., n$) on the plane, each with a positive weight w_i, the covering objective finds a location $X = (x, y)$ for a circle with a fixed radius that covers the most or least demand weights possible, depending upon the covering objective function.*

If the objective is to cover the most demand weights possible, when there are insufficient resources to cover all demand weights, then we call this objective the maximal covering objective. Insufficient resources are represented by putting a limitation on the coverage distance. Any facility within the coverage distance is covered but outside of the coverage distance is not covered. Mehrez and Stulman (1982) showed that the potential optimal locations for the maximal covering problem is a finite set of points which are the intersection points of circles with the radius of coverage distance drawn around demand points. The idea of identifying candidate points can be generalized for all types of distance norms. For a given distance norm, the corresponding unit ball blown up with a constant factor can be drawn around demand points to identify candidate points as intersection points. The maximal covering objective has been applied to various problems including use of different distance norms and probabilistic demand weights. This objective becomes the minimax objective if the radius is not fixed but the smallest radius is required.

When the objective is to cover the least possible demand weights for a given radius, the objective is called the minimal covering objective. If there is a region that the facility has to stay in, and if the facility has a fixed radius, then the minimal covering objective finds a location within the region that covers the least possible weights. As with the relation between the maximal covering objective and the minimax objective, this objective is related to the maximin objective. Allowing the radius to be as small as possible (i.e. a point) and letting the facility be as far away as possible from demand points reduces this objective to the maximin objective.

Chapter 2

Background

2.1 Introduction

The facility location literature can be divided into two research areas: Facility location problems and facility layout problems. In facility location problems, facilities are considered small relative to their location space. Layout problems on the other hand, assume that facilities are large relative to their space. Both facility location problems and layout problems can be discrete or continuous and can be modeled in a d-dimensional real space or on networks. For a detailed review of the facility location literature, the reader is referred to ReVelle and Eiselt (2005). Figure 2.1 is an overview of the facility location literature.

In this dissertation work, we deal with planar facility location problems under uncertainty and/or in the presence of barrier regions. Therefore, in Section 2.2 we review some of the literature on planar facility location problems under uncertainty. For a general state-of-the-art literature review of facility location problems under uncertainty, the user is referred to Snyder (2006). Section 2.3 deals with the literature review of planar facility location problems in the presence of barrier regions. In Section 2.4, we specify the gaps in the literature, provide the dissertation objectives, and give the outline of the thesis.

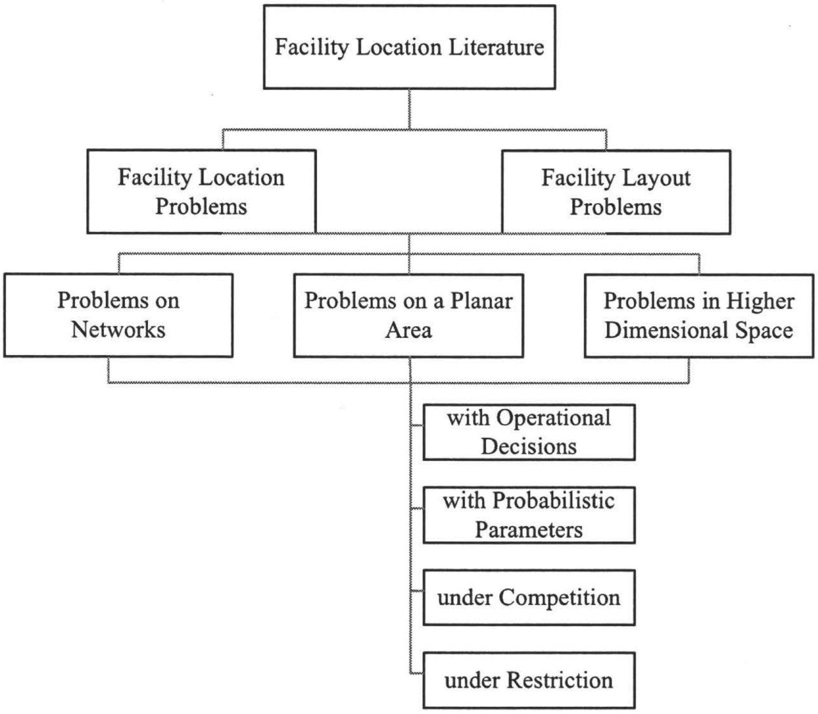


Figure 2.1: Facility Location Literature: An Overview

2.2 Planar Facility Location Problems Under Uncertainty

Planar facility location problems under uncertainty have been studied under two categories. The first category deals with problems that contain random parameters which follow certain probabilistic distributions. For example the weights attached to demand points could be associated with a known probability distribution. Various objective functions are considered. Facility layout problems with random parameters are also included in this area although the research on this topic is limited. The second category, on the other hand, deals with so called robust facility location problems where distributions of the random parameters are unknown. This type of parameters are either represented by interval values or by parameter estimators. In Table 2.1 we provide the cited research on planar facility location problems under uncertainty, classified by their main characteristics. For an alternative review for addressing facility location uncertainty in continuous space siting, the reader is referred to Murray (2003).

Wesolowsky (1977a) was one of the earliest papers that considered a facility location problem under uncertainty. The author considered a problem of locating a facility on a line in the presence of n demand points. The demand points have probabilistic weights that follow the multivariate normal distribution. The Weber objective was considered. Because the problem is one-dimensional, the solution method is the same for both rectangular and Euclidean distances. The probability of a facility being located on any point on the line is found. As in the result of the Hakimi (1965) property for the p-median problems, it is shown that only the locations at demand points have a nonzero probabilities for the optimal location of the facility. The author also determined the EVPI (Expected Value of Perfect Information) for the problem. EVPI is defined as the expected difference in costs between the best location for any outcome of weights and the location is found by using the expected values of the weights. Later, Drezner and Wesolowsky (1981) extended the work of Wesolowsky (1977a) by considering a similar problem on the plane with p-norm distances. The

Table 2.1: Planar Facility Location Problems under Uncertainty

Uncertain Parameter	Underlying Distribution	Objective Function	Distance Norm	Space	Study
Demand weights	Multivariate Normal	Expected Minisum	Rectilinear	One dimensional	Wesolowsky (1977a)
Location of demand points	Bivariate Normal	Expected Minisum	Rectilinear	Planar	Wesolowsky (1977b)
Location of demand points	Standard Normal	Min Expected Maximum	Rectilinear	Planar	Carbone and Mehrez (1980)
Demand weights	Multivariate Normal	Expected Minisum	p-norm	Planar	Drezner and Wesolowsky (1981)
Location of demand points	Bivariate Uniform	Max Expected Minimum	Rectilinear	One dimensional	Mehrez et al. (1983)
Location of demand points	Arbitrary	Min Expected Maximum	Arbitrary	Planar	Mehrez and Stulman (1984)
Demand weights	Triangular Fuzzy	Maximin Aspiration Level	Euclidean	Planar	Bhattacharya (1994)
Existence of demand points	Binomial	Min Expected Maximum	Euclidean	Planar	Berman et al. (2003a)
Demand weights	Uniform	Minimax probability with threshold	Euclidean	Planar	Berman et al. (2003b)
Location of demand points	Uniform	Expected Minimax	Euclidean	Planar	Foul (2006)
Product mix and product demand	Discrete	Expected Minisum	Euclidean	Planar Layout	Benjafaar and Sheikhzadeh (2000)
Demand weights	Unknown	Minimax Regret (Robust)	Rectilinear	Planar	Carrizosa and Nickel (2003)
Demand weights and locations	Unknown	Minimax Regret (Robust)	Rectilinear	Planar	Averbakh and Bereg (2005)
Demand weights	Arbitrary	Minimax probability with threshold	Arbitrary	Planar	Pelegrin et al. (2008)

property found in Wesolowsky (1977a) is no longer valid for this general case. This means that any point on the plane may have nonzero probability of a facility being located there.

In another study, Wesolowsky (1977b) proposed a solution to the single facility location problem with rectangular distances in which the locations of demand points have random coordinates that follow a bivariate normal distribution. It is shown that the objective function, which is the expected sum of the weighted rectilinear distances in x and y coordinates, is separable, and is thus not affected by correlation of demand point coordinates. Because the objective function is unimodal along each axis, the author proposed a rather easy method in which one can take the derivative of the objective function for each axis and apply an interval bisection method to find the values of coordinates that make the derivative zero.

There is a number of facility location papers that use the minimax criterion. Carbone and Mehrez (1980) was the first that studied the problem of minimizing the expected maximum distances where the coordinates of the demand points $(x_1, x_2, ..., x_n,\ y_1, y_2, ..., y_n)$ are identical, pairwise independent, and normally distributed random variables with mean 0 and variance 1. The authors showed that in this problem, the optimal location of the single facility is at the $(0, 0)$ point. Later, for the same problem, (Mehrez and Stulman, 1984) proposed a general statement that if the distance distribution between any demand point and a facility placed at coordinate (x, y) dominates the distribution of the distance between the same point and the facility placed at any other feasible coordinate, then (x, y) will be the dominating point and hence provides an optimal solution to the problem. A necessary and sufficient condition for the distribution $F(x)$ to dominate distribution $G(x)$ is that $F(x) \leq G(x)$ for all x. Only certain types of problems can be solved using this method under stringent assumptions, and actual values of objective functions may require extensive calculations. Berman et al. (2003a) approached the same expected maximum objective from a different perspective. In their model, the problem is designed to minimize the expected maximum distances where for each demand point, there is a probability associated with its existence. In their words, the discussed model aims at minimizing expected 'undesirability'. The model also separates itself from the minimax model by using expectation to balance 'damage equity' (using information from all demand points in the optimum solution).

Berman et al. (2003b) studied extensively a probabilistic version of the weighted minimax location problem on the plane where the weights of the demand points are uniformly distributed. The objective of their problem is to minimize the probability that the maximum distance to all demand points is greater than or equal to some pre-specified threshold value T. The authors proved that the problem is convex for certain parameters of the uniform distributions and therefore can be solved using standard optimization methods.

Foul (2006) studied a similar problem where the demand points have probabilistic locations that follow a bivariate uniform distribution. The best location for a facility is determined under the objective of minimizing the maximum expected weighted distance to all probabilistic demand points.

Recently, Pelegrin et al. (2008) argued that the 1-center problem on the plane with probabilistic weights has only been studied for a number of specific probability distributions and distance measures. The authors proposed a general framework where weights are associated with arbitrary probability distributions and distances are measured by any distance norm. Two objective functions were studied. The first maximizes the covering probability for all demand points within a given threshold, while the second satisfies a minimum allowed coverage probability. Two algorithms that provide global optimal solutions were tested with different values of parameters and both were found to be highly efficient.

Mehrez et al. (1983) analyzed the problem of locating a facility on a line, in the presence of n hazardous points that have probabilistic locations. The objective is to maximize the expected minimum distance from these hazardous points. The authors showed that even for $n = 2$, acquiring an analytical result is cumbersome, and therefore suggested a simulation model for solving the problem.

As a different approach to handle uncertainty, Bhattacharya (1994) presented a cost minimization model to locate multiple facilities on the plane where the cost per unit distances are not known precisely. Uncertainty in the cost is handled through the use of fuzzy numbers. The fuzzy model is tranformed into a crisp model by generating some aspiration levels (goals) by using different levels of the fuzzy numbers. A solution is determined through finding a compromise solution which maximizes the minimum aspiration level.

When it comes to the probabilistic facility layout problems, the research is very limited. One of the important papers published in this area is Benjaafar and Sheikhzadeh (2000). The authors proposed a model for the design of plant layouts under uncertainty. It is discussed in the paper that in manufacturing environments where product variety is high, the general practise is to use functional layouts where same type of resources share the same location. This approach is known to cause inefficiency and is also not a good fit for probabilistic environment. Thus there is a need for alternative customized layout plans that make the underlying process more flexible and more efficient. The authors presented a probabilistic layout model for the design of plant layouts which considers random product mix and product demand. Demand for each product is represented by a finite discrete distribution where demands can be correlated or independent from each other. It is also considered that there might be a duplicate or duplicates of the same department in the same facility. A set of scenarios involving combinations of different products and demands with a specified probability of occurrence is considered. The authors used a heuristic approach first to find a minimum cost flow allocation between departments in a fixed layout, then to find a minimum cost layout with fixed flow allocation.

Robust models are used when uncertainty can not be defined by known probability distributions. Robust facility location problems differ from probabilistic location problems where the latter have uncertainty associated with some distribution functions with known parameters, but the former have uncertainty associated with no known distribution functions and hence no known parameters. Because decisions are made in the presence of unknown parameters, and estimation of parameters needs to be used, researchers, in general, aim to find a minimax regret location in order to minimize the maximum loss. Research in the area is recent and mostly on the discrete facility location problems.

Carrizosa and Nickel (2003) considered the robust planar facility location problem when uncertainty in demand weights is high and only estimates of the weights are provided. The authors defined the robustness of the new facility location as the minimum deviation in the weights with respect to their estimates needed to exceed a given threshold on the total cost. The objective is then to find a location that maximizes the robustness which essentially finds the location where the weights can largely differ from their estimates with a minimum violation on the total cost.

Finally, Averbakh and Bereg (2005) recently investigated the case of rectilinear distances and uncertain weights and coordinates of demand points. The authors considered both median and centre objectives and proposed polynomial algorithms for 1-median and 1-centre problems.

2.3 Planar Facility Location Problems in the Presence of Barriers

Although facility location problems in the presence of barriers have more practical relevance than general facility location problems, they haven't been given much attention until lately, due to the computational complexities associated with these problems. Klamroth (2002) is an excellent book on the subject which discusses various aspects of the single facility location problems with barriers. Table 2.2 is an overview and classification of facility location problems in the presence of barrier regions studied in the literature. Note that our classification of the literature only accounts for the planar problems as we treat the barriers as obstacles on the plane that may have an affect on distance functions. However, network based models can also account for barriers as both untravellable or unlocatable regions can be defined in a network model.

Research in the area started with Katz and Cooper (1981) which was the first paper in this area that considered the Weber problem with Euclidean distances and barrier regions. The authors discussed the problem with one circular barrier and showed that the problem had a nonconvex objective function. A heuristic based solution approach is proposed with no guarantee of finding the global optimum. Some properties of the problem were later analyzed by Klamroth (2004). Klamroth developed some structural results that led to a model which handles the problem by dividing the feasible region into convex regions where the objective function is convex in each region. The number of such convex regions is bounded by $O(N^2)$ where N is the number of demand points. When N increases, construction of these convex regions becomes cumbersome. To deal with this difficulty, Bischoff and Klamroth (2007) proposed a genetic algorithm based solution to the problem. The algorithm proposed by the authors works only with polyhedral barriers and therefore the circular barrier was approximated by a 128-sided equilateral polygon.

Aneja and Parlar (1994) considered the Weber problem with Euclidean distances and convex or non-convex polyhedral barriers. The solution procedure proposed by the authors generates some candidate locations using simulated annealing and, for each candidate location, a visibility graph is constructed to find the shortest path network. The shortest path between any candidate location and existing facility location is found using Dijkstra's algorithm, which finds shortest paths on networks in polynomial time.

Butt and Cavalier (1996) developed an algorithm that finds local optima to the Euclidean distance Weber problem in the presence of some polyhedral barriers. The authors proposed a decomposition of the feasible region into subregions in which shortest barrier distance between two points remains constant throughout the region. The problem with this approach is that the boundaries of the subregions are generally nonlinear. To overcome this difficulty, Klamroth (2001a) suggested a different decomposition approach by applying visibility grids to the same problem.

A modified version of the Big Square Small Square (BSSS) method was proposed by McGarvey and Cavalier (2003) for Euclidean distance Weber problems with barriers. The BSSS method is a Branch and Bound (B&B) technique that divides the feasible region into square subregions and produces either a global optimal solution or a solution within a very small tolerance of the global optimum. The method was originally proposed by Hansen et al. (1981) for locating obnoxious facilities. In this method nonconvex polygonal barrier regions can also be considered. Butt (1994) showed that if no existing facility is located within this nonconvex forbidden region's convex hull, then the optimal location for a new facility can never be within this convex hull.

Larson and Sadiq (1983) is a seminal work that first considered using rectilinear distances for facility location problems in the presence of barrier regions. The authors examined the rectilinear distance p-median problem on the plane with polyhedral barrier regions and defined a special structured grid that contains nodes and edges. They discovered that this set of nodes provides a finite dominating set of solution points for the problem.

These fundamental results motivated some researchers who continued working on the same problem to provide some extensions. First, Batta et al. (1989) extended the work by considering both convex forbidden regions and arbitrarily shaped barriers. Second, findings in the PhD thesis by Segars (2000), and their extensions were published by Dearing and Segars (2002a) and Dearing and Segars (2002b). In the first paper, using the visibility idea, the authors showed that the barriers can be modified without affecting the objective value, thus allowing some nonconvex barrier shapes to be equivalent to convex ones. Also, the feasible region can be reduced by this modification, and it can be decomposed into rectangular cells. These rectangular cells can be partitioned into convex domains where the distance functions are convex and methods from convex optimization can be used to solve the problem. The second paper discusses this solution methodology and provides an example which gives an optimal solution on the nodes as in Larson and Sadiq (1983), also in a convex cell. This is important because one does not have to restrict oneself to nodes of the network to get an optimal value. These equivalence results are promising, but the authors should have taken into account the modification costs. If barriers are enlarged, they occupy some 'free to build' areas, which should incur some cost. Also, if barriers are reduced, this action may not be 'free of charge'.

Similar results, based on smart ways of decomposition of the planar feasible region, are provided by Dearing et al. (2002) for rectilinear center location problems with polyhedral barriers. They proposed an algorithm for the problem by considering a finite number of candidate sets, called dominating sets, to find the optimal location. Later, these results were extended by Dearing et al. (2005) using block norm distances in place of rectilinear distances. This work is also an extension to Hamacher and Klamroth (2000) who first considered block norm distances for the Weber problem with barriers.

There is also another body of research extending the studies of Larson and Sadiq (1983) and Batta et al. (1989). The first study is by Savas et al. (2002), who proposed a model for the finite size facility placement problem in the presence of some barriers and under the rectilinear distance norm. Interaction between a facility and demand points is handled through the facility's server point which is located on the facility's boundary. The demand points also interact with each other. The finite size facility, which has a fixed size and an arbitrary shape, acts as a barrier against the flow among the demand points. The authors provided objective function concavity results for facility location with a fixed orientation of the facility and also for facility orientation with a fixed location of the facility. Possible heuristics are suggested for simultaneous location and orientation decisions.

A special case of this problem in which the supply facility and the demand facilities have rectangular shapes was discussed by Wang et al. (2002) in a layout context. The objective was to determine optimal location of the new facility as well as optimal locations of input/output points on the demand facilities, in order to minimize total transportation costs. A similar problem with congested regions in which facility location is not allowed but where through-travel is possible at an additional cost per unit distance (weighted rectilinear) was discussed by Sarkar et al. (2005).

Another extension to Savas et al. (2002) is provided by Kelachankuttu et al. (2007), who also considered rectangular shaped facilities and barriers. The paper is devoted to the construction of contour lines, which are lines of equal objective value. Such contour lines provide alternatives for placement of the new facility.

Nandikonda et al. (2003) analyzed a similar problem with a minimax objective and point facility placement. Their work is extended by Sarkar et al. (2007) who addressed the finite size facility placement problem with only user-facility interactions.

In addition to the existing literature on Weber problems with polyhedral or circular barriers, a line barrier was introduced by Klamroth (2001b). The author considered a linear line barrier with a given number of passages, that divides the plane into two subplanes. Travelling from one subplane to the other has to be through one of these passages. The problem becomes a combinatorial problem when the number of passages is ≥ 2. Complexity of the problem grows exponentially with increasing number of passages but remains polynomial for a fixed number of passages.

From a different perspective, Frieß et al. (2005) conducted an experimental and simulation based study in which the authors used a wavefront approach for the minimax location problem with barriers. A physical experiment was conducted in a lab environment for the Euclidean distance case and based on the results from the experiment, a simulation model was developed using both Euclidean and Manhattan distances.

Table 2.2: Facility Location in the Presence of Barriers Literature Overview

	Distance	Objective	Interaction	Facility Shape	Barrier Shape	Barrier Type	Result
Katz and Cooper (1981)	Euclidean	minisum	user-facility	point	circular	fixed	local optimal
Larson and Sadiq (1983)	Rectilinear	minisum	user-facility	point	arbitrary	fixed	optimal
Batta et al. (1989)	Rectilinear	minisum	user-facility	point	arbitrary	fixed	optimal
Aneja and Parlar (1994)	Euclidean	minisum	user-facility	point	arbitrary	fixed	heuristic (SA)
Butt and Cavalier (1996)	Euclidean	minisum	user-facility	point	convex polygonal	fixed	local optimal
Butt and Cavalier (1997)	Weighted Rect.	minisum	user-facility	point	convex polygonal	fixed	optimal
Hamacher and Klamroth (2000)	Block	minisum	user-facility	point	convex polygonal	fixed	optimal
Klamroth (2001a)	any	minisum or center	user-facility	point	convex polygonal	fixed	optimal
Klamroth (2001b)	Euclidean	minisum	user-facility	point	line	fixed	optimal
Dearing et al. (2002)	Rectilinear	center	user-facility	point	convex polygonal	fixed	optimal
Dearing and Segars (2002a)	Rectilinear	minisum or center	user-facility	point	arbitrary	fixed	n/a
Dearing and Segars (2002b)	Rectilinear	minisum or center	user-facility	point	arbitrary	fixed	optimal
Klamroth and Wiecek (2002)	any	minisum and center	user-facility	point	line	fixed	pareto optimal
Savas et al. (2002)	Rectilinear	minisum	user-user and user-fac.	finite	arbitrary	variable	heuristic, optimal
Wang et al. (2002)	Rectilinear	minisum	user-user and user-fac.	point	rectangular	fixed	optimal
McGarvey and Cavalier (2003)	Euclidean	minisum	user-facility	point	convex polygonal	fixed	optimal
Nandikonda et al. (2003)	Rectilinear	center	user-facility	point	arbitrary	fixed	optimal
Klamroth (2004)	Euclidean	minisum	user-facility	point	circular	fixed	optimal
Dearing et al. (2005)	Block	minisum or center	user-facility	point	convex polygonal	fixed	optimal
Frieß et al. (2005)	Euclidean	center	user-facility	point	convex polygonal	fixed	heuristic
Sarkar et al. (2005)	Weighted Rect.	minisum	user-user and user-fac.	finite	rectangular	variable	optimal
Bischoff and Klamroth (2007)	Euclidean	minisum	user-facility	point	convex polygonal	fixed	heuristic (GA)
Kelachankuttu et al. (2007)	Rectilinear	minisum	user-user and user-fac.	rectangular	rectangular	variable	optimal
Sarkar et al. (2007)	Rectilinear	center	user-fac.	finite	arbitrary	variable	local optimal
Mustafa S. Canbolat Thesis							
Chapter 3	Rectilinear	minisum	user-facility	point	line	probabilistic	optimal
Chapter 4	Euclidean	minisum and center	user-facility	point	line with passages	fixed and variable	optimal
Chapter 5	Closest Rect.	minisum and center	user-user and user-fac.	rectangular	facility itself	variable	optimal
Chapter 6	Euclidean	minisum	user-facility	point	convex polygonal	fixed	optimal

2.4 Dissertation Objectives and Outline

This dissertation research consists basically of four essays exploring new facility location problems in the presence of barrier regions. Table 2.2 classifies and reviews the existing literature on the subject, and will be used to place these contributions in context.

The first essay, given in Chapter 3, deals with facility location problems in the presence of a probabilistic line barrier. To the best of our knowledge, there are no probabilistic models for facility location problems with barriers in the literature. We observe the fact that, in the real life, there might be a barrier to travel whose location may occur by chance and its' coordinates may follow some probability distribution. An example to such a barrier can be a road block, a natural disaster, an explosion, etc. The solution methodologies for Weber problems with barriers that have deterministic locations are already complicated. Therefore, for example, we expect the location problem with a polyhedral barrier that can occur randomly anywhere on the plane to be quite difficult. We therefore start with a very special type of barrier whose location can occur by chance and assume uniformity of time frames that it is located at any place. Our problem is to optimally locate a new facility in the presence of a rectangular or line barrier that occurs randomly on a given route on the plane. We analyze this problem in terms of the minisum objective and suggest a solution methodology.

In Chapter 4 we propose an extension of a problem proposed by Klamroth (2001b). Our contribution is threefold. First, in Section 4.3, we formulate the same minisum version of the problem as in Klamroth (2001b) as a Mixed Integer Nonlinear Programming (MINLP) model and provide an optimal solution methodology based on an Outer Approximation (OA) algorithm. Second, we discuss the minimax version of this problem for locating an emergency facility in Section 4.4 and use the same solution methodology as in Section 4.3 to solve the problem. We provide simple example problems and extensive computational results for both problems. In Section

4.5, we analyze an extension to a minimax version of the problem where locations of the passage points are also unknown. We also propose a one-infinity approximation approach for the same problem which yields a linear model.

The interrelated problems explored in Chapter 5 have also been of interest to some other researchers including Savas et al. (2002), Sarkar et al. (2005), and Kelachankuttu et al. (2007). The aforementioned studies investigate similar problems as described in the literature review but use a solution methodology that is different from ours. Their solution methodologies, in fact, are extensions to Larson and Sadiq (1983) and Batta et al. (1989). Our work is different from these studies in many aspects. In Section 5.2 we start with a simpler version of the problem given in Savas et al. (2002) by locating a barrier facility with a fixed orientation on a rectangular plane to minimize its interference to user-user interaction. In Section 5.3, we incorporate another objective into the problem such that the maximum of the closest distances from the demand points to the barrier is minimized. To the best of our knowledge, user-user and user-facility interactions under a minimax objective have never been explored. Also the 'closest distance family' has never been used for this type of problem. We provide a mixed integer programming formulation for the problem. In Section 5.4 we develop a Simulated Annealing (SA) heuristic for an extension of the problem where expropriation of existing facilities with some cost is also possible.

Finally, in Chapter 6, we present a new analog approach to the Weber problem in the presence of barriers by using a variant of the Varignon frame. We show through analysis that the same analog can also be used for certain Weber problems in the presence of barriers. We believe that the analog approach introduced in this study presents an easy way of solving such problems. We describe the problem, justify its usage through some analytical results and explain the analog process for the problem. Some examples from the literature are revisited through analog analysis and solutions are compared.

Chapter 7 is the conclusion and discussion for further research directions.

Chapter 3

The Planar Weber Problem with a Probabilistic Line Barrier

3.1 Introduction

Consider locating a facility on the plane to serve a finite set of existing demand points with different demand levels. One objective would be to find a location such that the sum of the distances from the facility to the demand points is minimum. Difficulties in solution methodologies occur when there are restrictions on location as well as restrictions on shortest travel paths. In the facility location literature, these restrictions take the form of barrier regions. Mountains, lakes, military zones, finite size existing facilities, railroads, highways etc. can be given as examples of barrier regions where neither travel through these regions nor placement in a region is possible. Despite the fact that facility location problems with barriers are more realistic, they have received relatively less attention from facility location researchers.

In this chapter, we address the problem of locating a single facility on the plane in the presence of a line barrier that occurs randomly on a given horizontal route on the plane. Note that we assume uniformity in time frames so that the barrier is located at any location. The existence of uncertainty in time can be incorporated into the model as a different probability distribution function but it may increase the underlying problem's complexity. This type of barrier generally occurs in some facility layouts where wagons on grooved rails are used for in-plant transportation. A wagon with negligible width compared to its length, can be anywhere on its route at anytime, and can interfere with the flow of material. Also accidents on transportation routes may cause sections of routes to be closed and these sections may not be transversable. Another interesting example can be given in the Robotics context as robots need to be designed to avoid collision with random obstacles while being in a least cost path. These lead to the question of where to optimally locate a new facility, in the presence of such a barrier. One objective might be to locate a new facility such that the sum of the expected rectilinear distances from the facility to the demand points in the presence of this probabilistic barrier is minimized. We analyze this problem.

Section 3.2 includes the definition of the problem and provides details on the formulation. A solution algorithm along with an example are given in Section 3.3. We discuss some possible extensions to the problem in Section 3.4 including a rectangular barrier case and a more general case where the barrier takes place on a non-horizontal linear route. Finally Section 3.5 summarizes the chapter with conclusions and further research directions.

3.2 Problem Definition and Formulation

Consider n demand points on the plane. Also consider a horizontal line barrier B_l with length l. Assume that the barrier has a fixed y coordinate at b and a probabilistic X_s coordinate (starting point of the line) where X_s is a continuous random variable with known parameters. Then the ending point of the line, X_e, will be $X_s + l$. The problem is to locate a facility on the plane so that the sum of the expected weighted rectilinear distances from the facility to the demand points is minimized:

$$\min \sum_{i=1}^{n} E\left[w_i l_1^B\left((x,y),(x_i,y_i)\right)\right] = \min \sum_{i=1}^{n} w_i E\left[l_1^B\left((x,y),(x_i,y_i)\right)\right]$$

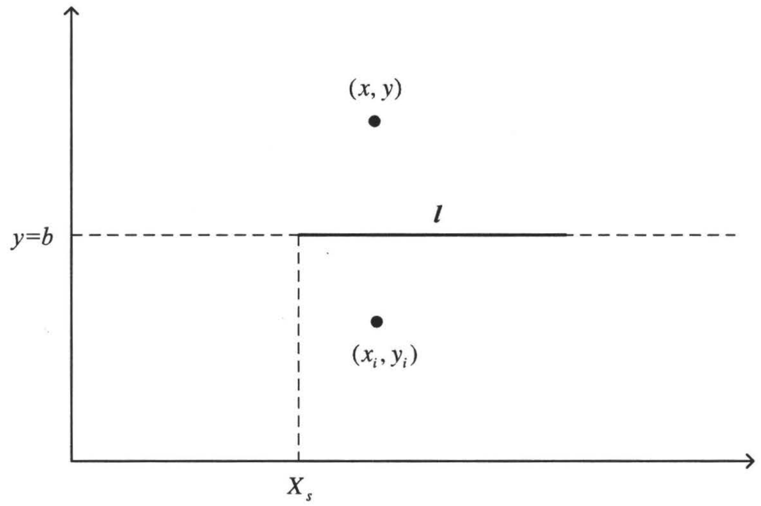


Figure 3.1: A Probabilistic Line Barrier on the Plane

The problem is illustrated in Figure 3.1. Without loss of generality and for notational convenience, we assume that the existing facilities are numbered such that their y-coordinate values are in increasing order:

$$y_1 \leq y_2 \leq ... \leq y_n.$$

For the barrier to have relevance, there will be some facilities that have smaller y-coordinate values than b and some others that have larger values:

$$y_1 \leq y_2 \leq ... \leq y_j < b < y_{j+1} \leq ... \leq y_n,$$

where $y_j = b$ is not allowed and j is the index such that $y_j < b < y_{j+1}$. Now we consider two different cases. Case 1 is where the optimal location of the new facility is at (x^*, y^*) where $y^* > b$ and Case 2 is where the optimal location of the new facility is at (x^*, y^*) where $y^* < b$. We will work only on Case 1, since the procedure is almost identical for Case 2.

Property 3.2.1 *Only the x-distance from the optimal facility location to the existing facilities is affected by the location of the barrier. That is, no matter where the* *barrier is located, the shortest y-path will remain the same as in the regular rectilinear distance.*

Proof. Assume that the width of the line barrier is negligible. The optimal facility location will either be 1-visible from an existing facility or invisible. If it is visible, then the distance will be the rectangular distance. If it is not visible, then the shortest path will be through one of the line barrier ends, which only affects the x-distance. Since the line barrier has a fixed y-coordinate, the y-distance will remain the same, no matter where the barrier is located on its route. ∎

This property also shows that the objective function is separable as in the traditional rectilinear distance Weber problem. Now we can write the objective function as:

$$\min \sum_{i=1}^{n} w_i E\left[l_1^B(x, x_i) + l_1^B(y, y_i)\right]$$

$$= \min \sum_{i=1}^{j} w_i E\left[l_1^B(x, x_i) + l_1^B(y, y_i)\right] + \sum_{i=j+1}^{n} w_i E\left[|x - x_i| + |y - y_i|\right]$$

$$= \min \sum_{i=1}^{j} w_i E\left[l_1^B(x, x_i) + l_1^B(y, y_i)\right] + \sum_{i=j+1}^{n} w_i \left(|x - x_i| + |y - y_i|\right).$$

We expand the first part of the formulation as:

$$\sum_{i=1}^{j} w_i E\left[l_1^B(x, x_i) + l_1^B(y, y_i)\right] = \sum_{i=1}^{j} w_i E\left[l_1^B(x, x_i)\right] + \sum_{i=1}^{j} w_i \left(|y - y_i|\right)$$

so that the objective function becomes:

$$\min \sum_{i=1}^{j} w_i E\left[l_1^B(x, x_i)\right] + \sum_{i=j+1}^{n} w_i \left(|x - x_i|\right) + \sum_{i=1}^{n} w_i \left(|y - y_i|\right).$$

The following lemma defines the conditions where the shortest path between x and x_i becomes a barrier distance. In other words, under which conditions are x and x_i invisible from each other. It also gives a formula for the shortest barrier distance.

Lemma 3.2.1 *Consider an existing facility i, where $i = 1, ..., j$. If the barrier is in effect, that is if facility x is not 1-visible from facility x_i, then $0 \leq x - X_s \leq l$ and $0 \leq x_i - X_s \leq l$. Furthermore, the shortest x-path between x and x_i is;*

$$l_1^B(x, x_i) = \min\{x + x_i - 2X_s, 2X_s + 2l - x - x_i\}$$

Proof. See Figure 3.2. Shaded areas represent the points that are not 1-visible from the new facility. It is clear that in order for the distance between x and x_i to become barrier distance, x and x_i have to be somewhere in between X_s and $X_s + l$. Conditions $0 \leq x - X_s \leq l$ and $0 \leq x_i - X_s \leq l$ together satisfy this requirement. In the first illustration x is less than X_s. Thus, the distance between these two points is rectangular. Indeed, there is no 1-shadow point for the existing facility in this case. In the second illustration, x_i is less than X_s which fails the barrier conditions. In the last illustration, both x and x_i are in between X_s and $X_s + l$ satisfying the barrier conditions. In this case the existing facility will be 1-shadow from the new facility; then the shortest l_1 path will follow the shortest route through one of the end points of the barrier:

$$l_1^B(x, x_i) = \min\{(|x - X_s| + |X_s - x_i|), (|x - X_e| + |X_e - x_i|)\} \Rightarrow$$

$$l_1^B(x, x_i) = \min\{x + x_i - 2X_s, 2X_s + 2l - x - x_i\}$$

■

We can see that the barrier goes into effect only under the conditions given in Lemma 3.2.1. We can also give these conditions as:

Lemma 3.2.2 *The barrier conditions $0 \leq x - X_s \leq l$, $0 \leq x_i - X_s \leq l$ can be represented in the following form:*

$$\max\{x - l, x_i - l\} \leq X_s \leq \min\{x, x_i\}$$

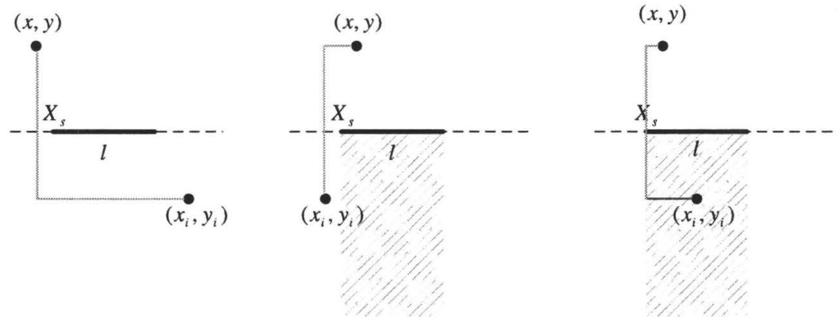


Figure 3.2: Visibility Conditions

Proof. From the barrier conditions, we know that $X_s \geq x - l$, $X_s \geq x_i - l$ and $X_s \leq x$, $X_s \leq x_i$. Therefore $\max\{x - l, x_i - l\} \leq X_s \leq \min\{x, x_i\}$. ■

The difficulty is that the shortest barrier distance is a random variable. Depending upon the underlying probability distribution of X_s and its parameters, the shortest barrier distance function will be different. For the sake of tractibility of the analysis, we assume that X_s is a uniformly distributed random variable with parameters $U(u_1, u_2)$ where $u_1 \leq \max\{x - l, x_i - l\}$ and $u_2 \geq \min\{x, x_i\}$. The probability density function of X_s is given as:

$$h(X_s) = \begin{cases} \frac{1}{u_2 - u_1} & ; u_1 \leq \mathrm{X_s} \leq u_2 \\ 0 & ; \text{otherwise} \end{cases}$$

The following theorem will help us in understanding the distribution function of the shortest barrier distance.

Theorem 3.2.1 *For given values of x and x_i, the distance function $l_1^B(x, x_i)$ is symmetric, piece-wise linear concave in X_s when $X_s \in \Psi : 0 \leq x - X_s \leq l$, $0 \leq x_i - X_s \leq l$ where Ψ is the set of X_s that satisfy these barrier conditions.*

Proof. First consider the case where $x > x_i$. Then $x - x_i < l$ and $X_s \in (x - l, x_i)$. In this case, the distance function is linearly increasing when $X_s \in (x - l, (x + x_i - l)/2]$, and it is linearly decreasing when $X_s \in [(x + x_i - l)/2, x_i)$.

Now consider the case where $x < x_i$. Then $x_i - x < l$ and $X_s \in (x_i - l, x)$. In this case, the distance function is linearly increasing when $X_s \in (x_i - l, (x + x_i - l)/2]$, and it is linearly decreasing when $X_s \in [(x + x_i - l)/2, x)$. ■

Lemma 3.2.3 *The expected value of the barrier distance when $X_s \in \Psi$ is:*

$$E[l_1^B(x, x_i)|X_s \in \Psi] = \frac{l + |x - x_i|}{2}.$$

Proof. See Figure 3.3 for illustration. The expected values of the barrier distance functions can easily be seen from the sub figures. Alternatively, one can calculate the expected value for $x > x_i$;

$$E[l_1^B(x, x_i)|X_s \in \Psi] =$$

$$\int\limits_{x-l}^{(x+x_i-l)/2} \frac{(2x_s + 2l - x - x_i)}{P(x - l \leq X_s \leq x_i)} h(x_s) dx_s + \int\limits_{(x+x_i-l)/2}^{x_i} \frac{(x + x_i - 2x_s)}{P(x - l \leq X_s \leq x_i)} h(x_s) dx_s =$$

$$\int\limits_{x-l}^{(x+x_i-l)/2} \frac{(2x_s + 2l - x - x_i)}{x_i - x + l} dx_s + \int\limits_{(x+x_i-l)/2}^{x_i} \frac{(x + x_i - 2x_s)}{x_i - x + l} dx_s =$$

$$= \frac{x - x_i + l}{2}$$

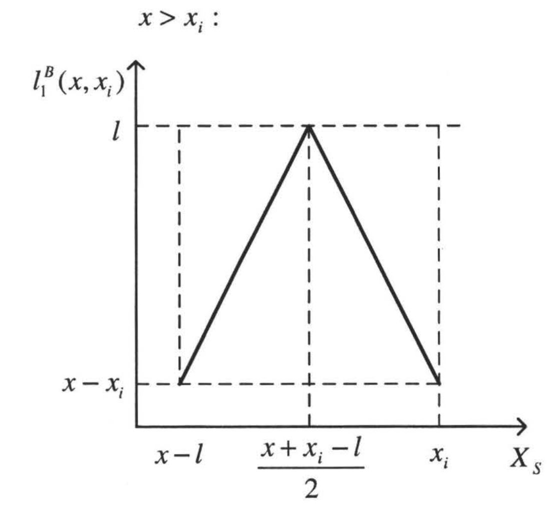
Calculations will be similar for $x < x_i$. ■

Let the probability that the barrier is in effect be

$$P(\max\{x - l, x_i - l\} \leq X_s \leq \min\{x, x_i\}) = \alpha(x),$$

where $\alpha(x)$ is a function of x. Then the expected distance between x and x_i for a given x will be;

$$E\left[l_1^B(x, x_i)\right] = \alpha(x)\frac{(|x - x_i| + l)}{2} + (1 - \alpha(x))\,(|x - x_i|)$$



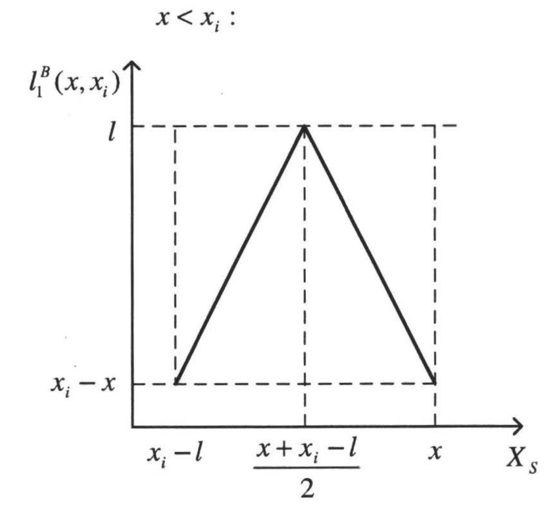


Figure 3.3: The distance function in the presence of the probabilistic line barrier

We know that when $|x - x_i| \geq l$, $\alpha(x) = 0$, thus the barrier will not have any affect on the rectilinear distance between x and x_i.

When $|x - x_i| < l$, first consider the case where $x \leq x_i$:

Then $\alpha(x)$ can be given as;

$$\alpha(x) = \frac{l + x - x_i}{r}, r = u_2 - u_1$$

Then consider the case where $x > x_i$:

Then $\alpha(x)$ becomes;

$$\alpha(x) = \frac{l - x + x_i}{r}, r = u_2 - u_1$$

In general,

$$\alpha(x) = \frac{l - |x - x_i|}{r}, r = u_2 - u_1.$$

Theorem 3.2.2 *Expected distance* $E\left[l_1^B(x, x_i)\right]$ *is a convex function of* x *for every existing location* x_i *where* $i \in \{1, 2, ..., j\}$ *or* $i \in \{j + 1, j + 2, ..., n\}$.

Proof. Obviously, expected distance is a convex function of x when $|x-x_i| \geq l$ because it is a rectilinear distance. We now look for the case when $|x-x_i| < l$.

$$E\left[l_1^B(x,x_i)\right] = \alpha(x)\frac{(|x-x_i|+l)}{2} + (1-\alpha(x))\left(|x-x_i|\right)$$

$$= \frac{(l-|x-x_i|)(l+|x-x_i|)}{2r} + \left(1-\frac{l-|x-x_i|}{r}\right)|x-x_i|$$

$$= \frac{(l-|x-x_i|)^2}{2r} + |x-x_i| = \frac{(x-x_i)^2}{2r} + \left(1-\frac{l}{r}\right)|x-x_i| + \frac{l^2}{2r}.$$

The first part of the expression is a convex function. Since $l < r$ by definition, second part of the expression is also a convex function. The last part is a constant term. Therefore the whole function is convex.

■

In general, we can write the expected distance of x from x_i as;

$$E\left[l_1^B(x,x_i)\right] = \left\{\begin{array}{cl} \frac{(l-|x-x_i|)^2}{2r} + |x-x_i| & ; |x-x_i| < l \\ |x-x_i| & ; |x-x_i| \geq l \end{array}\right\}$$

Given that the optimal location of the new facility will be in the rectilinear hull of demand points as being outside of this rectilinear hull will only increase the objective function, x-coordinate of the expected value function will have values between $\min_{1\leq k\leq n}\{x_k\}$ and $\max_{1\leq k\leq n}\{x_k\}$. We can rewrite the expected distance as;

$$E\left[l_1^B(x,x_i)\right] = \left\{\begin{array}{rr} -x+x_i & ; \min\limits_{1\leq k\leq n}\{x_k\} \leq x \leq x_i - l \\ \frac{(l+x-x_i)^2}{2r} - x + x_i & ; x_i - l < x < x_i \\ \frac{(l-x+x_i)^2}{2r} + x - x_i & ; x_i \leq x < x_i + l \\ x - x_i & ; x \geq x_i + l \leq \max\limits_{1\leq k\leq n}\{x_k\} \end{array}\right\}$$

This is a piecewise convex function of x and it reaches its global minimum at $x = x_i$. Figure 3.4 is an illustration of this function with given values of $l = 5$,$r = 10$,$x_i = 10$, and in the range of $[0, 20]$.

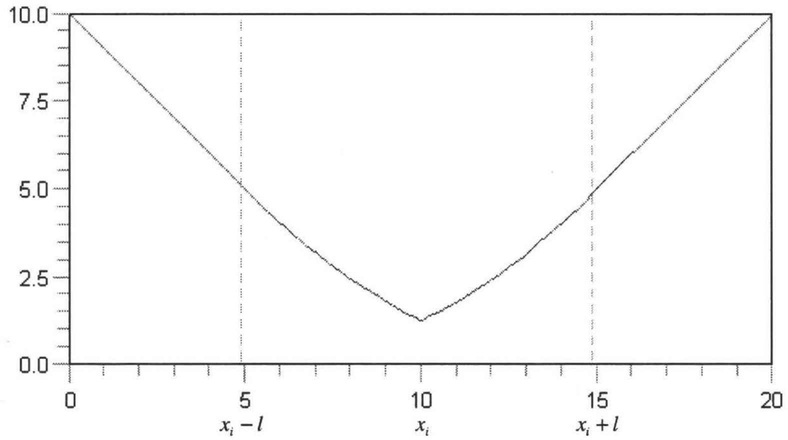


Figure 3.4: Expected distance function with given values

Since sum of the convex functions is also a convex function, it follows that the sum of the expected distances will be a convex function as well. We need to find the optimal value that minimizes the sum of these expected distances.

3.3 Solution Methodology

Let $f\left(X^*_{up}\right)$ be the objective value function of the problem when $y^* > b$ (Case 1) and $f\left(X^*_{down}\right)$ be the objective value of the problem when $y^* < b$ (Case 2). Then we can represent the problem in two subproblems. Algorithm 1 outlines the solution procedure for the problem. The procedure starts with sorting existing facilities by their y-coordinate in an increasing order. Since the barrier has no effect on the optimal y-coordinate, y^*, we find this value at the beginning. If y^* is above the barrier route, then the sum of the expected distances that belong to the existing facilities whose y-coordinates are smaller than b might be affected by the barrier (Case 1). Otherwise, the sum of the expected distances that belong to the existing facilities whose y-coordinates are larger than b might be affected by the barrier (Case 2). Furthermore, we also need to take into consideration that if more demand points are affected by the presence of the barrier in the opposite side of y^*, then it may be better to place the facility infinitesimally close (denote this distance as a very small number ϵ) to the barrier line in the opposite side, since locating the facility on the barrier route is not allowed. We define $f\left(X_{up}^{*c}\right)$ as the optimal objective function when the facility is located just below the barrier line and define $f\left(X_{down}^{*c}\right)$ as the optimal objective function when the facility is located just above the barrier line. Finally, if $y^* \in [t_1, t_2]$ where $t_1 < b < t_2$ then for Case 1, we need to have $y^* \in [t_1, b)$ and for Case 2, we need to have $y^* \in (b, t_2]$. The minimum of the two objective values will be the optimal objective value.

Algorithm 1:

Inputs:

- Demand points $(x_i, y_i), i = 1, ..., n$ where $y_1 \leq y_2 \leq ... \leq y_n$

- A line barrier, defined by its random x-coordinate (starting point) $X_s = U(u_1, u_2)$, y-coordinate b and fixed length l.

$$y^* = \arg\min\left\{\sum_{i=1}^{n} w_i\left(|y - y_i|\right)\right\}$$

1. If $y^* > b$ then;

$$f\left(X_{up}^{*}\right) = \min\sum_{i=1}^{j} w_i E\left[l_1^B(x, x_i)\right] + \sum_{i=j+1}^{n} w_i\left(|x - x_i|\right) + \sum_{i=1}^{n} w_i\left(|y^* - y_i|\right)$$

$$f\left(X_{up}^{*c}\right) = \min\sum_{i=j+1}^{n} w_i E\left[l_1^B(x, x_i)\right] + \sum_{i=1}^{j} w_i\left(|x - x_i|\right) + \sum_{i=1}^{n} w_i\left(|b - \epsilon - y_i|\right)$$

$$f\left(X^*\right) = \min\left\{f\left(X_{up}^{*}\right), f\left(X_{up}^{*c}\right)\right\}$$

$$\textbf{If } f\left(X^*\right) = f\left(X_{up}^{*c}\right) \Rightarrow y^* = (b - \epsilon)$$

$$x^* = \arg\min\left\{f\left(X^*\right)\right\}$$

2. If $y^* < b$ then;

$$f\left(X_{down}^{*}\right) = \min\sum_{i=j+1}^{n} w_i E\left[l_1^B(x, x_i)\right] + \sum_{i=1}^{j} w_i\left(|x - x_i|\right) + \sum_{i=1}^{n} w_i\left(|y^* - y_i|\right)$$

$$f\left(X^{*c}_{down}\right) = \min \sum_{i=1}^{j} w_i E\left[l_1^B(x, x_i)\right] + \sum_{i=j+1}^{n} w_i\left(|x - x_i|\right) + \sum_{i=1}^{n} w_i\left(|b + \epsilon - y_i|\right)$$

$$f\left(X^{*}\right) = \min\left\{f\left(X^{*}_{down}\right), f\left(X^{*c}_{down}\right)\right\}$$

$$\mathbf{If} f\left(X^{*}\right) = f\left(X^{*c}_{down}\right) \Rightarrow y^* = (b + \epsilon)$$

$$x^* = \arg\min\left\{f\left(X^*\right)\right\}.$$

3. If $y^* \in [t_1, t_2] \backslash b$ where $t_1 < b < t_2$ then;

$$f\left(X^{*}\right) = \min\left\{f\left(X^{*}_{up}\right), f\left(X^{*}_{down}\right)\right\}$$

$$x^* = \arg\min\left\{f\left(X^*\right)\right\}.$$

The initial value of y^* can easily be determined by using the technique proposed in (Love et al. (1988)). Because the objective function is convex as is proven in Theorem 2, we can determine the value of x^* by employing a one dimensional search technique or by using a nonlinear mixed integer programming (MINLP) solver. We can also construct some ranges for x^* in which the problem reduces to a pure convex linear or nonlinear program. Consider the case where $y^* > b$. The objective function for x-coordinate will be:

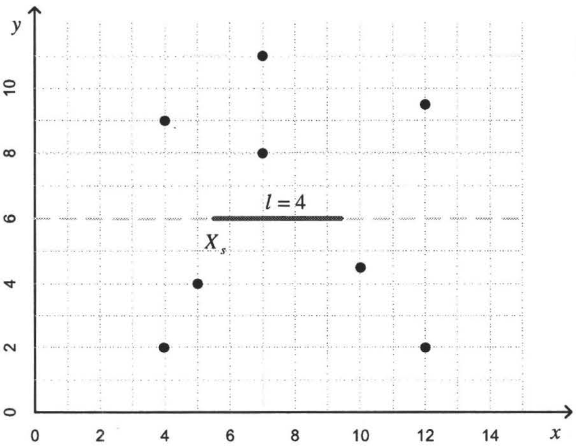
$$f\left(x^*\right) = f\left(x^*_{up}\right) = \min \sum_{i=1}^{j} w_i E\left[l_1^B(x, x_i)\right] + \sum_{i=j+1}^{n} w_i\left(|x - x_i|\right) =$$

$$\min \sum_{i=1}^{j} w_i \left\{ \begin{array}{cc} \frac{(l-|x-x_i|)^2}{2r} + |x - x_i| & ; |x - x_i| < l \\ |x - x_i| & ; |x - x_i| \geq l \end{array} \right\} + \sum_{i=j+1}^{n} w_i\left(|x - x_i|\right).$$

Let all existing facilities be shown at their projection points on x-axis. The break points for the objective function will be either at demand points $x_i, i = 1, ..., n$, or at the points $x_i - l$, $x_i + l$ where $i = 1, ..., j$. Between break points, the ranges determine where the objective function becomes a pure linear or nonlinear convex problem.

3.3.1 Example

Consider eight demand points and a line barrier on the plane. The line barrier has a probabilistic x-coordinate with parameters $U(0,12)$ and a fixed y-coordinate at $b=6$. Figure 3.5 illustrates the example and provides the data for the example.



i	x_i	y_i	w_i
1	4	2	1
2	12	2	1
3	5	4	1
4	10	4.5	1
5	7	8	1
6	4	9	1
7	12	9.5	1
8	7	11	1

Figure 3.5: Example 1

The optimal value of y can be determined easily. From Figure 3.5 we can see that its value is rather a range, $y^* \in [4.5, 8] \backslash 6$. We used the Excel Solver to identify the optimal value of x^*. Table 3.1 contains the results. Figure 3.6 is a surface plot for the objective function.

Table 3.1: Results for the example problem

x^*_{up}	y^*_{up}	$f\left(X^*_{up}\right)$	x^*_{down}	y^*_{down}	$f\left(X^*_{down}\right)$
7	(6,8]	46.25	7	[4.5,6)	47.375

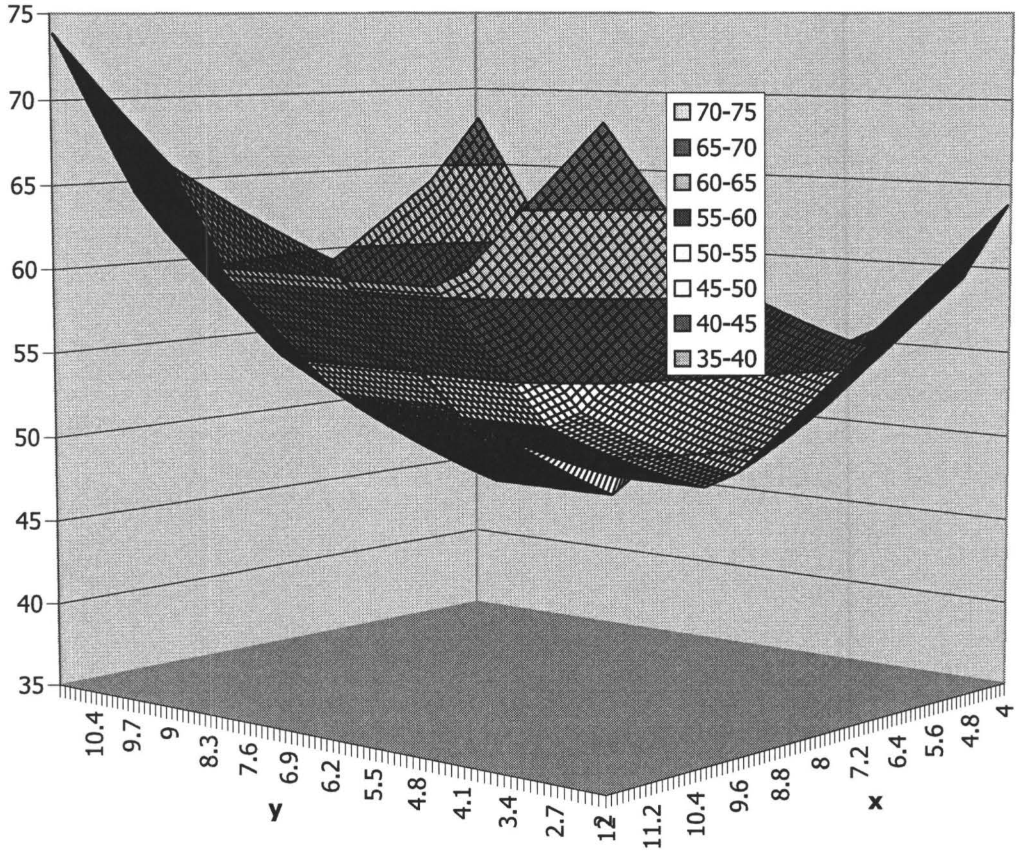


Figure 3.6: Surface Plot of the Objective Function

$$X^* = \arg\min\left\{f\left(X^*_{up}, X^*_{down}\right)\right\} = \arg\min\left\{(46.25, 47.375)\right\} = (7, (6, 8]).$$

We also calculated the optimal value of the objective function (See Table 3.2), when there is no barrier $f(nb)$, and when the barrier is at its expected location $f(fb)$, $(X_s = 6)$. Observe that the barrier has no effect on the objective value when it stays at its expected location. However this may not be the case in general.

Table 3.2: Results for $f(nb)$ and $f(fb)$

x^*_{nb}	y^*_{nb}	$f\left(X^*_{nb}\right)$	x^*_{fb}	y^*_{fb}	$f\left(X^*_{fb}\right)$
7	[4.5,8]	46	7	[4.5,8]	46

To find the value of x^* over some ranges, in which the problem is a pure linear or nonlinear convex function, we need to determine the break points as suggested before. Consider the case where $y^* \in (6, 8]$ in the above example. The break points in increasing order will be $\{x_1, x_6\}, x_3, x_4 - l, \{x_5, x_8\}, \{x_2 - l, x_1 + l\}, x_3 + l, x_4, \{x_2, x_7\}$. As an illustration, Table 3.3 shows the ranges, points affected by the barrier in each range, and relative location of x^* in each range for $f\left(X^*_{up}\right)$.

We find the objective values for all the ranges, then we identify the optimal value as $x^* = 7$ which has the lowest objective value of 46.25 for the above subplane (Case 1). Note that we do not need to calculate the objective function for range $[0, 4)$ as it falls out of the rectangular hull of the demand points.

3.4 Possible Extensions

One of the objectives of this study is to provide preliminary results and suggest avenues for future research. We have made simplifying assumptions to facilitate our analysis. One of the assumptions is the uniform distribution of the line barrier location. The first extension can be consideration of other probability distributions such

Table 3.3: Ranges for Optimal Facility Location for Case 1

	Range	Points in the Barrier Range	Relative Location of x^*	$f\left(X_{up}^*\right)$
1	$[4, 5)$	x_1, x_3	$x_1, x_6 \leq x^* \leq x_3$	51.43
2	$[5, 6)$	x_1, x_3	$x_3 \leq x^* \leq x_5, x_8$	48.78
3	$[6, 7)$	x_1, x_3, x_4	$x_3 \leq x^* \leq x_5, x_8$	46.46
4	$[\mathbf{7}, 8)$	x_1, x_3, x_4	$x_5, x_8 \leq x^* \leq x_4$	**46.25**
5	$[8, 9)$	x_2, x_3, x_4	$x_5, x_8 \leq x^* \leq x_4$	48.21
6	$[9, 10)$	x_2, x_4	$x_5, x_8 \leq x^* \leq x_4$	50.42
7	$[10, 11)$	x_2, x_4	$x_4 \leq x^* \leq x_2, x_7$	52.83
8	$[11, 12]$	x_2, x_4	$x_4 \leq x^* \leq x_2, x_7$	56.75

as the normal distribution. Also, we have considered a single probabilistic barrier, whereas in the real life, there might be more than one probabilistic barrier, or a mix of probabilistic and fixed barriers present on the plane. Other possible extensions, namely a probabilistic rectangular barrier, and a probabilistic line barrier on a non-horizontal linear route are explored in detail in the subsections.

3.4.1 Rectangular Barrier Case

If we thicken the line barrier (rectangular shape), the problem will hold the same properties (See Figure 3.7) but there is one exception: If there is any demand point located on the route of this rectangular barrier, we should take into consideration the probability that this demand point might be swallowed by the barrier, which makes the demand point unreachable by the new facility. However, in practice, such a situation will not exist, because the new facility can not be located on the route of the barrier.

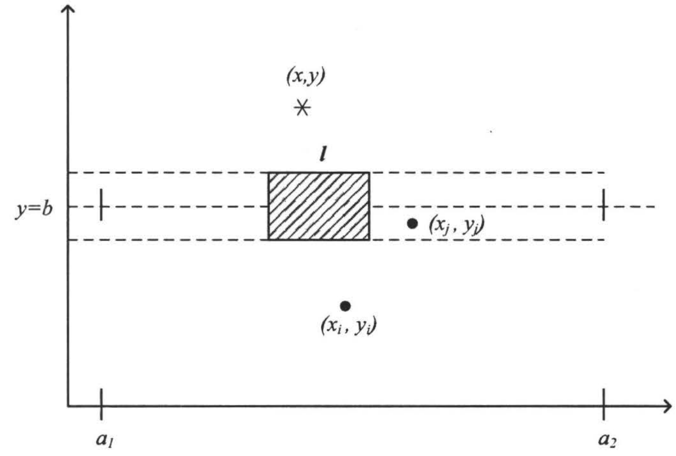


Figure 3.7: Probabilistic Rectangular Barrier on a Horizontal Route

3.4.2 Line barrier on a Linear Route

Assume now that the line barrier is on a linear route where $Y_s = \rho X_s + b$. In this case, Y_s as well becomes a random variable. Moreover, the minisum objective function is no longer separable, because location of the barrier can also affect the distance in the y direction. One of the tricks that can be used to simplify the problem is to perform a linear transformation in such a way that $Y_s' = b$. In this case, the distance function being used will not be the rectilinear distance anymore with respect to the original axes; instead, it will be a block norm distance function with four fundamental directions. Before advancing on the problem, we provide two lemmas given by Klamroth (2002), that have relevance.

Lemma 3.4.1 (Klamroth,(2002)) *Let $\|h\|$ be a block norm with four fundamental vectors namely b_1, b_2, $b_3 = -b_1$ and $b_4 = -b_2$. Let T be the linear transformation* *such that* $T(b_1) = Tb_1$ *and* $T(b_2) = Tb_2$. *T can be written as,*

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \frac{1}{b_{1y}b_{2x} - b_{1x}b_{2y}} \begin{pmatrix} b_{2x} & -b_{1x} \\ -b_{2y} & b_{1y} \end{pmatrix}.$$

Then for any two points $X, X_1 \in \Re$,

$$h(X, X_1) = l_1 (T(X), T(X_1))$$

Lemma 3.4.2 (Klamroth,(2002)) *Let h be a block norm with four fundamental vectors and let the linear transformation T be defined as in Lemma 3.4.1. Then for points* $X, X_1 \in F$,

$$h^B (X, X_1) = l_1^{T(B)} (T(X), T(X_1))$$

The lemmas state that whether there are barriers present on the plane or not, the distance between any two points before and after linear transformation will be the same but the underlying distance function will be different.

Let us now consider a special case where a probabilistic line barrier with length l located on a linear route represented by $Y_s = X_s$, in the presence of n existing points. We want to find a point, say X, that minimizes the sum of the rectilinear barrier distances to the existing points.

Consider the following linear transformation: $T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

For given two points $X = (x, y)$ and $X_1 = (x_1, y_1)$, the rectilinear barrier distance between these points will be equal to the Tchebychev barrier distance between the transformed locations of X and X_1 given as $T(X)$ and $T(X_1)$ respectively because of the lemmas provided above. Observe that this particular linear transformation essentially involves in rotating the axes by $-\pi/4°$, and scaling up the planar area by $\sqrt{2}$ as can be seen in Figure 3.8. This acknowledges that both minisum problems before and after the linear transformation will provide the same objective value.

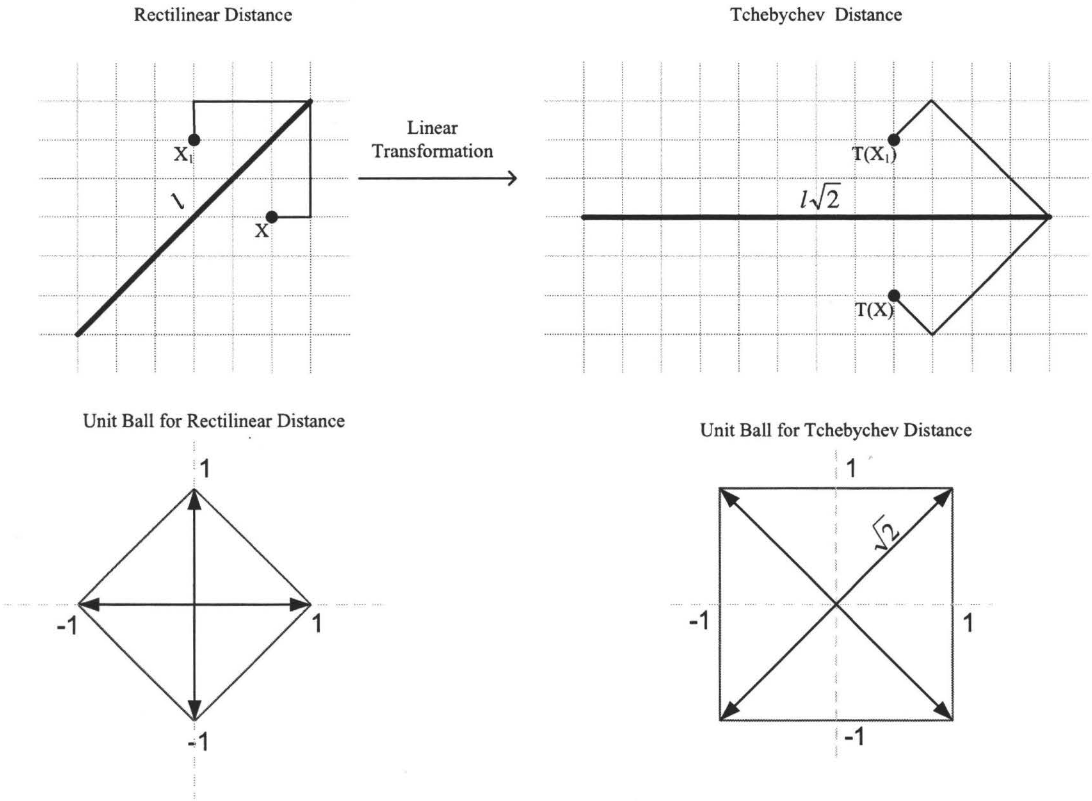


Figure 3.8: Change in the distance function after a linear transformation

We call this equivalent new problem as 'Finding the optimal minisum point in the presence of a probabilistic line barrier under the Tchebychev distance norm'. For this problem, the are affected by the presence of the barrier, hence the barrier conditions depend on both x and y coordinates of the new facility. Figure 3.9 shows shadow regions for points $X_1 = (x_1, y_1)$ and point $X_2 = (x_2, y_2)$. As future research, we would like to see if the convexity property holds under the Tchebychev distance norm or in general under block norms.

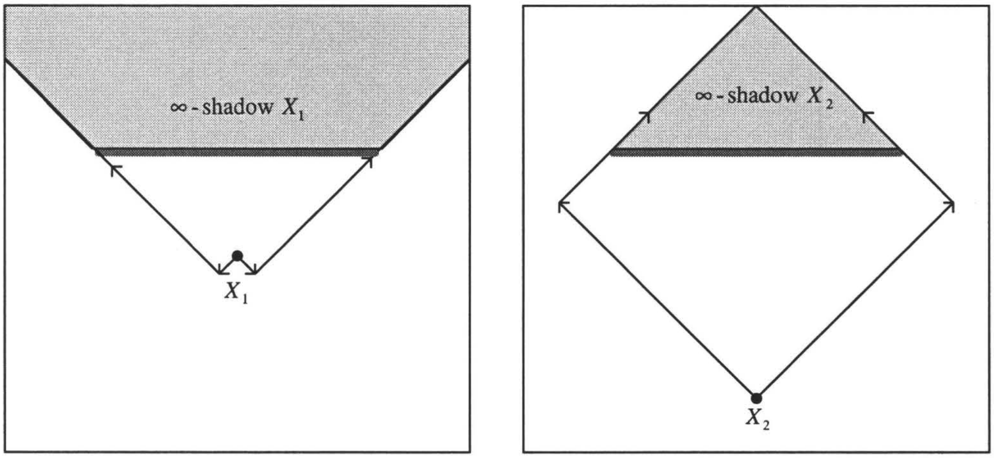


Figure 3.9: ∞-shadow regions for X_1 and X_2

3.5 Summary

In this chapter we investigated the problem of finding a minisum point for a facility in the presence of a line barrier whose coordinates are random. Despite the fact that the facility location problems with barriers are more realistic, they have received little attention from facility location researchers. Also, so far no one has considered the case where the location of a barrier is random. It is important to investigate these problems. Examples of these type of barriers can be transportation wagons in manufacturing facilities in a smaller scale and accidents or road constructions inside segments of transportation corridors in a bigger scale. We observed some properties of the problem and suggested a solution algorithm.

As future research we discussed some possible extensions of the model under the Weber objective. Some of these extensions will be investigated in detail. Another extension to the problem may be to use maximum 'regret' as an objective. The idea is to choose a location for the facility while observing the worst location of the barrier (i.e., the location of the barrier that maximizes the minimum distance).

Chapter 4

Optimal Location of a Facility and Passage Points in the Presence of a Line Barrier

4.1 Introduction

In Chapter 3 we introduced a Weber problem in the presence of a probabilistic line barrier with a given length. As discussed in Chapter 2, the concept of a line barrier was first introduced by Klamroth (1996) in a different context. A fixed linear line barrier that divides the planar region into two half-planes is considered. Crossing the line barrier is only allowed through a finite set of passage points located on the line barrier. The author proposed a solution algorithm which requires solving a number of convex optimization problems and selecting the best one among them. The time complexity of the proposed algorithm is given as $O\left(N\,(M \log M) + \binom{M+N-1}{N-1} T\right)$ where N is the number of passages, M is the number of demand points and T is the time complexity of the Weber problem. The complexity grows exponentially with an increasing number of passages but remains polynomial for a fixed number of passages. Although the suggested algorithm reduces the problem size significantly compared to a complete enumeration, it has a number of drawbacks. First of all, for $N > 2$, the algorithm requires finding a permutation that satisfies ordering of demand points based on their distance differences to the passage points. Second, the formulation is problem specific and it may not be suitable for some possible extensions such as considering different facility location objectives, including additional constraints, or an extension where locations of the passage points on the line barrier are also decision variables. Therefore, an alternative solution methodology that provides faster solutions and allows these extensions would be beneficial.

After providing brief preliminaries in Section 4.2, we consider the same minisum version of this problem as in Klamroth (2002) in Section 4.3 for locating a supply facility and propose a different solution methodology based on a Mixed Integer Nonlinear Programming (MINLP) model. We transform the initial MINLP formulation which contains the products of binary variables and the functions of continuous variables (Euclidean distances in this case) into a new formulation in which the binary variables are linearly associated with the continuous variables. To do that, we use a trick suggested by Glover (1975). We then successfully apply the Outer Approximation (OA) algorithm structure developed by Duran and Grossmann (1986) to the new formulation in GAMS (Rosenthal (1988)). Because of the convexity of the Euclidean distances, we show that the OA algorithm is guaranteed to find the optimal solution to the problem in a finite number of steps. In Section 4.4, we discuss the minimax version of this problem for locating an emergency facility and use the same solution methodology as in Section 4.3 to solve the problem. We provide simple example problems and extensive computational results for both problems.

In Section 4.5, we argue that using a MINLP formulation makes it possible to consider various extensions of these problems. One possible extension is to assume that the locations of the passage points on the line barrier are also decision variables. This problem finds itself a basis in a real life situation where bridges over highways or rivers need to be built simultaneously with a facility. Some other constraints can also be added to the formulation such as minimum distance requirements between passage points and capacity constraints on passage points. We briefly discuss the minimax version of this problem and provide a modified OA algorithm as a solution methodology. Finally, in Section 4.6, we approximate the Euclidean distance by one-infinity norm distance, which leads to a linear Mixed Integer Programming model. We believe that the new model may provide faster approximate solutions for larger problem instances.

4.2 Preliminaries

Definition 4.2.1 (Klamroth, (1996)) *Let* $L := \{(x, y) \in \Re^2 | y = ax + b\}$ *be a line and let* $\{P_j = (p_{xj}, p_{yj}) \in J | j \in J := \{1, ..., J\}\}$ *be a set of points on* L*. Then* $B_L := L\ \{P_1, ..., P_J\}$ *is called a* ***line barrier with passages****.*

The feasible region Ω is defined as the union of the two closed half planes Ω_1 and Ω_2 above and below B_L, respectively. We have a set of existing demand points each with a positive weight w_i with locations $X_i = (x_i, y_i), i \in I$ given in Ω.

We want to locate a facility in Ω under a given objective. To do that, first of all, we need to make an assumption that the location of the facility is in one of the subplanes. Any demand point that is not in the same subplane as the new facility will reach the new facility through one of the passage points on the line barrier. The other demand points, that are in the same subplane, are not affected by the presence of the line barrier.

Obviously, we can solve two subproblems by assigning the new facility to each subplane and take the minimum of these two solutions as the global optimum solution. Without loss of generality, in our formulations, we assume that the new facility is assigned to Ω_2. The solution methodology will be similar for the other case.

4.3 Optimal Location of a Supply Facility in the Presence of a Line Barrier

4.3.1 Problem Definition and Formulation

In this section we consider a supply facility location problem. The objective is to find a location $X \in \Omega_2$ for a new facility in the presence of the line barrier such that the sum of the weighted Euclidean distances between the new facility and the existing facilities is minimized.

The initial formulation for this problem will be (Formulation 4.1):

$$\text{Min} \quad \sum_{i \in \Omega_2} w_i l_2(X, X_i) + \sum_{i \in \Omega_1} \sum_{j \in J} u_{ij} w_i \left(l_2(X, P_j) + l_2(P_j, X_i) \right) \tag{4.1}$$

subject to

$$\sum_{j \in J} u_{ij} = 1, \forall i \in \Omega_1 \tag{4.2}$$

$$u_{ij} \in \{0, 1\}, \forall i \in \Omega_1, \forall j \in J \tag{4.3}$$

where u_{ij} is defined as,

$$u_{ij} = \left\{ \begin{array}{ll} 1 & \text{if passage j is assigned to demand point i} \\ 0 & \text{otherwise} \end{array} \right\}. \tag{4.4}$$

The objective function (4.1) aims to minimize the sum of the weighted Euclidean distances between the new facility and the demand points. Assuming that the optimal facility location is in Ω_2, the facility communicates with the demand points in Ω_2 directly. But for the demand points in Ω_1, the distances involve in finding the possible optimal assignments of demand points to the passage points. Constraint (4.2) is the assignment constraint to make sure each demand point in Ω_1 is served by one passage point. Constraint (4.3) imposes a binary restriction on the decision variable u_{ij}.

The difficulty with Formulation 4.1 is that it contains products of binary variables u_{ij} and continuous variables (Euclidean distance functions) in the objective function. To overcome this difficulty, using the linearization technique suggested by Glover (1975) and then modified by Torres (1991) for problems with nonlinearities in the objective functions, we can replace Formulation 4.1 with the following equivalent formulation (Formulation 4.2):

$$\text{Min} \quad \sum_{i\in\Omega_2} w_i l_2(X, X_i) + \sum_{i\in\Omega_1}\sum_{j\in J} \Upsilon_{ij} + \sum_{i\in\Omega_1}\sum_{j\in J} u_{ij} w_i l_2(P_j, X_i) \tag{4.5}$$

subject to

$$\Upsilon_{ij} \geq w_i l_2(X, P_j) - M^+_{1ij}(1 - u_{ij}), \forall i \in \Omega_1, \forall j \in J \tag{4.6}$$

$$\Upsilon_{ij} \geq 0, \forall i \in \Omega_1, \forall j \in J \tag{4.7}$$

$$\sum_{j\in J} u_{ij} = 1, \forall i \in \Omega_1 \tag{4.8}$$

$$u_{ij} \in \{0, 1\}, \forall i \in \Omega_1, \forall j \in J \tag{4.9}$$

where Υ_{ij} is a continuous variable that replaces the product of u_{ij} and $w_i l_2(X, P_j)$ and M^+_{1ij} is a bounding parameter that is larger than $w_i l_2(X, P_j)$ for any feasible location of $X \in \Omega_2$.

Lemma 4.3.1 *Formulation 4.1 is equivalent to Formulation 4.2*

Proof. For a given pair of i and j, when $u_{ij} = 1$, the model will force Υ_{ij} to be equal to $w_i l_2(X, P_j)$. When $u_{ij} = 0$, the model will force Υ_{ij} to be equal to 0. In both cases, the formulation becomes equivalent to Formulation 4.1. Note that, the determination of u_{ij} is also affected by the last term in the objective function which does not depend on X but depends only on a known constant term which is the Euclidean distance between passage point j and demand point i. ■

Property 4.3.1 determines the supremum upper bound values of the bounding parameter M^+_{1ij}. These values have to be as close as possible to their corresponding $w_i l_2(X, P_j)$ values. Using unnecessarily large values typically leads to very bad branch-and-bound trees in the solution process in terms of quality as it allows for many fractional values in the linear relaxation and causes to larger linear feasible regions (Bosch and Trick (2005)).

Property 4.3.1 *$M_{1ij}^{+} > l_2\left(X^e, P_j\right)$ where $X^e \in \Omega_2$ is defined as the farthest extreme point of the convex hull of the demand points and the passage points in Ω_2 to P_j.*

Proof. We know that by definition $M_{1ij}^{+} > w_i l_2\left(X, P_j\right)$. Then the farthest possible location of the new facility will be at the farthest extreme point of the convex hull of the demand points and the passage points. This is because of the fact that the new facility location will be in the convex hull of the demand points and the extreme points of the barrier regions (Klamroth (2002)). Therefore, $M_{1ij}^{+} > w_i l_2\left(X^e, P_j\right)$. ■

Transforming Formulation 4.1 into Formulation 4.2 gives us an advantage of solving this problem with an OA algorithm. In order to solve MINLP problems using an OA algorithm, one needs to satisfy two conditions: First, the model should have the convexity property with respect to the continuous variables. And second, the model should be linear in the binary or integer variables. In models that are suitable for the OA algorithm, the continuous space is formed by a finite number of convex regions where each region is defined by a different discrete parameter combination. The OA algorithm solves a finite sequence of MILP master programs and NLP subproblems. Solutions of the MILP master programs are lower bounds for the original problem, while solutions of the NLP subproblems are upper bounds for the original problem. The algorithm converges when these two bounds cross. For MILP master programs, linearity in the convex nonlinear continuous variables is introduced by outer approximation of their convex sets. For NLP subproblems, pure nonlinearity is obtained by fixing the discrete variables in the original problem. The algorithm can be considered a cutting plane method, where after each iteration, a combination of the discrete variables, and hence a convex region from the continous space, is eliminated through the introduction of integer cuts. Therefore, the algorithm is guaranteed to terminate in a finite number of steps. The worst case of the algorithm is the total enumeration of the discrete variables which is exponential.

4.3.2 An Outer Approximation Algorithm Approach for the Supply Facility Location Problem in the Presence of a Line Barrier

Let $(MasterS^k)$ be the k^{th} iteration of the master program with an objective function defined as $f^{(k)}$. Let $(SubS^k)$ be the k^{th} iteration of u_{ij}^k parameterized NLP sub program with an objective function defined as $f(u_{ij}^k)$.

START

Step 0 :

For a given facility location X^k, the **master** program $(MasterS^k)$ is shown as the following mixed integer linear programming (MILP) model:

$$f^{(k)} = \text{Min} \quad \chi + \sum_{j\in J}\sum_{i\in\Omega_1} \Upsilon_{ij} + \sum_{j\in J}\sum_{i\in\Omega_1} u_{ij} w_i l_2\left(P_j, X_i\right)$$

subject to

$$\chi \geq \sum_{i\in\Omega_2} w_i l_2\left(X^k, X_i\right) + \sum_{i\in\Omega_2} w_i \nabla l_2\left(X^k, X_i\right)\begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix}$$

$$\Upsilon_{ij} \geq w_i l_2\left(X^k, P_j\right) + w_i \nabla l_2\left(X^k, P_j\right)\begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} - M_{1ij}^{+}(1-u_{ij}), \forall i \in \Omega_1, \forall j \in J \tag{4.10}$$

$$\Upsilon_{ij} \geq 0, \forall i \in \Omega_1, \forall j \in J$$

$$\sum_{j\in J} u_{ij} = 1, \forall i \in \Omega_1$$

$$f^{(k-1)} \leq \chi + \sum_{j\in J}\sum_{i\in\Omega_1} \Upsilon_{ij} + \sum_{j\in J}\sum_{i\in\Omega_1} u_{ij} w_i l_2\left(P_j, X_i\right) \tag{4.11}$$

$$f^{(k)} < f^* \tag{4.12}$$

$$u_{ij} \in \{0, 1\}, \forall i \in \Omega_1, \forall j \in J$$

where,

$$\nabla l_2\left(X^k, X_i\right) = \left(\frac{x^k - x_i}{\sqrt{(x^k-x_i)^2+(y^k-y_i)^2+\epsilon}}, \frac{y^k - y_i}{\sqrt{(x^k-x_i)^2+(y^k-y_i)^2+\epsilon}}\right),$$

$$\nabla l_2\left(X^k, P_j\right) = \left(\frac{x^k - p_{xj}}{\sqrt{(x^k-p_{xj})^2+(y^k-p_{yj})^2+\epsilon}}, \frac{y^k - p_{yj}}{\sqrt{(x^k-p_{xj})^2+(y^k-p_{yj})^2+\epsilon}}\right),$$

$$f^* = \min\left\{f^*, f(u_{ij}^k)\right\} \text{ and,}$$

χ is a positive variable which becomes the largest linear approximation (first order linear Taylor series approximation) to $\sum_{i\in\Omega_2} w_i l_2\left(X, X_i\right)$ in the optimal solution to Formulation 4.2.

Similarly, constraint (4.10) contains the first order linear Taylor series approximation to nonlinear distance function $l_2\left(X, P_j\right)$ in constraint (4.6).

Constraint (4.11) tightens the lower bound after every iteration and acts as a weak cut to get a faster solution (Duran and Grossmann (1986)).

The ϵ term, which is a very small positive number is added into the denominator of the gradients to prevent the possibility of dividing by zero during the algorithmic process if any potential optimal location coincides with one of the demand points or the passage points. It can be taken as zero initially, and if the algorithm terminates due to a 'division by zero error', a very small value of ϵ such as $\epsilon = 10^{-5}$ can be used.

Step 1 :

Let the objective function be denoted as f. Set lower bound of the objective function $f^{(0)} = -\infty$, upper bound of the objective function $f^* = +\infty$ and $k = 1$. Make an arbitrary initial feasible selection of binary variables u_{ij}^k. These variables are initially selected as each demand point will use the closest passage to itself.

Step 2 :

Solve the u_{ij}^k parameterized NLP **sub** problem $SubS^k$:

$$f(u_{ij}^k) = \text{Min} \quad \sum_{i\in\Omega_2} w_i l_2(X, X_i) + \sum_{j\in J}\sum_{i\in\Omega_1} \Upsilon_{ij} + \sum_{j\in J}\sum_{i\in\Omega_1} u_{ij}^k w_i l_2(P_j, X_i)$$

subject to

$$\Upsilon_{ij} \geq w_i l_2(X, P_j) - M_{1ij}^{+}(1 - u_{ij}^k), \forall i \in \Omega_1, \forall j \in J$$

$$\Upsilon_{ij} \geq 0, \forall i \in \Omega_1, \forall j \in J$$

$$\Upsilon_{ij} \in \Re, \forall i \in \Omega_1, \forall j \in J$$

If problem $SubS^k$ has a finite optimal solution, update the current upper bound;

Estimate: $f^* = \min\left\{f^*, f(u_{ij}^k)\right\}$ and if $f^* = f(u_{ij}^k)$ set $u_{ij}^* = u_{ij}^k$, $X^* = X^k$;

Add an integer cut to $MasterS^k$ by introducing the following constraint to the model to eliminate previously used integer combinations u_{ij}^k from further consideration:

$$\sum_{ij\in Cut_1^k} u_{ij}^k - \sum_{ij\in Cut_0^k} u_{ij}^k \leqslant \left|Cut_1^k\right| - 1, \forall k$$

where $Cut_1^k = \left\{(i,j)\left|u_{ij}^k = 1\right.\right\}$ and $Cut_0^k = \left\{(i,j)\left|u_{ij}^k = 0\right.\right\}$ and $\left|Cut_1^k\right|$ is the cardinality of Cut_1^k.

Step 3 :

Solve the MILP master program $MasterS^k$, adding the integer cuts as constraints from Step 2.

If program $MasterS^k$ doesn't have a mixed integer feasible solution STOP. The optimal solution is $\left(u_{ij}^*, X^*\right)$. This is because of the fact that lack of a feasible solution suggests the violation of constraint (4.12) in the master program. The algorithm converges when the master problem has no solution, which indicates the crossing of the lower and upper bounds.

If program $MasterS^k$ has an optimal mixed integer solution, set $u_{ij}^{k+1} = u_{ij}$, $k = k + 1$ and return to **Step** 2.

END

The proof of the convergence of a general OA algorithm is given in Duran and Grossmann (1986) and is applicable to the algorithm given above since our model has the convexity property with respect to the Euclidean distance functions and it became linear in the binary assignment variables which are the two requirements of the convergence as we have discussed before. Convergence is proved based on two independent criteria. The first one is based on the bounding properties of the algorithm. The lack of a feasible solution due to constraint (4.12) implies the infeasibility for all of the remaining solutions because of a monotone nondecreasing sequence of lower bounds on the optimal value of the original problem as proven by the authors. The second one depends on the finiteness of the set of discrete variables as each set is generated only once. Figure 4.1 is a flowchart of the algorithmic process.

4.3.3 Example [Klamroth, (2002)]

We consider the example given in Klamroth (2002) to illustrate our OA approach. Let B_L be a line barrier defined by the following parameters:

$$B_L := \{X = (x, y) \in \Re^2 | y = 5\} / \{P_1 = (4,5), P_2 = (9,5)\},$$

B_L divides the planar feasible region into two subplanes (Ω_1 and Ω_2). Three existing facilities are located on each subplane. These are listed in Table 4.1 along with their corresponding weights and the subplanes that they belong to. We want to locate a supply facility on the plane to minimize the sum of the Euclidean distances to these demand points in the presence of the line barrier defined above.

We need to apply the OA algorithm two times, each time assuming the location of the supply facility to be in one of the subplanes. First, we assume that the facility is located on the upper subplane (Ω_1). In this case OA algorithm converges to the optimal solution of 50.405 for this subplane in 0.187 seconds and in four iterations on a PC Pentium 4 2.4 Ghz with 1 GB RAM. Then, we assume that the facility is located on the lower subplane (Ω_2). The OA algorithm converges to the optimal solution of 48.4623 for this subplane in 6.654 seconds and in three iterations. Figure 4.2 is an

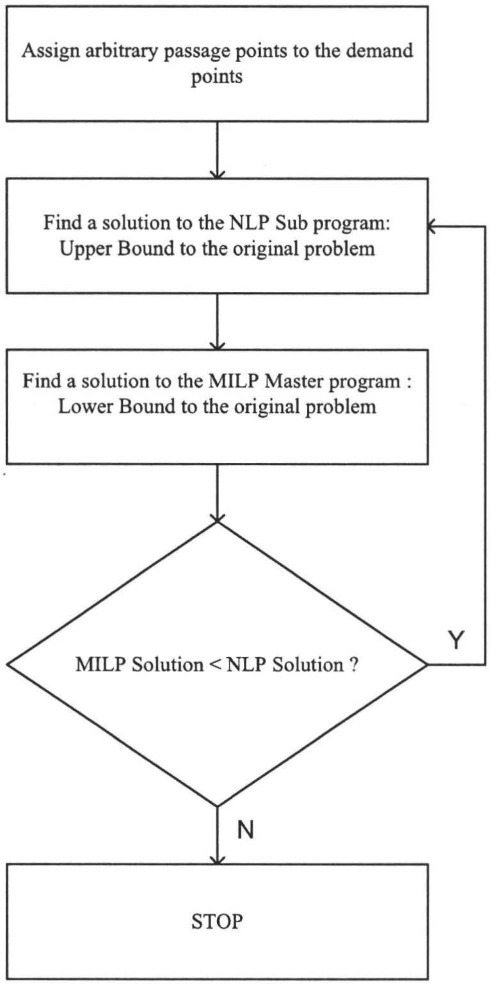


Figure 4.1: Outer Approximation Algorithm Flowchart

i	x_i	y_i	w_i	Ω_j
1	5	7	1	Ω_1
2	4.5	9	2	Ω_1
3	10.0	7.5	2	Ω_1
4	3.0	3.0	2	Ω_2
5	6.0	1.0	3	Ω_2
6	8.5	4	2	Ω_2

Table 4.1: Parameter Values for the Existing Facilities

illustration of the convergence. Therefore, we can identify the optimal objective value for the whole plane as 48.4623 with the facility located at $(5.676, 3.434)$. This result is also confirmed by solving the example using a general nonlinear global solver named BARON (Sahinidis (1996)) which gives the solution in 19.170 seconds. Klamroth (2002) reports her optimal objective function as 48.47 with the facility located at $(5.72, 3.43)$.

4.3.4 Further Computations

To test the performance of the proposed OA algorithm and validate its usage for possible extensions of the problem, we evaluated two different MINLP solvers namely DICOPT (Kocis and Grossmann (1989)) and BARON using Formulation 4.2. We compared their results with the proposed OA algorithm. DICOPT is a general MINLP solver which is also based on the OA method. BARON is a global branch and reduce optimizer, developed also at the same university, which can solve non convex optimization problems to global optimality. Our implementation of the OA algorithm uses XPRESSMP[1] to solve its MILP master program and MINOS (Murtagh and Saunders (1997)) to solve the relaxed MINLP sub program. We incorporated the bounding parameters discussed in the formulation phase for the OA algorithm

[1] http://www.dashoptimization.com

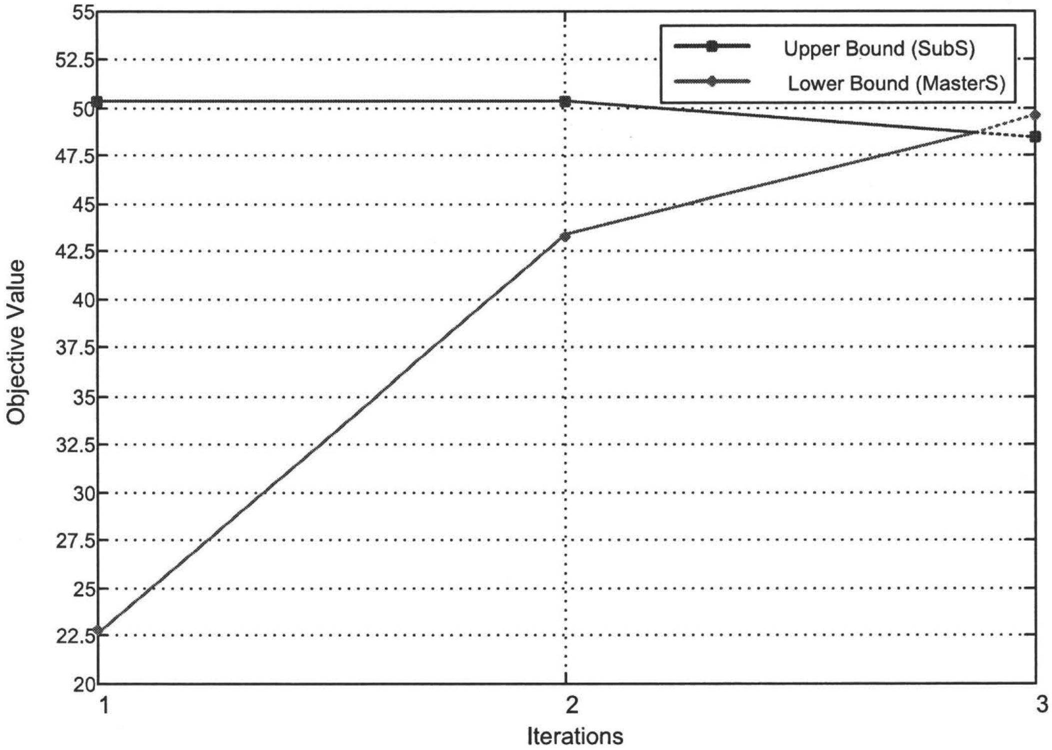


Figure 4.2: Convergence of the OA Algorithm for Example 4.3.3

and limited the search area to the convex hull of the demand points and the passage points for all models. Computations were performed on a PC Pentium 4 2.4 Ghz with 1 GB RAM. A time limit of 1000 seconds is defined for MINLP solvers whereas an iteration limit of 20 is defined for the OA algorithm. If a solver does not find an optimal solution within the defined time limit, we terminate the process and report the best integer solution, if there is one.

Test problems are formulated according to the following parameters listed in Table 4.2. Throughout the process, without loss of generality, we assume that the facility is located on the lower subplane (Ω_2). Problems can also be run in the other case as described earlier. Performance of the forementioned general MINLP solvers

Line Barrier Formula	$y = 5$
x–coordinates of the Demand Points in Ω_1	$Uniform(0, 10)$
x–coordinates of the Demand Points in Ω_2	$Uniform(0, 10)$
y–coordinates of the Demand Points in Ω_1	$Uniform(6, 10)$
y–coordinates of the Demand Points in Ω_2	$Uniform(0, 4)$
Weights of the Demand Points in Ω_1	$Uniform(1, 3)$
Weights of the Demand Points in Ω_2	$Uniform(2, 5)$

Table 4.2: Fixed and Random Parameters for the Test Problems

and the proposed AO algorithm and their solution times are reported in Table 4.3.

The OA algorithm theoretically should give the optimal result if master and sub problems give optimal results in every iteration. However, in our runs, the solver that we use to solve the master problem (XPRESSMP) sometimes resulted in the best integer solution due to the iteration limit. Therefore we can not guarantee global optimality although we have the best results compared to the other solvers. We can observe this from the computational results. As we have used the same set of demand points for the runs with 2 and 5 passages, and the locations of two passage points are the subset of the locations of five passage points, the objective values for the runs with 5 passage points should be at least as good as the runs with 2 passage points. However, we can see that the OA algorithm runs with 20 and 60 demand points for 5 passages have higher objective values. One remedy for this issue can be using the objective value of the lesser number of passage points as the upper bound. One can also try different MIP solvers. When we tried CPLEX for these two instances we were able to get lower objective values. But the solution time became longer.

The OA algorithm performs better compared to DICOPT and BARON in terms of running times and solution quality. DICOPT is faster than BARON but its solution quality is not as good as BARON's. Although DICOPT is using the same solvers as the OA algorithm for its subproblems, it gets stuck at a local optimum and is not able to find the global optimal solution for almost all of the runs. The

		DICOPT		BARON		OA Algorithm (Xpress + Minos)	
# of Passages	# of Points	Run.Time (sec)	Obj. Val.	Run.Time (sec)	Obj. Val.	Run.Time	Obj. Val.
2	20	26.54	191.62	445.66	191.62	1.05	191.62
$p_{xj} = 3, 7$	40	31.90	439.22	> 1000	436.54	3.94	424.14
	60	90.63	625.25	> 1000	626.49	12.16	617.39
	80	122.89	836.57	> 1000	836.58	14.82	826.56
	100	196.47	1074.08	> 1000	1074.63	23.92	1062.79
3	20	29.18	191.56	> 1000	191.56	10.18	191.56
$p_{xj} = 3, 6, 9$	40	50.86	441.23	> 1000	434.17	32.17	423.62
	60	82.86	636.90	> 1000	636.90	24.12	621.81
	80	133.10	856.71	> 1000	850.68	18.23	840.86
	100	181.69	1089.64	> 1000	1068.61	19.61	1068.61
4	20	24.52	197.81	> 1000	197.81	28.71	193.92
$p_{xj} = 2, 4, 6, 8$	40	76.54	426.96	> 1000	428.67	9.34	426.96
	60	96.10	624.79	> 1000	624.80	8.92	607.73
	80	149.50	830.29	> 1000	830.30	9.84	822.16
	100	210.14	1072.16	> 1000	1072.17	7.12	1050.13
5	20	29.17	197.81	> 1000	192.15	4.60	192.01
$p_{xj} = 1, 3, 5, 7, 9$	40	70.23	443.58	> 1000	434.91	8.53	420.96
	60	106.92	624.79	> 1000	645.28	11.57	622.65
	80	171.10	840.32	> 1000	848.49	14.36	826.37
	100	244.55	1072.16	> 1000	1095.69	18.78	1048.34

Table 4.3: Computational Results

reason for this may be its additional stopping criterion that terminates the algorithm when the NLP solution is worsening. For all but one run, BARON exceeded the time limit. We believe that if the time limit was higher, BARON would find better solutions. Overall, the OA algorithm finds the best solutions for all of the runs within reasonable computation times with the highest running time of 32 CPU seconds and an average running time of 14.1 CPU seconds. These computational results suggest that the OA algorithm can be an effective tool in solving the problem described in this section.

4.4 Optimal Location of an Emergency Facility in the Presence of a Line Barrier

4.4.1 Problem Definition and Formulation

In this section, the objective is to find a location X for a new facility such that the maximum of the weighted distances between the new facility and the existing facilities is minimized. This objective is suitable for locating emergency facilities as the most possible attention is given to the weakest (and/or farthest) facilities.

The formulation for this problem when $X \in \Omega_2$ will be (Formulation 4.3):

$$\text{Min} \quad z \tag{4.10}$$

subject to

$$z \geq w_i \sum_{j \in J} u_{ij} \left(l_2 \left(X, P_j \right) + l_2 \left(P_j, X_i \right) \right), \forall i \in \Omega_1 \tag{4.11}$$

$$z \geq w_i l_2 \left(X, X_i \right), \forall i \in \Omega_2 \tag{4.12}$$

$$\sum_{j \in J} u_{ij} = 1, \forall i \in \Omega_1 \tag{4.13}$$

$$u_{ij} \in \{0, 1\}, \forall i \in \Omega_1, \forall j \in J. \tag{4.14}$$

where z is a continuous variable.

The objective tries to minimize z while satisfying the constraints that z needs to be greater than or equal to all the weighted distances. This known trick finds the minimax objective function. Because z needs to be minimized, the binary variables u_{ij} will naturally be chosen such that the shortest path through one of the passage points will be selected. However, the solver may make an arbitrary allocation if constraint (4.12) becomes nonbinding due to the tightness of constraint (4.13).

As in Section 4.3, we need to manipulate the products of binary and continuous variables in the constraints of Formulation 4.3 according to Glover (1975).

Let ξ_{ij} be a continuous variable which represents the multiplication of assignment variable u_{ij} with the distance from the facility to a demand point via a passage point. Also define bounding parameters, $M^+_{2ij} > \sup_X \{l_2(X, P_j) + l_2(P_j, X_i)\}$ and $M^-_{2ij} < \inf_X \{l_2(X, P_j) + l_2(P_j, X_i)\}$ for all feasible locations of the new facility in Ω_2. Then we can replace Formulation 4.3 with the following equivalent formulation (Formulation 4.4):

Minimize z

subject to

$$z \geq w_i \sum_{j \in J} \xi_{ij}, \forall i \in \Omega_1 \tag{4.15}$$

$$\xi_{ij} \geq l_2(X, P_j) + l_2(P_j, X_i) - M^+_{2ij}(1 - u_{ij}), \forall i \in \Omega_1, \forall j \in J \tag{4.16}$$

$$\xi_{ij} \leq M^+_{2ij} u_{ij}, \forall i \in \Omega_1, \forall j \in J \tag{4.17}$$

$$\xi_{ij} \leq l_2(X, P_j) + l_2(P_j, X_i) - M^-_{2ij}(1 - u_{ij}), \forall i \in \Omega_1, \forall j \in J \tag{4.18}$$

$$\xi_{ij} \geq M^-_{2ij} u_{ij}, \forall i \in \Omega_1, \forall j \in J \tag{4.19}$$

$$z \geq w_i l_2(X, X_i), \forall i \in \Omega_2 \tag{4.20}$$

$$\sum_{j \in J} u_{ij} = 1, \forall i \in \Omega_1 \tag{4.21}$$

$$u_{ij} \in \{0, 1\}, \forall i \in \Omega_1, \forall j \in J \qquad \xi_{ij} \in \Re^+, \forall i \in \Omega_1, \forall j \in J \tag{4.22}$$

Lemma 4.4.1 *Formulation 4.3 is equivalent to Formulation 4.4*

Proof. For a given pair of i and j, when $u_{ij} = 1$, constraint sets (4.16) and (4.18) will force ξ_{ij} to be at least as large as $l_2(X, P_j) + l_2(P_j, X_i)$ which is the barrier distance through a passage point. When $u_{ij} = 0$, constraint sets (4.17) and (4.19) will force ξ_{ij} to be equal to 0. In both cases, the formulation becomes equivalent to Formulation 4.3. ■

Property 4.4.1 determines the minimum upper bound value and the maximum lower bound value of the multiplier parameters M^+_{2ij} and M^-_{2ij}.

Property 4.4.1 *$M^-_{2ij} < l_2(X_i, P_j)$ and $M^+_{2ij} > l_2(X^e, P_j) + l_2(X_i, P_j)$ where $X^e \in \Omega_1$ is defined as the farthest extreme point of the convex hull of the demand points and extreme passage points in Ω_1 to P_j.*

Proof. We know that by definition $M_{2ij}^{-} < \inf_{X} \{l_2(X, P_j) + l_2(X_i, P_j)\}$. Then the closest possible location of the new facility to P_j will be at P_j. Therefore $M_{2ij}^{-} < l_2(X_i, P_j)$. Also we know that by definition $M_{2ij}^{+} > \sup_{X} \{l_2(X, P_j) + l_2(X_i, P_j)\}$. Then the farthest possible location of the new facility will be at the farthest extreme point of the convex hull of the demand points and the extreme passage points in Ω_2. Therefore, $M_{2ij}^{+} > l_2(X^e, P_j) + l_2(X_i, P_j)$. ■

4.4.2 An Outer Approximation Algorithm Approach

The outer approximation algorithm to solve the emergency facility location problem with a line barrier and passages is as follows.

Let $(MasterC^k)$ be the k^{th} iteration of the master program with an objective function defined as $g^{(k)}$. Let $(SubC^k)$ be the k^{th} iteration of u_{ij}^k parameterized NLP sub program with an objective function defined as $g(u_{ij}^k)$.

START

Step 0 :

For a given facility location X^k, the **master** program $(MasterC^k)$ is shown as the following mixed integer linear programming (MILP) model:

$g^{(k)} = \text{Min} \quad z$

subject to

$$z \geq w_i \sum_j \xi_{ij}, \forall i \in \Omega_1$$

$$\xi_{ij} \geq l_2(X, P_j)^k + l_2(P_j, X_i) + \nabla l_2(X, P_j)^k \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} - M_{2ij}^{+}(1 - u_{ij}), \forall i \in \Omega_1, \forall j \in J$$

$$\xi_{ij} \leq M^{+}_{2ij}u_{ij}, \forall i \in \Omega_1, \forall j \in J$$

$$\xi_{ij} \leq l_2\left(X, P_j\right)^k + l_2\left(P_j, X_i\right) + \nabla l_2\left(X, P_j\right)^k \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} - M^{-}_{2ij}(1 - u_{ij}), \forall i \in \Omega_1, \forall j \in J$$

$$\xi_{ij} \geq M^{-}_{2ij}u_{ij}, \forall i \in \Omega_1, \forall j \in J$$

$$z \geq w_i \left(l_2\left(X^k, X_i\right) + \nabla l_2\left(X^k, X_i\right)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} \right), \forall i \in \Omega_2$$

$$\sum_j u_{ij} = 1, \forall i \in \Omega_1$$

$$g^{(k-1)} \leq z$$

$$g^{(k)} \leq g^*$$

$$u_{ij} \in \{0, 1\}, \forall i \in \Omega_1, \forall j \in J$$

$$\xi_{ij} \in \Re^+, \forall i \in \Omega_1, \forall j \in J$$

where,

$$\nabla l_2\left(X^k, P_j\right) = \left(\frac{x^k - p_{xj}}{\sqrt{(x^k - p_{xj})^2 + (y^k - p_{yj})^2 + \epsilon}}, \frac{y^k - p_{yj}}{\sqrt{(x^k - p_{xj})^2 + (y^k - p_{yj})^2 + \epsilon}} \right),,$$

$$\nabla l_2\left(X^k, X_i\right) = \left(\frac{x^k - x_i}{\sqrt{(x^k - x_i)^2 + (y^k - y_i)^2 + \epsilon}}, \frac{y^k - y_i}{\sqrt{(x^k - x_i)^2 + (y^k - y_i)^2 + \epsilon}} \right). \text{ and,}$$

$$g^* = \min\left\{g^*, g(u^k_{ij})\right\}.$$

Step 1 :

Set lower bound of the objective function $g^0 = -\infty$, upper bound of the objective function $g^* = +\infty$ and $k = 1$. Make an arbitrary initial feasible selection of binary variables u^k_{ij} by assigning the closest passage to each demand point.

Step 2 :

Solve the u_{ij}^k parameterized NLP **sub** problem $SubC^k$:

$$g(u_{ij}^k) = \text{Min} \quad z^k$$

subject to

$$z^k \geq w_i \sum_j \xi_{ij}, \forall i \in \Omega_1$$

$$\xi_{ij} \geq l_2(X, P_j) + l_2(P_j, X_i) - M_{2ij}^{+}(1 - u_{ij}^k), \forall i \in \Omega_1, \forall j \in J$$

$$\xi_{ij} \leq M_{2ij}^{+} u_{ij}^k, \forall i \in \Omega_1, \forall j \in J$$

$$\xi_{ij} \leq l_2(X, P_j) + l_2(P_j, X_i) - M_{2ij}^{-}(1 - u_{ij}^k), \forall i \in \Omega_1, \forall j \in J$$

$$\xi_{ij} \geq M_{2ij}^{-} u_{ij}^k, \forall i \in \Omega_1, \forall j \in J$$

$$z^k \geq w_i l_2(X, X_i), \forall i \in \Omega_2$$

$$\xi_{ij} \in \Re^+, \forall i \in \Omega_1, \forall j \in J$$

If problem $SubC^k$ has a finite optimal solution, update the current upper bound;

Estimate: $g^* = \min\{g^*, g(u_{ij}^k)\}$ and if $g^* = g(u_{ij}^k)$ set $u_{ij}^* = u_{ij}^k$, $X^* = X^k$;

Add integer cut constraints to $MasterC^k$ to eliminate u_{ij}^k from further consideration:

$$\sum_{ij \in Cut_1^k} u_{ij}^k - \sum_{ij \in Cut_0^k} u_{ij}^k \leqslant \left|Cut_1^k\right| - 1, \forall k$$

where $Cut_1^k = \{(i,j)\,|u_{ij}^k = 1\}$ and $Cut_0^k = \{(i,j)\,|u_{ij}^k = 0\}$ and $\left|Cut_1^k\right|$ is the cardinality of Cut_1^k.

Step 3 **:**

Solve the MILP master program $MasterC^k$, adding the integer cuts as constraints from Step 2.

If program $MasterC^k$ doesn't have a mixed integer feasible solution STOP. The optimal solution is (u_{ij}^*, X^*). As in Section 4.3, the algorithm converges when the master problem doesn't have a feasible solution. This indicates the crossing of the lower and upper bounds.

If program $MasterC^k$ has an optimal mixed integer solution, set $u_{ij}^{k+1} = u_{ij}^k$, $k = k + 1$ and return to **Step** 2.

END

4.4.3 Example

Consider a line barrier defined by:

$$B_L := \{X = (x, y) \in \Re^2 | y = 5\} / \{P_1 = (3, 5), P_2 = (4.5, 5), P_3 = (6, 5)\},$$

which divides the planar feasible region into two subplanes (Ω_1 and Ω_2). Five existing facilities are located on each subplane. These are listed in Table 4.4 along with their corresponding weights and the subplanes that they belong to. We would like to locate an emergency facility on the plane minimax to these demand points and in the presence of the line barrier defined above.

i	x_i	y_i	w_i	Ω_j
1	8.9	9.6	1	Ω_1
2	2.2	8.9	2	Ω_1
3	6.0	8.2	3	Ω_1
4	2.5	6.3	1	Ω_1
5	1.4	4.4	1	Ω_1
6	7.5	3.6	1	Ω_2
7	2.9	3.4	2	Ω_2
8	1.3	2.6	1	Ω_2
9	8.6	1.4	1	Ω_2
10	3.8	1.0	2	Ω_2

Table 4.4: Parameter Values for the Existing Facilities

We need to apply the OA algorithm two times, each time assuming the location of the emergency facility to be in one of the subplanes. First we assume that the facility is located on the upper subplane (Ω_1). In this case OA algorithm converges to the optimal solution of 9.114 for this subplane in three iterations. Figure 4.3 illustrates the convergence. This result is also confirmed by solving Formulation 4.3 using BARON. The optimal value for the lower subplane (Ω_1) is also found as 10.632 whereas BARON finds 10.602. The small difference in the objective value is due to the ϵ term. When we checked the passage point for optimality, we found the exact value as BARON did. Overall, the global optimal value for the whole plane is 9.114 and an optimal location for the new emergency facility will be on the upper subplane (Ω_1) at coordinates $(4.710, 5.449)$.

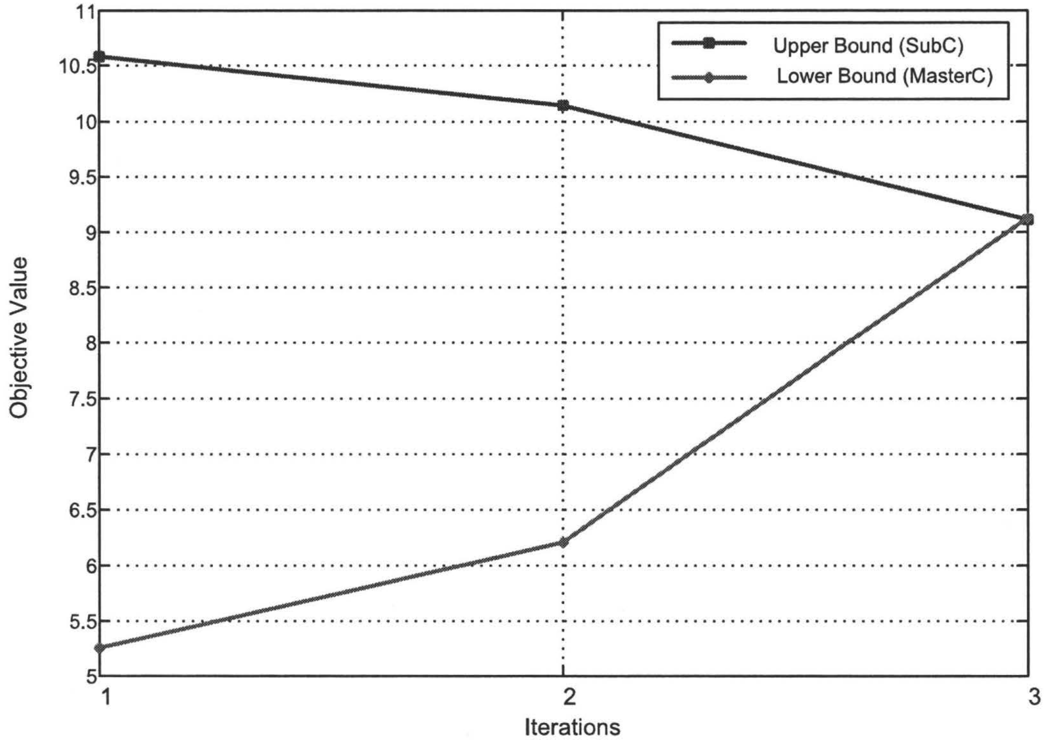


Figure 4.3: Convergence of the OA Algorithm for Example 4.4.3

Line Barrier Formula	$y = 5$
$x-$coordinates of the Demand Points in Ω_1	$Uniform(0, 10)$
$x-$coordinates of the Demand Points in Ω_2	$Uniform(0, 10)$
$y-$coordinates of the Demand Points in Ω_1	$Uniform(6, 10)$
$y-$coordinates of the Demand Points in Ω_2	$Uniform(0, 4)$
Weights of the Demand Points in Ω_1	$Uniform(2, 5)$
Weights of the Demand Points in Ω_2	$Uniform(5, 8)$

Table 4.5: Fixed and Random Parameters for the Test Problems

4.4.4 Further Computations

We have conducted a similar computational study to that in Section 4.3 to test the performance of the proposed OA algorithm for the emergency facility location problem in the presence of a line barrier and passages. We evaluated the general MINLP solvers DICOPT and BARON using Formulation 4.4 and compared their results with the proposed OA algorithm. The other settings remained the same as in Section 4.3. Test problems are formulated according to the following parameters listed in Table 4.5 and performance of the solvers and the proposed algorithm and solution results are reported in Table 4.6.

As can be seen from the results, the OA algorithm and BARON performs better than DICOPT in terms of finding the optimal solution for this problem. DICOPT is the fastest in this case as well, but it again gets stuck at a local optimum and is not able to find the global optimal solution for more than half of the runs. We can clearly see that it has suboptimal results for the 5 passages case, as these values should have been at least as good as the 2 passages case. BARON finds the global optima for all in reasonable running times, but its computation time is significantly higher than the proposed OA algorithm. Overall, the OA algorithm finds optimal solutions for all of the runs within reasonable computation times with the highest running time of 24.23 CPU seconds and an average running time of 4.8 CPU seconds. These compu-

# of Passages	# of Points	DICOPT Run.Time (sec)	DICOPT Obj. Val.	BARON Run.Time (sec)	BARON Obj. Val.	OA Algorithm (Xpress + Minos) Run.Time	OA Algorithm (Xpress + Minos) Obj. Val.
2	20	0.92	17.48	0.44	17.48	0.84	17.48
$p_{xj} = 3, 7$	40	1.01	20.89	1.38	19.45	1.19	19.45
	60	2.35	21.98	4.31	21.98	1.01	21.98
	80	0.96	21.83	5.36	21.83	1.13	21.83
	100	1.24	23.47	7.73	21.42	1.91	21.42
3	20	0.98	18.58	0.62	16.53	1.08	16.53
$p_{xj} = 3, 6, 9$	40	0.94	21.21	9.19	19.04	2.26	19.04
	60	0.95	24.06	4.31	21.98	1.71	21.98
	80	0.67	24.36	12.56	21.83	2.56	21.83
	100	0.89	23.47	19.72	21.06	4.41	21.06
4	20	1.23	16.53	1.19	16.53	1.61	16.53
$p_{xj} = 2, 4, 6, 8$	40	1.03	20.71	2.84	19.04	2.88	19.04
	60	0.88	22.02	46.94	21.99	5.28	21.99
	80	0.96	22.54	34.16	21.95	7.05	21.95
	100	1.12	21.72	19.52	21.06	10.81	21.06
5	20	1.19	17.48	1.16	16.56	3.08	16.56
$p_{xj} = 1, 3, 5, 7, 9$	40	1.30	19.38	12.27	19.24	6.34	19.24
	60	1.23	22.59	27.03	21.98	6.78	21.98
	80	1.28	22.69	13.83	21.83	10.03	21.83
	100	1.48	21.08	120.38	21.08	24.23	21.08

Table 4.6: Computational Results

tational results suggest that the OA algorithm can be an effective tool in solving the emergency facility location problem in the presence of a line barrier with passages.

4.5 Simultaneous Location of an Emergency Facility and Passage Points

4.5.1 Problem Definition and Formulation

Consider a case where locations of one or more passage points $P_j = (p_{xj}, p_{yj}), j \in J$ on barrier B_L and an emergency facility $X = (x, y)$ in Ω are determined under the objective of minimizing the maximum of the weighted distances from the facility to the demand points. As in the previous cases, we can solve two subproblems by assigning the new facility to the subplanes and take the minimum of these two solutions as the global optimum solution. Without loss of generality, we assume that the new facility is located in Ω_2. The problem has to be solved for the location of the emergency facility in Ω_1 in order to find the global optimum. Formulation for the problem will be the same as Formulation 4.3 and Formulation 4.4, with the addition that $P_{xj}, j \in J$ is a decision variable.

We can apply the OA algorithm in a similar way to that in Section 4.4.2. However, we now have $p_{xj}, j \in J$ as variables in our formulations, therefore, we should redefine the master problem to reflect the changes. Let $(NewMaster)$ be the master problem for this problem. For a given facility location X^k, and set of initial locations for passage points $P_j^k, j \in J$ define the kth master problem $(NewMaster^k)$ as:

$$
\begin{aligned}
& f^{(k)} = \text{Minimize} \quad z \\
& \text{subject to} \\
& z \geq w_i \sum_j \xi_{ij} \\
& \xi_{ij} \geq d_{ij}(X)^k + \nabla d_{ij}(X)^k \begin{pmatrix} x - x^k \\ y - y^k \\ p_{xj} - p_{xj}^k \end{pmatrix} - M_{2ij}^{+}(1 - u_{ij}) \\
& \xi_{ij} \leq M_{2ij}^{+} u_{ij} \\
& \xi_{ij} \leq d_{ij}(X)^k + \nabla d_{ij}(X)^k \begin{pmatrix} x - x^k \\ y - y^k \\ p_{xj} - p_{xj}^k \end{pmatrix} - M_{2ij}^{-}(1 - u_{ij}) \\
& \xi_{ij} \geq M_{2ij}^{-} u_{ij} \\
& z \geq w_i \left(l_2\left(X^k, X_i\right) + \nabla l_2\left(X^k, X_i\right)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} \right), \forall i \in \Omega_2 \\
& f^{(k)} \leq f^* \\
& \sum_j u_{ij} = 1, \forall i \in \Omega_1 \\
& u_{ij} \in \{0, 1\} \\
& \xi_{ij} \in \Re^+,
\end{aligned}
$$

where $d_{ij}(X) = l_2\left(X, P_j\right) + l_2\left(P_j, X_i\right)$.

The rest of the OA algorithm including the sub problem will be the same.

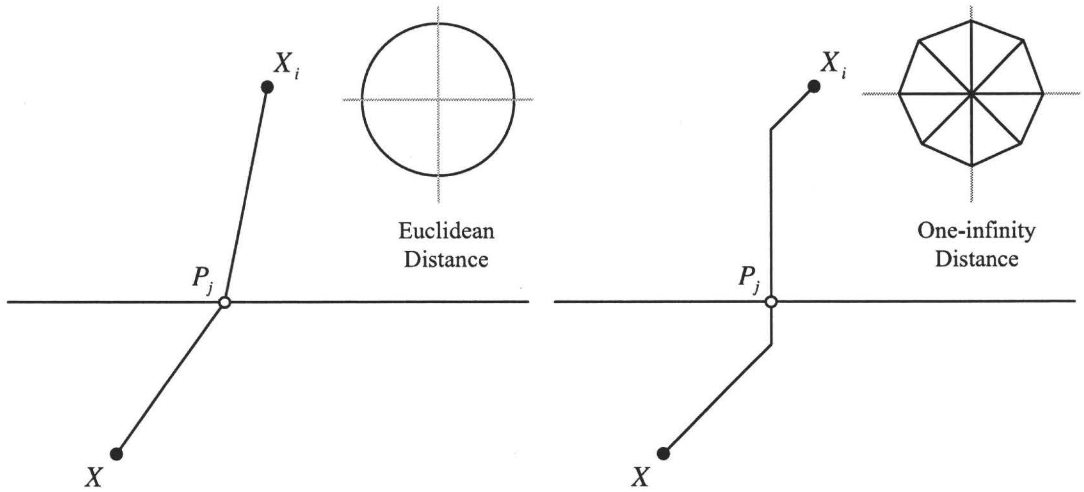


Figure 4.4: Euclidean and One-infinity norm distances in the presence of a line barrier

4.5.2 An Approximation Approach using Weighted One-Infinity Norms

Recall that in Formulation 4.3, we have the Euclidean distance terms bring nonlinearity into the problem. We know that an alternative distance function that we have discussed in Chapter 1 (pg. 7), called the weighted one-infinity norm distance, can provide a good approximation to the Euclidean distance norm. Ward and Wendell (1980) proposed this distance measure as 'a new norm which yields linear location problems'. The only problem with this distance function is that it is a weighted combination of rectilinear and maximum distances and it requires the determination of two parameters that controls the proportional usage of these two distance functions. Nevertheless, using this distance measure in place of the Euclidean distance terms in Formulation 4.3 will allow us to come up with a new mixed integer programming formulation that may provide a good approximation to Formulation 4.3.

Figure 4.4 is a representation of Euclidean and One-infinity norm distances between points X and X_i through passage point P_j. Let $\overline{d}_{ij}(X)$ be this distance.

We can write it as;

$$\begin{aligned}\overline{d}_{ij}(X) &= \psi_1\left(|x-p_{jx}|+|y-p_{jy}|\right)+\psi_2\sqrt{2}\max\left\{|x-p_{jx}|,|y-p_{jy}|\right\}+\\ &\quad \psi_1\left(|p_{jx}-x_i|+|p_{jy}-y_i|\right)+\psi_2\sqrt{2}\max\left\{|p_{jx}-x_i|,|p_{jy}-y_i|\right\}\end{aligned}$$

where ψ_1 and ψ_2 are weighting parameters for rectilinear and maximum distance functions respectively whose sum add up to $2\sqrt{2}-2$. In this study, we take $\psi_1 = \psi_2$. Which means, we assume that half of the travel from one point to another will be in diagonal directions (maximum distance), and the other half would be in rectilinear directions (rectilinear distance).

Similarly, we also approximate $l_2\left(X, X_i\right)$ with the following one-infinity norm distance;

$$l_{1-\infty}\left(X, X_i\right) = \psi_1\left(|x-x_i|+|y-y_i|\right)+\psi_2\sqrt{2}\max\left\{|x-x_i|,|y-y_i|\right\}$$

Then, we can write the following formulation (Formulation 4.5):

$$\begin{aligned}
&\text{Minimize} \quad z\\
&\text{subject to}\\
&z \geq w_i \sum_{j\in J}\xi_{ij}, \forall i \in \Omega_1\\
&\xi_{ij} \geq \overline{d}_{ij}(X) - M^{+}_{2ij}(1-u_{ij}), \forall i \in \Omega_1, \forall j \in J\\
&\xi_{ij} \leq M^{+}_{2ij}u_{ij}, \forall i \in \Omega_1, \forall j \in J\\
&\xi_{ij} \leq \overline{d}_{ij}(X) - M^{-}_{2ij}(1-u_{ij}), \forall i \in \Omega_1, \forall j \in J\\
&\xi_{ij} \geq M^{-}_{2ij}u_{ij}, \forall i \in \Omega_1, \forall j \in J\\
&z \geq w_i l_{1-\infty}\left(X, X_i\right), \forall i \in \Omega_2\\
&\sum_{j\in J} u_{ij} = 1, \forall i \in \Omega_1\\
&u_{ij} \in \{0,1\}, \forall i \in \Omega_1, \forall j \in J\\
&\xi_{ij}, \forall i \in \Omega_1, \forall j \in J \in \Re
\end{aligned}$$

To keep the formulation mixed integer linear, first, we need to transform the maximum functions in $\overline{d}_{ij}(X)$, and $l_{1-\infty}(X, X_i)$. Using the idea that $\max\{|a|, |b|\} = 1/2(|a+b| + |a-b|)$ as suggested by Ward and Wendell (1980), we can rewrite $\overline{d}_{ij}(X)$, and $l_{1-\infty}(X, X_i)$ as,

$$\begin{aligned}\overline{d}_{ij}(X) &= \psi_1(|x-p_{jx}| + |y-p_{jy}|) + \psi_2\frac{1}{\sqrt{2}}(|x-p_{jx}+y-p_{jy}| + |x-p_{jx}-y+p_{jy}|) + \\ &\quad \psi_1(|p_{jx}-x_i| + |p_{jy}-y_i|) + \psi_2\frac{1}{\sqrt{2}}(|p_{jx}-x_i+p_{jy}-y_i| + |p_{jx}-x_i-p_{jy}+y_i|)\end{aligned}$$

and

$$l_{1-\infty}(X, X_i) = \psi_1(|x-x_i| + |y-y_i|) + \psi_2\frac{1}{\sqrt{2}}(|x-x_i+y-y_i| + |x-x_i-y+y_i|).$$

Some commercial modeling languages such as Lindo, can handle the absolute value terms in linear models during preprocessing stage. But GAMS, the modeling language that we use, does not have that capability. Therefore, we need to transform the absolute value terms in these equations. Let $r^+_{kij}, r^-_{kij}, k = 1, ..., 8; i \in \Omega_1; j \in J$ and $s^+_{ki}, s^-_{ki}, k = 1, ..., 4; i \in \Omega_2$ be continuous positive variables. Then in Formulation 4.5, we replace $\overline{d}_{ij}(X)$ and $l_{1-\infty}(X, X_i)$ by,

$$\begin{aligned}\overline{d}_{ij}(X) &= \psi_1\left(r^+_{1ij} + r^-_{1ij} + r^+_{2ij} + r^-_{2ij} + r^+_{5ij} + r^-_{5ij} + r^+_{6ij} + r^-_{6ij}\right) + \\ &\quad \psi_2\frac{1}{\sqrt{2}}\left(r^+_{3ij} + r^-_{3ij} + r^+_{4ij} + r^-_{4ij} + r^+_{7ij} + r^-_{7ij} + r^+_{8ij} + r^-_{8ij}\right)\end{aligned}$$

$$l_{1-\infty}(X, X_i) = \psi_1\left(s^+_{1i} + s^-_{1i} + s^+_{2i} + s^-_{2i}\right) + \psi_2\frac{1}{\sqrt{2}}\left(s^+_{3i} + s^-_{3i} + s^+_{4i} + s^-_{4i}\right),$$

and add the following system of linear equations as a constraint set (ConstSet4.1) into the formulation:

$$
\begin{aligned}
&x - p_{jx} - r_{1ij}^{+} + r_{1ij}^{-} = 0 \\
&y - p_{jy} - r_{2ij}^{+} + r_{2ij}^{-} = 0 \\
&x - p_{jx} + y - p_{jy} - r_{3ij}^{+} + r_{3ij}^{-} = 0 \\
&x - p_{jx} + y - p_{jy} - r_{4ij}^{+} + r_{4ij}^{-} = 0 \\
&p_{jx} - x_i - r_{5ij}^{+} + r_{5ij}^{-} = 0 \\
&p_{jy} - y_i - r_{6ij}^{+} + r_{6ij}^{-} = 0 \\
&p_{jx} - x_i + p_{jy} - y_i - r_{7ij}^{+} + r_{7ij}^{-} = 0 \\
&p_{jx} - x_i + p_{jy} - y_i - r_{8ij}^{+} + r_{8ij}^{-} = 0 \\
&x - x_i - s_{1i}^{+} + s_{1i}^{-} = 0 \\
&y - y_i - s_{2i}^{+} + s_{2i}^{-} = 0 \\
&x - x_i + y - y_i - s_{3i}^{+} + s_{3i}^{-} = 0 \\
&x - x_i - y + y_i - s_{4i}^{+} + s_{4i}^{-} = 0
\end{aligned}
\tag{ConstSet4.1}
$$

The solution quality of the one-infinity approximation approach may differ with the selection of the weighting parameters ψ_1 and ψ_2. There have been a number of successful attempts (Ward and Wendell (1985), Love and Walker (1994)) to determine the best fitting parameter values using linear regression techniques. The resulting empirical models provided good approximations to the real life distances being considered. Therefore, in the case of a real life location problem in the presence of a line barrier and passages, it would be beneficial to conduct preliminary empirical study to determine the best fitting parameters.

Chapter 5

Locating a Finite Size Barrier on A Rectangular Plane

5.1 Introduction

Consider locating an area facility in a place where the size of the facility is not negligible with respect to the area of the place where it is to be located. An example for such a facility may be a new department in a factory layout. Various objectives can be considered. Recently, this problem family has been of interest to other researchers including Savas et al. (2002), Sarkar et al. (2005, 2007), and Kelachankuttu et al. (2007), who studied their corresponding problems in facility layout settings. Detailed descriptions of these studies and their contributions are discussed in Chapter 2, in the literature review. In this chapter we study a number of interrelated problems. In Section 5.2 we start with a simpler version of the problem given in Savas et al. (2002) by locating a finite size barrier facility with a fixed orientation on an isolated rectangular region. The goal is to minimize its interference to interaction among demand points under the minisum criterion. A solution algorithm is proposed. The restriction of demand points being in the region where the barrier facility is located is then relaxed and it is shown that the same algorithm can be used by partitioning the planar area into rectangular cells and running the algorithm in each cell. In Section 5.3, we incorporate another objective into the problem. The maximum of the closest rectilinear distances from the demand points to the barrier is minimized. To the best of our knowledge, interaction among demand points and between demand points and a facility under a centre objective have never been explored. Also, the 'closest distance family' has never been used for this type of problem. A practical example for such a bi-objective problem can be given in a city planning context. If a new city park is going to be opened in the city's downtown core, it has to be close enough to the farthest demand point that uses the park (centre objective) and its interference to the existing flows between demand points should be minimum. Since these two objectives may conflict with each other, a pareto optimal solution is sought. Finally, in Section 5.4 we develop a Simulated Annealing (SA) heuristic for an extension of the problem where expropriation of existing facilities with some cost is also possible.

5.2 Locating a Barrier Facility on the Plane to Minimize its Interference to Demand Point Interactions

5.2.1 Problem Definition and Formulation

In this section we consider the problem of locating a finite sized rectangularly shaped barrier on the plane to minimize the sum of the weighted distances of interactions among demand points. We call this problem the 'minimum rectangular interference problem'. To start with the simplest case we assume that the barrier is to be located in an isolated region where there is no demand point present in that region. For an illustration of the problem see Figure 5.1. We start with the following notation:

I: Set of existing demand points,

$X_i = (x_i, y_i)$: coordinates of demand point $i \in I$,

v_{ij}: volume shipped between demand point $i \in I$ and demand point $j \in I$,

$i > j$,

R^B: a user determined rectangular feasible region in which barrier B will be located,

(u_{1x}, u_{1y}): lower left coordinates of R^B,

(u_{2x}, u_{2y}): upper right coordinates of R^B,

$\Omega_k, k = 1, ..., 8$: rectangular regions surrounding region R^B,

$X = (x, y)$: coordinates of the lower-left corner of barrier B,

a: length of barrier B,

b: width of barrier B,

R: the feasible region where (x, y) is located, $R \subset R^B$,

$l_1^B (X, X_i, X_j) = l_1^B (x, x_i, x_j) + l_1^B (y, y_i, y_j)$: the barrier distance between X_i and X_j.

Let $f(X) = \sum\limits_{i \in I} \sum\limits_{j \in I} v_{ij} l_1^B (X, X_i, X_j)$. Our problem is to find (x^*, y^*) that minimize $f(X)$. The variables x and y become present in $l_1^B (X, X_i, X_j)$ when X_i and X_j are 1−invisible to each other due to the barrier facility. Therefore we need to determine the conditions in which this distance becomes a barrier distance. In this problem, we observe properties similar to those in Chapter 3; therefore, the reader will be referred to Chapter 3 whenever necessary. The following separability property is helpful in finding the optimal values of (x^*, y^*).

Property 5.2.1 *Let* $f_x(x) = \sum\limits_{i \in I} \sum\limits_{j \in I} v_{ij} l_1^B (x, x_i, x_j)$ *and* $f_y(y) = \sum\limits_{i \in I} \sum\limits_{j \in I} v_{ij} l_1^B (y, y_i, y_j)$. *Then* $f(X) = f_x(x) + f_y(y)$.

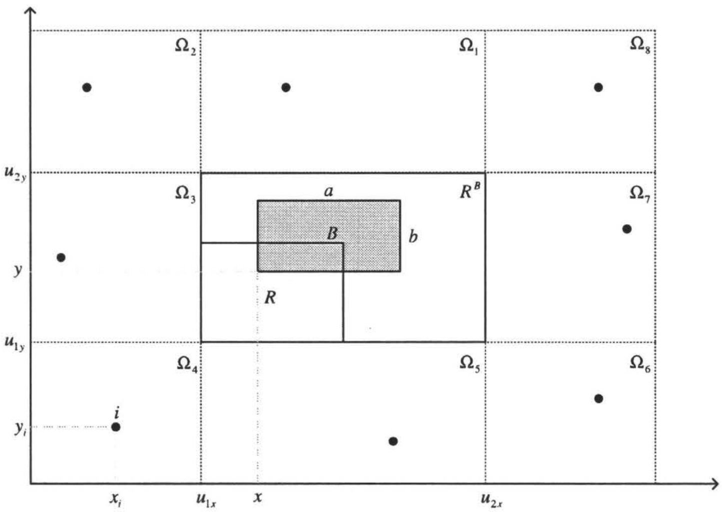


Figure 5.1: Illustration of the Problem

Proof. Assume that the barrier has a fixed x coordinate. In that case, a change in its y coordinate will not increase or decrease $f_x(x)$. Similarly, if the barrier has a fixed y coordinate, a change in its x coordinate will not increase or decrease $f_y(y)$. This proves that $f(X)$ is separable in x and y. ■

The solution procedure for finding an optimal x^* that minimizes $f_x(x)$ and finding an optimal y^* that minimizes $f_y(y)$ will be similar. When the barrier has no effect on the distance between two demand points, the barrier distance becomes a rectilinear distance which is a constant value. Therefore we can exclude the pair of demand points from the problem where the barrier can have no effect on their interaction. We can observe from Figure 5.1 that to find optimal x, it is sufficient to consider pairs of demand points from Ω_1 and Ω_5 and similarly, to find optimal y, it is sufficient to consider pairs of demand points from Ω_3 and Ω_7. Property 5.2.2 provides closed form barrier distance functions for these pairs.

Property 5.2.2 (Barrier Conditions) *For each pair (i, j) where $i \in \Omega_1$, $j \in \Omega_5$:*

If $0 < x_i - x < a$, $0 < x_j - x < a$ *and* $|x_i - x_j| < a$, *then*

$$l_1^B(x, x_i, x_j) = \min\{x_i + x_j - 2x, 2x + 2a - x_i - x_j\} \tag{5.1}$$

else

$$l_1^B(x, x_i, x_j) = |x_i - x_j|. \tag{5.2}$$

For each pair (i, j) *where* $i \in \Omega_3$, $j \in \Omega_7$:
If $0 < y_i - y < b$, $0 < y_j - y < b$ *and* $|y_i - y_j| < b$, *then*

$$l_1^B(y, y_i, y_j) = \min\{y_i + y_j - 2y, 2y + 2b - y_i - y_j\} \tag{5.3}$$

else

$$l_1^B(y, y_i, y_j) = |y_i - y_j|. \tag{5.4}$$

Proof. The proof is omitted as it is similar to the one of Lemma 3.2.1 ■

Remark 5.2.1 *When the barrier is in effect, the distance function* $l_1^B(x, x_i, x_j)$ *will be piecewise symmetric linear concave and reaches its maximum at* $(x_i + x_j - a)/2$ *where the maximum value is* a *and the distance function* $l_1^B(y, y_i, y_j)$ *will also be piecewise symmetric linear concave and reaches its maximum at* $(y_i + y_j - b)/2$ *where the maximum value is* b.

For a proof of Remark 5.2.1 the reader is referred to Theorem 3.2.1. As an example, the barrier distance function of x is illustrated in Figure 5.2.

We can now propose a solution algorithm for this problem which runs in $O(n \log n)$ time where n is the number of demand points. The algorithm finds all pairs of demand points whose x distances can potentially be affected by the presence of the barrier, and defines the barrier distance function between them. It then generates ranges for all these pairs such that the distance functions remain constant throughout any range. The smallest and the largest values of these ranges are then sorted and checked for optimality in x. Using the fact that the sum of the distance

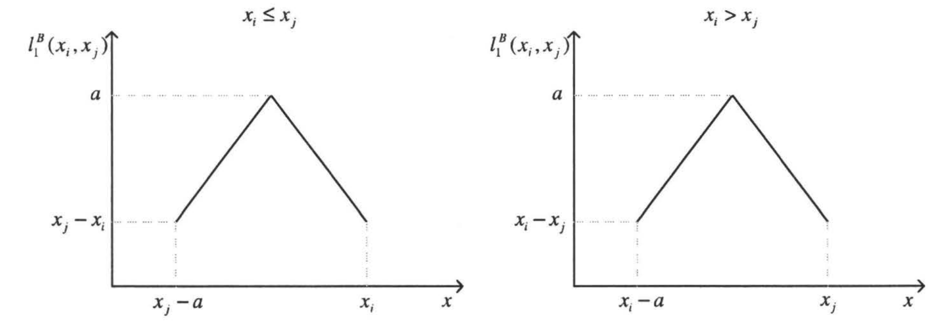


Figure 5.2: The Barrier Distance Function

functions in any given range will be a linear function and nonincreasing or nondecreasing function of x or a constant value, it will be sufficient to check starting and ending points of these ranges for the optimal barrier location as suggested by the following algorithm.

5.2.2 Solution Algorithm 5.1

`Step 1.` Find all pairs $(i, j), i \in \Omega_1, j \in \Omega_5$, where $|x_i - x_j| < a$

`Step 2.` For each successful pair (i, j):

If $x_i \leq x_j$

Determine $x_j - a$, x_i and $(x_i + x_j - a)/2$

else

Determine $x_i - a$, x_j and $(x_i + x_j - a)/2$

`Step 3.` Sort all these points in increasing order.

`Step 4.` Check each point for optimality in x.

`Step 5.` Find all pairs $(i, j), i \in \Omega_3, j \in \Omega_7$, where $|y_i - y_j| < b$

`Step 6.` For each eligible pair (i, j):

If $y_i \leq y_j$

Determine $y_j - b$, y_i and $(y_i + y_j - b)/2$

else

Determine $y_i - a$, y_j and $(y_i + y_j - b)/2$

`Step 7`. Sort all these points in increasing order.

`Step 8`. Check each point for optimality in y.

Theorem 5.2.1 explains the idea. This process is then repeated for y distances.

Theorem 5.2.1 (Reduction Result) *Algorithm 5.1 yields optimal values of x and y for Problem 5.2.*

Proof. Consider $\min_x f_x(x) = \sum_{i\in\Omega_1} \sum_{j\in\Omega_5} l_1^B(x, x_i, x_j)$. For each pair of demand points (i, j), and in each range, the weighted barrier distance $v_{ij} l_1^B(x, x_i, x_j)$ will be one of the followings:

- $v_{ij} |x_i - x_j|$
- $v_{ij}(x_i + x_j - 2x)$
- $v_{ij}(2x + 2a - x_i - x_j)$

Therefore $f_x(x)$ will be a mixed sum of these terms. We can easily observe that coefficient of x in $f_x(x)$ will have either a negative value, a positive value or zero. Therefore, because $f_x(x)$ is either a linear nonincreasing or nondecreasing function of x or is a constant value, it will be sufficient to check starting and ending points of each range for an optimal x^*. Similar reasoning apply for finding an optimal y^*. ■

Theorem 5.2.1 also suggests that if more than one point in a sequence results in the minimum objective value, then they are joined by a horizontal line segment, which implies that the whole range between these points is optimal.

5.2.3 Example Problem

Consider 20 demand points at fixed locations and a rectangular barrier on the plane. The length and width of the barrier is given as $a = 8$ and $b = 5$, respectively. All demand points have interactions between each other with equal volumes ($v_{ij} =$

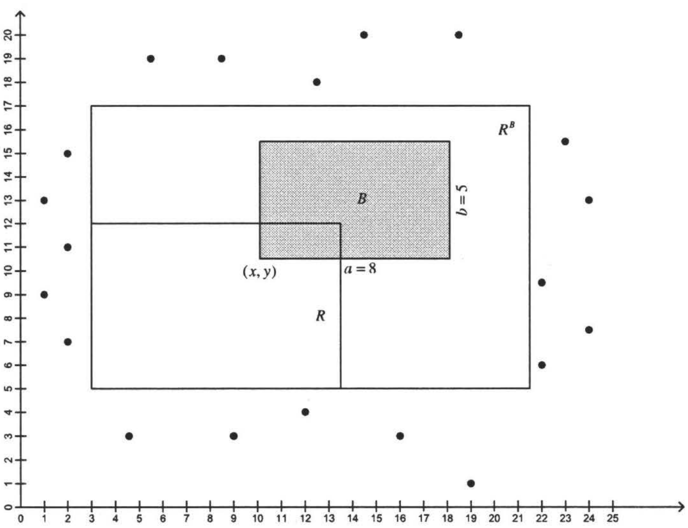


Figure 5.3: Example Problem

$1, \forall i, j, i < j$). We want to locate the barrier in a given rectangular area such that we minimize its interference to these interactions. Figure 5.3 illustrates the example.

We solved the problem using Algorithm 5.1. The optimal location for the facility's lower left corner is found as $x^* = 4$ and $y^* = 8$. Figures 5.4 and 5.5 are the objective function plots for x and y coordinates respectively. From Figure 5.4 we can see that the optimal x^* is a range between $[4, 4.5]$. If there were no barrier, the sum of the volume times distances would be a constant value of 102.5. But because of the presence of the barrier, our optimal objective value, which is the lowest possible sum of the volumes times distances, is found to be 123.5 in this problem.

Until now we considered the simplest version of the rectangular barrier location problem where the barrier is located inside an isolated region. What happens when we want to locate a barrier facility in a region where demand points are also present? Clearly, if there were no boundaries, we could locate the barrier facility far away from the demand points, thus not affecting their interaction. But assume that the barrier facility needs to be located inside the rectangular convex hull of the demand points. In that case, we are still able to use Algorithm 5.2.1 to determine the optimal

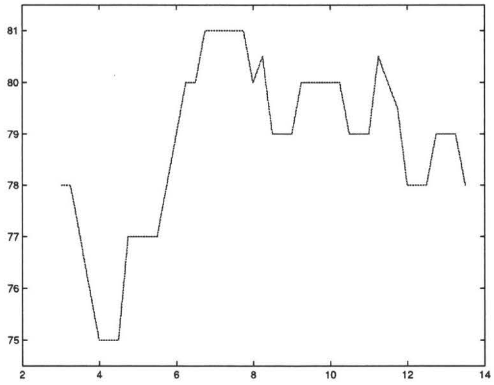


Figure 5.4: x-Distance Objective

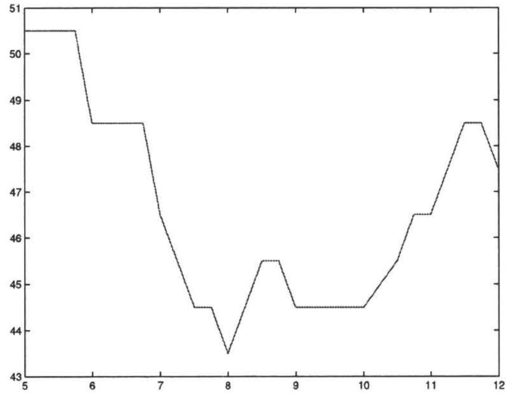


Figure 5.5: y-Distance Objective

barrier facility location. Consider the example presented in Figure 5.6. We want to locate a barrier facility in a region (R^B) where six demand points are present (Figure 5.6a). Location of the facility is defined by its lower left coordinate ($(x, y) \in R$). We assume that the edges or corners of the barrier facility can be located at a demand point. We can see that it is impossible to locate the barrier facility in the shaded areas (Figure 5.6b) without expropriating some demand points. Therefore, we need to remove these areas from further consideration. The rest of the region in R^B will be feasible for (x, y). We can divide this remaining region into some rectangular feasible regions for the barrier's corner point as R_1, R_2, R_3 and R_4 (Figure 5.6c). For each region, a new barrier area R^B and its surrounding regions ($\Omega_k, k = 1, ..., 8$) can be determined and Algorithm 5.2.1 can be run to find the optimal solution. A global optimal solution can be found by selecting the minimum of all optimal solutions.

5.2.4 Mixed Integer Linear Programming (MIP) Formulation for Problem 5.2

To provide a basis for more complex formulations, we propose a MIP formulation (Formulation 5.2.1) for this problem as an alternative to Algorithm 5.2.1. It uses

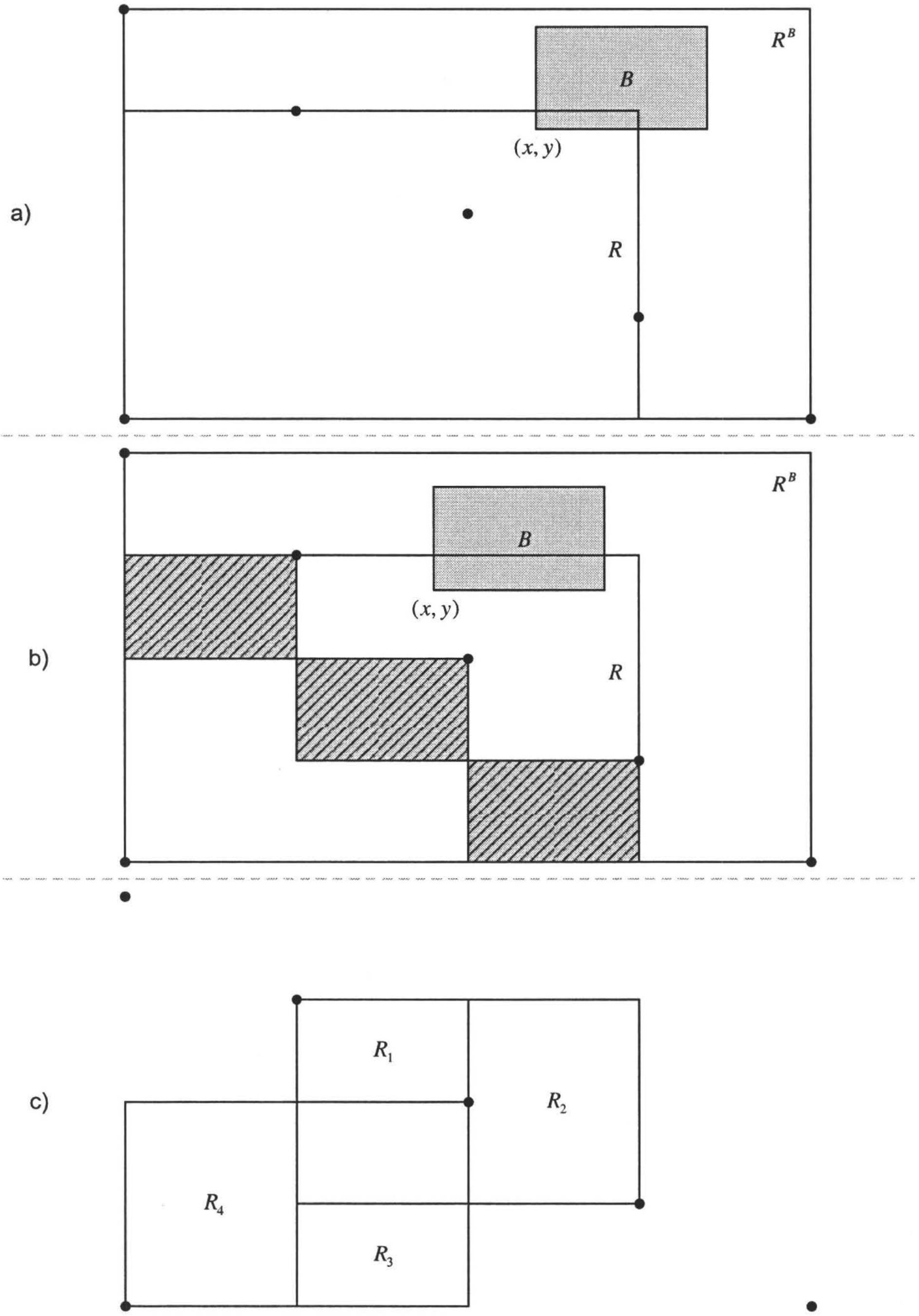


Figure 5.6: Partitioning the Region

the barrier conditions to determine the distance between demand points and runs separately for x and y and in each partitioned sub region described earlier. Although we only consider the x distance in Formulation 5.2.1, the same approach can be followed to find the optimal value of y. Because of the separability property of x and y, the two models can be combined together or run separately.
Formulation 5.2.1: For all pairs ($i \in \Omega_1$, $j \in \Omega_5$) which satisfy $|x_i - x_j| < a$,

$$\text{Min} \sum_{i\in\Omega_1}\sum_{j\in\Omega_5} v_{ij} l_1^B(x, x_i, x_j)$$

subject to

$$l_1^B(x_i, x_j) \geq |x_i - x_j|, \forall i \in \Omega_1, \forall j \in \Omega_5 \tag{5.5}$$

$$l_1^B(x_i, x_j) \geq x_i + x_j - 2x - M(\phi_{ij}^x + o_{ij}^x + \pi_{ij}^x), \forall i \in \Omega_1, \forall j \in \Omega_5 \tag{5.6}$$

$$l_1^B(x_i, x_j) \geq 2x + 2a - x_i - x_j - M(1 - [\phi_{ij}^x + o_{ij}^x + \pi_{ij}^x]), \forall i \in \Omega_1, \forall j \in \Omega_5 \tag{5.7}$$

$$x_i + x_j - 2x \leq 2x + 2a - x_i - x_j + M\phi_{ij}^x, \forall i \in \Omega_1, \forall j \in \Omega_5 \tag{5.8}$$

$$2x + 2a - x_i - x_j \leq x_i + x_j - 2x + M(1 - \phi_{ij}^x), \forall i \in \Omega_1, \forall j \in \Omega_5 \tag{5.9}$$

$$x \leq \max\{x_i - a, x_j - a\} + M(1 - o_{ij}^x), \forall i \in \Omega_1, \forall j \in \Omega_5 \tag{5.10}$$

$$x \geq \min\{x_i, x_j\} - M(1 - \pi_{ij}^x), \forall i \in \Omega_1, \forall j \in \Omega_5 \tag{5.11}$$

$$x \in R \tag{5.12}$$

$$\phi_{ij}^x, o_{ij}^x, \pi_{ij}^x \in \{0, 1\}, \forall i \in \Omega_1, \forall j \in \Omega_5 \tag{5.13}$$

The objective function aims for minimizing the sum of the weighted rectilinear distances for pairs of (x_i, x_j) which satisfy the condition $|x_i - x_j| < a$ and $\forall i \in \Omega_1, \forall j \in \Omega_5$. As it was mentioned, this is due to the fact that the barrier has no effect on the distance when the distance between two points is larger than the barrier size and only the points in Ω_1 and Ω_5 are affected by the barrier's presence. We introduce three binary variables $\phi_{ij}^x, o_{ij}^x, \pi_{ij}^x$ for each eligible pair of i and j:

$$\phi_{ij}^x = \begin{cases} 0 & ; x_i + x_j - 2x \leq 2x + 2a - x_i - x_j \\ 1 & ;\text{otherwise} \end{cases},$$

$$o_{ij}^{x} = \begin{cases} 0 & ; x > \max\{x_i - a, x_j - a\} \\ 1 & ;\text{otherwise} \end{cases},$$

and

$$\pi_{ij}^{x} = \begin{cases} 0 & ; x < \min\{x_i, x_j\} \\ 1 & ;\text{otherwise} \end{cases}.$$

The binary variables work jointly to make sure that if the barrier conditions are satisfied, the shortest path is through one of the end points of the barrier. If the barrier conditions are not satisfied, the rectilinear path is selected by constraint (5.5) as the shortest path.

The binary variables work as follows: Constraints (5.10) and (5.11) check if the barrier conditions are satisfied. We observe that $o_{ij}^{x} = 1$ in (5.10) forces $x \leq \max\{x_i - a, x_j - a\}$ while $\pi_{ij}^{x} = 1$ in (5.11) forces $x \geq \min\{x_i, x_j\}$. If both conditions (the barrier conditions) are satisfied, both o_{ij}^{x} and π_{ij}^{x} must be equal to 0.

Constraints (5.8) and (5.9) jointly determine the shorter path around the barrier when the barrier is in effect. If $x_i + x_j - 2x < 2x + 2a - x_i - x_j$, then $\phi_{ij}^{x} = 0$, in which case constraint (5.9) becomes inactive. Otherwise constraint (5.8) becomes inactive.

Since $l_1^B(x, x_i, x_j)$ is minimized, when the barrier is in effect, constraint (5.6) will become active and determines the barrier distance using $o_{ij}^{x} = 1$ or $\pi_{ij}^{x} = 1$ and (5.7) will become active and determines the barrier distance using $o_{ij}^{x} = \pi_{ij}^{x} = 0$.

Finally, constraint (5.12) makes sure that x is in the partitioned sub region R. Constraint (5.13) is the binary conditions.

5.3 Centre Location of a Rectangular Barrier Region: A Bi-objective Problem (Problem 5.3)

5.3.1 Problem Definition and Formulation

Consider a rectangular shaped barrier region on the plane that needs to be located optimally among a set of given demand points. Two conflicting objectives are considered. The first objective is to place the barrier shape in such a way that its interference to interactions among demand points is to be minimized as in Section 5.2. The second objective is to locate the barrier shape minimax to the demand points where the distance between a demand point and the barrier shape will be taken as the closest distance. Because of the conflicting natures of the objective functions, we generate a bi-objective model and define a weighting parameter α on the objectives to give the decision maker flexibility. Municipalities often encounter this type of problem. A fire station or a park needs to be located in a city, its location has to be minimax to the demand points but its presence might affect interaction among the demand points. The term 'closest distance' was first introduced to the facility location literature by Brimberg and Wesolowsky (2000) who discussed the minisum problem, and followed by Brimberg and Wesolowsky (2002) in the context of the centre problem. The authors analyzed the problems and proved that for any given distance norm, the closest distance between a point and a convex polyhedron is a convex function of x and y coordinates. Furthermore, they examined the special case where the distance considered is the rectangular distance and the customers are given as rectangular demand areas and found a closed form relation for the closest distance, which simplifies the solution procedure dramatically. We provide this special case below as we are going to use it in our formulation. Consider Figure 5.7. It shows the functional forms of the closest distances depending upon the location of barrier B relative to demand point $k \in K$ where K is the set of all demand points.

We can now write the closed form rectilinear distance function from a point to the rectangular barrier as;

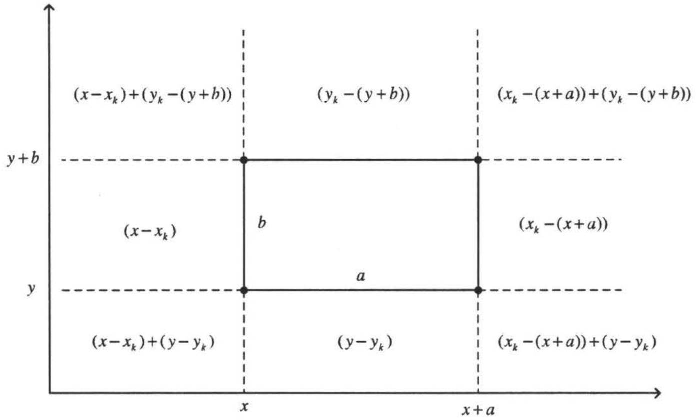


Figure 5.7: Closest Rectangular Distances to the Barrier

$$l_1^C(X, X_k) = \frac{1}{2}\left(|x_k - x| + |x_k - (x+a)| + |y_k - y| + |y_k - (y+b)| - (a+b)\right) \tag{5.14}$$

where x is x-coordinate of the lower left corner of the barrier facility and y is y-coordinate of the lower left corner of the barrier facility. This closed form function is given in Brimberg and Wesolowsky (2000) in the context of the closest distance from a point to a rectangular shape.

To locate the barrier minimax to the demand points, we need to find x and y that satisfy,

$$\text{Min} \quad z \qquad \text{(Formulation 5.3.1)}$$

subject to

$$z \geq w_k l_1^C(X, X_k), \forall k \in K \tag{5.15}$$

To solve this problem as an LP problem, we need to linearize the absolute value terms present in $l_1^C(X, X_k)$ as suggested by Brimberg and Wesolowsky (2002). To do that, for all $k \in K$ we need to replace constraint (5.15) by the system of eight linear constraints given below. The constraints correspond to the eight zones surrounding the barrier facility.

$$\frac{2z}{w_k} + (a+b) \geq -(x_k - x) - (x_k - (x+a)) + (y_k - y) + (y_k - (y+b)) \qquad \text{(ConstSet5.1)}$$
$$\frac{2z}{w_k} + (a+b) \geq (x_k - x) - (x_k - (x+a)) + (y_k - y) + (y_k - (y+b))$$
$$\frac{2z}{w_k} + (a+b) \geq (x_k - x) + (x_k - (x+a)) + (y_k - y) + (y_k - (y+b))$$
$$\frac{2z}{w_k} + (a+b) \geq (x_k - x) + (x_k - (x+a)) + (y_k - y) - (y_k - (y+b))$$
$$\frac{2z}{w_k} + (a+b) \geq (x_k - x) + (x_k - (x+a)) - (y_k - y) - (y_k - (y+b))$$
$$\frac{2z}{w_k} + (a+b) \geq (x_k - x) - (x_k - (x+a)) - (y_k - y) - (y_k - (y+b))$$
$$\frac{2z}{w_k} + (a+b) \geq -(x_k - x) - (x_k - (x+a)) - (y_k - y) - (y_k - (y+b))$$
$$\frac{2z}{w_k} + (a+b) \geq -(x_k - x) - (x_k - (x+a)) + (y_k - y) - (y_k - (y+b))$$

We can now combine Formulation 5.2.1 and Formulation 5.3.1 and generate a new bi-objective formulation. Formulation 5.2.1 is used for both x and y distances. Therefore we introduce three more binary variables $\phi^y_{ij}, o^y_{ij}, \pi^y_{ij}$ for calculating y distances. The new formulation, called Formulation 5.3.2, uses a user defined parameter α, to determine the relative weights of each objective function. When α is equal to zero, the problem reduces to a closest minimax rectangle problem and when α is equal to one, the problem reduces to a minimum rectangular interference problem.

Formulation 5.3.2: For all pairs ($i \in \Omega_1$, $j \in \Omega_5$) which satisfy $|x_i - x_j| < a$, and for all pairs ($i \in \Omega_3$, $j \in \Omega_7$) which satisfy $|y_i - y_j| < b$,

$$\text{Min} \quad \alpha \left\{ \sum_{i\in\Omega_1}\sum_{j\in\Omega_5} v_{ij}\left\{ l_1^B(x,x_i,x_j) \,||x_i - x_j| < a \right\} + \sum_{i\in\Omega_3}\sum_{j\in\Omega_7} v_{ij}\left\{ l_1^B(y,y_i,y_j) \,||y_i - y_j| < b \right\} \right\} +$$
$$(1-\alpha)z$$

subject to

$l_1^B(x_i,x_j) \geq |x_i - x_j|, \forall i \in \Omega_1, \forall j \in \Omega_5$

$l_1^B(x_i,x_j) \geq x_i + x_j - 2x - M(\phi_{ij}^x + o_{ij}^x + \pi_{ij}^x), \forall i \in \Omega_1, \forall j \in \Omega_5$

$l_1^B(x_i,x_j) \geq 2x + 2a - x_i - x_j - M(1 - [\phi_{ij}^x + o_{ij}^x + \pi_{ij}^x]), \forall i \in \Omega_1, \forall j \in \Omega_5$

$x_i + x_j - 2x \leq 2x + 2a - x_i - x_j + M\phi_{ij}^x, \forall i \in \Omega_1, \forall j \in \Omega_5$

$2x + 2a - x_i - x_j \leq x_i + x_j - 2x + M(1 - \phi_{ij}^x), \forall i \in \Omega_1, \forall j \in \Omega_5$

$x \leq \max\{x_i - a, x_j - a\} + M(1 - o_{ij}^x), \forall i \in \Omega_1, \forall j \in \Omega_5$

$x \geq \min\{x_i, x_j\} - M(1 - \pi_{ij}^x), \forall i \in \Omega_1, \forall j \in \Omega_5$

$l_1^B(y_i,y_j) \geq |y_i - y_j|, \forall i \in \Omega_3, \forall j \in \Omega_7$

$l_1^B(y_i,y_j) \geq y_i + y_j - 2y - M(\phi_{ij}^y + o_{ij}^y + \pi_{ij}^y), \forall i \in \Omega_3, \forall j \in \Omega_7$

$l_1^B(y_i,y_j) \geq 2y + 2b - y_i - y_j - M(1 - [\phi_{ij}^y + o_{ij}^y + \pi_{ij}^y]), \forall i \in \Omega_3, \forall j \in \Omega_7$

$y_i + y_j - 2y \leq 2y + 2b - y_i - y_j + M\phi_{ij}^y, \forall i \in \Omega_3, \forall j \in \Omega_7$

$2y + 2b - y_i - y_j \leq y_i + y_j - 2y + M(1 - \phi_{ij}^y), \forall i \in \Omega_3, \forall j \in \Omega_7$

$y \leq \max\{y_i - b, y_j - b\} + M(1 - o_{ij}^y), \forall i \in \Omega_3, \forall j \in \Omega_7$

$y \geq \min\{y_i, y_j\} - M(1 - \pi_{ij}^y), \forall i \in \Omega_3, \forall j \in \Omega_7$

$\phi_{ij}^x, o_{ij}^x, \pi_{ij}^x \in \{0,1\}, \forall i \in \Omega_1, \forall j \in \Omega_5$

$\phi_{ij}^y, o_{ij}^y, \pi_{ij}^y \in \{0,1\}, \forall i \in \Omega_3, \forall j \in \Omega_7$

$x, y \in R$

$+$ ConstSet5.1

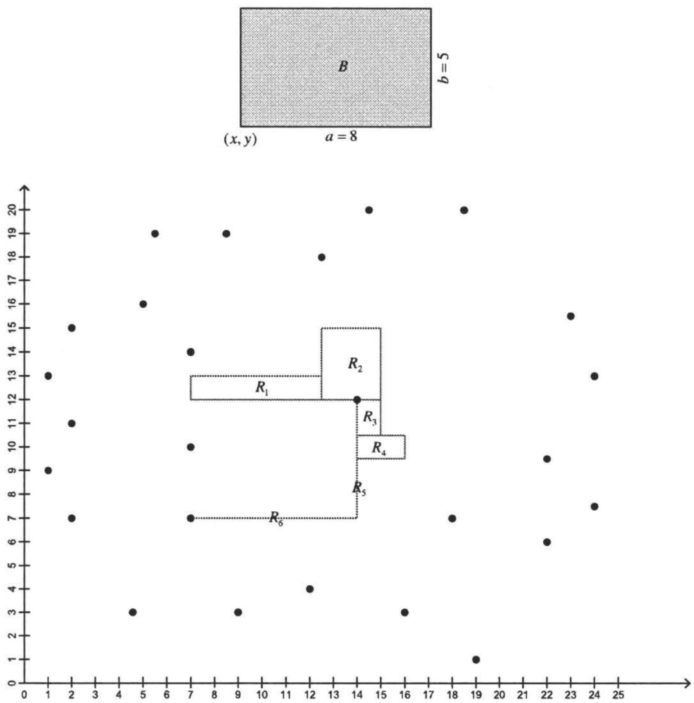


Figure 5.8: Example Problem

5.3.2 An Example Problem

Consider the example problem illustrated in Figure 5.8. We want to locate a rectangular facility (barrier) with dimensions $a = 8$ and $b = 5$ in a region where user to user interaction with equal unit weights among the existing facilities is present. Our objective is to minimize the closest distance between the rectangular facility and the existing points with minimum interference to their interaction. The feasible regions for the rectangular facility's corner point (x, y) are illustrated in the figure. The regions R_1, R_2, R_3 and R_4 have rectangular shapes whereas R_5 and R_6 are lines. For each region, we first solved the problem using one objective function at a time to determine the weighting parameter α values. These α values are normalized such that the lower objective value gets a higher weight. We, then, solved the problem for each region with normalized α values hoping that the magnitude of each objective function in the bi-objective problem is roughly equal to each other. Note that this procedure is not required as the decision maker can freely choose the α values based on the importance given to each objective function. Results are presented in Table 5.1 and the solution is illustrated in Figure 5.9.

	Closest Centre Obj.	Min. Inter. Obj.	Normalized Bi-Obj.
R_1	11.5	32.0	17.6
R_2	17.0	**23.0**	22.4
R_3	17.0	32.0	22.6
R_4	16.0	30.0	21.2
R_5	14.5	30.0	19.5
R_6	**10.0**	33.0	**15.8**
Opt. Loc.	$(8.5,7) \in R_6$	$(15,15) \in R_2$	$(8.5,7) \in R_6$

Table 5.1: Solution Results for Example 5.3.2

5.4 Simulated Annealing (SA) Heuristic for the Centre Location Problem of a Rectangular Barrier Region with Expropriation

In this section, we develop an SA heuristic for the problem given in Section 5.3. Although the SA heuristic may not guarantee the optimum solution, it has a number of advantages over the MIP formulation. First, the MIP formulation may not be efficient for larger problem instances. Second, using the SA heuristic, we do not need to determine the rectangular feasible regions in which we locate the facility. We designed the SA heuristic to work on the plane taking into account the expropriation costs of the demand points. When we are not allowed to expropriate the demand points, we simply assign very high expropriation costs to the demand points, therefore, the heuristic finds a location that does not expropriate any demand points. Reducing the expropriation costs of the demand points may allow the heuristic to choose a location with expropriation.

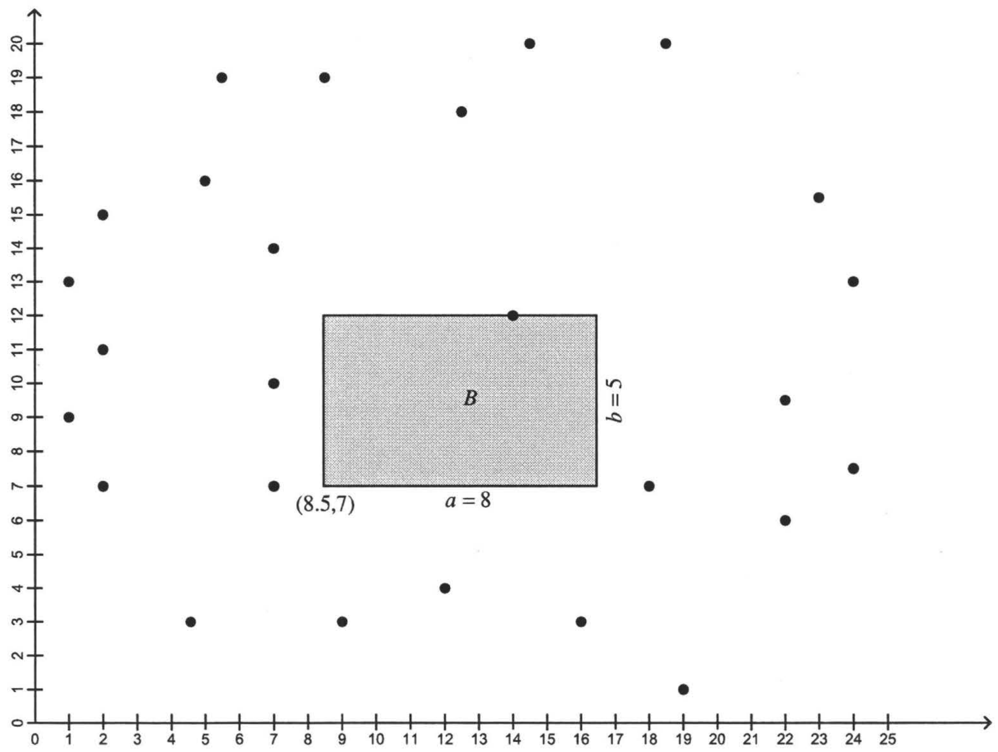


Figure 5.9: Optimal Bi-Objective Location of the Facility

The SA heuristic was proposed by Kirkpatrick *et al.*, (1983) as an adaptation of the Metropolis-Hastings algorithm (Metropolis *et al.*, 1953). The heuristic takes its name from the physical process of annealing in metallurgy. Annealing is a technique requiring heating and slow cooling of materials in order to achieve a minimum energy crystalline structure. The SA algorithm mimics this process with the aim of finding a good solution while providing the opportunity to escape from local optima. The opportunities to jump from local optima are greater early in the process when the 'temperature' is higher. The temperature 'cools' down after every iteration with a phase defined by the decision maker. As the process 'cools', the focus is on finding an optimal solution in the neighborhood, and the probability of a jump to a new neighborhood is reduced.

Previous research by Bennage and Dhingra (1995) has shown that SA has great potential for problems with mixed discrete and continuous variables. More recently, Arostegui et al. (2007) empirically evaluated several different heuristics in different location problems. While tabu search was better for some types of location problems, SA was slightly better statistically for some others. Given this context, we choose to explore the possibility of efficiently finding good solutions to this problem using an SA approach.

We used the following parameters in our SA process which are similar to those used in previous SA literature:

F_0 (Initial Objective Value)

T_0 (Initial temperature) = 100;

N (The number of iterations remaining) = 500;

γ (The proportion by which the temperature reduced after each iteration) = $1 - (5/N)$;

The SA algorithm for this problem is as follows:

The process starts with a parameter T_0 that represents a high temperature, which is reduced after each iteration. After an initial solution is generated, a random search is conducted to move from the current solution to a neighborhood solution. The selection of a neighborhood range is at the discretion of the user. Specifying the neighborhood range $N(Z)$ effectively is a key to the successful implementation of an SA algorithm which is mostly specific to the underlying problem. Initial trials have been made with a number of different neighbourhood sizes and the best one is kept for the example problem.

A new solution with a better objective value will always be accepted. There is also an opportunity to accept an inferior solution based on a probability p which is given by $p = \exp\left(-\frac{\Delta}{T}\right)$, where Δ is the difference between the new solution and the current solution, and T is the current temperature.

When the temperature is high, the probability of accepting a worse solution is

input : $T = T_0$, γ, $N = N_{max}$, F_0
output: Best heuristic solution

```
while N ≤ N_max do
    randomly choose Z̄ ∈ N(Z);
    if F(Z̄) ≤ F(Z) then
        Z̄ ← Z;
    else
        Δ = F(Z̄) − F(Z) ;
        p = exp(−Δ/T) ;
        Z̄ ← Z with probability p;
    end
    T ← Tγ;
end
```

Algorithm 1: The Simulated Annealing Algorithm

higher, so that in the initial steps, it is easier to escape from a local optimum. When the temperature decreases gradually, the probability will also decrease, so that it will be harder to move from the current solution. The algorithm terminates when the process 'cools' down.

At first, to find a solution for Example 5.3.2, expropriation costs are kept relatively high in order to find a location that does not overlap with any demand points. Figure 5.10 is the illustration of the solution iterations in which the algorithm converged to the optimal solution.

As we have stated before, the SA algorithm can find a good location for the rectangular facility by expropriating a number of demand points. Assume now that each demand point has a unit cost of expropriation. In that case the SA algorithm expropriates a demand point and finds a location that results with a lower objective value of 14.63 which is a 7.5% savings compared to the original solution. Figure 5.11 shows the new facility location with expropriation.

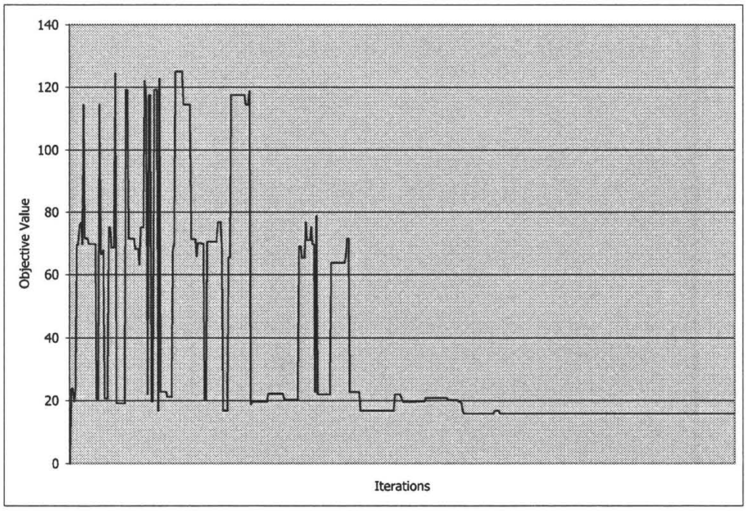


Figure 5.10: The SA Heuristic Solution Iterations

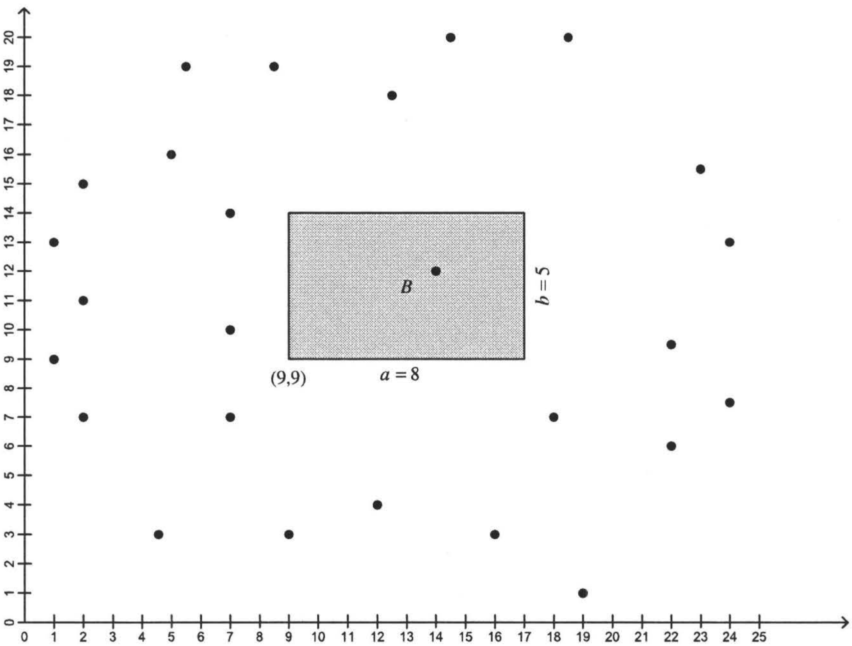


Figure 5.11: A New Location for the Facility with Expropriation

Chapter 6

On the Use of the Varignon Frame for Single Facility Weber Problems in the Presence of Barriers

6.1 Introduction

The Weber problem entails locating a facility on the plane to serve a finite set of existing demand points with different demand levels. It has been of interest to many researchers since as early as the 17th century, but its practical usage was identified by Weber in 1909. Its objective is to find a location such that the sum of the weighted distances from the facility to the demand points is minimized. One of the most interesting solution methodologies for the problem is based on using a mechanical device, called the 'Varignon Frame'. The Varignon frame is a mechanical system which consists of strings, weights and a board with holes drilled in it representing the locations of existing facilities. Weights, corresponding to the demands for the existing facilities, are attached to the strings that pass through the holes. The ends of the strings are all tied together in a knot on top of the board. Figure 6.1 is an illustration of a Varignon frame with 4 demand points.

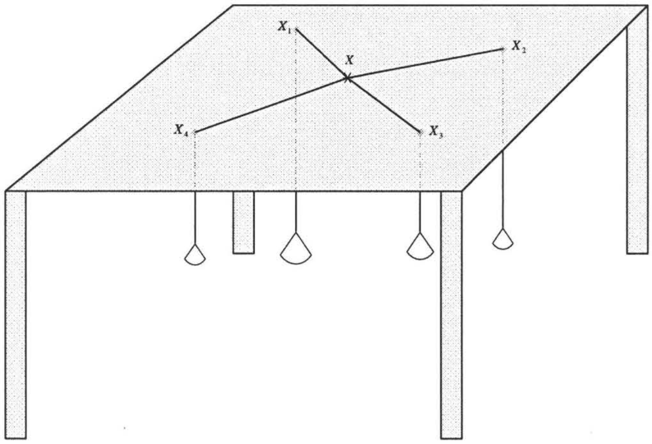


Figure 6.1: A Varignon frame with four demand points

Because of the equilibrium of forces, under the assumption of the absence of friction, the knot is going to have a final position at the optimal point. This result derives from the fact that the first order optimality conditions (partial derivatives) for the minisum function, which is known to be a convex function, are equivalent to a balance of force components at the optimal point in the x and y directions respectively. The only exception for this equivalency emerges when the location of an optimal point coincides with that of an existing point. The partial derivatives at the optimal point will not exist. The knot will stay on top of that hole, or go down into the hole. The condition for the knot to disappear down the hole is that the weight at that hole has to be larger than the sum of the other weights. However, an existing point does not have to satisfy this condition in order to be at an optimal location. It is necessary and sufficient if the net force exerted by the other points' weights is less than or equal to its own weight. The existing point will then be at an optimal location even if it has a very small weight (Drezner and Hamacher (2001)).

In this chapter we apply the same analog approach to the Weber problem in the presence of barriers. We will show through analysis that the analog can also be used for certain problems. We note that this is not a comprehensive and complete solution approach. The introduction of barriers into the Weber problem, results in non-convex distance functions which makes the problem more complicated. We believe that the analog approach introduced in this study presents a fast and easy way of exploring such problems and providing an intuitive basis for the analysis. From a practitioner's point of view, this approach has a number of advantages in conducting experimental what-if analysis. If a barrier happens to move from one location to another over time, the response of the optimal location of the facility can be investigated by a visual approach. Also, if there are some preferred facility location zones that have been occupied by the barrier, preliminary investigations for relocation or resizing of the barriers can be conducted. Further, depending upon the optimal location of the facility, identical solutions with a different set of barrier regions can be acquired.

Section 6.2 describes example problems and provides required preliminaries. In Section 6.3, we justify the usage of the Varignon frame for some problem types through analytical results and explain the analog solution process. Some further experimental examples are provided in Section 6.4. Section 6.5 is the conclusion and final remarks.

6.2 Background

Consider n demand points each with a nonnegative weight $w_i, i = 1, ..., n$ and a finite set of convex, closed polyhedral and pairwise disjoint barriers $B_1, ..., B_p$ present in the plane. Let B be a union of barrier sets in $\Re^2$ and $F = \Re^2 \backslash \mathrm{int}\,(B)$ be the feasible region in $\Re^2$. The barrier distance $l_p^B(X, X_i)$, between the points X and X_i is then defined as the infimum of the lengths of all permitted paths (the shortest path) between X and X_i. Two points $X = (x, y), X_i = (x_i, y_i) \in F$ are called p-visible from each other if $l_p^B(X, X_i) = l_p(X, X_i)$ and p-shadow if $l_p^B(X, X_i) > l_p(X, X_i)$. If the points are not p-visible from each other, then the distance between them becomes a barrier distance. In this chapter we consider the Euclidean distance case where $p = 2$, since the Varignon frame can only be used under this distance norm. For more information

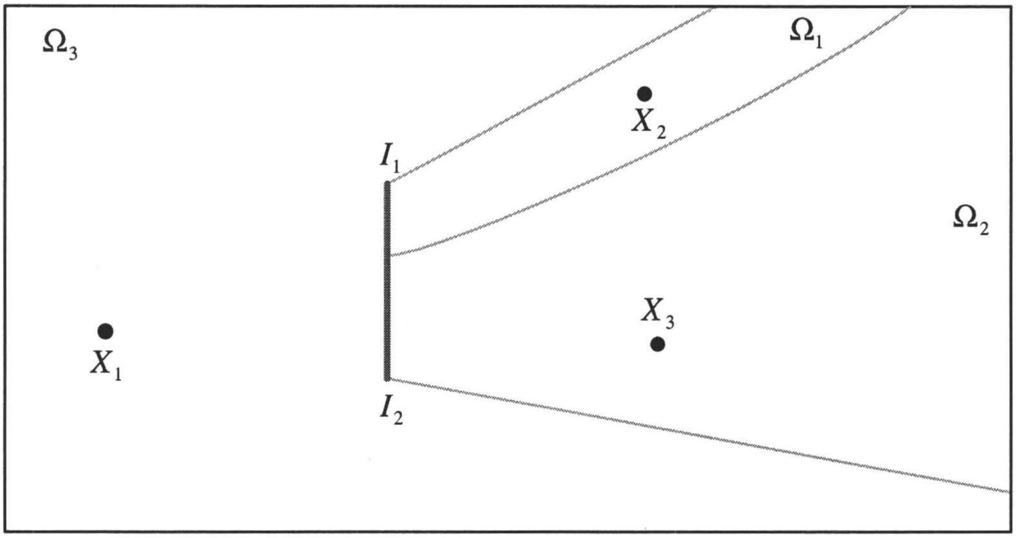


Figure 6.2: Partitioning with Respect to X_1

on visibility in the facility location literature, the user is referred to Chapter 1, Section 1.1.7.

Let $P(B)$ where $|P(B)| = K$ be the finite set of barrier extreme points. In the case where the points are not p-visible, there is an important and useful property called the Barrier Touching Property proven by Klamroth (2001) which states that there exists a shortest feasible path P that consists of line segments with breaking points only at extreme points of barriers. The barrier distance in this case will be through some intermediate points $I_j, j = 1, ..., K$, which are the extreme points of the barriers. The difficulty is to determine the assignment of these intermediate points to the demand points. Butt and Cavalier (1996) suggested a remedy which is based on the idea of partitioning the plane by demarcation lines.

Figure 6.2 shows an example with three demand points (X_1, X_2, and X_3) and one line barrier. This results in demarcation lines which partition the plane into three regions Ω_1, Ω_2 and Ω_3 with respect to X_1. The demarcation lines that go from the end points of the line barrier are as the same as the visibility lines. The function for the demarcation line that splits Ω_1 from Ω_2 can be found using the following equation:

$$l_2(X_1, I_1) + l_2(I_1, X) = l_2(X_1, I_2) + l_2(I_2, X)$$

The distance function from demand point X_1 to a point in any given cell is convex. That is, a facility located anywhere in this cell communicates with each demand point either directly, or through a constant extreme point of the barrier. For example, for any point in Ω_1 (i.e. X_2), the shortest path to X_1 is through intermediate point I_1, whereas, for any point in Ω_2 (i.e. X_3), the intermediate point is I_2. All points in Ω_3 communicate with X_1 directly.

The advantage of this approach is that once the plane is partitioned by the demarcation lines, the intermediate points to be used for each region become known. The disadvantage of the approach is that intersections of these lines form a cell structure which is mostly nonconvex with nonlinear boundaries. Also, the number of cells increases rapidly with the number of demand points and the number of barrier extreme points.

6.3 Solution Methodology

6.3.1 The Varignon Frame with a Line Barrier

In order to initiate the discussion, we first consider a Varignon frame with a line barrier (e.g. a stick). The following cases are studied: optimality of a demand point, optimality of a barrier extreme point (end point) and optimality of a general point.

Case 1: Optimality of a Demand Point To provide the conditions for the optimality of a demand point, we are going to follow a similar procedure to Drezner and Hamacher (2001, pg. 11) given for the unrestricted Varignon frame and start with a simple example. Consider the Varignon frame with a line barrier illustrated in Figure 6.3a.

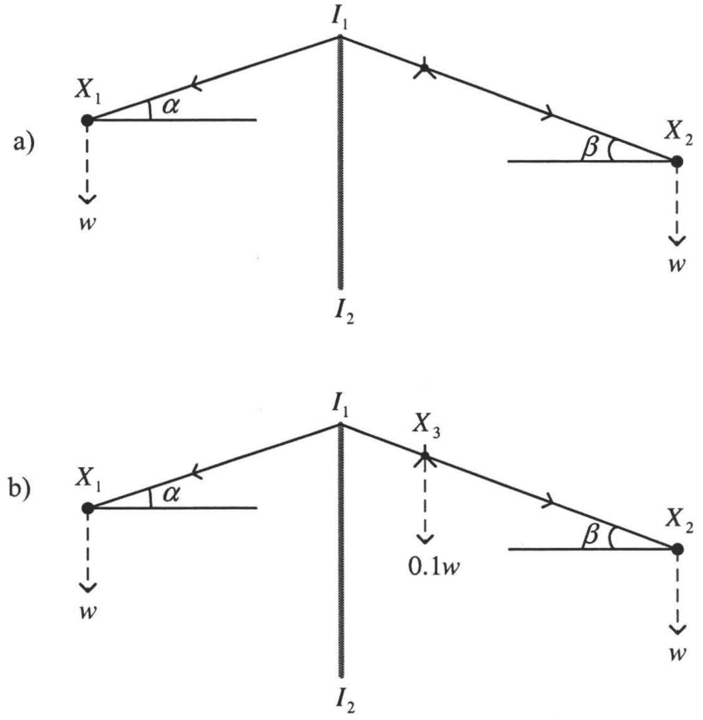


Figure 6.3: Optimality of a Demand Point

There are two demand points with equal w weights located on each side of the barrier. The weights are attached to the strings through the holes and ends of the strings are tied together in a knot on top of the board. There is a line barrier on the board with a given length and orientation and the strings pass through the upper end point of the barrier (I_1). Assume that the shortest distance between demand point locations X_1 and X_2 are through I_1. At equilibrium, the knot will stay at any point since the resultant force on the knot would be zero everywhere including I_1. Also any point on the route of the strings would be optimal for the Weber problem because the sum of the distances is constant.

However, if we drill a hole on the board at X_3 and attach a very small weight (e.g. $0.1w$) through it (Figure 6.3b), then the optimal location for the Weber problem will be at X_3 even though the partial derivatives of the minisum function for the location will not exist. We can also describe the problem in terms of the equilibrium of forces. When the knot is at X_3, the weight forces for X_1 and X_2 will cancel each other out and the knot will try to go down the hole of the small weight because there is no other force has an effect on it other than the small weight itself. But when it 'just passes' through the hole, if there is no friction, the forces of the two weights for X_1 and X_2 will be upwards and their combined force of $2w$ will move the knot up again. The conclusion is that, for a demand point to be optimal under this setting, it does not have to outweigh the sum of the other weights. If it does, then the knot will disappear down the hole of that weight. The conditions of optimality of a demand point in a more general linear barrier case can be given by the following property:

Property 6.3.1 *Let R be a set which is the union of the demand points visible from demand point X_r and the end points of the line barrier that are on the shortest paths to the points that are invisible from X_r. X_r is optimal if and only if:*

$$w_r \geqslant \left[\left(\sum_{i \in R} \frac{w_i \left(x_r - x_i\right)}{l_2 \left(X_r, X_i\right)} \right)^2 + \left(\sum_{i \in R} \frac{w_i \left(y_r - y_i\right)}{l_2 \left(X_r, X_i\right)} \right)^2 \right]^{1/2}$$

Proof. The proof is omitted as a similar one is given in Love et al. (1988, pg. 32). ■

The proof depends on the fact that if the force exerted by the weight at X_r is at least as much as the net combined forces in the x and y direction of the other demand points at X_r, then X_r will be optimal. For a demand point that is invisible from X_r, the force gets a new direction from one of the end points of the barrier. Therefore, if that end point is on the shortest path from X_r to the corresponding demand point, it is valid to consider that end point as the location of the demand point for practical purposes.

For our example illustrated in Figure 6.3b, optimality of X_3 can be given as;

$$w_3 \geqslant \left[\left(\frac{w(x_3 - x_{I_1})}{l_2(X_3, I_1)} + \frac{w(x_3 - x_2)}{l_2(X_3, X_2)}\right)^2 + \left(\frac{w(y_3 - y_{I_1})}{l_2(X_3, I_1)} + \frac{w(y_3 - y_2)}{l_2(X_3, X_2)}\right)^2\right]^{1/2} \Rightarrow$$

$$w_3 \geqslant w\left[(\cos\beta - \cos\beta)^2 + (\sin\beta - \sin\beta)^2\right]^{1/2} \Rightarrow$$

$$w_3 \geqslant 0.$$

Therefore, even a very tiny weight will make X_3 optimal.

Case 2: Optimality of a Line Barrier Extreme (End) Point

Assume that when we let the knot free, it settles at an end point of the line barrier where the forces generated by the weights are in equilibrium. The first case would be that the end point of the line barrier is an 'unconstraining point', i.e. removing the line barrier wouldn't change the location of the knot. In that case the problem reduces to the unrestricted Weber problem. If, however, there is a reaction force at the end point of the barrier, i.e. removing the line barrier would change the location of the knot, the situation will be different. Consider the following example illustrated in Figure 6.4.

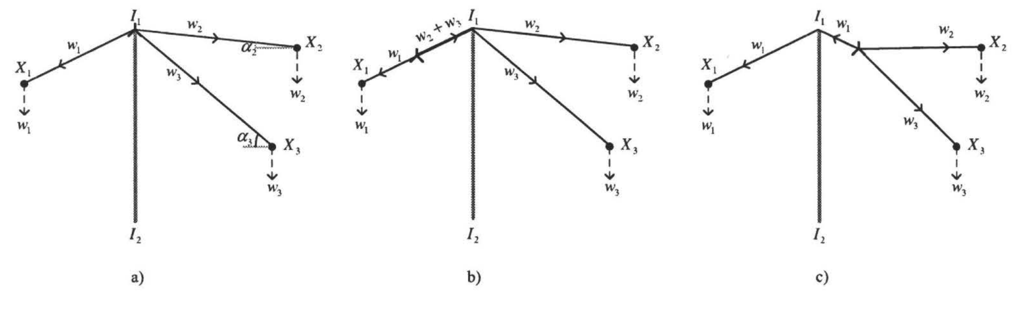


Figure 6.4: Illustration for the Optimality of I_1

We know that if $w_1 = w_2 + w_3$ the knot will stay at the end point (Figure 6.4a).

If we move the knot towards X_1, the knot will remain at the place where we move it (Figure 6.4b), because the forces will be equal to each other in opposite directions in the absence of friction. But if we move the knot towards X_2 and X_3 (Figure 6.4c), the knot will come back to the end point of the line barrier since the combined force of w_2 and w_3 in the opposite direction of w_1 will be less than w_1. Furthermore, we do not need the condition $w_1 = w_2 + w_3$ in order to keep the knot at the end point of the line barrier. This is explained by the following property.

Property 6.3.2 *I_1 will be the equilibrium point if:*

$$w_2 + w_3 > w_1 \geqslant \left((w_2 \cos\alpha_2 + w_3 \cos\alpha_3)^2 + (w_2 \sin\alpha_2 + w_3 \sin\alpha_3)^2\right)^{1/2}$$

Proof. The term $\left((w_2 \cos\alpha_2 + w_3 \cos\alpha_3)^2 + (w_2 \sin\alpha_2 + w_3 \sin\alpha_3)^2\right)^{1/2}$ given in the property is the net force of w_2 and w_3 just to the right of I_1. If the net force is less than or equal to w_1, then it will not be able to pull the knot from I_1 towards w_2 and w_3. Also, because of the condition $w_1 < w_2 + w_3$, w_1 will not be able to pull the knot towards itself. Therefore the knot will stay at I_1. ■

If the assignment of the line barrier end points to the corresponding demand points is optimal, then the solution of the Weber problem will also give this end point as the global optimal point. Because all of the demand points will be visible from the end point of the barrier, the distances from the end point to the demand points will be Euclidean. Any movement of the knot from the end point will not decrease the objective function because of the triangle inequality.

Case 3: Optimality of a Point not on the Barrier

In this part we consider the optimality of a point that is not on the barrier. When we let the knot free, it may settle at any point on the board where the forces of the demand points reach equilibrium. If we can prove the local optimality of this point under the Weber objective, then we can justify the use of the Varignon frame for the Weber problems with a line barrier.

Consider the example illustrated in Figure 6.5. There are four demand points located on the plane which has a line barrier. Assume that when the knot is freed, it settles at point $X^* = (x^*, y^*)$, which is an equilibrium point of the weight forces on the strings and it is not on the barrier.

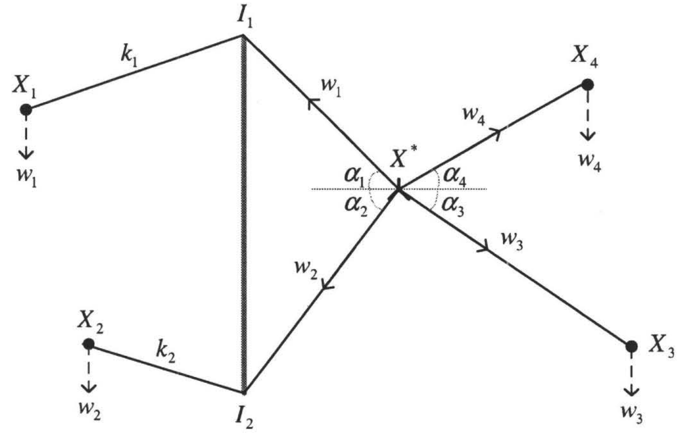


Figure 6.5: Optimality of a General Point

Let F be the net force on the knot and F_{x^*} be the horizontal component of the net force, F_{y^*} be the vertical component of the net force. We can write F_{x^*} and F_{y^*} as;

$$F_{x^*} = -w_1 \cos(\alpha_1) - w_2 \cos(\alpha_2) + w_3 \cos(\alpha_3) + w_4 \cos(\alpha_4) = 0$$

and

$$F_{y^*} = w_1 \sin(\alpha_1) - w_2 \sin(\alpha_2) - w_3 \sin(\alpha_3) + w_4 \sin(\alpha_4) = 0.$$

Now for a point X on the right hand side of the barrier, the Weber Problem can be written as the following MINLP program (Formulation 6.1):

$$f(X) = \text{Min} \quad \sum_{i=1}^{2} \sum_{j=1}^{2} u_{ij} w_i \left(l_2\left(X, I_j\right) + l_2\left(I_j, X_i\right)\right) + \sum_{i=3}^{4} w_i l_2\left(X, X_i\right)$$

subject to

$$\sum_{j=1}^{2} u_{ij} = 1, \forall i$$

$$u_{ij} \in \{0, 1\}, \forall i, j$$

where u_{ij} is defined as,

$$u_{ij} = \left\{ \begin{array}{ll} 1 & \text{if barrier end point j is assigned to demand point i} \\ 0 & \text{otherwise} \end{array} \right\}.$$

Assume that the planar area is partitioned in regions according to Butt and Cavalier (1996) as discussed in Section 6.2 and that the global optimal solution to the Weber problem in the presence of this line barrier is in the same region as X^*. Let this region be defined by Ω and let the optimal location for the Weber problem be defined by X^{**}. Figure 6.6 is the illustration of the partitioning in the convex hull of the demand points and the barrier end points. We restrict the location of the facility to the convex hull of the demand points and the barrier end points, due to the fact that the optimal facility location must lie within this convex hull (Butt and Cavalier (1996)). The demand points in Figure 6.6 have unit weights and are located at $X_1 = (-5.5, 2.5)$, $X_2 = (-4, -3.5)$, $X_3 = (10, -3.5)$,$X_4 = (9, 3)$ and the barrier end points are located at $I_1 = (0, 4.5)$ and $I_2 = (0, -4.5)$.

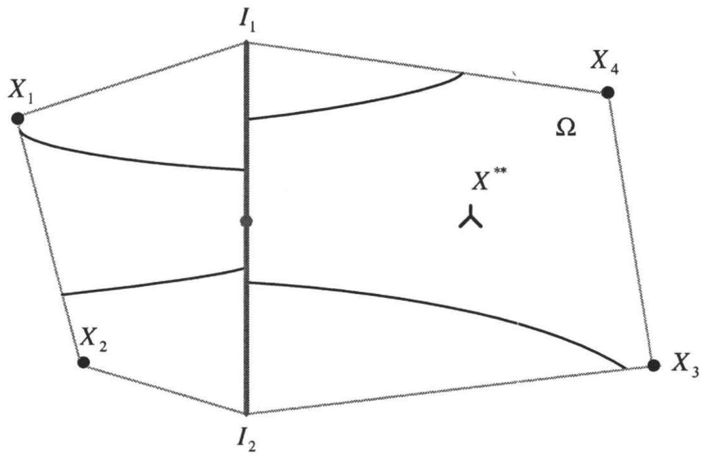


Figure 6.6: Partitioning the Region

For region Ω, barrier end point I_1 will be the optimal assignment to X_1 and barrier end point I_2 will be the optimal assignment for X_2.

The Weber problem will reduce to:

$$f(X) = \text{Min} \quad w_1(k_1 + l_2(X, I_1)) + w_2(k_2 + l_2(X, I_2)) + w_3 l_2(X, X_3) + w_4 l_2(X, X_4),$$

where $k_1 = l_2(I_1, X_1)$ and $k_2 = l_2(I_2, X_2)$ are constant terms.

This problem has the exact same minimizer as the unrestricted Weber problem, with the demand points located at I_1, I_2, X_3 and X_4, since k_1 and k_2 are constant values. Partial derivatives of $f(X^{**})$ are equal to the force components at X^*. Therefore, we can conclude that $X^{**} = X^*$ is a global minimum in Ω.

The optimum minisum location for the facility in Ω will be at $X^{**} = (5.51, 0.092)$ with the objective value of 34.497.

We should note that if the assignment of demand points to the barrier end points was not correct, then X^{**} would be a local minimum for the complete problem. For example, if we mistakenly assign I_2 to X_1, then the resulting minisum objective value will be 37.549 with the facility located at $(5.5, 0)$ which will be a local minimum for the complete problem.

6.3.2 The Varignon Frame with a Polyhedral Barrier

In this section we consider the Varignon frame with a convex polyhedral barrier. A convex polyhedral shape acting as a barrier has a set of extreme points (vertices) denoted as $P(B)$ and a set of facets (edges) denoted as $P(F)$ where $|P(B)|, |P(F)| \geqslant 3$ and $P(B) \subset P(F)$. The extreme points on each end of the facet are called the end points of the facet. Figure 6.7 shows a convex octagonal barrier. As we know from the barrier touching property, for any convex polyhedral shape, if one of the existing facilities is not visible from the new facility location, the shortest path to that existing facility, (in this case the length of the string), will pass through extreme points or facets of the barrier. The extreme points on this shortest path are called the intermediate points.

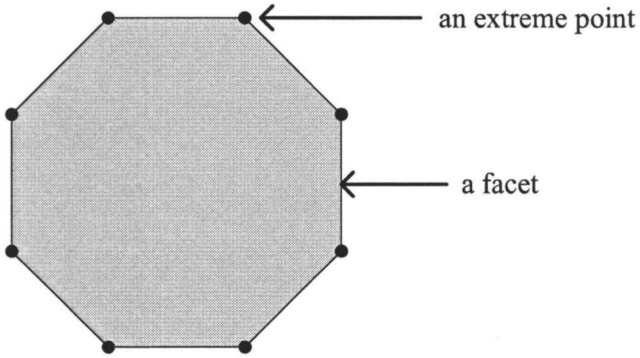


Figure 6.7: An Octagonal Barrier

Optimality of Barrier Facets (Edges)

Assume that the knot settles at some point on the barrier with a convex polyhedral shape. What can we say about optimality of this point and the optimality of the facet that the knot stays on? Consider the following example illustrated in Figure 6.8. There is an octagonal barrier with three demand points. The knot settles on the facet between I_1 and I_2. We provide the following propositions.

Proposition 6.3.1 *If the knot stays on a facet, then there can not be a weight in the half plane opposite the facet.*

Proof. If a knot position on the interior of the facet is stable then the opposing forces parallel to the facet are balanced. There can be no force component perpendicular to the facet because then the knot would be pulled off the facet up to a certain point at which equilibrium is reached. Thus there can not be a weight in the half plane opposite the facet. More generally, there can be no string bounding in that half plane. This means that the knot can be moved to any point on the facet including the end points because the forces will be balanced everywhere on the facet. ■

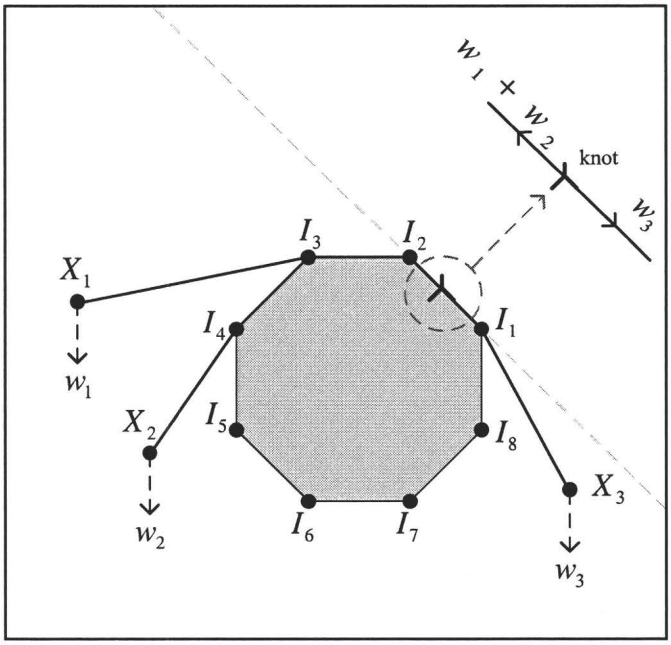


Figure 6.8: Optimality of a Barrier Facet

Proposition 6.3.2 *If the knot settles on a facet where there is no demand point present, then all the points on that facet including the end points are optimal for the Weber objective function.*

Proof. See Figure 6.8. If there is no demand point on the facet then any movement over the facet will generate a linear change in the competing forces $w_1 + w_2$ and w_3. Since these forces should be equal to each other because of their collinearity in opposite directions, any point on the barrier facet will provide the same objective value. Therefore if a settle point is optimal, then all the points on that facet including the end points will be optimal. ■

However, if there is a demand point present on the facet, then any movement of the knot towards the demand point will decrease the objective value. Consequently, the optimal location will be a single point, which is the demand point location.

Optimality of a Point on the Board

Finding the optimal solution of a point located anywhere on the board for the Weber problem in the presence of convex polygonal barriers is difficult analytically. There are two known solution methodologies. The first one suggested by Butt and Cavalier (1996) requires the partitioning of the region by demarcation lines and solving a number of problems with convex objective function and possibly noncovex constraints. The second one suggested by Klamroth (2002) requires the partitioning of the region by visibility grids and determining which barrier intermediate points are to be used. The Varignon frame approach may overcome these difficulties. We explain the idea on Figure 6.8. To reach X_2 from the knot, the decision maker only needs to determine if upper or lower side of the barrier is to be used. If the string goes through the upper side of the barrier as in the figure, then weight w_2 will automatically force the string to follow the shortest path which passes through intermediate points I_2, I_3 and I_4. If we move the string towards the lower side of the barrier, then to reach X_2, the string will pass naturally through I_1,I_8,I_7 and I_6. As we can see, the intermediate points do not have to be determined explicitly.

Once the knot settles at a point, then we can treat the first intermediate point that is visible from the knot as the corresponding demand point location. If the assignment of the barrier directions to the demand points is correct, then as in the line barrier case, the location of the knot will be the global optimal location for the Weber objective. Otherwise, the location of the knot will at least be the local minimum.

6.4 Examples

To provide outputs for our analysis of the analog approach, we conducted some experiments using two examples from the literature. The first example is from Butt and Cavalier (1996). The second example is from Katz and Cooper (1981), which is also used by Bischoff and Klamroth (2007). We did not attempt to achieve precision in our experiments. They were meant to be illustrative only. However, we did take measures to minimize friction. We used satin strings and the sides of the foam barriers were covered with plastic adhesive tape. The holes on the plywood board were lubricated.

6.4.1 Example 1: (Butt and Cavalier (1996))

In this example, we considered four existing facilities with equal weights, located at the coordinates given in Table 6.1.

i	1	2	3	4
x_i	1	15	9	3
y_i	12	0	9	4

Table 6.1: Parameters for Demand Points in Example 1

We also considered two polygonal barriers, with the following extreme points given in clockwise order:

$B_1 : ((0,7),(3,10),(6,10),(6,5))$ and, $B_2 : ((8,1),(9,4),(13,14),(14,4),(14,2))$.

When we let the knot free, it settled at a point very close to point $(6.857, 6.143)$ which was declared by Butt and Cavalier (1996) as optimal. Figure 6.9 is a picture from the experiment that shows the final location of the knot.

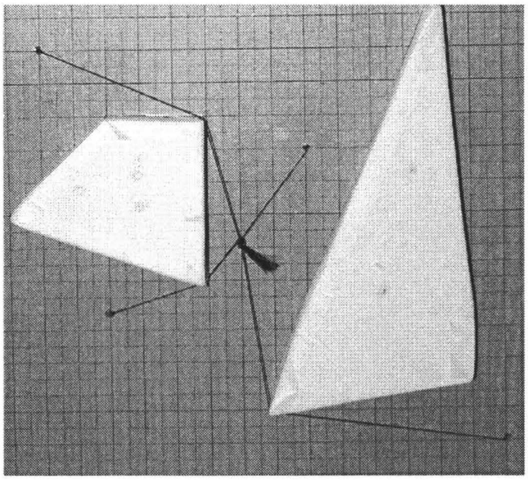


Figure 6.9: Example 1

6.4.2 Example 2: (Katz and Cooper, 1981)

In this second example we considered 10 demand points with equal weights, located at the coordinates given in Table 6.2. There is also a circular barrier with a centre at $(0, 0)$ and radius 3. This circular barrier is approximated by a 14-sided equilateral polygon from the outside.

i	1	2	3	4	5	6	7	8	9	10
x_i	8	5	6	-3	-6	-3	-5	-8	5	8
y_i	8	7	4	5	6	-4	-6	-8	-5	-8

Table 6.2: Example 2

In this experiment, the knot settled around $X_l^* = (-3.5, -0.4)$ (See Figure 6.10), when we let the knot free from the left side of the barrier, and $X_r^* = (3.4, 0.2)$ (See Figure 6.11), when we let the knot free from the right side of the barrier. The objective function for X_r^* is found as 88.4841 which is very close to the objective function reported by Bischoff and Klamroth (2007) (88.4689) for their best location when they approximated the circle from the outside with a 16-sided equilateral polygon.

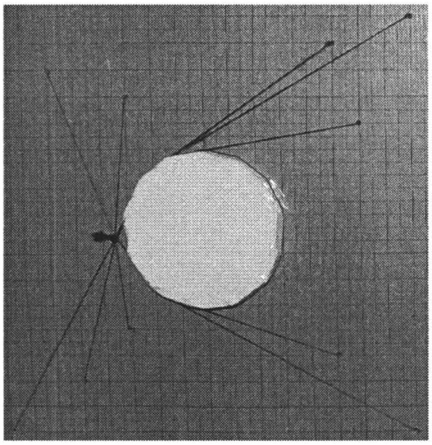


Figure 6.10: Example 2, Picture 1

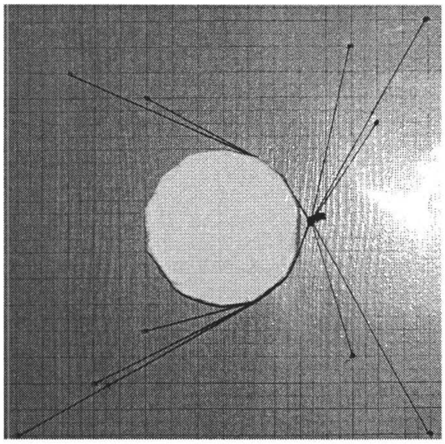


Figure 6.11: Example 2, Picture 2

6.5 Conclusions

We have shown that a modified Varignon frame can be used to find an approximate local optimal solution for some Weber problems in the presence of barriers. The underlying mathematical functions for these types of problems are in general non-convex and nonlinear, and it is difficult to find an optimal location. The method presented in this paper is an easy and traditional way of finding a 'good' solution in some circumstance. It is also a good way to illustrate the problem to lay persons.

Practical application is limited by factors such as friction, hole size, number of demand points, etc. Intervention such as selection of the intermediate points is another problem. If there is a limited number of intermediate points, these intermediate points can be selected through eye observation to find the shortest possible paths for a 'good' solution. Nevertheless, this approach can provide close to optimal solutions for problems involving barriers with a variety of convex polygonal shapes, which are normally an obstacle to providing analytical results.

Even though we have not discussed the use of the frame for problems with circular barriers, we may still be able to use the frame for these problems as circular shapes can be easily approximated by polygonal shapes. The frame is a valuable aid to visualization and could have applications in education and presentations. As a future research project, a computer program with a visual interface that uses the idea of equilibrium of force vectors could be developed. This would provide more accuracy and flexibility in applications.

Chapter 7

Conclusions and Future Research

This chapter summarizes the research performed in this thesis and suggests a number of research directions related to the problems explored.

7.1 Conclusions

This research examines four interrelated problems in the domain of restricted planar facility location and proposes new formulations and solution approaches.

Chapter 1 provides a brief introduction to planar facility location theory, and presents various distance and objective functions that are commonly used in the literature.

Chapter 2 presents literature reviews and classifications for planar facility location problems under uncertainty and planar facility location problems in the presence of barrier regions. The classification in the literature gives us a clear picture of the literature and allows us to identify the gaps in the literature for possible research directions.

Chapter 3 considers the problem of locating a facility on the plane and in the presence of a probabilistic line barrier whose location occurs by chance on a given horizontal route. The objective is to locate the facility such that the sum of the weighted expected distances between the facility and demand points is minimized in the presence of this barrier. We prove that, when the underlying probability distribution is Uniform, the objective function is a convex function of the new facility location. A solution algorithm is presented and possible extensions are discussed. The findings of this chapter have been published in the European Journal of Operational Research (Canbolat and Wesolowsky (2010)).

In Chapter 4, we start with the problem of locating a facility in a region where a fixed line barrier such as a borderline or a river divides the region into two. The sub-regions communicate with each other through a number of passage points located on the line barrier. First, we provide a new solution methodology to the previously studied minisum version of the problem. The problem is formulated as a mixed integer nonlinear programming (MINLP) program and an Outer Approximation (OA) algorithm is proposed. We show the efficiency of the algorithm on an example problem in the literature and through extensive computational work. We then apply the modified version of the OA algorithm to the minimax version of the problem and solve a number of randomly generated problems. We show that the OA algorithmic approach for both problem types is better than some general MINLP solvers, namely BARON and DICOPT. However, we also note that during the algorithmic process, we use a number of nonlinear programming (NLP) and mixed integer programming (MIP) solvers. Therefore, larger problem instances may still be difficult to solve.

Lastly, we propose using one-infinity distance norms instead of the Euclidean distance for a version of the problem where the locations of the passage points are also unknown. One-infinity distance norms approximate the original Euclidean distance problem and yield a linear model. We are planning to send the findings of this chapter to the journal of Computers and Operations Research.

In Chapter 5 we address the finite size barrier facility placement problem. Until recently, in facility location literature, barriers were thought of as static shapes that do not move. The robotics literature however has some research on planning the paths in the presence of moving obstacles (Latombe (1991)). But as far as we know, most of this research is dedicated to algorithmic approaches that aim for collision avoidance and there is no direct relation to facility location research. The finite size facility (acting as a barrier) placement problem has been of interest to some facility layout researchers including Savas et al. (2002), Sarkar et al. (2005), and Kelachankuttu et al. (2007). Our work is different from these studies in many aspects as outlined in the literature review.

In Section 5.2 we start with a simpler version of the problem given in Savas et al. (2002) by locating a barrier facility with a fixed orientation on a rectangular plane to minimize its interference to user-user interaction. We propose a solution algorithm that works in $O(n \log n)$ time where n is the number of existing facilities. In Section 5.3, we incorporate another objective into the problem. The maximum of the closest rectilinear distances from the demand points to the barrier is minimized. To the best of our knowledge, user-user and user-facility interactions under a minimax objective have not been explored. Also the 'closest distance family' has never been used for this type of problems despite its relative importance. We provide a mixed integer programming formulation for this new bi-objective problem. In Section 5.4 we develop a Simulated Annealing (SA) heuristic for an extension of the problem where expropriation of existing facilities with some cost is also possible.

The finite size facility placement problem has a practical importance in the urban planning context. Creation of public spaces in an existing downtown layout is a strategic decision that most municipalities may face. Locating such a public space inside an already filled up city layout may require expropriating some existing zones in exchange of finding a better place for the new public space. The resulting multiple criteria problem may produce bad decisions if not properly handled. For example, rezoning plans for downtown Brooklyn ignited a lot of controversy in 2004 where local residents and civic groups found the plans unacceptable in terms of the impacts of the plan on the surrounding residential areas, transportation, lack of open space, and displacement of existing residents and businesses' [1]. This clearly shows the practical importance of the problem. We aim to publish the findings of this chapter in the

[1] http://www.gothamgazette.com/article/landuse/20040119/12/841

Journal of Regional Science.

Finally, in Chapter 6, we present a new analog approach to the Weber problem in the presence of barriers by using a variant of the Varignon frame. We show through analysis that the same analog can also be used for Weber problems in the presence of convex polygonal barriers. The analog may have practical uses in education and presentations. It provides rapid solutions, allows for flexibility, enables one to visualize the problem and helps in conducting experimental what-if analysis. The approach also has some shortcomings. First, it is easily affected by physical conditions such as friction, hole size, etc. Also the decision of selecting the barrier intermediate points through which the string pass is sometimes cumbersome. When the number of barriers increases, this decision may not be made easily by eye observation. Nevertheless, this approach can provide close to optimal solutions in a short time and can allow the decision maker to conduct post-optimality analysis. We are hoping to publish these findings.

7.2 Future Research

Although the problems studied in this thesis are all in the same domain, they are not directly related to each other. Therefore we would like to discuss the possible future research directions for each chapter.

Chapter 3 is the first attempt to formulate a facility location problem in the presence of a probabilistic barrier. Therefore, we have made simplifying assumptions to facilitate our analysis. The first extension to the problem can be the consideration of other probability distributions for the location of the line barrier. The convexity of the objective function under a general probability distribution needs to be investigated in order to come up with a general statement about the convexity of the objective function. The probabilistic barrier can also be free of any restrictions, e.g. a barrier with a given shape and orientation can take place anywhere on the plane. These more advanced models will better represent real life problems but we presume that analytical solution methodologies for these models will be hard to develop. One way of tackling these advanced models is to use a simulation-optimization approach. There are commercial stochastic optimizers such as MS Excel add-in Risk Optimizer[2]. Risk Optimizer has been developed by the Palisade Decision Tools company to perform stochastic optimizations. It has been used to solve a variety of optimization problems. The tool combines Monte Carlo simulation techniques with a genetic algorithm heuristics for approximate optimization of mathematical models with random variables.

In Chapter 4 we discuss the problem of locating a facility in the presence of a line barrier with a number of passages. We discuss the problem with passages being on the line but the passages may not be arranged on the line in real life. A practical example for this problem involves border crossing points in Southern Ontario. International trade between the U.S. and Canada surpasses nearly 1.1 billion in goods alone that cross our border every day. In this context, logistics companies make strategic decisions such as on which side and where to build a transshipment point or a warehouse, as well as operational decisions such as which crossing point should be used given that each crossing point is a server with queuing issues. Furthermore, given the financial crises in the automotive industry, the companies may have to make simultaneous decisions of opening and closing of facilities. Opening and closing of facilities have been considered by Wang et al. (2003) for a facility location problem with budget constraints only.

For the problem family discussed in Chapter 5, we only consider the rectilinear distance norm. Depending upon the city road network, some other distance functions may better fit to the underlying problem. Therefore, for real life problems, it would be beneficial to conduct preliminary empirical study to determine the best fitting distance function to the problem. This brings the need for more studies on the problem using different distance functions. As it may be difficult to solve these new problems if the distance function is nonlinear, the BSSS algorithm can be investigated

[2]http://www.palisade.com/RISKoptimizer/

as an alternative solution methodology.

The analog approach suggested in Chapter 6 also requires further research. First, applicability of the frame to the facility location problems in the presence of circular and/or nonconvex barrier shapes needs to be explored. Second, as the analog approach can easily be affected by physical conditions, a computer program with a visual interface that uses the idea of equilibrium of force vectors could be developed in order to provide more accuracy and flexibility in applications.

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