UTILITY INDIFFERENCE VALUATION

OF

LIFE INSURANCE RISKS

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By

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Abstract

We address the problem of valuation of life insurance risks of different nature, market independent or equity–linked, under various assumptions regarding policyholders'mortality and the financial market. Given the incomplete nature of life insurance markets, an indifference valuation approach tailored to different models of the insurer's liability is applied. To be more specific, we propose three models for the insurer's liability: a single life insurance model, the individual risk model and the collective risk model. The last two models are generalizations of the aggregate loss models with the same name from actuarial mathematics.

First, we investigate the pricing problem of market independent life insurance risks under the assumption of random mortality, focussing on the effects of this latter assumption on the premium. We find that random mortality is an essential assumption especially when pricing in aggregate loss models. Then, we consider life insurance products with a more complex structure of the benefit, as equity–linked term life insurances. We price them via utility indifference in all liability models mentioned above, assuming deterministic mortality and a Black-Scholes market model. Comparing the results obtained, we observe that the collective risk model is computationally more efficient than the others, but at the cost of higher premium. Finally, we conclude by extending our pricing results for equity–linked term life insurance to a one factor stochastic volatility market model. We obtain that in a fast-mean-reverting volatility regime, the indifference premium can be well approximated by adjusted constant volatility results, previously derived.

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Introduction

The complexity of the life insurance products that have appeared on the market during these last decades have generated major pricing challenges to insurers. In this thesis, we address some of these challenges and try to answer them. It is fair to say that the complexity of these life insurance products arises from the risks embedded in their benefits, risks of different natures as we discuss below.

Clearly, all life insurance products embed mortality and/or longevity risk. Therefore, mortality models are essential for life insurance pricing and management. Mortality modeling constitutes the object of the first chapter of this thesis. Within this chapter, we do a survey of the development of mortality modeling, highlighting stochastic mortality models that admit affine mortality term structure.

Traditionally, life insurance was offering financial protection by means of a fixed and guaranteed lump-sum payable contingent on death or survival of the policyholder. The competition with other financial intermediaries, forced life insurers to introduce more attractive products, such as *equity–linked life insurances* that offer jointly mortality protection and equity investment opportunities. Accordingly, equity–linked life insurance products present both mortality and financial risk. The financial risk embedded in the benefit of these products challenged actuaries, since their standard actuarial pricing methods were not applicable. In chapter two, we present two financial methods for the pricing of equity–linked life insurance. The first approach was proposed by Brennan & Schwartz (1976) and Boyle & Schwartz (1977). The authors assume that the mortality risk is diversifiable and then, left only with the financial risk, they view equity–linked life insurance contracts as financial options and price them via Black-Scholes and Merton option pricing theory. An alternative approach for solving the pricing problem of equity-linked life insurance is *utility indifference pricing*. This approach requires solving two portfolio investment problems: "the insurer investment problem with no claim", the so called *Merton investment problem* and "insurer investment problem in the presence of insurance claim(s)". The first problem was initially solved by Merton (1969) and we recall it as a first illustration of using dynamic programming in solving portfolio investment problems. The second problem requires a specific model for the insurer loss and accordingly we proceed in introducing the liability models employed in this thesis. Essentially, these models represent generalizations of the individual and the collective risk models from actuarial mathematics.

The object of the third chapter is the pricing of market independent life insurance risks, specifically of pure endowments, term life insurances and endowments under the assumption of random mortality. This chapter represents a generalization of Young & Zariphopoulou (2002) and Ludkovski & Young (2008). We extend the results of Young & Zariphopoulou (2002) in several directions: first, we assume random mortality as opposed to deterministic mortality and then, we determine both the lump-sum and continuous premium for the contracts mentioned above, in all loss models. Clearly, our work has certain points in common with Ludkovski & Young (2008). The latter authors, consider the problem of calculating the lump-sum premium for pure endowments and life annuities under stochastic mortality and interest rates in a single life insurance model and in the individual risk model. However, the individual risk model proves to be not a feasible model to work with, since as we will show in the second section of chapter three, the premium for k life-insurance contracts require solving recursively k linear partial differential equations. In contrast, the collective model, that we propose, is numerically more efficient, since the premium for k life insurance contracts require solving only one linear partial differential equation.

In chapter four, we employ the utility indifference approach for pricing equity-linked term life insurances in a single life and in the individual and collective risk model. For tractability, we assume deterministic mortality and a Black-Scholes financial market model. The results obtained represent a generalization of Young (2003) to the aggregate liability models just mentioned. Specifically, we obtain that in all models, the premium per risk satisfies a second order partial differential equation similar to the Black-Scholes equation while at the same time reflecting the mortality risk and the risk preferences of the insurer.

Chapter five represents a natural extension of chapter four, to a stochastic volatility financial market model. Given the numerical efficiency of the collective risk model, in this chapter we choose to model from start the insurer's losses using this model. We find that the premium solves a nonlinear second order partial differential equation. However, we do not pursue in calculating the premium numerically, but instead following the singular perturbation technique proposed by Sircar & Zariphopoulou (2004), we derive an asymptotic approximation of the premium in a fast-mean-reverting volatility regime.

Chapter 1

Mortality modeling

Life insurance risks arise from the uncertain nature of human lifetime. Accordingly, the random variable of time until death and the corresponding mortality models became essential for life insurance pricing and management. In this chapter we describe several aspects of the development of mortality modeling, focusing on the important contributions in this field. We start by introducing the most important life functions in a static setting and then continue with life table models and law based mortality models. However, mortality changes over time and we illustrate this assertion by analyzing the U.S. mortality experience. This leads naturally to dynamic mortality and the need for mortality projections. Finally, we conclude the chapter with continuous time stochastic mortality models, highlighting the models that admit an affine mortality structure.

1.1 Death and survival probabilities

Let us begin by considering an individual aged a_0, generic for a cohort or population of this age. From now on, we refer to this individual by (a_0). Further, let τ_{a_0} denote the time until death of (a_0). Then, the cumulative distribution function of τ_{a_0} is denoted and defined by

$$F_{\tau_{a_0}}(t) = P(\tau_{a_0} \leq t), \tag{1.1}$$

that is $F_{\tau_{a_0}}(t)$ represents the probability that (a_0) will die within t years. Here, we assume that $F_{\tau_{a_0}}$ is continuous and has a probability density function denoted by $f_{\tau_{a_0}}$.

In this thesis, we adopt the international actuarial community notations. Accordingly, we use the symbol ${}_tq_{a_0}$ for the probability that (a_0) will die within t years. Similarly, we use the symbol ${}_tp_{a_0}$ for the probability that (a_0) will survive more then t years. Naturally,

$${}_tp_{a_0} = 1 - {}_tq_{a_0}. \tag{1.2}$$

Also, we mention that it is implicitly assumed that

$$\lim_{t \to \infty} {}_tp_{a_0} = 0 \text{ and } {}_0p_{a_0} = 1, \tag{1.3}$$

that is (a_0) will not live forever and respectively (a_0) is alive at the current time. Observe that the first condition is required for a correct definition of the cumulative distribution function $F_{\tau_{a_0}}$.

Then, we denote by ${}_sq_{a_0+t}$ and ${}_sp_{a_0+t}$ the probabilities of death within

s years and of survival more than s years respectively, conditional on survival at age $a_0 + t$. We have

$$ {}_s q_{a_0+t} = P(t < \tau_{a_0} \leq s + t | \tau_{a_0} > t) = \frac{F_{\tau_{a_0}}(s+t) - F_{\tau_{a_0}}(t)}{1 - F_{\tau_{a_0}}(t)} \tag{1.4} $$

and

$$ {}_s p_{a_0+t} = P(\tau_{a_0} > t + s | \tau_{a_0} > t) = \frac{1 - F_{\tau_{a_0}}(s+t)}{1 - F_{\tau_{a_0}}(t)}. \tag{1.5} $$

A useful identity in terms of the survival probabilities is

$$ {}_{t+s} p_{a_0} = P(\tau_{a_0} > t + s) = \frac{1 - F_{\tau_{a_0}}(s+t)}{1 - F_{\tau_{a_0}}(t)} (1 - F_{\tau_{a_0}}(t)) = {}_t p_{a_0} {}_s p_{a_0+t}. \tag{1.6} $$

Next, let ${}_{t|s} q_{a_0}$ be the probability that (a_0) will survive t years and die in the following s years. Then

$$ {}_{t|s} q_{a_0} = P(t < \tau_{a_0} < t + s) = {}_t p_{a_0} - {}_{t+s} p_{a_0}. \tag{1.7} $$

Using (1.6) we can further write

$$ {}_{t|s} q_{a_0} = {}_t p_{a_0} - {}_t p_{a_0} {}_s p_{a_0+t} = {}_t p_{a_0} {}_s q_{a_0+t}. \tag{1.8} $$

For convenience, we follow the convention to omit the prefix in the survival and death probabilities if it equals one. Accordingly, in this case we write $p_{a_0}, q_{a_0}, {}_{|s} q_{a_0}$.

At this point, we introduce a central life function for mortality modeling. This is the instantaneous death rate, referred to here as force of mortality.

We define the force of mortality of (a_0) at age $a_0 + t$ by

$$\lambda(a_0 + t) = \lim_{\delta t \to 0} \frac{P(t < \tau_{a_0} \leq t + \delta t | \tau_{a_0} > t)}{\delta t}. \tag{1.9}$$

We have

$$\begin{aligned} \lambda(a_0 + t) &= \lim_{\delta t \to 0} \frac{F_{\tau_{a_0}}(t + \delta t) - F_{\tau_{a_0}}(t)}{\delta t(1 - F_{\tau_{a_0}}(t))} \\ &= \frac{1}{1 - F_{\tau_{a_0}}(t)} f_{\tau_{a_0}}(t) \\ &= -\frac{d}{dt} \log(1 - F_{\tau_{a_0}}(t)). \end{aligned}$$

Integrating both sides, after some straightforward calculations, we obtain

$$_tp_{a_0} = e^{-\int_0^t \lambda(a_0+s)ds}. \tag{1.10}$$

As can be observed from its definition, the force of mortality is positive on all its domain. Additionally, as a direct consequence of the fact that $\lim_{t\to\infty} {}_tp_{a_0} = 0$, the force of mortality has to satisfy

$$\int_0^{\infty} \lambda(a_0 + s)ds = \infty. \tag{1.11}$$

1.2 Life Tables

The first specification of the distribution of τ_{a_0} was done through a life table by Sir Edmund Halley in 1693. A life table provides a discrete distribution of the random variable τ_{a_0} by the specification of the death probabilities q_{a_0} for ages $a_0 = 0, 1 \ldots \omega$. Here, ω denotes the limiting age of the life table and is typically between 110 and 120. For the limiting age, the corresponding death probability q_{a_0} is taken to equal 1. Besides the death probabilities q_{a_0}, a life table might also contain tabulations of the life functions l_{a_0}, d_{a_0}, e_{a_0} defined below and possibly other functions.

When constructing a life table, one starts with a group of newborns, say $l_0 = 100000$. Essential to the construction of a life table are the death probabilities q_{a_0} for $a_0 = 0, 1, \ldots \omega - 1$. Obtaining estimations of these probabilities is a statistical problem belonging to the area of survival analysis. We assume here that these death probabilities are known and refer the interested reader to Gerber (1997), chapter 11, for a review of the various estimation methods of these probabilities.

We denote by l_{a_0} the expected number of survivors to age a_0. Consequently, we have

$$l_{a_0} = l_0 S(a_0), \tag{1.12}$$

where $S(a_0) = P(\tau_0 > a_0) = {}_{a_0}p_0$. The function $S(a_0)$ is called the *survival function.*

Next, d_{a_0} denotes the expected number of deaths between ages a_0 and $a_0 + 1$. The probability that a newborn dies between ages a_0 and $a_0 + 1$ is $S(a_0) - S(a_0 + 1)$ and thus we can express d_{a_0} as follows

$$d_{a_0} = l_0 \left(S(a_0) - S(a_0 + 1) \right) = l_{a_0} - l_{a_0+1}. \tag{1.13}$$

An important measure of the level of health of a population is the *complete expectation of life.* The complete expectation of life of an age a_0 is denoted by e_{a0} and it is defined as

$$e_{a0} = E[\tau_{a0}]. \tag{1.14}$$

Further, we can write

$$e_{a0} = \int_0^\infty t f_{\tau_{a0}}(t)dt = \int_0^\infty t d(-{}_tp_{a0}) \tag{1.15}$$

and using integration by parts we obtain

$$e_{a0} = -t_t p_{a0}|_0^\infty + \int_0^\infty {}_tp_{a0}dt = \int_0^\infty {}_tp_{a0}dt. \tag{1.16}$$

Notice that the life functions l_{a0}, d_{a0} and e_{a0} can be all calculated recursively for all ages using only the one year death probabilities q_{a0}.

The *central death rate* is defined by

$$m_{a0} = \frac{l_{a0} - l_{a0+1}}{L_{a0}}, \tag{1.17}$$

where L_{a0} denotes the total expected number of years lived between ages a_0 and $a_0 + 1$ by survivors of the initial group of l_0 lives. L_{a0} can be expressed as follows

$$L_{a0} = l_{a0+1} + \int_0^1 t l_{a0+t}\lambda(a_0 + t)dt = l_{a0+1} - \int_0^1 t dl_{a0+t} \tag{1.18}$$

$$= l_{a0+1} - t l_{a0+t}|_0^1 + \int_0^1 l_{a0+t}dt = \int_0^1 l_{a0+t}dt \tag{1.19}$$

At this point, we would like to remark that depending on the data used, life tables can be classified as *period* or *cohort* life tables.

A period life table is generated using the mortality experience of a population over a short period of time, typically 1 to 3 years. In this type of table, the data for each age corresponds to different cohorts at a certain moment in time, common for all ages. Consequently, a period life table can be regarded as an excellent model for describing the mortality level of a population. However, it does not accurately reflect any particular cohort mortality.

On the other hand, cohort life tables are generated by using the entire experience of a generation. Since they require reliable data for a long period of time, cohort life tables are rare. The most recent complete cohort life tables are for generations born around 100 years ago. Cohort life tables are very important since they are more appropriate than period life tables for insurance pricing purposes. For example, if an insurance company has to calculate the premium for a life insurance product to be sold to (a_0) in the year y, then the relevant life table is the cohort life table for the year $y - a_0$.

We conclude by pointing out that a life table does not capture the entire information about an individual mortality since for example, it omits fractional age death rates and death rates for fractional durations. However, life tables offer a rich collection of data about human mortality, augmented periodically at census years. This fact together with the increasing reliability of mortality data, makes a life table a very popular mortality model in life insurance practice.

1.3 Trends in mortality

In what follows we present some of the most important tendencies of human mortality over the last century. For our analysis, we use American male mortality data, from the U.S. Social Security Area life tables, Actuarial Study No. 120.

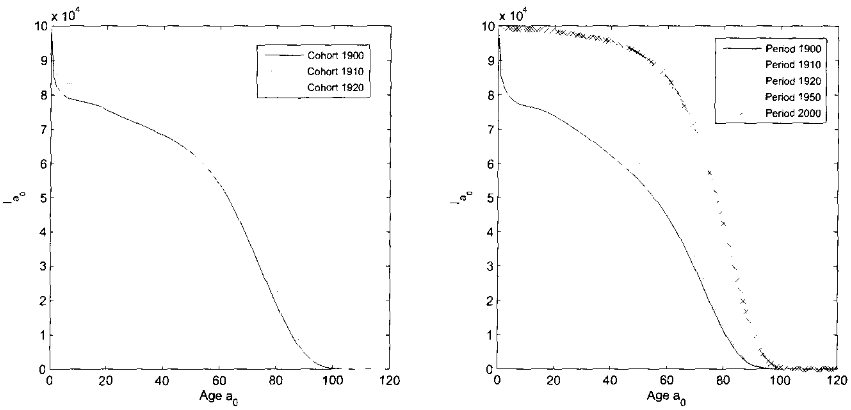

Figure 1.1: Survival function l_{a_0} for American males from selected generations/periods.

Figure (1.1) shows that the survival function l_{a_0} increases with generation/period for all ages. Notice that when passing from a generation/period to a successive one, the concentration of deaths at old ages increases. Consequently, the shape of l_{a_0} becomes increasingly rectangular. At the same time, observe the movement of l_{a_0} to very old ages. These two phenomena are known in the actuarial literature as "rectangularization" and "expansion" of the survival function.

In terms of the life function d_{a_0}, the features mentioned above are illustrated by the dispersion of the expected number of deaths around the mode that reduces with each generation/period and by the movement of the mode towards older ages.

An interesting fact that we would like to remark is the way that mortality shocks are reflected in the curve of deaths d_{a_0}. Obviously, when using period life tables for calculating d_{a_0}, the shocks in mortality are reflected by

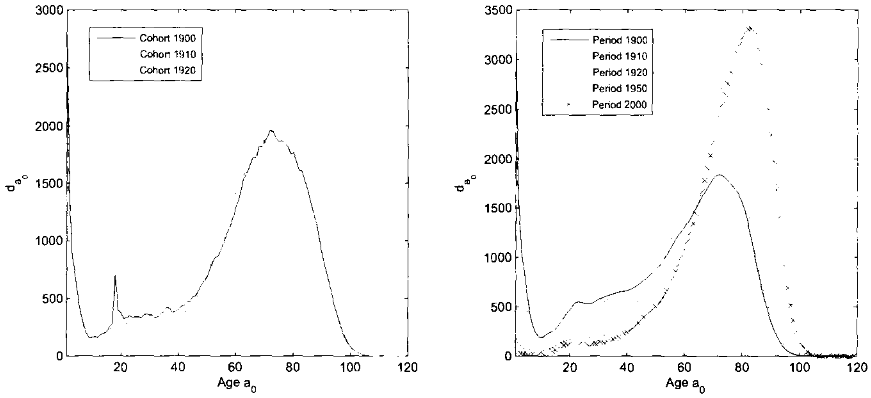

Figure 1.2: Curve of death d_{a0} for American males from selected generations/periods.

the entire curve of deaths. On the other hand, when using cohort life tables they are reflected as jumps in the curve of deaths as illustrated by figure (1.2). Here, the mortality shock is generated by the pandemic flu of 1918 and concretely, the jump appears in the curve of deaths for Cohort 1900.

Next, we signal the general decline in the death probabilities q_{a_0} with each generation/period. As figure (1.3) shows, the decline is uniform up to age 90.

The general trend in human mortality can also be illustrated via the complete expectation of life e_{a_0}. As can be observed from figure (1.4), both life expectancies e_0 and e_{65} improved during this century. Especially, notice the huge decline in infant mortality.

As figure (1.4) shows, life expectancy exhibits more fluctuations from year to year when based on period life tables. This happens because mortality shocks affect the entire population for a period of one or two years but affect just one or two years of the mortality experience of the cohorts alive during that period.

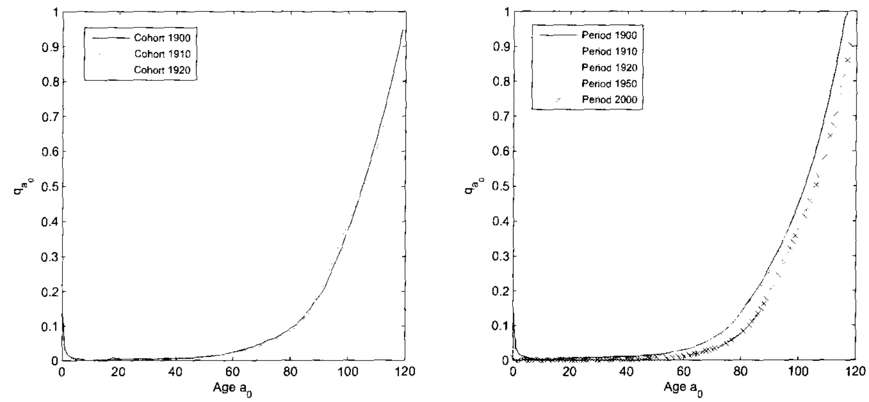

Figure 1.3: Death probability q_{a_0} for American males from selected generations/periods.

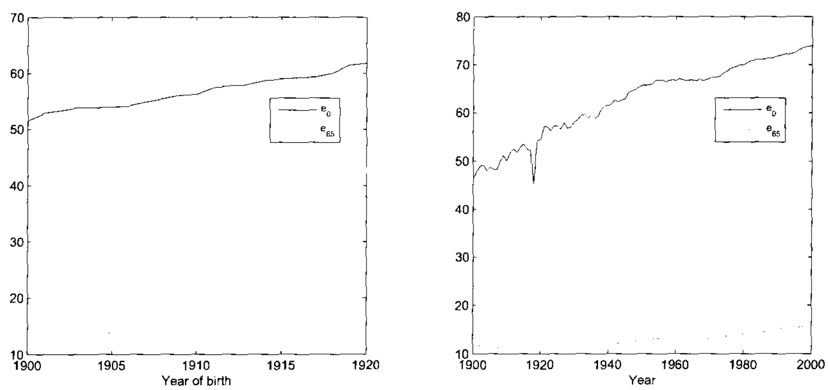

Figure 1.4: Life expectation at age 0 and at age 65 for American males, based on cohort (left) and period (right) life tables.

1.4 Law based mortality models

Many fundamental concepts in science are expressed through a law and we naturally question if there also exists a law for human mortality. Actually, this was a subject of continuous interest in actuarial literature and probably the first answer to our question was the one of Abraham De Moivre (1729), who proposed the following law for the force of mortality

$$\lambda(a_0) = \frac{1}{\omega - a_0}, \quad 0 \leq a_0 < \omega \tag{1.20}$$

where ω is the limiting age.

A century later, Benjamin Gompertz observing mortality data, suggested that a "law of geometric progression pervades" in mortality over a certain age and proposed a force of mortality of the form

$$\lambda(a_0) = \alpha e^{\beta a_0} \tag{1.21}$$

where $\alpha, \beta > 0$.

Gompertz's mortality law generates death rates that fit well to the actual ones for adult and old ages. However, Gompertz's law doesn't accurately capture the mortality at young and extremely old ages.

In 1860, William Makeham slightly generalized Gompertz's law, by adding a constant to better fit the mortality at extremely old ages. Specifically, Makeham's mortality law is

$$\lambda(a_0) = \gamma + \alpha e^{\beta a_0} \tag{1.22}$$

where $\alpha, \beta > 0$ and $\gamma \geq 0$.

Another important generalization of Gompertz's law is the one proposed by Thorvald Thiele in 1867. To capture the human mortality over the whole life, Thiele postulated the mortality law

$$\lambda(a_0) = \alpha_1 e^{-\beta_1 a_0} + \alpha_2 e^{-\beta(a_0 - \eta)^2} + \alpha_3 e^{\beta_3 a_0} \tag{1.23}$$

where α_1, β_1, α_2, β_2, η, α_3 and β_3 are positive constants. The first term in (1.23) models the decreasing mortality at very young ages, the second one the mortality hump at young-adult ages while the third one coincides with the Gompertz's law and as mentioned, captures the mortality at adult and old ages.

A model similar to the one of Thiele, was proposed in 1980 by Heligman and Pollard for the mortality odds, namely

$$\frac{q_{a_0}}{p_{a_0}} = A^{(a_0+B)^C} + De^{-E(\ln a_0 - \ln F)^2} + GH^{a_0}. \tag{1.24}$$

The significance of the three terms is the same as in Thiele's mortality law.

Observe that the last two mortality laws are more complex and reflect all age segments. However, they depend on many parameters that are highly correlated and therefore they are hard to fit to experienced mortality.

A criticism of Gompertz's mortality and its generalizations relates to the asymptotic behavior of the force of mortality. For all these models, it holds that

$$\lim_{a_0 \to \infty} \lambda(a_0) = \infty. \tag{1.25}$$

Naturally, this contradicts the empirical evidence. An interesting demographic argument, regarding the asymptotic nature of the force of mortality is that it is slowly increasing at extremely old ages, having a rather flat shape.

An essential feature of these models is that they are age continuous. They represent an important step forward from the life table model, having the advantage of being flexible, compact, highly interpretable and generalizable.

1.5 Dynamic deterministic mortality

So far we have been regarding mortality as dependent on age only and accordingly the mortality models introduced were all static. However, analyzing the mortality trends, we observed that human mortality changes not just from one age to another but also it changes over time. This suggests that human mortality has to be viewed as a function depending on both age and time and thus, modeled dynamically.

Clearly, in this context, the time until death random variable and all the life functions introduced in section 1.1 will be dependent on both age and time. Next, we define the most important life functions in a dynamic context.

We consider an individual age a_0 at time t and define his force of mortality at age $a_0 + u$ and time $t + u$ as follows

$$\lambda(a_0 + u, t + u) = \lim_{\delta u \to 0} \frac{P(u < \tau_{a_0,t} \leq u + \delta u | \tau_{a_0,t} > u)}{\delta u} \tag{1.26}$$

Here $u \geq 0$ and $\tau_{a_0,t}$ denotes the time until death of the individual under consideration.

Therefore, we have

$$\begin{aligned} \lambda(a_0 + u, t + u) &= \lim_{\delta u \to 0} \frac{F_{\tau_{a_0,t}}(u + \delta u) - F_{\tau_{a_0,t}}(u)}{(1 - F_{\tau_{a_0,t}}(u))\delta u} \\ &= \frac{f_{\tau_{a_0,t}}(u)}{1 - F_{\tau_{a_0,t}}(u)} = -\frac{d}{du} \log(1 - F_{\tau_{a_0,t}}(u)), \end{aligned}$$

where $F_{\tau_{a_0,t}}$ and $f_{\tau_{a_0,t}}$ denote the cumulative distribution function and the probability density function of the random variable $\tau_{a_0,t}$.

Consequently,

$$F_{\tau_{a_0,t}}(u) = 1 - e^{-\int_0^u \lambda(a_0+v,t+v)dv}.$$

1.6 Stochastic mortality

Maybe the beauty of the future consists of its randomness. The future may reserve unfortunate events such as wars, dangerous diseases, natural catastrophes but can also be better in some aspects; for example the continuous developments in science may provide the necessary technology and cure for many diseases, the lifestyle may improve around the world and past experience may help political leaders to be more wise in their decisions, avoiding wars and use of nuclear weapons.

These are just a few of the factors that suggest a random future mortality. Thus, a realistic mortality model should be both dynamic and stochastic. In what follows, we follow Cairns, Blake & Dowd (2008) and define the relevant life functions within a stochastic and dynamic setting for mortality.

We have shown that under deterministic mortality, the probability that an individual of age a_0 at time 0 will survive to age $a_0 + t$ at time t, is

$$P(\tau_{a_0,0} > t) = e^{-\int_0^t \lambda(a_0+s,s)ds}. \tag{1.27}$$

Clearly, when assuming random mortality, the future evolution of mortality intensity is unknown and one may argue that the survival probability above should be expressed by an expected value. Indeed, the probability that an individual age a_0 at the current time 0 will survive to age $a_0 + t$ at time t, can be expressed as follows

$$p(a_0, 0, t) = E[1_{\tau_{a_0,0}>t}] = E[E[1_{\tau_{a_0,0}>t}|\mathcal{F}_t^\lambda]]. \tag{1.28}$$

Here $\mathcal{F}_t^\lambda$ captures all the information about the mortality up to and including time t, for all ages. We assume that given $\mathcal{F}_t^\lambda$, it is possible to estimate accurately the mortality intensity up to time t. In reality, the mortality data is not readily available and an insurer will not always have enough data for an accurate estimation. However, for simplicity, we make this assumption. Accordingly, $E[1_{\tau_{a_0,0}>t}|\mathcal{F}_t^\lambda] = P(\tau_{a_0,0} > t|\mathcal{F}_t^\lambda)$ coincides with the survival probability in a deterministic setting and as expected we have

$$p(a_0, 0, t) = E[e^{-\int_0^t \lambda_{a_0+s,s}ds}], \tag{1.29}$$

in which $\lambda_{a_0+s,s}$ denotes the mortality intensity of the individual under consideration at age $a_0 + s$ and time s.

Now, let us derive the survival probability that an individual age a_0 at time 0, still alive at current time t will survive until time T. We have

$$\begin{aligned} p(a_0, t, T) &= E[1_{\tau_{a_0,0}>T}|\tau_{a_0,0} > t, \mathcal{F}_t^\lambda] \\ &= E[E[1_{\tau_{a_0,0}>T}|\tau_{a_0,0} > t, \mathcal{F}_T^\lambda]|\mathcal{F}_t^\lambda] \\ &= E[e^{-\int_t^T \lambda_{a_0+s,s}ds}|\mathcal{F}_t^\lambda] \end{aligned}$$

Further, observe that Jensen's inequality implies

$$p(a_0, 0, t) = E[e^{-\int_0^t \lambda_{a_0+s,s}ds}] \geq e^{-\int_0^t E[\lambda_{a_0+s,s}]ds}. \tag{1.30}$$

Consequently, if an insurer considers a deterministic estimate at time 0 of the future mortality intensity (for example its first moment) instead of a stochastic mortality intensity, then the survival probabilities obtained will be smaller than the actual ones and this will generate premiums too low for life products contingent on survival as pure endowments or annuities.

1.7 Mortality Projections

Mortality data for a population or cohort is readily available for past years and naturally one may wonder if this data can be used to infer future mortality. In mortality literature this subject is referred to as mortality projection.

One of the first projected mortality models was proposed by Blaschke in 1923. First, a dynamic version of the Makeham's law was defined as follows

$$\lambda(a_0, y) = \gamma(y) + \alpha(y)\beta(y)^{a_0}, \tag{1.31}$$

where y stands for calendar year.

The parameters $\alpha(y), \beta(y)$ and $\gamma(y)$ are estimated by fitting the death probabilities within one year corresponding to the calendar year y, $q_{a_0}(y)$, to the actual rates from the period life table for year y. Then, by extrapolation the values of the parameters for future years y of interest are obtained. This method is called "vertical" projection since the fitting of the theoretical death rates is done using the columns of the matrix $M_{a_0,y} = q_{a_0}(y)$.

Notice that this is a parametric mortality projection model and require a relatively small number of parameters for estimation. Certainly, that is an advantage. However, the model provides an extrapolative forecasting method but no one can be certain that the historical trend will continue in the future. Moreover, the independent extrapolation of the parameters is not able to capture the correlation that develops over time between them, leading to unrealistic future death rates.

One year later, the Institute of Actuaries in London proposed a projection mortality model for which the probability of death within one year of an age a_0 in year y, is given by

$$q_{a_0}(y) = A_{a_0} + B_{a_0} C_{a_0}^y \tag{1.32}$$

In this case, the model parameters are estimated by fitting the corresponding mortality profiles age by age. Since in this case the parameters are estimated using the rows of the matrix $M_{a_0,y}$, this method is referred to as "horizontal" projection. Notice, that (1.32) is a non-parametric projection model.

Also in this case, the projected mortality rates depend strongly on the trend within the fitting period. Another disadvantage of the model is the large number of parameters to be estimated, equal to the number of age groups times the number of parameters in each formula.

These models are both deterministic and thus unable to reflect the uncertain nature of future mortality. To overcome this disadvantage, L.Carter and R.D.Lee proposed in 1992 a stochastic mortality model for projecting the mortality in U.S. The model describes the central death rate of an individual age a_0 at time t in the following way:

$$\ln(m_{a_0,t}) = a_{a_0} + b_{a_0} k_t + \varepsilon_{a_0,t}, \tag{1.33}$$

where a_{a_0}, b_{a_0} and k_t are positive parameters and $\varepsilon_{a_0,t}$ are the error terms.

The signification of the parameters is as follows: a_{a_0} represents an average log mortality rate over time at age a_0, k_t is a stochastic process, sometimes referred to as mortality index measuring the general speed of mortality (improvement) over time, while b_{a_0} describes the way in which mortality varies at age a_{a_0} as a reaction to changes in the mortality index. The error terms $\varepsilon_{a_0,t}$ describe the age-time uncertainty not captured by the model and their statistical properties are estimated from the data.

Notice that the model (1.33) is invariant with respect to the transformations

$$\{a_{a_0}, b_{a_0}, k_t\} \longrightarrow \{a_{a_0}, cb_{a_0}, \frac{1}{c}k_t\} \quad \forall c \in \mathbb{R} - \{0\}$$
$$\{a_{a_0}, b_{a_0}, k_t\} \longrightarrow \{a_{a_0} - cb_{a_0}, b_{a_0}, k_t + c\} \quad \forall c \in \mathbb{R}.$$

Consequently the model admits more than one parametrization. For a unique parametrization, Lee and Carter impose the constraints

$$\sum_{a_0} b_{a_0} = 1 \quad \sum_{t=1}^{T} k_t = 0. \tag{1.34}$$

The constraint for the process k_t, implies the least square estimator

$$\hat{a}_{a_0} = \frac{1}{T}\sum_{t=1}^{T} \ln(\hat{m}_{a_0,t}), \tag{1.35}$$

where $\hat{m}_{a_0,t}$ are estimations of central death rates at age a_0 and time t.

Lee and Carter estimate the parameters b_{a_0} and k_t using the method of Singular Value Decomposition (SVD) applied to the matrix $R = \ln(\hat{m}_{a_0,t}) - \hat{a}_{a_0}$.

Then, they perform a "second stage estimation" that re-estimates k_t using the existing estimation for a_{a_0} and b_{a_0}. This second stage estimation is such that the total number of deaths for the year in observation is equal to that estimated from the model.

After this second stage estimation, Lee and Carter observe for different sets of mortality data that k_t declines linearly over time and has relatively constant variance. After testing several ARIMA specifications, Lee and Carter conclude that a random walk with drift is the most appropriate model for their data. Accordingly, they model k_t as follows

$$k_t = k_{t-1} + u + \xi_t \tag{1.36}$$

where u is a constant and $\xi_t \sim \mathcal{N}(0, \sigma_k^2)$. This variance of ξ_t shows the uncertainty of forecasting k_t over any time horizon.

Then, the Box-Jenkins approach is used to fit the ARIMA model to the empirical k_t data. Finally, the projected k_t together with the estimations for a_{a_0} and b_{a_0} are used to obtain forecasts of the central death rates and then of other life functions.

Currently, the Lee-Carter model is used to forecast the population mortality of many countries. The model has very appealing features: it is parsimonious, the parameters are easily interpretable thereby allowing further generalizations. Also, observe that the parameters are estimated together, eliminating the scenario of having implausible future death rates. Finally, the model is stochastic, generating stochastic projection intervals instead of point estimates as in the case of deterministic models.

However, the model has some limitations:

- Lee Carter model provides a extrapolative method for forecasting mortality and thus produce good forecasts as long the historic mortality trend continue in the future.

- Lee-Carter model gives a description of a population mortality considering a single mortality index, this meaning that the changes in mortality for all ages are perfectly correlated.

- Many people argue that the forecasted intervals for the projected central death rates are too narrow (Alho (1992))

- Observe that

$$d\ln(m_{a_0,t}) = \frac{dk_t}{dt} b_{a_0} = u b_{a_0}. \tag{1.37}$$

  Thus, the central death rates decline at their own exponential constant rate. However, Horiuchi and Wilmoth (1995) show that now in some countries mortality at older ages declines more rapidly than at lower ages, reversing the historical pattern. This suggests that the coefficients b_{a_0} change over time.

- As Cairns, Blake & Dowd (2008) illustrates, the model gives a poor fit for countries with pronounced cohort effect.

Many authors have proposed extensions of the Lee-Carter model, trying to answer to these criticisms. The possibility of imperfect correlations of mortality improvements was considered by Renshaw & Haberman (2003) who introduced a second time dependent factor to the model. Renshaw & Haberman (2006) propose an extension of the Lee-Carter model that incorporates a cohort effect.

1.8 Continuous time stochastic mortality models

One of the first continuous time mortality models was proposed by Milevky & Promislow (2001). They model the mortality intensity as follows

$$\lambda_t = \lambda_0 e^{gt+\sigma Y_t}, \quad \lambda_0, g, \sigma > 0. \tag{1.38}$$

Here, Y_t is an Ornstein-Uhlenbeck process with dynamics

$$\begin{cases} dY_t = -bY_t dt + dB_t \\ Y_0 = 0. \end{cases} \tag{1.39}$$

where B_t is standard Brownian motion.

Clearly, the model is an extension of Gompertz's law that allows for random future mortality. Given the mean reverting nature of the process Y, the mortality model (1.38) it is referred to in literature as "mean reverting Brownian Gompertz" (MRBG).

Observe that Y is mean reverting to a long run mean equal to 0 and mean reverting speed equal to b. Solving the stochastic differential equation (1.39), we obtain

$$Y_t = \int_0^t e^{-b(t-s)} dB_s. \tag{1.40}$$

Applying Dambis, Dubins-Schwartz Theorem, we obtain that there exists a Brownian motion $\tilde{B}_t$ such that $Y_t = \tilde{B}_{[Y,Y](t)} = \tilde{B}_{\sigma_Y^2}$, where

$$\sigma_Y^2 = \frac{1 - e^{-2bt}}{2b} \tag{1.41}$$

Accordingly, Y is a mean reverting process with variance smaller than that of the Brownian motion B_t. In particular, when $b \to 0$, we have $Y_t = B_t$ and

$$E[\lambda_t] = \lambda_0 e^{gt} M_{B_t}(\sigma) = \lambda_0 e^{gt + \frac{\sigma^2}{2} t} \tag{1.42}$$

that is, the expected mortality intensity coincides with the Gompertz's law.

Due to the mean-reverting nature of the process Y, when this takes negative values, it has the tendency to go up to the long run mean, which is 0. Accordingly, the mortality intensity has the tendency to go up to the Gompertz curve. Similarly, when $Y_t > 0$, then the process Y has the tendency to go down to 0 and consequently, the mortality intensity will have the tendency to go down to the Gompertz curve. Thus, λ_t randomly fluctuates around the Gompertz curve and therefore the model captures mainly unsystematic mortality risk. Systematic deviations may occur and they have to be captured by a mortality model. With this in mind, Ballotta & Haberman (2006) extend the Milevsky and Promislow model.

Ballotta & Haberman (2006) model the mortality intensity of an individual age a_0 at time 0 by a reduction factor model of the form

$$\lambda_{a_0+z,z} = \lambda_{a_0+z,0} RF(a_0 + z, z) \tag{1.43}$$

Here, the mortality for the age $a_0 + z$ and for the base year (year 0) is given

$$\lambda_{a_0+z,0} = a_1 + a_2 R + e^{b_1 + b_2 R + b_3 (2R^2 - 1)}, \quad R = \frac{a_0 + z - 70}{50} \text{ for } a_0 \geq 50. \tag{1.44}$$

Then, the factor $RF(a_0 + z, z)$ describes the change in mortality from time

0 to time z for an individual aged $a_0 + z$ and it will be referred as reduction factor. Concretely, Ballotta and Haberman take the reduction factor of the form

$$RF(a_0 + z, z) = e^{(\alpha+\beta(a_0+z))z+\tilde{\sigma}Y_z}, \tag{1.45}$$

where Y_z is modeled by an Ornstein-Uhlenbeck process of the form (1.39). Following the same arguments as in the Milevsky and Promislow mortality model, one can argue that the model (1.43) captures unsystematic mortality risk. Further, to capture systematic mortality risk, Haberman and Ballotta add another component to the mortality model (1.43); specifically, they consider the mortality intensity process $\lambda_{a_0+z,z;H(a_0)}$ dependent not just on age and time but also on a particular belief, hypothesis $H(a_0)$ regarding the future mortality trend for individuals aged a_0 at time 0.

Next, following Dahl (2004) we introduce affine mortality models.

Definition 1.8.1. If for a fixed cohort age a_0 at time 0, the survival probabilities $p(a_0, t, T)$ have the form

$$p(a_0, t, T) = e^{A(a_0,t,T)-B(a_0,t,T)\lambda_{a_0+t,t}} \tag{1.46}$$

for deterministic functions $A(a_0, t, T)$ and $B(a_0, t, T)$, then the model for the mortality intensity of the given cohort is said to have an affine mortality structure. Moreover, if (1.46) holds for all admissible ages, the model is said to have an affine mortality structure.

Observe that this definition is an analog in terms of mortality intensities of the affine term structure from interest rate theory. In fact, the idea of affine mortality models was motivated by the analogy between the survival probabilities $p(a_0, t, T)$ and zero-coupon bond prices. Taking advantage of this analogy, one can use results from interest rate theory to calculate survival probabilities.

The following proposition is an analogue of Proposition 17.2 (Affine term structure) from Björk (2004).

Proposition 1.8.1. *If the mortality intensity of a given cohort aged a_0 at time 0 is given by*

$$d\lambda_{a_0+t,t} = \mu^\lambda(a_0, t, \lambda_{a_0+t,t}) + \sigma^\lambda(a_0, t, \lambda_{a_0+t,t})dW_t^\lambda \tag{1.47}$$

where W^λ is a standard Brownian motion and μ^λ and σ^λ have the form

$$\begin{aligned} \mu^\lambda(a_0, t, \lambda_{a_0+t,t}) &= \alpha(a_0, t)\lambda_{a_0+t,t} + \beta(a_0, t) \\ \sigma^\lambda(a_0, t, \lambda_{a_0+t,t}) &= \sqrt{\gamma(a_0, t)\lambda_{a_0+t,t} + \delta(a_0, t)} \end{aligned}$$

then the model admits an affine mortality term structure of the form (1.46), *where A and B satisfy the system of Ricatti equations*

$$\begin{cases} B_t(a_0, t, T) + \alpha(a_0, t)B(a_0, t, T) - \frac{1}{2}\gamma(a_0, t)B^2(a_0, t, T) = -1 \\ B(a_0, T, T) = 0 \end{cases}$$

$$\begin{cases} A_t(a_0, t, T) = \beta(a_0, t)B(a_0, t, T) - \frac{1}{2}\delta(a_0, t)B^2(a_0, t, T) \\ A(a_0, T, T) = 0 \end{cases}$$

The proposition also acts as a necessary condition for an affine mortality structure if μ^λ and σ^λ are time independent.

One of the first affine mortality models is Dahl & Moller (2006). Concretely, Dahl & Moller (2006) propose a reduction factor mortality model similar to Ballotta & Haberman (2006)

$$\lambda_{a_0+t,t} = \lambda_{a_0+t,0}\xi_{a_0+t,t} \tag{1.48}$$

Here, the reduction factor is modeled via a time-inhomogeneous CIR process

$$d\xi_{a_0,t} = (\gamma^{\xi}(a_0,t) - \delta^{\xi}(a_0,t)\xi_{a_0,t})dt + \sigma^{\xi}(a_0,t)\sqrt{\xi_{a_0,t}}dW_t^{\xi}. \tag{1.49}$$

Applying Ito's formula, it follows that $\lambda_{a_0+t,t}$ has the dynamics

$$d\lambda_{a_0+t,t} = (\gamma^{\lambda}(a_0+t,t)-\delta^{\lambda}(a_0+t,t)\lambda_{a_0+t,t})dt+\sigma^{\lambda}(a_0+t,t)\sqrt{\lambda_{a_0+t,t}}dW_t^{\xi} \tag{1.50}$$

where γ^{λ}, δ^{λ} and σ^{λ} are as follows

$$\begin{aligned}
\gamma^{\lambda}(a_0,t) &= \gamma^{\xi}(a_0,t)\lambda_{a_0+t,0} \\
\delta^{\lambda}(a_0,t) &= \delta^{\xi}(a_0,t) - \frac{\frac{d}{dt}\lambda_{a_0+t,0}}{\lambda_{a_0+t,0}} \\
\sigma^{\lambda}(a_0,t) &= \sigma^{\xi}(a_0,t)\sqrt{\lambda_{a_0+t,0}}
\end{aligned}$$

Observe that the model satisfies the conditions of the proposition (1.8.1). Moreover, $\lambda_{a_0+t,t}$ is a CIR process, mean reverting to a long-run mean equal to $\frac{\gamma^{\lambda}}{\delta^{\lambda}}$.

The coefficients γ^{ξ}, δ^{ξ} and σ^{ξ} are assumed positive, bounded and satisfying the condition $(\sigma^{\xi}(a_0,t))^2 < 2\gamma^{\xi}(a_0,t)$ for all a_0 and t. This condition assures that the mortality intensity is positive. The form of the model allows flexibility when choosing the parameters of the reduction factor. Specifically, Dahl and Moller consider the following parameterizations

- $\delta^{\xi}(a_0, t) = \tilde{\delta}, \quad \gamma^{\xi}(a_0, t) = \tilde{\delta} e^{-\tilde{\gamma} t}, \quad \sigma^{\xi} = \tilde{\sigma}$
- $\delta^{\xi}(a_0, t) = \tilde{\gamma}, \quad \gamma^{\xi}(a_0, t) = \frac{1}{2}\tilde{\sigma}^2, \quad \sigma^{\xi} = \tilde{\sigma}$

A disadvantage of the model is its feature of mean reversion. However, calibrating the model to experienced mortality data, it is found that this feature of mean reversion is weak.

Luciano & Vigna (2005) further investigate affine mortality models. They calibrate different mean reverting affine models to historic mortality data and compare their performances. They find that these types of models are not able to capture essential features of the survival curve l_{a_0} such as "rectangularization" and "expansion" and moreover they are not consistent with historic mortality data. Intuitively, one would expect that mean reverting models are not appropriate to model mortality intensity, since this will imply that once mortality declines below the mean reverting level, it will have the tendency to go up. However, empirical evidence contradicts this fact, since for example if a progress in medicine generates a huge decline of mortality then this progress persists and the corresponding treatments are not suddenly forgot.

In Luciano & Vigna (2008), the mean reversion feature is dropped and the mortality intensity is modeled via non-mean reverting processes such as Ornstein-Uhlenbeck or Feller. It is found that the models fit well to different generation life tables and capture the essential features of mortality. Moreover, the models offer a simple and parsimonious description of mortality, are easy to implement and produce survival probabilities in closed form. This later fact greatly simplifies the valuation of mortality derivatives.

Motivated by Luciano & Vigna (2008) results, we model the mortality intensity of a cohort aged a_0 at time 0 by a non-mean reverting Ornstein-Uhlenbeck process

$$d\lambda_{a_0+t,t} = \mu^\lambda(a_0)\lambda_{a_0+t,t}dt + \sigma^\lambda(a_0)dW_t^\lambda, \tag{1.51}$$

where $\mu^\lambda(a_0) > 0$ and $\sigma^\lambda(a_0) \geq 0$.

Solving the stochastic differential equation (1.51), we obtain

$$\lambda_{a_0+t,t} = \lambda_{a_0,0}e^{\mu^\lambda t} + \sigma^\lambda \int_0^t e^{-\mu^\lambda(s-t)}dW_s^\lambda. \tag{1.52}$$

By the Dambis, Dubins-Schwartz theorem, there exists a Brownian motion $\tilde{W}_t^\lambda$ such that

$$\int_0^t e^{-\mu^\lambda(s-t)}dW_s^\lambda = \tilde{W}_{\nu(t)}^\lambda, \tag{1.53}$$

where $\nu(t) = \dfrac{1}{2\mu^\lambda}(e^{2\mu^\lambda t} - 1)$. Accordingly, we can express the mortality intensity as follows

$$\lambda_{a_0+t,t} = \lambda_{a_0,0}e^{\mu^\lambda t} + \sigma^\lambda \tilde{W}_{\nu(t)}^\lambda, \tag{1.54}$$

that is the mortality intensity is a process having its deterministic part given by the Gompertz's mortality law.

Observe that the mortality intensity can take negative values with positive probability. Concretely, we have

$$P(\lambda_{a_0+t,t} \leq 0) = P(\lambda_{a_0,0}e^{\mu^\lambda t} + \sigma^\lambda \tilde{W}_{\nu(t)}^\lambda \leq 0) = \phi\left(-\frac{\lambda_{a_0,0}e^{\mu^\lambda t}}{\sigma^\lambda\sqrt{\nu(t)}}\right) \tag{1.55}$$

where ϕ denotes the cumulative distribution function for the standard normal distribution. This is a major disadvantage of the model. However, in practical applications the probability (1.55) turns out to be negligible.

Naturally, for a biologically reasonable mortality model the survival probability $p(a_0, 0, t)$ has to be decreasing for all t. Unfortunately, the mortality model (1.51) implies that the survival probability is decreasing for $t < T^*$ and increasing for $t > T^*$, where

$$T^* = \frac{1}{\mu^\lambda} \ln \left[1 + \frac{(\mu^\lambda)^2 \lambda_{a_0,0}}{(\sigma^\lambda)^2} \left(1 + \sqrt{1 + \frac{2(\sigma^\lambda)^2}{(\mu^\lambda)^2 \lambda_{a_0,0}}} \right) \right]. \tag{1.56}$$

However, once the model is calibrated, it turns out that T^* is very large. In other words, the model is not reasonable for ages that exceed usual human survivorship.

Notice that the model admits an affine mortality term structure of the form (1.46) where $A(a_0, t, T)$ and $B(a_0, t, T)$ satisfy

$$\begin{cases} B_t + \mu^\lambda B = -1 \\ B(a_0, T, T) = 0 \end{cases} \qquad \begin{cases} A_t = -\frac{1}{2}(\sigma^\lambda)^2 B^2 \\ A(a_0, T, T) = 0. \end{cases} \tag{1.57}$$

Solving the systems of equations above, we obtain

$$A(a_0, t, T) = \frac{1}{2} \left(\frac{\sigma^\lambda}{\mu^\lambda} \right)^2 (T - t) + \frac{(\sigma^\lambda)^2}{(\mu^\lambda)^3} (1 - e^{\mu^\lambda (T-t)}) - \frac{(\sigma^\lambda)^2}{4(\mu^\lambda)^3} (1 - e^{2\mu^\lambda (T-t)}) \tag{1.58}$$

$$B(a_0, t, T) = -\frac{1}{\mu^\lambda} (1 - e^{\mu^\lambda (T-t)}). \tag{1.59}$$

Accordingly, the probability of survival from the current time 0 to time t, is given by

$$p(a_0, 0, t) = P(\tau_{a_0,0} > t) = e^{A(a_0,0,t) - B(a_0,0,t)\lambda_{a_0,0}}, \tag{1.60}$$

where $A(a_0, 0, t)$ and $B(a_0, 0, t)$ are calculated using (1.58) and (1.59).

1.9 Mortality model calibration

In what follows, we calibrate the mortality model (1.51) to the U.S. cohort life table for the Social Security Area, males, generation 1900. For determining the model parameters, we use the mean least square method (MLS), that is minimize the spread between the empirical survival probabilities ${}_tp_{a_0}$ and their theoretical counterparts calculated via (1.60). Specifically, for an American male, aged 45, we obtain the following values for the model's parameters

$$\mu^{\lambda}(45) = 0.07307, \quad \sigma^{\lambda}(45) = 0.00061, \quad \lambda_{45,0} = 0.00778. \tag{1.61}$$

Here, $\lambda_{45,0}$ is approximated by $-\ln(q_{45})$. This is a consequence of considering the mortality intensity constant over the base year.

As figure (1.5) illustrates, the fit using the estimates (1.61) is very good.

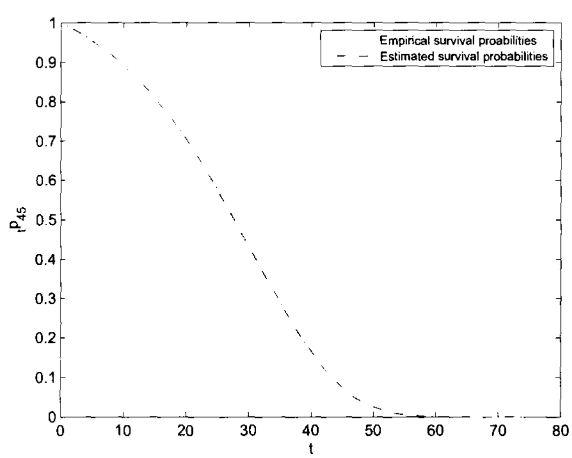

Figure 1.5: The probability of survival within t years, t=1:74, for an American male born in 1900.

Then, we calculate the probability that the force of mortality takes negative values. We find that the probabilities (1.55) for t=1:75 take very small values, their maximum value being of order 10^{-7}, which is negligible for practical applications.

As mentioned, the model generates increasing survival probabilities $p(a_0, 0, t)$ for $t > T^*$, with T^* given by (1.56). Considering the estimates above, we obtain that T^* is approximately 74 years. However, extremely rarely an individual age 45 survives for another 74 years, thus reaching age 119.

As a consequence, once the model calibrated, its theoretical disadvantages are irrelevant and the model can be used for practical applications.

Chapter 2

Financial Pricing Methods of Life Insurance Products

Traditionally, the valuation of life insurance products was performed using actuarial methods. The competition with other financial intermediaries, forced life insurers to introduce on the market new life insurance products, more attractive by their equity growth potential and often incorporating payment guarantees. The complex structure of these products, with the financial risk embedded in their benefits, made their pricing a real challenge for actuaries. Clearly, new pricing methods had to be applied. In what follows, we describe two financial approaches proposed for pricing this type of life insurance products.

2.1 The risk-neutral approach

For a long period, finance and life insurance were regarded as two completely separate fields. One reason for this view might be the different nature of financial and insurance risks. Secondly, the corresponding markets are totally different. With regards to the financial sector, there are organized financial markets trading standardized contracts. On the other hand, the markets for life insurance consists of insurance and reinsurance companies and here the contracts by their nature are unique, requiring individual assessment. Moreover, for a long time, life insurance and finance were offering products of different type - where in the financial sector contracts with variable payoff were common, in traditional life insurance, contracts with fixed benefit were predominant.

Another fact that is worth underlining and that separated finance and insurance sectors, was the way of investing, that is the distribution of the assets. Historically, insurance companies had most of their assets invested in bonds and mortgages and none or very low equity investments, given the existing regulations. For example, in the United States, until 1951, life insurers were not allowed to hold any equity investment; from 1951 equity investment was authorized up to a limit of 3% (Briys & de Varenne (2001)); then gradually, these restrictions became weaker.

From the late 1960s, given the changes on the financial markets and the competition with other financial intermediaries, life insurers started redesigning their product lines. They began to offer more attractive products such as life insurances with benefit linked to the performance of some stocks or stock market indices, the so called *equity-linked life insurance*. This allowed policyholders to enjoy the benefits of mortality protection together with those of equity investments. Moreover, to most of these contracts was attached a guarantee as a downside protection against a poor equity performance. Popular types of equity–linked life insurance contracts include segregated funds in Canada, variable annuities and equity indexed annuities in the United States and unit linked insurances in the United Kingdom.

However, traditional actuarial pricing methods were not able to solve the problem of pricing of these new types of products launched on the market. The first viable pricing method of equity–linked life insurance was proposed by Brennan & Schwartz (1976) and Boyle & Schwartz (1977). Their approach is a financial one and it is based on Black-Scholes and Merton option pricing theory. Moreover, this approach also provides a risk management strategy for the issuers of equity–linked life insurance. In what follows, we describe this method for equity–linked life insurance with death or maturity benefits.

Let us start with a market model consisting of a stock and a money market account with constant interest rate $r > 0$. We assume that the discounted stock (index) price follows a geometric Brownian motion

$$\begin{cases} dS_s = S_s((\mu - r)ds + \sigma dW_s) \\ S_0 = S \end{cases} \tag{2.1}$$

where $\mu > r > 0$, $\sigma > 0$ and W_t is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Here, the stock price is discounted for consistency with later chapters, but could as well be the actual price.

We examine equity–linked life insurance contracts with benefits as follows

$$B^{ELTL} = \begin{cases} \max(S_\tau, g(\tau)) & \text{if } \tau < T \\ 0 & \text{if } \tau \geq T, \end{cases} \qquad B^{ELPEnd} = \begin{cases} 0 & \text{if } \tau < T \\ \max(S_T, g(T)) & \text{if } \tau \geq T, \end{cases}$$

$$B^{ELEnd} = \begin{cases} \max(S_\tau, g(\tau)) & \text{if } \tau < T \\ \max(S_T, g(T)) & \text{if } \tau \geq T. \end{cases}$$

where τ is the policyholder's death time and $g(t)$ is the discounted minimum guaranteed amount (for example in our numerical experiments we consider $g(t) = Ge^{-rt}$). We choose the notations B^{ELTL}, B^{ELPEnd} and B^{ELEnd} since, as it can be observed, the first contract is an equity–linked term life insurance, the second one is an equity–linked pure endowment insurance and the last one is an equity–linked endowment insurance.

Remarks 2.1.1.
The benefits above are similar to guaranteed minimum death/maturity benefits provided by segregated funds and variable annuities. In fact, a guaranteed minimum death benefit and a guaranteed minimum maturity benefit are defined respectively as B^{ELTL} and B^{ELPEnd} where the risky underlying asset is the fund value and where the actual guaranteed amount, in its simplest form, is a certain proportion (typically between $75\% - 100\%$) of the premium.

An essential assumption of Brennan & Schwartz (1976) and Boyle & Schwartz (1977) is that the mortality risk is diversifiable. This implies that an insurer who sells a sufficiently large number of equity–linked life contracts, practically regards these contracts as carrying out just the financial risk. Accordingly, the equity–linked contracts given above can be viewed as financial options with a random exercise time, that is: the policyholder's death time, the contract maturity and the policyholder's death time or the contract maturity.

Now let us assume that the contracts defined above become active at a certain time s. Then, their discounted benefit at time s, here generically denoted by B_s, can be written in the following two alternative forms

$$B_s = g(s) + (S_s - g(s))^+ = S_s + (g(s) - S_s)^+ \tag{2.2}$$

that is, the benefit can be decomposed as the guaranteed amount plus the payoff of an European call option on the stock with strike $g(s)$ and maturity s or alternatively the benefit is given by the stock price plus the payoff of a European put option on the stock with strike $g(s)$ and maturity s. Consequently, the premium at time $t = 0$ is given by

$$P(0, S, s, g(s)) = g(s) + c(0, S, s, g(s)) = S + p(0, S, s, g(s)) \tag{2.3}$$

where $P(0, S, s, g(s))$ denotes the premium at time $t = 0$ for the claim (either B^{ELTL}, B^{ELPEnd} or B^{ELEnd}) maturing at time s, while $c(0, S, s, g(s))$ and $p(0, S, s, g(s))$ represent respectively the price at time $t = 0$ of a European call and put on the stock with strike price $g(s)$ and maturity s. One can recognize in (2.3) the put-call parity formula. Further, observe from (2.3) that the amount $p(0, S, s, g(s))$ is the premium for providing the guarantee.

At this point, Brennan & Schwartz and Boyle & Schwartz appeal to the Black-Scholes and Merton option price theory. Using the pricing formula for a European call, they find

$$P(0, S, s, g(s)) = g(s) + SN(d_1) - g(s)N(d_2) \tag{2.4}$$

where N denotes the cumulative distribution function for $\mathcal{N}(0, 1)$ and d_1 and

d_2 are given by

$$d_1 = \frac{\log \frac{S}{g(s)} + \frac{1}{2}\sigma^2 s}{\sigma\sqrt{s}}, \quad d_2 = d_1 - \sigma\sqrt{s}. \tag{2.5}$$

Additionally, from (2.3) and (2.4) the premium for the guarantee is

$$p(0, S, s, g(s)) = g(s) - SN(-d_1) - g(s)N(d_2). \tag{2.6}$$

The claim B^{ELTL} can become active at any time before maturity. Accordingly, assuming a discrete distribution of the time of death and considering that if death takes place in a certain year s the premium is payed at the end of the year, we have

$$P^{ELTL}(0, S) = \sum_{s=1}^{T} P(0, S, s, g(s))_{s-1|}q_{a_0} = \sum_{s=1}^{T} P(0, S, s, g(s))_{s-1}p_{a_0}q_{a_0+s-1} \tag{2.7}$$

while if the time of death has a continuous distribution $F_\tau(s) = 1 - e^{-\int_0^s \lambda(a_0+u)du}$, we obtain

$$P^{ELTL}(0, S) = \int_0^T P(0, S, s, g(s))\lambda(a_0 + s)_s p_{a_0} ds. \tag{2.8}$$

Accordingly, the premium for the guarantee is

$$\int_0^T p(0, S, s, g(s))\lambda(a_0 + s)_s p_{a_0} ds. \tag{2.9}$$

Clearly, in (2.7), (2.8) and (2.9), $P(0, S, s, g(s))$ and $p(0, S, s, g(s))$ denote the premium for the equity–linked term life insurance and the premium for the guarantee, respectively, given that the policy matures at the known date s.

For the equity–linked pure endowment contract, the benefit is provided at maturity if the policyholder survives to that time. Thus, we have

$$P^{ELPEnd}(0,S) = P(0,S,T,g(T))_T p_{a_0} \tag{2.10}$$

while the premium for the guarantee is

$$p(0,S,T,g(T))_T p_{a_0}. \tag{2.11}$$

Again, here $P(0,S,T,g(T))$ and $p(0,S,T,g(T))$ denote the premium for the equity–linked pure endowment insurance and the premium for the guarantee, respectively, given that the policy matures at time T.

Finally, given the additivity of the Black-Scholes and Merton pricing rule, the premium for the equity–linked endowment insurance is given by the sum of the two premiums above.

Next, for finding the optimal investment hedging strategy that a seller of equity–linked life insurance has to follow, Brennan & Schwartz and Boyle & Schwartz apply the Black-Scholes and Merton hedging arguments. In what follows, we employ these arguments for the hedging of the three contracts considered above.

A seller of the claim B^{ELTL}, at a certain time s (precisely at the beginning of the year s), $s = 0, 1 \ldots T-1$ (conditional on the contract being in force at that time) is short q_{a_0+s}, ${}_{1|}q_{a_0+s}$ $\cdots$ ${}_{T-1-s|}q_{a_0+s}$ European call options with maturities $s+1$, $s+2$ $\ldots$ T, respectively. Following the Black-Scholes and Merton hedging arguments, the amount to be invested in the stock under the riskless investment strategy at time s, is

$$\begin{aligned}\pi_s^{ELTL} &= S_s \sum_{u=0}^{T-1-s} {}_{u|}q_{a_0+s} \frac{\partial}{\partial S} c\left(s, S_s, u+s+1, g(u+s+1)\right) \\ &= S_s \sum_{u=0}^{T-1-s} {}_{u|}q_{a_0+s} N\left(d_1(s, u+s+1)\right)\end{aligned}$$

where $d_1(s,u) = \frac{\log \frac{S_s}{g(u)} + \frac{1}{2}\sigma^2(u-s)}{\sigma\sqrt{u-s}}$.

On the other hand, a seller of the claim B^{ELPEnd}, at a certain time s, $s = 0, 1 \ldots T-1$ (conditional on the contract being in force at time s) is short ${}_{T-s}p_{a_0+s}$ European call options with maturity T. Accordingly, applying the Black-Scholes and Merton hedging arguments, the amount to be invested in the stock at time s, under the riskless investment strategy, is

$$\pi_s^{ELPEnd} = S_s \frac{\partial}{\partial S} c(s, S_s, T, g(T))_{T-s}p_{a_0+s} = S_s N\left(d_1(s,T)\right)_{T-s}p_{a_0+s}. \tag{2.12}$$

Finally, a seller of the claim B^{ELEnd}, at time s, $s = 0, 1 \ldots T-1$ is short q_{a_0+s}, ${}_{1|}q_{a_0+s}$ $\cdots$ ${}_{T-1-s|}q_{a_0+s}$ European call options with maturities $s+1$, $s+2 \ldots T$ respectively and ${}_{T-s}p_{a_0+s}$ European call options with maturity T. Therefore, the riskless investment strategy requires to invest in the stock at time s the amount $\pi_s^{ELTL} + \pi_s^{ELPEnd}$.

Observe that here, for notational convenience, we assumed that the insurer rebalances his portfolio yearly; obviously, if the insurer rebalances his portfolio more frequently - for example monthly, the formulas above have to be appropriately adjusted. Also, observe that here mortality is assumed deterministic; clearly, we assume deterministic dynamic mortality but again, for notational convenience we chose to omit the time index in the life functions notations.

Numerical experiments

We conclude this section by several numerical experiments that refer to an American male policyholder, born in 1900 and having age $a_0 = 45$ years. We assume that his mortality is given by the deterministic version of the non-mean reverting Ornstein-Uhlenbeck process (1.51). Consequently,

$$\lambda(a_0 + t) = \lambda(a_0)e^{\mu^\lambda t}. \tag{2.13}$$

We approximate $\lambda(45)$ by $-\ln p_{45}$ and estimate μ^λ using the mean least squares method. In this way we obtain the values $\lambda(45) = 0.00778$ and $\mu^\lambda = 0.07204$. Then, we assume that $\sigma = 0.2$ and $r = 0.06$.

We calculate the premium for the claims B^{ELTL}, B^{ELPEnd} and B^{ELEnd}, assuming that $g(t) = Ge^{-rT}$, $G = 10$ and that S varies between 0 and 20. Then, the time to maturity of the contracts is varied between 5 and 20 years.

Figures 2.1 and 2.2 show that for all contracts, the premium increases as the spot price increases. This is expected, since the price of the call options embedded in the benefits is an increasing function of the spot price. Also observe that for very small values of S, premium increases very slowly, looking almost flat. That is because in this situation, the guarantee will be active; essentially this is a premium for providing the guarantee.

Notice that for the claims B^{ELPEnd} and B^{ELEnd}, the premium is a decreasing function of maturity time. For the claim B^{ELPEnd}, this is because the value of the put options embedded in the premium, decreases as maturity in-

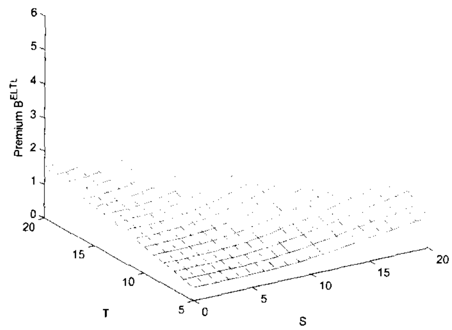

Figure 2.1: Premium for B^{ELTL} as a function of time to maturity and spot price.

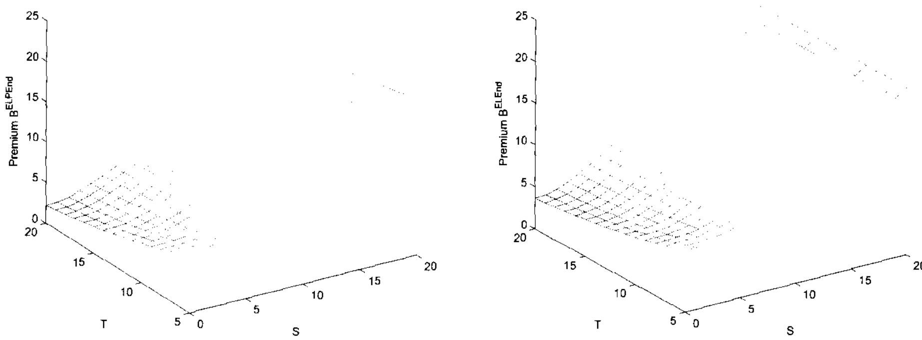

Figure 2.2: Premium for B^{ELPEnd} and B^{ELEnd} as a function of time to maturity and spot price.

creases and because of the survival effect. The same motivation also holds for the claim B^{ELEnd} which here, is similar to B^{ELPEnd} since the policyholder is a young adult. On the other hand, for the claim B^{ELTL} the premium increases with maturity, since mortality increases over time and dominates the decrease in the price of the put options embedded in the premium.

We conclude our analysis by discussing the validity of the assumptions made by Brennan & Schwartz (1976) and Boyle & Schwartz (1977). The essential assumption of the approach described above is that the mortality risk is diversifiable. Actually, this assumption is a consequence of the *Law of Large Numbers* that works as long as the risks involved are independent and identically distributed. The independence assumption regarding the mortality risks is valid conditional on knowing the individuals mortality. However, when mortality is uncertain, these risks may become dependent over time. Intuitively, the random nature of mortality can be explained by factors such as medical breakthroughs, natural catastrophes, wars etc. and many or even all policyholders will be effected by their affects, thus creating a dependency between their mortality. Naturally, given this dependency, the *Law of Large Numbers* can no longer be applied to prove that the standard deviation per policy vanishes in the limit. In fact, Milevky, Promislow & Young (2006) show that under uncertain mortality, the standard deviation per policy can be decomposed into two components: one component accounting for unsystematic or diversifiable mortality risk and one for the systematic mortality risk. As the number of policies increases the first component decreases, reaching zero for an infinite number of policies. In contrast, the second component does not vanishes when the number of policies approaches infinity and may even increase as the number of policies increases.

Another assumption of the Brennan & Schwartz (1976) and Boyle & Schwartz (1977) approach is that the stock price volatility is constant. Clearly, this is not a viable assumption, especially given the long term of the life insurance contracts.

Observe that if any one of the assumptions mentioned above is removed, the Brennan & Schwartz and Boyle & Schwartz approach cannot be applied.

That is because the market is incomplete and the Black-Scholes and Merton theory cannot be used. Accordingly, we need a pricing and hedging approach for incomplete markets and in what follows, we propose the *utility indifference pricing* approach.

2.2 Utility indifference pricing of life insurance claims

2.2.1 From expected utility theory to utility indifference pricing

The concept of "utility" goes back to Daniel Bernoulli (1738) who argued that often money cannot be appropriately measured by its monetary value and a better measure would be its "moral value" or its usefulness. Accordingly, he proposed that lotteries [1] have to be compared not by their fair price (i.e. expected value), which was commonly used, but instead by their expected utilities.

Expected utility theory came to life again due to Neumann & Morgenstern (1944). They proved that under certain axioms, there exists a utility function and a preference order between lotteries as suggested by Bernoulli, given by the comparison of the corresponding expected utilities. For a detailed exposure of this theory see also Föllmer & Schied (2004).

With regards to the utility function, it is natural to assume that this is strictly increasing and concave. The former feature is desirable since "rational"

[1] Lotteries are probability distributions over a set of outcomes. The outcomes could be of different nature: events, goods, money etc.

decision makers prefer more to less, while the latter is because decision makers are risk averse. The second feature is controversial since agents in certain conditions switch from risk averse to risk seeker. From now on, we consider the following definition for a utility function:

Definition 2.2.1. A function $U : \mathcal{S} \to \mathbb{R}$ is called a utility function if it is strictly concave, strictly increasing and twice continuously differentiable on $\mathcal{S}$.

Here the set $\mathcal{S}$ of monetary outcomes, can be either the whole real line or just the positive real line.

Now, assume that the decision maker is an insurer with wealth x_0 and utility function U. Further, assume that the insurer has the possibility to insure a risk B. Then, the insurer faces the following two scenarios: either he is not taking the risk or he accepts the risk, charging a premium P. Essentially, these two scenarios correspond to two lotteries and according to the preference order mentioned above, P should be such that

$$E[U(x_0 + P - B)] \geq U(x_0) \tag{2.14}$$

where the equality case

$$E[U(x_0 + P - B)] = U(x_0) \tag{2.15}$$

holds for the minimum premium to be asked. As can be observed from (2.15), this premium is such that the insurer is indifferent between accepting or not accepting the insurance risk. Equation (2.15) is called the *principle of equivalent utility* and the premium that solves this equation is called *indifference premium.*

Let P^{EU} denote the indifference premium for the insurance contract considered. Observe that by Jensen's inequality, we have

$$E[U(x_0 + P^{EU} - B)] \leq U(x_0 + P^{EU} - E[B]). \tag{2.16}$$

But P^{EU} solves (2.15) and taking into account that U is strictly increasing, we obtain that $P^{EU} \geq E[B]$.

We would like to underline the fact that any acceptable premium for the insurance contract considered is a premium that corresponds to a particular insurer, with preferences towards risk and wealth specified by his utility function. So, different insurers will charge different premiums for insuring the risk B. A decision maker's attitude towards risk can be described via the concept of *absolute risk aversion* (Arrow (1970) and Pratt (1964)) that is defined below.

Definition 2.2.2. The absolute risk aversion $r(x)$ of the utility function U, at a wealth x is given by

$$r(x) = -\frac{U''(x)}{U'(x)}. \tag{2.17}$$

In the subsequent chapters, we will show that the more risk averse an insurer is, the greater the premium to be charged. Further, we will see that for a risk neutral insurer, the premium approaches the fair premium.

Essential to expected utility theory is the agent utility function. The problem of determining an agent utility function is a delicate one that we do not pose here. Popular examples of utility functions are: exponential utility ($U(x) = -\gamma e^{-\gamma x},\ \gamma > 0$), power utility ($U(x) = x^c,\ x > 0,\ 0 < c \leq 1$), logarithmic utility ($U(x) = \log(\gamma + x),\ x > -\gamma$) and quadratic utility ($U(x) =$

$-(\gamma - x)^2, \ x \leq \gamma)$.

In this thesis, we assume that the agent's utility function is exponential. Concretely, we choose an exponential utility of the form

$$U(x) = -e^{-\gamma x}, \ \ \gamma > 0. \tag{2.18}$$

Observe, that in this case, the absolute risk aversion coefficient is constant and has the value $r(x) = \gamma$.

Then, with this choice of utility function, the premium P^{EU} can be readily calculated and is as follows

$$P^{EU} = \frac{1}{\gamma} \log E[e^{\gamma B}]. \tag{2.19}$$

Thus, the premium P^{EU} is wealth independent.

These two features of the exponential utility function - absolute risk aversion coefficient and indifference premium independent of wealth - might suggest that this utility is not realistic. However, exponential utility has major advantages such as mathematical tractability and intuitive premium formulas that are nice to interpret. Moreover, there is a connection between the probability of ruin and the insurer risk aversion (see Gerber (1976), page 135).

More recently, the principle of equivalent utility was adapted for derivative pricing in incomplete markets. The resulting pricing approach is called *utility indifference pricing* and it was introduced by Hodges & Neuberger (1989) for valuing European calls subject to transaction costs. Since then, utility indifference pricing has been applied in many areas: Musiela & Zariphopoulou (2003) examine the pricing of claims on a non-traded asset correlated to a tradable one, Sircar & Zariphopoulou (2004) study the pricing of European derivatives in financial markets with random volatility, Carmona (2009a) considers applications to weather derivatives and energy contracts, while Young & Zariphopoulou (2002), Young (2003) and Jaimungal & Young (2005) apply the approach for pricing and hedging life insurance products. These are just a few contributions to the field of utility indifference pricing. A comprehensive review of the theory regarding utility indifference pricing as well as of its further developments and applications is given in Carmona (2009b).

In order to illustrate how the indifference pricing approach works, we consider the following example. Assume that an insurer has the opportunity to sell a life insurance contract with maturity T to one or more individuals. The insurer has initial wealth x_0 and can trade between a risky stock and a money market account. Further, let X_T be the wealth generated from the initial wealth x_0 and corresponding to a self financing trading strategy $(\pi_t)_{0\leq t\leq T}$.

As in the static case presented earlier, when deriving the principle of equivalent utility, the insurer faces two possible scenarios: either he does not take any risk and receives no premium or he takes on the risk by accepting to write one or more life insurance contracts and receives a certain premium from each individual insured. In the former scenario, the insurer will invest, aiming to maximize his expected utility of terminal wealth. That is, he will have to solve the optimization problem

$$u^0(x_0) = \sup_{\pi} E[U(X_T)|X_0 = x_0]. \tag{2.20}$$

On the other hand, in the later scenario, the insurer aims to maximize his expected utility of terminal wealth, taking into account the benefit to be paid when the insurance policies generate claims. Clearly, the insurer has to choose a model for his liability and the models's choice will play an essential role for pricing and risk management purposes. We will refer to this issue in detail at the end of this section. For now, let us assume that the insurer's liability, denoted by L_T is to be paid at the maturity of the life insurance contracts. Then, the corresponding optimization problem is

$$u(x_0) = \sup_{\pi} E[U(X_T - L_T)|X_0 = x_0]. \tag{2.21}$$

Definition 2.2.3. The indifference premium of the insurer for the life insurance contract(s) is the amount P such that

$$u^0(x_0) = u(x_0 + P). \tag{2.22}$$

As can be observed, (2.22) represents a generalization of the principle of equivalent utility to a dynamic market setting. The premium defined by (2.22) it is called *indifference premium* and similarly, this is a generalization of the indifference premium in a static market setting.

Indifference pricing and the prices generated by this approach have several remarkable properties. First, in contrast to no-arbitrage pricing, indifference pricing is a nonlinear pricing rule; secondly, if the financial market is complete, the indifference price of an option is unique and equals its risk-neutral price. Not least, indifference prices are increasing functions of the risk aversion coefficient and of the claim size. A detailed exposure regarding the properties of the indifference prices is given by Becherer (2001).

Finally, we would like to remark that in contrast to the static case, the problem of calculating the indifference premium in a dynamic market setting is a delicate one. For this, first it is necessary to solve the optimization problems (2.20) and (2.21). This can be done using either dynamic programming or martingale theory arguments. In this thesis, we will use the first approach, also known as the *primal* approach. Below, we present a brief description of the dynamic programming approach and then we apply it for solving the insurer problem without the claim, referred to in financial literature as the *Merton investment problem.* Then, since insurer's liability models are essential for the second optimization problem, we conclude this chapter with a description of these models.

2.2.2 The dynamic programming approach

Dynamic programming is a powerful tool for solving *optimal control problems* introduced by R. Bellman in early 1950s. In what follows, we describe this approach for stochastic optimal control problems formulated on finite horizon. Our main references are Yong & Zhou (1999), Pham (2009) and Fleming & Soner (2006).

Let us assume that the state of a stochastic system is described by an Itô process X with dynamics

$$\begin{cases} dX_t = dX_t^{\pi} = a(t, X_t, \pi_t)dt + b(t, X_t, \pi_t)dW_t \\ X_0 = x_0. \end{cases} \tag{2.23}$$

Here W_t is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions. In this context, we refer to X as the *state process* and to π as the *control process*. Further, we assume that the control $\pi = (\pi_t)$ is $\mathbb{F}$ progressively measurable with values in a subset C of $\mathbb{R}$. With regards to the coefficients a and b we assume that they satisfy the uniform Lipschitz condition: there exists $K > 0$ such that for all $t \geq 0$, $x, y \in \mathbb{R}$ and $u \in C$

$$|a(t, x, z) - a(t, y, z)| + |b(t, x, z) - b(t, y, z)| \leq K|x - y|, \tag{2.24}$$

where K is a constant independent of (t, x, y, z).

Now, for a given time horizon $T > 0$, we introduce the *performance* or *gain function*

$$\mathcal{J}(x_0, \pi) = E[U(X_T^{\pi})|X_0 = x_0] \tag{2.25}$$

and consider the optimal control problem

$$\sup_{\pi \in \mathcal{A}[0,T]} E[U(X_T)|X_0 = x_0], \tag{2.26}$$

where U is a given utility function. Here, the family of *admissible* controls $\mathcal{A}[0, T]$ includes control processes π valued in C that are $\mathbb{F}$ progressively measurable and additionally satisfy

$$|a(\cdot, 0, \cdot)| + |b(\cdot, 0, \cdot)| \in L^2(0, T; \mathbb{R}). \tag{2.27}$$

It can be shown that for $\pi \in \mathcal{A}[0, T]$, under the uniform Lipschitz condition (2.24), the stochastic differential equation with random coefficients (2.23) has a unique solution (for details see e.g. Pham (2009), Theorem 1.3.15). Consequently, the performance function (2.25) is well defined.

The basic idea of dynamic programming is to embed the original problem (2.26), in a large family of optimal control problems, with different initial times and states, to establish a relationship between them, and finally to solve them all at once.

Observe that when taking a certain time and state (t, x), the state $X_t = x$ is a random variable in the original probability space. However, at time t, $\mathcal{F}_t$ gives us all the relevant information about X_t. So, basically, X_t is a.s. deterministic under the probability measure $P(\cdot\ |\mathcal{F}_t)$. Thus, essentially, when using dynamic programming for solving stochastic optimal control problems, one does not vary just the initial time and state as in the deterministic case, but varies as well the probability spaces. Accordingly, the problem (2.29) introduced below can be regarded as a weak formulation of the stochastic optimal control, while problem (2.26) can be viewed as a strong formulation.

Now, let $(t, x) \in [0, T] \times \mathbb{R}$ and consider the state equation

$$\begin{cases} dX_s = a(s, X_s, \pi_s)ds + b(t, X_s, \pi_s)dW_s \\ X_t = x \end{cases} \tag{2.28}$$

The corresponding optimal control problem is

$$\sup_{\pi \in \mathcal{A}[t,T]} E[U(X_T)|X_t = x], \tag{2.29}$$

where the family of *admissible* controls $\mathcal{A}[t, T]$ consists 5-tuples $(\Omega, \mathcal{F}, P, W, \pi)$ satisfying the following conditions

1. $(\Omega, \mathcal{F}, P)$ is a complete probability space. Here, $\mathcal{F}$ contains all the information available starting with time t while P is the original probability

measure given the information up to time t.

2. $(W_u)_{t \leq u \leq T}$ is a one dimensional Brownian motion defined on $(\Omega, \mathcal{F}, P)$ with $W_t = 0$ a.s and $\mathcal{F}_s^t = \sigma(W_u, t \leq u \leq s)$ augmented by the P null sets in $\mathcal{F}$.

3. $\pi : [t, T] \times \Omega \to \mathbb{C}$ is $(\mathcal{F}_s^t)_{s \geq t}$ progressively measurable and is such that the following integrability condition holds

$$|a(\cdot, 0, \cdot)| + |b(\cdot, 0, \cdot)| \in L^2(t, T; \mathbb{R}). \tag{2.30}$$

4. $U(X_T) \in L^1(\Omega; \mathbb{R})$.

Often, when clear from the context, for notational simplicity, we will write simply $\pi \in \mathcal{A}$ instead of $(\Omega, \mathcal{F}, P, W, \pi) \in \mathcal{A}[t, T]$.

Now, we consider the *performance* or *gain function*

$$\mathcal{J}(t, x; \pi) = E[U(X_T)|X_t = x] \tag{2.31}$$

and define the *value function*, as follows

$$u(t, x) = \sup_{\pi \in \mathcal{A}[t,T]} \mathcal{J}(t, x; \pi). \tag{2.32}$$

When a control process $\pi \in \mathcal{A}$ is adapted to the filtration generated by the state process X, we will refer to it as *feedback control.* If $\pi \in \mathcal{A}$ is of the form $\pi_s = f(s, X_s^x)$, then this type of control will be called *Markov control.* Clearly, any Markov control is a feedback control.

At this point, we can present a fundamental principle for the theory of stochastic control, known as *Bellman's principle of dynamic programming.*

Theorem 2.2.1. *Let $(t, x) \in [0, T) \times \mathbb{R}$ be given. Then, for every stopping time $\tau \in [t, T]$, we have*

$$u(t,x) = \sup_{\pi \in \mathcal{A}[t,T]} E[u(\tau, X_\tau)|X_t = x] \tag{2.33}$$

Thus, Bellman's principle states that the value function is a supermartingale for any admissible control π and it is a martingale if an optimal control π^* exists.

When the value function is smooth, as the stopping time τ approaches t, Bellman's principle together with stochastic calculus arguments generate a second order partial differential equation that describes the local behavior of the value function. This equation is called *the Hamilton-Jacobi-Bellman* (HJB) equation. A formal derivation of the HJB equation is given below.

For $\tau = t + h$, Bellman's principle of dynamic programming implies that

$$u(t,x) \geq E[u(t+h, X^{\boldsymbol{\pi}}_{t+h})|X_t = x], \tag{2.34}$$

where $\boldsymbol{\pi}$ is an arbitrary control in $\mathcal{A}[t, T]$.

Assuming that u is smooth enough we can apply Itô's lemma and obtain

$$\begin{aligned} u(t+h, X^{\boldsymbol{\pi}}_{t+h}) = u(t,x) + \int_t^{t+h} \left(u_s + a(s, X^{\boldsymbol{\pi}}_s, \boldsymbol{\pi}_s) u_x + \frac{1}{2} b^2(s, X^{\boldsymbol{\pi}}_s, \boldsymbol{\pi}_s) u_{xx} \right) ds \\ + \int_t^{t+h} u_x(s, X^{\boldsymbol{\pi}}_s) b(s, X^{\boldsymbol{\pi}}_s, \boldsymbol{\pi}_s) dW_s \end{aligned} \tag{2.35}$$

Substituting (2.35) in (2.34) and assuming that the stochastic integral in (2.35) is a martingale, we obtain

$$E_{t,x}\left[\int_t^{t+h} u_s + a(s, X_s^{\pi}, \boldsymbol{\pi}_s)u_x + \frac{1}{2}b^2(s, X_s^{\pi}, \boldsymbol{\pi}_s)u_{xx}ds\right] \leq 0.$$

Dividing by h and then taking the limit as h goes to 0, it follows that

$$u_t(t,x) + \mathcal{L}^{\pi}u(t,x) \leq 0.$$

Here $\pi = \boldsymbol{\pi}_t$ and $\mathcal{L}^{\pi}u(t,x) = a(t,x,\pi)u_x(t,x) + \frac{1}{2}b^2(t,x,\pi)u_{xx}(t,x)$. Since $\boldsymbol{\pi}$ was arbitrary chosen, the inequality above holds for all $\pi \in C$ and thus we have

$$u_t(t,x) + \sup_{\pi\in C}\mathcal{L}^{\pi}u(t,x) \leq 0. \tag{2.36}$$

On the other hand, assuming the existence of an optimal control $\boldsymbol{\pi}^*$, following arguments as above we obtain that

$$u_t(t,x) + \mathcal{L}^{\pi^*}u(t,x) = 0. \tag{2.37}$$

Combining (2.36) with (2.37), we obtain that

$$\begin{cases} u_t + \sup_{\pi\in C}\mathcal{L}^{\pi}u = 0 \\ u(T,x) = U(x) \end{cases} \tag{2.38}$$

This is the so called HJB equation.

It is very difficult to show just from the definition of the value function that this satisfies the regularity properties assumed above. Usually, we will derive formally the HJB equation. Then, we will try to solve and prove the existence of a smooth solution for the HJB equation. The next step is called

Verification step. Throughout this thesis we will use a version of a verification result by Duffie & Zariphopoulou (1993).

2.2.3 The Merton investment problem

In what follows, we apply the dynamic programming approach for solving the Merton investment problem. We assume a market with two securities: a money market account with interest rate $r > 0$ and a stock. We model the discounted price of the stock by a geometric Brownian motion

$$\begin{cases} dS_t = S_t\left((\mu - r)dt + \sigma dW_t\right) \\ S_0 = S > 0 \end{cases} \tag{2.39}$$

where $\mu > r > 0$, $\sigma > 0$ and W_t is a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Let π_t denote the amount to be invested in the stock at time t. Then, the discounted wealth process evolves as follows

$$\begin{cases} dX_t = \dfrac{\pi_t}{S_t} dS_t = \pi_t((\mu - r)dt + \sigma dW_t) \\ X_0 = x_0. \end{cases} \tag{2.40}$$

Recall that the Merton investment problem is

$$\max_{\pi} E[U(X_T)|X_0 = x_0]. \tag{2.41}$$

Remark that, in this case, X and π are the state and control process, respectively. It is straightforward to show that the drift and volatility of the state process satisfy the uniform Lipschitz condition (2.24).

As mentioned when describing the dynamic programming approach, we need to embed the original problem in a large family of problems, corresponding to different initial times, states and probability spaces. So, let $(t, x) \in [0, T] \times \mathbb{R}$ and consider the state equation

$$\begin{cases} dX_s = \pi_s((\mu - r)ds + \sigma dW_s) \\ X_t = x. \end{cases} \tag{2.42}$$

The corresponding value function is

$$u^0(x,t) = \sup_{\pi \in \mathcal{A}[t,T]} E[U(X_T)|X_t = x]. \tag{2.43}$$

Notice that in this case the set of admissible controls $\mathcal{A}[t,T]$ consists of controls π that are $(\mathcal{F}_s^t)_{s \geq t}$ progressively measurable and that satisfy the integrability condition $E\left[\int_t^T \pi_s^2 ds\right] < \infty$. Clearly, here we consider a filtered probability space as described in the conditions 1 and 2. Then, observe that in this case, the integrability condition just mentioned implies both existence and uniqueness of a solution for equation (2.42).

In this case, the HJB equation for u^0 is as follows

$$\begin{cases} u_t^0 + \max_\pi \left[(\mu - r)\pi u_x^0 + \frac{1}{2}\sigma^2 \pi^2 u_{xx}^0\right] = 0 \\ u^0(x,T) = U(x). \end{cases} \tag{2.44}$$

Let us assume that the solution of the HJB equation is concave in wealth. Then, the maximization term in (2.44) is also concave in wealth. Accordingly, the maximum in equation (2.44) is well defined and by the first order necessary condition, we have

$$\pi^*(x,t) = -\frac{\mu - r}{\sigma^2}\frac{u_x^0}{u_{xx}^0}. \tag{2.45}$$

Then, inserting $\pi^*(x,t)$ in (2.44), leads to the following equation

$$\begin{cases} u_t^0 - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{(u_x^0)^2}{u_{xx}^0} = 0 \\ u^0(x,T) = U(x). \end{cases} \tag{2.46}$$

Due to the assumption of exponential utility function, we consider an ansätz of the form $u^0(x,t) = -e^{-\gamma x}f(t)$. Substituting in (2.46), we obtain that f is a solution of the ordinary differential equation

$$\begin{cases} f'(t) - \frac{(\mu - r)^2}{2\sigma^2}f(t) = 0 \\ f(T) = 0. \end{cases} \tag{2.47}$$

Solving this equation, we obtain that $f(t) = e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}$. Therefore

$$u^0(x,t) = -e^{-\gamma x - \frac{(\mu-r)^2}{2\sigma^2}(T-t)}. \tag{2.48}$$

Observe that

$$\pi^*(x,t) = \frac{\mu - r}{\gamma\sigma^2}. \tag{2.49}$$

Straightforward calculations imply that $\pi^* \in \mathcal{A}$. Notice that $u^0 \in C^{1,2}([0,T] \times \mathbb{R})$ and additionally has the properties of concavity and exponential growth in x. The ansätz satisfies the conditions of the Verification theorem and we conclude that the value function coincides with (2.48) and the optimal control is given by (2.49).

We would like to point out that this analysis does not impose any solvency condition on the insurer's portfolio. That is, we not require that $X_T \geq 0$. In fact, the solvency condition will not be imposed neither when examining the insurer's investment problem in the presence of life insurance claims. In other words, we implicitly assume that the insurer can collect cash to meet its liabilities, if necessary, though at the cost of taking a bit hit in utility.

2.3 Liability modeling

By its nature, an insurance company is exposed to insurance losses and therefore, for pricing and risk management purposes, the company has to choose an appropriate model for these losses. In this thesis, we assume that the insurer can choose between *a single life insurance model, the individual risk model* and *the collective risk model.* The last two are aggregate models with long history in actuarial practice while the first one models the loss over a single policy and can be seen as a particular case of the individual risk model. In what follows, we introduce these loss models and then show that in certain circumstances, the collective risk model can be thought of as an approximation of the individual risk model.

The individual risk model

Consider a portfolio of n insurance policies. Within the individual risk model, the aggregate claim in a certain time interval is modeled as follows

$$L^{ind} = Y^1 + Y^2 + \ldots + Y^n, \tag{2.50}$$

where $Y^i, i = 1 \dots n$ denotes the payment on policy i in the time interval under study.

Clearly, the use of the individual risk model needs to be tailored to the specific nature of the portfolio of insurance contracts. For example, for a portfolio of pure endowments all maturing at time T, the insurer will be interested in modeling his losses just at time T since only at this time he may have losses. On the other hand, for a portfolio of term life insurances, the insurer will be interested in modeling his losses over time intervals prior to maturity.

In actuarial mathematics, see for example Bowers, Gerber, Hickman, Jones & Nesbit (1997) or Gerber (1997), the random variables Y^i, $i = 1 \dots n$ are assumed to be independent. However, this assumption it is not valid in certain situations and thus, imposing it will limit the use of the model. We motivate this assertion by the following example: consider a portfolio of pure endowments, each with benefit 1 if the policyholder survive to maturity T; then, the loss at time T will be given by the sum of the payments $Y^i_T = 1_{\{\tau_i > T\}}$, where τ_i denotes the time of death of the policyholder that owns the ith policy. The assumption that Y^i_T, $i = 1 \dots n$ are independent is valid when the random variables τ_i are themselves independent, for example when policyholders' mortalities evolve deterministically over time. However, if considering random mortality, the assumption above is no longer valid.

Collective risk models

Whereas in the individual risk model, one first looks at the loss over each individual policy and then by cumulating these losses obtains the total loss, in the collective model, one models the loss on the whole portfolio from the beginning. As time evolves, at random points in time, the portfolio generates claims. Then, the random sum of the claims generated in the time period under study gives the aggregate claim on that time period.

Accordingly, in the collective risk model, the total claim amount in a specified time interval, is modeled as follows

$$L^{coll} = Z_1 + Z_2 + \ldots + Z_N, \tag{2.51}$$

where N is a random variable counting the number of claims generated in the time interval under study and $Z_i, i = 1, 2 \ldots$ denote the severity of these claims.

In actuarial mathematics, the random variables $Z_i, i = 1, 2 \ldots$ are assumed independent and identically distributed and also independent of the random variable N. In this thesis, we do not impose these assumptions since again, this limits the use of the model.

Observe that no vanishing term appears in (2.51), since as mentioned, in this case the aggregate loss incorporates just actual claims. In contrast, in the individual risk model, many of the terms that determine the aggregate claim are zero, corresponding to policies that remained in force during the time period considered.

With regards to the distribution of the random variable N, this depends on the nature of the portfolio of insurance contracts. For example, in the case of a portfolio of term life insurance contracts, N will count the number of deaths (claims) over the time interval considered. If assuming deterministic mortality, an appropriate model for the number of deaths process is an inhomogeneous

Poisson process. Consequently, N will be Poisson distributed. On the other hand, for random mortality, a suitable model for the the number of deaths is a doubly stochastic Poisson process.

In certain circumstances the collective risk model can be thought of as an approximation of the individual risk model. We illustrate this assertion by considering a situation from life insurance. Concretely, assume a portfolio of n term life insurance contracts, all written at time 0 and maturing at a certain time T. Also, assume that these contracts are sold to a cohort of policyholders age a_0 at time 0 and that their mortality is deterministic. Within the individual risk model, we model the insurer's total loss over the time interval $[0, t),\ 0 < t \leq T$ as follows

$$L_t^{ind} = Y_t^1 + Y_t^2 + \ldots + Y_t^n. \tag{2.52}$$

We assume that

$$Y_t^i = X^i 1_{\{\tau_i < t\}}, \tag{2.53}$$

where τ_i denotes the time of death of the individual owning policy i and $X^i, i = 1 \ldots n$ denotes the claim that results from policy i. Let us further assume that $X^i, i = 1 \ldots n$ are independent and identically distributed random variables and moreover have a time-independent distribution.

The indicator random variables $1_{\{\tau_i < t\}}$ are Bernoulli$({}_tq_{a_0})$ distributed. Accordingly, we have

$$L_t^{ind} = \sum_{i=1}^{M_t} Z_i. \tag{2.54}$$

Here M_t denotes the number of claims (deaths) by time t in the individual risk model and has a Binomial$(n, {}_tq_{a_0})$ distribution. Then, Z_i denotes the severity of the ith claim that occurs prior to time t. In fact, $Z_i, i = 1, \ldots$ are payments $X^i, i = 1 \ldots n$ that correspond to actual claims. We assume that $Z_i, i = 1, \ldots$ are independent of the number of claims (deaths) random variable.

Now, let us return to the collective risk model and consider an inhomogeneous Poisson process $(N_t)_{0 \le t < T}$ counting the number of deaths (claims) by time t. We want that this process matches as well as possible the corresponding number of claims process from the individual risk model. Therefore, we assume that $(N_t)_{0 \le t < T}$ is such that $E[N_t] = E[M_t]$, $\forall\ t \in [0, T)$. This means that N_t is Poisson $(n_t q_{a_0})$ distributed. For a large portfolio and small probability of death ${}_t q_{a_0}$, by the virtue of the *Poisson Approximation to the Binomial Theorem*, the distribution of M_t can be approximated by the distribution of N_t.

Further, we have

$$\begin{aligned} M_{L_t^{ind}}(u) &= E[e^{uL_t^{ind}}] = E[e^{u\sum_{i=1}^{M_t} Z_i}] = E[E[e^{u\sum_{i=1}^{M_t} Z_i}|M_t]] \\ &= \sum_{m=1}^{n} E[e^{u\sum_{i=1}^{M_t} Z_i}|M_t = m]P(M_t = m) \\ &= \sum_{m=1}^{n} E[e^{u\sum_{i=1}^{m} Z_i}]P(M_t = m) \quad \text{since } Z_i, i = 1, \ldots \text{ and } M_t \text{ are ind.} \\ &\simeq \sum_{m=1}^{\infty} E[e^{u\sum_{i=1}^{m} Z_i}]P(N_t = m) = E[e^{uL_t^{coll}}] = M_{L_t^{coll}}(u), \end{aligned} \tag{2.55}$$

for n approaching ∞ and very small probability of death ${}_t q_{a_0}$.

Thus, if the size of the portfolio n and the probability of death are sufficiently large and small respectively, we have that $L_t^{ind} \overset{d}{\simeq} L_t^{coll}$. Here we assumed that by time t the total claim in both models consists of at least one claim. Otherwise, the approximation is trivial.

At this point, notice that

$$E[L_t^{coll}] = E[\sum_{i=0}^{N_t} Z_i] = E[E[\sum_{i=0}^{N_t} Z_i|N_t]] = \sum_{m=0}^{\infty} E[\sum_{i=0}^{m} Z_i|N_t = m]P(N_t = m)$$
$$= \sum_{m=0}^{\infty} mE[Z]P(N_t = m) = E[Z]E[N_t],$$

where Z is a random variable with the same distribution as Z_i.

Similarly, it can be shown that $E[L_t^{ind}] = E[Z]E[M_t]$. Accordingly, $E[L_t^{coll}] = E[L_t^{ind}]$.

On the other hand,

$$Var[L_t^{coll}] = E[Var[L_t^{coll}|N_t]] + Var[E[L_t^{coll}|N]]$$
$$= E[N_t Var[Z]] + Var[N_t E[Z]] = E[N_t]Var[Z] + E[Z]^2 Var[N_t].$$

But, $Var[N_t] > Var[M]$ and consequently we have $Var[L_t^{coll}] > Var[L_t^{ind}]$.

Accordingly, the collective model is riskier than the individual model given the greater variance of the total loss amount.

Now, given the assumption that N_t and M_t have the same mean, we have

$$\int_0^t \eta(a_0 + u)du = n_t q_{a_0}. \tag{2.56}$$

Consequently, the intensity of the Poisson process is given by

$$\eta(a_0 + t) = n_t p_{a_0} \lambda(a_0 + t). \tag{2.57}$$

As can be observed from (2.55), a necessary condition for this approximation of the individual risk model is that the number of deaths in the collective risk model approximates the one from the individual model. This happens if the size of the portfolio n and the probability of death are large and small enough, respectively. So, if the insurer wants to take advantage of this approximation, he first needs to check if the size of his portfolio and the probability of death have suitable values. Clearly, the insurer can try to increase the size of his portfolio by insuring other policyholders; however he can not do anything regarding the values of the probabilities of death of the policyholders.

Next, we consider a cohort $n = 10000$ policyholders and perform several numerical experiments to investigate for which values of the insurance contracts maturities the probabilities of death are sufficiently small to assure the validity of the approximations mentioned above. We assume that the policyholders are aged 45 at time $t = 0$, with force of mortality given by (2.13) where $\lambda(45) = 0.00778$ and $\mu^{\lambda} = 0.07204$. Then, for $t = 5$ years, $t = 10$ years, $t = 15$ years and $t = 20$ years, we consider a number of 10000 realizations of N_t and M_t and plot the corresponding histograms as well as the cumulative distribution functions.

Figure 2.3, Figure 2.4, Figure 2.5 and Figure 2.6 show that the approximation is very good for insurance contracts with maturity up to 10 years but after that the approximation is progressively less satisfactory. Moreover, as expected, observe from these histograms that the standard deviation of the number of deaths in the collective risk model is always greater than the standard deviation of the number of deaths in the individual risk model.

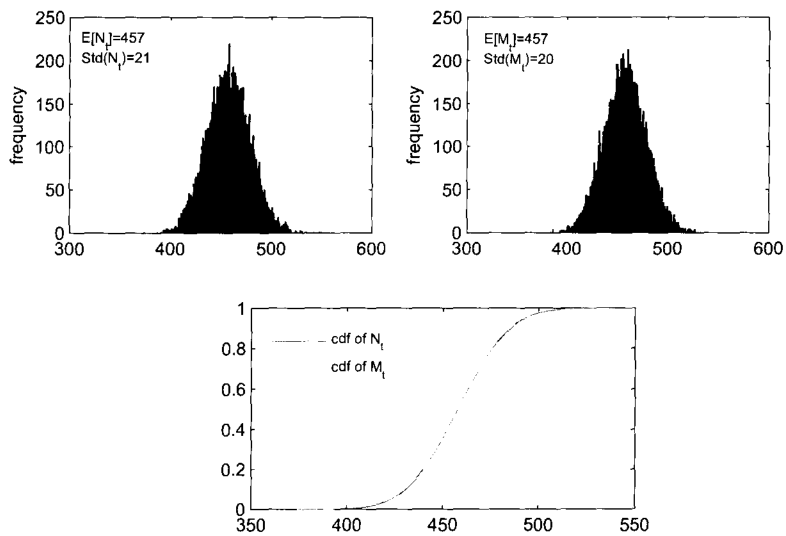

Figure 2.3: Histogram and c.d.f. of the number of deaths from time 0 to time t = 5 years in the collective (right) and individual (left) risk model.

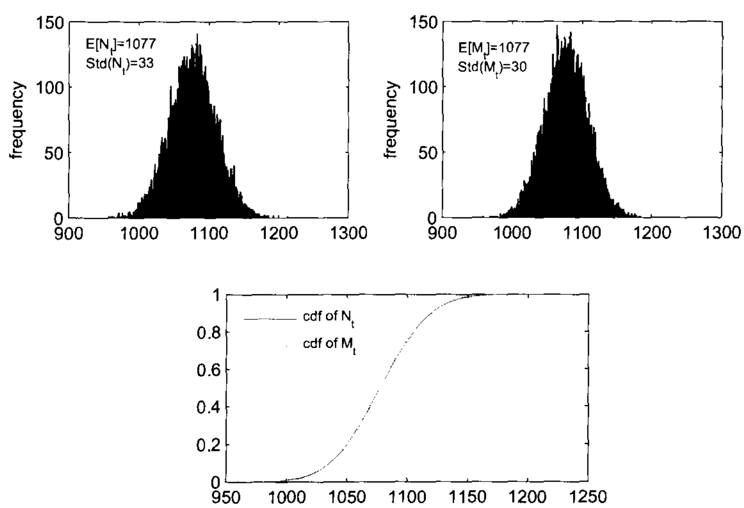

Figure 2.4: Histogram and c.d.f. of the number of deaths from time 0 to time t = 10 years in the collective (right) and individual (left) risk model.

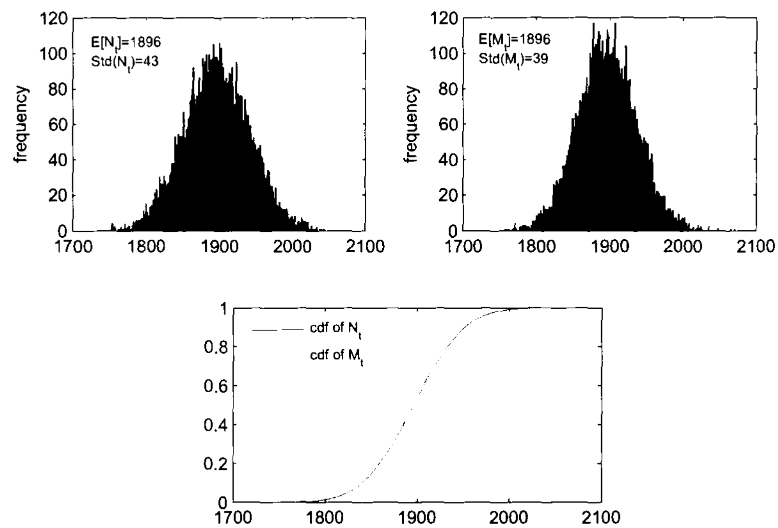

Figure 2.5: Histogram and c.d.f of the number of deaths from time 0 to time t = 15 years in the collective (right) and individual (left) risk model.

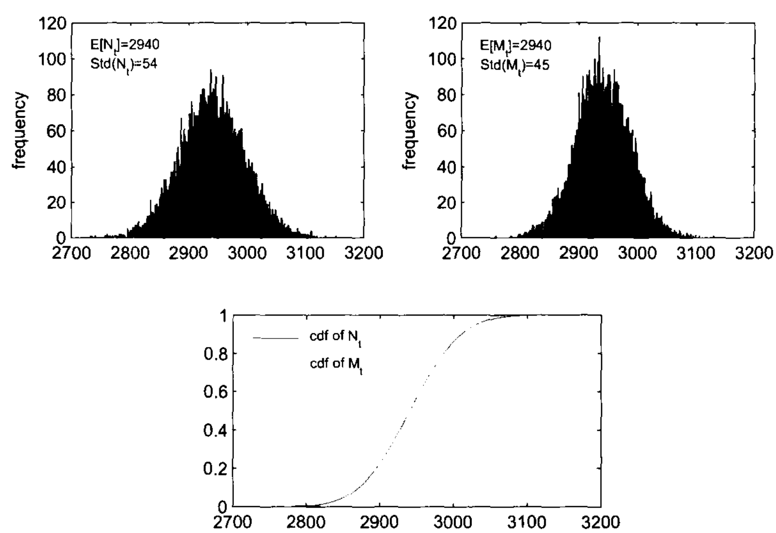

Figure 2.6: Histogram and c.d.f. of the number of deaths from time 0 to time t = 20 years in the collective (right) and individual (left) risk model.

Chapter 3

Utility Indifference Pricing of Market Independent Life Insurance Risks

Let $(\Omega, \mathcal{G}, \mathbb{G}, P)$ denote a filtered probability space with the filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ satisfying the usual conditions and containing all available information. Specifically, we define $\mathbb{G}$ as the natural filtration generated by two independent standard Brownian motions W and W^λ and a counting process for the number of deaths. In addition to $\mathbb{G}$, we consider the sub-filtrations $\mathbb{F}$, $\mathbb{F}^\lambda$ and $\mathbb{H}$ generated by W, W^λ and the number of deaths process, respectively.

Throughout this section, the financial market consists of a risky asset and a riskless money market account with constant interest rate r. The discounted price of the risky asset follows the geometric Brownian motion

$$\begin{cases} dS_t = S_t((\mu - r)dt + \sigma dW_t) \\ S_0 = S > 0, \end{cases} \tag{3.1}$$

where $\mu > r > 0$ and $\sigma > 0$.

Further, we assume an insurer that has the possibility of investing in the financial market defined above and additionally has the opportunity to sell life insurance contracts with discounted benefits of the form

$$B = \begin{cases} g_1(\tau) & \text{if } \tau < T \\ g_2(T) & \text{if } \tau \geq T, \end{cases} \tag{3.2}$$

where g_1 and g_2 are deterministic functions of time and τ is the policyholder's time of death. Observe that for $g_1 = g_2 \neq 0$ the life insurance contract is an endowment insurance, for $g_1 = 0$ and $g_2 \neq 0$ it is a pure endowment, while for $g_2 = 0$ and $g_1 \neq 0$ it is a term life insurance. In these particular situations we denote the insurance contract discounted benefit respectively by B^{End}, B^{PEnd} and B^{TL}.

Here, we model the mortality intensity of a policyholder age a_0 at a certain reference time 0 by a stochastic process $(\lambda_{a_0+t})_{t\geq 0}$ with dynamics given by

$$\begin{cases} d\lambda_{a_0+t} = \mu^\lambda \lambda_{a_0+t}dt + \sigma^\lambda dW_t^\lambda \\ \lambda_{a_0} = \lambda \end{cases} \tag{3.3}$$

where $\mu^\lambda > 0$, $\sigma^\lambda \geq 0$ and $\lambda > 0$.

In this chapter we apply the utility indifference valuation approach to solve the pricing problem for the general claim B from the insurer point of view, within a single life insurance model as well as in the individual and collective risk model. Additionally, we are interested in investigating qualitative and quantitative properties of the premium as: premium dependence on the model parameters, super-additivity of the premium as a function of the number of policies sold and comparison of the premiums in the three loss models mentioned above.

3.1 Indifference premium in a single life insurance model

3.1.1 Lump-sum premium

We consider an insurance model consisting of a single life, aged a_0 at time 0 and assume that the insurer accepts to sell to (a_0) a life insurance contract with discounted benefit B as defined by (3.2). In this case, we define the value function of the insurer as follows

$$u^B(x, \lambda, t) = \sup_{\pi \in \mathcal{A}} E[U(X_T - g_2(T)1_{\{\tau \geq T\}})|X_t = x, \lambda_{a_0+t} = \lambda]. \tag{3.4}$$

Here, the insurer's discounted wealth has the dynamics

$$\begin{cases} dX_s = \pi_s\left((\mu - r)ds + \sigma dW_s\right), \quad s \neq \tau \\ X_t = x \\ X_{\tau+} = X_{\tau-} - g_1(\tau), \text{ if } \tau < T \end{cases} \tag{3.5}$$

and the set of admissible controls $\mathcal{A} = \mathcal{A}[t, T]$ consists of controls π that are $(\mathcal{F}_s^t)_{s \geq t}$ progressively measurable and satisfy the integrability condition $E\left[\int_t^T \pi_s^2 ds\right] < \infty$.

Applying Bellman's principle of dynamic programming and Itô's lemma, we obtain that the value function $u^B(x, \lambda, t)$ satisfies the HJB equation

$$\begin{cases} u_t^B + \max_\pi[(\mu - r)\pi u_x^B + \frac{1}{2}\sigma^2\pi^2 u_{xx}^B] + \mu^\lambda \lambda u_\lambda^B + \frac{1}{2}(\sigma^\lambda)^2 u_{\lambda\lambda}^B \\ \qquad\qquad\qquad + \lambda(u^0(x - g_1(t), t) - u^B(x, \lambda, t)) = 0 \\ u^B(x, \lambda, T) = U(x - g_2(T)). \end{cases} \tag{3.6}$$

The last term of equation (3.6) appears due to a potential death of the policyholder. In this case the wealth drops by $g_1(t)$ and since there is no risk left to the insurer, the value function u^B switches to the value function in the Merton investment problem.

Now, we assume that $u_{xx}^B < 0$. This implies that the maximum in (3.6) is well defined. By the first order necessary condition, the maximum is attained in

$$\pi^* = -\frac{\mu - r}{\sigma^2}\frac{u_x^B(x, \lambda, t)}{u_{xx}^B(x, \lambda, t)}. \tag{3.7}$$

Further, due the assumption of exponential utility, we consider an ansätz of the form $u^B(x, \lambda, t) = u^0(x, t)f(\lambda, t)$, where u^0 represents the value function of the insurer in the Merton investment problem. Then, we have

$$\pi^* = -\frac{\mu - r}{\sigma^2}\frac{u_x^0}{u_{xx}^0} = \frac{\mu - r}{\gamma\sigma^2}. \tag{3.8}$$

Substituting π^* and the ansätz above in (3.6), leads to

$$\begin{cases} (u_t^0 - \frac{(\mu - r)^2}{2\sigma^2}u^0)f + u^0 f_t + \mu^\lambda \lambda u^0 f_\lambda + \frac{1}{2}(\sigma^\lambda)^2 u^0 f_{\lambda\lambda} + \lambda u^0(e^{\gamma g_1(t)} - f) = 0 \\ f(T) = e^{\gamma g_2(T)}. \end{cases} \tag{3.9}$$

Observe that the first bracket from (3.9) represents the HJB equation for u^0.

Since u^0 solves this equation, (3.9) reduces to the linear partial differential equation

$$\begin{cases} f_t + \mu^\lambda \lambda f_\lambda + \frac{1}{2}(\sigma^\lambda)^2 f_{\lambda\lambda} + \lambda(e^{\gamma g_1(t)} - f) = 0 \\ f(T) = e^{\gamma g_2(T)}. \end{cases} \tag{3.10}$$

Further, the Feynman-Kač formula leads to

$$\begin{aligned} f(\lambda, t) &= E_{t,\lambda}[e^{\gamma g_2(T) - \int_t^T \lambda_{a_0+s} ds}] + \int_t^T e^{\gamma g_1(s)} E_{t,\lambda}[\lambda_{a_0+s} e^{-\int_t^s \lambda_{a_0+u} du}] ds \\ &= e^{\gamma g_2(T)} p(a_0, t, T) + \int_t^T e^{\gamma g_1(s)} dq(a_0, t, s) \\ &= E_{t,\lambda}[e^{\gamma B}]. \end{aligned}$$

Notice that the expectations from the preceding formula are all bounded since (λ_{a_0+t}) is bounded a.e.

For well behaved benefit functions g_1 and g_2, the ansätz proposed is smooth. Additionally, since $u^0(x,t)$ has the properties of concavity and exponential growth in x, the ansätz inherits these two properties. Consequently, by the Verification Theorem the ansätz coincides with the value function. Accordingly, the value function is given by

$$u^B(x, \lambda, t) = u^0(x, t) E_{t,\lambda}[e^{\gamma B}]. \tag{3.11}$$

Also, the Verification Theorem implies that the optimal investment policy can be specified by the first order condition, and is given by (3.8). Observe that the optimal investment policy is the same as in the Merton problem. This result agrees with our intuition since the insurance claims are independent of the financial market and clearly, the insurer will not hedge them with instruments present in the financial market.

Now, let P^B denote the indifference premium for the insurance contract with discounted benefit B. P^B satisfies the equation

$$u^0(x,t) = u^B(x + P^B, \lambda, t). \tag{3.12}$$

By (3.11) and using the properties of the solution of the Merton problem, the indifference premium equation becomes

$$u^0(x,t) = u^0(x + P^B, t)E_{t,\lambda}[e^{\gamma B}] = u^0(x,t)e^{-\gamma P^B}E_{t,\lambda}[e^{\gamma B}]. \tag{3.13}$$

Therefore,

$$P^B(\lambda, t) = \frac{1}{\gamma} \ln E_{t,\lambda}[e^{\gamma B}]. \tag{3.14}$$

In particular, for an endowment insurance the premium becomes

$$\begin{aligned} P^{End}(\lambda,t) &= \frac{1}{\gamma} \ln E_{t,\lambda}[e^{\gamma B^{End}}] \\ &= \frac{1}{\gamma} \ln \left(e^{\gamma g_1(T)} p(a_0,t,T) + \int_t^T e^{\gamma g_1(s)} E_{t,\lambda}[\lambda_s e^{-\int_t^s \lambda_{a_0+u,u} du}] ds \right) \end{aligned}$$

while the premiums for a pure endowment and term life insurance are given by

$$\begin{aligned} P^{PEnd}(\lambda,t) &= \frac{1}{\gamma} \ln E_{t,\lambda}[e^{\gamma B^{PEnd}}] \\ &= \frac{1}{\gamma} \ln \left(e^{\gamma g_2(T)} p(a_0,t,T) + \int_t^T E_{t,\lambda}[\lambda_s e^{-\int_t^s \lambda_{a_0+u,u} du}] ds \right) \\ &= \frac{1}{\gamma} \ln \left(e^{\gamma g_2(T)} p(a_0,t,T) + q(a_0,t,T) \right) \end{aligned}$$

$$\begin{aligned} P^{TL}(\lambda,t) &= \frac{1}{\gamma}\ln E_{t,\lambda}[e^{\gamma B^{TL}}] \\ &= \frac{1}{\gamma}\ln\left(p(a_0,t,T) + \int_t^T e^{\gamma g_1(s)} E_{t,\lambda}[\lambda_s e^{-\int_t^s \lambda_{a_0+u,u}du}]ds\right), \end{aligned}$$

respectively.

At this point, we would like to emphasize that under a stochastic setting for the force of mortality, the value of a pure endowment is greater than in a deterministic setting. This is a direct consequence of the fact that the survival probability, when assuming stochastic mortality, is greater than its counterpart in a deterministic setting for mortality. Indeed,

$$\begin{aligned} P^{PEnd}(\lambda,t) &= \frac{1}{\gamma}\ln\left(1 + (e^{\gamma g_2(T)} - 1)p(a_0,t,T)\right) \\ &\geq \frac{1}{\gamma}\ln(1 + (e^{\gamma g_2(T)} - 1)_{T-t}p_{a_0+t}) = \frac{1}{\gamma}\ln E[e^{\gamma B^{PEnd}}] = P^{PEnd}(t) \end{aligned}$$

where $P^{PEnd}(t)$ denotes the premium for the pure endowment considering deterministic mortality.

Remarks 3.1.1.

- For deterministic mortality intensity, the indifference premium is simply the premium generated by the principle of equivalent utility in a static market setting. This fact is expected, since the insurance risks that we examine here are independent of the financial market and thus the dynamic market setting considered cannot have any influence on the premiums.
- On the other hand, the assumption of stochastic mortality is reflected in the premiums of the insurance contracts analyzed.

- The indifference premium is an increasing function of the risk aversion parameter γ. Indeed, taking $0 < \gamma_1 < \gamma_2$ and applying Holder's inequality, leads to

$$E_{t,\lambda}[e^{\gamma_1 B}] \leq \left(E_{t,\lambda}[e^{\gamma_2 B}]\right)^{\frac{\gamma_1}{\gamma_2}}.$$

Thus,

$$\ln E[e^{\gamma_1 B}] \leq \frac{\gamma_1}{\gamma_2} \ln E_{t,\lambda}[e^{\gamma_2 B}]$$

which implies that $P^B(\lambda, t; \gamma_1) \leq P^B(\lambda, t; \gamma_2)$.

- Indifference valuation is generally a nonlinear pricing rule. Observe that

$$B^{End} = B^{PEnd} + B^{TL}.$$

Then, we have

$$\begin{aligned} P^{End}(\lambda, t) &= \frac{1}{\gamma} \ln E_{t,\lambda}[e^{\gamma(B^{PEnd}+B^{TL})}] \\ &= \frac{1}{\gamma} \ln \left(E_{t,\lambda}[e^{\gamma B^{PEnd}}] E_{t,\lambda}[e^{\gamma B^{TL}}] + Cov(e^{\gamma B^{PEnd}}, e^{\gamma B^{TL}}) \right) \end{aligned}$$

but since the random variables B^{PEnd} and B^{TL} are negatively correlated, it follows that the premiums satisfy the relation

$$P^{End}(\lambda, t) \leq P^{PEnd}(\lambda, t) + P^{TL}(\lambda, t),$$

where the equality case corresponds to a risk neutral insurer.

- Note that $B^{End} \geq B^{PEnd}$ and also $B^{End} \geq B^{TL}$. Therefore

$$\frac{1}{\gamma} \ln E_{t,\lambda}[e^{\gamma B^{End}}] \geq \frac{1}{\gamma} \ln E_{t,\lambda}[e^{\gamma B^{PEnd}}]$$

$$\frac{1}{\gamma}\ln E_{t,\lambda}[e^{\gamma B^{End}}] \geq \frac{1}{\gamma}\ln E_{t,\lambda}[e^{\gamma B^{TL}}]$$

Hence, we have

$$P^{End}(\lambda,t) \geq P^{PEnd}(\lambda,t) \text{ and } P^{End}(\lambda,t) \geq P^{TL}(\lambda,t).$$

Accordingly, the indifference premium is an increasing function of the claim size.

- As the risk aversion of the insurer approaches zero, the indifference premium reduces to the net premium, as proved below:

$$\begin{aligned}
&\lim_{\gamma\to 0} P^B(\lambda,t)\\
&= \lim \frac{\ln(e^{\gamma g_2(T)}E_{t,\lambda}[e^{-\int_t^T \lambda_{a_0+s}ds}] + \int_t^T e^{\gamma g_1(s)}E_{t,\lambda}[\lambda_{a_0+s}e^{-\int_t^s \lambda_{a_0+u}du}]ds)}{\gamma}\\
&= \lim \frac{e^{\gamma g_2(T)}g_2(T)E_{t,\lambda}[e^{-\int_t^T \lambda_{a_0+s}ds}] + \int_t^T e^{\gamma g_1(s)}g_1(s)E_{t,\lambda}[\lambda_{a_0+s}e^{-\int_t^s \lambda_{a_0+u}du}]ds}{e^{\gamma g_2(T)}E_{t,\lambda}[e^{-\int_t^T \lambda_{a_0+s}ds}] + \int_t^T e^{\gamma g_1(s)}E_{t,\lambda}[\lambda_{a_0+s}e^{-\int_t^s \lambda_{a_0+u}du}]ds}\\
&= g_2(T)E_{t,\lambda}[e^{-\int_t^T \lambda_{a_0+s}ds}] + \int_t^T g_1(s)E_{t,\lambda}[\lambda_{a_0+s}e^{-\int_t^s \lambda_{a_0+u}du}]ds = E_{t,\lambda}[B]
\end{aligned}$$

We conclude our analysis by numerically illustrating the some of the analytical results obtained so far. We implement the indifference premium for endowment insurances, pure endowments and term life insurances, corresponding to an American male policyholder, aged 45 years and born in 1900. We model his force of mortality by the affine process (1.51), where the model's parameters are given by (1.61). Consequently, the survival probabilities can be calculated directly via (1.60), avoiding the use of Monte-Carlo methods. Also, this leads to an efficient computation of the premiums.

We consider endowment insurances, pure endowments and term life insurances, with discounted benefits as follows

$$B^{End} = \begin{cases} Ge^{-r\tau} & \text{if } \tau < T \\ Ge^{-rT} & \text{if } \tau \geq T \end{cases} \qquad B^{PEnd} = \begin{cases} 0 & \text{if } \tau < T \\ Ge^{-rT} & \text{if } \tau \geq T. \end{cases} \tag{3.15}$$

$$B^{TL} = \begin{cases} Ge^{-r\tau} & \text{if } \tau < T \\ 0 & \text{if } \tau \geq T. \end{cases} \tag{3.16}$$

In our first experiment, we illustrate the dependence of the premiums on the time to maturity for several risk aversion parameters. We consider that $G = 10$, $r = 0.06$, T varies between 10 and 20 years and that the insurer's risk aversion takes the values $\gamma = 0$, $\gamma = 0.05$ and $\gamma = 0.1$.

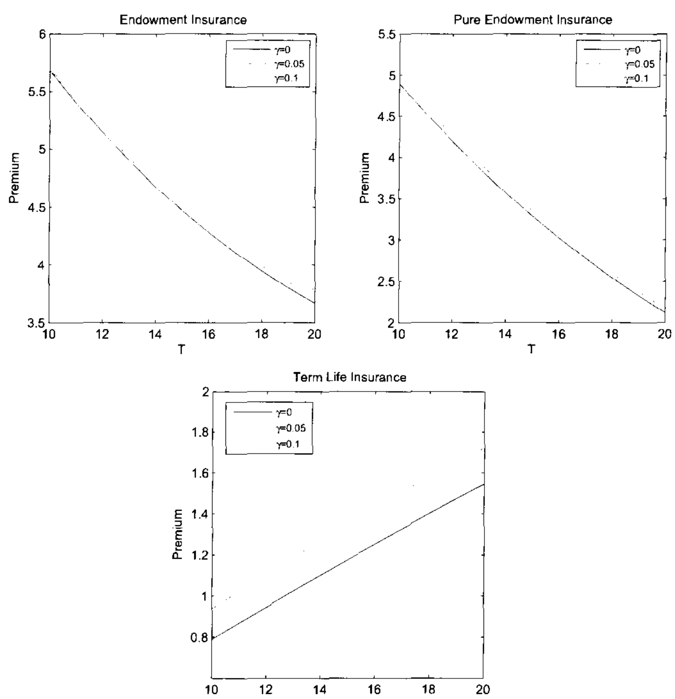

Figure 3.1: Lump-sum premium for a constant benefit endowment insurance, pure endowment and term life insurance, as a function of time to maturity, for different risk aversion parameters.

Notice from Figure 3.1 that the premium for endowment insurances and pure endowments decreases as the maturity time increases. For pure endowment insurances, that is due to the fact that the survival probabilities decrease as maturity increases. The same argument explains the behavior of the premium for endowment insurances, since for young adults, as in our case, the pure endowment component is dominant. On the other hand, for term life insurances the premium increases as maturity increases and that is because the probability of death increases with maturity. Then, as expected, observe that as the insurer's risk aversion increases, the premium for all contracts increases.

Recall that we proved that the indifference premium for endowment insurance is smaller than the sum of the indifference premiums for pure endowment insurance and term life insurance. Below, we demonstrate numerically this fact for an insurer with risk aversion parameter $\gamma = 0.05$. All the other parameters are the same as in the preceding numerical experiment.

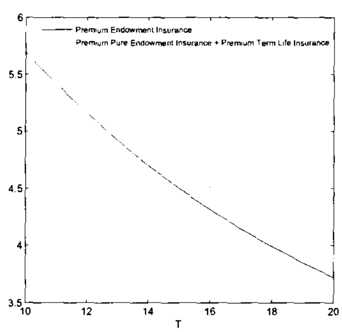

Figure 3.2: Comparison between the lump-sum premium for a constant benefit endowment insurance and the sum of the premiums for a constant benefit pure endowment and term life insurance.

3.1.2 Continuous premium

In what follows, we calculate the continuous premium rate for the insurance contract with benefit B defined by (3.2) . We assume that this premium rate is set when the insurance contract is signed and remains unchanged during the life of the policy. In this case, we define the value function of the insurer with the insurance risk as follows

$$u^B(x,\lambda,t;h^B) = \sup_{\pi\in\mathcal{A}} E[U(X_T - g_2(T)1_{\{\tau>T\}})|X_t = x, \lambda_{a_0+t} = \lambda] \tag{3.17}$$

where $h^B(\lambda,t)$ denotes the premium rate at time t. Here, the discounted wealth process has the dynamics

$$\begin{cases} dX_s = ((\mu - r)\pi_s + h^B e^{-rs})ds + \sigma\pi_s dW_s \text{ if } t < s < \tau \\ dX_s = \pi_s((\mu - r)ds + \sigma dW_s) \text{ if } \tau < s < T \\ X_t = x \\ X_{\tau+} = X_{\tau-} - g_1(\tau) \text{ if } \tau < T. \end{cases} \tag{3.18}$$

Applying Bellman's principle of dynamic programming and Itô's lemma, we obtain that u^B solves the HJB equation

$$\begin{cases} u_t^B + \max_\pi[(\mu - r)\pi u_x^B + \frac{1}{2}\sigma^2\pi^2 u_{xx}^B] + \mu^\lambda \lambda u_\lambda^B + \frac{1}{2}(\sigma^\lambda)^2 u_{\lambda\lambda}^B + h^B u_x^B e^{-rt} \\ \qquad + \lambda\left(u^0(x - g_1(t),\lambda,t) - u^B(x,\lambda,t)\right) = 0 \\ u^B(x,\lambda,T) = U(x - g_2(T)) \end{cases} \tag{3.19}$$

Following arguments similar to those applied in the lump-sum premium case, we obtain that the value function is given by $u^B(x,\lambda,t) = u^0(x,t)l(\lambda,t;h^B)$, where l satisfies the linear partial differential equation

$$\begin{cases} l_t - \gamma h^B l e^{-rt} + \mu^\lambda \lambda l_\lambda + \frac{1}{2}(\sigma^\lambda)^2 l_{\lambda\lambda} + \lambda(e^{\gamma g_1(t)} - l) = 0 \\ l(\lambda, T) = e^{\gamma g_2(T)} \end{cases} \tag{3.20}$$

Therefore, l can be calculated via Feynman-Kač formula, as follows

$$\begin{aligned} l(\lambda, t; h^B) &= e^{\gamma g_2(T) - \gamma \int_t^T h^B e^{-rs} ds} E_{t,\lambda}[e^{-\int_t^T \lambda_{a_0+u} du}] \\ &\quad + \int_t^T e^{\gamma g_1(s) - \gamma \int_t^s h^B e^{-ru} du} E_{t,\lambda}[\lambda_s e^{-\int_t^s \lambda_{a_0+u} du}] ds \\ &= e^{\gamma g_2(T) - \gamma \int_t^T h^B e^{-rs} ds} p(a_0, t, T) + \int_t^T e^{\gamma g_1(s) - \gamma \int_t^s h^B e^{-ru} du} dq(a_0, t, s) \\ &= E_{t,\lambda}[e^{\gamma B - \gamma H}], \end{aligned}$$

where H is a random variable denoting the total discounted premium paid during the life of the policy. H is defined as follows

$$H = h^B \int_t^T e^{-rs} ds. \tag{3.21}$$

Now, we introduce the function $V(\lambda, t; h^B) = \frac{1}{\gamma} \ln l(\lambda, t; h^B)$. Let us give an intuitive description of V. For this, following Bowers, Gerber, Hickman, Jones & Nesbit (1997), we define the concept of *benefit reserve.*

Definition 3.1.1. Suppose that an insurer assumes an insurance risk at a certain time t. The benefit reserve at time $s \geq t$ is the amount $V(s)$ that makes the insurer indifferent between continuing with the risk while receiving the premium and paying the amount $V(s)$ to a reinsurer to assume the risk.

In other words, the reserve at time $s \geq t$ is such that

$$u^0(X_s - V, s) = u^B(X_s, \lambda_{a_0+s}, s; h^B). \tag{3.22}$$

It is straightforward to verify that $V(\lambda_{a_0+s}, s; h^B)$ satisfies equation (3.22). Therefore, from now on we refer to $V(\lambda_{a_0+s}, s; h^B)$ as the benefit reserve at time s.

The indifference premium rate $h^B(\lambda, t)$ is such that the insurer at time t is indifferent between accepting or not accepting the insurance risk, that is

$$u^0(x, t) = u^B(x, \lambda, t; h^B). \tag{3.23}$$

Accordingly, h^B is given implicitly by the equation

$$l(\lambda, t; h^B) = 1 \tag{3.24}$$

or in terms of the benefit reserve, the premium rate is such that $V(\lambda, t; h^B) = 0$.

So, the indifference premium rate is such that the benefit reserve has zero value at the moment of writing the insurance contract. However, over time, due to changes in the policyholder's mortality, the premium rate might not coincide with the prevailing indifference premium rate and consequently the benefit reserve will no longer be zero.

In particular, for the claims B^{End}, B^{PEnd} and B^{TL}, the corresponding premium rates, denoted by h^{End}, h^{PEnd} and h^{TL} respectively, are such that

$$\frac{1}{\gamma}\ln(e^{\gamma g_1(T)-\gamma h^{End}\int_t^T e^{-rs}ds}p(a_0, t, T) + \int_t^T e^{\gamma g_1(s)-\gamma h^{End}\int_t^s e^{-ru}du}dq(a_0, t, s)) = 0$$

$$\frac{1}{\gamma}\ln(e^{\gamma g_2(T)-\gamma h^{PEnd}\int_t^T e^{-rs}ds}p(a_0,t,T)+\int_t^T e^{-\gamma h^{PEnd}\int_t^s e^{-ru}du}dq(a_0,t,s))=0$$
$$\frac{1}{\gamma}\ln(e^{-\gamma h^{TL}\int_t^T e^{-rs}ds}p(a_0,t,T)+\int_t^T e^{\gamma g_1(s)-\gamma h^{TL}\int_t^s e^{-ru}du}dq(a_0,t,s))=0.$$

Remarks 3.1.2.

- Multiplying (3.24) by $-e^{-\gamma x}$, leads to

$$U(x)=E_{t,\lambda}[U(x+H-B)] \tag{3.25}$$

Notice that (3.25) is nothing more than the principle of equivalent utility, modified in order to incorporate the continuous premium and the stochastic mortality assumptions.

- Observe that

$$l(\lambda,t;h^B)\geq e^{-\gamma h^B\int_t^T e^{-rs}ds}E_{t,\lambda}[e^{\gamma B}]$$

Taking into account that $l(\lambda,t;h^B)=1$, after some calculations, we obtain

$$(T-t)h^B\geq h^B\int_t^T e^{-rs}ds\geq\frac{1}{\gamma}\log E_{t,\lambda}[e^{\gamma B}] \tag{3.26}$$

and now, we recognize in the right hand side of (3.26) the lump-sum premium for the insurance contract with benefit B at time t.

Accordingly, if the insurance contract has one year maturity, the continuous premium rate is greater than the lump-sum premium. That is expected because in the case of continuous premium, the insurer receives the premium for a random period of time, contrary to the case of lump sum premium. Thus, the insurer in the former case takes more risk than in the latter and this fact will be reflected by the premiums.

- We showed that

$$E_{t,\lambda}[e^{\gamma(B-H)}] - 1 = 0. \tag{3.27}$$

Dividing by γ and taking the limit as γ goes to 0 in (3.27), leads to

$$E_{t,\lambda}[B - H] = 0. \tag{3.28}$$

Therefore, we have

$$E_{t,\lambda}[B] - h^B \int_t^T e^{-rs} p(a_0, t, T) - \int_t^T \int_t^s h^B e^{-ru} du \; dq(a_0, t, s) = 0. \tag{3.29}$$

After some calculations, we obtain that for a risk neutral insurer, the continuous premium rate is given by

$$h^B = \frac{E_{t,\lambda}[B]}{\int_t^T e^{-rs} p(a_0, t, s) ds}. \tag{3.30}$$

- We would like to point out that a similar analysis applies to a more complex choice of the premium, such as a time dependent premium $\tilde{h}(s; \lambda, t) = h^0(s; t) h(\lambda, t)$, where $h(\lambda, t)$ is the premium set at the initial time t and $h^0(s; t)$ is a "ramp-up" premium factor.

Next, we consider the same cohort of individuals as in the preceding subsection and implement the indifference premium rate for endowment insurances, pure endowments and term life insurances with benefits given by (3.15) and (3.16) respectively, when the time to maturity varies between 10 and 20 years, $r = 0.06$ and for three choices of the insurer's risk aversion parameter $\gamma = 0$, $\gamma = 0.05$ and $\gamma = 0.1$.

Parallel to the results obtained in the lump sum premium case, observe from Figure 3.3 that as the time to maturity increases, the premium rates for endowment insurances and pure endowment insurances increase, while the premium rate for term life insurances decreases. Also, as expected, the premium rate is an increasing function of the risk aversion parameter γ.

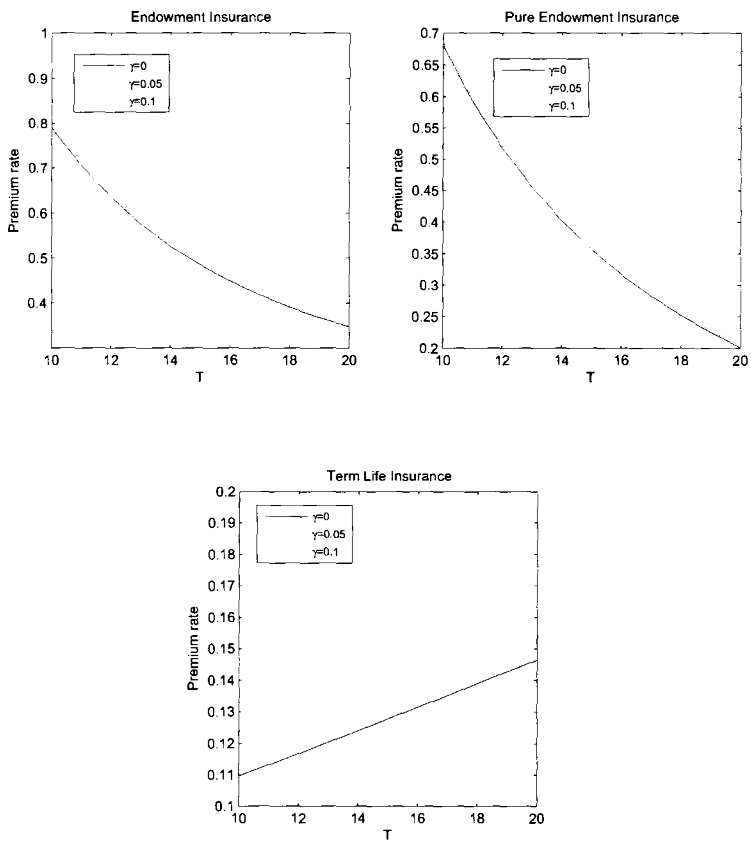

Figure 3.3: Indifference premium rate for a constant benefit endowment insurance, pure endowment and term life insurance for different risk aversion parameters.

3.2 Indifference premium in the individual risk model

In this section, we examine the pricing of life insurance risks described by (3.2) under the assumption that the insurer pools together a certain numbers of such risks. Concretely, we consider a portfolio consisting of n policies corresponding to a cohort of individuals aged a_0 at time 0 and having their force of mortality given by (3.3). Further, we model the losses on this portfolio using an individual risk model. In this context, in what follows, we apply the indifference valuation approach and calculate the lump-sum premium and the continuous premium rate for the insurance contracts mentioned above.

3.2.1 Lump-sum premium

Assume that at time $t = 0$, the insurer sells life insurance contracts with discounted benefit given by (3.2) to a cohort of individuals and that at a certain time $t \in [0, T)$, k individuals from the initial cohort are still alive. Then, the value function of the insurer is given by

$$u^{(k)}(x, \lambda, t) = \sup_{\pi \in \mathcal{A}} E[U(X_T - C_T) | X_t = x, \lambda_{a_0+t} = \lambda], \tag{3.31}$$

where the discounted wealth process satisfies the stochastic differential equation

$$\begin{cases} dX_s = \pi_s \left((\mu - r) ds + \sigma dW_s \right), \quad s \neq \tau_i \\ X_t = x \\ X_{\tau_i +} = X_{\tau_i -} - C_{\tau_i}, \ \text{ if } \tau_i < T. \end{cases} \tag{3.32}$$

Here τ_i, $i = 1 \dots k$ denote the times of death of the policyholders. We assume that these times of death are not necessarily distinct. Consequently, we choose to denote by C_{τ_i} the total discounted claim at time $\tau_i < T$, and not just the loss on the ith policy. Then, we denote by C_T the total discounted claim upon survival at time T.

Alternatively, the discounted wealth dynamics can be written as

$$\begin{cases} dX_s = \pi_s \left((\mu - r)ds + \sigma dW_s\right) - dL_s \\ X_t = x \end{cases} \tag{3.33}$$

Here L_s, $s \in (t, T)$ denotes the total loss on the insurer's portfolio on the time interval $[t, s)$ and we model it as follows

$$L_s = \sum_{i=1}^{k} g_1(\tau_i) 1_{\{\tau_i < s | \tau_i > t\}}. \tag{3.34}$$

Consequently, the total discounted claim at a certain time $\tau_i < T$ can be written as

$$C_{\tau_i} = L_{\tau_i +} - L_{\tau_i}. \tag{3.35}$$

Then, we model the total discounted claim at maturity by

$$C_T = g_2(T) \sum_{i=1}^{k} 1_{\{\tau_i > T | \tau_i > t\}}. \tag{3.36}$$

It follows that the HJB equation for $u^{(k)}$ is

$$\begin{cases} u^{(0)} = u^0 \\ \text{For } k \geq 1 \\ u_t^{(k)} + \max_{\pi\in\mathcal{A}}[(\mu - r)\pi u_x^{(k)} + \frac{1}{2}\sigma^2\pi^2 u_{xx}^{(k)}] + \mu^\lambda \lambda u_\lambda^{(k)} + \frac{1}{2}(\sigma^\lambda)^2 u_{\lambda\lambda}^{(k)} \\ \qquad\qquad + k\lambda\left(u^{(k-1)}(x - g_1(t), \lambda, t) - u^{(k)}(x, \lambda, t)\right) = 0 \\ u^{(k)}(x, \lambda, T) = U(x - kg_2(T)). \end{cases} \tag{3.37}$$

Assume that the solution of (3.37) is concave in wealth. Then, the maximum in (3.37) is well defined and attained at

$$\pi_t^* = -\frac{\mu - r}{\sigma^2}\frac{u_x^{(k)}}{u_{xx}^{(k)}}. \tag{3.38}$$

Inserting the the expression of π_t^* in the HJB equation for $u^{(k)}$, we obtain

$$\begin{cases} u^{(0)} = u^0 \\ \text{For } k \geq 1 \\ u_t^{(k)} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(u_x^{(k)})^2}{u_{xx}^{(k)}} + \mu^\lambda \lambda u_\lambda^{(k)} + \frac{1}{2}(\sigma^\lambda)^2 u_{\lambda\lambda}^{(k)} \\ \qquad\qquad + k\lambda\left(u^{(k-1)}(x - g_1(t), \lambda, t) - u^{(k)}(x, t)\right) = 0 \\ u^{(k)}(x, \lambda, T) = U(x - kg_2(T)). \end{cases} \tag{3.39}$$

Due to the assumption of exponential utility, we consider an ansätz of the form

$$u^{(k)}(x, \lambda, t) = u^0(x, t)f^{(k)}(\lambda, t). \tag{3.40}$$

Inserting the ansätz in (3.39), after some straightforward calculations, it follows that $f^{(k)}$ solves the linear partial differential equation

$$\begin{cases} f_t^{(k)} + \mu^\lambda \lambda f_\lambda^{(k)} + \frac{1}{2}(\sigma^\lambda)^2 f_{\lambda\lambda}^{(k)} + k\lambda\left(e^{\gamma g_1(t)} f^{(k-1)} - f^{(k)}\right) = 0 \\ f^{(k)}(\lambda, T) = e^{\gamma k g_2(T)} \end{cases} \tag{3.41}$$

with $f^{(0)} = 1$. Applying the Feynman-Kač formula, we obtain that

$$f^{(k)}(\lambda, t) = e^{\gamma k g_2(T)} E_{t,\lambda}[e^{-k\int_t^T \lambda_{a_0+u} du}] \\ + k \int_t^T e^{\gamma g_1(s)} E_{t,\lambda}[e^{-k\int_t^s \lambda_{a_0+u} du} \lambda_{a_0+s} f^{(k-1)}(\lambda_{a_0+s}, s)] ds,$$

which implies that $f^{(k)}$ can be calculated recursively.

Let $P^{(k)}$ denote the indifference premium for k insurance contracts with discounted benefit given by (3.2). $P^{(k)}$ solves the equation

$$u^0(x,t) = u^{(k)}(x + P^{(k)}, \lambda, t). \tag{3.42}$$

It follows that $P^{(k)}(\lambda, t)$ is given by

$$P^{(k)}(\lambda, t) = \frac{1}{\gamma} \ln f^{(k)}(\lambda, t). \tag{3.43}$$

Notice that the premium can be found by solving k recursively defined linear partial differential equations or alternatively by calculating the functions $f^{(k)}$ via a Monte-Carlo method.

Next, we determine the premium corresponding to a risk neutral insurer. First, let us write the equation for the premium $P^{(k)}$. This equation can be obtained from (3.41) and is as follows

$$\begin{cases} P_t^{(k)} + \mu^\lambda \lambda P_\lambda^{(k)} + \frac{1}{2}(\sigma^\lambda)^2 P_{\lambda\lambda}^{(k)} + \frac{1}{2}\gamma(\sigma^\lambda)^2 (P_\lambda^{(k)})^2 \\ \qquad\qquad + \frac{k\lambda}{\gamma}\left(e^{\gamma g_1(t) + \gamma P^{(k-1)} - \gamma P^{(k)}} - 1\right) = 0 \\ P^{(k)}(\lambda, T) = k g_2(T). \end{cases} \tag{3.44}$$

As the insurer's risk aversion γ approaches zero, (3.44) becomes

$$\begin{cases} P_t^{(k)} + \mu^\lambda \lambda P_\lambda^{(k)} + \frac{1}{2}(\sigma^\lambda)^2 P_{\lambda\lambda}^{(k)} + k\lambda\left(g_1(t) + P^{(k-1)} - P^{(k)}\right) = 0 \\ P^{(k)}(\lambda, T) = k g_2(T) \end{cases} \tag{3.45}$$

We expect that the solution of (3.45) is the net premium, that is $E_{t,\lambda}[kB]$. Next, we show that this is indeed the case.

First, observe that $E_{t,\lambda}[kB] = kP^1(\lambda, t) + kP^2(\lambda, t)$. Here, P^1 and P^2 are the premiums corresponding to a risk neutral insurer for a pure endowment insurance and a term life insurance with discounted benefits given by B^{PEnd} and B^{TL} respectively, as calculated in a single life insurance model. Consequently, P^1 and P^2 satisfy the equations

$$\begin{cases} P_t^1 + \mu^\lambda \lambda P_\lambda^1 + \frac{1}{2}(\sigma^\lambda)^2 P_{\lambda\lambda}^1 - \lambda P^1 = 0 \\ P^1(\lambda, T) = g_2(T) \end{cases}$$

$$\begin{cases} P_t^2 + \mu^\lambda \lambda P_\lambda^2 + \frac{1}{2}(\sigma^\lambda)^2 P_{\lambda\lambda}^2 + \lambda(g_1(t) - P^2) = 0 \\ P^2(\lambda, T) = 0. \end{cases}$$

Accordingly, $P(\lambda, t; k) := kP^1(\lambda, t) + kP^2(\lambda, t)$ solves the partial differential equation

$$\begin{cases} P_t + \mu^\lambda \lambda P_\lambda + \frac{1}{2}(\sigma^\lambda)^2 P_{\lambda\lambda} + k\lambda\left(g_1(t) - P^2 - P^1\right) = 0 \\ P^2(\lambda, T) = 0. \end{cases}$$

But, $-P^1 - P^2 = P(\lambda, t; k-1) - P(\lambda, t; k)$. Thus, $P^{(k)}(\lambda, t; \gamma = 0)$ coincide with $P(\lambda, t; k) = E_{t,\lambda}[kB]$ by uniqueness.

At this point, it is interesting to analyze the pricing of the life insurance contracts considered above, assuming deterministic force of mortality. In this situation, the value function $u^{(k)}$ is given by (3.31) and (3.32) or (3.33) eliminating the dependence of the parameter λ. In fact, given the deterministic nature of the force of mortality, the lambda dependence of the value function, here is built into the variable t. So the problem that we are going to analyze below is different of the pricing problem under stochastic force of mortality when the volatility parameter σ approaches 0.

In this case, the corresponding HJB equation can be obtained from (3.37), removing the λ dependence. Accordingly, we have

$$\begin{cases} u^{(0)}(x,t) = u^0(x,t) \\ \text{For } k \geq 1 \\ u_t^{(k)} + \max_\pi[(\mu - r)\pi u_x^{(k)} + \frac{1}{2}\sigma^2\pi^2 u_{xx}^{(k)}] \\ \qquad + k\lambda(a_0 + t)\left(u^{(k-1)}(x - g_1(t), t) - u^{(k)}(x,t)\right) = 0 \\ u^{(k)}(x,T) = U(x - kg_2(T)). \end{cases} \tag{3.46}$$

In this case, one can show that

$$u^{(k)}(x,t) = u^0(x,t) f_0(t)^k, \tag{3.47}$$

where f_0 satisfies the equation

$$\begin{cases} f_0' + \lambda(a_0+t)(e^{\gamma g_1(t)} - f_0) = 0 \\ f_0(T) = e^{\gamma g_2(T)}. \end{cases} \tag{3.48}$$

Solving (3.48), leads to

$$\begin{aligned} f_0(t) &= e^{\gamma g_2(T)} e^{-\int_t^T \lambda(a_0+u)du} + \int_t^T e^{\gamma g_1(s)} \lambda(a_0+s) e^{-\int_t^s \lambda(a_0+u)du} ds \\ &= e^{\gamma g_2(T)}{}_{T-t}p_{a_0+t} + \int_t^T e^{\gamma g_1(s)} \lambda(a_0+s)_{s-t}p_{a_0+t} ds = E[e^{\gamma B}]. \end{aligned} \tag{3.49}$$

Therefore, under the assumption of deterministic mortality, the premium for k life-insurance contracts with discounted benefit B, $P^{(k)}(t)$ is given by

$$P^{(k)}(t) = \frac{k}{\gamma} \ln f_0(t) = \frac{k}{\gamma} \ln E[e^{\gamma B}], \tag{3.50}$$

while the premium per risk coincides with the premium corresponding to a single life-insurance contract of benefit B, that is

$$\frac{1}{k} P^{(k)}(t) = \frac{1}{\gamma} \ln E[e^{\gamma B}]. \tag{3.51}$$

Indifference pricing is generally a non-additive pricing rule but notice that in the present case, $P^{(k)}$ acts as an additive function of k. Consequently, the premium per risk, $\left(\frac{1}{k}P^{(k)}\right)_{k\geq 1}$ is a constant sequence. In contrast, these two characteristics are not met by the indifference premium when assuming random mortality. Following Ludkovski & Young (2008) we show that in fact, in this context, the premium per risk for the insurance contracts analyzed above is an increasing sequence and the indifference price $P_B^{(k)}$ is a superadditive function of k. An important ingredient for proving this result is the theorem 3.2.1 below. Essentially, this theorem acts as comparison principle for parabolic PDEs on infinite domains and is a consequence of a more general result from Walter (1970), chapter 28.

First, we consider the notations $D = (0, \infty) \times [0, T]$ and $\mathcal{D} = C(\bar{D}) \cap C^{2,1}(D)$.

Theorem 3.2.1. *Let $\mathcal{L}$ a differential operator on $\mathcal{D}$ defined by*

$$\mathcal{L}v = v_t + \frac{1}{2}(\sigma^\lambda)^2 v_{\lambda\lambda} + H(\lambda, t, v, v_\lambda) \tag{3.52}$$

where H satisfies the conditions: for $v > w$

$$H(\lambda, t, v, y) - H(\lambda, t, w, z) \leq c(\lambda, t)(v - w) + d(\lambda, t)|y - z| \tag{3.53}$$

with the functions c and d such that

$$\begin{aligned} 0 \leq c(\lambda, t) &\leq K(1 + \lambda^2) \\ |d(\lambda, t)| &\leq K(1 + \lambda) \end{aligned} \tag{3.54}$$

for some $K \geq 0$ and for all $(\lambda, t) \in D$.

Suppose that $v, w \in \mathcal{D}$ satisfy the inequalities

$$v(\lambda, t) \leq e^{K\lambda^2} \quad \text{and} \quad w \geq -e^{K\lambda^2} \text{ for large } \lambda.$$

If $\mathcal{L}v \geq \mathcal{L}w$ on D and if $v(\lambda, T) \leq w(\lambda, T)$ for all $\lambda > 0$, then $v \leq w$ on D.

Proposition 3.2.1. *Under stochastic force of mortality, the price per risk* $\left(\frac{1}{k}P_B^{(k)}\right)_{k\geq 1}$, *is an increasing sequence.*

Proof. We have to show that

$$\frac{1}{k-1}P_B^{(k-1)}(\lambda,t) \leq \frac{1}{k}P_B^{(k)}(\lambda,t), \text{ for } \forall k \geq 2. \tag{3.55}$$

This reduces to proving that

$$(f^{(k-1)})^{\frac{1}{k-1}} \leq (f^{(k)})^{\frac{1}{k}}, \text{ for } \forall k \geq 2. \tag{3.56}$$

In what follows, we prove (3.56) by induction but first we need to consider several notations. For $k \geq 1$, we define the operator

$$\begin{aligned}\mathcal{L}^{(k)}u(\lambda,t) &= u_t + \mu^\lambda \lambda u_\lambda + \frac{1}{2}(\sigma^\lambda)^2 u_{\lambda\lambda} + k\lambda\left(e^{\gamma g_1(t)}f^{(k-1)} - u\right) \\ &= u_t + \frac{1}{2}(\sigma^\lambda)^2 u_{\lambda\lambda} + H^{(k)}(\lambda,t,u,u_\lambda)\end{aligned} \tag{3.57}$$

where $H^{(k)}(\lambda,t,u,v) = \mu^\lambda\lambda v + k\lambda\left(e^{\gamma g_1(t)}f^{(k-1)} - u\right)$. Then, according to (3.41), for every $k \geq 1$ we have $\mathcal{L}^{(k)}f^{(k)} = 0$.

Next, we apply the theorem 3.2.1 above. Clearly, first we will show that the operator $\mathcal{L}^{(k)}$ is as in the theorem just mentioned, that is $H^{(k)}(\lambda,t,u,v)$ satisfies the conditions (3.53) and (3.54).

For $u_1 > u_2$, we have

$$H^{(k)}(\lambda,t,u_1,v_1) - H^{(k)}(\lambda,t,u_2,v_2) = \mu^\lambda\lambda(v_1 - v_2) - k\lambda(u_1 - u_2) \leq \mu^\lambda\lambda(v_1 - v_2)$$

Observe that $c(\lambda,t) = 0$ and $d(\lambda,t) = \mu^\lambda\lambda$. Clearly, $0 \leq d(\lambda,t) = \mu^\lambda\lambda \leq$

$K(1+\lambda)$ where K is a constant such that $\mu^\lambda \leq K$. Hence, the conditions of the theorem 3.2.1 hold true.

Now, we show that (3.56) holds for $k=2$, that is

$$(f^{(1)})^2 \leq f^{(2)} \tag{3.58}$$

First, recall that $\mathcal{L}^{(2)} f^{(2)} = 0$. But on the other hand,

$$\mathcal{L}^{(2)}(f^{(1)})^2 = 2f^{(1)}\mathcal{L}^{(1)}f^{(1)} + (\sigma^\lambda)^2(f^{(1)}_\lambda)^2 = (\sigma^\lambda)^2(f^{(1)}_\lambda)^2 \geq 0. \tag{3.59}$$

Thus, $\mathcal{L}^{(2)}(f^{(1)})^2 \geq \mathcal{L}^{(2)} f^{(2)}$ while $(f^{(1)})^2(\lambda, T) = f^{(2)}(\lambda, T)$ and by the theorem mentioned above follows that $(f^{(1)})^2(\lambda, t) \leq f^{(2)}(\lambda, t)$ for all $(\lambda, t) \in D$.

Next, we assume that the inequality (3.56) holds for $k-2$ and we will show that it also holds for $k-1$, namely

$$f^{(k-1)} \leq (f^{(k)})^{\frac{k-1}{k}}. \tag{3.60}$$

We have

$$\begin{aligned}\mathcal{L}^{(k)}(f^{(k-1)})^{\frac{k}{k-1}} &= \frac{k}{k-1}(f^{(k-1)})^{\frac{1}{k-1}}\left(f^{(k-1)}_t + \mu^\lambda\lambda f^{(k-1)}_\lambda + \frac{1}{2}(\sigma^\lambda)^2 f^{(k-1)}_{\lambda\lambda}\right. \\ &\left.+(k-1)\lambda\left((f^{(k-1)})^{\frac{k-2}{k-1}}e^{\gamma g_1(t)} - f^{(k-1)}\right)\right)+\frac{1}{2}(\sigma^\lambda)^2\frac{k}{(k-1)^2}(f^{(k-1)})^{\frac{2-k}{k-1}}(f^{(k-1)}_\lambda)^2.\end{aligned}$$

But, $(f^{(k-1)})^{\frac{k-2}{k-1}} \geq f^{(k-2)}$ (this is the inequality (3.56) for $k-2$) and therefore

$$\begin{aligned}\mathcal{L}^{(k)}(f^{(k-1)})^{\frac{k}{k-1}} &\geq \frac{k}{k-1}(f^{(k-1)})^{\frac{1}{k-1}}\mathcal{L}^{(k-1)}f^{(k-1)} + \frac{k(\sigma^\lambda)^2}{2(k-1)^2}(f^{(k-1)})^{\frac{2-k}{k-1}}(f^{(k-1)}_\lambda)^2 \\ &= \frac{k(\sigma^\lambda)^2}{2(k-1)^2}(f^{(k-1)})^{\frac{2-k}{k-1}}(f^{(k-1)}_\lambda)^2.\end{aligned}$$

Hence, $\mathcal{L}^{(k)}(f^{(k-1)})^{\frac{k}{k-1}} \geq \mathcal{L}^{(k)} f^{(k)}$ and $(f^{(k-1)})^{\frac{k}{k-1}}(\lambda, T) = f^{(k)}(\lambda, T)$. Applying again theorem 3.2.1 we obtain that the inequality (3.60) is satisfied.

□

Proposition 3.2.2. *For every nonnegative integers k and l,*

$$P_B^{(k+l)} \geq P_B^{(k)} + P_B^{(l)}$$

Proof. Applying proposition (3.2.1), leads to

$$\frac{1}{k+l} P^{(k+l)} \geq \frac{1}{k} P^{(k)}, \quad \frac{1}{k+l} P^{(k+l)} \geq \frac{1}{l} P^{(l)}. \tag{3.61}$$

Therefore,

$$\frac{k}{k+l} P_B^{(k+l)} \geq P_B^{(k)}, \quad \frac{l}{k+l} P_B^{(k+l)} \geq P_B^{(l)} \tag{3.62}$$

and adding the last two inequalities, we obtain the result.

□

Accordingly, when mortality behaves randomly, an insurer who will pool together a large number of insurance risks, trying to enforce the law of large numbers, actually increases his total risk.

Intuitively, the super-additivity of the indifference premium with respect to the number of policies sold can be explained as follows: while the mortality risks are independent conditional on knowing the cohort mortality, upon removing this latter assumption they may become dependent over time, linked by a common factor - for example a certain disease, certain natural conditions, exposure to common social and economic factors, etc. Thus, essentially a positive correlation between individual risks develops over time and this induce a systematic component within the mortality uncertainty of the individuals, causing the total risk to be more dangerous than the sum of the individual risks.

In what follows, we perform several numerical experiments, considering the same cohort of individuals as in the numerical experiments from the first section of this chapter. Obviously, these numerical experiments regard the pricing of endowment insurances, pure endowments and term life insurances with discounted benefits given by (3.15) and (3.16)

In our first experiment, we consider a portfolio consisting of $k = 20$ policies and examine the dependence of the premium per risk on the time to maturity, for several choices of the risk aversion parameter γ. We assume that $G = 10$, $r = 0.06$, T varies between 10 and 20 years and the risk aversion parameter takes the values $\gamma = 0$, $\gamma = 0.05$ and $\gamma = 0.1$.

Notice from Figure 3.4 that consistent with our intuition, when the time to maturity increases, the premium per risk for endowments and pure endowments decreases while the premium for term life insurances increases. As expected, the plots resemble those for the lump sum premium in a single life insurance model. However, as already proved in proposition 3.2.1, the premiums per risk for all three types of insurance contracts are higher than the corresponding lump-sum premiums.

Our second numerical experiment can be regarded as a numerical illustration of 3.2.1. We calculate the premium per risk for the life insurance contracts mentioned above assuming that the the number of policies in in-

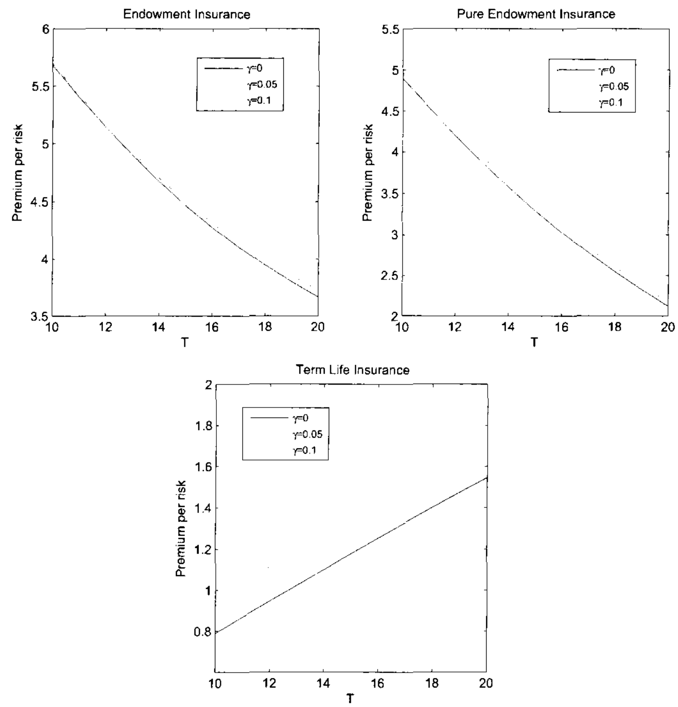

Figure 3.4: Premium per risk in the individual risk model for a constant benefit endowment insurance, pure endowment and term life insurance as a function of the time to maturity, for several choices of the risk aversion parameter.

surer's portfolio varies between 1 and 20 policies. Further, we consider $T = 10$ years, $G = 10$ and $r = 0.06$.

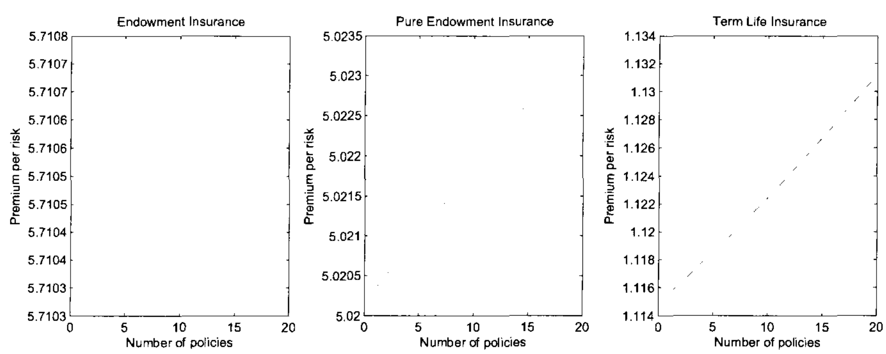

Figure 3.5: Premium per risk in the individual risk model for a constant benefit endowment insurance, pure endowment and term life insurance as a function of the number of policies in the insurer's portfolio.

Observe from Figure 3.5 that for all insurance contracts, the premium per risk increases linearly as the number of policies increases. The increase in the premium per risk is very slow because of the very small value of the volatility σ^λ. However, observe that this increase is more pronounced for pure endowments and term life insurances than for endowment insurances. This fact might be explained by the following argument: endowments insurance contracts are contingent on both events - death before maturity and survival to maturity; thus, if some insurance risks become positively correlated over time, this correlation is balanced by the negative correlation between mortality and survival risk. On the other hand, for pure endowments and term life insurances this phenomena does not happen since these contracts are contingent on only one of the events mentioned above.

3.2.2 Continuous premium

We continue to investigate the pricing problem of market independent life-insurance contracts with discounted benefits given by (3.2), but in contrast to the preceding subsection, here we assume that the premium is payable continuously, at a constant rate established when the insurance contracts are signed. As in the preceding subsection, we model the insurer's total loss using the individual risk model. Then, the value function of the insurer in the presence of insurance risks is given by

$$u^{(k)}(x, \lambda, t; h^{(k)}) = \sup_{\pi \in \mathcal{A}} E[U(X_T - C_T) | X_t = x, \lambda_{a_0+t} = \lambda], \tag{3.63}$$

where $h^{(k)}(\lambda, t)$ denotes the continuous rate per risk set at time t and which remains fixed over the life of all k policies. The discounted wealth process follows the dynamics

$$\begin{cases} dX_s = \left(\pi_s(\mu - r) + Z_s h^{(k)} e^{-rs}\right) ds + \sigma \pi_s dW_s, \quad s \neq \tau_i \\ X_t = x \\ X_{\tau_i +} = X_{\tau_i -} - C_{\tau_i} \ \text{if} \ \tau_i < T, \end{cases} \tag{3.64}$$

where Z_s represents the number of survivors at time s and τ_i, $i = 1 \ldots k$ the times of death of the policyholders. With regards to C_{τ_i} and C_T, they represent as in the lump-sum premium case, the total claim at time $\tau_i < T$ and the total claim upon survival to maturity time T.

Applying Bellman's principle of dynamic programming and stochastic calculus, we obtain that $u^{(k)}$ solves the HJB equation

$$\begin{cases} u^{(0)} = u^0 \\ \text{For } k \geq 1, \\ u_t^{(k)} + \max_\pi [(\mu - r)\pi u_x^{(k)} + \frac{1}{2}\sigma^2 \pi^2 u_{xx}^{(k)}] + \mu^\lambda \lambda u_\lambda^{(k)} + \frac{1}{2}(\sigma^\lambda)^2 u_{\lambda\lambda}^{(k)} \\ \qquad + k e^{-rt} h^{(k)} u_x^{(k)} + k\lambda \left(u^{(k-1)}(x - g_1(t), \lambda, t) - u^{(k)}(x, \lambda, t)\right) = 0 \\ u^{(k)}(x, \lambda, T) = U(x - k g_2(T)). \end{cases} \tag{3.65}$$

We look for a solution of the form

$$u^{(k)}(x, \lambda, t; h^{(k)}) = u^0(x, t) l^{(k)}(\lambda, t). \tag{3.66}$$

After some calculations, it follows that $l^{(k)}$ satisfies the linear partial differential equation

$$\begin{cases} l_t^{(k)} - \gamma k e^{-rt} h^{(k)} l^{(k)} + \mu^\lambda \lambda l_\lambda^{(k)} + \frac{1}{2}(\sigma^\lambda)^2 l_{\lambda\lambda}^{(k)} + k\lambda(e^{\gamma g_1(t)} l^{(k-1)} - l^{(k)}) = 0 \\ l^{(k)}(\lambda, T) = e^{\gamma k g_2(T)} \end{cases} \tag{3.67}$$

where $l^{(0)} = 0$. Applying the Feynman-Kač formula, $l^{(k)}$ admits the representation

$$\begin{aligned} l^{(k)}(\lambda, t) &= e^{\gamma k g_2(T) - \gamma k \int_t^T h^{(k)} e^{-rs} ds} E_{t,\lambda}[e^{-k \int_t^T \lambda_{a_0+u} du}] \\ &+ k \int_t^T e^{\gamma g_1(s) - \gamma k \int_t^s h^{(k)} e^{-ru} du} E_{t,\lambda}[e^{-k \int_t^s \lambda_{a_0+u} du} \lambda_{a_0+s} l^{(k-1)}(\lambda_{a_0+s}, s)] ds. \end{aligned}$$

The indifference premium rate is defined by the equation

$$u^0(x, t) = u^{(k)}(x, \lambda, t; h^{(k)}). \tag{3.68}$$

Inserting the ansätz (3.66) in equation (3.68), we obtain that $h^{(k)}(\lambda, t)$ is given implicitly by the equation

$$l^{(k)}(\lambda, t; h^{(k)}) = 1. \tag{3.69}$$

The indifference premium rate can also be characterized using the concept of benefit reserve defined in the preceding section. Let us define

$$V(\lambda, t; h^{(k)}) := \frac{1}{\gamma} \ln l^{(k)}(\lambda, t; h^{(k)}) \tag{3.70}$$

It is straightforward to check that $V(\lambda_{a_0+s}, s; h^{(k)})$, $s \geq t$ satisfies equation (3.22). Therefore $V(\lambda_{a_0+s}, s; h^{(k)})$ represents the benefit reserve at time $s \geq t$.

Combining (3.70) and (3.69) we obtain that the indifference premium rate is such that $V(\lambda, t; h^{(k)}) = 0$.

In particular, in the case of deterministic mortality intensity, it can be shown that

$$u^{(k)}(x,t;h^{(k)}) = u^0(x,t)l_0^k(t) \tag{3.71}$$

where l_0 solves the equation

$$\begin{cases} l_0' - \gamma e^{-rt}h^{(k)}l_0 + \lambda(a_0+t)(e^{\gamma g_1(t)} - l_0) = 0 \\ l_0(T) = e^{\gamma g_2(T)}. \end{cases} \tag{3.72}$$

Then,

$$l_0(t) = e^{\gamma g_2(T) - \gamma\int_t^T h^{(k)}e^{-rs}ds}{}_{T-t}p_{a_0+t} + \int_t^T e^{\gamma g_1(s) - \gamma\int_t^s h^{(k)}e^{-ru}du}\lambda(a_0+s){}_{s-t}p_{a_0+t}ds$$
$$= e^{\gamma g_2(T) + \frac{\gamma h^{(k)}}{r}(e^{-rT}-e^{-rt})}{}_{T-t}p_{a_0+t} + \int_t^T e^{\gamma g_1(s) + \frac{\gamma h^{(k)}}{r}(e^{-rs}-e^{-rt})}\lambda(a_0+u){}_{s-t}p_{a_0+t}ds.$$

The indifference premium equation implies that $l_0(t) = 1$ and thus $h^{(k)}$ is given implicitly by the equation

$$e^{\gamma g_2(T) + \frac{\gamma h^{(k)}}{r}(e^{-rT}-e^{-rt})}{}_{T-t}p_{a_0+t} + \int_t^T e^{\gamma g_1(s) + \frac{\gamma h^{(k)}}{r}(e^{-rs}-e^{-rt})}\lambda(a_0+u){}_{s-t}p_{a_0+t}ds = 1.$$

Consequently, as one would expect in a deterministic setting for mortality, $h^{(k)}(t)$ coincides with the continuous premium rate in a single life insurance model.

Remarks 3.2.1.

- A similar analysis applies to market independent life insurance contracts with benefits of the form (3.15) and (3.16) but with continuous premiums that evolve according to some time dependent schedule, fixed in advanced. For example, the premium can be taken of the form $\tilde{h}^{(k)}(s) = h^{(k)}(\lambda, t)h^0(s, t)$, where $h^0(s, t)$ is fixed in advance and $h^{(k)}(\lambda, t)$ is the factor to be calculated.

- As shown above, in a deterministic setting for mortality, the continuous premium rate is a constant function of k.
- In the case of stochastic mortality, the positive correlation that develops over time between policyholders'mortality suggests that the premium is an increasing function of k. However, since the premium is given implicitly via equations (3.69) and (3.67) we are not able to provide an analytical result regarding the increasing nature of the premium $h^{(k)}$ with respect to k, as we have shown in the case of lump-sum premium. Therefore, we numerically implement the premium and investigate its dependence on the number of policies sold.

In this numerical experiment we consider endowments, pure endowments and term life insurances with benefits of the form (3.15) and (3.16), where $G = 10$, the time to maturity is $T = 1$ year, $r = 0.06$ and the insurer's

Number of policies k	5	10	15	20	25
Endowment	9.8844	9.9420	10.0002	10.0587	10.1178
Pure endowment	9.7481	9.8058	9.864	9.9226	9.9817
Term Life	0.136317	0.136327	0.1363369	0.136346	0.136356

Table 3.1: Continuous premium rate for a constant benefit endowment, pure endowment and term life insurance in the individual risk model.

risk aversion is $\gamma = 0.1$.

Observe from Table 3.1 that the premium increases slowly and linearly for all contracts, as the number of policies k increases.

- The increasing nature of the premium with respect to the number of policies sold suggests that lapsation problem can be ignored. If eventually policyholders decide to walk away from the contract, this just makes things better for the insurer.

We conclude this section by several remarks regarding premium calculation in the individual risk model. As already observed, the calculation of the premium (either lump-sum or continuous) requires solving a number of recursively defined linear partial differential equations equal to the number of policies in the insurer's portfolio. Typically, this number is very large and this makes the numerical calculation of the premium not feasible. Accordingly, it is desirable to find a loss model more efficient from the point of view of premium calculation. In what follows, we show that the collective risk model satisfies this requirement.

3.3 Indifference premium in the collective risk model

We consider the same cohort as in the preceding section, but here, we choose to model the number of deaths within the cohort via a Poisson process. Obviously, the intensity of this Poisson process has the same nature as the policyholders'mortality intensity. Accordingly, we model the number of deaths by an inhomogeneous Poisson process when policyholders'mortality is assumed deterministic, while when assumed stochastic, we consider a doubly stochastic Poisson process.

Let us assume that at time $t = 0$, the cohort consists of m individuals, where m has a very large value. Further, assume that the insurer has the opportunity to sell to all these individuals life insurance contracts with discounted benefits of the form (3.2). The insurer is faced with the pricing problem of these claims. Next, we analyze the insurer's problem, assuming that the policyholders'mortality evolves randomly in time.

Let us denote by $(N_s)_{0 \leq s \leq T}$ a doubly stochastic Poisson process of intensity η_{a_0+s} which counts the number of deaths from time 0 up to time s. That is, conditionally on any particular trajectory $u \to \eta_{a_0+u}$, $u \in [0, s]$, (N_s) is an inhomogeneous Poisson process with parameter $\int_0^s \eta_{a_0+u} du$. Here, we assume that the intensity η_{a_0+s} satisfies the stochastic differential equation

$$d\eta_{a_0+s} = \mu^{\eta} \eta_{a_0+s} ds + \sigma^{\eta} dW_s^{\eta}. \tag{3.73}$$

Next, we model the aggregate loss on the time interval $[0, s)$, $s \in [0, T]$, as follows

$$L_s^{coll} = Y_1 + Y_2 + \ldots + Y_{N_s} = \sum_{i=1}^{N_s} g_1(\tau_i) \tag{3.74}$$

where $Y_i = g_1(\tau_i)$ and τ_i denote the ith claim to occur and the arrival time of this claim, respectively.

3.3.1 Lump-sum premium

We define the value function of the insurer in the presence of insurance risks as follows

$$u(x,\eta,n,t;m) = \sup_{\pi\in\mathcal{A}} E[U(X_T - (m - N_T)g_2(T))|X_t = x, \eta_{a_0+t} = \eta, N_t = n], \tag{3.75}$$

where the discounted wealth process has the dynamics

$$\begin{cases} dX_s = \pi_s\left((\mu - r)ds + \sigma dW_s\right) - dL_s^{coll} \\ X_t = x. \end{cases} \tag{3.76}$$

We implicitly assume that the size of cohort at any time $0 \le t \le T$ is very large. Then, the corresponding HJB equation for u is

$$\begin{cases} u_t + \max_{\pi\in\mathcal{A}}[(\mu - r)\pi u_x + \frac{1}{2}\sigma^2\pi^2 u_{xx}] + \mu^\eta \eta u_\eta + \frac{1}{2}(\sigma^\eta)^2 u_{\eta\eta} \\ \qquad\qquad +\eta\left(u(x - g_1(t), \eta, n+1, t) - u(x,\eta,n,t)\right) = 0 \\ u(x,\eta,n,T;m) = U(x - (m-n)g_2(T)). \end{cases} \tag{3.77}$$

Next, assume that $u_{xx} < 0$. Then, the maximum in (3.77) is well defined and attained at

$$\pi_t^* = -\frac{\mu - r}{\sigma^2}\frac{u_x}{u_{xx}}. \tag{3.78}$$

We assume an ansätz of the form $u(x,\eta,n,t;m) = u^0(x,t)F(\eta,n,t;m)$. Inserting the expression of π_t^* and of the ansätz in (3.77), we obtain that F satisfies the partial differential equation

$$\begin{cases} F_t + \mu^\eta \eta F_\eta + \frac{1}{2}(\sigma^\eta)^2 F_{\eta\eta} + \eta \left(F(\eta, n+1, t; m) e^{\gamma g_1(t)} - F(\eta, n, t; m) \right) = 0 \\ F(\eta, n, T; m) = e^{\gamma(m-n) g_2(T)}. \end{cases} \tag{3.79}$$

Straightforward calculations lead to

$$F(\eta, n, t; m) = e^{\gamma(m-n)g_2(T)} E_{t,\eta}[e^{\int_t^T \eta_{a_0+s}(e^{\gamma g_1(s)-\gamma g_2(T)}-1)ds}]. \tag{3.80}$$

Consequently, by the Verification Theorem, the ansätz is the unique smooth solution of (3.77) and coincides with the value function of the problem.

Next, the indifference premium satisfies the equation

$$u^0(x, t) = u(x + P_B, \eta, n, t; m). \tag{3.81}$$

Inserting the expression of the value function in (3.81), it follows that the premium is given by

$$P_B(\eta, n, t; m) = (m-n)g_2(T) + \frac{1}{\gamma} \ln E_{t,\eta}[e^{\int_t^T \eta_{a_0+s}(e^{\gamma g_1(s)-\gamma g_2(T)}-1)ds}]. \tag{3.82}$$

At this point, recall that in the individual risk model, the insurer's pricing problem requires solving a system of $k = m - n$ linear partial differential equations. On the other hand, observe from (3.80) that in the collective risk model, the insurer has to solve only one partial differential equation. Accordingly, the collective risk model offers a huge advantage in terms of tractability and computation time.

So far, we have examined the pricing of a life insurance contracts with general benefit of the form (3.2). However, observe that for term life insurances, we have an essential difference. In this case, the insurer at any point in time does not need to know the number of deaths; he just pays the claims as they arrive. That is because the Poisson process, contrary to the Binomial, does not need information about the number of deaths.

Accordingly, the value function corresponding to a term life insurance contract is given by (3.75) but removing the dependence of the number of deaths random variable. Then, it follows that the function $F(\eta, t)$ satisfies the linear differential equation

$$\begin{cases} F_t + \mu^\eta \eta F_\eta + \frac{1}{2}(\sigma^\eta)^2 F_{\eta\eta} + \eta F(\eta, t)\left(e^{\gamma g_1(t)} - 1\right) = 0 \\ F(\eta, T) = 1. \end{cases} \tag{3.83}$$

By the Feynman-Kač formula, $F(\eta, t)$ is given by

$$F(\eta, t) = E_{t,\eta}[e^{\int_t^T \eta_{a_0+s}(e^{\gamma g_1(s)}-1)ds}]. \tag{3.84}$$

Naturally, this equation is exactly (3.80) when $g_2 = 0$.

For deterministic mortality intensity, as we have already mentioned, we model the number of deaths by time s by an inhomogeneous Poisson process $(N_s)_{0\leq s\leq T}$ with intensity $\eta(a_0 + s)$. In this case, the insurer investment problem can be obtained from (3.75) removing the dependence on the mortality intensity. Accordingly, the value function can be written as

$$u(x, n, t; m) = u^0(x, t)F_0(n, t; m) \tag{3.85}$$

where $F_0(n,t;m)$ is given by

$$F_0(n,t;m) = e^{\gamma(m-n)g_2(T)+\int_t^T \eta(a_0+s)(e^{\gamma g_1(s)-\gamma g_2(T)}-1)ds}. \tag{3.86}$$

Therefore, the indifference premium at time $t=0$ is given by

$$P_B(n=0,t=0;m) = mg_2(T) + \frac{1}{\gamma}\int_0^T \eta(a_0+s)(e^{\gamma g_1(s)-\gamma g_2(T)}-1)ds. \tag{3.87}$$

In this case, the mortality intensity $\eta(a_0+s)$ such that the expected number of deaths from time $t=0$ up to any time $s<T$ is the same in the individual and collective model. This implies that

$$\eta(a_0+s) = m_s p_{a_0}\lambda_{a_0+s}. \tag{3.88}$$

Accordingly, the indifference premium can be written as

$$P_B(n=0,t=0;m) = m\left(g_2(T) + \frac{1}{\gamma}\int_0^T (e^{\gamma g_1(s)-\gamma g_2(T)}-1)_s p_{a_0}\lambda_{a_0+s}ds\right)$$

and consequently the premium per risk is constant. At this point, observe that the premium obtained is greater than the corresponding one from the individual model since

$$P_B^{coll} > mg_2(T) + \frac{m}{\gamma}\ln(1+\int_0^T (e^{\gamma g_1(s)-\gamma g_2(T)}-1)_s p_{a_0}\lambda_{a_0+s}ds) = P_B^{ind}.$$

3.3.2 Continuous premium

We continue to examine the pricing problem from the preceding subsection, but here we assume a continuous premium rate h that remains constant over time. This rate is set when the insurance contracts are written and applies to the whole cohort of policyholders.

We define the value function of the insurer in the presence of insurance risks as follows

$$u(x, \eta, n, t; m) = \sup_{\pi \in \mathcal{A}} E[U(X_T - (m - N_T)g_2(T))|X_t = x, \eta_{a_0+t} = \eta, N_t = n], \tag{3.89}$$

Here, the discounted wealth process follows the dynamics

$$\begin{cases} dX_s = ((\mu - r)\pi_s + he^{-rs})\, ds + \sigma \pi_s dW_s - dL_s^{coll} \\ X_t = x. \end{cases}$$

Observe from the wealth dynamics that we implicitly consider that the premium rate h will be payed until the maturity of the contracts. This, essentially is a consequence of the assumption that the size of the cohort at any time $0 \leq t \leq T$ is very large.

It follows that u solves the HJB equation

$$\begin{cases} u_t + \max_\pi[(\mu - r)\pi u_x + \frac{1}{2}\sigma^2\pi^2 u_{xx}] + he^{-rt}u_x + \mu^\eta \eta u_\eta + \frac{1}{2}(\sigma^\eta)^2 u_{\eta\eta} \\ \qquad + \eta\,(u(x - g_1(t), \eta, n+1, t) - u(x, \eta, n, t)) = 0. \\ u(x, \eta, n, T; m) = U(x - (m - n)g_2(T)) \end{cases} \tag{3.90}$$

Assuming an ansätz of the form $u(x, \eta, n, t; m) = u^0(x, t)G(\eta, n, t; m)$, it follows that $G(\eta, n, t; m)$ satisfies the partial differential equation

$$\begin{cases} G_t + \mu^\eta \eta G_\eta + \frac{1}{2}(\sigma^\eta)^2 G_{\eta\eta} - \gamma h e^{-rt} G + \eta(G(\eta, n+1, t)e^{\gamma g_1(t)} - G(\eta, n, t)) = 0 \\ G(\eta, n, T; m) = e^{\gamma(m-n)g_2(T)}. \end{cases} \tag{3.91}$$

Solving (3.91) with respect to G, leads to

$$G(\eta, n, t; m) = e^{\gamma(m-n)g_2(T) - \gamma \int_t^T h e^{-rs} ds} E_{t,\eta}[e^{\int_t^T \eta_{a_0+s}(e^{\gamma g_1(s) - \gamma g_2(T)} - 1)ds}]. \tag{3.92}$$

The indifference premium rate is such that the insurer is indifferent between accepting or not accepting the insurance risks, that is

$$u^0(x, t) = u(x, \eta, n, t; m). \tag{3.93}$$

This implies that $G(\eta, n, t; m) = 1$. Accordingly, the premium rate is given by

$$h = \frac{1}{\int_t^T e^{-rs} ds} \left((m-n) g_2(T) + \frac{1}{\gamma} \ln E_{t,\eta}[e^{\int_t^T \eta_{a_0+s}(e^{\gamma g_1(s) - \gamma g_2(T)} - 1)ds}] \right). \tag{3.94}$$

Thus, as expected, the total discounted premium payed coincides with the lump sum premium of the claims.

Remarks 3.3.1.

- Since in this case the premium rate for the whole cohort is set at the time of writing the contracts and remains constant for the life of the policies, it follows that the corresponding premium per risk will increase over time as the policyholders die. So, the premium per risk here has a different nature than in the individual risk model.

- An alternative idea regarding premium modeling would be to consider the following choice for the premium rate: $\tilde{h}(s) = h(\lambda, t)(m - N_s)e^{-rs}$. Here $h(\lambda, t)$ denotes the premium per risk set at the time t and it is assumed constant for the life of the policies. With this choice the nature of the premium per risk and of the premium rate is similar to the nature of the corresponding premiums in the individual risk model.

Chapter 4

Utility Indifference Pricing of Equity–Linked Term Life Insurance

Given their resemblance with financial options, it is no surprise that the first method used for valuation and hedging of equity–linked life insurance was a financial one. As mentioned in chapter 1, this method was introduced by Brennan & Schwartz (1976) and Boyle & Schwartz (1977) and essentially it is based on the Black-Scholes and Merton theory. The crucial assumption of this approach is that the mortality risk is diversifiable, that is by selling a large number of life insurance contracts, the insurer mortality exposure approaches zero. However, we learned from Milevky, Promislow & Young (2006) that if an insurer tries to sell more and more policies hoping to reach the concept of *large* mentioned above and if policyholder's mortality behaves stochastically, contrary to his expectation, the insurer's total exposure may even increase. Accordingly, it is desirable to use an approach that explicitly recognizes the mortality risk instead of assuming that it is diversifiable. As we already know, *the utility indifference approach* satisfies this requirement.

The first to apply the utility indifference approach for pricing and hedging equity–linked life insurance were Young (2003) and Moore & Young (2003). These were essential contributions to the problem of pricing equity–linked life products. They show that in a Black-Scholes market model and under deterministic mortality, the lump sum premium for a single life satisfies a non-linear partial differential equation similar to the Black-Scholes equation, except for a nonlinear term that reflects the mortality risk. Jaimungal & Young (2005) generalize the work of Moore & Young (2003) to a more realistic market model, where the stock price is modeled via a Lévy process. They obtain that the lump sum premium in a single life insurance model incorporates a significant correction in comparison to the one generated in the Black-Scholes market model. Another interesting work in the same area was done by Jaimungal & Nayak (2006). They consider equity–linked losses that continually arrive at Poisson times and examine within the same framework the valuation of equity–linked life insurance and reinsurance contracts.

Our contribution to the area of utility based pricing and hedging of equity–linked term life insurance consists of extending the results of Young (2003) to group benefits, by embedding the individual and collective risk model. Moreover, we study the problem of finding both the lump-sum and continuous premium in all models considered and provide numerical schemes for calculating these premiums.

We start with the same financial market model as in chapter 3 and consider equity–linked term life insurance contracts with discounted benefit as follows

$$B = \begin{cases} g(S_\tau, \tau) & \text{if } \tau < T \\ 0 & \text{if } \tau \geq T. \end{cases} \tag{4.1}$$

where, g is a positive bounded function on $[0, \infty) \times [0, T]$ and τ denotes the policyholder's time of death. In contrast to the preceding chapter, here we adopt a deterministic mortality model.

4.1 A single life insurance model

Below we examine the pricing and hedging of the insurance contract with benefit B via the utility indifference approach in a single life insurance model, following Young (2003).

4.1.1 Lump-sum premium

We consider a life insurance model consisting of a single life aged a_0 at time 0 and assume that (a_0) is willing to buy an equity–linked term life insurance with benefit given by (5.3). If the insurer accepts to write this claim, the insurer's value function is defined by

$$u^B(x, S, t) = \sup_{\pi \in \mathcal{A}} E[U(X_T) | X_t = x, S_t = S] \tag{4.2}$$

where X_t denotes the discounted wealth with dynamics given by

$$\begin{cases} dX_s = \pi_s \left((\mu - r) ds + \sigma dW_s \right), \quad s \neq \tau \\ X_{\tau+} = X_{\tau-} - g(S_\tau, \tau), \text{ if } \tau < T \\ X_t = x. \end{cases} \tag{4.3}$$

Applying Bellman's principle of dynamic programming and stochastic calculus arguments, u^B satisfies the HJB equation

$$\begin{cases} u_t^B + \max_\pi[(\mu - r)\pi u_x^B + \frac{1}{2}\sigma^2\pi^2 u_{xx}^B + \sigma^2\pi S u_{xS}^B] + \frac{1}{2}\sigma^2 S^2 u_{SS}^B + (\mu - r)S u_S^B \\ \qquad\qquad +\lambda(a_0 + t)\left(u^0(x - g(S,t), S, t) - u^B(x, S, t)\right) = 0 \\ u^B(x, S, T) = U(x). \end{cases} \tag{4.4}$$

Now, we consider an ansätz of a solution to the HJB equation of the form

$$u^B(x, S, t) = u^0(x, t)e^{\phi(S,t)}. \tag{4.5}$$

Observe that $u_{xx}^B < 0$ and therefore u^B is concave in wealth. This implies that the maximum in equation (4.4) is well defined and can be specified by the first order necessary condition, as follows

$$\pi^*(x, S, t) = \frac{\mu - r}{\gamma\sigma^2} + \frac{\phi_S}{\gamma}S. \tag{4.6}$$

Inserting the ansätz and the expression of π^* in the HJB equation, leads to

$$\begin{cases} \phi_t + \frac{1}{2}\sigma^2 S^2 \phi_{SS} + \lambda(a_0 + t)(e^{\gamma g(S,t) - \phi} - 1) = 0 \\ \phi(S, T) = 0. \end{cases} \tag{4.7}$$

For well behaved benefit function g, equation (4.7) has a smooth solution. Accordingly, by the Verification Theorem, the ansätz proposed coincides with the value function. Also, the Verification Theorem implies that the optimal control policy can be specified by the first order condition in (4.4). Remark that the optimal policy is wealth independent and is given by the optimal policy in the Merton problem plus the amount $\frac{\phi_S}{\gamma}S$. Recall that the optimal policy in the Merton problem represents the optimal amount to be invested in the stock in the absence of insurance risks. Thus, $\frac{\phi_S}{\gamma}$ can be interpreted as the optimal number of shares to be invested in the stock due to accepting the insurance risk. In what follows we refer to this amount as the *optimal excess hedge* and we will show that in fact this is analogous to the Black-Scholes delta hedge.

The indifference premium for the term life insurance contract satisfies the equation

$$u^0(x,t) = u^B(x+P,S,t). \tag{4.8}$$

Straightforward calculations lead to

$$P(S,t) = \frac{1}{\gamma}\phi. \tag{4.9}$$

Thus, the indifference premium solves the nonlinear second order partial differential equation

$$\begin{cases} P_t + \frac{1}{2}\sigma^2 S^2 P_{SS} - \frac{\lambda(a_0+t)}{\gamma}\left(1 - e^{-\gamma(P-g(S,t))}\right) = 0 \\ P(S,T) = 0. \end{cases} \tag{4.10}$$

Observe that the first two terms of equation (4.10) represent the discounted version of the Black-Scholes equation and they reflect the financial risk embedded in the benefit. On the other hand, the last term reflects the mortality risk and the risk preferences of the insurer.

At this point, notice that the optimal excess hedge $e^*(S,t)$ can be written as

$$e^*(S,t) = P_S(S,t) \tag{4.11}$$

and indeed, $e^*(S,t)$ is analogous to the Black-Scholes delta hedge.

Further, if the insurer risk aversion goes to zero, the premium equation becomes

$$\begin{cases} P_t + \frac{1}{2}\sigma^2 S^2 P_{SS} + \lambda(a_0+t)(g(S,t)-P) = 0 \\ P(S,T) = 0. \end{cases} \tag{4.12}$$

Using the Feynman-Kač formula, we obtain that

$$\begin{aligned} P(S,t) &= \int_t^T E_{t,S}^Q[g(S_s,s)]\lambda(a_0+s)e^{-\int_t^s \lambda(a_0+u)du}ds \\ &= \int_t^T E_{t,S}^Q[g(S_s,s)]d_{s-t}q_{a_0+t} = E[E_{t,S}^Q[B]]. \end{aligned}$$

Here the risk neutral measure Q is given by

$$Q(A) = E\left[1_A \exp\left(-\frac{\mu-r}{\sigma}W_T - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}T\right)\right], \quad A \in \mathcal{F}_T. \tag{4.13}$$

Under this measure the dynamics of the stock price process is as follows

$$\begin{cases} dS_s = \sigma S_s dW_s^Q \\ S_t = S \end{cases}$$

with $W_s^Q = W_s + \frac{\mu-r}{\sigma}s$.

Accordingly, the premium for a risk-neutral insurer coincides with the premium calculated using the Brennan & Schwartz (1976) approach. Similarly, the optimal excess hedge for a risk neutral insurer is given by the Brennan and Schwartz hedge.

In what follows, we implement the lump-sum premium for the term life insurance contract considered, assuming a benefit function $g(S_t, t)$ of the form

$$g(S_t, t) = \begin{cases} G_1 e^{-rt}, & \text{if} \quad S_t < G_1 e^{-rt} \\ S_t, & \text{if} \quad G_1 e^{-rt} \leq S_t \leq G_2 e^{-rt} \\ G_2 e^{-rt}, & \text{if} \quad G_2 e^{-rt} < S_t \end{cases} \tag{4.14}$$

where G_1 and G_2 are strictly positive constants. Notice that this benefit function represents the capped version of the GMBD type benefit considered in chapter 3.

As in the previous chapters, we assume that (a_0) is an American male 45 years old and born in 1900. Further, we consider that his force of mortality is deterministic and given by

$$\lambda(a_0 + t) = \lambda(a_0) e^{\mu^\lambda t}, \quad \text{where } \lambda(a_0) = 0.00778 \text{ and } \mu^\lambda = 0.07204. \tag{4.15}$$

For calculating the lump-sum premium, first we perform the change of variable $S_t = e^{z(t)}$ in equation (4.10) . Then, we discretize the equation obtained by employing a fully implicit finite difference scheme for the linear part, while treating the nonlinear part of the equation explicitly. Further, we truncate the domain $\mathbb{R} \times [0, T]$ to $[z_{min}, z_{max}] \times [0, T]$ and introduce the grid

$$z_m = z_{min} + m\Delta z, \quad m = 0, 1 \ldots M \qquad t_n = n\Delta t, \quad n = 0, 1 \ldots N. \tag{4.16}$$

Here, the values of z_{min} and z_{max} are chosen small and large enough respectively, such that they do not affect our domain of interest.

Let $P_m^n = P(z_m, t_n)$.

With regards to the boundary conditions, clearly for $t = T$ we have

$P_m^N = 0$. Given the flatness of the benefit function in a neighborhood of $z = z_{min}$ and $z = z_{max}$, we have Newmann boundary conditions, that is

$$P_0^n = P_1^n, \qquad P_{M-1}^n = P_M^n. \tag{4.17}$$

Alternatively, we can specify Dirichlet boundary conditions. Notice that if $z = z_{min}$ or $z = z_{max}$, the benefit at death equals to $G_1 e^{-r\tau}$ and $G_2 e^{-r\tau}$ respectively. Accordingly, the premium in these situations has to be calculated as for market independent term life insurance products. Therefore, we have

$$\begin{aligned} P(z_{min}, t) &= \frac{1}{\gamma} \log \left({}_{T-t}p_{a_0+t} + \int_t^T e^{\gamma G_1 e^{-rs}} \lambda(a_0 + s)_{s-t}p_{a_0+t} ds \right) \\ P(z_{max}, t) &= \frac{1}{\gamma} \log \left({}_{T-t}p_{a_0+t} + \int_t^T e^{\gamma G_2 e^{-rs}} \lambda(a_0 + s)_{s-t}p_{a_0+t} ds \right). \end{aligned}$$

In the experiments that follow, we assume that $G_1 = 5$, $G_2 = 10$, $r = 0.06$ and consider that the spot price varies between 0 and 20. In order to have an accurate solution, not altered by the truncation of the spatial domain, we choose $z_{min} = -25$ and $z_{max} = 25$.

Figure 4.1 illustrates the dependence of the lump sum premium on the insurer's risk aversion. In these two experiments, we assume that $\sigma = 0.2$ and that the contract maturity is 5 and 10 years respectively. Notice that consistent with our intuition, the premium increases as the risk aversion increases and also as the term of the contract increases.

Next, for these two contracts, we calculate the excess hedge for several risk aversion parameters. Taking a look at the premium in Figure 4.1, the form of the optimal excess hedge from Figure 4.2 is expected. Naturally, in the regions where the premium is asymptotically flat, the optimal excess hedge

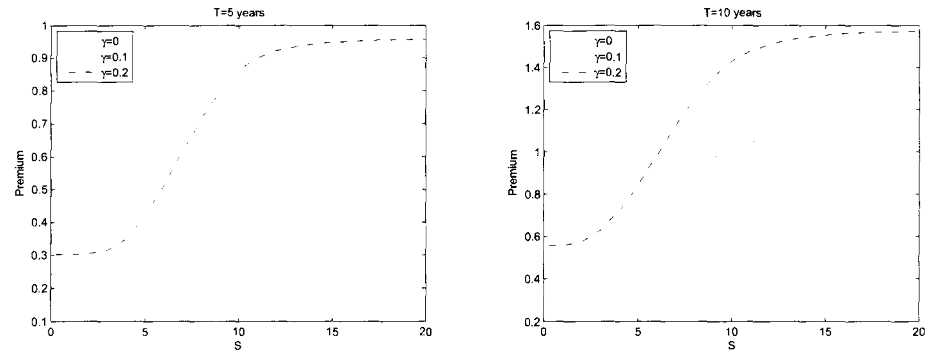

Figure 4.1: Lump-sum premium for an equity–linked term life insurance for different risk aversion parameters

approaches 0, while on the regions where the premium is a convex/concave function, the optimal excess hedge is an increasing/decreasing function of the stock price, respectively. Also, consistent with our intuition, observe that the optimal excess hedge increases as the insurer's risk aversion increases and as the term of the contract increases.

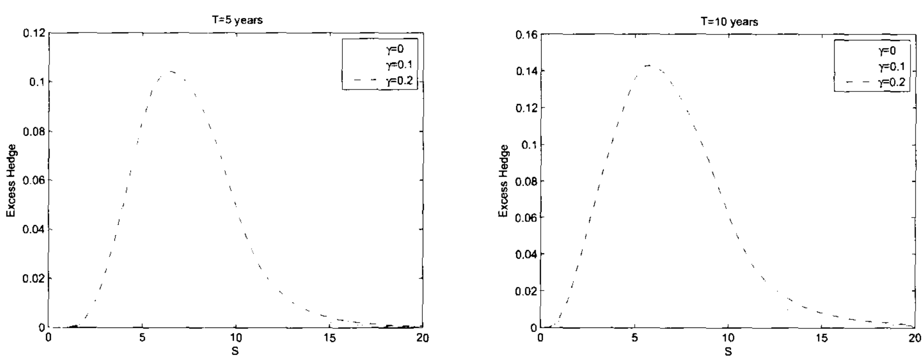

Figure 4.2: Excess-hedge for an equity–linked term life insurance for different risk aversion parameters.

In Figure 4.3 we show the dependence of the lump-sum premium on the volatility parameter. Observe that on the region where the premium is convex, the premium increases with volatility while on the region where it is concave, it decreases with volatility. Accordingly, if the stock behaves well and exhibits low volatility but the insurer assumes a high volatility parameter, as can be observed from Figure 4.3 this will result in charging a premium that is too small and thus results in a loss for the insurer.

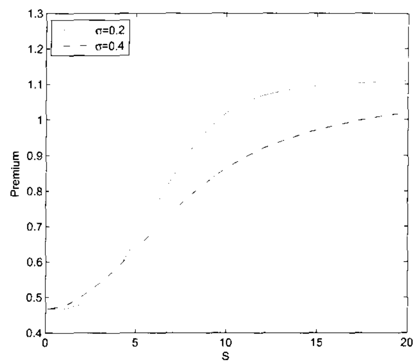

Figure 4.3: Lump-sum premium for an equity–linked term life insurance for different values of the volatility parameter.

4.1.2 Continuous premium

In this subsection we consider that the premium for the term life insurance contract is payable continuously at an annual rate h. We assume that this rate is set at the time of writing the contract and that it remains fixed during the life of the policy. Under these hypotheses, we define the value function of the insurer in the presence of the insurance risk, as follows

$$u^B(x, S, t; h(S,t)) = \sup_{\pi \in \mathcal{A}} E[U(X_T)|X_t = x, S_t = S], \tag{4.18}$$

where $h(S,t)$ denotes the premium rate set at time t. Here, the discounted wealth dynamics evolve as

$$\begin{cases} dX_s = ((\mu - r)\pi_s + he^{-rs})ds + \sigma\pi_s dW_s & \text{if} \quad t < s < \tau \\ dX_s = \pi_s\left((\mu - r)ds + \sigma dW_s\right) & \text{if} \quad \tau < s < T \\ X_{\tau+} = X_{\tau-} - g(S_\tau, \tau) \\ X_t = x, \end{cases}$$

where τ denotes the time of death of (a_0).

Then, it follows that the value function solves the HJB equation

$$\begin{cases} u_t^B + \max_\pi[(\mu - r)\pi u_x^B + \frac{1}{2}\sigma^2\pi^2 u_{xx}^B + \sigma^2\pi S u_{xS}^B] + he^{-rt}u_x^B \\ +(\mu - r)Su_S^B + \frac{1}{2}\sigma^2 S^2 u_{SS}^B + \lambda(a_0 + t)(u^0(x - g(S,t), t) - u^B(x, S, t)) = 0 \\ u^B(x, S, T; h^B) = U(x). \end{cases}$$

Following arguments similar to those applied in the lump-sum premium case, we obtain that the value function is given by $u^B(x, S, t; h) = u^0(x,t)e^{\gamma V(S,t;h)}$, where V satisfies the equation

$$\begin{cases} V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - he^{-rt} + \frac{\lambda(a_0 + t)}{\gamma}\left(e^{-\gamma(V - g(S,t))} - 1\right) = 0 \\ V(S, T; h) = 0. \end{cases} \tag{4.19}$$

Additionally, the optimal policy is given by

$$\pi^*(x, S, t; h) = \frac{\mu - r}{\gamma\sigma^2} + Sl_S. \tag{4.20}$$

Notice that

$$u^{(0)}(X_s - V(s, S_s; h), s) = u^B(X_s, S_s, s; h). \tag{4.21}$$

and therefore $V(t, S; h)$ represents the benefit reserve at time s.

The premium rate is such that, at the moment of writing the contract, the insurer is indifferent between accepting or not accepting the insurance risk, that is

$$u^0(x, t) = u^B(x, S, t; h). \tag{4.22}$$

Thus, the premium rate is given implicitly by the equation

$$V(S, t; h) = 0. \tag{4.23}$$

So, the indifference premium rate is such that the benefit reserve has zero value at the moment of writing the insurance contract. However, over time, due to the evolution of the stock price and of the individual's mortality, this premium rate might not coincide with the prevailing indifference premium rate and consequently the benefit reserve will no longer be zero. Concretely, by Itô's lemma, the evolution of the benefit reserve can be specified by the equation

$$\begin{cases} dV_s = (he^{-rs} + (\mu - r)V_S S - \frac{1}{2}\sigma^2 S^2 V_{SS} \\ \qquad\qquad - \frac{\lambda(a_0 + s)}{\gamma}(e^{-\gamma(V(S_s,s) - g(S_s,s))} - 1))dt + \sigma V_S S_s dW_t \\ V(t, S_t; h) = 0. \end{cases}$$

Further, observe that similar to the lump-sum premium case, the optimal excess hedge is analogous to the Black-Scholes delta hedge and is given by $e^*(S,t;h) = V_S(S,t;h)$.

For a risk neutral insurer, the partial differential equation for the benefit reserve becomes

$$\begin{cases} V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - he^{-rt} - \lambda(a_0+t)(V - g(S,t)) = 0 \\ V(S,T;h) = 0. \end{cases} \tag{4.24}$$

Applying the Feynman-Kač formula leads to

$$\begin{aligned} V(S,t;h) &= \int_t^T (E^Q[g(S_s,s)]\lambda(a_0+s) - he^{-rs})e^{-\int_t^s \lambda(a_0+u)du} ds \\ &= P(S,t) - \int_t^T he^{-rs}\,{}_{s-t}p_{a_0+t} ds \end{aligned}$$

where Q is the risk neutral measure given by (4.13) and $P(S,t)$ is the lump sum premium, corresponding to a risk neutral insurer. Combining the last equation with (4.23) we obtain

$$h = \frac{P(S,t)}{\int_t^T e^{-rs}\,{}_{s-t}p_{a_0+t} ds}. \tag{4.25}$$

Hence, for a risk neutral insurer, the premium rate is such that the lump sum premium coincides with the actuarial present value of the premium rate.

We conclude this subsection by implementing the premium rate for the term life insurance contract with benefit function given by (4.14). We assume that (a_0) is a 45 year old American male with force of mortality given by (4.15).

In order to calculate the indifference premium rate, first we discretize equation (4.19) by using an implicit-explicit finite difference scheme as the one applied in the lump-sum premium case. Then, the premium rate $h(S_0, 0)$ is obtained by varying its value until the reserve at time $t = 0$ has zero value.

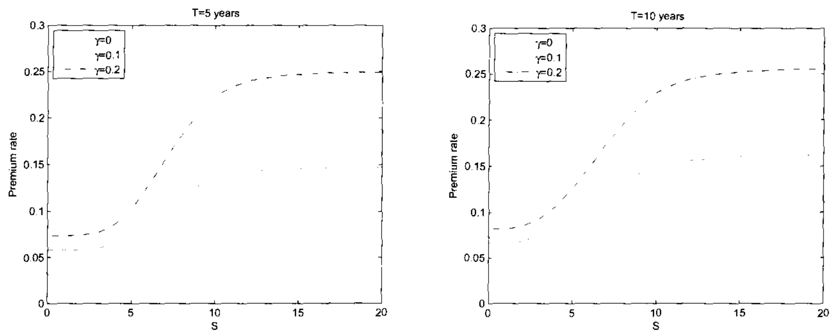

Figure 4.4: Continuous premium rate for an equity–linked term life insurance for different risk aversion parameters

Figure (4.4) depicts the behavior of the continuous premium rate with respect to the spot price for different levels of the insurer risk aversion when the term to maturity is 5 and 10 years, respectively. As expected, the premium rate increases when the risk aversion and the term to maturity increases.

4.2 The individual risk model

In this section, we study the pricing and hedging problem of the term life insurance contract (5.3) in a setting where the insurer's losses are modeled by the individual risk model. Clearly, in this situation we deal with a cohort of individuals. We assume that all these individuals are aged a_0 at time 0, with future lifetimes modeled as independent and identically distributed random variables. Further, we assume that they and have a (common) deterministic force of mortality.

4.2.1 Lump-sum premium

First, we analyze the insurer investment problem when accepting to sell term life insurances of the form mentioned above, to all individuals of the cohort. In this situation, the insurer's value function is given by

$$u^{(k)}(x,S,t)=\sup_{\pi\in\mathcal{A}}E[U(X_T)|X_t=x,S_t=S], \tag{4.26}$$

where k is a parameter specifying the number of individuals from the cohort alive at time t. Here, the discounted wealth process evolves as follows

$$\begin{cases} dX_s=\pi_s\left((\mu-r)ds+\sigma dW_s\right), & s\neq\tau_i \\ X_t=x \\ X_{\tau_i+}=X_{\tau_i-}-C_{\tau_i}, \ \text{if} & \tau_i<T, \end{cases}$$

where $\tau_i,\ i=1\dots k$ denote the times of death of the policyholders and C_{τ_i} denotes the discounted total claim at time τ_i. We assume that the times of death of the policyholders are not necessarily distinct.

Now, we model the insurer's loss on the time interval $[t,s),\ s\in(t,T]$ as a sum of the losses on each policy, that is

$$L_s^{ind}=\sum_{i=1}^{k}g(S_{\tau_i},\tau_i)1_{\{\tau_i<s|\tau_i>t\}}. \tag{4.27}$$

Accordingly, the discounted claim at time $\tau_i,\ t<\tau_i<T$ can be written as

$$C_{\tau_i}=L_{\tau_i+}-L_{\tau_i}. \tag{4.28}$$

Applying arguments similar to those used when pricing market independent insurance risks in an individual risk model, we obtain that $u^{(k)}$ solves the HJB equation

$$\begin{cases} u^{(0)}(x,S,t) = u^0(x,t) \\ \text{For } k \geq 1, \\ u_t^{(k)} + \max_\pi[(\mu - r)\pi u_x^{(k)} + \frac{1}{2}\sigma^2\pi^2 u_{xx}^{(k)} + \sigma^2\pi S u_{xS}^{(k)}] + (\mu - r)S u_S^{(k)} \\ +\frac{1}{2}\sigma^2 S^2 u_{SS}^{(k)} + k\lambda(a_0 + t)\left(u^{(k-1)}(x - g(S,t), S, t) - u^{(k)}(x,S,t)\right) = 0 \\ u^{(k)}(x,S,T) = U(x). \end{cases} \tag{4.29}$$

We consider an ansätz of the form $u^{(k)}(x,S,t) = u^0(x,t)e^{\gamma F^{(k)}(S,t)}$. Further, assuming that the optimal investment policy is given by the first order necessary condition, we have

$$\pi_t^* = \frac{\mu - r}{\gamma\sigma^2} + F_S^{(k)} S. \tag{4.30}$$

Inserting the expression of the ansätz and of π_t^* in (4.29), after some straightforward calculations, we obtain that $F^{(k)}$ satisfies the nonlinear partial differential equation

$$\begin{cases} F_t^{(k)} + \frac{1}{2}\sigma^2 S^2 F_{SS}^{(k)} - \frac{k\lambda(a_0 + t)}{\gamma}\left(1 - e^{-\gamma(F^{(k)}(S,t) - F^{(k-1)}(S,t) - g(S,t))}\right) = 0 \\ F^{(k)}(S,T) = 0. \end{cases} \tag{4.31}$$

with $F^{(0)} = 0$.

For a well-behaved benefit function $g(S,t)$, equation (4.31) has a smooth solution and the Verification Theorem implies that the ansätz proposed coincides with the value function. Also, the Verification Theorem confirms our initial assumption that the optimal investment policy is given by the first order necessary condition.

Now, let $P^{(k)}(S,t)$ denote the indifference premium for k policies at time t. The indifference premium solves the equation

$$u^0(x,t) = u^{(k)}(x + P^{(k)}(S,t), S, t). \tag{4.32}$$

Hence,

$$P^{(k)}(S,t) = F^{(k)}(S,t) \tag{4.33}$$

and consequently, the premium per risk $\tilde{P}^{(k)}(S,t) = \frac{1}{k}P^{(k)}(S,t)$ satisfies the equation

$$\begin{cases} \tilde{P}_t^{(k)} + \frac{1}{2}\sigma^2 S^2 - \frac{\lambda(a_0+t)}{\gamma}\left(1 - e^{-\gamma(k\tilde{P}^{(k)}(S,t)-(k-1)\tilde{P}^{(k-1)}-g(S,t))}\right) = 0 \\ \tilde{P}^{(k)}(S,T) = 0 \end{cases} \tag{4.34}$$

where $\tilde{P}^{(0)} = 0$ and where $\tilde{P}^{(1)}$ represents the lump sum premium in a single life insurance model. Notice that $\tilde{P}^{(k)}$ can be calculated by solving a system of k recursively defined partial differential equations.

Then, the optimal excess hedge corresponding to a portfolio consisting of k policies, $e^{*(k)}$ is given by

$$e^{*(k)} = P_S^{(k)} \tag{4.35}$$

and again notice the analogy with the Black-Scholes delta hedge.

At this point, it is interesting to analyze the dependence of the premium per risk on the number of policies in the insurer's portfolio. Recall, that we assumed that the policyholders have future lifetimes independent and identically distributed and moreover, their force of mortality evolves deterministically in time. Given these assumptions, one would expect that the premium per risk is constant and thus, coincides with the premium in a single life insurance model. Additionally, if the premium per risk is constant, by (4.35) the excess hedge per risk is also constant and equals the excess hedge in a single life insurance model.

Indeed, numerically implementing the premium per risk via equation (4.34) we obtain that when varying the number of policies in the portfolio, the premium per risk remains constant. Below, we exemplify our experiments by showing the plot of the premium per risk when the insurer's portfolio consists of $k = 1 \ldots 20$ policies. We assume that the force of mortality of all policyholders is given by (4.15). Additionally, we assume that all the insurance claims have a benefit function of the form (4.14), the time to maturity is $T = 10$ years, $\sigma = 0.2$, $r = 0.06$ and the insurer's risk aversion is $\gamma = 0.1$.

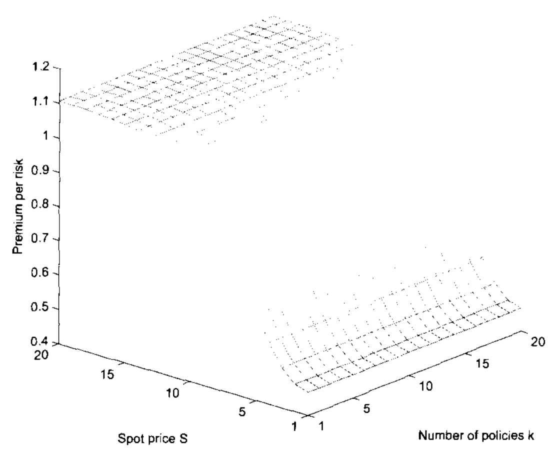

Figure 4.5: Premium per risk as a function of the spot price and of the number of policies in the insurers portfolio.

4.2.2 Continuous premium

In what follows, we examine the pricing and hedging of the term life insurance contract within the same loss model, but assumming that the premium is payable continuously, at a fixed annual rate $h^{(k)}$ set at the time of writing the contracts. In this situation, we define the value function of the insurer with the insurance risks by

$$u^{(k)}(x, S, t; h^{(k)}(S,t)) = \sup_{\pi \in \mathcal{A}} E[U(X_T)|S_t = S, X_t = x], \tag{4.36}$$

where the discounted wealth process has the dynamics given by

$$\begin{cases} dX_s = \left((\mu - r)\pi_s + Z_s e^{-rs} h^{(k)}\right) ds + \sigma \pi_s dW_s, \quad s \neq \tau_i \\ X_t = x \\ X_{\tau_i+} = X_{\tau_i-} - C_{\tau_i}, \quad \text{if } \tau_i < T. \end{cases} \tag{4.37}$$

Here τ_i, $i = 1 \ldots k$ represent the times of death of the policyholders, C_{τ_i} is the discounted total claim at time τ_i due to death of one or more policyholders and Z_s is a process recording the number of survivors at time s. With regards to C_{τ_i}, this is defined as in the lump-sum premium case.

Applying Bellman's principle of dynamic programming and stochastic calculus, it follows that $u^{(k)}$ satisfies the HJB equation

$$\begin{cases} u_t^{(k)} + \max_\pi [(\mu - r)\pi u_x^{(k)} + \frac{1}{2}\sigma^2 \pi^2 u_{xx}^{(k)} + \sigma^2 \pi S u_{xS}^{(k)}] + k e^{-rt} h^{(k)} u_x^{(k)} \\ \quad +(\mu - r) S u_S^{(k)} + \frac{1}{2}\sigma^2 S^2 u_{SS}^{(k)} + k\lambda(a_0 + t)\left(u^{(k-1)}(x - g(S,t), S, t) - u^k\right) = 0 \\ u^{(k)}(x, S, T; h^{(k)}) = U(x). \end{cases}$$

Following arguments similar to those applied in the lump-sum premium case, we obtain that

$$u^{(k)}(x, S, t; h^{(k)}) = u^0(x, t)e^{\gamma V^{(k)}(S,t;h^{(k)})} \tag{4.38}$$

where $V^{(k)}$ satisfies the partial differential equation

$$\begin{cases} V_t^{(k)} - ke^{-rt}h^{(k)} + \frac{1}{2}\sigma^2 S^2 V_{SS}^{(k)} + \frac{k\lambda(a_0 + t)}{\gamma}\left(e^{-\gamma(V^{(k)} - V^{(k-1)} - g(S,t))} - 1\right) = 0 \\ V^{(k)}(S, T) = 0. \end{cases} \tag{4.39}$$

Then, the optimal hedging policy is given by

$$\pi^*(S, t) = \frac{\mu - r}{\gamma\sigma^2} + V_S^{(k)} S. \tag{4.40}$$

Consequently the optimal excess hedge corresponding to a portfolio of k equity–linked term life insurance contracts is

$$e^{(k)*}(S, t) = V_S^{(k)}(S, t). \tag{4.41}$$

Observe that $V^{(k)}(S_s, s)$ represents the benefit reserve at time s since

$$u^{(0)}(X_s - V^{(k)}(S_s, s), t) = u^{(k)}(X_s, S_s, s). \tag{4.42}$$

The premium rate $h^{(k)} = h^{(k)}(S, t)$ is such that the insurer at time t is indifferent between accepting or not accepting the k insurance risks, that is

$$u^{(k)}(x, S, t) = u^{(0)}(x, t). \tag{4.43}$$

This implies that $V^{(k)}(S, t) = 0$. Clearly, as in a single life insurance model, the benefit reserve equals zero just at time t, when the contracts are written. Until the contracts' maturity, the policyholders' mortality changes as well as the financial market, thus rendering the insurer to change his initial attitude of indifference towards the insurance risks. Consequently, the benefit reserve is no longer zero.

Recall that in the lump sum premium case analyzed earlier we obtained that the premium per risk is independent of the number of policies in the insurer's portfolio. Naturally, the question arises of whether or not this property also applies to the premium rate.

Below, we numerically implement the premium rate when the insurer's portfolio consists of $k = 1 \dots 10$ policies. We assume that all the insurance claims have a benefit function of the form (4.14), the time to maturity is $T = 1$ year, $\sigma = 0.2$, $r = 0.06$ and the insurer's risk aversion is $\gamma = 0.1$.

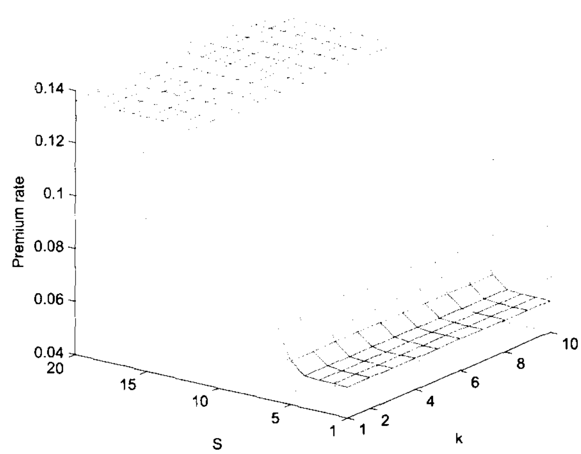

Figure 4.6: Premium rate as a function of the spot price and of the number of policies in the insurers portfolio.

Notice from Figure 4.6 that the premium per risk is independent of the number of policies in the insurer's portfolio.

4.3 The collective risk model

In this section, we analyze the pricing problem of term life insurances with benefits of the form (5.3) in a setting where insurer's losses are modeled via the collective risk model. We start with a cohort of policyholders as in section 4.2 and model the insurer's total loss on a time interval $[t, s)$, $s \in (t, T]$ as follows

$$L_s^{coll} = \sum_{i=1}^{N_s - N_t} g(S_{\tau_i}, \tau_i), \tag{4.44}$$

where $(N_t)_{0 \leq t \leq T}$ is a inhomogeneous Poisson process with intensity $\eta(a_0 + t)$. Specifically, N_t records the number of deaths within the cohort from time 0 up to time t and τ_i represents the arrival time of the ith (death) event.

An important assumption in this section is that the initial size of the cohort of policyholders is very large and consequently at any time $t < T$ the size of the remaining cohort is still arbitrary large.

Then, we connect the collective model to the individual risk model by requiring that on average, the number of deaths during any time interval $[0, t)$ is the same in both models. Thus,

$$E[N_t] = E[Y_t], \forall t \in (0, T]. \tag{4.45}$$

Here Y_t denotes the number of deaths from time 0 up to time t in the individual risk model. Accordingly, we have

$$\int_0^t \eta(a_0 + u) du = {}_k{}_t q_{a_0}, \tag{4.46}$$

where k represents the size of the cohort at time 0. Consequently,

$$\eta(a_0 + t) = k_t p_{a_0} \lambda(a_0 + t). \tag{4.47}$$

4.3.1 Lump-sum premium

We first consider the insurer investment problem when accepting the insurance risks and define the value function by

$$u(x, S, t) = \sup_{\pi \in \mathcal{A}} E[U(X_T) | X_t = x, S_t = S]. \tag{4.48}$$

Here the discounted wealth process evolves as follows

$$\begin{cases} dX_s = \pi_s \left((\mu - r)ds + \sigma dW_s \right) - dL_s^{coll}, \\ X_t = x. \end{cases} \tag{4.49}$$

Observe that the value function does not depend on the number of deaths by time t. That is because the term life insurance contracts that we analyze have the benefits payable at the moment of death of the policyholders and because for the Poisson process, used here to count the number of deaths, we have to specify just its intensity, not also the number of deaths or survivors. However, if the benefits were payable at maturity, we would have to include the dependence on the number of deaths by time t in the definition of the value function.

Applying Bellman's principle of dynamic programming and stochastic calculus arguments, we obtain that the value function solves the HJB equation

$$\begin{cases} u_t + \max_\pi[(\mu - r)\pi u_x + \frac{1}{2}\sigma^2\pi^2 u_{xx} + \sigma^2 S\pi u_{xS}] + (\mu - r)Su_S + \frac{1}{2}\sigma^2 S^2 u_{SS} \\ \qquad\qquad + \eta(a_0 + t)\left(u(x - g(S,t), S, t) - u(x, S, t)\right) = 0 \\ u(x, S, T) = U(x). \end{cases} \tag{4.50}$$

We consider an ansätz of the form $u(x, S, t) = u^0(x, t)e^{\gamma F(S,t)}$. Observe that $u_{xx} < 0$ and thus the maximum in (4.50) is well defined and attained at

$$\pi^*(S, t) = \frac{\mu - r}{\gamma\sigma^2} + SF_S(S, t). \tag{4.51}$$

Further, substituting the ansätz and the expression of $\pi^*(S, t)$ in (4.50), we obtain that F solves the linear partial differential equation

$$\begin{cases} F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} - \frac{\eta(a_0 + t)}{\gamma}\left(1 - e^{\gamma g(S,t)}\right) = 0 \\ F(S, T) = 0. \end{cases} \tag{4.52}$$

Applying the Feynman-Kač formula, we have that

$$F(S, t) = \int_t^T \frac{\eta(a_0 + s)}{\gamma}\left(E^Q[e^{\gamma g(S_s, s)}] - 1\right) ds \tag{4.53}$$

where Q is the risk neutral measure given by (4.13).

For well behaved benefit functions g, equation (4.62) has a smooth solution. Consequently, we can apply the Verification Theorem and obtain that the ansätz proposed coincides with the value function. Also, by the Verification Theorem the optimal policy is given by (4.51).

Now, let $P(S, t)$ denote the indifference premium at time t for the k

term life insurance contracts. $P(S,t)$ solves the equation

$$u^0(x,t) = u(x+P,S,t). \tag{4.54}$$

Solving equation (4.54) with respect to P, leads to $P(S,t) = F(S,t)$. Further by (4.53) and (4.47) it follows that

$$P(S,t) = \int_t^T \frac{\eta(a_0+s)}{\gamma}\left(E^Q[e^{\gamma g(S_s,s)}] - 1\right) ds. \tag{4.55}$$

Notice that also within this model, the excess hedge is analogous to the Black-Scholes delta hedge. Specifically, we have that $e^*(S,t) = P_S(S,t)$.

From (4.55), it follows that the premium at time $t=0$ for the k term life insurance contracts is

$$\begin{aligned} P(S,0) &= \int_0^T \frac{\eta(a_0+s)}{\gamma}\left(E^Q[e^{\gamma g(S_s,s)}] - 1\right) ds \\ &= \frac{k}{\gamma}\int_0^T \left(E^Q[e^{\gamma g(S_s,s)}] - 1\right) {}_sp_{a_0}\lambda(a_0+s)ds. \end{aligned}$$

Thus, the premium per risk $\bar{P}(S,0) = \frac{1}{k}P(S,0)$ is given by

$$\bar{P}(S,0) = \frac{1}{\gamma}\int_0^T \left(E^Q[e^{\gamma g(S_s,s)}] - 1\right) {}_sp_{a_0}\lambda(a_0+s)ds. \tag{4.56}$$

Taking the limit as $\gamma \to 0$ in equation (4.56), we obtain that the premium per risk at time $t=0$ for a risk neutral insurer is

$$\bar{P}(S,0) = \int_0^T E^Q[g(S_s,s)]_s p_{a_0}\lambda(a_0+s)ds. \tag{4.57}$$

Now, recognize that the expression from the left hand side of equation (4.57) represents the lump-sum premium for a risk neutral insurer in a single life insurance model, or as observed in subsection 4.2.1, the premium per risk for a risk neutral insurer in the individual risk model.

We conclude this subsection by several numerical experiments. First, we implement the premium per risk for the an equity–linked term life insurance with benefit (4.14) in the collective risk model and then, we compare it with the corresponding premium per risk in the individual risk model. We assume that the force of mortality of the policyholders is given by (4.15), the time to maturity is $T = 10$ years, $r = 0.06$, the spot price varies between 0 and 20 and the insurer's risk aversion is $\gamma = 0.1$. With regards to the benefit function g as mentioned above, this is chosen of the form (4.14), where $G_1 = 5$ and $G_2 = 10$. Notice from Figure 4.7 that the premium per risk in the collective

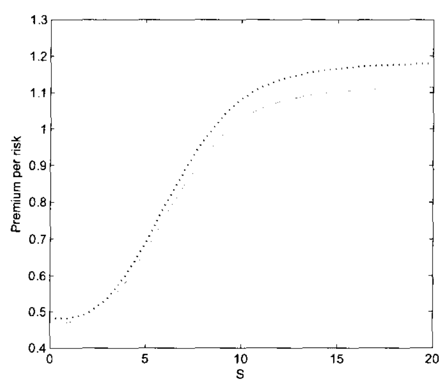

Figure 4.7: Premium per risk for an equity–linked term life insurance contract in the individual risk model (solid line) and in the collective risk model (dashed line).

risk model is greater than the premium in the individual risk model. Thus, the collective risk model proves to be more risky than the individual risk model. Next, we consider the same equity–linked term life insurance as in the preceding experiment and illustrate the dependence of the premium per risk on the insurer's risk aversion parameter.

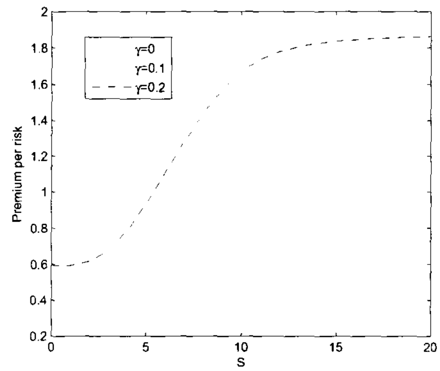

Figure 4.8: Premium per risk for an equity–linked term life insurance in the collective risk model, for different values of the insurer's risk aversion parameter.

As Figure 4.8 shows, the premium per risk increases as the risk aversion increases. The result is consistent with our intuition. Finally, in the last experiment, we show the dependence of the premium per risk on the time to maturity. Observe from Figure 4.9 that the premium per risk increases as the

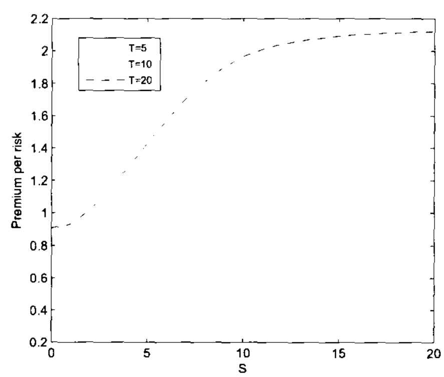

Figure 4.9: Premium per risk for an equity–linked term life insurance in the collective risk model, for different values of the time to maturity.

time to maturity increases. The result agrees to our intuition since as the time to maturity increases, increases the probability of death or in other words the probability that the insurer's portfolio will generate claims.

4.3.2 Continuous premium

In the following, we assume the same loss model as in the preceding subsection and examine the pricing of equity-linked term life insurance contracts in the situation where the premium is payable continuously, at a fixed annual rate h. However, here h represents the rate corresponding to the entire cohort and not the premium rate per risk as assumed in the individual risk model. In this case, we define the value function of the insurer with the insurance risks, as follows

$$u(x, S, t; h(S, t)) = \sup_{\pi \in \mathcal{A}} E[U(X_T)|X_t = x, S_t = S] \tag{4.58}$$

where $h(S, t)$ denotes the premium rate set at time t. Here, the discounted wealth process evolves according to the equations

$$\begin{cases} dX_s = ((\mu - r)\pi_s + he^{-rs})ds + \sigma\pi_s dW_s - dL_s^{coll}, \\ X_t = x. \end{cases} \tag{4.59}$$

Notice from the wealth equation that we implicitly assume that the premium rate will be paid until maturity. Essentially, this is a consequence of the assumption that the size of the cohort at any time $t < T$ is arbitrary large.

In this case, the HJB equation for u is

$$\begin{cases} u_t + \max_\pi[(\mu - r)\pi u_x + \frac{1}{2}\sigma^2\pi^2 u_{xx} + \sigma^2 S\pi u_{xS}] + (\mu - r)Su_S + \frac{1}{2}\sigma^2 S^2 u_{SS} \\ \qquad\qquad +he^{-rt}u_x + \eta(a_0 + t)\left(u(x - g(S,t), S, t) - u(x, S, t)\right) = 0 \\ u(x, S, T) = U(x). \end{cases} \tag{4.60}$$

We consider an ansätz of the form $u(x, S, t) = u^0(x, t)e^{\gamma V(S,t)}$. Notice that $u_{xx} < 0$ and applying the first order condition we obtain

$$\pi^*(S, t) = \frac{\mu - r}{\gamma\sigma^2} + SV_S(S, t). \tag{4.61}$$

Inserting the ansätz and the expression of π^* in (4.60), after some straightforward calculations we obtain that V satisfies the linear partial differential equation

$$\begin{cases} V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - he^{-rt} - \frac{\eta(a_0 + t)}{\gamma}\left(1 - e^{\gamma g(S,t)}\right) = 0 \\ V(S, T) = 0. \end{cases} \tag{4.62}$$

Accordingly, V has the Feynman-Kač representation

$$V(S, t) = \int_t^T \frac{\eta(a_0 + s)}{\gamma}\left(E^Q_{t,S}[e^{\gamma g(S_s,s)} - 1]\right) - he^{-rs}ds, \tag{4.63}$$

where Q is the risk neutral measure defined by (4.13). Further, we can write

$$V(S, t) = P^{Coll,LS}(S, t) - \int_t^T he^{-rs}ds, \tag{4.64}$$

where here $P^{Coll,LS}(S, t)$ denotes the lump-sum premium in the collective model for the term life insurance contract. Now, observe that $V(S_s, s)$ represents the benefit reserve at time s since it solves the equation

$$u^0(X_s - V(S_s, s), s) = u^B(X_s, S_s, s). \tag{4.65}$$

In this case the indifference premium equation is

$$u^0(x, t) = u(x, S, t; h(S, t)) \tag{4.66}$$

which implies that $V(S, t) = 0$. Accordingly, the premium rate at time t is given by

$$h(S, t) = \frac{P^{Coll,LS}(S, t)}{\int_t^T e^{-rs} ds}. \tag{4.67}$$

That is, the annual premium rate $h(S, t)$ is such that the total discounted premium paid during the life of the policies equals the lump-sum premium for the term life insurance contract.

From (4.67) it follows that the premium rate at time $t = 0$ is

$$h(S, 0) = \frac{kr \int_0^T \left(E^Q[e^{\gamma g(S_s, s)}] - 1 \right) {}_s p_{a_0} \lambda(a_0 + s) ds}{\gamma(1 - e^{-rT})}. \tag{4.68}$$

Remark 4.3.1.

In contrast to the individual risk model, observe that in the collective risk model the premium rate per risk increases over time as policyholders die. An alternative model for the premium rate is as follows: $\tilde{h}(s) = (k - N_s) h(S, t) e^{-rs}$, where $h(S, t)$ is a fixed premium rate per risk, set at time t. Indeed, with this choice the nature of the premium rate and of the premium rate per risk would be similar in the two risk models. However, this will introduce a new variable into our pricing analysis, namely the number of deaths by time t. The valuation of equity–linked term life insurances is a tractable problem within the collective risk model, essentially because of the independence of our pricing analysis on the number of deaths. Choosing the premium model just mentioned above, makes the problem not tractable since the dependence on the number of deaths cannot be factored out. Consequently, one will have to solve a system of PDEs.

Next, we implement the continuous premium rate for a term life insurance contract with benefit of the form (4.14) and examine its dependence on the insurer's risk aversion parameter. We assume a benefit function of the form just mentioned, where $G_1 = 5$, $G_2 = 10$, the spot price varies between 0 and 20 and $T = 10$ years. Then, we consider a cohort of $k = 10000$ policyholders whose force of mortality is given by (4.15).

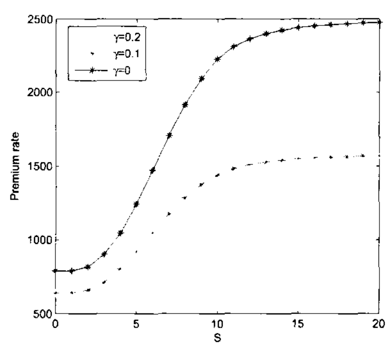

Figure 4.10: Continuous premium rate for an equity–linked term life insurance contract in the collective risk model, for different choices of the insurer's risk aversion parameter.

As expected, observe from Figure 4.10 that the premium rate increases as the risk aversion increases.

Chapter 5

Utility Indifference Pricing of Equity–Linked Term Life Insurance in Stochastic Volatility Market Models

In this chapter we study the pricing of equity–linked term life insurance in stochastic volatility market models. Unlike the Black-Scholes market model, considered in the preceding chapter, stochastic volatility market models have the advantage of assuming more realistic return distributions, with fatter tails and asymmetry. Moreover, stochastic volatility models are able to predict European option prices whose implied volatility "smiles".

Clearly, in this context, the combined insurance-financial market is incomplete since neither volatility nor individuals'mortality can be hedged. As in the preceding chapters, we propose utility indifference as a pricing approach.

In this chapter, we extend the results of Sircar & Zariphopoulou (2004) and price equity–linked term life insurance. We would like to mention that from the start we choose to model the insurer's losses via the collective model, this model being so far computationally more efficient than the others. In what follows, we show that the indifference premium for an equity linked term life insurance, in a fast-mean-reverting volatility regime can be well approximated by adjusted constant volatility results.

The financial-insurance market model

We consider a financial market consisting of two assets: a money market account with constant interest rate $r > 0$ and a risky stock or stock index. We assume that the discounted price of the stock (or stock index) satisfies the stochastic differential equation

$$\begin{cases} dS_s = S_s\left((\mu - r)ds + \sigma(Y_s)dW_s\right) \\ S_t = S > 0. \end{cases} \tag{5.1}$$

In the above, $\mu > r > 0$ and the volatility driving process (Y_s) is modeled as a correlated Markov diffusion

$$\begin{cases} dY_s = a(Y_s)ds + b(Y_s)d\tilde{Z}_s \\ Y_t = y \end{cases} \tag{5.2}$$

where $\tilde{Z}_s = \rho W_s + \sqrt{1-\rho^2}Z_s$ and $\rho \in [-1, 1]$. Here, the processes W and Z are independent Brownian motions on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F}$ is the augmentation of the natural filtration generated by W and Z.

Notice that by correlating the stock price with the volatility, the model explains the skew phenomenon present in option markets. Empirical evidence shows that typically, in equity markets the correlation coefficient ρ assume negative values; in other words, when volatility goes up stock prices go down and viceversa. This is called *leverage effect.*

With regards to the coefficients a and b and the function σ, we make the following assumption:

Assumption 5.0.1.

- σ, a and b are smooth and bounded with bounded derivatives.
- For all y, $\sigma(y) \geq L > 0$ for some $L < \infty$.

Now, as in chapter 4, we consider an insurer that has the opportunity to sell equity–linked term life insurance contracts with discounted benefit as follows

$$B = \begin{cases} g(S_\tau, \tau) & \text{if } \tau < T \\ 0 & \text{if } \tau \geq T, \end{cases} \tag{5.3}$$

in which τ denotes the policyholder's time of death and g is a positive, smooth and bounded function on $[0, \infty) \times [0, T]$.

In what follows, we examine the pricing of this life insurance contract using the utility indifference pricing approach. Recall that this approach requires solving two optimization problems. One of them is the Merton investment problem, that is maximizing the expected utility of terminal wealth by investing in the financial market. The second optimization problem consists of maximizing the expected utility of terminal wealth with the insurance risk by trading in the financial market.

Let us assume that the insurer's initial wealth is x and that he can actively trade in the financial market described above. Further let π_s denote the amount deemed to be invested in the stock at time s. Then, the discounted wealth of the insurer follows the dynamics

$$\begin{cases} dX_s = \pi_s \left((\mu - r)ds + \sigma(Y_s)dW_s \right) \\ X_t = x \end{cases} \tag{5.4}$$

At this point, we define the set of admissible policies $\mathcal{A}[t,T]$ as the set of processes π that are $\mathbb{F}$ progressively measurable and satisfy the integrability condition $\int_t^T \pi_s^2 ds < \infty$ $a.s.$ Notice that for $\pi \in \mathcal{A}[t,T]$, given the assumption that σ is bounded, the stochastic differential equation with random coefficients (5.4) has a unique solution.

5.1 The Merton problem in stochastic volatility market models

In this financial market setting, the Merton investment problem can be formulated as follows

$$u^0(x,y,t) = \sup_{\pi \in \mathcal{A}} E[U(X_T)|X_t = x, Y_t = y], \tag{5.5}$$

where the insurer's wealth process evolves according to (5.4).

The HJB equation for u^0 is

$$\begin{cases} u_t^0 + \max_\pi \left((\mu - r)\pi u_x^0 + \frac{1}{2}\sigma^2(y)\pi^2 u_{xx}^0 + \rho b(y)\pi\sigma(y) u_{xy}^0 \right) + a(y)u_y^0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{1}{2}b^2(y)u_{yy}^0 = 0 \\ u^0(x, y, T) = U(x) \end{cases} \tag{5.6}$$

We assume that u^0 is concave in wealth. Then, the maximum in (5.6) is well defined and it is attained in

$$\pi_t^* = -\frac{\mu - r}{\sigma^2(y)} \frac{u_x^0}{u_{xx}^0} - \frac{\rho b(y)}{\sigma(y)} \frac{u_{xy}^0}{u_{xx}^0}. \tag{5.7}$$

Inserting the expression of π_t^* in the HJB equation, leads to

$$\begin{cases} u_t^0 - \frac{(\mu - r)^2}{2\sigma^2(y)} \frac{(u_x^0)^2}{u_{xx}^0} - \frac{1}{2}b^2(y)\rho^2 \frac{(u_{xy}^0)^2}{u_{xx}^0} - \frac{\rho b(y)(\mu - r)}{\sigma(y)} \frac{u_{xy}^0 u_x^0}{u_{xx}^0} + a(y)u_y^0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{1}{2}b^2(y)u_{yy}^0 = 0 \\ u^0(x, y, T) = U(x) \end{cases} \tag{5.8}$$

Now, we consider the ansätz $u^0(x, y, t) = -e^{-\gamma x}\phi(y, t)$. Inserting the ansätz in equation (5.8), after some calculations, we obtain

$$\begin{cases} \phi_t - \frac{(\mu - r)^2}{2\sigma^2(y)}\phi - \frac{1}{2}b^2(y)\rho^2 \frac{(\phi_y)^2}{\phi} - \frac{\rho b(y)(\mu - r)}{\sigma(y)}\phi_y + a(y)\phi_y \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{1}{2}b^2(y)\phi_{yy} = 0 \\ \phi(y, T) = 1 \end{cases} \tag{5.9}$$

Further, we consider the transformation $\phi(y, t) = \psi^\delta(y, t)$, where $\delta = \frac{1}{1 - \rho^2}$ to be determined. This power transformation was introduced in the financial literature by Zariphopoulou (2001). Then, equation (5.9) becomes

$$\begin{cases} \psi_t + \left(a(y) - \dfrac{\rho b(y)(\mu - r)}{\sigma(y)}\right)\psi_y + \dfrac{1}{2}b^2(y)\psi_{yy} - \dfrac{(\mu - r)^2(1-\rho^2)}{2\sigma^2(y)}\psi = 0 \\ \psi(y,T) = 1 \end{cases} \tag{5.10}$$

Observe that essentially, the parameter δ that is called *distortion power* is chosen such that the partial differential equation (5.9) reduces to a linear equation.

Now, we would like to obtain a probabilistic representation of ψ from equation (5.10). Notice that for obtaining this, first we have to change the original probability space. We define the measure Q on $\mathcal{F}_T$ by

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \frac{\mu - r}{\sigma(Y_s)}dW_s - \frac{1}{2}\int_0^T \frac{(\mu - r)^2}{\sigma^2(Y_s)}ds\right) \tag{5.11}$$

Since σ is bounded, it is straightforward to show that Novikov's condition is satisfied. Accordingly, we can apply the Girsanov's theorem and obtain that Q is a probability measure on $\mathcal{F}_T$, $Q \sim P$ and

$$(\tilde{W}_s, Z_s) = \left(\int_0^s \frac{\mu - r}{\sigma(Y_u)}du + W_s, Z_s\right), \quad 0 \le s \le T \tag{5.12}$$

is a two dimensional Brownian motion under Q. Moreover, the dynamics of (S, Y) under Q are as follows

$$dS_s = \sigma(Y_s)S_s d\tilde{W}_s \tag{5.13}$$

$$dY_s = \left(a(Y_s) - \frac{\rho b(Y_s)(\mu - r)}{\sigma(Y_s)}\right)ds + b(Y_s)(\rho d\tilde{W}_s + \sqrt{1-\rho^2}dZ_s) \tag{5.14}$$

At this point, we can apply the Feynman-Kač formula and obtain that

$$\psi(y,t) = E^{Q}_{t,y}[e^{-\int_t^T \frac{(\mu-r)^2(1-\rho^2)}{2\sigma^2(Y_s)}}]. \tag{5.15}$$

Given the assumptions (5.0.1), the conditions of the Theorem 2.9.10 from Krylov (1980) are satisfied. Accordingly, (5.15) is the unique solution of equation (5.10) in the class of functions that are $C^{2,1}(\mathbb{R} \times [0,T])$ and satisfy a polynomial growth condition in y. Consequently, the ansätz proposed is a classical solution of the HJB equation (5.6) and therefore a viscosity solution of this equation. By the Verification Theorem the value function is the unique viscosity solution of equation (5.6) in the class of functions that are concave and of exponential growth in x and uniformly bounded in y. The ansätz considered satisfies these properties and therefore this coincides with the value function. Thus, the value function is given by

$$u^0(x,y,t) = -e^{-\gamma x}\left(E^{Q}_{t,y}[e^{-\int_t^T \frac{(\mu-r)^2(1-\rho^2)}{2\sigma^2(Y_s)}}]\right)^{\frac{1}{1-\rho^2}}. \tag{5.16}$$

5.2 The insurer's investment problem with the insurance risk

In what follows, we assume that the insurer accepts to sell to a cohort of individuals equity–linked term life insurance contracts with benefit given by (5.3). We consider that the contracts are written at time $t = 0$ and at that time the cohort consists of k individuals, where k has a very large value. Further, we assume that all these individuals have future lifetimes independent and identically distributed and that their mortality evolves deterministically in time.

Recall that when examining the pricing of equity–linked term life insurance in a Black-Scholes model, between all loss models, the collective model proved to be computationally more efficient, with the insurer's pricing problem reducing in this case to solving one linear partial differential equation. This motivates us to model from start, the insurer's losses via a collective risk model. Accordingly, we model the number of deaths within the cohort by a inhomogeneous Poisson process $(N_t)_{0\leq t\leq T}$ with intensity $\eta(a_0+t)$.

In this case, we define the value function of the insurer with the insurance risk by

$$u(x,y,S,t) = \sup_{\pi\in\mathcal{A}} E[U(X_T)|X_t = x, Y_t = y], \tag{5.17}$$

in which, the discounted wealth process evolves as follows

$$\begin{cases} dX_s = \pi_s\left((\mu - r)ds + \sigma(Y_s)dW_s\right) - dL_s^{coll} \\ X_t = x, \end{cases} \tag{5.18}$$

where $L_s = \sum_{i=1}^{N_s-N_t} g(\tau_i, S_{\tau_i})$ denotes the total loss of the insurer from time t to time s.

Now, let us express the value function $u(x,y,S,t)$ in terms of the certainty equivalent function for the insurance claim B, $c^B(y,S,t)$. We have

$$u(x,y,S,t) = -e^{-\gamma(x-c^B(y,S,t))}. \tag{5.19}$$

The value function u solves the HJB equation

$$\begin{cases} u_t + \max_\pi((\mu - r)\pi u_x + \frac{1}{2}\sigma^2(y)\pi^2 u_{xx} + \rho b(y)\sigma(y)\pi u_{xy} + \sigma^2(y)S\pi u_{xS}) \\ \qquad +a(y)u_y + \frac{1}{2}b^2(y)u_{yy} + (\mu - r)Su_S + \frac{1}{2}\sigma^2(y)S^2 u_{SS} + \rho b(y)S\sigma(y)u_{ys} \\ \qquad\qquad +\eta(a_0 + t)(u(x - g(S,t), y, S, t) - u(x, y, S, t)) = 0. \\ u(x, y, S, T) = 0 \end{cases} \tag{5.20}$$

Inserting (5.19) into the HJB equation, we conclude that the certainty equivalent function c^B satisfy

$$\begin{cases} c_t^B + \frac{1}{2}\sigma^2(y)S^2 c_{SS}^B + \rho b(y)S\sigma(y)c_{yS}^B + \frac{1}{2}b^2(y)c_{yy}^B + \left(a(y) - \frac{(\mu - r)\rho b(y)}{\sigma(y)}\right) c_y^B \\ \qquad +\frac{1}{2}b^2\gamma(1-\rho^2)(c_y^B)^2 - \frac{(\mu - r)^2}{2\gamma\sigma^2} + \frac{\eta(a_0 + t)}{\gamma}\left(e^{\gamma g(S,t)} - 1\right) = 0 \\ c^B(y, S, T) = 0 \end{cases} \tag{5.21}$$

Applying Theorem 5.8, chapter 5, from Carmona (2009b) it follows that equation (5.21) has a unique solution in the class of $C^{2,2,1}(\mathbb{R}\times\mathbb{R}^+\times[0,T])$ functions that satisfy a polynomial growth condition on $\mathbb{R}\times\mathbb{R}^+\times[0,T]$. Then, by the Verification Theorem, the value function is the unique viscosity solution of (5.20).

5.3 Indifference premium

Let $P(y, S, t)$ denote the indifference premium for the equity–linked term life insurance that we are studying. From the indifference premium equation it follows that

$$P(y, S, t) = c^B(y, S, t) - c^0(y, t), \tag{5.22}$$

where c^B and c^0 represent the certainty equivalent functions for the claim B and for the Merton problem, respectively.

We proved that the value function in the Merton problem can be written as follows

$$u^0(x,y,t) = -e^{-\gamma x}\psi(y,t)^{\frac{1}{1-\rho^2}}, \tag{5.23}$$

where ψ satisfy the equation (5.10). On the other hand, we have that $u^0(x,y,t) = -e^{-\gamma(x-c^0(y,t))}$ since c^0 is the certainty equivalent function for the Merton problem. Accordingly, we have that

$$\psi(y,t) = e^{\gamma(1-\rho^2)c^0(y,t)}. \tag{5.24}$$

Substituting (5.24) into equation (5.10), after some calculations, we obtain that c^0 solves the partial differential equation

$$\begin{cases} c^0_t + \left(a(y) - \dfrac{(\mu-r)b(y)\rho}{\sigma(y)}\right)c^0_y + \dfrac{1}{2}b^2(y)c^0_{yy} + \dfrac{1}{2}b^2(y)\gamma(1-\rho^2)(c^0_y)^2 \\ \qquad\qquad -\dfrac{\mu-r}{2\gamma\sigma^2(y)} = 0 \\ c^0(y,T) = 0 \end{cases} \tag{5.25}$$

Subtracting (5.25) from (5.21), we find that the indifference premium P, solves the following partial differential equation

$$\begin{cases} P_t + \left(a(y) - \dfrac{(\mu-r)b(y)\rho}{\sigma(y)} + b^2(y)\gamma(1-\rho^2)c^0_y\right)P_y + \dfrac{1}{2}b^2(y)P_{yy} \\ \quad +\dfrac{1}{2}b^2(y)\gamma(1-\rho^2)P_y^2 + \dfrac{1}{2}\sigma^2(y)S^2P_{SS} + \rho b(y)S\sigma(y)P_{yS} \\ \qquad\qquad +\dfrac{\eta(a_0+t)}{\gamma}(e^{\gamma g(S,t)}-1) = 0 \\ P(y,S,T) = 0 \end{cases} \tag{5.26}$$

5.4 Fast-mean-reversion asymptotics

In what follows, we examine the pricing of equity–linked term life insurance in a mean-reverting stochastic volatility model of the form

$$dS_t = (\mu - r)S_t dt + \sigma(Y_t)S_t dW_t \tag{5.27}$$

$$dY_t = \alpha(m - Y_t)dt + \beta(\rho dW_t + \sqrt{1-\rho^2}dZ_t). \tag{5.28}$$

Observe that Y is an Ornstein-Uhlenbeck process with rate of mean reversion α and long-run mean m. From (5.28) it follows that

$$Y_t \sim \mathcal{N}(m + (Y_0 - m)e^{-\alpha t}, \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})). \tag{5.29}$$

It is well-known that the invariant distribution of the process Y is $\mathcal{N}(m, \nu^2)$, where $\nu^2 = \frac{\beta^2}{2\alpha}$ (see Fouque, Papanicolau & Sircar (2000), page 68). From now on, given a function $f(Y)$, we denote its expectation with respect to the invariant distribution by $\langle f \rangle$.

Remarks 5.4.1. Observe that the invariant distribution of Y corresponds to the normal distribution from (5.29) in the limit cases $t \to \infty$ or $\alpha \to \infty$.

Now, we define the mean-square time averaged volatility by:

$$\overline{\sigma^2} := \frac{1}{t}\int_0^t \sigma^2(Y_u)du \tag{5.30}$$

The Ergodic Theorem (see for example Fouque, Papanicolau & Sircar (2000), pages 66-67) implies that

$$\overline{\sigma^2} \to \langle \sigma^2 \rangle \quad \text{as} \quad t \to \infty \quad \text{a.s} \tag{5.31}$$

As mentioned in the remark (5.4.1), the limits $t \to \infty$ or $\alpha \to \infty$ are the same in terms of distributions. Accordingly, the relation (5.31) holds true also when Y_t is a very fast-mean-reverting process.

At this point, one may wonder to what reduces the indifference premium in a very fast mean reverting volatility regime. We will show that actually, in this case, the premium coincides with the premium for the insurance claim B in a market model with constant volatility $\bar{\sigma} = \sqrt{\langle \sigma^2 \rangle}$. Now, let us prove this assertion in a simple case, specifically when the insurer is risk neutral.

Observe that as γ goes to zero, equation (5.26) becomes

$$\begin{cases} P_t + \left(a(y) - \dfrac{(\mu - r)b(y)\rho}{\sigma(y)} \right) P_y + \dfrac{1}{2} b^2(y) P_{yy} + \dfrac{1}{2}\sigma^2(y) S^2 P_{SS} + \rho b(y)\sigma(y) S P_{yS} \\ \qquad + \eta(a_0 + t) g(S,t) = 0 \\ P(y,S,T) = 0 \end{cases} \tag{5.32}$$

Applying the Faynman-Kač formula, we obtain that the indifference premium at time $t = 0$ for the k term life insurance contracts is

$$P(y,S,0) = \int_0^T E^Q[g(S_t,t)]\eta(a_0 + t)dt = k \int_0^T E^Q[g(S_t,t)]_s p_{a_0} \lambda(a_0 + s) ds \tag{5.33}$$

where the risk neutral measure Q is given by (5.11).

Accordingly, the price per risk is as follows

$$\bar{P}(y, S, 0) = \int_0^T E^Q[g(S_t, t)]_s p_{a_0} \lambda(a_0 + s) ds \tag{5.34}$$

It is straightforward to show that

$$\ln \frac{S_t}{S} \sim \mathcal{N}\left(-\frac{1}{2}\int_0^t \sigma^2(Y_u)du, \int_0^t \sigma^2(Y_t)dt\right) \quad \text{under } Q. \tag{5.35}$$

Consequently, in a very fast-mean-reverting volatility regime, we have that

$$\ln \frac{S_t}{S} \sim \mathcal{N}\left(-\frac{1}{2}\bar{\sigma}^2 t, \bar{\sigma}^2 t\right) \quad \text{under } Q. \tag{5.36}$$

Therefore, we conclude that the premium per risk for the claim B in a stochastic volatility market model, with a very fast mean reverting volatility factor, coincides with the corresponding premium in a market model with constant volatility $\bar{\sigma}$. Later on, we will show that this result holds for any risk aversion γ.

This convergence result motivates us to consider an asymptotic approximation technique for calculating the indifference premium. Concretely, we consider the regime $\alpha = \frac{1}{\varepsilon}$, where $\varepsilon \downarrow 0$. This implies that β scales as $\beta = \frac{\nu\sqrt{2}}{\varepsilon}$. Further, the coefficients of the volatility driving factor Y_t are as follows

$$a(y) = \alpha(m - y) = \frac{1}{\varepsilon}(m - y)$$

$$b(y) = \beta$$

Then, the equations (5.25) and (5.25) become

$$\begin{cases} c_t^0 + (\frac{1}{\varepsilon}(m-y) - \frac{(\mu-r)\nu\sqrt{2}\rho}{\sigma(y)\sqrt{\varepsilon}})c_y^0 + \frac{\nu^2}{\varepsilon}c_{yy}^0 + \frac{\nu^2}{\varepsilon}(1-\rho^2)\gamma(c_y^0)^2 - \frac{(\mu-r)^2}{2\gamma\sigma^2(y)} = 0 \\ c^0(y,T) = 0 \end{cases} \tag{5.37}$$

and

$$\begin{cases} c_t^B + \frac{1}{2}\sigma^2(y)S^2c_{SS}^B + \frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon}}S\sigma(y)c_{yS}^B + \frac{\nu^2}{\varepsilon}c_{yy}^B + (\frac{1}{\varepsilon}(m-y) - \frac{(\mu-r)\rho\nu\sqrt{2}}{\sigma(y)\sqrt{\varepsilon}})c_y^B \\ \qquad + \frac{\nu^2}{\varepsilon}\gamma(1-\rho^2)(c_y^B)^2 - \frac{(\mu-r)^2}{2\gamma\sigma^2(y)} + \frac{\eta(a_0+t)}{\gamma}\left(e^{\gamma g(S,t)-1}\right) = 0 \\ c^B(y,S,T) = 0 \end{cases} \tag{5.38}$$

In order to write the above equations in a compact form, we introduce the following operators

$$\mathcal{L}_0 u = \nu^2 u_{yy} + (m-y)u_y$$
$$\mathcal{L}_1 u = \sqrt{2}\nu\rho\left(S\sigma(y)u_{yS} - \frac{\mu-r}{\sigma(y)}u_y\right)$$
$$\mathcal{L}_2 u = u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}$$

Denoting

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2, \tag{5.39}$$

we see that c^0 and c^B satisfy

$$\begin{cases} \mathcal{L}^\varepsilon c^0 + \frac{\nu^2}{\varepsilon}(1-\rho^2)\gamma(c_y^0)^2 - \frac{(\mu-r)^2}{2\gamma\sigma^2(y)} = 0 \\ c^0(y,T) = 0 \end{cases} \tag{5.40}$$

and

$$\begin{cases} \mathcal{L}^{\varepsilon} c^{B} + \dfrac{\nu^2}{\varepsilon}(1-\rho^2)\gamma(c_y^B)^2 - \dfrac{(\mu-r)^2}{2\gamma\sigma^2(y)} + \dfrac{\eta(a_0+t)}{\gamma}(e^{\gamma g(S,t)}-1) = 0 \\ c^B(y,S,T) = 0 \end{cases} \tag{5.41}$$

5.4.1 Approximation for the certainty equivalents

We now consider the following formal expansions of c^0 and c^B in powers of $\sqrt{\varepsilon}$:

$$c^0 = \psi^{(0)} + \psi^{(1)}\sqrt{\varepsilon} + \psi^{(2)}\varepsilon + \psi^{(3)}\varepsilon\sqrt{\varepsilon} + \dots \tag{5.42}$$

$$c^B = \phi^{(0)} + \phi^{(1)}\sqrt{\varepsilon} + \phi^{(2)}\varepsilon + \phi^{(3)}\varepsilon\sqrt{\varepsilon}\dots \tag{5.43}$$

The asymptotic expansion for $c^0(y,t)$ was obtained in Sircar & Zariphopoulou (2004). To describe their results, we define $\dfrac{1}{\sigma_*^2} = \left\langle \dfrac{1}{\sigma^2} \right\rangle$ and let $F(y)$ the solution of the Poisson equation

$$\mathcal{L}_0 F = \frac{1}{\sigma(y)^2} - \frac{1}{\sigma_*^2}. \tag{5.44}$$

It then follows from item (ii) of Proposition 4.4 of Sircar & Zariphopoulou (2004) that setting

$$\psi^{(0)}(t) = -\frac{(\mu-r)^2}{2\sigma_*^2\gamma}(T-t) \tag{5.45}$$

$$\psi^{(1)}(t) = -\frac{\rho\nu(\mu-r)^3}{\sqrt{2\alpha}}\left\langle \frac{F'}{\sigma} \right\rangle (T-t), \tag{5.46}$$

leads to

$$|c^0(y,t) - \psi^{(0)} - \sqrt{\varepsilon}\psi^{(1)}(t)| = \mathcal{O}(\varepsilon), \tag{5.47}$$

for each point (y,t).

Our aim is to prove a similar result for $c^B(y,S,t)$ and consequently for $P(y, S, t)$. Inserting the formal expansion (5.43) into (5.41) and collecting terms of equal order in ε, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon}(\mathcal{L}_0\phi^{(0)}+\nu^2(1-\rho^2)\gamma(\phi_y^{(0)})^2)+\frac{1}{\sqrt{\varepsilon}}(\mathcal{L}_0\phi^{(1)}+\mathcal{L}_1\phi^{(0)}+2\nu^2(1-\rho^2)\gamma\phi_y^{(0)}\phi_y^{(1)}) \\
&+(\mathcal{L}_0\phi^{(2)}+\mathcal{L}_1\phi^{(1)}+\mathcal{L}_2\phi^{(0)}+\nu^2(1-\rho^2)\gamma((\phi_y^1)^2+2\phi_y^0\phi_y^{(2)})-\frac{(\mu-r)^2}{2\gamma\sigma^2(y)}+\frac{\eta(a_0+t)}{\gamma}(e^{\gamma g}-1)) \\
&+\sqrt{\varepsilon}(\mathcal{L}_0\phi^{(3)}+\mathcal{L}_1\phi^{(2)}+\mathcal{L}_2\phi^{(1)}+2\nu^2(1-\rho^2)\gamma(\phi_y^{(0)}\phi_y^{(3)}+\phi_y^{(1)}\phi_y^{(2)}))+\ldots=0
\end{aligned} \tag{5.48}$$

Observe from equation (5.48) that we will need to solve Poisson equations associated with $\mathcal{L}_0$, of the form

$$\mathcal{L}_0\chi + f = 0. \tag{5.49}$$

It can be shown, see Fouque, Papanicolau & Sircar (2000), pages 91 – 92 that equation (5.49) admits a solution unless the following condition holds

$$\langle f \rangle = 0. \tag{5.50}$$

Next, we will refer to (5.50) as *centering condition* and to the function f as the *source term*. Writing explicitly the expression of the differential operator $\mathcal{L}_0$ in (5.49), after some calculations, we obtain

$$\chi'(y) = \frac{1}{\nu^2\Phi(y)}\int_y^\infty f(z)\Phi(z)dz. \tag{5.51}$$

Therefore, if f is bounded then χ' is bounded and χ has at most logarithmic growth, i.e.

$$|\chi'(y)| \leq C_1 \tag{5.52}$$

$$|\chi(y)| \leq C_2(1 + \log(1 + |y|)). \tag{5.53}$$

These properties of the solutions of the Poisson equation (5.49) will be very useful when proving the order of approximation of the truncation error.

Now, we consider each order separately, as follows

$\mathcal{O}(1/\varepsilon)$: Matching terms of order $1/\varepsilon$, we have

$$\mathcal{L}_0\phi^{(0)} + \nu^2(1-\rho^2)\gamma(\phi_y^{(0)})^2 = 0. \tag{5.54}$$

For fixed (S, t) this corresponds to the nonlinear ordinary differential equation

$$\nu^2 f''(y) + (m-y)f'(y) + \nu^2(1-\rho^2)\gamma(f'(y))^2 = 0, \tag{5.55}$$

whose solution is

$$f(y) = \frac{1}{\gamma(1-\rho^2)} \log\left(1 + c_1 \int_{-\infty}^{y} e^{\frac{(m-z^2)}{2\nu^2}} dz\right) + c_2, \tag{5.56}$$

for constants c_1 and c_2. Observe that $\int_{-\infty}^{\infty} e^{\frac{(m-z^2)}{2\nu^2}} dz = \infty$. Since we are interested only in solutions that are well-behaved at ∞, we take $c_1 = 0$ and conclude that $\phi^{(0)}$ must be constant in y.

$\mathcal{O}(1/\sqrt{\varepsilon})$: The terms of order $1/\sqrt{\varepsilon}$ give

$$\mathcal{L}_0\phi^{(1)} + \mathcal{L}_1\phi^{(0)} + 2\nu^2(1-\rho^2)\gamma\phi_y^{(0)}\phi_y^{(1)} = 0 \tag{5.57}$$

Since $\phi^{(0)} = \phi^{(0)}(S,t)$, the equation above reduces to

$$\mathcal{L}_0\phi^{(1)} = 0 \tag{5.58}$$

For fixed (S,t), this corresponds to the linear differential equation

$$\nu^2 h''(y) + (m-y)h'(y) = 0 \tag{5.59}$$

Solving this equation, we obtain

$$h(y) = C_1 \int_{-\infty}^{y} e^{\frac{(m-z^2)}{2\nu^2}} dz + C_2, \tag{5.60}$$

for constants C_1 and C_2. We are interested only in well behaved solutions at ∞ and therefore we take $C_1 = 0$. This implies that $\phi^{(1)}$ is also constant in y.

$\mathcal{O}(\varepsilon)$: For these choices of $\phi^{(0)}$ and $\phi^{(1)}$, the order one terms give

$$\mathcal{L}_0\phi^{(2)} + \mathcal{L}_2\phi^{(0)} - \frac{(\mu-r)^2}{2\gamma\sigma^2(y)} + \frac{\eta(a_0+t)}{\gamma}(e^{\gamma g(S,t)} - 1) = 0 \tag{5.61}$$

For fixed (S,t), (5.61) is a Poisson equation for $\phi^{(2)}$ with respect to the operator $\mathcal{L}_0$, in the variable y. Equation (5.61) does not admit a solution unless the centering condition

$$\left\langle \mathcal{L}_2\phi^{(0)} - \frac{(\mu - r)^2}{2\gamma\sigma^2(y)} + \frac{\eta(a_0 + t)}{\gamma}(e^{\gamma g(S,t)} - 1) \right\rangle = 0 \tag{5.62}$$

is satisfied. $\phi^{(0)}$ is y independent and therefore $\langle \mathcal{L}_2\phi^{(0)}\rangle = \langle \mathcal{L}_2\rangle\phi^{(0)}$. Accordingly, the centering condition (5.62) holds true, provided we define $\phi^{(0)}$ as the solution of the linear parabolic equation

$$\begin{cases} \phi_t^{(0)} + \frac{1}{2}\bar{\sigma}^2 S^2 \phi_{SS}^{(0)} - \frac{(\mu - r)^2}{2\gamma\sigma_*^2} + \frac{\eta(a_0 + t)}{\gamma}(e^{\gamma g(S,t)} - 1) = 0 \\ \phi^{(0)}(S, T) = 0 \end{cases} \tag{5.63}$$

Now, subtracting (5.62) from (5.61), we obtain

$$\mathcal{L}_0\phi^{(2)} + (\mathcal{L}_2 - \langle \mathcal{L}_2\rangle)\phi^{(0)} - \frac{(\mu - r)^2}{2\gamma}\left(\frac{1}{\sigma^2(y)} - \frac{1}{\sigma_*^2}\right) = 0. \tag{5.64}$$

Consequently,

$$\mathcal{L}_0\phi^{(2)} = -\frac{1}{2}(\sigma^2(y) - \bar{\sigma}^2)S^2\phi_{SS}^{(0)} + \frac{(\mu - r)^2}{2\gamma}\left(\frac{1}{\sigma^2(y)} - \frac{1}{\sigma_*^2}\right). \tag{5.65}$$

Therefore, we can chose $\phi^{(2)}$ as follows

$$\phi^{(2)}(y, S, t) = -\frac{1}{2}f_1(y)S^2\phi_{SS}^{(0)} + \frac{(\mu - r)^2}{2\gamma}f_2(y), \tag{5.66}$$

where f_1 and f_2 are solutions of the Poisson equations

$$\mathcal{L}_0 f_1 = \sigma(y)^2 - \bar{\sigma}^2 \tag{5.67}$$

$$\mathcal{L}_0 f_2 = \frac{1}{\sigma^2(y)} - \frac{1}{\sigma_*^2}. \tag{5.68}$$

$\mathcal{O}(\sqrt{\varepsilon})$: The terms of order $\sqrt{\varepsilon}$ give

$$\mathcal{L}_0\phi^{(3)} + \mathcal{L}_1\phi^{(2)} + \mathcal{L}_2\phi^{(1)} = 0. \tag{5.69}$$

This is a Poisson equation for $\phi^{(3)}$ with respect to $\mathcal{L}_0$ in the variable y. In this case, the centering condition is

$$\langle \mathcal{L}_1\phi^{(2)} + \mathcal{L}_2\phi^{(1)} \rangle = 0. \tag{5.70}$$

Further, using (5.66), we have

$$\begin{aligned}\mathcal{L}_1\phi^{(2)} = \sqrt{2}\nu\rho \Bigg(& -\frac{1}{2}\sigma(y)f_1'(y)(2S^2\phi^{(0)}_{SS} + S^3\phi^{(0)}_{SSS}) \\ & + \frac{\mu - r}{2\sigma(y)} f_1'(y) S^2\phi^{(0)}_{SS} - \frac{(\mu - r)^3}{2\sigma(y)\gamma} f_2'(y)\Bigg).\end{aligned}$$

Consequently,

$$\begin{aligned}\langle \mathcal{L}_1\phi^{(2)} \rangle &= \sqrt{2}\nu\rho \left(-\frac{1}{2}\langle \sigma f_1' \rangle (2S^2\phi^{(0)}_{SS} + S^3\phi^{(0)}_{SSS}) + \frac{\mu - r}{2}\left\langle \frac{f_1'}{\sigma} \right\rangle S^2\phi^{(0)}_{SS} \right. \\ & \quad \left. - \frac{(\mu - r)^3}{2\gamma}\left\langle \frac{f_2'}{\sigma} \right\rangle \right) \\ &= S^2\phi^{(0)}_{SS}\left(-\sqrt{2}\nu\rho\langle \sigma f_1' \rangle + \frac{(\mu - r)\rho\nu}{\sqrt{2}}\left\langle \frac{f_1'}{\sigma} \right\rangle \right) - \frac{\nu\rho}{\sqrt{2}} S^3\phi^{(0)}_{SSS}\langle \sigma f_1' \rangle \\ & \quad - \frac{(\mu - r)^3\rho\nu}{\sqrt{2}\gamma}\left\langle \frac{f_2'}{\sigma} \right\rangle.\end{aligned}$$

Inserting this into (5.70), leads to

$$\begin{aligned}\langle \mathcal{L}_2 \rangle \phi^{(1)} = S^2\phi^{(0)}_{SS}\left(\sqrt{2}\nu\rho\langle \sigma f_1' \rangle - \frac{(\mu - r)\rho\nu}{\sqrt{2}}\left\langle \frac{f_1'}{\sigma} \right\rangle \right) + \frac{\nu\rho}{\sqrt{2}} S^3\phi^{(0)}_{SSS}\langle \sigma f_1' \rangle \\ + \frac{(\mu - r)^3\rho\nu}{\sqrt{2}\gamma}\left\langle \frac{f_2'}{\sigma} \right\rangle.\end{aligned}$$

We conclude that the centering condition (5.70) is satisfied provided we define $\widetilde{\phi^{(1)}}(S,t) = \sqrt{\varepsilon}\phi^{(1)}(S,t)$ as the solution of the linear differential equation

$$\begin{cases} \widetilde{\phi^{(1)}}_t + \frac{1}{2}\bar{\sigma}^2 S^2 \widetilde{\phi^{(1)}}_{SS} - A(S,t) = 0 \\ \widetilde{\phi^{(1)}}(S,T) = 0 \end{cases} \tag{5.71}$$

where

$$A(S,t) = C_1 + C_2 S^2 \phi^{(0)}_{SS} + C_3 S^3 \phi^{(0)}_{SSS}, \tag{5.72}$$

for constants

$$C_1 = \frac{(\mu - r)^3 \rho\nu\sqrt{\varepsilon}}{\gamma\sqrt{2}} \langle \frac{f_2'}{\sigma} \rangle \tag{5.73}$$

$$C_2 = \frac{\nu\rho\sqrt{\varepsilon}}{\sqrt{2}} \left(2\langle \sigma f_1' \rangle - (\mu - r) \left\langle \frac{f_1'}{\sigma} \right\rangle \right) \tag{5.74}$$

$$C_3 = \frac{\nu\rho\sqrt{\varepsilon}}{\sqrt{2}} \langle \sigma f_1' \rangle. \tag{5.75}$$

Now, subtracting (5.70) from (5.69) and using relations, we obtain that

$$\begin{aligned} \mathcal{L}_0\phi^{(3)} = -\frac{1}{2}(\sigma^2(y) - \bar{\sigma}^2) S^2 \phi^{(1)}_{SS} + \frac{\nu\rho}{\sqrt{2}} \Big((\sigma(y) f_1'(y) - \langle \sigma f_1' \rangle)(2S^2\phi^{(0)}_{SS} + S^3 \phi^{(0)}_{SSS}) \\ -(\mu - r)\left(\frac{f_1'(y)}{\sigma(y)} - \left\langle \frac{f_1'}{\sigma} \right\rangle \right) S^2 \phi^{(0)}_{SS} + \frac{(\mu - r)^3}{\gamma} \left(\frac{f_2'(y)}{\sigma(y)} - \left\langle \frac{f_2'}{\sigma} \right\rangle \right) \Big) \end{aligned} \tag{5.76}$$

which implies that we can chose $\phi^{(3)}$ as follows

$$\begin{aligned} \phi^{(3)}(y,S,t) = -\frac{1}{2} f_1(y) S^2 \phi^{(1)}_{SS} + \frac{\nu\rho}{\sqrt{2}} \Big(f_3(y)(2S^2\phi^{(0)}_{SS} + S^3\phi^{(0)}_{SSS}) \\ +(\mu - r) f_4(y) S^2 \phi^{(0)}_{SS} \frac{(\mu - r)^3}{\gamma} f_5(y) \Big) \end{aligned} \tag{5.77}$$

where the functions f_3, f_4 and f_5 are solutions of the Poisson equations

$$\mathcal{L}_0 f_3 = \sigma(y) f_1'(y) - \langle \sigma f_1' \rangle \tag{5.78}$$

$$\mathcal{L}_0 f_4 = \frac{f_1'(y)}{\sigma(y)} - \left\langle \frac{f_1'}{\sigma} \right\rangle \tag{5.79}$$

$$\mathcal{L}_0 f_5 = \frac{f_2'(y)}{\sigma(y)} - \left\langle \frac{f_2'}{\sigma} \right\rangle. \tag{5.80}$$

Now, considering the terms up to order 3 in $\sqrt{\varepsilon}$ in the expansion (5.43), we have

$$c^B(y,S,t) = \phi^{(0)}(S,t) + \sqrt{\varepsilon}\phi^{(1)}(S,t) + \varepsilon\phi^{(2)}(y,S,t) + \varepsilon\sqrt{\varepsilon}\phi^{(3)}(y,S,t) - E^{\varepsilon}(y,S,t). \tag{5.81}$$

Here $\phi^{(0)}$, $\phi^{(1)}$, $\phi^{(2)}$ and $\phi^{(3)}$ are defined via (5.63), (5.71), (5.66) and (5.77) respectively and $E^{\varepsilon}(y,S,t)$ denotes the error term that occurs due to considering just the first four terms of the expansion.

Inserting (5.81) in equation (5.41), we obtain that E^{ε} solves the quasi-linear parabolic partial differential equation

$$\begin{cases} \mathcal{L}^{\varepsilon} E^{\varepsilon} + 2\nu^2(1-\rho^2)\gamma(\phi_y^{(2)} + \sqrt{\varepsilon}\phi_y^{(3)})E_y^{\varepsilon} - \varepsilon I = \dfrac{\nu^2}{\varepsilon}(1-\rho^2)\gamma(E_y^{\varepsilon})^2 \\ E^{\varepsilon}(y,S,T) = \varepsilon J(y,S) \end{cases} \tag{5.82}$$

where I and J are given by

$$I(y,S,t) = \mathcal{L}_2\phi^{(2)} + \mathcal{L}_1\phi^{(3)} + \sqrt{\varepsilon}\mathcal{L}_2\phi^{(3)} + \nu^2(1-\rho^2)\gamma(\phi_y^{(2)} + \sqrt{\varepsilon}\phi_y^{(3)})^2 \tag{5.83}$$

$$J(y,S) = \phi^{(2)}(y,S,T) + \sqrt{\varepsilon}\phi^{(3)}(y,S,T). \tag{5.84}$$

Remarks 5.4.2.

- Observe that the assumption that the benefit function is smooth and bounded with bounded derivatives, implies that the kth order logarithmic derivatives of $\phi^{(0)}$, $D_k = S^k \frac{\partial^k \phi^{(0)}}{\partial S^k}$ are bounded for any $k \geq 1$. Moreover, this implies the same property for $\phi^{(1)}$.

- Also notice that (5.65) and (5.76) are Poisson equations for $\phi^{(2)}$ and $\phi^{(3)}$ respectively, with source terms bounded. Accordingly, $\phi^{(2)}$ and $\phi^{(3)}$ are at most logarithmically growing as functions of y and both have bounded derivatives with respect to y.

Given these remarks, it follows that $\phi^{(2)}$, $\phi^{(3)}$, I and J are bounded as functions of (S, t) and are at most logarithmically growing as functions of y.

In what follows we show that the error term E^ε is of order ε. To prove this, we closely follow the approach adopted by Sircar & Zariphopoulou (2004). Essentially, this approach consists in finding an upper and lower bound of the error term E^ε and prove that both are of order ε. In order to prove these results we will need the theorem below obtained from Walter (1970). The first part of the theorem provides us with existence and uniqueness of a solution of a quasilinear parabolic equation on unbounded domains. The second part of the theorem acts as a comparison principle.

First, let us denote by $D = \mathbb{R} \times \mathbb{R}^+ \times [0, T]$ and by $\mathcal{D} = C(\bar{D}) \cap C^{2,2,1}(D)$.

Theorem 5.4.1. *Let $\bar{\mathcal{L}}^\varepsilon$ a differential operator on $\mathcal{D}$ defined by*

$$\bar{\mathcal{L}}^\varepsilon u = u_t + \frac{1}{2}\sigma^2(y)S^2 u_{SS} + \frac{\sqrt{2}\nu\rho S\sigma(y)}{\sqrt{\varepsilon}} u_{yS} + \frac{\nu^2}{\varepsilon} u_{yy} + h(y, S, t, u, u_y, u_S), \quad (5.85)$$

where h satisfies the condition: for $w > v$,

$$h(y, S, t, w, w_1, w_2) - h(y, S, t, v, v_1, v_2) \leq a(y, S, t)(w - v) + b_1(y, S, t)|w_1 - v_1| + b_2(y, S, t)|w_2 - v_2|, \quad (5.86)$$

with the functions a, b_1 and b_2 satisfying the growth conditions

$$a(y, S, t) \leq C(1 + y^2 + (\ln S)^2) \quad (5.87)$$

$$|b_1(y, S, t)| \leq C(1 + |y| + |\ln S|), \ |b_2(y, S, t)| \leq C(1 + |y| + |\ln S|), \quad (5.88)$$

for a constant $C > 0$ and for all $(y, S, t) \in D$.

Then, the terminal value problem

$$\begin{cases} \bar{\mathcal{L}}^\varepsilon u = 0 \ \text{in } D \\ u(y, S, T) = \xi(y, S) \ \text{for all } y \in \mathbb{R} \text{ and } S \in \mathbb{R}^+ \end{cases} \quad (5.89)$$

has an unique solution belonging to the class $\mathcal{D}$ and satisfying the growth condition: there exists $C > 0$ such that

$$|u(y, S, t)| \leq e^{C(y^2 + (\ln S)^2)}, \ \text{for large } y \text{ and } S. \quad (5.90)$$

Moreover, if $u, v \in \mathcal{D}$ satisfy the growth condition (5.90) and $\bar{\mathcal{L}}^\varepsilon u \geq \bar{\mathcal{L}}^\varepsilon v$ in D while $u(y, S, T) \leq v(y, S, T)$ for all $y \in \mathbb{R}$ and $S \in \mathbb{R}^+$, then $u \leq v$ in D.

Proof. First, we consider the change of variable

$$S = e^z \quad \text{and} \quad \tau = T - t \tag{5.91}$$

and denote $\tilde{u}(y, z, \tau) = u(y, S, t)$ and $\tilde{h}(y, z, \tau, \tilde{u}, \tilde{u}_1, \tilde{u}_2) = h(y, S, t, u, u_1, u_2)$. Then, the differential operator $\bar{\mathcal{L}}^\varepsilon$ becomes

$$\bar{\mathcal{L}}^\varepsilon \tilde{u} = -\tilde{u}_\tau + \frac{1}{2}\sigma^2(y)\tilde{u}_{zz} + \frac{\sqrt{2}\nu\rho\sigma(y)}{\sqrt{\varepsilon}}\tilde{u}_{yz} + \frac{\nu^2}{\varepsilon}\tilde{u}_{yy} + H(y, z, \tau, \tilde{u}, \tilde{u}_y, \tilde{u}_z), \tag{5.92}$$

where $H(y, z, \tau, \tilde{u}, \tilde{u}_y, \tilde{u}_z) = \tilde{h}(\tau, y, z, \tilde{u}, \tilde{u}_y, \tilde{u}_z) - \frac{1}{2}\sigma^2(y)\tilde{u}_z$.

Moreover, via the transformation (5.91), the terminal value problem (5.89) becomes a initial value problem, as follows

$$\begin{cases} \bar{\mathcal{L}}^\varepsilon \tilde{u} = 0 \text{ in } D \\ \tilde{u}(y, z, 0) = \xi(y, z), \text{ for all } \quad (y, z) \in \mathbb{R}^2 \end{cases} \tag{5.93}$$

From (5.86), (5.87) and (5.88), we have: for $w > v$

$$H(y, z, \tau, \tilde{w}, \tilde{w}_1, \tilde{w}_2) - H(y, z, \tau, \tilde{v}, \tilde{v}_1, \tilde{v}_2) = \tilde{h}(y, z, \tau, \tilde{w}, \tilde{w}_1, \tilde{w}_2) - \tilde{h}(y, z, \tau, \tilde{v}, \tilde{v}_1, \tilde{v}_2)$$
$$-\frac{1}{2}\sigma^2(y)(\tilde{w}_2 - \tilde{v}_2) \le \tilde{a}(y, z, \tau)(\tilde{w} - \tilde{v}) + \tilde{b}_1(y, z, \tau)|\tilde{w}_1 - \tilde{v}_1| + (\tilde{b}_2(y, z, \tau) + \frac{1}{2}\sigma^2(y))|\tilde{w}_2 - \tilde{v}_2|$$

where

$$\tilde{a}(y, z, \tau) = a(y, S, t) \le C(1 + y^2 + (\ln S)^2) = C(1 + y^2 + z^2)$$
$$|\tilde{b}_1(y, z, \tau)| = |b_1(y, S, t)| \le C(1 + |y| + |\ln S|) = C(1 + |y| + |z|)$$
$$|\tilde{b}_2(y, z, \tau)| = |b_2(y, S, t)| \le C(1 + |y| + |\ln S|) = C(1 + |y| + |z|)$$

Accordingly, (5.93) is a initial value problem, of the form studied by Walter (1970), chapter 4, pages 211 – 215. Consequently, the result follows. □

Now, let us show the existence of an upper bound for the error term E^ε. First we notice that the left hand side of equation (5.82) can be written as

$$\mathcal{L}^\varepsilon E^\varepsilon + 2\nu^2(1-\rho^2)\gamma(\phi_y^{(2)} + \sqrt{\varepsilon}\phi_y^{(3)})E_y^\varepsilon - \varepsilon I = \bar{\mathcal{L}}^\varepsilon E^\varepsilon \tag{5.94}$$

where

$$h(y,S,t,u,u_1,u_2) = \left(\frac{m-y}{\varepsilon} - \frac{(\mu-r)\sqrt{2}\nu\rho}{\sqrt{\varepsilon}\sigma(y)} + 2\nu^2(1-\rho^2)\gamma(\phi_y^{(2)} + \sqrt{\varepsilon}\phi_y^{(3)})\right)u_1$$
$$-\varepsilon I(y,S,t).$$

For $w > v$, we have

$$h(y,S,t,w,w_1,w_2) - h(y,S,t,v,v_1,v_2) \leq b_1(y,S,t)|w_1 - v_1| \tag{5.95}$$

in which

$$b_1(y,S,t) = \left|\frac{m-y}{\varepsilon} - \frac{(\mu-r)\sqrt{2}\nu\rho}{\sqrt{\varepsilon}\sigma(y)} + 2\nu^2(1-\rho^2)\gamma(\phi_y^{(2)} + \sqrt{\varepsilon}\phi_y^{(3)})\right|. \tag{5.96}$$

Since $\sigma, \phi_y^{(2)}$ and $\phi_y^{(3)}$ are all bounded, we obtain that $b_1(y,S,t) \leq C(1+|y|)$ and this implies that h satisfies the condition (5.86).

Applying theorem 5.4.1, it follows that the terminal value problem

$$\begin{cases} \bar{\mathcal{L}}^\varepsilon E^\varepsilon = 0 \text{ in } D \\ E^\varepsilon(y,S,T) = \varepsilon J(y,S) \text{ for all } y \in \mathbb{R} \text{ and } S \in \mathbb{R}^+. \end{cases} \tag{5.97}$$

has a unique solution belonging to the class $\mathcal{D}$ and satisfying the growth condition (5.90). Let us denote this solution by U^ε. Observe that $\bar{\mathcal{L}}^\varepsilon U^\varepsilon \leq \bar{\mathcal{L}}^\varepsilon E^\varepsilon$ while $U^\varepsilon(y,S,T) = E^\varepsilon(y,S,T)$. Accordingly, invoking again theorem 5.4.1, we obtain that $U^\varepsilon \geq E^\varepsilon$.

Next, we look for a lower bound for the error term E^ε. Multiplying equation (5.82) by $A^\varepsilon = e^{-E^\varepsilon}$, we obtain that A^ε solves the equation

$$\begin{cases} \mathcal{L}^\varepsilon A^\varepsilon + 2\nu^2(1-\rho^2)\gamma(\phi_y^{(2)} + \sqrt{\varepsilon}\phi_y^{(3)})A_y^\varepsilon + \varepsilon I A^\varepsilon = \frac{1}{2}A^\varepsilon\left(\sigma(y)S\frac{A_S^\varepsilon}{A^\varepsilon} + \frac{\rho\nu\sqrt{2}A_y^\varepsilon}{\varepsilon A^\varepsilon}\right)^2 \\ A^\varepsilon(y,S,T) = e^{-\varepsilon J(y,S)} \end{cases} \tag{5.98}$$

Now, notice that the left hand side of equation (5.98) can be written as follows

$$\mathcal{L}^\varepsilon A^\varepsilon + 2\nu^2(1-\rho^2)\gamma(\phi_y^{(2)} + \sqrt{\varepsilon}\phi_y^{(3)})A_y^\varepsilon + \varepsilon I A^\varepsilon = \bar{\mathcal{L}}^\varepsilon A^\varepsilon \tag{5.99}$$

where

$$h(y,S,t,u,u_1,u_2) = \left(\frac{m-y}{\varepsilon} - \frac{(\mu-r)\sqrt{2}\nu\rho}{\varepsilon\sigma(y)} + 2\nu^2(1-\rho^2)\gamma(\phi_y^{(2)} + \sqrt{\varepsilon}\phi_y^{(3)})\right)u_1 \\ +\varepsilon I u.$$

Suppose that $w > v$, then

$$h(y,S,t,w,w_1,w_2) - h(y,S,t,v,v_1,v_2) \leq a(y,S,t)(w-v) + b_1(y,S,t)|w_1 - v_1| \tag{5.100}$$

where $a(y,S,t) = |\varepsilon I(y,S,t)|$ and $b_1(y,S,t)$ is given by (5.96).

Recall that $I(y,S,t)$ is bounded as a function of S and t and is at most logarithmically growing as a function of y. Thus, we have $I(y,S,t) \leq$

$C(1+\log(1+|y|))$. Note that $a(y, S, t)$ satisfies (5.87) if $\log(1+|y|) \leq y^2$.

Assumption 5.4.1. We assume that the coefficients of the volatility driving process Y are such that the growth condition (5.87) holds true.

Now, since $b_1(y, S, t)$ and $a(y, S, t)$ satisfy the growth conditions (5.88) and (5.87), we can apply theorem 5.4.1 and conclude that the terminal value problem

$$\begin{cases} \bar{\mathcal{L}}^{\varepsilon} A^{\varepsilon} = 0 \text{ in } D \\ A^{\varepsilon}(y, S, T) = e^{-\varepsilon J(y,S)} \text{ for all } y \in \mathbb{R} \text{ and } S \in \mathbb{R}^{+} \end{cases} \tag{5.101}$$

has a unique solution in $\mathcal{D}$ and satisfying the growth condition (5.90). We denote this solution by B^{ε} and by $L^{\varepsilon} = -\ln B^{\varepsilon}$. Notice that $\bar{\mathcal{L}}^{\varepsilon} A^{\varepsilon} \geq \bar{\mathcal{L}}^{\varepsilon} B^{\varepsilon}$ while $A^{\varepsilon}(y, S, T) = B^{\varepsilon}(y, S, T)$. Applying the second part of theorem 5.4.1 we obtain that $A^{\varepsilon} \leq B^{\varepsilon}$. Consequently, $e^{-E^{\varepsilon}} \leq e^{-L^{\varepsilon}}$ and therefore $E^{\varepsilon} \geq L^{\varepsilon}$.

At this point, remains to show that both U^{ε} and L^{ε} are of order $\mathcal{O}(\varepsilon)$. Observe that from (5.97) and (5.101), we have by the Feynman-Kač formula the following probabilistic representations for U^{ε} and L^{ε}:

$$U^{\varepsilon} = \varepsilon E^{Q^*}_{t,y,S}[J(\tilde{Y}_T, \tilde{S}_T)] - \varepsilon \int_t^T E^{Q^*}_{t,y,S}[I(\tilde{Y}_s, \tilde{S}_s, s)]ds \tag{5.102}$$

$$L^{\varepsilon} = -\ln B^{\varepsilon} = -\ln\left(E^{Q^*}_{t,y,S}[e^{-\varepsilon J(\tilde{Y}_T,\tilde{S}_T)+\varepsilon\int_t^T I(\tilde{Y}_s,\tilde{S}_s,s)ds}]\right). \tag{5.103}$$

Here the measure Q^* is a measure is defined by

$$\frac{dQ^*}{dP} = \exp\Big(-\int_0^T \frac{\mu - r}{\sigma(Y_s)} dW_s - \int_0^T \Gamma(Y_s, S_s, s) d\tilde{Z}_s - \frac{1}{2}\int_0^T \frac{(\mu-r)^2}{\sigma(Y_s)^2} + \Gamma(Y_s, S_s, s)^2 ds\Big)$$

where

$$\Gamma(Y_s, S_s, s) = \frac{(\mu - r)\sqrt{2}\nu\rho}{\sqrt{\varepsilon}\sigma(y)} - 2\nu^2(1-\rho^2)\gamma(\phi_y^{(2)} + \sqrt{\varepsilon}\phi_y^{(3)}).$$

Observe that the boundedness of σ, $\phi_y^{(2)}$ and $\phi_y^{(3)}$ implies that $\Gamma(Y_s, S_s, s)$ and $\dfrac{\mu - r}{\sigma(Y_s)}$ are both bounded and thus Novikov's condition is satisfied. Accordingly, by the Girsanov's theorem we have that $Q^* \sim P$ and moreover, the dynamics of S and Y under Q^* are as follows

$$\begin{aligned}
d\tilde{S}_s &= \sigma(\tilde{Y}_s)\tilde{S}_s dW_s^* \\
d\tilde{Y}_s &= \left(\frac{m - Y_s}{\varepsilon} - \Gamma(Y_s, S_s, s)\right) ds + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} d\tilde{Z}_s^*
\end{aligned}$$

in which $W_s^* = W_s + \int_0^s \frac{\mu - r}{\sigma(Y_u)} du$ and $\tilde{Z}_s^* = \tilde{Z}_s + \int_0^s \Gamma(Y_u, S_u, u) du$.

Recall that $I(y, S, t)$ and $J(y, S)$ are bounded as functions of S and t and are at most logarithmically growing as functions of y. Accordingly, we have

$$|U^\varepsilon| \le \varepsilon C_1 E_{t,y,S}^{Q^*}[1 + \ln(1 + |\tilde{Y}_T|)] + \varepsilon C_2 \int_t^T E_{t,y,S}^{Q^*}[1 + \ln(1 + |\tilde{Y}_s|)] ds$$

for some positive constants C_1 and C_2. Further, applying Young's inequality, we obtain

$$|U^\varepsilon| \le \varepsilon C_1 \left(1 + \ln(1 + E_{t,y,S}^{Q^*}[|\tilde{Y}_T|])\right) + \varepsilon C_2 \int_t^T 1 + \ln(1 + E_{t,y,S}^{Q^*}[|\tilde{Y}_s|]) ds. \tag{5.104}$$

Next, we will use the following proposition from Sircar & Zariphopoulou (2004).

Proposition 5.4.1. *There exists $\varepsilon_0 > 0$ and a constant $C(t,T,v,y)$ independent of ε, such that for any $s \in [t,T]$ and $\varepsilon \in (0,\varepsilon_0)$,*

$$E^{Q^*}_{t,y,S}[e^{v\tilde{Y}_s}] < C(t,T,v,y). \tag{5.105}$$

Notice that (5.4.1) implies that Y_s is Q^* integrable. Combining this with (5.104) it follows $U^\varepsilon = \mathcal{O}(\varepsilon)$.

Next, applying the same properties of I and J as above, we have

$$B^\varepsilon = E^{Q^*}_{t,y,S}[e^{-\varepsilon J(\tilde{Y}_T,\tilde{S}_T)+\varepsilon \int_t^T}] \leq E^{Q^*}[e^{\varepsilon C_1(1+\ln(1+|\tilde{Y}_T|))+\varepsilon C_2 \int_t^T \int_t^T 1+\ln(1+|\tilde{Y}_s|)ds}]$$

for some positive constants C_1 and C_2. Further, we have

$$\begin{aligned} B^\varepsilon &\leq \varepsilon E^{Q^*}_{t,y,S}[e^{C_1(1+\ln(1+|\tilde{Y}_T|))+C_2 \int_t^T 1+\ln(1+|\tilde{Y}_s|)ds}] + 1 \\ &\leq \varepsilon E^{Q^*}_{t,y,S}[e^{C_1(1+|\tilde{Y}_T|)+C_2 \int_t^T 1+|\tilde{Y}_s|ds}] + 1. \end{aligned}$$

Consequently, by proposition (5.4.1) we have that $B^\varepsilon = 1 + \mathcal{O}(\varepsilon)$ and thus $L^\varepsilon = \mathcal{O}(\varepsilon)$.

Piecing together the results obtained so far, we conclude that $E^\varepsilon = \mathcal{O}(\varepsilon)$.

5.4.2 Approximation for the indifference premium

Now, we use the asymptotic expansion just obtained for $c^B(y,S,t)$ and the one provided in Sircar & Zariphopoulou (2004) for $c^0(S,t)$ to find the first two terms in the following expansion for the indifference premium itself

$$P(y,S,t) = P^{(0)}(y,S,t) + \sqrt{\varepsilon}P^{(1)}(y,S,t) + \dots \tag{5.106}$$

From (5.22), we deduce that

$$P^{(0)}(S,t) = \phi^{(0)}(S,t) - \psi^{(0)}(t)$$
$$P^{(1)}(S,t) = \phi^{(1)}(S,t) - \psi^{(1)}(t).$$

Using (5.63) and (5.45), we conclude that $P^{(0)}(S,t)$ must be the solution of the linear parabolic equation

$$\begin{cases} P_t^{(0)} + \frac{1}{2}\bar{\sigma}^2 S^2 P_{SS}^{(0)} + \frac{\eta(a_0+t)}{\gamma}(e^{\gamma g(S,t)} - 1) = 0 \\ P^{(0)}(S,T) = 0 \end{cases} \tag{5.107}$$

Observe that $P^{(0)}(S,t)$ corresponds to the indifference premium of the equity–linked term life insurance contract with benefit (5.3) in a financial market with constant volatility $\bar{\sigma}$.

Next, using (5.71) and (5.46), we conclude that $\tilde{P^{(1)}}(S,t) = \sqrt{\varepsilon}P^{(1)}(S,t)$ must be the solution of the linear partial differential equation

$$\begin{cases} \widetilde{P^{(1)}}_t + \frac{1}{2}\bar{\sigma}^2 S^2 \widetilde{P^{(1)}}_{SS} - G(S,t) = 0 \\ \widetilde{P^{(1)}}(S,T) = 0 \end{cases} \tag{5.108}$$

where

$$G(S,t) = C_2 S^2 P_{SS}^{(0)} + C_3 S^3 P_{SSS}^{(0)} \tag{5.109}$$

and C_2 and C_3 are the constants defined by (5.74) and (5.75), respectively.

As shown in subsection 5.4.1, the first two terms of the asymptotic expansions for c^0 and c^B provide for these two functions an approximation of order $\mathcal{O}(\varepsilon)$. Consequently, finally we have

$$|P(y,S,t) - P^{(0)}(S,t) - \widetilde{P^{(1)}}| = \mathcal{O}(\varepsilon).$$

Conclusions and directions for future research

We solve the valuation problem of life insurance risks of different nature, market independent or equity–linked, under various assumptions regarding policyholders'mortality and the financial market. Given the incomplete nature of life insurance markets, an indifference valuation approach tailored to different models of the insurer's liability is applied.

First, we analyze market independent life insurance risks under the assumption of random mortality. Although the market independent life insurance products that we study have a simple structure of the benefit, the assumption of random mortality transforms their pricing problem in a very interesting one. We find that within the individual risk model, this assumption lead us to two important qualitative properties of the indifference premium, such as super-additivity of the premium with respect to the number of policies and an increasing nature of the premium per risk (lump-sum or continuous). Intuitively, these results can be explained by a positive correlation that may develop over time between policyholders'mortality. Thus, we conclude that random mortality is an essential assumption especially when pricing in aggregate loss models. With regards to the pricing problem in these models, we obtain that in the individual risk model, premium calculation requires solving a number of recursively defined linear partial differential equations equal to the size of the insurer's portfolio. Accordingly, we conclude that this is not a very useful model for an insurer. In contrast, the collective risk model, proves to be more efficient since premium calculation within this model requires solving only one linear partial differential equation.

Equity–linked term life insurances are attractive products since they provide mortality protection in conjunction to equity investment. So, these products incorporate both mortality and financial risk and as observed, in this thesis we explicitly recognize both these risks. Applying the utility indifference valuation approach we obtain that in all loss models, the premium solves a second order partial differential equation similar to the Black-Scholes equation while also reflecting the mortality risk. In this case, we assume deterministic mortality and as expected in such setting, we obtain that the premium per risk in all models is constant. We would like to underline that also here, the collective risk model proves to be computationally more efficient than the other loss models. The price to pay for this feature is higher premiums per risk than in the other models. However, for an insurer with small risk aversion, the premium per risk is very close to the one obtained in the individual risk model. An interesting generalization of these results, that we consider for future research, is considering the pricing problem of equity–linked term life insurance under the assumption of random mortality.

Finally, in the last chapter, we extend our analysis regarding the pricing of equity–linked term life insurance to a more realistic financial market model, where the volatility of the stock index is stochastic. Concretely, we consider a one factor mean-reverting model and treat from start our pricing problem in the collective risk model. Using the utility indifference valuation approach, we are led to a quasilinear partial differential equation for the indifference premium. However, we do not solve this equation numerically, but instead we propose an asymptotic approximation of the indifference premium, valid in a fast-mean-reverting volatility regime. Interestingly, it follows that the indifference premium can be well approximated by adjusted constant volatility results, already derived in the preceding chapter. In this case, an interesting direction to explore is to examine this pricing problem in a multiscale stochastic volatility model.

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