GENERALIZED LIPSCHITZ

ALGEBRAS

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By

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SCOPE AND CONTENTS: A class of Banach algebras which generalize the idea of the Lipschitz algebra on a metric space is studied. It is shown that homomorphisms of these algebras correspond to mappings of the underlying space which satisfy certain moduli of continuity. The relation is expressed in categorical terms, and application is made to the theory of quasiconformal mapping.

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PREFACE

This thesis deals with two problems:

(i) To generalize the concept of the Lipschitz algebra of a metric space.

(ii) To study the analytic properties of the mappings induced on the underlying metric spaces by homomorphisms of generalized Lipschitz algebras, and to see to what extent maps with these analytic properties induce algebra homomorphisms.

We commence by studying the lattice of convex and concave moduli of continuity on a closed real interval. Elements of this lattice, with suitable restrictions, are then used to give a family of Banach algebra norms on the algebra of bounded continuous complex-valued functions on any metric space. The algebras of functions bounded in these norms are the generalized Lipschitz algebras..

Again with certain restrictions, homomorphisms of these algebras are shown to correspond to space mappings which satisfy moduli of continuity. The collections of such space maps and of generalized Lipschitz algebras inherit a partial order from the lattice of moduli of continuity. Using this, we express the relationship between algebra homomorphisms and space maps in categorical terms. Finally, when the metric spaces in question are taken to be domains in real n-space, isomorphisms of generalized Lipschitz algebras are shown to induce quasiconformal mappings of the underlying domains. It is also established that quasiconformal maps induce homomorphisms, within fixed limits, of the generalized Lipschitz algebras.

TABLE OF CONTENTS

CHAPTER 0

GENERAL PRELIMINARIES

We collect here some well known results on function algebras and quasi-conformal mappings which will be given without specific references. Proofs and further discussion are to be found in [Loomis, (1)], [Royden, (2)] and [Gehring, (3)] along with bibliographies of further sources.

Let X be a topological space and A an algebra of continuous complex-valued functions on X. We assume that A contains the multiplicative identity $[f(x) = 1, \ x \in X]$ and that the functions in A separate the points of X.

The spectrum $\Sigma = \Sigma(A)$ of A is the set of all non-zero homomorphisms of A into the complex field C. The space X may be embedded in Σ by identifying with each x in X the homomorphism $f \to f(x)$ taking each function in A to its value at x. A is naturally isomorphic with the algebra $\hat{A}$ of functions on Σ defined by $\hat{f}(\pi) = \pi(f)$ for $\pi \in \Sigma$ and $f \in A$.

The algebra A then consists of the functions in $\hat{A}$ restricted to the space X.

Σ is usually topologized by using the weakest topology under which the elements of $\hat{A}$ are continuous. This will be called the Gelfand topology on Σ; in it, a neighbourhood basis for the point $\mathcal{O} \in \Sigma$ is given by sets:

$$N(\mathcal{O}; f_1, \ldots, f_n, \epsilon) = \{\pi \in \Sigma \mid |\pi(f_i)| < \epsilon, i = 1, \ldots n,\}$$

for $\epsilon > o$, $f_1, \ldots, f_n \in A$. The Gelfand topology is always Hausdorff and the natural embedding of X into Σ is continuous with respect to it; if X is compact, this embedding is a homeomorphism.

We can also associate with A the set $M = M(A)$ of maximal ideals of A. Since each element of Σ is uniquely determined by the maximal ideal which is its kernel, we can identify Σ with a subset of M.

A function algebra A on a space X is <u>weakly inverse closed</u> if it has the property:

(α_o) If $f \in A$ and $\sup_{x \in X} |f(x)| < 1$, then $(1-f)$ has an inverse in A. A is <u>inverse closed</u> on X if it has the property:

(α) If $f \in A$ and f is bounded away from 0 on X, then f has an inverse in A.

We note that α implies α_o; further, if A is closed under uniform convergence, then α_o holds. The following propositions relate the divisibility properties of A, its maximal ideal space and its spectrum.

<u>0.1</u> Let A be a weakly inverse closed algebra of bounded functions. Then the spectrum and maximal ideal space of A coincide with the spectrum of $\bar{A}$, the completion of A under uniform convergence on X. Further, the Gelfand topologies of $\Sigma(A)$ and $\Sigma(\bar{A})$ coincide.

<u>0.2</u> Let A be an inverse closed and self-adjoint (closed under complex conjugation) algebra of bounded functions which separates X. By 0.1, the maximal ideal space and the spectrum of A are identical, and if X is compact they coincide with X. If X is not compact, it is

homeomorphic, by the natural embedding, with a subset of the spectrum of A which is dense in that spectrum in the Gelfand topology.

Since the algebras with which we will be concerned satisfy the conditions of 0.2, we identify the spectrum and maximal ideal space and use the same symbol, M, for both. Elements of M corresponding to points of X under the natural embedding are called fixed ideals; other elements of M are called free ideals.

The association of the space on which an algebra of functions is defined with the spectrum of the algebra provides the basic connection between algebra homomorphisms and space maps. Given two algebras A and B on topological spaces X and Y, we let T be a homomorphism of A into B. Then for each homomorphism $\pi \in M(B)$ we have a homomorphism $t(\pi)$ in $M(A)$ defined by $t(\pi) = \pi \circ T$. Thus for each $y \in Y$ and $f \in A$ we have

$$f(t(y)) = (Tf)(y).$$

We call the mapping t the adjoint of T and sometimes write $t = T^*$. With respect to the Gelfand topology of $M(A)$, t is a continuous map of $M(B)$ into $M(A)$ and its restriction to Y is a continuous map of Y into $M(A)$. Further, if T maps A onto a dense subset of B, then t is a homeomorphism of $M(B)$ onto a closed subset of $M(A)$. If T maps A onto B, then t is 1-1, $M(B)$ into $M(A)$; if t maps $M(B)$ onto $M(A)$, then T is 1-1.

Conversely, if X and Y are topological spaces and t a continuous map of X into Y, then for each function $f \in C(Y)$, the algebra of all continuous complex valued functions on Y, we have $f \circ t \in C(X)$. This gives a unitary algebra homomorphism $T : C(Y) \to C(X)$

defined by

$$T\, f\, (x) = f \circ t\, (x) \qquad f \in C(Y)$$

We call T the adjoint of t and sometime write $T = t^*$
If t maps X onto Y, then T is 1-1; if T maps $C(Y)$ onto $C(X)$, then t is 1-1.

For algebras A and B on topological spaces X and Y, let T be a homomorphism of A into B.

When X, considered as a point set without topology, coincides with $M(A)$, the adjoint mapping t is a mapping of Y into X. If X is compact, by 0.2 its topology is the Gelfand topology and $t : Y \to X$ is continuous. If X is not compact, the mapping t may not be continuous with respect to the topology on X and may not carry Y into X as a subset of $M(A)$. We can give conditions which ensure that $t(Y) \subset X$; as follows:

Definition 0.3 Let A be an algebra of bounded functions defined on a topological space X with values in a normed space. The compact open topology on A is generated by the neighbourhood basis given by sets of form:

$$N(f; K_1, \ldots, K_n; \epsilon) = \{g \in A \mid \|g - f|_{K_i}\|_\infty < \epsilon,\ i = 1, \ldots, n\}$$

where $f \in A$, $\epsilon > 0$ and the K_i are any finite collection of compact sets in X.

0.4 Let A, B be point-separating algebras of continuous complex-valued bounded functions on the topological spaces X, Y respectively; such that X is dense in $M(A)$ and every compact open neighbourhood of the

unit of A contains a function of compact support in X.
Then for any unitary homomorphism $T : A \to B$ which is continuous in the compact-open topology the adjoint $t : M(B) \to M(A)$ carries Y into X.

<u>Proof</u>: Let $y \in Y$. $M_y = \{f \in B | f(y) = 0\}$ is closed in the compact open topology of B. Suppose $t y \notin X$.

$$M_{ty} = \{f \in A | f(ty) = Tf(y) = 0\}$$

is exactly $T^{-1}(My)$ and hence is compact-open closed in A. Since $t y \notin X$, all functions f in A of compact support in X are in M_{ty}, for if K is the support of f, X - K is dense in M(A) - X and thus $\hat{f}$ vanishes on M(A) - X. But every compact open neighbourhood of the unit, 1, of A contains a function of compact support. Hence $1 \in M_{ty}$, which is a contradiction of the fact that T is unitary, so $ty \in X$ for all $y \in Y$.

In other words, the conditions of <u>0.4</u> ensure that the mapping t carries fixed ideals of B to fixed ideals of A. An argument similar to this was given by Nakai [9] in the special case of the Royden ring of functions on a Riemann surface, but as far as we know this is its first statement as a general proposition.

We note that the compact-open topology on an algebra A of continuous functions on a space X depends strongly on the underlying space X. For instance, if Y is a dense subspace of X and B the function algebra obtained by restricting A to Y then the restriction map $T : A \to B$ is clearly an algebra isomorphism, and hence so is its inverse, but T^{-1} may fail to be compact open continuous, although T always is.

CHAPTER I

Moduli of Continuity

In this Chapter we discuss the lattice of real convex and concave moduli of continuity on a closed real interval. We define an equivalence relation on this lattice and introduce a subclass which will eventually be used to give norms for systems of Banach algebras.

SECTION O: Concavity and Convexity.

Definition 1.0 A real valued function f defined on a closed real interval I is concave on I if and only if for every $x_1, x_2 \in I$, $x_1 < x_2$, every point $(y, f(y))$ of the graph of f with $x_1 < y < x_2$ is above the line segment $[(x_1, f(x_1)), (x_2, f(x_2))]$.

This condition is equivalent to the following:

1.1 $f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$ for all $x_1, x_2, \in I$ with $x_1 < x_2$; all λ, $o < \lambda < 1$.

1.2 f is continuous on I, differentiable on the complement B of a set in I which is at most countable, and has non-increasing derivative on B.

Definition 1.3 A real valued function f defined on a closed real interval I is convex on I if and only if $(-f)$ is concave on I.

This is equivalent to the following:

1.4 For every $x_1, x_2 \in I$ with $x_1 < x_2$, every point $(y, f(y))$ of the graph of f with $x_1 < y < x_2$ is below the line segment

$$[(x_1, f(x_1)), (x_2, f(x_2))].$$

1.5 $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$ for all $x_1, x_2 \in I$, $x_1 < x_2$; all $\lambda \in (0,1)$.

1.6 f is continuous on I, differentiable on the complement B with respect to I of a set in I which is at most countable, and has non-decreasing derivative on B.

Concavity and convexity are thus dual properties.

We note those properties of concave functions which follow directly from the definitions, as shown in Bourbaki, [5].

1.7 For real λ, concave f:

$$f(\lambda x) \geq \lambda f(x) \qquad 0 \leq \lambda \leq 1.$$

$$f(\lambda x) \leq \lambda f(x) \qquad \lambda > 1.$$

1.8 For concave f, $a > o$, $f(x+a) \leq f(x) + f(a)$

1.9 Let $(f_i)_{i \in I}$ be a family of real concave functions on a real interval J. If the lower envelope g of this family, defined by

$$g(x) = \inf \{f_i(x) \mid i \in I\}.$$

is finite at all finite points of J, then g is concave on J.

1.10 Let f be a finite concave function on an interval J. Then at every interior point of J, f has finite right and left derivatives, f'_d and f'_g respectively with $f'_d(x) \leq f'_g(x)$ for all x where they are defined. f'_d and f'_g are non-increasing on the interior of J and f' is non-increasing wherever f is differentiable; i.e., on the complement with respect to J of at most a countable set in J.

The dual properties hold for convex functions, as follows:

1.11 For all real λ, convex f:

$$f(\lambda x) \leq \lambda f(x) \qquad 0 \leq \lambda \leq 1.$$

$$f(\lambda x) \geq \lambda f(x) \qquad \lambda > 1.$$

1.12 For convex f, $a > 0$, $f(x+a) \geq f(x) + f(a)$

1.13 Let $(f_i)_{i \in I}$ be a family of real convex functions on a real interval J. If the upper envelope g of this family, defined by:

$$g(x) = \sup \{f_i(x) \mid i \in I\}$$

is finite at all finite points of J, then g is convex on J.

1.14 Let f be a finite convex function on an interval J. Then at every interior point of J, f has finite right and left derivatives f'_d and f'_g respectively with $f'_d(x) \geq f'_g(x)$ for all x where both are defined. f'_g and f'_d are non-decreasing on the interior of J and f' is non-decreasing wherever f is differentiable; i.e., on the complement with respect to J of at most a countable set in J.

SECTION 1: Moduli of Continuity.

Definition 1.15 A real modulus of continuity α defined on a closed bounded real interval $[0,d]$ is a convex or concave homeomorphism of $[0,d]$ onto itself with $\alpha(0) = 0$. A real modulus of continuity α defined on the half line $[0,\infty]$ is a concave or convex homeomorphism of $[0,\infty]$ onto itself with $\alpha(0) = 0$ and $\lim_{x \to \infty} \frac{\alpha(x)}{x} = 1$.

Let CC be the set of all concave moduli of continuity on $[0,d]$ for fixed d, $0 < d \leq \infty$; CV the set of all convex ones; $C = CC \cup CV$. Note that for any d, $CC \cap CV$ consists of the identity map, $\alpha(x) = x$. (It has been necessary to use here a different and slightly more restrictive definition of modulus of continuity than that used by Glaeser [6], but the essential properties are the same).

We establish some properties of C which will be required in what follows:

1.16 Let α be a homeomorphism $[0,d] \to [0,d]$ with $\alpha(0) = 0$, differentiable except for a set in $[0,d]$ which is at most countable. Then $\alpha \in C$ if and only if the function $\alpha(x)-x$ attains exactly one relative extremum on $(0,d)$ which may be attained on an interval.

Proof Let $\alpha \in CC$. Since α is non-increasing where defined, $\alpha(x)-x$ must have exactly one relative maximum on $(0,d)$. For $\alpha \in CV$, α is non-decreasing and the same argument holds. Conversely, for α as per hypothesis, the existence of a single relative extremum for $\alpha(x)-x$ implies that the derivative of $\alpha(x)$ is monotone.

1.17 [Glaeser, [6]] For any family F of bounded, uniformly equicontinuous complex-valued functions on a metric space (X,d) of diameter (not necessarily finite) D, there exists a non-decreasing concave real valued function α, continuous at 0 with $\alpha(0) = 0$, such that:

$$|f(x) - f(y)| \leq \alpha(d(x,y)) \quad \text{for all } f \in F,\ x,\ y \in X.$$

Proof: Set $\lambda(t) = \sup_{f \in F} (|f(x) - f(y)| \; |d(x,y) \leq t)$.

λ is non-decreasing, continuous at 0 by the uniform equicontinuity of F and has $\lambda(0) = 0$. For α we take the function whose ordinate set is the closed convex hull in the plane of the ordinate set of λ. α is then concave and inherits the other required properties from λ.

1.18 For fixed d, $0 < d \leq \infty$, we define a partial order on C by: $\alpha_1 \leq \alpha_2$ if and only if $\alpha_1(x) \leq \alpha_2(x)$ for all $x \in [0,d]$. With this order, C is a lattice.

Proof: We note that if h is the identity map on $[0,d]$ then $\alpha \leq h$ for $\alpha \in CV$ and $\alpha \geq h$ for $\alpha \in CC$. If α_1, α_2 are not comparable in C, they must be both in CC or both in CV. Let $\alpha_1, \alpha_2 \in CC$ and not comparable. By 1.6, the lower envelope g of α_1, α_2, given by:

$$g(x) = \min [\alpha_1(x), \alpha_2(x)] \quad \text{for } x \in [0,d]$$

is concave. Suppose $g(x) = g(y)$ for $x < y$. Since α_1, α_2 are strictly increasing, this can happen only if:

$\min\ [\alpha_1(x), \alpha_2(x)] = \alpha_1(x) = \alpha_2(y) = \min\ [\alpha_1(y), \alpha_2(y)]$ or vice versa. But the firstequality implies:

$$\alpha_1(x) \leq \alpha_2(x) < \alpha_2(y)$$

by the monotonicity of α_2, so $g(x)$ is strictly increasing on $[0,d]$.

By definition, $g(x) = \alpha_1(x) \wedge \alpha_2(x)$ for $x \in [0,d]$ and $\wedge$ is a continuous operation. Hence

$$\lim_{x \to \infty} \frac{\alpha_1(x) \wedge \alpha_2(x)}{x} = \lim_{x \to \infty} \frac{\alpha_1(x)}{x} \wedge \frac{\alpha_2(x)}{x}$$

$$= (\lim_{x \to \infty} \frac{\alpha_1(x)}{x}) \wedge (\lim_{x \to \infty} \frac{\alpha_2(x)}{x})$$

$$= 1$$

Thus $\alpha_1 \wedge \alpha_2 \in CC$, on $[0,d]$.

By 1.10, α_1 and α_2 have a concave upper bound h which is continuous at 0 with $h(x) \leq d$ on $[0,d]$.

Suppose $h(x) = h(y) < d$ for $x < y$, d finite. Then the line segment $[(x,h(x)), (d,d)]$ lies above the graph of h at the point $(y, h(y))$ which contradicts the concavity of h. If d is infinite and $h(x) = h(y)$ for $x < y$, x,y finite, then there exists a point $z > y$ with $h(z) > h(y)$, so the line segment $[(x,h(x)), (z,h(z))]$ lies above the graph of h at the point $(y, h(y))$, again contradicting the concavity of h.

We note that for d infinite, $h(x)$ is always finite for finite x by definition. We have left the possibility that for finite d, $h(x) = d$ for $x < d$. Consider the point set:

$$\{(x,y) \mid o \leq x \leq d,\ o \leq y \leq h(x)\} \qquad 1.18.1$$

which consists by definition of points (x,y) lying on line segments both ends of which are in:

$$\{(x,y) \mid o \leq x \leq d,\ o \leq y \leq \max\ [\alpha_1(x), \alpha_2(x)]\} \qquad 1.18.2$$

including degenerate segments of one point only. By concavity, if $h(x) = d$ for $x < d$, then h is identically equal to d on $[x,d]$. Then the point (x,d) must be in (1.18.2) which contradicts the strict monotonicity of α_1 and α_2. Thus h is strictly increasing.

By the argument used for $\wedge$, we see that

$$\lim_{x \to \infty} \frac{\alpha_1(x) \vee \alpha_2(x)}{x} = 1$$

Considering the point sets 1.18.1 and 1.18.2, it is apparent that

$$\lim_{x \to \infty} \frac{h(x)}{x} = 1 \quad \text{as well.}$$

Hence $h(x) = \alpha_1 \vee \alpha_2\ (x)$ in CC.

We have established the existence in CC of $\alpha_1 \vee \alpha_2$, $\alpha_1 \wedge \alpha_2$ for all $\alpha_1, \alpha_2 \in$ CC. The dual formulations give $\alpha_1 \vee \alpha_2$, $\alpha_1 \wedge \alpha_2 \in$ CV for $\alpha_1, \alpha_2 \in$ CV.

SECTION 2: Composition of Moduli of Continuity.

<u>1.19</u> For $\alpha_1, \alpha_2 \in CC\ (CV)$, $\alpha_1 \circ \alpha_2 \in CC\ (CV)$.

<u>Proof</u>: Let $\alpha_1, \alpha_2 \in CC$ on $[0,d]$. Then for $x, x' \in [0,d]$, $x < x'$,

$$\alpha_i\ (\lambda x + (1-\lambda)x') \geq \lambda\ \alpha_2(x) + (1-\lambda)\alpha_i(x')$$

for all $\lambda \in (0,1)$, $i = 1,2$ by 1.1. Then:

$$\begin{aligned}\alpha_1 \circ \alpha_2\ (\lambda x + (1-\lambda)x') &= \alpha_1(\alpha_2(\lambda x + (1-\lambda)x'))\\ &\geq \alpha_1(\lambda\alpha_2(x) + (1-\lambda)\alpha_2(x'))\\ &\geq \lambda\ \alpha_1 \circ \alpha_2(x) + (1-\lambda)\alpha_1\circ\alpha_2(x')\end{aligned}$$

For finite d, this implies that $\alpha_1 \circ \alpha_2 \in CC$, since we have shown that composition preserves concavity and the other properties of a modulus of continuity are immediate. For infinite d, we have that:

$$\lim_{x \to \infty} \frac{\alpha_1(\alpha_2(x))}{x} = \lim_{x \to \infty} \frac{\alpha_1(\alpha_2(x))}{\alpha_2(x)} \cdot \frac{\alpha_2(x)}{x} = 1$$

So in any case, $\alpha_1, \alpha_2 \in CC$ imply $\alpha_1 \circ \alpha_2 \in CC$. The dual argument shows that for $\alpha_1, \alpha_2 \in CV$, $\alpha_1 \circ \alpha_2 \in CV$.

We note that for $\alpha_1 \in CV$, $\alpha_2 \in CC$, the composition $\alpha_1 \circ \alpha_2$, while still monotone increasing, need be neither concave nor convex. Consider:

$$\alpha_1(x) = \begin{cases} \frac{x}{2} & 0 \le x \le {}^2/3 \\ 2x-1 & \frac{2}{3} < x \le 1 . \end{cases}$$

$$\alpha_2(x) = \begin{cases} 4x & 0 \le x \le \frac{1}{8} \\ \frac{4x+3}{7} & \frac{1}{8} < x \le 1 . \end{cases}$$

Then $\alpha_1 \in CV$, $\alpha_2 \in CC$ on $[0,1]$, but

$$\alpha_1 \circ \alpha_2(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{8} \\ \frac{4x+3}{14} & \frac{1}{8} < x \le {}^5/12 . \\ \frac{8x+6}{7} - 1 & {}^5/12 < x < 1 . \end{cases}$$

which is neither concave nor convex on $[0,1]$.

1.20 For $\alpha \in CC$, α^{-1}, the inverse of α with respect to composition, is in CV and vice versa.

Proof. We have immediately, for finite d, that $\alpha^{-1}(0) = 0$, $\alpha^{-1}(d) = d$ and α^{-1} is a homeomorphism of $[0,d]$ onto itself.

By 1.1,

$$\alpha(\lambda\alpha^{-1}(x) + (1-\lambda)\alpha^{-1}(x')) \ge \lambda x + (1-\lambda)x' \text{ for } x < x', \lambda \in (0,1)$$

Acting α^{-1} on both sides:

$$\lambda\,\alpha^{-1}(x) + (1-\lambda)\alpha^{-1}(x') \ge \alpha^{-1}(\lambda x + (1-\lambda)x')$$

So $\alpha^{-1} \in CV$ by 1.1 and 1.3. The dual argument holds for $\alpha \in CV$.

We need to show that, in the case where d is infinite, $\lim\limits_{x \to \infty} \frac{\alpha^{-1}(x)}{x} = 1$.

For $\alpha \epsilon C$, we have:

$$\lim_{x \to \infty} \frac{\alpha(x)}{x} = 1.$$

Hence

$$\lim_{x \to \infty} \frac{\alpha(\alpha^{-1}(x))}{\alpha^{-1}(x)} = \lim_{x \to \infty} \frac{x}{\alpha^{-1}(x)} = 1$$

i.e. $\lim_{x \to \infty} \frac{\alpha^{-1}(x)}{x} = 1$, so $\alpha^{-1} \epsilon CV$ for $\alpha \epsilon CC$ and the dual argument gives the result in the opposite direction.

1.21. [10] Every $\alpha \epsilon C$ may be represented as:

$$\alpha(x) = \int_0^x p(t)dt$$

where p is a monotone right continuous function, essentially bounded on any finite interval.

Proof: By 1.10, α has a derivative almost everywhere and, since it is absolutely continuous, is equal to the indefinite integral of that derivative. The function p is the right derivative α_d' of α, which is monotone and equals the derivative of α a.e. by 1.10. For the right continuity, suppose α is concave. Then by an application of 1.1, for all $h > 0$,

$$\alpha_d'(x) \geq \frac{\alpha(x+h) - \alpha(x)}{h}$$

Keeping h fixed and passing to the limit as $x \to x_0^+$:

$$\lim_{x \to x_0^+} \alpha_d'(x) \geq \frac{\alpha(x_0+h) - \alpha(x_0)}{h}$$

by the continuity of α; the limit on the left exists by the monotonicity of α_d'. Now passing to the limit as $h \to 0^+$,

$$\lim_{x \to x_o^+} \alpha_d'(x) \geq \alpha_d'(x_o).$$

For concave α, α_d' is non-increasing, so

$$\lim_{x \to x_o^+} \alpha_d'(x) \leq \alpha_d'(x_o)$$

Thus $\lim_{x \to x_o^+} \alpha_d'(x) = \alpha_d'(x_o)$ for concave α; the analogous procedure establishes right continuity for convex α. Note that since α is 0 only at 0, $p(t)$ must be bounded away from 0 on any interval not containing 0.

Since $\alpha(x)$ is finite for finite x, $p(t)$ can be infinite on no set of measure greater than 0 and so is essentially bounded on every finite interval.

<u>1.22</u> Let $\alpha, \beta \in C$. Then there exists $\gamma \in C$ such that

$$\lim_{x \to o} \frac{\alpha o \beta(x)}{\gamma(x)} = 1.$$

<u>Proof</u>: Suppose $\alpha \circ \beta \notin C$. By 1.21,

$$\alpha(x) = \int_o^x p(t)dt \qquad \beta(x) = \int_o^x q(t)dt .$$

where p and q are monotone and right continuous. Then ([10],p.10)

$$\alpha \circ \beta(x) = \int_o^x p(\beta(t)) \quad q(t)dt$$
$$= \int_o^x g(t)dt.$$

where g is right continuous and $> o$ for any $t > o$. If, for some x_o, $g(t)$ is monotone on $o < x \leq x_o$, we can construct γ by:

$$\gamma(x) = \int_o^x g(t)dt \qquad x \leq x_o$$

and by defining $\gamma(x)$ in terms of a suitable line segment for $x > x_o$. If there is no neighbourhood of 0 on which g is monotone, we note that $o < g(o) < \infty$ since:

(i) if $g(o) = o$, g must be monotone increasing in some neighbourhood is 0, since $\alpha \circ \beta$ is zero only at 0.

(ii) if $g(o) = \infty$, g must be monotone decreasing in some neighbourhood of 0, since $\alpha \circ \beta$ is finite for finite x.

We then set $\gamma(x) = xg(o)$ in a neighbourhood $o < x \leq x_1$, and define it in terms of a suitable line segment elsewhere. By right continuity, for any $\epsilon > o$ we can find $x_o > o$ such that

$$|g(o) - g(t)| < \epsilon \quad \text{for} \quad o < t < x_o .$$

Then, for $x < x_o$,

$$1 - {}^{\epsilon}/g(o) < \frac{\int_o^x g(t)dt}{x.g(o)} < 1 + {}^{\epsilon}/g(o) .$$

So $$\lim_{x \to o} \frac{\alpha o \beta(x)}{\gamma(x)} = 1.$$

SECTION 3: The Class C_a

1.23 We now consider the class $C_a \subset C$ consisting of moduli of continuity α with the property:

$$o < \liminf_{x \to o} \frac{\alpha(\lambda x)}{\alpha(x)} \quad \text{for all } \lambda > o .$$

We note the following properties of C_a:

1.24 $CC \subset C_a$

Proof: Let $\alpha \epsilon CC$. Then for $o < \lambda \leq 1$, $\lambda \alpha(x) \leq \alpha(\lambda x)$ by 1.4. so

$$o < \lambda \leq \liminf_{x \to o} \frac{\alpha(\lambda x)}{\alpha(x)}$$

For $\lambda > 1$, $\alpha(\lambda x) \geq \alpha(x)$, so

$$o < 1 \leq \liminf_{x \to o} \frac{\alpha(\lambda x)}{\alpha(x)} .$$

1.25 For $\alpha \epsilon C_a$, $\frac{\alpha(\lambda x)}{\alpha(x)} > o$ for all x, all $\lambda > o$; we show that there exists $K > o$ with $\frac{\alpha(\lambda x)}{\alpha(x)} \geq K$ for all x.

Proof: By 1.15, this is immediate for $\alpha \epsilon CC$. For $\alpha \epsilon CV$, we represent α as in 1.21. Then:

$$\frac{\alpha(\lambda x)}{\alpha(x)} \geq \frac{\alpha(\lambda x)}{x} = \frac{1}{x} \int_o^{\lambda x} p dt$$

$$\geq \frac{1}{x} \int_{\frac{\lambda x}{2}}^{\lambda x} p dt$$

$$\geq \frac{1}{x} \left(\frac{\lambda x}{2}\right) p \left(\frac{\lambda x}{2}\right) = \frac{\lambda}{2} p\left(\frac{\lambda x}{2}\right) > 0 .$$

for x bounded away from 0. Since the definition of C_a ensures that $\frac{\alpha(\lambda x)}{\alpha(x)}$ is bounded away from 0 in a neighbourhood of 0, this gives the result.

<u>1.26</u> $CV \not\subset C_a$.

<u>Proof</u>: Consider

$$\alpha(x) = \begin{cases} e^{-\frac{1}{x^2}} & 0 < x \leq a \\ 0 & x = 0 . \end{cases}$$

$\alpha'(x) = \frac{2}{x^3} \; e^{-\frac{1}{x^2}}$ is strictly increasing in a neighbourhood of 0, since its derivative $\alpha''(x) = \frac{4-6x^2}{6} \; e^{-\frac{1}{x^2}}$ is positive for small x.

Thus α is convex for small x and so is:

$$\beta(x) = \begin{cases} \alpha(x) & 0 \leq x \leq 0.1 \\ \frac{10}{9}\,(1-e^{-100})x + \frac{10}{9}\,e^{-100} - \frac{1}{9} & 0.1 < x \leq 1 \end{cases}$$

However, for $x < 0.1$, $\lambda < 1$,

$$\frac{\beta(\lambda x)}{\beta(x)} = e^{-\frac{1}{(\lambda x)^2} + \frac{1}{x^2}}$$
$$= e^{\frac{1}{x^2}\left(1 - \frac{1}{\lambda^2}\right)}$$ which has limit 0 as $x \to 0$.

(Example due to B. Banaschewski.)

<u>1.27</u> The class of functions in CV which are of form $\alpha(x) = x^k$, $k \geq 1$ in a neighbourhood of 0 is properly contained in $CV \cap C_a$.

<u>Proof</u>: It is immediate that any function of this form is in C_a .

Consider the function defined by:

$$\alpha(x) = \begin{cases} 0 & x = 0 \\ -\dfrac{x}{\ln x} & 0 < x \leq \dfrac{1}{e} \end{cases}$$

Since $\frac{d}{dx}(\alpha(x)) = \frac{1}{(\ln x)^2} - \frac{1}{\ln x}$ which is monotone increasing on $[0, \frac{1}{e}]$, $\alpha \in$ CV on $[0, \frac{1}{e}]$.

Consider $\lim\limits_{x \to 0} \dfrac{\alpha(x)}{x^k}$, $k > 1$

$$= \lim_{x \to 0} \left[-\frac{x^{1-k}}{\ln x}\right]$$

Applying L'Hospital's rule, we have:

$$\lim_{x \to 0} \frac{\alpha(x)}{x^k} = \lim_{x \to 0} - \frac{(1-k)\, x^{-k}}{1/x}$$

$$= - \lim_{x \to 0} (1-k)\, x^{1-k} = \infty .$$

For $k = 1$, $\lim\limits_{x \to 0} - \dfrac{x^{1-k}}{\ln x} = \lim\limits_{x \to 0} - \dfrac{1}{\ln x} = 0$.

Thus $\alpha(x)$ is of comparable order at 0 with no function of form x^k, $k \geq 1$, in a neighbourhood of 0.

Finally, consider $\lim_{x \to 0} \frac{\alpha(\lambda x)}{\alpha(x)} = \lim_{x \to 0} \frac{\lambda x}{\ln(\lambda x)} \cdot \frac{\ln x}{x}$

$$= \lambda \lim_{x \to 0} \frac{\ln x}{\ln \lambda + \ln x} = \lambda > 0$$

for all $\lambda > 0$. Thus $\alpha \in CV \cap C_a$ on $[0, {}^1/e]$.

We now note that any $\alpha \in C$ on $[0,d]$ for any finite d can be extended to C on $[0,\infty]$ as follows:

For $\alpha \in C$, there exists x_1, $0 < x_1 < d$, at which $\alpha(x) - x$ attains a relative extremum, by 1.16. Define $\hat{\alpha}$ on $[0,\infty]$ by:

$$\hat{\alpha}(x) = \alpha(x) \qquad 0 \leq x \leq x_1$$

$$\hat{\alpha}(x) = x + \alpha(x_1) - x_1 \qquad x_1 < x < \infty .$$

Then $\hat{\alpha}$ is contained in CV, CC or C_a, as the case may be, with α.

From here on, except as otherwise noted, we will consider all α to be defined on $[0,\infty]$ as above.

1.28 For $\alpha \in CC$, by 1.7:

$$\lambda\alpha(x) \leq \alpha(\lambda x) \leq \alpha(x) \qquad \lambda \leq 1$$

$$\alpha(x) \leq \alpha(\lambda x) \leq \lambda\alpha(x) \qquad \lambda > 1 .$$

For $\alpha \in CV \cap C_a$:

$$\inf_{t \in [0,d]} \left[\frac{\alpha(\lambda t)}{\alpha(t)}\right] \alpha(x) \leq \alpha(\lambda x) \leq \lambda\alpha(x) \qquad \lambda \leq 1.$$

by 1.11 and 1.24, $0 < d \leq \infty$.

$$\lambda\, \alpha(x) \leq \alpha(\lambda x) \leq \sup_{t \epsilon [o,d]} [\frac{\alpha(\lambda t)}{\alpha(t)}]\, \alpha(x) \qquad \lambda \geq 1 .$$

$o < d \leq \infty$.

The supremum here is finite for any finite d, and the condition that $\lim_{x \to \infty} \frac{\alpha(x)}{x} = 1$ ensures that it is finite for infinite d.

1.29 For $\alpha_1, \alpha_2 \epsilon (CV \cap C_a)$, $\alpha_1 o \alpha_2 \epsilon (CV \cap C_a)$.

Proof: Let α_1, α_2 be in $CV \cap C_a$. Then, for $\lambda \leq 1$, all $x > o$,

$$\frac{\alpha_1(\alpha_2(\lambda x))}{\alpha_1(\alpha_2(x))} \geq \frac{\alpha_1(K_2\alpha_2(x))}{\alpha_1(\alpha_2(x))}$$

$$\geq \frac{K_1\alpha_1(\alpha_2(x))}{\alpha_1(\alpha_2(x))} = K_1 > o \text{ by } 1.25$$

For $\lambda > 1$, $\frac{\alpha_1(\alpha_2(\lambda x))}{\alpha_1(\alpha_2(x))} \geq \frac{\lambda\alpha_1(\alpha_2(x))}{\alpha_1(\alpha_2(x))}$

$= \lambda > o$ by convexity.

Thus for all $\lambda > o$, $\lim_{x \to o} \inf \frac{\alpha_1 o \alpha_2(\lambda x)}{\alpha_1 o \alpha_2(x)} > o$

Note that since $CV = (CC)^{-1}$, 1.25 implies that for $\alpha \epsilon CC$, α^{-1} is not necessarily in C_a. Also, 1.19 and the example following show that while CC and CV are closed under composition, C_a is not.

SECTION 4: The Equivalence Relation on C_a.

We now define a relation on the union of all $C_a\,[o,p]$ $0 < p \leq \infty$ as follows:

$\alpha_1 \sim \alpha_2$ if and only if:

$$o < \lim_{x \to o} \inf \frac{\alpha_1(x)}{\alpha_2(x)} \leq \lim_{x \to o} \sup \frac{\alpha_1(x)}{\alpha_2(x)} < \infty .$$

1.30 The relation so defined is an equivalence relation on C_a .

Proof: Reflexivity is obvious. Symmetry follows from the fact that if $\lim_{x \to o} \inf f(x) = a > o$, then $\lim_{x \to o} \sup \frac{1}{f(x)} = \frac{1}{a}$.

For transitivity we can write:

$$\lim_{x \to o} \inf \frac{\alpha_1(x)}{\alpha_3(x)} \geq \lim_{x \to o} \inf \frac{\alpha_1(x)}{\alpha_2(x)} \cdot \lim_{x \to o} \inf \frac{\alpha_2(x)}{\alpha_3(x)}$$

since the limits on the right are finite and non-zero for $\alpha_1 \sim \alpha_2$ and $\alpha_2 \sim \alpha_3$; similarly:

$$\lim_{x \to o} \sup \frac{\alpha_1(x)}{\alpha_3(x)} \leq \lim_{x \to o} \sup \frac{\alpha_1(x)}{\alpha_2(x)} \cdot \lim_{x \to o} \sup \frac{\alpha_2(x)}{\alpha_3(x)}$$

We note that $\alpha \sim \alpha \circ \lambda$ (where $\alpha \circ \lambda(x) = \alpha(\lambda x)$, λ real) if and only if $\alpha \in C_a$; also:

1.31 For $\alpha, \beta \in C_a$, there always exists $\gamma \in C_a$ with $\alpha \circ \beta \sim \gamma$.

Proof: We define γ as in 1.22. If γ is linear throughout, it is in C_a trivially; this is also the case for $\gamma \in CC$. The remaining possibility is that γ may be equal to $\alpha \circ \beta$ for $x \leq x_1$, $x_1 > o$, and linear elsewhere. Suppose $\alpha \in CC$, $\beta \in CV$. Then for $o < x \leq x_1$, a calculation as in 1.29 shows that

$$\liminf_{x \to o} \frac{\gamma(\lambda x)}{\gamma(x)} = \liminf_{x \to o} \frac{\alpha o \beta(\lambda x)}{\alpha o \beta(x)} > o \quad \text{for all} \quad \lambda > o .$$

so here also $\gamma \in C_a$. By construction, $\gamma \sim \alpha \circ \beta$.

Consequently, the set of functions C_a is closed under composition up to equivalence.

We note here that the composition defined in 1.22 is, in general, order preserving in C_a, in the sense that equivalence classes are preserved. This follows from 1.30 and the following:

1.32 The equivalence defined above on C_a is preserved under composition and inversion in C_a.

Proof: Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in C_a$, with $\beta_1 \sim \alpha_1$, $\beta_2 \sim \alpha_2$ and $\beta_1 \circ \beta_2$, $\alpha_1 \circ \alpha_2 \in C_a$. $\beta_i \sim \alpha_i$ implies, by 1.27 and 1.29, that there exist K_i, $K_i' > o$ and finite such that:

$$K_i \, \beta_i(x) \leq \alpha_i(x) \leq K_i' \, \beta_i(x) \quad \text{for all} \quad x, \; i = 1, 2, \text{ and}$$

R_i, $R_i' > o$ and finite such that

$$R_i \, \alpha_i(x) \leq \beta_i(x) \leq R_i' \, \alpha_i(x) \quad \text{for all} \quad x, \; i = 1, 2 .$$

Then $\lim_{x \to o} \inf \frac{\beta_1(\beta_2(x))}{\alpha_1(\alpha_2(x))} \geq \lim_{x \to o} \inf \frac{\beta_1(R_2\alpha_2(x))}{\alpha_1(\alpha_2(x))}$

$$\geq \lim_{x \to o} \inf \frac{R_1 R_2 \alpha_1(\alpha_2(x))}{\alpha_1(\alpha_2(x))} = R_1 R_2 > o .$$

Similarly, $\lim_{x \to o} \sup \frac{\beta_1 o \beta_2(x)}{\alpha_1 o \alpha_2(x)} < \infty$.

Now let α, α^{-1}; $\beta, \beta^{-1} \epsilon C_a$ with $\beta \circ \alpha$, $\beta^{-1} \circ \alpha^{-1} \epsilon C_a$ and $\alpha \sim \beta$.

We have R, R' and K, K' as above with:

$K \beta(x) \leq \alpha(x) \leq K' \beta(x)$ for all x and

$R \alpha(x) \leq \beta(x) \leq R' \alpha(x)$ for all x . Then:

$$\beta^{-1}(x) = \beta^{-1} \alpha^{-1} \alpha(x)$$

$$\geq \beta^{-1} \alpha^{-1} K' \beta(\alpha\alpha^{-1}(x))$$

$$\geq K_1 \beta^{-1}\alpha^{-1} K_2 \alpha(\alpha\alpha^{-1}(x))$$

$$\geq K_3 \beta^{-1}\alpha \alpha^{-1}(x) .$$

$$\geq K_3 \beta^{-1} K_4 \beta\alpha^{-1}(x)$$

$\geq K_5 \alpha^{-1}(x)$ using the above inequalities and 1.28, for suitable $K_5 > o$ and all x .

Consequently, $\lim_{x \to o} \inf \frac{\beta^{-1}(x)}{\alpha^{-1}(x)} \geq K_5 > o$, which in turn implies that

$$\lim_{x \to o} \sup \frac{\alpha^{-1}(x)}{\beta^{-1}(x)} < \infty .$$

Similarly, the inequalities arising from the equivalence of α and β can be used to show that

$$\lim_{x \to o} \inf \frac{\alpha^{-1}(x)}{\beta^{-1}(x)} > o \quad \text{and}$$

$$\lim_{x \to o} \sup \frac{\beta^{-1}(x)}{\alpha^{-1}(x)} < \infty .$$

The assumption here that $\alpha, \beta \in C_a$ implies $\alpha \circ \beta \in C_a$ is not necessary, since by 1.31, for any such composition, there is an equivalent element of C_a. Consequently, we have the result that for $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$, $\alpha_1 \circ \beta_1 \sim \alpha_2 \circ \beta_2$; and that $\alpha \sim \beta$ if and only if $\alpha^{-1} \sim \beta^{-1}$ in C_a.

As noted in the remarks following 1.27, we can extend any $\alpha \in C_a$ on $[o,d]$ to $\hat{\alpha}$ contained in C_a on $[o,\infty]$ and the extension preserves equivalences since $\alpha(x) = \hat{\alpha}(x)$ for small x. In a similar way we can set up an equivalence preserving correspondence between C_a on $[o,\infty]$ and C_a on $[o,d]$ for any d, $o < d < \infty$.

<u>1.33</u> For $0 < d < \infty$, there exists a 1-1 onto correspondence between the equivalence classes of C_a on $[o,d]$ and those of C_a on $[o,\infty]$.

<u>Proof:</u> As noted, we have a 1-1 correspondence between equivalence classes of $C_a\,[0,d]$ and those of $C_a\,[o,\infty]$. Let $\alpha \in C_a\,[o,\infty]$. Define $\bar{\alpha}$ as follows:

$$\bar{\alpha}(x) = \alpha(x) \qquad 0 \leq x \leq \alpha^{-1}(\tfrac{d}{2}) \;.$$

$$\bar{\alpha}(x) = \frac{d}{2}\left[\frac{x-\alpha^{-1}(\frac{d}{2})}{d-\alpha^{-1}(d/2)}\right] + d/2. \qquad x > \alpha^{-1}(\tfrac{d}{2})$$

Then $\bar{\alpha}$ is ϵ $C_a[0,d]$ and if $\alpha_1 \sim \alpha_2$ in $C[o,\infty]$, then $\bar{\alpha}_1 \sim \bar{\alpha}_2$ in $C_a[o,d]$, since $\alpha = \bar{\alpha}$ for small x.

CHAPTER II

The Generalized Lipschitz Algebras of a Metric Space.

In this chapter we will use the lattice of moduli of continuity discussed in Chapter I to define norms which determine a class of function algebras, the generalized Lipschitz algebras, on any metric space. We will discuss the separation properties of these algebras, and will define a set of metrics with respect to which a subclass of the generalized Lipschitz algebras on a given metric space are in fact Lipschitz algebras on that space.

SECTION 0: The Algebras L_α

Definition 2.0: Let (X,d) be any metric space with diameter $p \leq \infty$. For α fixed in $C_a[o,q]$, $q \geq p$ we denote by $L(X,d,\alpha) = L_\alpha$ the set of all continuous bounded complex valued functions f on X with:

$$||f||_{d,\alpha} = \sup_{\substack{x \neq y \\ x,y \in X}} \frac{|f(x)-f(y)|}{\alpha(d(x,y))} < \infty .$$

We define a norm on L_α by:

$$|| f || = || f ||_\infty + || f ||_{d,\alpha}$$

where $||f||_\infty$ denotes the uniform norm, $\sup_{x \in X} |f(x)|$. Note that each such function f has a unique extension $\tilde{f}$ to the completion of X with respect to d, and hence the $\tilde{f}$, $f \in L_\alpha$, make up the corresponding set of functions for the completion.

2.1 With this norm and with pointwise addition and multiplication, L_α is a semi-simple commutative Banach algebra with unit.

Proof: (i) L_α is a Banach space.

The triangle inequality is trivial and hence L_α is a linear space. We follow a procedure due to Mirkil [8] to show completeness.

Let $\{f_n\}$ be a Cauchy sequence in L_α. Then $\{f_n\}$ is also a Cauchy sequence in the space $C(X)$ of continuous bounded complex-valued functions on X with the uniform norm, so $\{f_n\}$ converges in this norm to $f \in C(X)$.

To show that the limit function f is in L_α, consider

$$\frac{|f(x)-f(y)|}{\alpha(d(x,y))} \leq \frac{|f(x)-f_n(x)|+|f_n(x)-f_n(y)|+|f_n(y)-f(y)|}{\alpha(d(x,y))}$$

for any n, all $x, y \in X$. Since $\{f_n\}$ is Cauchy in L_α, $\{f_n\}$ is bounded in L_α so there is some $M > 0$ with $\| f_n \|_{d,\alpha} < M$ for all n. Fixing x,y and letting $n \to \infty$, we see by the above inequality that $\| f \|_{d,\alpha}$ is also $\leq M$, so $f \in L_\alpha$.

We still have to show that $\{f_n\}$ converges to f in L_α. By considering the sequence $\{f_n - f\}$ and changing notation, we may assume without loss of generality that $f = 0$. We must show that $\{f_n\}$ is a zero sequence in L_α. Now since $\{f_n\}$ is Cauchy in L_α, given any $\epsilon > o$ we can find N such that $\|f_n - f_m\|_{d,\alpha} < \epsilon$ for $n,m > N$; i.e., for any $x, y \in X$, $x \neq y$:

$$\frac{|(f_n-f_m)(x) - (f_n-f_m)(y)|}{\alpha(d(x,y))} < \epsilon$$

Since the mapping $f \to A_x f$, where

$$A_x\, f(y) = \frac{f(x)-f(y)}{\alpha(d(x,y))}$$

is continuous from L_α into $C(X)$ for fixed x, we can fix x,y and n and let $m \to \infty$ in the above, obtaining:

$$\frac{|f_n(x) - f_n(y)|}{\alpha(d(x,y))} \leq \epsilon$$

in the above, for all $x, y \in X$ and all $n > N$.

Thus $\lim_{n \to \infty} ||f_n||_{d,\alpha} = 0$, so $\{f_n\}$ is a zero sequence in L_α and L_α is complete in the given norm.

(ii) L_α is a Banach algebra:

For $f, g \in L_\alpha$, $x \neq y$:

$$\frac{|f(x)g(x) - f(y)g(y)|}{\alpha(d(x,y))} \leq |f(x)| \left|\frac{g(x)-g(y)}{\alpha(d(x,y))}\right| + |g(y)| \left|\frac{f(x)-f(y))}{\alpha(d(x,y))}\right|$$

$$\leq ||f||_\infty ||g||_{d,\alpha} + ||g||_\infty ||f||_{d,\alpha}$$

hence $||fg||_{d,\alpha} \leq ||f||_\infty ||g||_{d,\alpha} + ||g||_\infty ||f||_{d,\alpha}$

So $||fg|| = ||fg||_\infty + ||fg||_{d,\alpha}$

$$\leq ||f||_\infty ||g||_\infty + ||f||_\infty ||g||_{d,\alpha} + ||g||_\infty ||f||_{d,\alpha} .$$

$$\leq (||f||_\infty + ||f||_{d,\alpha}) (||g||_\infty + ||g||_{d,\alpha})$$

$$= ||f|| \; ||g||$$

This shows that L_α is closed under multiplication, and that the given norm is a Banach algebra norm on L_α. Since L_α is a function algebra, it is semi-simple and since it evidently contains the constant functions, it has as its unit $f(x) \equiv 1$, $x \in X$.

We note that the norm $||\cdot||'$ on L_α, defined by :

$||f||' = \max\,[||f||_\infty,\ ||f||_{d,\alpha}]$ is boundedly

equivalent to the original norm $||\cdot||$, since

$$||f||' \leq ||f|| \leq 2\,||f||' \text{ for all } f \in L_\alpha .$$

For each $\alpha \in C_a$, L_α on (X,d) is obviously closed under complex conjugation (self-adjoint). L_α is also inverse closed, since if $|f(x)| \geq a > o$ for all $x \in X$, then

$$\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| \leq \frac{1}{a^2}\,|f(x) - f(y)|$$

By 0.2, the maximal ideal space of L_α coincides with its spectrum for each α. If L_α separates X, the space X corresponds homeomorphically with a subset of the spectrum Σ_α of L_α which is dense in the Gelfand topology; if X is compact, X coincides with Σ_α. Since $f_s \in L_\alpha$ for all $s \in X$ when L_α separates X, (see 2.5) the homeomorphic embedding of X in Σ_α means that for such L_α the metric topology on X coincides with the topology induced on X by the Gelfand topology on Σ_α.

<u>2.2</u> L_α is closed under truncation; i.e. for any real valued $f \in L_\alpha$, $a > o$, $f \wedge a \in L_\alpha$.

<u>Proof</u>: For any $a > o$, $||f \wedge a||_\infty \leq ||f||_\infty$.

By considering cases, it is readily seen that for any $x, y \in X$,

$$|f \wedge a(x) - f \wedge a(y)| \leq |f(x) - f(y)| \quad \text{so} \quad \| f \wedge a \|_{d,\alpha} \leq \|f\|_{d,\alpha}.$$

It follows, by (Sherbert) [7] that L_α is a regular algebra if it separates X .

SECTION I: Comparison of different L_α

The division of C_a into equivalence classes in Chapter 1 is now seen to provide a classification of the algebras L_α on a fixed metric space as α varies in C_a.

2.3 Let (X,d) be a metric space of diameter D, $\alpha_1 \sim \alpha_2$ in C_a. Then L_{α_1} and L_{α_2} contain the same elements and there exist $K, K' > o$ such that

$$K\,||f||_1 \le ||f||_2 \le K'\,||f||_1$$

where $||f||_i$ is the norm of f in α_i, $i = 1, 2$.

Proof: Let f be $\epsilon\ L_{\alpha_1}$. As noted in 1.32, there exist $R, R' > o$ such that

$$\therefore R'\alpha_1(x) \le \alpha_2(x) \le R\,\alpha_1(x) \quad \text{for all } x \in [o,D]$$

Then $$||f||_{d,\alpha_2} = \sup_{x \neq y} \left|\frac{f(x)-f(y)}{\alpha_2(d(x,y))}\right|$$

$$\le \frac{1}{R'} \sup \left|\frac{f(x)-f(y)}{\alpha_1(d(x,y))}\right| = \frac{1}{R'}\,||f||_{d,\alpha_1}.$$

i.e. $$||f||_{d,\alpha_2} \le {}^1/_{R'}\,||f||_{d,\alpha_1}$$

Also, the uniform norm of f is independent of α, so $L_{\alpha_2} \subseteq L_{\alpha_1}$.

Reversing the argument we have $L_{\alpha_1} = L_{\alpha_2}$ as sets of functions.

It now follows from the general theory that the norms of L_{α_1} and L_{α_2} are boundedly equivalent, but the above calculation shows this directly.

Corollary Suppose D is finite. Let $\hat{\alpha}$ be the extension of α to $[o,\infty]$ as in 1.33, for $\alpha \in C_a[o,D]$. Then $L(X,d,\alpha)$ and $L(X,d,\hat{\alpha})$ contain the same functions .

Proof: Since $\alpha \sim \hat{\alpha}$ by construction, we have for all $x \in [o,D]$,

$$K'\alpha(x) \leq \hat{\alpha}(x) \leq K\,\alpha(x) \qquad K, K' > o .$$

2.4 Let (X,d) be a locally compact metric space of diameter D. Let a new metric:

$$r(x,y) = \frac{d(x,y)}{1+d(x,y)} \qquad D = \infty$$

$$r(x,y) = \frac{1}{D}\, d(x,y) \qquad D < \infty$$

be introduced on X so that (X,r) has diameter 1 . Then for each $\alpha \in C_a[0,D]$ there exists $\beta \in C_a[0,1]$ such that $L(X,d,\alpha)$ and $L(X,r,\beta)$ contain the same functions.

Proof: For finite D, take $\hat{\alpha} \in C_a[o,\infty]$ and $\beta = \bar{\hat{\alpha}} \in C_a[o,1]$ as in the proof of 1.33. Then $\alpha \sim \hat{\alpha} \sim \beta$. By 1.25 and the definition of equivalence, we have for some $K_1, K_2 > 0$ and all $x,y \in x$;

$$\begin{aligned} K_1\, \alpha(d(x,y)) &\leq \beta(\tfrac{1}{D}\, d(x,y)) = \beta(r(x,y)) \\ &\leq K_2\, \alpha(d(x,y)). \end{aligned}$$

Thus the norms on bounded continuous functions on X given by α and β are boundedly equivalent, so the sets of functions bounded in these norms coincide; i.e., the algebras $L(X,d,\alpha)$ and $L(X,r,\beta)$ contain the same functions

For infinite D, take β as in the proof of 1.33. For $d(x,y) \leq 1$, we have

$$d(x,y) \geq \frac{d(x,y)}{1+d(x,y)} \geq \frac{d(x,y)}{2} .$$

So for $d(x,y) \leq 1$,

$$\beta(d(x,y)) \geq \beta\left(\frac{d(x,y)}{1+d(x,y)}\right) = \beta(r(x,y)) \geq \beta\left(\frac{d(x,y)}{2}\right) \geq K\,\beta(d(x,y)) .$$

for suitable $K > 0$. By construction, $\alpha(x) = \beta(x)$ for small x. Then for, say, $d(x,y) < \delta$, any function f on X:

$$\frac{|f(x)-f(y)|}{\alpha(d(x,y))} < \frac{|f(x)-f(y)|}{\beta(r(x,y))} < \frac{|f(x)-f(y)|}{K\alpha(d(x,y))}$$

For $d(x,y) \geq \delta$, any bounded f:

$$\frac{|f(x)-f(y)|}{\beta(r(x,y))} \leq \frac{2\|f\|_\infty}{\beta(\delta/1+\delta)} < \infty .$$

and

$$\frac{|f(x)-f(y)|}{\alpha(d(x,y))} \leq \frac{2\|f\|_\infty}{\alpha(\delta)} < \infty .$$

Consequently a bounded function f on X is bounded in the norm $\|\cdot\|_{d,\alpha}$ if and only if it is bounded in the norm $\|\cdot\|_{r,\beta}$. Since the uniform norm is unaffected by the change in metric, this again means that $L(X,d,\alpha)$ and $L(X,r,\beta)$ contain the same functions.

SECTION 2: Separation Properties.

We now consider the separating properties of L_α . For some metric spaces and some $\alpha \in C_a$, L_α may contain only the constant functions; for example, take $X = [0,1]$, $d(x,y) = |x-y|$ and $\alpha(d(x,y)) = (d(x,y))^2$. Let $f \in L_\alpha$; then :

$$\overline{\lim_{h \to 0}} \frac{|f(x+h)-f(x)|}{h^2} = \overline{\lim_{h \to 0}} \frac{|f(x+h)-f(x)|}{h} \cdot \frac{1}{h} < ||f||_{d,\alpha} < \infty .$$

So $\lim_{|x-y| \to 0} \frac{|f(x)-f(y)|}{|x-y|} = 0$ for all x,y . Consequently, f is constant on $[0,1]$. (Example based on that given by Sherbert, [7].)

We give a sufficient condition on α for L_α to separate X , as follows : (from [7])

2.5 Let (X,d) be a metric space; $\alpha \in C_a$ such that $\alpha \circ d$ is again a metric on X . Then L_α contains the function $f_s(x) = \min(\alpha(d(s,x)),1)$ for all $s \in X$.

Proof: Certainly $||f_s||_\infty \leq 1$ for all $s \in X$. Since $\alpha \circ d$ is a metric,

$$|\alpha(d(s,x)) - \alpha(d(s,y))| \leq \alpha(d(x,y)) \text{ for all } x,y,s \in X .$$

By considering cases, we see that $||f_s||_{d,\alpha} \leq 1$ as well, so $f_s \in L_\alpha$.

It follows that if α is concave, L_α contains f_s for all $s \in X$ and so always separates X ; this is also the case if d is an ultrametric, i.e. if:

$$d(x,y) \leq \max\ [d(x,z),\ d(y,z)] \text{ for all } x,\ y,\ z \in X ,$$

regardless of α , since the fact that any $\alpha \in C_a$ is a monotone increasing homeomorphism makes $\alpha \circ d$ an ultrametric as well.

On the other hand, it is not necessary for $\alpha \circ d$ to be a metric in order for L_α to contain f_s for all s . Consider

$$X = [0,1]$$

$$d(x,y) = |x-y|$$

$$\alpha(d(x,y)) = \tan\ (^{\pi}/4|x-y|)$$

For $\alpha(x) = \tan \frac{\pi x}{4}$, $\alpha(o) = o$, $\alpha(1) = 1$, $\alpha'(x) = {}^{\pi}/4 \sec^2 (\frac{\pi x}{4})$

which is strictly increasing on $[0,1]$, so $\tan \frac{\pi x}{4} \in CV \; [0,1]$.

Also $$\lim_{x \to o} \frac{\tan \frac{\lambda \pi x}{4}}{\tan \frac{\pi x}{4}} = \lambda \quad \text{for all} \quad \lambda > 0$$

so $\tan \frac{\pi x}{4} \in C_a$.

$$\tan (x+y) = \frac{\tan x + \tan y}{1-\tan x \tan y}$$

$$\geq \tan x + \tan y \text{ for } \tan x, \tan y > o$$

so $\tan \frac{\pi x}{4}$ is not a metric on $[0,1]$, since it is positive on this range. However, for this α on this space:

$$f_s(x) = \alpha(d(s,x))$$

$$= \tan {}^{\pi}/4 \, |s-x| \quad \text{and :}$$

$$\frac{f_s(x) - f_s(y)}{\alpha(d(x,y))} = \frac{\tan {}^{\pi}/4 \, |s-x| - \tan {}^{\pi}/4 \, |s-y|}{\tan {}^{\pi}/4 \, |x-y|}$$

$$= \tan {}^{\pi}/4 \, (|s-x| - |s-y|) \, (1 + \tan {}^{\pi}/4 \, |s-x| \, \tan {}^{\pi}/4 \, |s-y|) \, \frac{1}{\tan {}^{\pi}/4 |x-y|}$$

$$\leq 1 + \tan {}^{\pi}/4 \, |s-x| \, \tan {}^{\pi}/4 \, |s-y|$$

≤ 2 for all $x, s, y \in [0,1]$.

So L_α contains f_s for all s .

It is immediate that whenever $f_s \in L_\alpha$ for all s, L_α separates X; the converse also holds:

2.6 If L_α separates X then it contains the function f_s defined by:

$$f_s(x) = \alpha(d(x,s)) \wedge 1 \quad \text{for every } s \in X .$$

Proof: Let L_α separate X . Then for any $a, y \in X$, $x \neq y$ there is some $g \in L_\alpha$ with $|g(x)-g(y)| = a > 0$. We have $\frac{1}{a} g \in L_\alpha$ and :

$$\frac{|\frac{1}{a} g(x) - \frac{1}{a} g(y)|}{\alpha(d(x,y))} = \frac{\frac{1}{a} |g(x) - g(y)|}{\alpha(d(x,y))} = \frac{1}{\alpha(d(x,y))}$$

$$\geq \frac{|f_s(x) - f_s(y)|}{\alpha(d(x,y))}$$

since $f_s(x) \leq 1$ for all x . Thus $||f_s||_{s,\alpha}$ is finite for each s, so $f_s \in L_\alpha$ for all $s \in X$.

We can now prove a statement analogous to 2.2 .

2.7 Let (X,d) be any metric space of diameter $D \leq \infty$; let α, β be $\in C_a [0,D]$ such that $\liminf\limits_{x \to 0} \frac{\alpha(x)}{\beta(x)} = 0$ and

$$\limsup_{x \to 0} \frac{\alpha(x)}{\beta(x)} < \infty .$$

and such that L_β separates X. Then L_α is properly contained in L_β.

Proof: Since L_β separates X, f_s as defined in 2.5 belongs to L_β for all $s \in X$. For any s and all x such that $\beta(d(s,x)) < 1$, we have

$$\frac{|f_s(s) - f_s(x)|}{\beta(d(s,x))} = 1 = \frac{|f_s(s) - f_s(x)|}{\alpha(d(s,x))} \cdot \frac{\alpha(d(s,x))}{\beta(d(s,x))}$$

Then for any $M > 0$, there exists $\delta > 0$ such that for $d(s,x) < \delta$,

$$||f_s||_{d,\alpha} \geq \frac{|f_s(s) - f_s(x)|}{\alpha(d(s,x))} > M$$

i.e. $||f_s||_{d,\alpha}$ is infinite for each s and so f_s is not in L_α.

Let $f \in L_\alpha$. Since $\overline{\lim}_{x \to 0} \frac{\alpha(x)}{\beta(x)} < \infty$, there exists K such that $\alpha(x) \leq K \beta(x)$ for all $x \in [0,D]$. Consequently,

$$\frac{|f(x) - f(y)|}{\beta(d(x,y))} \leq \frac{K\,|f(x) - f(y)|}{\alpha(d(x,y))} \leq K||f||_{d,\alpha} \quad \text{for all}$$

$x, y \in X$. So $f \in L_\beta$. Note that $L_\alpha \subseteq L_\beta$ whether L_β separates X or not.

In particular, for $\alpha < \beta$ in the lattice order on $C_a\,[0,D]$ and $\underline{\lim}_{x \to 0} \frac{\alpha(x)}{\beta(x)} = 0$, $L_\alpha \subset L_\beta$ whenever L_β separates X.

SECTION 3: The Metrics σ_α.

For each $\alpha \in C_a$ such that L_α separates X, where (X,d) is a fixed metric space, we may introduce a metric σ_α on Σ_α as follows:

Let $\phi, \psi \in \Sigma_\alpha$; then:

$$\sigma_\alpha(\phi,\psi) = \sup \{|\phi(f) - \psi(f)| \; | f \in L_\alpha, \; ||f|| \leq 1\}$$

$$= \sup \{|\hat{f}(\phi) - \hat{f}(\psi)| \; | f \in L_\alpha, \; ||f|| \leq 1\}$$

Then σ_α induces a metric on the set $X \subseteq \Sigma_\alpha$:

$$\sigma_\alpha(x,y) = \sup \{|f(x) - f(y)| \; | f \in L_\alpha, \; ||f|| \leq 1\} \quad \text{for } x, y \in X .$$

<u>2.8</u> For any metric space (X,d) of diameter $D < \infty$ and any $\alpha \in C_a$ such that L_α separates X and the set $\{f_s | s \in X\}$ is bounded in L_α, the metric σ_α is boundedly equivalent on X to the distance given by $\alpha \circ d$; i.e., there exist $K, K' > 0$ with :

$$K' \alpha(d(x,y)) \leq \sigma_\alpha(x,y) \leq K\alpha(d(x,y))$$

for all $x, y \in X$.

<u>Proof</u>: By definition, $\sigma_\alpha(x,y) \leq \alpha(d(x,y))$, since if $f \in L_\alpha$ with $||f|| \leq 1$, then

$$|f(x) - f(y)| \leq \alpha(d(x,y)) \qquad \text{for all } x, y \in X .$$

We have $f_s(x) = \alpha(d(x,s)) \in L_\alpha$ for all $s \in X$, since L_α separates X and X is of finite diameter. By hypothesis, there exists some constant P, $0 < P < \infty$, such that $||f_s|| \leq P$ for all $s \in X$.

Let $x, y \in X$ be given and set $g(s) = \frac{1}{P} \; f_x(s)$ for $s \in X$. Then $g \in L_\alpha$ with $||g|| \leq 1$ and

$$\sigma_\alpha(x,y) \geq |g(x) - g(y)| = \frac{1}{P} \; \alpha\,(d(x,y)),$$

which completes the inequality as stated.

From this there follows immediately:

2.9 For any metric space (X,d) of finite diameter and any $\alpha \in C_a$ such that L_α separates X and $\{f_s | s \in X\}$ is bounded in L_α, there exists a metric on X such that L_α is precisely the associated Lipschitz algebra.

With respect to the boundedness of $\{f_s | s \in X\}$, we note that this is the case whenever the triangle inequality holds for $\alpha \circ d$ at all points in X. We have then:

$$\begin{aligned} |f_s(x) - f_s(y)| &= |\alpha(d(s,x)) - \alpha(d(s,y))| \\ &\leq \alpha(d(x,y)) \qquad \text{for all } x, s, y \in X, \end{aligned}$$

i.e. $||f_s||_{d,\alpha} \leq 1$ for all $s \in X$.

If the triangle inequality for $\alpha \circ d$ holds nowhere in X, we have

$$\begin{aligned} |f_s(x) - f_s(y)| &= |\alpha(d(s,x)) - \alpha(d(s,y))| \\ &\leq \alpha(d(s,x)) + \alpha(d(s,y)) \\ &< \alpha(d(x,y)) \qquad \text{for all } x, s, y \in X. \end{aligned}$$

so in this case also, $||f_s||_{d,\alpha} \leq 1$ for all $s \in X$. The case in which $\alpha \circ d$ satisfies the triangle inequality at some points of X and not at others remains open.

We note finally that the uniform structures U_α on X generated by the sets $\{(x,y) \mid \alpha(d(x,y)) \leq \epsilon\}$ are all equivalent to the metric uniformity determined by d (i.e., U_1). Hence for α with separating L_α and $\{f_s \mid s \in X\}$ bounded, the σ_α are uniformly equivalent metrics on X.

CHAPTER 3

SPACE MAPS AND ALGEBRA HOMOMORPHISMS.

Given two metric spaces and their associated collections of algebras L_α, we now consider the relations between continuous maps of one space into another and homomorphisms of the algebras. We will give a categorical characterization of these relations. In this Chapter we will consider all $\alpha \in C_a$ to be defined on $[0,\infty]$. and all metric spaces to be of finite diameter. By 2.4 , this involves no loss of generality.

SECTION 0: Adjoints.

Let (X,d) and (Y,r) be two metric spaces.

3.0 A continuous map t of X into Y will be said to be β-modally continuous if it satisfies the condition:

$$r(tx,tx) \leq K\,\beta(d(x,y)) \qquad (*)$$

for $\beta \in C_\alpha$, some $K > 0$ and all $x, y \in X$.

We will denote the fact that a continuous map t satisfies condition $(*)$ for a particular $\beta \in C_a$ by writing:

$$t : (X,d) \rightarrow (Y,r) \qquad (\beta)$$

We note here that if t is β-modally continuous and $\alpha \sim \beta$ in C_a, then t is also α-modally continuous.

Given any continuous map $t : X \rightarrow Y$, we consider T, the adjoint of t, to be the algebra homomorphism

$$T: C(Y) \rightarrow C(X)$$

defined as in Chapter 0 by

$$Tf(x) = f(tx) \qquad \text{for } x \in X .$$

3.1 Let $t: (X,d) \rightarrow (Y,r)$ be a β-modally continuous map. For $\alpha \in C_\alpha$, the restriction T_α of the adjoint T of t to L_α determines a compact-open continuous algebra homomorphism

$$T_\alpha: L_\alpha(Y) \rightarrow L_{\alpha \circ \beta}(X) .$$

We note that if $\alpha \circ \beta$ is not in C_a, we can replace it, by 1.31, with an equivalent function in C_a without loss of generality, and will consider this to have been done henceforth when the occasion arises.

Proof: For $f \in L_\alpha(Y)$:

$$\|Tf\|_\infty = \sup \{|f(tx)| \mid x \in X\} .$$

$$\leq \sup \{|f(y)| \mid y \in Y\}$$

$$= \|f\|_\infty$$

$$\|Tf\|_{d,\alpha o\beta} = \sup_{x \neq y} \{\frac{|Tf(x) - Tf(y)|}{\alpha o\beta(d(x,y))} \mid x, y \in X\} .$$

$$= \sup \{\frac{|f(tx) - f(ty)|}{\alpha o\beta(d(x,y))} \mid x, y \in X\} .$$

$$\leq \sup \{\frac{|f(tx) - f(ty)|}{K'\alpha(r(tx,ty))} \mid x, y \in X\} .$$

$$\leq K' \sup_{s \neq t} \{\frac{|f(s) - f(t)|}{\alpha(r(s,t))} \mid s, t \in Y\} = K'\|f\|_{r,\alpha} .$$

Thus $T(L_\alpha(Y)) \subseteq L_{\alpha o\beta}(X)$ and T is trivially a homomorphism. To show that T is compact open continuous, for $\epsilon > o$, $g \in L_\alpha(Y)$, K compact in X consider:

$$N(Tg, K, \epsilon) = \{f \in L_{\alpha o\beta}(X) \mid \|f-Tg|K\|_\infty < \epsilon\} .$$

and $$N(g, tK, \epsilon) = \{f \in L_\alpha(Y) \mid \|f-g|tK\|_\infty < \epsilon\} .$$

The continuity if t implies that tK us compact in Y, so $N(g,tK,\epsilon)$ is an element of the subbase of the compact open neighbourhood system of g and

$$T(N(g,tK,\epsilon)) \subseteq N(Tg,K,\epsilon)$$

since

$$\| f - g \mid tK \|_\infty = \| Tf - Tg \mid K \|_\infty$$

This establishes the compact-open continuity of T. In the case where Y is compact, T is continuous in the uniform topology of $L_\alpha(Y)$, since the two topologies coincide.

In order to get a partial converse to this, we need first to establish that the adjoint of a compact-open continuous homomorphism of L-algebras, $T:A \to B$, carries the underlying space of B into that of A; in other words, preserves fixed ideals. In view of 0.4, all we need to show is that for any X, any compact open neighbourhood of the unit of $L_\alpha(X)$ contains a function of compact support when $L_\alpha(X)$ separates X. Since L_α is regular when it separates X, and always self-adjoint and closed under truncation the following result will provide this:

<u>3.2</u> Let X be a locally compact metric space. If A is a regular subalgebra of $C(X)$ which is self-adjoint and closed under truncation, then every compact open neighbourhood of the unit of A contains a function $h \in A$ with compact support in X.

<u>Proof</u>: We need to show that for every compact set K in X, there exists a function $h \in A$ which is identically 1 on K and zero outside a compact set in X. Let K be any compact set in X,

U an open neighbourhood of K with compact closure. Let M(A) denote the maximal ideal space of A. For each $p \in M(A)-U$, we have by regularity a function $f_p \in A$ with :

$$f_p(p) = 1$$
$$\hat{f}_p(K) = 0$$

Set
$$V_p = \{q \mid |\hat{f}_p(q)| > \tfrac{1}{2}\}$$

Then $p \in V_p$, and the sets

$$U \text{ and } V_p \qquad p \notin U$$

form an open cover of M(A). By compactness we can select a finite number of these sets, $U, V_{p_1}, \ldots, V_{p_n}$

Now set
$$f = \sum_{i=1}^{n} f_{p_i} \overline{f}_{p_i} \in A$$

For $q \in X$, $q \notin U$ we have $q \in V_{p_j}$; for some j, so

$$f(q) \geq | f_{p_j}(q) |^2 > 1/4$$

For $q \in K$, $f(q) = 0$

Now set $h = 1 - (4f) \wedge 1 \in A$.

Then $h \mid K = 1$, while for $x \notin U$, $\hat{h}(x) = 0$. h is then the required function.

It follows that we can talk about the adjoint $t: Y \to X$ of a compact-open continuous algebra homomorphism

$$T: L_\alpha(X) \to L_\beta(Y)$$

when $L_\alpha(X)$ separates X and X is locally compact.

<u>3.3</u> Let (X,d), (Y,r), where (X,d) is locally compact, be metric spaces with $\alpha \in C_a$ such that $L_\alpha(X)$ separates X, $\alpha^{-1} \in C_a$ and the set $\{f_s \mid s \in X\}$ is bounded in $L_\alpha(X)$.

Let

$$T: L_\alpha(X) \to L_\beta(Y)$$

be a compact-open continuous homomorphism for some $\beta \in C_a$. Then its adjoint t is an $\alpha^{-1} \circ \beta$-modally continuous map of (Y,r) into (X,d).

<u>Proof:</u> By the preceding discussion, t carries Y into $X \subseteq M(L_\alpha)$. Since T is a Banach algebra homomorphism, the set

$$\{Tf_s \mid s \in X\}$$

is bounded in norm in L_β, say by K. For all $x, y \in Y$

$$K \geq \frac{|Tf_s(x) - Tf_s(y)|}{\beta(r(x,y))}$$

$$\geq \frac{|\alpha(d(s,tx)) - \alpha(d(s,ty))|}{\beta(r(x,y))}$$

Then for $s = ty$:

$$\alpha(d(tx,ty)) \leq K\, \beta\, (r(x,y))$$

$$\text{i.e., } d(tx,ty) \leq K' \alpha^{-1} \circ \beta\, (r(x,y))$$

for suitable $K' > 0$ since $\alpha^{-1} \in C_a$ by hypothesis. In the case where X is compact, we note that all three of the conditions of

3.3 are still necessary in order to reach the same conclusion. If $\alpha = \beta =$ the identity in C_a, this reduces to the case of the Lipschitz algebras on X and Y, as considered by Sherbert [7]. In this case, the separation of X by L_α, the inclusion of α^{-1} in C_a and the boundedness of $\{f_s \mid s \in X\}$ in L_α follow from the definition of L_α.

SECTION I: Algebra Isomorphisms.

From here on, we will consider all metric spaces to be locally compact.

3.4 A compact-open continuous homomorphism

$$T : L_1 (X,d) \rightarrow L_\alpha (Y,r)$$

where $\alpha^{-1} \in C_a$, $L_\alpha (Y)$ separates Y and the set $\{f_s \mid s \in Y\}$ is bounded in $L_\alpha (Y)$, is an isomorphism of $L_1 (X)$ onto $L_\alpha (Y)$ if and only if the adjoint $t: Y \rightarrow X$ is a homeomorphism of Y onto X satisfying

$$K' \alpha (r(x,y)) \leq d(tx,ty) \leq K \alpha (r(x,y)) \qquad (3.4.0)$$

for some $K, K' > 0$, all $x, y \in Y$.

Proof: Let T be a compact open topological isomorphism of $L_1 (X)$ onto $L_\alpha (Y)$. Since T is onto, t is 1-1, so

$$t^{-1} (M(X)) \rightarrow M(Y) \text{ is defined on } t(M(Y)).$$

Now suppose $t : M(Y) \rightarrow M(X)$ is not onto, so that there exists $\psi \in M(X)$, $\psi \notin t(M(Y))$. Since $M(Y)$ is compact and t is continuous, $t(M(Y))$ is compact in $M(X)$. By the regularity of L_1, there exists $f \in L_1$ with

$$\hat{f}(\psi) = 1$$

$$f(x) = 0 \qquad \text{for all } x \in t(M(Y))$$

So $$\hat{Tf}(\psi) = \hat{f}(t\psi) = 0 \quad \text{for all } \psi \in M(Y)$$

This contradicts the assumption that T is 1-1, so t must be onto, $M(Y) \to M(X)$. Likewise $t^{-1} : M(X) \to M(Y)$ is onto and

$$t^{-1}(X) \subsetneq Y \subseteq t^{-1}(M(X)) = M(Y) .$$

i.e. $X \subseteq tY$ since t is 1-1. By the compact open continuity of T, $tY \subseteq X$, so $tY = X$ and t is onto.

Now we define $d_o(x,y) = d(tx,ty)$ for $x, y \in Y$. Since t is 1-1, this gives a metric on Y. Set

$$L_o = L_1(Y, d_o)$$

Then for $f \in L_\alpha(Y)$, $f = Tg$ for some $g \in L_1(X)$ and :

$$\frac{|f(x) - f(y)|}{d_o(x,y)} = \frac{|g(tx) - g(ty)|}{d(tx,ty)}$$

$$\leq \|g\|_{d,1} \qquad \text{for all } x, y \in Y .$$

Thus $L_\alpha(Y) \subseteq L_o$; in particular, since L_α separates Y by hypothesis,

$$f_s(x) = \alpha(r(s,x))$$

is in L_o for all $s \in Y$, so for some constant $K_1 > 0$

$$\alpha(r(x,y)) \leq K_1(d(tx,ty))$$

with 3.1, this shows that t satisfies 3.4.0 for suitable $K, K' > o$.

Let $t : Y \to X$ be a homeomorphism satisfying 3.4.0 . Then the adjoint T is 1-1, $L_1(X) \to L_\alpha(Y)$ and is compact-open continuous by 3.1 . Then d_o as defined above is boundedly equivalent to $\alpha \circ r$; i.e., $L_1(Y,d_o)$ and $L_\alpha(Y,r)$ contain the same elements. Let $f \in L_\alpha(Y,d_o)$ and define

$$g(s) = f(t^{-1}s) \qquad \text{for } s \in X .$$

Then

$$\frac{|g(x) - g(y)|}{d(x,y)} = \frac{|f(t^{-1}x) - f(t^{-1}y)|}{d(x,y)} \quad \text{for } x, y \in X .$$

$$= \frac{|T^{-1}f(x) - T^{-1}f(y)|}{d(x,y)}$$

Since $T^{-1} : L_\alpha(Y) \to L_1(X)$, $g \in L_1(X)$ and $Tg = f$. Thus $T : L_1(X,d) \to L_\alpha(Y,d_o)$ is onto, so T maps $L_1(X,d)$ onto $L_\alpha(Y,r)$.

As far as the automorphisms of a particular $L_\alpha(X,d)$ are concerned, we can say:

3.5 Every compact open automorphism T of $L_\alpha(X,d)$, $\alpha \in C_a$, where $L_\alpha(X)$ separates X , $\alpha^{-1} \in C_a$ and $\{f_s \mid s \in X\}$ is bounded in L_α , is of the form:

$$Tf(x) = f(tx) \qquad f \in L_\alpha , x \in X .$$

where $t: X \to X$ is a homeomorphism onto satisfying

$$K'd(x,y) \leq d(tx,ty) \leq Kd(x,y) \quad x,y \in X, K, K' > 0$$

i.e. the adjoints of such automorphisms are homeomorphisms which satisfy a double Lipschitz condition .

Proof: Follows from applying the method of proof of 3.3 to the situation $T: L_\alpha \to L_\alpha$ as described. The only change needed is to set

$$L_0 = L_\alpha (Y, d_0)$$

instead of $L_1 (Y, d_0)$.

SECTION 2: Categorical Considerations.

We can express some aspects of the relationship between modally continuous space maps and compact-open continuous algebra homomorphisms in terms of suitable categories. We refer the reader to [11] for basic material about categories .

To begin with, we can describe the system of L_α algebras on a fixed metric space as follows:

<u>3.6</u> Let $C_a[o,d]$ be directed by the lattice order and assign to each $\alpha \in C_a$ the corresponding algebra L_α on X . For $\alpha \leq \beta$ in C_a let $h_\alpha^\beta : L_\alpha \to L_\beta$ be the mapping of L_α onto itself as a subset of L_β . $(L_\alpha, h_\alpha^\beta)$ then forms a direct system of Banach algebras and Banach algebra homomorphisms.

<u>Proof</u>: By 2.6 , for $\alpha \leq \beta$, $L_\alpha \subseteq L_\beta$ and

$$||f||_{d,\alpha} \geq ||h_\alpha^\beta(f)|| = ||f||_{d,\beta} .$$

Since $f(x) = h_\alpha^\beta(x)$ for all $x \in X$, $||f||_\infty = ||h_\alpha^\beta(f)||_\infty$, so h_α^β is a Banach algebra homomorphism, $L_\alpha \to L_\beta$.
$h_\alpha^\alpha : L_\alpha \to L_\alpha$ is the identity for every $\alpha \in C_a$; and for $\alpha \leq \beta \leq \gamma$, we have $h_\alpha^\gamma = h_\alpha^\beta \circ h_\beta^\gamma$. Thus $(L_\alpha, h_\alpha^\beta)$ forms a direct system.

Since by 1.17 every uniformly continuous function on (X,d) satisfies some modulus of continuity in C, the union of this system is the algebra of all uniformly continuous complex-valued bounded functions on X . It is conjectured that the uniform closure of this algebra is the direct limit of this system in the category of commutative semi-simple Banach algebras.

Now let G stand for the set of equivalence classes of C_a under the relation defined in Chapter I, Section 4. G inherits the lattice order of C_a and is closed under composition by 1.31. We have then associated with each metric space (X,d) the direct system of algebras $(L_\phi(X), h_\phi^\eta)_{\eta,\phi \in G}$ where $L_\phi(X)$ is the algebra with compact-open topology given by $\alpha \in C_a$ and ϕ is the equivalence class of α in G. $L_\phi(X)$ then consists of the same set of functions as $L_\alpha(X)$ for all $\alpha \subset \phi$, and it is easily seen that this is again a direct system of algebras over (X,d). Moreover, any β-modally continuous map with $\beta \in C_a$

$$t : (X,d) \to (Y,r) \qquad (\beta)$$

determines by its adjoint the algebra homomorphisms

$$T_\phi : L_\phi(Y) \to L_{\phi \circ \eta}(X)$$

for all $\phi \in G$, where η is the equivalence class of β in C_a.

This leads us to consider the category $\mathcal{B}$ defined as follows:

The objects are the direct systems of topological algebras:

$$(B_\phi, h_\phi^\eta)_{\phi,\eta \in G,\ \phi \leq \eta}$$

with the property that, for $\phi \leq \eta$, B_ϕ is a sub-algebra of B_η, and the algebra homomorphism

$$h_\phi^\eta : B_\phi \to B_\eta$$

is in fact the natural injection. The morphisms are pairs of the form:

$$\theta = ((T_\phi)_{\phi \in G}, \eta)_{\eta \in G}$$

where $T_\phi : B_\phi \dashrightarrow B_{\phi \circ \eta}$ is compact open continuous with

$$\theta \circ \theta' = ((T'_{\phi \circ \eta} \circ T_\phi), \eta \circ \eta')$$

where defined, and for any $\phi \leq \eta$

$$T_\eta \circ h^\eta_\phi = T_\phi$$

i.e. $T_\eta | B_\phi = T_\phi$

The composition defined is evidently associative and θ is the identity belonging to (B_ϕ, h^η_ϕ) if each T_ϕ is the identity homomorphism of B_ϕ and η is the equivalence class in G containing the identity map in C_a .

On the other hand, we consider the category $\mathcal{L}$ whose objects are the metric spaces (X,d) and whose morphisms are the η-modally continuous maps for $\eta \in G$; (t,η) such that:

$$t: (X,d) \rightarrow (Y,r) \quad (\beta)$$

for any β in the class η of G .

Again it is readily seen that

$$(t', \eta') \circ (t,\eta) = (t' \circ t, \eta \circ \eta')$$

where defined; this composition is associative and the identity for (X,d) is simply the pair (t,η) where η is the identity equivalence class in G and t is the identity mapping on X .

We can now define a category mapping $F : \mathcal{L} \to \mathcal{B}$ as follows:

$$F(X,d) = (L(X,d,\phi), h^{\eta}_{\phi})_{\phi, \eta \in G, \phi \leq \eta}$$

$$F(t,\eta) = ((T_{\phi})_{\phi \in G}, \eta)$$

on the objects and morphisms respectively of L .

<u>3.7</u> This mapping $F : \mathcal{L} \to \mathcal{B}$ (G) is a contravariant functor which maps the set of modally continuous

$$t: (X,d) \to (Y,r) \quad (\eta) \qquad \eta \in G$$

one to one onto the set of all morphisms

$$F(Y,r) \to F(X,d) .$$

<u>Proof</u>: Let $\theta = ((T_{\phi})_{\phi \in G}, \eta)$ for $\eta \in G$. In order to show that F is onto, we need to find (t,η) in $\mathcal{L}$ such that $F(t,\eta) = \theta$. We take $t = T_{\rho}^{*}$, the adjoint of an algebra homomorphisma

$$T_{\rho}: L_{\rho}(Y) \to L_{\rho \circ \eta}(X)$$

in θ such that $L_{\rho}(X)$ satisfies the requirements of L_{α} in 3.3. It is sufficient to take ρ the equivalence class of the identity map in C_a.

By 3.3, $t: (X,d) \to (Y,r)$ (η) .

Moreover, we claim that

$$T_{\rho}^{*}(x) = T_{\phi}^{*}(x)$$

for all $\phi \in G$, all $x \in X$ where T_{ρ}^{*} is defined. Suppose that $\phi \leq \rho$, so that $L_{\phi}(Y) \subseteq L_{\rho}(Y)$. Then

$$T_\phi = T_\rho \mid L_\phi$$

$$\text{i.e., } T_\phi^*(x) = T_\rho^*(x)$$

where defined, so the above assertion holds.

For $L_\rho(Y) \subseteq L(Y)$, we recall that the adjoint t of $T_\rho :$ $L_\rho(Y) \to L_{\rho \circ \eta}(X)$ carries all points (i.e., fixed maximal ideals) of X to points of Y. Since $L_\rho(Y)$ separates Y, if two ideals are identified by t, they correspond to the same point of Y. Suppose $t\,x_1 = t\,x_2$; consider the adjoint h of

$$T_\phi : L_\phi(Y) \to L_{\phi \circ \eta}(X)$$

Suppose $hx_1 \neq hx_2$. Then for all $s \in X$, $f_s(hx_1) \neq f_s(hx_2)$;

$$\text{i.e., } T_\phi f_s(x_1) \neq T_\phi f_s(x_2)$$

But since $f_s \in L_\rho(Y)$ for all $s \in Y$:

$$T_\phi f_s(x_1) = T_\rho f_s(x_1) = T_\rho f_s(x_2) = T_\phi f_s(x_2)$$

So we must have $hx_1 = hx_2 = ty_1$ for the adjoint of T_ρ. This takes care of all cases, since for $\phi, \rho \in G$ we have that $\phi \leq \rho$, $\rho \leq \phi$ or ϕ and ρ are not comparable. However, if ϕ and ρ are not comparable, we can find $\eta \in G$ with $\eta \geq \phi$ and $\eta \geq \rho$, and in this case $L_\eta(Y)$ will have the same properties as $L_\rho(Y)$. This reduces the matter to the two cases we have considered. It follows that the adjoint of every $T_\phi \in \theta$ is determined by that of T_ρ, so we can take $t = T_\rho^*$ and then $F(t,\eta) = \theta$ as required.

Finally, let

$$\theta_1 = ((T_{1\phi})_{\phi \in G}, \eta_1)$$

$$\theta_2 = ((T_{2\phi})_{\phi \in G}, \eta_2)$$

Then if $\theta_1 = \theta_2$, we have

$$\eta_1 = \eta_2$$

$$T_{1\phi} = T_{2\phi} \qquad \text{for all } \phi \in G .$$

In particular, $T_{1\rho} = T_{2\rho}$ for ρ as above, and since these homomorphisms are enough to determine t, θ_1 and θ_2 are the image under F of the same (t,η) in $\mathcal{L}$; so F is one to one as stated.

Since F is a functor, it takes isomorphisms of $\mathcal{L}$ to isomorphisms of $\mathcal{B}$. An isomorphism in $\mathcal{L}$ is a map

$$t : (X,d) \to (Y,r) \qquad (\eta)$$

such that t^{-1} exists with

$$t^{-1} : (Y,r) \to (X,d) \qquad (\eta^{-1})$$

where $\eta^{-1} \in G$. Such a map must correspond under F with

$$\theta = ((T_\phi)_{\phi \in G}, \eta)$$

where all T_ϕ are isomorphisms

$$T_\phi : L_\phi(Y) \to L_{\phi \circ \eta}(X)$$

We note that the existence of t and t^{-1} as above implies that t satisfies

$$r(tx,ty) \le K\,\alpha(d(x,y)) \qquad x,y \in X, \quad \alpha \in \eta$$

$$d(t^{-1}z, t^{-1}w) \leq C\beta(r(z,w)) \quad z,w \in Y \ , \quad \beta \in \eta^{-1}$$

Hence, for suitable $K_1, K_2 > 0$,

$$K_1 \alpha\, (d(x,y)) \leq \quad r(tx,ty) \leq K_2\, \alpha\, (d(x,y)) \ .$$

Thus the isomorphisms of $\mathcal{B}$ come from maps (t,η) in $\mathcal{L}$ such that t^{-1} is defined, $\eta^{-1} \in G$ and t satisfies the above double condition with respect to α , for any α in the equivalence class η .

CHAPTER 4

Quasiconformal Mappings.

In this chapter we consider the analytic properties of the space maps induced by compact-open continuous isomorphisms of generalized Lipschitz algebras, and see to what extent space maps of this type give rise to algebra homomorphisms.

SECTION 0: Quasiconformality.

The basic results on quasiconformal mappings in the plane are to be found in Kunzi [12]; the 3-dimensional case has been developed by Gehring [3] .

In real n-space, R^n, $n \geq 2$, a <u>ring</u> R is defined to be a finite connected doubly connected domain, that is, a domain whose complement with respect to extended n-space consists of two components C_o and C_1, such that C_1 contains the point at infinity. Let ∂C_i denote the boundary of C_i with R, i = 1,2 .

We define the <u>conformal capacity</u> of a ring R as follows:

$$C(R) = \inf_{u} \int_R |\Delta u|^n \, d\omega$$

where the infimum is taken over all functions $u = u(x)$, $x = (x_1, \ldots, x_n)$ which are continuously differentiable in R and have boundary values 0 on ∂C_o and 1 on ∂C_1 . We then define the <u>modulus</u> of R by means of the relation

$$\mathrm{mod}\, R = \left[\frac{1}{C(R)} \cdot \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \right]^{\frac{1}{n-1}}$$

where n is the dimension of the space and Γ is the gamma function. If R is the n-dimensional shell bounded by concentric balls of radii a and b , $a < b$, then

$$\text{mod } R = \log {}^{b}/a \ .$$

4.0 A homeomorphism of a domain D in R^n is said to be a K-quasiconformal mapping, $1 \leq K < \infty$, if the inequality

$$\frac{1}{K} \text{ mod } R \leq \text{ mod } t\ (R) \leq K \text{ mod } R$$

for fixed K and every bounded ring R with $\bar{R} \subset D$. A quasiconformal mapping is one which is K-quasiconformal for some K.

If t is a homeomorphism of a domain D in R^n, we define, for $x \in D$.

$$L(x,r) = \sup_{|x-y|=r} |tx-ty|$$

$$l(x,r) = \inf_{|x-y|=r} |tx-ty|$$

$$H(x) = \limsup_{r \to 0} \frac{L(x,r)}{l(x,r)}$$

The following result, in R^2 and R^3, is due to Gehring [3], [12]; its extension to R^n is simply a matter of following Gehring's procedure for R^3 in the n-dimensional case.

4.1 A topological mapping of a domain D in R^n is quasiconformal on D if and only if H(x) is bounded in D.

Quasiconformal maps may, of course, be characterized in analytic terms. In the present context we will be interested only in the extent to which they are Hölder continuous. The classical result on Hölder continuity of quasiconformal mappings is that given in its final form by Mori in the case of plane mappings:

4.2 [12] Let $w = H(z)$ be a K-quasiconformal plane map of $|z|<1$ onto $|w|<1$ with $H(0)= 0$. Then for each pair of points z_1, z_2 in $|z|<1$,

$$C^{-k}|z_1 - z_2|^k \leq | H(z_1) - H(z_2)| \leq C|z_1 - z_2|^{1/k}$$

where C is an absolute constant whose smallest possible value is 16.

More generally, we have the results of Callendar [4] :

4.3 Let $f(x)$ be a K-quasiconformal map of the open unit ball $|x|<1$ in R^n onto itself, with $f(0) = 0$. Then there exists an absolute constant H and an absolute exponent p such that for any pair of points x_1, x_2 in $|x|<1$

$$|f(x_1) - f(x_2)| \leq H |x_1 - x_2|^p$$

Here H and p depend on n and on K.

4.4 Let f be a K-quasiconformal map defined on a domain D of R^n, then on any compact subregion B of D

$$|f(x_1) - f(x_2)| \leq C |x_1 - x_2|^p$$

where x_1, x_2 are any two points of B, C depends on n, K and the distance from B to the boundary of A and p depends only on n and K.

SECTION 1: $\mathcal{B}$ - Homomorphisms and Quasiconformality.

We can now relate the algebra isomorphisms discussed in Chapter 3 to quasiconformal mappings, as follows:

4.5 Let X and Y be domains in real n-space. The direct systems $S(X)$ and $S(Y)$ of generalized Lipschitz algebras over X and Y are objects of the category $\mathcal{B}$ discussed in Section 2 of Chapter 4 . Let θ be a $\mathcal{B}$-isomorphism; $\theta : S(X) \to S(Y)$. Then the adjoint t of θ is a quasiconformal map of Y onto X.

Proof: By 3,4 and remarks following, t is a homeomorphism satisfying

$$K_1\, \alpha(|x-y|) \le |tx-ty| \le K_2\, \alpha(|x-y|) \qquad x, y \in Y$$

where α is in the equivalence class of η and $|x-y|$ is the usual metric in n-space. Then for every $y \in Y$:

$$H(y) = \lim_{r \to 0} \sup \frac{\sup |tx-ty|}{\inf |tx-ty|} \qquad |x-y| = r$$

$$\le \lim_{r \to 0} \frac{K_2\, \alpha(|x-y|)}{K_1\, \alpha(|x-y|)} \qquad |x-y| = r$$

$$= \frac{K_2}{K_1} < \infty \, .$$

So by 4.1, t is quasiconformal.

In general, a quasiconformal map of one domain onto another cannot be shown to give rise to an $\mathcal{B}$-isomorphism of the associated direct systems of algebras, since, even with the strong conditions of 4.2, such a map does not necessarily satisfy the double inequality

$$K_1\, \alpha(|x-y|) \leq |tx-ty| \leq K_2\, \alpha(|x-y|)$$

for any $\alpha \in C_a$. However, combining the results of Chapter 3 with 4.2, 4.3 and 4.4, we have :

4.6 Let t be a quasiconformal map of the open unit disc U in the plane onto itself, with $t(0) = 0$. The adjoint T of t determines a $\mathcal{B}$-morphism

$$\theta = ((T_\phi)_{\phi \in G}, \eta) \quad \text{i.e.} \quad \theta : S(U) \to S(U) \quad (\eta)$$

of the direct system $S(U)$ into itself, where η is the equivalence class of $\alpha(x) = x^{\frac{1}{K}}$. Moreover, for each $\phi \in G$, we have :

$$L_{\phi \circ \beta}(U) \subseteq T_\phi(L_\phi(U)) \subseteq L_{\phi \circ \alpha}(U)$$

where $\beta(x) = x^k$.

Proof: The existence of T and the fact that

$$T_\phi(L_\phi(U)) \subseteq L_{\phi \circ \alpha}(U)$$

follow from 3.1, 3.7, and the inequality in 4.2. The left hand inclusion comes by applying the proof of 3.1 to the left side of the inequality of 4.2 .

The last two results follow immediately from 3.1, 3.7, and the quasiconformality results 4.3 and 4.4 .

4.7 Let t be a quasiconformal mapping of the open unit ball U in R^n onto itself, with $t(0) = 0$. The adjoint T of t then determines a $\mathcal{B}$-morphism

$$\theta : S(U) \to S(U) \qquad (\eta)$$

of the direct system $S(U)$ into itself, where η is the equivalence class of $\alpha(x) = x^p$, p as in 4.3 .

4.8 Let t be a quasiconformal mapping defined on a domain D in R^n. Let B be any compact subregion of D such that B is separated from the boundary of D . Then the adjoint T of t determines a $\mathcal{B}$-morphism

$$\theta : S(tB) \to S(B) \quad (\eta)$$

where $t\,B$ is the image under t of B and η is the equivalence of x^p, p as in 4.4.

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