

BINDING AND SATURATION OF NUCLEAR MATTER

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SCOPE AND CONTENTS:

Using the reference spectrum method the G-matrix elements, and hence the binding energy of an infinite nucleus is calculated. Three modern potentials, two of which are hard core and one soft core are used. This formalism is then extended to a finite nuclei and spin orbit splittings around some closed shell nuclei are calculated. Both for binding energy and spin orbit splittings fairly good agreement with experiment is obtained.

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## CHAPTER I

### INTRODUCTION

Nuclear Matter is a hypothetical nucleus of infinite size. It contains an equal number of protons and neutrons, with no coulomb interaction present, and because of its infinite size no surface effect. Obviously this does not occur in nature. Still in a sense nuclei do resemble such an abstraction when seen from the liquid-drop model point of view. In this model the nucleus is supposed to have a central region of uniform density of particles, with a surface where the density falls away rapidly to zero.

There are two lines of evidence to show that this is a fairly accurate picture of a nucleus:

(1) Assuming this model the Bethe-Weizsacker semi-empirical mass formula can be obtained:

$$E(N, Z) = -a_1 A + a_2 A^{2/3} + a_3 Z^2 A^{-1/3} + \frac{1}{4} a_4 (N-Z)^2 / A + \dots \quad (\text{I-1})$$

The first term comes from the uniform density of the nucleus. Unless it is near the surface the particle sees the same surroundings everywhere, and has the same energy.

The second, third and fourth terms are respectively to take into account the surface effect, the coulomb effect, and the symmetry energy. There is also a fifth term to allow for the pairing energy. The parameters in this equation have been adjusted to give a good fit ( $\pm 500$  KeV) to the general tendencies of nuclear binding energies.

(2) Electron scattering experiments by Hofstadter show that nuclei  $A > 16$  have a uniform central density of about  $.168$  nucleons per  $F^3$  which falls from  $0.9$  to  $0.1$  of its maximum value over a distance of  $2.5 F$ .

For an infinite nucleus the binding energy per particle is given by the first term  $-a_1$  in Equation (I-1). Empirically this value has been determined by Green, and Cameron (1) to be between  $-15.8$  MeV and  $-17.2$  MeV. The goal of nuclear matter theory is to obtain this value from theory at the correct saturation density.

Thus, the model under consideration is an infinite number of nucleons filling co-ordinate space. Due to translational invariance, the single particle states are plane waves. Since nucleons are Fermi particles they will occupy the lowest available state according to the Pauli's exclusion principle, four nucleons going in each state. The filled states will form a sphere in momentum space of radius  $p_F$  (Fermi momentum) extending to the highest occupied state corresponding to the most energetic particle (Fermi energy). Generally we use  $\hbar = 1$  and talk in terms of the wave number  $k_F = p_F/\hbar$  as the Fermi momentum. To deal with this interacting Fermi gas we must use a many body formalism.

The details of the many body formalisms have been given in several places (2) and need not be repeated here in any detail. It will be sufficient to mention that the most usual procedure is to use the perturbation theory and seek an effective one-body potential from a given two particle interaction, as in the Hartree-Fock approach. Normally this method would be excellent, but nuclear forces are known to be strongly repulsive at short distances and the Hartree-Fock method would fail for them, because the matrix elements of the potential would be very large, if not infinite. The solution to this difficulty is in Brueckner's theory of nuclear structure. This theory has also been dealt in detail in several places (3). Essentially this procedure is an extension of the Hartree-Fock method to strong short range forces by allowing the interacting particles to interact any number of times before they return to the Fermi sea, so that the resulting effective two-body interaction is given by a matrix 'G' which is the "sum" of a series of matrix elements of 'v', which alternate in sign. The matrix elements of 'G' between the states of two particles of initial momentum  $m_1, n_1$  and of final momentum  $m, n$  are then written as

$$\langle m n | G | m_1, n_1 \rangle = \langle m n | v | m_1, n_1 - n_1, m_1 \rangle - \frac{1}{2} \sum_{m', n' > k_F} \langle m n | v | m' n' - n' m' \rangle \frac{1}{\epsilon_{m'} + \epsilon_{n'} - \epsilon_m - \epsilon_n} \langle m' n' | G | m_1, n_1 - n_1, m_1 \rangle \quad (\text{I-2})$$



or more formally

$$G = v - v \frac{Q}{e} G \quad (I-3)$$

where  $Q$  is the Pauli operator which requires that the intermediate states be above  $k_F$ .  $e$  is the energy denominator defined positive and given by

$$e = \epsilon_{m'} + \epsilon_{n'} - \epsilon_m - \epsilon_n \quad (I-4)$$

It will be seen from the definition of the G-matrix in Equation (I-3) that its structure very much resembles that of the scattering matrix  $t$  or  $K$  of two free particles. Actually it is the same, except for the operator  $Q$  which prohibits transitions to the occupied states which do not exist in the case of free scattering. This then is the main difference between free scattering of two nucleons and that of those in nuclear matter. The presence of this operator makes the solution for G-matrix elements very complicated and various simplifying means have to be developed. The other difference is that the energy denominator includes binding effects of the nuclear medium.

One of the most important applications of the 'G' matrix formalism has been to derive the effective single particle potential due to other nucleons experienced by a single nucleon in the Fermi sea, and hence its binding energy to the system. A simple extension

of Hartree-Fock procedure to the 'G'-matrix shows that a particle of momentum  $k_m$  experiences a potential  $U(m)$  given by

$$U(m) = \sum_{n < k_F} \langle mn | G | mn - nm \rangle \quad (I-5)$$

The average binding energy for a particle is then given by

$$\begin{aligned} \langle B.E. \rangle &= \langle K.E. \rangle + \frac{1}{2} \langle P.E. \rangle \\ &= \frac{\hbar^2 \langle k_m^2 \rangle}{2m} + \frac{1}{2} \bar{U}(m) \\ &= \frac{3}{5} \cdot \frac{\hbar^2 k_F^2}{2m} + \frac{1}{2} \bar{U}(m) \end{aligned} \quad (I-6)$$

A study of the binding energy characteristics according to density also leads us to a saturation density which can be compared with the nuclear density inside a large nucleus like  $Pb^{208}$ . In the next three chapters we give the details of our approach to this aim. In Chapter II we give the simplified methods to calculate the G-matrix elements and discuss the physical ideas of the origin of different terms and the contributions of different regions of the two body potentials. Chapter III is devoted to techniques of calculation. In Chapter IV we present the results of all this exercise and come to the conclusion that the hard core potentials give better results than before and that the soft core potentials may indeed lead to the right binding energy and saturation density when the three body clusters are taken into account. Only

the lowest order three body clusters are removed in Hartree-Fock type approach. Very briefly considering them merely represents the fact that for very short range strong forces like those in a nucleus, more than two particles may interact before returning to their normal situation in the Fermi sea. The procedure for taking them into account was given by Bethe (4) and is briefly given in Chapter II.

So far we have only considered the application of the G-matrix to an infinite nuclear system where only effective central forces exist. A finite system has a surface and therefore the tensor, spin-orbit and other forces can come into play. The extension to such a system was done by Brueckner, Gammel and Weitzner (5) under the local density approximation. In Chapter V we have extended the G-matrix calculations to study the spin-orbit splittings in a few closed shell  $\pm 1$  nuclei and obtain values in reasonable agreement with experiment. We also find that the spin-orbit force peaks about  $0.15 F$  inside the half density radius and go on to argue that the spin-orbit force is about  $.15 F$  inside the central force, confirming the recent works of several people (36). We also show that the two body spin-orbit force obtained directly from the two body potential by itself is sufficient to account for the splitting of the levels.

## CHAPTER II

### APPROXIMATE METHODS FOR SOLVING THE "G" MATRIX

There are essentially three ways to solve for the G-matrix. The first one was given by Brueckner (8) himself. It is a direct approach to the problem and at least in principle should lead to a very great accuracy. But the procedure is very lengthy and complicated and still requires an approximation for the core region. The other two procedures make approximations to the G-matrix, but give more physical insight into the problem and simplify the calculation. These are the Moszkowski-Scott separation method (9) and the Reference Spectrum method (10). The Moszkowski-Scott separation method has been modified by Bethe, Brandow and Petschek (10) (henceforth referred to as B.B.P.) to improve its accuracy and is then called the modified Moszkowski-Scott separation method. The methods were developed to give greater physical insight into the theory. In particular, one wishes to understand the role of the exclusion principle, the hard core, and the deviation of the energy denominator  $e(k)$  from a simple effective mass form. Earlier methods treated these things in various ways, but nothing was very clear.

In the Moszkowski-Scott method the long range part  $v_l$  of  $v$  and the short range part  $v_s$  were separated out, such that the

attraction due to  $v_s$  was completely balanced by the hard core repulsion and these together produced no phase shift. The distance ' $d$ ' at which this separation occurred was called the separation distance. Moszkowski-Scott then calculated  $G_s$  and  $G_\ell$  as if  $v_s$  and  $v_\ell$  were independent of each other and showed that the correction  $G - (G_\ell + G_s)$  are 10 to 20% of the main term.  $v_\ell$  is relatively weak and of long range and hence could mainly induce transitions within the Fermi sea which are prohibited by the exclusion principle. Thus  $G_\ell \simeq v_\ell - v_\ell \frac{Q}{e} v_\ell$ . The second order term was small. This also meant that the wavefunction  $\Psi$  differs little from the uncorrelated wavefunction  $\phi$  in the region  $v_\ell$ . Thus " $d$ " corresponded to the "healing" distance introduced by Gomez, Walecka and Weisskopf (11). For  $v_s$  the exclusion principle is of little importance, since it mainly permits transitions only beyond the Fermi sea. Thus the Moszkowski-Scott separation method brought out the role of the exclusion principle and other things more clearly and understandably from the physical point of view. To improve accuracy this method was slightly modified by B.B.P. We shall talk of it later.

### The Reference Spectrum Method

The understanding of Nuclear Matter given by Moszkowski and Scott was made use of by Bethe, Brandow and Petschek (10) to develop the "Reference Spectrum Method". Here one can calculate the 'G' matrix by a procedure which is more accurate and simpler than the

separation method. They showed, first, that given two types of 'G' matrices  $G^N$  and  $G^R$  which differ from each other only in the intermediate state projection and in the energy denominator, they are related by

$$G^N = G^R + G^{R\dagger} (P^{R\dagger} - P^N) G^N \quad (\text{II-1})$$

where  $P^R$  and  $P^N$  contain the intermediate state projection operator and the energy denominator for each of the G-matrices  $G^R$  and  $G^N$ . Now if  $G^N$  is the nuclear G matrix (Equation I-3) then  $P^N = Q/e^N$  where Q is the Pauli operator prohibiting transitions to the filled Fermi sea. Then

$$G^N = G^R + G^{R\dagger} \left( P^{R\dagger} - \frac{Q}{e^N} \right) G^N \quad (\text{II-2})$$

B.B.P. tried to find a  $G^R$  which is such a good approximation to  $G^N$  so that the second term in (II-2) is small. This would then permit the replacement of  $G^N$  in the second term by  $G^R$ . The Moszkowski-Scott method, and studies by Kohler (12) required that  $G^R$  should take a good approximate account of  $e^N(k)$  for  $k > k_F$  and of Q prohibiting transitions to states  $k < k_F$ . But simplicity and ease of calculations required that Q be ignored and  $e^N(k)$  be replaced by a quadratic form  $(a_1 + a_2 k^2)$ , because this is equal to the operator  $(a_1 - a_2 \nabla^2)$ . B.B.P. re-investigated Brueckner's single particle energies and found that for  $k_b \gg k_F$ ,  $U^N(b)$ , the single particle potential energies are very nearly

quadratic in  $k_b$ . Thus to a good approximation, one could replace for  $k_b \gg 1.5 k_F$ ,  $U^N(b)$  by a "Reference Spectrum" potential:

$$U^R(b) = A + B k_b^2 \quad (\text{II-3})$$

The intermediate state-particle energies in the "Reference Spectrum" approximation then becomes

$$E^R(b) = A + \frac{T(k_b)}{m^*} \quad (\text{II-4})$$

where

$$\frac{T(k)}{m^*} = T(k) + B k^2 \quad (\text{II-5})$$

and  $T(k)$  is the kinetic energy of the particle.

Brueckner's work had shown that for states inside the Fermi sea  $U_N(m)$  was also nearly quadratic in  $k_m$  and even though the effective mass for these states is different from the particle states, we can treat the difference as small and write

$$U^N(m) = A_0(m) + B k_m^2 \quad (\text{II-6})$$

and

$$\Delta(m) = \frac{M m^*}{\hbar^2 k_F^2} [A - A_0(m)] \quad (\text{II-7})$$

Thus 
$$E^R(m) = E^N(m) + \frac{\hbar^2 k_F^2}{M m^*} \Delta(m) \quad (\text{II-8})$$

Thus  $\Delta$  is the energy gap between occupied and intermediate energy spectra. It is a function of  $k_m$ , but for most purposes may be taken to be constant at its value at  $\sqrt{.6} k_F$ . We find its value to be around 0.6.

With these definitions of  $E^R$ , B.B.P. have shown that the energy denominator

$$\begin{aligned} e^R(k) &= E^R\left(\left| \frac{P+k}{2} \right|\right) + E^R\left(\left| \frac{P-k}{2} \right|\right) - E^N(m) - E^N(n) \\ &= \frac{\hbar^2}{M m^*} (k^2 + \gamma_{mn}^2) \end{aligned} \quad (\text{II-9})$$

where

$$\gamma_{mn}^2 = 2 \Delta k_F^2 - k_0^2 \quad (\text{II-10})$$

Thus  $e^R$  has the form desired of it.  $\gamma^2$  is large enough to strongly inhibit transitions to forbidden intermediate states.

#### Application of the Reference Spectrum Method

Thus, in the reference spectrum approximation, Equation (I-3) will become

$$G^R = v - v \frac{1}{e^R} G^R \quad (\text{II-11})$$



where  $e^R$  is given by (II-9). Introducing the two-body correlated wavefunction  $\psi^R$  and the uncorrelated wavefunction  $\phi$  satisfying

$$\langle \phi | G^R | \phi \rangle = \langle \phi | v | \psi^R \rangle \quad (\text{II-12})$$

i.e. defining

$$G \phi = v \psi \quad (\text{II-13})$$

we can write

$$\psi^R = \phi - \frac{1}{e^R} v \psi^R \quad (\text{II-14})$$

which with  $e^R$  as given above (Equation II-9) gives the reference wave equation

$$(\nabla^2 - \gamma^2) \mathcal{S}^R = -m^* v \psi^R \quad (\text{II-15})$$

where

$$\mathcal{S}^R = \phi - \psi^R \quad (\text{II-16})$$

is the wavefunction distortion. Equation (II-15) is the fundamental equation of the reference spectrum method. When expanded in partial waves for uncoupled states it leads to equations of the form

$$(\nabla_L^2 - \gamma^2) \chi_L = -m^* v u_L \quad (\text{II-17})$$

We use the convention of B.B.P.:

$$g_L(\rho) = \rho j_L(\rho) \quad (a)$$

$$\chi_L(x) = k_0 x g_L(x) \quad (b) \quad (II-18)$$

$$U_L(x) = k_0 x \psi_L(x) \quad (c)$$

and

$$\nabla_L^2 \chi_L = \frac{\partial^2 \chi_L}{\partial x^2} - \frac{L(L+1)}{x^2} \chi_L \quad (II-19)$$

$\gamma^2$  for the occupied initial states is given above (Equation II-10); for particle states B.B.P. have after considering the off-energy shell effect that

$$\gamma^2 = (3\Delta - 0.6) k_F^2 + 3 k_0^2 \quad (II-20)$$

Comparing (II-12) and (II-17), and making allowance for an infinite hard-core of radius  $c$ , we find

$$\begin{aligned} \langle k_0 | G^R | k_0 \rangle &= \frac{4\pi (\gamma^2 + k_0^2)}{m^* k_0^2} \left[ \int_0^c g_L^2(k_0 x) dx \right. \\ &\quad \left. + \int_c^\infty g_L(k_0 x) \chi_L^R(k_0, x) dx \right] \end{aligned} \quad (II-21)$$

$$= \frac{4\pi}{m^* k_0^2} \left[ (k_0^2 + \gamma^2) \int_0^c g_L^2(k_0 x) dx \right. \quad (II-22)$$

$$\left. + g_L (g_L' - \mathcal{H}_L') \Big|_c + m^* \int_c^\infty (g_L - \mathcal{H}_L) v U_L(x) dx \right]$$

with

$$\int_0^c g_L^2(k_0 r) dr = \frac{c}{2} \left[ g_L^2(k_0 c) - g_{L-1}(k_0 c) g_{L+1}(k_0 c) \right] \quad (\text{II-23})$$

Here

$$g_L(r) = g_L(k_0 c) H_L^{(-)}(\gamma r) / H_L^{(-)}(\gamma c) \quad (\text{II-24})$$

where

$$H_L^{(-)} = i^{L+1} (ix) h_L^{(+)}(ix)$$

is the solution of the problem with only the hard core potential;

$h_L^{(+)}(ix)$  is the outgoing wave spherical Hankel function. In

Equation (II-22) the interaction is separated into core volume, core surface, and outer contributions. Both Equations (II-21) and (II-22) are useful. For coupled partial waves, which occur in triplet spin states, the corresponding equations can be written in a compact matrix form (Appendix A).

To calculate the binding energy of nuclear matter we require the diagonal G-matrix element, averaged over the sixteen spin and iso-spin states of the interacting pair. With proper statistical weights, and exchange contributions, this is given by (Equations 6-14a and 6-14b B.B.P.):

$$\begin{aligned} \frac{1}{16} \sum_{(S,M,T,T_3)} \langle \phi_{S,T}^M | G^R | \phi_{S,T}^M \rangle &= \frac{\pi}{2k_0^2} \left[ \sum_{\text{odd } L} (2L+1) \int_0^\infty g_L v(S=0, T=0) u_L dr \right. \\ &+ 3 \sum_{\text{even } L} (2L+1) \int_0^\infty g_L v(S=0, T=1) u_L dr + \sum_{\substack{\text{even } L, L' \\ J}} (2L+1) \int_0^\infty g_L v_{L,L'}^J(S=1, T=0) u_{L,L'}^J dr \\ &\left. + 3 \sum_{\text{odd } L, L'} \sum_J (2J+1) \int_0^\infty g_L v_{L,L'}^J(S=1, T=1) u_{L,L'}^J dr \right] \quad (\text{II.25a}) \end{aligned}$$

$$\begin{aligned}
&= \frac{\pi(\gamma^2 + k_0^2)}{2m^*k_0^2} \left[ \sum_{\text{odd } L} (2L+1) \int_0^\infty g_L(k_0, z) \chi_L(k_0, z) (S=0, T=0) dz \right. \\
&+ 3 \sum_{\text{even } L} (2L+1) \int_0^\infty g_L(k_0, z) \chi_L(k_0, z) (S=0, T=1) dz \\
&+ \sum_{\text{even } L} \sum_J (2J+1) \int_0^\infty g_L(k_0, z) \chi_L(k_0, z) (S=1, T=0) dz \\
&\left. + 3 \sum_{\text{odd } L} \sum_J (2J+1) \int_0^\infty g_L(k_0, z) \chi_L(k_0, z) (S=1, T=1) dz \right]
\end{aligned} \tag{II.25b}$$

$$\begin{aligned}
&= \frac{4\pi}{k_0^2} \left[ \left\{ \frac{3}{4} \sum_{\text{even } L} + \frac{5}{4} \sum_{\text{odd } L} \right\} (2L+1) \left\{ \frac{\gamma^2 + k_0^2}{m^*} \int_0^c g_L^2(k_0, z) dz \right. \right. \\
&+ \left. \frac{g_L(k_0, z)}{m^*} (g_L' - \mathcal{H}_L') \Big|_{z=c} \right\} + \frac{1}{8} \left\{ \sum_{\text{odd } L} (2L+1) \int_c^\infty (g_L - \mathcal{H}_L) \psi (S=0, T=0) u_L dz \right. \\
&+ 3 \sum_{\text{even } L} (2L+1) \int_c^\infty (g_L - \mathcal{H}_L) \psi (S=0, T=1) u_L dz \\
&+ \sum_{\text{even } L, L'} \sum_J (2J+1) \int_c^\infty (g_L - \mathcal{H}_L) \psi_{LL'}^J (S=1, T=0) u_{L'L}^J dz \\
&\left. + \sum_{\text{odd } L, L'} \sum_J (2J+1) \int_c^\infty (g_L - \mathcal{H}_L) \psi_{LL'}^J (S=1, T=1) u_{L'L}^J dz \right\}
\end{aligned} \tag{II-25c}$$

Equation (II-25b) is obtained from Equation (II-25a) by using Equation (II-17).

After the reference  $G^R$  matrix is known, the correct  $G^N$  matrix can be determined by solving the equation (see Equation II-1)

$$G^N = G^R + G^{R+} \left( \frac{1}{e^R} - \frac{Q}{e^N} \right) G^N \quad (\text{II-26})$$

which was derived in their Appendix A by B.B.P. Assuming that  $G^R$  is a good first approximation to  $G^N$ , this may be solved by iteration

$$G^N - G^R = G^{R+} \left( \frac{1}{e^R} - \frac{Q}{e^N} \right) G^R + G^{R+} \left( \frac{1}{e^R} - \frac{Q}{e^N} \right) G^{R+} \left( \frac{1}{e^R} - \frac{Q}{e^N} \right) G^R + \dots \quad (\text{II-27})$$

The terms on the r.h.s. we refer to as the second, third, . . . order corrections to  $G^R$ . The second order terms were estimated by B.B.P. and by Brown Schappert and Wong (13), but the third order term was not seriously considered previously. The calculation of the second order term,  $\Delta^{(2)}G$  proceeds as follows. Consider an interacting pair of particles, denoting their momenta  $k_\ell, k_m$  by  $P \pm k_0$ , in a definite spin, isospin state. The interaction conserves P, S, and T, so the intermediate state sum is only over relative momentum  $k'$ . The energy denominators involve the kinetic and potential energies of the initial states ( $\ell$  and  $m$ ) and the intermediate states (a and b) of the two particles:

$$e^{R,N} = T(k_a) + T(k_b) - T(k_\ell) - T(k_m) \\ + U^{R,N}(k_a) + U^{R,N}(k_b) - U(k_\ell) - U(k_m) \quad (\text{II.28})$$

In units of  $\hbar^2/m = 41.469 \text{ MeV} \cdot \text{F}^2$  we have

$$\begin{aligned} T(k_a) + T(k_b) &= P^2 + k'^2 \\ T(k_e) + T(k_m) &= P^2 + k_0^2 \end{aligned} \quad (\text{II.29})$$

$$\begin{aligned} U(k_a) + U(k_b) &= U(P + k') + U(P - k') \\ &= 2U(\sqrt{P^2 + k'^2}) \end{aligned} \quad (\text{II.30})$$

The last step would be correct if  $U$  were really a quadratic function of momentum. For a given initial  $k_0$ , not all values of  $P$  are possible. We make the further approximation of replacing  $P^2$  in Equation (II.30) by its average value,

$$P_{av}^2 = \frac{3}{5} k_F^2 \left(1 - \frac{k_0}{k_F}\right) \left[1 + \frac{k_0^2 / k_F^2}{3(2 + k_0/k_F)}\right] \quad (\text{II.31})$$

so that

$$e^{R,N}(k') = k'^2 - k_0^2 + 2 \left[ U^{R,N}(\sqrt{P_{av}^2 + k'^2}) - U(\sqrt{P_{av}^2 + k_0^2}) \right] \quad (\text{II.32})$$

This is a slight improvement on the expression given in B.B.P., to which it reduces if  $P = 0$ . They then showed that the matrix element of the second order correction can be expressed as

$$\langle \phi_{ST}^m | G^{R+} \left( \frac{1}{e^R} - \frac{Q}{e^N} \right) G^R | \phi_{ST}^m \rangle = \int_0^\infty \mathcal{E}(k') \mathcal{F}_{ST}^m(k') dk' \quad (\text{II.33})$$

where

$$\mathcal{E}(k') = (1-a) e^R + a(e^N - e^R) \frac{e^R}{e^N} \quad (\text{II.34})$$

and the average of  $\mathcal{F}_{ST}^M$  over spin and isospin states is

$$\begin{aligned} \mathcal{F}_{av} \equiv \frac{1}{16} \sum_{(s,m,T,T_3)} \mathcal{F}_{ST}^M(k') &= \left( \sum_{\text{odd } L} + 3 \sum_{\text{even } L} \right) (2L+1) F_L(k')^2 \\ &+ \left( \sum_{\text{even } LL'} + 3 \sum_{\text{odd } LL'} \right) \sum_J (2J+1) F_{L'L}^J(k')^2 \end{aligned} \quad (\text{II.35})$$

$$F_{L'L}^J(k') \equiv \frac{1}{k_0} \int_0^\infty g_{L'}(k't) \chi_{L'L}^J(k_0, t) dt \quad (\text{II.36})$$

is a spherical Hankel transform of the wave function distortion  $\chi_{L'L}^J$  in the coupled state with total angular momentum  $J$ , taking the solution with dominant orbital momentum  $L$  and looking at the  $L'$  component. For example, the deuteron ( $J=1$ ) is mostly S-wave, so  $L=0$  and the large and small components are labelled by  $L' = 0, 2$  respectively. For uncoupled states, only the index  $L$  is required. The average  $\mathcal{F}_{av}$  is of course all one requires in binding energy calculations.

In Equation (II-33) the angular integration has been performed. This involves an angle-averaging over the Pauli operator  $Q$  and thus in Equation (II-34) it is this angle averaged operator which occurs. B.B.P. considered the case  $P = 0$  so  $Q$  was simply a step function at  $k' = k_F$ . We find that by taking  $Q(P_{0v}, k')$  the second order

corrections are reduced by about 10%. The angle averaged Pauli operator is given as

$$\begin{aligned}
 G(P, k') &= 0, & k' < \sqrt{k_F^2 - P^2} \\
 &= \frac{(k'^2 + P^2 - k_F^2)}{(2k'P)}, \sqrt{k_F^2 - P^2} < k' < k_F + P & \text{(II.37)} \\
 &= 1, & k' > k_F + P
 \end{aligned}$$

In our calculations the greatest part of the second order correction comes in the coupled  ${}^3S_1 - {}^3D_1$  partial waves, where the tensor force is important. In fact the sum of all other second order corrections is virtually zero. The small component  $\chi'_{20}$  of the "deuteron state" does not occur in the first order  $G^R$  calculation, but does enter Equation (II-35) for the corrections. Now  $F_{00}^1$ , the fourier transform of the major, S-wave component is negative at small  $k$  (from the long range attraction) and positive at moderate  $k \simeq$  several  $k_F$ , due to the short range repulsion. Empirically it is small near  $k_F$  due to these compensating effects.  $F_{20}^1$ , however is the transform of a function of one sign, zero inside the core.  $F_{20}^1$  is negative at all  $k$  upto about  $6k_F$ , and takes its maximum value near  $2k_F$ , as in Figure 1. Since it is large near the Fermi surface it makes a big contribution to both the Pauli and spectral correction



terms. Because so much of the correction came from this one state we evaluated the third order correction term, and found it to be reasonably small, but not negligible.

The evaluation is similar to that of the second order term. Taking the matrix element of  $\Delta^{(3)}G$  between states of definite P,  $k_0$ , S, M and T we have

$$\begin{aligned} & \langle \phi_{ST}^M | \Delta^{(3)}G | \phi_{ST}^M \rangle \\ &= \langle \phi_{ST}^M | (1 - \Omega^R)^+ e^R \left( \frac{1}{e^R} - \frac{Q}{e^N} \right) (1 - \Omega^R)^+ e^R \left( \frac{1}{e^R} - \frac{Q}{e^N} \right) e^R \\ & \qquad \qquad \qquad (1 - \Omega^R) | \phi_{ST}^M \rangle \end{aligned} \tag{II-38}$$

$$= \sum_{(k', k'', M', M'')} \langle k' M' | \hat{\xi}(k') | k' M' \rangle \langle k'' M'' | \xi(k'') | k'' M'' \rangle \tag{II-39}$$

$$\langle \phi_{ST}^M | (1 - \Omega^R)^+ | k' M' \rangle \langle k' M' | (1 - \Omega^R)^+ | k'' M'' \rangle \langle k'' M'' | (1 - \Omega^R) | \phi_{ST}^M \rangle$$

The wave operator  $\Omega^R$  has the property

$$\Omega^R \phi = \psi^R \tag{II-40}$$

$\xi(k)$  was defined in Equation (II-34) and  $\hat{\xi} = \xi / e^R(k)$ . There are seen to be two intermediate state sums, but as before P, S, and T are conserved by the interaction. Equation (II-39) is written for triplet

spin states, but the singlet case is an obvious simplification.

Using Equations (6.28), (6.27) and (6.31) of B.B.P. we write

(for  $S = 1, T = 0$ ):

$$\langle \hat{k}'' M'' | (1 - \Omega^R) | \Phi_{10}^M \rangle = 4\pi\sqrt{2} \sum_{\text{even } L_1, L_2} \sum_J f_{L_2 J}^{M''*} f_{L_1 J}^M \quad (\text{II-41})$$

$$\int_0^{-1} j_{L_2}(k''z) \chi_{L_2 L_1}^J(k_0 z) z dz \mathcal{D}_{M'' M}^J(\hat{k}_0 \rightarrow \hat{k}'')$$

$$\langle \Phi_{10}^M | (1 - \Omega^R)^+ | \hat{k}' M' \rangle = 4\pi\sqrt{2} \sum_{\text{even } L_1, L_2} \sum_J f_{L_2 J}^{M'} f_{L_1 J}^{M*} \quad (\text{II-42})$$

$$\int_0^{-1} j_{L_2}(k'z) \chi_{L_2 L_1}^J(k_0 z) z dz \mathcal{D}_{M' M}^J(\hat{k}' \rightarrow \hat{k}_0)$$

$$\langle \hat{k}' M' | (1 - \Omega^R)^+ | \hat{k}'' M'' \rangle = 4\pi \sum_{L_1, L_2} \sum_J f_{L_2 J}^{M''} f_{L_1 J}^{M'*} \quad (\text{II-43})$$

$$(\hat{k}')^{-1} \int_0^{\infty} j_{L_2}(k''z) \chi_{L_2 L_1}^J(k'z) z dz \mathcal{D}_{M' M''}^J(\hat{k}'' \rightarrow \hat{k}')$$

with  $f_{LJ}^M \equiv i^L \sqrt{2L+1} \langle L 1 0 M | J M \rangle \quad (\text{II-44})$

We need the identity

$$\int d^2 \hat{k}' d^2 \hat{k}'' \mathcal{D}_{M'' M}^{J_1}(\hat{k}_0 \rightarrow \hat{k}'') \mathcal{D}_{M' M''}^{J_2}(\hat{k}'' \rightarrow \hat{k}') \mathcal{D}_{M M'}^{J_3}(\hat{k}' \rightarrow \hat{k}_0) \quad (\text{II-45})$$

$$= \left( \frac{4\pi}{2J+1} \right)^2 \delta_{J_1, J_2} \delta_{J_1, J_3}$$

In view of these relations, Equation (II-38) gives

$$\sum_M \langle \Phi_{ST}^M | \Delta^{(3)} G | \Phi_{ST}^M \rangle = \int_0^\infty dk' dk'' \hat{\epsilon}(k') \epsilon(k'') F_{ST}(k'k'') \quad (\text{II-46})$$

where

$$F_{ST}(k'k'') = \sum_{MM'M''} 2(2\pi)^{-6} \int d^2\hat{k}' d^2\hat{k}'' k'^2 k''^2 (4\pi)^3$$

$$\sum_{\substack{\text{even } L_1, L_2, \\ L_5, L_6}} \sum_{L_3, L_4} (k''^2 k')^{-1} \sum_J f_{L_2, J}^{M''*} f_{L_1, J}^M f_{L_4, J}^{M'} f_{L_3, J}^{M*} f_{L_6, J}^{M''} f_{L_5, J}^{M'*} \quad (\text{II-47})$$

$$D_{M''M}^J(\hat{k}_0 \rightarrow \hat{k}'') D_{M'M}^J(\hat{k}' \rightarrow \hat{k}_0) D_{M'M''}^J(\hat{k}'' \rightarrow \hat{k}')$$

$$F_{L_2, L_1}^J(k'', k_0) F_{L_4, L_3}^J(k', k_0) F_{L_6, L_5}^J(k'', k')$$

The symbol  $F_{LL}^J$ , was defined in Equation (II-36), but here we specify both the initial and final momenta as parameters.

Equation (II-47) simplifies to

$$F_{ST} = \frac{3^2}{\pi} k' \sum_{\text{even } L, L_1, L_2} \sum_J (2J+1) F_{L, L}^J(k'', k_0) F_{L_2, L}^J(k', k_0) F_{L, L_2}^J(k'', k') \quad (\text{II-48})$$

For the correction to the average G-matrix element, we average (II.48) over S, T states. The T = 0 states are given a factor of 1/16 and the T = 1 states, 3/16. The T = 1 triplet states have a sum over odd L, of course. The singlet states have only a single sum since L = J. We have computed the third order corrections only for the  $^1S$  and ( $^3S_1 - ^3D_1$ ) states. The double integral in Equation (II-46)

is sufficiently complicated that we considered the Pauli operator to be a step function in  $k'$ ,  $k''$ ; i.e. we took  $P = 0$  in Equation (II-37).

### Effects of Three Body Clusters

For occupied states, the single particle potential energy is defined as

$$U(m) = \sum_{k_n < k_F} \langle mn | G | mn - nm \rangle \quad (\text{II-49})$$

This is similar to the Hartree-Fock theory, with the anti-symmetrised G-matrix element replacing the potential  $V$ . This definition was suggested by the third order (in G) hole-bubble diagram (Figure 2A) where a particle in the state  $m$  scatters forward from a particle in the occupied state  $n$ . It was shown by B.B.P., based on a suggestion of Brueckner and Goldman (14) that this G-matrix element should be calculated "on the energy shell".  $U(m)$  then compensates for a wider class of diagrams (Figure 2B) which are grouped together by the process of "generalised time ordering". Brandow (15) has greatly developed this concept.

For states  $k_b$  above the Fermi surface the situation is more complicated (Figure 2C). The G-matrix element for the interaction of states  $b$  and  $n$  must be calculated "off the energy shell", meaning that in solving Equation (I-3) the energy denominator involves the excitation energy of the rest of the diagram in addition to the energies of  $b$  and  $n$ .

B.B.P. showed that this insertion cannot be put on the energy shell by generalised time ordering. By a simple estimate of this effect, B.B.P. derived the estimates Equations (II-10) and (II-20) for the reference energy denominators. It is clear that  $\gamma^2$  for particle states (b-n interaction) is much larger than for hole states (m-n interaction) and this has considerable effect on raising the potential energies of particle states  $U(b)$ . In short, Equation (II-49) is also used for  $U(b)$ , but  $G(b,n)$  is calculated on a different basis.

The motive in defining a potential energy of particle states is to reduce higher order terms in the perturbation series. In the above, only the third order "bubble" diagrams were considered, because the third order "ring" diagram (Figure 2D) was known to be small. More recently it has been realized that all diagrams containing only three occupied state lines should be considered together as constituting a three body cluster. Rajaraman (16) first pointed out that all the diagrams of third order in  $G$  could be grouped together, considered as insertions in a particle line  $b$ , and therefore cancelled by a suitable definition of  $U(b)$ . Essentially there are two distinct third order diagrams, the "bubble" and the "ring", plus all their exchanges. Rajaraman arrived at a very simple prescription, namely that Equation (II-49) is used, but summing only over even partial waves of  $G$  and using statistical weight 1 in place of  $3/4$  as usually applied. This was done in Razavy's (17) calculation.

Rajaraman ( 18) also pointed out that there are diagrams of all orders in  $G$  containing only three occupied state lines. These are, therefore, of the same order in the density. Bethe ( 4 ) derived a Faddeev equation for the sum of all these diagrams and constructed an approximate solution. His result is that the sum of all three body cluster diagrams can be written as the lowest (third) order diagrams alone, with the middle  $G$ -interaction modified to  $(G.f.)$ . The "suppression factor"  $f(r)$  is one at large  $r$ , but falls to about  $1/3$  inside the hard core (Figure 3 ). It was, therefore, suggested by Bethe that the three body clusters be incorporated in the calculation by defining

$$U(b) = \sum_{k_n < k_F} \langle b n | [G f] | b n - n b \rangle \quad (\text{II-50})$$

Writing the relative momentum  $k_0$  in place of the individual state labels, we have for the matrix element in a single partial wave

$$\langle k_0 | [G f] | k_0 \rangle_L = \int_0^\infty g_L(k_0 r) v(r) f(r) U_L(k_0, r) dr \quad (\text{II-51})$$

$$= (k_0^2 + \gamma^2) \int_0^c g_L^2(k_0 r) f(r) dr + g_L(k_0 c) (g_L' - \mathcal{H}_L') f(c) \\ + \int_c^\infty [g_L(k_0 r) f(r) - \mathcal{H}_L(r) f(c)] v(r) U_L(k_0 r) dr \quad (\text{II-52})$$

The generalization to coupled states is easy. Consideration of exchange diagrams shows that Rajaraman's prescription to take only even waves

still applies. This seems clear because the complete sum is a weighted third order diagram alone.

Bethe estimated  $f(r)$  to take average values .384 inside the core and .86 outside. The effect of  $f(r)$  is thus to make the energies  $U(b)$  much more attractive than formerly, since the repulsive contributions are relatively more suppressed. It was estimated (19) that the three body clusters contribute of the order 1 MeV to the binding energy compared to about 10 MeV repulsion when the third order alone was considered; this is the source of the improvement in agreement with experiment.

In our calculation we took somewhat better account of the suppression factor  $f(r)$ , using Equation (II-51) and (II-52), than in Bethe's paper. For the hard core potentials we used his numerical values of  $f(r)$ , scaled to the appropriate core radius. In principle,  $f(r)$  should be evaluated for the actual potential used, and thus our work is subject to a correction on this point. M. W. Kirson (20) is considering this, while B. Day (21) has recently improved on Bethe's solution of the Faddeev equations. For the soft core Bressel potential, we evaluated  $f(r)$  on the same basis as Bethe's hard core function. The relevant equations are (4.4) to (4.6) and (3.24) of reference (4), which we reproduce here:

$$F_1(\chi_{23}) = \int d\tau, \eta_{12} (\eta_{12} + \eta_{13}) \quad (\text{II-53})$$

$$F(x_{23}) = \int d\tau_1 \eta_{12} (\Phi - \Psi^{(1)}(x_{12}, x_{13}, x_{23})) \quad (\text{II-54})$$

$$f(x_{23}) = F(x_{23}) / F_1(x_{23}) \quad (\text{II-55})$$

$$u_{12} = 1 - \mathcal{S}_{12} \quad (\text{II-56})$$

$$\Phi - \Psi^{(1)} = \frac{\eta_{12} u_{13} + \eta_{13} u_{12} - \eta_{23} (u_{12} + u_{13} - 2 u_{12} u_{13})}{u_{12} u_{13} + u_{12} u_{23} + u_{31} u_{32} - 2 u_{12} u_{23} u_{31}} \quad (\text{II-57})$$

The subscripts denote the argument of each function, so  $u_{12} = u(r_{12})$  etc.  $\eta$  and  $\mathcal{S}$  are the wave function distortions (more precisely  $\eta \cdot \emptyset$  and  $\mathcal{S} \cdot \emptyset$  are) for a typical hole-hole and particle-hole interaction respectively. One requires these two situations because the initial and final stage in any three body cluster is the interaction of two particles in occupied states. The remaining stages involve excited particles. Equation (II-57) is Bethe's approximate solution of the Faddeev equations.  $\Psi^{(1)}$  can be interpreted as the part of the three body wave function after any number of two body interactions, when particle 1 does not take part in the final interaction. The energy due to three body clusters is

$$W = \int G(x_{23}) F(x_{23}) d\tau_{23} \quad (\text{II-59})$$

while by considering the third order only one would have

$$W_1 = \int G(x_{23}) F_1(x_{23}) d\tau_{23} \quad (\text{II-60})$$



Using (II-55) it is obvious that

$$W = \int G \int f(r_{23}) F_1(r_{23}) d\tau_{23} \quad (\text{II-61})$$

which leads to the interpretation stated earlier.

In Bethe's calculation for  $f(r)$  several approximations were made. Only the short range part of the interaction was used, the long range part removed by a modified Moszkowski-Scott (10) separation. This is helpful because  $\eta$  and  $\mathcal{S}$  are then positive functions, and justified because only the strong short range force can make appreciable contribution in high order diagrams. This is done here. Next  $\mathcal{S}(r)$  was taken to be a real function of  $r$  only. Actually,

$$\mathcal{S}(r) \phi(r) \equiv \chi(r) \quad (\text{II-62})$$

where  $\chi = \phi - \Psi$  is the wave function distortion in Equation (II-16). Clearly  $\mathcal{S}$  is in general a complex function of  $r$  and  $\theta$ . If  $\phi$  and  $\chi$  are expanded in partial waves, and  $\mathcal{S}$  in a series of multipoles,

$$\mathcal{S}(r) = \sum_L (2L+1) i^L \mathcal{S}_L(r) P_L(\cos \theta) \quad (\text{II-63})$$

the  $\mathcal{S}_L$  may be determined by term by term comparison.  $\mathcal{S}_{L=0}$  is the angle average of  $\mathcal{S}$  and it is only in this approximation ( $\mathcal{S} = \mathcal{S}_0$ ) that  $\mathcal{S}$  will be a real function only of  $r$ . Still considering only spinless particles,

$$\mathcal{S}_0 = \sum_L (2L+1) \frac{\chi_L \phi_L}{(k_0 r)^2} \quad (\text{II-64})$$

This agrees with (II-62) in giving  $\mathcal{S} = 1$  inside a hard core.

Finally, Bethe did not calculate  $\mathcal{S}(r)$  for any particular potential, but adopted a simple and reasonable quadratic form for it.

In our calculation we constructed  $\chi(r)$  for the  $^1S_0$  Bressel potential using the Modified Moszkowski-Scott method. This was done for the two representative hole and particle cases.  $\eta$  and  $\mathcal{S}$  could then be taken from the  $L = 0$  parts of either (II-62) or (II-64). Since Equation (II-62) is too large near a zero of  $g_0(k_0 r)$ , but (II-64) is too small, it finally seemed most reasonable to use  $\mathcal{S}(r) = \chi_0(r)$  with no factor of  $\phi$ . Higher partial waves could be included in Equation (II-64) but actually this leads to difficulty if  $\eta(r)$  exceeds  $\mathcal{S}(r)$  in magnitude. Day's new solution may improve matters here. Having thus defined  $\eta$  and  $\mathcal{S}$  we used Equations (II-53) to (II-58) to evaluate  $f(r)$ . The result is shown in Figure 3.

It is seen that  $f(r)$  for the Bressel potential takes values about  $\frac{1}{2}$  as  $r \rightarrow 0$ . This arises because  $\Psi/\phi \simeq \frac{1}{2}$  at small  $r$ . In the case of an infinite hard core,  $\Psi/\phi = 0$  of course. Now it follows from Equation (II-58) that when  $\eta = \mathcal{S} = \frac{1}{2}$ ,  $\phi - \Psi^{(1)} = \frac{1}{2}$ , and by comparing (II-52) and (II-53) we obtain  $f(r) \simeq \frac{1}{2}$ . This contrasts to the hard core limit of  $1/3$ . We conclude that while a soft core is less repulsive than an infinite hard core, it is also less thoroughly suppressed in its effect on three body clusters. This effect will tend to minimize the difference between hard and soft core potentials in nuclear matter.

Modified Moszkowski-Scott Method

As mentioned in the beginning of this section in Moszkowski and Scott separation method beyond the separation distance  $d_F$  the wavefunction of two free nucleons interacting via  $v_S$  goes over to the unperturbed wavefunction. B.B.P. modified this condition to the effect that the wavefunction in the reference spectrum goes to the unperturbed wavefunction beyond  $d_R$ . This is more accurate than the MS condition because the reference G-matrix is much closer to the actual G than the free nucleon G-matrix. This condition means

$$\chi_{0S}^R(k_0, \epsilon) = 0 \quad r > d_R \quad (\text{II-65})$$

or equivalently

$$u_{0S}^R = g_0 - \chi_{0S}^R = g_0 \quad r > d_R \quad (\text{II-66})$$

Since  $\chi$  and  $\chi'$  are continuous, condition (II-65) is equal to

$$\chi(d-\epsilon) = \chi'(d-\epsilon) = 0 \quad (\text{II-67})$$

B.B.P. then go on to show that up to the terms in second order

$$G^N = G_S^R + v_\ell + (1-K)^{-1} G_S^{R+} \left( \frac{1}{e^R} - \frac{Q}{e^N} \right) (2v_\ell + G_S^R) - v_\ell \frac{Q}{e^N} v_\ell \quad (\text{II-68})$$

where  $K$  is a small quantity given by

$$K = \frac{1}{(2\pi)^3} \int d^3 k' \left[ (1-Q) - Q(e^R - e^N)/e^N \right] \langle k' | 1 - \Omega_S^R | k_0 \rangle \quad (\text{II-69})$$

where

$$\begin{aligned} \langle k' | 1 - \Omega_S^R | k_0 \rangle &= \frac{4\pi}{k' k_0} \int_0^\infty g_L(k'z) \chi_{LS}(k_0, z) dz & k' \neq 0 \\ &= \frac{4\pi}{k_0} \int_0^\infty z \chi_{LS}(k_0, z) dz & k' = 0 \end{aligned} \quad (\text{II-70})$$

Proceeding just as we did in the Reference Spectrum Method, we can easily obtain

$$\begin{aligned} \langle k_0 | G_S^R | k_0 \rangle &= \frac{4\pi(\gamma^2 + k_0^2)}{m^* k_0^2} \left[ \int_0^c g_L^2(k_0, z) dz \right. \\ &\quad \left. + \int_c^d g_L(k_0, z) \chi_S^R(k_0, z) dz \right] \end{aligned} \quad (\text{II-71})$$

$$\begin{aligned} \langle k_0 | \psi^l | k_0 \rangle &= \frac{4\pi}{k_0^2} \int_d^\infty g_L^2(k_0, z) \psi^l(z) dz \\ &= \frac{2\pi}{k_0^2} \int_d^\infty (1 - \cos 2k_0 z) \psi^l(z) dz \end{aligned} \quad (\text{II-72})$$

for  $L = 0$  wave alone.

The third term in (II-68) is given by

$$\begin{aligned}
 \langle k_0 | G_3^N | k_0 \rangle &= (1-K)^{-1} \int_0^\infty \langle k_0 | G_S^{R+} | k' \rangle \langle k' | \left( \frac{1}{e^R} - \frac{Q}{e^N} \right) (2V_2 + G_S^R) | k_0 \rangle d^3 k' \\
 &= (1-K)^{-1} \int_0^\infty d^3 k' \left[ \left\{ \frac{4\pi}{k_0 k'} \int_0^\infty g_0(k't) \chi_{0S}(k_0 t) dt \right\} \right. \\
 &\quad \left. \left\{ \left[ (1-Q) - Q \frac{e^R - e^N}{e^N} \right] \langle k' | 2V_2 + G_S^R | k_0 \rangle \right\} \right] \quad (\text{II-73}) \\
 &= \frac{16}{k_0^2 (1-K)} \left[ \int_0^{k_F} d^3 k' \left\{ \left( \int_0^\infty g_0(k't) \chi_0(k_0 t) dt \right) \left( \int_0^\infty g_0(k't) V_2 g_0(k_0 t) dt \right) \right\} \right. \\
 &\quad \left. - \int_{k_F}^\infty d^3 k' \frac{e^R - e^N}{e^N} \left\{ \left( \int_0^\infty g_0(k't) \chi_0(k_0 t) dt \right) \left( \int_0^\infty g_0(k't) V_2 g_0(k_0 t) dt \right) \right\} \right] \\
 &+ \frac{8\pi}{k_0^2 (1-K)} \left[ \int_0^{k_F} d^3 k' e^R \left| \int_0^\infty g_0(k't) \chi_{LS}(k_0 t) dt \right|^2 \right. \\
 &\quad \left. - \int_{k_F}^\infty d^3 k' \frac{e^R (e^R - e^N)}{e^N} \left| \int_0^\infty g_0(k't) \chi_{LS}(k_0 t) dt \right|^2 \right] \quad (\text{II-74})
 \end{aligned}$$

The fundamental equation for  $\chi_S$  is of the form (II-17)

$$\begin{aligned}
 \frac{d^2 \chi_{LS}}{dx^2} - \frac{L(L+1)}{x^2} \chi_{LS} - \gamma^2 \chi_{LS} &= -m^* V_{LS} U_{LS} \\
 &= -m^* V_{LS} (g_L - \chi_{LS}) \quad (\text{II-75})
 \end{aligned}$$

## CHAPTER III

### DESCRIPTION OF THE CALCULATION

#### (a) Reference Spectrum

The first order or reference spectrum calculation involves the solution of the reference wave equation (II-17) and the evaluation of the integrals in Equation (II-21). To simplify the discussion we consider singlet states and relegate the triplet equations to Appendix A. Equation (II-17) may be written as

$$\left( \frac{d^2}{dr^2} - \gamma^2 - \frac{L(L+1)}{r^2} - m^* v(r) \right) \chi_L(r) = -m^* v(r) j_L(k_0 r) \quad (\text{III-1})$$

$$\chi_L(k_0, r) = j_L(k_0 r) - u_L(k_0, r) \quad (\text{III-2})$$

For a hard core potential,  $u_L$  vanishes for  $r < c$  so  $\chi_L$  is equal to  $j_L$ . We solve Equation (III-1) for  $r > c$ , subject to  $\chi \rightarrow 0$  (heals) at large  $r$ . In practice we assume that  $v(r)$  is negligible for  $r$  greater than some large distance  $d$ . Beyond this distance,

$$\chi_L(r) = \mathcal{H}_L(r) \cdot N \quad (\text{III-3})$$

with  $N$  a constant and  $\mathcal{H}_L$  a decaying Hankel function as in Equation (II-24).

Explicitly

$$\begin{aligned}\mathcal{H}_0(x) &= e^{-x} \\ \mathcal{H}_1(x) &= \left(1 + \frac{1}{x}\right) e^{-x} \\ \mathcal{H}_{L+1}(x) &= \frac{2L+1}{x} \mathcal{H}_L(x) + \mathcal{H}_{L-1}(x) \\ x &= \gamma r\end{aligned}\tag{III-4}$$

Giving a name

$$\begin{aligned}\Gamma_L &= - \frac{d}{dx} \log \mathcal{H}_L(x), \\ \Gamma_0 &= \gamma\end{aligned}\tag{III-5}$$

to the log-derivative of  $\mathcal{H}_L$ , we can write the two point boundary conditions on Equation (III-1) as

$$\begin{aligned}\chi_L(c) &= g_L(k_0 c) \quad , \quad r = c \\ \chi_L' + \Gamma_L \chi_L(d) &= 0 \quad , \quad r = d\end{aligned}\tag{III-6}$$

In our work we generally used  $d \simeq 7.1F$ . This problem is conveniently solved by the method of E. C. Ridley (22) as suggested by Razavy (17). Actually, Razavy used a variant of this method starting the integration at  $r = c$ . By starting instead at  $r = d$  the procedure is greatly simplified, as described here.

In the Ridley method, auxiliary functions  $s$  and  $\omega$  are introduced, so that Equation (III-1) is factored into three first

order equations,

$$\frac{ds}{dr} = s^2 + g \quad (\text{III-7})$$

$$\frac{d\omega}{dr} = s\omega + h \quad (\text{III-8})$$

$$\frac{d\chi}{dr} = -s\chi + w \quad (\text{III-9})$$

with

$$g(r) = -\gamma^2 - v_L(r) m^* - L(L+1)/r^2 \quad (\text{III-10})$$

$$h(r) = -m^* v_L(r) g_L(k_0 r) \quad (\text{III-11})$$

The boundary condition at  $r = d$  is satisfied (see (III-9)) if  $s(d) = \sqrt{L}$  and  $\omega(d) = 0$ . Using these initial values, (III-7) and (III-8) are integrated inwards to  $r = c$ , constructing a table of  $s$  and  $\omega$  by the Gill-Runge-Kutta method (23). Finally, using  $\chi(c)$  from (III-6) as initial value, Equation (III-9) may be integrated back out to  $r = d$ , completing the solution. The mesh here is double that used on the inward integration, so  $s(r)$  and  $\omega(r)$  are available at the mid-points of the steps.

The values of  $\chi(r)$  are now used to construct the integrals in Equation (II-24) for  $r > c$ . These may be considered spherical



Hankel transforms, and conveniently evaluated by an extension of Filon's method (24). This is a generalization of Simpson's rule in which the wiggles of  $\sin(qr)$  are exactly accounted for, so the mesh required is determined by the smoothness of  $\chi(r)$ . Thus, for all  $q$  of interest we can use the same table of  $\chi(r)$ . Our computer routine FILON assumed  $\chi(r)$  to be tabulated in blocks of 24 steps, with the mesh size doubling between blocks. For the hard core potentials we used  $H = .04 F$  outside the core, and three blocks in all. For the soft core Bessel potential we used a fictitious hard core of  $0.01F$  as a device to start the integration. We fitted two blocks exactly inside the square core region and three blocks outside, being careful to pick up the very deep value of  $v(r)$  immediately beyond the square potential edge. For the region  $r > d$ , so far neglected, we added a correction. Assuming as before that  $v(r)$  is very small, it is easy to derive the approximation

$$\int_d^\infty J_L(k_0 r) \chi_L(k_0 r) dr = \frac{1}{r^2 + k_0^2} \left( J_L'(k_0 r) + J_L(k_0 r) r \right) \chi_{LL}^J \Big|_{r=d} \quad (\text{III-12})$$

For  $r < c$   $\chi_L = J_L$  and hence the core contribution is given by

$$\begin{aligned} \int_0^c J_L(k_0 r) \chi_L(k_0 r) dr &= \int_0^c J_L^2(k_0 r) dr \\ &= \frac{c}{2} \left[ J_L^2(k_0 c) - J_{L-1}(k_0 c) J_{L+1}(k_0 c) \right] \quad (\text{III-13}) \end{aligned}$$

The Reid hard core potential, given separately for each partial wave, is available only for the following states:

$${}^1S_0, \quad {}^3S_1 - {}^3D_1, \quad {}^1D_2, \quad {}^3D_2, \quad {}^1P_1, \quad {}^3P_0, \quad {}^3P_1, \quad {}^3P_2 - {}^3F_2.$$

For sake of comparison we calculated the binding energies for these very states for the Hamada-Johnston and the Bressel-Kerman potentials.

For higher partial waves we evaluated the OPEP contribution in Born approximation as first used by Razavy. For this purpose we assume that only the OPEP is effective for these waves and replace  $u_L$  by  $f_L$  as in Born approximation in (II-25a). This procedure is quite good since the centrifugal barrier allows these partial waves to be effective only at large distances for the energies in nuclear matter and the wave function is not too much distorted. The detailed procedure is this: We first assume that the OPEP acts in all partial waves and then subtract out the ones which we have taken into account more accurately. The OPEP is given by

$$V_{OPEP} = \frac{f^2}{b} \mu c^2 \frac{\tau_1 \cdot \tau_2}{3} \left[ \sigma_1 \cdot \sigma_2 + S_{12} \left( 1 + \frac{3}{\mu r} + \frac{3}{\mu^2 r^2} \right) \right] \frac{e^{-\mu r}}{\mu r} \quad (\text{III-14})$$

$$f^2 = 0.0758$$

$$\mu c^2 = \text{PION MASS}$$

Since the tensor part averages out to zero when we sum over all the J components of any partial wave we have writing

$$V(r) = V_0 \frac{e^{-\mu r}}{\mu r} \quad (\text{III-15})$$

$$S_{\text{even } L} = \frac{4\pi}{k_0^2} \sum_{\text{even } L} (2L+1) \int_0^\infty g_L^2(k_0 t) v(t) dt$$

$$= \left( \frac{2\pi}{\mu^3} + \frac{2\pi}{\mu(\mu^2 + 4k_0^2)} \right) V_0 \quad (a)$$

$$S_{\text{odd } L} = \left( \frac{2\pi}{\mu^3} - \frac{2\pi}{\mu(\mu^2 + 4k_0^2)} \right) V_0 \quad (b) \quad (\text{III-16})$$

Then by Equation (II-25a)

$$\frac{1}{16} \sum_{S, M, T, T_3} \langle \phi_{S, T}^M | G^R | \phi_{S, T}^M \rangle = -\frac{3\pi}{\mu(\mu^2 + 4k_0^2)} f^2 \mu c^2 \quad (\text{III-17})$$

From this if we subtract out the contributions in OPEP of the waves we have used, viz., those given on page 37 we shall obtain the contributions of the higher partial waves. For this we require the integrals (25)

$$\int_0^\infty \frac{e^{-\mu x} g_L^2(kx)}{x} dx = \frac{1}{2} Q_L \left( 1 + \frac{1}{2} \hat{a}^2 \right) \quad (\text{III-18})$$

$$\int_0^\infty \left( 1 + \frac{3}{\mu x} + \frac{3}{\mu^2 x^2} \right) \frac{e^{-\mu x}}{x} g_L^2(kx) dx = \frac{1}{2} Q_L + \frac{3}{2 \hat{a}^2 (2L+1)} (Q_{L-1} - Q_{L+1}) \quad (\text{III-19})$$

where  $\hat{a} = \frac{\mu}{k}$ ,  $g_L(kr) = kr j_L(kr)$  and  $Q$  are the Legendre functions of the second kind. Then the OPEP contribution is

$$\begin{aligned}
& \frac{1}{16} \sum_{S,M,T,T_3} \langle \phi_{S,T}^M | G^R | \phi_{S,T}^M \rangle \Big|_{\text{OPEP}} \\
&= 3\pi f^2 \mu c^2 \left[ -\frac{1}{\mu(\mu^2 + 4k^2)} + \frac{1}{2\mu k^2} \left( Q_0 - 3Q_1 + \frac{9Q_2}{2} + \frac{Q_3}{2} \right) \right. \\
&\quad \left. + \frac{1}{\mu^3} \left( \frac{1}{5} (Q_1 - Q_3) + \frac{2}{7} (Q_2 - Q_4) \right) \right] \tag{III-20}
\end{aligned}$$

Similarly if only the even waves from amongst those given on page 37 are considered the OPEP contribution is

$$\begin{aligned}
& \frac{1}{16} \sum_{S,M,T,T_3} \langle \phi_{S,T}^M | G^R | \phi_{S,T}^M \rangle \Big|_{\substack{\text{OPEP} \\ \text{even L}}} \\
&= \frac{3\pi f^2 \mu c^2}{2\mu} \left[ -\frac{1}{\mu^2} - \frac{1}{\mu^2 + 4k_0^2} + \frac{1}{k_0^2} \left( Q_0 + \frac{9}{2} Q_2 \right) + \frac{2}{5} \cdot \frac{1}{\mu^2} (Q_1 - Q_3) \right] \tag{III-21}
\end{aligned}$$

The procedure of calculation was as follows: For each  $k_F$  reference spectrum parameters  $\Delta$  and  $m^*$  were chosen. After some time it became clear that one can choose  $\Delta = 0.6$  and  $m^*$  (always near 1.0) was determined by making the reference spectrum cross the "nuclear" or computed spectrum at some  $k_0 \simeq 3 k_F$ . We chose seven values of  $k_0/k_F = 0.1, 0.2, 0.4, 0.54, 0.775, 0.9$  and 1.0 and for each  $k_0$  constructed  $G^R(k_0)$ . The  $\gamma^2$  for given  $k_0$  is determined by Equation (II-10). In the first order calculation,  $G^R$  depends only on  $k_0$ , not  $P$ , so Equation (II-49) is equivalent to

$$\begin{aligned}
U(k_m) &= \frac{16}{\pi^2} \int_0^{(k_F - k_m)/2} G(k_0) k_0^2 dk_0 \\
&\quad + \frac{2}{\pi^2 k_m} \int_{\frac{k_F - k_m}{2}}^{(k_F + k_m)/2} k_0 \left[ k_F^2 - (2k_0 - k_m)^2 \right] G(k_0) dk_0. \tag{III-22}
\end{aligned}$$

and the average potential energy of occupied states is

$$\begin{aligned}\bar{U} &= \frac{1}{\rho} \sum_{m < k_F} U(m) \\ &= 12\rho \int_0^{k_F} k^2 (k-1)^2 (k+2) G(k, k_F) dk\end{aligned}\quad (\text{III-23})$$

It turns out that  $\bar{U}$  is very nearly equal to  $U(m)$ , where  $\bar{k}_m = \sqrt{.6} k_F$  is the average momentum in the Fermi sea.

To determine  $U^N(k_0)$  we evaluated  $[Gf](k_0)$  by (II-52) at eight values of  $k_0$  from zero relative momentum up to  $k_0 = 3.5 k_F$ . In this case  $\gamma^2$  is taken from Equation (II-20). Only even partial waves of  $G$  are required. The sum over  $k_n$  in Equation (II-50) was carried out by a double quadrature,

$$U(k_0) = \frac{3}{2k_F^3} \int_0^{k_0} dk_n k_n^2 \int_{-1}^1 d(\cos \theta_{kn}) G\left(\left|\frac{k_0 - k_n}{2}\right|\right) \quad (\text{III-24})$$

To obtain  $G(k_0)$  at all the points required, a Lagrange interpolation was made in the table of  $G$ . The computed  $U^N(k_0)$  and  $U(k_m)$  were compared against the input reference spectrum parameters and the process repeated once or twice to obtain reasonable self consistency. A typical spectrum is shown in Figure 4.

#### (b) Second Order Corrections

The second order corrections were evaluated from Equation (II-33) to (II-37), which give diagonal matrix elements of  $\Delta^{(2)}G$ . Generally we used a "one-point" approximation in which we estimated

$$\frac{1}{2} \Delta \bar{U} \simeq \frac{1}{2} \rho \Delta^{(2)} G(k_0 = \sqrt{.3} k_F) \quad (\text{III-25})$$

where  $k_0$  is the average relative momentum between occupied states. Since  $\Delta^{(2)}G$  is not exactly a quadratic function of  $k_0$ , this is not exact. Later on we evaluated  $\Delta^{(2)}G$  at seven values of  $k_0$  and integrated accurately to find the change in binding, as in Equation (II-49). As shown in Table I, the one-point approximation is very good, with an error of order 5%. The angle averaged Pauli operator Equation (II-37) with the  $P_{av}$  appropriate to  $k_0$  was used in all cases. Compared to setting  $P = 0$ , (so  $Q$  is a step function), this decreases the corrections by about 10%.

We made one slight error in our use of Equation (II-32), by using  $U^R(k')$  in place of the argument  $\sqrt{P_{av}^2 + k'^2}$ . For large  $k'$  this makes extremely little difference and we feel there are greater uncertainties than this in the nuclear spectrum.

With  $P \neq 0$  the integration in Equation (II-33) breaks up into four regions, two Pauli and two spectral. The three short regions were covered by 9 point Simpson's rule, the longer region by 21 points. In our most accurate evaluation of the spectral correction we found some difficulty with taking  $k_0 = k_F$ . Due to the near continuity of the nuclear spectrum at  $k_F$ ,  $e^N$  tends to zero at the Fermi surface. However  $P_{av}$  also tends to zero in this limit, so  $Q$  is a step function. To evaluate the spectral correction we took the integrand at  $k = k_F$  as a polynomial extrapolation from the values above  $k_F$ . This is probably justified, and was the only numerical problem faced.

(c) Third Order Correction

The third order correction was evaluated from Equation (II-46) and (II-48). In this case we took  $Q$  to be a step function, to keep the work to a minimum. Further, we used the "one-point" approximation (III-25) to estimate the change in binding energy. It would be a very lengthy task to evaluate  $\Delta^{(3)}G$  at several initial  $k_0$ . Considering the coupled  ${}^3S_1 - {}^3D_1$  waves,  $J = 1$  and it turns out there are eight combinations of  $L$ -values in the sum (II-48). These are shown in Table II, in order of importance. The region of integration is shown in Figure 5. The upper limit was set at  $6k_F$  which we believe is adequate. The Pauli operator  $Q$  divides the square into four regions, so with the eight terms there are thirty-two contributions to the corrections. In practice only one or at most three of these contributions was significant, the others being several times smaller. This gives us confidence that the result obtained is meaningful, and indicates also that the series of correction terms is converging absolutely.

Simpson's rule was used in the integration, with eight points below  $k_F$  and 21 above. According to Equation (II-48) we must have 29 Hankel transforms of  $\chi(k_0, r)$  for the table of  $F_{L_1 L}^J(k'', k_0)$ . However, the final term in the sum would require 29 transforms of each 29 reference wave defect functions  $\chi(k', r)$ . This would be a large amount of computing and we made the following approximation.

We chose seven representative values of  $k'$ ,  $k'/k_F = .22, .56, .775, .937, 1.5, 3.125, \text{ and } 5.125$ . We found  $\chi^R$  for each of these and for each made a table of the transform. During the integration the required  $F(k'', k')$  was taken from the correct row of the closest available  $k'$  (above or below  $k_F$  as required). Some comparisons given in Table III show that even with only two  $k'$  - columns the results agree quite closely with each other. We believe that our evaluation of  $\Delta^{(3)} U$  is good to better than 10%.

We also calculated the third order correction for the  $^1S_0$  state for a few representative cases. It was of the order 0.2 MeV, so completely negligible.

#### (d) MMS Method

Since the corrections to the Reference Spectrum are most important in the S state we proceeded to apply the MMS method only in the  $^1S_0$  state, the  $^3S_1$  being taken care of by the third order correction. Equation (II-75) then becomes

$$\frac{d^2 \chi_{0s}}{dr^2} - \gamma^2 \chi_{0s} = -m^* v_{0s} (f_0 - \chi_{0s})$$

or 
$$\frac{d^2 \chi_{0s}}{dr^2} - (\gamma^2 + m^* v_{0s}) \chi_{0s} = -m^* v_{0s} f_0 \quad (\text{III-26})$$

The boundary conditions are given as

$$\chi_{0s} = f_0 \quad |_{r \leq c} \quad (\text{a})$$

$$= 0 \quad |_{r=d-\epsilon \quad \epsilon \rightarrow 0} \quad (\text{b}) \quad (\text{III-27})$$



The equation is again solved by the method of E. C. Ridley, but now, because the separation distance is not known at first, we must use its variant, as was done by Razavy.

Equation (III-24) is again factored into three first order equations:

$$\frac{ds}{dz} = - (1 + sgs) \quad (\text{III-28})$$

$$\frac{dw}{dz} = - s(gw - h) \quad (\text{III-29})$$

$$\frac{du}{dz} = g(su + w) - h \quad (\text{III-30})$$

$$\chi_{0s} = su + w \quad (\text{III-31})$$

and 
$$\chi'_{0s} = -u \quad (\text{III-32})$$

where 
$$g(z) = - (r^2 + m^* v_s) \quad (\text{III-33})$$

$$h(z) = - m^* v_s g_0(k_0 z) \quad (\text{III-34})$$

The equations are similar to (III-7, 8, 9, 10, 11) if we identify

$$s = \frac{1}{s} \quad (\text{III-35})$$

and 
$$w = \frac{\omega}{s}$$

The boundary condition (III-27a) is satisfied if we put  $S = 0$  and  $W = \int_0^{k_0 c} (k_0 c)$  at  $r = c$  (Equation III-31). With these initial values we integrate outwards for  $S$  and  $W$ , constructing a table for them in this operation, till  $W = 0$ . Here we then put  $U = 0$ , thus satisfying the other two boundary conditions (III-21b and c), and integrate inwards for  $U$ , constructing a table for it too.  $\chi_{os}$  is then calculated from Equation (III-31). The evaluation of  $\langle k_0 / G_s^R / k_0 \rangle$  is now given by Equation (II-71).

The evaluation of the other terms in (II-68) is quite obvious from (II-72) and (II-74). The effect on potential energy of the correction terms was evaluated as in the Reference Spectrum Method.

## CHAPTER IV

### DISCUSSION

We have carried out the nuclear matter calculations for three modern potentials, two of which have hard cores (Hamada-Johnston (26) and Reid (27)) and one has finite or "soft" repulsive cores (Bressel-Kerman (28)). The potentials are described in Appendix B. The results for binding and saturation are summarised in Tables IV and V. With the inclusion of three-body cluster effects, both binding and saturation are closer to the experimental values than in previous calculations. The result for the Bressel Kerman potential shows that it is feasible to hope for agreement with the experimental values of about 16 MeV per particle at  $k_F = 1.4 \text{ F}^{-1}$ .

The main surprise is the 2 MeV difference between the Reid and Hamada-Johnston potentials, and the four MeV gain of the soft core Bressel-Kerman potential over its parent, Hamada-Johnston, (Figure 6, Table V). By referring to Table IV, it can be seen that the gain in binding of Reid over Hamada-Johnston is in the  ${}^3D_2$  and the net  ${}^3P$  states. Evaluation of the phase shifts for these potentials shows that in the  ${}^3D_2$  state, Reid's potential is more

attractive, its phase shift being about 10 to 15% greater (and agreeing better with the most recent phase shift analyses). Since the average potential energy due to  ${}^3D_2$  interactions is about 15 MeV, we can expect Reid to give about 1 MeV extra binding on this account. A similar contribution from the  ${}^3P$  states accounts for the difference. The Bressel potential gain in binding is a more subtle effect. First of all, softening the core does give a gain of about 1 MeV binding in each of the S-states at a given density. The comparison of Bressel to Reid rather than Hamada-Johnston is undoubtedly the fairer one, since both Bressel and Reid fit the most recent data and we have just seen that differences of a few MeV can easily arise from fitting different data selections (i.e. giving different phase shifts). The remaining small gain in binding is due to saturation occurring at a higher density. This will be brought out more fully below.

Our work shows that at least for the hard core potentials the third order correction to the reference spectrum approximation is significant, being of order 1 MeV. This can be seen in Tables VI and VII. In the soft core Bressel potential it is much smaller and generally unimportant (Table VIII). This seems confirmed by preliminary calculations with another soft core potential. It can be seen in Tables VI and VII that calculations made from different reference spectrum parameters  $\Delta$  and  $m^*$ , if carried to third order,

agree remarkably well: within .5 MeV in all cases. This is in spite of up to 5 MeV differences in the first order result. We have also calculated the  $^1S_0$  state by the modified Moszkowski-Scott Method, obtaining similar values as shown in Table IX . This agreement gives confidence that the reference spectrum series is converging to a meaningful  $G^N$  matrix. It also indicates that only a modest degree of self consistency is required between  $U^R$  and  $U^N$ . Of course, if  $\Delta$  and  $m^*$  vary too greatly from reasonable values, the individual correction terms become large and their relative inaccuracy more dangerous. For example, in Table VII we calculated one case for the situation  $U(b) = 0$   $m^* = 1$ , with self consistency in  $\Delta$  ) which gives  $\Delta = 0.86$ . The second order corrections are huge (spectral = -23 MeV) and could easily be inaccurate by one or two MeV. The final result is -13.2 MeV binding at  $k_F = 1.36 F^{-1}$ , differing by about 2 MeV from our best value. A substantial third order  $^1S_0$  state contribution could contribute to the discrepancy.

It could be argued that once one decides to define  $U(b) = 0$  the spectral correction should not be included at all. Rather the contribution of three body cluster diagrams should be included as a perturbation. Removing the spectral correction would reduce the binding disastrously, and it is unlikely the three body cluster diagrams can account for the discrepancy. It appears that at least two or

three MeV binding are arising in our calculation from the iteration of three body clusters in higher order diagrams. This comes in when we define  $U(b)$  to be something non-zero. This point requires further study.

To complete the point we began, evaluation of the third order reference spectrum correction is quite simple, and is apparently an adequate method to get an accurate result from the tensor force.

Generally speaking, we see in Tables VI, VII and VIII a large compensation between the second order Pauli and spectral corrections, with a small net result. This is true until  $\Delta$  and or  $m^*$  is too far from rough self consistency. We have already noted that there is no such cancellation between parts of  $\Delta^{(3)}G$ . Thus from the point of view of absolute convergency it is fair to say the potential energy contributions from first second and third order are in the ratio 80 : 20 : 2, which is satisfactory.

It should be noted that decreasing  $\Delta$  makes the Pauli correction large, while increasing  $\Delta$  makes the spectral correction large. The Pauli correction is determined mainly by the magnitude of  $F_L(k')$  for  $k' < k_F$ . A small  $\Delta \approx 0.5$  makes  $\gamma^2$  small, so  $\chi^R(r)$  has a long tail. This makes  $F(k')$  large at small relative momentum. The spectral correction is sensitive to  $e^R - e^N$  in the region above but near  $k_F$ . This difference is nearly proportional to  $\Delta$ . Empirically, the net second order correction is small for  $\Delta \approx 0.6$ , which makes this the most useful value to employ for reference spectrum calculations.

Many details of the average potential energy per state are apparent in Table IV, and are more easily seen in Figures 7, 8 and 9. Over the range of densities displayed, the average kinetic energy  $T$  is roughly linear in  $k_F$ ; of course it is really quadratic. The potential energy due to  $^1S_0$  interactions is seen to be parallel to the kinetic energy ( $-T$ ) line. This shows the effect of the short range repulsion, in cutting down the  $^1S_0$  contribution from a  $k_F^3$  dependence to only  $k_F^2$ . The states with  $J > 2$ , interestingly, all have about the same dependence on  $k_F$  and vary more rapidly. Thus, the many accidental cancellations which occur, hold over a wide range of densities. The  $^3P$  states combine to have a very small net result, consistent with the usual interpretations that the P state force is primarily spin-orbit. The higher partial waves, here given by OPEP, make a substantial repulsive contribution which is highly density dependent. The  $^1P_1$  state repulsion is slightly stronger. Together these odd states roughly cancel the large attractive  $^3D_2$  contribution. The remaining attractive state,  $^1D_2$  is not shown as it lies just between the lines ( $-OPEP$ ) and ( $-^1P_1$ ). Thus, it would be in effect a good approximation to calculate only the  $^1S_0$ ,  $^1D_2$  and  $^3S_1 - ^3D_1$  states as done by Brueckner and Gammel.

Finally we note the most interesting feature, the decisive saturation effect of the tensor force in the coupled  $^3S_1 - ^3D_1$  states. It is the only contribution to  $U$  which shows definite saturation in this density range. The  $^1S_0$  state, where the force is purely central,

will saturate at a higher density comparable to the close packing of the hard cores.

A central force can act in first order, so that all states in the Fermi sea can contribute, and saturation occurs only after the average matrix element becomes small. A tensor force acts only in second order. Empirically it is a long range force, so can only scatter to intermediate states through a momentum transfer of order one inverse fermi. Thus the states which can take advantage of the tensor force are those in a shell at the top of the Fermi sea. Since the surface area of the Fermi sea increases less fast than the volume, the net contribution of the tensor force to the binding will increase less fast than that of the central force.

If only the  $^1S_0$  and  $^3S_1$  states were important, the mean potential energy  $\frac{1}{2}U$  would lie midway between the S-state curves in Figures 7 to 9. However, as we noted above, the  $^1D_2$  state can be regarded as the net result of all the higher states, and its contribution is seen to be anti-saturating. Its effect is thus to move the saturation point from about  $k_F = 1.3 F^{-1}$  (where it would be with only S-state forces) to the desired region near  $1.4 F^{-1}$ .

We can now consider the difference in binding between the Reid and Bressel potentials. Comparing Figures 7 and 9 we see that the major difference in the average potential energy curves is in the  $^3S_1 - ^3D_1$  state, where Bressel shows less tendency to saturate until a higher density. It appears in fact that the Bressel potential



has a stronger ratio of central to tensor force in this state, in agreement with the above discussion. This was checked in a rough manner by computing the phase shifts, for both potentials, using the central forces alone. Hence, the main difference is that Bressel fails to saturate until  $1.5 F^{-1}$ , due to a weaker  ${}^3S_1$  tensor force. But by postponing saturation until the higher density, it gains a small amount of binding from the other states (which one can say are in sum, the  ${}^1S_0$  and  ${}^1D_2$ ). In principle, the gain in binding this way could be greater, but it does not happen to be.

At this point one small defect in the work must be pointed out. In applying Equation (II-52) to the hard core forces, we made the error of using  $\mathcal{H}_L(r) f(r)$  in the final term, rather than  $f(c)$ . This makes the energies of excited states less attractive than they would otherwise be, and costs nearly 1 MeV of binding. Thus our hard core calculation is not strictly comparable to the soft core one where Equation (II-51) could be used directly. On the other hand, using Day's solution of the Faddeev equations will give compensating effects, slightly reducing the binding. These two corrections might well cancel, and neither is included as we hesitate to repeat all the work now.

Another point that should be pointed out here is that for the soft core potential (Bressel-Kerman) while calculating Bethe's suppression factor  $f(r)$  we assumed the particle state momentum to be about  $4F^{-1}$ . To test if this was reasonable we also calculated  $f(r)$

at  $3F^{-1}$  and  $5F^{-1}$  and found that what was to be gained in binding energy at  $3F^{-1}$  was lost at  $5F^{-1}$ . Thus the two effects were compensating and that  $4F^{-1}$  was a good value to use.

In Figure 10 we show the  $3S - 3D$  reference wave function distortion for the dominant S-channel solution.  $\chi_{20}$  vanishes inside the core, since a plane wave has no "small component". This is calculated for the average relative momentum  $k_0 = \sqrt{.3} k_F$ . The Fourier transform of  $\chi_{00}$  is small for  $k' \simeq k_F$  because of cancellation between the long range attractive and short range repulsive parts of  $\chi_{00}$ . This circumstance makes the second order correction from  $F_{00}$  small. However, it is clear that for  $\chi_{20}$  there is no cancellation, and its  $J_2$  Hankel transform is negative and quite substantial for  $k' \simeq k_F$  (Figure 1).  $\chi_{20}$  is made large by the strong tensor force. Thus, it is the tensor force which makes the second order correction terms so large in the coupled  $3S - 3D$  states. Only in this state is the third order correction appreciable. The relative smallness of the third order correction is due to geometrical effects: only at a few places in the region of integration do all factors in the integrand take on large values.

In Table X we tabulate  $G^N$  matrix elements for the Reid potential, state by state, with statistical weights included, at several sub-nuclear densities. Each line refers to a relative momentum  $k_0/k_F = .1, .2, .4, .545, .775, .9, 1.0$ . These results are of value for further work.

In Figure IV one can see the details of the potential energy and the reference spectrum. This is for the Reid potential,  $k_F = 1.36$  and  $\Delta = 0.6$ ,  $m^* = .949$ . The curve  $U$  is the potential energy of occupied states using  $G^R$ .  $U^C$  includes the second and third order corrections evaluated at several  $k_0$ . The agreement seems quite good.  $U^M$  is a linear interpolation between  $U(k_F)$  and  $U^N(2k_F)$ . We believe it is wrong to accept the computed values of  $U^N$  for  $k_b \simeq k_F$  for the following reason. The very great attraction just above the Fermi surface comes about because (i)  $G^R$  is modified by the suppression factor  $f$ , removing most of the hard core repulsion and (ii) we take only even waves into account, by Rajaraman's prescription. At least the second of these effects is suspect, since Rajaraman's argument (16) relied heavily on neglecting all hole-state momenta compared to the momentum of the state considered. This is clearly wrong until  $k_b$  is at least 2 or 3  $k_F$ . One can also say that the spectrum just above  $k_F$  will be sensitive to the long range forces, such as the tensor force. However, for these an improved treatment of the three body problem is needed, and has been developed by Dahlblom (29). One needs now two suppression factors  $f_c$  and  $f_T$ , and the calculation becomes more involved. Here, we have simply used Bethe's solution and applied a single  $f(r)$  to all parts of  $G$ . We believe that the actual spectrum will be closer to  $U^M$  than to  $U^N$  in this region. The spectral correction can be changed one or two MeV by moving  $U^M$  about, but no attempt was made to do this

to make the result look artificially better. When a better treatment of the three body problem and the energy spectrum is available it will be a simple matter to correct our result.

In conclusion, we believe the remaining uncertainty in our results lies in the physical approximations used and not in the numerical ones. We have just mentioned the necessary improvements in the treatment of three body effects and in the evaluation of the energy spectrum near the Fermi surface. The question of the energy spectrum for particle states is bound up with the convergency of the theory, and has recently been discussed very nicely by Brandow (15). These effects could certainly be a few MeV in magnitude.

Our result for the compressibility in Table V is of the same order of magnitude as in Brueckner's calculations. Remarkably, the soft core potential gives a greater value than the hard core. This is probably connected with the fact that saturation is not exclusively a property of the repulsive core. As discussed above, the Bressel potential saturates at a higher density, where the core can be playing a more important role in saturation, and this would account for the higher value of the compressibility coefficient. In any case, there is no experimental evidence against the present values which are of the magnitude required recently by Sorensen in his consideration of the isotope shift.

In Table XI we present the results for the re-arrangement energy calculated according to the relation

$$E_R = \frac{1}{A} E_{TOTAL} - E_{s.p.} (\bar{k}_m = \bar{k}_F) \quad (IV-1)$$

## CHAPTER V

### SPIN ORBIT FORCE

We have seen that for an infinite system like nuclear-matter the 'G' matrix formalism gives excellent results both for the Binding Energy and saturation density. An extension of this procedure to finite systems was initially suggested by Brueckner, Gammel and Weitzner ( 5 ) (hereafter referred to as B.G.W.). Here the main term in the effective interaction in a finite nucleus is the 'G' matrix, taken from infinite nuclear matter at that density. This procedure is commonly known as the "local density approximation". It has been followed more recently by Kuo and Brown ( 30 ) and to some extent justified by Brandow ( 31 ).

In nuclear matter, only the central force part of G is effective for binding and saturation. In a finite nucleus the spin-orbit, tensor and quadratic spin-orbit forces will also come into play. Obviously the balance of these various forces will be different in 'G' than in the original two body force. It is clearly of interest to know whether, indeed, G is made up of a "strong central-weak tensor" force, or just what the balance is.

In this section the spin-orbit force is considered in some detail, and the separation of the effective tensor interaction is only considered in a general way. Using the effective spin-orbit force, the one body spin orbit force of the shell model is evaluated in an approximation similar to that of Blin-Stoyle (32). Shell model level splittings are considered, reasonable values are found and certain problems of current interest are discussed (constancy of splitting among the isotopes of Ca, questions of the 'shape' of the one-body spin-orbit force of the shell and optical models).

In view of the satisfactory agreement with experiment and with other calculations, these techniques should be applied to the separation of the tensor and other parts of the two body force. Further, the question of the applicability of the Moszkowski-Scott (9) method to the spin-orbit force is considered.

Consider the G in its most general form:

$$\langle \hat{k}' | G | \hat{k} \rangle = a + ic (\underline{\sigma}_1 + \underline{\sigma}_2) \cdot \underline{N} + m (\underline{\sigma}_1 \cdot \underline{N} \underline{\sigma}_2 \cdot \underline{N}) + (g+h) (\underline{\sigma}_1 \cdot \underline{P} \underline{\sigma}_2 \cdot \underline{P}) \\ + (g-h) (\underline{\sigma}_1 \cdot \underline{K} \underline{\sigma}_2 \cdot \underline{K})$$

where  $\underline{N}$  and  $\underline{\sigma}$  are axial vectors,  $\underline{K}$  and  $\underline{P}$  are polar vectors:

$$\underline{N} = \frac{(\hat{k} \times \hat{k}')}{|\hat{k} \times \hat{k}'|}$$

$$\underline{P} = \hat{k}' + \hat{k}$$

$$\underline{K} = \hat{k}' - \hat{k}$$
(V-2)

where  $\underline{k}$ ,  $\underline{k}'$  are unit vectors in the incident and outgoing directions. For the spin-orbit force we are interested only in the coefficient  $c$  which is given, by geometrical arguments, to be

$$\begin{aligned} c &= \frac{1}{\sqrt{8}} \left( e^{i\phi} G_{10} - e^{-i\phi} G_{01} \right) \\ &= -\frac{1}{\sqrt{8}} \left( G_{-10} + G_{01} \right) e^{-i\phi} \end{aligned} \quad (V-3)$$

Details for other coefficients are given in Appendix C. Thus the evaluation of  $c$  requires calculation of matrix elements of  $G$  off diagonal in  $M$  (the magnetic quantum number). B.B.P. (10) have given the procedure for the diagonal matrix elements; the off-diagonal is a simple extension of that.

The free-particle wavefunctions for state  $S = 1$ ,  $T = 0$  are given by (B.B.P., Equation 6.2)

$$\phi_{1,0}^M(\underline{k}_0, \underline{x}) = \sqrt{2} (\underline{k}_0, \underline{x})^{-1} \sum_{\text{even } L} \left[ 4\pi(2L+1) \right]^{1/2} i^L g_L(\underline{k}_0, \underline{x}) Y_L^0 X_{S=1}^M \lambda_{T=0}^0 \quad (a)$$

$$= (8\pi)^{1/2} (\underline{k}_0, \underline{x})^{-1} \sum_{\text{even } L} \sum_J i^L (2L+1)^{1/2} C(L10M|JM) \quad (V-4)$$

$$g_L(\underline{k}_0, \underline{x}) | Y_{L,J}^M \rangle \lambda_0^0 \quad (b)$$

The nuclear matter "reference" wavefunctions are (B.B.P., Equation 6.3)

$$\psi_{1,0}^{M'} = (8\pi)^{1/2} (\underline{k}'_0, \underline{x})^{-1} \sum_{\text{even } L} \sum_J i^L (2L+1)^{1/2} C(L10M'|JM')$$

$$u_{L,J}^{M'}(\underline{k}'_0, \underline{x}) | Y_{L,J}^{M'} \rangle \lambda_0^0 \quad (V-5)$$

Then using (V-4a, 5)

$$\begin{aligned} \langle \phi_{10}^M | G^R | \phi_{10}^{M'} \rangle &= \langle \phi_{10}^M | v | \psi_{10}^{M'} \rangle \\ &= \frac{8\pi}{k_0 k_0'} \int \frac{1}{k^2} \sum_{\text{even } L, L'} \sum_J (-i)^L (i)^{L'} (2L+1)^{1/2} (2L'+1)^{1/2} \\ &\quad \cdot g_L(k_0, \epsilon) \chi_1^{M*} u_{L'J'}^{M'} y_L^0(k_0, \epsilon) v(\epsilon) | y_{L'J'}^{M'} \rangle c(L'10M' | J'M') d\epsilon \end{aligned} \quad (V-6)$$

Like B.B.P. Equations 6.8 and 6.10 we define

$$f_{LJ}^M = i^L (2L+1)^{1/2} c(L10M | JM) \quad (V-7)$$

and

$$f_{L'J'}^{M'} u_{L'J'}^{M'} = \sum_{L''} f_{L''J'}^{M'} u_{L'L''}^{J'}$$

Then

$$\begin{aligned} \langle \phi_{10}^M | G^R | \phi_{10}^{M'} \rangle &= \frac{8\pi}{k_0 k_0'} \int \frac{1}{k^2} \sum_{\text{even } L, L', L''} \sum_J (-i)^L (2L+1)^{1/2} g_L(k_0, \epsilon) \\ &\quad \chi_1^{M*} y_L^0 f_{L''J'}^{M'} u_{L'L''}^{J'} v(\epsilon) | y_{L'J'}^{M'} \rangle d\epsilon \end{aligned} \quad (V-8)$$

Using Equation (17) B.G.W.

$$\int d\Omega y_L^0(k_0, \epsilon) v(\epsilon) | y_{L'J'}^{M'}(k_0, \epsilon) \rangle = \sqrt{\frac{4\pi}{2L+1}} u_{LL'}^J y_{JL}^{M'}(\hat{k}_0, \hat{k}') \quad (V-9)$$



where 
$$v_{LL'}^J = \int d\Omega y_{JL}^{m'*} v y_{JL}^M \delta_{mm'} \quad (V-10)$$

$$\langle \phi_{10}^M | G^R | \phi_{10}^{M'} \rangle = \frac{16\pi^{3/2}}{k_0 k'_0} \int \sum_{\text{even } L, L', L''} \sum_J (-i)^L g_L(k_0, \tau) \chi_1^{M*} \int_{L''}^{M'} u_{L'L''}^J u_{LL'}^J y_{JL}^{M'}(k, k') d\tau \quad (V-11)$$

Using

$$\chi_1^{M*} y_{JL}^{M'}(k, k') = Y_L^{M'-M} C(L, M'-M, M | J, M') \quad (V-12)$$

and Equation (V-7) we finally obtain, using the reference G matrix  $G^R$  for G

$$\begin{aligned} G_{MM'}^R &\equiv \langle \phi_{10}^M(k_0) | G^R | \phi_{10}^{M'}(k'_0) \rangle \\ &= \frac{2(4\pi)^{3/2}}{k_0 k'_0} \sum_{\text{even } L, L', L''} \sum_J (i)^{L''-L} (2L''+1)^{1/2} \\ &\quad \cdot C(L'' 10 M' | J M') C(L 1 M'-M M | J M') \\ &\quad \cdot \int g_L(k_0, \tau) u_{L'L''}^J(k'_0, \tau) u_{LL'}^J d\tau Y_L^{M'-M} \end{aligned} \quad (V-13)$$

As a first use of this equation we apply it to the case  $M = M'$ , and sum over  $M$ . Then putting  $k_0 = k_0'$  and using the Clebsch-Gordan identity

$$\sum_M C(L S O M | J M) C(L'' S O M | J M) = \delta_{LL''} \frac{2J+1}{2L+1} \quad (V-14)$$

we obtain

$$\langle \phi_{10} | G^R | \phi_{10} \rangle = \frac{8\pi}{k_0^2} \sum_{\text{even } L, L'} \sum_J (2J+1) \int_0^\infty g_L u_{LL'} u_{L'L}^J dx \quad (V-15)$$

which agrees with the corresponding quantity in Equation (6.14a) of B.B.P.

The G-matrix elements given by Equation (V-13) are in the k-space. It is sometimes useful and necessary to go from the k-space to the r-space or vice-versa. It is convenient to know how this affects a given partial wave, where the Fourier Transform induces a radial Hankel transform. The rules are summarised below:

(1) To go from k space to r space:

$$\text{Multiply the matrix elements by } \frac{(4\pi)^2}{(2\pi^3)^2} \frac{kk'}{rr'} = \frac{1}{4\pi^4} \frac{kk'}{rr'}$$

Then take a double Hankel Fourier transform with respect to  $g_L(kr)$  and  $g_L(k'r')$  integrating over  $kk'$ . The result is the  $(rr')$  matrix element of G.

(2) To go from r space to k space.

$$\text{Multiply the coefficient by } \frac{(4\pi)^2 rr'}{kk'}$$

Then take a double Hankel Fourier transform with respect to  $J_L(kr)$  and  $J_L(k'r')$  integrating over  $rr'$ . The result is the  $(kk')$  matrix element of  $G$ .

While doing these transformations the following identities are useful:

$$\int_0^{\infty} J_L(kr) J_L(k'r) dr = \frac{\pi}{2} \delta(k-k') \quad (a)$$

$$\int_0^{\infty} J_L(kr) J_L(k'r) dk = \frac{\pi}{2} \delta(r-r') \quad (b) \quad (V-17)$$

Thus Equation (V-13) and the procedure for going from  $k$  space to  $r$  space and vice-versa we know all the matrix elements of  $G$  in either space.

Coming back now to the evaluation of 'c' given by Equation (V-3) we obtain by using Equation (V-13) in it

$$c = -\frac{1}{\sqrt{8}} (G_{-10} + G_{01}) e^{-i\phi} \quad (V-18)$$

$$= -\frac{(4\pi)^{3/2}}{k_0 k_0'} \sum_{\substack{\text{even} \\ L, L', L''}} \sum_J ASUM \int J_L(k_0 r) U_{LL'}^J U_{L'L''}^J(k_0' r) dr e^{-i\phi} Y_L'(\cos\theta)$$

where

$$ASUM = \frac{(i)^{L''-L} (2L''+1)^{1/2}}{\sqrt{2}} \left\{ c(L||-1|J_0) c(L''100|J_0) \right. \\ \left. + c(L||0|J_1) c(L''101|J_1) \right\} \quad (V-19)$$

Thus the second term in Equation (V-1) becomes

$$\begin{aligned}
& i (\underline{\sigma}_1 + \underline{\sigma}_2) \cdot \frac{(\underline{k} \times \underline{k}')}{kk' \sin(\hat{k}\hat{k}')} \cdot c \\
&= -\frac{i}{kk'} \frac{(\underline{\sigma}_1 + \underline{\sigma}_2)}{\sin(\theta)} (\underline{k} \times \underline{k}') \frac{(4\pi)^{3/2}}{k_0 k'_0} \sum_{\text{even } L, L''} \sum_J \text{ASUM} \\
&\quad \cdot \int g_L(k_0 r) U_{LL'}^J U_{L'L''}^J(k'_0 r) dr e^{-i\phi} Y_L'(\cos\theta) \quad (V-20)
\end{aligned}$$

Using the relation

$$\begin{aligned}
& e^{-i\phi} Y_L'(\cos\theta) \operatorname{cosec}\theta = -\{L(L+1)\}^{1/2} \frac{d}{d(\cos\theta)} Y_L^0(\cos\theta) \\
& i (\underline{\sigma}_1 + \underline{\sigma}_2) \cdot \underline{N} \cdot c = i (\underline{\sigma}_1 + \underline{\sigma}_2) \left( \frac{\underline{k}}{k} \times \frac{\underline{k}'}{k'} \right) \frac{(4\pi)^{3/2}}{k_0 k'_0} \\
& \quad \cdot \sum_{\text{even } L, L''} \sum_J \text{BSUM} \int g_L(k_0 r) U_{LL'}^J U_{L'L''}^J(k'_0 r) dr \frac{d}{d(\cos\theta)} Y_L^0(\cos\theta) \quad (V-21)
\end{aligned}$$

$$\text{where} \quad \text{BSUM} = \text{ASUM} \{L(L+1)\}^{1/2} \quad (V-22)$$

Now we have the identity

$$\begin{aligned}
& i (\underline{\sigma}_1 + \underline{\sigma}_2) \left( \frac{\underline{k}}{k} \times \frac{\underline{k}'}{k'} \right) \frac{d}{d(\cos\theta)} Y_L^0(\cos\theta) \\
&= \frac{1}{i} (\underline{\sigma}_1 + \underline{\sigma}_2) \underline{k} \times \nabla_{\underline{k}} Y_L^0 \\
&= 2 \underline{L} \cdot \underline{S} Y_L^0 \quad (V-23)
\end{aligned}$$

where  $L = \underline{r} \times \underline{k}$  and  $r = i \nabla_k$ .

The identity can easily be seen if we take the z-axis along the  $k'$  direction. Putting this result in Equation (V-21) we get  $B(k_0, k_0')$  as the coefficient of the L.S in the G-matrix as

$$B(k_0, k_0') = \frac{2(4\pi)^{3/2}}{k_0 k_0'} \sum_{\text{Even } L, L', L''} \sum_J \text{BSUM} \int g_L(k_0 x) v_{LL'}^J(k_0' x) dx Y_L^0(k_0') \quad (\text{V-24})$$

$$= \frac{4\pi}{k_0 k_0'} \sum_{\substack{\text{even} \\ L, L', L''}} \sum_J \text{CSUM} \int g_L(k_0 x) v_{LL'}^J v_{L'L''}^J(k_0' x) dx P_L(\cos \theta) \quad (\text{V-25})$$

where  $\text{CSUM} = 2 \cdot \text{BSUM} \cdot \sqrt{2L+1}$

$$= (i)^{L''-L} \sqrt{\frac{2(2L''+1)(2L+1)}{L(L+1)}} \left\{ \begin{array}{l} C(L'' 11 -1 | J_0) C(L'' 10 0 | J_0) \\ + C(L'' 0 1 | J_1) C(L'' 10 1 | J_1) \end{array} \right\} \quad (\text{V-26})$$

The above expression (Equation V-25) has been derived only for the  $T = 0$  case. For  $T = 1$  an exactly similar expression will be obtained, but the summation will be over odd  $L$ . The procedure adopted above is very similar to that of B.G.W. (5). They worked in  $r$ -space instead of  $k$  space as we are doing.

In order to evaluate  $B(k_0, k_0')$  we require the following Clebsch-Gordan coefficients:

$$\begin{aligned}
 C(L''-1|J_0) &= \begin{cases} \sqrt{\frac{L}{2(2L+1)}} & J=L+1 \\ \sqrt{\frac{1}{2}} & J=L \\ \sqrt{\frac{L+1}{2(2L+1)}} & J=L-1 \end{cases} & C(L''100|J_0) &= \begin{cases} \sqrt{\frac{L''+1}{2L''+1}} & J=L''+1 \\ 0 & J=L'' \\ -\sqrt{\frac{L''}{2L''+1}} & J=L''-1 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 C(L110|J1) &= \begin{cases} \sqrt{\frac{L(L+2)}{(2L+1)(L+1)}} & J=L+1 \\ \frac{1}{\sqrt{L(L+1)}} & J=L \\ -\sqrt{\frac{(L-1)(L+1)}{L(2L+1)}} & J=L-1 \end{cases} & C(L''101|J1) &= \begin{cases} \sqrt{\frac{L''+2}{2(2L''+1)}} & J=L''+1 \\ -\sqrt{\frac{1}{2}} & J=L'' \\ \sqrt{\frac{L''-1}{2(2L''+1)}} & J=L''-1 \end{cases}
 \end{aligned}$$

(V-27)

It is now a very simple matter to deduce that in Equation (V-26)

(i) if  $L \neq L''$  CSUM = 0

(ii) if  $L = L''$  CSUM =

$$\left\{ \begin{array}{ll} \frac{2L+3}{L+1} & L = J-1 \\ -\frac{2L+1}{L(L+1)} & L = J \\ -\frac{2L-1}{L} & L = J+1 \end{array} \right. \quad \text{(V-28)}$$

Thus, knowing the L.S coefficient for the  $T = 1, 0$  cases the total coefficient is given by

$$B(k_0, k'_0) = \frac{3}{4} B(k_0, k'_0) \Big|_{T=1} + \frac{1}{4} B(k_0, k'_0) \Big|_{T=0} + \frac{N-Z}{4A} \left( B(k_0, k'_0) \Big|_{T=1} - B(k_0, k'_0) \Big|_{T=0} \right) \quad (V-29)$$

which for a  $N = Z$  nucleus reduces to the usual  $\frac{3}{4}$  ( $T = 1$  contribution) +  $\frac{1}{4}$  ( $T = 0$  contribution).

The above expression for the coefficient of the spin-orbit operator has been obtained in the  $k$  space. For a practical application we must know its form in the  $r$ -space. The procedure for such a transformation has been given on page 61. Before we do this we must note that for physically interesting cases the coefficient of the operator is diagonal in the  $k$ -space, i.e.,  $k_0 = k'_0$ , or else energy will not be conserved. Let  $B(r, r')$  be the  $r$  space representation of the L.S coefficient.

There are two ways in which we can proceed to obtain  $B(r, r')$  from  $B(k_0, k_0)$ . Both procedures should at least in principle lead to the same result, though in actual practice it is more convenient and accurate to use one rather than the other. For physical insights the opposite is the case. We, therefore, mention both.

We know from our reference spectrum calculations

$$\sum_{\text{even } L'} \int g_L(k_0 r) U_{LL'}^J U_{L'L''}^J(k_0 r) dr = \frac{\gamma^2 + k_0^2}{m^*} \int g_L(k_0 r) \chi_{LL''}^J(k_0 r) dr \quad (\text{V-30})$$

Using this relation in (V-25)

$$B(k_0 k_0) = \frac{4\pi(\gamma^2 + k_0^2)}{m^* k_0^2} \sum_{\text{even } L, L''} \sum_J \text{CSUM} \int g_L(k_0 r) \chi_{LL''}^J(k_0 r) dr P_L(\cos\theta) \quad (\text{V-31})$$

with a similar expression for odd L. Now (see Equation (II-21))

$$\frac{4\pi(\gamma^2 + k_0^2)}{m^* k_0^2} \int g_L(k_0 r) \chi_{LL''}^J(k_0 r) dr$$

are simply the diagonal matrix elements in k space of  $G^R$  and can simply be found by the Bessel Fourier Transform of the wave defect obtained in the Nuclear Matter calculations. Thus we see that a suitable combination of the G-matrix elements, the combination being provided by CSUM (Equation V-26), we can very easily obtain the coefficient of the spin-orbit operator in k-space. To obtain its r-space representation  $B(r, r')$  it is then just a matter of two more Bessel Fourier Transforms according to the prescription on page 61.



When this procedure is adopted it is a trivial matter to see that

$$B(k, k') = B(k', k) \quad (V-32)$$

i.e., the coefficient of the spin-orbit operator is symmetrical in r-space (see Equation V-35).

Explicitly we can write for an  $N = Z$  nucleus after taking

$$\left\{ \frac{3}{4} (T=1) + \frac{1}{4} (T=0) \right\} \text{ of } B(k_0, k_0) \Big|_{T=0 \text{ or } T=1}$$

$$B(k_0, k_0) = \frac{4\pi (\gamma^2 + k_0^2)}{m^* k_0^2} \left[ \left( -G_{11}^0 - \frac{3}{2} G_{11}^1 + \frac{5}{2} G_{11}^2 \right) P_1(\cos \theta) \right. \\ \left. + \left( -\frac{3}{2} G_{21}^1 - \frac{5}{6} G_{21}^2 + \frac{7}{3} G_{21}^3 \right) P_2(\cos \theta) + \dots \right] \quad (V-33)$$

where

$$G_{LS}^J(k_0, k_0) = \int_0^\infty g_L(k_0, r) \chi_{LS}^J(k_0, r) dr \quad (V-34)$$

Looking at Equation (V-33) it is immediately seen that there is no hard core contribution to  $B(k_0, k_0)$  since  $\chi_{LS}^J \Big|_{r < c} = g_L(k_0, r) \Big|_{r < c}$  which is independent of  $J$ .

The r space representation for " $G_{LS}^J$ " is given by

$$G_{LS}^J(k, k') = \frac{1}{4\pi^4} \frac{1}{kk'} \iint g_L(k, r) G_{LS}^J(k, k') g_L(k', r) k k' dr \quad (V-35)$$

Another point that can be easily seen in this procedure is the absence of the second order correction to the G's. In fact all G matrix elements which are off diagonal in M have no second order correction. The proof goes as follows: The second order correction is given by (See Equation (II-27)):

$$G_2^R = \langle \phi_{ST}^M | G^{R\dagger} \left( \frac{1}{e^R} - \frac{Q}{e^N} \right) G^R | \phi_{ST}^{M'} \rangle \quad (V-36)$$

$$= \int_0^\infty \mathcal{E}(k'') \mathcal{F}_{ST}^{MM'} dk'' \quad (V-37)$$

where

$$\mathcal{E}(k'') = \langle k'' m'' | e^R \left( \frac{1}{e^R} - \frac{Q}{e^N} \right) e^R | k'' m'' \rangle \quad (V-38)$$

$$\mathcal{F}_{ST}^{MM'} = \sum_{m''} \frac{1}{(2\pi)^3} \int d^2 \hat{k}'' k''^2 \langle \phi_{ST}^M | (1 - \Omega^R)^\dagger | k'' m'' \rangle \cdot \langle k'' m'' | (1 - \Omega^R) | \phi_{ST}^{M'} \rangle \quad (V-39)$$

Considering, as a special case, the triplet even states we can write from Equations (II-41) and (II-42)

$$\begin{aligned}
 \mathbb{F}_{ST}^{MM'} &= \sum_{M''} \frac{32\pi^2}{(2\pi)^3} \int \frac{d^2 \hat{k}'' k''^2}{k_0^2} \sum_{\substack{\text{even } L', L'', L''', \\ L'', \text{ all } J, J'}} f_{L'' J}^{M''*} f_{L' J}^{M'} f_{L'' J}^{M''} f_{L''' J'}^{M*} \\
 &\cdot \int_0^\infty j_L(k'' z) \chi_{L' L'}^J(k_0 z) z dz \int_0^\infty j_{L'''}(k'' z) \chi_{L'' L'''}^J(k_0 z) z dz \quad (\text{V-40}) \\
 &\cdot \mathcal{D}_J^{M'' M'}(k_0 \rightarrow k'') \mathcal{D}_{J'}^{M' M''}(k'' \rightarrow k_0)
 \end{aligned}$$

Now

$$\begin{aligned}
 &\int d^2 \hat{k}'' \mathcal{D}_J^{M'' M'}(k_0 \rightarrow k'') \mathcal{D}_{J'}^{M' M''}(k'' \rightarrow k_0) \\
 &= \int d^2 \hat{k}'' \left( \mathcal{D}_J^{M' M''}(k'' \rightarrow k_0) \right)^* \mathcal{D}_{J'}^{M'' M'}(k'' \rightarrow k_0) \quad (\text{V-41}) \\
 &= \frac{4\pi}{2J+1} \delta_{JJ'} \delta_{M' M}
 \end{aligned}$$

Thus, since our G-matrix has only off-diagonal (in  $MM'$ ) elements, there is no second order correction. An exactly similar proof goes for the Triplet odd states.

The procedure outlined here to obtain the r-space representation for G is very simple. Unfortunately it is not a trivial matter to achieve accuracy. Firstly to obtain the final result we have to perform three Bessel Fourier transforms which all have to be done on a computer. Two of these transforms are over  $k$ , which requires knowledge of G for large  $k$ 's. The final accuracy is, therefore, poor. Our initial attempt to obtain  $\mathcal{B}(r, r')$  was by this method, but we had to

abandon it in favour of the procedure given below, for reasons mentioned and also because the  $k_0 > k_F$  contribution was masking the  $k_0 < k_F$  contribution which, of course, should be the more important one.

The following procedure is less direct and the physical picture is not clear; but it is more accurate since the total number of integrations are reduced to a bare minimum.

In Equation (V-25) the only quantity that changes over when  $B(k_0, k_0') \rightarrow B(r, r')$  is the integral

$$B \text{ INTG}(k_0, k_0') = \frac{1}{k_0 k_0'} \int g_L(k_0, x'') v_{LL'}^J(x'') u_{L'L''}^J(k_0, x'') dx'' \quad (\text{V-42})$$

Consider its Bessel Fourier transform according to the prescription on page 61. Then

$$\begin{aligned} B \text{ INTG}(r, r') &= \frac{1}{4\pi^4} \iint \frac{k_0 k_0'}{k_0 k_0'} \cdot \frac{1}{r r'} g_L(k_0, x) \int g_L(k_0, x'') v_{LL'}^J(x'') u_{L'L''}^J(k_0, x'') dx'' \\ &\quad \cdot g_L(k_0', x') dk_0 dk_0' \\ &= \frac{1}{4\pi^4} \iint \frac{1}{r r'} \cdot \frac{\pi}{2} \delta(x - x'') v_{LL'}^J(x'') u_{L'L''}^J(k_0', x'') g_L(k_0', x') dx dk_0' \\ &\quad \cdot g_L(k_0', x') \quad (\text{V-43}) \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{r r'} v_{LL'}^J(x) u_{L'L''}^J(k_0', x) g_L(k_0', x') dk_0' \end{aligned}$$

Thus

$$B(r, r') = \frac{4\pi}{(2\pi)^3} \cdot \frac{1}{r r'} \sum_{\substack{\text{even } LL'' \\ \text{all } J}} \text{CSUM} \int g_L(k_0', x') v_{LL'}^J(x) u_{L'L''}^J(k_0', x) dk_0' \quad (\text{V-44})$$

$\cdot P_L(\cos \theta)$

This then is the  $r$  space representation of the spin-orbit coefficient and differs from Equation (32) of B.G.W. by a factor of 2 because of our inclusion of the exchange term. They derived this equation more directly in the  $r$ -space itself.

We thus see that to obtain the  $r$ -space  $B(r, r')$  the number of integrals have been reduced from three to one. This will certainly improve accuracy. For a practical application of  $B(r, r')$  it is better to delay the integration until the end.

We now wish to consider the spin-orbit splittings of the shell model potential. As is well known a two body spin orbit force will lead directly to a one body splitting, while the tensor force will do so only in second order. According to B.G.W. the second order effect is negligible; here we consider only L.S because we want to see if it alone is adequate. We follow Blin-Stoyle (32) and others by considering a "closed shell  $\pm 1$  nucleon" nucleus, and ask for the one body force due to the interaction of the odd nucleon with the core. This procedure is followed because it will lead into the "usual"  $\frac{1}{r} \frac{d\rho}{dr}$  form of the L.S force, which will facilitate comparisons with the optical and shell model wells used by other authors. The novel feature here is that we take into account the nonlocality of the two-body <sup>force</sup> (G-matrix). Certain approximations have to be made to lead to a simple result. These are pointed out.

We, therefore, have to consider the expectation of the spin-orbit term

$$B(x, x') L \cdot S = \frac{1}{2} (\underline{\sigma}_1 + \underline{\sigma}_2) \cdot x_{12} \times p_{12} B(x_{12}, x'_{12}) \quad (V-45)$$

where

$$\underline{x} \equiv \underline{x}_{12} = \underline{x}_1 - \underline{x}_2 \quad \underline{x}' \equiv \underline{x}'_{12} = \underline{x}'_1 - \underline{x}'_2$$

(V-46)

In doing so we closely follow B.G.W.

Suppose that we are considering particle "1" which has a specified spin state. Then the expectation value of  $\sigma_2$  summed over the spin states of particle 2 will be taken to be zero if the states are all populated equally with spin up and down. In actual practice, however, this is not quite correct, since a single particle spin-orbit interaction splits two states with the same orbital motion, but opposing spins. This effect is of second order in the spin orbit potential strength and we shall ignore it since the spin-orbit potential is weak relative to the central potential. We shall also neglect any possible lack of spin-pairing in unfilled shells. Consequently our results will hold at and near closed shells.

To evaluate the expectation value of  $\underline{r}_{12} \times \underline{p}_{12}$ , we write  $\underline{L}_{12}$  as

$$\underline{x}_{12} \times \underline{p}_{12} = \frac{1}{2} (\underline{x}_1 \times \underline{p}_1 + \underline{x}_2 \times \underline{p}_2 - \underline{x}_1 \times \underline{p}_2 - \underline{x}_2 \times \underline{p}_1) \quad (V-47)$$

Since  $P = 0$

$$\underline{p}_2 \cdot B(\underline{x}_{12}, \underline{x}'_{12}) = -\underline{p}_1 \cdot B(\underline{x}_{12}, \underline{x}'_{12}) \quad (V-48)$$

$$\therefore \underline{x}_{12} \times \underline{p}_{12} = (\underline{x}_1 \times \underline{p}_1 - \underline{x}_2 \times \underline{p}_1) \quad (V-49)$$

Consider the expectation value of the first term. This is

$$\sum_j C_j \int \Psi_j^*(\underline{r}_2) \left[ \underline{r}_1 \times \underline{p}_1 B(\underline{r}_{12}, \underline{r}'_{12}) \right] \Psi_j(\underline{r}'_2) d\underline{r}_2 d\underline{r}'_2 \quad (\text{V-50})$$

where  $C_j$  are statistical weight factors for the states of particle "2" and  $\Psi_j$  its initial and final wavefunction. The operator  $\underline{p}_1$  of course operates only on  $B(r_{12}, r'_{12})$ . Because the centre of mass is fixed, i.e.,  $r_1 + r_2 = r'_1 + r'_2$ , we make use of the delta function on it which we have thus far ignored. Then we evaluate the integral over  $r'_2$  to obtain

$$\sum_j C_j \int \Psi_j^*(\underline{r}_2) \left[ \underline{r}_1 \times \underline{p}_1 B(\underline{r}_{12}, \underline{r}_{12} - 2(\underline{r}_1 - \underline{r}'_1)) \right] \Psi_j(\underline{r}_2 + \underline{r}_1 - \underline{r}'_1) d\underline{r}_2 \quad (\text{V-51})$$

If  $B$  were a local function, it would have a delta function on  $r_1 - r'_1$ . Then  $r_1 - r'_1$  would not occur in  $\Psi_j(r_2 + r_1 - r'_1)$  and we could remove  $r_1 \times p_1$  from the integral. But we know that the non-locality in  $B$  is of very short range and that we can imagine that  $\Psi_j$  is slowly varying relative to  $B$ . This will allow us to take  $r_1 \times p_1$  from the integral.

Now consider the expectation value of the second term.

This is

$$\sum_j C_j \int \Psi_j^*(\underline{r}_2) \left[ \underline{r}_2 \times \underline{p}_1 B(\underline{r}_{12}, \underline{r}'_{12}) \right] \Psi_j(\underline{r}'_2) d\underline{r}_2 d\underline{r}'_2 \quad (\text{V-52})$$

Since we are integrating over  $\underline{r}_2$  and  $\underline{r}_2'$  the result of this integration must be proportional to a vector constructed from  $\underline{r}_1$  and  $\underline{r}_1'$ . If the nonlocality in  $B$  is of very short range, it is accurate enough to take this vector to be  $\underline{r}_1$ . This approximation is similar to that already made in removing  $\underline{p}_1$  from the integral of Equation (V-51). We, therefore, retain only the component of  $\underline{r}_2$  along  $\underline{r}_1$ , i.e., we make the replacement

$$\underline{r}_2 \rightarrow (\underline{r}_1 \cdot \underline{r}_2) \underline{r}_1 / r_1^2 \quad (\text{V-53})$$

Then in the expectation value of the spin orbit term we can make the replacement

$$\langle \underline{r}_{12} \times \underline{p}_{12} \rangle \rightarrow \underline{r}_1 \times \underline{p}_1 \left\langle 1 - \frac{\underline{r}_1 \cdot \underline{r}_2}{r_1^2} \right\rangle \quad (\text{V-54})$$

Then writing

$$\frac{1}{2} \sigma_1 \underline{r}_1 \times \underline{p}_1 = \underline{L}_1 \cdot \underline{S}_1 \quad (\text{V-55})$$

and combining Equations (V-45), (V-49), (V-51), (V-52) and (V-54) we obtain finally the single particle spin-orbit potential as the coefficient of  $\underline{L}_1 \cdot \underline{S}_1$ :

$$\begin{aligned} \langle \underline{r}_1 | V^{(LS)} | \underline{r}_1' \rangle = & \sum_j C_j \int d\underline{r}_2 d\underline{r}_2' \Psi_j^*(\underline{r}_2) B(\underline{r}_{12}, \underline{r}_{12}') \\ & \cdot \left( 1 - \frac{\underline{r}_1 \cdot \underline{r}_2}{r_1^2} \right) \Psi_j(\underline{r}_2') \end{aligned} \quad (\text{V-56})$$



Recalling Equation (V-46) this can also be written as

$$(\underline{x}_1 | V^{(LS)} | \underline{x}_1') = \int d\underline{x}_2 d\underline{x}_2' \rho(\underline{x}_2, \underline{x}_2') B(\underline{x}_2, \underline{x}_2') \left( \frac{\underline{x}_1 \cdot \underline{x}_1'}{x_1^2} \right) \quad (V-57)$$

where we have introduced the density correlation function

$$\rho(\underline{x}_2, \underline{x}_2') = \sum_j c_j \psi_j^*(\underline{x}_2) \psi_j(\underline{x}_2') \quad (V-58)$$

We now assume that the nonlocality is of short range so that we can make a Taylor expansion of  $\rho(\underline{x}_2, \underline{x}_2')$  about  $\underline{x}_2$  and retain only the first two terms of the expansion. Thus using

$$\rho(\underline{x}_2 + \underline{h}) = e^{\underline{h} \cdot \nabla} \rho(\underline{x}_2) \quad (V-59)$$

we can write

$$\begin{aligned} \rho(\underline{x}_2, \underline{x}_2') &= e^{\frac{1}{2}(\underline{x}_2' - \underline{x}_2) \cdot \nabla} \rho(\underline{x}_2, \underline{x}_2') \\ &= \rho(\underline{x}_2, \underline{x}_2) + \frac{1}{2}(\underline{x}_2' - \underline{x}_2) \cdot \nabla_{\underline{x}_2} \rho(\underline{x}_2, \underline{x}_2) + \dots \end{aligned} \quad (V-60)$$

The extra factor of '1/2' that we have put in compensates for allowing the operator  $\nabla$  to act on both the coordinates  $\underline{x}_2$ . We now make

a second Taylor expansion between  $r_2$  and  $r_1$ :

$$\begin{aligned} \rho(\underline{x}_2) &= \rho(\underline{x}_1) + (\underline{x}_2 - \underline{x}_1) \cdot \nabla_{\underline{x}_1} \rho(\underline{x}_1) \\ &= \rho(\underline{x}_1) - \underline{x} \nabla_{\underline{x}_1} \rho(\underline{x}_1) + \dots \end{aligned} \quad (\text{V-61})$$

and

$$\begin{aligned} \nabla_{\underline{x}_2} \rho(\underline{x}_2) &= \nabla_{\underline{x}_2} \left[ \rho(\underline{x}_1) + (\underline{x}_2 - \underline{x}_1) \nabla_{\underline{x}_1} \rho(\underline{x}_1) + \dots \right] \\ &= \nabla_{\underline{x}_1} \rho(\underline{x}_1) \end{aligned} \quad (\text{V-62})$$

Substituting these expansions in (V-57)

$$\begin{aligned} (\underline{x}_1 | V^{(LS)} | \underline{x}'_1) &= \int d\underline{x} d\underline{x}' B(\underline{x}, \underline{x}') \left( \frac{\underline{x}_1 \cdot \underline{x}}{x_1^2} \right) \left\{ \rho(\underline{x}_1) - \underline{x} \nabla_{\underline{x}_1} \rho(\underline{x}_1) \right. \\ &\quad \left. + \frac{1}{2} (\underline{x}'_2 - \underline{x}_2) \nabla_{\underline{x}_1} \rho(\underline{x}_1) \right. \\ &= \int d\underline{x} d\underline{x}' B(\underline{x}, \underline{x}') \left( \frac{\underline{x}_1 \cdot \underline{x}}{x_1^2} \right) \left\{ \rho(\underline{x}_1) - \underline{x} \nabla_{\underline{x}_1} \rho(\underline{x}_1) + \frac{1}{2} \left( (\underline{x} - \underline{x}') \right. \right. \\ &\quad \left. \left. - (\underline{x}_1 - \underline{x}'_1) \right) \nabla_{\underline{x}_1} \rho(\underline{x}_1) \right\} \\ &= \int d\underline{x} d\underline{x}' B(\underline{x}, \underline{x}') \frac{\underline{x}_1 \cdot \underline{x}}{x_1^2} \left\{ \rho(\underline{x}_1) - \frac{1}{2} \underline{x} \nabla_{\underline{x}_1} \rho(\underline{x}_1) - \frac{1}{2} \underline{x}' \nabla_{\underline{x}_1} \rho(\underline{x}_1) \right. \\ &\quad \left. - \frac{1}{2} (\underline{x}_1 - \underline{x}'_1) \cdot \nabla_{\underline{x}_1} \rho(\underline{x}_1) \right\} \end{aligned}$$

Let us now consider a multipole expansion of  $B(r, r')$ . This is given in Equation (V-44). Let us write this equation in the form

$$B(r, r') = \sum_L B_L(r, r') \frac{2L+1}{4\pi} P_L(\cos \theta) \quad (V-64)$$

Within the limits of our nuclear matter calculations  $B_L(r, r')$  is known only for  $L = 1, T = 1$  and  $L = 2, T = 0$  states; we took all other states to be OPEP. Thus there is no experimental information which denies the assumption that all the multipoles  $B_L$  are equal; this assumption leads to a function  $B(r, r')$  which is a local operator in the angle variable. An alternative but no more reasonable assumption would make all the other multipoles zero; this would give maximum non-locality and have angular dependence  $P_1(\Omega)$  in place of  $\delta(\Omega - \Omega')$ . (Note: The  $L = 0$  multipole is irrelevant since it comes with the operator  $L, S$ ). Thus with the above assumptions we can write:

$$B(r, r') = B_{T=1}(r, r') \sum_{\text{odd } L} \frac{2L+1}{4\pi} P_L(\cos \theta) + B_{T=0}(r, r') \sum_{\text{even } L} \frac{2L+1}{4\pi} P_L(\cos \theta) \quad (V-65)$$

$$= B_{T=1}(r, r') \frac{1}{2} \left[ \delta(\Omega - \Omega') - \delta(\Omega + \Omega') \right] + B_{T=0}(r, r') \frac{1}{2} \left[ \delta(\Omega - \Omega') + \delta(\Omega + \Omega') \right] \quad (V-66)$$

where  $\Omega$ ,  $\Omega'$  are the angles which  $r$ ,  $r'$  make with a given arbitrary direction.

Let us first consider the  $T = 1$  state. By Equation (V-63) and (V-66) this becomes

$$\begin{aligned} \langle r_1 | V^{(LS)} | r_1' \rangle \Big|_{T=1} &= \int r^2 r'^2 dr dr' d\Omega d\Omega' B_{T=1}(r, r') \\ &\cdot \frac{1}{2} [\delta(\Omega - \Omega') - \delta(\Omega + \Omega')] \frac{r_1 \cdot r_1'}{r_1^2} \left\{ \rho(r_1) - \frac{1}{a} r_1 \cdot \nabla_{r_1} \rho(r_1) \right. \\ &\quad \left. - \frac{1}{2} r_1' \cdot \nabla_{r_1} \rho(r_1) - \frac{1}{2} (r_1 - r_1') \cdot \nabla_{r_1} \rho(r_1) \right\} \end{aligned} \quad (V-67)$$

Let us now take the arbitrary direction from which  $\Omega$  and  $\Omega'$  are measured to be along  $r_1$ . Then Equation (V-67) reduces to

$$\begin{aligned} \langle r_1 | V^{(LS)} | r_1' \rangle \Big|_{T=1} &= -\frac{1}{4} \int dr dr' d\Omega d\Omega' \left\{ r^2 r'^2 B_{T=1}(r, r') \right. \\ &\quad \left. \cdot \frac{r_1 \cdot r_1'}{r_1^2} \cdot r_1' \cdot \nabla_{r_1} \rho(r_1) \right\} (\delta(\Omega - \Omega') - \delta(\Omega + \Omega')) \\ &= -\frac{2\pi}{3} \int dr dr' r^3 r'^3 B_{T=1}(r, r') \frac{1}{r_1} \frac{d\rho}{dr} \end{aligned} \quad (V-68)$$

For the  $T = 0$  case we shall have

$$\begin{aligned} \langle r_1 | V^{(LS)} | r_1' \rangle \Big|_{T=0} &= -\frac{1}{4} \int dr dr' d\Omega d\Omega' B_{T=0}(r, r') \\ &\cdot \frac{1}{2} (\delta(\Omega - \Omega') + \delta(\Omega + \Omega')) \cdot \frac{r_1 \cdot r_1'}{r_1^2} \left\{ \rho(r_1) - \frac{1}{2} r_1 \cdot \nabla_{r_1} \rho(r_1) \right. \\ &\quad \left. - \frac{1}{2} r_1' \cdot \nabla_{r_1} \rho(r_1) - \frac{1}{2} (r_1 - r_1') \cdot \nabla_{r_1} \rho(r_1) \right\} \end{aligned} \quad (V-69)$$

$$= -\frac{1}{4} \int d\tau d\tau' d\Omega d\Omega' r^2 r'^2 B_{\tau_0}(r, r') \frac{r_1 \cdot r_1'}{r_1^2} r_1 \cdot \nabla_{r_1} \rho(r_1) \cdot (\delta(\Omega - \Omega') + \delta(\Omega + \Omega'))$$

$$\text{or } (r_1 | V_i^{(LS)} | r_1') \Big|_{T=0} = -\frac{2\pi}{3} \int d\tau d\tau' r^4 r'^2 B_{T=0}(r, r') \frac{1}{r_1} \frac{d\rho}{dr_1} \quad (\text{V-70})$$

$B(r, r') \Big|_{\substack{L=1, T=1 \\ \text{or } L=2, T=0}}$  cases are given by Equation (V-44). We

now assume that nonlocality in this function comes from inside the integral over  $k_0$  and that outside we can put the factor  $rr' = r^2$ .

This approximation is of the same order as made in connection with Equations (V-51) and (V-52). Thus using this approximation and Equations (V-44) and (V-28) we can write

$$B_{T=1}(r, r') = \frac{4\pi}{3} \cdot \frac{1}{2\pi^2} \cdot \frac{1}{r^2} \left[ \int \mathcal{G}_1(kr) \left\{ -v_{11}^0(r') U_{11}^0(k, r') - \frac{3}{2} v_{11}'(r') U_{11}'(k, r') + \frac{5}{2} (v_{11}^2(r') U_{11}^2(k, r') + v_{13}^2(r') U_{13}^2(k, r')) \right\} \right. \\ \left. \cdot dk \right] \quad (\text{V-71})$$

and

$$B_{T=0}(r, r') = \frac{4\pi}{5} \cdot \frac{1}{2\pi^2} \cdot \frac{1}{r^2} \left[ \int \mathcal{G}_2(kr) \left\{ -\frac{3}{2} (v_{22}'(r') U_{22}'(k, r') + v_{20}'(r') U_{02}'(k, r')) - \frac{5}{6} v_2^2(r') U_2^2(k, r') + \frac{7}{3} (v_{22}^3(r') U_{22}^3(k, r') + v_{24}^3(r') U_{42}^3(k, r')) \right\} \right. \\ \left. \cdot dk \right] \quad (\text{V-72})$$

If we now put these values of B's in Equations (V-68) and (V-70) we obtain the coefficient of the single-particle spin-orbit operator in r-space. In order to avoid the difficulty experienced earlier because of lack of knowledge of  $u$  for large  $k$  it is better to interchange the order of integration leaving the  $k$ -integration to the last. This will reduce considerably the apparent importance of large  $k$  and thus the contribution from inside the Fermi-sea will be dominant together with that of immediately outside. Further the number of integrations will reduce from 3 to 2. When this is done we require integrals of the type

$$\int x^n g_m(kx) dx$$

These can be easily evaluated by introducing exponentially decaying functions of the type  $e^{-\epsilon x}$ , integrating and then letting  $\epsilon \rightarrow 0$ .

The general case is a little difficult to work out, but in some particular cases they are quite simple. This procedure then gives

$$\begin{aligned} \int x^2 g_2(kx) dx &= \frac{8}{k^3} \\ \int x^3 g_2(kx) dx &= 0 \\ \int x g_1(kx) dx &= \frac{2}{k^2} \\ \int x^2 g_1(kx) dx &= 0 \\ \int x^3 g_1(kx) dx &= -\frac{8}{k^4} \end{aligned} \tag{V-73}$$

Thus combining (V-68) and (V-71) and (V-70) and (V-72), making the change in integration order as suggested above and then using (V-73) we shall finally obtain

$$\begin{aligned}
 (x_1 | V^{(LS)} | x'_1) \Big|_{T=1} &= -\frac{8}{9} \int \frac{dk}{k^2} \left[ \int dx' x'^3 \left\{ -v_1^0(x') u_1^0(k, x') \right. \right. \\
 &\quad \left. \left. - \frac{3}{2} v_1^1(x') u_1^1(k, x') + \frac{5}{2} \left( v_{11}^2(x') u_{11}^2(k, x') + v_{13}^2(x') u_{13}^2(k, x') \right) \right\} \right] \\
 &\quad \cdot \frac{1}{k} \frac{d\rho}{dx}
 \end{aligned} \tag{V-74}$$

and

$$\begin{aligned}
 (x_1 | V^{(LS)} | x'_1) \Big|_{T=0} &= -\frac{32}{15} \int \frac{dk}{k^3} \left[ \int dx' x'^2 \left\{ -1.5 (v_{22}^1(x') u_{22}^1(k, x')) \right. \right. \\
 &\quad \left. \left. + v_{20}^1(x') u_{02}^1(k, x') - \frac{5}{6} (v_2^2(x') u_2^2(k, x')) \right\} \right. \\
 &\quad \left. + \frac{7}{3} \left( v_{22}^3(x') u_{22}^3(k, x') + v_{24}^3(x') u_{42}^3(k, x') \right) \right] \frac{1}{k} \frac{d\rho}{dx}
 \end{aligned} \tag{V-75}$$

and finally for  $N = Z$  closed shell nucleus

$$(x_1 | V^{(LS)} | x'_1) = \frac{3}{4} (x_1 | V^{(LS)} | x'_1) \Big|_{T=1} + \frac{1}{4} (x_1 | V^{(LS)} | x'_1) \Big|_{T=0} \tag{V-76}$$

Equations (V-74, 75, 76) give the effective one body spin orbit force.

These equations give the correct result (namely that of Blin-Stoyle)

in the case of a local two body force. To see this we consider the Moszkowski Scott separation method. According to this the effective spin orbit part of  $G$  is just  $U^{LS}(r)$  for  $r > d$  and the wave function

$$u_{LL'}^J \rightarrow g_{L'} \text{ plane waves.}$$

The work follows:

Consider first Equation (V-74). In the Born approximation limit using

$v_{\ell}$  this becomes

$$\begin{aligned} \langle x_1 | V^{(LS)} | x_1' \rangle \Big|_{T=0} &= -\frac{8}{9} \int \frac{dk}{k^2} \int dx' x'^3 \cdot 6 v_{\ell}(x') g_1(kx') \\ &= -\frac{16}{3} \int_d^{\infty} dx' x'^3 v_{\ell}(x') \int_0^{\infty} \frac{dk}{k^2} g_1(kx') \end{aligned} \quad (\text{V-77})$$

The factor 6 with  $V_{\text{long}}$  is really 2 (2 $\ell$  + 1) and for  $T = 1$ , we have only

$L = 1$ . We now make use of the integral (33)

$$\int_0^{\infty} x^{\mu} J_{\nu}(xy) (xy)^{\nu/2} dx = 2^{\mu+\frac{1}{2}} y^{-\mu-1} \frac{\Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{3}{4})}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}\mu + \frac{1}{4})} \quad (\text{V-78})$$

$$- \text{Re } \nu - \frac{3}{2} < \text{Re } \mu < -\frac{1}{2}$$

$$\int_0^{\infty} \frac{dk}{k^2} g_1(kx') = \frac{\pi x'}{4} \quad (\text{V-79})$$



$$\left( x_1 | V^{(LS)} | x_1' \right) \Big|_{T=1} = - \frac{4\pi}{3} \int_d^\infty x'^4 v_\rho(x') dx' \frac{1}{x} \frac{d\rho}{dx} \quad (V-80)$$

Similarly

$$\begin{aligned} \left( x_1 | V^{(LS)} | x_1' \right) \Big|_{T=0} &= - \frac{32}{15} \int_0^\infty \frac{dk}{k^3} \int_d^\infty dx' x'^2 10 v_\rho g_2(kx') \cdot \frac{1}{x} \frac{d\rho}{dx} \\ &= - \frac{64}{3} \int_d^\infty dx' x'^2 v_\rho(x') \int \frac{dk}{k^3} g_2(kx') \cdot \frac{1}{x} \frac{d\rho}{dx} \quad (V-81) \\ &= - \frac{4\pi}{3} \int x'^4 v_\rho(x') dx' \frac{1}{x} \frac{d\rho}{dx} \end{aligned}$$

after using (V-78) to obtain

$$\int \frac{dk}{k^3} g_2(kx') = \frac{\pi x'^2}{16} \quad (V-82)$$

Thus

$$\left( x_1 | V^{(LS)} | x_1' \right) = \frac{3}{4} \left( x_1 | V^{(LS)} | x_1' \right) \Big|_{T=1} + \frac{1}{4} \left( x_1 | V^{(LS)} | x_1' \right) \Big|_{T=0} \quad (V-83)$$

Equations (V-80) and (V-81) correspond to the form of the force given by Blin-Stoyle.

Using Equations (V-80, 81 and 83) it is now possible to evaluate the one body spin-orbit force directly using only the outer part of the two-body spin orbit force. Taking this force from the Hamada Johnston Potential (Appendix B) we can then show the one-body force to be

$$\begin{aligned} (\chi_i | V^{(LS)} | \chi_i') = & - \pi d^2 \cdot \frac{\mu c^2}{\mu^3} \left[ 0.11043 \left( 1 + \frac{1}{\mu d} + \frac{1}{2\mu^2 d^2} \right) \right. \\ & \left. - 0.398262 \frac{e^{-\mu d}}{\mu d} \left( 1 + \frac{1}{3\mu d} \right) \right] e^{-2\mu d} \cdot \frac{1}{r} \frac{d\rho}{dr} \end{aligned} \quad (V-84)$$

where  $d$  is the distance beyond which the two body force is considered. We show below that it is equal to or near to the core radius.

Once the effective single particle spin orbit force is calculated the level splitting is given by  $\Delta E = E_{J=L+1/2} - E_{J=L-1/2}$  for an odd nucleon in state  $(n, l)$  by

$$\begin{aligned} \Delta E = & \int_0^\infty f_{ne}^*(r) V^{(LS)} L \cdot S f_{ne} r^2 dr \\ & = - (2L+1) \int_0^\infty |f_{ne}|^2 V^{(LS)} r^2 dr \end{aligned} \quad (V-85)$$

where  $f_{ne}$  is the radial wave function of the particle and depends upon the type of well chosen for the nuclear potential.

### Calculations, Results and Discussion

Before we go into the procedure of the calculations and give any results it should be pointed out that the effective single particle spin-orbit force consists of the product of two terms:

$\frac{1}{r} \frac{d\rho}{dr}$  and its coefficient. Let us call the coefficient by the name VLSCOEF. Thus the spin orbit force is given by

$$(\chi_i | V^{LS} | \chi'_i) = \text{VLSCOEF} \frac{1}{r} \frac{d\rho}{dr} \quad (\text{V-86})$$

In general VLSCOEF is a function of  $r$ , i.e. it is density dependent.

The aim of the present calculations is to find the spin-orbit splittings in nuclei near to the closed shell ones. The calculations are in two parts: first is the calculation of VLSCOEF and the second is using it to find the splittings. Both these are independent of each other and require quite different assumptions.

The calculation of the actual force is done by using Equations (V-74, 75, and 76). Equations (V-80, 81 and 83) are also used if the potential  $V_{LS}$  were used directly as the two body force, rather than the G-matrix. The  $u$ 's are the two body correlated wave-functions in nuclear matter. In Chapters II and III we have described how the wave function distortion  $\chi$  can be calculated in reference spectrum approximation. The  $u$  are then simply given by  $(g - \chi)$ . Thus knowing the two body potential in the  ${}^3P$  and  ${}^3D$  states it is very easy to calculate the spin-orbit force.

One of the main assumptions made so far is that the non-locality of  $B(r, r')$  is of much shorter range than that of  $P(r, r')$ . This was found to be true by B.G.W. Preliminary investigations showed this to be true, but no definite numerical values were obtained owing to the difficulties experienced in such a calculation as explained above. The main aim of our calculation, viz. evaluation of VLSCOEF was reached without explicitly calculating  $B(r, r')$ .

If the spin-orbit part beyond some separation distance  $d$  of the Hamada-Johnston potential is used, VLSCOEF can be calculated directly from Equation (V-84). It is found that almost the complete external potential must be used to get the correct splittings. B.G.W. made this assumption in their calculations for binding energy effects, Our justification is provided by Figure 11. Here we have plotted the two body Hamada-Johnston spin-orbit force as the effective two-body force calculated in the 'G' matrix approximation. This was obtained in the following manner after Brandow's suggestion (35) for defining effective interactions:

From Equations (V-25 and V-28) we see that in momentum space the effective two body spin-orbit interaction is given by (for  $T = 1$ ,  $L = 1$  state only and  $k_0 = k_0'$ )

$$B(k_0, k_0) = \frac{4\pi}{k_0^2} \left[ \int J_1(k_0, r) \left\{ \frac{5}{2} (v_{11}^2 u_{11}^2 + v_{13}^2 u_{31}^2) - \frac{3}{2} u_1^1 u_1^1 - v_1^0 u_1^0 \right\} dr \right] \quad (\text{V-87})$$

In Born approximation this reduces to

$$B(k_0, k_0) = \frac{4\pi}{k_0^2} \cdot 6 \int J_1^2(k_0, r) v_{LS}(r) dr \quad (\text{V-88})$$

Thus the effective spin-orbit interaction is

$$\frac{\frac{5}{2} (v_{11}^2 u_{11}^2 + v_{13}^2 u_{31}^2) - \frac{3}{2} u_1' v_1' - v_1^0 u_1^0}{6 J_1(k_0 r)} \quad (V-89)$$

From Figure 11 it is clear that the effective interaction and the two body Hamada-Johnston spin-orbit force are very much the same except at very small distances. It can be said that the "healing" of the spin-orbit amplitude is remarkably fast. The Hamada-Johnston spin-orbit force will, therefore, give the same splitting as the effective one only if almost all of it outside the hard core is used. From Equation (V-89) we also calculated the effective two body spin-orbit force using only the spin-orbit part of the Hamada-Johnston potential. This we found to be very close to the one calculated by using all of the potential. Use of the central part of the potential gave practically zero value throughout. Thus the tensor part contribution will be very small and we could conclude that the effective spin orbit interaction comes mainly from the spin orbit part of the potential. In Figure 11 the top most curve represents the difference of the contributions to the two body spin-orbit effective forces obtained by using the total potential and only the spin-orbit part of the potential in Equation V-89. This would be the tensor force contribution. It is seen to be small and vanishes very fast. These conclusions, however, are at best correct only in first approximation, since the u's involve the total potential. The

above conclusions are true only for the  $T = 1$  case. We have not investigated the  $T = 0$  case; the spin-orbit force here, however, is small.

One of the interesting points to be studied about VLSCOEF is its variation with density and the contributions to it from different regions of the momentum space. These are shown in Figures 12 and 13. Figure 12 gives the variations of the contributions to VLSCOEF from different regions of the momentum space for  $T = 0, 1$  components, and the total force for  $N = Z$  closed shell nucleus. It is seen that the  $T = 1$  part is virtually constant over the whole range of the density, though the contributions from above and below  $k_F$  change. It is interesting to note that the variation with  $k_F$  is linear and that as  $k_F$  increases thereby increasing the size of the occupied region, the contribution increases by the same amount as the outside sea contribution decreases. For  $T = 0$  this is not the case. The outside sea contribution is constant and negligibly small. The contribution from inside the sea decreases in magnitude as the density increases. The reason for such a variation can be seen from the  $T = 0, k < k_F$  contributions given at  $k_F = 1.36$  and  $0.8$ . This variation is similar to that in the central force where it occurs mainly due to the  $3D_2$  states, the  $3D_1$  and  $3D_3$  are very small and do not change total  $3D$  much. This variation occurs in  $3P$  states also, but is mutually cancelled by the three states.

In Table XII we give VLSCOEF at different  $k_F$  for  $T = 1$  and 0 states. The fourth column gives this coefficient for  $N = Z$  closed shell nuclei, while  $(N - Z)/4A$  times the number in fifth column must be added to the corresponding number in the fourth column to obtain VLSCOEF for  $N \neq Z$  closed shell nuclei. The numbers are in  $F^3$  and must be multiplied by 41.469 to get them in  $\text{MeV } F^5$ .

To obtain the spin-orbit force in the form 'constant  $\cdot \frac{1}{r} \frac{d\rho}{dr}$ ' we have made several assumptions. This form is useful since it facilitates comparison with the usual shell model and optical model potentials. B.G.W. and Brueckner, Lockett and Rotenberg (7) did not do this. B.G.W. directly used the non-local potential  $B(r, r')$  and very ingeniously reduced the integro-differential equation to a pair of differential equations of the form  $\Psi'' + F\Psi' + G\Psi = 0$ . Both  $F$  and  $G$  involve potential functions. They observed that the spin-orbit part of the  $G$  potential was not localized in the nuclear surface, but due to the presence of the additional potential  $F$  it is not clear that their work disagrees with the usual type of analysis.

In Figure 14 we give the VLSCOEF  $\cdot \frac{1}{r} \frac{d\rho}{dr}$  as well as  $100 \cdot \frac{1}{r} \frac{d\rho}{dr}$ . We have seen that the coefficient of  $\frac{1}{r} \frac{d\rho}{dr}$  is not a constant, but an increasing function of the local density. As discussed above, this density dependence comes from the  $T = 0$  part

of the two body potential. The force is seen to be very similar in shape to  $\frac{1}{r} \frac{d\rho}{dr}$ , but peaking about  $0.15 F$  inside the half density radius. From optical model analysis of proton scattering data several authors (36) have found that the radius parameter for the spin-orbit interaction was approximately  $.1F$  less than for the real central interaction.

Greenless, Pyle and Tang (6) proposed an explanation, based on the following assumptions, which we shall see are not entirely justified: (1) the potential well radius squared is estimated to be the mean square density radius plus the mean square two body force radius. (2) The two body central mean square radius was taken to be  $3F^2$ , and the spin-orbit radius zero. However, it is well known that the long range OPEP does not contribute to the extension of the nuclear force, because of its exchange character. According to Reid, the intermediate range two body force is primarily of range of about three pion masses, so that its mean square radius is below  $1F^2$ . On the other hand, the spin-orbit force has a range of order three to five pion masses (three in Hamada-Johnston potential) which is not dramatically shorter than that of the central force. Thus the crucial steps in the Greenless-Pyle and Tang's argument are not supported.

Our nuclear matter calculations lead easily to a one body spin-orbit force with a shift of the right magnitude inside the nuclear density. This is brought about by the density dependence of the spin-orbit coefficient VLSCOEF (see Figure 12). An argument similar



to that of the spin-orbit force if made for the central force would be rather rough, but would indicate the right tendency of the force. In Figure 14 we have also plotted the  $\rho(r)$  and  $U(r)$ ; the latter is the single particle potential energy and may be very roughly written as  $\rho(r) \cdot \text{constant}$ . It is seen that unlike the spin-orbit force  $U(r)$  is outside the density distribution. Such a behaviour is brought about by the saturation property of the nuclear force.

Now it is well known that the density distribution is well inside the optical potential well. Though never calculated from the first principles this is believed to be a manifestation of the finite range of the nuclear forces. It should be expected that such a smearing effect should be the same for the central as well as the spin-orbit potential. Thus, there would still be a displacement of at least .15 F between the spin-orbit radius and the central potential well radius.

The figure of  $100 \cdot \frac{1}{r} \frac{d\rho}{dr}$  which we found convenient to use in Figure 14 can be related to the usual "Thomas Term", if we assume a depth for the central well. Thus, writing

$$100 \times \frac{1}{r} \frac{d\rho}{dr} = X \cdot \frac{1}{2} (\bar{\lambda}_p)^2 \frac{V_0}{\rho_{Nm}} \frac{1}{r} \frac{d\rho}{dr}$$

we have

$$V_0 X = \frac{200 \rho_{Nm}}{(\bar{\lambda}_p)^2} = \frac{200 \cdot .168}{(.2103)^2} = 760$$

$$\therefore X = \frac{760}{V_0} = 19 \quad \text{if } V_0 = 40 \text{ Mev.}$$

The usual potential models require 20 to 30 times the Thomas Term.

Having thus far made a detailed study of VLSCOEF, the coefficient of  $\frac{1}{r} \frac{d\rho}{dr}$  in the effective one body force, we now consider the next part of the calculations: a study of the spin-orbit splittings in actual closed shell  $\pm 1$  nuclei. The splittings can readily be calculated using Equation (V-85). This involves knowledge of mainly two quantities: the radial wave functions  $f_{n\ell}$  and the density distribution in the nucleus. For the first one, viz., the radial wave functions, we as usual took spherical oscillator functions. There is by now ample evidence for the reasonableness of this assumption from several Hartree-Fock calculations on light nuclei. For particles deep inside the well this approximation is quite good. For density distributions we used both the shell model density distribution and the Fermi density distribution.

The shell model density distribution was used for  $O^{16}$ ,  $Ca^{40}$ , and  $Ca^{48}$ . It is given by

$$(1) \text{ For } O^{16} \quad \rho(r) = \frac{4\beta^3}{\pi^{3/2}} e^{-\beta^2 r^2} (2 + \beta^2 r^2) \quad (V-91)$$

$$(2) \text{ For } Ca^{40} \quad \rho(r) = \frac{4\beta^3}{\pi^{3/2}} e^{-\beta^2 r^2} (2.5 + 2\beta^4 r^4) \quad (V-92)$$

$\beta$  is a certain mean of the oscillator wave function size parameter determined by requiring that the mean square radius of the nucleus be given by  $\frac{1}{\beta^2} \left[ 2(n-1) + \ell + \frac{3}{2} \right]$ , where  $n$  takes the values = number of nodes (including origin)  $-\ell = 1, 2, 3 \dots$ . For  $\text{Ca}^{48}$  we took  $\beta$  to be that of  $\text{Ca}^{40}$  times  $(40/48)^{1/3}$ . Thus  $\beta = .566$  for  $\text{O}^{16}$ ,  $.4914$  for  $\text{Ca}^{40}$ ,  $.4624$  for  $\text{Ca}^{48}$ .

The Fermi density distribution is given by (37)

$$\rho(r) = \frac{A \rho_F}{1 + e^{(r-c)/a}} \quad (\text{V-93})$$

where  $c$  is the half-density radius; the surface thickness is given by

$$s = 4a \ln 3 \approx 4.40a \quad (\text{V-94})$$

and

$$\rho_F = \frac{3}{4\pi c^3} \left( 1 + \frac{\pi^2 a^2}{c^2} \right)^{-1} \quad (\text{V-95})$$

$c$  and  $s$  are the only quantities determined by electron scattering experiments.  $c$  is determined by the position of the first minimum in the diffraction pattern as a function of scattering angle and  $s$  from the depth of this minimum.  $c = 2.60$  for  $\text{O}^{16}$ ,  $3.64$  for  $\text{Ca}^{40}$ ,  $6.50$  for  $\text{Pb}^{208}$ . For  $\text{Ca}^{48}$   $c = c$  of  $\text{Ca}^{40}$  times  $(48/40)^{1/3}$ .  $s = 1.85$  for  $\text{O}^{16}$ ,  $2.5$  for  $\text{Ca}^{40}$  and  $\text{Ca}^{48}$  and  $2.3$  for  $\text{Pb}^{208}$ .

The spin-orbit splitting calculations have been carried out for  $1p_{1/2} - 1p_{3/2}$  and  $1d_{3/2} - 1d_{5/2}$  states around  $O^{16}$ ; for  $1p_{1/2} - 1p_{3/2}$ ,  $1d_{3/2} - 1d_{5/2}$ ,  $1f_{5/2} - 1f_{7/2}$  states around  $Ca^{40}$ ; for  $1d_{3/2} - 1d_{5/2}$ ,  $1f_{5/2} - 1f_{7/2}$  and  $2p_{1/2} - 2p_{3/2}$  states around  $Ca^{48}$  using both the shell model and Fermi density distributions. For lead we calculated the splittings for the states  $1h_{9/2} - 1h_{11/2}$ ,  $1i_{11/2} - 1i_{13/2}$ ,  $2f_{5/2} - 2f_{7/2}$ ,  $2g_{7/2} - 2g_{9/2}$ ,  $3d_{3/2} - 3d_{5/2}$  and  $3p_{1/2} - 3p_{3/2}$  using only the Fermi density distribution. The oscillator size parameter  $\alpha$  was varied around the mean value  $\beta$  as determined above. Thus while the wave function of the single particle was allowed to change its size the density distribution, both shell model as well as Fermi, were kept fixed. The result of these calculations are presented in Figures 15, 16, 17 and 18. Marked on these figures are the experimental values (39). The agreement with experiment is seen to be quite good especially when it is to be seen in view of the approximations we have made.

Firstly all our calculations are based on the Hamada-Johnston force. In nuclear matter this does not give enough binding and it would, therefore, seem too much to expect the correct spin orbit splitting. Secondly our calculations have been done in the Reference Spectrum approximation. We have shown that the second order correction in the spin-orbit case is zero, hence the leading correction is the third order one. However, from nuclear matter calculations we know that most of the correction comes from S-states and for spin-

orbit force this does not exist. On the whole we therefore expect results lying reasonably close to the experimental values. This is what we get.

The only general trend that can be seen for all the nuclei studied is that as the experimental values require the size parameter  $\alpha$  to be smaller, the smaller the quantum number  $n$ . In other words the smaller the quantum number  $n$  more spread out is the wavefunction.

APPENDIX A

Coupled States

The reference wave equation for coupled triplet states can also be solved by the Ridley method. We use the matrix notation developed by Razavy and Sprung (34). For a given  $J$ , there are two coupled orbital angular momenta,  $L = J \pm 1$ . A solution of the reference wave equation (A.4) consists of two amplitudes, one for each  $L$ . For example,  $J = 1$  has  $L = 0, 2$ .  $\chi_{00}$  and  $\chi_{20}$  are the "large" and "small" components of the deuteron state. The second index indicates that this solution is mostly S-wave. There is a second, dominant D-wave solution, with components  $\chi_{02}$ ,  $\chi_{22}$ . It is very useful to regard these as making up a "solution matrix"  $\chi^J$ , components  $\chi_{LL'}^J$ . Introducing the notation

$$\nabla_J^2 = \begin{pmatrix} d^2/dx^2 - J(J-1)/x^2 & 0 \\ 0 & d^2/dx^2 - (J+1)(J+2)/x^2 \end{pmatrix} \quad (\text{A-1})$$

$$U^J = \begin{pmatrix} U_{J-1, J-1} & U_{J-1, J+1} \\ U_{J+1, J-1} & U_{J+1, J+1} \end{pmatrix} \quad (\text{A-2})$$

$$J^J(k_0 t) = \begin{pmatrix} J_{J-1}(k_0 t) & 0 \\ 0 & J_{J+1}(k_0 t) \end{pmatrix} \quad (\text{A-3})$$

we can write the coupled reference wave equation (2.5) as

$$\left(\nabla_{\mathcal{J}}^2 - \gamma^2 - \nu^{\mathcal{J}}\right) \chi^{\mathcal{J}} = -m^* \nu^{\mathcal{J}} g^{\mathcal{J}} \quad (\text{A-4})$$

Of course, 
$$u^{\mathcal{J}} = g^{\mathcal{J}} - \chi^{\mathcal{J}} \quad (\text{A-5})$$

Inside a hard core, all components of  $u^{\mathcal{J}}$  vanish, so the off-diagonal components of  $\chi^{\mathcal{J}}$  do. We define  $\mathcal{H}^{\mathcal{J}}$  to be a diagonal matrix like (A-3) with  $\mathcal{H}_L(r)$  elements. By an argument similar to Equation (III-3) we will have, for  $r > d$ ,

$$\chi^{\mathcal{J}}(r) = \mathcal{H}^{\mathcal{J}}(r) \cdot N \quad (\text{A-6})$$

with  $N$  some normalization (2 x 2) matrix. It must be written on the right so that the off diagonal components of  $\chi^{\mathcal{J}}$  have the appropriate L-character. Setting

$$\Gamma_{LL'}^{\mathcal{J}} = -\frac{d}{dt} \log \mathcal{H}_L(r) \delta_{LL'} \quad (\text{A-7})$$

The boundary conditions on  $\chi$  are

$$\begin{aligned} \chi^{\mathcal{J}}(c) &= g^{\mathcal{J}}(k_0 c), \quad r = c \\ \chi^{\mathcal{J}'}(d) + \Gamma^{\mathcal{J}} \chi^{\mathcal{J}}(d) &= 0 \quad r = d \end{aligned} \quad (\text{A-8})$$

which correspond to Equation (III-6).

To apply the Ridley method we introduce  $s^J$  and  $\omega^J$  as  $2 \times 2$  matrices. Equations (III-7) to (III-11) may be interpreted as matrix equations. Due to symmetry of the potential (A.2), the  $g^J$  in (III-7) and (III-10) is symmetric, and so is  $s^J$ . The boundary condition at  $r = d$  is satisfied with  $\omega^J(d) = 0$  and  $s^J(d) = \Gamma^J$ . We can integrate the eight (seven are enough) coupled equations (III-10) and (III-11) into  $r = c$ , where  $\chi^J(c) = g^J$ . Thus the procedure is completely straightforward.



APPENDIX B

Potentials

The Hamada-Johnston Potential (26) is expressed as a sum of central, tensor, spin-orbit and  $(L L + (L S)^2)$  terms. The parameters are different in singlet even and odd and triplet even and odd states. The numerical values are given in Table XIII. The potential is

$$V(x) = V_c(x) + V_T(x) S_{12} + V_{LS}(x) (\underline{L} \cdot \underline{S}) + V_{LL}(x) L_{12}$$

$$L_{12} = \delta_{LJ} (1 + \underline{\sigma}_1 \cdot \underline{\sigma}_2) \underline{L}^2 + \underline{L} \cdot \underline{S} = \left\{ \delta_{L\sigma} + (\underline{\sigma}_1 \cdot \underline{\sigma}_2) \right\} \underline{L}^2 - (\underline{L} \cdot \underline{S})^2$$

$$V_c(x) = 0.08 \frac{\mu c^2}{3} (\underline{\tau}_1 \cdot \underline{\tau}_2) (\underline{\sigma}_1 \cdot \underline{\sigma}_2) Y(x) \left[ 1 + a_c Y(x) + b_c Y^2(x) \right]$$

$$V_T(x) = 0.08 \frac{\mu c^2}{3} (\underline{\tau}_1 \cdot \underline{\tau}_2) (\underline{\sigma}_1 \cdot \underline{\sigma}_2) Z(x) \left[ 1 + a_T Y(x) + b_T Y^2(x) \right]$$

$$V_{LS}(x) = \mu c^2 G_{LS} Y^2(x) \left[ 1 + b_{LS} Y(x) \right]$$

$$V_{LL}(x) = \mu c^2 G_{LL} \frac{Z(x)}{x^2} \left[ 1 + a_{LL} Y(x) + b_{LL} Y^2(x) \right] \quad (B-1)$$

$$Y(x) = \text{EXP}(-x)/x$$

$$Z(x) = \left( 1 + \frac{3}{x} + \frac{3}{x^2} \right) Y(x)$$

Here  $\mu c^2 = 139.4$  MeV is the pion mass.  $x = \mu r$ . The hard core radius  $c$  is given by  $\mu c = .343$  where  $\mu = 139.4/197.314 = .706481 \text{ F}^{-1}$ .

$$\langle y_{J'L'S}^M | \tilde{L} \cdot \tilde{S} | y_{JLS}^M \rangle = 0 \quad S=0$$

$$= \begin{cases} (J-1) \delta_{LL'} & L = J-1 \\ - \delta_{LL'} & L = J \\ -(J+2) \delta_{LL'} & L = J+1 \end{cases} \quad S=1$$

$$\langle y_{J,L,S}^M | S_{12} | y_{J,L',S}^M \rangle = 0 \quad S=0$$

$$= \begin{cases} -\frac{2(J-1)}{2J+1} \delta_{LL'} & L = J-1 \\ \frac{6\sqrt{J(J+1)}}{2J+1} & L \neq L' = J \pm 1 \\ -\frac{2(J+2)}{2J+1} \delta_{LL'} & L = J+1 \\ 2 \delta_{LL'} & L = J \end{cases} \quad S=1$$

$$\langle y_{J,L,S}^M | L_{12} | y_{J,L',S}^M \rangle = -2J(J+1) \quad S=0$$

$$= \begin{cases} J-1 & L = J-1 \\ 2J(J+1)-1 & L = J \\ -(J+2) & L = J+1 \end{cases} \quad S=1$$

The potential in Bressel's thesis was derived by replacing the infinite hard cores by finite square potentials. The meson mass was taken differently for the three states  $T = 1$  (pp),  $T = 1$  (np),  $T = 0$  (np), being 139.4 MeV, 135.28 MeV, 137.2 MeV respectively. This makes the potential charge symmetric, but not charge independent. However, we made the approximation of charge independence, using a single  $T = 1$  force with the correct average meson mass. This is correct to first order. The core radius is  $x_c = .4852$  meson Compton wave lengths and is, therefore, charge dependent. For  $x > x_c$  the potential is the same as the original Hamada Johnston definition, as a function of  $x = \mu r$ .

The core heights are given in Table XIV.

Recently Bressel improved the fit to the data and corrected an error in his programme. His revised potential has pion masses 137.4, 133.08 and 137.34 MeV, in the same order as above. The  $T = 1$  core heights were amended to the values given in brackets in Table XIV. Further, three of the Hamada Johnston potential parameters were amended:  $a_c(^1E) = 8.7075$  (from 8.7),  $a_c(^3O) = -11.2$  (from -9.07) and  $b_c(^3O) = 3.28$  (from 8.7). This potential was fitted directly to a selection of experimental data from 0 to 350 MeV, avoiding the correlated errors of fitting to phase shifts.

The Reid potential was fitted to phase shifts of the Yale groups.

The potential was given separately for each (J, L, S) state instead of being expressed as a sum of central, tensor, spin orbit, etc. forces.

It is:

$$x = \mu r, \quad \text{defined to be } 0.7 F^{-1}$$

$$h^2/M \quad \text{defined to be } 41.4700 \text{ MeV } F^{-2}$$

$${}^1S_0 \quad \text{hard core radius } x = 0.29614$$

$$V(x) = -10.463 (e^{-x} + 39.633 e^{-3x})/x \text{ MeV}$$

$${}^3S_1 - {}^3D_1 \text{ hard core radius } x = .35611$$

$$V = V_c + S_{12} V_T$$

$$V_c(x) = -10.463 (e^{-x} + 22.917 e^{-3x})/x$$

$$V_T(x) = -10.463 \left[ \left( 1 + \frac{3}{x} + \frac{3}{2} \right) e^{-x} \right. \\ \left. + \left( 71.98 - \frac{18}{x} - \frac{3}{2} \right) e^{-6x} \right. \\ \left. - 11.294 e^{-3x} \right] /x$$

All other states hard core radius  $x = 0.3$

$${}^1D_2 \quad V(x) = -10.463 (e^{-x} + 4.939 e^{-2x} + 154.7 e^{-6x})/x$$

$${}^3D_2 = -10.463 \left[ \left( 3 + \frac{6}{x} + \frac{6}{x^2} \right) e^{-x} + 28.45 e^{-2x} - \left( 93.6 + \frac{18}{x} + \frac{6}{x^2} \right) e^{-3x} \right] / x$$

$${}^3P_0 \quad V(x) = -10.463 \left[ \left( 1 + \frac{4}{x} + \frac{4}{x^2} \right) e^{-x} - \left( 545 + \frac{24}{x} + \frac{4}{x^2} \right) e^{-6x} + 16.2 e^{-2x} - 55.6 e^{-3x} \right] / x$$

$${}^3P_1 \quad V(x) = 10.463 \left[ \left( 1 + \frac{2}{x} + \frac{2}{x^2} \right) e^{-x} + \left( 175.1 - \frac{12}{x} - \frac{2}{x^2} \right) e^{-6x} - 1.1553 e^{-2x} - 8.722 e^{-3x} \right] / x$$

$${}^3P_2 - {}^3F_2 \quad V(x) = V_c + V_T S_{12} + V_{LS} \quad \underline{L.S.}$$

$$V_c = 10.463 \left( \frac{1}{3} e^{-x} - 13.8 e^{-3x} + 138 e^{-6x} \right) / x$$

$$V_T = 10.463 \left[ \left( \frac{1}{3} + \frac{1}{x} + \frac{1}{x^2} \right) e^{-x} - \left( \frac{6}{x} + \frac{1}{x^2} \right) e^{-6x} - 5.688 e^{-3x} \right] / x$$

$$V_{LS} = 10.463 (-250.9 e^{-6x}) / x$$

$${}^1P_1 \quad V(x) = 31.389 (e^{-x} - 11.08 e^{-2x} + 20.3 e^{-3x} + 465 e^{-6x}) / x$$

All the potentials are in MeV. The number 10.463 is the value adopted for  $f^2 \mu c^2$ , the strength of the O.P.E.P. In using the Reid potential we used his defined values for the physical constants, including the contribution from O.P.E.P. in high partial waves.

APPENDIX C

In this section we propose to give a simpler method to determine the various coefficients in Equation (V-1) reproduced here:

$$\begin{aligned} \langle k' | G | k \rangle = & a + ic (\underline{\sigma}_1 + \underline{\sigma}_2) \cdot \underline{N} + m (\underline{\sigma}_1 \cdot \underline{N} \underline{\sigma}_2 \cdot \underline{N}) + \\ & (g+h) (\underline{\sigma}_1 \cdot \underline{P} + \underline{\sigma}_2 \cdot \underline{P}) + (g-h) (\underline{\sigma}_1 \cdot \underline{K} \underline{\sigma}_2 \cdot \underline{K}) \end{aligned} \quad (C-1)$$

where  $\underline{N}$ ,  $\underline{P}$  and  $\underline{K}$  are given by (V-2). The G-matrix can also be written in the singlet-triplet representation as in Chapter III.

When this is done it is seen by geometrical arguments

$$\begin{aligned} a &= \frac{1}{4} (2G_{11} + G_{00} + G_{33}) \\ c &= \frac{1}{\sqrt{8}} (e^{i\phi} G_{10} - e^{-i\phi} G_{01}) \\ m &= \frac{1}{4} (G_{00} - G_{33} - 2e^{2i\phi} G_{1,-1}) \\ g &= \frac{1}{4} (G_{11} - G_{33} + e^{2i\phi} G_{1,-1}) \\ h &= \frac{1}{4 \cos \theta} (G_{11} - G_{00} - e^{2i\phi} G_{1,-1}) = \frac{1}{\sqrt{8} \sin \theta} (G_{10} e^{i\phi} + G_{01} e^{-i\phi}) \end{aligned} \quad (C-2)$$

and the relations

$$\begin{aligned}
 G_{11} &= G_{-1-1} \\
 G_{1,-1}(\theta, \phi) &= G_{-11}(\theta, -\phi) \\
 G_{10}(\theta, \phi) &= -G_{-10}(\theta, -\phi) \\
 G_{01}(\theta, \phi) &= -G_{0,-1}(\theta, -\phi)
 \end{aligned} \tag{C-3}$$

and the useful check relation

$$G_{11} - G_{00} - e^{2i\phi} G_{1,-1} = \sqrt{2} \cot \theta (e^{i\phi} G_{10} + e^{-i\phi} G_{01}) \tag{C-4}$$

The point which will be exploited here is that the G-matrix has exactly the same structure in terms of spin operators and rotational tensors as does the nucleon-nucleon scattering matrix. Thus we can simply take over the results of Stapp (40) who has given the partial wave expansion of  $M_{ij}$  in terms of phase shifts or more precisely the partial wave scattering amplitude  $\alpha_{ij}$ .

$$G_{ss} = f_c(\theta) + f_c(\pi - \theta) + \sum_{\text{even } L} 2(2L+1) \alpha_L P_L(\theta) / 2ik$$

$$G_{11} = f_c(\theta) - f_c(\pi - \theta) + \sum_{\text{odd } L} \left\{ (L+2) \alpha_L^{L+1} + (2L+1) \alpha_L^L + (L-1) \alpha_L^{L-1} - U \right\} \frac{P_L(\theta)}{2ik}$$

$$G_{00} = f_c(\theta) - f_c(\pi - \theta) + \sum_{\text{odd } L} \left\{ 2(L+1) \alpha_L^{L+1} + 2L \alpha_L^{L-1} + U \right\} \frac{P_L(\theta)}{2ik}$$

$$G_{01} = \sum_{\text{odd } L} \left\{ -\sqrt{2} \frac{(L+2)}{L+1} \alpha_L^{L+1} + \frac{\sqrt{2} (2L+1)}{L(L+1)} \alpha_L^L + \frac{\sqrt{2} (L-1)}{L} \alpha_L^{L-1} + V \right\} \frac{P_L'(\theta)}{2ik} e^{i\phi}$$

$$G_{10} = \sum_{\text{odd } L} \left\{ \sqrt{2} \alpha_L^{L+1} - \sqrt{2} \alpha_L^{L-1} + V \right\} \frac{P_L'(\theta)}{2ik} e^{-i\phi} \quad (\text{C-5})$$

$$G_{1-1} = \sum_{\text{odd } L} \left\{ \frac{\alpha_L^{L+1}}{L+1} - \frac{2L+1}{L(L+1)} \alpha_L^L + \frac{\alpha_L^{L-1}}{L} - \frac{\alpha^{L-1}}{\sqrt{L(L+1)}} - \frac{\alpha^{L+1}}{\sqrt{(L+1)(L+2)}} \right\} \frac{P_L^2(\theta)}{2ik} e^{-2i\phi}$$

where

$$U = \sqrt{(L+1)(L+2)} \alpha^{L+1} + \sqrt{L(L-1)} \alpha^{L-1}$$

$$V = \sqrt{\frac{L+2}{L+1}} \alpha^{L+1} - \sqrt{\frac{L-1}{L}} \alpha^{L-1}$$

For  $T = 0$ , summations over odd and even  $L$  interchange. For nuclear matter  $f$ 's are of course zero.

If we now combine Equations (C-5) and (C-2) for the coefficient  $c$  we shall have

$$c = \frac{1}{\sqrt{8}} \sum_{\text{odd } L} \left\{ \sqrt{2} \frac{2L+3}{L+1} \alpha_L^{L+1} - \sqrt{2} \frac{2L+1}{L(L+1)} \alpha_L^L - \sqrt{2} \frac{2L-1}{L} \alpha_L^{L-1} \right\} \frac{P_L'(\theta)}{2ik}$$

$$= \frac{1}{4ik} \sum_{\text{odd } L} \sqrt{\frac{4\pi L(L+1)}{2L+1}} \left\{ \begin{array}{l} -\frac{2L+3}{L+1} \alpha_L^{L+1} \quad J=L+1 \\ + \frac{2L+1}{L(L+1)} \alpha_L^L \quad J=L \\ + \frac{2L-1}{L} \alpha_L^{L-1} \quad J=L-1 \end{array} \right\} e^{-i\phi} \frac{1}{\sqrt{L}} P_L'(\theta) \quad (\text{C-6})$$



Using Equations (V-18), (V-19) and (V-27) we can also write after defining

$$\beta_L^J = \sum_{L'} \int g_L(k_0 r) v_{LL'}^J u_{L'L}^J(k_0 r) dr \quad (C-7)$$

$$c = \frac{4\pi}{2k_0^2} \sum_{\text{odd } L} \sqrt{\frac{4\pi L(L+1)}{2L+1}} \left\{ \begin{array}{l} -\frac{2L+3}{L+1} \quad J=L+1 \\ \frac{2L+1}{L(L+1)} \quad J=L \\ \frac{2L-1}{L} \quad J=L-1 \end{array} \right\} e^{-i\phi} Y_L^1 \cdot \beta_L^J \quad (C-8)$$

Thus comparing (C-6) and (C-8) it is immediately seen that

$$\frac{1}{2ik} \alpha_L^J = \frac{4\pi}{k_0^2} \beta_L^J \quad (C-9)$$

$$= \frac{4\pi}{k_0^2} \sum_{L'} \int g_L(k_0 r) v_{LL'}^J(r) u_{L'L}^J(k_0 r) dr \quad (\text{C-10})$$

$$\equiv \frac{4\pi (r^2 + k_0^2)}{m^* k_0^2} \int g_L(k_0 r) \chi_{LL}(k_0 r) dr \quad (\text{C-11})$$

$$\equiv \langle k_0 | G^J | k_0 \rangle \quad (\text{C-12})$$

With this identification between the scattering amplitudes and the nuclear matter G-matrix elements it is now possible to easily calculate the coefficients a, c, m, g and h in Equation (C-1).

As a first application of this procedure let us calculate the coefficient 'a'. This should give us Equation (II-25) (Equation 6-14, B.B.P.) when we take  $\frac{3}{4}$  (T = 1) +  $\frac{1}{4}$  (T = 0) states. Thus

$$a = \frac{1}{2} \left[ \sum_{eL} (2L+1) \alpha_L + \sum_{\text{odd } L} \left\{ (2L+3) \alpha_L^{L+1} + (2L+1) \alpha_L^L + (2L-1) \alpha_L^{L-1} \right\} \right] \frac{P_L(\theta)}{2ik} \quad (\text{C-13})$$

using Equations (C-2) and (C-5).

Putting in the values of  $\alpha_L$  from (C-9) we get

$$\begin{aligned}
 a \Big|_{T=1} &= \frac{4\pi}{2k_0^2} \left[ \sum_{\text{even } L} (2L+1) \beta_L + \sum_{\text{odd } L} (2L+3) \beta_L^{L+1} + (2L+1) \beta_L^L + (2L-1) \beta_L^{L-1} \right] P_L(\theta) \\
 &= \frac{4\pi}{2k_0^2} \left[ \sum_{\text{even } L} (2L+1) \int g_L(k_0 r) v_L(r) u_L(k_0 r) dr \right. \\
 &\quad \left. + \sum_{\text{odd } L, L'} \sum_J (2J+1) \int g_L(k_0 r) v_{LL'}^J(r) u_{L'L}^J(k_0 r) dr \right] \tag{C-14}
 \end{aligned}$$

Similarly for  $T = 0$  this becomes

$$\begin{aligned}
 a \Big|_{T=0} &= \frac{4\pi}{2k_0^2} \left[ \sum_{\text{odd } L} (2L+1) \int g_L(k_0 r) v_L(r) u_L(k_0 r) dr \right. \\
 &\quad \left. + \sum_{\text{even } L, L'} \sum_J (2J+1) \int g_L(k_0 r) v_{LL'}^J(r) u_{L'L}^J(k_0 r) dr \right] \tag{C-15}
 \end{aligned}$$

Now taking  $\frac{3}{4}$  of (C-14) and  $\frac{1}{4}$  of (C-15) we get

$$\begin{aligned}
 a &= \frac{4\pi}{8k_0^2} \left[ \sum_{\text{odd } L} (2L+1) \int g_L(k_0 r) v_L(r) u_L(k_0 r) dr \right. \\
 &\quad + 3 \sum_{\text{even } L} (2L+1) \int g_L(k_0 r) v_L(r) u_L(k_0 r) dr \\
 &\quad + 3 \sum_{\text{odd } L, L'} \sum_J (2J+1) \int g_L(k_0 r) v_{LL'}^J(r) u_{L'L}^J(k_0 r) dr \\
 &\quad \left. + \sum_{\text{even } L, L'} \sum_J (2J+1) \int g_L(k_0 r) v_{LL'}^J(r) u_{L'L}^J(k_0 r) dr \right] \tag{C-16}
 \end{aligned}$$

This corresponds exactly with Equation (II.25a) (Equation 6.14a B.B.P.).

Following the same procedure it is possible to determine the other coefficients m, g and h also. Accumulating in one place these coefficients are for  $T = 1$  (Interchange odd and even  $L$  for  $T = 0$ ):

$$a = \frac{1}{2} \sum_{\text{odd } L} \left\{ (2L-1)\alpha_L^{L-1} + (2L+1)\alpha_L^L + (2L+3)\alpha_L^{L+1} \right\} \frac{P_L(\theta)}{2ik} + \sum_{\text{even } L} (2L+1)\alpha_L \frac{P_L(\theta)}{2ik}$$

$$c = -\frac{1}{2} \sum_{\text{odd } L} \left\{ (L+1)(2L-1)\alpha_L^{L-1} + (2L+1)\alpha_L^L - L(2L+3)\alpha_L^{L+1} \right\} \frac{\partial P_L(\theta)}{\partial(\cos\theta)} \frac{\sin\theta}{2ik}$$

$$m = \frac{1}{2} \sum_{\text{odd } L} \left\{ \alpha_L^{L-1} - \alpha_L^L + \alpha_L^{L+1} + \frac{4u}{2L+1} \right\} (2L+1) \frac{P_L(\theta)}{2ik}$$

$$+ \sum_{\text{odd } L} \left\{ -(L+1)\alpha_L^{L-1} + (2L+1)\alpha_L^L - L\alpha_L^{L+1} + 4L(L+1)v \right\} \frac{1}{L(L+1)} \frac{\partial P_L(\theta)}{\partial(\cos\theta)} \cdot \frac{\cos\theta}{2ik}$$

$$- \frac{1}{2} \sum_{\text{even } L} (2L+1)\alpha_L \frac{P_L(\theta)}{2ik}$$

$$g = \frac{1}{2} \sum_{\text{odd } L} \left\{ -\alpha_L^{L-1} + (2L+1)\alpha_L^L + \alpha_L^{L+1} - 2u \right\} \frac{P_L(\theta)}{2ik}$$

$$+ \frac{1}{2} \sum_{\text{odd } L} \left\{ (L+1)\alpha_L^{L-1} - (2L+1)\alpha_L^L + L\alpha_L^{L+1} + 2L(L+1)v \right\} \frac{1}{L(L+1)} \frac{\partial P_L(\theta)}{\partial(\cos\theta)} \cdot \frac{\cos\theta}{2ik}$$

$$- \frac{1}{2} \sum_{\text{even } L} (2L+1)\alpha_L \frac{P_L(\theta)}{2ik}$$

$$h = \frac{1}{2} \sum_{\text{odd } L} \left\{ -(L+1)\alpha_L^{L-1} + (2L+1)\alpha_L^L - L\alpha_L^{L+1} - 4vL(L+1) \right\} \frac{1}{L(L+1)} \frac{\partial P_L(\theta)}{\partial(\cos\theta)} \cdot \frac{1}{2ik}$$

where  $\frac{\alpha_L^J}{2ik}$  are given by Equations (C-9 to C-12). Once the tensor or quadratic spin-orbit forces are scrambled in terms of  $m$ ,  $g$  and  $h$  it is, at least in principle, possible to determine them.

Table I

A comparison between second order corrections evaluated at seven points and the one-point approximation.  
Reid potential.

$k_F$ ( $F^{-1}$ )	One point Correction	Seven point Correction	(MeV)
0.7	2.21	2.19	
0.9	4.35	4.33	
1.1	6.11	6.16	
1.36	4.19	3.61	
1.43	5.87	6.20	

Table II

L values occurring in the summation for the third order correction,  $J = 1$ .

L	$L_1$	$L_2$	Term
0	2	0	$F_{20} F_{00} F_{20}$
0	0	0	$F_{00} F_{00} F_{00}$
0	0	2	$F_{00} F_{20} F_{02}$
0	2	2	$F_{20} F_{20} F_{22}$
2	0	0	$F_{02} F_{02} F_{00}$
2	2	0	$F_{22} F_{02} F_{20}$
2	0	2	$F_{02} F_{22} F_{02}$
2	2	2	$F_{22} F_{22} F_{22}$

Table III

## Third order correction.

Accuracy of the two- $k'$  approximation compared to the seven- $k'$  approximation for  $\langle k''' / \chi(k') \rangle$ . Line three is the most reasonable of the two-point values.

k' values		III order Correction (MeV)
$k' / k_F < 1$	$k' / k_F > 1$	
.95	1.0	1.36
.775	1.0	2.18
.55	1.0	2.03
.22, .56 )	1.5, 3.125 )	1.85
.775, .937 )	5.125 )	



Table IV

Tabulation of Binding Energy statewise for the (1) Reid, (2) Hamada-Johnston, and (3) Bressel-Kerman Revised potentials.  $\Delta = 0.6$  throughout, and  $m^*$  given for each potential in the above order.

$k_F$	1.2	1.36	1.43	1.5	1.6	1.7
$m^*$	.952, .952	.949, .949 .98	.957, .957 .97	.96, .96 .97	.959, .959 .97	.97
$1s_0$ (1)	-23.52	-29.92	-32.55	-35.47	-39.19	
(2)	-23.18	-29.48	-32.16	-34.86	-38.72	
(3)		-30.82	-34.00	-37.12	-41.55	-45.85
$1D_2$ (1)	-3.11	-5.40	-6.71	-8.19	-10.69	
(2)	-3.03	-5.32	-6.63	-8.15	-10.72	
(3)		-5.50	-6.83	-8.37	-10.99	-14.09
$3s_1$ (1)	-32.54	-37.24	-38.93	-40.68	-41.69	
(2)	-32.49	-36.88	-38.46	-39.73	-40.94	
(3)		-40.47	-42.79	-44.82	-47.30	-49.00
$3D_1$ (1)	1.11	2.21	2.82	3.55	4.83	
(2)	1.10	2.67	2.62	3.53	4.86	
(3)		2.04	2.73	3.48	4.72	6.23
$3D_2$ (1)	-5.34	-9.04	-11.05	-13.32	-17.00	
(2)	-4.58	-7.77	-9.53	-11.50	-14.74	
(3)		-8.50	-10.34	-12.48	-15.93	-19.84
$1P_1$ (1)	3.38	5.95	7.54	9.45	12.91	
(2)	4.01	6.27	7.54	9.01	11.51	
(3)		6.62	7.94	9.48	12.11	15.37

$3P_0$	(1)	-5.66	-7.96	-8.98	-10.04	-11.49	
	(2)	-4.62	-6.45	-7.27	- 8.10	-9.22	
	(3)		-6.90	-7.81	-8.73	-10.00	-11.18
$3P_1$	(1)	12.93	20.55	24.72	29.42	37.25	
	(2)	12.25	19.48	23.42	27.88	35.27	
	(3)		19.92	23.94	28.54	36.18	45.25
$3P_2$	(1)	-8.53	-14.14	-17.21	-20.71	-26.41	
	(2)	-8.00	-13.36	-16.35	-19.74	-25.30	
	(3)		-14.37	-17.55	-21.18	-27.12	-34.11
$3F_2$	(1)	-.54	-1.04	-1.33	-1.69	- 2.28	
	(2)	-.63	-1.24	-1.61	-2.05	-2.79	
	(3)		-1.35	-1.74	-2.19	-2.99	-3.95
OPEP	(1)	2.55	4.58	5.75	7.11	9.43	
	(2)	2.59	4.67	5.86	7.26	9.63	
	(3)		5.07	6.37	7.86	10.40	13.46
Total	(1)	-59.27	-71.44	-75.93	-80.48	-84.28	
	(2)	-56.23	-67.86	-72.33	-76.37	-81.06	
	(3)	-	-74.25	-80.09	-85.95	-92.45	-97.87
B.E.	(1)	-11.75	-12.90	-12.62	-12.46	-10.59	
	(2)	-10.30	-10.9	-10.71	-10.10	-8.5	
	(3)		-14.28	-14.79	-15.20	-14.64	-13.29

TABLE V

Summary of Binding Energy Results (MeV)

$k_F$ ( $F^{-1}$ )	Hamada- Johnston	Reid	Bressel Thesis	Bressel- Kerman Revised
1.0	-8.55	-9.56		
1.1	-9.47	-10.95		
1.2	-10.30	-11.75		
1.28		-12.54		
1.36	-10.90	-12.90	-14.71	-14.28
1.43	-10.71	-12.62		-14.79
1.5	-10.07	-12.46	-16.01	-15.20
1.6	-8.50	-10.59	-15.21	-14.64
1.7			-14.36	-13.29
Saturation Binding	-10.87	-12.93	-15.80	-15.12
Density ( $F^{-1}$ )	1.35	1.37	1.51	1.50
Compressi- bility	89.8	159	251	203

Table VI

Binding Energy in MeV for Reid potential comparing various  $\Delta$  and  $m^*$   
at several densities.

$K_F$	$m^*$	$\Delta$	Ist Order	(i)Pauli	IIInd Order (ii)Spectral	(iii)Total	Total (I+II)	IIIrd Order	Total
0.70	.95	0.600	-6.34	3.87	-2.77	1.10	-5.24	-0.29	-5.53
0.80	.95	0.600	-8.06	4.49	-2.92	1.56	-6.60	-0.42	-7.02
0.90	.95	0.600	-9.88	5.14	-2.57	2.56	-7.32	-0.73	-8.05
1.00	.95	0.600	-11.70	5.77	-3.31	2.46	-9.24	-0.32	-9.56
1.10	.95	0.600	-13.36	6.44	-3.58	2.86	-10.50	-0.45	-10.95
1.20	.952	0.600	-14.86	7.17	-4.27	2.90	-11.90	+0.15	-11.75
1.20	.966	0.600	-16.03	7.74	-3.37	4.37	-11.66	-0.20	-11.86
1.28	.95	0.600	-15.38	7.69	-5.41	2.28	-11.98	+0.56	-12.54
1.36	.949	0.600	-15.48	8.37	-6.69	1.68	-13.80	0.90	-12.90
1.36	.960	0.570	-17.81	9.47	-4.43	5.03	-12.78	0.14	-12.64
1.36	.962	0.600	-16.83	8.89	-5.40	3.49	-13.33	0.67	-12.66
1.36	.970	0.535	-20.80	11.06	-1.97	9.09	-11.71	-1.29	-13.00
1.43	.957	0.600	-16.00	9.46	-7.05	2.41	-13.58	0.96	-12.62
1.50	.960	0.600	-15.66	10.55	-8.59	1.96	-13.70	1.23	-12.46
1.50	.969	0.600	-16.79	10.91	-7.15	3.76	-13.03	1.08	-11.95
1.60	.959	0.600	-13.25	12.37	-11.25	1.12	-12.13	1.54	-10.59

TABLE VII

Hamada-Johnston Potential. Binding Energy vs density

$K_F$	$m^*$		Ist Order	IIInd Order			Total (I+II)	IIIInd Order	Total
				(i) Pauli	(ii) Spectral	(iii) Total			
1.0	.97	0.6	-11.31	6.20	-2.80	3.40	-7.91	-0.64	-8.55
1.1	.97	0.6	-12.68	6.84	-3.25	3.59	-9.09	-0.38	-9.47
1.2	.952	0.6	-12.13	6.79	-5.47	1.32	-10.81	0.51	-10.30
1.36	.949	0.6	-11.86	7.98	-8.40	-0.42	-12.28	1.21	-11.07
1.36	.98	0.6	-15.38	9.24	-4.89	4.35	-11.03	0.31	-10.72
1.36	1.00	0.86	-9.52	5.95	-11.48	-5.53	-15.05	1.74	-13.31
1.43	.957	0.6	-12.16	9.06	-9.08	-0.02	-12.14	1.29	-10.85
1.43	.98	0.6	-15.05	10.03	-6.20	3.84	-11.21	.64	-10.57
1.50	.99	0.6	-15.62	11.42	-6.24	5.18	-10.44	.62	-9.82
1.50	.96	0.6	-11.45	10.13	-10.56	-.43	-11.87	1.49	-10.38
1.60	.959	0.6	-8.62	11.90	-14.08	-2.18	-10.80	1.87	-8.93
1.60	.99	0.6	-13.54	13.15	-8.71	4.44	-9.10	1.04	-8.06

TABLE VIII

Binding Energy in MeV for Bressel's Revised Potential

$K_F$	$m^*$	$\Delta$	1st Order	(i) Pauli	(ii) Spectral	(iii) Total	Total I + II	IIIrd Order	Total
1.36	.949	.6	-18.86	10.16	-5.92	4.25	-14.61	+.31	-13.29
1.36	.98	.6	-20.75	11.21	-4.27	6.94	-13.81	+.20	-14.64
1.43	.97	.6	-20.85	11.70	-5.54	6.16	-14.69	+.03	-15.20
1.50	.97	.6	-21.26	12.66	-6.63	6.03	-15.23	-.10	-14.79
1.60	.97	.6	-21.12	14.30	-8.01	6.28	-14.84	+.11	-14.50
1.70	.97	.6	-19.93	16.33	-10.04	6.30	-13.60	-.47	-14.28

TABLE IX

Binding Energy for Nuclear Matter.  $^1S_0$  by M.M.S. method; others by Reference Spectrum Method. Third Order Correction for  $^3S_1 - ^3D_1$  is not included.

$$\Delta = 0.6$$

$k_F$	$m^*$	Ist Order	IIInd Order (Pauli + Spectral)	$(1-\kappa)^{-1} G_S^{R+} \left( \frac{1}{e^R} - \frac{Q}{e^N} \right) \cdot (2U_\ell + G_S^R)$	$U_\ell \frac{Q}{e} U_\ell$	Total
1.2	.966	-13.52	1.72	-.07	-.40	-12.27
1.36	.962	-14.08	0.83	-.36	-.40	-14.0
1.50	.969	-13.78	1.28	-.69	-.40	-13.19

$$\begin{aligned} \kappa &= -.04 \text{ at } k_F = 1.2 \\ &= -.06 \text{ at } k_F = 1.36 \\ &= -.04 \text{ at } k_F = 1.50 \end{aligned}$$

TABLE X

G-matrix elements for Reid Potential at various densities

KF = .7,      = .6      m\* = .95

$1s_0$	$1p_1$	$1d_2$	$3p_0$	$3p_1$	$3d_2$	$3s_1$	$3d_1$	$3p_2$	Total
-9.51	0.02	0.	-.047	.079	0.	-16.204	0.	-.027	-25.69
-9.27	.08	-.004	-.181	.305	-.007	-15.856	0.002	-.110	-25.04
-8.61	.27	-.055	-.625	1.057	-.097	-15.216	.030	-.438	-23.65
-8.05	.41	-.162	-.996	1.696	-.289	-14.881	.089	-.813	-22.87
-7.24	.58	-.457	-1.468	2.525	-.838	-15.185	.262	-1.589	-23.07
-6.93	.65	-.673	-1.669	2.848	-1.270	-17.024	.427	-2.128	-25.32
-6.33	.70	-.878	-1.864	2.987	-1.750	-19.091	1.213	-2.665	-27.15



TABLE X (continued)

$$K_F = .8 \quad m^* = .95 \quad = .6$$

$l_{S_0}$	$l_{P_1}$	$l_{D_2}$	$z_{P_0}$	$z_{P_1}$	$z_{D_2}$	$z_{S_1}$	$z_{D_1}$	$z_{P_2}$	Total
-9.160	0.030	-0.001	-0.064	0.108	-0.001	-14.425	0.000	-0.036	-23.548
-8.848	0.110	-0.007	-0.242	0.410	-0.013	-14.016	0.004	-0.146	-22.741
-7.925	0.341	-0.094	-0.784	1.363	-0.165	-14.033	0.052	-0.572	-20.750
-7.114	0.490	-0.260	-1.171	2.069	-0.460	-12.282	.144	-1.005	-19.436
-5.865	0.642	-0.645	-1.519	2.819	-1.174	-11.488	.366	-1.847	-18.394
-5.220	0.714	-0.883	-1.580	3.012	-1.651	-11.878	.542	-2.533	-18.931
-4.370	0.754	-1.087	-1.533	2.993	-1.778	-13.575	1.115	-2.928	-20.328

Table X (continued)

$$K_F = .9, \quad \Delta = .6 \quad m^* = .95$$

$l_{S_0}$	$l_{P_1}$	$l_{D_2}$	$z_{P_0}$	$z_{P_1}$	$z_{D_2}$	$z_{S_1}$	$z_{D_1}$	$z_{P_2}$	Total
-8.867	.039	-0.	-0.083	0.144	-0.001	-12.997	.0	-0.046	-21.81
-8.480	.143	-0.012	-0.308	0.538	-0.021	-12.529	.007	-0.184	-20.84
-7.312	.409	-0.146	-0.933	1.661	-0.256	-11.276	.080	-0.712	-18.38
-6.309	.554	-0.374	-1.298	2.384	-0.659	-10.268	.205	-1.275	-16.78
-4.802	.706	-0.814	-1.477	2.998	-1.467	-8.928	.449	-2.867	-15.10
-4.028	.795	-1.041	-1.411	3.094	-1.915	-8.752	.588	-2.861	-14.90
-3.431	.831	-1.217	-1.290	2.957	-2.350	-11.616	.661	-3.325	-18.07

TABLE X (continued)

$$K_F = 1.00, \Delta = .6 \quad m^* = .95$$

$l_{S_0}$	$l_{P_1}$	$l_{D_2}$	$\bar{z}_{P_0}$	$\bar{z}_{P_1}$	$\bar{z}_{D_2}$	$\bar{z}_{S_1}$	$\bar{z}_{D_1}$	$\bar{z}_{P_2}$	Total
-8.629	0.050	-0.001	-0.103	0.813	-0.002	-12.489	0.001	-0.057	-24.434
-8.163	0.177	0.019	-0.378	0.668	-0.033	-11.343	0.010	-0.224	-19.296
-6.770	0.429	-0.211	-1.064	1.938	-0.366	-9.881	0.114	-0.855	-16.479
-5.622	0.607	-0.496	-1.375	2.638	-0.870	-9.607	0.224	-1.503	-14.729
-3.959	0.790	-0.946	-1.381	3.133	-1.691	-7.119	0.514	-2.528	-12.635
-3.143	0.907	-1.155	-1.218	3.165	-2.074	-6.687	0.614	-3.025	-12.006
-2.744	0.931	-1.257	-1.024	2.948	-2.429	-8.376	0.315	-3.020	-15.463

Table X (continued)

 $K_F = 1.1$  $\Delta = .6$  $m^* = .95$ 

$1S_0$	$1P_1$	$1D_2$	$3P_0$	$3P_1$	$3D_2$	$3S_1$	$3D_1$	$3P_2$	Total
-8.437	0.062	-0.002	-0.127	0.224	-0.003	-10.886	.001	-0.067	-19.23
-7.888	0.215	-0.028	-0.451	0.806	-0.048	-10.281	.015	-0.266	-17.91
-6.287	0.527	-0.286	-1.174	2.190	-0.494	-8.639	.155	-0.999	-14.81
-5.028	0.665	-0.616	-1.408	2.845	-1.076	-7.380	.334	-1.714	-12.97
-3.270	0.910	-1.059	-1.256	3.264	-1.845	-5.647	.562	-2.814	-10.48
-2.444	1.057	-1.247	-1.021	3.271	-2.156	-5.044	.644	-3.246	-9.49
-2.032	1.064	-1.387	-0.789	3.016	-2.449	-6.467	.371	-3.783	-11.59

TABLE X (continued)

$$K_F = 1.2, \quad = .6, \quad m^* = .952$$

$1s_0$	$1p_1$	$1d_2$	$3p_0$	$3p_1$	$3d_2$	$3s_1$	$3d_1$	$3p_2$	Total
-8.261	0.075	-0.003	-0.151	0.270	-0.050	-10.120	0.001	-0.081	-18.272
-7.630	0.253	-0.040	-0.526	0.949	-0.068	-9.461	0.022	-0.311	-16.793
-5.844	0.585	-0.366	-1.260	2.399	-0.633	-7.696	0.199	-1.148	-13.518
-4.501	0.726	-0.728	-1.406	3.013	-1.265	-6.395	0.392	-1.922	-11.615
-2.689	1.044	-1.150	-1.118	3.384	-2.053	-4.656	0.584	-2.980	-8.830
-1.877	1.201	-1.335	-1.335	3.364	-2.203	-3.874	0.637	-3.346	-7.798
-1.431	1.162	-1.410	-1.410	3.069	-2.492	-4.909	0.312	-3.477	-9.415

Table X (continued)

$$K_F = 1.36 \quad \Delta = .6 \quad m^* = .95$$

$1s_0$	$1p_1$	$1d_2$	$3p_0$	$3p_1$	$3d_2$	$3s_1$	$3d_1$	$3p_2$	Total
-8.080	0.098	-0.005	-0.194	0.350	-0.008	-9.113	.003	-0.099	-17.05
-7.312	.316	-0.064	-0.647	1.184	-0.109	-8.350	.035	-0.387	-15.30
-5.259	0.657	-0.502	-1.351	2.714	-0.867	-6.424	.272	-1.382	-11.82
-3.811	0.858	-0.880	-1.350	3.224	-1.511	-5.095	.468	-2.225	- 9.73
-1.952	1.303	-1.283	-0.879	3.574	-2.059	-3.379	.619	-3.298	- 6.48
-1.166	1.460	-1.476	-0.530	3.535	-2.244	-2.745	.584	-3.759	- 5.34
-0.698	1.346	-1.640	-0.253	3.217	-2.487	-3.018	-.011	-4.225	- 6.69

TABLE X (continued)

$$K_F = 1.43, \quad = .6, \quad m^* = .957$$

$1S_0$	$1P_1$	$1D_2$	$3P_0$	$3P_1$	$3D_2$	$3S_1$	$3D_1$	$3P_2$	Total
-7.985	0.109	-0.066	-0.213	0.387	0.010	-9.232	0.003	-0.107	-16.437
-7.152	0.345	-0.077	-0.700	1.270	-0.131	-7.801	0.036	-0.422	-14.568
-4.993	0.696	-0.562	-1.373	2.828	-0.968	-5.817	0.304	-1.481	-11.004
-3.506	0.936	-0.936	-1.308	3.340	-1.597	-4.483	0.497	-2.344	-8.792
-1.637	1.434	-1.340	-0.765	3.668	-2.087	-2.789	.279	-3.386	-5.359
-.864	1.588	-1.533	-.397	3.617	-2.147	-2.147	.056	-3.797	-4.162
-.285	1.450	-1.707	-0.117	3.296	-2.277	-2.277	-.231	-3.690	-6.372

TABLE XI

Re-arrangement Energy in MeV Calculated According to Equation (IV-1)

$k_F$	$m^*$		$\frac{1}{A}$ E Total	$E_{s.p.} (k_m = k_F)$	$E_R$
1.43	.957	.6	-12.02	-24.21	12.19
1.36	.949	.6	-12.49	-26.18	13.69
1.10	.95	.6	-10.71	-21.95	11.24
0.90	.95	.6	-8.48	-17.48	9.00
0.70	.95	.6	-5.44	-12.36	6.92



TABLE XII

VLSCOEF the coefficient of  $\frac{1}{r} \frac{d\rho}{dr}$  in the spin orbit force as a function of  $k_F$ . All quantities are measured in  $F^3$  and must be multiplied by 41.469 to obtain them in MeV  $F^5$ .

$k_F$ (in $F^{-1}$ )	VLSCOEF $T=1$	VLSCOEF $T=0$	$\frac{3}{4}$ VLSCOEF $T=1$ $+\frac{1}{4}$ VLSCOEF $T=0$	VLSCOEF $T=1$ $-\text{VLSCOEF } T=0$
1.43	4.131	-1.111	2.820	5.242
1.36	4.114	-1.292	2.762	5.406
1.23	4.093	-1.896	2.595	5.989
1.10	4.056	-2.600	2.391	6.656
0.90	3.962	-4.230	1.913	8.192
0.80	3.920	-5.454	1.576	9.374
0.70	3.871	-7.165	1.112	11.036

TABLE XIII

Parameters of Hamada-Johnston Potential

State	$a_c$	$b_c$	$a_t$	$b_t$	$G_{LS}$	$b_{LS}$	$G_{LL}$	$a_{LL}$	$b_{LL}$
Singlet even	+8.7	10.6	--	--	--	--	-0.000891	+0.2	-0.2
Triplet odd	-9.07	+3.48	-1.29	+0.55	+0.1961	-7.12	-0.000891	-7.26	+6.92
Triplet even	+6.0	-1.0	-0.5	+0.2	+0.0743	-0.1	+0.00267	+1.8	-0.4
Singlet odd	-8.0	+12.0	--	--	--	--	-0.00267	+2.0	+6.0

Table XIV

## Bressel-Kerman Core Potentials

	T = 0		T = 1	
	S = 0	S = 1	S = 0	S = 1
Central	648	500	648(670)	436 (670)
Tensor		99		106 (-50)
S.L.		-160		-130 (-374)
$L_{12}$	-46	224	-46	224 (332)

These are the heights in MeV of square potentials of radius  $x = .4852$  meson Compton wave-lengths, for the Bressel Thesis Potential. The figures in brackets are for the Revised Potential.

CAPTION OF FIGURES

- Figure 1. Spherical Hankel Transforms  $F(k')$  of the reference wave function distortion, defined in Equation (II-36).  $F$  is plotted in units of  $F^2$ . Reid Potential,  $k_F = 1.36 F^{-1}$ ,  $k_0 = \sqrt{.3} k_F$ .  $F_{00}, F_{20}$  are the large and small components of the deuteron state.  $F_0$  is for the  $^1S_0$  state. Note the similarity of the S-states, and that  $F_{20}$  is large for momenta near  $k_F$ .  $k'$  is in units of  $k_F$ .
- Figure 2. Definition of the single particle potential energy for occupied states (m) and unoccupied states (b), Equations (II-49) and (II-50) In (A) and (B) a sum over occupied states (n) is implied. (B) takes account of generalised time ordering. In (C) and (D) a sum over (n) and some average over the other lines is required, since they are off the energy shell. The "suppression factor"  $f$  arises from summation of three body clusters.
- Figure 3. Suppression factor  $f(r)$  for hard core potentials as calculated by Bethe ( $c = 0.5F$ ) and for the soft core Bressel potential as calculated by us.
- Figure 4. Potential Energy of states  $k_m$  in nuclear matter.  $U$  is calculated by Equation (II-49) and  $U^N$  by Equation (II-50).  $U^C$  includes second and third order corrections to  $G^R$ .  $U^R$  is the reference spectrum,  $\Delta = 0.6$  and  $m^* = .949$ . Reid potential,  $k_F = 1.36 F^{-1}$ .  $U^M$  is the modified particle spectrum as explained in Section IV.

Figure 5. Region of Integration for the third order correction, Equation (II-46).

Figure 6. Binding Energy vs  $k_F$  for Reid, Hamada-Johnston and Bressel-Kerman potential.

Figure 7. Average Potential Energy for each state, and total  $U$  as a function of Fermi Momentum.

Figure 8. Same as Figure 7, but for

Figure 9. Same as Figure 7, but Bressel-Kerman Revised Potential.

Figure 10. Reference wave function distortion for large and small components of the deuteron state. See Figure 1 also.

$\chi$  is dimensionless.

Figure 11. A comparison of the two body spin orbit interaction in Hamada-Johnston potential (bottom most graph) and the effective two body spin orbit interaction obtained from G-matrix formalism (graph in the middle) using Equation (V-89). The top most curve is the difference of the effective spin orbit interactions calculated using the total potential and only the spin orbit potential in equation (V-89). This, of course, shows only in a first approximation that the effective spin-orbit interaction comes mainly from the two body spin orbit force, since  $u$ 's already contain the whole potential.

Figure 12. Variation of VLSCOEF vs density. Actually the variation is against  $k_F$  and density  $\rho = 2k_F^3/3\pi^2$ . VLSCOEF is the coefficient  $\frac{\partial}{\partial \lambda} \frac{1}{r} \frac{d\rho}{dr}$  in the one body spin orbit force.

Figure 12  
continued

The total VLSCOEF is given by the small dashed lines; the other top three lines refer to the  $T = 1$  state and the bottom three to  $T = 0$  state. The continuous line gives the total contribution from a particular  $T$ ; the big dashed lines are the contributions from outside the Fermi sea; the big dash small dash lines are the contributions from inside the sea.

Figure 13. Contributions to the VLSCOEF at densities corresponding to  $k_F = 1.36$  and  $0.8$ . The dashed lines are from  $k_F = 0.8F^{-1}$  and the continuous lines are from  $k_F = 1.36 F^{-1}$ . The top four lines correspond to  $T = 1$  component and the bottom lines to  $T = 0$  component.

Figure 14. Line 2 is  $100 \cdot \frac{1}{r} \frac{d\rho}{dr}$ .  
Line 1 is  $\text{VLSCOEF} \cdot \frac{1}{r} \frac{d\rho}{dr}$ , hence represents the spin orbit force.

Line 4 is the Fermi density distribution times 10.93.

Line 3 is the  $\frac{1}{40}$  (single particle potential energy) obtained from nuclear matter calculations.

Fermi density distribution for  $\text{Ca}^{40}$  has been assumed and all lines correspond to it.

Figure 15. The  $1p$  and  $1d$  splittings around  $O^{16}$  respectively, drawn as a function of the oscillator size parameter  $\alpha$ . The continuous lines are for the Fermi density distribution; the broken lines correspond to shell model density distribution.

Figure 15. The oscillator size parameter for it was kept fixed continued

at .566. Vertical lines show the range of the experimental results for n and p levels. The mean value of  $\alpha$  viz.  $\beta$  is marked at the bottom,

Figure 16. The 1d, 1f and 2p splittings around  $\text{Ca}^{40}$ . Notation same as in Figure 15.

Figure 17. The 1d, 1f and 2p splittings around  $\text{Ca}^{48}$ . Notation same as in Figure 15. The blobs show the neutron levels. These are the only ones known experimentally.

Figure 18. Spin orbit splittings around  $\text{Pb}^{208}$  variation with the size parameter is shown. Only neutron levels are shown. Fermi density distribution is assumed.

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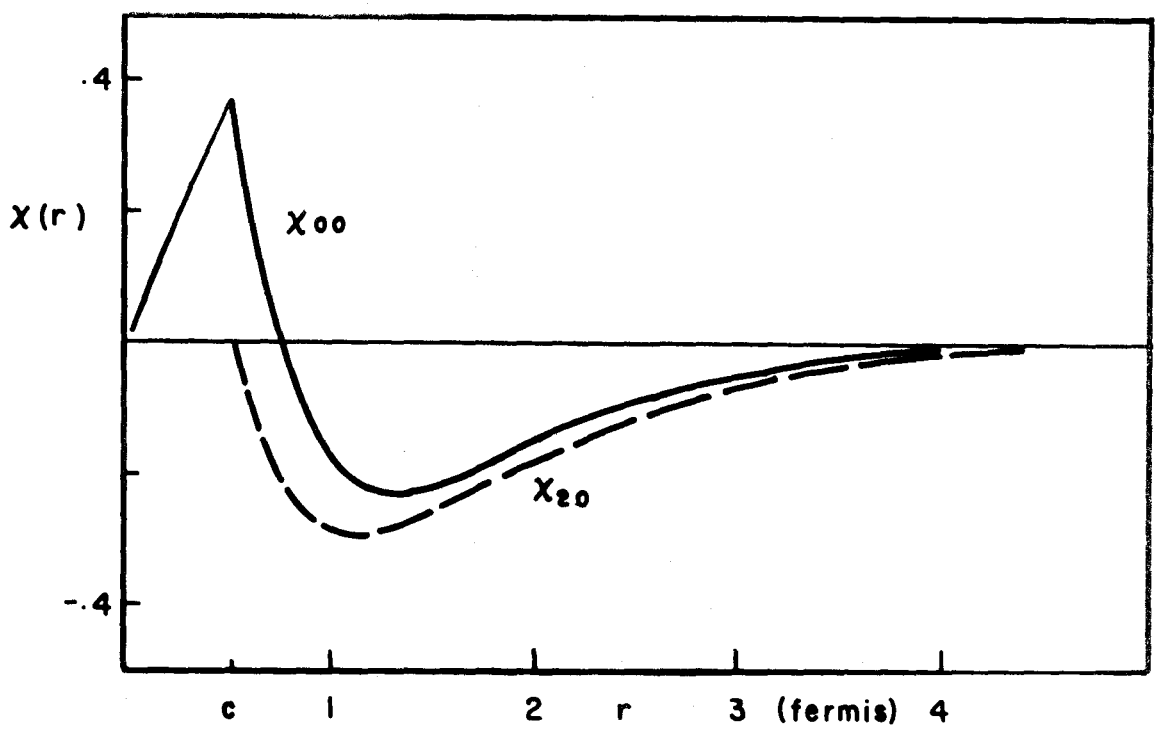


FIGURE 10

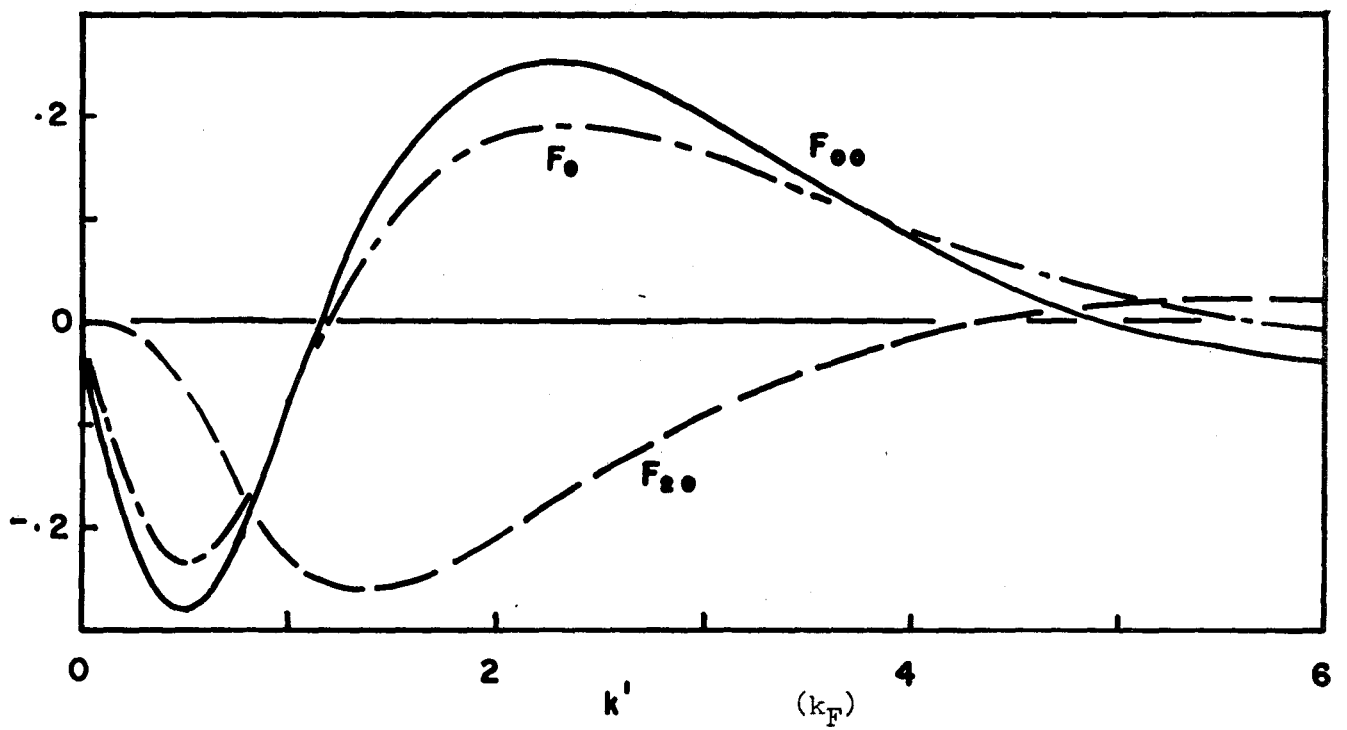


FIGURE 11

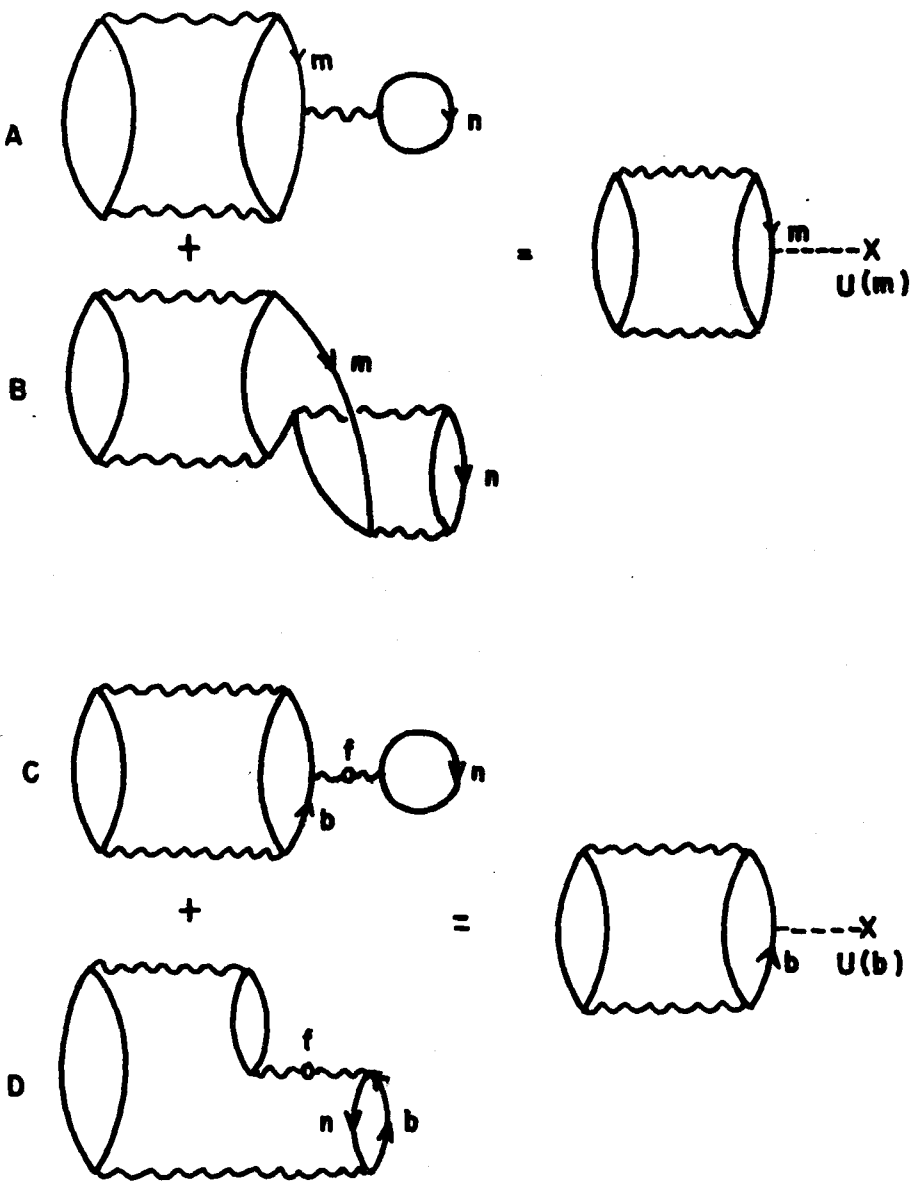


FIGURE 2

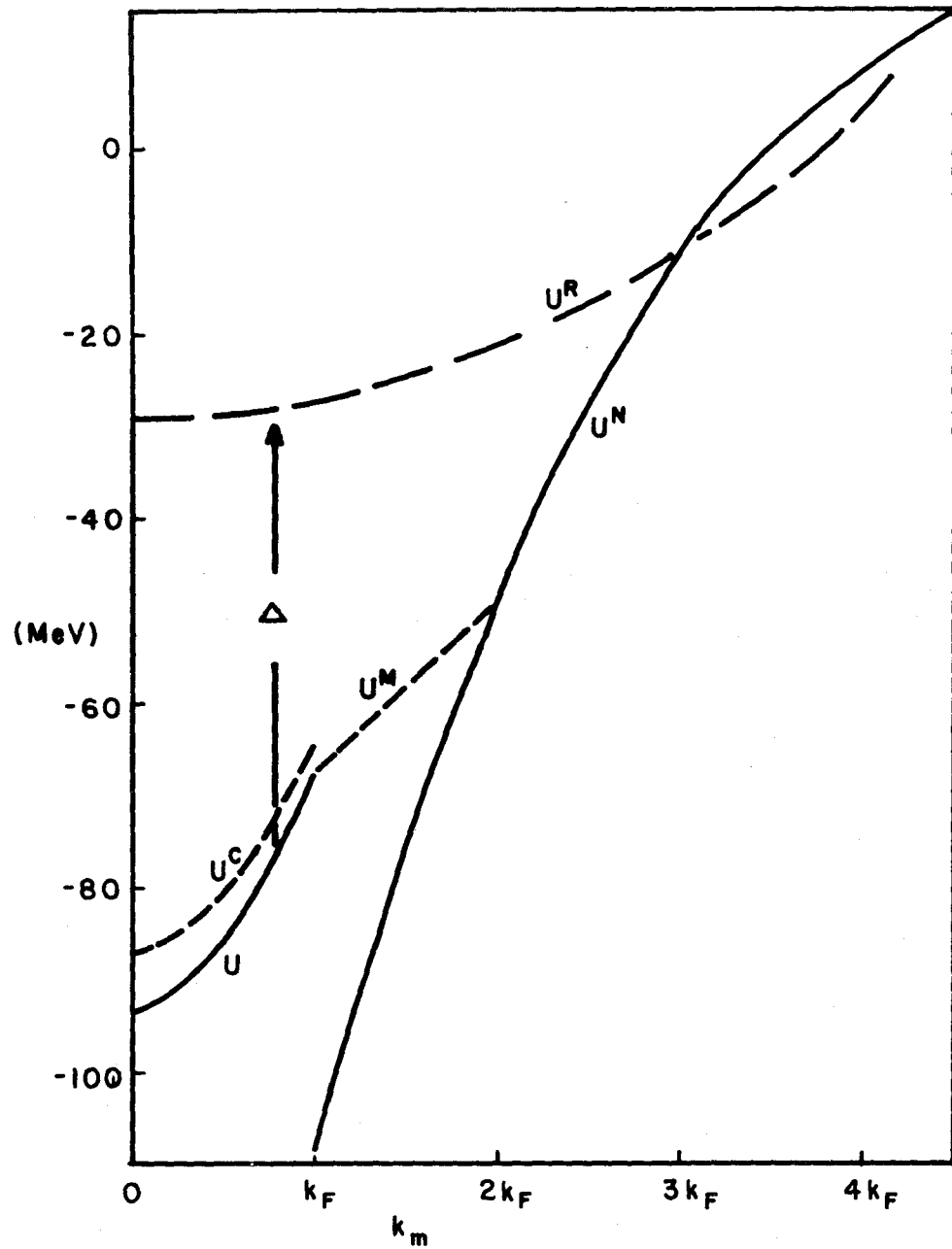


FIGURE 4

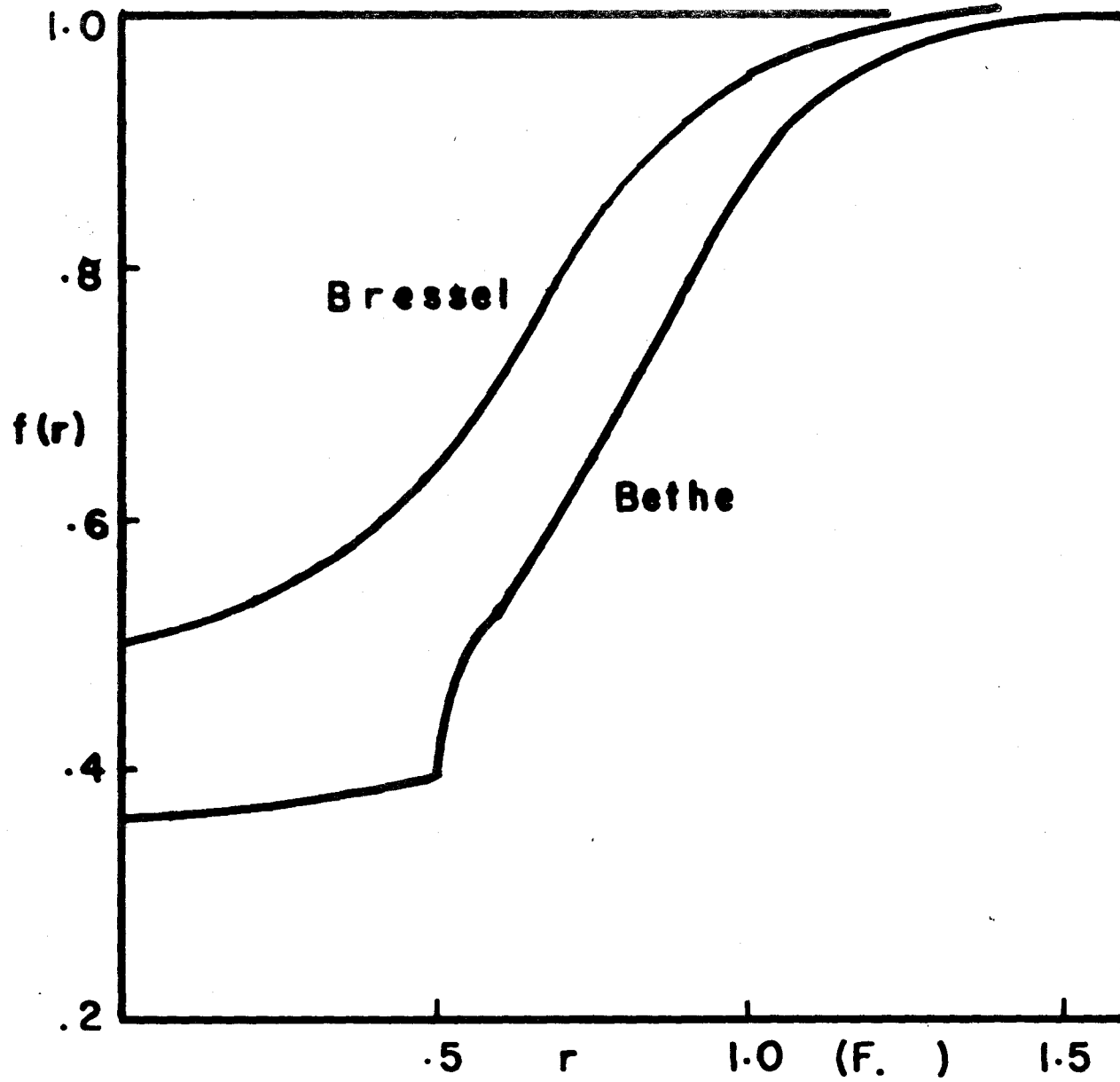


FIGURE 3

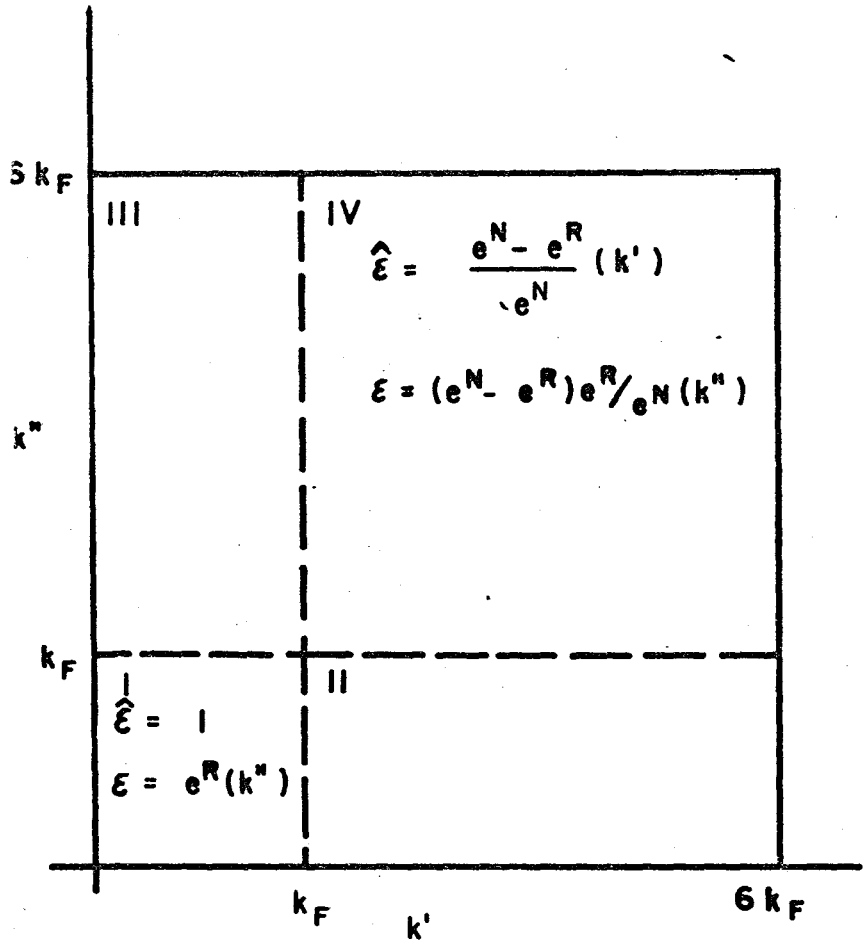


FIGURE 5

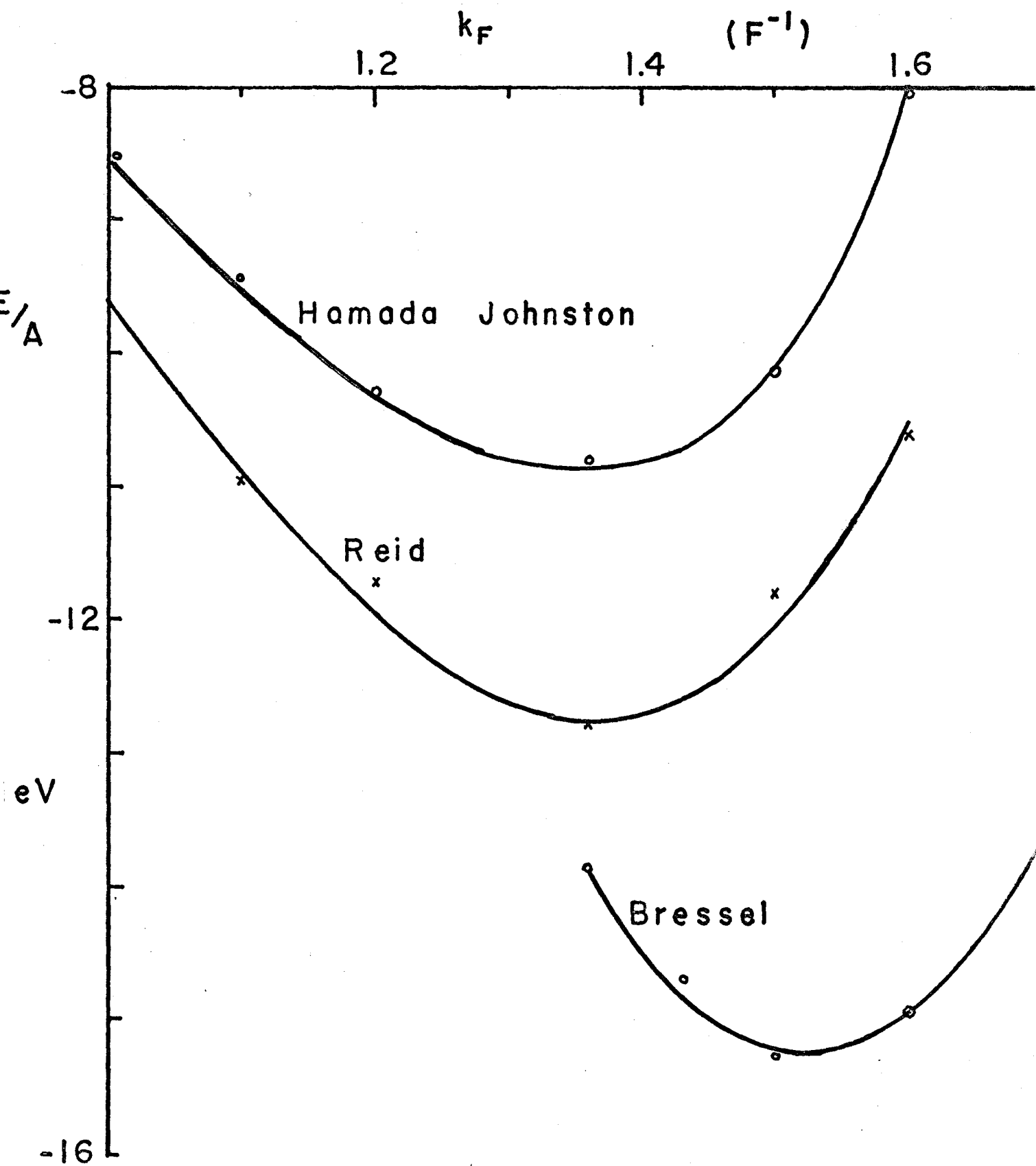


FIGURE 6

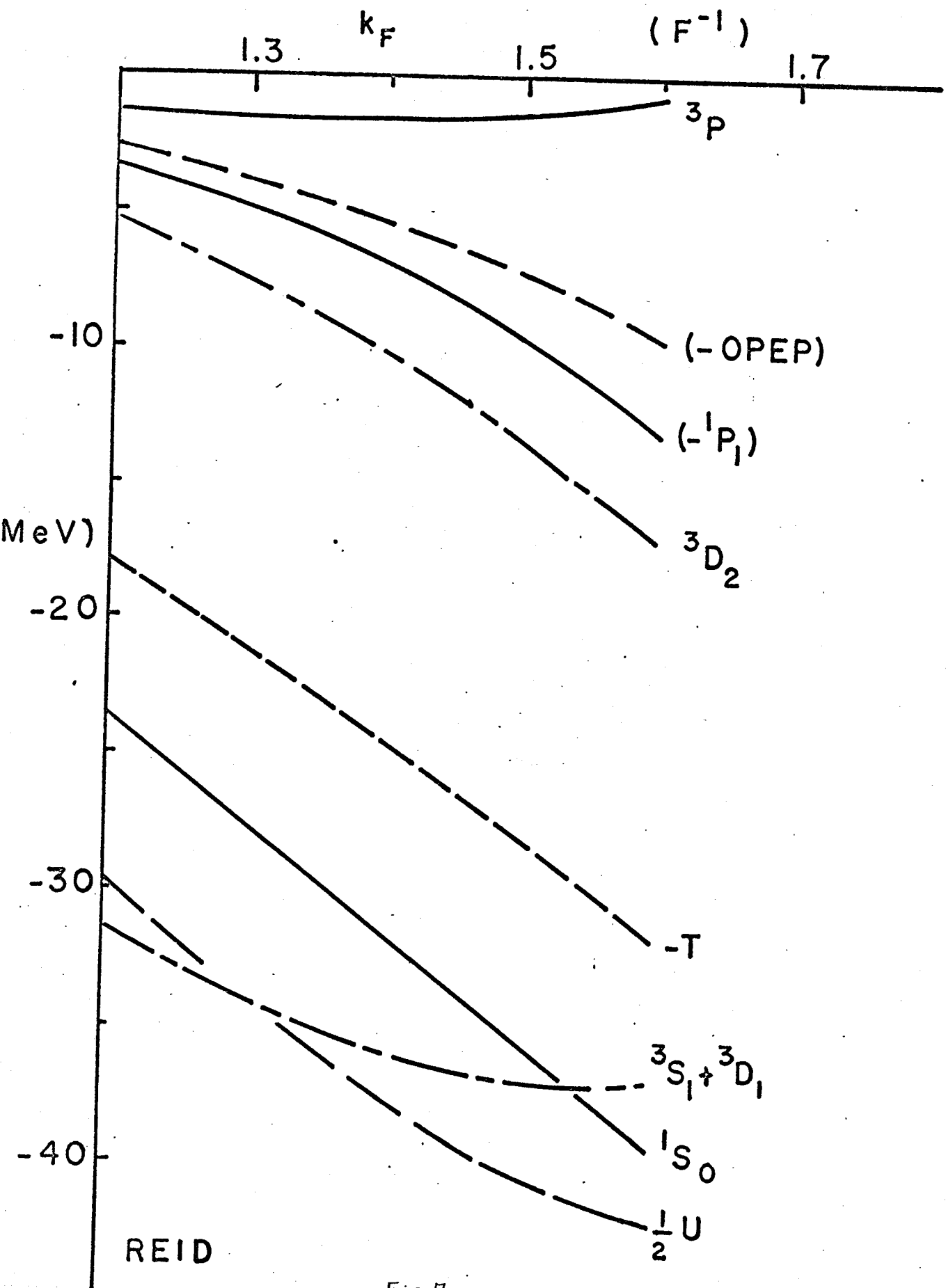
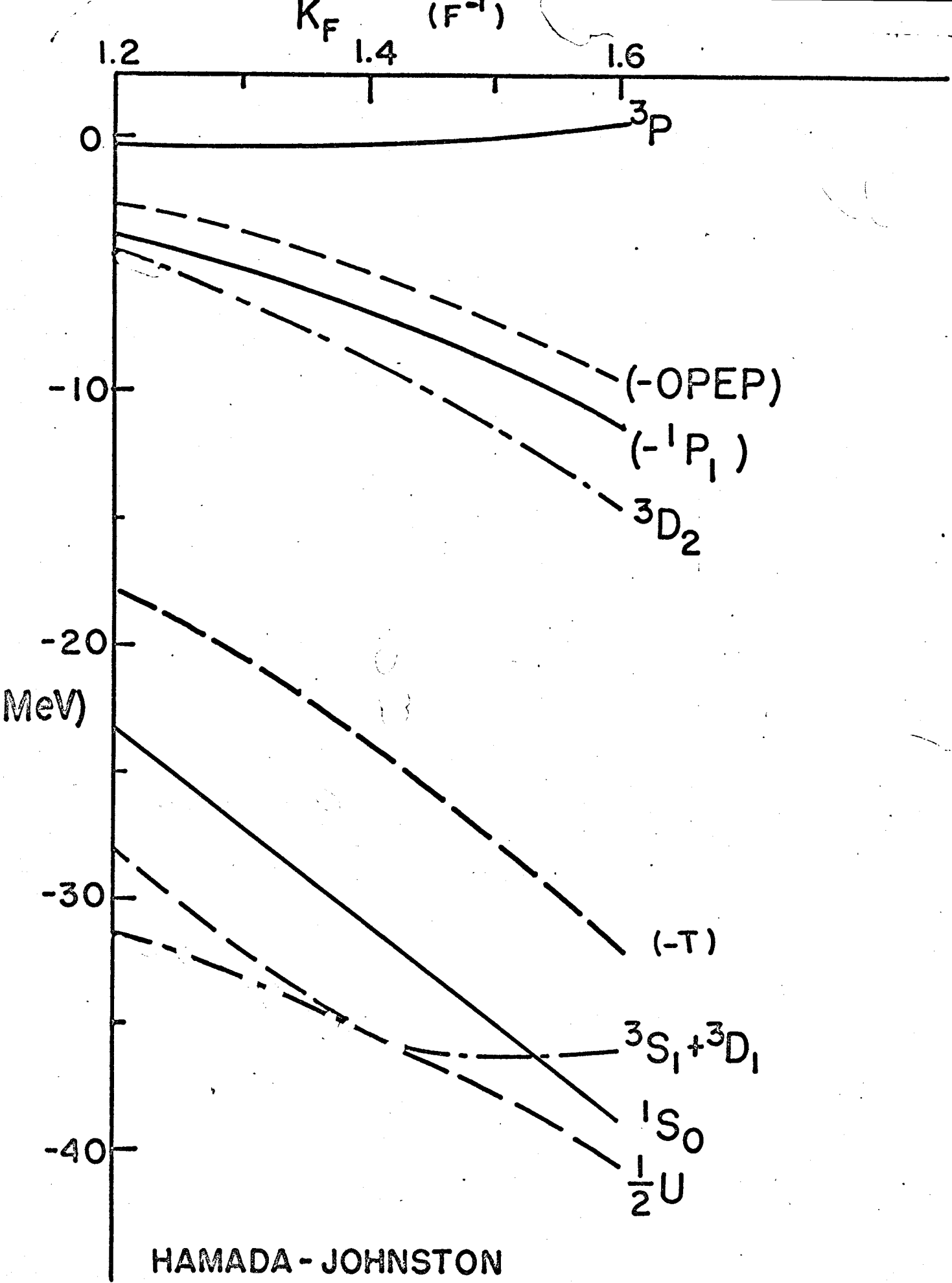


Fig 7





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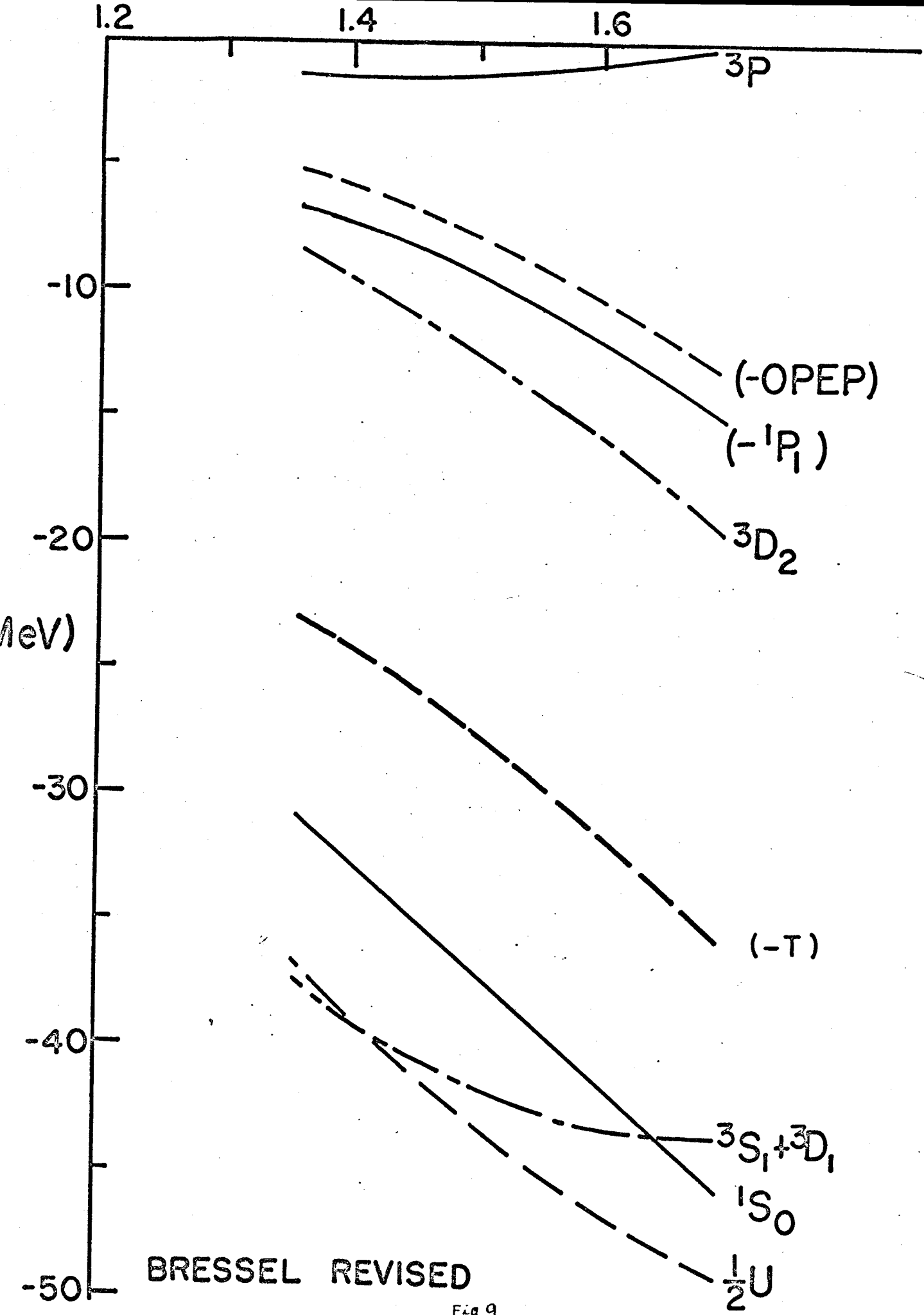
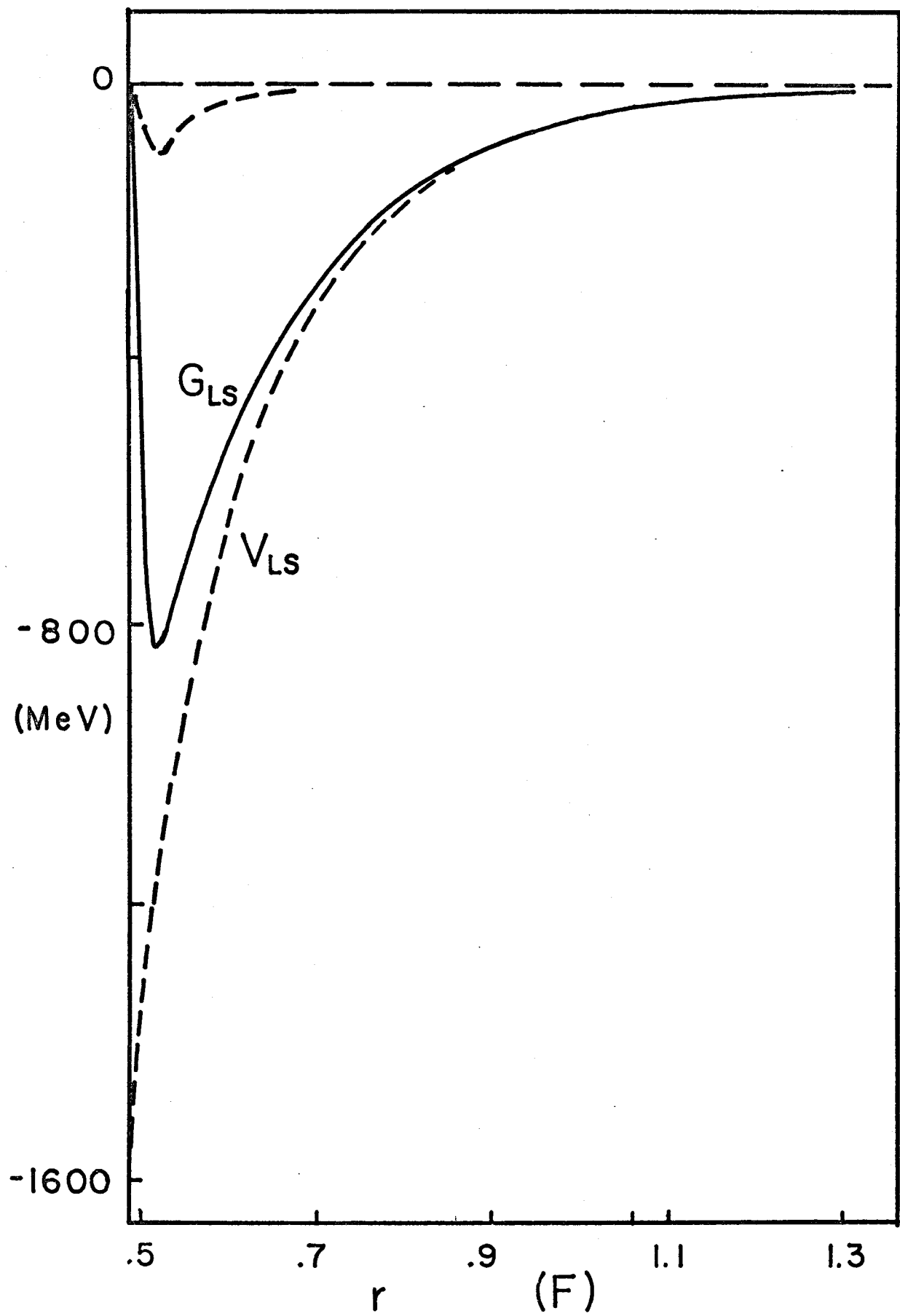
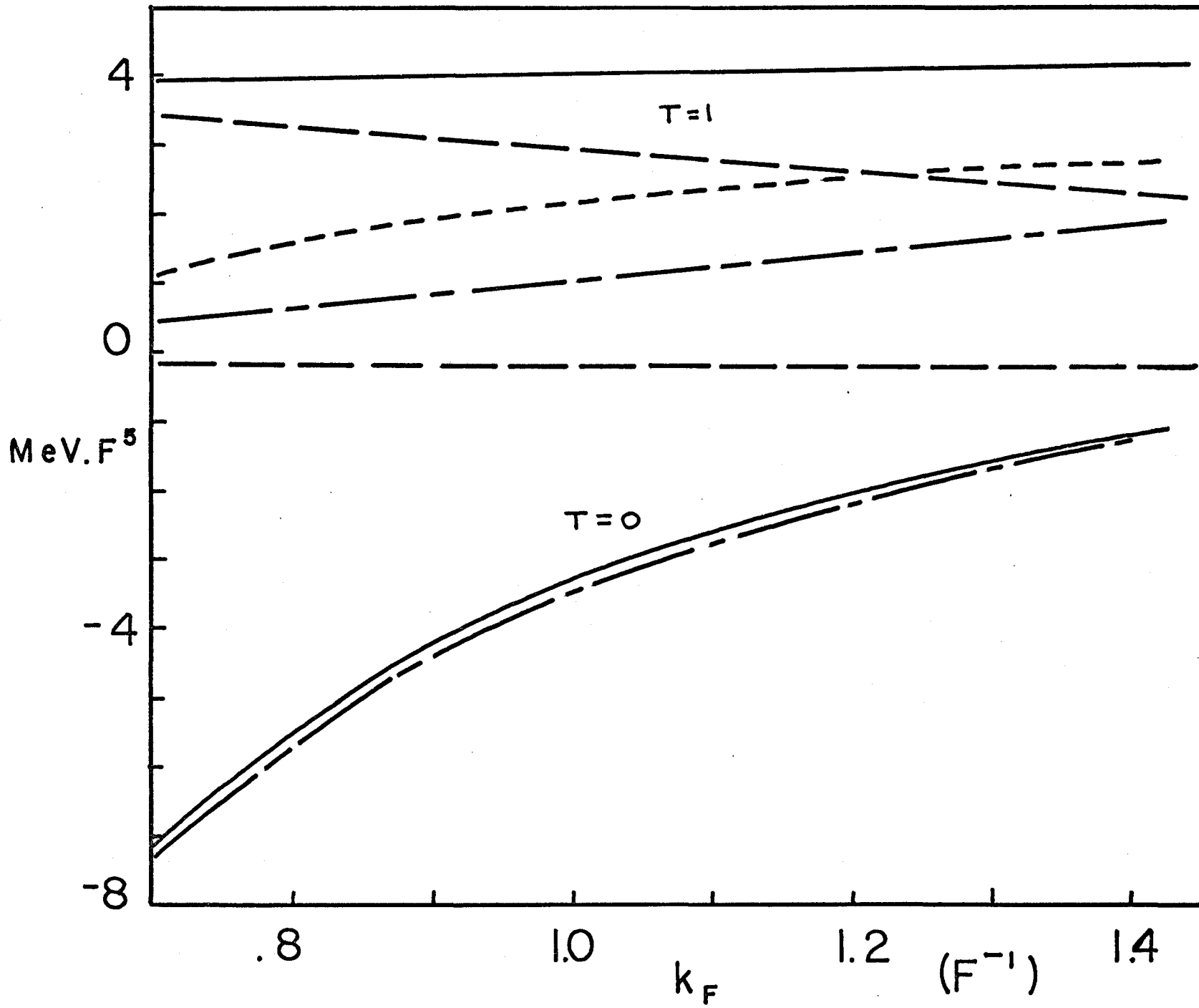
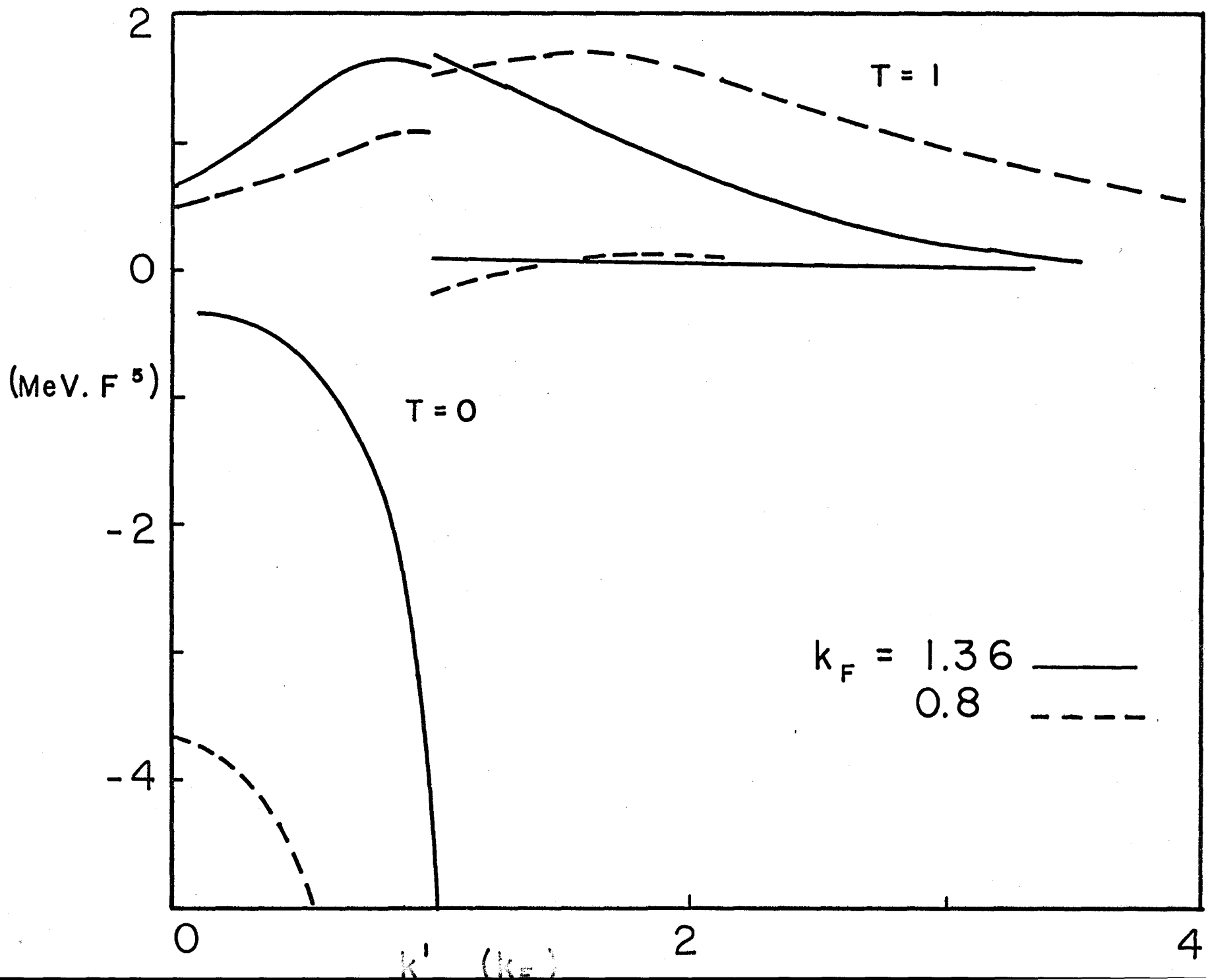
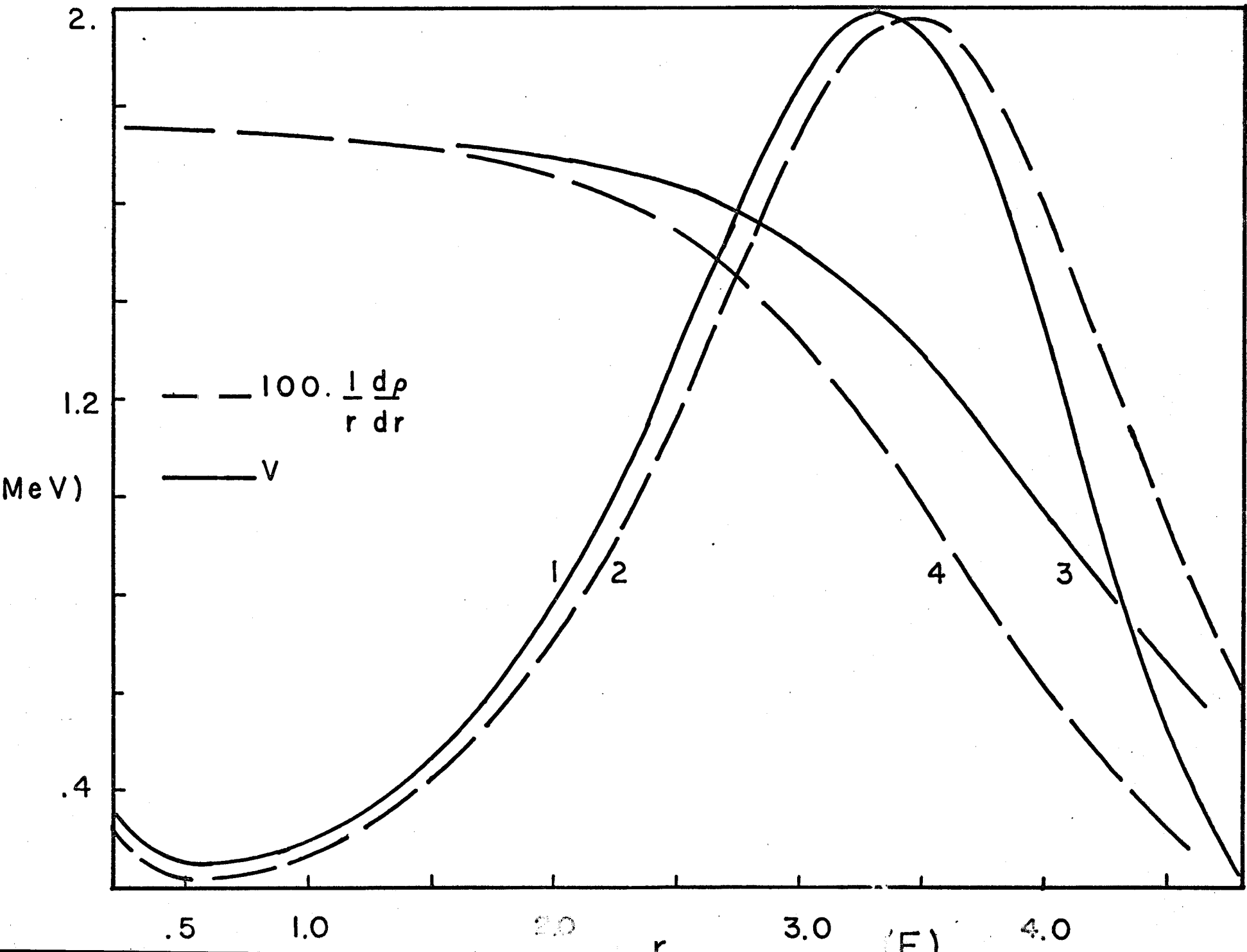


Fig 9









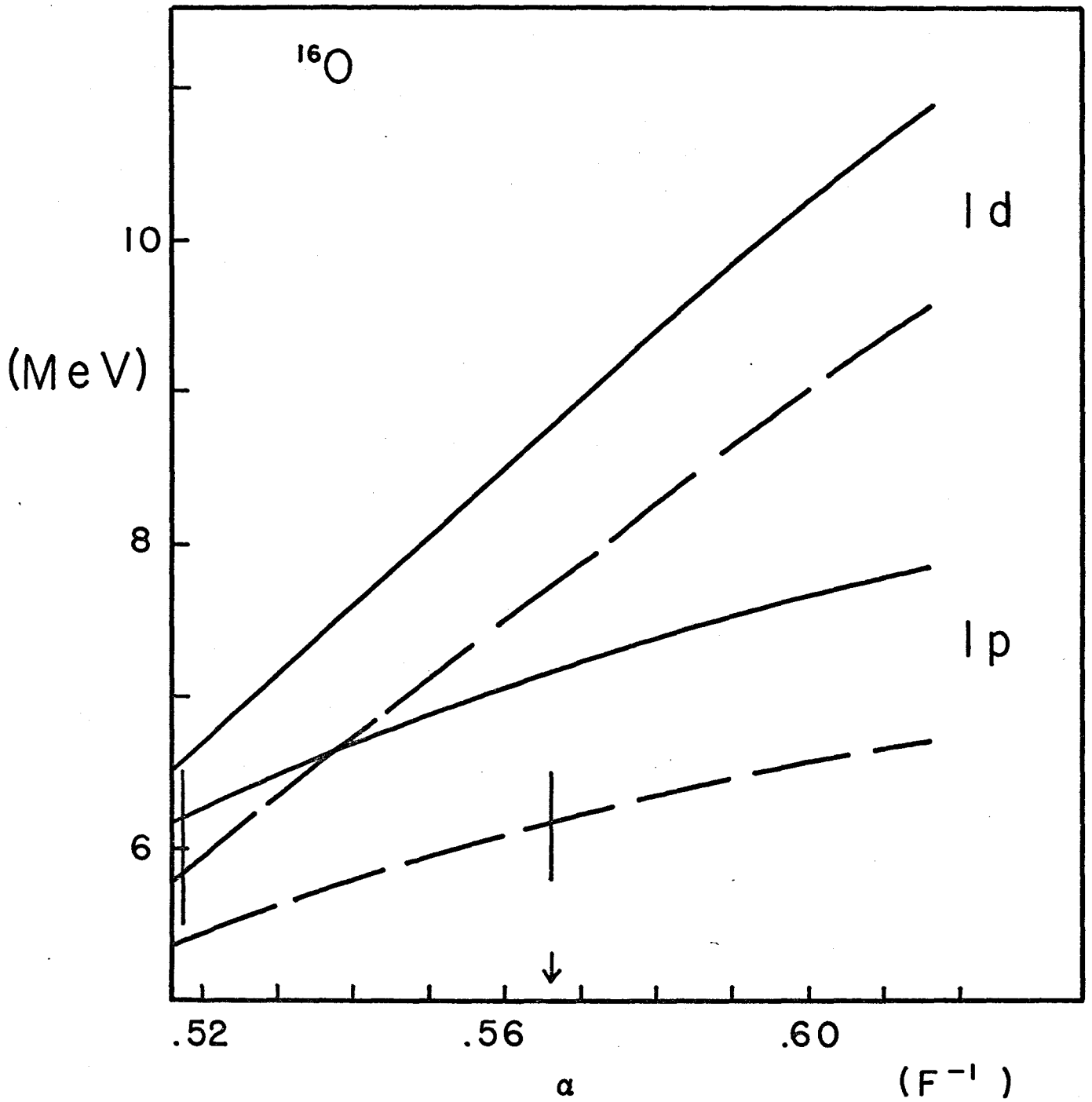


FIGURE 15





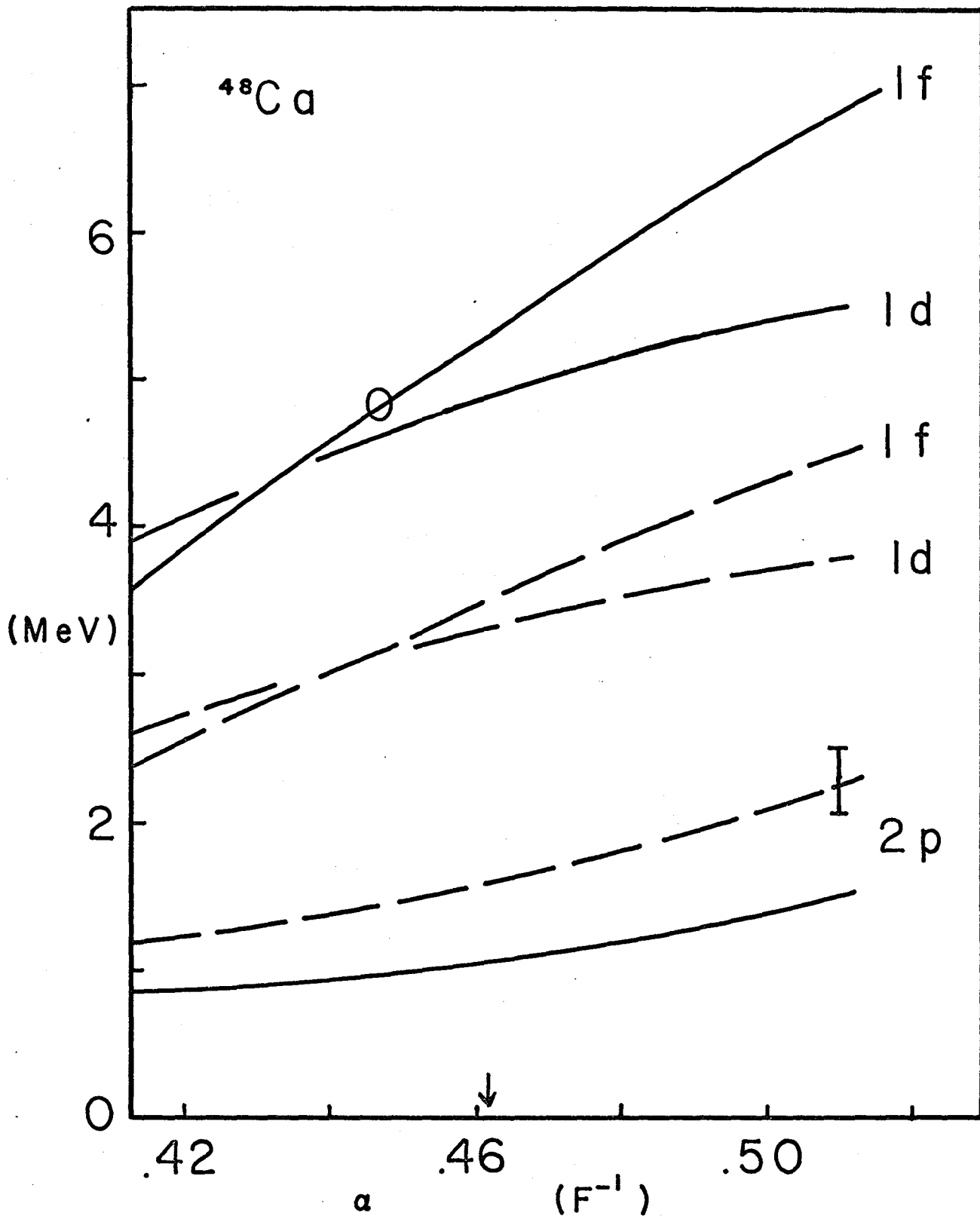


FIGURE 17

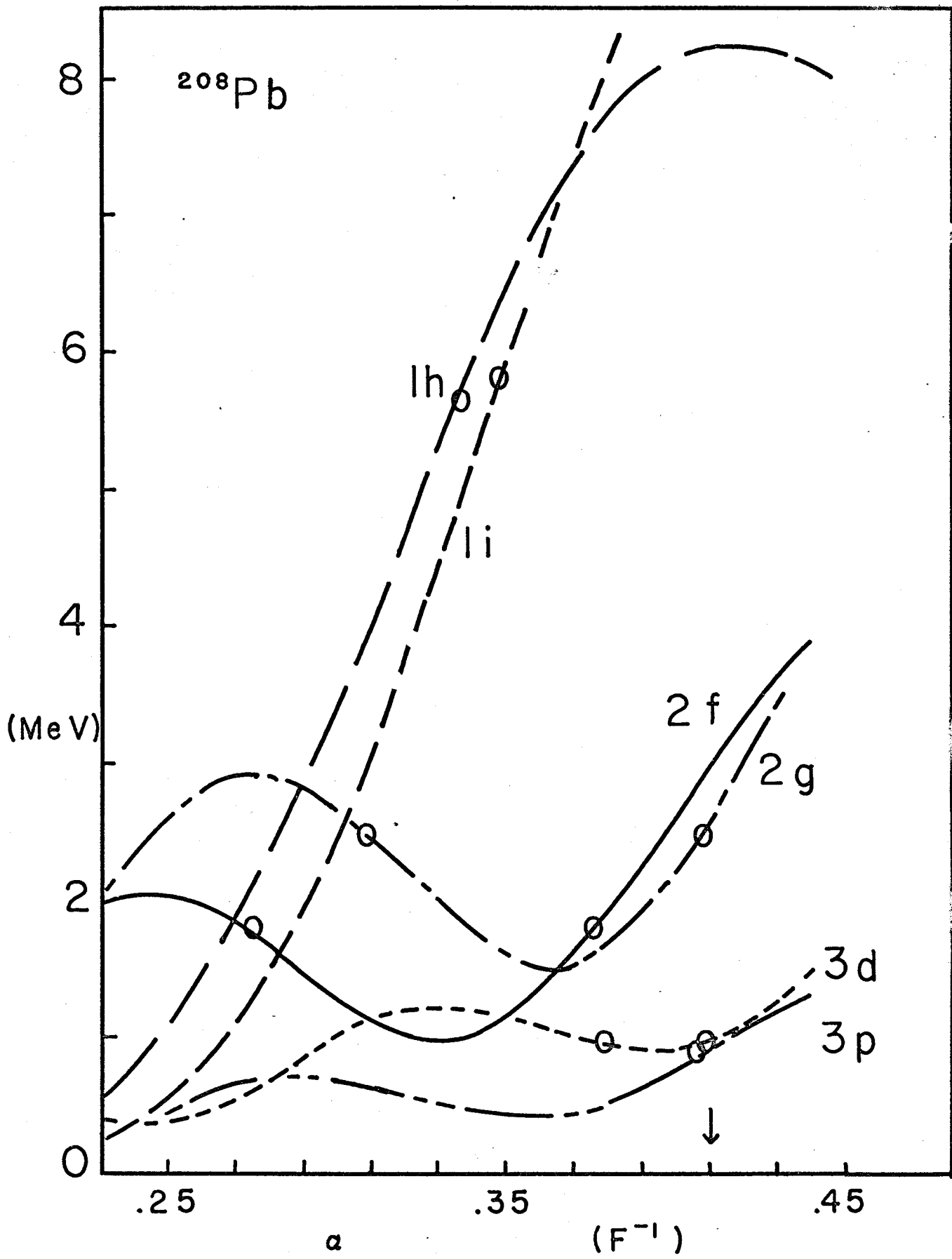


FIGURE 18