INJECTIVE AND PROJECTIVE TOPOLOGICAL LATTICES

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By

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SCOPE AND CONTENTS: This thesis gives characterizations for all the injective and projective objects in the categories of compact, totally disconnected, distributive topological lattices, and compact, distributive topological lattices. It also contains known results concerning distributive lattices and Hausdorff topological spaces.

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PREFACE

The purpose of this thesis is to provide simple characterizations of the injective and projective objects in the categories of compact, totally disconnected, distributive topological lattices, and of compact, distributive topological lattices, each with continuous lattice homomorphisms.

In Chapter 0, some of the less familiar concepts and theorems in the separate fields of Category Theory, Topology, Algebra, and Topological Algebra are given.

Chapter 1 presents known results on injectivity and projectivity in the categories of compact Hausdorff spaces and of distributive lattices.

Chapters 2 and 3 present new material dealing with the injectives and projectives as mentioned above.

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CHAPTER 0

Preliminaries

The question of injectivity is one which arises in many areas of Mathematics, and concerns the possibility of extending a given mapping defined on a substructure to the whole structure. More generally, if A is a structure which may be embedded into B, then when can a map $f:A\rightarrow C$ be extended to a map $\bar{f}:B\rightarrow C$?

The theory of categories has helped facilitate the study of such situations, and it will be the concern of this paper to investigate them in two categories of topological lattices.

Many of the ideas developed in the ensuing chapters will depend on known results from the areas of lattice theory and general topology, and some of these will be given, as well as some of the notions from general category to be used.

1. Category Theory

Definition 0.1: A category is a class $\mathcal{A}$, called the objects of the category, and such that:

1) to every pair (A,B) of objects there is associated a set M(A,B) of morphisms of A onto B. $u\epsilon M(A,B)$ is often denoted $u:A\longrightarrow B$.

2) for every triple (A,B,C) of objects, there exists a function $f: M(A,B) \times M(B,C) \longrightarrow M(A,C)$ denoted by $f(u,v) = vu$ and called the composition of the morphisms.

3) a) the composition function f is associative

b) for each object $A \varepsilon \mathcal{A}$, there exists a morphism $1_A \varepsilon M(A,A)$ such that $1_A u = u$ and $v 1_A = v$ whenever such compositions are defined.

4) for distinct pairs of objects $(A,B) \neq (A',B')$ $M(A,B) \cap M(A',B') = \emptyset$.

When the intention is clear, a category shall be denoted by its class of objects.

<u>Definition 0.2</u>: A category $\mathcal{A}'$ is a subcategory of the category $\mathcal{A}$ when:

1) $\mathcal{A}' \subseteq \mathcal{A}$

2) $M(A,B)_{\mathcal{A}'} \subseteq M(A,B)_{\mathcal{A}}$ for all $(A,B) \varepsilon \mathcal{A}' \times \mathcal{A}'$

3) the composition function in $\mathcal{A}$ extends the composition function in $\mathcal{A}'$.

4) for $A \varepsilon \mathcal{A}'$, 1_A in $\mathcal{A}'$ is the same as 1_A in $\mathcal{A}$.

<u>Definition 0.3</u>: A category $\mathcal{A}'$ is a full subcategory of $\mathcal{A}$ if $M(A,B)_{\mathcal{A}'} = M(A,B)_{\mathcal{A}}$ for all $(A,B) \varepsilon \mathcal{A}' \times \mathcal{A}'$.

<u>Definition 0.4</u>: The dual category of $\mathcal{A}$, denoted $\mathcal{A}^*$, has class of objects $\mathcal{A}$ and $M(A,B)_{\mathcal{A}^*} = M(B,A)_{\mathcal{A}}$. Composition uv in $\mathcal{A}^*$ is defined as vu in $\mathcal{A}$.

<u>Definition 0.5</u>: An object A in a category $\mathcal{A}$ is called a retract of $B \varepsilon \mathcal{A}$ iff for some morphism $u:A \longrightarrow B$, there exists a morphism $u':B \longrightarrow A$ such that $u'u = 1_A$.

<u>Definition 0.6</u>: A morphism $u \varepsilon M(A,B)$ is called a monomorphism, and denoted $u:A \rightarrowtail B$ if $uv = uw$ implies $v = w$ for all v,w having codomain A.

A morphism $u \varepsilon M(A,B)$ is called an epimorphism, and denoted $u:A \twoheadrightarrow B$ if $vu = wu$ implies $v = w$ for all v,w having domain B.

Epimorphism is called the dual notion of monomorphism because u is a monomorphism in $\mathcal{A}$ iff u is an epimorphism in $\mathcal{A}^*$.

<u>Definition 0.7</u>: For a family $\{A_i\}_I$ of objects in a category $\mathcal{A}$, a product for the family is a family of morphisms $\{p_i:A' \longrightarrow A_i\}_I$ such that for any family $\{f_i:A' \longrightarrow A_i\}_I$ there is a unique morphism $f:A' \longrightarrow A$ such that $p_i f = f_i$ for all $i \varepsilon I$. If the family $\{f_i\}_I$ is also a product, then it is easily seen that f is an isomorphism and the object A shall be denoted πA_i and referred to as the product of the family $\{A_i\}$.

<u>Definition 0.8</u>: An object $A \varepsilon \mathcal{A}$ is called injective iff for every diagram of the form

$$\begin{tikzcd} X \arrow[r, tail, "u"] \arrow[dr, "v"'] & Y \\ & A \end{tikzcd}$$

there exists a morphism $w:Y \longrightarrow A$ making the diagram commute.

<u>Definition 0.9</u>: An object $A \varepsilon \mathcal{A}$ is called projective iff for every diagram of the form

$$\begin{tikzcd} & A \arrow[d, "v"] \\ X \arrow[r, two heads, "u"'] & Y \end{tikzcd}$$

there exists a morphism $w:A \longrightarrow X$ making the diagram commute.

We now note that the question of extending the embedding u in Definition 0.8 from X to all of Y is precisely that of the injectivity of A in the category $\mathcal{A}$, while the dual category gives rise to the notion of projectivity.

We now give some theorems from the theory of categories, without proofs, which will be of use in the ensuing chapters. Proofs may be found in Mitchell [9] .

<u>Theorem 0.1</u>: Retracts and products of injective objects are injective; retracts of projective objects are projective.

<u>Definition 0.10</u>: An object $A \varepsilon \mathcal{A}$ is called a generator of the category $\mathcal{A}$ iff for distinct morphisms $f,g:B \longrightarrow C$, there is a morphism $u:A \longrightarrow B$ such that $fu \neq gu$. Dually, A is a cogenerator iff for distinct morphisms $f,g:B \longrightarrow C$, there is a morphism $u:C \longrightarrow A$ such that $uf \neq ug$.

Theorem 0.2: If a category $\mathcal{A}$ has an injective cogenerator I, then the injectives of $\mathcal{A}$ are precisely the retracts of powers of I.

2. Topology

Notation: For a subspace A of a topological space X,

$$A^{o} = \text{the interior of } A$$

$$A^{*} = \text{the closure of } A.$$

Definition 0.11: Given a topological space X, a compactification of X will be a pair $(\overline{X}, f)$ where $\overline{X}$ is compact and f is an embedding of X into $\overline{X}$ such that f(X) is dense in $\overline{X}$.

Definition 0.12: Given $(\overline{X}, f)$ a compactification of the space X, $(\overline{X}, f)$ has the extension property for compactifications of X iff for every continuous function g from X into a compact T_2-space Y, the function $gf^{-1}: f(X) \longrightarrow Y$ has a continuous extension to $\overline{X}$.

Theorem 0.3: A completely regular space X is homeomorphic to a subspace $\beta(X)$ of $I^{C[X,I]}$ where $I = [0,1]$ with the usual topology, and $C[X,I]$ is the set of continuous maps from X into I. $(\beta(X))^{*} \subseteq I^{C[X,I]}$ is a compactification for X, called the Stone-Cech Compactification of X.

Theorem 0.4: The Stone-Cech Compactification of a space has the extension property for compactifications.

Theorem 0.5: (Tietze's Extension Theorem): If X is a normal space, F a closed subset of X, and g a continuous function from F into I, the unit interval, then g has a continuous extension defined on all of X.

Theorem 0.6: If X is a compact totally disconnected T_2-space, then X has a base of closed-open sets.

3. Algebra

Our concern in this section, and in later related sections, will be with various types of lattices, but some definitions will be given in their more general form in the framework of universal algebra.

Definition0.13: An algebra of type t is said to be freely generated by a set S of generators if for any set map $f:S \longrightarrow T$ with T an algebra of type t, f can be extended to a homomorphism $\overline{f}:F \longrightarrow T$.

Theorem 0.7: The homomorphism $\overline{f}$ in the previous definition is unique.

Definition 0.14: An extension of an algebra A is a pair (B,f) with $f:A \longrightarrow B$, and f is one to one.

Definition 0.15: An extension (B,f) of A is called essential if for any $g:B \longrightarrow C$ such that gf is one to one, then g is one to one.

Definition 0.16: Given a partially ordered set P, a MacNeille Completion of P is a complete lattice M with P as sub-partially ordered set, and

1) every $a \in M$ is the join of elements of P

2) every $a \in M$ is the meet of elements of P.

Theorem 0.8: In the category of Boolean lattices and Boolean homomorphisms, a MacNeille Completion of a Boolean lattice is an essential extension.

Definition 0.17: In a lattice L, an ideal is a subset I of L with

1) $x \in I,\ y \leq x \Rightarrow y \in I$

2) $x, y \in I \Rightarrow x \vee y \in I$.

Definition 0.18: An ideal I of L is called prime iff for $x, y \in L$, and $x \wedge y \in I$, then one of x and y is in I.

Theorem 0.9: In a distributive lattice with 0 and e, if I is an ideal, F a dual ideal (filter), and $I \cap F = \emptyset$, then there exists a prime ideal P with $I \subseteq P$ and $P \cap F = \emptyset$.

Theorem 0.10: The set complement of a prime ideal in a lattice is a prime filter.

Theorem 0.11: Any complete Boolean lattice can be embedded in a power of the two element chain.

4. Topological Lattices

Definition 0.19: A topological lattice L is a T_2-topological space with a lattice structure on it such that the operations meet and join are continuous from $L \times L$ into L.

Notation: For a topological lattice L, $A \subseteq L$:

$$A \,\hat{\vee}\, L = \{a \hat{\vee} l : a \in A,\ l \in L\}$$

$$a \,\hat{\vee}\, L = \{a \hat{\vee} l : l \in L\}.$$

Theorem 0.12: For a topological lattice L, $A \subseteq L$, then if A is open, $A \,\hat{\vee}\, L$ are open. If A is compact, then $A \,\hat{\vee}\, L$ are closed.

Theorem 0.13: A compact topological lattice is complete.

Theorem 0.14 (Numakura [10]): Any compact, totally disconnected, distributive topological lattice can be embedded in a power of $2 = \{0,1\}$ with the discrete topology.

Theorem 0.15: Any compact topological Boolean lattice is iseomorphic (isomorphic and homeomorphic) with a power of 2.

This theorem is a corollary to a theorem of Kaplansky [7] on topological rings.

Proofs for the following two theorems may be found in [4].

<u>Theorem 0.16</u>: A compact distributive topological lattice L has enough characters to separate points iff for A a closed ideal, B a closed filter and $A \cap B = \emptyset$, there exists an open ideal U and an open filter V with $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$. (L is lattice-normal).

<u>Theorem 0.17</u>: If L is a compact topological lattice, the following conditions are equivalent:

1) L is lattice-normal
2) if $x \not> y$ then there exists $z \in L$ with $x \in \{t : t < z\}^{\circ}$ and $z \not\geq y$, and dually
3) every point of L has a base of neighbourhoods of the form $\{t : a \leq t \leq b\}$
4) every point of L has a base of open neighbourhoods which are convex open sublattices of L.

CHAPTER 1

Injectives and Projectives in Compact Hausdorff Spaces And Distributive Lattices

This chapter will deal entirely with known results in the fields of topology and lattice theory.

1. Compact Hausdorff Spaces

In this section we give a characterization of the injective and projective objects in the category $\mathcal{T}$ of compact Hausdorff spaces and their continuous maps.

Lemma 1.1: The unit interval I with the usual topology is an injective cogenerator for the category $\mathcal{T}$.

Proof: (i) Given the situation

$$\begin{tikzcd} A \arrow[r, tail, "j"] \arrow[dr, "f"'] & B \\ & I \end{tikzcd}$$

then $j(A)$ is compact in B, hence closed, and by Tietze's Extension Theorem, since compact T_2-spaces are normal, f can be extended to B, showing that I is injective.

(ii) Given $f,g:A \longrightarrow B$ with $f \neq g$, take $a \in A$ such that $f(a) \neq g(a)$. Then $\{f(a), g(a)\}$ is closed in B and discrete in the relative topology. Define $h:\{f(a), g(a)\} \longrightarrow I$ by $hf(a) = 0$, $hg(a) = 1$. Again using Tietze's Extension Theorem, h can be extended to $\bar{h}:B \longrightarrow I$ and $\bar{h}f \neq \bar{h}g$, showing that I is a cogenerator for $\mathcal{T}$.

<u>Theorem 1.2</u>: The injectives in $\mathcal{T}$ are the retracts of the powers of the unit interval.

<u>Proof</u>: This is a direct result of Lemma 1.1 and Theorem 0.2

<u>Lemma 1.2</u>: In $\mathcal{T}$, the epimorphisms are precisely the continuous onto maps.

<u>Proof</u>: If f is continuous and onto, it is clearly an epimorphism.

Conversely, suppose $f:A \longrightarrow B$ is an epimorphism. Then if $g,h:B \longrightarrow C$, $gf = hf \Rightarrow g = h$. Suppose f is not onto B. Then define:

$B' = f(A) \cup \{x\}$ for $x \in B-f(A)$, $\{x\}$ open in B'

$C' = C \cup \{x',y'\}$ $x',y' \notin C$, $\{x'\}$, $\{y'\}$ open sets in C'

$g':B' \longrightarrow C'$ by $g'(b) = g(b)$ for $b \in B$

$g'(x) = x'$

$h':B' \longrightarrow C'$ by $h'(b) = h(b)$ for $b \in B$

$h'(x) = y'$.

Then $g'f = h'f$ with $g' \neq h'$, which provides the desired contradiction.

<u>Lemma 1.3</u>: Let A and B be compact T_2-spaces, and $f:B \twoheadrightarrow A$ such that $f(B_o) \subsetneq A$ for each $B_o \subsetneq B$ and B_o closed. Then for any open set $C \subseteq B$, $f(C) \subseteq [A-f(B-C)]^*$.

<u>Proof</u>: If $C = \emptyset$, there is nothing to prove.

For $C \neq \emptyset$, take $a \in f(C)$ and V any open neighbourhood of a.

Claim: $V \cap (A-f(B-C) \neq \emptyset$.

Proof: $C \cap f^{-1}(V) \neq \emptyset$ and is open in B,

$\Rightarrow f(B-(C \cap f^{-1}(V))) \neq A$, by hypothesis.

Take $x \in A-f(B-(C \cap f^{-1}(V)))$

$\Rightarrow x \in A-f(B-C)$

$\Rightarrow x = f(y)$ for some $y \in B$; in fact $y \in C \cap f^{-1}(V)$

$\Rightarrow x = f(y) \in f(f^{-1}(V)) = V$

$\Rightarrow x \in V \cap (A-f(B-C)) \neq \emptyset$,

which gives the desired result.

Definition 1.1: A T_2-space is called extremally disconnected if the closure of every open subset is open.

Lemma 1.4: If X is extremally disconnected and U_1, U_2 are disjoint open subsets of X, then $U_1^* \cap U_2^* = \emptyset$.

Proof: $U_1^* \cap U_2 = \emptyset$ since U_2 is open.

$U_1^* \cap U_2^* = \emptyset$ since U_1^* is open.

Lemma 1.5: If A is an extremally disconnected, compact T_2-space, B is a compact T_2-space and $f: B \twoheadrightarrow A$ is continuous such that $f(B_o) \neq A$ for any closed set $B_o \subsetneq B$, then f is a homeomorphism.

Proof: It suffices to show that f is one to one. Let $x_1 \neq x_2$ in B. Let V_1, V_2 be disjoint open neighbourhoods of x_1 and x_2 respectively,

$\Rightarrow B - V_1$ and $B - V_2$ are closed and hence compact

$\Rightarrow f(B-V_i)$ are closed since f is onto, $i = 1,2$

$\Rightarrow A-f(B-V_i)$ are open, $i = 1,2$

Claim: $(A-f(B-V_1)) \cap (A-f(B-V_2)) = \emptyset$.

Proof: $B = (B-V_1) \cup (B-V_2)$

$\Rightarrow f(B-V_1) \cup f(B-V_2) = A$

$\Rightarrow (A-f(B-V_1)) \cap (A-f(B-V_2)) = \emptyset$

$\Rightarrow (A-f(B-V_1))^* \cap (A-f(B-V_2))^* = \emptyset$ by Lemma 1.4

$$\Rightarrow \left.\begin{array}{l} f(x_1) \in (A-f(B-V_1))^* \\ f(x_2) \in (A-f(B-V_2))^* \end{array}\right\} \text{ by Lemma 1.3}$$

$\Rightarrow f(x_1) \neq f(x_2)$.

Lemma 1.6: Let A and B be compact T_2-spaces, and $f:B \twoheadrightarrow A$ be continuous. Then B contains a compact subset B' such that $f(B') = A$, but $f(B_o') \neq A$ for any closed set $B_o' \subsetneqq B'$.

Proof: This is a well known result of Zorn's Lemma.

Theorem 1.3: In the category $\mathcal{T}$, the projective objects are precisely the extremally disconnected spaces.

Proof: (i)Let X be projective in $\mathcal{T}$, and let $A \subseteq X$ be open. For $p,q \notin X$, take $\{p,q\}$ with the discrete topology. Then in $X \times \{p,q\}$ let $Y = ((X-A) \times \{p\}) \cup (A^* \times \{q\})$, which is closed, and let i be the natural embedding of Y into $X \times \{p,q\}$. Let p be the projection of $X \times \{p,q\}$ onto X,

$\Rightarrow pi:Y \twoheadrightarrow X$ is continuous

and $1_X:X \longrightarrow X$ is continuous.

Then we have

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "1_X"] \\ Y \arrow[r, two heads, "pi"'] & X \end{tikzcd}$$

where, by the projectivity of X, f exists with pif = 1_X. Now pi is one to one on A X {p}.

$\Rightarrow f(x) = (x,q)$ for any $x \varepsilon A$

and $f(x) = (x,q)$ for any $x \varepsilon A^*$ since f is continuous. Similarly, if $x \notin A^*$, then $f(x) = (x,p)$. Thus $A^* = f^{-1}(A^* \times \{q\})$ and, since f is continuous and $A^* \times \{q\}$ is open in Y, we conclude that A* is open in X.

Conversely, let A be a compact, extremally disconnected T_2-space, and let B and C be compact T_2-spaces. Then, let the diagram be given in $\mathcal{T}$:

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, two heads, "g"'] & C \end{tikzcd}$$

Let $D = \{(a,b) \varepsilon A \times B: f(a) = g(b)\}$. Clearly D is closed, hence compact. Since g is onto, $p_1 : A \times B \longrightarrow A$ takes D onto A. Then by Lemma 1.6, there exists a subset D' of D such that $p_1(D') = A$, but $p_1(D_o') \neq A$ for D_o' a proper closed subset of D'. Let $h = p_1|_D$. Then by Lemma 1.5, h is a homeomorphism. Let $k = p_2 h^{-1}$, and take $a \varepsilon A$. Then $h^{-1}(a) \varepsilon D$. Hence $g(p_2(h^{-1}(a))) = f(p_1(h^{-1}(a))) = f(a)$, and thus $f = gp_2h^{-1} = gk$, showing the projectivity of A.

2. Distributive Lattices

This section provides a characterization of the injective objects in the category $\mathbb{D}$ of distributive lattices and their homomorphisms, as well as the well-known characterization of the projectives in any equational class.

<u>Definition 1.2</u>: A Boolean lattice B is said to be strictly generated by a lattice L iff:

1) B is the smallest Boolean lattice containing L as sublattice.
2) $\vee_B L = e_B$; $\wedge_B L = 0_B$.

<u>Lemma 1.7</u>: The monomorphisms in $\mathbb{D}$ are precisely the one to one maps.

<u>Proof</u>: Clearly any one to one map is a monomorphism.

Conversely, let $f:A \longrightarrow B$ be a monomorphism, and let $f(x) = f(y)$ for $x,y \in A$. The two element chain 2 is contained in $\mathbb{D}$. Then define:

$$g:2 \longrightarrow A \text{ by } g(0) = g(1) = x$$
$$h:2 \longrightarrow A \text{ by } h(0) = h(1) = y$$

Clearly $fh = fg \Rightarrow g = h \Rightarrow x = y \Rightarrow f$ is one to one.

<u>Lemma 1.8</u>: For L a sublattice of a Boolean lattice B, the smallest sub-Boolean lattice L' of B containing L is:

$$L' = \{ \bigvee_{i}^{1,n} (x_i \wedge y_i') : n \geq 1,\ x_i, y_i \in L \cup \{0,e\} \}$$

<u>Proof</u>: The proof of this theorem is computational in nature, and thus will be omitted here.

Lemma 1.9: For a distributive lattice L, the Boolean lattice strictly generated by L is an essential extension B of L.

Proof: Take $A \in \mathbb{D}$, and $f: B \longrightarrow A$ with $f|_L$ one to one. Then $f(B)$ is contained in $[f(0_B), f(e_B)] \subseteq A$. Then $f(B)$ is a Boolean lattice with smallest element $f(0_B)$, and largest element $f(e_B)$, and f is a Boolean homomorphism onto $f(B)$. There is no loss of generality in restricting our interest in A to $f(B)$; in other words, we assume A to be Boolean. Take $0 \neq b \in B$, and assume $f(b) = 0$. Then by Lemma 1.8, $b = \bigvee_i^{1,n} (x_i \wedge y_i')$ for $x_i, y_i \in L \cup \{0,e\}$.

$$f(x_i) \wedge f(y_i') = 0$$
$$\Rightarrow f(x_i) \wedge f(y_i)' = 0$$
$$\Rightarrow f(x_i) \leq f(y_i) \quad \text{for } i = 1,2,\cdots,n$$

Since $b>0$, $(x_i \wedge y_i')>0$ for some i, and so $x_i \neq 0$, $y_i \neq e$.

Case 1: $x_i, y_i \in L \Rightarrow x_i \leq y_i$ since f is one to one on L

$\Rightarrow x_i \wedge y_i' = 0$

$\Rightarrow b = 0$ which provides a contradiction.

Case2: $x_i = e$, $y_i \in L - \{e\}$.

Now $0 = f(x_i \wedge y_i') = f(y_i') = f(y_i)'$

$\Rightarrow f(y_i) = e$

$\Rightarrow f(y_i) = f(y_i \vee y)$ for all $y \in L$

$\Rightarrow y_i = y_i \vee y$ for all $y \in L$

$\Rightarrow y_i \geq y$ for all $y \in L$

$\Rightarrow y_i = e$ contradiction

Case 3: $x_i \in L - \{0\}$, $y_i = 0$, which is the dual of Case 2.

<u>Lemma 1.10</u>: The two element chain is injective in $\mathbb{D}$.

<u>Proof</u>: Let S, $L \varepsilon \mathbb{D}$, and $j:S \rightarrowtail L$, and $f:S \longrightarrow 2$ a lattice homomorphism. Define $A = f^{-1}(0)$, $B = f^{-1}(1)$. We may assume that $A \neq \emptyset \neq B$, and define

$$\overline{A} = \{x \varepsilon L : x \leq a \text{ for some } a \varepsilon A\}$$

$$\overline{B} = \{y \varepsilon L : b \leq y \text{ for some } b \varepsilon B\}.$$

Then clearly $\overline{A}$ and $\overline{B}$ are disjoint ideal and filter respectively, and, by Theorem 0.9, there exists a prime ideal P with $\overline{A} \subseteq P$, and $\overline{B} \cap P = \emptyset$. Now define $\overline{f}:L \longrightarrow 2$ by

$$\overline{f}(x) = \begin{cases} 0 & \text{if } x \varepsilon P \\ e & \text{if } x \not\varepsilon P \end{cases}$$

which yields the desired result.

<u>Corollary 1.1</u>: Every power set lattice is injective in $\mathbb{D}$.

<u>Proof</u>: Every power set lattice is a power of 2.

<u>Lemma 1.11</u>: Every complete Boolean lattice is injective in $\mathbb{D}$.

<u>Proof</u>: Let B be a complete Boolean lattice. Then B can be embedded in a power set lattice C. B is injective in the category of Boolean lattices, and thus B is Boolean retract of every extension. Hence B is a Boolean retract of C; i.e. there exists $f:C \longrightarrow B$ such that $f|_B$ is one to one. Let there be given in $\mathbb{D}$ the diagram

$$\begin{tikzcd} S \arrow[r, tail, "i"] \arrow[d, "g"'] & D \\ B \arrow[r, shift left, "j"] & C \arrow[l, shift left, "f"] \end{tikzcd}$$

where $fj = 1_B$. Then by the injectivity of C, $\bar{g}:D \longrightarrow C$ exists with $\bar{g}i = jg$, and hence $f\bar{g}i = fjg = g$.

Theorem 1.4: In the category $\mathbb{D}$, the following are equivalent:

1) B is complete Boolean
2) B is injective
3) B has no proper essential extensions.

Proof: $1 \Rightarrow 2$: is a result of Lemma 1.11

$2 \Rightarrow 3$: Let $i:I \longrightarrow X$ be an essential extension of the injective object I. Then in the diagram

$$\begin{tikzcd} I \arrow[r, tail, "i"] \arrow[d, "1_I"'] & X \\ I & \end{tikzcd}$$

$f:X \longrightarrow I$ exists with $fi = 1_I$

$\Rightarrow fi$ is one to one

$\Rightarrow f$ is one to one

$\Rightarrow X = I$.

$3 \Rightarrow 1$: From Lemma 1.9, we conclude that B is Boolean. Let C be a MacNeille Completion of B.

Claim: C is an essential extension of B.

Proof: Let $f:C \longrightarrow L$ and $f|_B$ be one to one

$\Rightarrow f(C)$ is Boolean

$\Rightarrow f$ is a Boolean homomorphism onto $f(C)$ and f is one to one since C is essential in the Boolean case

$\Rightarrow B = C$

$\Rightarrow B$ is complete.

<u>Theorem 1.5</u>: In an equational class $\mathcal{A}$ of algebras, the projectives are precisely the retracts of the free objects.

<u>Proof</u>: Let P be projective in $\mathcal{A}$; then there exists a free algebra F(n) on sufficiently many generators to be mapped homomorphically onto P. Let the diagram be given

$$\begin{tikzcd} & P \arrow[d, "1_P"] \\ F(n) \arrow[r, two heads, "f"'] & P \end{tikzcd}$$

Then $\overline{f}:P\longrightarrow F(n)$ exists with $f\overline{f} = 1_P$ by the projectivity of P, and hence P is a retract of F(n).

Conversely, let P be freely generated by a set S, and let the diagram be given

$$\begin{tikzcd} S \arrow[r, "g"] & F(S) \arrow[d, "h"] \\ A \arrow[r, two heads, "f"'] & B \end{tikzcd}$$

where g is the natural embedding of S into F(S). By the freeness of F(S), there exists $\overline{g}:F(S)\longrightarrow A$, and $f^{-}:S\longrightarrow A$ such that

$$\begin{aligned} &\overline{f}(s) \in f^{-1}(h(s)) \quad \text{for all } s \in S \\ \Rightarrow\ & f\overline{f}(s) = h(s) \\ \Rightarrow\ & f\overline{g}|_S = h|_S \\ \Rightarrow\ & f\overline{g} = h \end{aligned}$$

and we conclude that F(S) is projective in $\mathcal{A}$.

CHAPTER 2

Injectives and Projectives in Compact Totally Disconnected Distributive Topological Lattices

Definition 2.1: A topological lattice is said to have small lattices iff it has a topological base of open sublattices.

Lemma 2.1: A compact, totally disconnected topological lattice L has small lattices.

Proof: Since by Theorem 0.14, L may be embedded as a topological lattice in a power of $2 = \{0,1\}$ with the discrete topology, then we have $L \xrightarrow{i} 2^a \xrightarrow{p_i} 2 \xrightarrow{f} I$, where i is the embedding, p_i are the projections, and f takes 0 to 0 and 1 to 1. All three are clearly lattice-homomorphisms. Then for $x \neq y$ in L, there exists a projection p_j separating $i(x)$ from $i(y)$, and since fp_ji are continuous lattice-homomorphisms from L to I (characters), L has sufficient characters to separate points. Thus by Theorems 0.16 and 0.17, L has a base of open neighbourhoods consisting of convex open sublattices. In particular, L has small lattices.

Lemma 2.2: If L is a totally disconnected, compact, distributive topological lattice, then $\{V \subseteq L\text{: } V \text{ is convex closed-open}\}$ forms a topological base for L.

Proof: Let $a \in U \subseteq L$, with U open, and choose V, a closed-open set such that $a \in V \subseteq U$. Such a V exists by Theorem 0.6. Let $W = (L \wedge V) \cap (L \vee V)$. Then by Theorem 0.12, W is closed-open, and $W \subseteq V \subseteq U$. Also, for $x<y$ in W, if $x<z<y$:

$$\begin{aligned}
x \in W &\Rightarrow x = l_1 \vee v_1 \quad \text{for } l_1 \in L,\ v_1 \in V \\
&\Rightarrow x \geqslant v_1 \\
&\Rightarrow z \geqslant v_1 \\
&\Rightarrow z = z \vee v_1 \\
y \in W &\Rightarrow y = l_2 \wedge v_2 \quad \text{for } l_2 \in L,\ v_2 \in V \\
&\Rightarrow y \leqslant v_2 \\
&\Rightarrow z \leqslant v_2 \\
&\Rightarrow z = z \wedge v_2 .
\end{aligned}$$

Hence $z \in W$, and so W is convex.

Lemma 2.3: If L is a compact, totally disconnected, distributive topological lattice, then $\{V \subseteq L$: V is a closed-open convex sublattice$\}$ forms a topological base for L.

Proof: Let $a \in U$, with U a closed-open convex subset of L. Consider the family $\Phi = \{V \subseteq U$: V is a convex open sublattice, and $a \in V\}$. It is clear that Φ, ordered by inclusion is inductive, and hence, by Zorn's Lemma, has a maximal element V.

Claim: V is closed.

Proof: Since U is closed, $V^* \subseteq U$, and V^* is a convex closed sublattice (sublattice by the continuity of $\wedge, \vee$), then let $V^* = [a,b]$. Since $a,b \in U$, there exist open convex sublattices V_1 and V_2 with $a \in V_1$, $b \in V_2$, and $V_1 \subseteq U$, $V_2 \subseteq U$.

Let $W = (L \vee V_1) \cap (L \wedge V_2)$, which is clearly a convex open sublattice of L. Since U is convex, $W \subseteq U$. Also $V \subseteq W$, and hence $V = W$ since V is maximal in Φ. Now, $V^* = [a,b] \subseteq W = V$ and hence $V^* = V$.

<u>Lemma 2.4</u>: Let $\mathscr{D}$ be the category of compact, totally disconnected, distributive topological lattices and continuous lattice-homomorphisms. Then the monomorphisms of are precisely the one to one maps.

<u>Proof</u>: Let $f: A \longrightarrow B$ be a monomorphism. Then let $g, h: 2 \longrightarrow A$ be defined by $g(0) = g(1) = x$, and $h(0) = h(1) = y$, with $x \neq y$ in A. Suppose $f(x) = f(y)$; then $fg = fh$, but $g \neq h$ which contradicts f's being a monomorphism.

The converse is obvious.

<u>Lemma 2.5</u>: In the category $\mathscr{D}$, $2 = \{0,1\}$ with the discrete topology is injective.

<u>Proof</u>: Let the diagram

$$\begin{tikzcd} D \arrow[r, tail, "i"] \arrow[dr, "f"'] & L \\ & 2 \end{tikzcd}$$

be given in $\mathscr{D}$, and let $I' = f^{-1}(0)$, $J' = f^{-1}(1)$. The situation is trivial if $I' = J'$ and hence we may assume $I' \cap J' = \emptyset$, and it is clear that they are closed ideal and filter respectively.

Now let $I = j(I')$, $J = j(J')$. Clearly $I \cap J = \emptyset$. We also have that I and J are closed sublattices of L, hence compact, and so by Theorem 0.13, have greatest and least elements. Let a be the greatest element of I, b the least of J. Then clearly $b \not\leq a$, and so $a \in L-(L \vee b)$ which is open. Thus there exists a closed-open lattice M such that $a \in M$ and $M \subseteq L-(L \vee b)$.

We show $L \wedge M$ is a closed-open ideal, and $L \wedge I \subseteq L \wedge M$. $L \wedge M$ is clearly closed-open since M is compact open. Also if $x, y \in L \wedge M$, then $x \leq m$, $y \leq m'$ for some $m, m' \in M$

$\Rightarrow x \vee y \leq m \vee m' \in M$

$\Rightarrow L \wedge M$ is an ideal in L.

Take $x \in L \wedge I$ $\Rightarrow x \leq u$ for some $u \in I$

$\Rightarrow x \leq a$ = the maximum element of I

$\Rightarrow x \leq a \in M$

$\Rightarrow x \in L \wedge M$

$\Rightarrow L \wedge I \subseteq L \wedge M$.

Analogously, we have $b \in L-(L \wedge M)$, and so there exists a closed-open lattice N such that $b \in N \subseteq L-(L \wedge M)$, and $L \vee N$ is a closed-open filter.

Now we show $(L \wedge M) \cap (L \vee N) = \emptyset$, and $L \vee J \subseteq L \vee N$. If there exists $x \in (L \wedge M) \cap (L \vee N)$

$\Rightarrow n \leq x \leq m$ for some $n \in N$ and $m \in M$

$\Rightarrow n \leq m$ which provides a contradiction.

Also, $x \in L \vee J \Rightarrow x \geq k$ for some $k \in J$, but $b \in N$

$\Rightarrow J \subseteq N$

$\Rightarrow x \geqslant n$ for some $n \in N$

$\Rightarrow x \in L \vee N$.

Now consider the family $\Phi = \{V \subseteq L$: V is an open ideal containing $L \wedge M$ and $V \cap (L \vee N) = \emptyset\}$. Then $L \wedge M \in \Phi$, hence $\Phi \neq \emptyset$, and is inductive under inclusion. Let V be a maximal element of Φ and show V is a closed sublattice of L. That V is a sublattice is trivial. Let x be the greatest element of V^*. Then there exists a closed-open lattice W such that $x \in W \subseteq L-(L \vee N)$, since $L-(L \vee N)$ is an open neighbourhood of x. Now $a \in V \Rightarrow a \leqslant x$

$$\begin{aligned} &\Rightarrow a = a \wedge x \\ &\Rightarrow a \in L \wedge W \\ &\Rightarrow V \subseteq L \wedge W \\ &\quad \text{but } L \wedge M \subseteq V \\ &\Rightarrow L \wedge M \subseteq L \wedge W. \end{aligned}$$

Also $(L \wedge W) \cap (L \vee N) = \emptyset$, since $W \subseteq L-(L \vee N)$, and hence

$$\left.\begin{array}{l} L \wedge W \in \Phi \\ V \subseteq L \wedge W \\ V \text{ maximal in } \Phi \end{array}\right\} \Rightarrow V = L \wedge W.$$

Now since $x \in L \wedge W = V$, $x \in V$, and so $V^* \subseteq V$ which shows that V is closed.

Now $V = x \wedge L$ and we show x is $\wedge$-irreducible. Suppose $x = p \wedge q$. Then since $L \vee N$ is $\wedge$-closed, and $V \cap (L \vee N) = \emptyset$, W.L.O.G. $p \notin L \vee N$. Then there exists a closed-open sublattice W containing p, with $W \cap (L \vee N) = \emptyset$, since $L-(L \vee N)$ is open. Also

$$\begin{aligned} x \leq p \text{ and } V = W \wedge L & \\ & \Rightarrow p \in V \\ & \Rightarrow p = x \\ & \Rightarrow V \text{ is a closed-open prime ideal.} \end{aligned}$$

We now define $g: L \longrightarrow 2$ by $g(x) = 0$ if $x \in V$, $g(x) = 1$ if $x \notin V$, and g is clearly a continuous lattice-homomorphism extending f, showing that 2 is injective in $\mathcal{S}$.

<u>Lemma 2.6</u>: 2 is a cogenerator in $\mathcal{S}$.

<u>Proof</u>: Let $u, v: A \longrightarrow B$ be distinct morphisms in $\mathcal{S}$, and let I and J be the closed ideal and filter respectively generated by u(a) and v(a) in B, where $u(a) \not\geq v(a)$. Then $I \cap J = \emptyset$.

We now have a situation identical to that of the previous lemma, and arrive at a closed-open prime ideal V with $I \subseteq V$, $V \cap J = \emptyset$.

Then define $f: B \longrightarrow 2$ in the identical way, providing a morphism such that $fu \neq fv$, and hence 2 is a cogenerator for $\mathcal{S}$.

<u>Theorem 2.1</u>: The injective objects of $\mathcal{S}$ are precisely the powers of the cogenerator 2.

<u>Proof</u>: The injectives of $\mathscr{D}$ are the retracts of powers of 2 by Lemmas 0.2 and 2.6. But the retracts of 2 are compact Boolean lattices since 2 is, and hence, by Theorem 0.15 are iseomorphic with powers of 2.

<u>Lemma 2.7</u>: If D is a distributive lattice with the discrete topology, then the Stone-Cech Compactification βD is a compact, totally disconnected, distributive topological lattice.

<u>Proof</u>: By Theorem 0.14, D can be embedded in a compact Boolean lattice, and hence admits sufficient characters to separate points. Hence D can be lattice-homomorphically embedded in $I^{\mathrm{Hom}(D,I)}$. Since D has the discrete topology, $\mathrm{Hom}(D,I) \subseteq C(D,I)$, where $C(D,I)$ are the continuous maps from D into I, the unit interval. Since $I^{\mathrm{Hom}(D,I)}$ is compact, it is closed in $I^{C(D,I)}$, and so, under the embedding $j: D \longrightarrow I^{\mathrm{Hom}(D,I)}$, $j(D)^* = \beta D \subseteq I^{\mathrm{Hom}(D,I)}$.

<u>Claim</u>: βD is a compact, totally disconnected, distributive topological lattice in the relative topology of $I^{\mathrm{Hom}(D,I)}$.

<u>Proof</u>: I^a is a topological lattice for any cardinal a, so we restrict the partial order to βD. Since $j(D)$ is a distributive sublattice of $I^{\mathrm{Hom}(D,I)}$, so is $j(D)^* = \beta D$.

Meets and joins in βD, being restrictions of meets and joins in $I^{\mathrm{Hom}(D,I)}$, are continuous.

It is well known that the Stone-Cech Compactification of a discrete space is extremally disconnected and in particular totally disconnected.

Lemma 2.8: Let D be a distributive lattice, and βD its Stone-Cech Compactification. Then if $f:D \longrightarrow E$ is a lattice-homomorphism into a totally disconnected, compact, distributive topological lattice, there exists a unique continuous lattice-homomorphism $\overline{f}:\beta D \longrightarrow E$ extending f.

Proof: By giving D the discrete topology, we make f a continuous lattice-homomorphism. Then by Theorem 0.4, there exists a continuous extension $\overline{f}$ of f which is unique since D is dense in βD. If $x, y \in \beta D$, there are nets $\{x_\alpha : \alpha \in A\}$ and $\{y_\alpha : \alpha \in A\}$ in D converging to x and y respectively, and since f is a lattice-homomorphism and $\overline{f}$ is continuous, $\overline{f}(x \diamond y) = \overline{f}(x) \diamond \overline{f}(y)$, showing that $\overline{f}$ is a lattice-homomorphism.

Let $\mathbb{D}$ be the category of distributive lattices and their homomorphisms. Then define functors:

$$F: \mathbb{D} \longrightarrow \mathscr{S} \text{ by } F(D) = \beta D$$

$$U: \mathscr{S} \longrightarrow \mathbb{D} \text{ forgetting the topology}$$

Claim: F is covariant.

Proof: Given $f:D \longrightarrow D'$, and $g:D' \longrightarrow D''$. Then if D' has the discrete topology, f can be extended to $\overline{f}:\beta D \longrightarrow D'$, and the embedding j of D' into $\beta D'$ is continuous, hence $j\overline{f}$ is a continuous lattice-homomorphism from βD into $\beta D'$.

F(gf) is the unique extension $\overline{gf}$ of gf. But F(g)F(f) also extends gf and thus F(gf) = F(g)F(f) by uniqueness. The preservation under F of identities in $\mathbb{D}$ is clear.

<u>Lemma 2.9</u>: U is adjoint for F.

<u>Proof</u>: For $D \varepsilon \mathbb{D}$, $E \varepsilon \mathscr{E}$, we show that $\eta : [F(D),E] \longrightarrow [D,U(E)]$ is a natural equivalence of set-valued bifunctors.

Taking $f \varepsilon [F(D),E]$, define $\eta(f) = f|_D$, and dropping the continuity requirement, $\eta(f) \varepsilon [D,U(E)]$. Conversely, for $g \varepsilon [D,U(E)]$, we have

$$\begin{tikzcd} D \arrow[r, "j"] \arrow[d, "g"'] & F(D) \arrow[d, "\overline{g}"] \\ U(E) \arrow[r, "i"'] & E \end{tikzcd}$$

where $\overline{g}$ exists uniquely by Lemma 2.8. Then $\eta^{-1}(g) = \overline{g}$.

To check the naturality of η, take (D,E) and (D',E') in $\mathbb{D} \times \mathscr{E}$ and show

$$\begin{tikzcd} {[F(D),E]} \arrow[r, "\eta"] \arrow[d, "{[F(a),b]}"'] & {[D,U(E)]} \arrow[d, "{[a,U(b)]}"] \\ {[F(D'),E']} \arrow[r, "\eta"'] & {[D',U(E')]} \end{tikzcd}$$

commutes for any product morphism $(a,b):(D,E) \longrightarrow (D',E')$. By taking $f \varepsilon [F(D),E]$, $\eta[F(a),b](f) = \eta(bfF(a)) = (bfF(a))|_{D'}$ $= U(b)f|_D a$. But $[a,U(b)]\eta(f) = U(b)\eta(f)a = U(b)(f|_D)a$.

Hence η is natural in D and E, and we conclude that U is adjoint for F.

<u>Remark</u>: There exist natural transformations

i) $\eta : 1_{\mathbb{D}} \longrightarrow UF$

ii) $\varepsilon : FU \longrightarrow 1_{\mathscr{S}}$

<u>Proof</u>: For $D \varepsilon \mathbb{D}$, there is a natural embedding η of D with the discrete topology into F(D), and hence of D into UF(D). Then for a morphism $f:D \longrightarrow D'$, construct the diagram

$$\begin{tikzcd} D \arrow[r, "\eta"] \arrow[d, "f"'] & UF(D) \arrow[d, "UF(f)"] \\ D' \arrow[r, "\eta"'] & UF(D') \end{tikzcd}$$

Taking $x \varepsilon D$, $UF(f)\eta(x) = UF(f)(x) = F(f(x)) = f(x)$, since $f(x)$ is an element of D'. Hence $UF(f)\eta(x) = \eta f(x)$, and η is natural.

On the other hand, for any $A \varepsilon \mathscr{S}$, there exists a homomorphism $\varepsilon : FU(A) \longrightarrow A$ as follows:

$$\begin{tikzcd} U(A) \arrow[rr] \arrow[dr, "1_{U(A)}"'] & & FU(A) \arrow[dl, "\varepsilon"] \\ & A & \end{tikzcd}$$

and ε is uniquely determined by Lemma 2.8. Now for a morphism $f:A \longrightarrow A'$, construct the diagram

$$\begin{tikzcd} FU(A) \arrow[r, "\varepsilon"] \arrow[d, "FU(f)"'] & A \arrow[d, "f"] \\ FU(A') \arrow[r, "\varepsilon"'] & A' \end{tikzcd}$$

Then $\varepsilon U(f) : U(A) \longrightarrow A'$ has a unique extension from $FU(A)$ to A by Lemma 2.8. Taking $x \in U(A)$, $f\varepsilon(x) = f(x)$, and $\varepsilon FU(f)(x) = \varepsilon f(x) = f(x)$ since $f(x) \in A'$. Since both $f\varepsilon$ and $\varepsilon FU(f)$ extend $\varepsilon U(f)$, they are equal, and ε is natural.

<u>Lemma 2.10</u>: If P is projective in $\mathbb{D}$, then $F(P)$ is projective in $\mathscr{S}$.

<u>Proof</u>: Let there be given the diagram in $\mathscr{S}$

$$\begin{tikzcd} & F(P) \arrow[d, "g"] \\ A \arrow[r, two heads, "f"'] & B \end{tikzcd}$$

Then we have

$$\begin{tikzcd} & P \arrow[d, "\eta"] \arrow[ddl, "h"'] \\ & UF(P) \arrow[d, "U(g)"] \\ U(A) \arrow[r, two heads, "U(f)"'] & U(B) \end{tikzcd}$$

since U clearly preserves epimorphisms, and h exists making the diagram commute by the projectivity of P in $\mathbb{D}$.

Hence we get the diagram in $\mathscr{S}$

$$\begin{tikzcd}
 & F(P) \arrow[d, "F(\eta)"] \arrow[ddl, "F(h)"'] \\
 & FUF(P) \arrow[d, "FU(g)"] \\
FU(A) \arrow[r, "FU(f)"'] \arrow[d, "\varepsilon"'] & FU(B) \arrow[d, "\varepsilon"] \\
A \arrow[r, two heads, "f"'] & B
\end{tikzcd}$$

The two smaller diagrams commute, and hence so does the larger.

<u>Claim</u>: $\varepsilon FU(g)F(\eta) = g$, which provides the map $\varepsilon F(h)$, showing the projectivity of $F(P)$ in $\mathscr{S}$.

<u>Proof</u>: Since $\varepsilon : FU \longrightarrow 1_{\mathscr{S}}$ is a natural transformation, we have

$$\begin{tikzcd}
FUF(P) \arrow[r, "FU(g)"] \arrow[d, "\varepsilon"'] & FU(B) \arrow[d, "\varepsilon"] \\
F(P) \arrow[r, "g"'] & B
\end{tikzcd}$$

where $\varepsilon FU(g) = g\varepsilon$.

It is enough to show that $\varepsilon F(\eta) = 1_{F(P)}$, for then $\varepsilon FU(g)F(\eta) = g\varepsilon F(\eta) = g1_{F(P)} = g$.

It is known, [9], that in the adjoint situation $(\eta;F,U;\mathscr{S},\mathbb{D})$, one has $(\varepsilon F)(F\eta) = 1_F$, and hence we have $1_{F(P)} = \varepsilon F(\eta)$.

This completes the proof of the lemma.

<u>Theorem 2.2</u>: In the category $\mathscr{S}$, A is projective iff A is a retract of the Stone-Cech Compactification of a free distributive lattice with the discrete topology.

<u>Proof</u>: The free objects are the projectives in $\mathbb{D}$, by Theorem 1.5 and hence, by the last lemma, their Stone-Cech Compactifications are projective in $\mathscr{S}$, and so are the retracts.

Conversely, let A be projective in $\mathscr{S}$. Then U(A) is a distributive lattice, and let G be the free distributive lattice freely generated by the lattice U(A). By the freeness of G, there exists a lattice-epimorphism $f: G \rightarrow U(A)$. Then in the diagram

$$\begin{tikzcd} G \arrow[rr] \arrow[dr, "f"'] & & F(G) \arrow[dl, two heads, "\bar{f}"] \\ & A & \end{tikzcd}$$

$\bar{f}$ exists making the diagram commute, and by the projectivity of A there exists $g: A \longrightarrow F(G)$ making the following diagram commute:

$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[d, "1_A"] \\ F(G) \arrow[r, two heads, "\bar{f}"'] & A \end{tikzcd}$$

Hence A is a retract of the Stone-Cech Compactification of the free lattice F(G).

CHAPTER 3

Injectives and Projectives in Compact Distributive Topological Lattices

Theorem 3.1: In the category $\mathscr{L}$ of compact distributive topological lattices and continuous lattice-homomorphisms, there are no non-trivial injectives.

Proof: Let Q be injective in $\mathscr{L}$. Then $U(Q) \in \mathbb{D}$ and $FU(Q) \in \mathscr{B} \subseteq \mathscr{L}$. Now $FU(Q) \leq 2^a$ for some cardinal a, and so we get morphisms:

$$\begin{tikzcd}
U(Q) \arrow[r] \arrow[dr, "1_{U(Q)}"'] & FU(Q) \arrow[r] \arrow[d, "f"] & 2^a \arrow[dl, "g"] \\
& Q &
\end{tikzcd}$$

The morphism f exists by Lemma 2.8, and g by the injectivity of Q in $\mathscr{L}$.

Hence Q, a homomorphic image of 2^a, is a compact Boolean lattice, and so $Q = 2^b$ for some $b \leq a$. Suppose $b \geq 1$. Then 2 is injective in $\mathscr{L}$. But, letting $j: 2 \longrightarrow [0,1]$ be defined by $j(0) = 0$, $j(1) = 1$, we have the following diagram:

$$\begin{tikzcd}
2 \arrow[r, tail, "j"] \arrow[dr, "1_2"'] & {[0,1]} \arrow[d, "f"] \\
& 2
\end{tikzcd}$$

where f exists by the injectivity of 2, and the diagram commutes. This provides a contradiction, since $[0,1]$ cannot be decomposed into disjoint open sets, and we conclude that $b = 0$.

Lemma 3.1: Let D be a distributive lattice, and βD its Stone-Cech Compactification. Then if $f: D \longrightarrow E$ is a lattice-homomorphism into a compact, distributive topological lattice, there exists a unique continuous lattice-homomorphism $\bar{f}: \beta D \longrightarrow E$ extending f.

Proof: The hypothesis of total disconnectedness in Lemma 2.8 was not used in its proof.

Theorem 3.2: In the category $\mathcal{L}$, P is projective iff P is a retract of the Stone-Cech Compactification of a free distributive lattice with discrete topology.

Proof: Let P be projective in $\mathcal{L}$. Let U' be the forgetful functor which drops the topology from objects of $\mathcal{L}$. Then in the diagram

$$\begin{tikzcd} U'(P) \arrow[rr] \arrow[dr, "1_{U'(P)}"'] & & FU'(P) \arrow[dl, "f"] \\ & P & \end{tikzcd}$$

there exists a continuous lattice-homomorphism f from the Stone-Cech Compactification of U'(P) onto P by Lemma 3.1.

Now in the diagram

$$\begin{tikzcd} & P \arrow[dl, "\overline{f}"'] \arrow[d, "1_P"] \\ FU'(P) \arrow[r, two heads, "f"'] & P \end{tikzcd}$$

the morphism $\overline{f}:P \longrightarrow FU'(P)$ exists by the projectivity of P. Let $x,y \in P$, and $C_x = C_y$, where C_x and C_y are the connected components of x and y respectively. But connectivity is preserved by continuous maps; in particular $\overline{f}(C_x)$ and $\overline{f}(C_y)$ are connected in FU'(P), which is totally disconnected, implying $\overline{f}(C_x) = \overline{f}(x) = \overline{f}(y)$. Then $f\overline{f}(x) = f\overline{f}(y)$ which implies $x = y$. Hence P is totally disconnected, and projective in $\mathscr{L}$. Since $\mathscr{D}$ is a full subcategory of $\mathscr{L}$, we conclude that P is projective in $\mathscr{D}$, and, by Theorem 2.2, is a retract of the Stone-Cech Compactification of a free distributive lattice.

Conversely, let P be the Stone-Cech Compactification of a free distributive lattice L with discrete topology. Then P is an object of $\mathscr{D}$, and is projective in $\mathscr{D}$ by Theorem 2.2. Let S be the set of free generators of L, and let the following diagram be given in $\mathscr{L}$:

$$\begin{tikzcd} & P \arrow[d, "f"] \\ A \arrow[r, two heads, "g"'] & B \end{tikzcd}$$

Then in the diagram

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "k"'] & L \arrow[r, "j"] \arrow[d, "l"] & P \arrow[d, "f"] \arrow[dl, "m"'] \\ & A \arrow[r, two heads, "g"'] & B \end{tikzcd}$$

where i and j are the inclusion maps, define k, l and m as follows:

(i) for every $s \varepsilon S$, $fji(s) \varepsilon B$. Since g is onto, take $a_s \varepsilon g^{-1}(fji(s))$, and let $k(s) = a_s$. Then k is a set mapping, and

$$gk = fji \qquad (1)$$

(ii) By the definition of freeness, $l:L \longrightarrow A$ is a homomorphism extending k, and hence

$$li = k \qquad (2)$$

(iii) Since P is the Stone-Cech Compactification of L, by Lemma 3.2, $m:P \longrightarrow A$ is a continuous lattice-homomorphism extending l, and so

$$mj = l \qquad (3)$$

Thus, by (1), (2), and (3):

$$gmji = fji \qquad (4)$$

and since gmj and fj agree on S, they agree on L, and

$$gmj = fj \qquad (5)$$

Since gm and f are continuous homomorphisms from P into B agreeing on L which is dense in P, they agree on P, and so, by (5), gm = f, showing that P, and hence also its retracts, are projective in $\mathcal{L}$.

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