BASES AND CONES IN LOCALLY CONVEX SPACES

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By

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The major results of this work include an isomorphism theorem for B-complete barrelled spaces with similar bases and a theorem which shows that the cone associated with a separating biorthogonal system in a perfect C.N.S. has a basis. We also obtain some applications of the former result in the case of dual generalized bases and some results concerning Schauder bases in countably barrelled spaces.

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TO MY WIFE MARGARET

INTRODUCTION

The major results of this work fall into two main parts. The first part consists of an isomorphism theorem between B-complete barrelled locally convex spaces relating generalized bases and similarity, and an analogous isomorphism theorem concerning countably barrelled spaces and similar Schauder bases. We also give some applications of these theorems and use them to generalize some known results to a larger class of locally convex spaces, and to inductive limit spaces.

The second part consists of a theorem which gives sufficient conditions for the cone associated with a biorthogonal system to have a basis in perfect complete countably normed spaces. We also show that in perfect spaces every cone associated with a biorthogonal system where the functionals form a separating family is normal and hence in perfect complete countably normed spaces every cone of this type has a basis. We also obtain some results concerning the relation between type P and boundedly complete bases over a cone K.

The first chapter of this thesis is composed of those definitions and results from the theory of topological vector spaces, locally convex spaces and basis theory etc. which are needed in future chapters. In the introduction frequent use is made of the contents of Chapter I without explicit reference.

In Chapter II, section 1, we give some preliminary results concerning closed graph theorems and then obtain an isomorphism theorem (theorem 2) for B-complete barrelled spaces and similar generalized bases. This theorem has been studied with great interest by Arsove and

Edwards (8), and was shown for the case of complete metric linear spaces. It is worth noting that Retherford and Jones (24) have shown that even in complete barrelled spaces the theorem does not hold, and hence our result is in some sense the best possible. This result is then used to advantage to generalize some theorems of Arsove and Edwards. In section 2 we apply the theorem again to the case of inductive limits, and in section 3 we further some results by Davis (15), dealing with dual generalized bases.

In Chapter III, sections 1 and 2, we give some results concerning countably barrelled spaces and we show that the isomorphism theorem in Chapter II holds for countably barrelled spaces and similar Schauder bases, thus extending a result by Retherford and Jones (24). It is worth noting that in (24) the authors remark that the isomorphism theorem concerning similar Schauder bases is false if one of the spaces is not barrelled, however our result holds for countably barrelled spaces. We also show that the weak basis theorem holds for countably barrelled spaces.

In section 3 we obtain a generalization of Gelbaum's result (20) dealing with the following question: If $(x_n, f_n)_{n\epsilon N}$ is a Schauder basis in a linear topological space, can one give necessary and sufficient conditions for $(y_n)_{n\epsilon N}$, $y_n = \sum_{i=1}^{n} a_i x_i$, $(a_i)_{i\epsilon N}$ a sequence of scalars, to be a Schauder basis?

One of the main outstanding problems in basis theory is: does every separable Banach space have a Schauder basis? Many people have tried in vain to solve this problem and I suppose that a considerable part of basis theory has been motivated to some extent by this problem.

In considering this problem, the idea of showing the existence of a basis for a cone has been studied by many people including Fullerton, Gurevic, Gurarii, Singer, Levin and McArthur, see for example, (23), (26), (27) and (28). In Chapter IV we show that in perfect locally convex spaces every cone of a certain type (i.e. associated with a separating biorthogonal system) is normal, and then we obtain the result that in perfect complete countably normed spaces every separating biorthogonal system is a basis for the associated cone, and is in fact an unconditional basis. The result was previously known for a restricted class of Banach spaces (23), theorem 2. Finally we give some results relating type P and boundedly complete bases over a cone K, and an example to show that the theorem does not hold in complete countably normed spaces in general, and hence our result is in some sense the most general as yet in this direction.

CHAPTER I

Preliminaries

1. Terminology

Let X, Y be sets. We use the standard notation of $x \in X$, $x \notin X$, $X \subseteq Y$, $X = Y$, $X \cap Y$, $X \cup Y$ and $\{\ldots\}$. $X \times Y$ shall denote $\{(x, y): x \in X, y \in Y\}$. The empty set shall be denoted by $\emptyset$ and the zero of a vector space by Θ. A function or linear map between spaces will be denoted $f: E \longrightarrow F$. The closed linear span of a set $S \subseteq E$, E a topological vector space will be given by $[S]$, and if $(x_n)_{n \in N}$ is a sequence the closed linear span is given by $[x_n]$. The span of a sequence or net $(x_\alpha)_{\alpha \in I}$ will be given by $sp\{x_\alpha\}$. Spaces will generally be denoted by E, F, G, H. Given $f: E \longrightarrow F$ and $M \subseteq E$, the restriction of f to M is denoted $f|M$. We denote infimum and supremum by inf and sup. If a map f is one-one we denote its inverse by f^{-1}.

2. Locally convex spaces

Definition: E_u is a topological vector space (in short a TVS) over a given field F if E as a pointset is a topological space and a vector space over F such that the maps

$$(x, y) \longrightarrow x + y$$

and $$(\lambda, x) \longrightarrow \lambda x$$ are continuous in both variables together, where $x, y \in E$, $\lambda \in F$, and u is the topology.

In the following we shall deal with real spaces unless otherwise specified, and all spaces considered will be Hausdorff.

Definitions: Let E be a vector space.

(a) A subset $M \subseteq E$ is circled if for each $x \in M$, $\lambda x \in M$ for all $|\lambda| \leq 1$, λ real.

(b) A subset $A \subseteq E$ absorbs a subset $B \subseteq E$ if there exists $\alpha > 0$ such that $\lambda B \subseteq A$ for all $|\lambda| \leq \alpha$, $\lambda \neq 0$. A subset $B \subseteq E$ is absorbing if it absorbs each point of E.

(c) Let $M \subseteq E$, $M \neq \emptyset$. The smallest circled set containing M is called the circled hull of M.

(d) A subset $M \subseteq E$ is convex if for any $x, y \in M$, $\lambda x + (1-\lambda)y \in M$ for $0 \leq \lambda \leq 1$, λ real.

Theorem 1: Let E_u be a TVS. Then there exists a fundamental system $\mathcal{U}$ of u-closed neighborhoods of θ such that:

(a) Each $U \in \mathcal{U}$ is circled and absorbing.

(b) For each $U \in \mathcal{U}$ there exists a V in $\mathcal{U}$ such that $V + V \subseteq U$.

(c) $\bigcap_{U \in \mathcal{U}} U = \{\theta\}$.

Conversely, if E is a vector space and $\mathcal{F}$ a filter base satisfying (a), (b) and (c), then there exists a Hausdorff topology u on E such that E_u is a TVS and $\mathcal{F}$ is a fundamental system of neighborhoods of θ.

Proof: A proof may be found in (14), (Chapter 2, Page 14).

Definitons: (a) A TVS E_u is locally convex if there exists a fundamental system of convex neighborhoods of θ.

(b) Let E be a real vector space. A semi-norm p on E is a real valued function $p: E \longrightarrow R$ (R = reals) such that

$$p(\lambda x) = |\lambda| \; p(x) \quad \text{for all} \quad \lambda \in R, x \in E$$

and $$p(x + y) \leq p(x) + p(y).$$

We observe that $p(\Theta) = 0$ and $p(x) \geq 0$ for each $x \in E$. If $p(x) = 0$ implies $x = \Theta$ then p is called a norm.

<u>Theorem 2:</u> Let P be a nonempty family of semi-norms on a vector space E. For $p \in P$, define $V(p) = \{x \in E: \; p(x) < 1\}$. Let $\mathcal{U}$ be the family of all finite intersections

$$\alpha_1 V(p_1) \cap \alpha_2 V(p_2) \cap \ldots \cap \alpha_n V(p_n), \; \alpha_k \in R, \; \alpha_k > 0$$

and $p_k \in P$ for all $1 \leq k \leq n$. Then there is a unique topology u on E such that E_u is a locally convex TVS and $\mathcal{U}$ is a fundamental system of convex neighborhoods of the origin.

<u>Proof:</u> A proof may be found in (29), Page 146.

<u>Proposition 1:</u> (a) Let E_u be a locally convex TVS, and $M \subseteq E_u$ a subspace. Let u be given by the family of semi-norms $(q_\lambda)_{\lambda \in L}$. Then the quotient space E/M and M are locally convex spaces with the topologies defined by the family of semi-norms $(\dot{q}_\lambda)_{\lambda \in L}$, (where $\dot{q}_\lambda(\dot{x}) = \inf_{x \in \dot{x}} q_\lambda(x))$ and $(q_\lambda | M)_{\lambda \in L}$, respectively.

(b) Let $(E_\alpha)_{\alpha \in I}$ be a family of locally convex spaces. Then the product $\pi_{\alpha \in I} E_\alpha$ is also a locally convex TVS with the product topology.

<u>Proof:</u> A proof may be found in (13)(Chapter 2, § 5).

<u>Definitions:</u> (a) A locally convex TVS E_u is complete if every Cauchy net converges.

(b) A subset $B \subseteq E_u$ is bounded if it is absorbed by every neighborhood of the origin.

Theorem 3: Let E_u be a locally convex TVS, and $M \subseteq E_u$ a closed linear subspace. Suppose there exists $x \in E_u$, $x \notin M$. Then there exists an $f: E_u \longrightarrow R$ such that $f(y) = 0$ for $y \in M$ and $f(x) = 1$, f a continuous linear (i.e. $f(\lambda x + uy) = \lambda f(x) + uf(y)$) map.

Proof: A proof can be found in (29), (Chapter 1, § 7, Page 42).

Proposition 2: Let E_u be a locally convex TVS defined by $(q_\lambda)_{\lambda \in L}$. A subset $B \subseteq E_u$ is bounded if and only if every q_λ is bounded on B.

Proof: See (13), (Chapter 2, § 6).

Definitions: (a) Two families of semi-norms on a vector space E are equivalent if they define the same locally convex topology on E.

(b) A family $\mathcal{F}$ of semi-norms on a vector space is saturated if for any finite subfamily $(q_i)_{i \in J}$ of $\mathcal{F}$, the semi-norm $q(x) = \max_{i \in J} q_i(x)$ is also in $\mathcal{F}$.

Remark: If $(q_\lambda)_{\lambda \in L}$ is a family of semi-norms defining a locally convex topology on a vector space E, then the family obtained by taking, for every finite subfamily $(q_i)_{i \in J}$, the $\max_{i \in J} q_i$, is an equivalent saturated family of semi-norms.

3. Mappings and the dual

Definitions: (a) Let E, F be two vector spaces. The set of all linear maps $f: E \dashrightarrow F$ is a vector space with pointwise addition and scalar multiplication.

(b) $E^{\#}$, the set of all linear maps $f: E \longrightarrow R$ is called the algebraic dual. If E_u is a TVS, the set of all continuous linear maps $f: E_u \longrightarrow R$ is called the topological dual E'. Clearly $E' \subseteq E^{\#}$. The elements of $E^{\#}$ are called linear functionals, and those of E' are called continuous linear functionals.

Proposition 3: Let E_u, F_v be TVS's. A linear map $f: E_u \longrightarrow F_v$ is continuous if it is continuous at the origin.

Proof: See (30), (Chapter 10, § 5, Theorem 1).

Definitions: Let X_u, Y_v be topological spaces.

(a) A map $f: X_u \longrightarrow Y_v$ is almost continuous at $x \in X$ if for each v-neighborhood V of $f(x)$, $\overline{f^{-1}(V)}$ contains a u-neighborhood of x. f is almost continuous on X if it is so at each point.

(b) A map $f: X_u \longrightarrow Y_v$ is almost open if, for each u-neighborhood U of $x \in X_u$, $\overline{f(U)}$ contains a v-neighborhood of $f(x) \in Y_v$.

(c) Given a map $f: X_u \longrightarrow Y_v$, the subset $\{(x, f(x)): x \in X_u\}$ of $X \times Y$ is called the graph of f.

(d) Given two TVS's E_u, F_v a map $f: E_u \longrightarrow F_v$ is an isomorphism if f is linear, one-one, onto, continuous and open.

Remark: All isomorphisms will be onto in the sequel unless otherwise indicated. An isomorphism which is not onto is called an isomorphism into.

Proposition 4: Let E_u be a locally convex TVS where u is given by the

saturated family $(q_\lambda)_{\lambda\varepsilon I}$ of semi-norms, and F_v a locally convex TVS, v given by $(r_\lambda)_{\lambda\varepsilon L}$. A linear map $f: E_u \longrightarrow F_v$ is continuous if and only if for every r_λ there exists a q_α and $M > 0$ such that

$$r_\lambda(f(x)) \leq M\, q_\alpha(x) \quad \text{for all } x \varepsilon E_u.$$

<u>Proof</u>: See (13), (Chapter 2, § 5 page 97).

<u>Remark</u>: If a linear map $f: E_u \longrightarrow F_v$ is continuous, then it is uniformly continuous.

<u>Proposition 5</u>: Let E_u be a TVS and F_v a complete TVS and $A \subseteq E_u$ a subset. If $f: A \longrightarrow F_v$ is a uniformly continuous map, then there exists a uniformly continuous extension of f to $\overline{A}$.

<u>Proof</u>: See (13), (Chapter 2, § 9, Proposition 5).

4. Inductive limits

<u>Definitions</u>: (a) Let $(E_\alpha)_{\alpha\varepsilon I}$ be a family of locally convex spaces, E a vector space and $(f_\alpha)_{\alpha\varepsilon I}$ linear maps $f_\alpha: E_\alpha \longrightarrow E$ for each $\alpha \varepsilon I$. Suppose $E = \bigcup_{\alpha\varepsilon I} f_\alpha(E_\alpha)$. The inductive limit of $(E_\alpha)_{\alpha\varepsilon I}$ is E_u where u is the finest locally convex topology of E such that each f_α is continuous.

(b) If $I = N$ (the natural numbers), then E_u is called the generalized strict inductive limit of the E_n.

(c) If $I = N$ and each f_n is the natural injection of E_n i.e. E is the union of a strictly increasing sequence of its' subspaces, and if the inductive limit topology on E induces the same topology as that of E_n, then E_u is called the strict inductive limit of $(E_n)_{n\varepsilon N}$.

Remark: If E_u is the strict inductive limit of locally convex spaces $(E_n)_{n\varepsilon N}$, then E_u is Hausdorff if each E_n is.

Proposition 6: Let E_u be the strict inductive limit of $(E_n)_{n\varepsilon N}$. Then if each E_n is complete, so is E_u.

Proof: See (2), (Chapter 2, § 2, Exercise 9).

5. Barrelled spaces

Definitions: (a) Let E_u be a locally convex TVS. A closed, convex, circled, absorbing subset $B \subseteq E_u$ is called a barrel.

(b) E_u is called a barrelled space (sometimes t-space from the French espace tonnelé) if every barrel in E_u is a neighborhood of the origin.

(c) A subset $A \subseteq X_u$, X_u a topological space, is nowhere dense if the interior of its closure is $\emptyset$.

(d) A topological space X_u is a Baire space if it can't be expressed as the countable union of nowhere dense sets.

Proposition 7: A Baire locally convex TVS E_u is barrelled.

Proof: See (13), (Chapter 3, § 6, Proposition 3).

Remarks: (a) If E_u is a barrelled TVS and $M \subseteq E_u$ a subspace, then the quotient space E/M is barrelled.

(b) If $(E_\alpha)_{\alpha\varepsilon I}$ is a family of barrelled TVS's, then the product with the product topology is barrelled.

(c) A closed subspace of a barrelled space is not necessarily barrelled.

(d) Inductive limits of barrelled spaces are barrelled.

(e) Every Baire space hence Frechét space (complete metric locally convex TVS) is barrelled.

(f) A barrelled space is neither necessarily metrizable nor complete.

6. Duality theory and reflexive spaces

Let E_u be a locally convex TVS. We present here some topologies on E and E' which play an important role.

Definitions: (a) The coarsest locally convex topology on E such that the map $x \longrightarrow f(x)$ for each $f \in E'$ is continuous, is called the weak topology, $\sigma(E, E')$ on E.

(b) The weak* topology, $\sigma(E', E)$ is defined as the coarsest locally convex topology on E' such that the map $f \longrightarrow f(x)$ for each $x \in E$, is continuous.

Remark: $\sigma(E', E)$ is precisely the topology of pointwise convergence which is induced from the product topology on $R^E = \{f \text{ where } f: E \longrightarrow R\}$.

Definition: Let E_u, F_v be two TVS's, not necessarily Hausdorff. Consider the vector space $\mathcal{L}(E, F)$ of all continuous linear maps $f: E_u \longrightarrow F_v$ with pointwise addition and scalar multiplication. Let $\mathfrak{S}$ be a class of subsets of E_u. One can define a topology of uniform convergence over sets $M \in \mathfrak{S}$ as follows: Let V be a neighborhood of the origin in F_v and $M \in \mathfrak{S}$. Let $T(M, V)$ be all continuous linear maps $f \in \mathcal{L}(E, F)$ such that $f(M) \subseteq V$. The collection of all $T(M, V)$, $M \in \mathfrak{S}$, V a neighborhood of the origin, forms a subbase for a topology on $\mathcal{L}(E, F)$ called the $\mathfrak{S}$-topology. See (2), (Chapter 3, § 3).

Remarks: (a) $\mathcal{L}(E, F)$ with an $\mathfrak{S}$-topology is compatible with the vector space structure if and only if $f(M)$ is bounded in F_v for each $M \in \mathfrak{S}$ and $f \in \mathcal{L}(E, F)$.

(b) If F_v is locally convex, then the $\mathfrak{S}$-topology is also.

(c) If $\mathfrak{S}$ consists of bounded sets of E_u such that $\bigcup_{M \in \mathfrak{S}} M$ is total (the set of all finite linear combinations dense) in E_u and F_v is locally convex, then the $\mathfrak{S}$-topology is a Hausdorff locally convex topology.

Definitions: (a) Let E_u be a locally convex TVS. Then the strong topology β on E' is the $\mathfrak{S}$-topology where $\mathfrak{S}$ is the collection of all u-bounded sets of E_u. E'^{β}_u (or E'^{β}) is called the strong dual of E_u.

(b) The $\mathfrak{S}$-topology on a locally convex TVS E_u where $\mathfrak{S}$ consists of all convex, circled, $\sigma(E', E)$ - compact subsets of E' is called the Mackey topology, denoted by $\tau(E, E')$.

(c) A locally convex TVS, E_u is a Mackey space if $u = \tau(E, E')$.

Remark: A barrelled TVS is a Mackey space.

Theorem 4: (Mackey) Let E_u be a locally convex TVS. Then a subset $M \subseteq E_u$ is $\sigma(E, E')$ - bounded if and only if M is $\tau(E, E')$ - bounded.

Proof: A proof may be found in (14) (Chapter 2, § 9, Theorem 8).

Definitions: (a) The dual $E'^{\beta\prime}$ of E'^{β} is called the bidual of E.

(b) If $E'^{\beta\prime} = E$ (algebraically) then E_u is called semi-reflexive.

(c) If $E'^{\beta,\beta} = E$ (topologically) then E_u is called reflexive.

Clearly a locally convex TVS E_u is semi-reflexive if it is reflexive.

<u>Proposition 8</u>: Every barrelled TVS E_u is a Mackey space.

<u>Proof</u>: See (14), (Chapter 2, § 9, Proposition 16).

<u>Proposition 9</u>: A reflexive locally convex TVS is barrelled.

<u>Proof</u>: See (14), (Chapter 2, § 9).

<u>Proposition 10</u>: The strong dual E'^{β} of a reflexive TVS is also reflexive.

<u>Proof</u>: See (13), (Chapter 3, § 8, Proposition 7).

7. Bases

<u>Definitions</u>: Let E_u be a TVS.

(a) A biorthogonal system $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ in E_u is a family $(x_\alpha)_{\alpha\varepsilon I}$ of points in E (not necessarily countable) and a family $(f_\alpha)_{\alpha\varepsilon I}$, $f_\alpha \varepsilon E'$ for all $\alpha \varepsilon I$, such that $f_\alpha(x_\beta) = \delta_{\alpha\beta}$. i.e. $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$, 1 if $\alpha = \beta$.

(b) A countable family $(x_n)_{n\varepsilon N}$ in E_u is a basis if for each $x \varepsilon E_u$ there exists a unique sequence of scalars $(a_n)_{n\varepsilon N}$ such that $x = \sum_1^\infty a_n x_n$ i.e. $x = \lim_{n\to\infty} \sum_{i=1}^n a_i x_i$ in the topology u. Hence $[x_n] = E_u$. Given $x = \sum_1^\infty a_n x_n$, we can define f_n by $f_n(x) = a_n$. This gives a unique family of functionals called coefficient or coordinate functionals, and $f_n(x_m) = \delta_{nm}$. When all the $(f_n)_{n\varepsilon N}$ are continuous, we call $(x_n)_{n\varepsilon N}$ or $(x_n, f_n)_{n\varepsilon N}$ a Schauder basis.

<u>Remarks</u>: (a) The above definition of basis was introduced by Schauder (5) with E_u taken as a Banach space, and has subsequently been extended to TVS's in general.

(b) The coefficient functionals are not always continuous as the following example shows. Consider E^{∞} the space of all real sequences of finite support, with norm $\|y\| = \max\{|y_n|, : n = 1,2,\ldots\}$ where $y = (y_n)_{n\varepsilon N} \varepsilon E^{\infty}$. Define $(x_n)_{n\varepsilon N}$ by $x_1 = e_1$, $x_n = e_1 + \frac{1}{n} e_n$ where $e_n = (0, 0, \ldots, \overset{n^{th}}{1}, 0, \ldots)$. We claim $(x_n)_{n\varepsilon N}$ is a basis, since for $a = (a_n)_{n\varepsilon N} \varepsilon E^{\infty}$, suppose $a_n = 0$ for $n > r$. Let the coefficients be $t_1 = a_1 - \sum_{k=2}^{r} k\, a_k$, $t_k = k\, a_k$ for $k \geq 2$. Then $\sum_{k=1}^{r} t_k x_k = a$ since $\sum_{k=1}^{r} t_k x_k = (t_1, 0, 0, \ldots) + (t_2, \frac{1}{2} t_2, 0, \ldots) + (t_3, 0, \frac{1}{3} t_3, 0, \ldots) + \ldots + (t_r, 0, \ldots, \frac{1}{r} t_r, 0, \ldots) = (\sum_{k=1}^{r} t_k, \frac{1}{2} t_2, \frac{1}{3} t_3, \ldots, \frac{1}{r} t_r, 0, \ldots) = (a_1, a_2, \ldots, a_r, 0, \ldots) = a$. Also, the t_k are unique, since if $a = \sum_{k=1}^{\infty} b_k x_k$, then $b_n = 0$ for $n > r$, and $a_n = \frac{1}{n} b_n$. By repeating this we get $b_1 = a_1 - \sum_{k=1}^{r} b_k = a_1 - \sum_{k=1}^{r} a_k$, $b_2 = 2a_2, \ldots$. The coefficient f_1 given by $f_1(a) = t_1$ is not continuous. This is seen by observing that $x_n \longrightarrow x_1$ as $n \longrightarrow \infty$, and since $f_1(x_n) = 0$ for $n = 2, 3, \ldots$, while $f_1(x_1) = 1$, f_1 cannot be continuous.

(c) A TVS with a (Schauder) basis is separable.

The following theorem due to Newns shows that for a very large class of spaces, the coefficient functionals are continuous.

Theorem 5 (Newns): Let E_u be a complete metric TVS over the real or complex field. Then every basis for E_u is a Schauder basis.

Proof: See (20), (Chapter 9, § 4, Theorem 2)

The question of existence of a Schauder basis in Banach spaces is yet unsolved although many people have made an effort in this direction. The following shows that the important question of existence cannot be answered generally in the affirmative in a general TVS.

Example: Let S be a compact interval of the reals. Let $L_p(S)$ be the set of all equivalence classes of measurable functions $f: S \longrightarrow R$, with $\|f\| = \int_S |f(x)|^p dx < +\infty$. It is known (20) (Chapter 9, § 5, Theorem 2) that $L_p(S)$ with $\| \ \|$ is a complete quasi-normed TVS. (i.e. $\|f\| = 0$ if and only if $f = 0$, $\|f + g\| \leq \|f\| + \|g\|$). and $L_p(S)$, $0 < p < 1$, provides examples of complete quasi-normed spaces which have no Schauder basis, since $L_p(S)' = \{0\}$. This example is due to M. M. Day, see (20), Chapter 9, Corollary 3). However $L_p(S)$, with $\|f\| = \left(\int_S |f(x)|^p\right)^{\frac{1}{p}}$, $1 < p < \infty$, provides many examples of separable locally convex TVS's with a Schauder basis.

Definitions: (a) A biorthogonal system $(x_\alpha, f_\alpha)_{\alpha \varepsilon I}$ in a TVS E_u is called separating if $f_\alpha(x) = 0$ for all $\alpha \varepsilon I$ gives $x = \theta$. One can also refer to $(f_\alpha)_{\alpha \varepsilon I}$ as being separating.

(b) A biorthogonal system $(x_\alpha, f_\alpha)_{\alpha \varepsilon I}$ in a TVS E_u is a generalized basis if it is separating. If in addition to this, the $(x_\alpha)_{\alpha \varepsilon I}$ is total i.e. $[x_\alpha] = E_u$, then the system is called a Markuschevich basis.

(c) A biorthogonal system is called maximal if it is not contained strictly in any larger biorthogonal system.

<u>Proposition 11</u>: Let $(x_n, f_n)_{n\varepsilon N}$ be a biorthogonal system in a TVS, E_u. Then $(x_n, f_n)_{n\varepsilon N}$ a Schauder basis implies Markuschevich basis implies generalized basis implies maximal biorthogonal system.

<u>Proof</u>: The implications Schauder to Markuschevich to generalized basis follow directly from the definition. That a generalized basis is a maximal biorthogonal system follows easily. If $(x_n, f_n)_{n\varepsilon N}$ were not maximal, then there would exist a point $y \varepsilon E_u$, $y \neq \Theta$ and $g \varepsilon E'$ such that $g(y) = 1$ and $f_n(y) = 0$ for all $n \varepsilon N$. This is a contradiction since $f_n(y) = 0$ for all n gives $y = \Theta$, and $g \varepsilon E'$ gives $g(\Theta) = 0$. Q.E.D.

The implications are not reversible as the following examples show.

<u>Examples</u>: (a) Let E_u be the Frechét space of all functions analytic on the open unit disc $(|z| < 1)$ topologized by the metric of uniform convergence on compact sets. Choose "a" any complex number such that $0 < |a| < 1$. Set $x_n(z) = (z - a)^n$ for $n = 0, 1, \ldots$. As (7) shows, $(x_n)_{n\varepsilon N}$ forms a Markuschevich basis in E_u and the corresponding coefficient functionals are given by

$$f_n(g) = \frac{g^{(n)}(a)}{n!} \qquad n = 0, 1, \ldots .$$

$(x_n, f_n)_{n\varepsilon N}$ is not a Schauder basis since the series $\sum_{n=1}^{\infty} \frac{(z-a)^n}{(1-a)^{n+1}}$ corresponding to $(1-z)^{-1}$ diverges outside the circle $|z-a| \leq |1-a|$. That is, $\frac{1}{1-z}$ is analytic on $|z| < 1$ but can't be represented on $|z| < 1$ in terms of $(x_n, f_n)_{n\varepsilon N}$.

(b) Let I be a countably infinite set with the discrete topology and E_u the space of all bounded real valued functions on I. This is a Banach space with sup norm. For each $\alpha \varepsilon I$, defined x_α by $x_\alpha(\alpha) = 1$ and 0 elsewhere. The point functionals $f_\alpha(x) = x(\alpha)$ are all continuous and $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ forms a biorthogonal system. For $x \varepsilon E_u$ if $f_\alpha(x) = 0$ for all α then $x = \Theta$, hence $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ is a generalized basis. E_u is not separable (see (29), Chapter 3, § 2, Page 89), hence $(x_\alpha)_{\alpha\varepsilon I}$ cannot be total, so the system is not a Markuschevich basis.

(c) Let E_u be as in (a). Define $(x_n)_{n\varepsilon N}$ by

$$x_n(z) = 1 + z + z^2 + \ldots + z^n$$

and $(f_n)_{n\varepsilon N}$ by $f_n(g) = \dfrac{g^{(n)}(0)}{n!} - \dfrac{g^{(n+1)}(0)}{(n+1)!}$ for $n = 0, 1, \ldots$ and $g \varepsilon E$. Trivially $(x_n, f_n)_{n\varepsilon N}$ forms a biorthogonal system, but $(f_n)_{n\varepsilon N}$ is not separating. To see this we use the fact that $f_n(g) = 0$ for all $n \varepsilon N$ if and only if g is of the form $g(z) = c(1 + z + z^2 + \ldots + z^n + \ldots)$, where c is an arbitrary scalar. Hence $(x_n, f_n)_{n\varepsilon N}$ is not a generalized basis, but is maximal since if not there exists an $h \varepsilon E'$ such that $h(x_n) = 0$ for all n but $h \neq 0$ the zero functional. This is impossible since every $h \varepsilon E'$ has the representation $h(y) = \sum_{n=0}^{\infty} \dfrac{y^{(n)}(0)}{n!} h_n$, h_n suitably chosen complex numbers. Hence $(x_n, f_n)_{n\varepsilon N}$ is a maximal biorthogonal system.

The following results affirm the existence of the above types of bases.

<u>Theorem 6 (Markuschevich)</u>: There exists a Markuschevich basis for every separable Banach space.

Proof: See (20), (Chapter 9, §1, Page 116).

Theorem 7: Every separable Hilbert space (complete inner-product space) has a Schauder basis.

Proof: See (20), (Chapter 6, § 1, Theorem 1).

As mentioned before the problem is yet unsolved for Banach spaces, however the following is known.

Theorem 8: Each infinite dimensional Banach space E_u contains an infinite dimensional subspace with a basis.

Proof: A proof may be found in (20), (Chapter 3, § 3, Theorem 7).

Definition: Let $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ be a generalized basis in a complex or real TVS E_u. Let R^I be the linear space of all indexed families $(\alpha_i)_{i\varepsilon I}$, the operations being pointwise relative to the scalar field. Define $\Phi : E_u \to R^I$ by $\Phi(x) = (f_\alpha(x))_{\alpha\varepsilon I}$. Φ is called the coefficient map determined by the $(f_\alpha)_{\alpha\varepsilon I}$.

Remarks: (a) Φ is linear since

$$\begin{aligned}\Phi(ax + by) &= (f_\alpha(ax + by))_{\alpha\varepsilon I} = (a\, f_\alpha(x) + b\, f_\alpha(y))_{\alpha\varepsilon I} \\ &= (a\, f_\alpha(x))_{\alpha\varepsilon I} + (b\, f_\alpha(y))_{\alpha\varepsilon I} \\ &= a\,(f_\alpha(x))_{\alpha\varepsilon I} + b\,(f_\alpha(y))_{\alpha\varepsilon I} \\ &= a\,\Phi(x) + b\,\Phi(y).\end{aligned}$$

(b) The property of being a separating family of functionals is equivalent to Φ being a one-one map.

We now present some further facts about coefficient functionals and generalized bases.

<u>Theorem 9</u>: A biorthogonal system $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ in a TVS E_u is a generalized basis for E_u if and only if Φ is one-one.

<u>Proof</u>: The proof is evident from the definitions.

<u>Theorem 10</u>: A given family $(f_\alpha)_{\alpha\varepsilon I} \subseteq E'$ can be the family of coefficient functionals for at most one generalized basis in E_u.

<u>Proof</u>: If there is another generalized basis $(x'_\alpha, f_\alpha)_{\alpha\varepsilon I}$ for E_u, then $f_\alpha(x'_\beta) = \delta_{\alpha\beta} = f_\alpha(x_\beta)$ for $\alpha, \beta \varepsilon I$. Hence $f_\alpha(x'_\beta - x_\beta) = f_\alpha(x'_\beta) - f_\alpha(x_\beta) = 0$ for all $\alpha \varepsilon I$, so $x'_\beta = x_\beta$. Q.E.D.

<u>Theorem 11</u>: A generalized basis $(x_\alpha)_{\alpha\varepsilon I}$ for a locally convex TVS E_u has a unique family of coefficient functionals if and only if $(x_\alpha)_{\alpha\varepsilon I}$ is total in E_u.

<u>Proof</u>: Let $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ be a total generalized basis in E_u such that $(g_\alpha)_{\alpha\varepsilon I}$ is another distinct family of coefficient functionals. Then $f_\alpha(x_\beta) = g_\alpha(x_\beta) = \delta_{\alpha\beta}$ for $\alpha, \beta \varepsilon I$. Hence $(f_\alpha - g_\alpha)(x) = 0$ for $\alpha \varepsilon I$, $x \varepsilon \operatorname{sp}_{\alpha\varepsilon I}\{x_\alpha\}$. Since $(x_\alpha)_{\alpha\varepsilon I}$ is total and $f_\alpha - g_\alpha$ is continuous, $(f_\alpha - g_\alpha)(x) = 0$ for all $x \varepsilon E_u$; that is, $f_\alpha = g_\alpha$ for all $\alpha \varepsilon I$.

Conversely, suppose that $(f_\alpha)_{\alpha\varepsilon I}$ is unique, but $(x_\alpha)_{\alpha\varepsilon I}$ is not total. Hence by Theorem 3, there exists an $f \varepsilon E'$ such that $f(x_\alpha) = 0$ for $\alpha \varepsilon I$, and $f \neq 0$. Let $\beta \varepsilon I$, β fixed. Then observe $(f_\alpha + \delta_{\alpha\beta} f)(x_\gamma) = \delta_{\alpha\gamma}$. Hence $(x_\alpha, f_\alpha + \delta_{\alpha\beta} f)_{\alpha\varepsilon I}$ is a biorthogonal system. To see that the system is separating, assume $(f_\alpha + \delta_{\alpha\beta} f)(x) = 0$ for $\alpha \varepsilon I$, and some $x \varepsilon E_u$. Then $f_\alpha(x) = -f(x)\, f_\alpha(x_\beta) = f_\alpha(-f(x)\, x_\beta)$ for $\alpha \varepsilon I$. Hence $\Phi(x) = \Phi(-f(x)\, x_\beta)$.

But since $(x_\alpha)_{\alpha\varepsilon I}$ is a generalized basis, Φ is one-one, so $x = -f(x)\, x_\beta$. Therefore $f_\alpha(x) = f_\alpha(-f(x)\, x_\beta) = -f(x)\, f_\alpha(x_\beta) = 0$ for $\alpha \varepsilon I$ because $f(x) = 0$ for all $x \varepsilon \operatorname{sp}\{x_\alpha\}$. Since $(f_\alpha)_{\alpha\varepsilon I}$ is separating $x = \Theta$. So there exists another generalized basis $(x_\alpha, f_\alpha + \delta_{\alpha\beta} f)_{\alpha\varepsilon I}$. Contradiction! Hence $(x_\alpha)_{\alpha\varepsilon I}$ is total. Q.E.D.

<u>Remark</u>: By the label "weak" as applied to any of the types of bases previously considered, we shall mean that the topology on the linear space being considered is the weak topology.

<u>Theorem 12</u>: Every weak Schauder basis in a barrelled TVS E_u is a Schauder basis in the initial topology.

<u>Proof</u>: See (1) (Theorem 11).

<u>Theorem 12</u>: Let E_u be a locally convex TVS. Then every weak Markuschevich or generalized basis in E_u is one for the initial topology.

<u>Proof</u>: See (1), (Theorem 10).

8. Countably normed spaces

Let $\|\ \|_1, \|\ \|_2, \ldots$ be a countable system of norms defined in a linear space E. Using these norms we can introduce a topology on E as follows. Let p be an arbitrary positive integer and $\varepsilon > 0$. Define neighborhoods of $\Theta \varepsilon E$ by

$$U_{p,\varepsilon}(\Theta) = \left\{ x \varepsilon E:\ \|x\|_i < \varepsilon,\ 1 \leq i \leq p \right\}.$$

We observe that the following properties hold:

(a) $\Theta \varepsilon U_{p,\varepsilon}$ for all p, ε.

(b) Given U_{p_1,ε_1} and U_{p_2,ε_2} defined as above, there exists a third neighborhood contained in $U_{p_1\varepsilon_1} \cap U_{p_2,\varepsilon_2}$ i.e. $U_{p,\varepsilon} \subseteq U_{p_1,\varepsilon_1} \cap U_{p_2,\varepsilon_2}$ where $U_{p,\varepsilon}$ is such that $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $p = \max\{p_1, p_2\}$.

(c) Give $x_0 \neq \Theta$, there exists a neighborhood U of Θ such that $x_0 \notin U$. viz. $U = \{x: \|x\|_1 < \varepsilon = \|x_0\|\}$

(d) Given a neighborhood U of Θ, there exists a neighborhood W of Θ such that $W + W \subseteq U$. i.e. If $U = \{x \in E: \|x\|_i < \varepsilon \quad 1 \leq i \leq p\}$, one can take $W = \{x \in E: \|x\|_i < \frac{\varepsilon}{2}, 1 \leq i \leq p\}$. This follows directly from the triangular inequality.

(e) Given $x_0 \in U_{p,\varepsilon}$ there exists a neighborhood V of Θ such that $x_0 + V \subseteq U_{p,\varepsilon}$. Simply take $\max_{i \leq p} \|x_0\|_i = \eta$. We note that $\eta < \varepsilon$, so one can take $V = \{x \in E: |x|_i < \varepsilon - \eta, 1 \leq i \leq p\}$.

(f) Given a neighborhood $U_{p,\varepsilon}$ and $\alpha \neq 0$ a scalar, there exists a neighborhood W of Θ such that $\alpha W \subseteq U_{p,\varepsilon}$. $W = \{x \in E: \|x\|_i < \frac{\varepsilon}{|\alpha|} \quad 1 \leq i \leq p\}$ does the trick.

(g) Given a neighborhood $U_{p,\varepsilon}$ and $X_0 \in E$, there exists an α such that $\delta x_0 \in U_{p,\varepsilon}$ for $|\delta| < \alpha$. Let $\|x_0\|_j = \alpha_j \quad j = 1, 2, \ldots$ then put $\alpha = \frac{\varepsilon}{\max(\alpha_1, \ldots, \alpha_p)}$.

(h) For any neighborhood V of Θ there exists an $\varepsilon > 0$ such that $\delta V \subseteq V$ for $|\delta| < \varepsilon$. To see this we simply take $\varepsilon = 1$.

By Theorem 1, this family of all $U_{p,\varepsilon}$ defines a topology u on E such that E_u is a TVS, and the $\{U_{p,\varepsilon}\}_{p \in N, \varepsilon > 0}$ is a neighborhood basis of the origin.

<u>Definitions</u>: (a) Two norms $\| \|_1$, $\| \|_2$ on a linear space E are comparable if there is a $C > 0$ such that $\|x\|_1 \leq C \|x\|_2$ for all $x \in E$.

(b) $\| \|_1$, $\| \|_2$ on E are compatible if every sequence $(x_n)_{n \in N}$ which is cauchy with respect to both norms and converges to zero with respect to one of them also converges to zero with respect to the other. A countable family of norms is compatible if any two are compatible with each other.

<u>Remark</u>: In the case where one of $(E, \| \|_1) = E_1$ and $(E, \| \|_2) = E_2$ is not complete, they can be completed in the usual way. If $\| \|_2$ is the stronger norm i.e. $(\|x\|_1 \leq C \|x\|_2$ for $x \in E$, some $C > 0)$ then one can establish a map of E_2 into E_1 where every element $x \in E_2$ is defined by a sequence $(x_n)_{n \in N} \subseteq E$ which is cauchy with respect to $\| \|_2$, and hence with $\| \|_1$ and so defines a uniquely determined element $y \in E_1$. To insure that this map is one-one we require the norms to be compatible. Hence if there are two comparable and compatible norms $\| \|_1$, $\| \|_2$ on E and $\|x\|_1 \leq C\|x\|_2$ for $x \in E$, then the completions of E_1, E_2 with respect to $\| \|_1$, $\| \|_2$ can be considered to have the following relationship.

$\hat{E}_1 \supset \hat{E}_2 \supset E$. We note that under this identification every element $x \in E$ is carried into itself.

<u>Definition</u>: A countably normed space E_u is a locally convex TVS where the topology can be given as described previously.

<u>Remark</u>: The defining neighborhood system of the origin can be countable

by taking $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n} \ldots$ and hence the first axiom of countability is satisfied. Also, one can always consider the sequence of norms to be nondecreasing. i.e. $\|x\|_1 \leq \|x\|_2 \leq \ldots \leq \|x\|_p \leq \ldots$ for $x \in E$, by taking an equivalent family give by $\|x\|'_p = \max_{1 \leq i \leq p} \{\|x\|_i\}$.

<u>Definition</u>: A sequence in a countably normed space is cauchy if it is cauchy with respect to each norm.

One can consider the completion of each $(E, \| \|_i)_{i \in N}$ say E_i and we have that $E_1 \supset E_2 \supset \ldots \supset E_p \supset \ldots \supset E_i$ where E_i is the completion of E with respect to $\| \|_i$.

<u>Theorem 13</u>: The space E_u is complete if and only if $E = \bigcap_{p=1}^{\infty} E_p$ where E_p is the completion of E with respect to $\| \|_p$.

<u>Proof</u>: See (9) (Chapter 1, § 3).

In what follows we shall always deal with complete countably normed spaces, denoted C.N.S. unless otherwise indicated. We now give some nontrivial examples of such spaces.

<u>Examples</u>: (a) Let $K(a)$ be the set of all infinitely differentiable functions on $(-\infty, \infty)$, which vanish outside $|x| \leq a$, with norms given by $\|f\|_p = \max_{|x| \leq a} \{|f(x)|, |f'(x)|, \ldots, |f^{(p)}(x)|\}$, for $p = 0, 1, 2, \ldots$. $K(a)$ is a C.N.S., (9), (Chapter 1).

(b) Let $Z(D)$ be all analytic functions f in the region $D = \{z : |z| < a\}$ where a is a positive real number with the norms given by $\|f\|_p = \max \{|f(z)| : |z| \leq a_p\}$ where $(a_p)_{p \in N}$ is a monotone increasing sequence such that $\lim_{p \to \infty} a_p = a$. $Z(D)$ is also a C.N.S. See (9), (Chapter 1).

<u>Definition</u>: A countably Hilbert space (C.H.S.) E_u is a countably normed space where the topology is defined by a countable system of scalar products $(\langle\ \rangle_p)_{p\varepsilon N}$ which are compatible (the norms $\|x\|_p = \sqrt{\langle x,x\rangle_p}$, $x \varepsilon E$, $p = 0, 1, 2, \ldots$ are compatible). The topology is defined in the manner described previously.

We note that every Banach (Hilbert) space is a C.N.S. (C.H.S.) respectively.

<u>Theorem 14</u>: Let E_u be a complete C.H.S. Then E_u is reflexive.

<u>Proof</u>: See (10) (Chapter 1, § 3, Page 61).

CHAPTER II

Isomorphism Theorem and Applications

1. B-Completeness

Definitions: (a) A locally convex TVS E_u is B-complete if any linear, continuous and almost open map of E_u onto any locally convex TVS F_v is open

(b) A locally convex TVS E_u is B_r-complete if any linear continuous almost open one-one map of E_u onto any locally convex TVS F_v is open.

Proposition 1: Every B-complete space is B_r-complete.

Proof: The proof follows immediately from the definition.

Proposition 2: Every Fréchet space is B-complete.

Proof: See (14) (Chapter 3, § 2, Theorem 2).

Proposition 3: Each B_r-complete (hence also B-complete) locally convex TVS E_u is complete.

Proof: A proof may be found in (14) (Chapter 5, § 3, Proposition 10).

Proposition 4: Let E_u be a B-complete TVS and $M \leq E_u$ a closed subspace. Then M is B-complete.

Proof: See (14) (Chapter 4, § 1, Proposition 4).

Proposition 5: Let E_u, F_v be two locally convex spaces. Let $f: E_u \longrightarrow F_v$ be a linear, continuous, almost open onto map. If E_u is B-complete, so is F_v.

Proof: See (14) (Chapter 4, § 1, Proposition 5).

Corollary 1: Let E_u be a B-complete TVS and $M \subseteq E_u$ a closed subspace. Then E/M with the quotient topology is B-complete.

Proof: See (14) (Chapter 4, § 1, Proposition 5).

Proposition 6: Let E_u be a Frechét space. Then the topological dual E' is B-complete for any topology v such that $\tau(E', E) \supset v \supset c$ where $\tau(E', E)$ is the Mackey topology and c the σ-topology of all u-compact subsets.

Proof: See (14) (Chapter 4, § 1, Proposition 7).

Definition: Let X_u, Y_v be two topological spaces and $f: X_u \longrightarrow Y_v$ a map. The graph of f is $\{(x, f(x)): x \in X_u\}$, and is a subset of $X_u \times Y_v$. The graph of f is closed if it is a closed subset of $X_u \times Y_v$ with the product topology.

Lemma 1: Let X_u, Y_v be topological spaces. Then $f: X_u \longrightarrow Y_v$ is closed if and only if for any net $(x_\alpha)_{\alpha \in I} \subseteq X_u$ with $x_\alpha \longrightarrow x$ and $f(x_\alpha) \longrightarrow y$, it follows that $f(x) = y$.

Proof: See for example (30), (Chapter 11, § 1, Theorem 1).

Theorem 1: (Robertson and Robertson). Let E_u be a B-complete TVS and F_v a barrelled TVS. Then if $g: F_v \longrightarrow E_u$ is a linear map with closed graph, then g is continuous.

Proof: See (14) (Chapter 4, § 5, Theorem 8(6)).

Corollary 1: Let E_u, F_v be B-complete barrelled spaces. Then (a) any linear map $f: E_u \longrightarrow F_v$ with

closed graph is continuous

(b) any continuous linear mapping $f: E \longrightarrow F$ onto is open.

Proof: The proof of (a) follows trivially from the theorem. Q.E.D.

2. An isomorphism theorem

Definitions: Let E_u, F_v be TVS's. Then

(a) two Schauder bases $(x_n)_{n\varepsilon N} \subseteq E_u$ and $(Y_n)_{n\varepsilon N} \subseteq F_v$ are similar if for any sequence of scalars $(a_n)_{n\varepsilon N}$, $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges.

(b) Two generalized bases $(x_\alpha)_{\alpha\varepsilon I} \subseteq E_u$ and $(y_\alpha)_{\alpha\varepsilon I} \subseteq F_v$ are similar if the corresponding coefficient functionals $(f_\alpha)_{\alpha\varepsilon I}$, $(g_\alpha)_{\alpha\varepsilon I}$ and coefficient maps Φ and Ψ are such that

$$\Phi(E_u) = \Psi(F_v)$$

Proposition 7: Let E be a linear space with u, v two topologies on E such that E_u, E_v are TVS's. Assume there exists a separating family $(f_\alpha)_{\alpha\varepsilon I}$ of real valued functions such that each f_α is continuous with respect to u and v. Then the identity map $i: E_u \longrightarrow E_v$ has a closed graph.

Proof: Let $(x_\beta)_{\beta\varepsilon J}$ be a net in E_u such that $(x_\beta)_{\beta\varepsilon J}$ converges to $x \varepsilon E_u$ and $(x_\beta)_{\beta\varepsilon J}$ converges to $y \varepsilon E_v$. Since f_α is continuous with respect to u and v for all α, $(f_\alpha(x_\beta))_{\beta\varepsilon J}$ converges to $f_\alpha(y)$ and $f_\alpha(x)$ for all $\alpha \varepsilon I$. Since limits of converging nets in the reals with the usual topology are unique, $f_\alpha(x) = f_\alpha(y)$ for all $\alpha \varepsilon I$.

Since $(f_\alpha)_{\alpha\varepsilon I}$ is a separating family of functions, $f_\alpha(x-y) = 0$ for all $\alpha \varepsilon I$, implies $x-y = \Theta$ or $x = y$. Hence the graph is closed.

<u>Theorem 2</u>: Let E_u and F_v be a B-complete barrelled spaces. Let $(x_\alpha)_{\alpha\varepsilon I}$ be a generalized basis in E_u. If T is an isomorphism $T: E_u \longrightarrow F_v$ and $Tx_\alpha = y_\alpha$ for all $\alpha \varepsilon I$, then $(y_\alpha)_{\alpha\varepsilon I}$ is a generalized basis in F_v and $(x_\alpha)_{\alpha\varepsilon I}$ is similar to $(y_\alpha)_{\alpha\varepsilon I}$. Conversely if $(y_\alpha)_{\alpha\varepsilon I}$ is a generalized basis in F_v similar to $(x_\alpha)_{\alpha\varepsilon I}$, then there exists an isomorphism $T: E_u \longrightarrow F_v$ such that $y_\alpha = Tx_\alpha$ for $\alpha \varepsilon I$.

<u>Proof</u>: Assume T is an isomorphism of E_u onto F_v. Let $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ be a generalized basis in E_u. To show that $(Tx_\alpha)_{\alpha\varepsilon I}$ is a generalized basis we need to show that there exists a family $(g_\alpha)_{\alpha\varepsilon I} \subseteq F'$ such that $(Tx_\alpha, g_\alpha)_{\alpha\varepsilon I}$ is a biorthogonal family and $(g_\alpha)_{\alpha\varepsilon I}$ is separating. Define $(g_\alpha)_{\alpha\varepsilon I}$ by $g_\alpha = f_\alpha \circ T^{-1}$. Then the g_α's are continuous and linear since f_α and T^{-1} are and $g_\alpha(Tx_\beta) = (f_\alpha \circ T^{-1})(Tx_\beta) = f_\alpha(x_\beta) = \delta_{\alpha\beta}$, so $(Tx_\alpha, g_\alpha)_{\alpha\varepsilon I}$ is biorthogonal. Also if $g_\alpha(y) = 0$ for all $\alpha \varepsilon I$, then $(f_\alpha \circ T^{-1})(y) = 0$ for all $\alpha \varepsilon I$, and hence $T^{-1}(y) = \Theta$. Since T is an isomorphism, $y = \Theta$ in F_v. Also, $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ is similar to $(Tx_\alpha, g_\alpha)_{\alpha\varepsilon I}$ since $\Psi(y) = (g_\alpha(y))_{\alpha\varepsilon I} = (g_\alpha(Tx))_{\alpha\varepsilon I} = (f_\alpha(x))_{\alpha\varepsilon I} = \Phi(x)$.

For the converse, assume $(y_\alpha)_{\alpha\varepsilon I}$ is a generalized basis in F_v similar to $(x_\alpha)_{\alpha\varepsilon I}$ in E_u. Then $\Phi(E_u) = \Psi(F_v) = \mathcal{A}$ say, where Φ, Ψ are the corresponding coefficient maps. $\mathcal{A}$ is clearly a subspace of R^I under pointwise addition and scalar multiplication, and $\mathcal{A}$ is algebraically isomorphic to E and F respectively. Hence we can induce the topologies u and v on $\mathcal{A}$ as follows. We define

$A \subseteq \mathcal{A}$ to be open if and only if $\Phi^{-1}(A)$, $\Psi^{-1}(A)$ are open in E_u and F_v respectively. Hence $\mathcal{A}_u$ and $\mathcal{A}_v$ are B-complete barrelled spaces. Now define T by $T = \Psi^{-1} \circ \Phi$. Clearly T is a linear mapping from E to F and is one-one and onto since Ψ and Φ are. In Proposition 7, take $E = \mathcal{A}$ and $(h_\alpha)_{\alpha\varepsilon I}$ defined by $h_\alpha(z) = f_\alpha(x) = g_\alpha(y)$ where $z = \Phi(x) = \Psi(y)$. $(h_\alpha)_{\alpha\varepsilon I}$ is a separating family of linear functionals because $(f_\alpha)_{\alpha\varepsilon I}$ and $(g_\alpha)_{\alpha\varepsilon I}$ are and $h_\alpha(z) = 0$ implies $f_\alpha(x) = g_\alpha(y) = 0$ for $\alpha \varepsilon I$, where $z = \Phi(x) = \Psi(y)$. Hence x and y are θ in E_u, F_v respectively, so $z = \theta$ because Φ and Ψ are one-one. The h_α are continuous with respect to u and v because f_α and g_α are continuous with respect to u and v respectively. By Proposition 7, the identity $i: \mathcal{A}_u \longrightarrow \mathcal{A}_v$ has closed graph and by Corollary 1 of this chapter, u is finer than v and hence $u = v$. T is the desired isomorphism. That $Tx_\alpha = y_\alpha$ for all $\alpha \varepsilon I$ follows by similarity since for α fixed but arbitrary,

$$Tx_\alpha = (\Psi^{-1} \circ \Phi)(x_\alpha) = \Psi^{-1}(f_\beta(x_\alpha))_{\beta\varepsilon I} = \Psi^{-1}(g_\beta(y_\alpha))_{\beta\varepsilon I} = y_\alpha. \qquad \text{Q.E.D.}$$

<u>Remark</u>: Theorem 2 extends a result due to Arsove for metric complete linear spaces.

This result allows one to obtain the following, which was known only for complete metric linear spaces. (see (1) Theorem 4).

<u>Theorem 3</u>: Let E_u and F_v be B-complete barrelled spaces with $(x_\alpha)_{\alpha\varepsilon I}$, $(y_\alpha)_{\alpha\varepsilon I}$ total generalized bases in E_u and F_v. Let $I' \subseteq I$ such that $I - I'$ is finite. Define E_1, F_1 to be the closed linear spans of $(x_\alpha)_{\alpha\varepsilon I'}$, $(y_\alpha)_{\alpha\varepsilon I'}$ with the relative topologies of E_u and F_v respectively. If T_1 is an isomorphism $T_1: E_1 \longrightarrow F_1$ such that

$T_1 x_\alpha = y_\alpha$, $\alpha \varepsilon I'$, then T_1 can be extended to an isomorphism $T: E \longrightarrow F$ such that $Tx_\alpha = y_\alpha$ for all $\alpha \varepsilon I$.

<u>Proof</u>: Observe that by the definition of E_1 and F_1, $(x_\alpha)_{\alpha\varepsilon I'}$, $(y_\alpha)_{\alpha\varepsilon I'}$ are total generalized bases, and this follows by defining $f'_\alpha = f_\alpha | E_1$ and $g'_\alpha = g_\alpha | F_1$, where $(f_\alpha)_{\alpha\varepsilon I}$ and $(g_\alpha)_{\alpha\varepsilon I}$ are the corresponding coefficient functionals. Since T_1 is an isomorphism and $T_1 x_\alpha = y_\alpha$ for $\alpha \varepsilon I'$, by Theorem 2, $(x_\alpha)_{\alpha\varepsilon I'}$ is similar to $(y_\alpha)_{\alpha\varepsilon I'}$. Observe that $E = E_1 \oplus [x_\alpha]_{\alpha\varepsilon I-I'}$ and $F = F_1 \oplus [y_\alpha]_{\alpha\varepsilon I-I'}$. Hence for $x \varepsilon E$, $x = x_1 \oplus \sum_{i\varepsilon I-I'} \lambda_i x_i$, $x_1 \varepsilon E_1$ and

$$\Phi(x) = (f_\alpha(x_1 + \sum_{i\varepsilon I-I'} \lambda_i x_i))_{\alpha\varepsilon I}$$

By similarity of $(x_\alpha)_{\alpha\varepsilon I'}$ and $(y_\alpha)_{\alpha\varepsilon I'}$, there exists a $y_1 \varepsilon F_1$ such that $f_\alpha(x_1) = g_\alpha(y_1)$ for $\alpha \varepsilon I'$. Clearly for $\alpha \varepsilon I-I'$ $f_\alpha(x) = f_\alpha(x_1 + \sum_{i\varepsilon I-I'} \lambda_i x_i) = f_\alpha(x_1) + \lambda_\alpha$. Define $y = y_1 + \sum_{i\varepsilon I-I'} \lambda_i y_i$.

Observe that $g_\alpha(\sum_{i\varepsilon I-I'} \lambda_i y_i) = f_\alpha(\sum_{i\varepsilon I-I'} \lambda_i x_i)$ for all $\alpha \varepsilon I$.

Hence
$$\begin{aligned}\Phi(x) &= (f_\alpha(x_1 + \sum_{i\varepsilon I-I'} \lambda_i x_i))_{\alpha\varepsilon I} \\ &= (f_\alpha(x_1) + f_\alpha(\sum \lambda_i x_i))_{\alpha\varepsilon I} \\ &= (g_\alpha(y_1) + g_\alpha(\sum \lambda_i y_i))_{\alpha\varepsilon I} \\ &= \Psi(y). \text{ Hence } \Phi(E_u) \subseteq \Psi(F_v).\end{aligned}$$

By a similar argument one gets $\Psi(F_v) \subseteq \Phi(E_u)$. Thus $\Phi(E_u) = \Psi(F_v)$ i.e. $(x_\alpha)_{\alpha\varepsilon I}$ is similar to $(y_\alpha)_{\alpha\varepsilon I}$. By Theorem 2 of this chapter there exists an isomorphism $T: E_u \longrightarrow F_v$ such that $Tx_\alpha = y_\alpha$ for $\alpha \varepsilon I$. T clearly extends T_1 since they coincide on a total subset of E_1. Q.E.D.

As another application of Theorem 2, we obtain the following characterization concerning similar bases.

<u>Theorem 4</u>: Let E_u, F_v be B-complete barrelled spaces with u, v defined by the families of seminorms $(p_i)_{i\varepsilon K}$ and $(q_\lambda)_{\lambda\varepsilon L}$ respectivly. Let $(x_\alpha)_{\alpha\varepsilon I}$, $(y_\alpha)_{\alpha\varepsilon I}$ be total generalized bases in E_u and F_v respectively. Then $(x_\alpha)_{\alpha\varepsilon I}$ is similar to $(y_\alpha)_{\alpha\varepsilon I}$ if and only if for any $i \varepsilon K$ there exists $\lambda \varepsilon L$ and $M > 0$ such that

$$\text{(a)} \qquad p_i \left(\sum_{n=1}^{m} a_n x_{i_n} \right) \leq M\, q_\lambda \left(\sum_{n=1}^{m} a_n y_{i_n} \right)$$

and

$$\text{(b)} \qquad q_i \left(\sum_{n=1}^{m} a_n y_{i_n} \right) \leq M\, p_\lambda \left(\sum_{n=1}^{m} a_n x_{i_n} \right)$$

hold for all finite sequences $i_1, \ldots, i_m$ in I and all finite sequences of scalars $a_1, \ldots, a_m$, where M does not depend upon m.

<u>Proof</u>: For the necessity, assume $(x_\alpha)_{\alpha\varepsilon I}$ is similar to $(y_\alpha)_{\alpha\varepsilon I}$. By Theorem 2 there exists an isomorphism $T: E_u \longrightarrow F_v$ such that $Tx_\alpha = y_\alpha$ for $\alpha \varepsilon I$. Hence by Proposition 4, Chapter 1, (a) and (b) are satisfied. For the sufficiency, assume (a) and (b) hold. Let G be the linear span of $(x_\alpha)_{\alpha\varepsilon I}$ with the relative topology induced by u. Define $T: G \longrightarrow F_v$ by $T\left(\sum_{n=1}^{m} a_n x_{i_n} \right) = \sum_{n=1}^{m} a_n y_{i_n}$. Clearly T is linear. By (b) and Proposition 4 of Chapter 1, T is continuous. Hence T can be extended to a linear map $T: G \longrightarrow F_v$ (see (29) Chapter 3, § 8)) which is continuous. T is one-one on G since for $x \varepsilon G$, $x = \sum_{i=1}^{n} \lambda_i x_i$ and $Tx = T\left(\sum_{1}^{n} \lambda_i x_i\right) = 0$ implies $\sum_{1}^{n} \lambda_i y_i = 0$.

Hence $g_\alpha(\sum_1^n \lambda_i y_i) = 0$ for $\alpha \varepsilon I$ gives $\lambda_i = 0$ for $1 \leq i \leq n$. Thus $x = \theta$. Let $(f_\alpha)_{\alpha\varepsilon I}$ $(g_\alpha)_{\alpha\varepsilon I}$ be the functionals associated with $(x_\alpha)_{\alpha\varepsilon I}$ $(y_\alpha)_{\alpha\varepsilon I}$ respectively. Then for $x \varepsilon G$, $x = \sum_{n=1}^m a_n x_{i_n}$ and $g_i(T_x) = g_i(\sum_{n=1}^m a_n y_{i_n}) = a_i = f_i(x)$ for all $i \varepsilon I$. For $x \varepsilon \overline{G} = E_u$, $x = \lim z_\beta$, $z_\beta \varepsilon G$. Since f_α, g_α and T are uniformly continuous, $g_\alpha(Tx) = g_\alpha(T \lim_\beta z_\beta) = \lim_\beta g_\alpha(Tz_\beta) = \lim_\beta f_\alpha(z_\beta) = f_\alpha(x)$. Hence $\Phi(E_u) \subseteq \Psi(F_v)$. By repeating the argument and using (a), we get $\Psi(F_v) \subseteq \Phi(E_u)$. Therefore $\Phi(E_u) = \Psi(F_v)$ and the bases are similar.

<u>Remark</u>: It is worth noting that Theorem 2 of this chapter cannot be strengthened to the case of complete barrelled spaces. As an example consider the following: Let E_u be an infinite dimensional Banach space with $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ a generalized basis and let F_v be E with the strongest locally convex topology (13) (Chapter 2, § 4, Example 3). Then E_u and F_v are complete barrelled spaces by (14) (Chapter 4, § 1, Proposition 1) $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ is also a generalized basis in F_v and is similar trivially. However E_u and F_v are not isomorphic since v is strictly finer than u. (14) (Chapter 4, § 1, Proposition 1).

<u>Theorem 5</u>: Let E_u be a B_r-complete TVS and F_v barrelled. Let $f: E_u \longrightarrow F_v$ be a linear onto map with closed graph. Then f is open.

<u>Proof</u>: See (14) (Chapter 4, § 4, Theorem 7).

<u>Definitions</u>: Let $\mathcal{J}$ be the class of all barrelled spaces. A locally convex TVS E_u is a $B(\mathcal{J})$ $[B_r(\mathcal{J})]$ space if for each locally convex space $F_v \in \mathcal{J}$, a linear continuous and almost open map $f: E_u \longrightarrow F_v$ onto is open [a linear continuous one-one almost open map $f: E_u \longrightarrow F_v$ onto is open].

<u>Theorem 6</u>: (T. Husain) Every barrelled space E_u which is a $B_r(\mathcal{J})$ space is B_r-complete.

<u>Proof</u>: A proof may be found in (14) (Chapter 7, § 4, Theorem 5).

<u>Definitions</u>: (McIntosh) A locally convex TVS E_u is a (C)-space [resp. (C_r)-space] if every linear subspace [resp. dense linear subspace] D of E' $(\sigma = \sigma(E', E))$ whose intersection with each $\sigma(E', E)$ bounded subset $B \subseteq E'$ is $\sigma(E', E)$ closed in B, is necessarily closed in E'^{σ}.

We note that every (C)-space is a (C_r)-space and every B-complete (resp. B_r-complete) space is a (C) (resp. (C_r)) space.

<u>Theorem 7</u>: (McIntosh) Every barrelled (C)-space (resp. (C_r)-space) is B-complete (resp. B_r-complete).

<u>Proof</u>: See (22) (Page 399).

<u>Remarks</u>: (a) Closed graph theorems have been proven for $B_r(\mathcal{J})$ and (C_r) spaces, (14) (Chapter 7, § 4, Theorem 5) and (22) (Theorem 3). However an attempt to extend Theorem 2 of this Chapter to $B_r(\mathcal{J})$ or (C_r) barrelled spaces does not lead to a generalization as Theorems 6 and 7 show, so in this sense Theorem 2 is most general.

(b) Theorems 2 and 3 of this Chapter also hold for barrelled $B_r(\mathcal{J})$ spaces (in particular for barrelled B_r-complete spaces) because of the closed graph theorem for these spaces and since a B_r-complete space is complete.

3. Inductive limits

Theorem 8: (Robertson and Robertson) Let E_u be a generalized strict inductive limit of B-complete locally convex spaces and F_v an inductive limit of Baire locally convex spaces. Then

(a) Any linear continuous map $f: E_u \longrightarrow F_v$ onto is open.

(b) Any linear map $g: F_v \longrightarrow E_u$ with closed graph is continuous.

Proof: See (26).

Theorem 9: Let E_u and F_v be generalized strict inductive limits of B-complete Baire locally convex spaces. Let $(x_\alpha)_{\alpha\varepsilon I}$ be a generalized basis in E_u. If T is an isomorphism $T: E_u \longrightarrow F_v$ such that $Tx_\alpha = y_\alpha$, $\alpha \varepsilon I$, then $(y_\alpha)_{\alpha\varepsilon I}$ is a generalized basis in F_v and is similar to $(x_\alpha)_{\alpha\varepsilon I}$. Conversely if $(y_\alpha)_{\alpha\varepsilon I}$ is a generalized basis in F_v similar to $(x_\alpha)_{\alpha\varepsilon I}$, then there exists an isomorphism $T: E_u \longrightarrow F_v$ such that $Tx_\alpha = y_\alpha$ for all $\alpha \varepsilon I$.

Proof: The proof is essentially the same as that of Theorem 2 section 1 of this chapter except that one must apply Theorem 8 above, instead of Theorem 1. Q.E.D.

Theorem 10: Let E_u be the generalized strict inductive limit of a sequence $(E_n)_{n\varepsilon N}$ of locally convex spaces, and let $(x_\alpha)_{\alpha\varepsilon I}$ be a family of points in E_1. Then

(a) If $(x_\alpha)_{\alpha\varepsilon I}$ is a total generalized basis in each E_n, then $(x_\alpha)_{\alpha\varepsilon I}$ is a total generalized basis in E_u.

(b) If $(x_\alpha)_{\alpha\varepsilon I}$ is an extended Schauder basis in each E_n, then it is an extended Schauder basis in E_u. Extended basis means an arbitrary indexing set, the basis expansions to be carried out according to some fixed linear ordering of the indexing set.

Proof: See (1), (Theorem 7).

Theorem 11: Let E_u, F_v be strict inductive limits of B-complete Baire locally convex spaces, the topologies defined by $(p_i)_{i\varepsilon K}$ and $(q_\lambda)_{\lambda\varepsilon L}$. Let $(x_\alpha)_{\alpha\varepsilon I}$ and $(y_\alpha)_{\alpha\varepsilon I}$ be total generalized bases in E_u, F_v respectively. Then $(x_\alpha)_{\alpha\varepsilon I}$ is similar to $(y_\alpha)_{\alpha\varepsilon I}$ if and only if for any $i \varepsilon K$ there exists a $\lambda \varepsilon L$, $M > 0$ such that

(a) $$p_i\left(\sum_{n=1}^{m} a_n x_{i_n}\right) \leq M\, q_\lambda\left(\sum_{n=1}^{m} a_n y_{i_n}\right)$$

and (b) $$q_i\left(\sum_{n=1}^{m} a_n y_{i_n}\right) \leq M\, p_\lambda\left(\sum_{n=1}^{m} a_n x_{i_n}\right)$$

hold for all finite sequences $i_1, \ldots, i_m$ of indices in I and scalars $a_1, \ldots, a_m$.

Proof: Assume $(x_\alpha)_{\alpha\varepsilon I}$ is similar to $(y_\alpha)_{\alpha\varepsilon I}$. Then by Theorem 9 this chapter there exists an isomorphism $T: E_u \longrightarrow F_v$ such that $Tx_\alpha = y_\alpha$, $\alpha \varepsilon I$, and hence we get (a) and (b). The converse follows as in Theorem 4 of this chapter. Q.E.D

Remark: Because of Theorem 10, one can get theorems analogous to Theorems 9 and 11 by letting E_u, F_v to be generalized strict inductive limits of $(E_n)_{n\varepsilon N}$ and $(F_n)_{n\varepsilon N}$ where $(x_\alpha)_{\alpha\varepsilon I}$ and $(y_\alpha)_{\alpha\varepsilon I}$ are total generalized bases in E_n and F_n respectively. The proof would be the same, except for applying Theorem 10.

4. Dual generalized bases and further applications

Definition: A dual generalized basis in a TVS E_u is a biorthogonal family $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ such that $[x_\alpha] = E_u$.

The next theorem due to Klee answers the question of existence.

Theorem 12: (Klee) If E_u is a separable locally convex space, then there exists a dual generalized basis for E_u.

Proof: Since E_u is separable, there is a sequence $(y_i)_{i\varepsilon N}$ such that $[y_i] = E_u$. Without loss of generality, let $(y_i)_{i\varepsilon N}$ be such that every finite subset is linearly independent. Let E_n be the linear span of $\{y_1, \ldots, y_n\}$. Let $x_1 = y_1$, $x_2 = y_2$. Then there exists continuous linear functionals f_1, f_2 such that $f_1(x_1) = f_2(x_2) = 1$ and $f_1(x_2) = f_2(x_1) = 0$. Also, we observe that f_1, f_2 have continuous linear extensions to all of E_u, and $f_i(x_j) = \delta ij$ for $i, j = 1, 2$. Recursively, let $\{x_1, \ldots, x_n, \bar{f}_1, \ldots, \bar{f}_n\}$ be such that $sp\ \{x_1, \ldots, x_n\} = E_n$ and $\bar{f}_i(x_j) = \delta ij$ for $1 \leq i, j \leq n$. The biorthogonality gives $\{x_i, \ldots, x_n\}$ linearly independent. Now define $x_{n+1} = y_{n+1} - \sum_{i=1}^{n} \bar{f}_i\ (y_{n+1})\ x_i$. Linear independence of $\{x_1, \ldots, x_n, x_{n+1}\}$ follows from the biorthogonality, and $sp\ \{x_1, \ldots, x_n, x_{n+1}\} = E_{n+1}$. As above, there exists $\bar{f}_{n+1} \varepsilon E'$ such that $\bar{f}_{n+1}(x) = 0$ for $x \varepsilon E_n$

and $\bar{f}_{n+1}(x_{n+1}) = 1$. Also, $\bar{f}_i(x_j) = \delta ij$ for $1 \leq i, j \leq n+1$. Hence by induction we can obtain a biorthogonal system $(x_i, \bar{f}_i)_{i\varepsilon N}$ for E_u such that $[x_i] = E_u$, and we therefore have a dual generalized basis.

<u>Example</u>: Let E_u be the Frechet space of all functions analytic on the open disc $|z| < 1$, topologized by the metric of uniform convergence on compact sets. Define $(x_n, f_n)_{n\varepsilon N}$ by $x_n(z) = 1 + z + z^2 + \ldots + z^n$ and f_n by $f_n(g) = \frac{g^{(n)}(0)}{n!} - \frac{g^{(n+1)}(0)}{(n+1)!}$ for $n = 0, 1, 2, \ldots$.

As we showed in Chapter 1, section 6, $(x_n, f_n)_{n\varepsilon N}$ is not a generalized basis, but since $(x_n)_{n\varepsilon N}$ is total, it is a dual generalized basis.

<u>Remark</u>: Dieudonné (see [16]), gives the following criterion for a biorthogonal system in a TVS E_u to be maximal: $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ is maximal if and only if $[x_\alpha]^1 \subseteq [f_\alpha]$ where $[x_\alpha]^1 = \{f \varepsilon E' : f([x_\alpha]) = 0\}$, or if and only if $\bigcap_{f_\alpha} \operatorname{Ker} f_\alpha \subseteq [x_\alpha]$.

<u>Lemma 2</u>: (Davis) Let $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ be a biorthogonal system for a TVS E_u. Let φ be the quotient map $\varphi : E_u \longrightarrow E/\operatorname{Ker}\Phi$, Φ the coefficient map associated with $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$. Then:

(a) $(\varphi(x_\alpha), \tilde{f}_\alpha)_{\alpha\varepsilon I}$ is a generalized basis for $E/\operatorname{Ker}\Phi$ where $\tilde{f}_\alpha$ is defined by $\tilde{f}_\alpha(\varphi(x)) = f_\alpha(x)$.

(b) If $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ is a dual generalized basis, then $(\varphi(x_\alpha), \tilde{f}_\alpha)_{\alpha\varepsilon I}$ is a Markuschevich basis for $E/\operatorname{Ker}\Phi$ with the quotient topology.

(c) If $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ is a maximal biorthogonal system for E_u and E_u is locally convex, then $(\varphi(x_\alpha), \tilde{f}_\alpha)_{\alpha\varepsilon I}$ is a Markuschevich basis for $E/\mathrm{Ker}\,\Phi$ if and only if $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ is a dual generalized basis for E_u.

<u>Proof</u>: (a) Let $\mathcal{A}$ denote the range of Φ. Define $\tilde{\Phi}: E/\mathrm{Ker}\,\Phi \longrightarrow \mathcal{A}$ by $\tilde{\Phi} \circ \varphi = \Phi$. This is well defined since φ is onto. Let $\mathcal{A}$ have the topology induced from $E/\mathrm{Ker}\,\Phi$ by $\tilde{\Phi}$. Hence Φ is continuous with respect to this topology. Also, $\tilde{\Phi}$ is the coefficient map associated with the continuous linear functionals $(\tilde{f}_\alpha)_{\alpha\varepsilon I}$, and $\mathrm{Ker}\,\tilde{\Phi} = \{\tilde{\theta}\}$. Hence Φ is one-one and $(\varphi(x_\alpha), \tilde{f}_\alpha)_{\alpha\varepsilon I}$ is a generalized basis. Q.E.D.

(b) Since $(x_\alpha)_{\alpha\varepsilon I}$ is total and φ a linear continuous onto map, $(\varphi(x_\alpha))_{\alpha\varepsilon I}$ is total. Combined with (a) this implies (b).

(c) Assume $(\varphi(x_\alpha), \tilde{f}_\alpha)_{\alpha\varepsilon I}$ is a Markuschevich basis. Hence $(\varphi(x_\alpha))_{\alpha\varepsilon I}$ is total. Since $(x_\alpha)_{\alpha\varepsilon I}$ is maximal, by the previous remark, $\mathrm{Ker}\,\Phi \subseteq [x_\alpha]$, so for $f \varepsilon [x_\alpha]^1$, $f \varepsilon (\mathrm{Ker}\,\Phi)^1$. Hence each such f induces a map $\tilde{f}: E/\mathrm{Ker}\,\Phi \longrightarrow \mathcal{A}$ by $f = \tilde{f} \circ \varphi$. Therefore $\tilde{f}(\varphi(x_\alpha)) = f(x_\alpha) = 0$ and since $(\varphi(x_\alpha))_{\alpha\varepsilon I}$ is total we have $\tilde{f} = 0$. Hence $f = 0$ and so $(x_\alpha)_{\alpha\varepsilon I}$ is total and therefore $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ is a dual generalized basis. Q.E.D.

<u>Theorem 13</u>: Let E_u, F_v be B-complete barrelled spaces and $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$, $(y_\alpha, g_\alpha)_{\alpha\varepsilon I}$ similar biorthogonal systems for E_u, F_v respectively. Then there exists an isomorphism $T: E/\mathrm{Ker}\,\Phi \longrightarrow F/\mathrm{Ker}\,\Psi$ such that $T(\varphi(x_\alpha)) = \varphi'(y_\alpha)$ for $\alpha \varepsilon I$, where φ, φ' are the quotient maps and Φ, Ψ the coefficient maps respectively.

<u>Proof</u>: By part (a) of Lemma 2 this section, $(\varphi(x_\alpha), \tilde{f}_\alpha)_{\alpha\varepsilon I}$ and $(\varphi'(y_\alpha), \tilde{g}_\alpha)_{\alpha\varepsilon I}$ are generalized bases for $E/\mathrm{Ker}\,\Phi$ and $F/\mathrm{Ker}\,\Psi$ respectively. These bases are similar since $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ is similar to $(y_\alpha, g_\alpha)_{\alpha\varepsilon I}$, because for $\tilde{\Phi}, \tilde{\Psi}$ the coefficient maps associated with the above and $\dot{x} \varepsilon E/\mathrm{Ker}\,\Phi$, we have $\tilde{\Phi}(\dot{x}) = \tilde{\Phi}(x + \mathrm{Ker}\,\Phi) = (\tilde{f}_\alpha(x + \mathrm{Ker}\,\Phi))_{\alpha\varepsilon I} = (\tilde{f}_\alpha(\varphi(x)))_{\alpha\varepsilon I} = (f_\alpha(x))_{\alpha\varepsilon I}$. Similarly $\tilde{\Psi}(\dot{y}) = (g_\alpha(y))_{\alpha\varepsilon I}$. But by similarity, $(f_\alpha(x))_{\alpha\varepsilon I} = (g_\alpha(y))_{\alpha\varepsilon I}$. Hence $\tilde{\Phi}(\dot{x}) = \tilde{\Psi}(\dot{y})$ and therefore $\tilde{\Phi}(E/\mathrm{Ker}\,\Phi) \subseteq \tilde{\Psi}(F/\mathrm{Ker}\,\Psi)$. Similarly $\tilde{\Psi}(F/\mathrm{Ker}\,\Psi) \subseteq \tilde{\Phi}(E/\mathrm{Ker}\,\Phi)$, and we obtain $\tilde{\Psi}(F/\mathrm{Ker}\,\Psi) = \tilde{\Phi}(E/\mathrm{Ker}\,\Phi)$. Now we topolgize $\Phi(E)$ and $\Psi(F)$ with the topologies induced from $E/\mathrm{Ker}\,\Phi$ and $F/\mathrm{Ker}\,\Psi$ by $\tilde{\Phi}, \tilde{\Psi}$ respectively. Hence Φ and Ψ are continuous, so $\mathrm{Ker}\,\Phi$, $\mathrm{Ker}\,\Psi$ are closed subspaces of E_u, F_v respectively. The quotient with a closed subspace is hereditary, so, $E/\mathrm{Ker}\,\Phi$ and $F/\mathrm{Ker}\,\Psi$ are B-complete barrelled spaces with similar generalized bases. Hence by Theorem 2 section 1 of this chapter, there exists an isomorphism $T: E/\mathrm{Ker}\,\Phi \longrightarrow F/\mathrm{Ker}\,\Psi$ such that $T(\varphi(x_\alpha)) = \varphi'(y_\alpha)$ for all $\alpha \varepsilon I$. Q.E.D.

<u>Remark</u>: Since the closed graph theorem holds for B_r-complete barrelled spaces, the above holds for same.

<u>Lemma 3</u>: Let $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ be a biorthogonal system for a TVS E_u and $J: E_u \longrightarrow E'^{\beta,\beta}$ the canonical map. Then if $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ is a dual generalized basis for E_u, it follows that $(f_\alpha, J(x_\alpha))_{\alpha\varepsilon I}$ is a generalized basis for $E'^{,\beta}$.

<u>Proof</u>: Let $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ be a dual generalized basis for E. For

$f \in \bigcap_{\alpha \in I} \operatorname{Ker} J(x_\alpha)$, $J(x_\alpha)(f) = f(x_\alpha) = 0$ for $\alpha \in I$, and since $[x_\alpha] = E_u$, $f = 0$. Therefore, since $J(x_\alpha)(f_\beta) = f_\beta(x_\alpha) = \delta_{\beta\alpha}$, $(f_\alpha, J(x_\alpha))_{\alpha \in I}$ is a generalized basis in E'^{β}.

The above lemma justifies the term "dual".

<u>Definition</u>: Let E_u, F_v be TVS's, and $(x_\alpha, f_\alpha)_{\alpha \in I}$, $(y_\alpha, g_\alpha)_{\alpha \in I}$ biorthogonal systems for E_u, F_v respectively. If $\hat{\Phi}$, $\hat{\Psi}$ are the coefficient maps associated with $(J(x_\alpha))_{\alpha \in I}$, $(J(y_\alpha))_{\alpha \in I}$, then the systems $(x_\alpha, f_\alpha)_{\alpha \in I}$ and $(y_\alpha, g_\alpha)_{\alpha \in I}$ are *-similar if $\hat{\Phi}(E') = \hat{\Psi}(F')$.

Using the above definition, the following is a corollary of Theorem 2 of this chapter.

<u>Lemma 4</u>: Let E_u, F_v be TVS's such that E'^{β}, F'^{β} are B-complete barrelled. If $(x_\alpha, f_\alpha)_{\alpha \in I}$ and $(y_\alpha, g_\alpha)_{\alpha \in I}$ are *-similar dual generalized bases in E_u, F_v respectively, then there exists an isomorphism $S: E'^{\beta} \longrightarrow F'^{\beta}$, such that $S(f_\alpha) = g_\alpha$, $\alpha \in I$.

<u>Proof</u>: Since $(x_\alpha, f_\alpha)_{\alpha \in I}$ and $(y_\alpha, g_\alpha)_{\alpha \in I}$ are dual generalized bases, by Lemma 3, $(f_\alpha, J(x_\alpha))_{\alpha \in I}$ and $(g_\alpha, J(y_\alpha))_{\alpha \in I}$ are generalized bases for E'^{β}, F'^{β} respectively. Also, we observe that they are similar since $(x_\alpha, f_\alpha)_{\alpha \in I}$ and $(y_\alpha, g_\alpha)_{\alpha \in I}$ are *-similar. Hence by Theorem 2 of this chapter, there exists an isomorphism $S: E'^{\beta} \longrightarrow F'^{\beta}$ such that $S(f_\alpha) = g_\alpha$, for all $\alpha \in I$. Q.E.D.

<u>Remark</u>: The lemma holds for reflexive C.N.S., since they are Fréchet, hence the strong duals are B-complete and barrelled.

We now prove a theorem dealing with *-similar bases which is analogous to Theorem 2 of this chapter.

<u>Theorem 14</u>: Let E_u, F_v be reflexive spaces with $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$, $(y_\alpha, g_\alpha)_{\alpha\varepsilon I}$ *-similar dual generalized bases for E_u, F_v respectively, and such that E'^β, F'^β are B-complete spaces. Then there exists an isomorphism $T: E_u \longrightarrow F_v$ such that $Tx_\alpha = y_\alpha$ for all $\alpha \varepsilon I$.

<u>Proof</u>: Let J_E, J_F be the respective canonical maps of E_u and F_v with $E'^{\beta,\beta}$ and $F'^{\beta,\beta}$. Since E_u, F_v are reflexive, J_E, J_F are isomorphisms. Since reflexive spaces are barrelled and the strong dual of a reflexive space is reflexive, by Lemma 4 there exists an isomorphism $S: E'^\beta \longrightarrow F'^\beta$ such that $S(f_\alpha) = g_\alpha$ for $\alpha \varepsilon I$. Recall that $S = \hat{\Psi}^{-1} \circ \hat{\Phi}$ where $\hat{\Phi}$, $\hat{\Psi}$ are the coefficient maps associated with $(J_E x_\alpha)_{\alpha\varepsilon I}$ and $(J_F y_\alpha)_{\alpha\varepsilon I}$ respectively. Consider the conjugate map $(S^{-1})^*$. Then for $g \varepsilon F'$, $\alpha \varepsilon I$,

$$[(S^{-1})^* (J_E x_\alpha) - J_F y_\alpha] (g)$$

$$= (S^{-1})^* (J_E x_\alpha)(g) - (J_F y_\alpha)(g) = [(\hat{\Phi}^{-1} \circ \hat{\Psi})^* (J_E x_\alpha) - (J_F y_\alpha)] (g)$$

$$= (J_E x_\alpha)(\hat{\Phi}^{-1} \circ \hat{\Psi}) (g) - (J_F y_\alpha)(g) = 0$$

by *-similarity of $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ and $(y_\alpha, g_\alpha)_{\alpha\varepsilon I}$. Thus $(S^{-1})^*(J_E x_\alpha) = J_F y_\alpha$. Hence $(S^{-1})^* \mid [J_E x_\alpha] = [J_F y_\alpha]$. Now define T by $T = J_F^{-1} \circ (S^{-1})^* \circ J_E$. Clearly T is an isomorphism since each of the maps J_F, S^{-1}, J_E are. Also

$$Tx_\alpha = (J_F^{-1} \circ (S^{-1})^* \circ J_E)(x_\alpha)$$

$$= J_F^{-1} \; J_F(y_\alpha) = y_\alpha \qquad \text{for } \alpha \varepsilon I.$$

Q.E.D.

<u>Theorem 15</u>: Let E_u, F_v be reflexive B-complete spaces such that E'^β, F'^β are B-complete. If $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ is a generalized basis system which is simultaneously similar and *-similar to a dual generalized

basis $(y_\alpha, g_\alpha)_{\alpha\varepsilon I}$ in F_v, then there exists an isomorphism $T: E_u \longrightarrow F_v$ such that $Tx_\alpha = y_\alpha$ for $\alpha \varepsilon I$, and both are Markuschevich bases.

<u>Proof</u>: Since E_u, F_v are B-complete and barrelled, and $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ is similar to $(y_\alpha, g_\alpha)_{\alpha\varepsilon I}$, hence by Theorem 13 of this chapter, there exists an isomorphism. $S: E/\mathrm{Ker}\,\Phi \longrightarrow F/\mathrm{Ker}\,\Psi$ such that $S(\varphi(x_\alpha)) = \varphi'(y_\alpha)$ for $\alpha \varepsilon I$, where φ, φ' are the respective quotient maps and Φ, Ψ the coeffiecient maps. By Lemma 2 (b), $(\varphi'(y_\alpha))_{\alpha\varepsilon I}$ is a Markuschevich basis for $F/\mathrm{Ker}\,\Psi$. Now $S^{-1}(\varphi'(y_\alpha)) = S^{-1}(S(\varphi(x_\alpha))) = \varphi(x_\alpha)$ for all $\alpha \varepsilon I$. Hence $(\varphi(x_\alpha))_{\alpha\varepsilon I}$ is a Markuschevich basis for $E/\mathrm{Ker}\,\Phi$. But $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ a generalized basis gives $\mathrm{Ker}\,\Phi = \{\theta\}$, hence $E/\mathrm{Ker}\,\Phi$ is isomorphic to E_u. This gives $(x_\alpha)_{\alpha\varepsilon I}$ a Markuschevich basis for E_u. This implies $(x_\alpha)_{\alpha\varepsilon I}$ is a dual generalized basis. By *-similarity of $(x_\alpha, f_\alpha)_{\alpha\varepsilon I}$ and $(y_\alpha, g_\alpha)_{\alpha\varepsilon I}$ and Theorem 14, we get the desired isomorphism $T: E_u \dashrightarrow F_v$ such that $Tx_\alpha = y_\alpha$ for all $\alpha \varepsilon I$. Q.E.D.

<u>Corollary</u>: Theorem 15 is true for the following pairs of locally convex spaces with the same hypothesis:

(a) When E, F are reflexive Fréchet spaces

(b) When E, F are reflexive Banach spaces

(c) When E, F are Montel, Fréchet spaces

<u>Proof</u>: The proof is trivial since these spaces are reflexive, B-complete and their strong duals are B-complete. Q.E.D.

<u>Remark</u>: In regard to *-similar bases, it has been observed by O.T. Jones that similar Schauder bases in barrelled spaces are *-similar.

CHAPTER III

Countably barrelled spaces and bases

1. Countably barrelled spaces

Definitions: (a) Let E_u be a locally convex TVS. E_u is countably barrelled if each $\sigma(E', E)$ bounded subset of E' which is the countable union of equicontinuous subsets of E' is itself equicontinuous (Recall that for E_u, F_v two TVS's, $H \subseteq \mathcal{L}(E_u, F_v)$ is equicontinuous if for each neighborhood V of θ in F_v, $\bigcap_{f \varepsilon H} f^{-1}(V)$ is a neighborhood of θ in E_u).

(b) Let $A \subseteq E_u$ be a subset of a TVS. The polar of A, denoted A° is $\{f \varepsilon E' : |f(x)| \leq 1$ for all $x \varepsilon A\}$. The bipolar of A, denoted $A^{\circ\circ}$ is $\{x \varepsilon E : |f(x)| \leq 1$ for all $f \varepsilon A^\circ\}$.

Remark: Countably barrelled spaces were introduced and studied by T. Husain [15]. Most of the results in sections 1 and 2 are due to T. Husain and are given here for the sake of completeness.

Proposition 1: Let A, B, $(A_\alpha)_{\alpha \varepsilon I}$ be subsets of a locally convex TVS E_u.

(a) If $A \subseteq B$, then $B^\circ \supseteq A^\circ$.

(b) If $\lambda \neq 0$, $(\lambda A)^\circ = \lambda^{-1} A^\circ$.

(c) $(\bigcup_{\alpha \varepsilon I} A_\alpha)^\circ = \bigcap_{\alpha \varepsilon I} A_\alpha^\circ$

(d) For each set A, A° is a $\sigma(E', E)$ closed convex set containing Θ.

(e) If A is circled, so is A°.

Proof: See (14), (Chapter 2, § 8, Proposition 12).

Theorem 1: A locally convex TVS E_u is countably barrelled if and only if each barrel B which is the countable intersection of convex, circled closed nieghborhoods of the origin Θ is itself a neighborhood of Θ.

Proof: For a proof see (15), (§ 3, Theorem 1).

The following provides numerous examples of countably barrelled spaces.

Corollary 1: Every barrelled TVS is countably barrelled. In particular, every Frechet space and each Banach space is countably barrelled.

Proof: See (15) (§ 3, Theorem 1, Corollaries 1 and 2).

Theorem 2: (T. Husain [15]) Let E_u be a countably barrelled TVS. Let $(f_n)_{n\varepsilon N}$ be a $\sigma(E', E)$ bounded sequence of functionals in E'. If $(f_n)_{n\varepsilon N}$ converges to a linear functional f pointwise, then $f \varepsilon E'$ and $(f_n)_{n\varepsilon N}$ converges to f uniformly on each precompact subset of E_u.

Proof: Observe that each singleton $\{f_n\}$ is equicontinuous, and $\bigcup_{n=1}^{\infty} \{f_n\} = \{f_n : n\varepsilon N\}$ is $\sigma(E', E)$ bounded by assumption. Since E_u is countably barrelled, $\bigcup_{n=1}^{\infty} \{f_n\}$ is equicontinuous, and by (2) (Chapter 3, § 3, Proposition 4), $f \varepsilon E'$. Hence by (2) (Chapter 3, § 3

Proposition 5), $(f_n)_{n\varepsilon N}$ converges to f uniformly on each precompact subset of E_u. Q.E.D.

<u>Proposition 2</u>: Let E_u be a countably barrelled TVS, and F_v a locally convex TVS. Let $(H_n)_{n\varepsilon N}$ be a sequence of equicontinuous sets of linear maps of E_u into F_v such that $\bigcup_{n=1}^{\infty} H_n$ is simply bounded. Then $\bigcup_{n=1}^{\infty} H_n$ is equicontinuous.

<u>Proof</u>: For all $n \varepsilon N$, $f_n \varepsilon H_n$ is linear and continuous, hence there exists a unique transpose f_n': $E' \longrightarrow E'$, defined by $< f_n(x), y' > = < x, f_n'(y') >$ for $x \varepsilon E_u$, $y' \varepsilon F'$. $< f_n(x), y' >$ simply means $y'(f_n(x))$. Let $H_n' = \{f_n' : f_n \varepsilon H_n\}$. Since H_n is equicontinuous and f_n', y' are continuous linear maps, we have for each equicontinuous subset $M' \subseteq F'$, $H_n'(M') = \{f_n' \circ y' : f_n' \varepsilon H_n', y' \varepsilon M'\}$ is equicontinuous in E'. Since $\bigcup_{n=1}^{\infty} H_n$ is simply bounded, $\bigcup_{n=1}^{\infty} H_n'(M')$ is $\sigma(E', E)$ bounded. Therefore since E_u is countably barrelled, $\bigcup_{n=1}^{\infty} H_n'(M')$ is an equicontinuous subset of E', so $\bigcup_{n=1}^{\infty} H_n$ is equicontinuous.

<u>Corollary 1</u>: Let E_u be a countably barrelled and F_v a locally convex TVS. If $(f_n)_{n\varepsilon N}$ is a simply bounded sequence of continuous linear maps of E_u into F_v, then $(f_n)_{n\varepsilon N}$ is equicontinuous.

<u>Proof</u>: Follows from Proposition 2 by taking $H_n = \{f_n\}$ for each n.

<u>Corollary 2</u>: Let E_u be a countably barrelled and F_v a locally convex TVS. If $(f_n)_{n\varepsilon N}$ is a sequence of continuous linear maps of E_u into F_v such that $(f_n)_{n\varepsilon N}$ converges pointwise to a map f: $E_u \longrightarrow F_v$, then f is continuous and linear.

Proof: Trivially f is linear and $(f_n)_{n\varepsilon N}$ is pointwise bounded. By corollary 1, $(f_n)_{n\varepsilon N}$ is equicontinuous. Hence by (2) (Chapter 3, §3, Proposition 4) f is continuous. Q.E.D.

Proposition 3: Let E_u be a metrizable locally convex TVS. Then E'^{β} is countably barrelled.

Proof: See (15), (§ 4, Proposition 1 and 4).

2. Permanence properties

Theorem 3: Let $(E_\alpha)_{\alpha\varepsilon I}$ be a family of countably barrelled spaces and $(f_\alpha)_{\alpha\varepsilon I}$ a family of linear maps into a linear space E. Let E be endowed with the finest locally convex topology u such that each f_α is continuous for each $\alpha \varepsilon I$. Then E_u is countably-barrelled.

Proof: See (15), (§ 8, Theorem 8).

Corollary 1: Let $(E_\alpha)_{\alpha\varepsilon I}$ be a family of countably barrelled spaces and E_u it's inductive limit. Then E_u is also countably barrelled.

Proof: This is just a special case of Theorem 3.

Corollary 2: Let E be a countably barrelled space and $M \subseteq E$ a closed subspace. Then the quotient E/M is also countably barrelled.

Proof: See (15), (§ 8, Theorem 8, Corollary 14).

Corollary 3: Let $(E_\alpha)_{\alpha\varepsilon I}$ be a family of countably barrelled spaces. Then its direct sum is also countably barrelled.

Proof: See (15), (§ 8, Theorem 8, Corollary 13).

3. The Isomorphism theorem

In chapter II, Theorem 2, we showed that the isomorphism theorem did not hold for generalized bases even if both spaces were complete and barrelled, however it did hold for B-complete barrelled spaces. In the case of Schauder bases the theorem can be much improved, as the following shows.

Theorem 4: Let E_u, F_v be countably barrelled spaces and let $(x_n)_{n\varepsilon N}$, $(y_n)_{n\varepsilon N}$ be Schauder bases in E, F respectively. Then $(x_n)_{n\varepsilon N}$ is similar to $(y_n)_{n\varepsilon N}$ if and only if there exists an isomorphism $T: E_u \longrightarrow F_v$ such that $Tx_n = y_n$ for all $n \varepsilon N$.

Proof: If T is an isomorphism such that $Tx_n = y_n$ for all $n \varepsilon N$, then clearly if $\sum_{n=1}^{\infty} a_n x_n$ converges, then $T(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n=1}^{\infty} a_n y_n$ also converges and vice-versa, hence one gets similarity. For the converse, assume the bases are similar. Let $(f_n)_{n\varepsilon N} \subseteq E'$ be such that $(x_n, f_n)_{n\varepsilon N}$ is a biorthogonal system. Then for $x \varepsilon E_u$, $x = \sum_{n=1}^{\infty} f_n(x)x_n$.

Define T_m by

$$T_m(x) = \sum_{n=1}^{m} f_n(x)\, y_n \qquad m = 1, 2, \ldots,$$

and T by

$$T(x) = \sum_{n=1}^{\infty} f_n(x)\, y_n.$$

T is well defined, since by similarity of the bases $\sum_{n=1}^{\infty} f_n(x)\, y_n$ is convergent. The T_m's are continuous and linear since f_n $n\varepsilon N$ are continuous and linear. $Tx = \theta$ implies $\sum_{n=1}^{\infty} f_n(x)\, y_n = \theta$, and since

$(y_n)_{n\varepsilon N}$ is a Schauder basis, $f_n(x) = 0$ for all $n \varepsilon N$, hence $x = \theta$ because $(f_n)_{n\varepsilon N}$ is separating. Therefore T is one-one. T is onto since for $y \varepsilon F_v$, $y = \sum_{n=1}^{\infty} b_n y_n$, and by similarity $\sum_{n=1}^{\infty} b_n x_n = \sum_{n=1}^{\infty} f_n(x) x_n$ converges in E_u for some $x \varepsilon E$. Also $T_m(x) \longrightarrow T(x)$ for $x \varepsilon E$ by definition. Hence $(T_m)_{m\varepsilon N}$ is pointwise bounded on E_u and T is in the pointwise closure of $(T_m)_{m\varepsilon N}$. By Proposition 2, Corollary 2 of section 1 of this chapter, T is linear and continuous. Likewise T^{-1} is also continuous. Hence T is the desire isomorphism E_u onto F_v. Q.E.D.

<u>Remark</u>: In view of the facts that inductive limits, direct sums and products of countably barrelled spaces are also countably barrelled, the theorem holds true for these cases.

<u>Theorem 5</u>: Let E_u be a countably barrelled TVS. Then every weak Schauder basis in E_u is a Schauder basis for the initial topology u.

<u>Proof</u>: Let $(x_n, f_n)_{n\varepsilon N}$ be a weak Schauder basis. Hence $(x_n, f_n)_{n\varepsilon N}$ is a weak Markuschevich basis, so $[x_n] = E_u$ in the initial topology by (1) Theorem 10. For each $x \varepsilon E$, choose a sequence $(y_n)_{n\varepsilon N} \subseteq E$ such that $y_n \longrightarrow x$, and $y_n \varepsilon \operatorname{sp}\{x_i : 1 \leq i \leq n\}$. Consider the sequence $(T_n)_{n\varepsilon N}$ of linear maps $T_n : E \longrightarrow E$ by $T_n(x) = \sum_{i=1}^{n} f_i(x) x_i$ for $x \varepsilon E$, $n = 1, 2, \ldots$. Observe that $\lim_{n \longrightarrow \infty} T_n(x) = x$ in the weak topology. Hence $\{T_n(x)\}_{n\varepsilon N}$ is weakly bounded, hence bounded in the initial topology. By Proposition 2 §1 of this chapter, $(T_n)_{n\varepsilon N}$ is equicontinuous in the inital topology. Hence

$$x = x + \lim_n T_n(x - y_n)$$

$$= x + \lim_n \sum_{i \leq n} f_i(x - y_n)x_i$$

$$= \lim_n y_n + \lim_n \left(\sum_{i \leq n} f_i(x)\, x_i - y_n \right)$$

$$= \lim_n \sum_{i \leq n} f_i(x)x_i,$$

and so $(x_n, f_n)_{n \varepsilon N}$ is a Schauder basis for the initial topology. Q.E.D.

4. Bases in locally convex spaces

Definition: Let $(x_n, f_n)_{n \varepsilon N}$ be a basis in a locally convex TVS E_u, u defined by $(p_\lambda)\ \lambda \varepsilon L$. Then it is monotone if for each $x \varepsilon E_u$, λ fixed but arbitrary, $p_\lambda \left\{ \sum_{i=1}^{n} f_i(x)\, x_i \right\}_{n \varepsilon N}$ is a nondecreasing sequence of n.

Lemma 1: Let E_u be a locally convex TVS and f a linear functional on E. Let $g: E \longrightarrow E$ be given by $g(x) = f(x)a$, $a \varepsilon E$, a arbitrary, and $a \neq \Theta$. Then g continuous implies f is continuous.

Proof: If f is not continuous, then since f is linear, f is not continuous at the origin Θ. Hence there exists a net $(x_\alpha)_{\alpha \varepsilon I}$ with $x_\alpha \longrightarrow \Theta$ and $f(x_\alpha) \not\longrightarrow 0$. Assume $|f(x_\alpha)| \geq \varepsilon > 0$. Now g continuous gives $g(x_\alpha) \longrightarrow g(\Theta) = 0$. Hence $a = \dfrac{g(x_\alpha)}{f(x_\alpha)} \longrightarrow 0$. This gives $a = \Theta$. Contradiction, hence f is continuous. Q.E.D.

Proposition 4: If E_u is a locally convex TVS, u given by $(p_\lambda)_{\lambda \varepsilon L}$, and E_u has a monotone basis, $(x_n, f_n)_{n \varepsilon N}$, then the coordinate functionals, are continuous and hence the basis is Schauder.

Proof: Let $n \varepsilon N$ arbitrary. Then define $g: E \longrightarrow E$ by

$g(x) = f_n(x)x_n$. For any $\lambda \varepsilon L$,

$$p_\lambda(g(x)) = p_\lambda\left(\sum_{i=1}^{n} f_i(x)x_i - \sum_{i=1}^{n-1} f_i(x)x_i\right)$$

$$\leq p_\lambda\left(\sum_{i=1}^{n} f_i(x)x_i\right) + p_\lambda\left(\sum_{i=1}^{n-1} f_i(x)x_i\right) \leq 2\, p_\lambda(x).$$

Hence g is continuous, so f_n is continuous for each n by the above lemma. Q.E.D.

<u>Theorem 6</u>: Let E_u be a locally convex TVS, u given by $(p_\lambda)\ \lambda \varepsilon L$. Let $(x_n, f_n)_{n\varepsilon N}$ be a basis in E. Then if for any λ, n there exists $M > 0$ such that $p_\lambda(s_n(x)) \leq M\, p_\lambda(x)$ for all $x \varepsilon E$, $s_n(x) = \sum_{i=1}^{n} f_i(x)\, x_i$, then $(x_n, f_n)_{n\varepsilon N}$ is a Schauder basis. i.e. f_n is continuous for each n.

<u>Proof</u>: To do this we define a new family of semi norms p'_λ by $p'_\lambda(x) = \sup_n p_\lambda\left(\sum_{i=1}^{n} f_i(x)x_i\right)$. This is well defined since $(x_n, f_n)_{n\varepsilon N}$ having a basis it follows $s_n(x) \longrightarrow x$ and p_λ continuous implies $p_\lambda(s_n(x)) \longrightarrow p_\lambda(x)$ for each $\lambda \varepsilon L$. Hence $\sup_n p_\lambda(s_n(x))$ exists. We observe the following:

$$p'_\lambda(\theta) = \sup_n p_\lambda(S_n(\theta)) = 0,$$

$$p'_\lambda(x + y) = \sup_n p_\lambda\left(\sum_{i=1}^{n} f_i(x + y)x_i\right)$$

$$\leq \sup_n p_\lambda\left(\sum_{i=1}^{n} f_i(x)x_i\right) + \sup_n p_\lambda\left(\sum_{i=1}^{n} f_i(y)\, x_i\right)$$

$$= p'_\lambda(x) + p'_\lambda(y),$$ and for α a scalar,

$$p'_\lambda(\alpha x) = \sup_n p_\lambda \left(\sum_{i=1}^{n} f_i(\alpha x) x_i \right) = |\alpha| \sup_n p_\lambda \left(\sum_{i=1}^{n} f_i(x)\, x_i \right)$$

$$= |\alpha| \; p'_\lambda(x).$$

Observe that $(p'_\lambda)_{\lambda \varepsilon L}$ and $(p_\lambda)_{\lambda \varepsilon L}$ are equivalent families of semi-norms since $p_\lambda(s_n(x)) \longrightarrow p_\lambda(x)$ implies that for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for $n \geq N(\varepsilon)$,

$$-\varepsilon + p_\lambda(x) \leq p_\lambda(s_n(x)) \leq \varepsilon + p_\lambda(x).$$

Taking sup we get

$$-\varepsilon + p_\lambda(x) \leq \sup_{n \geq N(\varepsilon)} p_\lambda(s_n(x)) \leq \sup_{n \geq 1} p_\lambda(s_n(x))$$

$$= p'_\lambda(x).$$

Since $\varepsilon > 0$ was arbitrary, let $\varepsilon \longrightarrow 0$, hence $p_\lambda(x) \leq p'_\lambda(x)$. Also, by assumption, for any λ, n there exists $M > 0$ such that $p_\lambda(s_n(x)) \leq M\, p_\lambda(x)$. Hence $\sup_n p_\lambda(s_n(x)) \leq M\, p_\lambda(x)$. Therefore $p'_\lambda(x) \leq M\, p_\lambda(x)$, and hence the two families are equivalent. Also, $(x_n, f_n)_{n \varepsilon N}$ is monotone with respect to $(p'_\lambda)_{\lambda \varepsilon L}$, since for λ arbitrary,

$$p'_\lambda \left(\sum_{i=1}^{m-1} \alpha_i x_i \right) = \sup_n p_\lambda \left(\sum_{i=1}^{n} \alpha_i x_i \right) = \sup_{1 \leq n \leq m-1} p_\lambda \left(\sum_{i=1}^{n} \alpha_i x_i \right)$$

$$\leq \sup_{1 \leq n \leq m} p_\lambda \left(\sum_{i=1}^{n} \alpha_i x_i \right) = p'_\lambda \left(\sum_{i=1}^{m} \alpha_i x_i \right).$$

By applying the previous proposition, f_n's are continuous for all $n \varepsilon N$. Q.E.D.

<u>Corollary</u>: If E is a normed space then the theorem holds.

In his paper (7) Gelbaum examined the following problem. Let $(x_n, f_n)_{n \varepsilon N}$ be a Schauder basis in a Banach space B, and $(a_n)_{n \varepsilon N}$

any sequence of scalars. Consider $y_n = \sum_{i=1}^{n} a_i x_i$ for each n.

When does $(y_n)_{n\varepsilon N}$ become a Schauder basis? Here we present a generalization of his result to the case of locally convex sequentially complete spaces (Theorem 6).

<u>Definition</u>: A TVS E_u is sequentially complete if any cauchy sequence in E_u converges.

In metric spaces sequential completeness and completeness coincide, but not in general.

<u>Definition</u>: Let $\sum_{i=1}^{\infty} x_i$ be a series in a locally convex TVS E_u, u given by $(p_\lambda)_{\lambda\varepsilon L}$. Then $\sum_{i=1}^{\infty} x_i$ is absolutely convergent if for any $\lambda \varepsilon L$, $\sum_{i=1}^{\infty} p_\lambda(x_i) < +\infty$.

<u>Remark</u>: In general, absolute convergence and convergence are not related. As examples consider first E_u = Reals with the usual topology. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent but not absolutely convergent. Next consider $E_u = E^{\infty}$ (the space of all real sequences of finite support) with the norm of C_0 (all zero sequences). Let $\delta_n = (0, 0, 0, \overset{n^{th}}{1}, 0, \ldots)$. $\sum_{n=1}^{\infty} \frac{\delta n}{n^2}$ converges absolutely but does not converge in E. The connection between the two notions is as follows.

<u>Lemma 2</u>: Let E_u be a sequentially complete locally convex TVS, defined by $(p_\lambda)_{\lambda\varepsilon L}$. Then absolute convergence implies convergence.

<u>Proof</u>: Let $\sum_{i=1}^{\infty} x_i$ be an absolutely convergent series in E_u.

Then for $m > n$, λ fixed but arbitrary, $p_\lambda\left(\sum_{i=1}^{m} x_i - \sum_{i=1}^{n} x_i\right) = p_\lambda\left(\sum_{i=n+1}^{m} x_i\right) \leq \sum_{i=n+1}^{m} p_\lambda(x_i)$. By assumption $\sum_{i=1}^{\infty} x_i$ converges absolutely, hence $\sum_{n+1}^{m} p_\lambda(x_i) < \varepsilon$ for all m, n sufficiently large. Since the space is sequentially complete, the series converges. Q.E.D.

<u>Proposition 5</u>: Let E_u be a locally convex TVS and $(x_n, f_n)_{n\varepsilon N}$ any countable biorthogonal system. Let $(a_n)_{n\varepsilon N}$ be any sequence of scalars. Let $y_n = \sum_{i=1}^{n} a_i x_i$ for any n. Then $[x_n] = [y_n]$ if and only if $a_n \neq 0$ for all n.

<u>Proof</u>: Assume $a_n \neq 0$ for all $n \varepsilon N$. Then $x_1 = \frac{y_1}{a_1}$, and by substitution each x_n is expressible in terms of $y_1, \ldots, y_n$. Hence $[x_n] \subseteq [y_n]$. By the definition of y_n, $[y_n] \subseteq [x_n]$. Hence $[x_n] = [y_n]$. For the converse assume $[x_n] = [y_n]$. If $a_k = 0$ for some k, then $f_k(x_k) = 1$ and $f_k(y_n) = f_k\left(\sum_{i=1}^{n} a_i x_i\right) = a_k = 0$. for all n. This implies $x_k \notin [x_n]$, for if $x_k \varepsilon [x_n] = [y_n]$ then $x_k = \lim_{z_n \varepsilon sp(x_n)} z_n$ and $f_k(x_k) = 1 = \lim_{z_n \varepsilon sp(x_n)} f_k(z_n) = 0$, contradiction. Q.E.D.

<u>Proposition 7</u>: Let E_u be a locally convex TVS, u defined by $(p_\lambda)_{\lambda\varepsilon L}$, and $B \subseteq E_u$ a subset. Then B is bounded if and only if $p_\lambda(B)$ is bounded for each $\lambda \varepsilon L$.

<u>Proof</u>: See (30), (Chapter 12, § 1, Theorem 2).

<u>Theorem 7</u>: Let E_u be a sequentially complete locally convex TVS, u given by $(p_\lambda)_{\lambda\varepsilon L}$. Let $(x_i, f_i)_{i\varepsilon N}$ be a Schauder basis such that for each fixed $\lambda \varepsilon L$ there exists $M > 0$ such that $p_\lambda(x_i) \leq M_\lambda$ for all $i \varepsilon N$. Let $(a_i)_{i\varepsilon N}$ be any sequence of nonzero scalars. Consider $y_n = \sum_{i=1}^{n} a_i x_i$. Then $(y_n)_{n\varepsilon N}$ is a Schauder basis if and only if $p_\lambda\left(\frac{y_n}{a_{n+1}}\right)_{n\varepsilon N}$ is a bounded sequence for each $\lambda \varepsilon L$.

<u>Proof</u>: Assume $(y_n)_{n\varepsilon N}$ is a Schauder basis and $\left\{p_\lambda\left(\frac{y_n}{a_{n+1}}\right)\right\}_{n\varepsilon N}$ is not a bounded sequence for each $\lambda \varepsilon L$. Then there exists $\lambda \varepsilon L$ such that $p_\lambda\left(\frac{y_{n_i}}{a_{n_i+1}}\right) = c_{n_i} \longrightarrow \infty$ for some subsequence (n_i). We choose (n_i) such that $c_{n_i} \geq n_i$. Hence $p_\lambda\left(\sum_{i=1}^{\infty} \frac{x_{n_i}}{\sqrt{c_{n_i}}}\right) \leq \sum_{1}^{\infty} p_\lambda\left(\frac{x_{n_i}}{\sqrt{c_{n_i}}}\right)$ $\leq M_\lambda \sum_{i=1}^{\infty} \frac{1}{n_i^2} < +\infty$. Thus, with this choice of (c_{n_i}), $\sum_{i=1}^{\infty} \frac{x_{n_i}}{c_{n_i}}$ is absolutely convergent. By Lemma 2, it is convergent to z, say. Then $f_n(z) = 0$ if $n \neq n_i$ and $f_{n_i}(z) = \frac{1}{\sqrt{c_{n_i}}}$ if $n = n_i$. Hence $p_\lambda\left(\frac{f_{n_i}(z)\, y_{n_i}}{a_{n_i+1}}\right) = \sqrt{c_{n_i}} \longrightarrow \infty$ as $n_i \longrightarrow \infty$. Consider $g_n = \frac{f_n}{a_n} - \frac{f_{n+1}}{a_{n+1}}$.

We claim that $(y_n, g_n)_{n\varepsilon N}$ is a biorthogonal family. Clearly each g_n is a continuous linear functional since each f_n is. Also

$$g_n(y_m) = \left(\frac{f_n}{a_n} - \frac{f_{n+1}}{a_{n+1}}\right)(y_m) = \left(\frac{f_n}{a_n} - \frac{f_{n+1}}{a_{n+1}}\right) \sum_{i=1}^{m} a_i x_i$$

$$= \frac{1}{a_n} \sum_{i=1}^{m} a_i\, f_n(x_i) - \frac{1}{a_{n+1}} \sum_{i=1}^{m} a_i\, f_{n+1}(x_i)$$

$$= 0 \text{ if } n \neq m \text{ and } 1 \text{ if } n = m.$$

Since $a_i \neq 0$ for all i, $[x_n] = [y_n]$ by the previous proposition. Hence from Chapter I, section 7, Theorem 11 $(y_n, g_n)_{n\varepsilon N}$ form a Schauder basis. Observe that

$$(1) \qquad \sum_{m=1}^{n} g_m(z)\, y_m = \sum_{m=1}^{n} f_m(z)\, x_m - \left(\frac{f_{n+1}(z)}{a_{n+1}}\right)(y_n)$$

This follows by induction, since for $n = 1$, we have

$$g_1(z)\, y_1 = \left(\frac{f_1}{a_1} - \frac{f_2}{a_2}\right)(z)\,(a_1 x_1) = f_1(z)\, x_1 - \frac{f_2}{a_2}(z)\, y_1.$$

Assume (1) holds for some n. Then

$$\sum_{m=1}^{n+1} g_m(z)\, y_m = \sum_{m=1}^{n} g_m(z)\, y_m + g_{n+1}(z)\, y_{n+1}$$

$$= \sum_{m=1}^{n} f_m(z)\, x_m - \left(\frac{f_{n+1}(z)}{a_{n+1}}\right) y_n + \left(\frac{f_{n+1}}{a_{n+1}} - \frac{f_{n+2}}{a_{n+2}}\right)(z)\, y_{n+1}$$

$$= \sum_{m=1}^{n+1} f_m(z)\, x_m - \left(\frac{f_{n+2}}{a_{n+2}}\right)(z)\, y_{n+1}.$$

Hence by induction, (1) holds for all n. Observe that $\sum_{m=1}^{\infty} g_m(z)\, y_m$ does not converge. If it did, then since $(x_n, f_n)_{n\varepsilon N}$ and $(y_n, g_n)_{n\varepsilon N}$

are Schauder bases, using (1), $p_\lambda\left(\frac{f_{n+1}}{a_{n+1}}(z)\, y_n\right) \longrightarrow 0$ as $n \longrightarrow \infty$.

In particular $p_\lambda\left(\frac{f_{n_{i+1}}}{a_{n_{i+1}}}(z)\, y_{n_i}\right) \longrightarrow 0$ for the subsequence (n_i).

Contradiction! Hence for each $\lambda \in L$, $p_\lambda\left(\frac{y_n}{a_{n+1}}\right)$ is bounded.

For the converse, assume $p_\lambda\left(\frac{y_n}{a_{n+1}}\right)$ is a bounded sequence of n for λ fixed but arbitrary. Observe that since $(x_n, f_n)_{n\in N}$ is a basis, $f_n(z) \longrightarrow 0$ as $n \longrightarrow \infty$, for each $z \in E_u$. We claim that $\sum_{m=1}^{n} g_m(z)\, y_m \longrightarrow z$ as $n \longrightarrow \infty$, for $z \in E_u$. For $\lambda \in L$,

$$p_\lambda\left(\sum_{m=1}^{n} g_m(z)\, y_m - z\right) = p_\lambda\left(\sum_{m=1}^{n} f_m(z)\, x_m - z - \left(\frac{f_{n+1}}{a_{n+1}}\right)(z)\, y_n\right)$$

$$\leq p_\lambda\left(\sum_{m=1}^{n} f_m(z)\, x_m - z\right) + p_\lambda\left(\frac{f_{n+1}}{a_{n+1}}(z)\, y_n\right) .$$

Since $(x_n, f_n)_{n\in N}$ is a basis and p_λ is continuous, $p_\lambda\left(\sum_{m=1}^{n} f_m(z)\, x_m - z\right) \longrightarrow$ as $n \longrightarrow \infty$, and since $f_n(z) \longrightarrow 0$ as $n \longrightarrow \infty$ and $p_\lambda\left(\frac{y_n}{a_{n+1}}\right)$ is bounded for all n, $p_\lambda\left(\frac{f_{n+1}(z)\, y_n}{a_{n+1}}\right) = |f_{n+1}(z)|\, p_\lambda\left(\frac{y_n}{a_{n+1}}\right) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $\sum_{m=1}^{\infty} g_m(z)\, y_m = z$. Therefore $(y_n, g_n)_{n\in N}$ is a basis. Q.E.D.

<u>Corollary 1</u>: Let E_u be a sequentially complete locally convex TVS, u given by $(p_\lambda)_{\lambda\in L}$. Let $(x_n, f_n)_{n\in N}$ be a Schauder basis in E_u

such that $(x_n)_{n\varepsilon N}$ is bounded. Then $y_n = \sum_{i=1}^{n} x_i$ is a basis if and only if $p_\lambda\left(\sum_{i=1}^{n} x_i\right)_{n\varepsilon N}$ is a bounded sequence.

Proof: Follows directly form the theorem by taking $a_n = 1$ for all n.

Remark: The condition $\sum_{i=1}^{n} x_i \;_{n\varepsilon N}$ being bounded is a generalization of the notion of type P basis due to Singer (28).

Proposition 3: Let $(x_n, f_n)_{n\varepsilon N}$ be a Schauder basis in a locally convex TVS E_u and $(c_n)_{n\varepsilon N}$ any sequence of nonzero scalars. Then $(c_n x_n)_{n\varepsilon N}$ is also a Schauder basis.

Proof: The proof is trivial, since by assumption, each $x \varepsilon E_u$ has a unique expansion $x = \sum_{n=1}^{\infty} b_n x_n = \sum_{n=1}^{\infty} \frac{b_n}{c_n} c_n x_n$, and the uniqueness of $(b_n)_{n\varepsilon N}$ gives the uniqueness of the sequence $\left(\frac{b_n}{c_n}\right)_{n\varepsilon N}$. Q.E.D.

We can now present the following corollary to Theorem 7.

Corollary 2: If E_u is a sequentially complete locally convex TVS, then $(y_n)_{n\varepsilon N}$, where $y_n = \frac{\sum_{i=1}^{n} x_i}{n}$, is a Schauder basis if and only if z_n, where $z_n = \sum_{i=1}^{n} x_i$, is a Schauder basis.

Proof: Follows directly from Theorem 7 and the above proposition. Q.E.D.

CHAPTER IV

Normal cones and bases in countably normed spaces

1. Cones

In this section we introduce the notion of cones and normal cones, and give some fundamental facts about them. We then give some results relating bases and cones.

Definition: Let E_u be a TVS. A cone $K \subseteq E_u$ is a convex subset of E, with the origin θ as an extreme point i.e. A convex set such that

(c_1) $K + K \subseteq K$.

(c_2) $\alpha K \subseteq K$, $\alpha \geq 0$, α real.

(c_3) $K \cap (-K) = \theta$.

Definition: An ordered linear space E is one with a partial ordering (i.e. reflexive transitive and antisymmetric), denoted $\leq$, such that

(O_1) $x \leq y$ implies $x + z \leq y + z$, $x, y, z \in E_u$.

(O_2) $x \leq y$ implies $\alpha x \leq \alpha y$, for real $\alpha \geq 0$.

Remarks: (a) We take the liberty of denoting the usual ordering of the reals by $\leq$, but we will make a distinction where confusion may arise.

(b) Given a cone $K \subseteq E_u$, we can define a partial ordering $\leq$ in E_u, by $x \leq y$ if and only if $y - x \in K$, and given an ordered TVS E_u one can define a cone $K \subseteq E_u$ by $K = \{x: x \geq \theta\}$.

(c) The definition of cone is the same for linear space, but since we are dealing with topological vector spaces, we give the definition in this context.

<u>Definition</u>: (a) Given a locally convex TVS E_u with a separating biorthogonal system $(x_n, f_n)_{n\varepsilon N}$, we can define a cone $K(x_n, f_n) = \{x \varepsilon E: f_n(x) \geq 0, n \varepsilon N\}$. That this is a cone and is closed follows trivially from the linearity and continuity of the $(f_n)_{n\varepsilon N}$. This cone is called the cone associated with the biorthogonal system $(x_n, f_n)_{n\varepsilon N}$. When there is no possibility of confusion we shall denote $K(x_n, f_n)$ by K for convenience.

(b) Given an ordered TVS E_u, with ordering $\leq$, $K = \{x \varepsilon E_u: x \geq \theta\}$ is called the positive cone.

Observe that the cone $K(x_n, f_n)$ defined above is the positive cone with respect to the ordering given by $x \leq y$ if and only if $f_n(x) \leq f_n(y)$ for all $n \varepsilon N$.

<u>Definition</u>: Let $A \subseteq E_u$ be a subset of an ordered TVS E_u, K the positive cone. The full hull of A is defined to be $FH(A) = \{z \varepsilon A: x \leq z \leq y, x, y \varepsilon A\}$ i.e. $FH(A) = (A + K) \cap (A - K)$. If $A = FH(A)$, we call A full.

<u>Definition</u>: Let E_u be an ordered TVS with $K \subseteq E_u$ the positive cone. Then K is normal in E_u if there exists a neighborhood basis of θ consisting of full sets.

<u>Proposition 1</u>: If E_u is an ordered TVS with $K \subseteq E_u$ the positive cone, then the following are equivalent:

(a) K is normal.

(b) There exists a neighborhood basis of θ consisting of sets V for which $\theta \leq x \leq y$, $y \in V$, implies $x \in V$.

(c) For any two nets $(x_\beta)_{\beta \in I}$, $(y_\beta)_{\beta \in I}$ in E_u such that $0 \leq x_\beta \leq y_\beta$ for $\beta \in I$, if $y_\beta \longrightarrow \theta$, then $x_\beta \longrightarrow \theta$, convergence taken with respect to u.

(d) Given a neighborhood V of θ in E_u, there exists a neighborhood W of θ in E_u such that $\theta \leq x \leq y$, $y \in W$, implies $x \in V$.

<u>Proof</u>: See (23), (Chapter 2, § 1, Proposition 3).

<u>Definitions</u>: (a) Given $x, y \in E_u$ an ordered TVS, $[x, y] = \{z \in E_u, x \leq z \leq y\}$.

(b) $B \subseteq E_u$ is order bounded if there exists $x, y \in E_u$ such that $B \subseteq [x, y]$.

<u>Proposition 2</u>: Let E_u be an ordered TVS, and $K \subseteq E_u$ the positive cone. If K is normal, then every order bounded subset in E_u is u-bounded.

<u>Proof</u>: See (23), (Chapter 2, § 3, Proposition 4).

<u>Proposition 3</u>: Let E_u be an ordered locally convex TVS, $K \subseteq E_u$ the positive cone. Then the following statements are equivalent:

(a) K is normal in E_u.

(b) There exists a family $(p_\alpha)_{\alpha \in A}$ of semi-norms generating the topology u such that $\theta \leq x \leq y$ implies $p_\alpha(x) \leq p_\alpha(y)$ for all $\alpha \in A$.

<u>Proof</u>: See (23), (Chapter 2, § 3, Proposition 5).

<u>Definitions</u>: (a) Let E_u be an ordered TVS. Then E_u is a vector lattice if for any $x, y \in E_u$, $\sup\{x, y\}$ and $\inf\{x, y\}$ exist.

(b) A subset $B \subseteq E_u$, E_u a vector lattice is solid if for $x \in B$ and $|x| \leq |y|$ for some $y \in E_u$, gives $y \in B$. Note, $|x| = \sup\{x, -x\}$.

(c) A linear subspace M of a vector lattice E_u is a lattice ideal if M is a solid subset of E_u.

<u>Remarks</u>: (a) Let E_u be an ordered TVS, $K \subseteq E_u$ the positive cone. If $M \subseteq E_u$ is a linear subspace, then M is an ordered TVS for the order determined by the cone $K \cap M$.

(b) If $(E_\alpha)_{\alpha \in I}$ is a family of ordered TVS's, $K_\alpha \subseteq E_\alpha$ the positive cone for each α, then the product space πE_α is an ordered TVS for the cone $K = \pi K_\alpha$, and πK_α is normal in πE_α if and only if K_α is normal in E_α for all $\alpha \in I$.

<u>Proposition 4</u>: Let E_u be an ordered TVS and $K \subseteq E_u$ a normal cone. Let $M \subseteq E_u$ be a linear subspace. Then $K \cap M$ is a normal cone in M for the subspace topology. In addition if E_u is a vector lattice and $M \subseteq E_u$ a lattice ideal, then the canonical image of K in E/M is a normal cone for the quotient topology.

<u>Proof</u>: See (23), (Chapter 2, § 1, Proposition 10).

<u>Definition</u>: A series $\sum_{i=1}^{\infty} x_i$ in a locally convex TVS is unconditionally convergent if for any permutation p of the indices, the series $\sum x_{p(i)}$ converges.

Remark: Every series of real numbers is unconditionally convergent iff it is absolutely convergent.

Definition: Let $(x_n, f_n)_{n\varepsilon N}$ be a separating biorthogonal system in a locally convex TVS E_u. Let $K \subseteq E_u$ be the cone associated with $(x_n, f_n)_{n\varepsilon N}$. Then $(x_n, f_n)_{n\varepsilon N}$ is a basis for K if every $x \varepsilon K$ has the representation $x = \sum_{i=1}^{\infty} f_i(x)\, x_i$.

Theorem 1: Let E_u be a locally convex, sequentially complete TVS, and $(x_n, f_n)_{n\varepsilon N}$ a separating biorthogonal system such that $(x_n, f_n)_{n\varepsilon N}$ is a basis of the associated cone K. Consider the following statements.

(a) For any $x \varepsilon K$, $\sum_{i=1}^{\infty} f_i(x)\, x_i$ is unconditionally convergent.

(b) For any $x \varepsilon K$, $\sum_{i=1}^{\infty} f_i(x)\, x_i$ is weakly unconditionally convergent.

(c) K is normal.

(d) For $x \varepsilon K$, $P_x = \{y \varepsilon E_u : \theta \leq y \leq x\}$ is bounded.

(e) For $x \varepsilon K$, P_x is homeomorphic to a cube,

then the following implications hold: (a) $\Longrightarrow$ (b), (b) $\Longrightarrow$ (d), (c) $\Longrightarrow$ (a), (c) $\Longrightarrow$ (d), (a) $\Longrightarrow$ (e).

Proof: (a) $\Longrightarrow$ (b). If $\sum_{1}^{\infty} f_i(x)\, x_i$ is unconditionally convergent, then it is weakly unconditionally convergent, hence the sequence of partial sums is weakly unconditionally cauchy (see (4), Chapter 4 §1).

(b) $\Longrightarrow$ (d) Take $x, y \varepsilon K$, $y \leq x$. Then trivially $f_i(y) \leq f_i(x)$ for all $i \varepsilon N$. Since $(x_n, f_n)_{n\varepsilon N}$ is a basis of K, $y = \sum_{i=1}^{\infty} f_i(y)\, x_i$.

For $f \varepsilon E'$, $|f(y)| = \left|\sum_{i=1}^{\infty} f_i(y)\, f(x_i)\right| \leq \sum_{i=1}^{\infty} f_i(y)\, |f(x_i)| \leq$

$\sum_{i=1}^{\infty} f_i(x) \left| f(x_i) \right|$. Since a weakly unconditionally cauchy sequence in a sequentially complete space is unconditionally convergent and in the reals the latter is equivalent to absolute convergence, we get

$\left| f(y) \right| \leq \sum_{i=1}^{\infty} f_i(x) \left| f(x_i) \right| = M_{f,x} < +\infty$ for each $f \in E$ where $M_{f,x}$ is a positive real number depending upon f and x only. Hence for each $x \in K$, $P_x = \left\{ y: \Theta \leq y \leq x \right\}$ is weakly bounded, hence bounded in the inital topology by the Mackey theorem.

(c) $\Longrightarrow$ (a) Assume K is normal. Then by Proposition 3 of this chapter, there exists a family of semi-norms $(p_\alpha)_{\alpha \in A}$ generating u such that $\Theta \leq x \leq y$ implies $p_\alpha(x) \leq p_\alpha(y)$ for all $\alpha \in A$. Take $x \in K$, $\varepsilon > 0$. Since $(x_n)_{n \in N}$ is a basis of K for each α, there exists $N_\alpha(\varepsilon)$ such that $p_\alpha \left(\sum_{i=n}^{\infty} f_i(x)\, x_i \right) < \varepsilon$ for all $n \geq N_\alpha(\varepsilon)$. Let $\sum_{i=1}^{\infty} f_{n_i}(x)\, x_{n_i}$ be any subseries of $\sum_{i=1}^{\infty} f_i(x)\, x_i$. Choose i_0 such that $n_i \geq N_\alpha(\varepsilon)$ for $i \geq i_0$. Then for $p, q \geq i_0$, $\Theta \leq \sum_{p}^{q} f_{n_i}(x)\, x_{n_i} \leq \sum_{i=N_\alpha(\varepsilon)}^{\infty} f_i(x) x_i$.

Since K is normal we have

$$0 \leq p_\alpha \left(\sum_{p}^{q} f_{n_i}(x)\, x_{n_i} \right) \leq p_\alpha \left(\sum_{N_\alpha(\varepsilon)}^{\infty} f_i(x)\, x_i \right) < \varepsilon$$

Hence $\left\{ \sum_{i=1}^{m} f_{n_i}(x)\, x_{n_i} \right\}_{m \geq 1}$ is cauchy with respect to each α, and hence is cauchy with respect to the original topology u. Since E_u is sequentially complete, $\sum_{i=1}^{\infty} f_{n_i}(x)\, x_{n_i}$ converges. But the subseries was arbitrary, hence by (4), (Chapter 4, § 1, Page 60), $\sum_{i=1}^{\infty} f_i(x)\, x_i$

is unconditionally convergent.

(e) $\Longrightarrow$ (d) This follows trivially since a cube, being the product of compact sets is compact, hence bounded.

(a) $\Longrightarrow$ (e) Assume (a) holds. That P_x is homeomorphic to a cube follows from (6), Theorem 3. Q.E.D.

<u>Remark</u>: A cone K is called minihedral if for any $x, y \in K$, $\sup\{x, y\}$ exists. If $(x_n, f_n)_{n\in N}$ is an unconditional basis for the cone K associated with it, then K is minihedral. For if $x, y \in K$, then $\sum_{i=1}^{\infty} (f_i(x) + f_i(y))x_i$ is unconditionally convergent by assumption, hence $\sum_{i=1}^{\infty} \sup\{f_i(x), f_i(y)\} x_i$ is also unconditionally convergent because $\sup\{f_i(x), f_i(y)\} \leq f_i(x) + f_i(y)$ and $f_i(x), f_i(y)$ are ≥ 0 for all $i \in N$, so $\sum_{i=1}^{\infty} \sup\{f_i(x), f_i(y)\} x_i = \sup\{x, y\}$.

2. Normal Cones and bases.

In this section we show (Theorem 7) that normality of a cone can force existence of a basis for the cone. This is done in perfect complete countably normed spaces, thus extending a result known previously for a restricted class of Banach spaces see (21), Theorem 2. We also show that every cone associated with a separating biorthogonal system is normal in a perfect C.N.S.

For the definition of complete countably normed space we refer the reader to chapter I.

Remark: A C.N.S. E_u can be made into a metric space by $d(f, g) = \sum_{p=1}^{\infty} \frac{1}{2^p} \frac{\|f-g\|_p}{1+\|f-g\|_p}$, and the metric topology is identical with the original topology. Recall that by C.N.S. we mean complete countably normed space.

Theorem 2: If a countably normed space E_u satisfies the first axiom of countability, then E' is complete in the strong topology β.

Proof: See (9), (Chapter 1, § 5).

Definition: A locally convex TVS is perfect if all bounded sets are relatively compact.

Theorem 4: Initial and weak convergence coincide in a perfect C.N.S.

Proof: See (9), (Chapter 1, § 5).

Corollary: A perfect C.N.S. is complete relative to weak convergence.

Proof: Follows as in (9), (Chapter 1, § 6).

Theorem 5: If E_u is a perfect C.N.S. then weak and strong convergence coincide in E'.

Proof: For a proof see (9), (Chapter 1, § 6).

Definition: A series $\sum_{i=1}^{\infty} x_i$ in a TVS E_u is weakly convergent if for any $f \in E'$, $\sum_{1}^{\infty} |f(x_i)| < +\infty$. A series of this type doesn't always converge to an element in the space. This is true only if the space is weakly complete.

<u>Proposition 5</u>: (Orlicz) In a normed space B, a series $\sum_{i=1}^{\infty} x_i$ converges weakly unconditionally if and only if there exists $M > 0$ such that for any finite sequence of indices $n_1 < n_2 \ldots < n_k$,

$$\left\| x_{n_1} + \ldots + x_{n_k} \right\| < M.$$

<u>Proof</u>: See (8), Theorem Page 241.

<u>Theorem 6</u>: Let E_u be a perfect C.N.S. and $(x_n, f_n)_{n \in N}$ a separating biorthogonal system. Let $K \subseteq E_u$ be the associated cone. Then if (1) $K' - K' = E'$, K' the positive span of (f_n), or

(2) For each (y_n), (z_n) where $\theta \leq y_n \leq z_n$, (z_n) bounded implies (y_n) bounded, then K is normal.

<u>Proof</u>: Let $(y_n)_{n \in N}$, $(z_n)_{n \in N}$ be two sequences in E_u such that $\theta \leq y_n \leq z_n$ for $n \in N$, and $z_n \longrightarrow \theta$. We want to show that $y_n \longrightarrow \theta$ and then apply Proposition 1 of this chapter to get K normal. Since E_u is perfect, it is sufficient to show $y_n \longrightarrow \theta$ weakly. Since $z_n \longrightarrow \theta$, then $f_j(z_n) \longrightarrow f_j(\theta) = 0$ for all $j \in N$. Also $\theta \leq y_n \leq z_n$ for all n gives $y_n, z_n, z_n - y_n \in K$, $n \in N$. This gives $f_j(z_n - y_n) \geq 0$ or $f_j(z_n) \geq f_j(y_n)$ for all $j, n \in N$. So $f_j(y_n) \longrightarrow 0$ for each fixed j. For $g \in K'$, $g = \sum_{i=1}^{k} \alpha_i f_i$, $\alpha_i \geq 0$. Observe that $g(y_n) \longrightarrow 0$ as $n \longrightarrow \infty$, since

$$* \quad \left(\sum_{1}^{k} \alpha_i f_i\right)(y_n) \leq \left| \sum_{i=1}^{k} (\alpha_i f_i)(y_n) \right| \leq \sum_{i=1}^{k} \left| (\alpha_i f_i)(y_n) \right| < \varepsilon$$

for suitably large n. By (1), $f \in E'$, $f = f_1 - f_2$, $f_1, f_2 \in K'$. By the above $f_1(y_n) \longrightarrow 0$ and $f_2(y_n) \longrightarrow 0$, hence $(f_1 - f_2)(y_n) = f(y_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $y_n \longrightarrow \theta$ and K is normal. If (2) holds we observe that * holds for any $g \in sp\,(f_n)$. For

$f \in E'$, $f = \lim_{\beta} g_\beta$, $g_\beta \in sp\,(f_n)$ in the weak topology. This follows since (f_n) is separating, hence weakly total by [2], Page 51. Hence $f(y_n) = (f - g_\beta + g_\beta)(y_n) \leq \left|(f - g_\beta)(y_n)\right| + \left|g_\beta(y_n)\right|$. Given $\varepsilon > 0$, since $g_\beta(y_n) \longrightarrow 0$, there exists $N(\varepsilon, \beta)$ such that for $n \geq N(\varepsilon, \beta)$, $\left|g_\beta(y_n)\right| < \varepsilon/2$. Since $z_n \longrightarrow \Theta$, by assumption (2), (y_n) is bounded, and since E_u is perfect, $g_\beta \longrightarrow f$ uniformly on bounded sets, so there exists a γ such that for $\beta \geq \gamma$, $\left|(f - g_\beta)(y_n)\right| < \varepsilon/2$ for all y_n. Hence $\left|f(y_n)\right| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$ for $n \geq N(\varepsilon, \beta)$, $\beta \geq \gamma$. Therefore $y_n \longrightarrow \Theta$ weakly and K is normal. Q.E.D.

<u>Definition</u>: Let $(x_n, f_n)_{n\in N}$ be a separating biorthogonal system in a locally convex TVS E_u, and $K \subseteq E_u$ the associated cone. Then $(x_n, f_n)_{n\in N}$ is boundedly complete over K if for any sequence $\{a_i\}_{i\in N}$ of scalars such that $a_i \geq 0$ for all i, $\left\{\sum_{i=1}^{n} a_i x_i\right\}_{n\in N}$ bounded implies that $\sum_{i=1}^{\infty} a_i x_i$ converges.

<u>Theorem 7</u>: Let E_u be a perfect C.N.S. and $(x_n, f_n)_{n\in N}$ a separating biorthogonal system. Let $K \subseteq E_u$ be the associated cone. If K is normal then $(x_n, f_n)_{n\in N}$ is an unconditional basis of K which is also boundedly complete on K.

<u>Proof</u>: To show $(x_n, f_n)_{n\in N}$ is boundedly complete, assume $\left\{\sum_{i=1}^{n} a_i x_i\right\}_{n\in N}$ $a_i \geq 0$, is bounded. Take $J \subseteq N$ arbitrary but finite. Then $J \subseteq [1, n]$ for some n. Then $\Theta \leq \sum_{i\in J} a_i x_i \leq \sum_{i=1}^{n} a_i x_i$ from the definition of K. Since K is normal by the previous Theorem it follows from Proposition 3 of this chapter that there exists an equivalent countable family of semi-norms $(q_n)_{n\in N}$ such that for

$\theta \leq x \leq y$, we have $q_n(x) \leq q_n(y)$ for all $n \in N$. Let $\{\| \ \|_p\}_{p \in N}$ be the defining family of norms for the C.N.S. E_u. Hence for p fixed, there exists q_k and $N_k > 0$ such that $\| \sum_{i \in J} a_i x_i \|_p \leq N_k q_k \left(\sum_{i \in J} a_i x_i \right) \leq$

$N_k q_k \left(\sum_{i=1}^{n} a_i x_i \right) \leq N_k M_k$ where M_k is the bound of $\left\{ \sum_{i=1}^{n} a_i x_i \right\}_{n \in N}$

with respect to q_k. Hence taking $M = N_k M_k$ and applying Proposition 5 we get that $\sum_{i=1}^{\infty} | f(a_i x_i) | < +\infty$ for any $f \in E'_p$. Since $E' = \bigcup_{p=1}^{\infty} E'_p$

(9) (Chapter 1 §5, Page 45), $\sum_{i=1}^{\infty} | f(a_i x_i) |$ converges for each $f \in E'$.

Hence $\sum_{i=1}^{\infty} a_i x_i$ converges weakly, and because E_u is perfect, $\sum_{i=1}^{\infty} a_i x_i$ converges in the initial topology. Since the weak topology coincides with the initial topology (Theorem 4), $\sum_{i=1}^{\infty} a_i x_i$ converges in E_u, and since K is complete, it converges to some element in K. Hence $(x_n, f_n)_{n \in N}$ is boundedly complete over K.

To show $(x_n, f_n)_{n \in N}$ is a basis for K, observe that since $(x_n, f_n)_{n \in N}$ is a biorthogonal system, for $x \in K$ and using the definition of K,

$$\theta \leq \sum_{i=1}^{n} f_i(x) x_i \leq x \quad \text{for all } n.$$

Since K is normal, for n arbitrary and fixed arbitray p, there exists q_k and $N_k > 0$ such that

$$\| \sum_{i=1}^{n} f_i(x) x_i \|_p \leq N_k q_k \left(\sum_{i=1}^{n} f_i(x) x_i \right) \leq N_k q_k (x) < +\infty.$$

Hence $\left\{ \sum_{i=1}^{n} f_i(x) x_i \right\}_{n \in N}$ is bounded in E_u, and from the above

$\sum_{i=1}^{\infty} f_i(x)\, x_i$ converges. Since $(f_n)_{n\varepsilon N}$ is separating, $\sum_{i=1}^{\infty} f_i(x)\, x_i = x$.

For if $\sum_{i=1}^{\infty} f_i(x)\, x_i = y$ then for j fixed but arbitrary,

$$f_j(y - x) = f_j\left(\sum_{i=1}^{\infty} f_i(x)\, x_i - x\right) = 0$$

hence $y - x = \theta$ or $y = x$. This shows that $(x_n, f_n)_{n\varepsilon N}$ is a basis of K which is boundedly complete over K. Also by Theorem 1 of section 1, $(x_n, f_n)_{n\varepsilon N}$ is an unconditional basis since K is normal.

<u>Corollary 1</u>: Let $(x_n, f_n)_{n\varepsilon N}$ be a separating biorthogonal system in a perfect C.N.S. E_u. Let $K \subseteq E_u$ be the associated cone. If K is normal and generating, then $(x_n, f_n)_{n\varepsilon N}$ is a basis for E_u.

<u>Proof</u>: Since K is normal, $(x_n, f_n)_{n\varepsilon N}$ is a basis for K, and since K is generating, for $x \varepsilon E_u$, $x = y_1 - y_2$ $x_1, x_2 \varepsilon K$. Hence

$$y_1 = \sum_{i=1}^{\infty} f_i(y_1)\, x_i, \quad y_2 = \sum_{i=1}^{\infty} f_i(y_2)\, x_i \quad \text{therefore} \quad x = \sum_{i=1}^{\infty} f_i(y_1 - y_2) x_i,$$

and the coefficients are unique because for $x = \sum_{i=1}^{\infty} \beta_i\, x_i$, we have

$$\sum_{i=1}^{\infty} (f_i(y_1 - y_2) - \beta_i) x_i = \theta \quad \text{and hence} \quad f_i(x) = f_i(y_1 - y_2) = \beta_i$$

for $i \varepsilon N$. Q.E.D.

3. Boundedly complete and type P bases, and some examples

In this section we present some results about boundedly complete and type "P" bases for cones.

<u>Proposition 6</u>: Let E_u be a semi-reflexive TVS and $(x_n, f_n)_{n\varepsilon N}$ a separating biorthogonal system with the associated cone K. If $(x_n, f_n)_{n\varepsilon N}$ is a basis for K, then $(x_n, f_n)_{n\varepsilon N}$ is boundedly complete over K.

<u>Proof</u>: Assume $\left\{\sum_{i=1}^{n} a_i x_i\right\}_{n\varepsilon N}$, $a_i \geq 0$ scalars is a bounded subset of E_u. Let C be the circled convex closure of $\left\{\sum_{i=1}^{n} a_i x_i\right\}_{n\varepsilon N}$. Then C is weakly compact since E_u is semi-reflexive, see (18) (Chapter 5, § 20). Hence $\left\{\sum_{i=1}^{n} a_i x_i\right\}_{n\varepsilon N}$ has a weak cluster point y, say. Clearly $y \varepsilon K$, and we have $f_n(y) \geq 0$ for all $n \varepsilon N$. Since $(x_n, f_n)_{n\varepsilon N}$ is a basis for K, there exists a unique sequence of scalars $\alpha_i \geq 0$ such that $y = \sum_{i=1}^{\infty} \alpha_i x_i$. Since C is weakly compact, for any $k \varepsilon N$, there exists a subsequence $(n_{m(k)})$ such that $f_k\left(\sum_{i=1}^{n_{m(k)}} a_i x_i\right) \to f_k(y) = \alpha_k$ as $n_{m(k)} \to \infty$. But $f_k\left(\sum_{i=1}^{n_{m(k)}} a_i x_i\right) = a_k$ if $k \leq n_{m(k)}$ otherwise zero, hence $a_k = \alpha_k$. Hence $\sum_{i=1}^{\infty} a_i x_i$ converges and $(x_n, f_n)_{n\varepsilon N}$ is boundedly complete over K. Q.E.D.

<u>Definition</u>: Let $(x_n, f_n)_{n\varepsilon N}$ be a biorthogonal system in a locally convex TVS E_u. Then it is of type P if

(a) There exists a neighborhood V of zero such that $x_n \notin V$ for all n, and

(b) $\left\{\sum_{i=1}^{n} x_i\right\}_{n\varepsilon N}$ is bounded.

<u>Proposition 7</u>: Let $(x_n, f_n)_{n\varepsilon N}$ be a biorthogonal system in a locally convex TVS E_u such that $(x_n, f_n)_{n\varepsilon N}$ is a basis for the associated cone K. If $(x_n, f_n)_{n\varepsilon N}$ is of type P, then it is not boundedly complete over K.

<u>Proof</u>: Assume $(x_n, f_n)_{n\varepsilon N}$ is of type P. Then there exists a neighborhood V of θ such that $x_n \notin V$ for all n. Let W be a

closed convex, circled, neighborhood of the origin such that $W + W \subseteq V$. If $(x_n, f_n)_{n\varepsilon N}$ is boundedly complete over K since $\left\{\sum_{i=1}^{n} x_i\right\}_{n\varepsilon N}$ is bounded, $\sum_{i=1}^{\infty} x_i$ converges, say to x. Then $x_n = \left(\sum_{i=1}^{n} x_i - x\right) + \left(x - \sum_{i=1}^{n-1} x_i\right) \varepsilon\ W + W \subseteq V$ for sufficiently large n. Contradiction! Hence $(x_n, f_n)_{n\varepsilon N}$ is not boundedly complete over K.

<u>Remark</u>: Theorem 6 does not hold in a general C.N.S. For example take E_u to be the closed hyperplane $\left\{x = (a_n)\ \varepsilon\ \ell_1 : \sum_{i=1}^{\infty} a_i = 0\right\}$ in ℓ_1, and let $x_n = e_n - e_{n-1}$, where $e_n = (0, 0, 0, \ldots, \underset{n^{th}}{1}, 0, 0, \ldots)$. Then $(x_n)_{n\varepsilon N}$ is a basis of E_u with the associated sequence of coefficient functionals $f_n(x) = \sum_{i=1}^{n} a_i$, $x = (a_n)_{n\varepsilon N}$ by (28), Page 364. The associated cone $K = \left\{x = (a_n)_{n\varepsilon N}\ \sum_{i=1}^{n} a_i \geq 0\ \ n \in N\right\}$ is generating but not normal since if it were, $(x_n, f_n)_{n\varepsilon N}$ would then be an unconditional basis but $(x_n, f_n)_{n\varepsilon N}$ is not unconditional, see (28), Page 364.

BIBLIOGRAPHY

1. Arsove, M. G. and Edwards, R. E. Generalized bases in topological linear spaces, Studia. Math. 19, 95-113 (1960).

2. Bourbaki, N. Espaces Vectorials Topologiques, Chapters B-5, Livre 5, Paris, Hermann and Cie. 1955.

3. Davis, William J. Dual generalized bases in linear topological spaces, Proc. Amer. Math. Soc. 17, 1057-1063 (1966).

4. Day, M. M. Normed Linear Spaces, Academic Press, New York, 1962.

5. Dieudonne, J. On biorthogonal systems, Michigan Math. J. 2, 7-20, (1954).

6. Fullerton, R. E. Geometric structures of absolute basis systems in a linear topological space. Pac. J. Math. 12, 137-147, (1962).

7. Gelbaum, B. R. Notes on Banach spaces and bases, Acad. Bras. des Ciencias, 29-36, (1958).

8. Gelfand, I. M. Abstrackte Funktionen und lineare Operatoren, Math. Sb. 46, 235-284 (1938).

9. Gelfand, I. M. and Shilov, G. E. Generalized Functions Vol. 2. Academic Press, 1964.

10. Gelfand, I. M. and Shilov, G. E. Generalized Functions, Vol. 4. Academic Press, 1964.

11. Gurarii, V. I. On bases for sets in Banach spaces, Rev. Roum. Math. Pures. Appl. 10, 1235-1240, (1965).

12. Gurevic, L. A. Conic tests for bases of absolute convergence, Problems of Mathematical Physics and Theory of Functions, 2, 12-21, (Nankova Dumka, Kiev) (1964).

13. Horvath, John, Topological Vector Spaces and Distributions, Addison-Wesley, 1966.

14. Husain, T. The Open Mapping and Closed Graph Theorems in Topological Vector Spaces, Oxford Mathematical Monographs, Claredon Press, Oxford, 1965.

15. Husain, T. Two New Classes of Locally Convex Spaces, Math. Ann. 166, 289-299, (1966).

16. Jones, O. T. Continuity of semi-norms and linear maps on a space with a Schauder basis (to appear).

17. Kelly, John, General Topology, Van Nostrand 1955.

18. Kelly, J. and Namioka I. Linear Topological Spaces, Van Nostrand, Princeton, New Jersey, 1963.

19. Markuschevich, A. I. On bases in the space of analytic functions, Math. Sb. 17, 1945.

20. Marti, T. Jurg, Introduction to the Theory of Bases, Springer Tracts in Natural Philosopy, 18, 1969.

21. McArthur, C. W. Singer I. and Levin M. On the cones associated with biorthogonal systems and bases in Banach spaces, Canadian Journal of Math. 21, 1206-1217 (1969).

22. McIntosh, A. On the closed graph theorem, Proc. Amer. Math. Soc. 20, 397-404 (1969).

23. Peressini, A. L. Ordered Topological Vector Spaces, Harper and Row, 1967.

24. Retherford J. and Jones, O. On similar bases in barrelled spaces, Proc. Amer. Math. Soc. 18, 677-680 (1967).

25. Retherford, J. R. Bases, basic sequences and reflexivity of a linear topological space, Math. Ann. 164, 280-285 (1966).

26. Robertson, A. P. and Robertson, W. On the closed graph theorem, Proc. Glas. Math. Ass. 3, 9-12 (1956).

27. Robertson, A. P. and Robertson, W. On the closed graph theorem, Proc. London Math. Soc. 8, 242-57 (1958).

28. Singer, I. Basic Sequences and Reflexivity in Banach spaces, Studia Math. 21, 351-369 (1962).

29. Taylor, A. Introduction to Functional Analysis, John Wiley and Sons, 1954.

30. Wilansky, A. Functional Analysis, Blaisdell Publishing Company, 1964.