HJELMSLEV PLANES

AND

TOPOLOGICAL HJELMSLEV PLANES

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by

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SCOPE AND CONTENTS:

In this thesis we examine a generalized notion of ordinary two dimensional affine and projective geometries The first six chapters deal very generally with coordinatization methods for these geometries and a direct construction of the analytic model for the affine case. The last two chapters are concerned with a discussion of these structures viewed as topological geometries.

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Introduction

Each subsection is usually prefixed with a brief discussion concerning content and motivation. Hence we shall restrict ourselves to general considerations here. Our basic desire is to examine objects known as Hjelmslev planes. These were introduced by J. Hjelmslev in the late twenties, but from a modern point of view, this discussion was initiated in 1954 by W. Klingenberg [cf. [K1], [K2] and [K3]] . The subject has gained much appeal and has been studied extensively, especially by B. Artmann and D. Drake.

To a geometer, a Hjelmslev plane can be thought of as a geometry where more than one line may pass through two distinct points. To an algebraist , a Hjelmslev plane is to an ordinary plane, as a local ring is to a division ring.

Chapter one introduces the affine and projective Hjelmslev planes and considers certain groups of mappings associated with each.

Chapter two summarizes known algebraic results which we shall employ later. The definition of a local monoid is new.

In Chapters three and four we deal with a generalization of the results of E. Artin [cf. [A2]] which was initiated by Klingenberg in 1954 [cf. [K2]] and continued by Lüneburg in 1962. [cf. [L1]] . Lüneburg, however, was more interested in generalizing the results of André for ordinary planes [cf. [A1]] . We reprove some of Lüneburg's and Klingenberg's results in the Artin setting utilizing dilatations which Artin employed in 1958 in [A2] and Lüneburg in 1962 in [L1].

Our main concern in Chapter five is motivating and then constructing the analytic model of an affine H-plane. In [K1], Klingenberg constructed the projective model but not the affine one directly. We shall discuss this problem in the preliminary comments for Section 5.3.

Chapter six introduces the ternary field of a Hjelmslev plane. A very detailed introduction to this generalization is found in Section 6.1. The results of this section are used extensively in Chapter eight.

Again except for a few results in Section (7.3), the material on 0-connectedness and some additional results in Section (7.2), Chapter seven is a compilation of known results in topology. We shall employ them in Chapter eight.

Lastly Chapter eight commences a study of topological Hjelmslev planes. The works of H. Salzmann [cf. [S1], [S2] and [S3]] are the primary source which motivates the results in this chapter. We obtain generalizations of the fact each ordinary topological affine or projective plane is connected or totally disconnected. Finally, we consider the topological properties of the group of translations and the ring of trace preserving endomorphisms of an affine H-plane, which, to my knowledge, has not been done for the ordinary case.

Notation

The following is a compilation of notational usage within the thesis, which is not described internally.

The complement of a subset A of a set X is written, $X \setminus A$ or $\complement A$. The notation $A \subsetneqq B$ denotes A is strictly contained in B.

With regards to an equivalence relation θ on a set X, the equivalence class containing the point x is written $[x]$, $[x]_\theta$ or $\bar{x}$. $x\theta y$ means $(x, y) \varepsilon \theta$ and $x \not\theta y$ is its negation. X/θ is the set of all equivalence classes.

If f and g are two functions, f g will designate their composition.

CHAPTER 1

§1.1. Affine Hjelmslev Planes

Definition (1.1.1). $<\mathbb{P}, \mathcal{L}, I, \parallel>$ is called an incidence structure with parallelism iff

(a) $\mathbb{P}$ and $\mathcal{L}$ are sets.

(b) $I \subseteq \mathbb{P} \times \mathcal{L}$.

(c) $\parallel \subseteq \mathcal{L} \times \mathcal{L}$ is an equivalence relation.

$\parallel$ is called parallelism.

The elements of $\mathbb{P}$ are called points and are denoted by P, Q, R, ... The elements of $\mathcal{L}$ are called lines and are denoted by ℓ, m, n ... . $(\ell, m) \in \parallel$ is written $\ell \parallel m$ and is read 'ℓ is parallel to m'. $(P, \ell) \in I$ is written $PI\ell$ and reads 'P lies on ℓ'. $P \not{I} \ell$ means $(P, \ell) \notin I$ and $\ell \nparallel m$ means $(\ell, m) \notin \parallel$.

Definition (1.1.2). $P, QI\ell, m$ means $P, QI\ell$ and P, QIm.

$$g \wedge h = \{P \mid P \in \mathbb{P} \text{ such that } P\ Ig, h\}.$$

$$g \vee h = \{P \mid P \in \mathbb{P} \text{ such that } PIg \text{ or } PIh\}.$$

$PIg \vee h$ means PIg or PIh.

If A is a subset of $\mathbb{P}$ and $\ell \varepsilon \mathcal{L}$, then

$$A \wedge \ell = \{P | P \varepsilon A \text{ and } PI\ell\}.$$

$|A|$ is the cardinality of the set A.

<u>Definition (1.1.3)</u>. Let P, Q$\varepsilon \mathbb{P}$ and ℓ, m$\varepsilon \mathcal{L}$, where $<\mathbb{P}, \mathcal{L}, I, \|>$ is an incidence structure with parallelism.

(a) <u>(P, Q)$\varepsilon o_{\mathbb{P}}$</u> iff there exist ℓ, m$\varepsilon \mathcal{L}$, $\ell \neq m$, such that P, QIℓ, m.

If (P, Q)$\varepsilon o_{\mathbb{P}}$, we write $Po_{\mathbb{P}}Q$ and say P and Q are <u>neighbouring points</u>.

(b) (ℓ, m)$\varepsilon o_{\mathcal{L}}$ iff for every PIℓ there exists a QIm such that $Po_{\mathbb{P}}Q$, and for every QIm there exists a PIℓ such that $Qo_{\mathbb{P}}P$.

If (ℓ, m)$\varepsilon o_{\mathcal{L}}$, write $\ell o_{\mathcal{L}} m$ and say ℓ and m are <u>neighbouring lines</u>.

(c) $P \not{o}_{\mathbb{P}} Q$ means (P, Q)$\notin o_{\mathbb{P}}$ and $\ell \not{o}_{\mathcal{L}} m$ means (ℓ, m)$\notin o_{\mathbb{P}}$.

<u>Definition (1.1.4)</u>. [L1] $\mathcal{H} = <\mathbb{P}, \mathcal{L}, I, \|>$ is called an <u>affine Hjelmslev Plane</u> or <u>affine H-Plane</u>

iff the following axioms are satisfied.

(A1) For any two points P and Q, there exists $\ell \varepsilon \mathcal{L}$ such that P, QIℓ.

The symbol PQ means $P \not{o}_{\mathbb{P}} Q$ and PQ is the unique line through P and Q.

(A2) There exist three points $\{P_1, P_2, P_3\}$ such that $P_iP_j \not{o}_{\mathcal{L}} P_iP_k$, $i \neq j \neq k \neq i$; $i, j, k = 1, 2, 3$.

(A3) $o_{\mathbb{P}}$ is a transitive relation on $\mathbb{P}$.

(A4) If PIg, h, then $g \not{o}_{\mathcal{L}} h$ iff $|g \wedge h| = 1$.

(A5) If $g \not{o}_{\mathcal{L}} h$; $P o_{\mathbb{P}} Q$; P, RIg; and Q, RIh; then $R o_{\mathbb{P}} P, Q$.

(A6) If $g o_{\mathcal{L}} h$; $j \not{o}_{\mathcal{L}} g$; PIg,j; and QIh,j; then $P o_{\mathbb{P}} Q$.

(A7) If $g \| h$; PIj, g; and $g \not{o}_{\mathcal{L}} j$; then $j \not{o}_{\mathcal{L}} h$ and there exists Q such that QIh, j.

(A8) For every $P \varepsilon \mathbb{P}$ and for every $\ell \varepsilon \mathcal{L}$, there exists a unique line $h \varepsilon \mathcal{L}$ such that PIh and $\ell \| h$.

Clearly if $o_{\mathbb{P}}$ is a transitive relation on $\mathbb{P}$ then $o_{\mathcal{L}}$ is a transitive relation on $\mathcal{L}$.

From now on assume we are dealing with an affine H-plane $\mathcal{H}$.

$\mathcal{L}$ will be called <u>proper</u> iff $o_{\mathcal{P}}$ is not the identity relation on $\mathcal{P}$. If $\mathcal{L}$ is proper, clearly $o_{\mathcal{L}}$ is also different from the identity relation on $\mathcal{L}$; cf. (A2).

<u>Definition (1.1.5)</u>. We define the map $L: \mathcal{P} \times \mathcal{L} \to \mathcal{L}$, where $L(P, \ell)$ is the unique line through P parallel to ℓ.

<u>Notation</u>: We will write PoQ for $Po_{\mathcal{P}}Q$ and ℓom for $\ell o_{\mathcal{L}} m$. $\{P = Q\} oR$ means $P = Q$ and QoR.

<u>Lemma (1.1.1)</u>. [L1]

(1) <u>For every</u> $\ell \varepsilon \mathcal{L}$, <u>there exists</u> $P \varepsilon \mathcal{P}$ <u>such that</u> $PI\ell$.

(2) <u>For every</u> $P \varepsilon \mathcal{P}$, <u>there exist</u> $\ell, m, \ell \neq m$ <u>such that</u> $PI\ell, m$.

<u>Proof</u>: (1) From (A2) there exists $R \varepsilon \{P_1, P_2, P_3\}$ such that $R \not{I} \ell$. Let it be P_1. Define $m = L(P_1, \ell)$. Since $P_1P_3 \not{o} P_1P_2$, then $P_1P_3 \not{o} m$ or $P_1P_2 \not{o} m$ by (A3). Suppose $P_1P_3 \not{o} m$. Then $P_1 I P_1P_3, m$; and $m \| \ell$ implies $\ell \not{o} P_1P_3$ and there exists $QI\ell, P_1P_3$ by (A7). A similar argument holds for $P_1P_2 \not{o} m$.

(2) From (A2) it follows that there exist i, j; $i \neq j$, such that $P \not{o} P_i, P_j$, where $P_i, P_j \varepsilon \{P_1, P_2, P_3\}$. Let $i = 1, j = 2$. Define $\ell_1 = PP_1$ and $\ell_2 = PP_2$. Choose ℓ_3 such that $P, P_3 I \ell_3$ by (A1). If $\ell_1 \neq \ell_2$, then we are finished. If $\ell_1 = \ell_2$, then $\ell_3 \neq \ell_1$, otherwise $P_1, P_2, P_3 I \ell_3$. Contradiction.

Corollary. $o_{\mathbb{P}}$ and $o_{\mathcal{L}}$ are equivalence relations.

Proof: It suffices to show this for $o_{\mathbb{P}}$, as the other follows immediately from it. $o_{\mathbb{P}}$ is reflexive by Lemma (1.1.1). It is clearly symmetric, and is transitive by (A3).

Lemma (1.1.2). The following are equivalent.

(1) (A5) *and* (A3).

(2) *If* PoQ *and* $R\emptyset P$, *then* $R\emptyset Q$ *and* $PRoQR$.

Proof: *(1)⟹(2):* PoQ and $R\emptyset P$ implies $R\emptyset Q$ by (A3). If $PR\emptyset QR$, then $P, RIPR$; $Q, RIQR$; and PoQ implies RoP, Q by (A5). Contradiction.

(2)⟹(1): (A3) is obvious. To show (A5), let PoQ; P, RIg; Q, RIh; and $g\emptyset h$. If $R\emptyset P$, then PoQ implies $R\emptyset Q$ and $\{PR = g\}\ o\ \{QR = h\}$. Contradiction. Similarly we may show RoQ.

Notation: (A5)* will denote condition (2) of Lemma (1.1.2).

Lemma (1.1.3).

(1) *If* $g \parallel h$, *then* $g \wedge h = \emptyset$ *or* $g = h$.

(2) *If* $g \wedge h = \emptyset$, *or* goh, *then there exists* j *such that* $j \parallel h$, jog *and* $j \wedge g \neq \emptyset$.

Proof: (1) This is an immediate consequence of (A8).

(2) Assume $g \wedge h = \emptyset$ or $g \circ h$. Choose PIg by Lemma (1.1.1). Let $j = L(P, h)$ by (A8). Hence $g \wedge j \neq \emptyset$. Then $j \circ g$, otherwise $j \not\circ g$; PIj, g; $j \parallel h$ imply that $g \not\circ h$ and $g \wedge h \neq \emptyset$ by (A7). Contradiction.

Definition (1.1.6) (a) $\Pi = \langle \mathcal{P}, \mathcal{L}, I, \parallel \rangle$ is called an affine plane iff the following axioms hold:

(A1)$^{\circ}$ For any two distinct points P, Q, there exists a unique line through P and Q.

(A2)$^{\circ}$ For each pair (P, ℓ), there exists a unique line m such that PIm and $m \parallel \ell$.

(A3)$^{\circ}$ There exist 3 non-collinear points.

(b) $\Pi = \langle \mathcal{P}, \mathcal{L}, I, \parallel \rangle$ is called an ordinary affine plane iff Π is an affine plane such that $\ell \parallel m$ is equivalent to $\ell = m$ or $\ell \wedge m = \emptyset$.

The next remark assures us that every ordinary affine plane is an affine H-plane and indicates the reason for (A7).

Remark (1.1.1). Let Π be an affine plane. The following are equivalent.

(1) Π is an ordinary affine plane.

(2) If $P = g \wedge j$; $j \neq g$; and $g \parallel h$; then $j \neq h$ and $j \wedge h \neq \emptyset$.

<u>Proof</u>: <u>$(1) \Longrightarrow (2)$</u>: Let $P = g \wedge j$; $j \neq g$; and $g \parallel h$.

<u>Claim</u>. $j \neq h$ and $j \wedge h \neq \emptyset$.

Suppose $j = h$. Then $j \parallel g$ and $j \neq g$ imply $j \wedge g = \emptyset$ by Definition (1.1.6)(b). Contradiction.

Next suppose $j \wedge h = \emptyset$. Then $j \parallel h$ by Definition (1.1.6)(b). But $g \parallel h$ and hence $j \parallel g$ since $\parallel$ is transitive. Hence, since $j \neq g$, $j \wedge g = \emptyset$. Contradiction.

<u>$(2) \Longrightarrow (1)$</u>: If $\ell \parallel m$, then $\ell \wedge m = \emptyset$ or $\ell = m$ is an immediate consequence of $(A2)^0$. Now suppose $\ell \wedge m = \emptyset$ or $\ell = m$. If $\ell = m$, then $\ell \parallel m$ since $\parallel$ is reflexive. If $\ell \wedge m = \emptyset$, choose $P I \ell$; hence $P \not{I} m$. Such a P exists by $(A1)^0$-$(A3)^0$ and (2); cf. Lemma (1.1.1)(1).

Let $n = L(P, m)$.

<u>Claim</u>. $n = \ell$.

If this is false, then $\ell \neq n$; $P = \ell \wedge n$; and $n \parallel m$ imply that $\ell \neq m$ and $\ell \wedge m \neq \emptyset$ by (2). Contradiction. Hence $n = \ell$ and so $m \parallel \ell$.

<u>Corollary</u>. <u>Every ordinary affine plane is an affine H-plane where $o_{\mathcal{L}}$ and $o_{\mathfrak{P}}$ are the identity relations</u>

on $\mathcal{L}$ and $\mathcal{P}$ respectively.

Notation. From now on in this section, $\{P_1, P_2, P_3\}$ will be the points of (A2) and $p_{i_j} = P_iP_j$.

Lemma (1.1.4). If (i, j, k) *is a permutation of* $\{1, 2, 3\}$, *then* $P_i \not o X$ *for every* XIp_{j_k}.

Proof. If this is false, then there exists X such that XIp_{j_k} and XoP_i. Then $p_{j_k} \not o p_{i_k}$ and $P_i oX$ imply $P_k oP_i$ by (A5). Contradiction.

Lemma (1.1.5). For every $P \varepsilon \mathcal{P}$, *there exist* $i, j; i \neq j$, *such that* $PP_i \not o PP_j$.

Proof: Case (1): $P = P_i$. Then $P_iP_j \not o P_iP_k$.

Case (2): PIp_{i_j} *and* $P \not\varepsilon \{P_1, P_2, P_3\}$ *for some* $i, j \varepsilon \{1, 2, 3\}$.

Since $P_i \not o P_j$, we may assume $P \not o P_i$. Hence $PP_i = p_{i_j}$. Then by Lemma (1.1.4), $P_k \not o X$ for every XIp_{i_j}, $k \neq i, j$ and $k \varepsilon \{1, 2, 3\}$. Thus $PP_k \not o \{p_{i_j} = PP_i\}$.

Case (3): $P \not I p_{i_j}$, *for* $i, j \varepsilon \{1, 2, 3\}$.

Without loss of generality we may assume $P \not o P_1, P_2$. We consider two possibilities: (i) PoP_3 and (ii) $P \not o P_3$. (i) If PoP_3, then $P_3 \not o P_1$ implies $P_3P_1 o PP_1$ by (A5)* and $P_3 \not o P_2$ implies $P_3P_2 o PP_2$ by (A5)*. Since $P_1P_3 \not o P_3P_2$ it follows by (A3) that $PP_1 \not o PP_2$.

(ii) Assume $P\emptyset P_3$. If our claim is false, then $PP_1oPP_2oPP_3$. We then show $PP_1oP_1P_2$ and $PP_1oP_1P_3$ by contradiction. If $PP_1\emptyset P_1P_2$, then PP_1oPP_2 implies P_1oP_2 by (A6). Contradiction. Similarly $PP_1\emptyset P_1P_3$ implies that P_1oP_3. Thus by (A3), $P_1P_2oP_1P_3$. Contradiction. Hence our claim is true.

<u>Lemma (1.1.6)</u>. <u>For each</u> $\ell \varepsilon \mathfrak{L}$, <u>there exists</u> $P\varepsilon\{P_1, P_2, P_3\}$ <u>such that</u> $P\emptyset X$ <u>for every</u> $XI\ell$.

<u>Proof</u>: Suppose our lemma is false. Then there exist Q_i, $Q_iI\ell$ such that Q_ioP_i; $i = 1, 2, 3$. Now choose $SI\ell$. By Lemma (1.1.5), there exist j, k such that $SP_j\emptyset SP_k$. Now P_joQ_j and $S\emptyset P_j$ imply that SQ_joSP_j by (A5)*. Similarly SQ_koSP_k. But $SQ_k = SQ_j$, and hence SP_joSP_k by (A3). Contradiction.

<u>Lemma (1.1.7)</u>. [L1] <u>If</u> $g \| h$; PIg; Q, RIh; PoQ; <u>and</u> $Q\emptyset R$, <u>then</u> goh.

<u>Proof</u>: PoQ and $Q\emptyset R$ imply $RPoRQ$ by (A5)*. Let $h = QR$ and $j = RP$. Hence joh and RIj, h. Thus by (A7), jog and by (A3), goh.

<u>Lemma (1.1.8)</u>. <u>Let</u> $P_i = g_i \wedge j$; $i = 1, 2$. <u>Let</u> QIg_1 <u>such that</u> $Q\emptyset P_1$ <u>and</u> $g_1 \| g_2$. <u>The following are then eqivalent</u>.

(1) $P_1 o P_2$.

(2) $g_1 o g_2$.

Proof: (1)$\Longrightarrow$(2): If $P_1 o P_2$, then $Q \not o P_1$ implies $g_1 o g_2$ by Lemma (1.1.7).

(2)$\Longrightarrow$(1): If $g_1 o g_2$, then by (A4), $j \not o g_i$; i = 1, 2. We obtain $P_1 o P_2$ by (A6).

Lemma (1.1.9) [L1] For each $\ell \varepsilon \mathcal{L}$, there exist P, $QI\ell$ such that $P \not o Q$.

Proof: We choose $PI\ell$ by Lemma (1.1.1)(1). Then by Lemma (1.1.6) there exists S such that $S \not o X$ for each $XI\ell$. Define j = PS. Then $j \not o \ell$ and $j \wedge \ell = P$ by the choice of S and (A4). By Lemma (1.1.6), there exists R such that $R \not o Y$ for each YIj. Define h = L(R, j). Then by (A7), $h \not o \ell$ and there exists Q such that $Q = h \wedge \ell$. By the choice of R, $h \not o j$. Hence by Lemma (1.1.8), $P \not o Q$.

Lemma (1.1.10) [L1] Let $g_1 \parallel g_2$. Then the following are equivalent.

(1) $g_1 o g_2$.

(2) There exist $P_i I g_i$, i = 1, 2, such that $P_1 o P_2$.

Proof: This follows immediately from Lemmas (1.1.7) and (1.1.9).

Lemma (1.1.11). Let $P_i = g_i \wedge j$; $i = 1, 2$, such that $g_1 \| g_2$. Then the following are equivalent.

(1) $g_1 o g_2$.

(2) $P_1 o P_2$.

Proof: This is an immediate consequence of Lemmas (1.1.8) and (1.1.9).

Lemma (1.1.12). [L1] For every P, there exist ℓ_i; $i = 1, 2, 3$, such that $P I \ell_i$ and $\ell_i \not{o} \ell_j$; $i \neq j$; $i, j = 1, 2, 3$.

Proof: By Lemma (1.1.5), we may assume without loss of generality that $PP_1 \not{o} PP_2$.

Claim. $P_1 P_2 \not{o} PP_i$; $i = 1, 2$.

If $P_1 P_2 o P P_i$, then since $PP_1 \not{o} PP_i$, it follows that $P o P_i$; $i = 1, 2$, by (A6). Contradiction.

Define $j = L(P, P_1 P_2)$. Then by (A7), $j \not{o} PP_i$; $i = 1, 2$. Hence j, PP_1 and PP_2 are our desired lines.

Definition (1.1.6). $\Lambda \subseteq \mathcal{L}$, is called a pencil of lines iff Λ is an equivalence class with respect to $\|$. $\Lambda g = \{\ell \mid \ell \in \mathcal{L}$ and $\ell \| g\}$.

Remark (1.1.2). Let Λ_1 and Λ_2 be two pencils and $t_1 \in \Lambda_1$. If $t_1 \not{o} t_2$ for every $t_2 \in \Lambda_2$, then $t_1 \wedge t_2 \neq \emptyset$ for each $t_2 \in \Lambda_2$.

Proof: Suppose this is false. Then there exists $\tilde{t}_2 \varepsilon \Lambda_2$ such that $t_1 \wedge \tilde{t}_2 = \emptyset$. Hence by Lemma (1.1.3)(2), there exists $t_2' \varepsilon \Lambda_2$ such that $t_2' o t_1$. Contradiction.

Conditions (3) and (4) of the next lemma are due to Lüneburg.

Lemma (1.1.13). *Let Λ_1 and Λ_2 be two pencils. The following are equivalent.*

(1) *For each pair* $\{\ell_1, \ell_2\}$, $\ell_i \varepsilon \Lambda_i$, $\ell_1 \wedge \ell_2 = \emptyset$ *or* $\ell_1 o \ell_2$.

(2) *There exist* $\ell_i \varepsilon \Lambda_i$; $i = 1, 2$, *such that* $\ell_1 \wedge \ell_2 = \emptyset$.

(3) *There exist* $\ell_i \varepsilon \Lambda_i$; $i = 1, 2$, *such that* $\ell_1 o \ell_2$.

(4) *For each* $\ell_1 \varepsilon \Lambda_1$, *there exists* $\ell_2 \varepsilon \Lambda_2$ *such that* $\ell_1 o \ell_2$.

Proof: (1) $\Longrightarrow$ (2). Take any pair $\{\ell_1, \ell_2\}$ such that $\ell_i \varepsilon \Lambda_i$, $i = 1, 2$. If $\ell_1 \wedge \ell_2 = \emptyset$ we are finished. If $\ell_1 o \ell_2$, then by Lemma (1.1.6), there exists P such that $P \not{o} X$, $X I \ell_1 \vee \ell_2$. Define $j = L(P, \ell_1)$. Clearly $j \neq \ell_1$ and $j \not{o} \ell_1, \ell_2$ by the choice of P. Then $j \parallel \ell_1$ and hence $j \wedge \ell_1 = \emptyset$ by Lemma (1.1.3)(1).

Claim. $j \wedge \ell_2 = \emptyset$.

If $j \wedge \ell_2 \neq \emptyset$, then since $j \not{o} \ell_2$ and $j \parallel \ell_1$, we have $\ell_1 \wedge \ell_2 \neq \emptyset$ and $\ell_1 \not{o} \ell_2$ by (A7). Contradiction.

Hence $j\varepsilon\Pi_1$, $\ell_2\varepsilon\Pi_2$ and $j \wedge \ell_2 = \emptyset$.

<u>(2) ⟹ (3)</u>. Let $\ell_i\varepsilon\Lambda_i$ such that $\ell_1 \wedge \ell_2 = \emptyset$. Then by Lemma (1.1.3)(2), there exists $j \| \ell_1$ such that $jo\ell_2$. Since $j\varepsilon\Lambda_1$ and $\ell_2\varepsilon\Lambda_2$, (3) is satisfied.

<u>(3) ⟹ (4)</u>. Assume there exist $\ell_i\varepsilon\Lambda_i$ such that $\ell_1 o \ell_2$.

<u>Claim</u>. <u>For each</u> $t_1\varepsilon\Lambda_1$ <u>there exists</u> $t_2\varepsilon\Lambda_2$ <u>such that</u> $t_1 o t_2$. If this is false, then there exists $t_1\varepsilon\Lambda_1$ such that $t_1 \not o t_2$ for each $t_2\varepsilon\Lambda_2$. Then by Remark (1.1.2), $t_1 \wedge t_2 \neq \emptyset$ for each $t_2\varepsilon\Lambda_2$. Thus, in particular, $t_1 \wedge \ell_2 \neq \emptyset$ and $t_1 \not o \ell_2$. But $t_1 \| \ell_1$. Hence by (A7), $\ell_1 \wedge \ell_2 \neq \emptyset$ and $\ell_1 \not o \ell_2$. Contradiction.

<u>(4) ⟹ (1)</u>. Take any pair $\{\ell_1, \ell_2\}$ such that $\ell_i\varepsilon\Lambda_i$, $i = 1, 2$. Then there exists t_2 such that $t_2 o \ell_1$ and $t_2\varepsilon\Lambda_2$. If $\ell_1 \not o \ell_2$ and $\ell_1 \wedge \ell_2 \neq \emptyset$, then by (A7), $\ell_i \wedge t_2 \neq \emptyset$ and $\ell_1 \not o t_2$. Contradiction. Thus $\ell_1 o \ell_2$ or $\ell_1 \wedge \ell_2 = \emptyset$.

<u>Definition (1.1.7)</u>. Let Λ_1 and Λ_2 be two pencils. Then $\Lambda_1 o_\Lambda \Lambda_2$ iff one of the conditions of Lemma (1.1.13) holds. There is no danger of ambiguity if we write o for o_Λ .

Lemma (1.1.14). The following are true.

(1) $\Lambda_g \emptyset \Lambda_h$ *iff* $g \emptyset h$ *and* $g \wedge h \neq \emptyset$ *iff* $|g \wedge h| = 1$.

(2) o_Λ *is an equivalence relation on* $\{\Lambda \mid \Lambda$ *is a pencil of lines*$\}$.

Proof: (1) The first part is just the negation of condition (1) of Lemma (1.1.13) and the second is (A4).

(2) This is an immediate consequence of conditions (3) and (4) of Lemma (1.1.13).

Corollary. For any two pencils Λ_1, Λ_2, *there exists* Λ_3 *such that* $\Lambda_3 \emptyset \Lambda_1, \Lambda_2$.

Proof: Let $g_i \varepsilon \Lambda_i$, $i = 1, 2$. Hence $\Lambda_{g_i} = \Lambda_i$; $i = 1, 2$. Take PIg_1. Then there exist j_1, j_2 such that PIj_1, j_2; $g_1 \emptyset j_1, j_2$; and $j_1 \emptyset j_2$ by Lemma (1.1.12). Thus $g_1 \emptyset j_i$ and $g \wedge h_i \neq \emptyset$; $i = 1, 2$. Hence by (1) of the Lemma $\Lambda_{g_1} \emptyset \Lambda_{j_i}$; $i = 1, 2$. Similarly $\Lambda_{j_1} \emptyset \Lambda_{j_2}$. Since o_Λ is an equivalence relation, we have $\Lambda_{j_1} \emptyset \Lambda_{g_2}$ or $\Lambda_{j_2} \emptyset \Lambda_{g_2}$. Thus $\Lambda_{j_1} \emptyset \Lambda_{g_1}, \Lambda_{g_2}$ or $\Lambda_{j_2} \emptyset \Lambda_{g_1}, \Lambda_{g_2}$.

Notation. For each $P \varepsilon \mathcal{P}$, $\bar{P} = \{Q \mid Q \varepsilon \mathcal{P}$ and $QoP\}$.

Definition (1.1.8) [1] $\mathcal{A}$ is a *uniform* affine H-plane iff goh; PIg, h; QIg; and PoQ imply QIh. Equivalently, goh and PIg, h imply $\bar{P} \wedge g = \bar{P} \wedge h$.

Remark (1.1.5). If $\mathcal{H}$ is a proper affine H-plane, that is, $o_{\mathbb{P}}$ is not the identity relation on $\mathbb{P}$, then $|\bar{P} \wedge \ell| > 1$ if $PI\ell$.

Proof: Since $\mathcal{H}$ is proper there exists Q such that $Q \neq P$ and $Q\, oP$. If $Q\, I\ell$, we are finished. Suppose $Q \not{I}\ell$. By Lemma (1.1.12), we may choose m such that PIm, $m \not{o} \ell$ and $Q\, Im$. Thus $m \neq L(Q\, ,\, m)$. PoQ then implies $moL(Q\, ,\, m)$ by Lemma (1.1.10). By (A7), there exists S such that $S = L(Q\, ,\, m) \wedge \ell$ and $P \neq S$. By Lemma (1.1.11), PoS. Thus $|\bar{P} \wedge \ell| > 1$.

Definition (1.1.9). Define an incidence structure on $\bar{P}$ as follows. $\mathcal{H}_p = \langle \bar{P},\ \mathcal{L}_p,\ I_p \rangle$ where $\ell \wedge \bar{P} \varepsilon\ \mathcal{L}_p$ iff $\ell \varepsilon \mathcal{L}$, and $QI_p \ell \wedge \bar{P}$ iff QoP and $QI\ell$.

The next theorem characterizes proper uniform affine H-planes.

Theorem (1.1.1). [L1] The following are equivalent:

(1) $\mathcal{H}$ is a proper uniform affine-H-plane.

(2) Each $\mathcal{H}_p$ is an ordinary affine plane for every $P \varepsilon \mathbb{P}$.

Proof: (1)$\Longrightarrow$(2): We show $\mathcal{H}_p$ satisfies Definition (1.1.5). $(A1)^{o}$. Let $Q \neq R$; $Q,\ R \varepsilon \bar{P}$ such that $Q,\ RIg,\ h$. By (A4), goh. Then by uniformity $g \wedge \bar{P} = h \wedge \bar{P}$.

$(A2)^{o}$. Let $Q \varepsilon \bar{P}$ and $g \wedge \bar{P} \varepsilon\ \mathcal{L}_p$.

By (A8), there exists h such that $h \parallel g$ and QIh. Lemma

(1.1.3)(1), implies that $h \wedge g = \emptyset$ and hence $(h\wedge\overline{P}) \wedge (g \wedge \overline{P}) = \emptyset$. Moreover $QI_p\overline{P} \wedge h$. We must show $h \wedge \overline{P}$ is unique. Suppose $(f \wedge \overline{P}) \wedge (g\wedge\overline{P}) = \emptyset$ and $QI_p f\wedge\overline{P}$. Choose $RIg \wedge \overline{P}$ and hence RoQ. Since $h \parallel g$; RIg; and QIh, we have hog by Lemma (1.1.10). Now we claim that foh and hence by uniformity, $f\wedge\overline{P} = h\wedge\overline{P}$. Suppose $f\not{o}h$. Since QIf, h and $g \parallel h$, there exists S such that $S = g \wedge f$ by (A7). Then $(f\wedge\overline{P})\wedge(g\wedge\overline{P}) = \emptyset$ implies that $R\not{o}S$. Because hog, it follows that QoS by Lemma (1.1.11). Hence $R\not{o}Q$. Contradiction.

<u>$(A3)^o$</u>. By Lemma (1.1.12) there exist $(\ell_i)_{i=1}^{3}$ such that $PI\ell_i$ and $\ell_i \not{o} \ell_j$; $i \neq j$; $i, j = 1, 2, 3$. By Remark (1.1.3), there exist T_i such that $T_i \neq P$, $T_i oP$ and $T_i I \ell_i$; $i = 1, 2, 3$. Clearly $\{T_1, T_2, T_3\}$ satisfy $(A3)^o$.

<u>$(2) \Rightarrow (1)$</u>: Let $\mathfrak{A}_p$ be an ordinary affine plane for each P. Take PoQ; goh; P, QIg; and PIh. (A4) implies that there exists R, $R \neq P$, such that RIg, h. If $R\not{o}P$, then $g = h$ and hence QIh. If RoP, then by $(A1)^o$, $g\wedge\overline{P} = h\wedge\overline{Q}$. Thus QIh. It follows that $\mathfrak{A}$ is uniform. $(A3)^o$ clearly implies that $\mathfrak{A}$ is proper.

§1.2. Homomorphisms of Affine-H-Planes

Definition (1.2.1). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two affine-H-planes, such that $\mathcal{H}_i = \langle \mathcal{P}_i, \mathcal{L}_i, I_i, \|_i \rangle$; $i = 1, 2$.

(a) $f = (\phi, \psi)$: $\mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a homomorphism from $\mathcal{H}_1$ into $\mathcal{H}_2$ iff the following conditions hold:

(i) ϕ: $\mathcal{P}_1 \longrightarrow \mathcal{P}_2$ and ψ: $\mathcal{L}_1 \longrightarrow \mathcal{L}_2$ are functions.

(ii) $P I_1 \ell$ implies that $\phi(P) I_2 \psi(\ell)$.

(iii) $\ell_1 \|_1 \ell_2$ implies that $\psi(\ell) \|_2 \psi(m)$.

Notation. For the sake of convenience we shall write I and $\|$ for both I_i and $\|_i$; $i = 1, 2$, in the above definition unless ambiguity arises. Similarly, we shall put L for L_i; $i=1,2$; cf. Definition (1.1.5).

(b) $f = (\phi, \psi)$: $\mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a epimorphism iff both ϕ and ψ are surjective.

(c) $f = (\phi, \psi)$: $\mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a monomorphism iff both ϕ and ψ are injective.

(d) $f = (\phi, \psi)$: $\mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is an isomorphism iff f is a monomorphism and an epimorphism. If $\mathcal{H}_1 = \mathcal{H}_2$, then $f = (\phi, \psi)$ is called an automorphism.

Notation. $\mathcal{H}_1 \cong \mathcal{H}_2$ means that $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic.

Lemma (1.2.1). [L1] *Let* $\Pi = \langle \mathcal{P}, \mathcal{L}, I, \parallel \rangle$ *with the property that for each* $\ell \in \mathcal{L}$, *there exist* P, Q *such that* $P, QI\ell$ *and* $P \neq Q$. *Define* $\Pi^* = \langle \mathcal{P}, \mathcal{L}^*, I^*, \parallel^* \rangle$ *as follows:* $\ell^* \in \mathcal{L}^*$ *iff there exists* $\ell \in \mathcal{L}$ *such that* $\ell^* = \{P \mid PI\ell\}$;

$$PI^*\ell^* \text{ iff } P\in\ell^*;$$

$$\ell^* \parallel^* m^* \text{ iff } \ell \parallel m.$$

Then Π^* *is an incidence structure with parallelism such that* $\Pi^* \cong \Pi$.

Proof: Π^* is obviously an incidence structure with parallelism. Define $f = (\phi, \psi): \Pi \to \Pi^*$ by

$\phi: \mathcal{P} \to \mathcal{P}$ is the identity map.

$\psi: \mathcal{L} \to \mathcal{L}^*$ such that $\psi(\ell) = \ell^*$.

Clearly f is an epimorphism and ϕ is injective. We show ψ is injective. Suppose $\psi(\ell) = \psi(m)$ or $\{P \mid PI\ell\} = \{P \mid PIm\}$. Let $P, QI\ell$ such that $P \neq Q$, by assumption.

Then P, QIm. Hence ℓ = PQ = m. Hence f is an isomorphism.

<u>Remark (1.2.1)</u>. Since an H-plane has the property of Lemma (1.2.1), we may assume from now on that $\mathcal{L} \subseteq P(\mathcal{P})$ = the power set of $\mathcal{P}$. That is, $\ell \varepsilon \mathcal{L}$ is the set $\{P|PI\ell\}$.

<u>Lemma (1.2.2)</u>. <u>Let</u> g, h$\varepsilon \mathcal{L}$. <u>Then there exists a bijective map</u> ϕ: g$\to$h <u>with the property</u>

$$SoR \text{ \underline{iff} } \phi(S)o\phi(R).$$

<u>Proof</u>: Consider Λ_g and Λ_h. By the corollary of Lemma (1.1.14), there exist $\Lambda_j \not{o} \Lambda_g$, Λ_h. By Lemma (1.1.14)(1), j$\not{o}$g, h and $j \wedge g \neq \emptyset \neq j \wedge h$. Define ϕ: g$\to$h by $\phi(S) = L(S, j) \wedge h$. Because of (A4), ϕ is clearly a function from g_1 into h since L(S, j)$\not{o}$h and $L(S, j) \wedge h \neq \emptyset$ by (A7). Similarly X: h$\to$g, defined by $X(R) = L(R, j) \wedge g$, is also a function. Clearly $\phi X = X \phi$ = the identity map and so ϕ is bijective. Finally if S, RIg, then SoR iff L(S, j)oL(S, j) iff $\phi(S)o\phi(R)$ follows from Lemma (1.1.11).

<u>Corollary.</u> [L1] <u>Each line has the same cardinality</u>.

Lemma (1.2.3). Let $g \varepsilon \mathcal{L}$. Then there exist $j \varepsilon \mathcal{L}$, $P \varepsilon \mathcal{P}$ such that $P = j \wedge g$ and a bijective map ψ: $\Lambda_g \to j$ with the property for h, $f \varepsilon \Lambda_g$, hof iff $\psi(h) o \psi(f)$.

Proof: Such a P and j exist by Lemma (1.1.12). Define ψ: $\Lambda_g \to j$ by $\psi(h) = h \wedge j$. ψ is a function since $P = g \wedge j$ implies by (A7) and (A4) that $|h \wedge j| = 1$. Clearly X: $j \to \Lambda_g$ defined by $X(S) = L(S, g)$ is also a function. Moreover simple calculation shows ψ X = $X \wedge \psi$ = the identity map. Hence ψ is bijective . From Lemma (1.1.11), we have $h_1 o f_2$ iff $\psi(h) o \psi(f)$.

Corollary. [L1] $|\Pi_g| = |g|$ for each $g \varepsilon \mathcal{L}$.

Proof: This is an immediate consequence of the Lemma and the corollary of Lemma (1.2.2).

Lemma (1.2.4). Let $f = (\phi, \psi)$: $\mathcal{S}_1 \to \mathcal{S}_2$ be a homomorphism. The following statements are true.

(1) If ψ is injective then PoQ implies $\phi(P) o \phi(Q)$.

(2) $\psi(L(P, \ell)) = L(\phi(P), \psi(\ell))$.

(3) If $P \not o Q$ and $\phi(P) \not o \phi(Q)$, then $\psi(PQ) = \phi(P)\phi(Q)$.

(4) If $\Lambda_{\psi(\ell)} \not o \Lambda_{\psi(m)}$, and $\Lambda_\ell \not o \Lambda_m$, then $\psi(\ell) \wedge \psi(m) = \phi(\ell \wedge m)$.

Proof: (1) Let PoQ Thus there exist $\ell_1, \ell_2 \varepsilon \mathcal{L}$ such that $\ell_1 \neq \ell_2$ and $P, QI\ell_1, \ell_2$. Since f is a homomorphism $\phi(P), \phi(Q)I\psi(\ell_1), \psi(\ell_2)$. Because ψ is injective $\psi(\ell_1) \neq \psi(\ell_2)$. Hence $\phi(P)o\phi(Q)$.

(2) Since f is a homomorphism, $\psi(\ell) \parallel \psi(L(P, \ell))$ and $\phi(P)I\psi(L(P,\ell))$. Hence by (A8), $\psi(L(P,\ell)) = L(\phi(P), \psi(\ell))$.

(3) Let $P \not{o} Q$ and $\phi(P)\not{o}\phi(Q)$. Since f is a homomorphism, $\phi(P), \phi(Q)I\psi(PQ)$. Thus $\psi(PQ) = \phi(P)\phi(Q)$.

(4) Let $\Lambda_{\psi(\ell)} \not{o} \Lambda_{\psi(m)}$ and $\Lambda_{\ell} \not{o} \Lambda_m$. By Lemma (1.1.14)(1) there exist P and Q such that $Q = \psi(\ell) \wedge \psi(m)$ and $P = \ell \wedge m$. Since f is a homomorphism, $PI\ell, m$ implies $\phi(P)I\psi(\ell), \psi(m)$. Thus $\phi(P) = Q$ or $\phi(\ell \wedge m) = \psi(\ell) \wedge \psi(m)$.

Theorem (1.2.1). Let $f = (\phi, \psi): \mathcal{Q}_1 \longrightarrow \mathcal{Q}_2$ be a homomorphism. The following are then equivalent.

(1) $PI\ell$ iff $\phi(P)I\psi(\ell)$.

(2) f is a monomorphism.

(3) ψ is injective.

Proof: (1) $\Longrightarrow$ (2). We show ϕ is injective. Assume $\phi(P) = \phi(Q)$ and $P \neq Q$. Choose $\ell \varepsilon \mathcal{L}$ such that

$PI\ell$, but $Q\not{I}\ell$. Now $\phi(P) = \phi(Q)I\psi(\ell)$, because f is a homomorphism. This implies that $QI\ell$ by (1). Contradiction. Similarly if $\psi(\ell) = \psi(m)$ and $\ell \neq m$, there exists P, $PI\ell$, such that $P\not{I}m$. Then $\phi(P)I\psi(\ell)$. But $\psi(\ell) = \psi(m)$ and hence $PI\ell$ by (1). Contradiction.

<u>(2) $\Longrightarrow$ (3)</u>. Obvious.

<u>(3) $\Longrightarrow$ (1)</u>. Suppose Ψ is injective. We must show $\Phi(P)I\Psi(\ell)$ implies $PI\ell$. Suppose $P\not{I}\ell$. Then $\ell \neq L(P,\ell)$. Since ψ is injective $\psi(\ell) \neq \psi(L(P,\ell)$. But $\psi(L(P,\ell)) = L(\phi(P);\psi(\ell))$ by Lemma (1.2.4)(2). Thus $\psi(\ell) \wedge L(\phi(P), \psi(\ell)) = \emptyset$ by Lemma (1.1.3)(1). However, $\phi(P)I\psi(\ell)$, $L(\phi(P), \psi(\ell))$. Contradiction.

<u>Lemma (1.2.5)</u>. <u>The following are true when</u> $f = (\phi, \psi): \mathcal{P}_1 \to \mathcal{P}_2$ <u>is a homomorphism</u>.

(1) <u>If</u> ϕ <u>is surjective and</u> ψ <u>is injective, then</u> $\psi(\ell) = \{\phi(P) | PI\ell\}$.

(2) <u>If</u> ϕ <u>is surjective and</u> ψ <u>is injective, then</u> $\ell o m$ <u>implies</u> $\psi(\ell) o \psi(m)$.

<u>Proof</u>: (1) Since f is a homomorphism, $\{\phi(P) | PI\ell\} \subseteq \psi(\ell)$. Now we show the reverse inclusion. Let $RI\psi(\ell)$. Since ϕ is onto there exists P such that $\phi(P) = R$.

<u>Claim</u>. $PI\ell$.

If this is false, then $PI\ell$. Define $m = L(P,\ell)$. Hence $m \neq \ell$. Since ψ is injective $\psi(m) \neq \psi(\ell)$. But $\psi(m) = \psi(L(P,\ell)) = L(\phi(P), \psi(\ell))$ by (2) of Lemma (1.2.4), and so $\psi(m) \parallel \psi(\ell)$. By Lemma (1.1.3)(1), $\psi(m) \wedge \psi(\ell) = \emptyset$. But $R = \phi(P) I \psi(\ell), \psi(m)$. Contradiction.

(2) follows from (1) and the definition of 0_{ℓ}.

<u>Lemma (1.2.6)</u>. <u>Let</u> $f = (\phi, \psi)\colon \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ <u>be a monomorphism</u>. <u>Then</u> $\ell \parallel m$ <u>iff</u> $\psi(\ell) \parallel \psi(m)$.

<u>Proof</u>: If $\ell \parallel m$, then $\psi(\ell) \parallel \psi(m)$ by definition. Conversely suppose $\psi(\ell) \parallel \psi(m)$. Without loss of generality $\ell \neq m$ and hence since ψ is injective $\psi(\ell) \neq \psi(m)$. Thus $\psi(\ell) \wedge \psi(m) = \emptyset$. It follows that $\ell \wedge m = \emptyset$, by Theorem (1.2.1) (1). Now assume $\ell \nparallel m$. Since $\ell \wedge m = \emptyset$, there exists j such that $j \parallel m$, $jo\ell$ and $j \wedge \ell \neq \emptyset$ by Lemma (1.1.3)(2). Since $\ell \nparallel m$, it follows that $j \neq \ell$. Thus $\psi(j) \neq \psi(\ell)$, $\psi(j) \parallel \psi(m)$ and $\psi(j) \wedge \psi(\ell) \neq \emptyset$. But $\psi(\ell) \parallel \psi(m)$ and hence $\psi(\ell) \parallel \psi(j)$. Thus $\psi(\ell) \wedge \psi(j) = \emptyset$. Contradiction.

<u>Remark</u>. If (ϕ, ψ) is an isomorphism, then (ϕ^{-1}, ψ^{-1}) is a homomorphism; cf. Definition (1.2.1), Theorem (1.2.1) and Lemma (1.2.6).

<u>Theorem (1.2.2)</u>. <u>Let</u> $f = (\phi, \psi)\colon \mathcal{L}_1 \to \mathcal{L}_2$ <u>be a homomorphism</u>. <u>The following are equivalent</u>.

(1) f <u>is an isomorphism</u>.

(2) ϕ <u>is surjective and</u> ψ <u>is injective</u>.

<u>Proof</u>: <u>(1)$\Longrightarrow$(2)</u>. Obvious.

<u>(2)$\Longrightarrow$(1)</u>. By Theorem (1.2.1), ϕ is injective.

Now to show ψ is onto, choose $\ell_2 \varepsilon \mathcal{L}_2$. Then choose P_2, $Q_2 I \ell$ such that $P_2 \not{o} Q_2$. Since ϕ is onto there exist P_1 and Q_1 such that $\phi(P_1) = P_2$ and $\phi(Q_1) = Q_2$. By Lemma (1.2.4)(1), $P_1 \not{o} Q_1$. Define $\ell_1 = P_1Q_1$ and $\ell_1 \varepsilon \mathcal{L}_1$. Then by Lemma (1.2.4), $\psi(\ell_1) = \psi(P_1Q_1) = \phi(P_1)\phi(Q_1) = P_2Q_2 = \ell_2$.

Definition (1.2.2). Let $\mathcal{H}$ be an affine H-plane Aut $\mathcal{H}$ = $\{f | f$ is an automorphism of $\mathcal{H}\}$.

Notation: For convenience, let $f \varepsilon \text{Aut}\,\mathcal{H}$ be $f = (f, f)$.

Theorem (1.2.3). Aut $\mathcal{H}$ is a group under functional composition.

Proof: Clearly if $f, g \varepsilon \text{Aut}\,\mathcal{H}$, then $fg \; \varepsilon \; \text{Aut}\,\mathcal{H}$. For if $PI\ell$, then $g(P)Ig(\ell)$ and so $fg(P)I\,fg(\ell)$. Similarly $\ell \| m$ implies $fg(\ell) \| fg(m)$. Since composition is associative and the identity map is the unit it is enough to show $f^{-1} \in \text{Aut}\,\mathcal{H}$ for each $f \in \text{Aut}\,\mathcal{H}$, where $f^{-1} = (f^{-1}, f^{-1})$.

Let $f \varepsilon \text{Aut}\,\mathcal{H}$. By Theorem (1.2.1), $PI\ell$ iff $f(P)If(\ell)$, and $\ell \| m$ iff $f(\ell) \| f(m)$ by Lemma (1.2.6). Hence $f^{-1} \varepsilon \text{Auto}\,\mathcal{H}$.

Definition (1.2.3). $\overline{\mathcal{H}} = \langle \overline{\mathcal{P}}, \overline{\mathcal{L}}, \overline{I}, \overline{\|} \rangle$ is defined as follows. $\overline{\mathcal{P}} = \{\overline{P} | \overline{P}$ is an equivalence class with respect to $o_{\mathbb{P}}$ }. $\overline{\mathcal{L}} = \{\bar{\ell} \mid \bar{\ell}$ is an equivalence class with respect to $o_{\mathcal{L}}$ }. $\bar{P}\bar{I}\bar{\ell}$ iff there exist S and m such that SoP, $mo\ell$ and SIm. Equivalently $\bar{P}\bar{I}\bar{\ell}$ iff there exist S such that $SI\ell$ and SoP. $\bar{\ell}\,\bar{\|}\,\bar{m}$ iff $\bar{\ell} = \bar{m}$ or $\bar{\ell} \wedge \bar{m} = \emptyset$.

<u>Lemma (1.2.7)</u>. <u>Let</u> $\overline{\mathcal{X}} = \langle \overline{\mathbb{P}}, \overline{\mathcal{L}}, \bar{I}, \bar{\|} \rangle$ <u>be as in the above definition. Then the following conditions hold</u>.

(1) <u>If</u> $\ell_1 \| \ell_2$, <u>then</u> $\bar{\ell}_1 \,\bar{\|}\, \bar{\ell}_2$.

(2) <u>If</u> $\ell_1 \wedge \ell_2 = \emptyset$, <u>then</u> $\bar{\ell}_1 \,\bar{\|}\, \bar{\ell}_2$.

(3) $\bar{\ell}_1 \,\bar{\|}\, \bar{\ell}_2$ <u>iff there exist</u> m_1, m_2 <u>such that</u> $m_1 o \ell_1$, $m_2 o \ell_2$ <u>and</u> $m_1 \| m_2$.

<u>Proof:</u> (1) Let $\ell_1 \| \ell_2$. We assume $\bar{\ell}_1 \wedge \bar{\ell}_2 \neq \emptyset$ and show $\bar{\ell}_1 = \bar{\ell}_2$. Let $\bar{P}\bar{I}\ \bar{\ell}_1 \wedge \bar{\ell}_2$. Hence there exist S_i such that $S_i oP$ and $S_i I \ell_i$; $i = 1, 2$. Thus $S_1 o S_2$ and hence by Lemma (1.1.10), $\ell_1 o \ell_2$. Therefore $\bar{\ell}_1 = \bar{\ell}_2$.

(2) Let $\ell_1 \wedge \ell_2 = \emptyset$. By Lemma (1.1.3)(2), there exists m such that $mo\ell_1$ and $m \| \ell_2$. By (1), $\bar{m}\,\bar{\|}\,\bar{\ell}_2$. But $\bar{m} = \bar{\ell}_1$. Hence $\bar{\ell}_1 \bar{\|}\ \bar{\ell}_2$.

(3) Assume $\bar{\ell}_1 \wedge \bar{\ell}_2 = \emptyset$ or $\bar{\ell}_1 = \bar{\ell}_2$. Thus $\ell_1 \wedge \ell_2 = \emptyset$ or $\ell_1 o \ell_2$. By Lemma (1.1.3)(2), there exists m such that $m_1 o\ \ell_1$ and $m_1 \| \ell_2$. Let $m_2 = \ell_2$. Then m_1 and m_2 satisfy the conditions. Conversely if

$m_1 o \ell_1$, $m_2 o \ell_2$ and $m_1 \parallel m_2$, then by (1) $\overline{m_1} \,\overline{\parallel}\, \overline{m_2}$ and so $\overline{\ell_1} \,\overline{\parallel}\, \overline{\ell_2}$.

Lemma (1.2.8). $\overline{\mathcal{A}}$ *is an incidence structure with parallelism.*

Proof: $\overline{\parallel}$ is clearly reflexive and symmetric by Lemma (1.2.7)(3) and the fact o and $\parallel$ are equivalence relations. Now let $\overline{\ell_1} \,\overline{\parallel}\, \overline{\ell_2} \,\overline{\parallel}\, \overline{\ell_3}$. We must show $\overline{\ell_1} \,\overline{\parallel}\, \overline{\ell_3}$. Without loss of generality we may assume $\overline{\ell_1} \wedge \overline{\ell_2} = \overline{\ell_2} \wedge \overline{\ell_3} = \emptyset$ such that $\overline{\ell_1} \neq \overline{\ell_2} \neq \overline{\ell_3}$. Thus $\ell_1 \wedge \ell_2 = \ell_2 \wedge \ell_3 = \emptyset$. By Lemma (1.1.3)(2) there exist j_1, j_3 such that $j_1 o \ell_1$, $j_1 \parallel \ell_2$, $j_3 o \ell_3$ and $j_3 \parallel \ell_2$. If $\ell_1 \wedge \ell_3 = \emptyset$, then $\overline{\ell_1} \,\overline{\parallel}\, \overline{\ell_2}$ by Lemma (1.2.7)(2). If there exists P such that $PI\ell_1, \ell_3$, then, since $j_1 o \ell_1$ and $j_3 o \ell_3$, there exist $X_i I j_i$; i = 1, 3, such that X_1, $X_3 o P$. Hence $X_1 o X_3$. But $j_1 \parallel j_3$ and thus by Lemma (1.1.10), $j_1 o j_3$. Thus $\overline{\ell_1} = \overline{j_1} = \overline{j_3} = \overline{\ell_3}$.

The next two theorems of Lüneburg establish the fundamental relationships between affine H-planes and ordinary affine planes.

Theorem (1.2.4). [L1] $\overline{\mathcal{A}}$ *is an ordinary affine plane. Moreover* $X = (X_{\mathcal{P}}, X_{\mathcal{L}}): \mathcal{A} \longrightarrow \overline{\mathcal{A}}$ *defined by* $X_{\mathcal{P}}(P) = \overline{P}$ *and* $X_{\mathcal{L}}(\ell) = \overline{\ell}$ *is an epimorphism with the properties*

(i) $\chi_{\mathcal{P}}(P) = \chi_{\mathcal{P}}(Q)$ iff PoQ.

(ii) $\chi_{\mathcal{L}}(\ell) = \chi_{\mathcal{L}}(m)$ iff $\ell o m$.

(iii) If $\ell \wedge m = \emptyset$, then $\chi_{\mathcal{L}}(\ell) \mathbin{\overline{\|}} \chi_{\mathcal{L}}(m)$.

Proof: We verify the axioms of Definition (1.1.6).

$(A1)^o$. Let $\bar{P}_1 \neq \bar{P}_2$. Hence $P_1 \not o P_2$. Clearly $\bar{P}_1, \bar{P}_2 \bar{I} \overline{P_1P_2}$. We must show $\overline{P_1P_2}$ is unique. Let $\bar{P}_1$, $\bar{P}_2 \bar{I} \bar{m}$. Then there exist X_1, X_2 such that $X_1 o P_1$, $X_1 Im$, $X_2 o P_2$ and $X_2 Im$. Since $P_1 \not o P_2$, it follows that $X_1 \not o X_2$. Thus $m = X_1X_2$. Now by (A5)*, $P_1P_2 o P_2X_1$ and $X_1X_2 o P_2X_1$. Thus $\{\ell = P_1P_2\} o \{X_1X_2 = m\}$. Hence $\bar{\ell} = \bar{m}$.

$(A2)^o$. Let $\bar{P} \varepsilon \bar{\mathcal{P}}$ and $\bar{\ell} \varepsilon \bar{\mathcal{L}}$. From (A8) there exists m such that PIm and $m \| \ell$. Hence $\bar{P} \bar{I} \bar{m}$ and $\bar{m} \mathbin{\overline{\|}} \bar{\ell}$ by Lemma (1.2.7)(1). We must show $\bar{m}$ is unique. Let $\bar{t} \varepsilon \bar{\mathcal{L}}$ such that $\bar{P} \bar{I} \bar{t}$ and $\bar{t} \| \bar{\ell}$. Since $\overline{\|}$ is an equivalence relation by Lemma (1.2.8), $\bar{t} \| \bar{m}$. But $\bar{P} \bar{I} \bar{t}$, $\bar{m}$ and hence $\bar{t} = \bar{m}$.

$(A3)^o$. Let $\{P_1, P_2, P_3\}$ be the points of (A2). Then $\{\bar{P}_1, \bar{P}_2, \bar{P}_3\}$ satisfy $(A3)^o$.

Now clearly $PI\ell$ implies $\bar{P}\bar{I}\bar{\ell}$. Also $\ell_1 \| \ell_2$ implies $\chi(\ell_1) \mathbin{\overline{\|}} \chi(\ell_2)$ by Lemma (1.2.7)(1). Thus χ is a homomorphism. Clearly χ is an epimorphism. Properties (i) and (ii) follow from the definition of χ

and property (iii) is just Lemma (1.2.7)(2).

Notation: Throughout this thesis, $\chi = (\chi_{\mathbb{P}}, \chi_{\mathcal{L}})$ will refer to the map of Theorem (1.2.4).

Theorem (1.2.5). [L1] Let $\mathcal{H}$ be an incidence structure with parallelism. The following are then equivalent.

(1) $\mathcal{H}$ is an affine H-plane.

(2) $\mathcal{H}$ satisfies axioms (A1), (A4) and (A8) and there exists an ordinary affine plane $\bar{\mathcal{H}}$ and a epimorphism $\chi = (\chi_{\mathbb{P}}, \chi_{\mathcal{L}}): \mathcal{H} \longrightarrow \bar{\mathcal{H}}$ with the properties

(i) $\chi(P) = \chi(Q)$ iff $P \circ Q$.

(ii) $\chi(\ell) = \chi(m)$ iff $\ell \circ m$.

(iii) If $\ell \wedge m = \emptyset$, then $\chi(\ell) \bar{\parallel} \chi(m)$.

Proof. (1)$\Longrightarrow$(2). This is just Theorem (1.2.4).

(2)$\Longrightarrow$(1). We must verify axioms (A2), (A3), (A5), (A6) and (A7). Since χ is a homomorphism, we may use the properties in Lemma (1.2.4).

(A2). By $(A2)^{o}$, there exist three non-collinear points $\{\chi_{\mathbb{P}}(P_i) \mid i = 1, 2, 3\}$ since $\chi_{\mathbb{P}}$ is onto.

Claim. $\{P_1, P_2, P_3\}$ satisfy (A2). By (i), $\chi_{\mathbb{P}}(P_2) \neq \chi_{\mathbb{P}}(P_j)$ iff $P_i \not\circ P_j$ $i \neq j$. Also $\{\chi_{\mathcal{L}}(P_iP_j) = \chi_{\mathbb{P}}(P_i)\chi_{\mathbb{P}}(P_j)\} \neq \{\chi_{\mathbb{P}}(P_i)\chi_{\mathbb{P}}(P_k) = \chi_{\mathcal{L}}(P_iP_k)\}$ iff

iff $P_iP_j \not{o} P_iP_k$ by (ii).

Next we show (A5)* in place of (A3) and (A5). (cf. Lemma (1.1.2)).

<u>(A5)*</u> Let PoQ and $Q\not{o}R$. Hence $\chi_{\mathbb{P}}(P) = \chi_{\mathbb{P}}(Q) \neq \chi_{\mathbb{P}}(R)$ and so $P\not{o}R$ by (i). Thus $\chi_{\mathcal{L}}(PR) = \chi_{\mathbb{P}}(P)\chi_{\mathbb{P}}(R) = \chi_{\mathbb{P}}(Q)\chi_{\mathbb{P}}(R) = \chi_{\mathcal{L}}(QR)$. Hence $PRoQR$ by (ii).

<u>(A6).</u> Let goh; $j\not{o}g$; PIg, j; and QIh, j. Then $\chi_{\mathcal{L}}(h) = \chi_{\mathcal{L}}(g) \neq \chi_{\mathcal{L}}(j)$, by (ii). Also $P = g \wedge j$ and $Q = h \wedge j$ by (A3) and (A4). Moreover, $\chi_{\mathbb{P}}(P) = \chi_{\mathcal{L}}(g) \wedge \chi_{\mathcal{L}}(j)$ and $\chi_{\mathbb{P}}(Q) = \chi_{\mathcal{L}}(h) \wedge \chi_{\mathcal{L}}(j)$. Thus $\chi_{\mathbb{P}}(P) = \chi_{\mathcal{L}}(g) \wedge \chi_{\mathcal{L}}(j) = \chi_{\mathcal{L}}(h) \wedge \chi_{\mathcal{L}}(j) = \chi_{\mathbb{P}}(Q)$. Hence PoQ by (i).

<u>(A7).</u> Let $g\not{o}j$; $g \wedge j \neq \emptyset$; and $g \parallel h$. Then $\chi_{\mathcal{L}}(g) \neq \chi_{\mathcal{L}}(j)$, $\chi_{\mathcal{L}}(g) \wedge \chi_{\mathcal{L}}(j) \neq \emptyset$ and $\chi_{\mathcal{L}}(g) \bar{\parallel} \chi_{\mathcal{L}}(h)$. Hence $\chi_{\mathcal{L}}(g) \not{\bar{\parallel}} \chi_{\mathcal{L}}(j)$. If hog, then $\chi_{\mathcal{L}}(h) = \chi_{\mathcal{L}}(j)$ and thus $\chi_{\mathcal{L}}(g) \bar{\parallel} \chi_{\mathcal{L}}(h)$. Contradiction. If $h \wedge j = \emptyset$, then $\chi_{\mathcal{L}}(h) \bar{\parallel} \chi_{\mathcal{L}}(j)$ by (iii). Hence $\chi_{\mathcal{L}}(g) \bar{\parallel} \chi_{\mathcal{L}}(j)$. Contradiction. Hence $h\not{o}j$ and $h \wedge j \neq \emptyset$.

§1.3. Projective Hjelmslev Planes

Definition (1.3.1). [Kl] $\mathcal{H} = \langle \mathbb{P}, \mathcal{L}, I \rangle$ is a projective Hjelmslev plane or projective H-plane iff the following axioms are satisfied.

(P1) For every $P, Q \in \mathbb{P}$, there exists $\ell \in \mathcal{L}$ such that $P, Q I \ell$.

(P2) For every $\ell, m \in \mathcal{L}$, there exists $P \in \mathbb{P}$ such that $P I \ell, m$.

We define $P o_{\mathbb{P}} Q$ iff there exist $\ell, m \in \mathcal{L}$, $\ell \neq m$, such that $P, Q I \ell, m$ and $\ell o_{\mathcal{L}} m$ iff there exist $P, Q \in \mathbb{P}$, $P \neq Q$, such that $P, Q I \ell, m$. $P \not{o}_{\mathbb{P}} Q$ and $\ell \not{o}_{\mathcal{L}} m$ mean that $P o_{\mathbb{P}} Q$ and $\ell o_{\mathcal{L}} m$ respectively are false. PQ has the same meaning as in Definition (1.1.3). We note that the definition of $\ell o m$ differs from that of Definition (1.1.3).

(P3) There exist four points $\{P_1, P_2, P_3, P_4\}$ such that $P_i \not{o} P_j$ and $P_iP_j \not{o} P_iP_k$; $i \neq j \neq k \neq i$; $i, j, k = 1, 2, 3, 4$.

(P4) If $P I \ell, m, n$ such that $\ell o m$ and $m \not{o} n$, then $\ell \not{o} n$.

(P5) If $\ell o m$; $m \not{o} n$; $P I m, n$ and $Q I \ell, n$, then $P o Q$.

(P6) If $P o Q$; $Q \not{o} R$; $Q, R I \ell$; and $P, R I m$, then $\ell o m$.

Clearly a projective H-plane is an ordinary projective plane iff $o_{\mathbb{P}}$ and $o_{\mathcal{L}}$ are the identity relations on $\mathbb{P}$ and $\mathcal{L}$ respectively. A Projective H-plane also has a dual structure just as in the ordinary case.

Definition (1.3.2). Let $\mathcal{H} = < \mathbb{P}, \mathcal{L}\ I>$ be a projective H-plane. For each $P \varepsilon \mathbb{P}$, $\phi_P = \{\ell | PI\ell\}$ is called the pencil of lines through P. $\mathcal{H}^* = < \mathbb{P}^*, \mathcal{L}^*, I^*>$ where $\mathbb{P}^* = \mathcal{L}$, $\mathcal{L}^* = \{\phi_P | P\varepsilon \mathbb{P}\}$ and $\ell I^* \phi_P$ iff $PI\ell$ is called the dual of $\mathcal{H}$.

It is obvious that $\mathcal{H}^*$ is also a projective H-plane. Thus any theorem concerning points and lines has a dual statement in terms of lines and points.

We now state some results, due to Klingenberg. We omit the proofs, as they follow along the same lines as the analogous theorems for affine H-planes.

Theorem (1.3.1). [Kl] Let $\mathcal{H}$ be a projective H-plane.

(1) For each $\ell \varepsilon \mathcal{L}$, there exist X_1, X_2, X_3 such that $X_i \not{o} X_j$; $i \neq j$; $i, j = 1, 2, 3$ and $X_1, X_2, X_3 I \ell$.

(2) $o_{\mathbb{P}}$ and $o_{\mathcal{L}}$ are equivalence relations.

(3) $\overline{\mathcal{H}} = <\overline{\mathbb{P}}, \overline{\mathcal{L}}, \overline{I}>$, defined by

$\overline{P}\ \varepsilon \overline{\mathbb{P}}$ iff $\overline{P}$ is an equivalence class of $o_{\mathbb{P}}$,

$\overline{\ell} \varepsilon \overline{\mathcal{L}}$ iff $\overline{\ell}$ is an equivalence class of $o_{\mathcal{L}}$

$\overline{P} I \overline{\ell}$ <u>iff</u> <u>there</u> <u>exists</u> R, m <u>such</u> <u>that</u> RoP, moℓ <u>and</u> RIm,

<u>is</u> <u>an</u> <u>ordinary</u> <u>projective</u> <u>plane</u>.

(4) <u>The</u> <u>map</u> $\chi = (\chi_{\mathbb{P}}, \chi_{\mathscr{L}}): \mathscr{H} \to \overline{\mathscr{H}}$ <u>defined</u> <u>by</u>

$\chi_{\mathbb{P}}(P) = \overline{P}$ and $\chi_{\mathscr{L}}(\ell) = \overline{\ell}$ is a epimorphsim

<u>with</u> <u>the</u> <u>properties</u>

(a) $\chi_{\mathbb{P}}(P) = \chi_{\mathbb{P}}(Q)$ <u>iff</u> PoQ.

(b) $\chi_{\mathscr{L}}(\ell) = \chi_{\mathscr{L}}(m)$ <u>iff</u> ℓom.

<u>Notation</u>: Let $\chi_{\ell}: \ell \to \ell/0$ be $\chi_{\mathbb{P}}$ restricted to ℓ, for any $\ell \in \mathscr{L}$.

Lemma (1.3.1). [A1] *Let* $\{P_1, P_2, P_3\}$ *be three points such that* $P_i \not{o} P_j$; $i \neq j$, *and* $P_iP_j \not{o} P_iP_k$; $i \neq j \neq k \neq i$; $i, j, k = 1, 2, 3$. *Let* $p_k = P_iP_j$ *where* (i, j, k) *is a permutation of* $\{1, 2, 3\}$. *Clearly such points exist by* (P3). *Then we have:*

(1) *For each* $P \varepsilon \mathcal{P}$, *there exists* $g \varepsilon \{p_1, p_2, p_3\}$ *such that* SIg *implies* $P \not{o} S$.

(2) *Dually, for each* $\ell \varepsilon \mathcal{L}$, *there exists* $P \varepsilon \{P_1, P_2, P_3\}$ *such that* PIk *implies* $k \not{o} \ell$. *Moreover for any such* P, $P \not{o} S$ *for each* $SI\ell$.

Proof: (1) Suppose there exist R_1 and R_2 such that $R_1 Ip_1$, PoR_1, R_2Ip_2 and PoR_2.

Claims. (i) R_1oR_2.

(ii) R_1oP_3.

(iii) R_2oP_3 and PoP_3.

(i) This follows since R_1 and R_2 are both neighbouring points of P.

(ii) If $R_1 \not{o} P_3$ then from (i) and (P6) we have $R_2P_3oR_1P_3$ or p_1op_2. Contradiction.

(iii) This follows immediately from (i) and (ii).

We now show that p_3 is our desired line. Let R_3Ip_3. We must show $R_3 \not{o} P$. If R_3oP, then $P_1 \not{o} R_3$, for otherwise P_1oR_3; R_3oP and PoP_3 imply that P_1oP_3. Contradiction.

Now PoP_3 and $P_1 \not o P_3$ imply $p_2 o P_1 P$. Also $P_1 \not o R_3$ and $R_3 oP$ imply $p_3 o PP_1$. Hence $p_2 o p_3$. Contradiction.

(2) The 1st part follows by duality. Now suppose the second part is false. Then there exists S such that $SI\ell$ and PoS. Choose $RI\ell$ such that $R\not o S$ by Theorem (1.3.1)(1). Then $R\not o S$ and PoS imply PRoSR by (P6). But ℓ = SR. Contradiction.

Corollary. If $P\varepsilon\mathcal{P}$ and $\ell\varepsilon\mathcal{L}$, then there exists $Q\varepsilon\mathcal{P}$ such that $P\not o Q$ and $Q\not o S$ for each $SI\ell$.

Proof. Let $\{P_1, P_2, P_3, P_4\}$ be four points satisfying (P3). By the Lemma there exists $R\ \varepsilon\{P_1, P_2, P_3\}$ such that $R\not o X$ for each XIP. Let $R = P_1$, for instance. Then applying the Lemma again, there exists $Q\varepsilon\{P_2, P_3, P_4\}$ such that $Q\not o X$ for each $XI\ell$. Since $Q\not o R$ it follows that $P\not o Q$ or $P\not o R_j$ and this is our desired point.

Lemma (1.3.2). If ℓ, $m\varepsilon\mathcal{L}$, the following are equivalent:

(1) ℓom.

(2) For each $QI\ell$, there exists PIm such that PoQ.

(3) For each PIm, there exists QIℓ such that PoQ.

Proof: We first show (1) is equivalent to (2).

<u>(1) $\Longrightarrow$(2)</u>. Let $\ell o m$. Take $QI\ell$. Choose $g\epsilon\phi_Q$ such that $g\not o\ell$ by the dual of Theorem (1.3.1)(1). Hence $Q = g \wedge \ell$. Then $\ell o m$ and $\ell \not o g$ imply there exists P such that $P = m \wedge g$ and PoQ by (P5).

<u>(2) $\Longrightarrow$(1)</u>. Let $RI\ell$, m. Now by Theorem (1.3.1)(1) there exists Q such that $QI\ell$ and $Q\not o R$. By (2) there exists P such that PIm and PoQ. By (P5), PRoQR or $m o \ell$.

Clearly in the same fashion we may show (1) is equivalent to (3) and hence our Lemma is proved.

<u>Remark (1.3.1)</u>. <u>If</u> $\mathscr{H}$ <u>is an affine H-plane, then in the above lemma</u>, (1) <u>is equivalent to</u> (2) <u>and</u> (3) <u>combined</u>.

<u>Lemma (1.3.3)</u>. <u>If</u> ℓ, $m\epsilon \mathscr{L}$, <u>then there exists</u> R <u>such that for each</u> $k\epsilon\phi_R$, $k\not o\ell$ <u>and</u> $k\not o m$. <u>Moreover</u> $R\not o X$ <u>for each</u> $XI\ell \vee m$.

<u>Proof</u>. We consider two cases.

<u>Case (1)</u>: $\ell o m$.

By Lemma (1.3.1)(2), there exists R such that $k\epsilon\phi_R$ implies $k\not o\ell$. Since $\ell o m$, $k\not o m$ is also true.

<u>Case (2)</u>: $\ell \not o m$.

By the dual of Theorem (1.3.1)(1), there exists

$\{t_1, t_2, t_3\}$ such that $t_i \not o t_j$; $i \neq j$; $i, j = 1, 2, 3$. Now there exists $t \in \{t_1, t_2, t_3\}$ such that $t \not o \ell, m$. For suppose $t_1 o \ell$. Then $t_2, t_3 \not o \ell$. If $t_2 o m$, then $t_3 \not o m$.

Hence choose $t \in \Phi_A$ such that $t \not o \ell, m$. By Lemma (1.3.2), there exists B such that $B \not o S$ for each SIt and $BI\ell$. Choose C such that CIt and $C \not o A$.

<u>Claim</u>. $\{A, B, C\}$ satisfy the conditions of Lemma (1.3.1).

By choice $A \not o B \not o C \not o A$. Also by choice $\ell \not o t$ or $AB \not o AC$. We must show $BC \not o \ell, t$. Suppose $BCot$. Then $t \not o \ell$ implies AoB by (P5). Contradiction. Similarly $BCo\ell$ implies the contradiction, AoC. Hence by Lemma (1.3.1)(2), C fulfils the demands of the lemma.

<u>Definition (1.3.3)</u>. For each $\ell \in \mathcal{L}$, define $\Sigma(\ell) = \{P \mid \text{there exists } m \in \mathcal{L} \text{ such that } mo\ell \text{ and } PIm\}$. and $\mathcal{X}(\ell) = \langle \mathcal{P}(\ell), \mathcal{L}(\ell), I, \| \rangle$ where

$$\mathcal{P}(\ell) = \mathcal{P} \setminus \Sigma(\ell), \qquad \mathcal{L}(\ell) = \{m \wedge \mathcal{P}(\ell) \mid m \in \mathcal{L}\},$$

$m \wedge \mathcal{P}(\ell) \| n \wedge \mathcal{P}(\ell)$ iff there exists P such that $PI\ell, m, n$ and $PIm \wedge \mathcal{P}(\ell)$ iff PIm and $P \in \mathcal{P}(\ell)$.

<u>Remark (1.3.2)</u>. $\Sigma(\ell) = \{P \mid \text{there exists } Q \text{ such that } QoP \text{ and } QI\ell\}$.

Proof: Let $\{P \mid$ there exists Q such that QoP and $QI\ell\} = T$. From Lemma (1.3.2), it immediately follows that $\Sigma(\ell) \subseteq T$. Conversely let $P \varepsilon T$. Hence there exists Q, QoP, such that $QI\ell$. Suppose $P \notin \Sigma(\ell)$. Then for each $t \varepsilon \phi_P$, $t \not o \ell$. It follows from Lemma (1.3.1)(2), that $P \not o S$ for each $SI\ell$. Contradiction.

Lemma (1.3.4). Let $\ell \varepsilon \mathcal{L}$. Then we have

(1) If $P \varepsilon \Sigma(\ell)$ and $X \varepsilon \mathbb{P}(\ell)$, then $P \not o X$.

(2) If $mo\ell$, then $\Sigma(m) = \Sigma(\ell)$.

(3) $m \wedge \mathbb{P}(\ell) = \emptyset$ iff $mo\ell$.

Proof: (1) an immediate consequence of Remark (1.3.2).

(2) This follows immediately from Lemma (1.3.2).

(3) If ℓom, then $m \wedge \mathbb{P}(\ell) = \emptyset$ because of (2).
Conversely suppose $\ell \not o m$. Then by Lemma (1.3.2), there exists P such that PIm and $P \not o S$ for each $SI\ell$. Hence $P \varepsilon m \wedge \mathbb{P}(\ell)$.

Theorem (1.3.2). [K1] Let $\mathcal{H}$ be a projective H-plane. Then for each $\ell \varepsilon \mathcal{L}$, $\mathcal{A}(\ell)$ is an affine H-plane.

Proof: We must show $\mathcal{A}(\ell)$ satisfies (A1) to

(A8). (Ai) follows from (P_i), i = 1, 2, 3. (A3) follows from Theorem (1.3.1)(2). For the rest of the proof, let $g' = g \wedge \mathcal{P}(\ell)$.

(A4). Let $g' \wedge h' \neq \emptyset$. Put XIg', h'. Then $g' \not{o} h'$ iff $g \not{o} h$. For, clearly $g \not{o} h$ implies $g' \not{o} h'$. Conversely if $g' \not{o} h'$ there exists QIg' such that $Q \not{o} S$ for each SIh'. Thus by Lemma (1.3.4)(1), $Q \not{o} S$ for each SIh. Hence $g' \not{o} h'$ iff $g \not{o} h$ iff $g \wedge h = X$.

(A5) and (A6) follow easily from (P6) and (P5).

(A7). Let $g' \parallel h'$; PIg', j'; and $g' \not{o} j'$. From (A4), $P = g' \wedge j'$. We show $h \not{o} j$. If hoj, then $\bar{h} = \bar{j}$. Hence $P = \bar{j} \wedge \bar{g} = \bar{h} \wedge \bar{g}$. Now $g' \parallel h'$ implies there exists R such that RIg, h, ℓ. Hence $\bar{R} = \bar{g} \wedge \bar{h} = \bar{P}$ or PoR. But by Lemma (1.3.4)(1), $P \not{o} R$. Thus $h \not{o} j$ and so $h' \not{o} j'$. Let $S = h \wedge j$. We must show $S \notin \Sigma(\ell)$, and hence $h' \wedge j' \neq \emptyset$. Assume $S \in \Sigma(\ell)$. Then there exists m, $mo\ell$, such that SIm. Hence $\bar{S} = \bar{\ell} \wedge \bar{h}$. But $RI\ell, h$. Hence RoS. Since $P \not{o} R$, it follows that $PRoPS$ or jog. Contradiction.

(A8). Let $P \in \mathcal{P}(\ell)$ and $g' \in \mathcal{L}(\ell)$. Then there exists T, such that $T = g \wedge \ell$, by Lemma (1.3.4)(3). Then $P \not{o} T$. Let $m = PT$. Hence $m' \parallel g'$ and PIm'. To show m' is unique, let $t' \parallel g'$ and PIt'. Then from the properties of Lemma (1.3.4), $g \wedge m = t \wedge g = T$, and hence $m = PT = t$.

§1.4. Projectivities of Projective H-planes

In this section, we generalize a result found in P1 on page 9.

Definition (1.4.1). Let ℓ, $m \in \mathcal{L}$. ϕ^R is called a perspectivity with centre R from ℓ to m iff $k \not\phi \ell$, m for each $k \in \phi_R$ and ϕ^R: $\ell \to m$ is the mapping $\phi^R(P) = PR \wedge m$.

ϕ^R is defined since by Lemma (1.3.3), $R \not\phi X$ for each $X I \ell \vee m$. Moreover ϕ^R is clearly a bijective map whose inverse is $(\phi^R)^{-1}$: $m \to \ell$, $(\phi^R)^{-1}(Q) = QR \wedge \ell$.

Lemma (1.4.1). For any two lines ℓ, m, there exists a perspectivity ϕ^R: $\ell \to m$.

Proof: This is an immediate consequence of Lemma (1.3.3).

Lemma (1.4.2). Each perspectivity ϕ^R: $\ell \to m$ has the property $X o Y$ iff $\phi^R(X) o \phi^R(Y)$.

Proof: Let XoY. By (P5), RXoRY. From the choice of R, we have $RX \not o m$. Hence $RX \wedge m o RY \wedge m$ or $\phi^R(X) o \phi^R(Y)$. Since $(\phi^R)^{-1}$ is essentially the same as ϕ^R, structurally, we also have $\phi^R(X) o \phi^R(Y)$ which implies XoY.

Definition (1.4.2). $\phi: \ell \to m$ is called a projectivity of order n iff ϕ is a finite chain of n perspectivities $\{\phi_i\}_{i=1}^n$ where $\phi_i: \ell_{i-1} \to \ell_i$, $i = 1, \ldots, n$. $\phi: \ell \to m$ is called a projectivity iff ϕ is a projectivity of order n for some n.

$PJ(\ell) = \{\phi \mid \phi: \ell \to \ell$, is a projectivity$\}$.

$PJ(\ell/o) = \{\phi \mid \phi: \ell/o \to \ell/o$, is a projectivity$\}$

where $\ell/o = \{\bar{P} \mid \bar{P} I \bar{\ell}\}$.

Remark (1.4.1). Each projectivity has the property XoY iff $\phi(X) o \phi(Y)$.

Proof: This is an immediate consequence of Definition (1.4.2) and Lemma (1.4.2).

Theorem (1.4.1). The following are true.

(A) $PJ(\ell)$ is a group under composition, for each $\ell \in \mathcal{L}$.

(B) The map $h: PJ(\ell) \to PJ(\ell/o)$, defined by $h(\phi) = \bar{\phi}$ such that $\bar{\phi}(\bar{P}) = \overline{\phi(P)}$, is a onto group homomorphism. Moreover, $\chi_\ell o \phi = \bar{\phi} o \chi_\ell$, for each $\phi \in PJ(\ell)$.

(C) The kernel of $h = K(\ell) = \{\phi \mid \phi(P) o P$, for each $P I \ell\}$. Hence $PJ(\ell)/K(\ell) \cong PJ(\ell/o)$.

Proof: (A) $PJ(\ell)$ is clearly closed under composition and is associative. Since each perspectivity is (1 - 1) onto, so is each projectivity. Finally if $R \not I X$ for each $X I \ell$, then ϕ^R = the identity map.

(B) ϕ is well defined, since if P_1, $P_2 I \ell$, $P_1 o P_2$, then $\phi(P_1) o \phi(P_2)$, and so $\bar{\phi}(\bar{P}_1) = \bar{\phi}(\bar{P}_2)$.

Let $\phi = (\phi^{R_n} \ldots \phi^{R_1}) \varepsilon PJ(\ell)$. By induction on n, and some easy computations, it follows that $h(\phi^{R_n} \ldots \phi^{R_1}) = (\bar{\phi}^{\bar{R}_n} \ldots \phi^{\bar{R}_1})$. Hence $h(\phi) \varepsilon PJ(\ell/o)$. Since

$$h(\phi_1 \phi_2) = h(\phi^{S_n} \ldots \phi^{S_1} \phi^{R_m} \ldots \phi^{R_1})$$

$$= (\phi^{S_n} \ldots \phi^{S_1} \phi^{R_m} \ldots \phi^{R_1}),$$

by the above remark, it follows that h is a homomorphism. h is onto, since if $\phi = (\phi^{\bar{R}_n} \ldots \phi^{\bar{R}_1}) \varepsilon PJ(\ell/o)$, then $R_i \not I \ell_{i-1} \vee \ell_i$ and hence $R_i \not I X$ for each $X I \ell_i \vee \ell_{i+1}$, $i = 1, \ldots, n$. Thus $\phi^{R_i} \varepsilon PJ(\ell)$, $i = 1, \ldots, n$. It then follows that $(\phi^{R_n} \ldots \phi^{R_1}) \varepsilon PJ(\ell)$ and $h(\phi^{R_n} \ldots \phi^{R_1}) = (\phi^{R_n} \ldots \phi^{R_1})$. Finally, $(\chi_\ell \; \phi)(P) = \chi_\ell(\phi(P)) = \overline{\phi(P)} = \bar{\phi}(\bar{P}) = (\bar{\phi} \; \chi_\ell)(P)$.

(C) This follows by some easy calculations and a well known theorem from group theory.

Definition (1.4.3). Let G be a group of automorphisms on a set X. Let θ be an equivalence relation on X. G is n-ply-transitive with respect to θ iff for each pair of n-tuples, $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ of X, such that $a_i \not\theta a_j$; $b_i \not\theta b_j$; $i \neq j$; $i, j = 1, \ldots, n$, there exists $g \varepsilon G$ such that $g(a_i) = b_i$, $i = 1, \ldots, n$.

Theorem (1.4.2). Let $\ell, \ell' \varepsilon \mathcal{L}$. Let $A, B, CI\ell$ and $A', B', C'I\ell'$, such that $A\phi B\phi C\phi A$ and $A'\phi B'\phi C'\phi A'$. Then there exists a projectivity Λ of order ≤ 4 such that $\Lambda(A) = A'$, $\Lambda(B) = B'$ and $\Lambda(C) = C'$.

Proof: We consider two cases, each with three subcases.

Case (1): $\ell\phi\ell'$.

(IA): $A = A'$. This implies that $B\phi B'$ and $C\phi C'$. For suppose BoB'. Then $A'\phi B'$ implies $A'B'oA'B$. Since $A = A'$, we obtain $\ell'o\ell$. Contradiction. Similarly $C\phi C'$.

Claim (1). (a) $BB'\phi\ell,\ell'$ and $CC'\phi\ell,\ell'$.

(b) $BB'\phi CC'$.

(a) If $BB'o\ell$, then $\ell\phi\ell'$ implies that $\ell \wedge \ell' o \ell' \wedge BB'$. Hence $A'oB'$. Contradiction. The rest of (a) follows similarly.

(b) If $BB' o CC'$, then $BB' \not{o} \ell$, by (a), It follows that $BB' \wedge \ell o CC' \wedge \ell$ and so BoC. Contradiction.

In view of Claim (1)(b), we may define $S = BB'' \wedge CC'$.

Claim (2). (a) $S \not{o} B, B'$.

(b) $S \not{o} X$, for each $XI \ell \vee \ell'$.

(a) If SoB, then $B \not{o} C$ implies $BCoSC$ or $\ell o CC'$. Contradiction to Claim (1)(a). Similarly $S \not{o} B'$.

(b) Suppose there exists X, $XI\ell$, such that SoX. Since $S \not{o} B$, we have $SBoXB$ or $BB' o \ell$. Contradiction to Claim (1)(a). Similarly if there exists X, $XI\ell'$, such that SoX, then $BB' o \ell'$, which again contradicts Claim (1)(a).

The perspectivity $\phi^S: \ell \to \ell'$ then satisfies the claim of the lemma.

(IB): $A \not{o} A'$. Since $\ell \not{o} \ell'$, we have $AA' \not{o} \ell'$ or $AA' \not{o} \ell$. Without loss of generality let us assume $AA' \not{o} \ell'$. Then let $P = \ell \wedge \ell'$. Thus $A \not{o} P$, for otherwise $A \not{o} A'$ implies $AA' o A'P$ or $AA' o \ell'$. Now choose $\alpha \varepsilon \phi_A$ such that $\alpha \not{o} AA'$, AB. Hence $\alpha \not{o} A'B'$. Otherwise, $\alpha \wedge \ell o \ell \wedge \ell'$ or AoP. Contradiction. Choose $S_1 I AA'$ such that $S_1 \not{o} A, A'$.

Claim (4). $S_1 \not{o} X$, for each $XI\ell' \vee \alpha$.

If there exists $XI\ell'$ such that $S_1 o X$, then $S_1 \not{o} A'$ implies $S_1 A' o XA'$ or $AA' o \ell'$. Contradiction.

Also if $S_1 o X$, $X I \alpha$, then $AA' \not o \alpha$ implies $A o S_1$. Contradiction.

Thus we may define $\phi^{S_1}: \ell' \to \alpha$. Clearly $\phi^{S_1}(A') = A$.

Claim (5). (a) $B \not o X$ for each $X I \alpha$ and $C \not o X$ for each $X I \alpha$.

(b) $\phi^{S_1}(B') \not o B$ and $\phi^{S_1}(C') \not o C$.

(c) $B\phi^{S_1}(B') \not o C\phi^{S_1}(C')$.

(a) If there exists X, $X I \alpha$, such that $B o X$, then $\alpha \not o \ell$ implies $A o B$. Contradiction. Similarly if there exists X, $X I \alpha$ such that $C o X$, then $A o C$.

(b) This follows immediately from (a), since $\phi^{S_1}(B')$, $\phi^{S_1}(C') I \alpha$.

(c) Suppose $B\phi^{S_1}(B') o C\phi^{S_1}(C')$. Since $\ell \not o \alpha$, it follows that $B\phi^{S_1}(B') \not o \alpha$ or $B\phi^{S_1}(B') \not o \ell$. If $B\phi^{S_1}(B') \not o \alpha$, then $\phi^{S_1}(B') o \phi^{S_1}(C')$. Hence $B' o C'$ by Remark (1.4.1). Contradiction. Similarly if $B\phi^{S_1}(B') \not o \ell$, then $B o C$. Contradiction. Thus (c) is proved.

In view of Claim (5), we may define $S_2 = B\phi^{S_1}(B') \wedge C\phi^{S_1}(C')$. Let $j_1 = B\phi^{S_1}(B')$ and $m_1 = C\phi^{S_1}(C')$.

Claim (6). $S_2 \not o B, C$; $S_2 \not o \phi^{S_1}(B'), \phi^{S_1}(C')$.

If $S_2 o B$, then $B \not o C$ implies $\ell o S_2 C$ and so $\ell o m_1$. Then

$\alpha\not o \ell$, implies $\ell \wedge \alpha o m_1 \wedge \alpha$. Thus $A o \phi^{S_1}(C')$. Since $(\phi^{S_1})^{-1}(A) = A'$ and $(\phi^{S_1})^{-1}\phi^{S_1}(C') = C'$ we have $A'oC'$ by Remark (1.4.1). Contradiction. Similarly $S_2 \not o C$. Now suppose $S_2 o \phi^{S_1}(C')$. Since $B'\not o C'$, $\phi^{S_1}(B')\not o \phi^{S_1}(C')$ and so $S_2\phi^{S_1}(C') o \alpha$. Thus $m_1 o \alpha$. Then $\ell \not o \alpha$ implies that $\ell \wedge \alpha o \ell \wedge m$, and so AoC. Contradiction. Similarly $S_2 o \phi^{S_1}(C')$ implies the contradiction AoB.

<u>Claim (7)</u>. (a) $j_1 \not o m_1$.

(b) $S_2 \not o X$ for each $XI\ell \vee \alpha$.

(a) If $j_1 o m_1$, then since $\ell \not o \alpha$, j_1, $m_1 \not o \ell$ or j_1, $m_1 \not o \alpha$. If j_1, $m_1 \not o\ \alpha$, then BoC. If j_1, $m_1 \not o \alpha$, then $\phi^{S_1}(B') o \phi^{S_1}(C')$. In both cases we obtain a contradiction, since $B\not o C$, and $\phi^{S_1}(B')\not o \phi^{S_1}(C')$ by Claim (6).

(b) If there exists X, $XI\ell$, such that $S_2 o X$, then $S_2 \not o B$ by Claim (6), implies $S_2 B o B X$ or $j_1 o \ell_1$. By (a), $j_1 \not o m_1$ and hence $j_1 \wedge m_1 o \ell \wedge m_1$. Thus $S_2 o C$. Contradiction. If there exists X, $XI\alpha$, such that $S_2 o X$, then since $S_2 \not o \phi^{S_1}(B')$, by Claim (6), we have $S_2\phi^{S_1}(B') o Y \phi^{S_1}(B')$. Hence $j_1 o \alpha$. Since $j_1 \not o m_1$, it follows that $j_1 \wedge m o \alpha \wedge m_1$ and so $S_2 o \phi^{S_1}(C')$. Contradiction.

Thus, in view of Claim (7), we may define the perspectivity $\phi^{S_2}: \ell \to \alpha$. It easily follows that $\phi^{S_2}(A) = A$; $\phi^{S_2}(B) = \phi^{S_1}(B')$; and $\phi^{S_2}(C) = \phi^{S_1}(C')$.

Now, finally define $\Lambda = (\phi^{S_1})^{-1} o \phi^{S_2}: \ell \to \ell'$. Hence $\Lambda(A) = (\phi^{S_1})^{-1}(A) = A'$; $\Lambda(B) = (\phi^{S_1})^{-1}(\phi^{S_1}(B)) = B'$ and $\Lambda(C) = (\phi^{S_1})^{-1}(\phi^{S_1}(C')) = C'$.

<u>(IC)</u>: <u>AoA' but $A \neq A'$</u>. Choose $\alpha \varepsilon \phi_B$ such that $\alpha \not o \ell, \ell'$. Let A'', $C'' I \alpha$ such that $B \not o A''$, C'' and $A'' \not o C''$. Apply (IA) to (B, A, C) and (B, A'', C''). Hence there exists Λ_1: $\ell \to \alpha$ such that $\Lambda_1(B) = B$, $\Lambda_1(A) = A''$ and $\Lambda_1(C) = C''$. Now $A'' \not o A'$; for if $A' o A''$, then AoA' implies AoA''. But $A'' \not o B$, and so $A''BoAB$. Thus $\ell o \alpha$. Contradiction. Thus we may apply (IB) to (A', B', C') and (A'', B, C''), to obtain Λ_2: $\alpha \to \ell'$ such that $\Lambda_2(A'') = A'$, $\Lambda_2(B) = B'$ and $\Lambda_2(C'') = C'$. Hence $\Lambda = \Lambda_2 o \Lambda_1$ is our desired projectivity.

<u>Case (II)</u>: <u>$\ell o \ell'$</u>. Choose $\alpha \varepsilon \phi_A$ such that $\alpha \not o \ell$.

<u>(IIA)</u>: <u>$A = A'$</u>.

Choose T, such that $T \not o X$, for each $X I \ell \vee \alpha$ by Lemma (1.3.3). Then define $B'' = \alpha \wedge TB$ and $C'' = \alpha \wedge TC$, which exist by the choice of T. Apply (IA) to (A, B, C) and (A, B'', C'') to obtain Λ_1: $\ell \to \alpha$ such that $\Lambda_1(A) = A$, $\Lambda_1(B) = B''$ and $\Lambda_1(C) = C''$. We must show $A \not o B'' \not o C'' \not o A$, however, to use (IA). Now since $B \not o C$, then $\Lambda_1(B) \not o \Lambda_2(C)$ and so $B'' \not o C''$. Also $A \not o B$, C, implies $\Lambda_1(A) \not o \Lambda_1(B)$, $\Lambda_1(c)$, and so $A \not o$, B'', C''. Again by (IA), there exists Λ_2: $\alpha \to \ell'$ such that $\Lambda_2(A) = A$, $\Lambda_2(B'') = B'$ and $\Lambda_2(C'') = C'$. Thus $\Lambda = \Lambda_2 . \Lambda_1$ is our desired projectivity.

<u>(IIB)</u>. <u>$A \not o A'$</u>. Choose A'' such that $A'' \not o X$ for each $X I \ell$. Define $\alpha = BA''$. By the choice of A'', $A'' \not o A$ and $\alpha \not o \ell$. Choose $C'' I \alpha$ such that $C'' \not o A''$, B. Apply (IA) to (A, B, C) and (A'', B, C'') to obtain Λ_1: $\ell \to \alpha$ such that $\Lambda_1(A) = A''$, $\Lambda_1(B) = B$, and $\Lambda_1(C) = C''$. Now since $\ell o \ell'$, $A'' \not o X$ for each $X I \ell'$. Thus in particular $A'' \not o A'$. Also $\alpha \not o \ell'$. Thus by (IB) there exists Λ_2: $\alpha \to \ell'$ such that $\Lambda_2(A'') = A'$, $\Lambda_2(B) = B'$ and $\Lambda_2(C'') = C'$. Hence $\Lambda = \Lambda_2 o \Lambda_1$ is our desired projectivity.

<u>(IIC)</u>. <u>$A o A'$ but $A \neq A'$</u>. We choose A'', α and C'' as in (IIB). Then we use (IA) and (IC) to obtain our desired projectivity as in (IIB).

<u>Corollary</u>. $PJ(\ell)$ <u>is triply-transitive with respect to</u> $o_{\mathfrak{P}}$, <u>for each</u> $\ell \varepsilon \mathcal{L}$.

CHAPTER 2

Algebraic Prerequisites

In this section we list, for convenience, algebraic results we will quote later. We shall give proofs only when the result is new.

2.1. Monoids.

Definition (2.1.1). (a) A pair $(M,\cdot)$ is called a monoid iff M is a set, $\cdot$ is an associative binary operation and there exists $1 \varepsilon M$ such that $x \cdot 1 = 1 \cdot x = x$ for each $x \varepsilon M$. 1 is called the unit of M, and is uniquely determined by this property. We write xy for $x.y$.

(b) y is a zero of a monoid M iff $xy = yx = y$ for each $x \varepsilon M$.

If M has a zero, it is clearly unique.

(c) S is a submonoid of M iff $S \cdot S \subseteq S$ and $1 \varepsilon S$.

(d) S is a right (left) ideal of M iff $SM \subseteq S$ $(MS \subseteq S)$. S is called an ideal iff S is both a left and right ideal. S is called a proper ideal iff $S \neq M$, or equivalently $1 \not\varepsilon S$.

(e) S is called a *maximal left (right) ideal or a maximal ideal* iff (i) S is proper.

(ii) If I is a left (right) ideal or an ideal such that $S \subseteq I \subseteq M$, then $I = S$ or $I = M$.

(f) If M is a monoid, $M^* = \{m \mid \text{there exists } s \in M \text{ such that } sm = ms = 1\}$ is called the *set of units*. Clearly for each $m \in M$ there exists at most one s such that $sm = ms = 1$. If $ms = sm = 1$, we write $s = m^{-1}$.

M^* is clearly a group.

Lemma (2.1.1). Let M be a monoid and $\{I_\alpha\}_{\alpha \in I}$ a family of left (right) ideals or ideals. Then $\bigcup_\alpha J_\alpha$ and $\bigcap_\alpha J_\alpha$ are both left (right) ideals or ideals.

Lemma (2.1.2). Let M be a monoid. Then every proper left (right) ideal or ideal is contained in a maximal left (right) ideal or ideal.

Definition (2.1.2). M is called a *local monoid* iff M has a unique maximal ideal.

The notion does not appear in the literature, I believe, but it parallels the concept of a local ring.

Lemma (2.1.3). Let M be a monoid and $\mathfrak{m} = \complement M^*$. If $\mathfrak{m}$ is an ideal, then M is a local monoid and $\mathfrak{m}$ is

<u>its unique maximal ideal.</u>

<u>Proof</u>: Since $\mathfrak{M}$ is an ideal and $1 \notin \mathfrak{M}$, there exists a maximal ideal J such that $\mathfrak{M} \subseteq J$. Hence maximal ideals exist.

<u>Claim</u>. If J is any maximal ideal, then $J = \mathfrak{M}$. We show $J \subseteq \mathfrak{M}$. If this is false there exists x, $x \in J \setminus \mathfrak{M}$. Hence $x \in M^*$ and so $1 = xx^{-1} \in JM \subseteq J$. Contradiction. Thus $J \subsetneq \mathfrak{M} \subseteq M$ and so $J = \mathfrak{M}$.

<u>Definition (2.1.3)</u>. Let M and L be monoids $f\colon M \to L$ is a <u>monoid homomorphism</u> iff

(i) $f(m_1, m_2) = f(m_1)f(m_2)$ for each $m_1, m_2 \in M$.

(ii) $f(1) = 1$.

<u>Lemma (2.1.4)</u>. <u>Let</u> $f\colon M \to L$ <u>be a monoid homomorphism</u>. <u>Then</u> (1) $f[M^*] \subseteq L^*$. <u>In fact</u> $f(m^{-1}) = f(m)^{-1}$.

(2) <u>If</u> S <u>is a submonoid of</u> M_*, <u>then</u> $f[S]$ <u>is a submonoid of</u> L.

(3) $\operatorname{Ker} f = \{m \mid f(m) = 1\}$ <u>is a submonoid of</u> M.

(4) $f|M^*\colon M^* \to L^*$ <u>is a group homomorphism</u>.

We next introduce the concept of an n-ary algebra, which is just a special universal algebra. The next result on universal algebras in general can be found in Grätzer's book, [G1].

Definition (2.1.4). The pair (K, T) is called an n-ary-algebra iff K is a set and T is a map from K^n into K. S is a sub-algebra iff $T[S^n] \subseteq S$.

Clearly a monoid is a 2-ary-algebra.

Definition (2.1.5). Let $\mathcal{A} = (K, T)$ be an n-ary-algebra. $\theta \subseteq K \times K$ is a congruence on $\mathcal{A}$ iff:

(i) θ is an equivalence relation;

(ii) If $a_i \, \theta \, b_i$, $i = 1, \ldots, n$, then $T(a_1, \ldots, a_n) \, \theta \, T(b_1, \ldots, b_n)$. Let $[x] = \{y \mid (x, y) \varepsilon \, \theta\}$.

Lemma (2.1.5). *Let θ be a congruence on an n-ary-algebra $\mathcal{A} = (K, T)$. Then $\mathcal{A}/\theta = (K/\theta, T_\theta)$ is also an n-ary-algebra with operations defined by*

$$T_\theta([a_1], \ldots, [a_n]) = [T(a_1 \ldots a_n)] .$$

If $\mathcal{A}$ is a monoid, then the unit of $\mathcal{A}/\theta$ is [1].

Definition. f: $(K, T) \to (L, H)$ is an n-ary-algebra homomorphism iff $f(T(a_1, \ldots, a_n)) = H(f(a_1), \ldots, f(a_n))$.

Lemma (2.1.6). *Let $f: \mathcal{A} = (K, T) \to \mathcal{L} = (L, H)$*

be an n-ary-algebra homomorphism. Define a $\theta_f b$ iff $f(a) = f(b)$. Then θ_f is a congruence and $\mathcal{A}/\theta_f \cong f[\mathcal{A}]$.

Corollary. If f: $M \to L$ is a monoid homomorphism, then $M^*/\mathrm{Ker}(f) \cap M^* \cong f[M^*]$.

Proof: This follows from the lemma and Lemma (2.1.4)(4).

Terminology: We give the definition of a universal algebra from [G1] for later use.

A universal algebraic type is a family $\lambda = (\lambda_\alpha)_{\alpha \in I}$ of ordinal numbers. An algebra of type λ is a pair (A, f), where A is a set and $f = (f_\alpha)_{\alpha \in I}$ is a family of maps f_α: $A \to A$.

Each f_α is called a λ_α-ary-operation. If $\lambda_\alpha = 0$, then $A^0 = \{\emptyset\}$ and usually one writes $a = f_0(\emptyset)$ for f_0. For example, in a group G, the unit, e, is a 0-ary operation.

2.2. Local Rings

The following results are taken from Lambek, [LO]. Throughout this section L is an associative ring with $0 \neq 1$.

Definition (2.2.1). Let L be an associative ring such that $0 \neq 1$.

(a) $h \in L$ is a left (right) sided zero divisor iff there exists $m \in L$, $m \neq 0$, such that $mh = 0$ $(hm = 0)$.

(b) $h \in L$ is left (right) invertible iff there exists $m \in L$ such that $mh = 1$ $(hm = 1)$ of L.
U_- (U_+) is the set of left (right) invertible elements. h is called a unit of L iff h is both right and left invertible. U is the set of units.

(c) I is a left (right) ideal of L iff

(i) $I + I \subseteq I$.

(ii) $LI \subseteq I$ $(IL \subseteq I)$.

I is an ideal iff it is both a right and left ideal. I is a proper left (right) ideal or ideal iff $I \neq L$; i.e., iff $1 \notin I$.

(d) I is a maximal left (right) ideal iff

(i) I is a proper ideal;

(ii) If J is a left (right) ideal or ideal such that $I \subsetneq J \subseteq L$, then $J = I$ or $J = L$.

<u>Notation</u>: D_+ = set of right-sided zero divisors.

D_- = set of left-sided zero divisors.

$D_0 = D_+ \cap D_-$.

$\mathcal{N}_+$ = set of non-right invertible elements.

$\mathcal{N}_-$ = set of non-left invertible elements.

$\mathcal{N}$ = set of non-units.

<u>Lemma (2.2.1). The following are true.</u>

(1) $U = U_+ \cap U_-$ <u>and</u> $\mathcal{N} = \mathcal{N}_+ \cup \mathcal{N}_-$.

(2) <u>For every proper left (right) ideal</u> I, $I \subseteq \mathcal{N}_-$ $(I \subseteq \mathcal{N}_+)$.
<u>Hence for every proper ideal</u> I, $I \subseteq \mathcal{N}$.

(3) $D_- \subseteq \mathcal{N}_+$; $D_+ \subseteq \mathcal{N}_-$ <u>and</u> $D_0 \subseteq \mathcal{N}$.

<u>Definition (2.2.2)</u>. $J(L) = \bigcap R$, (R is a maximal right ideal), is called the <u>Jacobson radical of</u> L.

<u>Theorem (2.2.1). The following are true.</u>

(1) $J(L) = \{r \mid 1 - rs \notin \mathcal{N}_+$ <u>for each</u> $s \in L\}$.

(2) J(L) <u>is a proper ideal</u>.

(3) J(L) <u>is the largest ideal</u> K <u>such that for each</u> $r \in K$, $1 - r$ <u>is a unit</u>.

(4) $J(L) = \bigcap M$, (M <u>is a maximal left ideal</u>)
$= \{r \mid 1 - sr \notin \mathcal{N}_-$ <u>for each</u> $s \in L\}$.

The next theorem is stated, but not proved, in Lambek's book. We shall exhibit a proof here and use it to derive some additional results; cf. [LO], p. 75 .

Theorem (2.2.2). The following are equivalent.

(1) $L/J(L)$ *is a division ring.*

(2) L *has a unique maximal right ideal.*

(3) *There exists a proper ideal* I *such that* $\mathfrak{N} \subseteq I$.

(4) $\mathfrak{N}$ *is a proper ideal.*

(5) *For each* $h \in L$, *either* $h \notin \mathfrak{N}$ *or* $1 - h \notin \mathfrak{N}$.

(6) *For each* $h \in L$, *either* $h \notin \mathfrak{N}_+$ *or* $1 - h \notin \mathfrak{N}_+$.

Proof. (1)$\Longrightarrow$(2). We show for every maximal right ideal, R, $R = J(L)$. If this is false, there exists a maximal right ideal R such that $J(L) \subsetneq R$. Hence we may choose $x \in R \setminus J(L)$. Thus $x + J(L) \neq J(L)$. Since $L/J(L)$ is a division ring, there exists y such that $xy + J(L) = 1 + J(L)$. Thus $1 - xy \in J(L) \subseteq R$. But $xy \in RL \subseteq R$ and so $1 \in R$. Contradiction.

(2)$\Longrightarrow$(3). Let R be the unique maximal right ideal. Hence $R = J(L)$.

Claim. R is a maximal left ideal. Let I be a left ideal such that $R \subsetneq I \subseteq H$. Then there exists $x \in I \setminus R = I \setminus J(L)$. Hence by Theorem (2.2.1)(4), there exists $y \in L$ such that $1 - yx \in \mathfrak{N}_-$.

Claim. $1 - yx \in \mathfrak{M}_+$.

If this is false, then $1 - yx \notin \mathfrak{M}_+$. Hence there exists $u \in L$ such that $(1 - yx)u = 1$ or $1 - u = (-yx)u$. We now claim that $u \notin \mathfrak{M}_+$. If $u \in \mathfrak{M}_+$, then uL is a proper right ideal. Hence $u \in uH \subseteq R$. Thus $1 - u = (-yx)u \in LR \subseteq R$. This implies $1 \in R$. Contradiction. Therefore there exists $v \in L$ such that $uv = 1$. It follows that

$$
\begin{aligned}
v = 1 \cdot v &= ((1 - yx)u)v = (1 - yx)(uv) \\
&= 1 - yx.
\end{aligned}
$$

Thus $u(1 - yx) = uv = 1$ which implies $1 - yx \in \mathfrak{M}_-$. Contradiction.

From the claim it follows that $(1 - yx)L$ is a proper right ideal and hence $(1 - yx) \in (1 - yx)L \subseteq R \subseteq I$. But $yx \in LI \subseteq I$. Hence $1 \in I$ and so $I = L$. Hence R is a unique maximal left ideal also.

Next we show $\mathfrak{M} \subseteq J(L)$. Since $J(L)$ is a proper ideal by Theorem (2.2.1)(2), our result will be proved. Let $x \in \mathfrak{M}$. Then $x \in \mathfrak{M}_+$ or $x \in \mathfrak{M}_-$ Hence xH is a proper right ideal and $xH \subseteq J(L)$ or Hx is a proper left ideal and $Hx \subseteq J(L)$. In both cases, $x \in J(H)$.

<u>(3) $\Rightarrow$ (4)</u>. Let I be a proper ideal such that $\mathfrak{M} \subseteq I$. If $\mathfrak{M} \subsetneq I$, then there exists $x \varepsilon I \setminus \mathfrak{M}$. Hence $x \varepsilon U$ and so $1 = xx^{-1} \varepsilon IH \subseteq I$. Contradiction.

<u>(4) $\Rightarrow$ (5)</u>. Let $\mathfrak{M}$ be a proper ideal. Suppose there exists $h \varepsilon L$ such that $h \varepsilon \mathfrak{M}$ and $1 - h \varepsilon \mathfrak{M}$. Then $1 \varepsilon \mathfrak{M}$. Contradiction.

<u>(5) $\Rightarrow$ (6)</u>. This follows easily since $\mathfrak{M}_+ \subseteq \mathfrak{M}$.

<u>(6) $\Rightarrow$ (1).</u> Assume condition (6).

<u>Claim.</u> (i) $\mathfrak{M}_+ \subseteq J(L)$ and so $J(H) = \mathfrak{M}_+$.

(ii) $\mathfrak{M}_- \subseteq J(L)$ and so $\mathfrak{M}_- = J(L)$.

Let $x \varepsilon \mathfrak{M}_+$. Then $xy \varepsilon \mathfrak{M}_+$ for each $y \varepsilon L$. By (6), $1 - xy \notin \mathfrak{M}_+$ for each $y \varepsilon L$. Hence $x \varepsilon J(L)$ by Theorem (2.2.1)(1). To show $\mathfrak{M}_- \subseteq J(L)$ it is enough, in view of Theorem (2.2.1)(4), to show for each $h \varepsilon L$, $h \notin \mathfrak{M}_-$ or $1 - h \notin \mathfrak{M}_-$.

Let $h \varepsilon L$. By (6), $h \notin \mathfrak{M}_+$ or $1 - h \notin \mathfrak{M}_+$. Suppose $1 - h \notin \mathfrak{M}_+$. Then there exists $u \varepsilon L$ such that $(1 - h)u = 1$. Hence $1 - u = -hu$. We now show $u \notin \mathfrak{M}_+$. If $u \varepsilon \mathfrak{M}_+$, then $u \varepsilon J(L)$ by (i). Hence $-hu = 1 - u \varepsilon J(L)$ and so $1 \varepsilon J(L)$. Contradiction. Hence there exists v such that $uv = 1$. Thus $v = 1\,v = (1 - h)u\,v = (1 - h)(uv) = 1 - h$. Therefore $u(1 - h) = uv = 1$ and so $1 - h \notin \mathfrak{M}_-$. Similarly $h \notin \mathfrak{M}_+$ implies $h \notin \mathfrak{M}_-$. It follows that $h \notin \mathfrak{M}_-$

or $1 - h \notin \mathfrak{N}_-$. Now we show (1). Let $h + J(L) \neq J(L)$, and so $h \notin J(H)$. By Theorem (2.2.1) parts (1) and (4), and the above claims, there exists y_1 such that $1 - xy_1 \in \mathfrak{N}_+ = J(L)$ and there exists y_2 such that $1 - y_2x \in \mathfrak{N}_- = J(L)$. Hence $xy_1 + J(L) = 1 + J(L) = y_2x + J(L)$. Therefore $L/J(L)$ is a division ring.

Definition (2.2.3). L is called a local ring iff one of the conditions of Theorem (2.2.2) is satisfied.

The next theorem is a consequence of Theorem (2.2.2). Since it is not explicitly stated in [10], we prove it here.

Theorem (2.2.3). The following statements are true.

(1) L is local iff $\mathfrak{N}$ is a unique maximal ideal.

(2) If L is local, $\mathfrak{N}_+ = \mathfrak{N}_- = \mathfrak{N} = J(L)$.

(3) If L is local, $L/\mathfrak{N}$ is a division ring.

Proof: (1) If $\mathfrak{N}$ is a unique maximal ideal, then L is local by condition (4) of Theorem (2.2.2). Now let L be a local. By condition (4) of Theorem (2.2.2), $\mathfrak{N}$ is a proper ideal. Let $\mathfrak{M}$ be any maximal ideal. By Lemma (2.2.1)(2), $\mathfrak{M} \subseteq \mathfrak{N}$. Since $\mathfrak{N}$ is a proper ideal and $\mathfrak{M}$ is maximal, $\mathfrak{M} = \mathfrak{N}$. Thus $\mathfrak{N}$ is a unique maximal ideal.

(2) Let L be local. From (6)$\Longrightarrow$(1), of

Theorem (2.2.2), $\mathcal{N}_+ = \mathcal{N}_- = J(L)$. Thus since $\mathcal{N} = \mathcal{N}_+ \cup \mathcal{N}_-$, we have our desired result.

(3) This follows from (2) and condition (1) of Theorem (2.2.2).

CHAPTER 3

Dilatations of Affine H-Planes

In this section we quote some results from [L1], as well as adding some new ones. We will include proofs of theorems from [L1] only when the proof as well as the statement of the theorem is to be used later in this thesis, or when we have a new or improved proof.

Notation: Λ will denote a pencil henceforth.

3.1. Dilatations

Definition (3.1.1). Let $\mathcal{H} = \langle \mathbb{P}, \mathcal{L}, I, \| \rangle$ be an affine H-plane. A function $\sigma : \mathbb{P} \to \mathbb{P}$ is called a dilatation iff for each $g \in \mathcal{L}$, if $P, Q I g$, then $P^{\sigma} I L(Q^{\sigma}, g)$, where for any $X \in \mathbb{P}$, X^{σ} is the image of X under σ.

Definition (3.1.2). Let $\mathcal{H}$ be an affine H-plane. Then $i : \mathbb{P} \to \mathbb{P}$ is the identity map. For each $P \in \mathbb{P}$, $0_P : \mathbb{P} \to \mathbb{P}$ is the map $Q^{0_P} = P$ for each $Q \in \mathbb{P}$.

Clearly i and 0_p, for any p, are dilatations. Throughout this chapter we are dealing exclusively with an affine H-plane $\mathcal{H}$. We next prove an elementary lemma, which was never explicitly stated in [L1], even though it was used there.

Lemma (3.1.1). The following are true.

(1) If σ is a dilatation and $S_1 o S_2$, then $S_1^\sigma o S_2^\sigma$.

(2) If σ is a dilatation with an inverse σ^{-1}, then σ^{-1} is also a dilatation.

Proof: (1) Let $S_1 o S_2$. Then there exist g, h, $g \neq h$, such that $S_1, S_2 I g, h$. Since σ is a dilatation, $S_1^\sigma \; I L(S_2^\sigma, g), L(S_2^\sigma, h)$. Let $\tilde{g} = L(S_2^\sigma, g)$ and $\tilde{h} = L(S_2^\sigma, h)$. Then $\tilde{g} \neq \tilde{h}$, otherwise $g \| h$ and hence $g \wedge h = \emptyset$. Contradiction. Thus $S_1^\sigma, S_2^\sigma \; I g, h$ and so $S_1^\sigma o S_2^\sigma$.

(2) Let $Q_1, Q_2 I g$. We must show $Q_1^{\sigma^{-1}} \; I \; (Q_2^{\sigma^{-1}}, g)$.

Case (1): $Q_1 \not{o} Q_2$. Then since σ is surjective there exist P_1, P_2 such that $P_i^\sigma = Q_i$; $i = 1, 2$. Also $P_1 \not{o} P_2$ by (1). Since σ is a dilatation, $P_1 P_2 \| Q_1 Q_2$ and so $P_1 = Q_1^{\sigma^{-1}} \; I L(Q_2^{\sigma^{-1}}, g)$.

Case (2): $Q_1 o Q_2$. Choose $Q_3 I g$ such that $Q_3 \not{o} Q_i$; $i = 1, 2$. Then there exist, as in Case (1), P_1, P_2, P_3 such that $P_3 \not{o} P_1, P_2$ and $P_i^\sigma = Q_i$; $i = 1, 2, 3$. Thus by Case (1), $P_1 P_3 \| Q_1 Q_3$ and $P_2 P_3 \| Q_2 Q_3$. But

$Q_1Q_3 = Q_2Q_3$. Hence $P_1P_3 \parallel P_2P_3$ and so $P_1P_3 = P_2P_3$.

Thus $L(Q_2^{\sigma^{-1}}, g) = L(P_2, g) = L(P_2, Q_2Q_3) = P_2P_3 = P_1P_3$ and so $\{Q_1^{\sigma^{-1}} = P_1\} I L(Q_2^{\sigma^{-1}}, g)$.

Remark (3.1.1). Assume $\ell = PQ$ and $S \not o X$ for each $X I \ell$. Then $SP \not o SQ$.

Proof: If $SPoSQ$, then since $\ell \not o SP$, by the choice of S, it follows that PoQ by (A6). Contradiction.

Remark (3.1.2). Assume $\ell = PQ$; $S \not o X$ for each $X I \ell$; and RoM for some $M I \ell$, where R is any point. Then $R \not o X$ for each $X I PS$ or $R \not o X$ for each $X I QS$.

Proof. Assume our claim is false. Then there exists X, $X I PS$, such that XoR and there exists Y, $Y I QS$ such that YoR. Thus XoY. Since $PS \not o QS$ by Remark (3.1.1), it follows that So X, Y by (A6). Since RoX, it follows that RoS. But RoM. Hence SoM for $M I \ell$. Contradiction.

The next theorem determines in its proof the structure of a dilatation.

Theorem (3.1.1). [L1]. Every dilation σ is uniquely determined by its action on any two points P, Q such that $P \not o Q$.

Proof: Let R be any point such that $R \neq P, Q$. Let $\ell = PQ$.

Case (1): $R\emptyset X$ for each $XI\ell$. Hence $R\emptyset P, Q$. Define $g = PR$ and $h = QR$. By Remark (3.1.1), $g\emptyset h$. From Lemma (1.1.14), $\Lambda_g \emptyset \Lambda_h$. Now define $g = L(P^\sigma, g)$ and $h = L(Q^\sigma, h)$. Hence $\Lambda_g \emptyset \Lambda_h$ and so there exists S such that $S = g \wedge h$. Since σ is a dilatation, $R^\sigma I\ \ g, h$. Thus $R^\sigma = S$.

Case (2): There exists M such that RoM. Choose S such that $S\emptyset X$ for every $XI\ell$, by Lemma (1.1.6). From Remark (3.1.1) and Remark (3.1.2), $R\emptyset X$ for each XIPS or $R\emptyset X$ for each XIQS. Then we may apply Case (1) to P and S or Q and S to find T such that $T^\sigma = R$.

Theorem (3.1.2) [L1]. Let σ be a dilatation and $P\emptyset Q$.

(1) If $P^\sigma \emptyset Q^\sigma$, then σ is bijective.

(2) If $P^\sigma o Q^\sigma$, then $P^\sigma o S^\sigma$ for every S.

Proof: We shall not prove (1). We will deal with (2) as it is here Lemma (3.1.1) was used, even though it was not mentioned in [L1]. If (2) is false, then there exists S such that $P^\sigma \emptyset S^\sigma$. By Lemma (3.1.1)(1), $P\emptyset S$. Thus by (1) and Lemma (3.1.1)(2), σ has an inverse σ^{-1}, which is a dilatation. Then $P^{\sigma} o Q^{\sigma}$ implies by Lemma (3.1.1)(1) that $\{P = (P^{\sigma})^{\sigma^{-1}}\} o \{(Q^{\sigma})^{\sigma^{-1}} = Q\}$. Contradiction.

From the above theorem we prove the following additional corollaries.

Corollary (1). Let σ be a dilatation such that $P \not o Q$. *The following are then equivalent.*

(1) $P^{\sigma} \not o Q^{\sigma}$.

(2) σ *is bijective.*

(3) σ *is surjective.*

Proof: *(1) ⟹ (2).* This is the theorem, part (1).

(2) ⟹ (3). Obvious.

(3) ⟹ (1). Suppose $P^{\sigma} o Q^{\sigma}$. By (2) of the theorem, $P^{\sigma} o S^{\sigma}$ for each S. Choose R, $R \not o P^{\sigma}$. Since σ is onto, there exists S such that $S^{\sigma} = R$. Thus $S^{\sigma} \not o P^{\sigma}$. Contradiction.

Corollary (2). If σ is not surjective then $S_1^{\sigma} o S_2^{\sigma}$ for each S_1, S_2.

Proof: This follows from Corollary (1) and part (2) of the theorem.

Definition (3.1.3). Let σ be a dilatation. Then

(1) g is called a *trace of* σ iff $g^{\sigma} \subseteq g$.

(2) σ is called degenerate iff σ is not surjective.

Otherwise σ is called non-degenerate.

Remark (3.1.3). [L1]. Let σ be a dilatation. g is a trace of σ iff there exists P, PIg, such that $P^{\sigma}Ig$.

Theorem (3.1.3). [L1]. Let σ be a dilatation. Then

(1) If g and h are traces of σ, and $P = g \wedge h$, then P is a fixed point of σ.

(2) If each line of $\mathfrak{A}$ is a trace of σ, then $\sigma = i$.

(3) P is a fixed point of σ iff all lines through P are traces of σ.

Lemma (3.1.2). Let σ, a dilatation, have no fixed points. If g and h are traces of σ, then $\Lambda_g o \Lambda_h$.

Proof: Suppose $\Lambda_g \not{o} \Lambda_h$. Then by Lemma (1.1.14) there exists $X = g \wedge h$. Hence by Theorem (3.1.3)(1), X is a fixed point. Contradiction.

Theorem (3.1.4). [L1]. If a dilatation σ has no fixed points, then σ is non-degenerate.

Proof: Assume σ is degenerate. By Corollary (2) of Theorem (3.1.2), $S_1^{\sigma} o S_2^{\sigma}$ for all pairs S_1, S_2.

Choose P and Q such that $P^\sigma \not o Q$. Hence $P^\sigma o Q^\sigma$ and so $Q^\sigma \not o Q$. Let $g = QQ^\sigma$. By Lemma (1.1.12), there exists h, $h \not o g$, such that $Q^\sigma Ih$. Thus $\Lambda_g \not o \Lambda_h$. Now choose R, RIh such that $R \not o P^\sigma$, by Lemma (1.1.9). Since $R^\sigma o P^\sigma$ we have $R \not o R^\sigma$. Let $j = RR^\sigma$. Next we show hoj. For if this were false, then $h \not o j$ and $R^\sigma o Q^\sigma$ would imply RoR^σ by (A5). Contradiction. Thus by Lemma (1.1.13), $\Lambda_h o \Lambda_j$. Also by Lemma (3.1.2), $\Lambda_g o \Lambda_j$. Hence $\Lambda_g o \Lambda_h$. Contradiction.

<u>Theorem (3.1.5)</u>. <u>If</u> σ <u>is degenerate with a fixed point</u> P, <u>then there exists</u> P_1, $P_1 \neq P$, $P_1 oP$ <u>such that</u> $P_1^\sigma = P$.

<u>Proof</u>: Without loss of generality we may assume $\sigma \neq o_P$. Now choose Q such that $Q \not o P$. By Corollary (2) of Theorem (3.1.2), PoS^σ for each S. Choose X such that $X \not o T$ for each TIPQ by Lemma (1.1.6). Define $g_1 = PQ$ and $h_1 = PX$. Then $X^\sigma IPX$ by Theorem (3.1.3)(3). By the choice of X, $g_1 \not o h_1$. Since PIg_1, h_1, it follows by Lemma (1.1.14) that $\Lambda_{g_1} \not o \Lambda_{h_1}$. Since PoQ^σ, there exist g_1, g_2; $g_1 \neq g_2$, $g_1 o g_2$ such that P, $Q^\sigma Ig_1, g_2$. Similarly PoX implies the existence of h_1, h_2; $h_1 \neq h_2$, $h_1 o h_2$ such that P, $X^\sigma Ih_1, h_2$. Hence $\Lambda_{h_1} o \Lambda_{h_2}$ and $\Lambda_{g_1} o \Lambda_{g_2}$ by Lemma (1.1.13). Thus $\Lambda_{h_2} \not o \Lambda_{g_2}$. Define $h = L(X, h_2)$

and $g = L(Q, g_2)$. Hence $\Lambda_h \not\circ \Lambda_g$. Thus there exists P_1, $P_1 = h \wedge g$. $P \neq P_1$ otherwise $h_2 = h_1 = h$. Contradiction. Moreover $P_1^\sigma I L(X^\sigma, h), L(Q^\sigma, g)$. Hence $L(X^\sigma, h) = h_2$ and $L(Q^\sigma, g) = g_2$. Thus $P_1^\sigma = h_2 \wedge g_2 = P$. Finally $P_1 \circ P$. For if this were not so, then $\sigma = O_P$ by Theorem (3.1.1). Contradiction.

<u>Notation</u>. D = the set of dilatations.

M = set of degenerate dilatations.

$D_P = \{\sigma | \sigma \varepsilon D$ such that $P^\sigma = P\}$.

$M_P = D_P \cap M$.

If σ_1, $\sigma_2 \epsilon D$, we write for their composition $P^{\sigma_1 \sigma_2} = (P^{\sigma_2})^{\sigma_1}$.

<u>Theorem (3.1.6)</u>. <u>The following are true</u>.

(1) D <u>is a local monoid with</u> M <u>as its unique maximal ideal, under functional composition</u>. $M = \bigcup_P M_P$.

(2) D_P <u>is a local monoid with a zero element</u> o_P, <u>and</u> M_P <u>its unique maximal ideal</u>.

(3) $M_P = \{\sigma | \sigma \varepsilon D_P$ <u>such that</u> $P o Q^\sigma$ <u>for all</u> $Q\}$.

(4) <u>If</u> $P \not\circ Q$, then $D_P \cap D_Q = \{i\}$.

<u>Proof</u>: (1) Let σ_1, $\sigma_2 \varepsilon D$. Take P, QIg. Then $\sigma_2 \varepsilon D$ implies $P^{\sigma_2} IL(Q^{\sigma_2}, g)$. Let $g_2 = L(Q^{\sigma_2}, g)$. Then $\sigma_1 \varepsilon D$ implies

$$P^{\sigma_1\sigma_2} = (P^{\sigma_2})^{\sigma_1} I\{L(Q^{\sigma_2\sigma_1}, g_2) = L(Q^{\sigma_1\sigma_2}, g)\}.$$

Hence $\sigma_1\sigma_2 \varepsilon D_1$. Also $i\varepsilon D$ and so D is a monoid.

To show D is local we invoke Lemma (2.1.3). Since $D^* = M$, we show M is an ideal. Let $\sigma_1 \varepsilon M$ and $\sigma_2 \varepsilon D$. Choose P, Q, such that $P \not\circ Q$. Since $\sigma_1 \varepsilon M$, $P^{\sigma_1} o Q^{\sigma_1}$ by Corollary (2) of Theorem (3.1.2). By Lemma (3.1.1)(1), $(P^{\sigma_1})^{\sigma_2} o (Q^{\sigma_1})^{\sigma_2}$. Thus $P^{\sigma_2\sigma_1} o Q^{\sigma_2\sigma}$ and so $\sigma_2\sigma_1 \varepsilon M$ by Corollary (1) of Theorem (3.1.2). Thus M is a left ideal. Since $\sigma_1 \varepsilon M$, $(P^{\sigma_2})^{\sigma_1} o (Q^{2})^{1}$ by Corollary (2) of Theorem (3.1.2). Hence $P^{\sigma_1\sigma_2} o Q^{\sigma_1\sigma_2}$ or $\sigma_1\sigma_2 \varepsilon M$. Hence M is a right ideal, and thus an ideal.

Clearly Theorem (3.1.4) implies $M = \bigcup_{P \varepsilon \mathbb{P}} M_P$.

(2) Clearly D_P is a submonoid of D. Since M is an ideal and D_P is a sub monoid we obtain that M_P is an ideal as follows:

$$M_P D_P = (M_n D_p) D_P \subseteq MD_P \cap D_P^2 \subseteq MD \cap D_P \subseteq M \cap D_P = M_P,$$

$$D_P M_P = D_P(M \quad D_P) \subseteq D_P M \cap D_P^2 \subseteq M \cap D_P = M_P.$$

o_P is a zero element since for each $\sigma \varepsilon D_P$, and each Q,

$$Q^{o_P o \sigma} = (Q^{\sigma})^{o_P} = P \text{ and } Q^{\sigma o_P} = (Q^{o_P})^{\sigma} = P^{\sigma} = P,$$

(3) follows from Corollary (2) of Theorem (3.1.2) and (4) follows from Theorem (3.1.2).

The next theorem was proved essentially by Lüneburg for $J(T,\Pi)$ type planes, which we will mention later. We give a different proof in our context.

Theorem (3.1.7). If $\mathfrak{X}$ is uniform, then $M_P^2 = \{o_P\}$ for each P. [cf. Definition (1.1.8).]

Proof. Let $\sigma_1, \sigma_2 \varepsilon M_P$. Hence by Corollary (2) of Theorem (3.1.2), PoQ^{σ_i} for each Q; i = 1, 2.

Now choose g_1, g_2, $g_1 \not{o} g_2$ such that PIg_1, g_2 by Lemma (1.1.12). Select $P_i I g_i$ such that $P \not{o} P_i$; i = 1, 2, by Lemma (1.1.9). Thus $g_i = PP_i$; i = 1, 2. Define $Q_i = P_i^{\sigma_i}$; i = 1, 2. Thus $P \text{ o } Q_i$ and $Q_i I g_i$; i = 1, 2. Now $P \not{o} P_2$ and $Q_1 o P$ implies $PP_2 o P_2 Q_1$ by (A5)*. Define $\ell_1 = P_2 Q_1$ and $\ell_2 = L(Q_2, \ell_1)$. Thus $\ell_1 o g_2$. Since $g_1 \not{o} g_2$, it follows that $\ell_1 \not{o} g_1$.

Since $P \not{o} P_1$, it suffices to show $P_1^{\sigma_2\sigma_1} = P$ by Theorem (3.1.1). Now we have

$$P_1^{\sigma_2\sigma_1} = (P_1^{\sigma_1})^{\sigma_2} = Q_1^{\sigma_2} I\{L(P, g_1) = g_1, L(Q_2, \ell_1) = \ell_2\},$$

since $\sigma_2\sigma_1 \varepsilon D$. Because $\ell_1 o g_2$; $\ell_1 \parallel \ell_2$; and $P_2 I \ell_1$, g_2

it follows from (A7) that $\ell_2 o g_2$. Since $\sigma_2 \varepsilon M_p$,

$Q_1^{\sigma_2} o \{P_2^{\sigma_2} = Q_2\}$ by Corollary (2) of Theorem (3.1.2).

Then $\ell_2 o g_2$; $Q_2 I \ell_2$, g_2; $Q_1^{\sigma_2} I \ell_2$; and $Q_2 o Q_1^{\sigma_2}$ imply

$Q_1^{\sigma_2} I g_2$ by uniformity. Hence $Q_1^{\sigma_2} I g_1$, g_2. Thus we have

$$P_1^{\sigma_2 \sigma_1} = Q_1^{\sigma_2} = g_1 \wedge g_2 = P.$$

§3.2. Translations

Definition (3.2.1). [L1] Let $\sigma \varepsilon D$. Then

(1) σ is called a quasi-translation iff σ has no fixed points or $\sigma = i$.

(2) σ is called translation iff (i) σ is a quasi-translation, (ii) If g is a trace of σ and $h \| g$, then h is a trace of σ.

Notation. $D^* = D \setminus M$.

$\tilde{T} = \{\tilde{\tau} | \tilde{\tau}$ is a quasi-translation$\}$.

$T = \{\tau | \tau$ is a translation$\}$.

We note that D^* is a group.

(3) Λ is called a direction of τ iff Λ is a pencil of traces of τ.

$D_\tau = \{\Lambda | \Lambda$ is a direction of $\tau\} = \{\Lambda_g | g$ is a trace of $\tau\}$.

$T_\Lambda = \{\tau | \tau \varepsilon T$ such that $\Lambda \varepsilon D_\tau\}$.

The next theorem of Klingenberg's is proved in a slightly different manner, as we shall use the actual structure of the proof later on.

Theorem (3.2.1). [K2] Each $\tau \varepsilon T$ is uniquely determined by its action on one point P.

Proof. Let g be any line such that $P, P^{\tau} I g$. Let Q be any point, $Q \neq P$. Take h any line such that $P, Q I h$.

Case (1): $h \not o g$. Since $P I g, h$, we have $\Lambda_g \not o \Lambda_h$. Thus $\Lambda_{L(Q,g)} \not o \Lambda_{L(P\ ,h)}$. Since $\tau \varepsilon D$, $Q^{\tau} I L(P^{\tau}, h)$. Because g is a trace of τ, $\tau \varepsilon T$, it follows that $L(Q, g)$ is a trace of τ and hence $Q^{\tau} I L(Q, g)$. Thus by Lemma (1.1.14), $Q^{\tau} = L(Q, g) \wedge L(P^{\tau}, h)$.

Case (2): $h o g$. By definition, there exists Y, $Y I g$ such that $Y o Q$. By Lemma (1.1.10), $g o L(Q, g)$. Choose R such that $R \not o X$ for each $X I g$. Since $g o L(Q, g)$, $R \not o X$ for each $X I L(Q, g)$, and so $L(R, g) \not o L(Q, g)$. By Lemma (1.1.10), $Q \not o X$ for each $X I L(R, g)$. Thus

$$QR \not o L(R, g) \quad \text{(I)}.$$

Now since g is a trace of τ, $L(R, g)$ is one also and hence $R^{\tau} I L(R, g)$.

By the choice of R,

$$RP \not o g \quad \text{(II)}.$$

Applying Case (1) to (II), we may determine R^{τ} from

P and P^{τ}. Applying Case (1) to (I) we may determine Q^{τ} from R and R^{τ}.

Definition (3.2.2). Let G be a group and $T \subseteq G$. T is called a normal subset of G iff $gTg^{-1} = T$ for each $g \varepsilon G$.

The next theorem, except for parts (2) and (5), is due to Lüneburg in [L1].

Theorem (3.2.2). The following are true.

(1) T is a normal subset of D^*.

(2) If $\tilde{\tau} \varepsilon \tilde{T}$, then $\tilde{\tau}^{-1} \varepsilon \tilde{T}$.

(3) T is a normal subset of D^* and $D_{\tau} = D_{\sigma^{-1}\tau\sigma}$ for each $\tau \varepsilon T$ and $\sigma \varepsilon D^*$.

(4) T_{Λ} is a normal subset of D^*.

(5) If $\tau \varepsilon T$, then $\tau^{-1} \varepsilon T$ and $D_{\tau} = D_{\tau^{-1}}$.

Proof: We prove only (2) and (5).

(2) If $P^{\tau^{-1}} = P$, then $P = P^{\tau}$. Hence $\tau = i$ and so $\tau^{-1} = i$.

(5) From (2), $\tau \varepsilon T$. However,

$$P,\ P^{\tau} I g \text{ iff } P^{\tau^{-1}\tau},\ P^{\tau} I g$$

and

$$P,\ P^{\tau^{-1}} I g \text{ iff } P^{\tau\tau^{-1}},\ P^{\tau^{-1}} I g.$$

Thus τ and τ^{-1} have the same traces. Therefore $D_\tau = D_{\tau^{-1}}$, and so $\tau^{-1} \varepsilon T$.

Comment (3.2.1). It is not known, in general, whether T or $\tilde{T}$ are groups. From Theorem (3.2.2), we see that T has all the properties of a group except it is not closed under its binary operation. Of course for ordinary affine planes $T = \tilde{T}$ and T is a group (cf. [A2]). We shall see later that there exist planes such that $\tilde{T} \neq T$.

Definition (3.2.3).

(1) $\tilde{N} = \{\tilde{\tau} \mid \tilde{\tau} \varepsilon \tilde{T}$ such that $P o P^{\tilde{\tau}}$ for each $P\}$, is called the set of neighbour quasi-translations. $N = \tilde{N} \cap T$ is called the set of neighbour translations.

(2) $\bar{D}$ is the set of dilatations of $\bar{\mathcal{H}}$.
$\bar{T}$ is the set of translations of $\bar{\mathcal{H}}$.

Theorem (3.2.3). Define the map ϕ: $D \to \bar{D}$ by $\phi(\sigma) = \bar{\sigma}$ where $(\bar{P})^{\bar{\sigma}} = \overline{P^{\sigma}}$. Then ϕ is a monoid homomorphism and Ker $\phi = \{\sigma \mid Q o Q^{\sigma}$ for each $Q\}$. Moreover $\chi_{\mathbb{P}} \sigma = \bar{\sigma} \cdot \chi_{\mathbb{P}}$.

Proof: We first show that $\bar{\sigma}$ is a function. Let $\bar{P} = \bar{Q}$ and so $P o Q$. By Lemma (3.1.1)(1), $P^{\sigma} o Q^{\sigma}$. Thus $(\bar{P})^{\bar{\sigma}} = \overline{P^{\sigma}} = \overline{Q^{\sigma}} = (\bar{Q})^{\bar{\sigma}}$.

<u>Claim.</u> <u>$\sigma\varepsilon D$</u>. Let P_1, $P_2 I \ell$. Hence there exist Y_1, Y_2 such that $Y_i I \ell$ and $Y_i o P_i$; $i = 1, 2$. Since $\sigma\varepsilon D$, $Y_1 I L(Y_2, \ell)$. Because χ is a homomorphism, then using the results of Lemma (1.2.4), we obtain

$$\{(\bar{P}_1)^{\bar{\sigma}} = (\bar{Y}_1)^{\bar{\sigma}} = \chi(Y_1^{\sigma})\} I \chi(L(Y_2^{\sigma}, \ell)),$$

$$\chi(L(Y_2^{\sigma}, \ell)) = L(\chi(Y_2^{\sigma}), \chi(\ell)) = L(\overline{Y_2^{\sigma}}, \bar{\ell}) = L((\bar{Y}_2)^{\bar{\sigma}}, \bar{\ell})$$

$$= L((\bar{P}_2)^{\bar{\sigma}}, \bar{\ell}).$$

Hence $\sigma\varepsilon D$.

Next we show ϕ is a monoid homomorphism. It is enough to show $\overline{\sigma_1\sigma_2} = \bar{\sigma}_1\bar{\sigma}_2$ and $\bar{i}$ is the identity map of $\bar{D}$. But for any P,

$$(\bar{P})^{\overline{\sigma_1\sigma_2}} = \overline{P^{\sigma_1\sigma_2}} = \overline{(P^{\sigma_2})^{\sigma_1}} = \overline{(P^{\sigma_2})}^{\bar{\sigma}_1} = ((\bar{P})^{\bar{\sigma}_2})^{\bar{\sigma}_1}$$

$$= (P)^{\sigma_1\sigma_2}$$

and

$$(\bar{P})^{\bar{i}} = \overline{P^{i}} = \bar{P}.$$

Next we have $\sigma\varepsilon\mathrm{Ker}\ \phi$ iff $\phi(\sigma) = i$ iff $(\bar{Q})^{\bar{\sigma}} = \bar{Q}$ for all Q, iff $\overline{Q^{\sigma}} = \bar{Q}$ for all Q iff $Q^{\sigma} o Q$ for all Q.

Finally for any P,

$$P^{\bar{\sigma}\circ\chi_{\mathfrak{P}}} = (P^{\chi_{\mathfrak{P}}})^{\bar{\sigma}} = (\bar{P})^{\bar{\sigma}} = \overline{P^{\sigma}} = (P^{\sigma})^{\chi_{\mathfrak{P}}} = P^{\chi_{\mathfrak{P}}\circ\sigma}.$$

Corollary. (a) $D/\theta_{\phi} \cong \phi[D]$.

(b) Ker $\phi \cap D^*$ *is a normal subgroup of* D^* *and* $D^*/\mathrm{Ker}\ \phi \cap D^* \cong \phi[D^*]$.

Proof: (a) This follows from the corollary of Lemma (2.1.6).

(b) This follows from the first isomorphism theorem of group theory and Lemma (2.1.4)(4).

Lemma (3.2.1). [K2] *The following are true*.
(1) $N = \{\tau \mid \tau \in T$ *and there exists* P *such that* $P \circ P^{\tau}\}$.
(2) $\phi[T] \subsetneq \bar{T}$.

Lemma (3.2.2). *Let* $\tilde{\tau} \in \tilde{T}$. *If any two traces of* $\tilde{\tau}$ *are parallel, then* $\tilde{\tau} \in T$.

Proof: Let g be a trace of $\tilde{\tau}$ and $h \parallel g$. Let QIh. Choose $\tilde{h}$ such that $Q\,Q^{\tilde{\tau}} I \tilde{h}$. Thus $\tilde{h}$ is a trace of $\tilde{\tau}$ and so $\tilde{h} \parallel g$. Hence $h = \tilde{h}$ since $QIh, \tilde{h}$. Thus h is a trace of $\tilde{\tau}$.

Lemma (3.2.3). *Let* $\tau \in T$. *The following are equivalent*.
(1) $\tau \notin N$.
(2) *If* h *and* g *are traces of* τ, *then* $h \parallel g$.
(3) $|D_{\tau}| = 1$.

<u>Proof</u>: <u>(1)$\Longrightarrow$(2).</u> Let $\tau \notin N$. Hence by Lemma (3.2.1)(1), $P \not o P^{\tau}$ for each P. Let h and g be traces of τ. Thus there exist P and Q such that $g = PP^{\tau}$ and $h = QQ^{\tau}$. Define $\tilde{h} = L(Q, g)$. Then $\tilde{h}$ is a τ-trace and so $Q^{\tau} I \tilde{h}$. Thus $\tilde{h} = QQ^{\tau}$ and so $h \parallel g$.

<u>(2)$\Longrightarrow$(3)</u>. Obvious.

<u>(3)$\Longrightarrow$(1)</u>. Suppose $\tau \in N$. Hence there exists P such that $P o P^{\tau}$. Thus there exist g_1, g_2, $g_1 \neq g_2$; $g_1 o g_2$, such that P, $P^{\tau} I g_1$, g_2. Now g_1 and g_2 are traces, but $g_1 \nparallel g_2$. Hence $\Lambda_{g_1} \neq \Lambda_{g_2}$ and so $|D_{\tau}| > 1$.

<u>Lemma (3.2.3)</u>. <u>Let</u> $\mathcal{X}$ <u>be</u> <u>uniform</u>. <u>Let</u> $\tau \in N$ <u>and</u> $\Lambda \in D_{\tau}$. <u>Then</u> $\tilde{\Lambda} \in D_{\tau}$ <u>iff</u> $\tilde{\Lambda} o \Lambda$.

<u>Proof</u>: If $\tilde{\Lambda} \in D_{\tau}$, then by Lemma (3.1.2), $\tilde{\Lambda} o \Lambda$. Conversely, assume $\tilde{\Lambda} o \Lambda$. Let $g \in \Lambda$ and $\tilde{g} \in \tilde{\Lambda}$, such that PIg, $\tilde{g}$. Since $\tilde{\Lambda} o \Lambda$ and $g \wedge \tilde{g} \neq \emptyset$, it follows that $g o \tilde{g}$ by Lemma (1.1.13). Since g is a trace of τ, $P^{\tau} I g$. Also $P o P^{\tau}$ since $\tau \in N$. Thus $P o P^{\tau}$; $g o \tilde{g}$; PIg, $\tilde{g}$ and $P^{\tau} I g$ imply $P^{\tau} I \tilde{g}$ by uniformity. Hence g is a τ-trace and so $\Lambda \in D_{\tau}$.

<u>Corollary</u>. (1) <u>Let</u> $\mathcal{X}$ <u>be</u> <u>uniform</u> <u>and</u> $\tau \in N$. <u>If</u> g <u>is</u> <u>a</u> <u>trace</u> <u>of</u> τ <u>and</u> hog <u>or</u> $h \wedge g = \emptyset$, <u>then</u> h <u>is</u> <u>a</u> <u>trace</u> <u>of</u> τ.

Proof: From Lemma (1.1.13), $\Lambda_g o \Lambda_h$. $\Lambda_g \varepsilon D_\tau$. The result then follows from the lemma.

Corollary. (2) If $\mathcal{H}$ is uniform and $\Lambda_1 o \Lambda_2$, then $T_{\Lambda_1} \cap N = T_{\Lambda_2} \cap N$.

Proof: This follows easily from the lemma.

Definition (3.2.4). An affine H-plane $\mathcal{H}$ is called a T-plane iff T is a group.

Theorem (3.2.4). [K2] Let $\mathcal{H}$ be a T-plane. Then

(1) T is a normal subgroup of D^*.

(2) T_Λ is a normal subgroup of T.

(3) N is a normal subgroup of T and $T/N \cong \phi[T]$.

Theorem (3.2.5). [L1] Let $\mathcal{H}$ be a T-plane. Moreover suppose there exist Λ_1, Λ_2, $\Lambda_1 \not{o} \Lambda_2$ such that $T_{\Lambda_i} \cap \complement N \neq \emptyset$; $i = 1, 2$. Then T is abelian.

We end this section with two technical lemmas we shall use later.

Lemma (3.2.4). Let $\Lambda_1 \not{o} \Lambda_2$. Then $T_{\Lambda_1} \cap T_{\Lambda_2} = \{i\}$.

Proof: Let $\tau \varepsilon T_{\Lambda_1} \cap T_{\Lambda_2}$. Then $\Lambda_1, \Lambda_2 \varepsilon D_\tau$. Thus by Lemma (3.1.2), $\Lambda_1 o \Lambda_2$. Contradiction.

Lemma (3.2.5). Assume the conditions of Theorem (3.2.5). Furthermore let $\Lambda_1 o \Lambda_2$; $\tau_i \varepsilon T_{\Lambda_i}$; PIg_i; and $g_i \varepsilon \Lambda_i$; $i = 1, 2$. Then there exists $\tau_3 \notin N$, where $D_{\tau_3} = \{\Lambda_3\}$ [cf. Lemma (3.2.3)] with the properties

(a) $\Lambda_3 \emptyset \Lambda_1, \Lambda_2$.

(b) If g_{13} is a trace of $\tau_1\tau_3$, then $g_{13}\emptyset g_1, g_2$.

Proof: By our assumptions there exist two pencils Λ and $\tilde{\Lambda}$ such that $\Lambda \emptyset \tilde{\Lambda}$ and $T_\Lambda \cap \complement N \neq \emptyset \neq T_{\tilde{\Lambda}} \cap \complement N$. Since $\Lambda_1 o \Lambda_2$, one of Λ and $\tilde{\Lambda}$, say Λ_3, has the property $\Lambda_3 \emptyset \Lambda_1, \Lambda_2$. Choose $\tau_3 \varepsilon T_{\Lambda_3} \cap \complement N$. Hence $D_{\tau_3} = \{\Lambda_3\}$. It is clear that (a) is satisfied. We show (b) next, recalling by Theorem (3.2.5) that T is abelian.

Assume (b) is false. Hence there exists g_{13}, a trace of $\tau_1\tau_3$, such that $g_{13} o g_2$ or $g_{13} o g_1$. Let $g_{13} o g_2$. Let g_3 be a trace of τ_3 through P.

Define $h = L(P, g_{13})$ and $j = L(P^{\tau_1}, g_3)$.

Claims. (1) $h o g_2$.

(2) $(P^{\tau_1})^{\tau_3} Ij$.

(3) $j \emptyset g_1$.

(4) $P^{\tau_3\tau_1} Ih$.

(1) Suppose $h \not o g_2$. Hence PIh, g_2 and $g_3 \| h$ imply $g_{31} o g_2$ by (A2). Contradiction.

(2) Since $j \| g_3$, then j is a trace of τ_3 through P^{τ_1}.

Hence $(P^{\tau_1})^{\tau_3} I j$.

(3) Since $\{\Lambda_j = \Lambda_3\} \not o \{\Lambda_1 = \Lambda_{g_1}\}$, by (a), we have $j_1 \not o g_1$ by Lemma (1.1.14).

(4) h is a trace of $\tau_1\tau_3$ through P. Since T is abelian our result follows.

Then $h o g_2$; $j \not o g_i$; $P^{\tau_3\tau_1} I j$, h; and $P^{\tau_1} I j$, g_1 imply that

$$\{P^{\tau_3\tau_1} = (P^{\tau_1})^{\tau_3}\} o P^{\tau_1} \text{ by (A6).}$$

Hence $\tau_3 \varepsilon N$ by Lemma (3.2.1). Contradiction. Hence $g_{13} \not o g_2$. Similarly $g_{13} \not o g_1$.

CHAPTER 4

Minor Desarguesian Planes

§*4.1*. *A brief Discussion of* $J(T, \beta)$ *types planes*.

In [L1], Lüneburg defines an incidence structure with parallelism, $J(T, \beta)$, where T is a group and β is a set of subgroups (called components); as follows: Points are the elements of T; lines are the right cosets of the components; incidence is given by inclusion; lines are taken to be parallel iff they are cosets of the same components. Lüneburg then proved the following theorems in [L1].

Theorem (4.1.1). [L1] $J(T, \beta)$ *is an affine* H-*plane iff the following conditions hold.*

(1) *The components cover* T.

(2) *If* $A, B \in \beta$ *such that* $A \cap B = 1$, *then* $T = AB$.

(3) *There exist* $A, B \in \beta$ *with* $A \cap B = 1$.

(4) *The set* $N = \{\tau \in T \mid \tau \circ 1\}$ *is a normal subgroup of* T.

(5) *If* $A \in \beta$, *then* $A \not\leq N$.

(6) *If* $A \cap B = 1$, *then* $N = NA \cap NB$.

(7) *If* $A \cap B \neq 1$, *then* $NA = NB$.

Definition (4.1.1). Let $J(T, \beta)$ be an affine H-plane. For each $\tau\epsilon T$, define τ^*: $T\to T$ by $\sigma^{\tau^*} = \sigma\tau$ for each $\sigma\epsilon T$. Clearly the set $T^* = \{\tau^* | \tau\epsilon T\}$ is the set of quasi-translations of $J(T, \beta)$ [cf. Definition (3.2.1)] It easily follows that f: $T\to T^*$ defined by $f(\tau) = \tau^*$ is a group isomorphism.

Remark (4.1.1). T^* is a transitive group and each τ^* is uniquely determined by its action on 1.

Proof. Let $\tau_1, \tau_2\epsilon T$. Next consider $(\tau_2\tau_1^{-1})^*$. This clearly maps τ_1 onto τ_2. Moreover the last part follows since $(1)^{\tau^*} = \tau$ for each $\tau\epsilon T$.

Notation. If $J(T, \beta)$ is an affine H-plane, then let $T(J)$ be its set of translations.

We then obtain the following important result.

Theorem (4.1.2). [L1] If $\mathfrak{X}$ is a T-plane such that T is a transitive group, then T is abelian; and there exists a collection β of subgroups of T such that $\mathfrak{X} \cong J(T, \beta)$. Moreover if $J(T, \beta)$ is an affine H-plane, then $T(J) = T^*$ iff T is abelian.

We prove the following corollary.

Corollary. If $J(T, \beta)$ is an affine H-plane, then the following are equivalent.

(1) T(J) is a transitive group.

(2) $T(J) = T^*$.

(3) T is abelian.

Proof: (1)$\Longrightarrow$(2). We first show $T \cong T(J)$. Since T(J) is transitive, for each $x \varepsilon T$, let τ_x be the unique translation mapping 1 to x. The uniqueness of τ_x follows from Theorem (3.2.1). Define $g: T \to T(J)$ by $g(x) = \tau_x$. Clearly $\tau_x = x^*$ from Remark (4.1.1). g is (1 - 1) and onto from the uniqueness of τ_x and the transitivity of T(J) respectively. To show g is a homomorphism it is enough to show $\tau_x \tau_y = \tau_{xy}$. Now $(1)^{\tau_y \tau_x} = (1^{\tau_x})^{\tau_y} = x^{\tau_y} = x^{y^*} = xy$, and hence $\tau_x \tau_y = \tau_{xy}$. Thus $T = T(J)$. From the theorem T(J) is abelian. Hence T is abelian and thus also from the theorem, $T^* = T(J)$.

(2)$\Longrightarrow$(3). Since $T(J) = T^*$ and T^* is transitive by Remark (4.1.1), T(J) is transitive, and hence abelian by the theorem. Hence as in (1)$\Longrightarrow$(2), $T(J) = T$ and so T is abelian.

(3)$\Longrightarrow$(1). This follows from the theorem.

§4.2. *The ring of trace preserving homomorphisms of a minor Desarguesian affine H-plane*

Definition (4.2.1). [K2] $\mathfrak{X}$ is called a *minor Desarguesian affine H-plane* iff it satisfies the axiom

(A9): T *is a transitive group.* If $\mathfrak{X}$ is minor Desarguesian we say it is a *M.D. plane.* Lüneburg calls minor Desarguesian planes, *translation planes.* He studied their structure, in the form of $J(T, \Pi)$ planes, in view of Theorem (4.2.1), just as André did in [A1] for ordinary planes.

We shall however proceed in the manner of Artin in [A2], in the general geometric form.

Notation: If $\mathfrak{X}$ satisfies (A9), and τ is the unique translation taking P to Q, we write $\tau = \tau_{PQ}$.

Lemma (4.2.1). [K2] *Let* $\mathfrak{X}$ *be a* M.D. *plane. Then* $\phi[T] = \vec{T}$ *and* $T/N \cong \vec{T}$.

Proof: From the Corollary of Lemma (3.2.1), $\phi[T] \subseteq \vec{T}$. Conversely, if $\vec{\tau} \in \vec{T}$, then $\vec{\tau}$ is uniquely determined by its action on any point $\vec{P}$. Let $\vec{Q} = (\vec{P})^{\vec{\tau}}$. By (A9), there exists $\tau \in T$ such that $\tau = \tau_{PQ}$. Since

$$(\bar{P})^{\bar{\tau}} = \overline{(P^{\tau})} = \bar{Q} = (\bar{P})^{\underline{I}},$$

it follows that $\bar{\tau} = \underline{I}$.

The last part follows from the first and Theorem (3.2.4)(3).

Now let us recall some results from group theory.

Theorem (4.2.1). *Let G be an abelian group. Let A and B be subgroups such that $A \cap B = 1$ and $G = AB$. Then, (a) Every element of G has a unique representation as a product of an element of P and an element of Q.*

(b) $G/A \cong B$.

Definition (4.2.2). Let G be a group. Then $G = A \oplus B$ iff (i) A and B are subgroups of G.

(ii) $G = AB$.

(iii) $A \cap B = 1$.

(iv) Each element has a unique representation as a product of elements of A and B.

By Theorem (4.2.1), (iv) is obviously redundant, if G is Abelian.

Theorem (4.2.2). *Let $\mathcal{R}$ be a M.D. plane. Then*

(1) *There exist Λ_1, Λ_2, $\Lambda_1 \not\parallel \Lambda_2$ such that $T_{\Lambda_i} \cap \mathbb{C}N \neq \emptyset$; $i = 1, 2$. Hence T is abelian.*

(2) *If $\Lambda_1 \not\parallel \Lambda_2$, then $T = T_{\Lambda_1} \oplus T_{\Lambda_2}$.*

Proof: (1) We invoke Theorem (3.2.5). Select any point P. Choose g_1, g_2 such that PIg_1, g_2 and $g_1 \not\!/ g_2$. Take $Q_i I g_i$ such that $g_i = PQ_i$; $i = 1, 2$. By (A9), $\tau_i = \tau_{PQ_i}$ exists; and $\tau_i \varepsilon N$ since $P \not\!/ Q_i$; $i = 1, 2$. It follows that $\Lambda_{g_1} \not\!/ \Lambda_{g_2}$ and $T_{\Lambda_{g_i}} \cap N \neq \emptyset$; $i = 1, 2$.

(2) From Lemma (3.2.4), $T_{\Lambda_1} \cap T_{\Lambda_2} = 1$.

Now we must show $T \subseteq T_{\Lambda_1} \cdot T_{\Lambda_2}$.

Let $\tau \varepsilon T$ such that $\tau \neq i$. By (A9), let $\tau = \tau_{PQ}$. Choose g_1 and g_2 as in (1). Define $h_1 = L(Q, g_1)$ and $h_2 = L(P, g_2)$. Since $\Lambda_{g_1} \not\!/ \Lambda_{g_2}$, we have $\Lambda_{h_1} \not\!/ \Lambda_{h_2}$ and hence there exists T such that $T = h_1 \wedge h_2$. Define $\tau_1 = \tau_{PT}$ and $\tau_2 = \tau_{TQ}$. Then $\tau_i \varepsilon T_{\Lambda_i}$; $i = 1, 2$, and $\tau = \tau_1 \tau_2$.

(3) Choose Λ such that $\Lambda \not\!/ \Lambda_1$, Λ_2 by the corollary to Lemma (1.1.14). By (2) and (b) of Theorem (4.2.1), $T/\Lambda_1 \cong T_\Lambda = T/\Lambda_2$.

Definition (4.2.3). Let $\mathcal{R}$ be a M.D. plane. δ: $T \to T$ is a trace preserving endomorphism iff

(i) δ is a group endomorphism of T.

(ii) $D_\tau \subseteq D_{\tau^\delta}$ for each $\tau \varepsilon T$, where τ^δ is the image of τ under δ. Let H be the set of these endomorphisms.

Theorem (4.2.4). [K2] If $\mathcal{P}$ is a M.D. plane, then H is a ring with unit in the following manner:

$$\tau^{\delta_1\delta_2} = (\tau^{\delta_2})^{\delta_1}; \quad \tau^{(\delta_1+\delta_2)} = \tau^{\delta_1}\cdot\tau^{\delta_2};$$

1 : $\tau^1 = \tau$ for each $\tau\varepsilon T$, is the unit of H;

0 : $\tau^0 = i$ for each $\tau\varepsilon T$, is the zero of H.

Lüneburg, then showed for $J(T, \mathcal{E})$ in [L1], that D_p is a rin and D_p is isomorphic to H. The proof proceeds beautifully, due to the fact the group of quasi-translations, T*, coincides with the group of translations, which is not true in general. We shall generalize Artin's proof of this fact, in his setting. Then we obtain much nicer proofs of the properties of H, obtained by Klingenberg in [K2], as well as see more clearly how each dilatation is related to a unique trace preserving endomorphism.

Definition (4.2.3). Let $\mathfrak{N}$ be the set of non-units of H.

Before we prove the main result of this section we require the following technical lemma.

Lemma (4.2.2). Let $\sigma\varepsilon D_p$ and Q any point such that

$P \not\parallel Q$. Let $Q^{\sigma} = R$ and $g = PQ$ Further let $\tau = \tau_{PS}$ and ℓ be a τ-trace through P such that $\ell \not\parallel g$. Then $\sigma \cdot \tau = (\tau_{PS^{\sigma}})\,\sigma$.

Proof: From Theorem (3.1.1), it suffices to show the two dilatations map P and Q identically. Now

$$P^{\sigma o \tau} = (P^{\tau})^{\sigma} = S^{\sigma} = P^{\tau_{PS^{\sigma}}} = (P^{\sigma})^{\tau_{PS^{\sigma}}} = P^{(\tau_{PS^{\sigma}})\sigma}.$$

It remains to show they coincide on Q. By Case (1) of the proof of Theorem (3.1.1), we have, since $g \not\parallel \ell$,

$$(R)^{\tau_{PS^{\sigma}}} = L(S^{\sigma}, g) \wedge L(R, \ell). \qquad \text{(I)}$$

Let $h = L(S, g)$ and $m = L(Q, \ell)$. Hence $h \not\parallel m$. Thus by Case (1) of the proof of Theorem (3.2.1), we have

$$Q^{\tau} = h \wedge m.$$

Now $(Q^{\tau})^{\sigma} I \{L(S^{\sigma}, h) = L(S^{\sigma}, g)\ ,\ L(Q^{\sigma}, m) = L(R, \ell)\}$. Thus by (I), we have

$$Q^{\sigma o \tau} = (Q^{\tau})^{\sigma} = (R)^{\tau_{PS^{\sigma}}} = (Q^{\sigma})^{\tau_{PS^{\sigma}}} = Q^{(\tau_{PS^{\sigma}}) o \sigma}.$$

Theorem (4.2.5). Let $\mathcal{D}$ be a M.D. plane and P any point. Then (I). For each $\delta\varepsilon H$ there exists a unique σ such that (i) $\sigma\varepsilon D_P$;

(ii) $(\tau_{PS})^{\delta} = \tau_{PS}\sigma$ for each $\tau_{PS}\varepsilon T$.

Let $\sigma(\delta)$ be this unique dilatation.

(II). ϕ_P: $H\to D_P$, defined by $\phi_P(\delta) = \sigma(\delta)$, is a monoid isomorphism.

Hence $\delta\varepsilon\,\Pi$ iff $\sigma(\delta)\varepsilon M_P$.

Proof: (I). We first show the uniqueness of σ. Let σ have properties (i) and (ii). Select any point Q. Then $P^{\sigma} = P$ and $Q^{\sigma} = P^{\tau_{PQ}\sigma} = P^{(\tau_{PQ})^{\delta}}$, which is independent of σ. Now we show the existence of σ.

Define $\sigma: \mathcal{P} \to \mathcal{P}$, by $S^{\sigma} = P^{(\tau_{PS})^{\delta}}$. Now $P^{\sigma} = P^{(\tau_{PP})^{\delta}} = P^{i^{\sigma}} = P^{i} = P$. To show $\sigma\varepsilon D_P$, take S, M such that S, MIg. Since g is a trace of τ_{MS}, $L(M^{\sigma}, g)$ is a τ_{MS} trace through M^{σ}. Thus $L(M^{\sigma}, g)$ is a τ_{MS}^{δ} trace through M^{σ}. Hence $(M^{\sigma})^{\tau_{MS}^{\delta}}$ $IL(M^{\sigma}, g)$. But

$$(M^{\sigma})^{\tau_{MS}^{\delta}} = (P^{\tau_{PM}^{\delta}})^{\tau_{MS}^{\delta}} = P^{\tau_{MS}^{\delta}\cdot\tau_{PM}^{\delta}}$$

$$= P^{(\tau_{MS}\tau_{PM})^{\delta}} = P^{\tau_{PS}^{\delta}} = S^{\sigma}.$$

Therefore $S^{\sigma}IL(M^{\sigma}, g)$. Property (ii) is easily satisfied since $\tau_{PS}\sigma = \tau_P(P^{\tau^{\delta}_{PS}}) = \tau^{\delta}_{PS}$.

(II) ϕ_P is a function by (I). To show ϕ_P is a monoid homomorphism, it is enough to show $\sigma(\delta_1)\ \sigma(\delta_2) = \sigma(\delta_1\delta_2)$ and $\sigma(1) = i$.

$$\tau_{PS}^{\delta_1\delta_2} = (\tau_{PS}^{\delta_2})^{\delta_1} = (\tau_{PS}\sigma(\delta_2))^{\delta_1} = \tau_{P_{(S^{\sigma(\delta_2)})^{\sigma(\delta_1)}}}$$

$$= \tau_{PS}\sigma(\delta_1)\sigma(\delta_2).$$

Clearly $\sigma(\delta_1)$, $\sigma(\delta_2)\varepsilon D_P$. Thus from the uniqueness of (I), $\sigma(\delta_1)\sigma(\delta_2) = \sigma(\delta_1\delta_2)$. Finally $\tau_{PS}^{\sigma(1)} = \tau_{PS}^{1} = \tau_{PS}$ implies $S^{\sigma(1)} = S$ for any S, and so $\sigma(1) = i$.

<u>Claim (1)</u>. ϕ_P is injective. Now let $\sigma(\delta_1) = \sigma(\delta_2)$. Then $\tau_{PS}^{\delta_1} = \tau_{PS}\sigma(\delta_1) = \tau_{PS}\sigma(\delta_2) = \tau_{PS}^{\delta_2}$ for each S. Hence $\tau^{\delta_1} = \tau^{\delta_2}$ for each $\tau\varepsilon T$ and so $\delta_1 = \delta_2$. Therefore ϕ_P is (1 - 1).

<u>Claim (2)</u>. ϕ_P is surjective. Let $\sigma\varepsilon D_P$. Choose Q such that $Q\not{o}P$. Let $Q^{\sigma} = R$ and $g = PQ$. Define δ: $T\to T$ by $\tau_{PS}^{\delta} = \tau_{PS}\sigma$; for each $S\varepsilon\mathcal{P}$. We must show (a) $(\tau_2\tau_1)^{\delta} = \tau_2^{\delta}\tau_1^{\delta}$ and (b) $D_{\tau} \subseteq D_{\tau}\delta$.

(a) Let $\tau_i = \tau_{PT_i}$; $i = 1, 2$. Thus we obtain

$$
\text{(I)}\quad
\begin{cases}
\tau_2\tau_1 = \tau_{PS} \text{ such that } S = T_1^{\tau_2} \\
\qquad\qquad \text{and} \\
(\tau_2\tau_1)^\delta = \tau_{PS^\sigma} = \tau_{P(T_1^{\tau_2})^\sigma} = \tau_{PT_1}(\sigma\circ\tau_2). \\
\tau_2^\delta\tau_2^\delta = \tau_{PT_2^\sigma}\cdot\tau_{PT_1^\sigma} = \tau_{PM} \text{ such that } M = (T_1^\sigma)^{\tau_{PT_2^\sigma}}.
\end{cases}
$$

Since T is abelian we also obtain

$$
\text{(II)}\quad
\begin{cases}
\tau_2\tau_1 = \tau_1\tau_2 = \tau_{PS} \text{ such that } S = T_2^{\tau_1}. \\
(\tau_2\tau_1)^\delta = (\tau_1\tau_2)^\delta = \tau_{PS^\sigma} = \tau_{P(T_2^{\tau_1})^\sigma} = \tau_{PT_2^{\sigma\tau_1}}. \\
\tau_2^\delta\tau_1^\delta = \tau_1^\delta\tau_2^\delta = \tau_{PT_1^\sigma}\tau_{PT_2^\sigma} = \tau_{PM}
\end{cases}
$$

such that $M = (T_2^\sigma)^{\tau_{PT_1^\sigma}}$. Thus by (I) $(\tau_2\tau_1)^\delta = \tau_2^\delta\tau_1^\delta$ iff $T_1^{\sigma\circ\tau_2} = (T_1^\sigma)^{\tau_{PT_2^\sigma}} = T_1^{\tau_{PT_2^\sigma}\circ}$ and by (II)

$(\tau_2\tau_1)^\delta = \tau_2^\delta\tau_1^\delta$ iff $T_2^{\sigma\tau_1} = (T_2^{\tau_1})^{\tau_{PT_1^\sigma}} = T_2^{\tau_{PT_1^\sigma}\cdot\sigma}$.

Hence it suffices to show (A) $\sigma\,\tau_2 = \left(\tau_{PT_2^\sigma}\right)\sigma$ or

(B) $\sigma\,\tau_1 = \left(\tau_{PT_1^\sigma}\right)\sigma$.

<u>Case (1): At least one of</u> τ_i <u>has a trace</u> g_i <u>through</u> P <u>such that</u> $g_i \not o g$; i = 1, 2.

If $g_1 \not o g$, then by Lemma (4.2.2), (A) is satisfied.
If $g_1 \not o g$, then similarly (B) is satisfied.

<u>Case (2)</u>: $g_i o g$; i = 1, 2. Thus $\Lambda_{g_i} o \Lambda_g$; i = 1, 2, and so $\Lambda_{g_1} o \Lambda_{g_2}$. By Lemma (3.2.5), there exists $\tau_3 \in N$ such that $D_{\tau_3} = \{\Lambda_3\}$; $g_3 \not o g_1$, g_2 where g_3 is a τ_3 trace through P; and g_{31} is a trace of $\tau_3 \tau_1$ through P such that $g_{31} \not o g_1$, g_2. Since $g_i o g$ and g_3, $g_{31} \not o g_i$; i = 1, 2, it follows that $g_3, g_{31} \not o g$. Then applying Case (1), we obtain

(1) $(\tau_3 \tau_1)^\delta \tau_2^\delta = ((\tau_3 \tau_1)\tau_2)^\delta$ since $g_{31} \not o g$.

(2) $\tau_3^\delta (\tau_1 \tau_2)^\delta = \tau_3 (\tau_1 \tau_2)^\delta$ since $g_3 \not o g$.

(3) $(\tau_3 \tau_1)^\delta = \tau_3^\delta \tau_1^\delta$ since $g_3 \not o g$.

Hence we finally obtain, using (3), (1) and (2)

$$\begin{aligned} \tau_3^\delta \tau_1^\delta \tau_2^\delta &= (\tau_3^\delta \tau_1^\delta)\tau_2^\delta \\ &= (\tau_3 \tau_1)^\delta \tau_2^\delta \\ &= ((\tau_3 \tau_1)\tau_2)^\delta \end{aligned}$$

$$= \tau_3(\tau_1\tau_2)^\delta.$$

Hence $\tau_1^\delta\tau_2^\delta = (\tau_1\tau_2)^\delta$.

(b) Let $\Lambda\varepsilon D$. Take $h\varepsilon\Lambda$ such that PIh. Define $T = P^\tau$ and so $P^\tau Ih$. To show $\Lambda\varepsilon D_{\tau^\delta}$, it is enough to show $P^{\tau^\delta} Ih$. Now $\tau^\delta = \tau^\delta_{PT} = \tau_{PT^\sigma}$ implies $P^{\tau^\delta} = T^\sigma$. But $T^\sigma I\{L(P^\sigma, h) = L(P, h) = h\}$.

Thus we have finally shown that $\delta\varepsilon H$ and clearly $\phi_P(\delta) = \sigma$. To complete the proof we see from Lemma (2.4.1)(1) and the fact ϕ_P is an isomorphism, that $\delta\varepsilon\, \mathfrak{M}$ iff $\sigma(\delta)\varepsilon M_P$.

The following corollaries, except for (1) and (2) are theorems from [K2]. We exhibit new proofs using the above theorem. The assumptions for the corollaries are the same as for the theorem.

Corollary (1). [K2] *Each $\delta\varepsilon H$ is uniquely determined by its action on one τ, $\tau\notin N$.*

Proof: Take P any point. Let $Q = P^\tau$. Thus $P\not{o}Q$ since $\tau\notin N$. Put $\tau^\delta = \tau_{PR}$. Now choose $\tilde{\delta}\varepsilon H$ such that $\tau^{\tilde{\delta}} = \tau_{PR}$. Then by the theorem, there exist $\sigma = \sigma(\delta)$ and $\tilde{\sigma} = \tilde{\sigma}(\tilde{\delta})$ with properties (i) and (ii). Thus

$$\tau_{PQ}^{\ \delta} = \tau_{PQ^\sigma} = \tau_{PR} \quad \text{and}$$

$\tau_{PQ}^{\tilde{\delta}} = \tau_{PQ^{\tilde{\sigma}}} = \tau_{PR}$. Thus $Q^{\tilde{\sigma}} = Q^{\sigma}$ and since $P\not{o}Q$, $\tilde{\sigma} = \sigma$ by Theorem (3.1.1). Because ϕ_P is injective $\tilde{\delta} = \delta$.

Corollary (2). If $\tau \notin N$ and $\delta \varepsilon H$, then
(a) $\tau^{\delta} = 1$ implies $\sigma = 0$.
(b) $\tau^{\delta} = \tau^{\beta}$ implies $\delta = \beta$.

Proof: (a) follows from Corollary (1) and (b) from (a).

Corollary (3). $N^{\delta} \subseteq N$ for each $\delta \varepsilon \Pi$.

Proof: Let $\delta \varepsilon \Pi$. Hence there exists $\sigma \varepsilon M_P$ such that $\tau_{PS}^{\delta} = \tau_{PS^{\sigma}}$. Let $\tau_{PS} \varepsilon N$ and so PoS. By Lemma (3.1.1)(1), PoS^{σ} and consequently $\tau_{PS^{\sigma}} \varepsilon N$.

Corollary (4). Let $\delta \varepsilon H$. The following are equivalent.
(1) There exists τ, $\tau \notin N$, such that $\tau^{\delta} \varepsilon N$.
(2) $T^{\delta} \subseteq N$.
(3) $\delta \varepsilon \Pi$.

Proof: (1)$\Longrightarrow$(2). Let τ, $\tau \notin N$ be chosen such that $\tau^{\delta} \varepsilon N$. Let $\tau = \tau_{PQ}$ and so $P\not{o}Q$. Then there exists $\sigma \varepsilon D_P$ such that $\tau_{PS}^{\delta} = \tau_{PS^{\sigma}}$. Now let $\tau_{PQ}^{\delta} = \tau_{PR} \varepsilon N$ and so $Q^{\sigma} = R$ and PoR. Hence $\sigma \varepsilon M_P$. Let τ_{PS} be any translation. Since $\sigma \varepsilon M_P$, PoS^{σ}, by (2) of Theorem (3.1.2) and so $\tau_{PS}^{\delta} = \tau_{PS^{\sigma}} \varepsilon N$.

<u>(2)$\Longrightarrow$(3)</u>. Assume $T^{\delta} \subseteq N$. Let $\sigma = \sigma(\delta)$. Take $\tau_{PQ} \notin N$ and so $P \not o Q$. Then $\tau^{\delta}_{PQ} = \tau_{PQ^{\sigma}} \varepsilon N$. Put $Q^{\sigma} = R$. Hence PoR. Thus by Theorem (3.1.2), Corollary (2), $\sigma \varepsilon M_P$ and hence $\delta \varepsilon \mathcal{T}$.

<u>Corollary (5)</u>. <u>If</u> $\delta \varepsilon \mathcal{T}$, <u>there exists</u> $\tau \epsilon N$, $\tau \neq i$, <u>such that</u> $\tau^{\delta} = i$.

<u>Proof</u>: Let $\delta \varepsilon \mathcal{T}$ and hence $\sigma = \sigma(\delta) \varepsilon M_P$. By Theorem (3.1.5), there exists P_1, $P_1 \neq P$, $P_1 o P$ such that $P_1^{\sigma} = P$. Then $\tau = \tau_{PP_1} \neq i$ and $\tau \varepsilon N$. Finally $\tau^{\delta} = \tau_{PP_1^{\sigma}} = \tau_{PP} = i$.

<u>Corollary (6)</u>. H <u>is a local ring</u>.

<u>Proof</u>: Since ϕ_P is a monoid isomorphism, and D_P is a local monoid, with maximal ideal M_P, by Theorem (3.1.6)(2), it follows that $\mathcal{T} H \subseteq \mathcal{T}$ and $H \mathcal{T} \subseteq \mathcal{T}$. Now we show $\mathcal{T} + \mathcal{T} \subseteq \mathcal{T}$. Take δ, $\xi \varepsilon \mathcal{T}$. By Corollary (4), $T^{\delta} \subseteq \mathcal{T}$ and $T^{\xi} \subseteq \mathcal{T}$. Hence $(T)^{\delta+\xi} \subseteq T^{\delta} \cdot T^{\delta} \subseteq \mathcal{T}^2 \subseteq \mathcal{T}$ and so $\delta + \xi \varepsilon \mathcal{T}$ by Corollary (4).

We complete this chapter with the following result. Parts (i) and (ii) are essentially the devices used to prove Theorem (4.1.2).

<u>Notation</u>. If G is a group, End G is the <u>set of endomorphisms of</u> G.

For a fixed P, $\{P^{\tau}/\tau\varepsilon T\} = \mathbb{P}$, since T is transitive on $\mathbb{P}$.

Theorem (4.2.6). Let $\mathfrak{X}$ be a M.D. plane and P a fixed point. Then, (i) $\mathbb{P}$ is a group under the following multiplication: $P^{\tau_1} P^{\tau_2} = P^{\tau_1\tau_2}$; $\tau_1, \tau_2 \in T$.

$$P^{i} = P \text{ is clearly the unit of } \mathbb{P}$$

(ii) $f_P: \mathbb{P} \to T$, defined by $f_P(P^{\tau}) = \tau$, is a group isomorphism with inverse f_P^{-1} defined by $\tau_{PX}^{f_P^{-1}} = X$.

(iii) $D_P \subseteq \operatorname{End} \mathbb{P}$.

(iv) $\phi_P^{-1}: D_P \to H$ has the form,

$\phi_P^{-1}(\sigma) = f_P \sigma f_P^{-1}$, for each $\sigma\varepsilon D_P$.

Proof: (i) This follows immediately from the fact T is a group.

(ii) We need only show f_P is a homomorphism, and clearly

$$f_P(P^{\tau_1}P^{\tau_2}) = f_P(P^{\tau_1\tau_2}) = \tau_1\tau_2 = f_P(\tau_1)f_P(\tau_2).$$

(iii) Let $\sigma\varepsilon D_P$. Then we have

$$(P^{\tau_1\tau_2})^{\sigma} = P^{\tau_P}(P^{\tau_1\tau_2})^{\sigma}, \qquad \text{(A)}$$

$$\tau_{P(P^{\tau_1\tau_2})\sigma} = (\tau_{PP^{\tau_1\tau_2}})^{\delta(\sigma)} = (\tau_1\tau_2)^{\delta(\sigma)} = \tau_1^{\delta(\sigma)}\tau_2^{\delta(\sigma)}. \quad \text{(B)}$$

Combining (A) and (B), we obtain

$$(P^{\tau_1\tau_2})\sigma = P^{\tau_1^{\delta(\sigma)}\tau_2^{\delta(\sigma)}} = P^{\tau_1^{\delta(\sigma)}} \cdot P^{\tau_2^{\delta(\sigma)}}$$

$$= P^{\tau_{P(P^{\tau_1})\sigma}} \cdot P^{\tau_{P(P^{\tau_2})\sigma}}$$

$$= (P^{\tau_1})\sigma \cdot (P^{\tau_2})\sigma.$$

Hence $\sigma \in \mathrm{End}\ \mathbb{P}$.

(iv) Now $\phi_P^{-1}(\sigma) = \delta(\sigma)$ such that $\tau_{PS}^{\delta(\sigma)} = \tau_{PS^\sigma}$.

But

$$(\tau_{PS})^{f_P\sigma f_P^{-1}} = (S^\sigma)^{f_P} = (P^{\tau_{PS^\sigma}})^{f_P}$$

$$= \tau_{PS^\sigma} = \tau_{PS}^{\delta(\sigma)}.$$

CHAPTER 5

Desarguesian Affine H-planes

In [K2], this discussion was initiated without the concept of a dilatation. We shall employ dilatations and continue it in the fashion of Artin.

§5.1. Desarguesian affine H-planes

Definition (5.1.1). We define the following axioms.

(A10). If $D_{\tau_1} \subseteq D_{\tau_2}$, $\tau_1 \notin N$, then there exists $\delta \varepsilon H$ such that $\tau_1^{\delta} = \tau_2$.

(A10)(P:$\not\emptyset$). For each collinear triple (PQR) such that $P \emptyset Q$ and $P \emptyset R$, there exists $\sigma \varepsilon D_P$ such that $Q^{\sigma} = R$.

(A10)(P:o). For each collinear triple (PQR), $P \emptyset Q$, $P o R$, there exists $\sigma \varepsilon D_P$ such that $Q^{\sigma} = R$.

(A10)(P). For each collinear triple (PQR) such that $P \emptyset Q$, there exists $\sigma \varepsilon D_P$ such that $Q^{\sigma} = R$.

Notation: If (A10)(P) is valid, let $\sigma = \sigma[PQR]$ be the unique dilatation mapping P to P and Q to R.

Comment (5.1.1). (A10) is a generalization of Artin's axiom 4b and (A10)(P) of Artin's axiom 4bP. [cf. [A2].] Klingenberg, in [K2], defined (A10) without the stipulation $\tau_1 \notin N$. However this is sufficient and, in view of Corollary (1) to Theorem (4.2.5), quite natural.

Theorem (5.1.1). Let $\mathfrak{X}$ be a M.D. plane. The following are equivalent.

(1) (A10).

(2) (A10)(P) holds for every P.

(3) For each set $\{P, Q, R, S \mid P \not\simeq Q$ and there exist m, $R, S I m$, such that $m \parallel PQ\}$ there exists $\sigma \varepsilon D$ such that $P^{\sigma} = R$ and $Q^{\sigma} = S$.

(4) There exists P_0 such that for each set $\{P_0, Q, R, S \mid P_0 \not\simeq Q$ and there exists m; $R, S I m$, such that $m \parallel PQ\}$, there exists $\sigma \varepsilon D$ such that $P_0^{\sigma} = R$ and $Q^{\sigma} = S$.

Proof: (1)$\Longrightarrow$(2). Take (PQR) such that $P \not\simeq Q$. Define $\tau_1 = \tau_{PQ}$ and $\tau_2 = \tau_{PR}$. Thus $\tau_1 \varepsilon N$. Since P, Q and R are collinear, $D_{\tau_1} \subseteq D_{\tau_2}$. (A10) then implies there exists $\delta \varepsilon H$ such that $\tau_1^{\delta} = \tau_2$. From Theorem (4.2.5), let $\sigma = \sigma(\delta)$. Hence $\sigma \varepsilon D_P$ and $S^{\sigma} = P^{\tau_{PS}}$. Thus

$$Q^{\sigma} = P^{\tau_{PQ}^{\delta}} = P^{\tau_1^{\delta}} = P^{\tau_2} = R.$$

(2)$\Longrightarrow$(3). Select $\{P, Q, R, S\}$ such that $P \not\in Q$ and there exists m; $R, S I m$, such that $m \parallel PQ$. Define $\tau = \tau_{PR}$ and $S^{\tau^{-1}} = T$. Hence

$$\{T = S^{\tau^{-1}}\} I \{L(R^{\tau^{-1}}, m) = L(P, m) = \ell\}.$$

Thus there exists $\sigma = \sigma[PQT]$. Define $\tilde{\sigma} = \tau\sigma$. This is our desired dilatation since

$$P^{\tilde{\sigma}} = P^{\tau\sigma} = (P^{\sigma})^{\tau} = P^{\tau} = R$$

and

$$Q^{\tilde{\sigma}} = Q^{\tau\sigma} = (Q^{\sigma})^{\tau} = T^{\tau} = R.$$

(3)$\Longrightarrow$(4). Obvious.

(4)$\Longrightarrow$(1). Let $D_{\tau_1} \subseteq D_{\tau_2}$ such that $\tau_1 \notin N$. Let $\tau_1 = \tau_{P_0T_1}$ and $\tau_2 = \tau_{P_0T_2}$ by (A9). Thus $P_0 \not\in T$. Since $D_{\tau_1} \subseteq D_{\tau_2}$, we have $T_2 I P_0 T_1$. By (A10)(P), there exists $\sigma = \sigma[P_0 T_1 T_2]$. Define $\delta = \phi_{P_0}^{-1}(\sigma)$; thus $\sigma = \sigma(\delta)$; cf. Theorem (4.2.5). Then clearly we have

$$\tau_1^{\delta} = \tau_{P_0T_1}^{\delta} = \tau_{P_0T_1^{\sigma}} = \tau_{P_0T_2} = \tau_2.$$

Corollary. (A10)(P:o) holds iff $D_{\tau_1} \subseteq D_{\tau_2}$ such that $\tau_1 \notin N$, $\tau_2 \in N$ implies there exists $\delta \in \Pi$ such that $\tau_1^{\delta} = \tau_2$.

Proof: The results follows from Corollary (4) of Theorem (4.2.5) and the above theorem.

Definition (5.1.2). Let $\mathcal{L}$ be a M.D. plane. $\mathcal{L}$ is called a Desarguesian plane or D-plane iff one of the conditions of Theorem (5.1.1) holds.

Lemma (5.1.1). Let $\mathcal{L}$ be a D-plane. Then ϕ: $D \to \overline{D}$ is onto and $D/\theta_{\phi} = \overline{D}$. [cf. Theorem (3.2.3)].

Proof: Let $\delta \in \overline{D}$. If $\delta = 0_{\overline{P}}$ for some $\overline{P}$, then $\phi(0_P) = 0_{\overline{P}}$. Hence we may assume δ is non-degenerate. Choose $\overline{P}_1$, $\overline{P}_2$ such that $\overline{P}_1 \neq \overline{P}_2$. Let $(\overline{P}_i)^{\delta} = \overline{Q}_i$; $i = 1, 2$. Since $\delta \in \overline{D}^*$, $\overline{Q}_1 \neq \overline{Q}_2$ and $\overline{P}_1\overline{P}_2 \parallel \overline{Q}_1\overline{Q}_2$. By Lemma (1.2.7)(3), there exist m_1, m_2, $m_1 o \ell_1$, $m_2 o \ell_2$ such that $m_1 \parallel m_2$. Thus there exist X_1, X_2; X_1, $X_2 I m_1$, such that $X_1 o P$ and $X_2 o P_2$. Hence $X_1 \not o X_2$. Also there exist Y_1, Y_2; Y_1, $Y_2 I m_2$ such that $Y_1 o Q_1$ and $Y_2 o Q_2$. Now consider $\{X_1, X_2, Y_1, Y_2\}$. Clearly $X_1 \not o X_2$ and Y_1, $Y_2 I m_2$ such that $m_2 \parallel X_1X_2$.

By condition (3) of Theorem (5.1.1), there exists $\sigma \in D$ such that $X_1^{\sigma} = Y_1$ and $X_2^{\sigma} = Y_2$. We next show

$\phi(\sigma) = \bar{\sigma} = \delta$. Now $(\bar{P}_i)^{\bar{\sigma}} = (\bar{X}_i)^{\bar{\sigma}} = \overline{X_i^{\sigma}} = \bar{Y}_i = \bar{Q}_i$; $i = 1, 2$. Hence the result follows from Theorem (3.1.1). The last statement is a consequence of Theorem (3.2.3).

Corollary.

$\phi|D_P: \; D_P \to \bar{D}_P$ is surjective

and

$D_P/\theta \cap (D_P x D_P) \cong \bar{D}_P$.

The next result, was proved by in [L1] for Desarguesian $J(T, \beta)$ structures. We will present a proof in our context.

Theorem (5.1.2). Let $\mathcal{H}$ be a M.D. plane with (A10)(P:0). The following conditions are equivalent.

(1) $\mathcal{H}$ is uniform.
(2) $M_P^2 = \{0_P\}$ for each P.
(3) $\pi^2 = \{0\}$.

Proof: From Theorem (4.2.5), it follows that (2) and (3) are equivalent. From Theorem (3.1.7), (1)$\Longrightarrow$(2). It remains only to show (2)$\Longrightarrow$(1). Let $Q_2 I g_2$, ℓ_2; $g_2 o \ell_2$, PIg_2; and PoQ_2. We must prove $PI\ell_2$.

By the Corollary of Lemma (1.1.14), there exists Λ_{g_1} such that $\Lambda_{g_1} \not\! o \Lambda_{g_2}$, Λ_{ℓ_2}, and PIg_1. Thus $P = g_1 \wedge g_2$ and $S = g_1 \wedge \ell_2$ exists. Choose $P_2 I g_2$ such that $P \not\! o P_2$; define $\ell_1 = L(P_2, \ell_2)$. Then $\Lambda_{\ell_1} \not\! o \Lambda_{\ell_2}$ and so $Q_1 = \ell_1 \wedge g_1$. Select P_1, $P_1 Ig$, such that $P_1 \not\! o P$. Since $\ell_2 o g_2$ and $\ell_i \not\! o g_2$; $i = 1, 2$, then by (A7), $\ell_1 o g_2$. Thus $g_1 \not\! o g_2$ implies $g_1 \not\! o \ell_1$. However, $g_1 \not\! o g_2$; $\ell_1 o g_2$; PIg_1, g_2; and $Q_1 I \ell_1$, g_1, imply PoQ_1 by (A6). Let $\sigma_i = \sigma_i [PP_iQ_i]$; $i = 1, 2$, which exist by (A10)(P:0). Moreover $\sigma_i \varepsilon M_P$, by Corollary (1) of Theorem (3.1.2). Hence by (2), $\sigma_2\sigma_1 = 0_P$ and thus

$$\{P = P_1^{\sigma_2\sigma_1} = (P_1^{\sigma_1})^{\sigma_2} = Q_1^{\sigma_2}\} I \{L(Q_2, \ell_1) = \ell_2\}.$$

Dembowski, in [D1], remarks that $\mathfrak{X}$ is Desarguesian iff H is transitive on T_Λ for each Λ. The following remark shows this is a hasty generalization of Artin's axiom 4b, which is exactly what Dembowski quoted.

<u>Remark (5.1.1)</u>. Let $\mathfrak{X}$ be a M.D. plane. The following are equivalent:

(1) $\mathfrak{X}$ is an ordinary Desarguesian affine plane.

(2) H is transitive on T_Λ for each Λ.

Proof: (1)$\Longrightarrow$(2). This follows immediately from Theorem (5.1.1).

(2)$\Longrightarrow$(1). It is enough to show $N = \{i\}$. Suppose there exists $\tau \varepsilon N$, $\tau \neq i$.

Choose $\tau_1 \notin N$ such that $D_{\tau_1} \subseteq D_{\tau_2}$. Hence $\tau_i \varepsilon T_\Lambda$; $i = 1, 2$, for some Λ. (2) then implies there exists $\delta \varepsilon H$ such that $\tau_2^\delta = \tau_1$. But then $\tau_1 = \tau_2^\delta \varepsilon N^\delta \subseteq N$, by Corollary (3) of Theorem (4.2.5). Contradiction. //

The next result is, in a sense, a converse of Theorem (3.1.5).

Theorem (5.1.2). Let $\mathcal{H}$ be an affine H-plane with (A10)(P:0). Then if $S_1 \neq S_2$, $S_1 o S_2$, there exist P and $\sigma \varepsilon M_P$, $\sigma \neq 0_P$, such that $S_1^\sigma = S_2^\sigma$.

Proof: Choose g such that S_1, $S_2 Ig$. Take P such that $P \not{o} X$ for each XIg. Define $h = PS_1$ and $f = PS_2$. Now $h o f$, otherwise PIh, f; $S_1 Ih$; $S_2 If$; and $S_1 o S_2$ imply PoS_1, S_2 by (A6). Contradiction. Now hof implies there exists R, $R \neq P$, such that RIh, f. By definition, PoR. Thus by (A10)(P:0), there exists $\sigma = \sigma[PS_1R] \varepsilon M_P$. $\sigma \neq 0_P$ since $P \neq R$.

Claim. $S_2^\sigma = R$. Define $g = L(R, g)$.

By the choice of P, $h \not{o} g$. Hence $g \not{o} h$ by (A7) and so

$g \not\phi f$. Thus $R = g \wedge f$. But $\sigma \varepsilon M_P$ implies

$$S_2^\sigma I\{L(S_1^\sigma, g) = L(R, g) = g\}, \{L(P, j) = f\}.$$

Hence $S_2^\sigma = R$. //

The following two corollaries were proved in [K2] where $\mathcal{X}$ was a M.D. plane with (A10).

Corollary [K2]. For each $\tau \neq i$, $\tau \varepsilon N$, there exists $\delta \varepsilon \mathcal{N} \ni \delta \neq 0$, such that $\tau^\delta = i$, if $\mathcal{X}$ is a M.D. plane with (A10)(P:0).

Proof: Let $\tau \neq i$; where $\tau \varepsilon N$. Choose any point S_1 and put $S_2 = S_1^\tau$. Then $\tau \neq i$, $\tau = \tau_{S_1 S_2} \varepsilon N$ implies $S_1 o S_2$ and $S_1 \neq S_2$. By the theorem there exist P and σ, $\sigma \varepsilon M_P$, such that $S_1^\sigma = S_2^\sigma$ and $\sigma \neq 0_P$. Put $\delta = \phi_P^{-1}(\sigma)$ and hence $\sigma = \sigma(\delta)$. Thus $\delta \neq 0$, since $\sigma \neq 0_P$. Then

$$\tau^\delta_{PS_1} = \tau_{PS_1^\sigma} = \tau_{PS_2^\sigma} = \tau^\delta_{PS_2} = (\tau_{PS_1} \cdot \tau_{S_1 S_2})^\delta$$

$$= \tau^\delta_{PS_1} \cdot \tau^\delta_{S_1 S_2}.$$

Hence $\tau^\delta_{S_1 S_2} = \tau^\delta = i$.

Corollary (2). [K2] . If $\mathcal{X}$ is a M.D. plane

<u>with</u> (A10)(P:0), <u>then</u> $\mathcal{N} = D_0$. [cf. Definition (2.2.1)].

<u>Proof</u>: In any ring, $D_0 \subseteq \mathcal{N}$. We show the converse. Let $\delta \varepsilon \mathcal{N}$. By Corollary (5) of Theorem (4.2.5), there exists $\tau \in N$, $\tau \neq i$, such that $\tau^{\delta} = i$. Select $\tilde{\tau} \notin N$ such that $D_{\tilde{\tau}} \subseteq D_{\tau}$. Then by the Corollary to Theorem (5.1.1), there exists $\rho \varepsilon H$ such that $\tilde{\tau}^{\delta} = \tau$. Then since $\tilde{\tau} \notin N$,

$$\tilde{\tau}^{\delta\rho} = (\tilde{\tau}^{\rho})^{\delta} = \tau^{\delta} = i$$

implies $\delta\rho = 0$ by Corollary (2) to Theorem (4.2.5). Hence $\delta \varepsilon D_+$. Now $\tilde{\tau}^{\delta} \varepsilon N^{\delta} \subseteq N$ by Corollary (3) of Theorem (4.2.5). Moreover $\tilde{\tau}^{\delta} \neq i$, in view of Corollary (1) to Theorem (4.2.5). By Corollary (1), there exists $\xi \varepsilon \mathcal{N}$, $\xi \neq 0$ such that $i = (\tilde{\tau}^{\delta})^{\xi} = \tilde{\tau}^{\xi\delta}$. Hence $\xi\delta = 0$ and so $\delta \varepsilon D_-$. Thus $\delta \varepsilon D_+ \cap D_- = D$. //

It would be nice to know whether or not the various systems of axioms are independent. For ordinary affine planes we know A(10)(P) implies (A9) but (A10) does not. I tried to show (A10)(P) implies (A9), but I had to assume $T = \tilde{T}$. However, for $J(T, \Pi)$ type planes, this result is meaningless in view of the Corollary to Theorem (4.1.2).

<u>Remark (5.1.2)</u>. The following are equivalent.

(1) Every line ℓ has three points S_1, S_2, S_3 such that $S_1 \not{o} S_2 \not{o} S_3 \not{o} S_1$.

(2) There exists ℓ with three points S_1, S_2, S_3 such that $S_1 \not{o} S_2 \not{o} S_3 \not{o} S_1$.

<u>Proof</u>: This is an immediate consequence of Lemma (1.2.2).

<u>Remark (5.1.3)</u>. If $T = \tilde{T}$, then T is a group.

<u>Proof</u>: From Comment (3.1.1), it suffices to show T is closed under composition. Let $\tau_1, \tau_2 \varepsilon T = \tilde{T}$. If $\tau_1\tau_2$ has a fixed point P then $P^{\tau_2} = P^{\tau_1^{-1}}$. Hence by Theorem (3.2.1), $\tau_2 = \tau_1^{-1}$ or $\tau_2\tau_1 = i$. Thus $\tau_1\tau_2 \varepsilon T$.

<u>Theorem (5.1.3)</u>. <u>Let</u> $\mathcal{H}$ <u>be an affine H-plane with</u> $T = \tilde{T}$. <u>Also there exist</u> ℓ, <u>and</u> $S_1, S_2, S_3 I \ell$ <u>such that</u> $S_1 \not{o} S_2 \not{o} S_3 \not{o} S_1$. <u>Then</u> (A10)(P) <u>implies</u> $\mathcal{H}$ <u>is a</u> M.D. <u>plane</u>.

<u>Proof</u>: By Remark (5.1.3), T is a group. It suffices to prove (A9) for $P \not{o} Q$. For it PoQ, choose R, $R \not{o} P$, Q. Then since T is a group, $\tau_{PQ} = \tau_{PR} \cdot \tau_{RQ}$.

Now let $g = PQ$. Let $R \bar{I} g$ such that $R \not{o} P \not{o} Q \not{o} R$. By (A10)(R), there exists $\sigma_1 = \sigma[RPQ] \; \varepsilon D_R^*$.

Now choose T such that $T \not{o} X$ for each XIg. Let $T^{\sigma_1} = S$; $\ell = PT$ and $h = RT$. Then S $I\{L(R, RT) = h\}$.

Define $j = L(T, g)$ and $k = L(Q, \ell)$. By the choice of T, $j \neq g$, $g \not o \ell$ and so $\Lambda_g \not o \Lambda_\ell$. Hence $\Lambda_j \not o \Lambda_k$. Let $M = j \wedge k$. Since $P \not o T$ and $\sigma_1 \varepsilon D_R^*$, it follows by Corollary (2) to Theorem (3.1.2) that $P^{\sigma_1} \not o T^{\sigma_1}$, or equivalently, $Q \not o S$. Let $a = TQ$.

<u>Claims</u>. (a) $a \not o g, \ell, k$.

(b) $T \not o M$.

(a) From the choice of T, $a \not o g$. If $a o \ell$, then $a \not o g$; $PI\ell, g$; and QIa, g implies PoQ by (A6). Contradiction. Since $k \parallel \ell$, $a \not o k$ follows from (A7).

(b) Suppose ToM. Then $a \not o k$; T, QIa; and M, QIk imply QoT, by (A5). Contradiction.

Now define $\sigma_2 = \sigma[QSM]$ by (A10)(0). Define $\sigma = \sigma_2\sigma_1$. Clearly $P^\sigma = Q$. We need only show $\sigma \varepsilon \tilde{T}$. Suppose there exists X such that $X^\sigma = X$.

<u>Claims</u>. (a) If X, PIr, then XIg.

(b) If X, TIn, then XIj.

(a) Let X, PIr. Then $X^\sigma IL(P^\sigma, r)$, or equivalently, $XIL(Q, r)$. Then $XIr, L(Q, r)$ implies $r = L(Q, r)$. Since $P \not o Q$, $r = PQ$ and so XIg.

(b) The proof is the same as (a) using the facts that $T^\sigma = M$ and $T \not o M$.

Hence XIg, j. But $g \neq j$ and $g \parallel j$. Contradiction. Hence $\sigma \varepsilon \tilde{T}$.

5.2. Coordinates in Desarguesian Affine H-planes

We shall assume $\mathcal{H}$ is Desarguesian throughout this section. Klingenberg coordinatized the plane in [K2], generalizing the methods of Artin. We shall introduce coordinates for lines and divide the lines into two kinds. We shall then study the interaction of the two line kinds and this will motivate us in our construction of an analytic model of a Desarguesian affine H-plane.

Theorem (5.2.1). [K2] Let $\Lambda_1 \not\sim \Lambda_2$. Let $\tau_i \varepsilon T_{\Lambda_i} \cap \complement N$; $i = 1, 2$. Then for each $\tau \varepsilon T$ there exist $\delta, \beta \varepsilon H$ such that $\tau = \tau_1^{\delta} \tau_2^{\beta}$.

Proof: Take $\tau \varepsilon T$. By (2) of Theorem (4.2.2), there exists a unique representation of τ such that $\tau = \tau_1 \tau_2$, $\tau_i \varepsilon T_{\Lambda_i}$; $i = 1, 2$.

Now $D_{\tau_i} \subseteq D_{\tilde{\tau}_i}$ and $\tau_1 \notin N$. Hence by (A10) and Corollary (2) to Theorem (4.2.5), there exists a unique $\delta_i \varepsilon H$ such that $\tau_i^{\delta_i} = \tau_i$; $i = 1, 2$. Hence $\tau = \tau_1^{\delta_1} \tau_2^{\delta_2}$

is a unique representation. //

The next result is never stated explicitly in [K2], but seems to be unconsciously employed, as in Theorem (5.2.2).

Lemma (5.2.1). Let τ_1 and τ_2 be as in Theorem (5.2.1). Let $\tau = \tau_1^{\alpha}\tau_2^{\beta}$ be the unique representation of τ. The following are equivalent.

(1) $\tau \notin N$.

(2) $\alpha \notin \mathfrak{N}$ or $\beta \notin \mathfrak{N}$.

Proof: (1)$\Longrightarrow$(2). Suppose $\alpha, \beta \in \mathfrak{N}$. Thus $\tau = \tau_1^{\alpha}\tau_2^{\beta} \in N.N \subseteq N$ by Corollary (3) of Theorem (4.2.5). Contradiction.

(2)$\Longrightarrow$(1). Suppose $\tau \in N$. Without loss of generality, assume $\alpha \notin \mathfrak{N}$. Define $\tilde{\tau} = \tau^{\alpha^{-1}}$. Then

$$\tilde{\tau} = \tau^{\alpha^{-1}} = (\tau_1^{\alpha}\tau_2^{\beta})^{\alpha^{-1}} = \tau_1\tau_2^{\alpha^{-1}\beta}. \qquad \text{(I)}$$

But $\tau = \tau^{\alpha^{-1}} \in N^{\alpha^{-1}} \subseteq N$, by Corollary (3) of Theorem (4.2.5). Choose $\tau_3 \notin N$ such that $D_{\tau_3} \subseteq D_{\tilde{\tau}}$. By (A10), there exists $\delta \in H$ such that $\tau_3^{\delta} = \tilde{\tau}$. Since $\tau_3 \notin N$ and $\tilde{\tau} \in N$, it follows that $\delta \in \mathfrak{N}$ by Corollary (4) of Theorem (4.2.5). Now $\mathfrak{N} = D_0$ by Corollary (2) of Theorem (5.1.2). Hence $\delta \in D_0$ and so there exists $z \neq 0$, $z \in H$ such that

$$z\delta = 0. \qquad \text{(II)}$$

By Theorem (5.2.1), there exist unique $x, y \varepsilon H$ such that $\tau_3 = \tau_1^x \tau_2^y$. Then we obtain from (I)

$$\tau_1^1 \tau_2^{\alpha^{-1}\beta} = \tilde{\tau} = \tau_3^\delta = (\tau_1^x \tau_2^y)^\delta = \tau_1^{\delta x} \tau_2^{\delta y}. \qquad \text{(III)}$$

From (III), since τ has a unique representation, we have $\delta x = 1$. Combining this with (II) we obtain

$$0 = z\delta = (z\delta)x = z(\delta x) = z.1 = z.$$

Contradiction.

Introduction of Coordinates.

Let $\{0, X, Y\}$ be choosen as in (A2). Define $\tau_1 = \tau_{0X}$ and $\tau_2 = \tau_{0Y}$. Let $\Lambda_1 = \Lambda_{0X}$ and $\Lambda_2 = \Lambda_{0Y}$. Since $0 \not\in X$, $0 \not\in Y$ and $0X \cap 0Y = 0$, we have $\tau_i \notin N$; $i = 1, 2$, and $\Lambda_1 \not\in \Lambda_2$. We then apply Theorem (5.2.1). Let P be any point. Then $\tau_{0P} = \tau_1^\alpha \tau_2^\beta$ such that α and β are unique. We associate P with (α, β). Conversely if $\alpha, \beta \varepsilon H$, define $\tau = \tau_1^\alpha \tau_2^\beta$ and $P = 0^\tau$. Then $\tau_{0P} = \tau = \tau_1^\alpha \tau_2^\beta$ and hence P is associated with (α, β).

Definition (5.2.1). For every point P, the elements α, β.

obtained above are called its coordinates. $\{0, X, Y\}$ is called a coordinate system. $0X$ is the X-axis and $0Y$ is the Y-axis. For convenience, we write $P = (\alpha, \beta)$.

Throughout this section $\{0, X, Y\}$ is a fixed coordinate system. Clearly $0 = (0, 0)$, $X = (1, 0)$ and $Y = (0, 1)$.

In view of the remark preceding Lemma (5.2.1), we shall include a proof of the following theorem. Our proof is more direct than Klingenberg's.

Theorem (5.2.2). [K2] Let $P = (\delta, \beta)$ and $Q = (\alpha, \gamma)$. The following are equivalent.

(1) PoQ.

(2) $\delta - \alpha, \beta - \gamma \varepsilon \mathcal{N}$.

Proof: (1)$\Longrightarrow$(2). Let PoQ. Select a τ_1-trace ℓ through P and a τ_2-trace m through Q. Since $\Lambda_1 \not o \Lambda_2$, $\Lambda_\ell \not o \Lambda_m$ and hence there exists $R = \ell \wedge m$. Then

$(PP^{\tau_1}R)$ and $(QQ^{\tau_2}R)$ are both collinear triples. Now $\ell \not o m$; $R, PI\ell$; R, QIm; and PoQ imply RoP, Q by (A6). Thus $\tau_{QR}, \tau_{RP} \varepsilon N$. Now $\tau_{Q0} = (\tau_{0Q})^{-1} = \tau_1^{-\alpha}\tau_2^{-\beta}$ and $\tau_{0P} = \tau_1^{\delta}\tau_2^{\beta}$. Also $\tau_{QP} \varepsilon N$ since PoQ. Hence

$$\tau_{QP} = \tau_{Q0}\tau_{0P} = \tau_1^{\delta-\alpha}\tau_2^{\beta-\gamma} \varepsilon N.$$

By Lemma (5.2.1), $\delta - \alpha$, $\beta - \gamma \in \mathcal{N}$.

<u>(2)$\Longrightarrow$(1)</u>. Assume $\delta - \alpha$, $\beta - \gamma \in \mathcal{N}$. Then $\tau_{QO} = \tau_1^{-\alpha}\tau_2^{-\gamma}$ and $\tau_{OP} = \tau_1^{\delta}\tau_2^{\beta}$ as in (1)$\Longrightarrow$(2). Since $\delta - \alpha$, $\beta - \gamma \in \mathcal{N}$, then $\tau_1^{\delta-\alpha}$, $\tau_2^{\beta-\gamma} \in N$ by Corollary (3) of Theorem (4.2.5).

Hence we obtain:

$$\tau = \tau_{QP} = \tau_{QO}\tau_{OP} = (\tau_1^{-\alpha}\tau_2^{-\gamma})(\tau_1^{\delta}\tau_2^{\beta})$$

$$= \tau_1^{\delta-\alpha}\ \tau_2^{\beta-\alpha} \in N^2 \subseteq N.$$

Hence QoP. //

Again in the next theorem we give a slightly more direct proof.

<u>Theorem (5.2.3)</u>. [K2] <u>Let ℓ be any line and</u> $\{P = (c, d)\} I \ell$. <u>Choose</u> $\tau = \tau_1^a\tau_2^b \in N$ <u>such that</u> ℓ <u>is a</u> τ- <u>trace through</u> P. <u>Then</u> $\ell = \{(ta + c, tb + d) \mid t \in H\}$.

<u>Proof</u>. Let $QI\ell$. Thus $D_\tau \subseteq D_{\tau_{PQ}}$ and $\tau \notin N$. Hence there exists $t \in H$ such that $\tau^t = \tau_{PQ}$. Thus

$$\tau_{OQ} = \tau_{OP}\tau_{PQ} = (\tau_1^{c}\tau_2^{d})(\tau_1^{ta}\tau_2^{tb}) = \tau_1^{ta+c}.\tau_2^{tb+d}.$$

Hence $Q = (ta + c, tb + d)$.

Conversely, let $Q = (ta + c, tb + d)$. Then $\tau_{PQ} = \tau_{PO}\tau_{OQ}$. Thus

$$\tau_{PQ} = (\tau_1^{-c}\tau_2^{-d})(\tau_1^{ta+c}\tau_2^{tb+d}) = \tau_1^{ta}\tau_2^{tb} = (\tau_1^a\tau_2^b)^t = \tau^t.$$

Now since ℓ is a τ-trace, ℓ is a τ^t-trace and hence $\{P^{\tau^t} = Q\}I\ell$.

<u>Definition (5.2.2)</u>. If ℓ is any line and $\tau = \tau_1^a\tau_2^b \varepsilon N$ has ℓ as a τ-trace through P, where $P = (c, d)$, then $\{(ta + c, tb + d) | t\varepsilon H\}$ <u>is the unique line through</u> P <u>with direction</u> τ, and we call it $\ell(\tau, P)$.

<u>Remark (5.2.1)</u>. If $P = (a, b)$ and $Q = (c, d)$ such that $P \not\!\!\; Q$, then $PQ = \ell(\tau, P)$ such that $\tau_1^{c-a}\tau_2^{d-b} = \tau$.

<u>Proof</u>: $P \not\!\!\; Q$ implies $a - c \,\varepsilon\, \mathcal{T}$ or $b - d \notin \mathcal{T}$. Hence by Lemma (5.2.1), $\tau \notin N$. Then $\ell(\tau, P)$ defines a line containing P and Q, by Theorem (5.2.3). Since $P \not\!\!\; Q$, $PQ = \ell(\tau, P)$.

<u>Definition (5.2.3)</u>. ℓ is called a <u>line of the first kind</u> iff $\ell = [m, n]_I = \{(tm + n, t) | t\varepsilon H$ and $m\varepsilon\mathcal{T}\}$. ℓ is called a <u>line of the second kind</u> iff

$$\ell = [m, n]_{II} = \{(t, tm + n) | t\varepsilon H\}.$$

Theorem (5.2.4). For every line ℓ there exists $m, n \varepsilon H$ such that $\ell = [m, n]_I$ or $[m, n]_{II}$. Conversely given $m, n \varepsilon H$, there exists a line ℓ such that $\ell = [m, n]_I$ or $\ell = [m, n]_{II}$.

Proof: Let $\ell = \ell(\tau, P)$ such that $\tau = \tau_1^a \tau_2^b \notin N$ and $P = (c, d)$. Since $\tau \notin N$, $a \notin \mathcal{N}$ or $b \notin \mathcal{N}$ by Lemma (5.2.1). If $a \notin \mathcal{N}$, then $\ell = [a^{-1}b, d - ca^{-1}b]_{II}$. If $a \varepsilon \mathcal{N}$, then $b \notin \mathcal{N}$ and so $b^{-1}a \varepsilon H \mathcal{N} \subseteq \mathcal{N}$. Therefore $\ell = [b^{-1}a, c - db^{-1}a]_I$.

Now choose $m, n \varepsilon H$. If $m \varepsilon \mathcal{N}$, define $P = (0, n)$ and $\tau = \tau_1^1 \tau_2^m$. $\tau \notin N$ since $1 \notin \mathcal{N}$ by Lemma (5.2.1). Hence $\ell(\tau, P) = [m, n]_{II}$. If $m \varepsilon \mathcal{N}$, define $P = (0, n)$ and $\tau = \tau_1^1 \tau_2^m$. Consequently $\ell(P, \tau) = [m, n]_{II}$. Also if $Q = (n, 0)$ and $\tilde{\tau} = \tau_1^m \tau_2^1$, then $\ell(Q, \tilde{\tau}) = [m, n]_I$.

Corollary (1). If ℓ is a line through 0, then $\ell = [m, 0]_I$ or $[m, 0]_{II}$.

Corollary (2). If $\tau = \tau_1^a \tau_2^b$ such that $a \notin \mathcal{N}$, then $\ell(X, \tau) = [a^{-1}b, - a^{-1}b]_{II}$. In particular $0X = [0, 0]_{II}$. If $a \varepsilon \mathcal{N}$, then $\ell(X, \tau) = [b^{-1}a, 1]_I$.

Corollary (3). If $\tau = \tau_1^a \tau_2^b$ and $a \notin \mathcal{N}$, then $\ell(Y, \tau) = [a^{-1}b, 1]_{II}$. If $a \varepsilon \mathcal{N}$, then $\ell(Y, \tau) = [b^{-1}a, -b^{-1}a]_I$. In particular, $0Y = [0, 0]_I$.

Theorem (5.2.5). [Kl]. If $a, b \varepsilon \mathcal{N}$, then $a \varepsilon bH$ or $b \varepsilon aH$.

Proof: Let P = (a, b). Hence there exists ℓ such that P, $0 I \ell$. By Corollary (1) of Theorem (5.2.4), $\ell = [m, 0]_I$ or $[m, 0]_{II}$ and our result then follows.

Notation. $[m, n]$ refers to an arbitrary line whose kind is not stipulated.

Theorem (5.2.6). The following are true.

(1) $[m, n]_I \wedge [u, v]_{II} = ((vm + n)(1 - um)^{-1}, (nu + v)(1 - mn)^{-1})$. Hence $[m, n]_I \not\parallel [u, v]_{II}$ and $[m, n]_I \wedge [u, v]_{II} \neq \emptyset$.

(2) If $[m, n] \wedge [u, v] = \emptyset$, then $[m, n]$ and $[u, v]$ are of the same kind and $m - u \in \mathfrak{M}$.

Proof: (1) By definition, $m \in \mathfrak{M}$. Hence $um \in H\mathfrak{M} \subseteq \mathfrak{M}$ and $mu \in \mathfrak{M}H \subseteq \mathfrak{M}$. Since H is local, $1 - um \in \mathfrak{M}$ and $1 - mu \notin \mathfrak{M}$ by Theorem (2.2.2). Let $P = (x, y)$. Then

$$P I [m, n]_I \wedge [u, v]_{II}$$
$$\Updownarrow$$
$$x = ym + n \text{ and } y = xu + v$$
$$\Updownarrow$$
$$x(-1 - um) = vm + n \text{ and } y(1 - mn) = nu + v$$
$$\Updownarrow$$
$$x = (vm + n)(1 - um)^{-1} \text{ and } y = (nu + v)(1 - mn)^{-1}.$$

(2). By (1), the lines must be of the

same kind. If both are of the first kind then $m, n \in \mathfrak{N}$. Hence $m - u \in \mathfrak{N}$. If we have $[m, n]_{II}$ and $[u, v]_{II}$, and $m - u \notin \mathfrak{N}$, define

$$x = (v - n)(m - u)^{-1} \text{ and } y = xm + n.$$

Then $P = (x, y) I [m, n]_{II} \wedge [u, v]_{II}$. Contradiction.

Theorem (5.2.7). The following are equivalent.
(1) $\left| [m, n]_{II} \wedge [u, v]_{II} \right| = 1$.
(2) $m - u \notin \mathfrak{N}$.

Proof: (1)$\Longrightarrow$(2). Let $P = (\mathfrak{a}, \mathfrak{b}) = [m, n]_{II} \wedge [u, v]_{II}$. Then $\mathfrak{b}$ is the unique solution of the equation

$$x(m - u) = v - n. \qquad \text{(I)}$$

If $m - u \in \mathfrak{N}$, then since $\mathfrak{N} = D_0$, there exists $t_0 \neq 0$ such that $t_0(m - n) = 0$. Define $\tilde{\mathfrak{b}} = \mathfrak{b} + t_0$, $\tilde{\mathfrak{b}} \neq \mathfrak{b}$ since $t_0 \neq 0$. Then we obtain

$$\tilde{\mathfrak{b}}(m - u) = \mathfrak{b}(m - u) + t_0(m - u) = v - n + 0 = v - n.$$

Hence $\tilde{\mathfrak{b}} \neq \mathfrak{b}$ is a solution of (I). Contradiction.

(2)$\Longrightarrow$(1). Let $m - u \notin \mathfrak{N}$. Then we have $P = (x, y) I [m, n]_{II} \wedge [u, v]_{II}$ iff $x(m - u) = v - n$

iff $x = (v - n)(m - u)^{-1}$. Then it follows that

$[m, n]_{II} \wedge [u, v]_{II} = ((v - n)(m - u)^{-1}, (v - n)(m - u)^{-1}m + n)$.

Corollary. If $[m, n]_{II} \wedge [u, v]_{II} \neq 0$, then $[m, n]_{II} \circ [u, v]_{II}$ iff $m - u \varepsilon \mathcal{T}$.

Theorem (5.2.8). The following are equivalent.

(1) $[m, n] \circ [u, v]$.

(2) $[m, n]$ and $[u, v]$ are of the same kind and $m - u$, $n - v \varepsilon \mathcal{T}$.

Proof: $(1) \Longrightarrow (2)$. By (1) of Theorem (5.2.6), both lines are of the same kind.

Case (1): Both lines are of the second kind. Hence by Theorem (5.2.7), $m - u \varepsilon \mathcal{T}$. We must show $n - v \varepsilon \mathcal{T}$. Now $P = (0, n) I [m, n]_{II}$. Hence there exists $Q = (\mathfrak{a}, \mathfrak{b}) I [u, v]_{II}$ such that $Q \circ P$, or equivalently, $\mathfrak{b} = \mathfrak{a}u + v$ and $n - \mathfrak{b}, \mathfrak{a} \varepsilon \mathcal{T}$. Hence $\mathfrak{a}u \; \varepsilon \mathcal{T} H \subseteq \mathcal{T}$. Thus

$$n - \mathfrak{b} = n - \mathfrak{a}u - v = (n - v) - \mathfrak{a}u \varepsilon \mathcal{T}.$$

Hence $n - v \varepsilon \mathcal{T}$.

Case (2): Both lines are of the first kind. By definition $m, u \varepsilon \mathcal{T}$. Hence $m - u \varepsilon \mathcal{T}$. We show $n - v \varepsilon \mathcal{T}$ in the same fashion as we did for Case (1), utilizing the point $(n, 0) I [m, n]_{I}$.

<u>(2)$\Longrightarrow$(1)</u>. Consider the case where both lines are of the second kind. Let $P = (a, b)I[m, n]_{II}$ and so $b = am + n$. Define $x = (n - v) + a$ and $y = xu + v$. Clearly $Q = (x, u)I[u, v]_{II}$. We must show QoP. Now

$$x - a = n - v \varepsilon \mathcal{T}$$

and

$$\begin{aligned} b - y &= am + n - [\{(n - v) + a\}u + v] \\ &= am + n - (n - v)u - au - v \\ &= a(m - u) + (n - v)(1 - u) \\ &\varepsilon H\mathcal{T} + \mathcal{T}H \subseteq \mathcal{T} + \mathcal{T} \subseteq \mathcal{T} . \end{aligned}$$

Thus PoQ. Similarly for each $QI[u, v]_{II}$, there exists $PI[m, n]_{II}$ such that PoQ. Hence $[u, v]_{II} o [m, n]_{II}$. A similar argument works for lines of the first kind.

<u>Corollary (1)</u>. <u>If</u> $[m, n]_I \wedge [u, v]_I \neq \emptyset$, <u>then</u> $[m, n]_I o [u, v]_I$.

<u>Proof</u>: By definition, $m, u \varepsilon \mathcal{T}$ and hence $m - u \varepsilon \mathcal{T}$. Now by our assumption there exists P, $P = (a, b)I[m, n]_I \wedge [u, v]_I$. Then

$$a = bm + n = bu + v.$$

Thus we obtain

$$v - n = b(m - u) \varepsilon H\mathcal{T} \subseteq \mathcal{T} .$$

Corollary (2).

$$\left| [m, n]_I \wedge [u, v]_I \right| = 0$$

or

$$\left| [m, n]_I \quad [u, v]_I \right| > 1.$$

Proof: This follows immediately from Corollary (1) and (A3).

Corollary (3). The following are equivalent.

(1) $\left| [m, n] \wedge [u, v] \right| = 1$.

(2) $[m, n]$ and $[u, v]$ are of the second kind and $m - u \notin \Pi$ or the lines are of different kinds.

Proof: This is an immediate consequence of Corollary (2), Theorem (5.2.6)(1) and Theorem (5.2.7).

Corollary (4). If both lines $[m, n]$ and $[u, v]$ are of the same kind and $m - u \varepsilon \Pi$, then $\left| [m, n] \wedge [u, v] \right| = 0$ or $\left| [m, n] \wedge [n, v] \right| > 1$.

Proof: This follows from Corollaries (2) and (3).

Corollary (5). $[m, n]$ is a line of the second kind iff $\Lambda_{[m,n]} \not\subset \Lambda_{oY}^{-}$.

Proof: By Corollary (3)

$$\Lambda_{[m, n]} \not\subset \Lambda_{oY}$$

$$\Updownarrow$$

$$\left| [m, n] \wedge [0, 0]_I \right| = 1$$

$$\Updownarrow$$

$[m, n]$ is of the second kind.

<u>Theorem (5.2.9)</u>. <u>The following are equivalent</u>.

(1) $[m, n] \parallel [u, v]$.

(2) $[m, n]$ <u>and</u> $[u, v]$ <u>are of the same kind and</u> $m = u$.

<u>Proof</u>: From Lemma (1.1.3) and Theorem (5.2.6) the lines must be of the same kind. Suppose both are of the first kind. Let $\tau_m = \tau_1^m\tau_2^1$ and $\tau_u = \tau_1^u\tau_2^1$ be the directions of $[m, n]$ and $[u, v]$ respectively. By Lemma (5.2.1), $\tau_u, \tau_m \varepsilon N$. Thus

$$[m, n]_I \parallel [u, v]_I$$

$$\Updownarrow$$

$$D_{\tau_u} = D_{\tau_m}$$

$$\Updownarrow$$

there exists $t \varepsilon H$ such that $\tau_u^t = \tau_m$

$$\Updownarrow$$

$$(\tau_1^u\tau_2^1)^t = \tau_1^m\tau_2^1$$

$$\Updownarrow$$

$$\tau_1^{tu}\tau_2^t = \tau_1^m\tau_2^1.$$

By Theorem (5.2.1), this is equivalent to

$$tu = m \text{ and } t = 1$$

$$\Updownarrow$$

$$m = u.$$

A similar argument holds for lines of the second kind.

§5.3. The Analytic Model of an Affine H-plane

Let us first give the following definition.

Definition (5.3.1). [Kl] H is a projective Hjelmslev ring or H-ring iff H has the following properties.

(1) H is a local ring with a maximal ideal $\mathfrak{N}$.

(2) $\mathfrak{N} = D_0$.

(3) If a, $b \in \mathfrak{N}$, then $a \in bH$ or $b \in aH$.

(4) If a, $b \in \mathfrak{N}$, then $a \in Hb$ or $b \in Ha$.

Klingenberg actually called this an H-ring. He then constructed the analytic model of a projective H-plane in the following fashion.

$$P(H) = \langle \mathcal{P}, \mathcal{L}, I \rangle \quad \text{where}$$

$$P(x_0,x_1,x_2) \in \mathcal{P} \text{ iff } P(x_0,x_1,x_2) = \{(sx_0,sx_1,sx_2) \mid s \in H\},$$

such that at least one $x_i \notin \mathfrak{N}$, i = 0, 1, 2.

$$\ell[u_0,u_1,u_2] \in \mathcal{L} \text{ iff } \ell[u_0,u_1,u_2] = \{(u_0t,u_1t,u_2t) \mid t \in H\}.$$

such that at least one of the $u_i \notin \mathfrak{N}$, and $P(X_0, x_1, x_2)$ $I\ell[u_0, u_1, u_2]$ iff $\sum_{i=0}^{2} x_i u_i = 0$. For each $\ell[u_0, u_1, u_2]$, $P(H)(\ell)$ is an affine H-plane by Theorem (1.3.2). Define $P(H)(\ell) = A(H: \ell)$.

Dembowski states in [D1], on page 299, that given a projective H-ring, Klingenberg constructed an affine H-plane A(H), and then embedded it in P(H) such that $A(H) \cong A(H: \ell)$ for ℓ, a line of P(H).

However in [K1] and [K2], Klingenberg constructs P(H) and then considers $A(H: \ell)$, when H, in fact, is assumed to be commutative. In [K3], Section 5.3, page 20-21, he refers to the construction $A(H: \ell)$ again. He does not construct A(H) directly.

I shall now proceed to construct A(H) over a non-commutative ring which has properties (1), (2) and (3) but not property (4). One could not construct P(H) over this type of ring as it is property (4) which allows one to prove axiom (P2) of a projective H-plane.

<u>Definition (5.3.2)</u>. H is called an <u>affine Hjelmslev ring</u> or AH-<u>ring</u> iff the following conditions are valid

(i) H is local with a maximal ideal $\mathfrak{N}$.

(ii) $\mathfrak{N} = D_0$

(iii) For every $a, b \in \mathfrak{N}$, $a \in bH$ or $b \in aH$.

Comment (5.3.1). If H is commutative then clearly H is projective H-ring iff H is an A H -ring. However if H is not commutative, it is not known if they are still equivalent. If they were, there would be no need to construct A(H), directly as we could construct P(H) and then A(H: ℓ).

Lemma (5.3.1). Let H be an A H -ring. Then

(1) $\mathcal{T}_{+} = \mathcal{T}_{-} = \mathcal{T}$.

(2) $D_{+} = D_{-} = D_{0}$.

Proof: (1) This follows from Theorem (2.3.2)(2).

(2) $D_{+} \subseteq \mathcal{T}_{-} = D_0 = D_{+} \cap D_{-} \subseteq D_{+}$ and so $D_0 = D_{+}$.

Similarly $D_0 = D_{-}$. //

Construction of the analytic model over H, A(H) where H is an A H -ring.

Define $A(H) = \langle \mathcal{P}, \mathcal{L}, I, \| \rangle$ as follows:

$$\mathcal{P} = \{(a, b) \mid a, b \in H\}.$$

$$\mathcal{L} = \mathcal{L}_{I} \cup \mathcal{L}_{II}$$

such that

$$\mathcal{L}_{I} = \{[m, n]_{I} \mid m \in \mathcal{T} \text{ and } n \in H\}$$

and

$$\mathcal{L}_{II} = \{[m, n]_{II} \mid m, n \varepsilon H\},$$

where

$$[m, n]_{I} = \{(tm + n, t) \mid t \varepsilon H\} \text{ such that } m \varepsilon \pi$$

and

$$[m, n]_{II} = \{(t, tm + n) \mid t \varepsilon H\}.$$

$\mathcal{L}_I$ is the set of lines of the first kind. $\mathcal{L}_{II}$ is the set of lines of the second kind. We write $[m, n] \varepsilon \mathcal{L}$, for an arbitrary line.

$$P = (a, b) I \ell \Longleftrightarrow P \varepsilon \ell.$$

In view of Theorem (5.2.9), we finally define $[m, n] \parallel [s, t]$ iff both lines are of the same kind and $m = s$.

<u>Remark (5.3.1)</u>. Each line ℓ has the form $\ell = \{(ta + c, tb + d) \mid t \varepsilon H\}$, such that $a \notin \pi$ or $b \notin \pi$. Conversely each set of the above form is a line.

<u>Proof</u>: Obviously $[m, n]_I$ has $b = 1$ and $[m, n]_{II}$ has $a = 1$. The converse is shown as in Theorem (5.2.4).

<u>Remark (5.3.2)</u>. Let $P = (a, b)$ and $Q = (c, d)$. Then $P, Q I [m, n]_I$ iff $n = a - bm$ and $a - c = (b - d)m$ and $P, Q I [m, n]_{II}$ iff $n = b - am$ and $b - d = (a - c)m$.

<u>Lemma (5.3.2)</u>. A(H) <u>is an incidence structure with parallelism satisfying</u> (A1) <u>and</u> (A8).

<u>Proof</u>: The first part is obvious. Next we show

<u>(A1)</u>. Let $P = (a, b)$ and $Q = (c, d)$.

<u>Case (1)</u>: $a - c \notin \mathfrak{N}$ or $b - d \notin \mathfrak{N}$.
Then $\ell = \{(t(a - c) + a,\ t(b - d) + b) \mid t \in H\}$ is a line containing both P and Q, by Remark (5.3.1).

<u>Case (2)</u>: $a - c,\ b - d \in \mathfrak{N}$. From (iii) of Definition (5.3.2), there exists m_1 such that $a - c = (b - d)m_1$ or there exists m_2 such that $b - d = (a - c)m_2$. Define $n_1 = a - bm_1$, $\tilde{n}_1 = b - am_1^{-1}$ if $m_1 \in \mathfrak{N}$ and $n_2 = b - am_2$ if $m_1 \notin \mathfrak{N}$. Then $P, Q I [m_1, n_1]_I$; $P, Q I [m_1^{-1}, n_1]_{II}$; or $P, Q I [m_2^{-1}, m_2]_{II}$, by Remark (5.3.2).

Finally, we show

<u>(A8)</u>. Let $P = (\mathfrak{a}, \mathfrak{b})$ and $\ell = [m, n]_I$ ($[m, n]_{II}$).

Define $t = [m, \mathfrak{a} - \mathfrak{b}m]_I([m, \mathfrak{b} - \mathfrak{a}m]_{II})$
$= [m, \tilde{n}]_I([m, \tilde{n}]_{II})$.

Clearly, $t \parallel \ell$ and PIt. Let $\tilde{t} = [u, v] \parallel \ell$ and PIt. Hence $u = m$ and $\mathfrak{a} = \mathfrak{b}u + v$ ($\mathfrak{b} = \mathfrak{a}u + v$). But $\mathfrak{a} = \mathfrak{b}m + \tilde{n}$ ($\mathfrak{b} = \mathfrak{a}m + \tilde{n}$). Hence $v = \tilde{n}$ and so $\tilde{t} = t$.

<u>Lemma (5.3.3)</u>. (1) $[m, n]_I \wedge [u, v]_{II} = ((vm + n)$

(2) If $[m, n] \wedge [u, v] = \emptyset$, then $m - u \in \mathfrak{N}$.

Proof: It is the same as that of Theorem (5.2.6).

Lemma (5.3.4). Let $P = (a, b)$ and $Q = (c, d)$. The following are equivalent.

(1) PoQ.

(2) $a - c,\ b - d \in \mathfrak{N}$.

Proof: (1) $\Longrightarrow$ (2). Let PoQ. Hence there exist ℓ, m, $\ell \neq m$, such that $P, Q I \ell, m$. By Lemma (5.3.3), the lines are of the same kind. Let $\ell = [m, n] I$ and $m = [s, t]_I$. By Remark (5.3.2) we obtain

$$a - c = (b - d)m,$$
$$a - c = (b - d)s.$$

Hence $(b - d)(m - s) = 0$. But $m - s \neq 0$, for otherwise $m = s$ and then $\ell \parallel m$. By (A8), $\ell \wedge m = \emptyset$. Contradiction. Since by Lemma (5.3.1)(2), $D_+ = D_0$, we obtain

$$b - d \in D_+ = D_0 = \mathfrak{N}$$

and

$$a - c = (b - d)m \in \mathfrak{N} H \subseteq \mathfrak{N}.$$

Thus $b - d,\ a - c \in \mathfrak{N}$.

(2) $\Longrightarrow$ (1). Let $a - c,\ b - d \in \mathfrak{N}$. Since (A1) is

valid by Lemma (5.3.2), there exists $\ell = [m, n]$ such that P, $QI\ell$.

<u>Case (1)</u>: $b - d = 0$. Here we may assume $\ell = [0, b]_{II}$. Since $a - c \varepsilon \mathcal{T} = D_0$, there exists $t_0 \neq 0$ such that $(a - c)t_0 = 0$. Define $h = [t_0, b - at_0]_{II}$. Because $t_0 \neq 0$, $h \neq \ell$. It clearly follows from Remark (5.3.2) that P, QIh. Hence PoQ.

<u>Case (2)</u>: $b - d \neq 0$. Suppose $\ell = [m, n]_I$. Thus

$$n = a - bm \text{ and } a - c = (b - d)m . \qquad \text{(I)}$$

by Remark (5.3.2). Now $b - d \varepsilon \mathcal{T} = D_0$ implies there exists $t_0 \neq 0$ such that $(b - d)t_0 = 0$. Since $b - d \neq 0$, and $D_- = D_0 = \mathcal{T}$ by Lemma (5.3.1), we have $t_0 \varepsilon \mathcal{T}$. Because $\ell = [m, n]_I$, we have $m \varepsilon \mathcal{T}$ and hence $m + t_0 \varepsilon \mathcal{T} + \mathcal{T} \subseteq \mathcal{T}$. Thus we may define $h = [m + t_0, n - bt_0]_I$. $h \neq \ell$ since $t_0 \neq 0$. Also, by (I),

$$a - b(m + t_0) = a - b_m - bt_0 = n - bt_0$$

and

$$(b - d)(m + t_0) = (b - d)m + (b - d)t_0$$
$$= (b - d)m + 0 = a - c.$$

From Remark (5.3.2), P, QIh. Similarly, if $\ell = [m, n]_{II}$ we may find $h \neq \ell$, such that P, QIh. Hence PoQ.

Lemma (5.3.5). *The following are equivalent.*

(1) $|[m, n]_{II} \wedge [u, v]_{II}| = 1$.

(2) $m - u \notin \mathfrak{N}$.

Proof: The proof is exactly the same as that of Theorem (5.2.7).

Remark (5.3.3). $P = (\mathfrak{a}, \mathfrak{b}) \in [m, n]_I \wedge [u, v]_I$ *iff the following conditions hold.*

(i) $\mathfrak{a} = \mathfrak{b}m + n$

(ii) $\mathfrak{b}(m - u) = v - n$.

Lemma (5.3.6). *If* $[m, n]_I \wedge [u, v]_I \neq \emptyset$, *then*

$$|[m, n]_I \wedge [u, v]_I| > 1.$$

Proof: Let $P = (\mathfrak{a}, \mathfrak{b}) I [m, n]_I \wedge [u, v]_I$. By Remark (5.3.3), $\mathfrak{a} = \mathfrak{b}m + n$ and $\mathfrak{b}(m - u) = v - n$. Since both lines are of the first kind, $m, u \in \mathfrak{N}$ and so $m - u \in \mathfrak{N} = D_0$. Thus there exists $t_0 \neq 0$ such that $t_0(m - u) = 0$. Define $\tilde{\mathfrak{b}} = \mathfrak{b} + t_0$, and $\tilde{\mathfrak{a}} = \tilde{\mathfrak{b}}m + n$. Clearly $\tilde{\mathfrak{b}} \neq \mathfrak{b}$ and so $(\tilde{\mathfrak{a}}, \tilde{\mathfrak{b}}) \neq (\mathfrak{a}, \mathfrak{b})$. Now

$$\tilde{\mathfrak{a}} = \tilde{\mathfrak{b}}m + n$$

and

$$\tilde{\mathfrak{b}}(m - u) = (\mathfrak{b} + t_0)(m - u)$$
$$= \mathfrak{b}(m - u) + t_0(m - u) = \mathfrak{b}(m - u) = v - n.$$

Hence by Remark (5.3.3), $(a, b)I[m, n]_I \wedge [u, v]_1$.
Thus $[m, n]_I \wedge [u, v]_I > 1$.

Lemma (5.3.7). The following are equivalent.

(1) $[m, n] \circ [u, v]$.

(2) *Both lines are of the same type and* $m - u,\ n - v \in \Pi$.

Proof: Since Theorems (5.2.2), (5.2.6) and (5.2.7) are exactly the same as Lemmas (5.3.4), (5.3.3) and (5.3.5) respectively, the proof of this Lemma is the same as that of Theorem (5.2.8).

Lemma (5.3.8). Let $\ell = \{(ta + u, tb + v) \mid t \in H\}$; $a \notin \Pi$ or $b \notin \Pi$; *and* $h = \{(tc + u, td + v) \mid t \in H\}$; $c \notin \Pi$ *or* $b \notin \Pi$; *such that* $P = (u, v)I\ell, h$. *The following are equivalent.*

(1) $\ell \not\circ h$.

(2) *There exists* $t^* \in H$ *such that for each* $t \in H$, $t^*a - tc \notin \Pi$ *or* $t^*b - td \notin \Pi$

OR

there exists $\tilde{t} \in H$ *such that for each* $t \in H$, $\tilde{t}c - ta \notin \Pi$ *or* $\tilde{t}d - tb \notin \Pi$.

Proof: This follows immediately from the definition of ℓom and Lemma (5.3.4).

Lemma (5.3.9). Let P, ℓ *and* h *be chosen as in Lemma* (5.3.8). *The following are equivalent.*

(1) $P = \ell \wedge h$.

(2) $\ell \not\! o\, m$.

Proof: (1)$\Longrightarrow$(2). Let $P = \ell \wedge h$; $\ell = [m, n]$, $h = [r, s]$. By Lemma (5.3.7) it is sufficient to show the lines are of different kinds or $m - r \notin \mathfrak{N}$. If the lines are of the first kind, then $|\ell \wedge h| > 1$ by Lemma (5.3.6). Contradiction. Hence we may assume $\ell = [m, n]_{II}$ and $h = [r, s]_{II}$. The result then follows from Lemma (5.3.5).

(2)$\Longrightarrow$(1). Let $\ell \not\! o\, m$. Without loss of generality we may assume, by Lemma (5.3.8), there exists $t^* \varepsilon H$ such that for each $t \varepsilon H$

$$t^*a - tc \notin \mathfrak{N} \text{ or } t^*b - tb - td \notin \mathfrak{N} . \qquad \text{(I)}$$

We must show $\ell \wedge m = P$. Let $QI\ell, m$.

Thus

$$Q = (t_1 a + u,\ t_1 b + v) = (t_2 c + u,\ t_2 d + v),$$

and so

$$t_1 a = t_2 c \text{ and } t_1 b = t_2 d. \qquad \text{(II)}$$

Claim. $t_1 = 0$. Without loss of generality let $c \notin \pi$. Then from (I) $t_2 = t_1 ac^{-1}$ and so

$$t_1 b = t_2 d = t_1 ac^{-1} d. \qquad \text{(III)}$$

From (III), we obtain

$$t_1(b - ac^{-1}d) = 0, \text{where } t_1 \neq 0. \qquad \text{(IV)}$$

Hence by (IV), $b - ac^{-1}d \varepsilon D_- = D = \pi$. Therefore

$$t^*(b - ac^{-1}d) = t^*b - (t^*ac^{-1})d \varepsilon H\pi \subseteq \pi. \qquad \text{(V)}$$

Hence by (I), $t^*a - (t^*ac^{-1})c = t^*a - t^*a = 0 \notin \pi$. Contradiction.

Thus $t_1 = 0$, and so $Q = (u, v) = P$.

Theorem (5.3.1). *A(H) is an affine H-plane.*

Proof: We invoke Theorem (1.2.5). Now A(H) satisfies (A1), and (A8) from Lemma (5.3.2). From Lemma (5.3.9), A(H) satisfies (A4). Since H is a local ring, H/π is a division ring by Theorem (2.2.3)(3). Then it is well known that $A(H/\pi) = \langle \bar{\mathcal{P}}, \bar{\mathcal{L}}, I \rangle$, where $\bar{\mathcal{P}} = \{(a + \pi, b + \pi \mid a, b \varepsilon H\}$; the lines $\bar{\mathcal{L}}$ are of the form $[\pi, n + \pi]_I$ and $[m + \pi, n + \pi]_{II}$ and incidence is inclusion, is an ordinary affine plane. (cf. [A2]).

Define χ: $A(H) \to A(H/\pi)$ by

$\chi(a, b) = (a + \pi, b + \pi)$ and $\chi([m, n])$ $= [m + \pi, n + \pi]$.

χ is clearly an epimorphism. Also by Lemma (5.3.4), $\chi((a, b)) = \chi((c, d))$ iff $(a + \pi, b + \pi) = (c + \pi, d + \pi)$ iff $a - c$, $b - d \varepsilon \pi$ iff $P \circ Q$. From Lemma (5.3.7), $\chi([m, n]) = \chi([u, v])$ iff $[m + \pi, n + \pi] = [u + \pi, v + \pi]$ iff $m - u$, $n - v \varepsilon \pi$ and $[m, n]$ and $[u, v]$ are of the same kind iff $[m, n] \circ [u, v]$. Finally if $[m, n] \cap [u, v] = \emptyset$, then by Lemma (5.3.3)(2), $m - u \varepsilon \pi$ or equivalently $m + \pi = u + \pi$. Hence

$\{\chi([m, n]) = [m + \pi, n + \pi]\} \parallel \{[u + \pi, v + \pi]$ $= \chi(u, v)\}$.

<u>Theorem (5.3.2)</u>. <u>The following are equivalent</u>.

(1) σ <u>is a dilatation of</u> $A(H)$.

(2) <u>There exists</u> $C = (c, d)$ <u>and</u> $a \varepsilon H$ <u>such that</u>

$$(x, y)^{\sigma} = (ax + c, ay + d).$$

Proof: We first show every map of the form $(x, y)^\sigma = (ax + c, ay + d)$, is a dilatation. Choose $\ell = [m, n]_I$ and let $(x, y), (s, t) I [m, n]_I$.

Claim. $(s, t)^\sigma I L((x, y)^\sigma, [m, n]_I)$.

From the proof of Lemma (5.3.2), we obtain

$$L((x, y)^\sigma, [m, n]_I) = [m, (ax + c) - (ay + d)m]_I$$

$$= [m, \tilde{n}]_I.$$

Since $(s, t)^\sigma = (as + c, at + d)$ it suffices to show

$$as + c = (at + d)m + \tilde{n}.$$

But

$$(at + d)m + \tilde{n} = atm + dm + ax + c - aym - dm$$

$$= a(t - y)m + ax + c$$

$$= a(s - x) + ax + c$$

$$= as + c.$$

The same argument holds for $[m, n]_{II}$.

Conversely, we show every dilatation σ is of this form. Let $(0, 0)^\sigma = (c, d) = C.(0, 0)(1, 0) = [0, 0]_{II}$.

<u>Claim</u>. $(1, 0)^\sigma = (u,d)$ for some $u\varepsilon H$. Since σ is a dilatation, $(1, 0)^\sigma I\{L((c,d), [0, 0]_{II}) = [0, d]_{II}\}$. Let $(1, 0)^\sigma = (u, v)$. Hence $v = 0.u + d = d$.

Now define $a = u - c$. Then the map $\tilde{\sigma}$ defined by $(x, y)^{\tilde{\sigma}} = (ax + c, ay + d)$ is a dilatation. Now

$$(0, 0)^{\tilde{\sigma}} = C \text{ and } (1, 0)^{\tilde{\sigma}} = (a + c, d) = (u, d).$$

Since $(0, 0)\phi(1, 0)$, it follows that $\tilde{\sigma} = \sigma$, by Theorem (3.1.1).

<u>Notation</u>. $\sigma(a, C)$ is the dilatation defined by $(x, y)^{\sigma(a,C)} = (ax + c, ay + d)$, where $C = (c, d)$. Also we write, in general,

$$X + Y = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

and $aX = a(x, y) = (ax, ay)$.

<u>Theorem (5.3.3)</u>. <u>Let</u> $\sigma = \sigma(a, C)$. <u>Then</u>
(1) σ <u>is non-degenerate iff</u> $a \notin \mathfrak{N}$.
(2) σ <u>has a unique fixed point iff</u> $1 - a \notin \mathfrak{N}$.

<u>Proof</u>: (1) Let $a \notin \mathfrak{N}$. Then clearly $\sigma^{-1} = \sigma(a^{-1}, C)$. Conversely, suppose σ is non-degenerate. Then there exists σ^{-1} such that $\sigma^{-1} = \sigma(b, D)$. Let

Let $D = (p, q)$. Now $\sigma^{-1}\sigma = i$. Thus

$$(0, 0) = (0, 0)^{\sigma^{-1}\sigma} = (c, d)^{\sigma^{-1}}$$

$$= (bc + p, bd + q).$$

Hence $bc + p = 0$ and $bd + q = 0$. (I)

Then

$$(1, 0) = (1, 0)^{\sigma^{-1}o\sigma} = (a + c, d)^{\sigma^{-1}}$$

$$= (b(a + c) + p, bd + q)$$

$$= (ba + bc + p, bd + q).$$

By (I), this is equal to $(ba, 0)$. Hence $ba = 1$. Thus $a \notin \Pi_{-} = \Pi$ by Lemma (5.3.1)(1).

(2) Let $1 - a \notin \Pi$. Define $P = (1 - a)^{-1}C$.

Then

$$P^{\sigma} = a(1 - a)^{-1}C + C = (a(1 - a)^{-1} + 1)C$$

$$= (a(1 - a)^{-1} + (1 - a)(1 - a)^{-1})C$$

$$= (1 - a)^{-1}C = P.$$

P is unique since if $Q = (x, y)$ is any fixed point of σ, then $x = ax + c$ and $y = ay + d$. Hence we obtain since $1 - a \notin \Pi$, $x = (1 - a)^{-1}c$ and $y = (1 - a)^{-1}d$. Thus

$Q = P$.

Conversely, assume σ has a unique fixed point, $P = (x_0, y_0)$. Then the equations

$$(1 - a)x = c \text{ and } (1 - a)y = d \qquad \text{(I)}$$

have (x_0, y_0) as their unique solution.

If $1 - a \varepsilon \mathfrak{N} = D_0$ then there exists $t_0 \neq 0$ such that $(1 - a)t_0 = 0$. Define $\tilde{x}_0 = x_0 + t_0$ and $\tilde{y}_0 = y_0 + t_0$. Hence $(\tilde{x}_0, \tilde{y}_0) \neq (x_0, y_0)$. Let $\tilde{P} = (\tilde{x}_0, \tilde{y}_0)$. Now

$$(1 - a)\tilde{x}_0 = (1 - a)x_0 + (1 - a)\ t_0 = (1 - a)x_0 = c.$$

$$(1 - a)\tilde{y}_0 = (1 - a)y_0 + (1 - a)\ t_0 = (1 = a)y_0 = d.$$

Hence $\tilde{P}$ is another solution of (I). Contradiction. Thus $1 - a \notin \mathfrak{N}$.

<u>Theorem (5.3.4)</u>. <u>Let</u> $\sigma = \sigma(a, C)$.

(1) <u>If</u> σ <u>is</u> <u>a</u> <u>quasi-translation</u>, <u>then</u> $1 - a \varepsilon \mathfrak{N}$.

(2) <u>If</u> $1 - a \varepsilon \mathfrak{N}$ <u>and</u> $C \neq 0$, <u>then</u> σ <u>is</u> <u>a</u> <u>quasi-translation</u>.

<u>Proof</u>: (1) This follows immediately from Theorem (5.3.3)(2).

(2) Suppose σ has a fixed point $P =$

(x, y). Then

$$x = ax + c \text{ and } y = ay + d.$$

Hence since $1 - a \varepsilon \mathfrak{N}$,

$$(1 - a)x = c \varepsilon \mathfrak{N} H \subseteq \mathfrak{N} \text{ and } (1 - a)y = d \varepsilon \mathfrak{N} H \subseteq \mathfrak{N}.$$

Hence Co0. Contradiction. //

The next theorem was proved in [K2] for $A(H: \ell)$. The proof for $A(H)$ is naturally the same.

Theorem (5.3.5). τ *is a translation iff there exists* $C = (c, d)$ *such that* $(x, y)^{\tau} = (x, y) + (c, d)$.

Notation. $\tau(C)$ is the translation, $(x, y)^{\tau} = (x, y) + C$.

Corollary. $A(H)$ *is a minor Desarguesian plane. Hence* T *is an abelian transitive group.*

We next give necessarily and sufficient conditions for $T = \tilde{T}$ in $A(H)$.

Theorem (5.3.6). The following are equivalent in $A(H)$.

(1) $T = \tilde{T}$.

(2) $|\Pi| = 1$.

(3) H *is a division ring*.

(4) $A(H)$ *is an ordinary affine plane*.

Proof: *(1)$\Longrightarrow$(2)*. Since $T = \tilde{T}$, then $\tilde{T}$ is an abelian group by the Corollary to Theorem (5.3.5). Suppose $|\Pi| > 1$. Then there exists a, $a \neq 0$, such that $a \varepsilon \Pi$. Let $b = 1 - a$. Hence $1 - b = a \varepsilon \Pi$. Define $C = (1, 0)$. Clearly $C \not\varnothing 0$. Then by Theorem (5.3.4)(2), $\sigma = \sigma(b, C) \varepsilon \tilde{T}$. But $\tilde{T} = T$ and so $\sigma(b, C) = \tau(C)$. Hence $b = 1$ and so $a = 0$. Contradiction.

(2)$\Longrightarrow$(3). Obvious.

(3)$\Longrightarrow$(4). This is a well known result. [cf. [A2]].

(4)$\Longrightarrow$(1). This is also a result from [A2].

Comment (5.3.1). *The above theorem is not true for an arbitrary Desarguesian H-plane*.

Proof: To see this take an A H -ring H such that $|\Pi| > 1$. We shall see presently that such rings exist. Then A(H) is Desarguesian such that $T \neq \tilde{T}$ and not an ordinary affine plane by Theorem (5.3.6). By Theorem (4.1.2), there exists $J(T, \Pi)$

such that $A(H) \cong J(T, \Pi)$. Thus $J(T, \Pi)$ is a Desarguesian H-plane but it is not an ordinary affine plane. But by the Corollary to Theorem (4.1.2). $T(J) = T^*$. //

Klingenberg showed in [K2], that $A(H: \ell)$ was Desarguesian, by proving his variation of axiom (A10) [cf. Comment (5.1.1)]. We next shall show A(H) satisfies (A10)(0).

<u>Theorem (5.3.7)</u>. A(H) <u>is</u> <u>Desarguesian</u>.

<u>Proof</u>: We show A(H) satisfies (A10)(0). Choose three collinear points 0, (c_1, d_1) and (c_2, d_2), such that $0 \not\sim (c_1, d_1)$. Hence $c_1 \notin \pi$ or $d_1 \notin \pi$. Define $T_i = (c_i, d_i)$; $i = 1, 2$.

<u>Case (1)</u>: $c_1 \notin \pi$. Define $a = c_2 c_1^{-1}$. Let $\sigma = \sigma(a, 0)$. Then

$$T_1^{\sigma(a,0)} = (ac_1, ad_1) = (c_2, c_2 c_1^{-1} d_1). \qquad \text{(I)}$$

Now $0(c_1, d_1) = [c_1^{-1} d_1, 0]_{II}$. Then $(c_2, d_2) I [c_1^{-1} d_1, 0]_{II}$ implies $d_2 = c_2 c_1^{-1} d_1$. Hence from (I) we obtain

$$T_1^{\sigma(a, 0)} = (c_2, d_2) = T_2.$$

<u>Case (2)</u>: $c_1 \in \pi$ and $d_1 \notin \pi$. Let $a =$

$d_2 d_1^{-1}$ and define $\sigma = \sigma(a, 0)$. Now $0(c_1, d_1) = [d_1^{-1} c_1, 0]_I$, since $d_1^{-1} c_1 \varepsilon H \mathfrak{N} \subseteq \mathfrak{N}$. Since $(c_2, d_2) \, I [d_1^{-1} c_1, 0]_I$ we have $c_2 = d_2 d_1^{-1} c_1$, Hence as in Case (1),

$$T_1^{\sigma(a,0)} = T_2.$$

Therefore A(H) is Desarguesian.

We next state a result, shown in [K2] for A(H: ℓ), which we shall use later.

Theorem (5.3.8). Let $\tilde{H}$ be the ring of trace preserving endomorphisms of A(H), with $\tilde{\mathfrak{N}}$ its unique maximal ideal. Then

(1) $\tilde{\delta} \varepsilon \tilde{H}$ iff there exists $c \varepsilon H$ such that

$$\tau^{\tilde{\delta}}(a, b) = \tau(ca, cb).$$

We write $\tilde{\delta} = \tilde{\delta}(c)$, where $\tau^{\tilde{\delta}}(1, 0) = \tau(c, 0)$.

(2) $\tilde{\delta}(c) \varepsilon \mathfrak{N}$ iff $c \varepsilon \mathfrak{N}$.

(3) The map f: $\tilde{H} \to H$ defined by $f(\tilde{\delta}(c)) = c$ is a ring isomorphism.

Corollary. The coordinate division ring of $A(H/\mathfrak{N})$ is isomorphic to $\tilde{H}/\tilde{\mathfrak{N}}$. //

We end this section with the following result, mentioned in [D1], without the stipulation that H be commutative and quoted as being a result from [K2], which it, in fact, is not.

<u>Theorem (5.3.9).</u> <u>If</u> H <u>is</u> <u>a</u> <u>commutative</u> A H - <u>ring</u> <u>then</u> A(H) <u>may</u> <u>be</u> <u>embedded</u> <u>into</u> P (H).

<u>Proof:</u> Since H is commutative, H is a projective H-ring and P (H) can be constructed. Then 0 = (0, 0, 1), E = (1, 1, 1), X = (1, 0, 0), Y = (0, 1, 0) and XY = [0, 0, 1] form a coordinate structure for P (H). That is 0, E, X, Y satisfy axiom (P3). Then let ℓ = [0, 0, 1]. We define

$$g: A(H) \to A(H: \ell)$$

by

$$g((a, b)) = (a, b, 1),$$

$$g([m, n]_I) = [m, -1, n]$$

and

$$g([m, n]_{II}) = [-1, m, n].$$

This is an isomorphism, and so our Theorem is proved.

Comment. As mentioned before it is not known whether an A.H. ring is necessarily a projective-H-ring. Thus we cannot say that every Desarguesian affine-H-plane can be embedded in a projective-H-plane.

§5.4. Examples of H-rings and affine H-planes.

I. Let $H = \mathbb{Z}/p^n$ = integers modulo p^n, where p is a prime number. This is a projective H-ring and hence A(H) is an Desarguesian affine H-plane.

II. We exhibit a uniform Desarguesian affine H-plane such that $\tilde{T} \neq T$. The example is from [L1], but we may show the fact that $\tilde{T} \neq T$ in an easier fashion using Theorem (5.3.6). The construction is as follows.

Let D be a division ring. Then define $H(D) = \{(a, b) | a, b \in D\}$ with the following operations;

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2),$$

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1 a_2).$$

Then H(D) is an A.H. ring with $\mathfrak{N} = \{(0, y) | y \in D\}$. It is in fact a projective H-ring. Clearly $\mathfrak{N}^2 = 0$. Thus from Section 5.3 and Theorem (5.1.2) A(H(D)) is a uniform Desarguesian H-plane. Since $|\mathfrak{N}| = |D|$, A(H(D)) has $T \neq \tilde{T}$ provided $|D| > 1$. Finally H(D) is commutative iff D is commutative. We shall show this as it is not mentioned in [L1].

H(D) is commutative iff (a, b)(c, d) = (c, d)(a, b)

iff (ac, ad + bc) = (ca, cb + da)

iff ac = ca and ad + bc = cb + da for all a, b, c, $d \varepsilon D$

iff ac = ca for all a, $b \varepsilon D$.

III. The next example is due to Kleinfeld, and found in [K3].

Let F be a field and $\alpha \varepsilon \mathrm{Aut}\ F$.

Let $H(F) = \{(a, b) | a, b \varepsilon F\}$.

Addition is defined as in Example (I). Multiplication is $(a, b)(c, d) = (ac, ad + bc^{\alpha})$.

Then H(F) is a projective H-ring such that $\mathcal{T}^2 = 0$. Moreover H(F) is commutative iff $\alpha = i$.

IV. <u>We exhibit a non-uniform Desarguesian affine H-plane</u>.

The example is found in [K1].

Let K be a field. K[x] is the ring of polynomials. (x^n) is the ideal generated by x^n. Let $K[x]/(x^n) = K(n)$. Let $[P]_n$, where $P \varepsilon K[x]$, represent an arbitrary member of K(n). Then $\mathcal{T} = D_0 = \{[a_1 x + \ldots + a_{n-1}x^{n-1}]_n\}$, such that $\pi^{n-1} \neq 0$ but $\pi^{n} = 0$. Thus for $n > 2$, $A(K(n))$ is a non-uniform Desarguesian H-plane. Again $K(n)$ is in fact a projective H-ring. I can find no A H -ring which is not a projective H-ring.

§5.5. The Fundamental Theorem of a Desarguesian affine H-plane.

In this section we generalize a result of Artin's in [A2], for ordinary Desarguesian affine H-planes. Throughout this section $\{0, X_0, Y_0\}$ is a fixed coordinate system for a Desarguesian H-plane $\mathcal{H}$. Let $0 = (0, 0)$, $X_0 = (1, 0)$ and $Y_0 = (0, 1)$.

Remark (5.5.1). The set of points of a Desarguesian H-plane may be regarded as a left H-module M over the local ring H in the obvious manner, namely;

$$(a, b) + (c, d) = (a + c, b + d) \text{ and}$$

$$\alpha(a, b) = (\alpha a, \alpha b) \text{ for each } \alpha \varepsilon H.$$

Remark (5.5.2). Take $P(a, b)$ such that $P ø 0$. Then $Q \, I \, OP$ iff $Q = tP$ for some $t \varepsilon H$.

Proof: Since $P ø Q$, the result follows immediately from Remark (5.2.1).

Lemma (5.5.1). Let $P = (a, b)$ and $Q = (c, d)$. Assume (i) $P \not\approx 0$,

(ii) $Q \not\approx X$ for each $XIOP$.

Then P and Q are linearly independent with respect to the module structure on the points of $\mathfrak{X}$ [cf. Remark (5.5.1)].

Proof: From (i), (ii) and Remark (5.5.2), we obtain the relations

$$a \notin \mathfrak{N} \quad \text{or} \quad b \notin \mathfrak{N} \qquad \text{(I)}$$

and for each $t \in H$,

$$c - ta \notin \mathfrak{N} \quad \text{or} \quad d - tb \notin \mathfrak{N}\,. \qquad \text{(II)}$$

Now assume $\lambda_1 P + \lambda_2 Q = 0$. Hence

$$\left.\begin{array}{l} \lambda_1 a + \lambda_2 c = 0 \text{ (a)} \\ \lambda_1 b + \lambda_2 d = 0 \text{ (b)}, \end{array}\right\} \text{(III)}$$

From (I), assume $a \notin \mathfrak{N}$. Hence from (III)(a), we obtain

$$\lambda_1 = -\lambda_2 c a^{-1}.$$

Substituting in III(b), we obtain

$$-\lambda_2 ca^{-1}b + \lambda_2 d = 0 \text{ or } \lambda_2\, d - (ca^{-1})b = 0. \qquad \text{(IV)}$$

Let $t_0 = ca^{-1}$. Then

$$c - t_0 a = c - ca^{-1}a = 0.$$

Thus (II) yields $d - ca^{-1}b \notin \mathfrak{N}$. Hence from (IV), $\lambda_2 = 0$. Then III reduces to

$$\lambda_1 a = \lambda_1 b = 0.$$

Hence (I) implies $\lambda_1 = 0$. A similar argument applies in the case $b \notin \mathfrak{N}$. Hence P and Q are linearly independent.

Lemma (5.5.2). Let 0, PIg *and* 0, QIh. *If* gøh, *then* $P + Q = L(P, h) \wedge L(Q, g)$.

Proof: Let $\tau = \tau(P)$. Then by Case (1) of Theorem (3.2.1),

$$P \dotplus Q = Q^{\tau(P)} = L(P, g) \wedge L(Q, g).$$

Lemma (5.5.3). Let 0, PIg; 0, QIh; *such that* gøh. *Further, let* f *be an automorphism* of $\mathfrak{X}$ *such that* $f(0) = 0$.

Then $f(P + Q) = f(P) + f(Q)$

Proof: Since f is an automorphism and $f(0) = 0$, we have by Lemma (1.2.5)(2), 0, $f(P) I f(g)$; 0, $f(Q) I f(h)$ and $f(g) \phi f(h)$. Thus by Lemma (5.5.2), we obtain

$$f(P) + f(Q) = L(f(P), f(h)) \wedge L(f(Q), f(g)).$$

By Lemma (1.2.4) and Lemma (5.5.2), we also obtain

$$f(P + Q) = f(L(P, h) \wedge L(Q, h)) = f(L(P, h)) \wedge f(L(Q, h))$$

$$= L(f(P), f(h)) \wedge L(f(Q), f(h)).$$

Hence $f(P) + f(Q) = f(P + Q)$.

Lemma (5.5.4). If PIg, h then there exists R, $R \phi X$, for each $XIg \vee h$.

Proof: By Lemma (1.1.12), there exists $f \varepsilon \phi_P$ such that $f \phi g$, h. Choose R, RIf, such that $R \phi P$. If there exists X, XIg, such that $R \circ X$, then since $g \phi f$, $R \circ P$ by (A6). Contradiction. Hence $R \phi X$ for each XIg. Similarly $R \phi X$ for each XIh.

Theorem (5.5.1). Let $f \varepsilon Aut\ \mathfrak{X}$. The following are equivalent.

(1) $f(0) = 0$.

(2) $f(P + Q) = f(P) + f(Q)$.

Proof: (2)$\Longrightarrow$(1). $f(0) = f(0 + 0) = f(0) + f(0)$. Hence $f(0) = 0$.

(1)$\Longrightarrow$(2). Let $0, PIg$ and $0, QIh$.

Case (1): $g\not oh$. This follows immediately from Lemma (5.5.3).

Case (2): goh. Choose ℓ such that $0, P + QI\ell$. By Lemma (5.5.4), we may select R such that $R\not oX$ for each $XI\ell \vee h$. Choose m and t such that $0, RIm$ and $0, Q + RIt$.

Claim. (i) $m\not oh, \ell$.

(ii) $g\not ot$.

(i) This follows immediately from the choice of R.

(ii) It suffices to show that $Q + R\not oX$ for each XIh, since goh. Because by (i), $h\not om$, it follows from Lemma (5.5.2), that

$$Q + R = L(R, h) \wedge L(Q, m). \qquad \text{(I)}$$

By the choice of R, $L(R, h)\not oh$. But from (I), $Q + RIL(R,h)$. Hence by Lemma (1.1.10), $Q + R\not oX$ for each XIh.

By applying Case (i) to the three situations of the above claim, we obtain

$$f(Q + R) = f(Q) + f(R) \qquad \text{(a)}$$

$$f[(P + Q) + R] = f(P + Q) + f(R) \qquad \text{(b)}$$

$$f(P + [Q + R]) = f(P) + f(Q + R) \qquad \text{(c)}.$$

Combining (a), (b) and (c), we obtain

$$f(P + Q) + f(R) = f([P + Q] + R) = f(P + (Q + R))$$

$$= f(P) + f(Q + R) = f(P) + f(Q) + f(R).$$

Hence $f(P + Q) = f(P) + f(Q)$.

To formulate our main result in algebraic terms we need the following definition and two remarks.

<u>Definition (5.5.1)</u>. $\mathrm{Aut}\,(\mathcal{X} : 0) = \{f \mid f \varepsilon \mathrm{Aut}\ \mathcal{X}$ such that $f(0) = 0\}$. $\mathrm{Aut}\ H = \{\phi \mid \phi$ is a ring automorphism of $H\}$. $f \varepsilon G.L.(\mathcal{X} : 0)$ iff $f \varepsilon \mathrm{Aut}\ \mathcal{X}$ and $f : \mathbb{P} \to \mathbb{P}$ is an isomorphism with respect to the left H-module structure on $\mathbb{P}$. Then $f : \mathbb{P} \to \mathbb{P}$ is a member of the general linear group of this left H-module structure.

Remark (5.5.3). Aut ($\mathcal{X}$: 0), Aut H and G.L.($\mathcal{X}$: 0) are all groups under functional composition.

Proof: Aut $\mathcal{X}$ is a group by Theorem (1.2.3). Aut ($\mathcal{X}$: 0) and G.L.($\mathcal{X}$: 0) are easily seen to be a subgroup of Aut $\mathcal{X}$. It is well known that Aut H is a group.

Remark (5.5.4). If PoQ and $Q\not o 0$, then $P - Q \not o 0$.

Proof: Let $P = (a, b)$ and $Q = (c, d)$. By our assumptions, $c \notin \mathfrak{n}$ or $d \notin \mathfrak{n}$. $P - Q = (a - c, b - d)$. Suppose $P - Qo0$. Then $a - c, b - d \in \mathfrak{n}$. But $a, b \in \mathfrak{n}$. Hence $c, d \in \mathfrak{n}$ and so $Qo0$. Contradiction.

We may now state the fundamental theorem.

Theorem (5.5.2). [Fundamental Theorem]

(I) If $f \in$ Aut ($\mathcal{X}$: 0), then

(a) $\{f(X_0), f(Y_0)\}$ is a basis of M.

(b) There exists a unique ring isomorphism $\phi \in$ Aut H such that $f(aP) = \phi(a)f(P)$ for each P, $P\not o 0$. Moreover $f(a, b) = \phi(a)f(X_0) + \phi(b)f(Y_0)$. Let ϕ_f denote this unique ring isomorphism.

Let h: Aut ($\mathcal{X}$: 0)$\to$Aut (H) be the map $h(f) = \phi_f$.

(II) The map h: Aut $(\mathcal{X} : 0) \to \mathrm{Aut}\ H$ is an onto group homomorphism, whose kernel is $G.L(\mathcal{X} : 0)$. Hence $\mathrm{Aut}\ (\mathcal{X} : 0)/G.L.(\mathcal{X} : 0) \cong \mathrm{Aut}\ H$.

Proof: (I) (a) By the choice of $\{0, X_0, Y_0\}$, and Lemma (1.1.4), we obtain $0 \not{I} X_0$ and $Y_0 \not{I} T$ for each $T I 0X_0$. Since $f \in \mathrm{Aut}\ (\mathcal{X} : 0)$, we have from Lemma (1.2.4),

$$f(0X_0) = 0f(X_0) \text{ and } f(Y_0) \not{I} T \text{ for each } T I 0f(X_0).$$

Thus (a) follows from Lemma (5.5.1).

(b) We first show the uniqueness of ϕ. Suppose $\tilde{\phi}$ has this property. Then choose P such that $f(P) = (1, 1)$. Let $(1, 1) = E$. Clearly $E \not{I} 0$. Then

$$f(aP) = \tilde{\phi}(a)E = (\tilde{\phi}(a), \tilde{\phi}(a)).$$

Also

$$f(aP) = \phi(a)E = (\phi(a), \phi(a)).$$

Hence $\tilde{\phi}(a) = \phi(a)$.

Now we show the existence of ϕ. Choose $P \not{I} 0$. For each $a \in H$, $aPI0P$. Hence $0 \not{I} f(P)$ and $f(aP) I 0f(P)$, and f(aP)I0f(P). By Remark (5.5.2),

$$f(aP) = \phi(a, P)f(P).$$

<u>Claim (1)</u>. $\phi(a, P)$ <u>is independent of the choice of</u> P, P$\not o$0. Choose Q, Q$\not o$0. Let h = 0Q and g = 0P.

<u>Case (1)</u>: g$\not o$h. By (A6), it follows that Q$\not o$X for each XIg. Thus by Lemma (5.5.1), P and Q are independent.

$$P + Q = L(P, h) \wedge L(Q, g), \text{ by Lemma (5.5.1).}$$

By Lemma (1.1.10), L(Q, g)$\not o$g, and so P + Q$\not o$X for each XIg, in particular, P + Q$\not o$0. Thus

$$\phi(a, P)f(P) + \phi(a, Q)f(Q) = f(aP) + f(aQ)$$

$$= f(a(P + Q)) = \phi(a, P + Q)f(P + Q)$$

$$= \phi(a, P + Q)f(P) + \phi(a, P + Q)f(Q).$$

Hence we obtain

$$\phi(a, P) = \phi(a, Q) = \phi(a, P + Q).$$

<u>Case (2)</u>: goh. Choose R such that R$\not o$X for

for each XIg. Since goh, R$\not{o}$X for each XIh. Choose f such that 0, RIf. By the choice of R, f$\not{o}$g, h. By Case (1), we obtain

$$\phi(a, R) = \phi(a, P) \text{ and } \phi(a, R) = \phi(a, Q).$$

Hence $\phi(a, P) = \phi(a, Q)$.

Thus we may replace $\phi(a, P)$ by $\phi(a)$ and obtain

$$f(aP) = \phi(a)f(P) \text{ for each } P\not{o}0.$$

Similarly,

$$f^{-1}(aP) = \chi(a)f^{-1}(P) \text{ for each } P\not{o}0.$$

<u>Claim (2)</u>. <u>ϕ is a ring isomorphism.</u> For each $a\varepsilon H$,

$$\phi(\chi(a))X_0 = \phi(\chi(a))f(f^{-1}(X_0)) = f(\chi(a)f^{-1}(X_0))$$

$$= f(f^{-1}(aX_0)) = aX_0.$$

Hence $\phi(\chi(a)) = a$. Similarly $\chi(\phi(a)) = a$.

Hence ϕ is a (1 - 1) onto map with inverse χ.

Now choose P such that $f(P) = X_0$. Then

$$\phi(a + b)X_0 = \phi(a + b)f(P) = f((a + b)P) = f(aP + bP)$$

$$= f(aP) + f(bP) = \phi(a)f(P) + \phi(b)f(P)$$

$$= \left[\phi(a) + \phi(b)\right]f(P) = (\phi(a) + \phi(b))X_0.$$

Hence $\phi(a + b) = \phi(a) + \phi(b)$. Also

$$\phi(ab)X_0 = \phi(ab)f(P) = f(abP) = f(a(bP))$$

$$= \phi(a)f(bP) = \phi(a)\phi(b)f(P) = \phi(a)\phi(b)X_0.$$

Hence $\phi(ab) = \phi(a)\phi(b)$. Thus ϕ is a ring isomorphism. Finally,

$$f((a, b)) = f(aX_0 + bY_0) = f(aX_0) + f(bY_0)$$

$$= \phi(a)f(X_0) + \phi(b)f(Y_0).$$

(II) From (a), h is a mapping. To show h is a homomorphism, it is enough to show $\phi_{f_1} \phi_{f_2} = \phi_{f_1 f_2}$. Now for $P \neq 0$,

$$(f_1 f_2)(aP) = f_1(f_2(aP)) = f_1(\phi_{f_2}(a)f_2(P))$$

$$= \phi_{f_1}(\phi_{f_2}(a))f_1(f_2(P)) = (\phi_{f_1} \phi_{f_2})(a)(f_1 f_2)(P).$$

Hence by the uniqueness of (I), $\phi_{f_1} \phi_{f_2} = \phi_{f_1 f_2}$.

To show h is onto, choose $\phi \varepsilon \mathrm{Aut}\ H$. Define $f\colon \mathcal{X} \to \mathcal{X}$ by

$$f(P) = \phi(x)f(X_0) + \phi(y)f(Y_0), \text{ where } P = (x, y),$$

and

$$f([m, n]) = [\phi(m), \phi(n)].$$

It is easy to show that $f \varepsilon \mathrm{Aut}\ (\mathcal{X} : 0)$. Moreover

$$f(aP) = f((ax, ay)) = \phi(ax)f(X_0) + \phi(ay)f(Y_0)$$

$$= \phi(a)\phi(x)f(X_0) + \phi(a)\phi(y)f(Y_0)$$

$$= \phi(a)\left[\phi(x)f(X_0) + \phi(y)f(Y_0)\right]$$

$$= \phi(a)f(P).$$

Hence $h(f) = \phi$.

Finally, $f \varepsilon \mathrm{Ker}\ H \Longleftrightarrow \phi_f = i$. Let $f \varepsilon \mathrm{Ker}\ h$. To show $f \varepsilon G.L.(\mathcal{X} : 0)$, we must prove

$$f(aP) = af(P) \text{ for all } P \varepsilon \mathcal{P}.$$

<u>Case (1)</u>. $P \neq 0$. Immediately the definition of ϕ_f yields

$$f(aP) = \phi_f(a)f(P) = af(P).$$

<u>Case (2)</u>: $P o 0$. Choose $Q \not o 0$. By Remark (5.5.4), $P - Q \not o 0$. Thus by applying Case (2) to Q and $P - Q$ we obtain

$$f(aP) = f\left[a\left[(P - Q) + Q\right]\right] = f\left[a(P - Q) + aQ\right]$$

$$= f\left[a(P - Q)\right] + f(aQ) = af(P - Q) + af(Q)$$

$$= af(P) - af(Q) + af(Q) = af(P).$$

Conversely if $f \in G.L.(\mathcal{X} : 0)$, then this yields for any P

$$f(aP) = af(P).$$

But by definition,

$$f(aP) = \phi_f(a)f(P).$$

Hence $\phi_f = i$.

The last statement of the theorem then follows immediately from group theory.

CHAPTER 6

The Ternary Ring of an Affine H-plane

§6.1. Introduction

The ternary ring of an ordinary projective or affine plane was first introduced by M. Hall and Skornyakov in [M1] and [S4]. We will generalize these results, which are collected nicely in [BO]. Let us first indicate the results for the ordinary case, and discuss to what extent they have been generalized.

Definition (6.1.1). A pair (Γ, T) is called a ternary field iff the following properties are valid.

(T0) (Γ, T) is a 3-ary algebra; cf. Definition (2.1.4).

(T1) There exist two distinct elements of Γ called 0 and 1.

(T2) $T(a, 0, c) = T(0, b, c) = c$.

(T3) $T(a, 1, 0) = T(1, a, 0) = a$.

(T4) $T(x, m, n) = T(x, m', n')$ has a unique solution for x if $m \neq m'$.

(T5) $T(a, x, y) = b$ and $T(a', x, y) = b'$ has a unique solution for (x, y) if $a \neq a'$.

(T6) $T(a, m, x) = c$ has a unique solution for x.

The first main result was

<u>Theorem (6.1.1).</u> <u>Let</u> A <u>be an ordinary affine plane. Each quadruple of points</u> $\{0, E, X, Y\}$ <u>such that</u> $\{0, X, Y\}$ <u>are three non-collinear points and</u> $E = L(Y, 0X) \wedge L(X, 0Y)$ <u>determines a ternary ring</u> (OE, T).

Conversely, if (Γ, T) is a ternary ring, then $A(T) = \langle \mathcal{P}, \mathcal{L}, I \rangle$ is an ordinary affine plane where

$\mathcal{P} = \Gamma \times \Gamma$

and

$$\mathcal{L} = \{\{(x, y) \mid y = T(x, m, n)\} \mid m, n \in \Gamma\} \cup \{\{(a, y) \mid y \in \Gamma\} \mid a \in \Gamma\}.$$

Also, because of (T2), $[a] = \{(x, y) \mid x = T(x, 0, a)\}$. $[m, n]_{II} = \{(x, y) \mid y = T(x, m, n)\}$ is called a line of the second kind and $[0, a]_I = \{(x, y) \mid x = T(x, 0, a)\}$ a line of the first kind. //

Given (OE, T) as in Theorem (6.1.1), addition and multiplication were defined as follows:

$$x + y = T(x, 1, y) \text{ and } x.y = T(x, y, 0). \tag{6.1.1}$$

The relations between the configuration theorems of the geometry and the algebraic properties of the ternary field were then studied extensively.

Klingenberg in [Kl] was the first to consider the generalizations of these problems. However, while he did define an addition and a multiplication, he did not introduce a ternary ring. It was done as follows for H-planes.

Let g, g' be chosen such that $0 = g \wedge g'$. Select $P_1 I g$, $P_1' I g'$ such that $0 \not\phi P_1, P_1'$. Define $g^* = L(P_1', g)$, and $g_1 = OP_1$. Let a, b, c ... be the elements of g. Then let

$$\begin{aligned} h_a &= aP_1' \\ g_a &= L(a, g_1) \\ a' &= g_a \wedge g' \\ g_a^* &= L(a, g') \\ a^* &= g_a^* \wedge g^*. \end{aligned}$$

Hence $g_a^* = aa^*$. Then taking $a, b \varepsilon OP_1$ we define

$$\left.\begin{aligned} a + b &= OP_1 \wedge L(b^*, h_a), \\ &\text{and} \\ a.b &= OP_1 \wedge L(b', h_a). \end{aligned}\right\} \qquad (6.1.2).$$

Next Klingenberg defined configuration theorems in [Kl], generalizing the minor Desarguesian and

Pappian configurations of ordinary affine planes.

He did not, or could not, define a configuration theorem for the Desarguesian planes.

In order to state Klingenberg's results, we make the next two definitions.

Definition (6.1.2). A pair $(L, +)$ is called a loop iff the following conditions are valid:

(1) L is a set and + is a binary relation.

(2) There exists $0 \varepsilon L$ such that $a+0 = 0+a = a$ for each $a \varepsilon L$.

(3) Each equation $x_1 + x_2 = x_3$ can be solved uniquely for x_i, if we are given x_j, x_k where (i, j, k) is a permutation of $\{1, 2, 3\}$.

We now summarize Klingenberg's results, from [Kl] using the notation just introduced after Theorem (6.2.1).

Definition (6.1.3): Let $\mathcal{H}$ be an affine H-plane. Choose g, g' such that $0 = g \wedge g'$. Choose PIg and P'Ig'. Define addition, +, and multiplication, ., as in equations (6.1.2), page 164. Finally we define

$\mathcal{M}_0 = \{a | a \varepsilon 0P, \text{ such that } a o 0\}$,

D_0 = the set of two sided zero divisors of (0p, .).

Theorem (6.1.2). Let $\mathcal{X}$ be an affine H-plane. Choose 0, P, P' as in Definition (6.1.3). Then

(1) $(0P_1, +)$ is a loop.

(2) (i) $a.1 = 1.a = a$.

(ii) $a.0 = 0.a = 0$.

(3) If $a \not\sigma 0$, and $b \in 0P_1$, then there exist unique x, y such that $xa = b$ and $ay = b$.

(4) If $a\sigma 0$, then there exist $b \neq 0$ and $c \neq 0$ such that $ab = 0$ and $ca = 0$.

(5) $\pi_0 = \{a | a\sigma 0\}$ is an ideal of $(0P_1, +, .)$ and $\pi_0 = D_0$. That is, $\pi_0 + \pi_0 \subseteq \pi_0$, $\pi(0P_1) \subseteq \pi_0$ and $(0P_1)\pi_0 \subseteq \pi_0$.

Theorem (6.1.2)[K1] Let $\mathcal{X}$ be minor Desarguesian [cf.[K1], Definition D12]. Then $(0P_1, +, .)$ has the properties

(1) $(0P_1, +)$ is an abelian group.

(2) $a(b + c) = ab + ac$.

Theorem (6.1.3). Let $\mathcal{X}$ be Pappian. [cf.[K1], Definition D13] Then $(0P_1, +, .)$ is a commutative projective H-ring with maximal ideal π_0.

In [K1], coordinates for lines and points were then introduced in a Pappian plane. It is not known how to construct an affine or projective H-plane over what one would naturally call an H-ternary-ring.

Let us recall the following definition from [A3].

Definition (6.1.4). [A3] A pair (Γ, T) is called a generalized ternary ring iff it satisfies (T0), (T1), (T2), (T3) and (T6) of Definition (6.1.1).

Artmann, in [A3], has taken a modular lattice with a normalized basis of order 3, constructed a generalized ternary ring with respect to this basis and defined addition and multiplication as in equations (6.1.1). By assuming certain related groups of the lattice to be transitive in some manner, he builds up algebraic properties on this ring. Then in (A3), and (A4) Artmann defined the notion of a H-lattice, a special type of modular lattice with normalized 3 basis. He then showed:

(A) Every H-lattice, L, determines a uniform projective H-plane, $\mathcal{H}(L)$.

(B) Every uniform projective H-plane determines an H-lattice $L(\mathcal{H})$.

(C) Every ordinary projective plane, Σ, may be extended to a uniform projective H-plane $\mathcal{H}(\Sigma)$, such that $\overline{\mathcal{H}(\Pi)} \cong \Sigma$.

Finally he studied the structure of the generalized ternary ring of $L(\mathcal{H}(\Sigma))$, where Σ is an ordinary projective-plane.

In this section I will do the following:

(A) Introduce a ternary ring in the sense of Artmann of an affine H-plane. (cf. Definition (6.1.3)).

(B) Define addition and multiplication as in equations (6.1.1).

(C) Coordinatize the points and the lines of $\mathcal{X}$, before introducing any configuration theorems or their equivalents, as in [K1].

(D) Investigate more closely the relations between the algebraic structure of the ternary ring and the configuration theorems. Here we will see the basic difficulty in generalizing the construction of A(T).

Our addition and multiplication is structurally different than that of Klingenberg's [cf. equations (6.1.2)] but we shall not exhibit proofs of results which are the same as Klingenberg's, as they are essentially the same. In fact we are primarily interested in applying these results to our next chapter on topological affine and projective H-planes.

§6.2. *The ternary ring of an affine H-plane* $\mathcal{H}$.

Lemma (6.2.1). *Let* $\{0, X, Y\}$ *be a coordinate system of* $\mathcal{H}$; *that is*, $\{0, X, Y\}$ *satisfies* [A2]. *Then:*

(1) *There exists* $E = L(Y, 0X) \wedge L(X, 0Y)$.

(2) $L(Y, 0X) ø 0X$ *and* $L(X, 0Y) ø 0Y$. *Hence* $E ø S$ *for each* $S I 0X \vee 0Y$.

(3) $\Lambda_{0E} ø \Lambda_{0X}, \Lambda_{0Y}$.

Proof: This follows directly from Lemmas (1.1.4), (1.1.10) and (1.1.11).

Notation: Let $\{0, X, Y\}$ be a coordinate system. Then $g = 0X$; $h = 0Y$; $E = L(X, h) \wedge L(Y, g)$. The elements of $0E$ are written $a, b, c. \ldots$. We fix $\{0, X, Y\}$ now throughout this section.

Lemma (6.2.2). *The map* h_2: $0E \times 0E \to \mathcal{P}$ *defined by* $h_2((a, b)) = L(a, h) \wedge L(b, g)$, *is bijective with inverse*, $h_2^{-1}(P) = (0E \wedge L(P, h), 0E \wedge L(P, g))$.

Proof: h_2 and h_2^{-1} are defined from Lemma (6.2.1). The claim of the Lemma is then easily verified by straightforward calculations.

<u>Definition (6.2.1)</u>. Let P be any point. <u>The coordinates of</u> P <u>with respect to</u> {0, E, X, Y}, are x and y, where $h_2^{-1}(P) = (x, y)$. <u>We write</u> P = (x, y).

From Lemma (6.2.1), x and y are unique and if x, $y \varepsilon 0E$, then there exist a unique $P \varepsilon \mathbb{P}$ such that P = (x, y).

<u>Remark (6.2.1)</u>. (A). 0 = (0, 0); X = (1, 0); Y = (0, 1); E = (1, 1); (B). PI0E iff P = (P, P); PIXE iff P = (1, m); PI0Y iff P = (0, y); and PI0X iff P = (x, 0).

<u>Lemma (6.2.3)</u>. <u>Let</u> $P_i = (a_i, b_i)$; i = 1, 2. <u>Then the following are equivalent</u>.

(1) $P_1 o P_2$.

(2) $a_i o b_i$, i = 1, 2.

<u>Proof</u>: <u>(1)$\Rightarrow$(2)</u>. Let $P_1 o P_2$. By Lemma (1.1.10), $L(P_1, h) o L(P_2, h)$ and $L(P_1, g) o L(P_2, g)$. The result then follows from Lemma (1.1.11).

<u>(2)$\Rightarrow$(1)</u>. Assume $a_i o b_i$; i = 1, 2. Define $P = L(P_1, g) \wedge L(P_2, h)$. Then $b_1 o b_2$ implies $L(b_1, h) o L(b_2, h)$ by Lemma (1.1.10). By Lemma (1.1.11), $P o P_1$. Similarly $a_1 o a_2$ implies $P o P_2$. Hence $P_1 o P_2$.

Corollary. For each ℓ such that $\Lambda_\ell \not\!\!/ \Lambda_h$, there exists a unique $s = L(0,\ell) \wedge XE$.

Definition (6.2.2). (i) XE is called the line of slopes (ii) ℓ is called a line of the second kind iff $\Lambda_\ell \not\!\!/ \Lambda_h$. Otherwise ℓ is a line of the first kind. Let $\mathcal{L}_i$ = set of lines of the i^{th} kind; $i = 1, 2$.

Lemma (6.2.4). The map

$$g_2 : \mathcal{L}_2 \to OE \times OE \text{ defined by}$$

$$g_2(\ell) = [OE \wedge L(L(0,\ell) \wedge XE,\ g),\ OE \wedge L(\ell \wedge OY,\ g)]$$

is bijective with inverse g_2^{-1} defined by

$$g_2^{-1}([m,\ n]) = L((0,\ n),\ 0(1,\ m)).$$

Proof: g_2 and g_2^{-1} are defined due to Lemma (6.2.1) and the Corollary to Lemma (6.2.3). The rest is direct calculation.

Definition (6.2.3). Let $\ell \varepsilon \mathcal{L}_2$. The coordinates of ℓ are m, n where $g_2(\ell) = [m,\ n]$. [cf. Lemma (6.2.4).] We write $\ell = [m,\ n]_{II}$. Clearly $\ell \wedge OY = (0,\ n)$ and $(1,\ m) = L(0,\ \ell) \wedge XE$. m is called the slope of ℓ and $(0,\ n)$ the Y-intercept.

Lemma (6.2.5). Let $\ell_i = [m_i, n_i]_{II}$; $i = 1, 2$. The following are equivalent, (1) $m_1 = m_2$.

(2) $[m_1, n_1]_{II} \parallel [m_2, n_2]_{II}$.

Proof: Let $[m_1, n_1]_{II} \parallel [m_2, n_2]_{II}$. Then

$$(1, m_1) = L(0, \ell_1) \wedge XE = L(0, \ell_2) \wedge XE = (1, m_2),$$

and so $m_1 = m_2$.

Conversely, if $m = m_i$; $i = 1, 2$, then

$$(1, m) = L(0, \ell_1) \wedge XE = L(0, \ell_2) \wedge XE$$

implies $L(0, \ell_1) = L(0, \ell_2)$. Hence $\ell_1 \parallel \ell_2$.

Remark (6.2.2). If $m_1 o m_2$, then $0(1, m_1) o 0(1, m_2)$.

Proof: $m_1 o m_2$ implies $(1, m_1) o (1, m_2)$ by Lemma (6.2.3). Now $0 \not o (1, m_1)$. Hence by (A5)*, $0(1, m_1) o 0(1, m_2)$.

Lemma (6.2.6). Let $\ell_i = [m_i, n_i]_{II}$; $i = 1, 2$. Then $\ell_1 o\, \ell_2$ iff $m_1 o m_2$ and $n_1 o n_2$.

Proof: (1)$\Longrightarrow$(2). Let $\ell_1 o \ell_2$. Then $0(1, m_i) = [m_i, 0]$; $i = 1, 2$. By Lemma (6.2.5), $[m_i, 0] \parallel [m_i, n_i]$; $i = 1, 2$.

Thus $\Lambda_{[m_1,0]} o \Lambda_{[m_2,0]}$. Since $0I[m_1, 0] \wedge [m_2, 0]$, $[m_1, 0] o [m_2, 0]$ by Lemma (1.1.13). Since $[m_i, 0] \not\emptyset XE$; $i = 1, 2$, we have $(1, m_1)o(1, m_2)$ by (A6). By Lemma (6.2.3), $m_1 o m_2$. Finally since $\ell_i \not\emptyset 0Y$; $i = 1, 2$, and $\ell_1 o \ell_2$, $(0, n_1)o(0, n_2)$ by (A6). Hence $n_1 o n_2$.

<u>(2)$\Longrightarrow$(1)</u>. Let $m_1 o m_2$ and $n_1 o n_2$. Thus by Remark (6.2.2), $[m_1, 0] o [m_2, 0]$. Hence $\Lambda_{\ell_1} o \Lambda_{\ell_2}$. By Lemma (1.1.13), $\ell_1 o \ell_2$ or $\ell_1 \wedge \ell_2 = \emptyset$. Suppose $\ell_1 \wedge \ell_2 = \emptyset$. By Lemma (1.1.3), there exists ℓ_3, $\{\ell_3 = L((0, n_2), \ell_1)\} o \ell_2$. Also $(0, n_1)o(0, n_2)$. Hence by Lemma (1.1.11), $\ell_3 o \ell_1$. Hence $\ell_1 o \ell_2$.

<u>Definition (6.2.4)</u>. Define T: $0E^3 \to 0E$ by $T(x, m, n) = 0E \wedge L(L((0, n), 0(1, m)) \wedge L(x, h), g)$. This is defined since $L((0, n), 0(1, m)) = [m, n]_{II}$ and $h = 0Y$. $(0E, T)$ is called <u>the associated ternary ring of $\mathcal{H}$ with respect to</u> $\{0, E, X, Y\}$.

<u>Lemma (6.2.7)</u>. $P = (x, y)I[m, n]_{II}$ <u>iff</u> $y = T(x, m, n)$.

<u>Proof</u>: Let (x, m, n) be given and $\ell = [m, n]_{II}$ and $PI\ell$. Then $x = 0E \wedge L(P, h)$; $y = 0E \wedge L(P, g)$: $P = \ell \wedge L(x, h)$ and $\ell = L((0, n), 0(1, m))$. Hence we obtain

$$y = 0E \wedge L(L((0, n), 0(1, m)) \wedge L(x, h), g) = T(x, m, n)$$

<u>Lemma (6.2.8).</u> (0E, T) <u>is a generalized ternary ring</u>.

<u>Proof</u>: We verify the axioms of Definition (6.1.4). (T0) and (T1) are obvious; let $E = 1$.

<u>(T2)</u> $T(a, 0, n) = 0E \wedge L(L((0, n), g) \wedge L(a, h), g)$

$= 0E \wedge L((0, n), g) = n$

and

$T(0, a, n) = 0E \wedge L(L((0, n), 0(1, a)) \wedge h, g)$

$= 0E \wedge L((0, n), g) = n.$

<u>(T3)</u> $T(a, 1, 0) = 0E \wedge L(L(0, 0E) \wedge L(a, h), g)$

$= 0E \wedge L(a, g) = a$

and

$T(1, a, 0) = 0E \wedge L(L(0, 0(1, a)) \wedge EX, g)$

$= 0E \wedge L((1, a), g) = a.$

<u>(T6)</u> Take $a, m, b \varepsilon 0E$. Let $P = (a, b)$ and $\ell = [m, 0]_{II}$. Then $L(P, \ell) \wedge 0Y = (0, n)$ for some n. $L(P, \ell) = [m, n]_{II}$. Thus $b = T(a, m, n)$. Suppose $\tilde{n}$

also has the property $T(a, m, \tilde{n}) = b$. Hence $P = (a, b)$ $I[m, \tilde{n}]_{II}$. Since $[m, n]_{II} \parallel [m, \tilde{n}]_{II}$, it follows that $[m, n]_{II} = [m, \tilde{n}]_{II}$ and so $\tilde{n} = n$.

<u>Theorem (6.2.1)</u>. o <u>is a congruence on</u> (OE, T). [cf. Definition (2.1.5).]

<u>Proof</u>: Let $x_1 o x_2$; $m_1 o m_2$, and $n_1 o n_2$. By Lemma (1.1.10), $L(x_1, h) o L(x_2, h)$. By Lemmas (6.2.3) and (6.2.6), $O(1, m_1) o O(1, m_2)$ and $(0, n_1) o (0, n_2)$ respectively. Define $T_i = L((0, n_i), O(1, m_i)) \wedge L(x_i, h)$; $i = 1, 2$. Since $O(1, m_1) o O(1, m_2)$, $L((0, n_1), O(1, m_1)) o L((0, n_1), O(1, m_2))$ by Lemma (1.1.13). Let $S_1 = L((0, n_1), O(1, m_2)) \wedge L(x_1, h)$. By (A5)

$$S_1 o T_1 . \tag{I}$$

Now let $\tilde{T}_1 = L((0, n_2), O(1, m_2)) \wedge L(x_1, h)$. Since $L(x_1, h) o L(x_2, h)$ (A6) yields

$$\tilde{T}_1 o T_2 . \tag{II}$$

Also by Lemma (1.1.11), $L((0, n_1), O(1, m_1)) o L((0, n_2), O(1, m_1))$. Since $L(x_1, h) \not{o} L((0, n_1), O(1, m_1))$, Lemma (1.1.11) yields

$$S_1 o \tilde{T}_1 . \qquad \text{(III)}$$

Combining (I), (II) and (III), we obtain $T_1 o T_2$. Hence by Lemma (1.1.10), $L(T_1, g) o L(T_2, g)$. Since $0E \not\in L(T_i, g)$; $i = 1, 2$, we obtain by Lemma (1.1.11),

$$0E \wedge L(T_1, g) o 0E \wedge L(T_2, g),$$

or equivalently, $T(x_1, m_1, n_1) o T(x_2, m_2, n_2)$. (Theorem (6.2.1) also follows from Lemma 6.2.9 (below) and Lemma (2.1.6).)

Remark (6.2.3). $\{\bar{0}, \bar{X}, \bar{Y}\}$ is a coordinate system of $\overline{\mathcal{R}}$. Let $P = (a, b)$ and $\ell = [m, n]_{II}$. Then $\bar{P} = (\bar{a}, \bar{b})$ and $\ell = [\bar{m}, \bar{n}]_{II}$.

Proof: The first part follows easily. Now let $\bar{P} = (\bar{x}, \bar{y})$. Then

$$\bar{x} = \bar{0}\bar{E} \wedge L(\bar{P}, \bar{h}) = \chi_{\mathcal{L}}(0E) \wedge \chi_{\mathcal{L}}(L(P, h))$$

$$= \chi_{\mathbb{P}}(0E \quad L(P, h)) = \chi_{\mathbb{P}}(a) = \bar{a}.$$

Similarly $\bar{y} = \bar{b}$ and $\bar{\ell} = [\bar{m}, \bar{n}]_{II}$.

Lemma (6.2.9). Let $(\bar{0}\bar{E}, \bar{T})$ be the associated ternary ring of $\overline{\mathcal{R}}$ with respect to $\{\bar{0}\bar{E}, \bar{X}, \bar{Y}\}$. Then the map χ_{0E}: $0E \to \overline{0E}$, defined by $\chi_{0E}(a) = \bar{a}$ is an onto homomorphism. Hence $(0E/\circ, T) \cong (\bar{0}\bar{E}, \bar{T})$.

Proof: χ_{0E} is clearly onto. Moreover by Lemma (1.2.4),

$$\chi_{0E}(T(x, m, n)) = \chi_{0E}(0E \wedge L(L((0, n), 0(1, m)) \wedge L(x, h), g))$$

$$= \chi_{\mathfrak{L}}(0E) \wedge \chi_{\mathfrak{L}}(L(L((0, n), 0(1, m)) \wedge L(x, h), g))$$

$$= \bar{0}\bar{E} \wedge L(L(\overline{0, n}), \bar{0}(\overline{1, m})) \wedge L(\bar{x}, \bar{h}), \bar{g})$$

$$= \bar{T}(\bar{x}, \bar{m}, \bar{n}) = \bar{T}(\chi_{0E}(x), \chi_{0E}(m), \chi_{0E}(n)).$$

Hence χ_{0E} is a homomorphism. Since $\theta_{\chi_{0E}} = \circ$, the result follows from Lemma (2.1.6).

Corollary. $\chi_{0E}\, T = T\, \chi_{0E}^3$.

We next coordinatize lines of the first kind. Notice that for ordinary planes, this is no problem. It is in fact the lines of the first kind which cause the difficulty in generalizing the construction A(T).

Coordinatization of lines of the first kind. Let $\ell \in \mathfrak{L}_I$. Then $\Lambda_\ell \circ \Lambda_{0Y}$. Hence $\Lambda_\ell \not\circ \Lambda_{0X}$. We then proceed in exactly the same fashion as we did for $\mathfrak{L}_2$ with X replaced by Y. Define

$$\ell \wedge 0X = (n, 0); \; L(0, \ell) \wedge YE = (m, 1).$$

m and n are the coordinates of ℓ and we write $\ell = [m, n]_I$.

Similarly we obtain a new ternary ring, (OE, T_1), with respect to $(0, E, Y, X)$,

$$T_1(y, m, n) = OE \wedge L(L((n, 0), 0(m, 1)) \wedge L(y, g), h)$$

where

$$(x, y) I [m, n]_I \text{ iff } x = T_1(y, m, n).$$

Lemma (6.2.10). The following statements are valid.

(1) $[m_1, n_1]_I \parallel [m_2, n_2]_I$ iff $m_1 = m_2$.

(2) $[m_1, n_1]_I \, o \, [m_2, n_2]_I$ iff $m_1 o m_2$ and $n_1 o n_2$.

Proof: This proof is completely analogous to the proofs of Lemmas (6.2.5) and (6.2.6).

Lemma (6.2.11). If $\ell = [m, n]_I$, then $m o 0$. Conversely if $m, n \varepsilon OE$ such that $m o 0$, then there exists a unique $\ell \varepsilon \mathcal{L}_1$, such that $\ell = [m, n]_I$.

Proof: $\Lambda_\ell o \Lambda_{0Y}$. Hence $\Lambda_{L(m,h)} o \Lambda_{L((m,1),\ell)}$. But $(m, 1) I L(m, h) \wedge L((m, 1), \ell)$. Hence by Lemma (1.1.13), $L(m, h) o L((m, 1), \ell)$. Since $\Lambda_{0E} \not\!o \Lambda_h$, we have

$\Lambda_{0E} \not\text{o} \Lambda_{L((m,\ 1),\ell)}$ and hence by (A6), mo0. Conversely, choose m, nε0E such that mo0. Define $\ell = L((m,\ 0),\ 0(m,\ 1))$. To show $\ell \varepsilon \mathcal{L}_1$, it suffices to prove $L((m,\ 1),\ell)oL(m,\ h)$. Suppose this is false. Since $(m,\ 1)IL((m,\ 1),\ell) \wedge L(m,\ h)$ and mo0, (A5) yields (m, 1)o0. Contradiction.

<u>Lemma (6.2.12)</u>$(0E, T_1)$ <u>has the same properties as</u> $(\circ E, T)$

<u>Lemma (6.2.13)</u>. <u>If</u> $a_1 \not\text{o} a_2$, <u>then</u> $(a_1,\ b_1)(a_2,\ b_2) = \ell$ <u>is a line of the second kind</u>.

<u>Proof</u>: By Lemma (6.2.3), $(a_1,\ b_1)\not\text{o}(a_2,\ b_2)$, for all $b_1,\ b_2 \varepsilon 0E$. Suppose $\ell \varepsilon \mathcal{L}_1$. Hence $\ell = [m,\ n]_I$ such that mo0, by Lemma (6.2.10). Thus $a_i = T_1(b_i,\ m,\ n)$; i = 1, 2. Hence $T_1(b_1,\ m,\ n)\not\text{o}T_1(b_2,\ m,\ n)$. Now since 0 is a congruence on $(0E,\ T_1)$, by Lemma (6.2.12), mo0 implies $T_1(b_i,\ m,\ n)oT_1(b_i,\ 0,\ n)$, i = 1, 2. But by (T2), $T_1(b_i,\ 0,\ n) = n$; i = 1, 2. Hence $T_1(b_1,\ m,\ n)oT_1(b_2,\ m,\ n)$. Contradiction.

<u>Lemma (6.2.14)</u>.

(1) $\left|[m;\ n]_I \wedge [u,\ v]_{II}\right| = 1$.

(2) If $[m_1,\ n_1] \wedge [m_2,\ n_2] = \emptyset$, <u>then both lines are of the same kind and</u> $m_1 o m_2$.

<u>Proof</u>:

(1) Since $\Lambda_{[m,n]_I} \not\text{o} \Lambda_{0Y}$ and $\Lambda_{[u,v]_{II}} \not\text{o} \Lambda_{0Y}$, we have

$^{\Lambda}[m,n]_{I}\,{}^{\phi\Lambda}[u,v]_{II}$.

(2) By (1), both are of the same kind. Let ℓ_i = $[m_i, n_i]_I$. Hence by Theorem (1.2.4), $\bar{\ell}_1 \| \bar{\ell}_2$. By Remark (6.2.3), $\bar{m}_1 = \bar{m}_2$ and so $m_1 o m_2$. A similar argument applies to lines of the second kind.

We may now state the main properties of (OE, T, T_1, O, E)

<u>Theorem (6.2.2)</u>. <u>The universal algebra</u> (cf. Page 52) (OE, T, T_1, 0, E) <u>has the properties</u>,

(HT1) (OE, T) <u>and</u> (OE, T_1) <u>are</u> 3-<u>ary algebras where</u> E = 1, <u>and</u> Eϕ0.

(HT2) T(a, 0, n) = T(0, a, n) = n.

(HT3) T(a, 1, 0) = T(1, a, 0) = a.

(HT4) T(a, m, n) = b <u>has a unique solution for</u> n.

(HT5) T(a_i, m, n) = b_i, i = 1, 2, <u>has a unique solution for</u> (m, n) <u>if</u> $a_1 \phi a_2$.

(HT6) T(a, m_i, n_i) = b; i = 1, 2, <u>has a unique solution for</u> (a, b) <u>if</u> $m_1 \phi m_2$.

(HT7) <u>For every choice of</u> (a_i, b_i); i = 1, 2, <u>either</u> T(a_i, m, n) = b_i <u>or</u> T_1(b_i, m, n) = a_i <u>has a solution</u> (m, n).

(HT8) $T(a, m, n) = b$ <u>has a unique solution for a if</u> $m \not\parallel 0$.

(HT9) $T(a, m, n) = b$ <u>has a unique solution for</u> m <u>if</u> $a \not\parallel 0$.

<u>Proof</u>: (HT0) to (HT4) were shown in Lemma (6.2.8).

<u>(HT5)</u> Since $a_1 \not\parallel a_2$, $(a_1, b_1)(a_2, b_2) = [m, n]_{II}$ by Lemma (6.2.13), where m, n are clearly unique. Hence $b_i = T(a_i, m, n)$; $i = 1, 2$.

<u>(HT6)</u> Let $\ell_i = [m_1, n_1]_{II}$; $i = 1, 2$. Since $m_1 \not\parallel m_2$, $\ell_1 \wedge \ell_2 \neq \emptyset$, by Lemma (6.2.14)(2), and $\ell_1 \not\parallel \ell_2$ by Lemma (6.2.6). Hence by (A4), $|\ell_1 \wedge \ell_2| = 1$.

<u>(HT7)</u> This is just the algebraic statement of (A1).

<u>(HT8)</u> By (HT2), $T(a, m, n) = T(a, 0, b) = b$. Since $m \not\parallel 0$, the result follows from (HT5).

<u>(HT9)</u> By (HT3), $T(0, m, n) = n$. Then $T(a, m, n) = b$ and $a \not\parallel 0$ implies the result by (HT6).

<u>Definition (6.2.4)</u>. Define the maps $\mathcal{Z}$, M, N as follows:

(i) $\mathcal{Z}(m_1, n_1, m_2, n_2) = x$ iff $T(x, m_i, n_i) = y$; $i = 1, 2$, for some y, where $m_1 \not\parallel m_2$.

(ii) $M(a_1, b_1, a_2, b_2) = m$ iff $T(a_i, m, n) = b_i$; $i = 1, 2$, for some $n \varepsilon 0E$, where $a_1 \not\parallel a_2$.

(iii) $N(x, m, y) = n$ iff $T(x, m, n) = y$.

These 3 maps are <u>called the inverses of</u> T.

<u>Lemma (6.2.15)</u>.

(1) $N(x, m, y) = 0E \wedge L(L((x, y), 0(1, m) \wedge h, g)$.

(2) $M(a_1, b_1, a_2, b_2) = 0E \wedge L[L(0, (a_1 b_1)(a_2, b_2] \wedge h, g)$, <u>where</u> $a_1 \not{o} a_2$.

(3) $Ƶ(m_1, n_1, m_2, n_2) = 0E \wedge L([m, n]_{II} \wedge [m_2, n_2]_{II}, h)$, <u>where</u> $m_1 \not{o} m_2$.

<u>Proof</u>: (1) follows from the proof of (HT4) in Lemma (6.2.8).(2) and (3) follow from the proof of Theorem (6.2.2).

<u>Lemma (6.2.16)</u>. <u>Let</u> Ƶ, M, N <u>be the inverses of</u> T. <u>Then</u>

(i) $X_{0E}\, N = N\, X_{0E}^3$.

(ii) $X_{0E}\, M = M\, X_{0E}^3$.

(iii) $X_{0E}\, Ƶ = Ƶ\, X_{0E}^4$.

<u>Proof</u>: The proofs follow from Lemma (6.2.15) by direct computations.

<u>Definition (6.2.5)</u>. If (0E, T) is the associated ternary ring of (0, E, X, Y), We define <u>addition</u> and <u>multiplication in</u> (0E, T) by

(i) $a + b = T(a, 1, b)$,

(ii) $a.b = T(a, b, 0)$.

T is called linear iff $T(x, m, n) = xm + n$. Similarly, we may define addition and multiplication for (OE, T_1). We shall write this as

(iii) $a +_1 b = T_1(a, 1, b)$.

(iv) $a ._1 b = T_1(a, b, 0)$.

Comment (6.2.1). We wish to describe all lines in terms of T. At present we have

$$[m, n]_{II} = \{(x, y) | y = T(x, m, n)\},$$

$$[m, n]_I = \{(x, y) | x = T_1(y, m, n)\}, \text{ where } m o 0.$$

Now in the ordinary case, $\mathcal{T}\!\!\mathcal{L}_0 = \{0\}$. Hence by (HT2), $[m, n]_I = \{(x, n) | x = T(x, 0, n)\}$. However in an arbitrary affine H-plane, our problem is to show for $m \varepsilon \mathcal{T}\!\!\mathcal{L}_0$,

$$T(x, m, n) = T_1(x, m, n).$$

I will show this for a Desarguesian plane. However, one would hope this would be true for at least uniform planes.

We may now restate Klingenberg's results in our setting, and add some additional results.

Theorem (6.2.3).

(1) (OE, +) is a loop. To be precise,

the unique solutions of $x + a = b$ *and* $a + y = b$ *are*

$$x = 0E \wedge L(L(S, h) \wedge g, h)$$

where

$$S = L((0, a), 0E) \wedge L(b, g)$$

and

$$y = 0E \wedge L(L((a, b), 0E) \wedge h, g).$$

(2) $a.1 = 1.a = a$ *and* $a.0 = 0.a = a$.

(3) *If* $a \not\circ 0$ *and* $b \in 0E$, *there exist unique* x, y *such that* $xa = b$ *and* $ay = b$.

(4) Π_0 *is an ideal of* $(0E, +, .)$ *and* $\Pi_0 = D_0$.

(5) $\circ$ *is a congruence of* $(0E, +)$ *and* $(0E, .)$.

(6) *If* $y \not\circ 0$, *then* $xy = xz$ *or* $yz = zx$ *implies* $y = z$.

(7) *If* $x \not\circ 0$, *then* $(xy)\circ(xz)$ *or* $(yx)\circ(zx)$ *implies* $y \circ z$.

Proof. (1) to (4) are esssentially the same as

Theorem (6.1.2). The precise statements of (3) are easily verified. (5) is an immediate consequence of Theorem (6.2.1). (6) is a special case of (HT8) and (HT9) where n = 0. Now we show (7). First suppose $(xy)o(xz)$. Hence $L(xy, g)oL(xz, g)$ by Lemma (1.1.10). Define $A_1 = 0(1, y) \wedge L(x, h)$ and $A_2 = 0(1, z) \wedge L(x, h)$. Since $L(x, h)\not{o}L(xy, g)$, (A5) yields $A_1 o A_2$. Now $A_1 \not{o} 0$, otherwise $0E\not{o}L(x, h)$ implies $xo0$. Contradiction. Thus $0A_1 o 0A_2$ by (A5)*. Since $0A_1 = [y, 0]_{II}$ and $0A_2 = [z, 0]_{II}$, the result follows from Lemma (6.2.6).

Secondly assume $yxozx$. Let $A_1 = L(y, h) \wedge [x, 0]_{II}$ and $A_2 = L(z, h) \wedge [x, 0]_{II}$. Now $x\not{o}0$ implies $[x, 0]_{II}\not{o}[0, 0]_{II}$ and hence (A7) yields, $[x, 0]_{II}\not{o}L(A_i, g)$; i = 1, 2. Then utilizing Lemma (1.1.11) several times we obtain,

$$(zx)o(yx) \Longleftrightarrow L(zx, g)oL(yz, g) \Longleftrightarrow L(A_1, g)oL(A_2, g)$$

$$\Longleftrightarrow A_1 o A_2 \Longleftrightarrow L(A_1, h)oL(A_2, h) \Longleftrightarrow yoz.$$

Corollary (1). Let Π_- and Π_+ be the set of non-left invertible and non-right invertible elements of (0E, .). Then $\Pi_0 = \Pi_- = \Pi_+$.

Proof: From (3) of the theorem, $\Pi_-, \Pi_+ \subseteq \Pi_0$. Conversely suppose $x \varepsilon \Pi_0$. By (4) of the theorem, $xy \varepsilon \Pi_0$ for each $y \varepsilon 0E$. Then if $x \notin \Pi_+$, there exists $y \neq 0$ such that $xy = 1$. Hence $1 \varepsilon \Pi_0$. Contradiction.

Corollary (2). *If* $ao0$ *and* $b\varepsilon 0E$, *then* $(a + b)$, $(b + a)o(ab)$.

Proof: Since $ao0$ and bob we have from (5) of the theorem,

$$(a + b)o(0 + b) = b \text{ and } (b + a)o(0 + b) = b.$$

Corollary (3). *If* boa, *then there exists* $y\varepsilon \Pi_0$ *such that* $a + y = b$.

Proof: From (3) of the theorem, we have

$$y = 0E \wedge L(M, g) \text{ such that } M = L(S, 0E) \wedge h, \text{ and } S = (a, b).$$

Since aob, we have $So(a, a)$ by Lemma (6.2.3). Hence by Lemma (1.1.10), $L(S, 0E)o0E$. Then $L(M, g)\not o 0E$, implies Moy by (A6). Similarly by (A6) $h\not o 0E$ implies $Mo0$. Hence $yo0$ and so $y \in \Pi_0$.

Corollary (4). *For each* $a\varepsilon 0E$, $\bar{a} = a + \Pi_0$, *where* $\bar{a} = \{b \mid boa$ *and* $b\varepsilon 0E\}$. *Hence* $0E/o = \{a + \Pi_0 \mid a\varepsilon 0E\}$.

Proof: Let $b\varepsilon a + \Pi_0$. Hence there exists $no0$ such that $b = a + n$. By Corollary (2) of Theorem (6.2.3), $a + noa$ and so $b\varepsilon\bar{a}$.

Conversely let $b\varepsilon\bar{a}$. Hence boa. By Corollary

(3) of Theorem (6.2.3), there exists $y \varepsilon \Pi_0$ such that $a + y = b$. Hence $b \varepsilon a + \Pi_0$.

Corollary (5). The map ${}_a\phi$: $0E \to 0E$ *defined by* ${}_a\phi(b) = a + b$ *is* (1 - 1) *onto. Its inverse is*

$$({}_a\phi)^{-1}(b) = 0E \wedge L\left[L\{(a, b), 0E\} \wedge h, g\right].$$

Similarly ϕ_a : $0E \to 0E$ *defined by* $\phi_a(b) = b + a$ *is a* (1 - 1) *onto map whose inverse is*

$$(\phi_a)^{-1}(b) = 0E \wedge L\left[L(S, h) \wedge g, h\right]$$

where

$$S = L((0, a), 0E) \wedge L(b, g)).$$

Proof: This follows easily from (1) of Theorem (6.2.3).

Corollary (6). Let a^r *be the unique solution of* $a + x = 0$. *Then the* map η_r: $0E \to 0E$ *defined by* $\eta_r(a) = a^r$ *is*

$$\eta_r(a) = 0E \wedge L(L((a, 0), 0E) \wedge h, g).$$

Proof: This is just a special case of property (HT4).

<u>Corollary (7).</u> <u>If</u> $ab \in \Pi_0$, <u>then</u> $a \in \Pi_0$ <u>or</u> $b \in \Pi_0$.

<u>Proof</u>: Suppose $a \notin \Pi_0$. Since $a.0 = 0$ it follows that $a.0oab$ and hence by (7) of the theorem, $b \in \Pi_0$.

<u>Corollary (8).</u> <u>The unique solution of</u> $ax = 1$ <u>is</u>

$$x = 0E \wedge L[XE \wedge 0(a, 1), g].$$

<u>Proof</u>: This is just a special case of (3) of the theorem.

Let us now consider the configuration theorems defined in [K2].

<u>Definition (6.2.6).</u> [K1] <u>A minor Desarguesian configuration</u> C_1 [see Figure (6.2.1)] is a set of six points and 3 lines satisfying the conditions:

(i) $g_i \in \Lambda$; $i = 1, 2, 3$.

(ii) $P_i, Q_i I g_i$; $i = 1, 2, 3$.

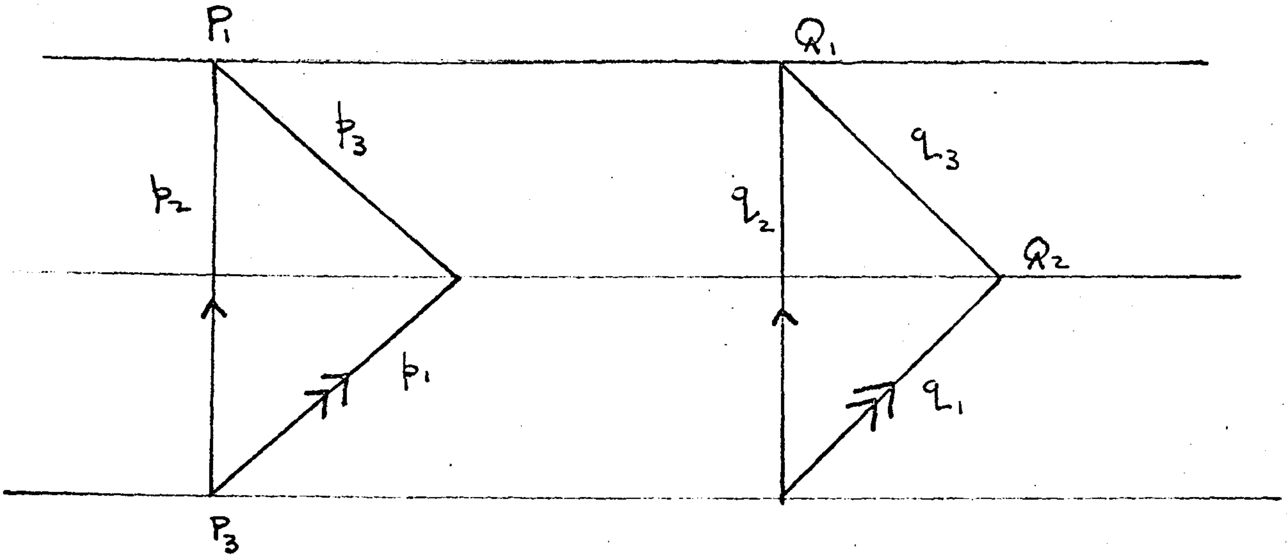
(iii) $P_i, P_h I p_k$ and $Q_i, Q_j I q_k$ if (i, j, k) is a permutation of $\{1, 2, 3\}$.

(iv) $p_2 \parallel q_2$ and $p_1 \parallel q_1$.

(v) $p_1, p_2 \not{o} g_3$.

(vi) $P_1 \not{o} P_2$.

Figure (6.2.1)



<u>Lemma (6.2.17)</u>. [K1]. <u>Let</u> C_1 <u>be a minor Desarguesian configuration.</u> <u>Then</u>

(a) $p_1 \not{o}\, g_2$ <u>and</u> $p_2 \not{o}\, g_1$.

(b) $q_1 \not{o}\, g_2$, g_3 <u>and</u> $q_2 \not{o}\, g_1$, g_3.

<u>Lemma (6.2.18)</u>.[K1]<u>Let</u> C_1 <u>be a minor Desarguesian configuration.</u> <u>Then</u>

(a) $Q_1 \not{o}\, Q_2$.

(b) <u>If</u> $g_1 o g_2$, <u>then</u> $p_3 o g_1$, g_2; $p_3 \not{o}\, p_1$, p_2; $P_3 \not{o}\, P_1$, P_2; $g_3 \not{o}\, g_1$, g_2; $Q_3 \not{o}\, Q_1$, Q_2; $p_1 \not{o}\, p_2$ <u>and</u> $q_1 \not{o}\, q_2$ <u>and</u> $Q_1 \not{o}\, Q_2$.

<u>Comment (6.2.2)</u>. If C_1 is a minor Desarguesian configuration, the previous lemma says the line q_3 is uniquely determined and is Q_1Q_2.

<u>Lemma (6.2.19)</u>. <u>Let</u> C_1 <u>be a minor Desarguesian configuration.</u> <u>Then</u> $p_3 \not{o}\, g_3$.

<u>Proof</u>: <u>Case (1)</u>: $g_1 \not{o}\, g_2$. If $p_3 o g_3$, then $p_3 o g_1$, g_2 by (A7). Hence $g_1 o g_2$. Contradiction.

<u>Case (2)</u>: $g_1 o g_2$. From (b) of Lemma

(6.2.18) we have $p_3 o g_1$, g_2; and $g_3 \not o g_1$, g_2. Hence $p_3 \not o g_3$.

Definition (6.2.7). We say $\mathcal{X}$ has the property D_1 iff for each minor Desarguesian configuration C_1, $p_2 \parallel q_3$.

Remark (6.2.4).

(1) If one line has three pairwise non-neighbouring points, then each line has this property.

(2) If one line has three pairwise non-neighbouring points then each pencil has three pairwise non-neighbouring lines.

Proof:

(1) follows immediately from Lemma (1.2.2).

(2) follows from (1) and Lemma (1.2.3).

In [K1], a plane with property D_1 was called minor Desarguesian. In [K2], a plane with (A9) was called minor Desarguesian. We shall show if $\mathcal{X}$ is a T-plane, these two definitions are both equivalent to each ternary ring being linear.

Theorem (6.2.4). Let $\mathcal{X}$ be a T-plane, having a line with three pairwise non-neighbouring points; cf. Definition (3.2.4). Then the following are equivalent.

(1) <u>Every ternary ring</u> (OE, T) <u>is linear</u>.

(2) D_1 <u>is valid</u>.

(3) T <u>is a transitive group</u>.

<u>Proof:</u> <u>$(1) \Rightarrow (2)$</u>. Let C_1 be a minor Desarguesian configuration. From Lemma (6.2.19), $g_3 \not I p_3$ and so $g_3 \not I L(P_3, p_3)$. Let $g = L(P_3, p_3)$. Choose $\tilde{p}_2$ such that $P_3 I \tilde{p}_2$ and $\tilde{p}_2 \not I g_3$, g by Lemma (1.1.12). Thus g_3, g and $\tilde{p}_2$ may be regarded as a coordinate system such that

$$P_3 = (0,\ 0),\ g_3 = 0Y,\ g = 0X \text{ and } \tilde{p}_2 = 0E.$$

Let (OE, T) be the associated ternary ring. Thus $Q_3 = (0, n)$ for some $n \in 0E$. Let $p_2 = [m, 0]_{II}$ and $p_1 = [m_2, 0]_{II}$. Hence $q_2 = [m, n]_{II}$ and $q_1 = [m_2, n]_{II}$. Thus

$$P_1 = (x,\ xm_1);\ Q_1 = (x,\ T(x,\ m_1,\ n_1));$$

and

$$P_2 = (a,\ am_2);\ Q_2 = (a,\ T(a,\ m_2,\ n)).$$

Now $\{P_1P_2 = p_3\} \parallel g$ implies $xm_1 = am_2$. Thus the linearity of T implies that

$$T(x, m_1, n) = xm_1 + n = am_2 + n = T(a, m_2, n).$$

Hence Q_1 and Q_2 have the same y-coordinates. Thus $Q_2 I L(Q_1, g)$ and so $Q_1Q_2 \parallel g$.

(2)$\Longrightarrow$(3). Choose P_3, $Q_3 \varepsilon \mathfrak{P}$. Without loss of generality we may assume $P_3 \not{o} Q_3$. For if $P_3 o Q_3$, select X such that $X \not{o} P_3$, Q_3. Then since $\mathfrak{A}$ is a T-plane, and $X \not{o} P_3$, Q_3, $\tau = \tau_{P_3X}\, \tau_{XQ_3}$ would be our desired translation.

Now let $g = P_3Q_3$. From Remark (6.2.4)(2) there exist g_1, $g_2 \varepsilon \Lambda g_3$ such that $g_i \not{o} g_j$; $i \neq j$; i, j = 1, 2, 3. Let $\Lambda = \Lambda_{g_3}$. Choose $P_i I g_i$; i = 1, 2.

From Lemma (1.1.10), $P_i \not{o} P_j$; i, j = 1, 2, 3. Let $p_i = P_jP_k$ where (i, j, k) is a permutation of (1, 2, 3).

Claim (1). $\Lambda_{p_i} \not{o}\ \Lambda$; i = 1, 2, 3. Let i = 1.

If $p_1 o g_3$ then there exists $T I g_3$ such that $T o P_2$. But $P_2 I g_2$, and hence by Lemma (1.1.10), $g_2 o g_3$. Contradiction. The rest follows in a similar manner.

Thus, in particular, $\Lambda_{L(Q_3, p_1)} \not{o} \Lambda_{g_2}$ and $\Lambda_{L(Q_3, p_2)} \not{o} \Lambda_{g_1}$ by (A7). Then we may define

$$Q_1 = L(Q_3, p_2) \wedge g_1 \text{ and } Q_2 = L(Q_3, p_1) \wedge g_2.$$

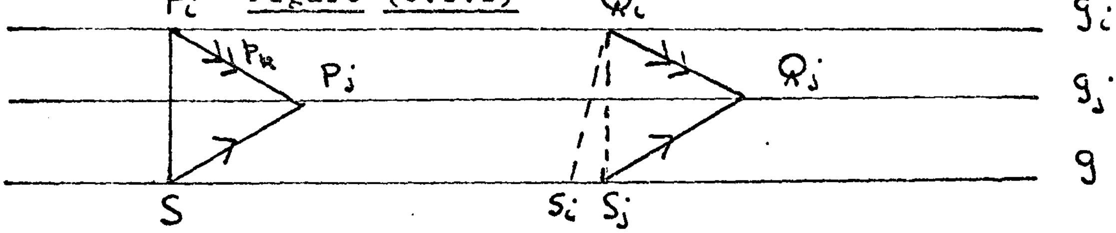
Next define the maps τ_i, $i = 1, 2, 3$ as follows:

$$S^{\tau_i} = L(S,\ g_i) \wedge L(Q_i,\ P_iS),\ \text{if } S \not I X \text{ for each } X I g_i.$$

This is clearly defined from our choice of S and (A7).

Claim (2). If $S \not I X$ *for each* $X I g_i \vee g_j$; $i \neq j$; $i, j = 1, 2, 3,$ *then* $S^{\tau_i} = S^{\tau_j}$. Let $g = L(S,\ g_i)$ and $k \neq i, j$ such that $k \in \{1, 2, 3\}$. From our choice of S and Claim (1), P_i, $P_j \not I S$; and p_k, $P_jS \not I g_j$.

Figure (6.2.2)



Now $\quad S_i = g \wedge L(Q_i,\ P_iS) = g \wedge S_iQ_i$

and

$$S_j = g \wedge L(Q_j,\ P_jS) = g \wedge S_iQ_j$$

exist by Lemma (1.1.10) and Claim (2). Thus Figure (6.2.2) is a C_1 configuration and hence by D_1, $P_iS \parallel Q_iS_j$. But $P_iS \parallel Q_iS_i$ and so $Q_iS_j \parallel Q_iS_i$. Hence $Q_iS_j = Q_iS_i$. But $Q_i \not I X$ for each XIg by Lemma (1.1.10), and so

$$S_j = Q_iS_j \wedge g = Q_iS_i \wedge g = S_i.$$

Claim (3). For each S, *there exists* $i \in \{1, 2, 3\}$ *such that* $S \not I X$ *for each* SIg_i. If this is false, then there exists $X_j I g_j$ such that $S o X_j$; $j = 1, 2, 3$. Hence for $i \neq j$, $X_i o X_j$ and hence $g_i o g_j$ by Lemma (1.1.10). Contradiction.

Now define $\tau: \mathbb{P} \to \mathbb{P}$ as follows: For each $S \varepsilon \mathbb{P}$

$$S^{\tau} = S^{\tau_i}, \text{ if } S \not o X \text{ for each } S I g_i.$$

In view of claims (2) and (3), τ is well defined. Clearly $P_3^{\tau} = P_3^{\tau_2} = Q_3$. We must now show $\tau \varepsilon T$. By Lemma (3.2.2) it suffices to show:

(i) $\tau \varepsilon \tilde{T}$.

(ii) $S \not o S^{\tau}$ for each S.

(iii) Any two traces of τ are parallel.

<u>(i)</u> We first show $\tau \varepsilon D$. Let X, Y be any two points and g any line such that X, YIg.

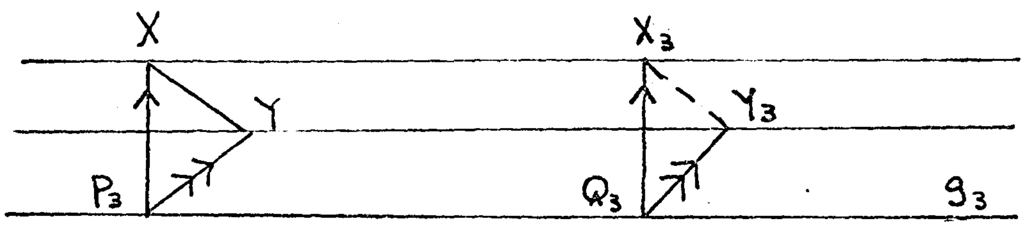
<u>Claim (4)</u>. <u>There exists</u> $i \varepsilon \{1, 2, 3\}$ <u>such that for each</u> $S I g_i$, $S \not o X, Y$. If this were false, then there would exist $X_1 I g_1$ such that $X o X_1$ or $Y o X_1$. Assume $X o X_1$. Also there exists $X_2 I g_2$ such that $X o X_2$ or $Y o X_2$. Hence $X_2 o Y$; otherwise $X_1 o X_2$ and thus by Lemma (1.1.10), $g_1 o g_2$. Contradiction. Finally there exists $X_3 I g_3$ such that $X_3 o X$ or $X_3 o Y$. If $X_3 o X$ then $X_1 o X_3$ and so

$g_1 o g_2$ by Lemma (1.1.10). Contradiction. Similarly $X_3 o Y$ implies the contradiction $g_2 o g_3$.

Let us assume without loss of generality then that $S \not o X, Y$ for each SIg_3. Hence $XP_3, YP_3 \not o g_3$.

<u>Case (1)</u>: $X \not o Y$. Let $X^{\tau_3} = X_3$ and $Y^{\tau_3} = Y_3$.

<u>Figure (6.2.3)</u>



Since $X \not o Y$ and $XP_3, YP_3 \not o g_3$, Figure (6.2.3) is a C_1 configuration and hence $XY \parallel X_3Y_3$, or equivalently, $Y_3 I L(X_3, XY)$.

<u>Case (2)</u>: XoY. Choose zIg such that $z \not o X, Y$. Thus $g = XZ = YZ$. By Case (1), we have

$$Z^{\tau} I\{L(Y^{\tau}, XZ) = L(Y^{\tau}, g)\} \text{ and so } L(Y^{\tau}, g) = L(Z^{\tau}, g).$$

Using Case (1) again we obtain

$$X^{\tau} I\{L(Z^{\tau}, g) = L(Y^{\tau}, g)\}.$$

Now from the definition of τ it is obvious τ has no fixed points. Hence $\tau \in \tilde{T}$.

(ii) Let S be any point. Let us assume that $S \not{o} X$ for each SIg_3. Hence $S^\tau = L(S, g_3) \wedge L(Q_3, SP_3)$.

Thus from the choice of S, (A7) and Lemma (1.1.10) we have

$$S = P_3S \wedge L(S, g_3) \text{ and } S^\tau = Q_3S^\tau \wedge L(S, g_3).$$

Hence Lemma (1.1.11) yields

$$P_3 \not{o} Q_3 \Longleftrightarrow P_3S \not{o} L(Q_3, P_3S) \Longleftrightarrow S \not{o} S^\tau.$$

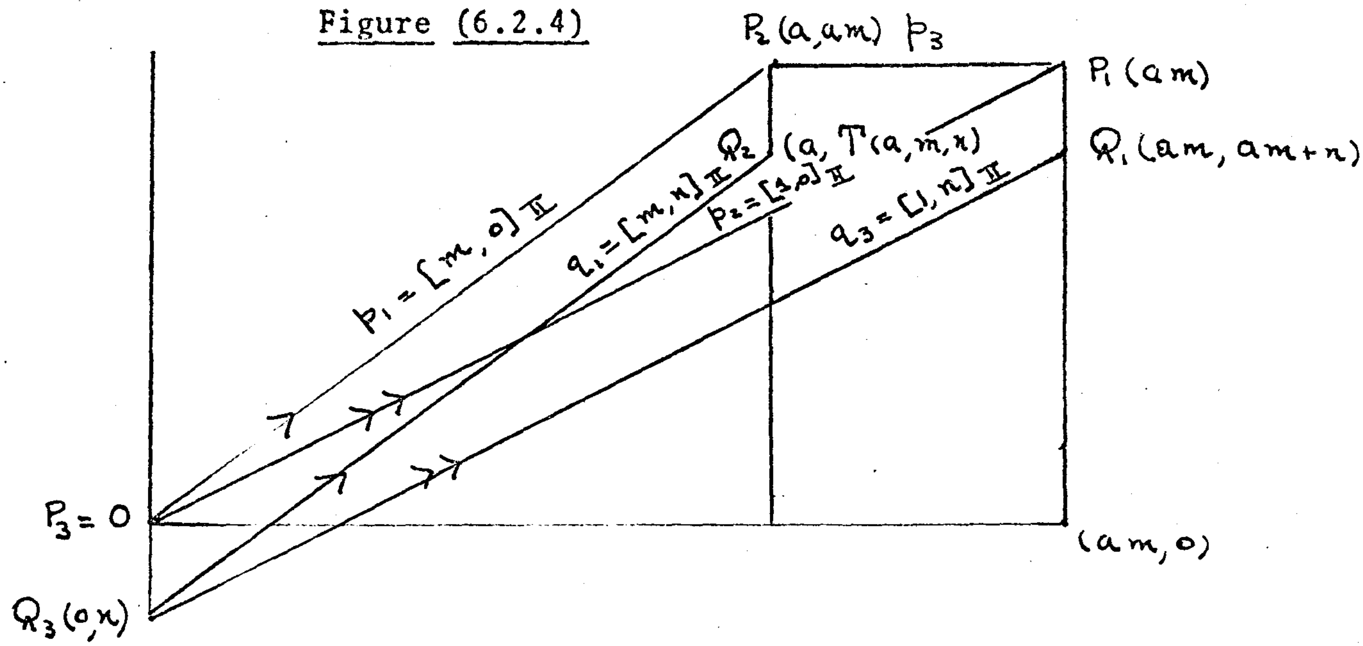
In particular $P_i \not{o} Q_i$; $i = 1, 2$. Thus if we replace g_3 by g_1 or g_2 we may use the identical argument to show $S \not{o} S^\tau$ if $S \not{o} X$ for each XIg_i; $i = 1, 2$.

(iii) Choose h any trace of τ.

It is sufficient to show $h \parallel g_3$. Let $S, S^\tau Ih$. Then by Claim (3) there exists $i \varepsilon \{1, 2, 3\}$ such that $S \not{o} X$ for each XIg_i. Thus $S^\tau = L(S, g_i) \wedge L(Q_i, SP_i)$. Hence we obtain $S, S^\tau\ IL(S, g_i)$. Since by (ii), $S \not{o} S^\tau$, we have $\{h = L(S, g_i)\} \parallel g_3$.

(3)$\Longrightarrow$(1). We must show $T(a, m, n) = am + n$.

<u>Figure</u> <u>(6.2.4)</u>



Let $P_3 = 0$; $Q_3 = (0, n)$; $P_2 = (a, am)$; $P_1 = am$;

$Q_1 = (am, am + n)$; $Q_2 = (a, \ (a, m, n))$; $p_1 = [m_1, 0]_{II}$;

$q_1 = [m, n]_{II}$, $p_2 = [1, 0]_{II}$; $q_2 = [1, n]_{II}$ and

$p_3 = L(P_2, g)$. In view of (HT2) and (HT3) we may

assume $m \neq 0, 1$ and $n \neq 0$. Consider figure (6.2.4).

By the definition of $\mathcal{L}_2$, $p_1, p_2 \emptyset h$. Let $\tau = \tau_{P_3Q_3}$.

Then by Case (1) of the proof of Theorem (3.2.1),

$$P_2^\tau = Q_2 \text{ and } P_1^\tau = Q_1.$$

Now $T(a, m, n) = am + n$ iff $Q_1 I L(Q_2, g)$. But

$\{Q_1 = P_1^\tau\} I \{L(P_2^\tau, p_3) = L(Q_2, g)\}$.

Comment (6.2.2). Notice in the above proof of (3) $\Rightarrow$ (1), we could not invoke D_1 from figure (6.2.4), since we do not know $P_2 \not o P_1$. In fact, this is true iff $am \not o a$, which is not true in general.

Theorem (6.2.5). [K1] *Let $\mathcal{X}$ be minor Desarguesian. Then*

(1) $(OE, +)$ *is an abelian group*.

(2) $a(b + c) = ab + ac$.

Proof: It is essentially the same as Theorem (6.1.2).

Theorem (6.2.6). Let $\mathcal{X}$ be minor Desarguesian. Then $a + b = a +_1 b$.

Proof: Now $a +_1 b = OE \wedge L(L((b, 0), OE) \wedge L(a, g), h))$. Let $T = L((b, 0), OE) \wedge L(a, g)$. Now since $OE \not o oY$, $L((b, 0), OE) \varepsilon \mathcal{L}_2$. By Corollary (6) of Theorem (6.2.3) and the fact $(OE, +)$ is an abelian group we obtain,

$$L((b, 0), OE) \wedge oY = (0, -b).$$

Hence $L((b, 0), OE) = [1, -b]_{II}$. Clearly $T = (x, a)$ for some x. But $T I L((b, 0), OE)$. Hence $x = a + b$. Thus,

$$a +_1 b = 0E \wedge L((a + b, a), h) = a + b. \;//$$

We next wish to obtain the statement of Theorem (6.1.3), by replacing the assumption that $\mathcal{K}$ is Pappian with the assumption that $\mathcal{K}$ is Desarguesian. Now as mentioned before no Desarguesian configuration has been defined. I can define one and show it is equivalent to (A10)(P: $\not{I}$). However the proof is very long and technical and so we shall omit it. Moreover we actually need the full force of (A10)(P), not just (A10)(P:$\not{I}$).

Remark (6.2.5). Let $\sigma \varepsilon D_P$; $P \not{I} Q$ and $Q^\sigma = R$. If S is any point such that S, PIf; S, QIj and $f \not{I} j$ then $S^\sigma = f \wedge L(R, j)$.

Proof: This follows immediately from Case (1) of the proof of Theorem (3.1.1).

Lemma (6.2.19). Let $\mathcal{K}$ be Desarguesian. Then $a(b + c) = ab + ac$.

Proof: $(0, c)(1, b + c) = [b, c]_{II}$ by Lemma (6.2.13). Let $\sigma = \sigma[0, (1, c), (a, ac)]$, which exists by (A10)(0) since $0 \not{I} (1, c)$. Thus since $XE \not{I} [b + c, 0]_{II}$ and $h \not{I} [0, c]_{II}$ Remark (6.2.5) yields

$$(1,\ b + c)^{\sigma} = (a,\ a(b + c)),$$

and

$$(0,\ c)^{\sigma} = (0,\ ac).$$

Hence

$$(a,\ a(b + c))\, I \{L((0,\ ac),\ [b,\ c]_{II}) = [b,\ ac]_{II}\},$$

and so

$$a(b + c) = ac + ac.$$

<u>Lemma (6.2.20).</u> <u>Let</u> $\mathcal{D}$ <u>be</u> <u>Desarguesian.</u> <u>Then</u>

$$(ab)c = a(bc).$$

<u>Proof</u>: It is enough to show this for $b \notin \Pi_0$. For if $b \in \Pi_0$, then $b^* = b - 1 \notin \Pi_0$. Thus we obtain from Theorem (6.2.5)(2), and Lemma (6.2.19),

$$(ab)c = (a(b^* + 1))c = (ab^* + a)c$$

$$= (ab^*)c + ac = a(b^*c) + ac$$

$$= a[b^*c + c] = a[bc - c + c] = a(bc).$$

Now we consider 3 cases for $b \notin \Pi_0$.

Case (1): $b \not o c$. Choose j such that (1, b), (b, bc)Ij.

Claim. $j \not o [c, 0]_{II}$. Suppose $j o [c, 0]_{II}$. Since $b \not o c$, we have $[c, 0]_{II} \not o [b, 0]_{II}$. Thus (A6) yields (1, b)o(0, 0). Contradiction.

Now let $\sigma_1 = \sigma[0, b, ab]$, which exists since $b \notin \Pi_0$. Since $L(b, h) \not o [c, 0]_{II}$ and $[0, b]_{II} \not o [b, 0]_{II}$, Remark (6.2.5) yields

$$(b, bc)^{\sigma_1} = (ab, (ab)c)$$

and

$$(1, b)^{\sigma_1} = (a, ab).$$

Thus (a, ab)IL((ab, (ab)c), j), since $\sigma_1 \varepsilon D$.

Now define $\sigma_2 = \sigma[0, (1, b), (a, ab)]$. Then since $XE \not o [bc, 0]_{II}$ and $j \not o [c, 0]_{II}$ by the above claim, we obtain

$$(1, bc)^{\sigma_2} = (a, a(bc))$$

and

$$(b, bc)^{\sigma_2} = (ab, (ab)c).$$

Hence $(a, a(bc)) I \{L((ab,(ab)c), [0, bc]_{II}) = L((ab, (ab)c), g)\}$. Thus $(a, a(bc))$ and $(ab, (ab)c)$ must have the same y-coordinate and so our result follows.

Case (2): $b = c$. Now $b \phi b - 1$, otherwise $1 \varepsilon \Pi_0$. Thus by Case (1) we obtain

$$(ab)b = ab(b - 1 + 1) = ab(b - 1) + ab$$

$$= a(b(b - 1)) + ab = a(bb - b) + ab$$

$$= a(bb) - ab + ab = a(bb).$$

Case (3): $b \neq c$ but $b o c$. Now $b o c$ implies $c = b + n$ for some $n \varepsilon \Pi_0$. Hence $b \phi n$. Thus by Case (1) and Case (2) we obtain

$$(ab)c = (ab)(b + n) = (ab)b + (ab)n$$

$$= a(bb) + a(bn) = a(bb + bn)$$

$$= a(b(b + n)) = a(bc).$$

Theorem (6.2.6). *Let* $\mathcal{X}$ *be Desarguesian. Then*

(1) *If* $a \notin \Pi_0$, *then*

$$a^{-1} = OE \wedge L(XE \wedge O(a, 1), g) \text{ where}$$

$$XE \wedge O(a, 1) = (1, a^{-1}).$$

(2) $a \cdot_1 b = ab$.

(3) <u>If</u> $m \varepsilon \Pi_0$, $T(a, m, n) = T_1(a, m, n)$ <u>and</u> <u>hence</u>

$[m, n]_I = \{(T(y, m, n), y) | y \varepsilon 0E\}$, <u>for</u> <u>each</u> $[m, n]_I \varepsilon \mathcal{L}_1$.

(4) $(0E, +, .)$ <u>is</u> <u>an</u> A H -<u>ring</u>.

(5) <u>If</u> H <u>is</u> <u>the</u> <u>ring</u> <u>of</u> <u>trace</u> <u>preserving</u> <u>endomorphisms</u> then $(0E, +, .) \cong H$.

<u>Proof</u>: (1) This follows from Corollary (1) and Corollary (8) of Theorem (6.2.3) and Lemma (6.2.20).

(2) Now $a.b = 0E \wedge L(0(1, b) \wedge L(a, h), g)$ and

$$a \cdot_1 b = 0E \wedge L(0(b, 1) \wedge L(a, g), h).$$

<u>Case (1)</u>: $b \notin \Pi_0$. Thus from Lemma (6.2.13), $0(b, 1) \varepsilon \mathcal{L}_2$. By (1), $0(b, 1) \wedge XE = (1, b^{-1})$. Hence $0(b, 1) = [b^{-1}, 0]_{II}$. Let $T = [b^{-1}, 0]_{II} \wedge L(a, g)$. Hence $T = (x, a)$ for some x such that $a = xb^{-1}$.

By Lemma (6.2.20), $x = ab$. Thus

$$a \cdot_1 b = 0E \wedge L(T, h) = 0E \wedge L[(ab, a), h] = ab.$$

<u>Case (2)</u>: $b \in \Pi_0$. Define $b^* = b - 1$. Clearly $b^* \notin \Pi_0$. By Theorem (6.2.6),

$$x +_1 y = x + y. \qquad \text{(I)}$$

Then using (I) and Case (1), we obtain

$$a \cdot_1 b = a \cdot_1 (b^* + 1) = a_1 . b^* +_1 a$$

$$= a.b^* + a = a(b^* + 1) = a.b.$$

(3) This follows immediately from Theorem (6.2.6) (2) and the fact that T is linear.

(4) We have already shown all the properties of an A H -ring.

(5) We may consider $\mathcal{H}$ as A(0E). The result then follows from Theorem (5.3.8)(3).

CHAPTER 7

Topological Prerequisites

In this chapter we list known results as well as proving some new results which we shall utilize in the next chapter.

Notation. (1) Let $(X_\alpha)_{\alpha\varepsilon I}$ be a family of topological spaces. Then $\prod_{\alpha\varepsilon I} X\alpha$ is the set theoretic product endowed with the product topology; i.e., if $pr_\alpha: \prod X_r \to X_\alpha$ is the α projection map, $p_{r_\alpha}((x_\beta)) = x_\alpha$, the sets $\{p_{r_\alpha}^{-1}(U_\alpha) | U_\alpha \text{ open in } X_\alpha\}$ form a subbase for the product topology.

If we have just two spaces, X_1, X_2 we write $X_1 \times X_2$ for the product.

(2) If X is a space and $x \varepsilon X$, we use $\mathcal{N}(x)$ or $\Omega(x)$ to represent neighbourhood filters about x.

(3) X is T_2 means X is Hausdorff and X is T_1 means X is a Fréchét space.

(4) If $S \subseteq X$, then $\Gamma(S)$ is the closure of S in X and I(S) is the interior of S. If

$A \subseteq S \subseteq X$, then $\Gamma_S(A)$ and $I_S(A)$ are the relative closure and interior of A with respect to S. It is well known that $\Gamma_S(A) = \Gamma(A) \cap S$ and $I(A) \cap S \subseteq I(A) \subseteq I_S(A)$.

§7.1. Quotient Topology

The results may all be found in [K00].

Definition (7.1.1). Let X be a topological space and R an equivalence relation on X. Let $X/R = \{[x] \mid [x] = \{y \mid (x, y) \in R\}\}$ be the quotient space, and let $f: X \to X/R$ be the quotient map, $f(x) = [x]$. We define a topology on X/R as follows:

U is open in X/R iff $f^{-1}(U)$ is open in X. This is called the quotient topology of X/R.

Theorem (7.1.1). Let $f: X \to X/R$ be as in definition (7.2.1). Then

(1) f is a continuous map.

(2) C is closed in X/R iff $f^{-1}(C)$ is closed in X.

(3) The quotient topology is the largest topology on X/R such that f is continuous.

Theorem (7.1.2). If X is a topological space, R is an equivalence relation on X, and X/R is endowed with the quotient topology, then

(1) If X/R is T_2, R is closed in X x X.

(2) If $f: X \to X/R$ is open and R is closed, then X/R is T_2.

(3) *If* R *is closed in* $X \times X$, *then* $[x]$ *is a closed set in* X.

§7.2. Connectedness

The well known results may be found in [E 1].

Definition (7.2.1). (1) A topological space X is connected iff it is not the disjoint union of two open (closed) sets. Equivalently the only sets which are both open and closed are $\emptyset$ or X.

If X is not connected, it is called disconnected.

(2) If $A, B \subseteq X$, then the pair (A, B) is called separated iff $\Gamma(A) \cap B = A \cap \Gamma(B) = \emptyset$.

Theorem (7.2.1).

(1) Let $C \subseteq X$. C is connected iff for each separated pair (A, B) in X such that $C = A \cup B$, $A = \emptyset$ or $B = \emptyset$.

(2) If $C \subseteq X$ and C is connected, then for each separated pair (A, B) such that $C \subseteq A \cup B$, we have $C \subseteq A$ or $C \subseteq B$.

(3) If $(X_\alpha)_{\alpha \in I}$ is a family of connected subspaces of X such that $\bigcap_\alpha X_\alpha \neq \emptyset$, then $\bigcup_\alpha X_\alpha$ is connected.

(4) If $(X_\alpha)_{\alpha \in I}$ is a family of connected subspaces such that $X_\alpha \neq \emptyset$, then $\prod_{\alpha \in I} X_\alpha$ is connected iff X_α is connected

for each α .

(5) The continuous image of a connected set is connected.

(6) If C is connected, then $\Gamma(C)$ is connected.

Theorem (7.2.2). Let R be an equivalence relation on X.

(1) If X is connected, so is X/R.

(2) If X/R is connected and each $[x]$ is connected, then X is connected.

Definition (7.2.2). Let X be a topological space and $x \in X$. $C(x)$ = largest connected subset containing x, is called the component of x. X is called totally disconnected iff $C(x) = \{x\}$ for each x.

$$Q(x) = \bigcap_{x \in A} A \text{ (A is open-closed) is called the}$$

quasi-component of x.

Theorem (7.2.3). The following are true.

(1) If $(x_\alpha)_{\alpha \in I}$ is a family of topological spaces, then $C((x_\alpha)) = \prod_{\alpha \in I} C(x_\alpha)$.

Hence $\prod X_\alpha$ is totally disconnected iff each X_α is totally disconnected.

(2) $C(x)$ and $Q(x)$ are closed sets.

(3) $C(x) \subseteq Q(x)$.

(4) If $Q(x) = X$, then X is connected.

Theorem (7.2.4). *Let* X *be a space. Then* $\{C(x) | x \in X\}$ *forms a disjoint partition. Let* C *be the corresponding equivalence relation. Then* X/C *is totally disconnected.*

Lemma (7.2.1). *Let* X *be a space. Let* $A \subseteq X$, *be closed such that* $x \in A$ *implies* $C(x) \subseteq A$.

Then $C_A(x) = C(x)$ *for each* $x \in A$.

Proof. It is enough to show that for each $x \in A$,

(a) $C(x)$ is a connected subspace of A.

(b) $C_A(x)$ is connected in X.

(a) Suppose $C(x)$ is disconnected in A, as by assumption $C(x) \subseteq A$. Hence there exist C_1, C_2, non-void, closed in A such that $C(x) = C_1 \cup C_2$, $C_1 \cap C_2 = \emptyset$. Now $C_i = S_i \cap A$, where S_i is closed in X; $i = 1, 2$. Since A is closed, in X, C_1 and C_2 are closed in X. Hence $C(x)$ is disconnected in X. Contradiction.

(b) Suppose $C_A(x)$ is disconnected in X. Then there exist C_1, C_2 closed and non-void such that $C_1 \cap C_2 = \emptyset$ and $C_A(x) = C_1 \cup C_2$. Hence $C_i \cap A = C_i$ is closed in A. Thus $C_A(x) = (C_1 \cap A) \cup (C_2 \cap A)$ implies $C_A(x)$ is disconnected in A. Contradiction.

Lemma (7.2.2). Let θ be an equivalence relation on X, such that $C(x) \subseteq \overline{x}$ where $\overline{x} = \{y \mid (x, y) \varepsilon R\}$. Moreover each $\overline{x}$ is closed in X. Then

$$C_{\overline{X}}(y) = C(y) \text{ for each } y \varepsilon \overline{x}.$$

Proof: Since $\overline{x}$ is closed and for each $y \varepsilon \overline{x}$, $C(y) \subseteq \{\overline{y} = \overline{x}\}$, the result follows from Lemma (7.3.1).

The next result is an exercise on page 261 of (E1). The proof may be found in [KO].

Theorem (7.2.5). Let $X = \prod_{\alpha \varepsilon I} X_\alpha$ and $(x_\alpha)_{\alpha \varepsilon I} = x \in X$

Then $Q(x) = \prod_{\alpha \varepsilon I} Q(x_\alpha)$.

The next result, or to be precise, the idea for its proof, is used in [P1] and [S1] to show that a topological projective plane is connected or totally disconnected. We shall prove the theorem in its most general setting.

Theorem (7.2.6). Let X be a topological space. Suppose G is a set of homeomorphisms from X into X with the property: for any two pairs of points (x, y), (x, z) such that $x \neq y$ and $x \neq z$, there exists $f \varepsilon G$, such that $f(x) = x$ and $f(y) = z$. Then X is connected or totally disconnected.

Proof: Suppose X is not connected. By Theorem (7.2.3)(4), $Q(x) \neq X$.

Claim. $Q(x) = \{x\}$. If this is false, there exists $y \in Q(x)$ such that $y \neq x$. Since $Q(x) \neq X$, there exists $z \in \complement Q(x)$. Hence there exists $f \in G$ such that $f(x) = x$ and $f(y) = z$. Thus $f[Q(x)] = Q(x)$ and so $f(y) = z \in Q(x)$. Contradiction. Thus $Q(x) = \{x\}$ and so by Theorem (7.2.3)(3), $C(x) = \{x\}$.

Next we consider a new concept of connectedness of a space X with respect to an arbitrary equivalence relation R.

Definition (7.2.1). Let X be a space and θ an equivalence relation on X. Let $[x]$ be any equivalence class (1) X is called θ-disconnected iff $X = U_1 \cup U_2$ such that U_1, U_2 are non-void open sets with the property, $x \in U_1$, $y \in U_2$ implies $x \not\theta y$.

Clearly $U_1 \cap U_2 = \emptyset$ and each U_i; $i = 1, 2$, is saturated with respect to θ; i.e., $x \in U_i$ implies $[x] \subseteq U_i$, $i = 1, 2$. Clearly we may replace open by closed.

If X is not θ-disconnected, we say X is θ-connected.

(2) A pair (A, B) is called θ-separated in X iff $x \in \Gamma(A)$ and $y \in B$ or $x \in A$ and $y \in \Gamma(B)$ implies $x \not\theta y$.

Let θ be an arbitrary equivalence relation on a space X, for the rest of this section.

Remark (7.2.1). If (A, B) is a pair such that A, B are open and $x \in U_1$, $y \in U_2$, implies $x \not\theta y$, then (A, B) is θ-separated.

Proof: Take $x \in \Gamma(u_1)$ and $y \in U_2$, such that $x \theta y$. Then since each U_i is saturated, $[x] = [y] \subseteq U_2$. Thus $x \in \Gamma(u_1) \cap U_2$ and hence $U_1 \cap U_2 \neq \emptyset$.

Remark (7.2.1). If θ is the identity relation then θ-connectedness is connectedness. Also θ-disconectedness implies disconnectedness.

Comment (7.2.1). We may now obtain results for θ-connectedness which are completely analogous to the well known results on connectedness. Since the proofs, as in Remark (8.3.1), are essentially the same, we shall not include them except where the generalization is not obvious.

Theorem (8.3.7). The following are equivalent, for $C \subseteq X$

(1) C is θ-connected.

(2) For each θ-separated pair (A, B) in X, such that $C = A \cup B$, $A = \emptyset$ or $B = \emptyset$.

(3) The only open-closed set V saturated with respect to θ is $\emptyset$ or X.

Corollary (1). If C is θ-connected and $C \subseteq U_1 \cup U_2$ such that (U_1, U_2) are θ-separated, then $C \subseteq U_1$ or $C \subseteq U_2$.

Corollary (2). If $\{C_i\}_{i \varepsilon I}$ is a family of θ-connected sets such that $\bigcap C_i \neq \emptyset$, then $\bigcup_{i \varepsilon I} C_i$ is θ-connected.

Corollary (3). If for each $x, y \varepsilon X$ there exists a θ-connected set C such that $x, y \varepsilon C$, then X is θ-connected.

Lemma (7.2.2). If $C \subseteq X$ is θ-connected, then $\Gamma(C)$ is θ-connected.

Lemma (7.2.3). If C_1 and C_2 are θ-connected, then $C_1 \times C_2$ is $(\theta \times \theta)$-connected, where (x_1, x_2) $(\theta \times \theta)(y_1, y_2)$ iff $x_1 \theta y_1$ and $x_2 \theta y_2$.

Proof: We invoke Corollary (3) of Theorem (7.3.6). Let (x_1, x_2), (y_1, y_2) be two points of $C_1 \times C_2$. Define $C = C_1 \times \{x_2\} \cup \{y_1\} \times C_2$. Since C_1 and C_2 are θ-connected, $C_1 \times \{x_2\}$ and $\{y_1\} \times C_2$ are $(\theta \times \theta)$-connected. Since $(y_1, x_2) \varepsilon C_1 \times \{x_2\} \cup \{y_1\} \times C_2$, C is $(\theta \times \theta)$-connected by Corollary (2) of Theorem (7.3.6). Since (x_1, x_2),

$(y, y_2) \epsilon C$, the result follows.

Lemma (7.2.4). Let C be θ-connected in X, and $f\colon\ C \times C \to X$ be a homeomorphism such that (a_1, a_2) $(\theta \times \theta)(b_1, b_2)$ iff $f(a_1, a_2)\ \theta\ f(b_1, b_2)$. Then X is θ-connected.

Proof: If V is open-closed and saturated with respect to θ, then $f^{-1}(V)$ is open-closed and saturated by the assumptions of the Lemma. Hence $f^{-1}(V) =$ $C \times C$ or $\emptyset$ and so $V = X$ or $\emptyset$.

Theorem (7.2.8). If X is θ-connected, then X/θ is connected. If $f\colon\ X \to X/\theta$ is open, the converse is true.

Proof: Let X be θ-connected. Choose V open-closed in X/θ. Then $f^{-1}(V)$ is open-closed in X and saturated with respect to θ. Hence $f^{-1}(V) = \emptyset$ or X, and so $V = \emptyset$ or X/θ. Conversely if f is open let $X = U_1 \cup U_2$ such that U_1, U_2 are non-void open and $x \epsilon U_1$, $y \epsilon U_2$ implies $x \not\theta y$. Hence $f(X) = f(U_1) \cup f(U_2)$ such that $f(U_i)$ are open non-void and $f(U_1) \cap f(U_2) = \emptyset$.

Definition (7.2.2). For each $x \epsilon X$, define,

(1) $A(x) = \{V | V$ is open-closed, saturated with respect to θ and $x \epsilon V\}$, and

$T(x) = \{y \mid \text{For each } V \in A(x), \bar{y} \cap V \neq \emptyset\}$

$= \{y \mid \text{For each } V \in A(x), \bar{y} \subseteq V\}.$

$= \bigcap V (V \in A(x)).$

(2) If $A \subseteq X$, $A_A(x) = \{V \mid V$ is open-closed, saturated in A such that $x \in V\}$ and $T_A(x) = \{y \mid \bar{y} \cap V \neq \emptyset$ for each $V \in A_A(x)\}$.

(3) X is called <u>totally θ-disconnected</u> iff $T(x) = \bar{x}$ for each $x \in X$.

<u>Theorem (7.2.9). The following are true, where</u> $f\colon X \to X/\theta$ is the quotient map.

(1) $T(x)$ <u>is a closed set</u>.

(2) <u>If</u> $T(x) = X$, <u>then</u> X <u>is</u> θ-<u>connected</u>.

(3) $x \subseteq T(x)$ <u>and</u> $Q(x) \subseteq T(x)$.

(4) $T(x) \subseteq f^{-1}(Q(x))$.

(5) <u>If</u> X/θ <u>is totally disconnected</u>, <u>then</u> X <u>is totally</u> θ-<u>disconnected</u>.

<u>Proof</u>: (1) is obvious. (2) is proved essentially the same as Theorem (7.3.3)(4). (3) is easily shown. (4) follows from the continuity of f and the fact that $f^{-1}(A)$ is saturated for any A. (5) follows immediately from (4).

<u>Lemma (7.2.5). Let</u> $A \subseteq X$; $f\colon X \to A$ <u>a continuous onto map such that</u> $x\theta y$ <u>implies</u> $f(x)\theta f(y)$. <u>Then</u>

$$T(x) \subseteq f^{-1}\, T_A(f_{(x)})\ .$$

<u>Proof:</u> Let C be open-closed in A and saturated such that $f(x)\varepsilon C$. Then $x\varepsilon f^{-1}(C)$ and $f^{-1}(C)$ is open-closed. Also if $y\varepsilon f^{-1}(C)$ and $z\theta y$, then $f(z)\theta f(y)$ and $f(y)\varepsilon C$. Hence $f(z)\varepsilon C$ or $z\varepsilon f^{-1}(C)$. Thus $f^{-1}(C)\varepsilon A(x)$. Hence it easily follows that $T_{(x)} \subseteq f^{-1}(T_A(f_{(x)})$.

§7.3. Miscellaneous Results

Theorem (7.3.1). Let X *be a topological space which is not indiscrete. Let* G *be a doubly transitive set of homeomorphisms from* X *to* X*. Then* X *is* T_1.

Proof: Let $x \neq y$. Since X is not indiscrete, there exist $s, t \in X$, $s \neq t$, and $V \in \Omega(s)$ such that $t \notin V$. There exists $f \in G$ such that $f(x) = s$ and $f(y) = t$. Since f is continuous, there exists $W \in \Omega(x)$ such that $f[W] \subseteq V$. Then clearly $y \notin W$, otherwise $f(y) = t \in V$. Contradiction. Similarly there exists $U \in \Omega(y)$ such that $x \notin U$. Thus the theorem is proved.

The next two results are easy to show. We omit their proofs.

Theorem (7.3.2). Let $f\colon X \to Y$; $g\colon Y \to Z$; $\tilde{f}\colon Y \to Z$ *and* $\tilde{g}\colon X \to Y$, *where* X, Y, Z *are topological spaces. Then*

(1) *If* $\tilde{f}\,\tilde{g}$ *is continuous and* $\tilde{g}$ *is open-onto, then* $\tilde{f}$ *is continuous.*

(2) If $\tilde{f}\,\tilde{g}$ is open and $\tilde{g}$ is continuous onto then $\tilde{f}$ is open.

(3) If $g\ f$ is continuous and g is open-(1 - 1), then f is continuous.

(4) If $g\ f$ is open and g is continuous-onto, then f is open.

Theorem (7.3.3). Let X, Y be topological spaces and $f\colon X \to Y$. If for each $x \in X$, there exists an open set $U \in \Omega(x)$, such that $f|U\colon U \to Y$ is continuous, then f is continuous.

Definition (7.3.1). Let $A \subseteq X$ and X is a topological space. Then $\partial(A) = \Gamma(A) \cap \Gamma(\complement A)$ is called the boundary of A.

From page 37 of [E2], we quote the following.

Lemma (7.3.1). [E2] Let $A \subseteq X$ and $B \subseteq X$, X a topological space.

(1) $\partial(A) = \emptyset$ iff A is open-closed.

(2) $\Gamma(A) = A \cup \partial(A)$.

(3) $\partial(A \cap B) \subseteq \partial(A) \cup \partial(B)$.

We end this chapter by proving the following technical lemma we shall use later.

Lemma (7.3.2). Let X be a topological space; $Q \subseteq X$; $V \subseteq X$. Assume (i) Q is closed in X,

(ii) V is open-closed in $X \setminus Q$.

Then $\partial(V) \subseteq Q$.

Proof: Claim. (a) V is open in X.

(b) $V \cap \partial(V) = \emptyset$.

(a) Since Q is closed, $X \setminus Q$ is open. Hence by (ii) V is open in X.

(b) From (a), V is open and hence $\Gamma \complement V = \complement V$. Then

$$V \cap \partial(V) = V \cap \left[\Gamma(V) \cap \Gamma(\complement V)\right]$$

$$= V \cap \Gamma(V) \cap \complement V = \emptyset.$$

Now V closed in $X \setminus Q$ implies $V = C \cap (X \setminus Q)$ for some closed set C in X. Hence $V = C \cap (X \cap \complement Q) = C \cap \complement Q$. Therefore we obtain

$$V \cup Q = (C \cap \complement Q) \cup Q = (C \cup Q) \cap (Q \cup \complement Q) = C \cup Q.$$

Since C and Q are closed in X, $C \cup Q$ is closed in X, and so $V \cup Q$ is closed. Now $\Gamma(V) = V \cup \partial(V)$ by Lemma (7.5.1)(2), and since $\Gamma(V)$ is the smallest closed set containing V, we obtain

$$\partial(V) \subseteq V \cup \partial(V) = \Gamma(V) \subseteq V \cup Q.$$

But by Claim (b), $V \cap \partial(V) = \emptyset$. Hence $\partial(V) \subseteq Q$.

§7.4. The compact-open topology

Definition (7.4.1). Let X and Y be topological spaces. $C(X, Y) = \{f \mid f: X \to Y \text{ is a continuous map}\}$.

We define two topologies on $C(X, Y)$ as follows:

Let $\mathfrak{F}$ = set of finite subsets of X.
$\mathfrak{C}$ = set of compact subsets of X.
$\mathcal{O}$ = open sets of Y.

The sets $T(F, U) = \{f \mid f \in C(X, Y) \text{ such that } f[F] \subseteq U\}$, where $F \in \mathfrak{F}$ ($\mathfrak{C}$) and $U \in \mathcal{O}$, form a subbase for a topology called the topology of pointwise convergence (the compact-open topology). We denote the former by p and the latter by c. If we wish to make explicit which topology we are considering, we write $C_p(X, Y)$ or $C_c(X, Y)$. Moreover, we write, $f_\alpha \xrightarrow{p} f$ and $f_\alpha \xrightarrow{c} f$ if we are talking about convergence in p or c respectively. If, $A \subseteq C(X, Y)$, then $\Gamma_c(A)$ and $\Gamma_p(A)$ refer to the closure of A with respect to c or p.

The following results may be found in [B2].

Theorem (7.4.1). Let X and Y be two topological spaces.

(1) $p \subseteq c$ in $\mathcal{C}(X, Y)$.

(2) $\Gamma_c(A) \subseteq \Gamma_p(A)$ for each $A \subseteq C(X, Y)$.

(3) p is the product topology on $C(X, Y)$.

(4) If Y is T_2, then $C_c(X, Y)$ and $C_p(X, Y)$ are T_2.

Definition (7.4.1). Let X, Y, Z be topological spaces. Let f: $X \times Y \to Y$. Define $\tilde{f}$: $X \to C_c(Y, Z)$ by $\tilde{f}(x) = f_x$ where f_x: $Y \to Z$ is the map $f_x(y) = f(x, y)$.

Theorem (7.4.2). Let X, Y, Z, f and $\tilde{f}$ be as in Definition (7.1.1). The following are true.

(1) If f is continuous, then $\tilde{f}$ is continuous.

(2) If Y is locally compact T_2, the map

H: $C_c(X, Y) \times C_c(Y, Z) \to C_c(X, Z)$ defined by $H(g, f) = g\ f$ is continuous.

§7.5. Topological groups and rings

Definition (7.5.1). Let G be a group. G is a topological group iff G is a topological space such that

(i) The map g_1: $G \times G \to G$, $g_1(x, y) = xy$ is continuous.
(ii) g_2: $G \to G$, $g_2(x) = x^{-1}$ is continuous.

If G is only a monoid under g_1, then G is a topological monoid iff g_1 is continuous.

Theorem (7.5.1). [P1] Every T_2 topological group is completely regular.

Theorem (7.5.2). [P1] Let X be a topological space and G a topological group. Then $C_c(X, G)$ is a topological group with the operation $(f.g)(x) = f(x).g(x)$.

Theorem (7.5.3). [B2] If X is a locally compact space, then $C_c(X, X) = C_c(X)$ is a topological monoid under composition.

Proof: This follows immediately from Theorem

(7.1.2)(2).

Definition (7.5.2). (R, +, .) is a topological ring iff (i) R is a set with a topology.

(ii) (R, +) is a topological group.

(iii) {R \ {0} .) is a topological monoid.

If R is a division ring, then R is a topological division ring iff R is a topological ring and $(R \setminus \{0\}, .)$ is a topological group.

Definition (7.5.3). Let G and H be topological groups. Hom(G, H) = {f | f: G→H is a continuous group homomorphism}. End G = Hom(G, G).

Theorem (7.5.4). [P1] Let G and H be topological groups. Then

(1) If H is T_2, $\mathrm{Hom}_c(G, H)$ is closed in $C_c(G, H)$.

(2) If H is abelian, Hom(G, H) is a topological subgroup of $C_c(G, H)$.

Theorem (7.5.5). [P1] Let G be a locally compact T_2 topological additive abelian group. Then $\mathrm{End}_c\, G$ is a topological ring with the operations; $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(g(x))$.

Theorem (7.5.6). [P1] Let G_1 and G_2 be topological groups. Let $h : G_1 \to G_2$ be a topological group isomorphism. Then ϕ_h: $\mathrm{End}_c\, G_1 \to \mathrm{End}_c(G_2)$ defined by $\phi_h(f) = h\, f\, h^{-1}$, is a monoid isomorphism with respect to composition and a homeomorphism.

Corollary. Let G_1 and G_2 be locally compact T_2 groups. Then if h: $G_1 \to G_2$ is a topological group isomorphism ϕ_h: End $G_1 \to$ End G_2 is a topological monoid isomorphism.

Definition (7.5.4). [H1] G is called a semi-topological group iff g_1: $G \times G \to G$, $g_1(x, y) = xy$ is continuous in both variables separately.

Theorem (7.5.7). [E2] Every locally compact T_2 semi-topological group is a topological group.

Theorem (7.5.8). [H1] Let G be a semi-topological group and N^a normal subgroup. Then

(1) G/N is a semi-topological group.

(2) The quotient map f: $G \to G/N$ is open.

The theorem remains true if we replace semi-topological group with topological group.

Theorem (7.5.9). [H1]. Let G and H be topological groups. Let f: $G \to H$ be an open-continuous onto homomorphism. Then $G/K \cong H$, where $\cong$ is a topological group isomorphism.

CHAPTER 8

Topological H-planes

In this chapter we initiate a study of topological affine and projective H-planes.

The theory for the ordinary cases is of course due to Salzmann and Skornyakov. We shall obtain generalizations of these results.

§8.1. Topological affine H-planes

Definition (8.1.1). $\mathcal{B} = \langle \mathbb{P}, \mathcal{L}, I \rangle$ is a *topological incidence structure* iff $\mathbb{P}$ and $\mathcal{L}$ are topological spaces and $I \subseteq \mathbb{P} \times \mathcal{L}$. Such a structure is said to have a topological property "P" iff $\mathbb{P}$ has this property.

Notation. If Λ is an incidence structure, then for $P \in \mathbb{P}$, $\phi_P = \{\ell \mid \ell \in \mathcal{L}$ and $P I \ell\}$, and $\mathbb{P}_\ell = \{P \mid P I \ell\}$.

Lemma (8.1.1). *Let Λ be a topological incidence structure satisfying* (P1) [cf. Definition (1.3.1)].

Then

(1) $\mathbb{P}$ is a indiscrete (T_1) space iff each $\mathbb{P}_\ell$ is an indiscrete (T_1) space, where $\ell \varepsilon \mathcal{L}$.

(2) If each $\mathbb{P}_\ell$ is connected, then $\mathbb{P}$ is connected.

Proof: (1) If $\mathbb{P}$ is indiscrete or T_1, clearly $\mathbb{P}_\ell$ is also. Conversely let each $\mathbb{P}_\ell$ be indiscrete (T_1). By (P1), $\mathbb{P} = \bigcup_{\ell \varepsilon \phi_P} \mathbb{P}_\ell$ for any P.

First assume U is open in $\mathbb{P}$, $U \neq \emptyset$. Select $P \varepsilon U$. Hence $U = \bigcup_{\ell \varepsilon \phi_P} [U \cap \mathbb{P}_\ell]$. Since $\mathbb{P}$ is indiscrete, $U \cap \mathbb{P}_\ell = \mathbb{P}_\ell$. Hence $U = \mathbb{P}$. Secondly suppose each $\mathbb{P}_\ell$ is T_1. Hence $\Gamma_{\mathbb{P}_\ell} \{P\} = \Gamma\{P\} \cap \mathbb{P}_\ell = \{P\}$, and so $\Gamma\{P\} = \bigcup_{\ell \varepsilon \phi_P} [\Gamma\{P\} \cap \mathbb{P}_\ell] = \{P\}$.

(2) Since $\bigcap_{\ell \varepsilon \phi_P} \mathbb{P}_\ell \neq \emptyset$ and $\mathbb{P} = \bigcup_{\ell \varepsilon \phi_P} \mathbb{P}_\ell$, the result follows from Theorem (7.3.1)(3).

Definition (8.1.2). Let $\mathcal{H} = \langle \mathbb{P}, \mathcal{L}, I, \| \rangle$ be an affine H-plane. Let $\underline{\mathbb{P}}^2 = \mathbb{P} \times \mathbb{P} \setminus \circ_{\mathbb{P}}$ and $\underline{\mathcal{L}}^2 = \mathcal{L} \times \mathcal{L} \setminus \{(\ell, m) \mid \Lambda_\ell \circ \Lambda_m\}$.

Define $\phi_1 : \underline{\mathbb{P}}^2 \longrightarrow \mathcal{L}$ by $\phi_1(P, Q) = PQ$ and $\phi_2 : \underline{\mathcal{L}}^2 \rightarrow \mathbb{P}$ by $\phi_2(\ell, m) = \ell \wedge m$. Recall, $\chi = (\chi_{\mathbb{P}}, \chi_{\mathcal{L}}) : \mathcal{H} \longrightarrow \mathcal{H}/\circ$ is the quotient map. ϕ_1, ϕ_2, L are called the associated maps of the plane $\mathcal{H}$. Let $\overline{\phi}_1$, $\overline{\phi}_2$, $\overline{L}$

be the associated maps of $\bar{\mathcal{X}} = \mathcal{X}/\circ$.

Notation: If $f\colon X \to Y$, $g\colon X \to Y$ and $h\colon X \times Y \to Z$ then for any integer n

f^n is the map $f^n(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$,

$(f \times g)$ is the map $(f \times g)(x, y) = (f(x), g(y))$, and h^x is the map $h^x(y) = h(x, y)$.

Remark (8.1.1). The following identities hold:

(1) $\phi_1 \chi^2_{\mathcal{P}} = \chi_{\mathcal{L}} \phi_1$.

(2) $\phi_2 \chi^2 = \chi_{\mathcal{P}} \phi_2$.

(3) $\phi \chi^2 = \chi_{\mathcal{L}} L$.

Proof: These all are easily shown using the fact χ is a homomorphism.

Definition (8.1.3). Let $\mathcal{P}^P = \mathcal{P} \setminus \bar{P}$ and $A(j) = \mathcal{L} \setminus \{\ell \mid \Lambda_\ell o \Lambda_j\}$ where $P \varepsilon \mathcal{P}$ and $j \varepsilon \mathcal{L}$. Define

$\phi_1^P\colon \mathcal{P}^P \to \mathcal{L}$ by $\phi_1^P(Q) = PQ$.

$\phi_2^j\colon A(j) \to \mathcal{P}$ by $\phi_2^j(\ell) = \ell \wedge j$.

$L^j\colon \mathcal{P} \to \mathcal{L}$ by $L^j(P) = L(P, j)$

$L^P\colon \mathcal{L} \to \mathcal{L}$ by $L^P(\ell) = L(P, \ell)$.

Definition (8.1.4). $\mathcal{H} = \langle \mathcal{P}, \mathcal{L}, I, \parallel \rangle$ is a *topological affine H-plane* iff $\mathcal{H}$ is an affine H-plane with the properties

(TA1). $\mathcal{H}$ is a topological incidence structure.

(TA2). The maps ϕ_1, ϕ_2 and L are continuous.

Clearly ϕ_1^p, ϕ_2^j, L^p and L^j are also continuous.

Lemma (8.1.2). *If $\mathcal{H}$ is an affine-H-plane satisfying (TA1) and having ϕ_2 and L continuous then for any ternary field* $\{0E, X, Y\}$, h_2: $0E \times 0E \to \mathcal{P}$ (cf. Lemma (6.2.2 *a homeomorphism. In general, for any* $\ell \in \mathcal{L}$, $\ell \times \ell$ *is homeomorphic to* $\mathcal{P}$.

Proof: We may clearly choose a coordinate system $\{0, E, X, Y\}$ such that $\ell = 0E$. From Lemma (6.2.2), there is a (1 - 1) onto map h_2: $0E \times 0E \to \mathcal{P}$ such that

$$h_2 = \phi_2 \ (L^h \times L^g) \text{ and } h_2^{-1} = (\phi_2^{0E} \ L^h) \times (\phi_2^{0E} \ L^g).$$

Hence since ϕ_2 and L are continuous, h_2 is a homeomorphism.

Definition (8.1.5).

(1) (L, .) is a *topological loop* iff L is a topological space and is a continuous map. If (L, .) is a loop, then ϕ_a and

${}_a\phi$ are the maps $\phi_a(x) = xa$ and ${}_a\phi(x) = ax$.

(2) (Γ, T) is a *topological ternary ring* iff T and its inverses are continuous. [cf. Definitions (6.1.1) and (6.2.4)].

Lemma (8.1.3). If $(L, .)$ is a topological loop such that ϕ_a and ${}_b\phi$ are homeomorphisms, then for any $a \in L$, $\Omega(a) = \{aU\} = \{Ua\}$ *where* $\{U\} = \Omega(1)$.

Proof: This follows immediately since ϕ_a and ${}_a\phi$ are homeomorphisms.

Theorem (8.1.1). Let $\mathcal{H}$ be a topological affine H-plane and $(0, E, X, Y)$ any coordinate system, with $\{0E, T\}$ its associated ternary ring. Then

(1) T and its inverses are continuous maps.

(2) $(0E, +)$ is a topological loop, and multiplication is also continuous.

(3) The maps ϕ_a and ${}_a\phi$ in $(0E, +)$ are homeomorphisms.

(4) If $\Omega(0) = \{U\}$, then $\{U + a\} = \{a + U\} = \Omega(a)$. In particular if, $A \subseteq 0E$ then for any open set U, $U + A$ is also open.

Proof:

(1) From Definition (6.2.4), we obtain $T = (\phi_2^{0E}.L^g.\phi_2).(L^h \times L).(i \times h_2^0 \times (\phi_1^0.h_2^E)$. From (TA2) and Lemma (6.1.2), T is clearly composed of products and compositions of continuous maps. From Lemma (6.2.15),

we see the same is true for N, M and X. Hence (1) is proved.

(2) Since T is continuous, this follows immediately.

(3) This follows immediately from Corollary (5) of Theorem (6.2.5).

(4) This comes from (2), (3) and Lemma (6.1.3).

Finally $U + A = \bigcup_{a\varepsilon A}(U + a)$ is open since each $U + a$ is open. //

<u>Notation</u>:

(1) Recall $\bar{\ell} = \{m|mo\ell\}$. We let $\ell/o = \{\bar{P}|\bar{P}I\bar{\ell}\}$. In view of Corollary (4) of Theorem (6.2.3), we may write, for any ternary ring (OE, T),

$$OE/o = OE/\pi_0 = \{a + \pi_0 | a\varepsilon OE\}.$$

Also $\bar{\mathcal{P}} = \mathcal{P}/o$ and $\bar{\mathcal{L}} = \mathcal{L}/o$ are used interchangeably.

(2) $\bar{\mathcal{P}}$, $\bar{\mathcal{L}}$ and ℓ/o <u>will all be endowed with their respective quotient topologies</u>. //

For the rest of this section $\mathcal{X}$ is a topological affine H-plane unless otherwise specified.

<u>Theorem (8.1.2)</u>. <u>For each</u> $\ell\varepsilon\mathcal{L}$, <u>the map</u> χ_ℓ: $\ell\to\ell/o$ <u>is open</u>.

<u>Proof</u>: We may assume for some coordinate system $(0, E, X, Y)$, that $\ell = 0E$.

<u>Claim</u>. If $U \subseteq 0E$, then

$$\{x \mid x + \mathcal{N}_0 = u + \mathcal{N}_0, \text{ for } u \in U\} = \{x \mid x = u+n, n \in \mathcal{N}_0, u \in U\}.$$

Since $0 \in \mathcal{N}_0$, we have the inclusion $\supseteq$. Now suppose $x = u + n$. Since $n \in \mathcal{N}_0$ we have $(u + n) \sigma (u)$ by Corollary (2) of Theorem (6.2.3). Hence we obtain from Corollary (4) of Theorem (6.2.3)

$$x + \mathcal{N}_0 = (u + n) + \mathcal{N}_0 = \overline{u + n}$$

$$= \bar{u} = u + \mathcal{N}_0.$$

Moreover again by Corollary (4) we have $\chi_{0E}(a) = a + \mathcal{N}_0$. Let U be open in $0E$. Then $\chi_{0E}(U)$ is open in $0E/\sigma$ iff $\chi^{-1}_{0E}\chi_{0E}(U)$ is open in $0E$. But by the above claim

$$\chi^{-1}_{0E}\chi_{0E}(U) = \{x \mid x + \mathcal{N}_0 = u + \mathcal{N}_0, u \in U\} = U + \mathcal{N}_0,$$

which is open by (4) of Theorem (8.1.1).

Corollary (1). Let $(0, E, X, Y)$ be a coordinate system, with T, M, N, $\mathcal{Z}$ its associated ternary operator and inverses. Let $\bar{T}$, $\bar{M}$, $\bar{N}$, $\bar{\mathcal{Z}}$ be the associated ternary operator and inverses of $(\bar{0}, \bar{E}, \bar{X}, \bar{Y})$ for $\mathcal{L}$. Then $\bar{T}$, $\bar{M}$, $\bar{N}$ and $\bar{\mathcal{Z}}$ are continuous maps. Thus $(0E/\theta, \bar{T})$ is a topological ternary ring where $\phi_{\bar{a}}$ and ${}_{\bar{a}}\phi$ are homeomorphisms.

Proof: Since χ_{0E} is open, continuous, onto, the results follow from the Corollary to Lemma (6.2.9), Lemma (6.2.16), Theorem (7.5.2) and the fact $\phi_{\bar{a}}\, \chi_{0E} = \chi_{0E}\, \phi_a$.

Corollary (2). $\mathcal{P}$ is not compact and each $\ell \varepsilon \mathcal{L}$ is not compact.

Proof: Let $\ell = 0E$. From page 48, (7.9) of [S3], no topological ternary field may be compact and so $(0E/\theta, \bar{T})$ is not compact. But if $0E$ is compact, then $0E/\theta$ is compact since χ_{0E} is continuous. Also if $\mathcal{P}$ is compact, then since $\mathcal{P} \cong 0E \times 0E$, $0E$ is compact by the Tychonoff theorem, and again $0E/\theta$ is compact. Contradiction.

Theorem (8.1.3). Let $(0E, T)$ and $(\bar{0}\bar{E}, \bar{T})$ be the associated ternary fields of $\{0, E, X, Y\}$ and $\{\bar{0}, \bar{E}, \bar{X}, \bar{Y}\}$ respectively. Then we have,

(1) <u>$0E/\Pi_0$ is T_1 iff Π_0 is closed in $0E$.</u>

(2) <u>The following are equivalent.</u>

(a) <u>$0E/\Pi_0$ is discrete.</u>

(b) <u>Π_0 is open.</u>

(c) <u>$\{\bar{0}\}$ is open in $0E/\Pi_0$.</u>

<u>Proof</u>: (1) If $0E/\Pi_0$ is T_1, then $\{\Pi_0\}$ is closed in $0E/\Pi_0$. Hence $\chi_{0E}^{-1}(\{\Pi_0\}) = \Pi_0$ is closed in $0E$. Conversely if Π_0 is closed, then $a + \Pi_0$ is closed, since a^ϕ is a homeomorphism by Theorem (8.1.1)(3). Thus $0E \setminus (a + \Pi_0)$ is open. Because χ_{0E} is open, $\chi_{0E}(0E \setminus (a + \Pi_0)) = 0E/\Pi_0 \setminus \{a + \Pi_0\}$ is open. Hence $\{a + \Pi_0\}$ is closed for each $a \varepsilon 0E$ and so $0E/\Pi_0$ is T_1.

(2) <u>(a)$\Longrightarrow$(b)</u> If $0E/\Pi_0$ is discrete, then $\{a + \Pi_0\}$ is open for each $a \varepsilon 0E$. Thus $\{\Pi_0\}$ is open. Hence $\chi_{0E}^{-1}(\{\Pi_0\}) = \Pi_0$ is open.

<u>(b)$\Longrightarrow$(c)</u>. Let Π_0 be open. Since χ_{0E} is open, $\chi_{0E}(\Pi_0) = \{\Pi_0\} = \{\bar{0}\}$ is open.

<u>(c)$\Longrightarrow$(d)</u>. This follows since ϕ_a is a homeomorphism, from Corollary (1) of Theorem (8.1.2).

<u>Corollary</u>. <u>If $P I \ell$ and $\{P\}$ is open in ℓ/ρ, then ℓ/ρ is discrete.</u>

The next theorem was shown in [S1], for ordinary topological planes. The proof for H-planes is exactly the same.

Theorem (8.1.4). Each $\ell \varepsilon \mathfrak{L}$ is a regular space. $\mathbb{P}$ is a regular space.

Proof: Choose $\ell = 0E$. Take $U \varepsilon \Omega(0)$. Since addition is continuous, there exists $V \varepsilon \Omega(0)$ such that $V + V \subseteq U$. Now for each $x \varepsilon 0E$, $f_x^{-1}(V) = M(x; V) \varepsilon \Omega(x)$, since the function, $f_x(t)$ = the unique solution of $y + t = x$, is continuous by Theorem (6.2.3)(1), $f_x(x) = 0$ and $f_x^{-1}(V) = M(x; V)$. Hence $\Gamma(V) \subseteq u$. The result follows from Lemma (8.1.3)(4). $\mathbb{P}$ is regular as $\mathbb{P} \cong 0E \times 0E$.

Corollary. Each $\ell/0$ is a regular space.

Theorem (8.1.5). Let $\mathcal{H}$ satisfy (TA1) with ϕ_2 and L continuous. Then

(1) If ℓ, $m \varepsilon \mathfrak{L}$, there exists a homeomorphism f: $\ell \to m$ such that $SoT \Leftrightarrow f(s) o f(T)$. Moreover if ℓ/o, $m/o \varepsilon \mathfrak{L}/o$, then they are homeomorphic.

(2) If $\ell \| \ell'$; $A, B I \ell$ and $A', B' I \ell'$ such that $A \not{o} B$ and $A' \not{o} B'$, then there exists a homeomorphism, f: $\ell \to \ell'$ such that $f(A) = A'$, $f(B) = B'$ and $XoY \Leftrightarrow f(X) o f(Y)$.

Proof: (1) The map f from Lemma (1.2.2) clearly satisfies the first statement since $f(X) = L(j, X) \wedge \ell$ and $f^{-1}(X) = L(j, X) \wedge m$. We prove the last part as follows. Define $\bar{f}$: $\ell/o \to m/o$ by $\bar{f}(\bar{s}) = \overline{f(s)}$. $\bar{f}$ is well defined since $SoT \Leftrightarrow f(s)of(T)$. Also $\chi_m f = \bar{f} \chi_\ell$. By Theorem (6.1.2), χ_m and χ_ℓ are open onto maps. $\bar{f}$ is easily seen to be bijective.

Finally since f is a homeomorphism, $\bar{f}$ is one also by Theorem (7.5.2).

(2) We consider two cases.

Case (1): $\ell \not o \ell'$. By Lemma (1.1.10), A, B, A', B' are pairwise non-neighbouring points. Define $t = A'B$, $j = A'A$ and $k = BB'$. By Lemma (1.1.10) and the assumptions of the theorem, we obtain $j \not o t, \ell$, and $k \not o t, \ell'$. Hence we may define the following maps. f_1: $\ell \to t$ by $f_1(X) = t \wedge L(X, j)$ and f_2: $t \to \ell'$, such that $f_2(X) = \ell' \wedge L(X, k)$. Just as in the proof of (1) we see both f_1 and f_2 are homeomorphisms with the property $XoY \Leftrightarrow f_i(X) o f_i(Y)$; $i = 1, 2$. Then $f = f_2 f_1$: $\ell \to \ell'$ is a homeomorphism such that $XoY \Leftrightarrow f(X) o f(Y)$ and $f(A) = A'$, $f(B) = B'$.

Case (2): $\ell o \ell'$. Choose $\ell'' \parallel \ell$ such that $\ell'' \not o \ell, \ell'$. Let A'', $B'' I \ell''$ such that $A'' \not o B''$. Then by Case (1), there exists a homeomorphism f_1: $\ell \to \ell''$

such that $f_1(A) = A''$, $f_1(B) = B''$ and XoY iff $f_1(X)$o$f_1(Y)$. Similarly we obtain a homeomorphism f_2: $\ell'' \to \ell$ with the same property such that $f_2(A'') = A'$ and $f_2(B'') = B'$. Thus $f = f_2\ f_1$ is our desired homeomorphism.

Let the following corollaries (1) and (2) have the same assumptions as the above theorem.

Corollary (1). Each ℓ/o has a doubly transitive set of homeomorphisms.

Proof: Take $\bar{A}$, $\bar{B}$, $\bar{A}'$, $\bar{B}'$ I ℓ/o, such that $\bar{A} \neq \bar{B}$ and $\bar{A}' \neq \bar{B}'$. Hence there exist X, Y, X', Y' such that XoA, YoB, X'oA' and Y'oB', and X, Y, X', Y'Iℓ. By the theorem, since $\ell \parallel \ell$, there exists a homeomorphism f such that $f(X) = X'$ and $f(Y) = Y'$. Define $\bar{f}$: $\ell/o \to \ell/o$ by $\bar{f}(\bar{M}) = \overline{f(M)}$. Then as in the proof of (1) from Theorem (8.1.5), f is a homeomorphism. Moreover, $\bar{f}(\bar{A}) = \bar{f}(\bar{X}) = \overline{f(X)} = \overline{X'} = \overline{A'}$. Similarly $\bar{f}(\bar{A}') = \bar{B}'$.

Corollary (2). Each ℓ/o is connected or totally disconnected.

Proof: This follows from Corollary (1) and Theorem (7.2.5).

Corollary (3). If $\Lambda = <\mathcal{P}, \mathcal{L}, I>$ is an ordinary affine plane satisfying (TA1) with ϕ_2 and L *continuous, then $\mathcal{P}$ is connected or totally disconnected.*

Proof: Since $\circ$ = identity relation, this follows from Corollary (2).

For the rest of this chapter assume there exists $\ell/\circ \in \bar{\mathcal{L}}$ such that $\ell/\circ$ is neither indiscrete nor discrete. In view of Theorem (8.1.5)(1), each $m/\circ$ then has this property.

Remark (8.1.1). (1) $\mathcal{P}/\circ$ is neither discrete nor indiscrete.
(2) Each ℓ is neither discrete nor indiscrete and the same holds for $\mathcal{P}$.

Proof: (1) This follows from Lemma (8.1.1) and the fact a subspace of a discrete space is discrete.
(2) In view of the proof of (1) it is enough to show this for $\mathcal{L}$. Since $\ell/\circ$ is not indiscrete, there exists an open set $\bar{U} \neq \emptyset$ such that $\bar{U} \subsetneq \ell/\circ$. Now $\chi^{-1}(U)$ is open in ℓ, and $\chi_\ell^{-1}(U) \neq \emptyset$. If ℓ is indiscrete, $\chi_\ell^{-1}(U) = \ell$, and so $U = \ell/\circ$. Contradiction. Since χ_ℓ is open, it easily follows that ℓ is not discrete.

Lemma (8.1.4). *Let $\Lambda = <\mathcal{P}, \mathcal{L}, I>$ be an ordinary affine plane satisfying (TA1) such that ϕ_2 and L are continuous. Then $\mathcal{P}$ is either connected or* *totally disconnected.*

Proof: This follows immediately from Corollary (1) of Theorem (8.1.5) and Theorem (7.3.5).

Lemma (8.1.5). If $\{0, E, X, Y\}$ *is a coordinate system and* $0 = P(0, 0)$, *then* $h_2\left[(\bar{0} \cap 0E) \times (\bar{0} \cap 0E)\right] = \bar{0}$. *Thus in general if* $\ell \in \phi_P$, $(\bar{P} \cap \ell) \times (\bar{P} \cap \ell) \cong \bar{P}$.

Proof: Now $h_2\colon\ 0E \times 0E \rightarrow \mathfrak{P}$ is a homeomorphism by Lemma (8.1.2).

Since $h_2((\bar{0} \wedge 0E) \times (\bar{0} \wedge 0E)) = \bar{0}$, by Lemma (6.2.3), the result follows. //

It should be noted that until now we do not know if $\bar{\mathfrak{A}}$ is an ordinary topological affine plane. We next consider T_2 planes and end this section by determining necessary and sufficient conditions for $\bar{\mathfrak{A}}$ to be a topological plane.

Definition (8.1.6). $\mathfrak{A} = \langle \mathfrak{P}, \mathfrak{L}, I, \parallel \rangle$ is a T_2 *topological affine H-plane* iff the following conditions are valid

(a) $\mathfrak{P}$ and $\mathfrak{L}$ are T_2 spaces.

(b) $0_{\mathfrak{P}}$ and $0_{\mathfrak{L}}$ are closed in $\mathfrak{P} \times \mathfrak{P}$ and $\mathfrak{L} \times \mathfrak{L}$ respectively.

Comment (8.1.1). If $\mathcal{X}$ is an ordinary topological affine plane, conditions (a) and (b) in the above definition are equivalent.

Remark (8.1.2). If $\mathcal{X}$ is T_2, then each $\bar{\ell}$ is closed in $\mathcal{L}$.

Proof: This follows as a consequence of Theorem (7.1.2)(3).

Definition (8.1.7). Let $\bar{h}_2$: $OE/\sigma \times OE/\sigma \longrightarrow \mathcal{P}/\sigma$ be the map $\bar{h}_2(\bar{a}, \bar{b}) = \bar{P}$ such that $\bar{a}$, $\bar{b}$ are the coordinates of $\bar{P}$ in $\bar{\mathcal{X}}$ with respect to $\{\bar{O}, \bar{E}, \bar{X}, \bar{Y}\}$.

Remark (8.1.2). The following are true.

(1) $\chi_{\mathcal{P}} \ h_2 = \bar{h}_2 \ \chi^2_{OE}$.

(2) $\bar{h}_2$ is a (1 - 1) onto continuous map.

(3) $\chi_{\mathcal{P}}$ is open iff h_2 is open.

Proof: (1) is an easy calculation, (2) and (3) follow from Theorem (7.3.2) since $\chi_{\mathcal{P}}$ is open, continuous onto and h_2 is a homoemorphism.

Remark (8.1.3). The map g_2: $\mathcal{L}_2 \to OE \times OE$ defined by $g_2(\ell) = [m, n]_{II}$ is a homeomorphism.

Proof: This follows immediately from Lemma (6.2.4)

Remark (8.1.4). If X is a topological space such that for each pair $x \neq y$ there exists a T_1 subspace $S(x, y)$ such that $x, y \in S(x, y)$, then X is T_1.

Theorem (8.1.6). The following are equivalent,

(1) $\mathfrak{P}$ is T_1.

(2) $\mathfrak{L}$ is T.

(3) ℓ is a closed set of $\mathfrak{P}$ for each $\ell \in \mathfrak{L}$.

Proof: (1)⟹(2). We invoke Remark (8.1.4). Choose $\ell \neq m$. Select $j \in \mathfrak{L}$ such that $\Lambda_j \not\ni \Lambda_\ell, \Lambda_m$. Choose a coordinate system $\{0, E, X, Y\}$ such that $j = 0Y$. Then $\ell, m \in \mathfrak{L}_2$. From Remark (8.1.3), $\mathfrak{L}_2$ is homeomorphic to $0E \times 0E$ and hence is T_1.

(2)⟹(3). Let $\ell \in \mathfrak{L}$ and $\{P_\alpha\}$ be a net in ℓ such that $P_\alpha \to P$. Hence $(P_\alpha, \ell) \to (P, \ell)$ and so $L(P_\alpha, \ell) \to L(P, \ell)$. But $L(P_\alpha, \ell) = \ell$ for each α. Thus $L(P, \ell) \in \Gamma\{\ell\}$. But since $\mathfrak{L}$ is T_1, $\Gamma\{\ell\} = \ell$. Therefore $L(P, \ell) = \ell$ and so $PI\ell$.

(3)⟹(1). Let $P \in \mathfrak{P}$. Choose $\ell, m \in \phi_P$ such that $P = \ell \wedge m$. Since ℓ and m are closed sets, so is $\{P\}$.

Lemma (8.1.6). If $\mathfrak{L}$ is a T_1 space, then each pencil Λ is closed in $\mathfrak{L}$.

Proof: Let $\{m_\alpha\}$ be a net in Λ such that $m_\alpha \to m$. Take $\ell \in \Lambda$. Choose $PI\ell$. Define $F_\alpha = (P, m_\alpha)$ and $F = (P, m)$. Hence $L(F_\alpha) \to L(F)$. But $L(F_\alpha) = \ell$ for each α. Thus since $\mathcal{L}$ is T_1, $L(F) = \ell$ and so $\ell \parallel m$.

Theorem (8.1.7). The following are valid if $\mathcal{L}$ is T_2.

(1) Each line is a closed set of $\mathcal{L}$.

(2) Each parallel pencil is a closed set of $\mathcal{L}$.

(3) Each set of lines ϕ_P is closed in $\mathcal{L}$. (cf. Notation page 227).

Proof: (1) and (2) follow from Theorem (8.1.6) and Lemma (8.1.6) respectively.

(3) Let $\{\ell_\alpha\}$ be a net in ϕ_P such that $\ell_\alpha \to \ell$. Hence $L(P, \ell_\alpha) \to L(P, \ell)$. But $L(P, \ell_\alpha) = \ell_\alpha$. Since $\mathcal{L}$ is T_2, $L(P, \ell) = \ell$ and so $\ell \in \phi_P$.

Definition (8.1.8). If $g \in \mathcal{L}$ and $P \in \mathcal{P}$, we define

(i) $H(g) = \{P \mid P \not\circ X$ for each $XIg\}$.

(ii) $H(P) = \{g \mid P \not\circ X$ for each $XIg\}$.

(iii) $\Lambda(g) = \{m \mid \Lambda_g \not\circ \Lambda_m\}$.

Lemma (8.1.7). Let $\mathcal{L}$ be T_2. Then

(1) $H(h)$ is open in $\mathcal{P}$, for each line g.

(2) $H(P)$ and $\Lambda(g)$ are open in $\mathcal{L}$ for each point P and line ℓ.

Proof: We show the complements of each of these sets is closed. Let $\{P_\alpha\}$ be a set in $\complement H(g)$ such that $P_\alpha \to P$. Hence there exists $X_\alpha I g$ such that $X_\alpha o P$. To show there exists XIg such that XoP it suffices to show $L(P, g)og$, by Lemma (1.1.10). Now $P_\alpha \to P$ implies $L(P_\alpha, g) \to L(P, g)$. Since $X_\alpha o P$, $L(P_\alpha, g)og$ for each α. Thus by Remark (8.1.2), $L(P, g) \varepsilon g$ or $L(P, g)og$.

Next we show $\complement H(P)$ is closed. Let $\{g_\alpha\}$ be a net in $\complement H(P)$ such that $g_\alpha \to g$. Hence there exists $X_\alpha I\, g_\alpha$ such that $X_\alpha o P$ and so $L(P, g_\alpha)og$, for each α. Now $L(P, g_\alpha) \to L(P, g)$. Since $0_{\mathfrak{P}}$ is closed, $(L(P, g), g) \varepsilon 0_{\mathfrak{L}}$ or $L(P, g)og$. Thus $g \varepsilon \complement H(P)$.

Finally we show $\complement A(g)$ is closed in $\mathfrak{L}$. Let $\{m_\alpha\}$ be a net in $\complement A(g)$ such that $m_\alpha \to m$. Let $\Lambda_{m_\alpha} = \Lambda_\alpha$. Hence $\Lambda_\alpha o \Lambda_g$ for each α. We must show $\Lambda_m o \Lambda_g$. By Lemma (1.1.13), it is enough to show gom or $g \wedge m = \emptyset$. Suppose PIg, m. Choose $\Lambda_j \not{o}\, \Lambda_g, \Lambda_m$, such that PIj. Then $P = j \wedge m$ and $\Lambda_j \not{o} \Lambda_\alpha$ for each α. Define $P_\alpha = j \wedge m_\alpha$. Since ϕ_2^j is continuous, $P_\alpha \to P$. Now $\Lambda_\alpha o \Lambda_g$ implies there exists ℓ_α, $\ell_\alpha \| m_\alpha$ such that $\ell_\alpha og$, by Lemma (1.1.3). Since $(P_\alpha, m_\alpha) \to (P, m)$, we have $L(P_\alpha, m_\alpha) \to L(P, m)$. But $L(P_\alpha, m_\alpha) = L(P_\alpha, \ell_\alpha) = \ell_\alpha$ and $L(P, m) = m$ and so $\ell_\alpha \to m$. Since $\ell_\alpha og$ and $\bar{g}$ is closed, we obtain the desired result, gom. //

We next show the topology on $\mathcal{L}$ is essentially determined by the topology on $\mathcal{P}$ or the topology on ℓ for each $\ell \varepsilon \mathcal{L}$.

Notation: $\Omega_{PU}(V)$ is the relative neighbourhood filter of V with respect to PU.

Definition (8.1.9). Let $\ell \varepsilon \mathcal{L}$, such that $\ell = UV$, $U \not\sim V$. Select P such that $P \not{o} X$ for each $XI\ell$. Then define β_1 and β_2 as follows:

(1) $W \varepsilon \beta_1$ iff there exists $W_1 \varepsilon \Omega(U)$ and $W_2 \varepsilon \Omega(V)$ such that $W = \{RS | R \varepsilon W_1 \text{ and } S \varepsilon W_2\}$.

(2) $W \varepsilon \beta_2$ iff there exists $U_1 \varepsilon \Omega_{PU}(u)$ and $U_2 \varepsilon \Omega_{PV}(V)$ such that $P \notin U_1, U_2$ and

$$W = \{RS | R \varepsilon \Omega_{Pu}(u) \text{ and } S \varepsilon \Omega_{PV}(V)\}.$$

Clearly β_1 and β_2 are filter bases.

(3) Let $\Omega_1(\ell: u, v)$ be the filter generated by β_1 and $\Omega_2(\ell: u, v, P)$ be the filter generated by β_2.

Theorem (8.1.8). Suppose $\mathcal{X}$ is T_2. Let $\ell = UV$ and $P \not{o} X$ for each $XI\ell$. Then

$$\Omega(\ell) = \Omega_1(\ell: U, V) = \Omega_2(\ell: U, V, P).$$

<u>Proof:</u> <u>(i).</u> $\Omega(\ell) \subseteq \Omega_1(\ell: U, V)$. Take $V \varepsilon \Omega(\ell)$. By the choice of P, PU, $PV \not\ni \ell$. Thus $\ell = (PU \wedge \ell)(PV \wedge \ell) = \phi_1(U, V)$. Hence there exist $W_1 \varepsilon \Omega(u)$ and $W_2 \varepsilon \Omega(v)$ such that $\phi_1[W_1 \times W_2 \cap \mathbb{P}^2] \subseteq V$. Define $V_1 = \{RS | R \varepsilon W_1$ and $S \varepsilon W_2\}$. Clearly $V_1 \varepsilon \Omega_1(\ell: U, V)$. Also $RS \varepsilon V_1$ implies $(R, S) \varepsilon W_1 \times W_2 \cap \mathbb{P}^2$. Thus $\{\phi_1(R, S) = RS\} \varepsilon V$. Therefore $V_1 \subseteq V$ and so $V \varepsilon \Omega_1(\ell: U, V)$.

<u>(ii).</u> $\Omega_1(\ell: U, V) \subseteq \Omega_2(\ell: U, V, P)$. Let $V_1 \varepsilon \Omega_1(\ell: U, V)$. Thus there exist $W_1 \varepsilon \Omega(u)$ and $W_2 \varepsilon \Omega(v)$ such that $V_1 = \{RS | R \varepsilon W_1$ and $S \varepsilon W_2\}$. Now $P \neq U$ and $\mathbb{P}$ being T_2 imply there exists $\tilde{W}_1 \varepsilon \Omega(u)$ such that $P \notin \tilde{W}_1$. Similarly there exists $\tilde{W}_2 \varepsilon \Omega(v)$ such that $P \notin \tilde{W}_2$.

Define $W_1' = W_1 \cap \tilde{W}_1 \cap PU$ and $W_2' = W_2 \cap \tilde{W}_2 \cap PV$. Then $W_1' \varepsilon \Omega_{PU}(u)$ and $W_2' \varepsilon \Omega_{PV}(V)$. Then $V_2 = \{RS | R \varepsilon W_1'$ and $S \varepsilon W_2'\} \varepsilon \Omega_2(\ell: U, V, P)$ and $V_2 \subseteq V_1$ imply $V_1 \varepsilon \Omega_2 (\ell: U, V, P)$.

<u>(iii).</u> $\Omega_2(\ell: U, V, P) \subseteq \Omega(\ell)$. Let $V_2 \varepsilon \Omega_2(\ell: U, V, P)$. Then there exist $W_1 \varepsilon \Omega(U)$, $W_2 \varepsilon \Omega(V)$, such that $P \notin W_1 \cup W_2$ and $V_2 = \{RS | R \varepsilon W_1 \cap PU$ and $S \varepsilon W_2 \cap PV\}$. By the choice of P, $\Lambda_\ell \not\ni \Lambda_{PU}, \Lambda_{PV}$. Then $U = \phi_2^{PU}(\ell)$ and $V = \phi_2^{PV}(\ell)$. Thus there exist $T \varepsilon \Omega(\ell)$ and $S \varepsilon \Omega(\ell)$ such that $\phi_2^{PU}[T \cap A(PU)] \subseteq W_1$ and $\phi_2^{PV}[S \cap A(PV)] \subseteq W_2$. Clearly $\ell \varepsilon H(P) \cap A(PU) \cap A(PU)$ and each of these sets is open by Lemma (8.1.7). Thus

$V = T \cap S \cap H(P) \cap A(Pu) \cap A(PV) \varepsilon \Omega(\ell)$.

Claim. $V \subseteq V_2$ and hence $V_2 \varepsilon \Omega(\ell)$. By Remark (2.2.2) $PU \not{o} PV$. Take $m \varepsilon V$. Since $m \varepsilon A(PU)$ there exists $R = m \wedge PU$ and $m \varepsilon A(PV)$ implies there exists $S = m \wedge PV$. Since $m \varepsilon H(P)$, $P \not{o} R$, S. Next $R \not{o} S$. **For if this is false,** then RoS; $PU \not{o} PV$; P, $RIPU$; and P, $SIPU$ imply PoR, S by (A5). Contradiction. Finally since $m \varepsilon T \cap S$ we have $\{R = \phi_2^{PU}(m)\} \varepsilon \phi_2^{PU}\left[T \cap A(PU)\right] \subseteq W_1$. Similarly $S \varepsilon W_2$. Hence $m = RS \varepsilon V_2$.

Corollary. ϕ_1 is an open map if $\mathcal{X}$ is T_2. Recall the following Lemma from topology.

Lemma (8.1.8). Let X be a set. Let B be a filter base such that $X = \bigcup_{B \varepsilon B} B$. Then B is a base for a topology on X.

Lemma (8.1.9). Let $\mathcal{X}$ be a T_2 plane. Define B on $\mathcal{L}/o$ as follows:

$W[\bar{U}: \bar{V}] \varepsilon B$ iff $\bar{U}$, $\bar{V}$ are open in $\mathcal{P}/o$ such that

$$\{\bar{X}\bar{Y} \mid \bar{X} \varepsilon \bar{U} \text{ and } \bar{Y} \varepsilon \bar{V}\} = W[\bar{U}; \bar{V}].$$

Then B is a base for a topology $\mathcal{U}$ on $\mathcal{L}/o$ if $X_{\mathcal{P}}$ is open.

Proof: It is easy to see that B is a filter base. We must show $\mathcal{L}/o \subseteq \bigcup_{B\varepsilon \mathcal{B}} B$. Select $\bar{\ell}\varepsilon\ \mathcal{L}/o$. Let $\bar{\ell} = \bar{P}\bar{Q}$. Since $o_{\mathcal{P}}$ is closed and $X_{\mathcal{P}}$ is open, $\mathcal{P}/o$ is T_2 by Theorem (7.1.2)(2). Hence there exist open sets, $\bar{U}\varepsilon\Omega(\bar{P})$ and $\bar{V}\varepsilon\Omega(\bar{Q})$ such that $\bar{U} \cap \bar{V} = \emptyset$. Thus $\bar{\ell}\varepsilon W[\bar{U}:\ \bar{V}]$.

Now recall the following facts from [KO].

Definition (8.1.10). Let X be a topological space. Let Y be a set and f: $X \to Y$, a map. Then define $\mathcal{U}$ to beopen in Y iff $f^{-1}(\mathcal{U})$ is open in X. This is a topology called the quotient topology of Y with respect to f.

If θ is an equivalence relation on X, then the quotient topology of X/θ with respect to the quotient map f: $X \to X/\theta$ is just the usual quotient topology of X/θ. [cf. Definition (7.1.1)].

Theorem (8.1.9). [KO] Let $\mathcal{V}$ be a topology on a set Y. Let X be a topological space and f: $X \to Y$ an onto, open continuous map with respect to $\mathcal{V}$. Then $\mathcal{V}$ is the quotient topology of Y with respect to f.

Lemma (8.1.10). Let $X_{\mathcal{P}}$ be open and $\mathcal{X}$ a T_2 plane. Let $\mathcal{U}$ be the topology of Lemma (8.1.9). Then $\mathcal{U}$ is the quotient topology of $\mathcal{L}/o$. Moreover $X_{\mathcal{L}}$ is open.

Proof: We invoke Theorem (8.1.9). First we show $\chi_{\mathfrak{L}}$ is continuous. Let $W[U: V] \in \mathcal{U}$ be an open neighbourhood of $\bar{\ell} = \bar{R}\bar{S}$. Since $\chi_{\mathfrak{P}}$ is open, and $0_{\mathfrak{P}}$ is closed, $\mathfrak{P}/o$ is T_2 by Theorem (7.1.2)(2). Hence we may assume $\bar{U} = \chi_{\mathfrak{P}}(U)$, $V = \chi_{\mathfrak{P}}(V)$, and $\bar{U} \cap \bar{V} = \emptyset$ where U and V are open sets such that $R \in U$ and $S \in V$. Thus $U \times V \subseteq \mathfrak{P}^2$. By Theorem (8.1.8), $M = \{XY \mid X \in U$ and $Y \in V\} \in \Omega(RS)$. It easily follows that $M \subseteq \chi^{-1}(W[\bar{U}: \bar{V}])$ and so $\chi_{\mathfrak{L}}$ is continuous.

Next we show $\chi_{\mathfrak{L}}$ is open. Let $\ell = XY$ be any line. Select $W = \{RS \mid R \in U$ and $S \in V\}$ an arbitrary neighbourhood of ℓ by Theorem (8.1.8), where $U \in \Omega(X)$ and $V \in \Omega(Y)$.

Claim. $\chi_{\mathfrak{L}}(M) = \{\bar{R}\bar{S} \mid \bar{R} \in \chi_{\mathfrak{P}}(U)$ and $\bar{S} \in \chi_{\mathfrak{P}}(V)$. Let $\{\bar{R}\bar{S} \mid \bar{R} \in \chi_{\mathfrak{P}}(U)$ and $\bar{S} \in \chi_{\mathfrak{P}}(V)\} = A$. Clearly $\chi_{\mathfrak{L}}(M) \subseteq A$, since $\chi = (\chi_{\mathfrak{P}}, \chi_{\mathfrak{L}})$ is a homomorphism. Conversely let $\bar{R}\bar{S} \in A$. Then $\bar{R} = \bar{W}$ such that $W \in U$ and $\bar{S} = \bar{Z}$ where $Z \in V$. Hence $WZ \in M$. Then we obtain, since χ is a homomorphism, $\chi_{\mathfrak{L}}(WZ) = \bar{W}\bar{Z} = \bar{R}\bar{S} \in \chi_{\mathfrak{L}}(M)$.

Since $\chi_{\mathfrak{P}}$ is open, and U and V are open it follows by the claim that $\chi_{\mathfrak{L}}(M)$ is a neighbourhood of $\bar{\ell} = \bar{X}\bar{Y}$ with respect to the topology $\mathcal{U}$. Thus $\chi_{\mathfrak{L}}$ is open and our result follows.

Theorem (8.1.10). If $\mathfrak{R}$ is a T_2 plane then the following are equivalent.

(1) $\chi_{\mathbb{P}}$ *is an open map.*

(2) $\chi_{\mathbb{P}}$ *and* $\chi_{\mathcal{L}}$ *are open maps.*

(3) $\overline{\mathfrak{X}}$ *is an ordinary topological affine plane.*

(4) $\bar{h}_2$ *is an open map.* [cf. Definition (8.1.7)].

Proof: *(1)⟹(2).* This is just Lemma (8.1.10).

(2)⟹(3). This follows from the equations of Remark (8.1.1), the fact $\chi_{\mathbb{P}}$ and $\chi_{\mathcal{L}}$ are open, continuous onto and Theorem (7.3.2).

(3)⟹(4). This follows from Lemma (8.1.2).

(4)⟹(1). Remark (8.1.2)(3) yields this result immediately. //

It is not know, in general, whether topological planes are completely regular. However, we may prove the following.

Theorem (8.1.11). Let $\mathfrak{X}$ *be minor Desarguesian. Then*

(1) *If* {0, E, X, Y} *is a coordinate system,* (0E, +) *is a topological group.*

(2) *If* $\mathfrak{X}$ *is* T_2, *then* $\mathbb{P}$ *is a completely regular space.*

Proof: (1) From Theorem (8.1.1), (0E, +) is a topological loop. Since (0E, +) is a group by Theorem

(6.2.5), Corollary (6) of Theorem (6.2.3) yields $-a = (\phi_2^{0E}\ L^g\ \phi_2^h\ L^{0E}\ h_2^0)(a)$. Hence (0E, +) is a topological group.

By a famous theorem of Pontrjagin , we have by (1) that (0E, +) is a completely regular space. Since $0E \times 0E \cong \mathcal{P}$, and the product of completely regular spaces is completely regular we have our result.

Theorem (8.1.12). Let $\mathcal{P}$ be Desarguesian. Then

(1) (0E, +, .) is a topological ring. Moreover the map $f\colon\ 0E \setminus \pi_0 \to 0E$ defined by $f(x) = x^{-1}$ is continuous.

(2) $0E/\pi_0$ is a topological division ring and $0E/\pi_0 \cong \overline{0E}$.

Proof: (1) From Theorem (8.1.11), (0E, +) is a topological group. By Theorem (6.2.6) and Theorem (8.1.1)(2) (0E, +, .) is a topological ring. Finally by Corollary (8) of Theorem (6.2.3)

$$f(x) = (\phi_2^{0E}\ L^g\ \phi_2^{XE}\ \phi_1^0\ h_2^E)(x)$$

and hence f is continuous.

(2) It is well known if 0E is a topological ring then $0E/\pi_0$ is also a topological ring. Now $0E/\pi_0$ is a division ring

by Theorem (2.2.3). To show $0E/\pi_0$ is a topological division ring we must show the map

$$\bar{f}: 0E/\pi_0 \setminus \{\pi_0\} \to 0E/\pi_0, \ \bar{f}(x+\pi_0) = (x+\pi_0)^{-1}$$

is continuous. Let f be the map of (1). Then clearly $\bar{f}\,\chi_{0E} = \chi_{0E}\, f$. Since f is continuous and χ_{0E} is a open, continuous, onto map, the result follows from Theorem (7.3.2).

Since χ_{0E} is open, continuous, onto, the last result follows in the same fashion as Theorem (7.5.9) for topological groups.

<u>Notation</u>. Let $C(\mathfrak{P}, \mathfrak{P}) = C(\mathfrak{P})$. [cf. Definition (7.4.1)].

<u>Theorem (8.1.13)</u>. <u>Let</u> $\mathcal{X}$ <u>be</u> <u>a</u> T_2 <u>plane</u>. <u>Then</u> <u>every</u> <u>dilatation</u> <u>is</u> <u>continuous</u>. <u>Hence</u> $D \subseteq C(\mathfrak{P})$.

<u>Proof</u>: We invoke Theorem (7.3.3). Take $\sigma\varepsilon D$. Let R be any point. Choose a line g such that $R\not{o}X$ for each XIg. Select P, QIg such that $P\not{o}Q$. By Lemma (8.1.7), H(g) is an open set containing R. From Case (1) of the proof of Theorem (3.1.1) we have for each $X\varepsilon H(g)$,

$$X^\sigma = L(P^\sigma, PX) \wedge L(Q^\sigma, QX)$$
$$= \phi_2\,(L^{P^\sigma}\,\phi_1^P \times L^{Q^\sigma}\phi_1^Q)(X).$$

Let $f = \phi_2 \,(L^{P\sigma}\, \phi_1^P \times L^{Q^\sigma}\, \phi_1^Q)$. Clearly σ restricted to $H(g)$ is f. Hence our result is proved.

<u>Theorem (8.1.14). Let</u> $\mathfrak{X}$ <u>be a</u> T_2 <u>plane. The following are true.</u>

(1) D <u>is closed in</u> $C_p(\mathbb{P})$ <u>and hence</u> in $C_c(\mathbb{P})$.

(2) M <u>is closed in</u> $C_p(\mathbb{P})$ <u>and hence</u> in $C_c(\mathbb{P})$.

(3) D_p <u>is closed in</u> D <u>and</u> M_p <u>is closed in</u> D_p.

(4) N <u>and</u> T_Λ <u>are closed in</u> T.

<u>Proof</u>: (1) Let $\{\sigma_\alpha\}$ be a net in D such that $\sigma_\alpha \to f$. Choose P, QIg. Since $\sigma_\alpha \varepsilon D$, $P^{\sigma_\alpha} I L(Q^{\sigma_\alpha}, g)$ for each α. Then $L(P^{\sigma_\alpha}, g) = L(Q^{\sigma_\alpha}, g)$. Now $\sigma_\alpha \to f$ implies $P^{\sigma_\alpha} \to P^f$ and $Q^{\sigma_\alpha} \to Q^f$. Hence $L(P^{\sigma_\alpha}, g) \to L(P^f, g)$ and $L(Q^{\sigma_\alpha}, g) \to L(Q^f, g)$. Since $\mathfrak{X}$ is T_2, we have $L(P^f, g) = L(Q^f, g)$ and so $P^f I L(Q^f, g)$. The last part follows by Theorem (7.3.1).

(2) Let $\{\sigma_\alpha\}$ be a net in M such that $\sigma_\alpha \to f$. By (1) $f \in D$. Choose $P \not o Q$. Since $\sigma_\alpha \varepsilon M$, $P^{\sigma_\alpha} o Q^{\sigma_\alpha}$ by Corollary (2) of Theorem (3.1.2). $P^{\sigma_\alpha} \to P^\sigma$ and $Q^{\sigma_\alpha} \to Q^\sigma$ and so $(P^{\sigma_\alpha}, Q^{\sigma_\alpha}) \to (P^\sigma, Q^\sigma)$. Since $0_{\mathbb{P}}$ is closed, $(P^\sigma, Q^\sigma) \varepsilon 0_{\mathbb{P}}$ or $P^\sigma o Q^\sigma$. Hence $\sigma \varepsilon M$.

(3) This follows easily since $\mathfrak{X}$ is T_2.

(4) We first show T_Λ is closed in T. Let $\{\tau_\alpha\}$ be a net in T, such that $\tau_\alpha \to \tau$. Let $h \varepsilon \Lambda$ such that PIh.

Since $\tau_\alpha \varepsilon T_\Lambda$, $P^{\tau_\alpha} I h$. But $P^{\tau_\alpha} \to P^\tau$. Since $\bar{h}$ is closed by Theorem (8.1.7)(1), $P^\tau I h$. Hence $\tau \varepsilon T_\Lambda$.

Now take $\{\tau_\alpha\}$, a net in N, such that $\tau_\alpha \to \tau$. Let P be any point. Thus $P^{\tau_\alpha} o P^\tau$ for each α, by Lemma (3.2.1). We must show $P^\tau o P$. Select ℓ such that P, $P^\tau I \ell$. Choose $h \varepsilon \phi_P$ such that $P = \ell \wedge h$. Let $g = L(P^\tau, h)$. By Lemma (1.1.11), $P^{\tau_\alpha} o P$ iff $g o h$. $P^{\tau_\alpha} o P$ implies $g_\alpha o h$.

Let $g_\alpha = L(P^{\tau_\alpha}, h)$. Now $P^{\tau_\alpha} \to P^\tau$. Hence $g_\alpha \to g$. Since $\bar{h}$ is closed, by Remark (8.1.2) and $g_\alpha \varepsilon \bar{h}$, we have $g \varepsilon \bar{h}$ or equivalently $g o h$.

<u>Theorem (8.1.15)</u>. <u>Let</u> $\mathcal{X}$ <u>be</u> T_2 <u>minor Desarguesian</u>. <u>Let</u> $P \varepsilon \mathbb{P}$. <u>Then</u>,

(1) $\mathbb{P}$ <u>is a</u> T_2 <u>semi-topological group</u> [cf. <u>Theorem</u> (4.2.6)].

(2) $\bar{P} = \{P^\tau | P^\tau o P$, <u>and</u> $\tau \varepsilon T\}$ <u>is a closed normal subgroup of</u> $\mathbb{P}$.

<u>Proof</u>: (1) Let P^{τ_0} be any point of $\mathbb{P}$. Let f_0 be the map $f_0(P^\tau) = P^\tau . P^{\tau_0}$. Clearly $f_0 = \tau_0$, which is continuous by Theorem (8.1.3). Hence $\mathbb{P}$ is a semi-topological group.

(2) Since N is a normal subgroup of T, $\bar{P}$ is clearly a normal subgroup of $\mathbb{P}$ by Theorem (4.2.6)(ii). $\bar{P}$ is closed by Theorem (8.2.1)(4).

Corollary. $\mathbb{P}/\bar{P} = \mathbb{P}/o$.

Theorem (8.1.16). Let $\mathfrak{X}$ be T_2 minor Desarguesian. Then

(1) $\chi_{\mathbb{P}}$ is an open map and $\mathbb{P}/\bar{P} \cong \bar{\mathbb{P}}$.

(2) $\bar{\mathfrak{X}}$ is a T_2 ordinary topological plane.

Proof: (1) From Theorem (8.1.15) and its Corollary, $\chi_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{P}/\bar{P}$ is the quotient map of the groups $\mathbb{P}$ and P. The result then follows from Theorem (7.5.8).

(2) From (1) and Theorem (8.1.10), $\bar{\mathfrak{X}}$ is an ordinary topological plane. Since $o_{\mathbb{P}}$ and $o_{\mathfrak{X}}$ are closed and $\chi_{\mathbb{P}}$ and $\chi_{\mathfrak{X}}$ are open by Theorem (8.1.10), $\bar{\mathfrak{X}}$ is T_2 by Theorem (7.1.2)(2).

Theorem (8.1.17). Let $\mathfrak{X}$ be a T_2 minor Desarguesian plane. Then for each P, f_P is an open map where T has the compact open topology. [cf. Theorem (4.2.6)(ii)].

Proof: Let $f_P(X) = \tau_0 = \tau_{PX}$. Choose $W \in \Omega(P^{\tau})$, W open. Thus $W = U.P^{\tau_0}$ where U is an open neighbourhood of P.

Now $U = \{P^{\tau_{PU}} | u \in U\}$. It easily follows that

$$f_P\ U.P^{\tau_0} = \{\tau_{PU}.\tau_0 | u \in U\}.$$

Claim. $f_P[U.P^{\tau_0}]\varepsilon\Omega(\tau_0)$. Let $M = \{P\}$ and $V = (U)^{\tau_0}$. M is clearly compact and V is open since τ_0 is a homeomorphism by Theorem (8.1.13). Thus $T(M, V)\varepsilon\Omega(\tau_0)$. If we show $T(M, V) \subseteq f_P[U.P^{\tau_0}]$ our claim and the theorem will be proved. Let $\tau\varepsilon T(M, V)$. Then P^τ V and so $P^\tau = Q^{\tau_0}$ such that $Q\varepsilon U$. Thus, $P^\tau = (P^{\tau_{PQ}})^{\tau_0} = P^{\tau_{PQ}\tau_0}$. Consequently

$$\tau = \tau_{PU}\tau_0 \varepsilon f_P\ [U.P^{\tau_0}].$$

§8.2. Connectedness in topological affine H-planes

In this section we obtain generalizations of the result by Salzmann that each plane is connected or totally disconnected.

Theorem (8.2.1). The following statements are valid.

(1) For each $\ell \varepsilon \mathfrak{L}$, $\ell/\circ$ is a connected or totally disconnected regular T_2 space.

(2) If $\ell \varepsilon \phi_p$, $\bar{P} \cap \ell$ is closed in ℓ and $I_\ell(\bar{P} \cap \ell) = \emptyset$.

(3) $\mathfrak{P}/\circ$ is T_2.

(4) $\bar{P}$ is closed and $I(P) = \emptyset$.

Proof: (1) From Corollary (1) of Theorem (8.1.4) and Corollary (2) of Theorem (8.1.5), $\ell/\circ$ is a connected or totally disconnected space. From Theorem (8.3.1), $\ell/\circ$ is T_1. Since T_1 is equivalent to T_2 in regular spaces our result follows.

(2) Some ℓ/o is T_2, $o_{\mathfrak{P}} \cap (\ell \times \ell)$ is closed in $\ell \times \ell$ by Theorem (7.1.2)(1). Hence by (3) of the same theorem, $\bar{P} \cap \ell$ is closed.

If $I_\ell(P \cap \ell) \neq \emptyset$, there exists an open set $U \subseteq \bar{P} \cap \ell$, $U \neq \emptyset$. Since χ_ℓ is open, $\chi_\ell(U) = \{P\}$ is open. Hence by the Corollary to Theorem (6.1.3), $\ell/0$ is discrete. Contradiction.

(3) $\mathfrak{P}/o$ is T_1 from (1) and Lemma (8.1.1).

(4) Since $\mathfrak{P}/o$ is T_1, $\chi_\ell^{-1}(\{P\}) = \bar{P}$ is closed. Finally, if $I(P) \neq \emptyset$, then without loss of generality there exists an open set $U \in \Omega(P)$. Hence for $\ell \in \phi_P$, $U \cap \ell$ is open in ℓ and $U \cap \ell \subseteq \bar{P} \cap \ell$. Hence $I_\ell(P \cap \ell) \neq \emptyset$. Contradiction to (2) of the theorem.

<u>Theorem (8.2.2)</u>. (1) <u>Each $\ell \in \mathfrak{L}$ is connected or $Q_\ell(P) \subseteq \bar{P} \cap \ell$ for each $PI\ell$.</u>

(2) <u>$\mathfrak{P}$ is connected or</u> $O(P) \subseteq \bar{P}$.

<u>Proof</u>: (1) Assume ℓ is disconnected. Let $PI\ell$.

<u>Claim</u>. $\mathfrak{C}(Q_\ell(P)) \cap \mathfrak{C}(\bar{P} \cap \ell) \neq \emptyset$. If this is false then $\mathfrak{C}(Q_\ell(P)) \subseteq \bar{P} \cap \ell$. Since $Q_\ell(P)$ is closed, $\mathfrak{C}(Q_\ell(P))$ is open. By (2) of Theorem (8 .2.1),

$I_\ell(\bar{P} \cap \ell) = \emptyset$. Hence $\mathbb{C}(Q_\ell(P)) = \emptyset$ and so $Q_\ell(P) = \ell$. Thus by (4) of Theorem (7.2.3), ℓ is connected. Contradiction.

Now we show $Q_\ell(P) \subseteq \bar{P}$. If this is false, there exists $Y \varepsilon Q_\ell(P)$ such that $Y \not I P$. By the above claim, there exists $Z \varepsilon \mathbb{C}\, Q_\ell(P)$ such that $ZI\ell$ and $Z \not I P$. By Theorem (8.1.5)(2) there exists a homeomorphism f such that $f(P) = P$ and $f(Y) = Z$. Hence we obtain $\{Z = f(Y)\} \varepsilon \{f(Q_\ell(P)) = Q_\ell(P)\}$. Contradiction.

(2) Assume $\mathcal{P}$ is disconnected. By Lemma (8.1.1)(2) and Theorem (8.1.5)(1), each line ℓ is disconnected. Let $P \varepsilon \mathcal{P}$. Choose a coordinate system $\{0, E, X, Y\}$ such that $P = 0$. Let $\ell = 0E$. By Theorem (7.3.5), $Q((0, 0)) = Q_\ell(0) \times Q_\ell(0)$. By Lemma (8.1.5), Lemma (8.1.2) and (1) we obtain

$$Q(0) = h_2(Q_\ell(0) \times Q_\ell(0))$$

$$\subseteq h_2(\bar{0} \cap \ell \times \bar{0} \cap \ell) = \bar{0}.$$

<u>Corollary</u>. <u>If</u> ℓ/o <u>is totally disconnected</u>, <u>then</u> $Q_\ell(P) \subseteq \bar{P} \cap \ell$ <u>and</u> $Q(P) \subseteq \bar{P}$.

Next we obtain another generalization using o-connectedness. [cf. Definition (7.2.1)].

Theorem (8.2.3).

(1) Each ℓ is $\circ$-connected or $\circ$-totally disconnected.

(2) $\mathfrak{P}$ is $\circ$-connected or $\circ$-totally disconnected.

Proof: (1) Suppose ℓ is $\circ$-disconnected. Then let $PI\ell$.

Claim. $\mathbb{C}(T_\ell(P)) \cap \mathbb{C}(\bar{P} \cap \ell) \neq \emptyset$. This follows, in view of Theorem (7.2.9)(1) and (2), essentially the same as in the claim in the proof of Theorem (9.2.2). By Theorem (7.2.9)(3), $\bar{P} \subseteq T_\ell(P)$. To show the converse, we use the above claim and employ essentially the same argument as we did to show $O_\ell(P) \subseteq \bar{P}$ in Theorem (9.2.2).

Lemma (8.2.1). The following are equivalent.

(1) There exists P_0 such that $\bar{P}_0$ is connected.

(2) There exists P_0 such that $\bar{P}_o \cap \ell$ is connected for each $\ell \varepsilon \phi_P$.

(3) $\bar{P}$ is connected for each P.

(4) $\bar{P} \cap \ell$ is connected for each P and each $\ell \varepsilon \phi_P$.

Proof: By Lemma (8.1.2), $(\bar{P} \cap \ell) \times (\bar{P} \cap \ell) \cong \mathfrak{P}$. Thus from Theorem (7.3.1)(4), (1) is equivalent to (2) and (3) is equivalent to (4). Obviously (4) implies (2). We have only to show (2) implies (4).

Let A be any point, $A \neq P_0$. Choose ℓ such that $P, P_0 I \ell$. Select a coordinate system $\{0, E, X, Y\}$ such that $P_0 = 0$ and $\ell = 0E$. Let $a = A$ in $0E$.

<u>Claim</u>. $\bar{a} \cap 0E = {}_a\phi(\bar{0} \cap 0E)$. Using the fact, $\mathcal{T}_0$ is an ideal, it easily follows that.

$${}_a\phi(\bar{0} \cap 0E) \subseteq \bar{a} \cap 0E.$$

Utilizing Corollary (3) of Theorem (6.2.3), the converse inclusion is easily shown.

Since ${}_a\phi$ is a homeomorphism, the result follows.

<u>Lemma (8.2.2)</u>. <u>Assume there exists P_0 such that $\bar{P}_0$ is connected. Then $\mathcal{P}$ is connected iff $\mathcal{P}/\circ$ is connected and ℓ is connected iff $\ell/\circ$ is connected.</u>

<u>Proof</u>: In view of Lemma(10.3.1), the result follows immediately from Theorem (7.2.2).

<u>Theorem (8.2.4)</u>. <u>Assume there exists P_0 such that $\bar{P}_0$ is connected. Then</u>

(1) $\mathcal{P}$ <u>is connected or</u> $Q(P) = C(P) = P$ <u>for each</u> $P \in \mathcal{P}$.

(2) ℓ <u>is connected or</u> $Q_\ell(P) = C_\ell(P) = \bar{P} \cap \ell$ <u>for each</u> $P I \ell$.

(3) *$\mathbb{P}/\circ$ is connected or totally disconnected.*

Proof: (1) If $\mathbb{P}$ is disconnected, then $Q(P) \subseteq P$ by Theorem (8.2.2). Since $\bar{P}$ is connected for each P by Lemma (8.2.1) the result follows.

(2) By Lemma (8.2.1), $\bar{P} \cap \ell$ is connected. The claim then follows as in (1).

(3) If $\mathbb{P}/\circ$ is disconnected, then $\mathbb{P}$ is disconnected. Hence by (1), $C(P) = \bar{P}$ for each P. The result then follows by Theorem (7.3.4).

Corollary. Assume there exists P_0 such that $\bar{P}_0$ is connected. Then the following are equivalent.

(1) *$\mathbb{P}$ is connected* $[Q(P) = C(P) = P]$.

(2) *ℓ is connected for each $\ell \in \mathcal{L}$.* $[Q_\ell(P) = C_\ell(P) = \bar{P} \cap \ell]$.

(3) *$\ell/\circ$ is connected for each $\ell \in \mathcal{L}$* [*$\ell/\circ$ is totally disconnected*].

(4) *$\mathbb{P}/\circ$ is connected* (*$\mathbb{P}/\circ$ is totally disconnected*),

Proof: We prove the first part. From Lemma (9.2.3), (1) is equivalent to (4) and (2) is equivalent to (3). Also (1) is equivalent to (2) since $\ell \times \ell \cong \mathbb{P}$.

Now we prove for the second set of assumptions.

(1) $\Rightarrow$ (2). If $Q(P) = C(P) = P$, then (2) follows from the theorem and Lemma (8.1.1).

(2)$\Longrightarrow$(3). This follows from Theorem (7.2.4).

(3)$\Longrightarrow$(4). If $\mathfrak{P}/\circ$ is not totally disconnected, then by (3) of the theorem, $\mathfrak{P}/\circ$ is connected. Hence by Lemma (8.2.3), $\mathfrak{P}$ is connected and so ℓ is connected since $\ell \times \ell \cong \mathfrak{P}$. Thus $\ell/\circ$ is connected. Contradiction.

(4)$\Rightarrow$(1). Let $\mathfrak{P}/\circ$ be totally disconnected. If (1) is false, then by the theorem, $\mathfrak{P}$ is connected and hence $\mathfrak{P}/\circ$ is connected. Contradiction.

We next examine uniform planes.

Notation. Let $\mathcal{H}$ be a uniform affine H-plane. Then each $\mathcal{H}_P$ is an ordinary affine plane. Let ϕ_{P1}, ϕ_{P2} and L_P be the associated maps of $\mathcal{H}_P$.

Comment (8.2.1). Let $\mathcal{H}$ be a uniform affine H-plane. Then

(1) $\phi_{P1}(Q, R) = f \cap \bar{P}$ such that $Q, R I f$.

(2) $\phi_{P2}(f \cap \bar{P}, g \cap \bar{P}) = \phi_2(f, g)$.

(3) $L_P(Q, g \cap \bar{P}) = L(Q, g) \cap \bar{P}$.

Proof: This follows from the proof of Theorem (1.2.4).

Comment (8.2.2). Let $\mathcal{H}$ be uniform. Each $\mathcal{H}_P$ is a topological incidence structure with the following topologies. $\bar{P}$ has the relative topology of $\mathcal{P}$, and the neighbourhood filter of any line in $\mathcal{L}_p$, $\bar{f} \cap \bar{P}$, $\Omega_{\bar{P}}(f \cap \bar{P})$ is defined as $V_p \varepsilon \Omega_p(f \cap \bar{P})$ iff there exists $V \varepsilon \Omega(f)$ such that

$$V_{\bar{P}} = \{h \cap \bar{P} \mid h \varepsilon V\}.$$

Theorem (8.2.5). Let $\mathcal{X}$ be uniform. Then

(1) Each $\mathcal{X}_p$ satisfies (TA1). Also ϕ_{p2} and L_p are continuous.

(2) Each $\bar{P}$ is connected or each $\bar{P}$ is totally disconnected.

(3) $\mathcal{P}$ is connected, totally disconnected or

$$C(P) = Q(P) = \{P\}.$$

Proof:

(1). From Remark (8.2.2) and the equations of Comment (8.2.1), the result easily follows.

(2) This follows from (1), Corollary (3) of Theorem (6.1.5) and Lemma (8.2.1).

(3) Now by (2), each $\bar{P}$ is connected or totally disconnected. Also $\bar{P}$ is closed by Theorem (8.2.1).

If $\mathcal{P}$ is disconnected, then by Theorem (8.2.2),

$$C(P) \subseteq Q(P) \subseteq \bar{P}. \qquad \text{(I)}$$

Lemma (7.3.2), then yields,

$$C_{\overline{P}}(P) = C(P) \text{ for each } P. \tag{II}$$

Hence if each $\overline{P}$ is connected, $C(P) = Q(P) = \overline{P}$ by (I) and if each P is totally disconnected, $\mathcal{P}$ is totally disconnected by (II).

Corollary. *If $\mathcal{X}$ is uniform and $\mathcal{P}$ is disconnected, then $\mathcal{P}$ is totally disconnected or $C(P) = Q(P) = \overline{P}$. In fact,*

(1) $C_{\overline{P}}(P) = C(P)$ *for each* P.

(2) $\mathcal{P}$ *is totally disconnected iff each* $\overline{P}$ *is totally disconnected*.

(3) $C(P) = Q(P) = P$ *iff each* $\overline{P}$ *is connected*.

Proof: This follows from the proof of the theorem.

§8.3. Topological Projective H-planes

In [S3], Salzmann defines an ordinary projective plane, $\Lambda = < \mathcal{P}, \mathcal{L}, I>$, to be topological iff $\mathcal{P}$ and $\mathcal{L}$ are Hausdorff spaces and the maps $\phi_1(P, Q) = PQ$ and $\phi_2(\ell, m) = \ell \wedge m$ are continuous. In terms of the neighbourhood relations $o_{\mathcal{P}}$ and $o_{\mathcal{L}}$. [in this case $\Delta_{\mathcal{P}}$ and $\Delta_{\mathcal{L}}$, the identity relations on $\mathcal{P}$ and $\mathcal{L}$ respectively] this is identical to saying $o_{\mathcal{P}}$ and $o_{\mathcal{L}}$ are closed in $\mathcal{P} \times \mathcal{P}$ and $\mathcal{L} \times \mathcal{L}$ respectively. Thus it is quite natural to make the following definition.

Definition (8.3.1). $\mathcal{H} = < \mathcal{P}, \mathcal{L}, I>$ is a topological projective-H-plane iff $\mathcal{H}$ is a projective H-plane with the properties; (TP1). $\mathcal{H}$ is a topological incidence structure.

(TP2). $o_{\mathcal{P}}$ and $o_{\mathcal{L}}$ are closed sets in $\mathcal{P} \times \mathcal{P}$ and $\mathcal{L} \times \mathcal{L}$ respectively.

(TP3) The maps ϕ_1: $\mathcal{P}^2 \setminus o_{\mathcal{P}} \to \mathcal{L}$, defined by $\phi_1(P, Q) = PQ$ and ϕ_2: $\mathcal{L}^2 \setminus o_{\mathcal{L}} \to \mathcal{P}$ defined by $\phi_2(\ell, m) = \ell \wedge m$ are continuous.

The maps ϕ_1 and ϕ_2 are called the *associated maps of* $\mathcal{H}$. ϕ_1^P and ϕ_2^ℓ, $P \in \mathcal{P}$ and $\ell \in \mathcal{L}$ are also continuous map defined by $\phi_1^P(Q) = PQ$ and $\phi_2^\ell(m) = \ell \wedge m$.

As with the affine case $\mathcal{P}/\circ$, $\mathcal{L}/\circ$ and $\ell/\circ$ are endowed with their quotient topologies.

Lemma (8.3.1).

(1) *Each $\overline{P}$ and $\overline{\ell}$ are closed sets in $\mathcal{P}$ and $\mathcal{L}$ respectively.*

(2) *$\mathcal{P}/\circ$ and $\mathcal{L}/\circ$ are T_1 spaces.*

Proof:

(1) This follows immediately from Theorem (7.2.2)(3).

(2) Since $\chi_{\mathcal{P}}^{-1}(\{P\}) = \overline{P}$ and $\chi_{\mathcal{L}}^{-1}(\{\ell\}) = \overline{\ell}$ our result follows from (1) and Theorem (7.1.1)(2).

Theorem (8.3.1). *$\mathcal{H}$ *, the dual of $\mathcal{H}$, is also a topological projective-H-plane.*

Proof: $\mathcal{P}^* = \mathcal{L}$ is already a topological space. We define a topology on $\mathcal{L}^* = \{\phi_P | P \in \mathcal{P}\}$ as follows. Let $h: \mathcal{L}^* \to \mathcal{P}$ be the map $h(\phi_P) = P$. Then U^* is open in $\mathcal{L}^*$ iff $h(u^*)$ is open in $\mathcal{P}$. Then $\mathcal{L}^*$ is a topological space and h is a homeomorphism. The associated maps of $\mathcal{H}^*$ are $\phi_1^*(\ell, m) = \phi_{\ell \wedge m}$ and $\phi_2^*(\phi_P, \phi_Q) = PQ$. Then $\phi_1^* = h^{-1}\,\phi_1$ and $\phi_2^*(h \times h) = \phi_1$. Since ϕ_2 and h^{-1} are continuous, ϕ_1^* is continuous. Because $h \times h$ is a homeomorphism, and ϕ_1 is continuous, ϕ_2^* is continuous from Theorem (7.3.2) (1).

Lemma (8.3.2). For each $\ell \varepsilon \mathfrak{L}$, $\Sigma(\ell)$ is a closed set. [cf. Definition (1.3.3)].

Proof: Let $\{P_\alpha\}$ be a net in $\Sigma(\ell)$ such that $P_\alpha \to P$. Hence there exist $X_\alpha I \ell$ such that $P_\alpha o X_\alpha$.

By the Corollary of Lemma (1.3.1), there exists X such that $X \not{o} Y$ for each $YI\ell$ and $X \not{o} P$. Hence $X \not{o} P_\alpha$ for each α. Now $P_\alpha \to P$, implies $(X, P_\alpha) \to (X, P)$. Since ϕ_1 is continuous, $XP_\alpha \to XP$. Let $h_\alpha = P_\alpha X$ and $h = PX$. By the choice of X and Lemma (1.3.2) $h \not{o} \ell$, and $h_\alpha \not{o} \ell$ for each α. Hence there exist $T_\alpha = h_\alpha \wedge \ell$ and $T = h \wedge \ell$. Then $T_\alpha o P_\alpha$, otherwise $T_\alpha o X$ would imply $h_\alpha o \ell$ by (P5). Contradiction. Now $h_\alpha \to h$ implies $(h_\alpha, \ell) \to (h, \ell)$. Since ϕ_2 is continuous, $T_\alpha \to T$. Hence $(T_\alpha, P_\alpha) \to (T, P)$. But $(T_\alpha, P_\alpha) \varepsilon o_{\mathfrak{P}}$, for each α. Then because $o_{\mathfrak{P}}$ is closed by (TP2), $(T, P) \varepsilon o_{\mathfrak{P}}$ and hence ToP. Since $TI\ell$, it follows that $P \varepsilon \Sigma(\ell)$.

Theorem (8.3.2). For each $\ell \varepsilon \mathfrak{L}$, $\mathfrak{X}(\ell)$ is a topological affine H-plane and $\mathfrak{P}(\ell)$ is open.

Moreover if $\{p_1, p_2, p_3\}$ are lines with the properties of those in Lemma (1.3.1)(1), then $\mathcal{P} = \bigcup_{i=1}^{3} \mathcal{P}(p_i)$.

Proof: By Lemma (8.3.2), $\mathcal{P}(\ell)$ is an open set. Let the associated maps of $\mathcal{K}(\ell)$ be $\phi_{\ell 1}$, $\phi_{\ell 2}$ and L_ℓ. Since ϕ_1 and ϕ_2 are continuous it easily follows that $\phi_{\ell 1}$ and $\phi_{\ell 2}$ are continuous. Finally, $L_\ell(P, g) = P(\ell \wedge g) = \phi_1 (i \times \phi_2^\ell)(P, g)$ and so L_ℓ is continuous. The last part is just a restatement of Lemma (1.3.1)(1).

Corollary. If $\bar{\ell} \in \bar{\mathcal{L}}$, then $\bar{\mathcal{P}}(\bar{\ell}) = \bar{\mathcal{P}} \setminus \ell/o$ is an open set in $\bar{\mathcal{P}}$.

Proof: Since $\chi^{-1}(\bar{\mathcal{P}}(\bar{\ell})) = \mathcal{P}(\ell)$ the result follows by Theorem (7.1.1)(2).

Theorem (8.3.3). For each ℓ, $\chi_\ell : \ell \to \ell/o$ is an open map.

Proof: From Theorem (8.3.2) we may choose $\{p_1, p_2, p_3\}$ such that $\mathcal{P} = \bigcup_{i=1}^{3} \mathcal{P}(p_i)$. From Theorem (8.1.2) we have $\chi_\ell | \mathcal{P}(p_i) = \chi_\ell^i : \ell \cap \mathcal{P}(p_i) \to \ell \cap \mathcal{P}(p_i)/o$ is an open map. Then if $U \subseteq \ell$ is open in ℓ, $\chi_\ell(U) = \chi_\ell(\bigcup_{i=1}^{3} U \cap \mathcal{P}(p_i)) = \bigcup_{i=1}^{3} \chi_\ell^i(U \cap \mathcal{P}(p_i))$. Let $U_i = \chi_\ell^i(U \cap \mathcal{P}(p_i))$; $i = 1, 2, 3$. Then U_i is open in $\ell \cap \mathcal{P}(p_i)/o$. But $\ell \cap \mathcal{P}(p_i)/o = \ell/o \cap \bar{\mathcal{P}}(\bar{p}_i)$ and $\bar{\mathcal{P}}(\bar{p}_i)$ is open in $\bar{\mathcal{P}}$ from the Corollary to Theorem

(8.3.2). Hence $\ell \cap \mathcal{P}(p_i)/\rho$ is open in ℓ/ρ and so U_i is open in ℓ/ρ. Hence $\chi_\ell(u) = \bigcup_{i=1}^{3} U_i$ is open in ℓ/ρ.

<u>Lemma (8.3.3).</u> <u>Every perspectivity is a homeomorphism. Hence each projectivity is a homeomorphism.</u>

<u>Proof:</u> Let $\phi^R: \ell \to m$ be a perspectivity with centre R. Then $\phi^R = \phi_2^m \phi_1^R$ and $(\phi_R)^{-1} = \phi_2^\ell \phi_1^R$. Hence ϕ^R is a homeomorphism.

<u>Theorem (8.3.4).</u>

(1) <u>For each ℓ, PJ(ℓ) is a triply-transitive group of homeomorphisms with respect to $\circ_{\mathcal{P}}$. Moreover each PJ(ℓ/ρ) is a triply-transitive group of homeomorphisms.</u>

(2) <u>Any two lines of $\mathcal{X}$ are homeomorphic. Moreover any two lines of $\overline{\mathcal{X}}$ are homeomorphic.</u>

<u>Proof:</u>

(1) The first part follows from the Corollary to Theorem (1.4.2) and Lemma (8.3.3). If $\overline{f} \in PI(\ell/\rho)$ there exists $f \in PJ(\ell)$ such that $\chi_\ell\, f = \overline{f}\, \chi_\ell$, by Theorem (1.4.1). Since f is a homeomorphism and χ_ℓ is open onto, f is a homeomorphism by Theorem (7.3.3).

(2) Let $\ell, m \in \mathcal{L}$. By Lemma (1.4.1) there exists a projectivity $\phi^R: \ell \to m$. By Lemma (8.3.3) our result follows.

Next choose $\ell/\circ$, $m/\circ$ in $\mathfrak{L}/\circ$. Choose $\overline{R}$ such that $\overline{R} \not{I} \overline{\ell} \vee \overline{m}$. Hence $R \not{o} X$ for each $X I \ell \vee m$. Thus ϕ^R: $\ell \to m$ is defined. Since ϕ^R has the property $\phi^R(X) o \phi^R(Y)$ iff XoY by Lemma (1.3.7), $\phi^{\overline{R}}$ is well defined and $\phi^{\overline{R}} \chi_\ell = \chi_m \phi^R$. Since χ_m and χ_ℓ are open, continuous onto and ϕ^R is a homeomorphism, $\phi^{\overline{R}}$ is a homeomorphism by Theorem (7.3.3).

Corollary. Each $\ell/\circ$ is connected or totally disconnected.

Proof: This follows from Theorem (7.3.6) and the fact $PJ(\ell/\circ)$ is a triply transitive set of homeomorphisms.

Remark (8.3.1). $\ell/\circ$ is discrete iff there exists $PI\ell$ such that $\{P\}$ is open in $\ell/\circ$.

Proof: Assume $\{\overline{P}\}$ is open. Take $\overline{Q} \neq \overline{P}$. By Theorem (8.3.4), $PJ(\ell/\circ)$ is a triply transitive group of homeomorphisms. Hence there exists $\overline{f} \varepsilon PJ(\ell/\circ)$ such that $\overline{f}(\overline{P}) = \overline{Q}$. Hence $\{\overline{Q}\}$ is open.

From now on we assume there exists $\ell/\circ$ such that it is neither discrete or indiscrete. By Theorem (8.3.4)(2) each line in $\overline{\mathfrak{X}}$ has this property. In *view of Remark (8.3.1) each $\ell/\circ$ has no isolated points.*

<u>Theorem (8.3.5)</u>. <u>The following are valid</u>.

(1) <u>Each</u> $\ell \varepsilon \mathfrak{L}$ <u>has no isolated points</u>.

(2) $\mathfrak{P}$ <u>has no isolated points</u>.

(3) $I(\bar{P}) = I(\bar{\ell}) = \emptyset$ <u>for each</u> $P \varepsilon \mathfrak{P}$ <u>and</u> $\ell \varepsilon \mathfrak{L}$.

(4) $I(\Sigma(\ell)) = \emptyset$ <u>for each</u> $\ell \varepsilon \mathfrak{L}$.

<u>Proof</u>:

(1) If $\{P\}$ is open in ℓ, then since χ_ℓ is open $\chi_\ell(\{P\}) = \{\bar{P}\}$ is open and so by Remark (8.3.1), ℓ/o is discrete. Contradiction.

(2) If $\{P\}$ is open in $\mathfrak{P}$, then $\{P\}$ is open in ℓ, for $\ell \varepsilon \phi_P$. Contradiction to (1).

(3) If $I(\bar{P}) \neq \emptyset$, there exists an open set $U \subseteq \bar{P}$, such that without loss of generality $P \varepsilon U$. Select $\ell \varepsilon \phi_P$. Then $\chi_\ell(U \cap \ell)$ is open in ℓ/o. But clearly $\chi_\ell(U \cap \ell) = \{\bar{P}\}$. Contradiction.

(4) Suppose $I(\Sigma(\ell)) \neq \phi$. Hence there exists an open set U such that $\emptyset \neq U \subseteq \Sigma(\ell)$. Thus there exists $P \varepsilon U$ and m such that PIm and $mo\ell$. Thus $\Sigma(m) = \Sigma(\ell)$. Choose $j \varepsilon \phi_P$ such that $j \not o m$. Thus $P \varepsilon U \cap j$.

<u>Claim</u>. $\Sigma(m) \cap j \subseteq \bar{P}$. If $X \varepsilon \Sigma(m) \cap j$, then there exist k and Y such that YoX and kom. Since kom and $j \not o m$, we have $j \wedge moj \wedge k$ or equivalently PoX. Hence $X \varepsilon \bar{P}$. Therefore $P \varepsilon U \cap j \subseteq \Sigma(m) \cap j \subseteq \bar{P}$. Thus $\chi_j(U \cap j) = \{\bar{P}\}$ is open in j/o . Contradiction.

<u>Lemma (8.3.4)</u>. <u>Each</u> ℓ <u>is</u> $\circ_{\mathfrak{P}}$ <u>-connected</u> <u>or</u> <u>totally</u> $\circ$-<u>disconnected</u>.

<u>Proof</u>: Because PJ(ℓ) is a triply transitive group of homeomorphisms the proof is the same as that of Theorem (8.2.3).

<u>Lemma (8.3.5)</u>. <u>The following are equivalent</u>.

(1) ℓ <u>is</u> $\circ$-<u>connected</u> (<u>totally</u> $\circ$-<u>disconnected</u>).

(2) $\ell/\circ$ <u>is connected</u> (<u>totally disconnected</u>).

<u>Proof</u>: Since χ_ℓ is open, the first part follows from Theorem (7.2.8). Now if $\ell/\circ$ is totally disconnected then ℓ is totally $\circ$-disconnected by Theorem (7.2.9) (5). Conversely if ℓ is totally $\circ$-disconnected, then ℓ is totally disconnected. Otherwise by the Corollary of Theorem (8.3.4), ℓ is connected and hence $\ell/\circ$ is $\circ$-connected by the first part of the lemma.

<u>Definition (8.4.2)</u>. {O, E, U, V} is called a <u>complete quadrangle</u> iff {O, E, U, V} have the properties of the points in (P3).

<u>Lemma (8.3.6)</u>. <u>Let</u> {O, E, U, V} <u>be a complete quadrangle</u>. <u>Then</u> $\mathfrak{P} \setminus \Sigma(uv) \cong (OE \setminus \bar{W}) \times (OE \setminus \bar{W})$, <u>where</u> $W = OE \wedge UV$.

<u>Proof</u>: It is easy to see that $OE \setminus \bar{W}$ is just

a line of the affine plane $\mathcal{R}$ (UV). The result then follows from Lemma (8.1.2).

We now may prove the main generalization on connectedness in topological projective H-planes. The technique is motivated by Saltzmann's proof in the ordinary case in [S1].

Theorem (8.3.6). The following are equivalent.

(1) $\mathfrak{P}$ is $o_{\mathfrak{P}}$ -connected.

(2) $\mathfrak{L}$ is $o_{\mathfrak{L}}$ -connected.

(3) Each $\ell \varepsilon \mathfrak{L}$ is $o_{\mathfrak{P}}$ -connected.

(4) ℓ/c is connected.

(5) ϕ_P is $o_{\mathfrak{L}}$ -connected.

(6) $\ell \setminus \overline{S}$ is $o_{\mathfrak{P}}$ -connected for each $SI\ell$ and $\ell \varepsilon \mathfrak{L}$.

(7) $\mathfrak{P}(\ell)$ is $o_{\mathfrak{P}}$ -connected for each $\ell \varepsilon \mathfrak{L}$.

Proof: By duality (1) is equivalent to (2) and (3) is equivalent to (5). Lemma (8.3.5) yields (4) equivalent to (3). Thus it suffices to show (1)$\Longrightarrow$(3)$\Longrightarrow$(6)$\Longrightarrow$(7)$\Longrightarrow$(1).

(1)$\Longrightarrow$(3). Suppose ℓ is $o_{\mathfrak{P}}$-disconnected. Let $PI\ell$. By Lemma (8.3.4), $T_\ell(P) = \overline{P} \wedge \ell$. Choose Q such that $Q \not{o} X$ for each $XI\ell$. Define $T^*(P) = T_{\mathfrak{P} \setminus \overline{Q}}(P)$ and κ: $\mathfrak{P} \setminus \overline{Q} \to \ell$ by $\kappa(X) = XQ \wedge \ell$. Clearly $\kappa|\ell = i$.

Claims.

(1) κ is a continuous onto map such that XoY implies $\kappa(X) o \kappa(Y)$.

(2) If $A \subseteq \ell$, $\kappa^{-1}(A) = \bigcup_{X\varepsilon A} [QX \setminus \bar{Q}]$.

(3) $\bigcup_{X\varepsilon \bar{P}_\wedge \ell} [QX \setminus \bar{Q}] \subseteq \Sigma(PQ) \setminus \bar{Q}$.

(4) $T^*(P) \subseteq \Sigma(PQ) \setminus \bar{Q}$.

(1) $\kappa = \phi_1^Q \phi_2^\ell$ and so is continuous onto. If XoY, then X, Y$\not{o}$Q implies QXoQY. But QX$\not{o}\ell$ by the choice of Q. Hence $\kappa(X)o\kappa(Y)$.

(2) Obvious.

(3) If $A\varepsilon \bigcup [QX \setminus \bar{Q}]$, then there exists XI$\ell$ such that XoP, A$\not{o}$Q and AIQX. But P$\not{o}$Q and so PQoQX. Thus $A\varepsilon\Sigma(PQ)$ by the definition of $\Sigma(PQ)$.

(4) Claims (1), (2), (3) and Lemma (7.2.5) yield

$$T^*(P) \subseteq \Sigma(PQ) \setminus \bar{Q}.$$

Now choose R such that R$\not{o}$X for each RIPQ and so $R \notin \Sigma(PQ)$. Since P$\not{o}$Q it follows that $Q \notin \Sigma(PR)$.

Claim. There exists $V\varepsilon A_{\mathcal{P}\setminus\bar{Q}}(P)$ such that $\bar{R} \cap V = \emptyset$.

Moreover $\partial(V) \subseteq \bar{Q}$. If this is false, then $R\varepsilon T^*(P) \subseteq \Sigma(PQ) \setminus \bar{Q}$. Contradiction. Since Q is closed Lemma (8.3.2) yields the last statement.

We may easily interchange the roles of Q and R to obtain $W\varepsilon A_{\mathcal{P}\setminus\bar{R}}(P)$ such that $\bar{Q} \cap W = \emptyset$ and $\partial(W) \subseteq \bar{R}$.

Claim. $\partial(V \cap W) = \emptyset$. By the previous claim and Lemma (7.5.1)(3) we obtain $\partial(V \cap W) \subseteq \bar{Q} \cup \bar{R}$. It is enough to show $\partial(V \cap W) \cap \bar{Q} = \partial(V \cap W) \cap \bar{R} = \emptyset$. Let XoQ and $X \varepsilon \partial(V \cap W)$. Hence for each $U \varepsilon \Omega(X)$, $T \cap W \neq \emptyset$. Also $T \cap \complement W \neq \emptyset$ since if not, $T \subseteq W$ and W saturated implies $\bar{Q} \subseteq W$. Contradiction. Thus $X \varepsilon \partial(W) \subseteq \bar{R}$. Thus XoR. But QoX then implies RoQ. Contradiction. Similarly $\partial(V \cap W) \cap \bar{R} = \emptyset$. Thus by Lemma (7.3.1)(1), $V \cap W$ is open-closed in $\mathfrak{P}$. $V \cap W$ is saturated since V and W are. Finally $P \varepsilon V \cap W$ but $R \notin V \cap W$. Hence $\mathfrak{P}$ is 0-disconnected.

<u>(3)$\Longrightarrow$(6).</u> Assume $\ell \setminus \bar{s}$ is o-disconnected. Since $\ell \setminus \bar{s}$ is a line of some affine plane $\mathfrak{A}(m)$, Theorem (8.2.3) yields $T_{\ell \setminus \bar{s}}(P) = \bar{P} \cap (\ell \setminus \bar{s})$, for $PI\ell \setminus \bar{s}$. Choose P, $QI\ell \setminus \bar{s}$ such that $P \not o Q$. Hence there exists $V \varepsilon A_{\ell \setminus \bar{s}}(P)$ such that $\bar{Q} \cap V = \emptyset$. Moreover $\partial(V) \subseteq \bar{S}$ by Lemma (7.3.2). Similarly there exists $W \varepsilon A_{\ell \setminus \bar{Q}}(P)$ such that $\bar{S} \cap V = \emptyset$ and $\partial(W) \subseteq \bar{Q}$. Hence just as in (1)$\Longrightarrow$(3), $\partial(V \cap W) = \emptyset$ and so ℓ is $0_{\mathfrak{P}}$ -disconnected.

<u>(6)$\Longrightarrow$(7).</u> Assume each affine line $\ell \setminus \bar{s}$ is o-connected. Choose a complete quadrangle $\{0, E, X, Y\}$ such that $s \doteq U$. Then by Lemma (8.3.6), $\mathfrak{P}(\ell) = (\ell \setminus \bar{s}) \times (\ell \setminus \bar{s})$. Moreover this homeomorphism is just the map h^2 with respect to $\mathfrak{A}(\ell)$. [cf. Lemma (8.1.2) .]

h_2 clearly satisfies the assumptions of Lemma (7.2.4) and so our result follows.

(7)$\Longrightarrow$(1). By Theorem (8.3.5)(4) and the fact $\complement\mathcal{P}(\ell) = \Sigma(\ell)$ we obtain $I\complement\mathcal{P}(\ell) = \complement\Gamma\mathcal{P}(\ell) = \emptyset$, or equivalently, $\Gamma\mathcal{P}(\ell) = \mathcal{P}$. The result then follows from Lemma (7.2.2).

Theorem (8.3.7). Each of the sets $\mathcal{P}$, $\mathcal{L}$, ℓ, ϕ_P, $\ell\setminus\bar{s}$ and $\mathcal{P}(\ell)$ is $\circ$-connected or totally $\circ$-disconnected.

Proof: We have already shown this for ℓ, $\ell\setminus\bar{s}$ and $\mathcal{P}(\ell)$. ϕ_P is true by duality. Then by duality it suffices to show this for $\mathcal{P}$. Suppose $\mathcal{P}$ is $\circ$-disconnected. Then ℓ is $\circ$-disconnected by Theorem (8.3.6) and Lemma (8.3.4). If $\bar{P} \subsetneq T(P)$, then there exists $Q\emptyset P$ such that $Q\varepsilon V$ for each $V\varepsilon A(P)$. Choose $\ell\varepsilon\emptyset_P$ such that $Q\emptyset X$ for each $XI\ell$. Then from the proof of (1)$\Longrightarrow$(2) in Theorem (8.3.6), there exists $W\varepsilon A(P)$. Contradiction. //

We finish this section with results analogous to the affine case for uniform projective H-planes.

Notation. If $\mathcal{H}$ is uniform, $\mathcal{H}_P = \langle\bar{P}, \mathcal{L}_P, I\rangle$ such that $\ell\wedge\bar{P}\varepsilon\mathcal{L}_P \Longleftrightarrow \ell\varepsilon\mathcal{L}$ and $QI\ell\wedge\bar{P}$ iff $QI\ell$ and QoP. $\mathcal{H}_P$ is rendered a topological incidence structure in the usual fasion. [cf. Remark (8.2.2).] //

From [A3] we obtain the following result which we have shown in the affine case.

Lemma (8.3.7). $\mathcal{H}$ is uniform iff each $\mathcal{H}_P$ is an ordinary affine plane.

Corollary. Let $\mathcal{H}$ be uniform. Let ϕ_{P1}, ϕ_{P2} and L_P be the associated maps of $\mathcal{H}_P$. Then, $\phi_{P1}(Q, R) = f \wedge \bar{P}$ *such that* $Q, R I f$,

$$\phi_{P2}(\ell \wedge \bar{P}, m \cap \bar{P}) = \ell \wedge m,$$

$L_P(Q, \ell \wedge \bar{P}) = RQ \cap \bar{P}$ *such that* R *is any point with the properties* $R \not{o} P$ *and* $R I \ell$. //

Thus we obtain the following.

Lemma (8.3.8). If $\mathcal{H}$ is uniform then

(1) Each $\mathcal{H}_P$ is a topological incidence structure such that ϕ_{P2} and L_P are continuous.

(2) $\bar{P}$ is connected or totally disconnected.

Proof: The proof is the same as that of Theorem (8.2.5)(1) and (2) except for showing L_P is continuous. Now from Theorem (8.3.2), $\mathbb{P} = \bigcup_{i=1}^{3} \mathbb{P}(p_i)$. Also $\mathcal{H}(p_i)$ is a uniform affine H-plane. Let L^i be the parallel map for $\mathcal{H}(\ell_i)$ and L^i_p the one for $\mathcal{H}_p(p_i)$. L^i_p are continuous; $i = 1, 2, 3$, by Theorem (8.2.5).

We will show L_p restricted to $\bar{P} \cap \mathfrak{P}(p_i)$ is essentially L_p^i. Since $\mathfrak{P}(p_i)$ is open, the result follows from Theorem (7.3.3). Using Comment (8.2.1), the Corollary to Lemma (8.3.7) and the fact $(\ell \wedge p_i) I \ell$ but $(\ell \wedge \ell_i) \not{I} P$ we obtain

$$L_p^i\left[0,\ \ell \wedge \mathfrak{P}\ (p_i) \wedge \bar{P}\right] = L^i\left[0,\ \ell \wedge \mathfrak{P}\ (p_i)\right] \cap \bar{P}$$

$$= Q(\ell \wedge p_i) \cap \bar{P} = L_p\left[0,\quad \bar{P} \cap \ell\right].$$

<u>Definition (8.3.3)</u>. $\mathfrak{X}$ is a T_1 <u>plane</u> iff both $\mathfrak{P}$ and $\mathfrak{L}$ are T_1 spaces.

<u>Lemma (8.3.9). If $\mathfrak{X}$ is a T_1 plane then each $\ell \varepsilon \mathfrak{L}$ is closed in $\mathfrak{P}$.</u>

<u>Proof</u>: Let $\ell = PQ$. Choose $X \varepsilon \ell$. Hence without loss of generality $X \not{I} P$. Then $PX \neq \ell$ and since $\mathfrak{X}$ is T_1, we have $V \varepsilon \Omega(PX)$ such that $\ell \notin V$. Since ϕ_1^P is continuous and $\bar{P}$ is closed, there exists $U \varepsilon \Omega(X)$ such that $\phi_1^P[U] \subseteq V$. Then $U \wedge \ell = \emptyset$ since $S \varepsilon U \wedge \ell$ implies $PS = \ell \varepsilon V$. //

Notice for the next lemma we need to use the fact that $\mathfrak{X}$ is T_1, whereas we could prove it in general in the affine case utilizing the ternary ring.

Lemma (8.3.10). Let $\mathcal{X}$ be a T_1 plane. The following are equivalent.

(1) There exists P_0 such that $\bar{P}_0$ is connected.

(2) There exists P_0 such that $\bar{P}_0 \wedge \ell$ is connected for each $\ell \varepsilon \phi_{P_0}$.

(3) $\bar{P} \wedge \ell$ is connected for each $\ell \varepsilon \mathcal{L}$ and $PI\ell$.

(4) $\bar{P}$ is connected for each P.

Proof: (1)$\Longrightarrow$(2). Select Q such that $Q \not{o} P_0$. Then the map κ: $\mathcal{P} \setminus \bar{P} \to \ell$, $\kappa(X) = PX \wedge \ell$ is continuous and XoY implies $\kappa(X) o \kappa(Y)$, as in the proof of Theorem (8.3.6). Since $\bar{P}$ is closed, $\bar{P}_0$ is connected in $\mathcal{P} \setminus \bar{P}$. Because κ is continuous with the above property, $\kappa(\bar{P}_0) = \bar{P}_0 \wedge \ell$ is connected in ℓ. By Lemma (8.4.9), ℓ is closed, and hence $\bar{P}_0 \wedge \ell$ is connected in $\mathcal{P}$.

(2)$\Longrightarrow$(3). Since $PJ(\ell)$ is triply transitive with respect to $\upsilon_{\mathcal{P}}$, there exists a homeomorphism f such that $f(P) = Q$ and $f[\bar{P} \wedge \ell] = \bar{Q} \wedge \ell$.

Since $\bar{Q} = \bigcup_{\ell \varepsilon \phi_Q} [(\bar{Q} \wedge \ell)]$, (3)$\Longrightarrow$(4) is a result of Theorem (7.3.1)(3). Finally (4)$\Longrightarrow$(1) is obvious.

Lemma (8.3.11). If $\mathcal{X}$ is a uniform T_1 plane, and there exists P_0 such that $\bar{P}_0$ is connected, then for

ℓ and $\mathcal{P}$, $\circ$-connectedness is equivalent to connectedness.

Proof: By Lemma (8.3.10), each $\overline{P} \wedge \ell$ is connected. The result for ℓ then follows from Theorem (7.3.2) and Lemma (8.3.5).

Finally, by Theorem (8.3.6), and the above $\mathcal{P}$ is $\circ$-connected iff ℓ is $\circ$-connected iff ℓ is connected. The result for $\mathcal{P}$ then follows from Lemma (8.1.1).

Theorem (8.3.8). Let $\mathcal{X}$ be a T_1 uniform plane. Then $\mathcal{P}$ and each ℓ are connected, $\circ$-connected, $C(P) = Q(P) = T(P) = P$, *or totally disconnected.*

Proof: By Lemmas (8.3.8) and (8.3.10), each $\overline{P}$ is connected or totally disconnected.

Case (1): Each $\overline{P}$ is connected. Then from Lemma (8.3.11) and Theorem (8.3.7), ℓ and $\mathcal{P}$ are connected or $C(P) = Q(P) = T(P) = \overline{P}$.

Case (2): Each P is totally disconnected. Then since $\circ$-disconnectedness implies

$$C(P) \subseteq Q(P) \subseteq T(P) = P$$

and $C(P) = C_{\overline{P}}(P)$ by Lemma (7.3.2), ℓ and $\mathcal{P}$ are $\circ$-connected or totally disconnected.

8.4. Locally compact H-planes

Salzmann has shown, in [S2] and [S3], that every locally compact T_2 ordinary plane is metrizable. Moreover each locally compact projective plane is σ-compact with the 2nd axiom of countability. The proof of this result for H-planes, except for a few minor points which we shall exhibit, is exactly the same.

$\mathfrak{X}$ will be a locally compact T_2 plane throughout this section unless otherwise specified.

Theorem (8.4.1). (1) *If X is a σ-compact, locally compact metric space, then X has the second axiom of countability.*

(2) *Every regular T_2 space with the second axiom of countability is metrizable.*

Proof: (1) This follows immediately from Theorem (5.6) page 137 and Theorem (7.2) page 239 of [D2].

(2) is well-known result. [cf. [K0]].

The next Lemma appears in [S2] with only (b) changed. We shall prove, then only (b).

Lemma (8.4.1). Let $\mathcal{X}$ be a locally compact T_2 affine plane and (OE, T) a ternary fieldof $\mathcal{X}$. Then

(a) OE is locally compact T_2.

(b) There exists a sequence $\{a_n\} \subseteq OE$ such that $a_n \not\sim 0$ and $a_n \to 0$.

(c) If $W \in \Omega(0)$ and K is compact then there exists $V \in \Omega(0)$ such that $KV \subseteq W$.

(d) If $a_n \to 0$ $(a_n \neq 0)$, and $U \in \Omega(0)$, such that $\Gamma(u)$ is compact then $\{U.a_n\}$ is a neighbourhood basis for 0.

Proof: Since $OE/\sim$ is not discrete, then OE is not discrete by Remark (8.1.2)(2). Hence each neighbourhood has infinitely many points since $I(\overline{P} \cap OE) = \emptyset$.

Claim: If $P \not\sim 0$, then there exist a compact $C \in \Omega(P)$ such that $0 \cap C = \emptyset$. Since $OE/\sim$ is T_2, and OE is regular there exist closed neighbourhoods $U \in \Omega(P)$ and $W \in \Omega(0)$ such that $X \in W$ and $Y \in U$ implies $X \not\sim Y$. Hence $\overline{0} \cap U = \emptyset$. By (a), there exists a compact neighbourhood C of P. Hence since U is closed and C is closed, $U \cap C$ is compact and $\overline{0} \cap (U \cap C) = \emptyset$.

Now take C is in the above claim. Thus C is infinite. Select a sequence $\{b_n\}$ in C and hence $b_n \neq 0$. Since C is compact, $\{b_n\}$ has a cluster point $b \in C$. Hence there exists a subsequence $\{C_n\}$ such that $C_n \to b$. Finally $a_n = c_n - b \to 0$ since addition and its inverses are continuous.

Theorem (8.4.2). *Every locally compact T_2 affine plane satisfies the first axiom of countability.*

Proof: By (b) and (d) of Lemma (8.4.1), 0E has the first axiom of countability. Hence since $0E \times 0E \cong \mathcal{P}$, our result follows.

Theorem (8.4.2). *Let $\mathcal{X}$ be a locally compact T_2 affine H-plane. Then*

(1) *Each ternary field {0E,T} is metrizable.*

(2) *$\mathcal{X}$ is metrizable.*

Proof: (1) In view of Lemma (8.4.1)(b), the proof is exactly the same as in the ordinary case [cf. 7.8 page 48 of [S3]].

(2) follows immediately from (1) since $0E \times 0E \cong \mathcal{P}$.

Definition (8.4.1). Let (0E, T) be a ternary field of an affine H-plane. We say (0E, T) has *inversion near zero* iff $a_n \to 0$ $(a_n \neq 0)$ and $a_n^{-1} b_n = b$ implies $b_n \to 0$.

Lemma (8.4.2). *Let $\mathcal{X}$ be a projective H-plane.*

Let $\{0, E, X, Y\}$ be choosen as in (P3). Put $W = 0E \wedge XY$. Then the ternary field of the affine H-plane $\mathcal{X}(XY)$, $T: (0E \setminus \bar{W})^3 \to 0E \setminus \bar{W}$ has the form

$$T(x, m, n) = \{(Xm \wedge 0Y)\left[(Xm \wedge YE)0 \wedge XY\right]\} \wedge Yx)X \wedge 0E.$$

Proof: This follows easily from Definition (6.2.4) and the definition of the associated maps of $\mathcal{X}(XY)$.

Lemma (8.4.3). Let our assumptions be the same as those of Lemma (8.4.2). Define the maps

$$g_1: (0E \setminus \bar{W}) \setminus \bar{0} \to (0E \setminus \bar{W}) \setminus \bar{0} \text{ by}$$

$g_1(z)$ is the unique solution of $x.z = 1$, and for $b \in 0E$

$$g_b: (0E \setminus \bar{W}) \setminus \bar{0} \to (0E \setminus \bar{W}) \setminus \bar{0} \text{ by}$$

$g_b(x)$ is the unique solution of $x.z = b$. The following hold:

(1) $g_1(z) = 0E \wedge Y\left[XE \wedge 0(YE \wedge Xz)\right]$ and $g_1(0) = W$.

(2) $g_b(x) = 0E \wedge \{YE \wedge \left[Yx \wedge Xb\right]0\}X$ and $g_b(W) = 0$.

Proof: This follows from the previous Lemma and some straightforward calculations.

Lemma (8.4.4). (1) If $\mathcal{X}$ is a Desarguesian affine plane then each ternary field has inversion near zero.
(2) If $\mathcal{X}$ is a projective H-plane, then the ternary fields of each associated affine H-plane, $\mathcal{X}(\ell)$ have inversion near zero.

Proof: (1) This follows easily since multiplication is then associative.
(2) Let $a_n \to 0$ $(a_n \neq 0)$, and $a_n^{-1} b_n = b$.

Now take the maps g_1 and g_b of Lemma (8.4.2). Then define $g = g_b . g_1$. Clearly from Lemma (8.4.2), g_b and g_1 are continuous and hence so is g. Moreover we also obtain from Lemma (8.4.2) that

$$g(0) = g_b g_1(0) = g_b(W) = 0 \qquad \text{(I)}$$

and

$$g(a_n) = b_n \text{ for each } n. \qquad \text{(II)}$$

Hence the continuity of g plus (I) and (II) implies $b_n \to 0$.

Theorem (8.4.3). If $\mathcal{X}$ is a locally compact T_2 affine H-plane and (OE, T) has inversion near zero, then,

(1) 0E *is σ-compact and so is* $\mathcal{X}$.

(2) 0E *and* $\mathcal{X}$ *have the second axiom of countability.*

Proof: Choose $W \in \Omega(0)$ such that $\Gamma(W)$ is compact. By Lemma (8.4.1)(b), there exists $\{a_n\}$ converging to zero such that $a_n \emptyset 0$.

Claim. $0E = \bigcup_{i=1}^{\infty} a_n^{-1}\Gamma(W)$.

Since multiplication is continuous, $a_n^{-1}\Gamma(W)$ is compact. Clearly $0 \in 0E$. Take $b \in 0E$ such that $b \neq 0$. Since $a_n^{-1} \emptyset 0$, there exists $\{b_n\}$ such that $a_n^{-1}\, b_n = b$ by Theorem (6.2.3)(3). Since we have inversion near zero, $b_n \to 0$ and $b_n \neq 0$. Since $W \in \Omega(0)$ there exists $b_{n_0} \in W$. Thus $b \in a_{n_0}^{-1}\Gamma(W)$. Since $\mathcal{P} \cong 0E \times 0E$ our result follows.

(2) Since $0E \times 0E \cong \mathcal{P}$ the result follows from (1), Theorem (8.4.2) and Theorem (8.4.1)(1).

Corollary (1). If $\mathcal{X}$ *is a Desarguesian locally compact* T_2 *affine H-plane, then each line* ℓ *and* $\mathcal{P}$ *are σ-compact metric spaces with the second axiom of countability.*

Corollary (2). If $\mathcal{X}(\ell)$ *is the associated affine H-plane of a locally compact* T_2 *projective H-plane, then* $\mathcal{X}(\ell)$ *is a σ-compact metric space with the second* *axiom of countability.*

Theorem (8.4.4). Every locally compact T_2 projective H-plane is a σ-compact metric space with the second axiom of countability.

Proof: Now by Theorem (8.3.2), each $\mathfrak{P}(\ell)$ is open and there exists $\{p_1, p_2, p_3\}$ such that $\mathfrak{P} = \bigcup_{i=1}^{3} \mathfrak{P}(p_i)$. Since an open set of a locally compact space is locally compact, each $\mathfrak{P}(p_i)$ is a σ-compact metric space with the second axiom of countability by Corollary (2) of Theorem (8.4.3). Hence it follows, since the $\mathfrak{P}(p_i)$ are open, that $\mathfrak{P}$ is a σ-compact space with the second axiom of countability. Thus by Theorem (8.4.1)(2) and the fact that a locally compact T_2 space is regular, we have our final result.

Notation. For each P, f_P: $\mathfrak{P} \to T$ is the map $f_P(P^\tau) = \tau$ and $f_{\bar{P}}$: $\mathfrak{P} \to \bar{T}$ is the map $f_{\bar{P}}(P^{\bar{\tau}}) = \bar{\tau}$.

Theorem (8.4.5). Let $\mathcal{X}$ be minor Desarguesian. Then

(1) $\mathfrak{P}$ *is a topological group.* (cf. Theorem (4.2.6)).

(2) D *is a topological monoid with the compact-open topology.*

(3) f_P: $\mathfrak{P} \to T$ *is a homeomorphism and so is* $f_{\bar{P}}$: $\tilde{\mathfrak{P}} \to \bar{T}$, *where* T *and* $\bar{T}$ *have the compact-open topology.*

(4) <u>T and $\bar{T}$ are locally compact T_2 topological groups with the compact-open topology and</u>

$$T/N \cong \bar{T}.$$

<u>Proof</u>: (1) In view of Theorem (8.1.15), this follows immediately from Theorem (7.5.7).

(2) This follows from Theorem (7.5.3).

(3) From Theorem (8.1.17), f_P is open. We must show it is continuous. Define f: $\mathbb{P} \times \mathbb{P} \to \mathbb{P}$ by $f(P^{\tau_1}, P^{\tau_2}) = P^{\tau_1 \tau_2}$. f is continuous by (1). Hence by Theorem (7.4.2), the map $\tilde{f}$: $\mathbb{P} \to C_c(\mathbb{P})$, where $\tilde{f}(P^{\tau_0}) = \phi_{P^{\tau_0}}$ such that $\phi_{P^{\tau_0}}(P^{\tau}) = f(P^{\tau_0}, P^{\tau_0})$ is continuous.

<u>Claim</u>. $\tilde{f} = f_P$. We must show $\phi_{P^{\tau}} = \tau$ for each $\tau \in T$. Now for any Y,

$$\phi_{P^{\tau}}(Y) = \phi_{P^{\tau}}(P^{\tau_{P_Y}}) = f(P^{\tau}, P^{\tau_{P_Y}})$$

$$= P^{\tau}.P^{\tau_{P_Y}} = (P^{\tau_{P_Y}})^{\tau} = Y^{\tau}.$$

Hence by our claim f_P is continuous.

Since $\bar{\mathcal{R}}$ is a topological plane by Theorem

(8.1.6), $f_{\bar{p}}$ is also a homeomorphism.

(4) This follows from (1), (3) Theorem (7.5.9) and Theorem (3.2.4), if we can show ϕ: $T \to \bar{T}$ [cf. Theorems (3.2.3) and (3.2.4)] is open. But it is easy to show $\phi . f_p = \bar{f}_{\bar{p}} . \chi_{\mathfrak{P}}$. Hence (3) and the fact that $\chi_{\mathfrak{P}}$ is open-onto yields this result using Theorem (7.3.2).

<u>Theorem (8.4.6).</u> <u>Let</u> $\mathfrak{X}$ <u>be minor Desarguesian.</u> <u>Then every trace preserving endomorphism is continuous.</u>

<u>Proof:</u> By the previous theorem f_p is a homeomorphism. Take $\delta \varepsilon H$. By Theorem (4.2.5), there exists $\sigma \varepsilon D_p$ such that $\tau_{PS}^{\delta} = \tau_{PS^{\sigma}}$. It is then easy to show

$\sigma\, f_p^{-1} = f_p^{-1}\, \delta$. Now δ is continuous by Theorem (8.1.13). Hence our result follows from Theorem (7.3.2).

<u>Theorem (8.4.7).</u> <u>H is a closed topological subring of</u> End $_C$ T. [cf. Definition (7.5.3)].

<u>Proof:</u> By Theorem (7.5.5), End $_C$ T is a topological ring, and so H is a topological ring. It is enough to show H is closed in End $_P$ T. Let $\{\delta_\alpha\}$ be a net in H such that $\delta_\alpha \to f$, $f \varepsilon$End T. By Theorem (4.2.5) there exists $\{\sigma_\alpha\}$ in D_p such that $\tau_{PS}^{\delta_\alpha} = \tau_{PS^{\sigma_\alpha}}$. Define

σ: $\mathfrak{P} \to \mathfrak{P}$ by $S^{\sigma} = P^{\tau_{PS}^{f}}$.

Claim. $\sigma_\alpha \to \sigma$ (Hence $\sigma \varepsilon D_p$ by Theorem (8.1.14) (3)). Take $s \varepsilon \mathfrak{P}$. Then,

$$\lim_\alpha (S^{\sigma_\alpha}) = \lim_\alpha (P^{\tau^{\delta_\alpha}_{PS}}) = P^{\tau^{f}_{PS}} = S^{\sigma}$$

since $\delta_\alpha \to f$ implies $\tau^{\delta_\alpha}_{PS} \to \tau^{f}_{PS}$, which in turn implies $P^{\tau^{\delta_\alpha}_{PS}} \to P^{\tau^{f}_{PS}}$.

Since $\sigma \varepsilon D_p$, there exists a unique $\delta \varepsilon H$ such that $\delta = \sigma(\delta)$ [cf. Theorem (4.2.5)]. We finish the proof by showing $f = \delta$. Since f_p is a homeomorphism by Theorem (8.9.5)(3),

$$\tau^{f}_{PS} = \lim_\alpha (\tau^{\delta_\alpha}_{PS}) = \lim_\alpha (\tau_{PS^{\sigma_\alpha}})$$

$$= \lim_\alpha (f_p(S^{\sigma_\alpha})) = f_p(\lim_\alpha (S^{\sigma_\alpha}))$$

$$= f_p(S^{\sigma}) = \tau_{PS^{\sigma}} = (\tau_{PS})^{\delta}.$$

Theorem (8.4.8). D_p is a topological monoid and ϕ_p: $H \to D_p$ is a topological monoid isomorphism. (cf. Theorem (4.2.

Proof: The first part follows from Theorem (8.4.5)(2). Now f_p: $\mathfrak{P} \to T$ is a homeomorphism by Theorem (8.4.5)(3). By Theorem (7.5.6), the map ϕ_h: End $\mathfrak{P} \to$ End T defined by $\phi_h(g) = f_p \cdot g \cdot f_p^{-1}$ is a topological monoid isomorphism. But by Theorem

(4.2.6), $D_p \subseteq \mathrm{End}\ \mathfrak{P}$, $\phi_h/D_p = \phi_p^{-1}$ and $\phi_h[D_p] = H$.
Hence our result follows.

Corollary, Π is closed in H.

Proof: M_p is closed in D_p and $\phi_p[M_p] = \Pi$.

Theorem (8.4.9). Let $\mathfrak{X}$ be Desarguesian. Then

(1) The map f: $OE \to H$ defined by $f(a) = \delta(a)$ is a topological ring isomorphism. [cf. Theorem (5.3.8)].

(2) T, H and each D_p are locally compact σ-compact metric spaces with the second axiom of countability.

Proof: (1) We know f is an algebraic isomorphism by Theorem (5.3.8). We must show it is a homeomorphism.

Now define $T_{OE} = \{\tau_{0a} \mid a \in OE\}$. Let f_1 be the map,

$$f_1: OE \to T_{OE} \text{ by } f_1(a) = \tau_{0a}.$$

f_1 can be shown to be a topological group isomorphism in the same fashion that f_p was. [cf. Theorem (8.4.5)(3)]. Let f_0: $T \to \mathfrak{P}$ be the map of Theorem (8.4.5)(3) such that $P = 0$. Let $h_2: \mathfrak{P} \to OE \times OE$ be the map of Lemma (8.1.2). Let $f_2 = h_2$. Define f_3: $OE \times OE \to T_{OE} \times T_{OE}$ by $f_3 = f_1 \times f_1$. Then the map g: $T \to T_{OE} \times T_{OE}$ defined by

$$g(\tau_{0(a,b)}) = (\tau_{0a}, \tau_{0b})$$

has the form $g = f_3\ f_2\ f_0$ and hence is a topological group isomorphism. Now define $\tilde{\delta}(a)$: $T_{0E} \times T_{0E} \to T_{0E} \times T_{0E}$ by $(\tau_{0x}, \tau_{0y})^{\tilde{\delta}(a)} = (\tau_{0(ax)}, \tau_{0(ay)})$. Let $\tilde{H} = \{\tilde{\delta}(a) | a \varepsilon 0E\}$. By Theorem (7.5.6), the map ϕ_g: Endo $(T_{0E} \times T_{0E}) \to$ End T is a topological monoid isomorphism; i.e., $\phi_g(h) = g^{-1}hg$. It is easy to show $\phi g(\tilde{\delta}(a)) = \delta(a)$ and hence $\tilde{H}$ is homeomorphic to H.

Now define h: $T_{0E} \to \tilde{H}$ by $h(\tau_{0a}) = \tilde{\delta}(a)$. Clearly $f = \phi_g\ h\ f_1$. Thus in order to show f is continuous it suffices to show h is continuous. Now define

$$k\colon\ T_{0E} \times (T_{0E} \times T_{0E}) \to T_{0E} \times T_{0E}$$

by

$$k(\tau_{0c}, (\tau_{0a}, \tau_{0b})) = (\tau_{0(ca)}, \tau_{0(cb)}).$$

Since multiplication is continuous in 0E and f_1 is continuous, we have that k is continuous. Then by Theorem (7.4.2), the map $\tilde{k}$: $T_{0E} \to C_c\ (T_{0E} \times T_{0E})$ defined by $k(\tau_{0c}) = \phi_{0c}$, is continuous. A simple calculation shows $\tilde{k} = h$ and our result is proved.

Finally we must show f is open. Let $W \varepsilon \Omega_{0E}(a)$.

Then $W = U \cap OE$ such that $U \varepsilon \Omega(a)$. Now define $\kappa = \{\tau_{0E}\}$ and $V = f_0^{-1}[U]$. Hence since $\tau_{0E}^{\delta(a)} = \tau_{0a}$, $T[K, V] \varepsilon \Omega(\delta(a))$. We show $T[K, V] \subseteq h(W)$. Now $\delta(b) \varepsilon T[K, V]$ implies $\tau_{0E}^{\delta(b)} = h(u) = \tau_{0a}$, $u \varepsilon U$. But $\tau_{0E}^{\delta(b)} = \tau_{0b}$ and so $b = u$. Since $0E$ is a trace of τ_{0E}, it is a trace of τ_{0b}. Hence $b \varepsilon 0E$. Consequently $\delta(b) \varepsilon h(W)$.

(2). By Corollary (1) of Theorem (8.4.3), $0E$ and $\mathfrak{P}$ are σ-compact metric spaces with the second axiom of countability. By the theorem, H also has this property. By Theorem (8.4.5), T has these properties. Finally by Theorem (8.4.3), each D_p has this property.

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