STUDY OF THE THREE-BODY NUCLEAR FORCE
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by

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SCOPE AND CONTENT:

Until now, the nuclear many body problem has been restricted almost entirely to considerations of the two-body force. However the meson theory of nuclear forces predicts that the exchange of mesons between three or more particles will give rise to a three-body or many-body force. The meson theory which has successfully explained the main features of the phenomenological nucleon-nucleon potential, is expected to provide a good basis for the study of three-body nuclear forces. Three-body nuclear forces can occur among baryons such as $N$, $\Lambda$, $\Sigma$ and $\Xi$. So far, however only bound states of nucleons (nuclei) and of nucleons and $\Lambda$ (hypernuclei) have been observed experimentally. Hence only the three-nucleon force and the $\Lambda NN$ force are considered. It will be seen that the $\Lambda NN$ force plays a more important role than the three-nucleon force. Thus in the present work the $\Lambda NN$ force will be studied in greater detail than the three-nucleon force. First the long and intermediate range parts of the $\Lambda NN$ force are derived from meson theory. Their effects on the binding energies of $^3_\Lambda H$, $^5_\Lambda$ He and nuclear matter are then estimated.
Since the short range part of the force is not known, no definite conclusion can be drawn. However it is found that the three-body $\Lambda NN$ force can play an important role in the nuclear structure problem. The effects of the three-nucleon force in $^3\Lambda H$ and in nuclear matter are also briefly discussed.
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CHAPTER 1

INTRODUCTION - THREE-BODY NUCLEAR FORCES

One of the major trends in nuclear physics is an attempt to explain nuclear structure in terms of the two-body nuclear forces which are determined from nucleon-nucleon scattering data. This program relies on the hope that three-body and many-body forces are unimportant. On the other hand, from the meson theory of nuclear forces, where it is believed that forces are due to the exchange of mesons, it is expected that the exchange of mesons between three particles or more will give rise to a three-body or many-body force.

The meson theory of the two-body nucleon-nucleon (N-N) force has been extensively studied by many authors. The one-pion-exchange potential, OPEP, well established theoretically, gives a good description of the N-N interaction at large distances, distances greater than about 2 fm (Iwade et al, 1956; Cziffra et al, 1959). For interaction distances less than about 2 fm, the potential due to the exchange of more mesons or heavier mesons has to be considered. Thus the two-pion-exchange potential, TPEP, has been calculated and is fairly well understood theoretically (Konuma et al, 1957; Cottingham and Vinh Mau, 1963). The potential due to the exchange of heavier mesons than p has also been considered, giving rise to the one-boson-exchange potentials, OBEF, (Arndt et al, 1966; Bryan and Scott, 1967). The OBEF, TPEP together with the OPEP reproduce the main features of the long and intermediate range part of the
phenomenological N-N potential. The meson theory of nuclear forces has been extended to the interaction between various baryons N-Λ, N-Σ, Λ-Λ etc... However experimental data are as yet too meager to provide a proper comparison with theoretical predictions. Since meson theory gives a good account for the two-body nuclear interaction, in particular for the N-N force, it is expected that meson theory will provide a good basis for the study of the three-body nuclear forces. Accordingly the exchange of mesons can take place between three particles and can give rise to a "meson theoretical" three-body force.

Three-body nuclear forces can occur among baryons such as N, Λ, Σ and Ξ. So far, however, only bound states of nucleons (nuclei) and of nucleons and Λ (hypernuclei) have been observed experimentally. Thus the three-body nuclear forces which are expected to play a role in nuclear structure, are the three-nucleon force and the ΛNN force. As will be seen the three-body ΛNN force is stronger than the three-nucleon force. On the other hand the charge symmetry of the strong interaction forbids the exchange of one pion in the Λ-N interaction, thus the longest ranged two-body Λ-N force arises due to TPE and has a shorter range than the N-N force. Therefore the ΛNN force will be relatively more important than the three-nucleon force and then it will be studied in greater detail.

In principle, if two-body forces are completely known and if many-body problems can be solved exactly with the two-body interaction, then any discrepancy between the theoretical predictions and experimental data, can be claimed to be due to many-body forces. In practice, the two-body interaction is not very well known and even if it is well known, it is practically impossible to solve the many-body problems exactly.
Nevertheless there seems to be some evidence for such discrepancy, in particular in the case of the $\Lambda$-N interaction.

The best source of information about the $\Lambda$-N interaction has come so far from the phenomenological analyses of the binding energies for light hypernuclei, mainly s-shell hypernuclei. Recently $\Lambda$-N scattering experiments have provided more direct information. If there is no many-body force, these two approaches should lead to the same results. Extensive analyses, in terms of the binding energies of s-shell hypernuclei, lead to an s-state $\Lambda$-N potential with strong spin dependence: it is much more attractive in the spin singlet state than in the spin triplet state. These results have been well described by Dalitz (1966). The most direct information, derived from $\Lambda$-p scattering experiments, is now becoming available (Alexander et al., 1966, 1968). The s-wave scattering lengths and effective ranges, determined from Alexander et al.'s experiments, are not very different for the singlet and triplet states, if the intrinsic range of the force is chosen to be around 2 fm or slightly greater (Alexander et al., 1966; Ali et al., 1967). This is contrary to the results of the previous analyses which utilised the binding energy calculations for light hypernuclei. This suggests a possible importance of the three-body $\Lambda\Lambda N$ force and that the $\Lambda$-N force determined from the $\Lambda$-p scattering will not give the right binding energies.

It has been shown that a $\Lambda$-N potential, reproducing the data of Alexander et al., 'overbinds' light hypernuclei, in particular $^5_\Lambda\text{He}$ (Shaduri et al., 1967; Hordon and Tang, 1967). Thus there is the possibility that a three-body repulsive $\Lambda\Lambda N$ force could play an important role in these hypernuclei. It was first pointed out by Weitzner (1958).
that for strong repulsive ΛΝΝ forces, the binding energies of light
hypernuclei could be accounted for with a two-body Λ-N force without
strong spin dependence. Recently, the effects of ΛΝΝ forces in s-shell
hypernuclei, in connection with the new scattering data, have been
discussed by Gal (1966). If such ΛΝΝ forces exist, then the low-energy
scattering parameters previously extracted from the binding energy data
of s-shell hypernuclei, assuming only a two-body Λ-N force, are the
result of some "effective Λ-N interaction". "This effective Λ-N force"
cannot be directly compared with the free two-body Λ-N force.

On the other hand it is interesting to have an accurate estimate
of the binding energy of a Λ in nuclear matter, since it is the same as
the depth of the average one-body potential, \( U_{\Lambda} \), in which it moves and
this depth gives a measure of the strength of the "effective Λ-N inter-
action". It was shown by Ali et al. (1967) that the Λ-N potential that
fits the Λ-p scattering data overbinds the Λ in nuclear matter. Here
again, one of the effects which may reduce the depth, \( U_{\Lambda} \), in nuclear
matter, may arise from the presence of a strong three-body ΛΝΝ force.

At the present time, the three-body ΛΝΝ force can only be cal-
culated using a meson field theoretical approach. This method has
already been applied to the determination of a "theoretical" Λ-N force
by several authors (Nogami et al., 1964; Deloff and Wrezecinio, 1964;
Rimpault and Vinh Mau, 1965). In such a calculation, the longest range
part of the three-body ΛΝΝ force, arising from TPE, has the same range
as that of the TPE Λ-N force which has been shown to be important. The
definition or the interpretation of the three-body force is not free from
ambiguity. In fact it is still a matter of serious controversy (Brown et al., 1963;
However in the present work the static meson theory is taken throughout.

The TPE ANN force is derived in the static approximation in chapter 2 and is compared with the previously obtained ANN forces. Only the p-wave and s-wave $\pi-\Lambda$ interaction are considered here, higher partial waves being ignored. Then the ANN force, $W$, consists of two parts, arising from the p- and s-wave $\pi-\Lambda$ interaction:

$$W = W_p + W_s \quad \ldots (1.1)$$

For $W_s$, a "suppression factor" of the s-wave $\pi-\Lambda$ interaction is introduced in analogy to the corresponding situation of the s-wave $\pi-N$ interaction. It will be shown that for a reasonable suppression factor, $W_s$ is unimportant. $W_p$ consists of a central and a tensor term. The tensor term appears as a product of two tensor operators and depends on the angle between the two $\Lambda N$ vectors. This term contributes significantly to the potential energy of the system.

Unfortunately, the ANN force, that is derived, is of a very singular nature at short distances and cannot be taken literally. In any case, in the short range region ($r < 0.7$ fm) processes other than TPE will become important so that the contribution of TPE should not be taken alone. Therefore only the tail of the TPE potential is considered here and this potential is set equal to zero when the $\Lambda-N$ distance is less than a cut-off distance, $d_{\Lambda N}$. For $d_{\Lambda N}$ greater than 1 fm only the TPE potential is expected to contribute significantly to the ANN force.

The contribution of this TPE ANN force to the binding energy of a $\Lambda$, in $^3\Lambda H$, $^5\Lambda$ He and in nuclear matter, is examined in chapter 3. The effect in $^3\Lambda H$ is found to be relatively small and can be compensated by
slightly changing the two-body \( \Lambda-N \) force. The \( \Lambda-N \) potential in \( ^{5}_{\Lambda} \) He due to \( W_p \) is repulsive but its strength is sensitive to \( d_{\Lambda N} \). The contribution from the tensor part of \( W_p \) is large but finite in the limit of \( d_{\Lambda N} \to 0 \). For a reasonable \( d_{\Lambda N} \), the contribution of \( W_p \) is still quite substantial and used in conjunction with a \( \Lambda-N \) force that fits the scattering data of Alexander et al., gives a \( B_{\Lambda} \) for \( ^{5}_{\Lambda} \) He much closer to the experimental value than that obtained by Ehsani et al. (1967). The effect on \( B_{\Lambda} \) in nuclear matter is considered using perturbation theory. It is found that the contribution from \( W_p \), mainly repulsive, is dominant and is extremely sensitive to \( N-N \) correlation. However, as pointed out above, processes other than TPE, contributing to the intermediate range part of the \( \Lambda N N \) force, may modify these results appreciably.

The three-pion-exchange three-body \( \Lambda N N \) force may be important in the intermediate range part since it is the next lowest order contribution after TPE and since it has been shown that the TPE \( \Lambda-N \) force is important (Nogami et al., 1964; Deloff and Wrezecinio, 1964; Rimpault and Vinh Mau, 1965). In the case of the meson theory of the \( N-N \) interaction, it has been shown that the OPEP gives a good description of the \( N-N \) interaction at distances greater than 2 fm (Cziffra et al., 1959; Breit and Hull, 1960). For distances smaller than this, the OPEP is dominated by the TPEP. It is of interest therefore to investigate at what distance the TPE \( \Lambda N N \) force becomes dominated by the three-pion-exchange \( \Lambda N N \) force. It will be also possible to say to what extent the previous results, on the effect of the TPE three-body \( \Lambda N N \) force in light hypernuclei and in nuclear matter, are reliable.

The three-pion-exchange \( \Lambda N N \) force, \( P \), is derived in the static
approximation in chapter 4. In chapter 5 the effects of P on $B^\Lambda$ in $^3_\Lambda^H$, in $^5_\Lambda^\Lambda$ and in nuclear matter are examined. In $^3_\Lambda^H$ it is found that P depends strongly on the $\Lambda$-N distance and that the contribution of P to $B^\Lambda$ in $^3_\Lambda^H$ is attractive and larger than that of the TPE $\Lambda$NN force, $W_p$. For $d \Lambda_N \approx 1$ fm, the overall effect (P and $W_p$) is found to be attractive and relatively small. The $\Lambda$-$\alpha$ potential in $^5_\Lambda^\Lambda$ due to P, $P(\Lambda r)$, is evaluated. The potential $P(\Lambda r)$ is found to be repulsive (except for $d \Lambda_N = 1$ fm and $r_\Lambda \approx 0$ fm) and its magnitude is sensitive to $d \Lambda N$. It further reduces the binding energy of $^5_\Lambda^\Lambda$ and for $d \Lambda N \approx 1$ fm the effect of P is smaller than that of $W_p$. A first order perturbation calculation is done to estimate the contribution of P to $B^\Lambda$ in nuclear matter. This contribution is found to be repulsive and its magnitude depends strongly on $d \Lambda N$ and is larger than that of $W_p$. Even for $d \Lambda N \approx 1$ fm, the overall effect (P and $W_p$) is found to be large and repulsive.

In the case of the N-N interaction, there are also discrepancies between theoretical predictions and experimental data when many-body problems are solved with the two-body interaction. Such a disparity occurs in triton, $^3\Lambda H$, where many authors have evaluated the binding energy due to a two-body N-N potential that fits N-N scattering data. So far the value obtained is always smaller than the experimental value by an amount of the order of 1 Mev or more (Delves and Blatt, 1967; Davies, 1967). There is a similar situation in nuclear matter where the binding energy obtained using a two-body N-N force is smaller than the "experimental value" extrapolated from heavy nuclei by an amount of the order of 3 Mev (Brueckner and Masterson 1962; Bhargava and Sprung, 1967). So it is possible that the three nucleon force may also play an important role in
the study of nuclear structure.

The long range part of the three nucleon force, arising from two-pion-exchange, is derived in chapter 6 and its effects on the binding energies of triton and nuclear matter are examined. The effect of the (controversial) \( I = J = 0 \) dipion resonance (the \( \sigma \)-meson) is considered.

The three-body potential consists of three parts:

\[
F = F_p + F_s + F_\sigma
\]  ...(1.2)

where \( F_\sigma \) is due to the (virtual) \( \pi-N \) scattering via the \( \sigma \)-meson, while \( F_p \) and \( F_s \) are due respectively to the p-wave and s-wave \( \pi-N \) scattering via 'direct interactions'. The direct s-wave \( \pi-N \) interaction is set up so that, together with the \( \pi-N \) interaction via the \( \sigma \)-meson, it reproduces the observed s-wave \( \pi-N \) scattering length. Since the short range part of the three-nucleon force is not known, the three-body potential is taken to be zero for \( N-N \) distances less than a cut-off distance \( d \). The triton wave function is taken from a variational calculation for a hard-core two-nucleon potential (hard-core radius \( D \)). The result is sensitive to \( d \) and also to the hard-core radius \( D \). For instance, at \( D = 0.4 \) fm, the contribution to the binding energy can be as big as about 1 Mev (attractive). The effect in nuclear matter is evaluated by a first order perturbation theory. The contribution is found to be order of a few Mev, between 0.7 and 4.0 Mev, depending on the value of the cut-off \( d \) and on the \( N-N \) correlation function.

A discussion of the results is given in chapter 7.

Throughout the whole calculation, the units \( \hbar = c = 1 \) are used.
2.1 Evaluation in the static approximation using Miyazawa formalism

There is an arbitrariness in the definition of the $\Lambda$-$N$ and $\Lambda NN$ potentials which is connected with the number of channels considered in the solution of the Shrödinger equation. For the two-body interaction if two channels are taken, $\Lambda$-$N$ and $\Sigma$-$N$, then correspondingly the wave function has two components. The potential is then a $2 \times 2$ matrix:

$$
V = \begin{bmatrix}
V(\Lambda N \rightarrow \Lambda N) & V(\Sigma N \rightarrow \Lambda N) \\
V(\Lambda N \rightarrow \Sigma N) & V(\Sigma N \rightarrow \Sigma N)
\end{bmatrix}
$$

As shown by Uehara (1960), the $\Sigma$-$N$ component of the wave function can be eliminated and the one-channel formalism can be used. Then the $\Lambda$-$N$ potential in the one-channel formalism which is of course different from $V(\Lambda N \rightarrow \Lambda N)$ in (2.1), is energy-dependent, but this energy-dependence is negligible if the energy of the system is well below the $\Sigma$-$N$ threshold. The $\Lambda$-$N$ and $\Lambda NN$ potentials, in the two-channel formalism, have been discussed by Uehara (1960). Since only the bound state is considered it is sufficiently accurate to use the one-channel formalism. The two-pion-exchange $\Lambda NN$ force, arising from the diagram shown in Fig. 2.1, is derived in the static approximation where the nucleons and $\Lambda$ are considered to be at rest.

---

† Here the effect of the pion-pion interaction ($\pi$-$\Lambda$ interaction via the $\sigma$-meson) has not been considered. It will be done for the three-nucleon force (chapter 6).
Following Miyazawa (1956, 1957), the S matrix element corresponding to the diagram 2.1 can be written \( c = \frac{1}{2} = 1 \):

\[
S = - \frac{4\pi f_N \mathbf{r} \cdot \mathbf{r}}{6 \pi} \int \frac{3}{dp} \frac{3}{dq} \frac{1}{p} \frac{2}{q} \frac{2}{q} \frac{2}{q} <p| S| q> e^{-i(p \cdot \mathbf{r}_1 - q \cdot \mathbf{r}_2)}
\]

\[
\cdots(2.2)
\]

Here \( f_N \) is the pseudo-vector \( ^{3}\mathrm{N}\) coupling constant \( f_N^2 = 0.08 \), \( p \) and \( q \) are the momenta of the exchanged pions, \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) are the coordinates of the two nucleons and \( \mu \) is the pion mass. The scattering matrix,

\[
<p| S| q>
\]

for the zero energy pion scattering by \( \Lambda \), is given by

\[
<p| S| q> = 2i\pi \delta(0) (\Lambda(0) \sigma^\Lambda p^\Lambda q^\Lambda c^\Lambda (0) \sigma^\Lambda q^\Lambda p^\Lambda)
\]
where \( x_{\Lambda} \) is the coordinate of \( \Lambda \). The functions \( A_{\Lambda} \), \( C_{\Lambda} \) and \( D_{\Lambda} \) are obtained by setting \( p_0 \) equal to zero in the dispersion relations for the p-wave and s-wave \( \pi-\Lambda \) scattering. From Nogami and Bloore (1964), they are given by:

\[
A_{\Lambda}(p_0) = C_{\Lambda}(-p_0) = \frac{4\pi (f_{\Lambda}/\mu)^2}{\Delta + p_0 - i\epsilon} + \frac{1}{2\pi} \int_{0}^{\infty} \frac{dk}{k} \frac{\sigma_3(k)}{\omega} + \frac{1}{6\pi} \int_{0}^{\infty} \frac{dk}{k} \frac{2\sigma_1(k) + \sigma_3(k)}{\omega + p - i\epsilon}, \ldots (2.4)
\]

\[
D_{\Lambda}(p_0) = 2\pi a_{\Lambda} + \frac{2}{2\pi} \int_{0}^{\infty} \frac{dk}{k} \frac{\sigma_3(k)}{\omega} \left( \frac{1}{\omega - p - i\epsilon} + \frac{1}{\omega + p - i\epsilon} \right), \ldots (2.5)
\]

Here \( f_{\Lambda} \) is the \( \pi-\Lambda \) pseudo-vector coupling constant, \( \omega_k = (k^2 + \mu^2)^{1/2} \), \( \Delta = m_{\Sigma} - m_{\Lambda} \) and \( \sigma_1 \) and \( \sigma_3 \) are the p-wave \( \pi-\Lambda \) scattering cross sections in the states with \( J = 1/2 \) and \( 3/2 \) respectively. In the following, the p-wave \( \pi-\Lambda \) scattering is assumed to be dominated by the \( Y_{1}^{N}(1385) \) resonance in \( \sigma_3 \), and \( \sigma_1 \) is ignored. In (2.5), \( a_{\Lambda} \) is the scattering length and \( \sigma_s \) the cross section of the s-wave \( \pi-\Lambda \) scattering. If expression (2.3) is substituted in eq. (2.2) \( S \) is obtained in the form

\[
S = -2\pi i \delta(0) W. \quad \text{The quantity } W, \text{ which is interpreted as the } \Lambda N N \text{ potential arising from diagram 2.1, is given by :}
\]

\[
W = W_p + W_s, \ldots (2.6)
\]
\[ W_p = -\frac{1}{6} \mathcal{C}_{\Lambda} \mathbf{r}_m \cdot \mathbf{r}_m \left\{ \sigma_{\Lambda} \cdot \sigma_{\Lambda} + S_{\Lambda} (x) T (x) \right\}, \]

\[ \sigma_{\Lambda} \cdot \sigma_{\Lambda} + S_{2\Lambda} (y) T (y) \}

\[ Y(x) Y(y), \quad \ldots (2.7) \]

\[ W_\varepsilon = C_{\Lambda} \mathbf{r}_m \cdot \mathbf{r}_m \sigma_{\Lambda} \cdot \sigma_{\Lambda} \cdot \chi_{\Lambda} (\mu x + 1) (\mu y + 1) Y(x) Y(y)/(\mu x y). \]

\[ \ldots (2.8) \]

Here, \( \{ \Lambda, B \} = AB + BA \), \( x = r_1 - r_\Lambda \), \( y = r_2 - r_\Lambda \) (see Fig. 2.2) and

\[ C_{\Lambda} = \frac{2(\mu r_\Lambda)}{3} \left( \frac{2}{\Delta} + \frac{2}{\mu} \int_0^{\infty} \frac{\sigma \, 3 (k)}{2} \right), \quad \ldots (2.9) \]

\[ C_{\Lambda} = - \left( \mu r_\Lambda \right)^2 N_{\Lambda} \left( 0 \right) / (2\pi), \quad \ldots (2.10) \]

\[ S_{\Lambda} (x) = 3 \frac{1}{\sigma_{\Lambda} \cdot \sigma_{\Lambda} \cdot \chi_{\Lambda}} \sigma_{\Lambda} \cdot \sigma_{\Lambda} + \frac{1}{\sigma_{\Lambda} \cdot \sigma_{\Lambda}}, \quad \ldots (2.11) \]

\[ T(x) = 1 + \frac{3}{\mu x} + \frac{3}{\mu x}, \quad Y(x) = \frac{e^{-\mu x}}{\mu x} \quad \ldots (2.12) \]
In the following chapter, the potentials $W_p$ and $W_q$ will be considered in the case where the nucleons and $\Lambda$ are in the $s$-shell. The expectation values of the $\tau$'s and $\sigma$'s are denoted by brackets $\langle \rangle$. The expectation value $\langle \sigma \cdot x \sigma \cdot y \rangle$ should be proportional to $x \cdot y$ for any two vectors $x$ and $y$. The constant of proportionality can be found by putting $x = y$ and doing the angular integration. Therefore:

$$\langle \tau_1 \cdot \tau_2 \sigma_1 \cdot \sigma_2 \cdot x \cdot y \rangle = \frac{1}{3} x \cdot y \langle \tau_1 \cdot \tau_2 \sigma_1 \cdot \sigma_2 \rangle \quad \ldots (2.13)$$

and using (2.13)

$$\langle \tau_1 \cdot \tau_2 \left\{ \sigma_1 \cdot \sigma_2 , s_1 \cdot \Lambda \cdot (y) \right\} \rangle = 0,$$
\[ < I_\Lambda \cdot I_\Lambda > \left\{ S_{1\Lambda}(x), \sigma \cdot \sigma \right\}^2 = 0 \]  
\[ \ldots (2.14) \]

and

\[ < I_\Lambda \cdot I_\Lambda > \left\{ S_{1\Lambda}(x), S_{2\Lambda}(y) \right\}^2 \geq 2 (3 \cos \theta_{xy} - 1) \]

\[ x < I_\Lambda \cdot I_\Lambda \sigma \cdot \sigma > , \ldots (2.15) \]

where \( \cos \theta_{xy} = x \cdot y / (xy) \). Thus, for the nucleons and \( \Lambda \) in the \( s \)-shell, \( W_p \) and \( W_s \) are reduced to:

\[ W_p = - \frac{C_{p\Lambda}}{3} \left[ 1 + (3 \cos \theta_{xy} - 1) T(x) T(y) \right] Y(x) Y(y) < I_\Lambda \cdot I_\Lambda \sigma \cdot \sigma > \]

\[ \ldots (2.16) \]

\[ W = - \frac{C_s}{3} \cos \theta_{xy} (\mu x + 1)(\mu y + 1) Y(x) Y(y) / (\mu xy) < I_\Lambda \cdot I_\Lambda \sigma \cdot \sigma > \]

\[ \ldots (2.17) \]

When \( \Lambda \) is far from the nucleons, the angle \( \theta_{xy} \) is small, hence \( W_p \) is repulsive and \( W_s \) is attractive (it will be seen that \( > - 3 \)).

The coefficient \( C_{p\Lambda} \) and \( C_{s\Lambda} \) are now evaluated. For the first term in \( C_{p\Lambda} \) (2.9) various estimates of \( f_{p\Lambda}^2 \) have been done indicating that \( f_{p\Lambda}^2 \) is slightly smaller than \( f_N^2 \) (Martin and Wali, 1963; Roman, 1966; Kwan Kim, 1967; Chen and Neale, 1968). It is assumed here that \( f_{p\Lambda}^2 = f_N^2 = 0.08 \), although this may be an overestimate of \( f_{p\Lambda}^2 \) by a factor \( \approx 2 \).
As in Nogami and Bloore (1964), the second term in \( C_{\Lambda \Lambda} \) has been evaluated assuming a Breit-Wigner resonance formula for \( Y_1^0 (1385) \). Thus it is found that:

\[
C_{\Lambda \Lambda} = 1.43 \text{ MeV} \quad \ldots (2.18)
\]

The first term in (2.9) constitutes 73% of \( C_{\Lambda \Lambda} \), while the rest comes from \( Y_1^* \). If the value \( f_\Lambda^2 = 0.04 \) is taken, \( C_{\Lambda \Lambda} \) is found to be equal to 0.89 MeV.

The true value of \( C_{\Lambda \Lambda} \) will be between 1.43 and 0.89 MeV.

If the two-channel formalism is used, the first term in (2.9) should be dropped, because it is interpreted as a repetition of the OPE two-body potential (Nogami and Bloore, 1964). In other words, the \( \Lambda N \) force in the one-channel formalism contains effects of two-body forces in the two-channel formalism. The potential \( V_p \) is much stronger than the TPE three-nucleon force, derived, for instance, by Fujita and Miyazawa (1957), which corresponds to the \( \Lambda NN \) force in the two-channel rather than the one channel formalism.

For \( C_{S \Lambda} \), since \( a_\Lambda \) and \( \sigma_s \) are not known experimentally, theoretical estimates are necessary. The lowest order perturbation theory gives

\[
D_{\Lambda} (0) \approx 2\pi a_\Lambda^2 \ , \ a_\Lambda = -2 \left( f_\Lambda / \mu \right) x (m_\Lambda + m_\Lambda) ,
\]

which results in

\[
C_{S \Lambda} \approx - \left( \mu f_N \right) a_\Lambda = 29.5 \text{ MeV} .
\]

However, such a simple perturbation calculation of \( a_\Lambda \) is very poor approximation. There is a corresponding problem in the three-nucleon force (Fujita and Miyazawa, 1957; Quang-Ho-Kim, 1966). There, \( a_\Lambda \) is replaced by \((a_{1/2} + 2a_{3/2}) / 3\), where \( a_{1/2} \) and \( a_{3/2} \) are the s-wave \( \pi N \) scattering lengths in the \( I = 1/2 \) and \( 3/2 \) states, respectively. According to Hamilton and Woolcock (1963) experimental values are \( a_{1/2} = (0.171 + 0.005) / \mu \) and \( a_{3/2} = (-0.088 \pm 0.004) / \mu \). Hence
\[ 0 \leq - \left( a + 2a \right) / 3 \leq 0.0 \, \text{1/}\mu. \]

On the other hand, it has been shown that for the soft pion (pion of zero four momentum) \( a + 2a = 0 \) (Weinberg 1966). However, the lowest-order perturbation calculation gives \( - (a + 2a) / 3 = 4 \pi m \left( f / \mu \right) = \frac{2}{1 \cdot 3} \frac{m_N}{N} N \). This is an overestimate by a factor of 200 or more. Various mechanisms have been considered for this "pair suppression" (Amati and Rubini, 1962). These mechanisms all seem to be applicable also to the \( s \)-wave \( \pi - \Lambda \) interaction. Thus a suppression factor of the same order of magnitude is expected there. This factor is tentatively assumed to be 100, namely \( a_\Lambda = - 2 \left( f_\Lambda / \mu \right) \left( m_\Sigma + m_\Lambda \right) / 100 \). It has been also shown that for the soft pion \( a_\Lambda = 0 \) (Weinberg, 1966; Tomozawa, 1966).

In analogy with the \( \pi - N \) case the integral containing \( \sigma_B \) in (2.5) is assumed to be negligible. Then, it is found that:

\[ C_{B\Lambda} = 0.30 \text{ MeV} \quad \ldots (2.19) \]

As will be seen in the following chapter, results will not be altered even when \( C_{B\Lambda} \) is as large as 3 MeV.

2.2 Comparison with the previously obtained \( \Lambda N N \) forces.

The \( \Lambda N N \) potential, obtained in the previous section, will be compared with those so far derived. The TPE \( \Lambda N N \) force has been evaluated, in various approximation, by Weitzner (1958), Spitzer (1958), Bach (1959), Uehara (1960), Chalk and Downs (1963). Weitzner's and Spitzer's results differ from all the later works and it is not easy to retrace their calculation.

In Bach's work, the \( \Lambda N N \) force is calculated in the static approximation
by perturbation theory using time-ordered diagrams. There are three types of diagrams, as shown in Fig. 2.3, which are called NB, B and SB.

The diagrams NB and B are due to the p-wave and SB is due to the s-wave $\pi$–$\Lambda$ interaction. The ANN potentials, arising from these diagrams, are denoted by $V_{NB}$, $V_B$ and $V_{SB}$, respectively. The mass difference, $\Delta = m_\Sigma - m_\Lambda$, is neglected in the energy denominators except in the second intermediate state of B. However, if $\Delta$ is not neglected and if the sum of the contributions from all the 16 NB diagrams and 8 B - diagrams is done, the potential that results is the same as $V_p$ but without the contribution from $V_1$. This result was not obtained in Bach's work, perhaps because $V_{NB}$ and $V_B$ were calculated separately. The diagram of Fig. 2.1 is not time-ordered and contains both NB and B. The same calculation was repeated by Chalk and Downs and Bach's $V_{NB}$ and $V_B$ were confirmed. The value of the coupling constant $f_\Lambda^2$, taken by Bach, is slightly smaller than that of $f_\Lambda^2$. 

Fig. 2.3
used here. Incidentally Bach's results have been misquoted by Dalitz (1965) and by Gal (1966). The central part of Bach's $V_B$ is quoted as $\sim 2 \text{MeV} \times Y(x) Y(y)$, which is actually 2 times Bach's original $V_B$.

Cal's strength parameter $C_G$, corresponding to $C_{PL}$, is taken equal to 17 MeV which is much stronger than the estimate of $C_{PL}$ done here. It is argued that the contribution from the $Y^*$ intermediate process may considerably modify the value of $C_G$. However, $C_{PL}$ includes contribution from $Y_L$. Also in Cal's work only the central part of $W$ is considered and it will be shown, in the next chapter, that the effect of the tensor part of $W_P$ dominates over that of the central part. Uehara's ANN potential is for the two-channel formalism and is obtained from $W_P$ by dropping the first term in $C_{PL}$ (2.9).

For the $s$-wave diagram $SB$, as was noted by Chalk and Downs, Bach's $V_{SB}$ has a wrong factor. The diagram $SB$ was calculated by Chalk and Downs by perturbation theory, firstly in static approximation and then in a relativistic way. Their result in the static approximation agrees with $W_S$, without the suppression factor. Their relativistic calculation shows that the recoil effect is unimportant for the long range part of the ANN potential. Since the suppression factor was not introduced their potential is much stronger than $W_S$. 
CHAPTER 3
EFFECTS OF THE TPE ΛΝΝ FORCE

3.1 Effect in \(^3\)Λ

The effects of the potentials \(W_p\) (2.16) and \(W_b\) (2.17) on \(^3\)Λ in \(^3\)Λ are estimated by perturbation theory. The unperturbed wave function is taken to be the wave function obtained by Downs, Smith, and Truong (1963) from a variational calculation of the binding energy of \(^3\)Λ.

It has the form:

\[
\psi = N^{1/2} f(x) f(y) g(z) \xi \chi 
\]

where

\[
\xi = \xi_1 - \xi_2, \quad \chi = \chi_2 - \chi_1, \quad z = \frac{z_1 - z_2}{2}
\]

and

\[
f(r) = 0 \quad \text{for } r < D
\]

\[
= \exp \left[ -\alpha (r - D) \right] - \exp \left[ -\beta (r - D) \right] \quad \text{for } r > D,
\]

\[
\xi(r) = 0 \quad \text{for } r < D,
\]

\[
= \exp \left[ -\gamma (r - D) \right] - \exp \left[ -\delta (r - D) \right] \quad \text{for } r > D.
\]

Here \(D\) is the hard-core radius of the Λ-Ν and Ν-Ν forces and the factor \(N^{-1/2}\) normalizes \(\psi\) to unity. The function \(\xi\) is the isospin singlet wave function for the two nucleons and \(\chi\) is the spin wavefunction of \(^3\)Λ.

The optimum variational parameters, obtained by Downs et al. (1963), are listed in Table 3.1 for the hard-core radii \(D = 0.2, 0.4,\) and \(0.6\) fm.
together with the corresponding normalization factors.

### TABLE 3.1

Parameters for the wavefunction, with the corresponding normalization factors $N$, for three different hard-core radii $D$. (Downs et al., 1963)

<table>
<thead>
<tr>
<th>$D$ ($\text{fm}$)</th>
<th>$\alpha$ ($\text{fm}^{-1}$)</th>
<th>$\beta$ ($\text{fm}^{-1}$)</th>
<th>$\gamma$ ($\text{fm}^{-1}$)</th>
<th>$\delta$ ($\text{fm}^{-1}$)</th>
<th>$N$ ($\text{fm}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.297</td>
<td>7.36</td>
<td>0.547</td>
<td>6.52</td>
<td>3.825</td>
</tr>
<tr>
<td>0.4</td>
<td>0.327</td>
<td>6.94</td>
<td>0.578</td>
<td>4.55</td>
<td>3.658</td>
</tr>
<tr>
<td>0.6</td>
<td>0.389</td>
<td>11.28</td>
<td>0.606</td>
<td>4.79</td>
<td>2.928</td>
</tr>
</tbody>
</table>

The expectation values of $W_p$ (2.16) and $W_s$ (2.17) with respect to $\Psi$ (3.1) are denoted by $\langle W_p \rangle$ and $\langle W_s \rangle$ respectively. The expectation value of the $\tau$'s and $\sigma$'s is:

$$
\langle \mathbf{\tau} \cdot \mathbf{\tau} \rangle \equiv -3
$$

$\ldots (3.4)$

† The normalization factor $N$ listed in table 3.1 differs from Chalk and Downs' $N$ (listed in Table 1 in Chalk and Downs (1963)) by a factor $8 \pi$. Their wavefunction is normalized as:

$$
8 \pi \int dx \, dy \, dz \, xyz \, \Psi(x, y, z, \tau) = 1
$$

whereas here the factor $8 \pi$ of the volume element is dropped throughout.
Then $<W_p>$ and $<W_s>$ can be written:

$$<W> = <W(I)> + <W(II)> \quad \ldots (3.5)$$

with

$$<W(I)> = c_p \int_{\Lambda} \left[ Y(x) Y(y) \right] \quad \ldots (3.6)$$

$$<W(II)> = c_p \int_{\Lambda} \left[ (3 \cos \theta - 1) T(x) T(y) Y(x) Y(y) \right] \quad \ldots (3.7)$$

and

$$<W>_s = -c_s \int_{\Lambda} \left[ \cos \theta (\mu x + 1) (\mu y + 1) Y(x) Y(y)/(\mu xy) \right] \quad \ldots (3.8)$$

where

$$I[\cdot] = N^{-1} \int dx \, dy \, dz \, xyz \left\{ f(x) f(y) g(z) \right\}^2 [\cdot] \quad \ldots (3.9)$$

The integration domain of eq. (3.9) is such that $x$, $y$ vary from $d_{\Lambda N}$ (cutoff of the ANN potential) to infinity and $z$ takes values from $D$ to infinity (assuming $d_{\Lambda N} \geq D$), subject to the triangular inequalities:

$$x + y \geq z, \quad y + z \geq x, \quad z + x \geq y.$$

The integral (3.6) has been done analytically and for the integrals (3.7) and (3.8) first the $z$ integration was done analytically and then the $xy$-integration numerically.

The results are shown in table 3.2 for five different values of the cutoff $d_{\Lambda N}$ and three different values of the hard-core radius $D$. It is found that $<W_s>$ is always negative (attractive) and smaller than $<W_p(I)>$ and $<W_p(II)>$. The 'central part' $<W_p(I)>$ is always positive (repulsive). The tensor part $<W_p(II)>$ is predominant and changes sign depending on $d_{\Lambda N}$ and $D$. 
### Table 3.2

**Expectation Values of the Different Parts of the $\Lambda N$ Potential in MeV for Different Cutoff $\Lambda$ and Hard-Core Radius $D$**

<table>
<thead>
<tr>
<th>$\Lambda N$ (fm)</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>1.0</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; W_0 &gt;$ in MeV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D = 0.2 , \text{fm}$</td>
<td>-0.023</td>
<td>-0.022</td>
<td>-0.018</td>
<td>-0.009</td>
<td>-0.004</td>
</tr>
<tr>
<td>$D = 0.4 , \text{fm}$</td>
<td>-0.015</td>
<td>-0.014</td>
<td>-0.009</td>
<td>-0.004</td>
<td></td>
</tr>
<tr>
<td>$D = 0.6 , \text{fm}$</td>
<td>-0.014</td>
<td>-0.010</td>
<td>-0.004</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&lt; W_p &gt;$ in MeV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D = 0.2 , \text{fm}$</td>
<td>$&lt; W_p(I) &gt;$</td>
<td>0.091</td>
<td>0.084</td>
<td>0.064</td>
<td>0.030</td>
</tr>
<tr>
<td></td>
<td>$&lt; W_p(II) &gt;$</td>
<td>-0.725</td>
<td>-0.231</td>
<td>0.354</td>
<td>0.297</td>
</tr>
<tr>
<td>$D = 0.4 , \text{fm}$</td>
<td>$&lt; W_p(I) &gt;$</td>
<td>0.074</td>
<td>0.068</td>
<td>0.034</td>
<td>0.012</td>
</tr>
<tr>
<td></td>
<td>$&lt; W_p(II) &gt;$</td>
<td>-0.700</td>
<td>-0.432</td>
<td>0.130</td>
<td>0.090</td>
</tr>
<tr>
<td>$D = 0.6 , \text{fm}$</td>
<td>$&lt; W_p(I) &gt;$</td>
<td>0.084</td>
<td>0.044</td>
<td>0.015</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$&lt; W_p(II) &gt;$</td>
<td>-1.010</td>
<td>-0.075</td>
<td>0.074</td>
<td></td>
</tr>
</tbody>
</table>

According to Chalk and Down (1963) the contribution of the two-body $\Lambda N$ force to the binding energy of $^3\Lambda H$ is given by:

$$2 < V_{\Lambda N} > = \begin{cases} 24.2 \text{ MeV} & \text{for } D = 0.4 \text{ fm} \\ 70.8 \text{ MeV} & \text{for } D = 0.6 \text{ fm} \end{cases} \hspace{1cm} (3.10)$$

The relative importance of the $\Lambda N$ force will be given by the ratio:

$$R = \frac{< W_p + W_0 >}{2 < V_{\Lambda N} >} = \begin{cases} 2.6\% & \text{for } D = d = 0.4 \text{ fm} \\ 1.3\% & \text{for } D = \Lambda N = 0.6 \text{ fm} \end{cases} \hspace{1cm} (3.11)$$
This small ratio justifies the use of a perturbation calculation to estimate the effect of the \( \Lambda NN \) force.

In view of the small binding energy of \( \Lambda \) in \( ^3_H \lambda \) (Gajerowski et al. 1967) the absolute value of \( (W_p + W_s) \) can be quite substantial depending on \( d_{\Lambda N} \) and \( D \). As was discussed in the introduction, however, a reasonable value of the cutoff radius \( d_{\Lambda N} \) may be taken as 1 fm. This follows from the fact that the TPE \( \Lambda NN \) potential, \( W \), is not very meaningful at short distances where processes other than TPE are likely to be important. On the other hand, if the same core radii for \( \Lambda - N \) and \( N - N \) are assumed, it will be reasonable to take \( D \approx 0.4 \) fm. Then it is found that \( <W_p + W_s> \approx 0.155 \) Mev for \( d_{\Lambda N} = 1 \) fm and \( D = 0.4 \) fm. Thus the effect is small, and can probably be compensated for by a slight change in the \( \Lambda - N \) force.

Finally, the results are compared with those of previous works. The expectation value of \( V_B(\S 2.2) \) which approximately corresponds to \( W_p \), has been evaluated by Bach (1959) who found that the ratio \( <V_B>/(2<W_{\Lambda N}>) \approx 2\% \) for a cutoff radius of 0.55 fm. However the wavefunction which was used in Bach's work does not satisfy proper boundary conditions. Bach's calculation has been refined in the work of Abou-Hadid (1962) where a proper hard-core wavefunction was used. Abou-Hadid's conclusion essentially agrees with Bach's. It was concluded, as in the present work, that the effect of the \( \Lambda NN \) force in \( ^3_H \lambda \) is small. The expectation value \( <W_s> \) has been evaluated by Chalk and Downs (1963). Since no suppression factor was introduced, their \( <W_s> \) is about 100 times as large as the \( <W_s> \) obtained here. Still their \( <W_s> \) is at most 5\% of the expectation value of the two-body \( \Lambda - N \) force. The expectation value \( <W_p> \) has not been
evaluated.

3.2 Effect in $^5\text{He}$

The calculation of $B$ in $^5\text{He}$, using only a two-body s-state $\Lambda$-$N$ potential that fits the low-energy data of Alexander et al (1966, 1968), is first briefly described. The details of the calculation can be found in the work of Bhaduri et al (1967). The low-energy parameters of the $\Lambda$-$N$ potential from $\Lambda$-$p$ scattering data of Alexander et al are:

$$a_s = -2.46 \text{ fm}, a_t = -2.07 \text{ fm}, r_s = 3.87 \text{ fm}, r_t = 4.50 \text{ fm} \ldots (3.12)$$

These are the most probable values. In order to bypass the construction of a complete hard-core (or soft-core) potential which makes binding-energy calculations rather cumbersome, the following form of the $\Lambda$-$N$ potential is chosen:

$$V_{s,t}(r) = 0 \quad \text{for } r < d_{s,t}$$

$$= -A_{s,t} t e^{-\nu r} \quad \text{for } r > d_{s,t} \ldots (3.13a)$$

Here the subscripts $s,t$ stand for singlet and triplet spin states respectively. The range parameter is taken as $\nu = 1.3992 \text{ fm}^{-1}$, corresponding to TPE, while

$$d_s = 1.017 \text{ fm}, A_s = 204.1 \text{ MeV}, d_t = 1.180 \text{ fm}, A_t = 223.3 \text{ MeV} \ldots (3.13b)$$

are determined by fitting parameters (3.12). Justification for choosing such a form for the $\Lambda$-$N$ potential is given in the work of Bhaduri et al (1967). Since this $\Lambda$-$N$ potential consists of a fairly weak attractive tail, the one-body average field that the $\Lambda$ experiences can be obtained by folding in the nucleon density distribution with this potential:

$$U^{(2)}(r_\Lambda) = \int \rho(r_1) \ V(|r_1 - r_\Lambda|) \ d^3r_1 \ldots (3.14)$$

where $V = V_s + 3V_t$, $\rho(r_1)$ is the density distribution of the nucleons in
\( \alpha \) and all the vectors are measured from the center of \( \alpha \). The superscript \( (2) \) on \( U \) refers to the fact that this part of the average field originates from a two-body \( \Lambda \)-N force. The complete \( U \) should also contain contributions from three-body \( \Lambda \mathrm{NN} \) forces. The normalized density distribution for the nucleons is taken as:

\[
\rho(r_1) = \left( \frac{\beta}{\pi^{1/2}} \right)^3 \exp \left( - \beta \frac{r_1^2}{\beta} \right) \quad \cdots (3.15)
\]

where \( \beta = 0.85056 \text{ fm}^{-1} \). In this calculation the \( \alpha \) is assumed to be undistorted in the presence of the \( \Lambda \). The binding, \( B_\Lambda \), due to \( U^2(r_\Lambda) \) was found to be 6.45 MeV (see Table 3.3); whereas the experimental value is 3.08 MeV (Gajerocki et al., 1967).

In the presence of \( \Lambda \mathrm{NN} \) forces \( W_p(2.16) \) and \( W_s(2.17) \), the average field \( U(r_\Lambda) \) will be modified. The average fields generated by \( W_p \) and \( W_s \) are denoted by \( U_p^3(r_\Lambda) \) and \( U_s^3(r_\Lambda) \), respectively. Then

\[
U_p^3(r_\Lambda) = 6 \chi \int_{p\Lambda} 3 \int_1^3 r_1 \rho(r_1) \rho(r_2) W(p, 1, \Lambda, 2, \Lambda) \frac{r_1 - r_2}{r_1^{1/2}} \quad \cdots (3.16)
\]

where the factor 6 comes from the six possible \( \Lambda \mathrm{NN} \) bonds and all the other quantities have been defined in chapter 2. The expectation value of the expression in \( \tau 's \) and \( \sigma 's \) in \( W_p(3.16) \) can be shown to be identical for \( \Lambda \) He and \( \Lambda \) H. Thus the value given in (3.4) is used in \( U_p(3.16) \).

Putting \( x = x_1 - x_\Lambda, x = x_2 - x_\Lambda \) (see Fig. 2.2), eq. (3.16) can be written:

\[
U_p^3(r_\Lambda) = 6 \chi \int_{p\Lambda} 3 \int x \int y \rho(\vec{x_\Lambda} + \vec{x}_1) \rho(\vec{x_\Lambda} + \vec{x}_1) W_p(x, y) \quad \cdots (3.17)
\]

It is shown, in Appendix 1, that the \( x_\Lambda \) and \( x_\Lambda \) integration can be separated to yield:
\[ U = U^{(I)} + U^{(II)} \] \[ \tag{3.18} \]

where:

\[ U^{(I)} = 6 \int \rho^3 \Lambda^3 \left[ \int d^3 \rho^3 \Lambda^3 \times X^2 \right] Y(x)^2 \] \[ \tag{3.19} \]

\[ U^{(II)} = 3 \int \rho^3 \Lambda^3 \left[ \int d^3 \rho^3 \Lambda^3 \times X^2 \right] (3 \cos \theta - 1) Y(x)^2 \] \[ \tag{3.20} \]

with \( \cos \theta = \frac{x \cdot x^\Lambda}{|x|^\Lambda} \). Only \( U^{(I)} \) has been considered previously by other authors (Dalitz, 1965; Gel, 1966). However \( U^{(II)} \) is the dominant term in eq. (3.18), as can be seen from Fig. 3.1. It is also seen from (3.19) and (3.20) that \( U^{(3)} \) is always repulsive in character. A similar analysis for \( W \), the \( \Lambda N N \) force arising from the \( s \)-wave interaction, yields

\[ U^{(s)} = -6 \int _s \Lambda^3 \left[ \int d^3 \rho^3 \Lambda^3 \times X^2 \right] \cos \theta (1 + \mu x) Y(x)/(\mu x)^2 \] \[ \tag{3.21} \]

which is always attractive. In the Appendix 1, it is shown that the angular interactions in \( x^\Lambda \) can easily be done in (3.20) and (3.21) and the problem reduces to the numerical integration of one-dimensional integrals to give \( U^{(3)}_p \) and \( U^{(3)}_s \). With \( c^{(3)}_{s\Lambda} = 0.30 \) MeV, \( U_s \) turns out to be completely negligible compared to \( U^{(3)}_p \). The situation would not change appreciably even if \( c^{(3)}_{s\Lambda} \) were ten times larger. In Fig. 3.1, \( U^{(3)}_p \) and \( U^{(3)}_s \) are plotted for a cutoff \( \Lambda^N = 1 \) fm. In Fig. 3.2, \( U^{(2)}_p \) and \( U = U^{(2)}_p + U^{(3)}_p + U^{(3)}_s \) are plotted for the same scale to show how the average field
is modified by \( \Lambda NN \) forces.

### TABLE 3.3

<table>
<thead>
<tr>
<th>( B ) in ( ^5\text{He} ) with ( U^{(2)} ) only = 6.45 MeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B ) (in MeV) when three-body forces are included</td>
</tr>
<tr>
<td>Cutoff ( d ) (fm)</td>
</tr>
<tr>
<td>( (2) ) ( U + U ) (I) ( p )</td>
</tr>
<tr>
<td>( (2) ) ( U + U ) (I) + ( U ) (II) ( p )</td>
</tr>
<tr>
<td>( (2) ) ( U + U + U ) ( p )</td>
</tr>
</tbody>
</table>

In table 3.3, the binding energy \( B \) in \( ^5\text{He} \) is tabulated with and without \( \Lambda NN \) forces for various cutoffs. \( U \) is calculated with the parameters given in (3.13 b) and shows considerable overbinding. It is seen that this overbinding is drastically reduced when \( U^{(3)} \) (II) is included in the calculation. This repulsive effect is probably overestimated since \( N-N \) correlations in the \( \alpha \)-particle are neglected. It is quite clear from
table 3.3 that three-body forces cannot be ignored in the calculation of $E$ in $^5\text{He}$. An effective $\Lambda$-$N$ force that is extracted purely from an analysis of $^3\text{He}$ and $^5\text{He}$ bindings, would therefore be much less attractive in the triplet s-state than the free $\Lambda$-$N$ interaction.

3.3 Effect in nuclear matter

The binding energy $U^{(3)}_\Lambda$ of a $\Lambda$ in nuclear matter due to the $\Lambda$NN force $W$ is calculated by first-order perturbation theory in a way similar to that formulated by Bodmer and Sampanthan (1962). The wavefunction of nuclear matter in the Fermi-gas model is:

$$
\psi = \psi(\Lambda) \phi(1, \ldots, A) \quad ...(3.22)
$$

where $A$ is the number of nucleons ($A\to\infty$ for nuclear matter) and

$$
\psi(\Lambda) = \frac{1}{k^\Lambda} \frac{1}{\sqrt{\Omega}} e^{\frac{i}{k^\Lambda} \Xi^\Lambda} \quad ...(3.23)
$$

$$
\phi = \frac{1}{A^\Lambda} \frac{1}{\sqrt{\Omega}} \text{Det} [ \phi(i) ] , \quad ...(3.24)
$$

$$
\phi(i) = \frac{1}{p^\Lambda} e^{\frac{i}{p^\Lambda} \Xi^\Lambda} \text{spin function} \times \text{isospin function} \quad ...(3.25)
$$

The function $\Xi^\Lambda$ represents the spin state of $\Lambda$ and $\phi_p(i)$ is the wave function of the $i$-th nucleon in the state $p$ where $p$ represents the momentum, the spin and isospin state of the nucleon. The quantity $\Omega$ is the volume of nuclear matter, $\Omega\to\infty$ with $A$ and $A^\Lambda=\rho$ where $\rho$ is the density of nuclear matter. The expectation value of the potential $W(i, j, \Lambda)$ is
\[ U_{\Lambda}^{(3)} = \frac{1}{2} \sum_{i,j} \int \psi_{(\Lambda)} (1) \phi (1) \phi (2) W(1, 2, \Lambda) \]

\[ x(\phi (1) \phi (2) - \phi (2) \phi (1))_{i}^{3} \frac{d r}{p_{i}} \frac{d r}{p_{j}} \frac{d r}{1} \frac{d r}{2} \Lambda \]

The binding energy \( U_{\Lambda}^{(3)} \) consists of two contributions: a direct term and an exchange term. The direct term vanishes identically because of the spin-isospin saturation. The expectation value of the \( T \)'s and \( \sigma \)'s, for the exchange term, is:

\[ \langle T \cdot T \cdot \sigma \cdot \sigma \rangle = 36 \]

The evaluation of the spatial part of \( U_{\Lambda}^{(3)} \) will now be considered. It can be shown that the function \( K(z) \) defined by:

\[ K(z) = K(\frac{z_{1}}{1} - \frac{z_{2}}{2}) = \frac{1}{2} \sum_{i,j} A \frac{p_{i} \cdot x_{1} e^{-i p_{i} \cdot z_{1}}}{p_{i} \cdot z_{2} e^{-i p_{i} \cdot z_{2}}} \]

\[ \times e^{-i p_{j} \cdot z_{1} e^{-i p_{j} \cdot z_{2}}} \]

\[ \times \frac{d r}{p_{i}} \frac{d r}{p_{j}} \frac{d r}{1} \frac{d r}{2} \Lambda \]

\[ = \frac{\rho}{32} D(k_{F} z) \]

Here \( \rho = 0.170 \text{ fm}^{-3} \) (equilibrium density of nucleon matter), \( k_{F} \) is the fermi momentum and
$$D(k z) = \frac{3j(k z)}{1 f} \frac{1 f}{k z} \frac{1 f}{r} = \frac{\sin r}{r} \frac{\cos r}{r}.$$ 

Then $U^{(3)}_\Lambda$ can be written

$$U^{(3)}_\Lambda = U^{(I)}_\Lambda + U^{(II)}_\Lambda + U^{(s)}_\Lambda$$

where $U^{(I)}_\Lambda$ is the contribution from $U^{(I)}_\Lambda$, $U^{(II)}_\Lambda$ originates from $U^{(II)}_\Lambda$ and $U^{(s)}_\Lambda$ from $U^{(s)}$. From eqs. (3.26) to (3.29) it can be seen that:

$$U^{(I)}_\Lambda = - \frac{3}{8} \rho \int D(k |x - y|) Y(x) Y(y) \frac{3}{3} \frac{3}{3}$$

$$U^{(II)}_\Lambda = - \frac{3}{8} \rho \int D(k |x - y|) (3 \cos \theta - 1) \frac{2}{2} \frac{2}{2}$$

$$U^{(s)}_\Lambda = - \frac{3}{8} c_s \rho \int D(k |x - y|) \frac{1}{1} \frac{1}{1}$$

Here $\cos \theta_{xy} = x \cdot y / |xy|$ which is shown in Fig. 2.2, the variables $x$ and $y$ are the $\Lambda$-$N$ distances, and $|x - y|$ is the $N$-$N$ distance. All the integrands, including that in (3.31), are well-behaved even for the limit $x, y \to 0$. An inspection of $U^{(II)}_\Lambda$ in eq. (3.31) now reveals why it is so sensitive to the $N$-$N$ correlation. The function $(3 \cos^2 \theta_{xy} - 1)$ is positive for $\theta_{xy} = 0$ to $55^\circ$ and then turns negative. Small values of $\theta_{xy}$ correspond to the case when the two nucleons are relatively close to each other, and this is the part most affected by the $N$-$N$ correlation function. The potential $U^{(II)}_\Lambda$
is not positive definite, unlike $U_\Lambda (I)$, and tends to turn negative if the N-N correlation is strongly repulsive, thereby excluding smaller values of $\theta_{xy}$. In order to study the effect of N-N correlation, a step-function has been used:

$$\theta(|x - y| - d_{NN}) = 0 \quad \text{for } |x - y| < d_{NN}$$

$$= 1 \quad \text{for } |x - y| > d_{NN}$$

...(3.33)

Here it is not worthwhile to use a more sophisticated correlation function because:

(a) It is not known sufficiently well. In particular, it depends sensitively on the form of the N-N potential chosen, whether it contains a hard-core or a soft-core, and if it is state-dependent.

(b) There is already considerable uncertainty in $W$ due to the $\Lambda$-N cutoff $d_{\Lambda N}$.

The integrals (3.30), (3.31) and (3.32) are evaluated numerically after putting the N-N correlation function $\theta(|x - y| - d_{NN})$ in the integrand. The results are shown in table 3.4 and graphically in Fig 3.3.

Roughly speaking, $d_{NN}$ should be chosen halfway between the N-N hard-core radius (assuming hard-core N-N potential) and the healing distance, which is state-dependent. A reasonable value of $d_{NN}$, on this basis, is about 0.7 fm. From Fig. 3.3 it can be seen that corresponding to this value of $d_{NN}$, $U^{(3)}_\Lambda = 1.8$ Mev for $d_{\Lambda N} = 0.6$ fm and is about 8.7 Mev for $d_{\Lambda N} = 1.0$ fm.
TABLE 3.4

EFFECT OF THE TPE THREE-BODY ANN POTENTIALS IN NUCLEAR MATTER. THE
NOTATIONS ARE EXPLAINED IN THE TEXT. $d_{^{\Lambda}N}$ IS THE CUTTOFF FOR THE ANN
POTENTIAL, WHILE $d_{NN}$ IS THE CUTTOFF FOR A STEP-FUNCTION TYPE N-N CORRELATION.

<table>
<thead>
<tr>
<th>$d_{^{\Lambda}N}$</th>
<th>0.6 fm</th>
<th>1. fm</th>
<th>1. fm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{NN}$</td>
<td>0.4 fm</td>
<td>0.6 fm</td>
<td>1 fm</td>
</tr>
<tr>
<td>$u(s)^{(s)}_{^{\Lambda}(MeV)}$</td>
<td>-0.534</td>
<td>-0.474</td>
<td>-0.290</td>
</tr>
<tr>
<td>$u(I)^{(I)}_{^{\Lambda}(MeV)}$</td>
<td>1.362</td>
<td>1.299</td>
<td>1.057</td>
</tr>
<tr>
<td>$u(II)^{(II)}_{^{\Lambda}(MeV)}$</td>
<td>13.950</td>
<td>4.687</td>
<td>6.049</td>
</tr>
<tr>
<td>$^3_u^{(3)}_{^{\Lambda}(MeV)}$</td>
<td>14.777</td>
<td>5.512</td>
<td>6.082</td>
</tr>
</tbody>
</table>

However, with a slightly larger value of $d_{NN}$, $u_{^{\Lambda}}^{(3)}$ may well turn negative for the case where $d_{^{\Lambda}N}$ is 0.6 fm. A repulsion of 3.535 Mev is obtained for $d_{^{\Lambda}N} = d_{NN} = 1$ fm, which is close to what Nyman (1967) obtained. The main point to emerge, however, is that $u_{^{\Lambda}}^{(3)}$ is extremely sensitive to $d_{^{\Lambda}N}$ as well as the N-N correlation function. If the very long-range part of the ANN potential (for $d_{^{\Lambda}N} > 1$ fm) were being considered then its effect is definitely repulsive and can be as large as 10 Mev. However, it would
be misleading to quote any definite number and therefore, the point of view adopted here, disagrees in spirit to Nyman's work.

Recently Gal (1966, 1967) introduced in his phenomenological analysis of light hypernuclei, a $\Lambda NN$ force of the type $W_p(I)$, which is completely central. Only Gal's $C_p$ was much larger than the $C_p$ used here and was taken to be 17 Mev as compared to 1.43 Mev used here. The contribution to $U_3^\Lambda$ of such a central potential is rather insensitive to $N-N$ correlation, as was already clear from the work of Bodmer and Sampanthar (1962). It behaves very differently from the dominant non-central term $W_p(II)$. For example, with $d_{\Lambda N} = 0.6$ fm and $d_{NN} = 0.7$ fm, its contribution to $U_3^\Lambda$ is 16 Mev, compared to a value of 2 Mev from $W_p(II)$ but for $d_{\Lambda N} = 1$ fm and $d_{NN} = 0.7$ fm it is about 10 Mev, nearly the same as the value found here. By taking a completely central phenomenological $\Lambda NN$ force like Gal's, there is the danger that the repulsive effect may be overestimated in calculating $B_\Lambda$.

Before concluding this chapter, it would be in order to mention that an extensive analysis of the effects of the TPE $\Lambda NN$ force, $W$, in $^{\Lambda C}_{13}$, $^{\Lambda 0}_{17}$, $^{\Lambda Si}_{29}$ and $^{\Lambda Ca}_{41}$ has been done by Friesen and Tomusiak (1968). Their conclusion is very similar to that of the present work in $^5\Lambda He$.

Namely, in all cases considered, they found that the $\Lambda NN$ force results in an appreciable repulsion between the $\Lambda$ and the core nucleus. Of course their conclusion is subject to the ambiguities which have been discussed in the case of $^5\Lambda He$. 
Fig. 3.3 (a)
Fig 3.3 (b)
CHAPTER 4

EVALUATION OF THE INTERMEDIATE RANGE THREE-PION EXCHANGE ANN FORCE

As previously explained in chapter 2 the $\Lambda N$ and $\Lambda NN$ potentials can be worked out in the one-or two-channel formalism. Since only the bound state is considered it is sufficiently accurate to use the one-channel formalism. The diagrams which contribute to the three-pion exchange $\Lambda NN$ force are shown in Fig. 4.1c and 4.1d. Diagrams 4.1c and 4.1d, equally important, are as yet too complicated to be evaluated exactly. Therefore the contribution of diagram 4.1d is approximated by the sum of the contributions of diagrams 4.1a and 4.1b. The force arising from diagrams 4.1a and 4.1b is derived in the static approximation. It was stated by Uehara (1960) that the potential due to Fig. 4.1c has the asymptotic tail of the OPEP. This statement was incorrect and therefore this potential is written down and the correct estimate of the range in the asymptotic region is given.

First the contributions arising from diagrams 4.1a and 4.1b are considered. The contribution from the diagram of Fig. 4.2 has to be subtracted once since it is included in both 4.1a and 4.1b. The $S$ matrix element is given by:

$$s = s_{1a} + s_{1b} - s_{2} + \tilde{s}_{1a} + \tilde{s}_{1b} - \tilde{s}_{2} ...(4.1)$$

where the suffix refers to the diagram number and the tilde indicates that nucleons 1 and 2 have been interchanged. Following Miyazawa (1956,1957) the $S$ matrix element corresponding to the diagram 4.1a can be written
\((c = \gamma = 1)\) : 

\[
S = - \frac{1}{(2\pi)^4} \sum_{\alpha, \beta, \gamma} \int \frac{4}{d} \frac{4}{d} \frac{4}{d} \frac{4}{d} \delta(k) \delta(p) \delta(q) \delta(k) \delta(p - \omega) \delta(q - \omega) \delta(k - \omega)
\]

\[
\frac{1}{2} \frac{2}{2} \frac{2}{2} \frac{2}{2} \frac{2}{2} \frac{2}{2}
\]

\[
\frac{<\gamma, -k|S|\alpha, -p> \Gamma \sigma \cdot k}{\gamma} \frac{1}{k - p + i\epsilon}
\]

\[
\frac{\exp \left[ i (q \cdot x - k \cdot x) \right]}{p \times x}
\]

The subscripts \(\alpha, \beta\) and \(\gamma\) refer to the index of the third component of the isotopic spin, \(f_N\) is the pseudo-vector \(\pi N\) coupling constant \((f_N^2 = 0.08)\), \(p\), \(q\) and \(k\) (which have fourth components \(p_0\), \(q_0\) and \(k_0\)) are the four momenta of the exchanged pions whose energies have been labelled \(\omega_p\), \(\omega_q\) and \(\omega_k\) respectively. The vectors \(r_1\) and \(r_2\) are the coordinates of the two nucleons and \(\mu\) is the pion mass. The matrix elements for the \(\pi - N\) and \(\pi - \Lambda\) scattering parts are given by (Nogami and Bloore, 1964):
\[ x \exp \left[ i \left( \frac{p - q}{\Lambda} \right) \cdot r \right] v \cdot v , \quad \cdots \tag{4.4} \]

where \( r \) is the coordinate of \( \Lambda \). The function \( v \) is a cutoff factor which is chosen to be \( \exp (-p^2/2p_m^2) \) where \( p_m \) is the momentum corresponding to the nucleon mass; it is introduced for the convenience of computation (Chew and Low, 1956; Nogami and Bloore 1964). The functions \( A_N, B_N, C_N, A_\Lambda \) and \( C_\Lambda \) are given in terms of the \( \pi N \) and \( \pi \Lambda \) scattering cross sections by the dispersion relations in the static approximation:

\[
A_N (p) = C_N (-p) = \frac{4\pi \left( f_N / \mu \right)}{2 \sqrt{2} p - i\epsilon} + \frac{1}{4\pi} \int_0^\infty \frac{\sigma (k)}{\omega} \frac{dk}{k} - \frac{1}{2} \left( \frac{\omega + p - i\epsilon}{k o} \right), \quad \cdots \tag{4.5a} \]

\[
B_N (p) = B_N (-p) = \frac{1}{12\pi} \left( \frac{\omega + p - i\epsilon}{k o} \right) \int_0^\infty \frac{\sigma (k) + 2\sigma (k)}{\omega - p - i\epsilon} \frac{dk}{k}, \quad \cdots \tag{4.5b} \]

\[
A_\Lambda (p) = C_\Lambda (-p) = \frac{4\pi \left( f_\Lambda / \mu \right)}{2 \sqrt{2} \Delta + p - i\epsilon} + \frac{1}{2\pi} \int_0^\infty \frac{\sigma (k)}{\omega} \frac{dk}{k} - \frac{1}{2} \left( \frac{\omega + p - i\epsilon}{k o} \right), \quad \cdots \tag{4.5c} \]
Here $f_{\Sigma}$ is the renormalized $\pi$-$\Sigma$ coupling constant, $\Delta$ is the mass difference between $\Sigma$ and $\Lambda$, $\sigma_{\text{2I}, J}$ is the total cross section of the $p$-wave $\pi$-$\Sigma$ scattering in the state $(I, J)$ and $\sigma_{2J}$ is the total cross section of the $p$ wave $\pi$-$\Lambda$ scattering in the state with angular momentum $J$ ($\frac{1}{2}$ or $\frac{3}{2}$).

If expressions (4.3) and (4.4) are substituted in eq. (4.2) $S_{1a}$ is obtained in the form $S_{1a} = 2\pi i \delta(0) V_a$. The quantity $V_a$ which is interpreted as the $\Lambda NN$ potential arising from diagram 4.1a is given by:

\[
V_a = \frac{i \frac{4\pi f_N}{(2\pi)^2 \mu}}{10} \int_0^\infty dk \int_0^\infty dp \int_0^\infty dq \int_0^\infty dk \frac{2\sigma(k) + \sigma_2(k)}{\omega + p - i\epsilon}
\]

\[
\cdots (4.6)
\]

where

\[
P_a = \{ \Lambda \} \quad \delta_{\alpha, \beta, \gamma} \quad \delta_{\tau, \tau'} \quad \sigma \cdot q \cdot \bar{q} \cdot \gamma \quad <\gamma, -k \mid T(p) \mid \alpha, -p> \quad \frac{2}{2} \quad \frac{2}{2} \quad \alpha \cdot \bar{q} \cdot k\text{.}
\]

\[
\cdots (4.8)
\]

with eq. (4.3)

\[
<\gamma, -k \mid T(p) \mid \alpha, -p> = 2\pi i \delta(p - k) \exp[-i(p - k) \cdot x] V V
\]

\[
x <\gamma, -k \mid T(p) \mid \alpha, -p> \text{.}
\]

\[
\cdots (4.9)
\]
and from eq. \((4.4)\)

\[
\{\Lambda\} = A \Lambda (p_o) \sum_{\sigma} p \sum_{\alpha} q + C(p_o) \sum_{\sigma} q \sum_{\alpha} p.
\]  

... \((4.10)\)

Then,

\[
\rho = \rho' + \rho'',
\]

... \((4.11)\)

with

\[
\rho' = \frac{1}{2} \sum_{\alpha} \sum_{p} \{\Lambda\} \sum_{\alpha} \sum_{p} \sum_{k} \rho_k,
\]

\[
\rho'' = 2 \sum_{\alpha} \sum_{p} \{\Lambda\} \sum_{\alpha} \sum_{p} \sum_{k} \rho_k
\]

... \((4.12)\)

Similar expressions are obtained for the potential \(V\) arising from diagram \(a\). The potential \(V (S = -2m \delta(0) V_2)\) could be obtained by replacing

\[
<\gamma, -k | T_1 (p_o) | \alpha, -p> \text{by} \left(\frac{1}{2} \pi \mu^2\right) \frac{1}{N} \frac{1}{r} \frac{1}{\gamma} \sum_{\alpha} \sum_{\mu} \sum_{k} \frac{1}{p_o - i}\]

expression for \(V_a\) \((4.7)\). However, it is simpler to notice that diagram \(4.2\) results only in terms proportional to \(I_{N}^4\). Therefore \(V_a + V_b - V_2\) can be replaced by \(V = V_a + V_b\) with terms proportional to \(I_{N}^4\) divided by two.

The part of \(V_a\) corresponding to \(P', \rho',\) can be written using eqs. \((4.7)\) and \((4.12)\),
\[ V^\prime = U(x) \cdot V(z), \quad \ldots (4.14) \]

with

\[
V(z) = \frac{4\pi \frac{1}{2} \frac{1}{2} f^N}{(2\pi)^2} \int \frac{3 \frac{1}{2} \frac{1}{2} \cdot \sigma \cdot k \cdot \sigma \cdot k \cdot \exp(ik \cdot z)}{2} \approx \frac{2}{k} \omega
\]

\[
= \frac{4\pi \frac{1}{2} \frac{1}{2} f^N}{(2\pi)^2} \frac{1}{3} \frac{1}{2} \frac{1}{2} \cdot \sigma \cdot \nabla_z \cdot \sigma \cdot \nabla_z \int \frac{3 \exp(ik \cdot z)}{2} \omega \]

\[ V(z) \text{ is simply the OPEP for the N-N potential:} \]

\[
V(z) = \frac{4\pi \frac{1}{2} \frac{1}{2} f^N}{3} \int \frac{2 \cdot \sigma \cdot \sigma + T(z) \cdot S(z)}{N} Y(z) \quad \ldots (4.15)
\]

where \( S_{12}(z), T(z) \) and \( Y(z) \) have been defined in chapter 2 (eqs. (2.11) and (2.12)).

Then,

\[
U(x) = \frac{1}{(2\pi)^4} \int 3 \int 3 \int 3 \frac{1}{P_1^N \sigma \cdot q \cdot \sigma \cdot p \cdot \exp \left[ i(q - p) \cdot x \right]} \quad \ldots (4.16)
\]

\[
(p - \omega) (p - \omega) (p - i\epsilon)
\]

That is,

\[
U(x) = \left( \sigma^\Lambda \cdot \nabla_{x_1} \cdot \sigma^\Lambda \cdot \nabla_x \cdot \sigma^\Lambda \cdot \nabla_x \cdot \sigma^\Lambda \cdot \nabla_x \cdot \sigma^\Lambda \cdot \nabla_x \cdot \sigma^\Lambda \cdot \nabla_x \cdot \sigma^\Lambda \cdot \nabla_x \cdot \sigma^\Lambda \cdot \nabla_x \right) \cdot \left( \sigma^\Lambda \cdot \nabla_{x_1} \cdot \sigma^\Lambda \cdot \nabla_x \cdot \sigma^\Lambda \cdot \nabla_x \cdot \sigma^\Lambda \cdot \nabla_x \cdot \sigma^\Lambda \cdot \nabla_x \right)
\]
After reduction of the differential operators, the potential \( U(x) \) can be expressed as a sum of central (spin-independent and spin-dependent terms) and tensor parts:

\[
U(x) = V(x) + V(x) \sigma_{s}^{\Lambda} \sigma_{t}^{1} + V(x) S(x)
\]  \[(4.19)\]

where

\[
V_{o}(x) = \left[ \left( \frac{2}{3} + \frac{\partial}{\partial r} \frac{\partial}{\partial r'} \right) \frac{\partial}{\partial r} \left( \frac{2}{3} + \frac{\partial}{\partial r} \frac{\partial}{\partial r'} \right) \right] I \left[ (A_{\Lambda} + C_{\Lambda}) P_{N}^{1} \right]_{N} \quad r = r' = x
\]  \[(4.20a)\]

\[
V_{s}(x) = \left[ \frac{2}{3} \left( \frac{1}{x} + \frac{\partial}{\partial r} + \frac{\partial}{\partial r'} \right) \frac{\partial}{\partial r} \right] I \left[ (A_{\Lambda} - C_{\Lambda}) P_{N}^{1} \right]_{N} \quad r = r' = x
\]  \[(4.20b)\]
and

$$V' = V(z) U(x)$$

Thus $V' = V'_a + V'_b = \{ U(x), V(z) \}$, where $\{ \, , \}$ indicates the anticommutator and

$$V' = \frac{1}{T_a \cdot T_b} \frac{2}{N} \left\{ (V(x) + V(x) \sigma \cdot \sigma + V(x) s(x)) \right\} + (\sigma \cdot \sigma + T(z) s(z)) \right\} Y(z).$$

That is,

$$V' = \frac{2}{3} \frac{1}{T_a \cdot T_b} \mu^2 \left[ V(x) \sigma \cdot \sigma + V(x) \sigma \cdot \sigma + V(x) (1 - T(z)) s(x) \right] + (V(x) + V(x) - V(x)) T(z) s(z) + V(x) T(z) \Sigma (x, z) \right]\ Y(z),$$

$\cdots (4.23)$
where

\[ \Sigma_{\Lambda} (x, z) = g_{x, z} \left( \frac{\Lambda \cdot x \cdot z^2}{x^2 z^2} \right) - \frac{\Lambda^2}{x \cdot z} \cdots (4.24) \]

The part of \( \bar{V} \) corresponding to \( \bar{P}_{a^2} \), \( \bar{V}_{a^2} \), can be written using eqs. (4.7) and (4.13)

\[ \bar{V}' = 2 \sum_{a} \frac{1}{r \cdot r^*} \frac{N}{\mu} \left( \frac{\Lambda \cdot \nabla \cdot z \cdot \nabla \cdot z^2}{r \cdot r^*} \right) \cdot I(\Lambda \cdot \bar{P}_{a^2}) \]

\[ + \frac{\Lambda}{\mu} \cdot \nabla \cdot z \cdot \nabla \cdot z^2 \cdot I(\Lambda \cdot \bar{P}_{a^2} \cdot z) \cdot \bar{Y}(z) \cdot r = r' = x \cdots (4.25a) \]

Similarly

\[ \bar{V}' = 2 \sum_{b} \frac{1}{r \cdot r^*} \frac{N}{\mu} \left( \frac{\Lambda \cdot \nabla \cdot z \cdot \nabla \cdot z^2}{r \cdot r^*} \right) \cdot I(\Lambda \cdot \bar{P}_{b^2}) \]

\[ + \frac{\Lambda}{\mu} \cdot \nabla \cdot z \cdot \nabla \cdot z^2 \cdot I(\Lambda \cdot \bar{P}_{b^2} \cdot z) \cdot \bar{Y}(z) \cdot r = r' = x \cdots (4.25b) \]

Because \( I(f(p), r, r') \) is a symmetric function of \( r \) and \( r' \) eq. (4.18), \( r \)

and \( r' \) can be exchanged in eq. (4.25a) and then adding eq. (4.25b) gives

\[ \bar{V}' = \bar{V}_{a^2} + \bar{V}_{b^2} = \sum_{a} \frac{1}{r \cdot r^*} \frac{N}{\mu} \left( \frac{\nabla \cdot z \cdot \nabla \cdot z^2}{r \cdot r^*} \right) \cdot I(\Lambda \cdot \bar{P}_{a^2} \cdot z) \cdot \bar{Y}(z), \]

\[ r = r' = x \cdots (4.26) \]
that is

\[ V'' = -\frac{4}{9} \frac{\mathbf{r} \cdot \mathbf{r}}{\mathbf{r} \cdot \mathbf{r}} \mu \mathbf{r} N \left\{ \begin{array}{c}
2 \left( \frac{1}{2} + \frac{2}{\mathbf{r} \cdot \mathbf{r}} \right) \mathbf{\sigma} \cdot \mathbf{\sigma} + \frac{3}{2} T(z) S(z) \\
+ 3 \left( \frac{1}{\mu z} + \frac{1}{\mu z} \right) \left( \frac{1}{2} - \frac{2}{\mathbf{r} \cdot \mathbf{r}} \right) S(x) - T(z) \left( \frac{1}{2} - \frac{2}{\mathbf{r} \cdot \mathbf{r}} \right) \Sigma(x, z) \\
\end{array} \right. \]

\[ \frac{2}{\mathbf{r} \cdot \mathbf{r}} \frac{d}{d\mathbf{r} \cdot \mathbf{r}'} \left[ \left( \mathbf{A} + \mathbf{C}_\Lambda \right) P'' \right]_N \left( x, z \right) . \]

\[ r = r' = x \]

\[ \ldots (4.27) \]

In the following sections these potentials will be considered in the cases where the nucleons and \( \Lambda \) are in the s-shell.

Therefore, with \( i, j = 1, 2, \Lambda \)

\[ \langle \mathbf{r} \cdot \mathbf{r} \mathbf{\sigma} \cdot \mathbf{\sigma} \mathbf{z} \rangle = \frac{1}{3} x \mathbf{z} \left( \langle \mathbf{r} \cdot \mathbf{r} \mathbf{\sigma} \cdot \mathbf{\sigma} \mathbf{z} \rangle \right), \]

\[ \ldots (4.28a) \]

\[ \langle \mathbf{r} \cdot \mathbf{r} \mathbf{\Sigma} (x, z) \rangle = (3 \cos \theta - 1) \langle \mathbf{r} \mathbf{\sigma} \mathbf{\sigma} \mathbf{z} \rangle, \]

\[ \ldots (4.28b) \]

\[ \langle \mathbf{r} \cdot \mathbf{r} \mathbf{S} (x) \rangle = 0, \]

\[ \ldots (4.28c) \]

where \( \cos \theta = x \mathbf{z} / x \mathbf{z} \) as shown in Fig. 4.3. Now \( V = V' + V'' \) can be written as :
\[ V(x, z) = \left[ S(x) + (3 \cos \frac{\theta}{xz} - 1) T(z) W(x) \right] Y(z) \quad \ldots (4.29) \]

with

\[ S(x) = - \frac{4}{3} \mu^2 \left[ \frac{1}{x^2} \sigma \cdot \sigma \right] \left( \frac{1}{2} + \frac{\partial^2}{\partial r^2} \right) \frac{1}{2} \frac{\partial^2}{\partial r^2} Z(r, r') \]

\[ + \frac{\Lambda}{2} \frac{1}{2} \frac{\partial^2}{\partial r^2} \frac{1}{x} \frac{\partial^2}{\partial r \partial r'} D(r, r') \]

\[ W(x) = - \frac{4}{3} \mu^2 \left[ \frac{1}{x^2} \sigma \cdot \sigma \right] \left( \frac{1}{2} + \frac{\partial^2}{\partial r^2} \right) \frac{1}{2} \frac{\partial^2}{\partial r^2} x(r, r') \]

\[ + \frac{\Lambda}{2} \frac{1}{2} \frac{\partial^2}{\partial r^2} \frac{1}{x} \frac{\partial^2}{\partial r \partial r'} D(r, r') \]

\[ r = r' = x \quad \ldots (4.30) \]

where

\[ Z(r, r') = I \left[ (\Lambda + C) \left( \frac{3p^1 + 2x^1}{N} \right) \right] = I \left[ (\Lambda + C) \left( \frac{A - 6B + 9C}{N} \right) \right] \]

\[ D(r, r') = I \left[ \left( \Lambda - C \right) \frac{p^1}{N} \right] = I \left[ \left( \Lambda - C \right) \left( \frac{A - 4B + 3C}{N} \right) \right] \]

\[ \ldots (4.32a) \]

\[ \ldots (4.32b) \]
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and

\[ X(r, r') = \frac{1}{2} \left[ (A + C_N) P_N \right] = \frac{1}{2} \left[ (A + C_N) \right] \left( -A + 3B \right) N_N \]

\[ \ldots (4.32c) \]

In order to evaluate these expressions from eqs. (4.5), (4.6) and (4.18), it was assumed that the \( \pi N \) and \( \pi \Lambda \) scattering were dominated by the \( N^* \) (1238) and the \( Y^* \) (1385) resonances respectively, that is \( \sigma^l \), \( \sigma \) and \( \sigma_1 \) were set equal to zero in expressions (4.5) and (4.6). Also the integrals in (4.5) and (4.6) were evaluated replacing \( \sigma(k) \) by

\[ 12\pi \frac{g^2}{k} \delta(\omega_k - \omega_x) \text{ with } g^2 = \Gamma(k)/2k^3 \text{, } \Gamma(k) \text{ being the width at the resonance energy \text{ (Fubini 1956). The values } g^2_N = 0.057, \omega_N = 1.27 \text{ fm}^{-1} \text{ for } \sigma_3^3(k) \text{ and } g^2_{\Lambda} = 0.074 \text{ with } \omega_{\Lambda} = 1.24 \text{ fm}^{-1} \text{ for } \sigma_3^3(k) \text{ were used.} \]

The diagram shown in Fig. 4.4, which is included in diagrams 4.1a and 4.1b does not contribute to the three-body force. Therefore its contribution, which is proportional to \( 1/(p_0 + i\epsilon) \) is subtracted by suppressing the term

\[ - \frac{4\pi \pi^2}{N_N^*} (p_0 + i\epsilon) \text{ in } C_N(p_0) \text{ (4.5a). Now the expressions (4.32)} \]

can be calculated, and their explicit forms are given in the appendix 2.

Then the three-pion exchange potential corresponding to the S matrix (4.1) is:

\[ P(x, y, z) = V(x, z) + \tilde{V}(y, z) \]

It can be seen that

\[ P(x, y, z) = P_c(x, y, z) + P_T(x, y, z) \]

\[ \ldots (4.33) \]

with

\[ P_c(x, y, z) = (S(x) + \tilde{S}(y)) Y(z) \]

\[ \ldots (4.34) \]

and

\[ P_T(x, y, z) = \left[ (3\cos^2 \theta_{xz} - 1) W(x) + (3\cos^2 \theta_{yz} - 1) W(y) \right] T(z) Y(z) \]

\[ \ldots (4.35) \]
In the above equations the tilde indicates that superscript 1 and 2 in equations (4.31) are interchanged.

The contribution arising from diagram 4.1c is now considered.

Following Miyazawa (1956, 1957) the S matrix corresponding to this diagram can be written

\[ S_c = \left( \frac{1}{(2\pi)^3} \right) \sum_{\alpha, \beta, \gamma} \int \frac{4}{dp} \frac{4}{dq} \frac{4}{dk} \frac{<\beta, q | S_\Lambda | \alpha, p>}{(p - \omega)} \frac{<\alpha, p | S_1 | \gamma, k>}{(q - \omega)} \]

The notation is the same as that used in eq. (4.2). \( S_c \) is obtained in the form \( S_c = -2\pi i \delta(0) V_c \), which is interpreted as the potential arising from diagram 4.1c, can be written

\[ V = \left( \frac{1}{(2\pi)^3} \right) \sum_{\alpha, \beta, \gamma} \int \frac{3}{dp} \frac{3}{dq} \frac{3}{dk} \frac{<\beta, q | T_\Lambda (p) | \alpha, p>}{(p - \omega)} \frac{<\alpha, p | T (p) | \gamma, k>}{(p - \omega)} \]

As in Uehara (1960), only the Born terms are considered; that is

\[ V = \left( \frac{1}{6} \right) \left[ \delta - \left( \sum_{\alpha, \beta, \gamma} \frac{1}{N} \sum_{x} \sum_{y} \sum_{z} \sigma^x_{\alpha \beta} \sigma^y_{\gamma} \sigma^z_{\Lambda} \right) \right] f f f \]

\[ \frac{<\beta, q | T_2(p_0) | \gamma, k>}{(p - \omega) (p - \omega) (p - \omega)} \exp\left( -iq \cdot y + ip \cdot x - ik \cdot z \right) \]

\[ \text{for } \sigma^x, \sigma^y, \sigma^z \text{ and } \Lambda \text{ acting on } x, y, z \text{ and } \Lambda \text{ respectively.} \]
\[ + \sigma \cdot \nabla \sigma \cdot \nabla \frac{1}{x} \frac{1}{x} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{2}{2} \frac{2}{y} \left( x, y, z \right) I - \left( (x, y, z) - Y \right) \Delta, 0, 0 \]

\[ - \left( (x, y, z) - Y \right) \Delta, 0, 0 \]

\[ I (x,y,z) = \frac{3}{\pi x y z} \int_{0}^{\infty} \frac{\sin \left[ p (x + y + z) \right]}{\omega p} \frac{\exp \left[ \frac{2}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right]}{(x + y + z)} \frac{2}{(\omega + \alpha) (\omega + \beta) (\omega + \gamma)} \]  

\[ \alpha, \beta, \gamma \]

\[ I (x,y,z) = \frac{3}{\pi x y z} \int_{0}^{\infty} \frac{\sin \left[ p (x + y + z) \right]}{\omega p} \frac{\exp \left[ \frac{2}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right]}{(x + y + z)} \frac{2}{(\omega + \alpha) (\omega + \beta) (\omega + \gamma)} \]  

\[ \alpha, \beta, \gamma \]

\[ Y (x,y,z) = \frac{\exp \left( \frac{2}{(\omega + \alpha) (\omega + \beta) (\omega + \gamma)} \right)}{(x + y + z)} \]  

\[ \alpha, \beta, \gamma \]

If the mass difference \( \Delta \) is neglected in (4.38) and if the repetition term corresponding to \( Y (x,y,z) \) is subtracted, \( V_c \) can be written:

\[ V_c = - \frac{4}{\mu^6} \left( 6 - \left( \frac{1}{x} \frac{2}{2} \frac{2}{2} \right) \right) x y z \frac{2}{(\omega + \alpha) (\omega + \beta) (\omega + \gamma)} \]  

\[ \alpha, \beta, \gamma \]
\begin{align*}
\frac{\nabla \cdot \nabla}{x z} + \frac{\nabla \cdot \nabla}{y x} + \frac{1}{z y} + (x, y, z) \quad \cdots(4.40)
\end{align*}

with
\begin{align*}
I_{0,0,0} = \frac{3}{4 \mu} \frac{x + y + z}{x y z} \exp \left[ - \mu (x + y + z) \right] \quad \cdots(4.41)
\end{align*}

Therefore the asymptotic form of $V_c$ is proportional to $I_{0,0,0}$. If the equilateral triangle $x = y = z = r$ is considered, the asymptotic form is proportional to $\exp\left(-3\mu r/\mu r^2\right)$ which has the range expected for the exchange of three-pions. There is no term with the asymptotic tail of the OPEP. In Uehara (1960) the function $I_{\alpha, \beta, \gamma}(x, y, z)$ was miscalculated. The poles of the equation (4.37) have to be treated carefully. After the $p_0$ integration, $V_c(4.37)$ is proportional to $J$, where

\begin{align*}
J = \int \int \int \limits_0^{\infty} dp \, dq \, dk \, p \sin px \, q \sin qy \, k \sin kz \left( \frac{1}{\omega (\omega - \omega^1) (\omega - \omega^2)} \right) + \text{cyclic permutation of } p, q \text{ and } k.
\end{align*}

The integral $J$ was evaluated by assigning small, unequal imaginary masses to the mesons; that is $\omega \rightarrow (p^2 + \mu^2 + 1\epsilon)^{1/2}$ etc. This leads to eq.† (4.39a). If poles are left on the real axis incorrect asymptotic behavior results.

† This result agrees with that of a similar integral calculated by S.D. Drell and K. Huang, Phys. Rev. 91 (1953), 1527.
Fig. 4.2

Fig. 4.3

Fig. 4.4
CHAPTER 5

EFFECTS OF THE THREE-PION-EXCHANGE ANN FORCE

5.1 Effect in \(^3\Lambda\)

In a calculation similar to that of the TPE ANN force which has been done in § 3.1, the effects of the potentials \(P_c(4.34)\) and \(P_T(4.35)\) on \(^3\Lambda\) are estimated by perturbation theory.

First the dependence of the three-pion exchange potential \(P(4.33)\) on the \(\Lambda-N\) distance, \(x\), is compared with the \(x\)-dependence of the TPE ANN force, \(W_p\). For \(^3\Lambda\) it has been shown that \(W_p\) has the form (2.16 and 3.4):

\[
W_p = C_p \left[ 1 + \left( 3 \cos \theta_{xy} - 1 \right) T(x) T(y) \right] Y(x) Y(y) \quad \ldots (5.1)
\]

with \(C_p = 1.43\) Mev and \(\cos \theta_{xy} = (x \cdot y) / xy\) as shown on Fig. 2.1. The functions \(Y\) and \(T\) have been defined in chapter 2, eqs. (2.12). The \(x\)-dependences of the central part and of the tensor part of \(W_p\) are denoted by \(C_2\) and \(T_2\) respectively, where from eq. (5.1):

\[
C_2 = C_p Y(x), \quad \ldots (5.2)
\]

\[
T_2 = C_p Y(x) T(x). \quad \ldots (5.3)
\]

Similarly the \(x\)-dependences of \(P_c(4.34)\) and \(P_T(4.35)\) respectively can be written:

\[
C_3 = S(x), \quad \ldots (5.4)
\]

\[
T_3 = W(x). \quad \ldots (5.5)
\]

where \(S(x)\) and \(W(x)\) are given by the expressions (4.30) and (4.31) with the corresponding expectation values of the \(\tau's\) and \(\sigma's\) replaced by their
values for $H$:

$$\begin{align*}
\langle \sigma_{\alpha} \cdot \sigma_{\beta} T_{\alpha} T_{\beta} \rangle &= -3, \\
\langle \sigma_{\alpha} \cdot \sigma_{\alpha} T_{\alpha} T_{\alpha} \rangle &= 6 \quad i = 1, 2.
\end{align*}$$

...(5.6)

The $x$-dependences of the central parts and tensor parts are plotted in Fig. 5.1 and Fig. 5.2 respectively.

It can be seen that $c_3$ depends strongly on $x$ and is larger than $c_2$ for $x \lesssim 1.2$ fm. The curve $T_3$ is also sensitive to $x$, but is of comparable strength to $T_2$, which becomes larger than $T_3$ only for $x \gtrsim 7$ fm.

Therefore for $d_{AN} \lesssim 1$ fm the central part $P_c$ is expected to give a negative contribution to $D_{\Lambda}(\text{repulsive})$ larger than that of the TPE $\Lambda\Lambda$ force. The contribution of the tensor part, $P_T$, will depend on the average of the angular dependence $''3 \cos \theta_{xz} - 1''$.

The effect of the three-pion exchange potential $P(\Lambda, 3.35)$ on $D_{\Lambda}$ will now be considered. The notation is the same as that used in § 3.1. The expectation values of $P_c(x, y, z)$ (4.34) and $P_T(x, y, z)$ (4.35) with respect to $W(3.1)$ are denoted by $\langle P_c \rangle$ and $\langle P_T \rangle$ respectively. Because of the complete symmetry of all the expressions in $x$ and $y$, $\langle P_c \rangle$ and $\langle P_T \rangle$ can be written:

$$\langle P_c \rangle = 2 I \left[ S(x) Y(z) \right],$$

...(5.8)

$$\langle P_T \rangle = 2 I \left[ (3 \cos \theta_{xz} - 1) W(x) T(z) Y(z) \right],$$

...(5.9)

where $I [\ldots]$ has been defined by eq. (3.9). The integration domain has also been explained in § 3.1.

The results are shown in table 5.1 for five different values of the cutoff $d_{AN}$ and three different values of the hard core radius $D$ together with the results obtained for the central part $\langle W_p (I) \rangle$ and tensor part $\langle W_p (II) \rangle$ of
the TPE ANN force (§ 3.1). It can be seen that for \( d_{\Lambda N} \leq 1 \) fm the central part \( <P_c> \) is always positive (repulsive) and larger than \( <W_p(I)> \), that the tensor part \( <P_T> \) is always negative (attractive) and larger than \( <W_p(II)> \) and that \( <P_T> \) is predominant. For \( d_{\Lambda N} \leq 1 \) fm the total effect \( <P_c + P_T + W_p> \) is always negative (attractive), for instance for \( d_{\Lambda N} = 1 \) fm and \( D = 0.4 \) fm, \( <W_p> = 0.164 \) Mev and \( <W_p + P_c + P_T> = -0.132 \) Mev. As in § 3.1 (eqs. 3.10 and 3.11), the relative importance of the \( \Lambda NN \) force will be given by the ratio:

\[
R = \frac{<P_c + P_T>}{2 <V_{\Lambda N}>} = \begin{cases} 8.5 \% & \text{for } D = d_{\Lambda N} = 0.4 \text{ fm} \\ 1.5 \% & \text{for } D = d_{\Lambda N} = 0.6 \text{ fm} \end{cases} \quad \ldots (5.10)
\]

This small ratio justifies the use of the perturbation calculation of the effect of the \( \Lambda NN \) force. For the case of the TPE \( \Lambda NN \) force the corresponding ratios were found to be 2.6\% and 1.5\% respectively. Thus although the previous TPE results are modified, the overall effect is still small and can probably be taken into account by a suitable modification of the \( \Lambda N \) force.

5.2 Effect in \( ^5_\Lambda He \)

The \( \Lambda -\alpha \) potential in \( ^5_\Lambda He \) due to three-pion-exchange potential \( P, P(x) \) is estimated in a way similar to that used for the TPE \( \Lambda NN \) force (§ 3.2). First as in the previous section, the \( x \)-dependence of \( P(4,33) \) in \( ^5_\Lambda He \) is compared with that of the TPE \( \Lambda NN \) force. The \( x \)-dependence of \( P_c(4,34) \) and \( P_T(4,35) \) respectively can be written:

\[
C_3' = S'(x), \quad \ldots (5.11)
\]

\[
T_3' = W'(x), \quad \ldots (5.12)
\]

where \( S'(x) \) and \( W'(x) \) are given by the expressions (4.30) and (4.31) with
the expectation values of the $T'$'s and $\sigma$'s replaced by their values for $^{5}\text{He}$:

$$
\langle \sigma_1 \sigma_2 \rho \text{, } \lambda \rangle = -3, \\
\langle \sigma_i \rho \text{, } \lambda \rangle = 0 \quad i = 1, 2.
$$

Eq. (5.13b) follows from the spin saturation of the nucleons. Because of the difference in the expectations values of the $T'$'s and $\sigma$'s for $^{3}\text{H}$ and $^{5}\text{He}$, $C_3'$ and $T_3'$ differ from $C_3$ and $T_3$, respectively. The curves $C_3'$ and $T_3'$ are plotted in Fig. 5.1 and Fig. 5.2. It can be seen that $C_3'$ depends strongly on $x$, is larger than $C_3$ for $x \lesssim 1.2$ fm but smaller than $C_3$ for $x \gtrsim 1.6$ fm.

The tensor part $T_3'$ is not very sensitive to $x$, becomes negative for $x > 0.4$ fm and is always smaller in absolute value than $T_2$ and $C_3$. Therefore the $\Lambda$-n potential due to $C_3'$ will be positive (as for $C_2$) and larger than that due to $T_3'$.

The average one-body field $P(r_\Lambda)$, that the $\Lambda$ experiences in $^{5}\text{He}$ will now be estimated. As in (§ 3.2) $P(r_\Lambda)$ can be written:

$$
P(r_\Lambda) = 6 \int d^3x_1 d^3x_2 \rho(r_1) \rho(r_2) P(x_1 - x_\Lambda, x_2 - x_\Lambda, x_1 - x_2),
$$

(5.14)

where the factor 6 comes from the six possible $\Lambda NN$ bonds and $\rho(r_\Lambda)$ is the normalized density distribution for the nucleons which has been defined by eq. (3.15). The vectors $(x_1, y, z)$ have been replaced by

$$(x_1 - x_\Lambda, x_2 - x_\Lambda, x_1 - x_2)$$

(see Fig. 4.3). The potential $P(x_1 - x_\Lambda, x_2 - x_\Lambda, x_1 - x_2)$ is obtained from eq. (4.32) using eqs. (5.13a) and (5.13b). In this calculation the $\alpha$ was assumed to be undistorted by the presence of the $\Lambda$.

The potential $P(r_\Lambda)$ can be written:

$$
P(r_\Lambda) = P_c(r_\Lambda) + P_T(r_\Lambda)
$$

(5.15)

where

$$
P_c(r_\Lambda) = 12 \int d^3x d^3z \rho|\tilde{x}_\Lambda \rangle \langle \tilde{x}_\Lambda | \rho|\tilde{x}_\Lambda + \tilde{z}_\Lambda - \tilde{z}_\Lambda | \rangle S'(x) \chi(z)
$$

(5.16)
and

\[ P_T(x_\Lambda) = 12 \int d^3x \, d^3z \, \rho(\mathbf{l} \cdot \mathbf{r}_\Lambda) \, \rho(\mathbf{l} \cdot \mathbf{r}_\Lambda - \mathbf{z} \cdot \mathbf{I}) \, (3 \cos^2 \theta_{xz} - 1) \, W(x) \, T(z) \, Y(z) \]

where \( \cos \theta_{xz} = \mathbf{x} \cdot \mathbf{z} / xz \). The additional factor of two is due to the symmetry of the expressions (4.34) and (4.35) in \( x \) and \( y \).

Equation (5.17) can be reduced to a four-dimensional integration which is done numerically. Similarly the integral (5.18) can be split into four-dimensional and five-dimensional integrals, both of which are done numerically. Details about the reduction of the integrals are given in appendix 3. Since the short range part is not known, the spatial integration on the \( N-N \) distance \( z \) is done with \( z \) varying from a cutoff distance \( d_{NN} \) to infinity. As it will be seen, results are quite insensitive to \( d_{NN} \).

In fact the integrals (5.16) and (5.17) are even convergent for \( z \) starting from zero.

The results for five different values of \( x_\Lambda \), two different values of the cutoff \( d_{\Lambda N} \), and two different values of \( d_{NN} \) are shown in table 5.2. It can be seen that the results are sensitive to \( d_{\Lambda N} \) but not to \( d_{NN} \), and that the potential due to the central part is repulsive and larger than that due to the tensor part which is mostly attractive and small (for \( x_\Lambda \approx 2 \text{ fm} \), \( P_T(x_\Lambda) \) is repulsive). The potentials \( P_c(x_\Lambda) \) and \( P_T(x_\Lambda) \) are plotted on Fig. 5.3 for \( d_{\Lambda N} = 1 \text{ fm} \) and \( d_{NN} = 0.6 \text{ fm} \) and compared with the \( \Lambda-N \) potential, \( U^3_p(\text{I}) \) (central) and \( U^3_p(\text{II}) \) (tensor), of the TPE \( \Lambda NN \) force. It can be seen that \( P(x_\Lambda) \) is smaller than \( U^3_p(x_\Lambda) \) \( ( = U^3_p(\text{I}) + U^3_p(\text{II})) \) and then reduces the overbinding of the \( \Lambda \) in \( ^5 \text{He} \) by an amount smaller than the reduction obtained using \( U^3_p(x_\Lambda) \). Therefore in the case of \( ^5 \text{He} \), for \( d_{\Lambda N} \approx 1 \text{ fm} \), the three-pion-exchange \( \Lambda NN \) force only slightly modifies the
results obtained using the TPE $\Lambda NN$ force.

5.3 Effect in Nuclear Matter

As in (§ 3.3) the binding energy $P_\Lambda$ of a $\Lambda$ particle in nuclear matter due to $P_\Lambda$ is calculated by first order perturbation theory in a way similar to that formulated by Bodmer and Sampanthar (1962). Because two nucleons are involved there is a direct term and an exchange term. The direct term vanishes identically because of the spin isospin saturation. As in the case of $^5$He

$$<g_\Lambda . g_\Lambda > = i f_\Lambda \cdot f_\Lambda > = 0 .$$

Therefore the comparison between the $x$-dependence of the three-pion-exchange potential and the two-pion one is essentially the same as that discussed in the previous section. The only difference is a common multiplicative factor arising from the fact that

$$<g_\Lambda . g_\Lambda > = 36 . \quad \cdots (5.18)$$

Therefore the contribution of the central part $P_c(4,3\frac{1}{2})$ to the binding energy, $P_\Lambda c$, is again expected to be predominant.

$P_\Lambda$ can be written

$$P_\Lambda = P_\Lambda c + P_\Lambda T \quad \cdots (5.19)$$

where $P_\Lambda T$ is the contribution from the tensor part $P_T(4,3\frac{1}{2})$.

A straightforward calculation yields

$$P_\Lambda c = \frac{3}{4} p^2 \int \frac{d^3 x}{f} \frac{d^3 y}{f} S(x) X(|x - y|) \cdots (5.20)$$
\[ P_{\Lambda T} = \frac{3}{4} \rho \int_{D} \int_{D} (k \cdot (x - y)) U(x)(3\cos \theta - 1)Y(|x - y|)^{3} \rho d\mathbf{x} d\mathbf{y} \]

\[ \rho, D, k \text{ have been defined in } \S 3.3 \text{ and } \cos \theta = \frac{x \cdot z}{x z} \text{. As in the case of the TPE } \Lambda NN \text{ force a step function, (eq. 3.3), is used as a nucleon-nucleon correlation function.} \]

The integrals (5.20) and (5.21) are then evaluated numerically. The results are shown in table 5.3 for two different values of \( d_{\Lambda N} \) and four different values of \( d_{NN} \), together with the results obtained for the central part \( U_{\Lambda} (I) \) and tensor part \( U_{\Lambda} (II) \) of the TPE \( \Lambda NN \) force ( \( \S 3.3 \)). It can be seen that the effect is repulsive and large, that \( P_{\Lambda n} \) is larger than \( P_{\Lambda T} \), \( U_{\Lambda} (I) \) and \( U_{\Lambda} (II) \) and that for \( d_{NN} \approx 1 \text{ fm} \), \( P_{\Lambda NN} \) is smaller than \( U_{\Lambda} (II) \). The main qualitative feature of the effect is that it is found to be repulsive, for instance with a reasonable value for \( d_{NN} \) of about 1 fm and that for \( d_{NN} = 1 \text{ fm} \), \( P \) gives a repulsive contribution of 7.43 Mev. For the same values of \( d_{NN} \) and \( d_{NN} \), the TPE \( \Lambda NN \) force was found to give a repulsive contribution of 3.78 Mev. For \( d_{NN} \approx 1 \text{ fm} \), the three-pion-exchange three-body \( \Lambda NN \) force reduces therefore the binding energy of a \( \Lambda \) in nuclear matter by an amount greater than that obtained with the TPE \( \Lambda NN \) force. However the large values of the results obtained in table 5.3 may indicate that the use of a first order perturbation theory is not sufficient, and then their magnitude should not be taken too seriously. Nevertheless the calculation shows that the effect of the three-pion-exchange \( \Lambda NN \) force in nuclear matter can be very important.
TABLE 5.1

EXPECTATION VALUES IN $^{3}\Lambda$ OF THE THREE-PION-EXCHANGE $\Lambda N N$ POTENTIAL IN MEV FOR DIFFERENT VALUES OF CUTOFF DISTANCE $d_{\Lambda N}$ AND HARD-CORE RADIUS $D$. THE RESULTS OBTAINED WITH THE TPE $\Lambda N N$ FORCE (§ 3.1) ARE SHOWN IN PARENTHESIS.

<table>
<thead>
<tr>
<th>$D$(fm)</th>
<th>$d_{\Lambda N}$(fm)</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>1.</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>$&lt;P_c&gt;$</td>
<td>2.97</td>
<td>1.432</td>
<td>0.454</td>
<td>0.096</td>
<td>0.019</td>
</tr>
<tr>
<td></td>
<td>$(&lt;w_p(I)&gt;)$</td>
<td>(0.091)</td>
<td>(0.084)</td>
<td>(0.064)</td>
<td>(0.030)</td>
<td>(0.011)</td>
</tr>
<tr>
<td>0.4</td>
<td>$&lt;P_c&gt;$</td>
<td>0.603</td>
<td>0.384</td>
<td>0.093</td>
<td>0.018</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(&lt;w_p(I)&gt;)$</td>
<td>(0.074)</td>
<td>(0.068)</td>
<td>(0.034)</td>
<td>(0.012)</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>$&lt;P_c&gt;$</td>
<td>0.368</td>
<td>0.112</td>
<td>0.019</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(&lt;w_p(I)&gt;)$</td>
<td>(0.084)</td>
<td>(0.044)</td>
<td>(0.015)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>$&lt;P_T&gt;$</td>
<td>-8.438</td>
<td>-8.416</td>
<td>-4.639</td>
<td>-0.569</td>
<td>-0.078</td>
</tr>
<tr>
<td></td>
<td>$(&lt;w_p(II)&gt;)$</td>
<td>(-0.706)</td>
<td>(0.231)</td>
<td>(0.354)</td>
<td>(0.297)</td>
<td>(0.118)</td>
</tr>
<tr>
<td>0.4</td>
<td>$&lt;P_T&gt;$</td>
<td>-2.659</td>
<td>-2.186</td>
<td>-0.390</td>
<td>-0.057</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(&lt;w_p(II)&gt;)$</td>
<td>(-0.700)</td>
<td>(-0.432)</td>
<td>(0.130)</td>
<td>(0.090)</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>$&lt;P_T&gt;$</td>
<td>-1.435</td>
<td>-0.369</td>
<td>-0.056</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(&lt;w_p(II)&gt;)$</td>
<td>(-1.010)</td>
<td>(-0.075)</td>
<td>(0.074)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE 5.2

THE $\Lambda-\alpha$ POTENTIAL IN $^5$He DUE TO THE THREE-PION-EXCHANGE POTENTIAL FOR DIFFERENT VALUES OF $r_\Lambda$.

THE NOTATION IS EXPLAINED IN THE TEXT.

<table>
<thead>
<tr>
<th>$d_{\Lambda N}(\text{fm})$</th>
<th>(r_\Lambda(\text{fm}))</th>
<th>0.6</th>
<th>1.0</th>
<th>2.0</th>
<th>3.0</th>
<th>4.0</th>
<th>0.6</th>
<th>1.0</th>
<th>2.0</th>
<th>3.0</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{\Lambda N} = 0.4\text{ fm}$</td>
<td>(d_{\Lambda N} = 0.4\text{ fm})</td>
<td>35.48</td>
<td>19.71</td>
<td>3.05</td>
<td>0.15</td>
<td>0.066</td>
<td>4.36</td>
<td>3.30</td>
<td>1.07</td>
<td>0.093</td>
<td>0.007</td>
</tr>
<tr>
<td>$d_{\Lambda N} = 0.6\text{ fm}$</td>
<td>(d_{\Lambda N} = 0.6\text{ fm})</td>
<td>33.10</td>
<td>19.01</td>
<td>3.33</td>
<td>0.131</td>
<td>0.069</td>
<td>4.52</td>
<td>3.21</td>
<td>1.10</td>
<td>0.101</td>
<td>0.007</td>
</tr>
<tr>
<td>$d_{\Lambda N} = 0.4\text{ fm}$</td>
<td>(d_{\Lambda N} = 0.4\text{ fm})</td>
<td>-22.57</td>
<td>-0.395</td>
<td>0.279</td>
<td>-0.061</td>
<td>-0.005</td>
<td>-6.19</td>
<td>-0.809</td>
<td>0.274</td>
<td>-0.041</td>
<td>-0.005</td>
</tr>
<tr>
<td>$d_{\Lambda N} = 0.6\text{ fm}$</td>
<td>(d_{\Lambda N} = 0.6\text{ fm})</td>
<td>-22.10</td>
<td>-0.436</td>
<td>0.286</td>
<td>-0.061</td>
<td>-0.005</td>
<td>-6.11</td>
<td>-0.807</td>
<td>0.274</td>
<td>-0.041</td>
<td>-0.005</td>
</tr>
</tbody>
</table>
Fig. 5.3
TABLE 5.3
EFFECT OF THE THREE-PION-EXCHANGE ANN POTENTIALS IN NUCLEAR MATTER. THE RESULTS
OBTAINED WITH THE TPE ANN FORCE (§ 3.3) ARE SHOWN IN PARENTHESIS.
THE NOTATION IS EXPLAINED IN THE TEXT.

<table>
<thead>
<tr>
<th>( d_{\Lambda N} ) (fm)</th>
<th>0.6</th>
<th>1.</th>
<th>0.6</th>
<th>1.</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_{NN} ) (fm)</td>
<td>0.4</td>
<td>0.6</td>
<td>1.</td>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>( P_\Lambda^c ) (Mev) ( (U_\Lambda(I)) )</td>
<td>41.11</td>
<td>35.81</td>
<td>23.66</td>
<td>9.71</td>
<td>8.40</td>
</tr>
<tr>
<td>( (U_\Lambda(II)) )</td>
<td>(1.36)</td>
<td>(1.30)</td>
<td>(1.06)</td>
<td>(0.84)</td>
<td>(0.81)</td>
</tr>
<tr>
<td>( P_{\Lambda T} ) (Mev) ( (U_\Lambda(II)) )</td>
<td>13.75</td>
<td>8.52</td>
<td>2.06</td>
<td>6.38</td>
<td>4.47</td>
</tr>
<tr>
<td>( (U_\Lambda(II)) )</td>
<td>(13.95)</td>
<td>(4.69)</td>
<td>(-6.85)</td>
<td>(12.53)</td>
<td>(9.95)</td>
</tr>
<tr>
<td>( P_\Lambda^c + P_{\Lambda T} ) ( (U_\Lambda(I) + U_\Lambda(II)) )</td>
<td>54.86</td>
<td>44.33</td>
<td>25.72</td>
<td>16.09</td>
<td>12.87</td>
</tr>
<tr>
<td>( (U_\Lambda(I) + U_\Lambda(II)) )</td>
<td>(15.31)</td>
<td>(5.99)</td>
<td>(-5.79)</td>
<td>(13.37)</td>
<td>(10.76)</td>
</tr>
</tbody>
</table>
6.1 Derivation of the three-nucleon (3N) force

The long-range part of the three-body force arises mainly due to the TPE among three nucleons. The TPE 3N potential has been derived by Fujita and Miyazawa (1957) (FM) in the static approximation using a technique from dispersion theory (See also: Smith and Sharp, 1960; Fujita et al, 1962; Coury and Frank, 1963; Quang Ho-Kim, 1966). The TPE 3N potential consists of three terms,

\[ F = F(1) + F(2) + F(3) \] \hspace{1cm} (6.1)

where \( F(3) \) is due to the process depicted in Fig. 6.1.
The other two terms are obtained by cyclic permutations of 1, 2 and 3. FM's expression for $F(i)$ is further divided into two parts:

$$F(i) = F_p(i) + F_s(i)$$

where $F_p(i)$ and $F_s(i)$ are due respectively to the p- and s-wave scatterings of the virtual pion from the i-th nucleon. In order to obtain a rough idea of the effect of the TPE 3N force, $F$, in triton, it was assumed by FM that triton consists of an equilateral triangle with side 1.3 fm. The 3N potential energy, $F$, for this configuration was found to be -0.22 MeV (attractive). The contribution of $V_s(i)$ was found to be negligible.

The pion-pion interaction was not considered by FM. However, in addition to the well established $I = J = 1$ resonance, or the $\rho$-meson, there has accumulated in the last few years considerable, although not quite conclusive, evidence for an $I = J = 0$ resonance. This is the so-called $\sigma$-meson, with mass of about 410 MeV (Rosenfeld et al, 1967). The 3N force, which arises from the pion-nucleon interaction via the $\sigma$-meson as shown in Fig. 6.2, was examined by Harrington (1963).

![Fig. 6.2](image)
The effects of this $3N$ force in triton and in nuclear matter were then estimated. The effects turned out to be repulsive and quite substantial, especially in nuclear matter. As Harrington himself was aware of, however, it is dangerous to consider the diagram of Fig. 6.2 alone. This is clear if the $s$-wave $\pi$-$N$ scattering at low energy is considered. If only the diagram (a) of Fig. 6.3 is considered, where the pion interacts with the nucleon via the $\sigma$-meson, it would yield a very large scattering length in drastic disagreement with experiment. There must be other "direct interactions", here represented by the diagram (b) in Fig. 6.3, which cancel the contribution from (a) so that the experimentally observed extremely small scattering length is reproduced.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.3}
\caption{Fig. 6.3}
\end{figure}
Including the effect of the $\sigma$-meson, $F(i)$ can be written:

$$F(i) = F_p(i) + F_s(i) + F_{\sigma}(i) \quad \ldots(6.3)$$

in place of (6.2). Here $F_\sigma(i)$ and $F_s(i)$ arises through the diagram (a) and (b) of Fig. 6.3 respectively. The potential $F_\sigma(i)$ is the one discussed by Harrington. The potential $F_s(i)$ has the same form as, but stronger than, that in (6.2). The potentials $F_\sigma(i)$ and $F_s(i)$ have opposite signs and, in fact, it will be seen that they almost completely cancel each other in the Fermi-gas model of nuclear matter. Strictly speaking, the diagram (a) in Fig. 6.3 contributes also to the p-wave $\pi$-N scattering, hence $F_p(i)$ has to be re-adjusted too. However, this is a very small effect, and the same $F_p(i)$, as was given by FM, is used here. According to FM, $F_p(i)$ and $F_s(i)$ are given by

$$F_p(3) = -(c / 8 \mu_p) \left[ 5 \left( \frac{1}{3} \cdot \frac{1}{3} \right)(\frac{3}{3} \cdot \frac{3}{3}) + 3 \left( \frac{2}{2} \cdot \frac{2}{2} \right)(\frac{3}{3} \cdot \frac{1}{1}) \right]$$

$$\times \left( \frac{1}{3} \cdot \frac{1}{3} \right)(\frac{2}{2} \cdot \frac{2}{2}) Y(x) Y(y) + (1 \cdot 1, \frac{2}{2} \cdot \frac{2}{2} x \cdot \frac{2}{2} y), \quad \ldots(6.4)$$

$$F_s(3) = c \left( \frac{1}{3} \cdot \frac{1}{3} \right)(\frac{3}{3} \cdot \frac{3}{3}) Y(x) Y(y) \quad \ldots(6.5)$$

with

$$\hat{x}_i = x_i - \frac{x_i}{x_i} \cdot \frac{x_i}{x_i}, \quad \hat{x}_i = x_i - \frac{x_i}{x_i} \cdot \frac{x_i}{x_i}, \quad \ldots(6.6)$$

$$Y(x) = \exp\left(-\mu x \right) / \left( \mu x \right). \quad \ldots(6.7)$$

Here $x_i$ is the coordinate of the $i$-th nucleon, and $\mu$ is the pion mass ($c = \gamma = 1$).
The coefficient \( C_{pN} \) is given by

\[
C_{pN} = \frac{f_N^2}{9\pi^2} \left( \frac{\sigma_{33}(p)}{\omega_p} \right) \quad \text{dp} = 0.46 \text{ Mev} \quad \ldots(6.8)
\]

where \( f_N = 0.08 \) is the \( \pi - N \) coupling constant and \( \sigma_{33} \) is the total cross section for p-wave \( \pi - N \) scattering in the \( I = J = 3/2 \) state. If the \( \sigma \)-meson is not considered, \( C_{sN} \) is related to the s-wave \( \pi - N \) scattering lengths \( a_1 \) and \( a_3 \), by

\[
C_{sN} = - (f_N)^2 \left( a_1 + 2a_3 \right)/3 \quad \ldots(6.9)
\]

This relation has to be modified if the contribution of the \( \sigma \)-meson is included.

For the \( \sigma \)-meson an effective interaction density for the \( \sigma - N \) and \( \sigma - \pi \) is assumed to be:

\[
H = \left( 4\pi \right)^{1/2} \left( g_\pi \phi_\pi + \frac{1}{2} g_\sigma \phi_\sigma \right) \quad \ldots(6.10)
\]

where \( \phi_\pi \) and \( \phi_\sigma \) are the nucleon, pion and \( \sigma \)-meson fields, respectively, and \( g_\pi \) and \( g_\sigma \) are dimensionless coupling constants. Then as was

\[\text{For the TPE \( \Lambda NN \) force, the corresponding strength factor is}
\]

\[C_{p\Lambda} \approx 1.43 \text{ Mev (S 2.1). For the TPE \( \Lambda NN \) force contributions come from}
\]

the \( \Sigma \) and \( Y_1^\ast \) intermediate state, whereas for the \( 3N \) force only the \( N \) intermediate state contributes. The difference is thus mainly due to the \( \Sigma \)-contribution. However, there is only one \( \Lambda NN \)-bond in \( \Lambda \), compared with \( 3N \)-bonds in \( \Lambda \). Moreover, the average distance between particles in \( \Lambda \) is shorter than in \( \Lambda \). Thus it is expected that the effect of the \( 3N \) force in \( \Lambda \) is more than that of the TPE \( \Lambda NN \) force in \( \Lambda \).
shown by Harrington, $F_\sigma(3)$ may be written as

$$F_\sigma(3) = \frac{\lambda_m}{4} \frac{1}{4 \pi \mu} \left( \mathbf{\tau} \cdot \mathbf{\tau} \mathbf{\tau} \cdot \mathbf{\tau} \mathbf{\tau} + \mathbf{\tau} \cdot \mathbf{\tau} \mathbf{\tau} \right).$$

\[ x \int d^3 r \frac{\exp(-\mu_1 s_1 - \mu_2 s_2 - ms_3)}{s s s} \]

where $s_i = |\mathbf{r}_i - \mathbf{r}_{i+1}|$, $i = 1, 2, 3$, $m$ is the mass of the $\sigma$-meson, and

$$\lambda = \frac{2}{f_{\sigma}^2} \mu / m \quad (6.12)$$

The choice of $C_{sN}$ and $\lambda$ will be discussed in the next section.

6.2 Effects of the TPE 3N force in triton

Here it is assumed that the triton is in the $s$-state of complete spatial symmetry. Then the spin-isospin averages of $F_p(1)(6.4)$ and $F_p(1)(6.5)$ become:

$$F_p(3) = c_p \frac{1}{2} \left( 3 \cos \theta_{xy} - 1 \right) M(x) M(y), \quad (6.13)$$

$$F_p(3) = -c_p \cos \theta_{xy} G(x) G(y) \quad (6.14)$$

with

$$M(x) = \left( 1 + \frac{3}{\mu x} + \frac{3}{\mu x} \right) \gamma(x), \quad G(x) = \left( 1 + \frac{1}{\mu x} \right) \gamma(x), \quad (6.15)$$

$$\cos \theta_{xy} = \frac{(x \cdot y)}{(x y)} \quad (6.16)$$
Other terms with \( i = 1 \) or 2 can be obtained from the above formulae by cyclic permutations of 1, 2 and 3.

As was shown by Harrington, \( F_\sigma (3) \) (6.11) for the triton may be written as

\[
F_\sigma (3) = - \frac{\lambda}{4\pi \mu} \int \frac{d^3 q_1 \, d^3 q_2 \, q_1 \cdot q_2}{(q + \mu)(q + \mu)} \frac{\exp \left[ i(q_1 \cdot x + q_2 \cdot y) \right]}{(q + q)^2 + m^2}.
\]

...(6.17)

Now, for the s-wave \( \pi\n\) scattering, the sum of the contributions of the diagrams (a) and (b) of Fig. 6.3 should give the observed (isospin-even part of) scattering length. This is achieved if \( C_8 \) in (6.9) is replaced by \( C_{\eta N} - \lambda \), namely

\[
C_{\eta N} - \lambda = - \frac{1}{3} (\mu f_N^2) \frac{2}{(a + 2 \alpha_\%)} \quad \text{...(6.18)}
\]

This can be seen in the following way. First, it is noted that the s-wave \( \pi\n\) scattering length does not depend on the shape of the source of the interaction, hence it is independent of \( m \), provided that \( \lambda \) is kept constant. Therefore the limit \( m \rightarrow \infty \) can be taken without affecting the scattering length. In this limit \( F_\sigma (3) \) becomes

\[
F_\sigma (3) \rightarrow \lambda \mu \frac{\nabla_x \cdot \nabla_y X(x) \cdot Y(y)}{x y} = \cos \theta \ G(x) \ G(y) \quad \text{...(6.19)}
\]

which is the same form as (6.14). In fact the \( \sigma \) - line in Fig. 6.2 or in diagram (c) of Fig. 6.3 shrinks to a point in this limit, and becomes indistinguishable from the 'direct interaction' diagram (b) of Fig. 6.3.
Thus (6.18) replaces (6.9) in the limit \( m \to \infty \). However, since the two sides of (6.18) are independent of \( m \), this relation should hold for any value of \( m \). Experimental values for the scattering lengths are \((a_1 + 2a_2) = (-0.035 \pm 0.012)/\mu\) (Samarayanske and Woolcock 1965). Substituting \( a_1 + 2a_2 = -0.035/\mu \) into (6.18) it can be seen that:

\[
C - \lambda = 0.13 \text{ MeV}
\]

In (6.17) the long-range part of the potential is determined mainly by the part of the integrand with small momenta \( q_1 \) and \( q_2 \). Since \( m \sim 410 \text{ MeV} \) is much larger than \( \mu \), the propagator of the \( \sigma \)-meson may be approximated by

\[
\left\{ \frac{2}{1} \frac{2}{2} \frac{2}{2} \frac{2}{1} \frac{1}{2} \right\} - 1 \left( \frac{q + m}{1} \right) \left( \frac{q + m}{2} \right).
\]

Then

\[
F_\sigma (3) = \lambda' \cos \theta \left\{ G(x) - G'(x) \right\} \left\{ G(y) - G'(y) \right\}, \ldots (6.22)
\]

where

\[
\lambda' = \frac{\lambda m}{(m - \mu)}, \ldots (6.23)
\]

and

\[
G'(x) = (\frac{m - \mu}{\mu x}) e^{-\mu x}. \ldots (6.24)
\]

For the parameters \( g \), \( h \) and \( m \), Harrington assumed

\[
g = 10, \quad h = 6, \quad g h > 0, \quad m = \frac{h}{\mu}. \ldots (6.25a)
\]
which give

\[ C = 5.48 \text{ Mev}, \quad \lambda = 5.35 \text{ Mev} \]

Here \( g \) and \( m \) were taken from an analysis of the \( N-N \) scattering in terms of the one-boson-exchange model (Bryan and Scott 1964). The \( \pi-\sigma \) coupling constant \( h \) was determined assuming the width of the \( \sigma \)-meson to be \( \Gamma = -\mu \). The sign \( g h > 0 \) was suggested from the analysis of the two-pion contribution to the \( \pi-N \) scattering. A more recent analysis of the \( N-N \) scattering gives much smaller \( g^2 (2.3 \text{ to } 2.8) \) and slightly smaller \( m \) (420 to 470 Mev) (Arndt et al., 1966). Therefore the following set is also considered here:

\[ g = 2.5, \quad h = 6, \quad g h > 0, \quad m = 3\mu \]

In this case, it can be seen that:

\[ C = 4.88 \text{ Mev}, \quad \lambda = 4.75 \text{ Mev} \]

Now the expectation values of \( F_p(3), F_s(3) \) and \( F_\sigma(3) \) in the triton are estimated by perturbation theory. Since the short range part of the three nucleon force is not known, the three-body potentials \( F_p(3), F_s(3) \) and \( F_\sigma(3) \) are taken to be zero for \( N-N \) distances less than a cutoff distance \( d \). The unperturbed wavefunction is taken to be the wavefunction given by Ohmura (1959). Its spatial part is given by

\[ \Psi(x,y,z) = \sum_{i=1}^{N} f(x) f(y) f(z) \]

where \( x, y \) and \( z \) are distances between the three nucleons and

\[ f(x) = 0 \quad \text{for } x < D \]

\[ = \exp \left[ -\alpha(x - D) \right] - \exp \left[ -\beta(x - D) \right] \quad \text{for } x > D \]

\[ \cdots (6.28) \]
Here $D$ is the hard-core radius of the $N-N$ force. The factor $N$ normalizes $\gamma$ to

$$\int \gamma(x,y,z) \frac{2}{xyz} dx dy dz = 1 \quad \ldots (6.29)$$

where it is understood that $x, y, z$ satisfy the triangular relations $x + y \geq z$ etc... The values of the variational parameters $\alpha, \beta$ and the normalization factor $N$ are listed in table 6.1 for the hard-core radii $D = 0.2, 0.4$ and $0.6$ fm. The expectation values of various parts of the $3N$ potential are listed in table 6.2, where the notation is:

$$\Delta E_p = 3 \langle F_p(3) \rangle = \int \frac{p}{p} (x,y,z)^2 \frac{2}{xyz} dx dy dz \quad \ldots (6.30)$$

$$\Delta E_s = 3 \langle F_s(3) \rangle, \quad \Delta E = 3 \langle F(3) \rangle, \quad \Delta E_p = \Delta E + \Delta E + \Delta E$$

<table>
<thead>
<tr>
<th>$D$ (fm)</th>
<th>$\alpha$ (fm)</th>
<th>$\beta$ (fm)</th>
<th>$N$ (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.462</td>
<td>5.03</td>
<td>1.084</td>
</tr>
<tr>
<td>0.4</td>
<td>0.457</td>
<td>4.09</td>
<td>1.676</td>
</tr>
<tr>
<td>0.6</td>
<td>0.450</td>
<td>4.20</td>
<td>2.718</td>
</tr>
</tbody>
</table>

**TABLE 6.1**

**THE PARAMETERS OF THE TRITON WAVE FUNCTION**

\[ \dagger \]

This is taken from table 2 of the work of Omura (1959), for the case of exponential $N-N$ potentials and $r_{os} = 2.7$ fm.
TABLE 6.2
THE EXPECTATION VALUE IN MEV OF THE 3N POTENTIAL IN THE TRITON AS DEFINED
BY EQ.(6.30), FOR DIFFERENT HARD-CORE RADIUS D, CUTOFF DISTANCE d AND
\(\sigma\)-MESON MASS m .

<table>
<thead>
<tr>
<th>D(fm)</th>
<th>0.2</th>
<th>0.6</th>
<th>1.0</th>
<th>0.4</th>
<th>0.6</th>
<th>1.0</th>
<th>0.6</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>d(fm)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\Delta E_p)</td>
<td>-1.07</td>
<td>-0.14</td>
<td>0.27</td>
<td>-0.76</td>
<td>-0.60</td>
<td>0.01</td>
<td>-0.41</td>
<td>-0.14</td>
</tr>
<tr>
<td>(\Delta E_s)</td>
<td>-3.28</td>
<td>-2.85</td>
<td>-1.37</td>
<td>-1.53</td>
<td>-1.52</td>
<td>-1.06</td>
<td>-0.91</td>
<td>-0.80</td>
</tr>
<tr>
<td>(\Delta E_\sigma)</td>
<td>2.21</td>
<td>2.03</td>
<td>1.18</td>
<td>1.27</td>
<td>1.86</td>
<td>0.94</td>
<td>0.83</td>
<td>0.74</td>
</tr>
<tr>
<td>(\Delta E)</td>
<td>-2.94</td>
<td>-0.96</td>
<td>0.06</td>
<td>-1.02</td>
<td>-0.86</td>
<td>-0.11</td>
<td>-0.49</td>
<td>-0.20</td>
</tr>
<tr>
<td>(m = 4\mu)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\Delta E_s)</td>
<td>-2.92</td>
<td>-2.54</td>
<td>-1.22</td>
<td>-1.50</td>
<td>-1.35</td>
<td>-0.94</td>
<td>-0.81</td>
<td>-0.71</td>
</tr>
<tr>
<td>(\Delta E_\sigma)</td>
<td>1.44</td>
<td>1.34</td>
<td>0.84</td>
<td>0.90</td>
<td>0.89</td>
<td>0.69</td>
<td>0.71</td>
<td>0.56</td>
</tr>
<tr>
<td>(\Delta E)</td>
<td>-3.35</td>
<td>-1.34</td>
<td>-0.11</td>
<td>-1.22</td>
<td>-1.06</td>
<td>-0.24</td>
<td>-0.51</td>
<td>-0.29</td>
</tr>
</tbody>
</table>

\(\Delta E\) does not involve \(m\). For \(\Delta E_s\) and \(\Delta E_\sigma\), two cases are considered ,
namely \(m = 4\mu\) and \(m = 3\mu\). \(\Delta E_s\) and \(\Delta E_\sigma\) are both quite large but they tend
to cancel each other. \(\Delta E_s + \Delta E_\sigma\) is negative (attractive) and is more
appreciable for smaller mass \(m\). In the limit \(m \to \infty\), as it has been
discussed before, \(\Delta E_s\) and \(\Delta E_\sigma\) becomes indistinguishable. The value of
\(\Delta E_s + \Delta E_\sigma\) in this limit, is obtained by multiplying \(\Delta E_s\) for \(m = 4\mu\) by
0.13/5.48 = 0.02\% : For example if D = d = 0.4 fm, \( \Delta E_b + \Delta E_\sigma \) becomes - 0.04 Mev, which is much smaller than \( \Delta E_\rho \). Results are all sensitive to d as well as to D. Presumably the most reasonable hard-core radius will be D = 0.4 fm. \( \Delta E_\rho \) is most sensitive to d and can be much more appreciable than FM's simple estimate - 0.22 Mev. The total contribution to the binding energy can easily be as large as or even larger than 1 Mev (attractive).

The expectation value of \( F \) in triton was calculated by Pask (1967) who used a wave function for \( ^3\text{H} \) obtained from detailed variational calculations done by Davies (1967). The result is of the same sign and same order of magnitude as the one obtained here (- 1.38 Mev, attractive).

6.3 **Effect in nuclear matter**

The effect of \( F_\sigma \) and \( F_\rho \) in nuclear matter is first considered. The contribution of \( F_\sigma \) has been estimated by Harrington (1966) who calculated the effective two-body potential obtained by averaging the \( 3\text{N} \) potential \( F_\sigma \) over the coordinate of the third particle. This potential was found to be repulsive and strong enough to dominate the OEP at distances less than 2 fm. However, it will be shown here, that \( F_\rho \) yields a similar effective two-body potential which largely cancels the effect of \( F_\sigma \), leaving a weakly attractive potential.

The effective two-body potential due to \( F_\sigma \) (3) (6.11) is given by

\[
V_\sigma (12) = \rho \int d^3 r \ F_\sigma (3)
\]

\[
= -\rho \lambda \ \mu \ \left( \tau^1 _\lambda \ \tau^2 _\mu \right) \left( \sigma^1 _\alpha \ \sigma^2 _\lambda \right) \ \sigma^2 _\mu \ \nu _\rho \ x
\]
where $\rho$ is the density of nucleons in nuclear matter. On the other hand, the effective two-body force due to $F_\rho(3)$ (6.5) is:

$$V_{s}(12) = \rho C_{sN} \mu \frac{2}{\sqrt{\pi}} \frac{1}{r} \frac{1}{2} \frac{2}{\sqrt{\pi}} \frac{2}{\sqrt{\pi}} \int \frac{d r}{3} Y(x) Y(y)$$

The integrals in (6.31) and (6.32) are identical. Since $\lambda \propto C_{sN}(6.20)$, $V_{\sigma}(12)$ and $V_{s}(12)$ almost completely cancel each other.

The expectation value of $F$ in nuclear matter, using the Fermi gas model will now be estimated. The calculation is similar to that of §3.3. There is a direct term which vanishes identically because of spin isospin saturation. There are two exchange terms and the binding energy due to $F_\rho$ in nuclear matter can be written:

$$U = U_1 + U_2$$

The single exchange term $U_1$ is

$$U_1 = \frac{A}{4} C_{sN} \rho \int \frac{d (k z)}{2} \left[ 1 + (3 \cos \theta - 1) T(x) T(y) \right]$$

$$x Y(x) Y(y) \frac{3}{d x} \frac{3}{d y},$$

and the double exchange term $U_2$ is:

$$\int \frac{d x}{3} \frac{d y}{3}$$
\[
U = -\frac{\Lambda}{2} + \frac{c}{4} \rho \int \frac{D(k x) D(k y) D(k z)}{f} \rho \frac{D(k x) D(k y) D(k z)}{f} \rho
\]
\[
\frac{1}{2} \left[ 1 + \frac{1}{4} (3 \cos \theta - 1) T(x) T(y) \right] Y(x) Y(y) d x d y.
\]

...(6.35)

The function \( D(r) \) has been defined in \( \S \ 3.3 \). The binding energies \( U_1 \) and \( U_2 \) consist of a central and a tensor term. The central terms \( U_1(I) \) and \( U_2(I) \) are given by the first terms of the integrals (6.34) and (6.35) respectively, and the tensor terms \( U_1(II) \) and \( U_2(II) \) by the second terms of the same integrals. The integral in \( U_1 \) is the same as the integral in \( U^\Lambda(I) + U^\Lambda(II) \) (3.30 and 3.31) and

\[
\frac{U_1(I)}{U^\Lambda(I)} = \frac{U_1(II)}{U^\Lambda(II)} = \frac{2 \rho N}{3} \frac{c_p}{c^\Lambda} A = 0.2165 A \quad \ldots(6.36)
\]

As in the case of the TPE ANN force a step function is used as a N-N correlation function

\[
\delta(x - y) - c) = 0 \quad \text{for } |x - y| < c,
\]
\[
= 1 \quad \text{for } |x - y| > c.
\]

...(6.37)

The energies \( U_1(I) \) and \( U_1(II) \) are then obtained by multiplying \( U^\Lambda(I) \) and \( U^\Lambda(II) \) by the factor defined in (6.36). The integral (6.35) is evaluated numerically after putting in the N-N correlation function \( \delta(|x - y| - c) \) in the integrand. The results are shown in table 6.3 for two different
values of \( d \) and three different values of \( c \).

**TABLE 6.3**

EFFECT OF THE TPE 3N POTENTIALS IN NUCLEAR MATTER. THE NOTATION IS EXPLAINED IN THE TEXT. \( d \) IS THE CUTOFF FOR THE 3N FORCE, WHILE \( c \) IS THE CUTOFF FOR A STEP-FUNCTION TYPE N-N CORRELATION.

<table>
<thead>
<tr>
<th>( d )</th>
<th>0.6</th>
<th>1.0</th>
<th>0.6</th>
<th>1.0</th>
<th>1.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>0.4</td>
<td>0.6</td>
<td>1.0</td>
<td>0.6</td>
<td>1.0</td>
</tr>
<tr>
<td>( U_1(I)/A )</td>
<td>0.292</td>
<td>0.278</td>
<td>0.227</td>
<td>0.173</td>
<td>0.143</td>
</tr>
<tr>
<td>( U_1(II)/A )</td>
<td>2.930</td>
<td>1.060</td>
<td>-1.469</td>
<td>2.130</td>
<td>0.667</td>
</tr>
<tr>
<td>( U_2(I)/A )</td>
<td>-0.164</td>
<td>-0.164</td>
<td>-0.137</td>
<td>-0.060</td>
<td>-0.060</td>
</tr>
<tr>
<td>( U_2(II)/A )</td>
<td>0.437</td>
<td>0.437</td>
<td>0.718</td>
<td>0.159</td>
<td>0.159</td>
</tr>
<tr>
<td>( U/A )</td>
<td>3.558</td>
<td>1.559</td>
<td>-0.660</td>
<td>2.405</td>
<td>0.910</td>
</tr>
</tbody>
</table>

For the values of \( d \) and \( c \) considered here, \( U_1(I)/A \) and \( U_2(II)/A \) are small and positive (repulsive), \( U_2(I)/A \) is small and negative (attractive) and \( U_1(I)/A \) is dominant and either positive (repulsive) or negative (attractive). For \( d = 0.6 \) fm, \( c = 0.4 \) fm the overall effect \( U/A \) is 3.558 Mev (repulsive) and for \( d = 0.6 \) fm, and \( c = 1 \) fm \( U/A = -0.660 \) Mev (attractive). In summary the effect is small and sensitive to the N-N correlation function.
The contribution of the three-nucleon force, in nuclear matter, has been estimated by several authors (FM; Smith and Sharp, 1960). The results obtained by FM are:

\[ U = -0.3 \text{ Mev} \times A \quad \text{(attractive)} \]
\[ U = +0.1 \text{ Mev} \times A \quad \text{(repulsive)} \]

No details of the calculation are given. In the calculation of Smith and Sharp an angular average of the tensor part is taken and it is found that:

\[ 1 \text{ Mev} \times A \leq U + U \leq 3 \text{ Mev} \times A \quad \text{(repulsive)} \]

These results cannot be directly compared to those of the present work since they are obtained with a different N-N correlation function.

As was mentioned in chapter 2, the contribution of the pion-pion interaction via the \( \sigma \)-meson has not been considered in the calculation of the TPE \( \Lambda NN \) force. However, from the results of the present chapter it can be expected that this force will give an appreciable effect in hypertriton as \( \Sigma \) does in triton. In the case of \( ^5 \Lambda \) He and nuclear matter, since the contribution of the \( p \)-wave part (\( \pi - \Lambda \) interaction) of the TPE \( \Lambda NN \) force is dominant over that of the \( s \)-wave part, the effect of the \( \sigma \)-meson is expected to be small.
DISCUSSION AND CONCLUSION

Three-body and many-body forces are predicted from the meson theory of nuclear forces. In the present work the long and intermediate range parts of the ANN force due to the two- and three-pion-exchange have been derived using the static approximation. Their effects on the binding energies of $^3\Lambda H$, $^5\Lambda He$ and nuclear matter have been estimated. Only the tail of these potentials was considered, since at short distances heavier meson exchanges may become important. The ANN force was arbitrarily taken to be zero for $\Lambda$-$N$ distances less than a cutoff distance $d_{\Lambda N}$. The contributions of the two-pion-exchange part of the three-nucleon ($3N$) force to the binding energy of the triton and nuclear matter have also been considered. Here again the force was taken to be zero for $N$-$N$ distances less than a cutoff distance $d$. It was found that the effects of the ANN force can be important and also that the contribution of the $3N$ force is appreciable, although these results depend on the cutoff distances quite sensitively.

In deriving the TPE ANN force, $W$, the $\pi$-$\Lambda$ interaction was considered to be dominated by the $p$- and $s$-wave. Then $W$ can be written:

$W = W_p + W_s$, where $W_p$ and $W_s$ arise from the $p$- and $s$-wave $\pi$-$\Lambda$ interaction, respectively. As in the case of the $s$-wave $\pi$-$N$ interaction, a "suppression factor" of the $s$-wave $\pi$-$\Lambda$ interaction was here introduced for $W_s$. For a reasonable "suppression factor", $W_s$ was found to be unimportant. $W$ consist of central and tensor terms. The tensor term
depends on the angle $\theta_{xy}$ as shown in Fig. 2.2. This term was found to be dominant.

In $^3\Lambda H$, the contribution of $W$ to the binding energy of $\Lambda A, \Lambda B$, can be repulsive or attractive, depending on the cutoff $d_{AN}$ and the wavefunction used. A reasonable value for the $\Lambda-N$ hard-core radius $D$ may be taken to be $0.4$ fm, and the cutoff radius for the $\Lambda NN$ force about $1$ fm. In this case $W$ gives a repulsive contribution of about $0.16$ Mev. Although this is substantial in view of the small value of $B_{\Lambda}$ in $^3\Lambda H$, this could easily be accommodated for, by slightly changing the two-body $\Lambda-N$ force.

In $^5\Lambda He$, the overall effect is much larger, because there are six $\Lambda NN$ bonds and also the average $\Lambda-N$ distance is smaller than that of $^3\Lambda H$. Here the contribution to $B_\Lambda$ from the $\Lambda NN$ forces is always repulsive, and is about $2$ Mev for a cutoff of $1$ fm. However, this repulsive effect is probably overestimated since no $N-N$ correlation in the $\alpha$-particle has been considered. This repulsive contribution of the $\Lambda NN$ force substantially reduces the overbinding of the $\Lambda$ in $^5\Lambda He$ as was obtained by using the two-body forces alone (Bhaduri et al, 1967).

In nuclear matter, the contribution, $U_\Lambda^{(3)}$, of $W$ to $B_\Lambda$ was estimated by using perturbation theory. The effect is extremely sensitive to $d_{AN}$ as well as the $N-N$ correlation function. For instance, with a cutoff value $d_{NN} = 0.7$ fm, for a step function type $N-N$ correlation, $U_\Lambda^{(3)} = 1.8$ Mev for $d_{AN} = 0.6$ fm and is about $8.7$ Mev for $d_{AN} = 1.0$ fm (Fig. 3.3). However, with a slightly larger value of $d_{NN}$, $U_\Lambda^{(3)}$ may well turn negative for the case where $d_{AN}$ is $0.6$ fm. For $d_{AN} = d_{NN} = 1$ fm, a repulsion of $3.535$ Mev is obtained. If only the very
long-range part of \( W (d > 1) \) is considered, the effect is definitely repulsive and can be as large as 10 Mev.

In the short-range region, processes other than TPE will become important, so that the above results will be modified. The next lowest order contribution to the \( \Lambda NN \) force, after TPE, arises from three-pion-exchange. The three-pion-exchange \( \Lambda NN \) force has been derived. Its effects on light hypernuclei and nuclear matter have been compared with those of \( W_p \), in order to see for what value of the cutoff \( d \Lambda N \), the TPE \( \Lambda NN \) force, \( W_p \), is dominated by the three-pion-exchange \( \Lambda NN \) force.

The three-pion-exchange \( \Lambda NN \) force arises from diagrams 4.1c and 4.1d which are as yet too complicated to be evaluated exactly. The force arising from diagram 4.1d was approximated by the sum of the forces arising from diagrams 4.1a and 4.1b. This \( \Lambda NN \) force, \( P \), derived in the static approximation, consists of a central and a tensor term, \( P = P_c + P_T \), where the tensor term, \( P_T \), depends on the angle \( \theta_{xz} \) as shown in Fig. 4.3. The effect of the potential due to diagram 4.1c was not examined, because of its complexity.

In \( \Lambda N \), \( P \) depends strongly on the \( \Lambda N \) distance and the contribution to \( B_\Lambda \) of \( P_T \) is larger than that of \( P_c \). For \( d \Lambda N < 1 \text{ fm} \), \( P_c \) always gives a positive contribution (repulsive), which is larger than that of the central part of \( W_p \). \( P_T \), always gives a negative contribution (attractive) which is larger in absolute value than that of the tensor term of \( W_p \). For \( d \Lambda N < 1 \text{ fm} \) the contribution to \( B_\Lambda \) of \( P \) is larger than that of \( W_p \). For a reasonable value of 0.4 fm for the \( \Lambda N \) hard-core radius, and \( d \Lambda N = 1 \text{ fm} \), \( P \) gives an attractive contribution of 0.3 Mev. The contribution of \( W_p \) was previously shown to be repulsive (0.16 Mev), therefore the total contribution is attractive (0.14 Mev). In fact for \( d \Lambda N < 1 \text{ fm} \),...
the overall effect is found to be always attractive.

The $\Lambda - \alpha$ potential in $^5\Lambda$He due to $P$, which is denoted by $P(r_{\Lambda})$, was evaluated. The contribution $P_c(r_{\Lambda})$ (central part) is always repulsive and larger in absolute value than the contribution $P_T(r_{\Lambda})$ (tensor part) which is attractive (except for $r_{\Lambda} \approx 2$ fm). The overall contribution of $P$ is thus repulsive (except for $d_{\Lambda} = 1$ fm and $r_{\Lambda} \approx 0$ fm) and it further reduces the overbinding of $^5\Lambda$He. Results are sensitive to $d_{\Lambda}$ but not to $d_{NN}$. For $d_{\Lambda} \approx 1$ fm, $P(r_{\Lambda})$ is smaller than $U^3(r_{\Lambda})$, the $\Lambda - \alpha$ potential of $W_p$, and therefore only slightly modifies the contribution of that force.

The contribution of $P$ to the binding energy of $\Lambda$ in nuclear matter depends strongly on the distance $d_{\Lambda}^N$, but its dependence on $d_{NN}$ is not as pronounced. The results in nuclear matter are more sensitive to $d_{NN}$ than those of $^5\Lambda$He (see table 5.3 and 5.2, respectively). This can be explained by the fact that the nuclear matter expressions (5.20) and (5.21) have a stronger dependence on NN distance than the corresponding $^5\Lambda$He expressions (5.16) and (5.17). It can be also seen that in nuclear matter the tensor part contribution $P_{\Lambda T}$ depends less strongly on $d_{NN}$ than the corresponding contribution $U_{\Lambda}^{\Lambda\Lambda}$ (II) of the TPE $\Lambda NN$ force (table 5.3). This is due to the difference in angular dependence of the two forces: $P_{\Lambda T}$ depends on the angle $\theta_{xz}$ (sensitive to the $\Lambda-N$ distance) whereas the tensor part of $W_p$ depends on the angle $\theta_{xy}$ (sensitive to the NN distance) (see Fig. 4.3). The contribution of the central part $P_c$ is found to be repulsive and larger than that of the tensor part $P_T$ which is also repulsive. This total repulsive contribution is larger than that of $W_p$. For instance, with a cutoff for a step function type NN correlation of 1 fm and for
due to \( P \) is \(-7.43\) Mev and \( B \) due to \( W \) was found to be \(-3.78\) Mev. As in the case of \( W \), \( P \) may have been overestimated because the coupling constant \( f_A^2 \) was chosen equal to \( f_N^2 (f_A^2 = f_N^2 = 0.08) \). Various estimates of \( f_A^2 \) have been done which indicated that \( f_A^2 \) is slightly smaller than \( f_N^2 \) (§ 2.1). However, \( P \) does indicate the order of magnitude of the intermediate range part of the \( \Lambda N N \) force.

In summary, the contribution of the long-range \( \Lambda N N \) force, \( W \), and the intermediate range \( \Lambda N N \) force, \( P \), to the binding energy of a \( \Lambda \)-particle in \( s \)-shell hypernuclei, has been shown to be important. In \( ^3 \Lambda H \), \( P \) was found to be attractive; for \( d \), \( \Lambda N \approx 1 \) fm the contribution of \( P \) is greater than that of \( W \), although the overall effect \( (P + W) \) is still relatively small and is attractive. The force \( P \) further reduces the binding energy of \( ^5 \Lambda \text{He} \); for \( d \), \( \Lambda N \approx 1 \) fm \( P \) only slightly modifies the results obtained using \( W \) alone. In nuclear matter the contribution of \( P \) is repulsive and greater than that of \( W \); even for \( d \), \( \Lambda N \approx 1 \) fm the overall effect \( (P + W) \) is quite large and is repulsive. Although there are ambiguities due to the unknown short-range part of the \( \Lambda N N \) force, it can be seen that the effects of the three-body \( \Lambda N N \) force cannot be ignored. Also, the 'effective' \( \Lambda N \) force extracted from binding energy data, assuming only a two-body force, can be significantly different from the 'free' \( \Lambda N \) force observed in two-body scattering.

The contributions of the long range part of the \( 3N \) force arising from two-pion-exchange to the binding energies of the triton and nuclear matter have been estimated. The effect of the pion-pion interaction was taken into account by the consideration of the contribution, \( P_\sigma \), of the (virtual) \( \pi-N \) scattering via the \( \sigma \)-meson (the \( \sigma \)-meson is a controversial
I = J = 0 dipion resonance). Then the 3N potential can be written, 
\[ F = F_p + F_s + F_\sigma, \]
where \( F_p \) and \( F_s \) are due respectively to the p- and s-wave \( \pi-N \) scattering. The s-wave "direct" \( \pi-N \) interaction was formulated so that, together with the \( \pi-N \) interaction via the \( \sigma \)-meson it reproduces the observed \( \pi-N \) scattering length. It was found that the potentials \( F_s \) and \( F_\sigma \) have opposite signs and that they tend to cancel each other.

The contribution of \( F_p \) to the binding energy of the triton has been estimated by perturbation theory. The triton wavefunction is taken from a variational calculation for a hard-core two-nucleon potential (hard-core radius \( D \)). The results were found to be sensitive to the cutoff of the 3N force, \( d \), as well as to \( D \). The contribution of \( F_\sigma \) and \( F_s \) are large but their sum is relatively small and negative (attractive). The effect due to \( F_p \) is found to be mainly attractive. At \( D = 0.4 \text{ fm} \) and \( d \approx 0.4 \text{ fm} \), the binding due to \( F \) is around 1 Mev (attractive). A perturbation calculation was done to estimate the effect of \( F_p \) in nuclear matter. This contribution is of the order of a few Mev, between -0.7 Mev (attractive) and 4.0 Mev (repulsive), depending on the value of \( d \) and on the nucleon-nucleon correlation function. As in the case of the ANN force, the three-pion-exchange 3N force will probably modify the above results.

Since, in both cases (ANN and 3N forces), the short range part of the force is unknown it is not possible to draw any definite conclusion. Nevertheless the results of the present work clearly show that the three-body force can play an important role in nuclear structure problems.
APPENDIX 1

DETAILS OF THE CALCULATION OF THE EFFECT OF THE TPE ANN FORCE ON $^5\text{He}$

In this appendix some calculational details on $^5\text{He}$ (§ 3.2) are given, in particular the angular integration in (3.20) and (3.21). The z-axis is taken along $\Lambda$ and polar coordinates for $x$ and $y$ are introduced as follows

$$x = (x, \theta, \phi), \quad y = (y, \theta, \phi) \quad \ldots (\text{A1.1})$$

Then the $\phi$-integration of $\cos^2 \theta$ becomes

$$\int_0^{2\pi} d\phi \int_0^{2\pi} d\phi \cos \theta \cos \phi \sin \theta \sin \phi (\phi - \phi)^2$$

$$= \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi (\cos \theta \cos \phi + \sin \theta \sin \phi \cos \phi) \ldots (\text{A1.2})$$

Thus the angular integral of the tensor part of $W (2.16)$ becomes

$$\int_0^{2\pi} d\phi \int_0^{2\pi} d\phi \cos \theta \cos \phi + \sin \theta \sin \phi \cos \phi \ldots (\text{A1.2})$$
\[
\frac{1}{2} \left\{ \int \frac{3}{d \times F(x)} (3 \cos \theta - 1) \right\}^2 = \ldots (A1.3)
\]

which explains (3.20).

The angular integral of $W$ is simpler. Since the second term in

\[
\cos \theta = \cos \theta \cos \phi + \sin \theta \sin \phi \cos (\varphi - \varphi)
\]

vanishes after $\varphi$-integration, it can be see that

\[
\int \frac{3}{d \times M(x)} M(y) \cos \theta = \left( \int \frac{3}{d \times M(x)} \cos \theta \right)^2 = \ldots (A1.4)
\]

which gives (3.21).

The radial integrations in (3.19) - (3.21) can be done as follows.

For (3.19) it can be seen that:

\[
\int \frac{3}{d \times \rho(\frac{x}{\Lambda} + \frac{y}{\Lambda}) Y(x)} = \frac{2\pi}{2} \rho(r) \int dx \frac{2^2}{\beta \mu r} \left\{ \begin{array}{c}
e^{-\frac{\mu r}{2}} \\
e^{\frac{\mu r}{2}} \end{array} \right\} \left\{ \begin{array}{c}
e^{-\frac{\mu r}{2}} \\
e^{\frac{\mu r}{2}} \end{array} \right\} \left\{ \begin{array}{c}
e^{-\frac{\mu r}{2}} \\
e^{\frac{\mu r}{2}} \end{array} \right\} \left[1 - erf\left(\beta \frac{d - r}{\Lambda \Lambda} + \frac{\mu}{2 \beta} \right) \right] + \ldots (A1.5)
\]
where
\[
\text{erf}(x) = \left( \frac{2}{\pi} \right)^{1/2} \int_0^x \exp(-t^2) \, dt \quad \ldots \text{(Al.6)}
\]

The integral (Al.5) is not very sensitive to the cutoff \( d \).

For (3.20) it can be seen that:
\[
\int_0^3 d x \; \rho \left( \frac{|x_i|}{\Lambda N} + \frac{x_j}{\Lambda N} \right) \left( 3 \cos \theta - 1 \right) T(x) Y(x)
\]
\[
= \frac{4\pi \rho(x)}{2 \beta \mu r} \int_0^{2\beta \mu x} 2 \beta x - \mu x \; T(x) \left\{ \left( 1 + \frac{3}{2} \right) \sinh z - \frac{3 \cosh z}{z} \right\},
\]
\[
\ldots \text{(Al.7)}
\]

where \( z = 2\beta \mu x \). \ldots \text{(Al.8)}

For small \( r \), \{ \} in (Al.7) may be expanded as \{ \} = \frac{3}{15} + \ldots .

Therefore, the integral (Al.7) vanishes at \( r = 0 \), as it should be.

Finally, for (3.21) it can be seen that
\[
\int_0^3 d x \; \rho \left( \frac{|x_i|}{\Lambda N} + \frac{x_j}{\Lambda N} \right) \cos \theta \left( 1 + \mu x \right) Y(x)/(\mu x)
\]
\[
= \frac{2\pi \rho(x)}{2 \beta \mu x} \int_0^{\infty} \frac{2 \beta x}{1 + \mu x} \; T(x) \left\{ \cosh z - \frac{\sinh z}{z} \right\},
\]
\[
\ldots \text{(Al.9)}
\]
where \( z \) is defined by (A1.8). For small \( r \), \( \{ \} \) in (A1.9) may be expanded as
\[
\{ \} = z^{2/3} + \ldots
\]
Therefore (A1.9) vanishes at \( r = 0 \).

The integrals (A1.7) and (A1.9) have been evaluated numerically.
APPENDIX 2

EXPRESSIONS NEEDED IN THE CALCULATION OF THE THREE-PION-EXCHANGE NN Force

The explicit form of the expressions (4.32) are given by

\[
Z(r, r') = - \frac{(4\pi)^2}{3} \left[ 3 f f \left( \frac{F(r, r')}{N} \right) + \frac{1}{\Delta} G \left( \frac{r, r'}{N} \right) \right] +
\]

\[
6 f f \left( \frac{F(r, r')}{N} \right) + \frac{1}{\Delta} G \left( \frac{r, r'}{N} \right) + 32 f f \left( \frac{F(r, r')}{N} \right) + \frac{1}{\Delta} G \left( \frac{r, r'}{N} \right) +
\]

\[
64 g g \left( \frac{F(r, r')}{N} \right) + \frac{1}{\Delta} G \left( \frac{r, r'}{N} \right) \right] \quad \ldots (A2.1),
\]

\[
D(r, r') = \frac{(4\pi)^2}{3} \left[ 3 f f \left( \frac{G(\Delta)}{N} \right) - \frac{1}{\Delta} G \left( \frac{r, r'}{N} \right) \right] +
\]

\[
8 g g \left( \frac{N}{\Delta} \right) - 8 g g \left( \frac{N}{\Delta} \right) \right] \quad \ldots (A2.2)
\]

and

\[
X(r, r') = \frac{(4\pi)^2}{3} \left[ 3 f f \left( \frac{F(r, r')}{N} \right) + \frac{1}{\Delta} G \left( \frac{r, r'}{N} \right) \right] + 6 f g \left( \frac{F(r, r')}{N} \right) +
\]

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\[
\frac{1}{\omega^\Delta} \mathcal{G}(r,r') + \frac{2}{\Delta} \frac{\mathcal{N}^0}{\omega^\Delta} \left[ \mathcal{G}(r,r') + \frac{\mathcal{N}^0}{\omega^\Delta} \mathcal{G}(r,r') \right] \]

\[\text{with}\]

\[
\mathcal{F}_{\alpha\beta}(r,r') = \frac{1}{2(2\pi)^3} \frac{1}{2} \left[ \begin{array}{c} k \sin k(r + r') \\ 0 \end{array} \right] \frac{2}{\omega (\omega + \alpha)(\omega + \beta)} \ldots (A2.4)
\]

and

\[
\mathcal{G}_{\alpha\beta}(r,r') = \frac{1}{2(2\pi)^3} \frac{1}{3} \frac{\alpha + \beta}{(\omega + \alpha)(\omega + \beta)} \left[ \begin{array}{c} k \sin k(r + r') \\ 0 \end{array} \right] \frac{2}{\omega} \ldots (A2.5)
\]

\[\text{As pointed out previously terms proportional to } f \text{ are divided by a factor of two.}\]
The reduction of the integrals (5.16) and (5.17) can be done as follows. The $z$ integration is first considered. Take $x$ as the Z-axis and the plane defined by $x$, $x_{z}$ as the Z-X plane. In this frame, polar coordinates for $z$ are defined by

$$z = (z, \theta, \phi)$$

...(A3.1)

Next the $x$ integration is considered. $x_{\Lambda}$ is chosen as new Z-axis. Polar coordinates for $x$ are:

$$x = (x, \theta, \phi)$$

...(A3.2)

Consequently the product $\rho(r_{1}) \rho(r_{2})$ can be written:

$$\rho(|x_{\Lambda} + x|) \rho(|x_{\Lambda} + x - z|) = \left(\frac{2}{\pi}\right) e^{\beta \frac{z^{2}}{2}} e^{-2\beta (r_{\Lambda} + x)}$$

$$= e^{-4\beta x_{\Lambda} \cos \theta} e^{-\beta z} e^{-2\beta b z}$$

...(A3.3)

where

$$b = -z \left(\sin \theta x_{z} \cos \phi + \cos \theta x_{z} \cos \phi\right) x_{z}$$

...(A3.4)
Then the integrals (5.16) and (5.17) become

\[
P_{\text{c}}(r_\Lambda) = 48\pi \left(\frac{\beta^3}{\pi}\right) e^{-2\beta r_\Lambda} \int_{-\infty}^{\infty} dx x s'(x) \int_{-1}^{1} dw e^{-4\beta x r_\Lambda w} \int_{0}^{\pi} \int_{-1}^{+1} d\phi d\psi I, \tag{A3.5}
\]

Here \( v = \cos \theta \), \( w = \cos \phi \) and (with unit \( \mu = 1 \))

\[
I = \int_{cz}^{\infty} dz \left[ 2 - \beta z + (2\beta b - 1)z \right] e^{-\beta d_{NN}} (2\beta b - 1) \frac{2}{4\beta} \int_{cz}^{\infty} dz \left[ 2 + \sqrt{\pi} (2\beta b - 1) \right], \tag{A3.7}
\]

\[
I = I + I_{Tz} \tag{A3.8}
\]
\[ I = \int_{-\infty}^{\infty} \frac{d\varphi}{N} \left( z^2 + 2 \beta z + (2 \beta^2 - 1) z \right) \]

\[ = \frac{2}{4 \beta} e^{-\beta} \int_{-\infty}^{\infty} \frac{d\varphi}{N} \left( 2 + \sqrt{\beta^2 (2 \beta - 1) + 6 \beta} \right), \quad \ldots (A3.9) \]

\[ I = 3 \int_{-\infty}^{\infty} \frac{d\varphi}{z} \left( -\beta z + (2 \beta^2 - 1) z \right) \quad \ldots (A3.10) \]

Thus the integral (5.16) is reduced to a four-dimensional integration (A3.5 with A3.7) which is done numerically. Similarly the integral (5.17) is split into a four-dimensional (A3.6 with A3.9) and five-dimensional (A3.6 with A3.10) integral, both of which are done numerically.
APPENDIX 4

THE VALUES OF NUMERICAL CONSTANTS USED THROUGHOUT THE COMPUTATION ARE GATHERED HERE

The mass of the pion is taken to be
\[ \mu = 137.28 \text{ MeV} = 0.6939 \text{ fm} \]

The units \( \not h = c = 1 \) are used.

Values of other constants are:

\[ f^2_N = 0.08 \]
\[ f^2_\Lambda = 0.08 \]
\[ \Delta = 0.389 \text{ fm} \]
\[ g^2_\Lambda = 0.047 \]
\[ g^2_N = 0.057 \]
\[ \omega_\Lambda = 1.24 \text{ fm} \]
\[ \omega_N = 1.27 \text{ fm} \]
\[ p = 6 \mu = 4.286 \text{ fm} \]
\[ \beta = 0.85056 \text{ fm} \]

\[ k^2 = 1.36 \text{ fm} \]
\[ \rho = 0.170 \text{ fm}^{-3} \]
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