NUMERICAL SOLUTION OF THE
LAMINAR BOUNDARY LAYER EQUATIONS

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ABSTRACT

An implicit finite difference technique has been developed for the solution of the steady two dimensional boundary layer equations.

The numerical method is free of stability limitations and similarity assumptions. Use has been made of Wu-type starting profiles which enable one to start the calculation from the leading edge.

Attractive features of the technique are its simplicity, flexibility and applicability to a wide range of boundary layer problems. In addition, results obtained from several case studies indicate that the numerical procedure is both accurate and fast.

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1. INTRODUCTION

The Navier-Stokes equations of motion constitute the corner stone of the science of fluid mechanics. Much research work has been devoted to the solution of these equations in two limiting cases of considerable practical importance: the flow of fluids at small Reynolds numbers and the flow of fluids of small viscosity at high Reynolds numbers.

The boundary layer theory has proved quite useful in investigating flow types of the latter case. Much time and effort has been spent over the past decades to obtain solutions of the boundary layer equations. Various methods of solution have been examined: analytical, approximate and numerical.

The purpose of this report is to describe a numerical technique for the solution of the steady two dimensional boundary layer equations and evaluate the convergence, accuracy and speed of the finite-difference method.

This report is divided into three main sections. In the first section the underlying principles of the boundary layer theory are outlined, and methods of solution proposed earlier are discussed. In the second section, the implicit finite-difference method is described, and in the third section, several boundary layer problems are examined.

2. THEORETICAL BACKGROUND

2.1 THE BOUNDARY LAYER CONCEPT

In the beginning of this century, L. Prandtl (R1) showed how it was possible to analyze viscous flow problems, precisely in cases where viscous effects are important. He suggested that the flow about a solid body could be divided into two regions: a very thin layer in the neighbourhood of the body where friction plays an important part and the remaining region outside this layer where friction effects may be neglected. Several simple experiments performed by him confirmed the predictions based on this hypothesis.

The boundary layer concept as set forth by Prandtl proved quite effective a tool in the investigation of viscous flows. One can justifiably say that the rapid developments in aerodynamics during the past decades are largely due to the better understanding of boundary layer type flows.

2.1.a The Velocity Boundary Layer

Consider the motion of water along a thin flat plate (Figure 1):

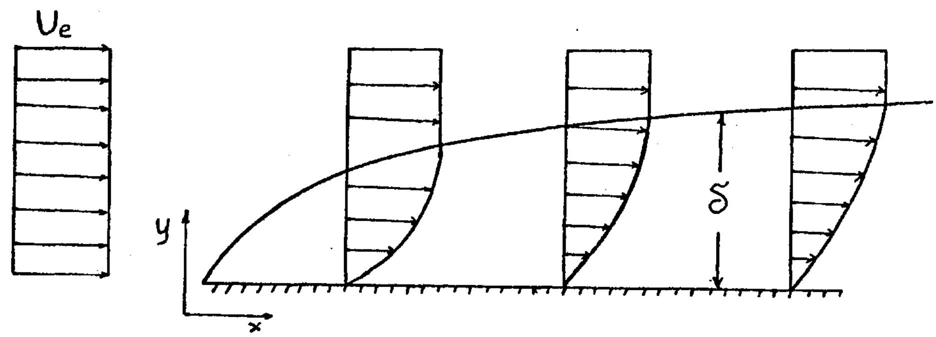

Figure 1

Illustration of Boundary Layer Development

Experiments (for example, sprinkling of particles on the surface of the water, to make the streamlines visible) show that there is a very thin layer around the plate, in which the velocity is considerably smaller than the free stream velocity. The thickness of this layer increases in the downstream direction along the plate.

This layer where the fluid moves with a smaller velocity than that of the main portion of the fluid away from the plate, is called the (velocity) boundary layer.

The thickness of this boundary layer decreases with decreasing viscosity. However, even with very small viscosities, the frictional shearing stresses in the boundary layer are considerable, because of the large velocity gradients across the flow, whereas outside the boundary layer these stresses are very small. This picture of the flow field suggests that for purposes of mathematical analysis, the flow of fluids of small viscosities around solid bodies can be divided into two regions as outlined in the beginning. From the mathematical viewpoint, such a division enables one to simplify considerably the equations describing flow situations of the type discussed. Solutions of the simplified equations have been found to describe with adequate accuracy the motion of fluids of small viscosity around solid bodies, and in a variety of cases the results have been verified experimentally.

2.1.b The Temperature Boundary Layer

Whenever the main stream temperature differs from the temperature of the rigid boundary, one can talk of a temperature (thermal) boundary layer. This means that in such a case the temperature field is considered to be of the boundary layer type, i.e., there exists a narrow zone in the neighbourhood of the body where temperature effects are of importance, whereas the regions beyond this zone are essentially unaffected by the different body temperature. In particular, this is the case when the thermal conductivity of the fluid is small (gases, liquids). In such cases there is a steep temperature gradient at the wall, and the heat flux due to conduction is of the same order of magnitude as that due to convection, only in a thin layer adjacent to the wall.

A temperature boundary layer develops also in the case where a body is immersed in a fluid of equal temperature, flowing at a high Reynolds number. The high velocity gradients across the boundary layer lead to degradation of energy through dissipation and, consequently, to large temperature gradients. In this case it is to be expected that temperature gradients will be important only within the velocity boundary layer, because the quantity of mechanical energy that is transformed into heat through friction is important only there.

The concept of a thermal boundary layer that is formed in conjunction with the velocity boundary layer, facilitates the analysis of the flow situation, in that it makes possible a simplification of the energy equation, similar to that of the equations describing the motion of the fluid.

2.2 RECENT DEVELOPMENTS

For clarity, the underlying concepts of the boundary layer theory, which enable one to simplify the equations describing the flow field of a fluid of small viscosity past a solid body, are summarized below (similar remarks apply to the temperature field):

There are two regions to be considered:

a) A very thin layer in the immediate vicinity of the body in which the velocity gradient normal to the wall is very large (boundary layer). In this region the very small viscosity of the fluid exerts an essential influence in so far as the shearing stresses $\tau = \mu(\partial u/\partial y)$ may assume large values.

b) In the remaining region, no such large velocity gradients occur and the influence of viscous forces is unimportant. In this region, the flow is frictionless and the flow field is the same as that of the undisturbed flow.

The division between the above two regions is not very sharp, and indeed, there is no exact definition of the thickness δ of the boundary layer. It (the thickness) is conceived as the distance from the body, measured in the direction normal to it, beyond which viscous effects can be neglected. A definition of δ, suitable for mathematical purposes is: δ is the distance from the body where the velocity is equal to 99% of its free stream value. As a first approximation it can be shown (R2) that

$$\delta \propto \sqrt{\nu}$$

The curve $u(x) = U_e$ defines what is known as the outer edge of the boundary layer.

The fluid is considered to adhere to the wall, i.e., there is no slip at the wall. The transition from zero longitudinal velocity at the wall to the full magnitude at some distance from it takes place smoothly in the boundary layer.

It is worth noting that Prandtl's boundary layer assumptions, as outlined before, have been subjected to re-examination lately (R3). It was found that with increasing speed of flow, one had to consider the influence of the curvature of the wall (longitudinal and transversal curvature of the wall are neglected in the original formulation), of vorticity of the outer flow, re-examine the condition of no slip at the wall, and also consider the possibility of a temperature jump at the wall.

An attempt to put Prandtl's original suggestions in a more rigorous mathematical formulation was made by Langestrom and Van Dyke (R4, R5). The basic idea remains that of stronger viscous forces in the neighbourhood of the wall. It is assumed that the solution of the boundary layer equations can be achieved by use of two expansions in a parameter $\varepsilon = 1/\sqrt{Re}$. One of the expansions is based on the wall situation and one on the conditions far from the body. Both expansions should be convergent and have a region of overlap, where the results obtained from both expansions have to be matched. There are still no proofs that the matching of such two series can always be achieved (R5, R6). Interesting results concerning the influence of curvature, of outer vorticity, of slip velocity and temperature jump at the wall have been obtained. Clearly, however, the formulation of such a "second order" boundary layer theory has not been completed as yet, although Goldstein (R7, R8) has already applied successfully a "third order" theory for the solution of the boundary layer flow over a semi-infinite flat plate.

Prandtl's theory is viewed upon as a "first order" theory, in the respect that the equations derived represent the first term in a series expansion in ε. Terms of order ε or higher are neglected, while in the second order theory, terms of $O(\varepsilon)$ are retained. It is then expected that solutions based on the second order theory will be valid for lower Reynolds numbers (R9).

The numerical solution of the boundary layer equations presented in this report is based on the boundary layer concept as formulated by Prandtl. Furthermore, the discussion will be limited to Newtonian fluids.

2.3 THE BOUNDARY LAYER EQUATIONS

The equations describing the steady two dimensional flow of a compressible Newtonian fluid are:

Equation of motion

x-direction

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left(\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y}\right) \qquad (1)$$

y-direction

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x} \left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y} \left(\mu \frac{\partial v}{\partial y}\right) \qquad (2)$$

Energy equation

$$\rho C_p \left(u\frac{T}{x} + v\frac{T}{y}\right) = \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \quad \phi \qquad (3)$$

where ϕ is the dissipation function defined as

$$\phi = 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right. + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 \qquad (4)$$

Continuity equation

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (5)$$

Equation of state. For an ideal gas

$$p = \rho RT \qquad (6)$$

The derivation of the boundary layer equations from the set of equations (1-6) will not be shown here. Schlichting (R2) derives them by estimating the order of magnitude of the individual terms. The possibility exists to arrive at the same simplified equations directly, in a purely mathematical way without the adoption of physically plausible concepts (R9, R10, R11).

The basic assumptions involved are:

1. The thickness δ of the boundary layer is very small compared to a characteristic length L of the body.

$$\delta \ll L \Rightarrow \overset{*}{\delta} = \frac{\delta}{L} \ll 1 \quad (7)$$

2. The Reynolds number based on the main flow conditions $Re = U_e L/\nu_e$ is very large and of the order

$$Re = O(1/\overset{*}{\delta}{}^2)$$

This assumption is necessary in order that the viscous forces in the boundary layer can become of the same order of magnitude as the inertial forces.

The following conclusions can be drawn (flow in the x-direction):

a) Equation of motion (Equations 1, 2)

1. $\partial^2 u/\partial y^2$ is of the order $O(1/\overset{*}{\delta}{}^2)$ whereas $\partial^2 u/\partial x^2$ is $O(1)$. Therefore $\partial^2 u/\partial x^2$ in (1) is neglected against $\partial^2 u/\partial y^2$.

2. Inferring that $\partial p/\partial y$ is $O(\overset{*}{\delta})$, the pressure gradient across the boundary layer which would be obtained by integrating (2), all of the terms of which are $O(\overset{*}{\delta})$ or less, would be $O(\overset{*}{\delta}{}^2)$, i.e., very small. Therefore, the pressure across the boundary layer is considered constant and hence equal to its value for the frictionless main flow.

3. Since all terms of the equation of

motion in the y-direction (2) are $O(\overset{*}{\delta})$ or less, the whole equation is discarded.

b). Equation of energy (Equation 3)

1. The term $\partial^2 T/\partial x^2$ can be neglected against $\partial^2 T/\partial y^2$.
2. In order that the remaining conduction term can become of the same order of magnitude as the convection term, the group RePr (where Pr is the Prandtl number based on the free stream conditions, $Pr = K_e/Cp_e\mu_e$) must be $O(1/\overset{*}{\delta}{}_T^2)$, where $\overset{*}{\delta}_T$ is the dimensionless thickness of the thermal boundary layer, $\overset{*}{\delta}_T = \delta_T/L$.
3. In the expression for the dissipation function (4) only the term $(\partial u/\partial y)^2$ is significant.
4. The frictional term $u\,\partial p/\partial x$ is important only if the Eckert number $E = U_e^2/\,Cp_e(T_W - T_e)$ is $O(1)$ whereas the other frictional term $v\,\partial p/\partial x$ is neglected.

The simplified equations, known as the boundary layer equations are:

Continuity

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (8)$$

Motion, x-direction

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y}) \qquad (9)$$

Energy

$$Cp\,\rho(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}) = u\frac{dp}{dx} + \frac{\partial}{\partial y}(K\frac{\partial T}{\partial y}) + \mu(\frac{\partial u}{\partial y})^2 \qquad (10)$$

State

$$\rho = p/RT \qquad (11)$$

These equations are supplemented by expressions for the viscosity and the thermal conductivity as functions of temperature of the form

$$\mu = f_1(T) \qquad (12)$$

$$K = f_2(T) \qquad (13)$$

Equations (8) through (10) form a system of partial differential equations of the parabolic type (R12, R13) in three unknowns u, v, T. (The pressure is considered known and ρ, μ, K can be replaced by their corresponding functions of T.) Theoretically, the above system of three differential equations in three unknowns, along with appropriate boundary conditions can be solved to yield expressions for u, T, and v. However, the exact analytical solution of this set of equations without any assumptions as to the nature of the solution is in practice a formidable task. Only in a limited number of special cases has an analytical solution become available (R2, R14).

Before proceeding further, the boundary layer equations for incompressible flow will be given here, in order to avoid ambiguity, since many a time reference will be made to these equations.

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (8a)$$

Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = - \frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2} \qquad (9a)$$

Boundary conditions

$y = 0 \qquad u = 0 \qquad v = 0$

$y \rightarrow \infty \qquad u = U_e$

2.4 BOUNDARY CONDITIONS

The usual boundary conditions imposed on the equations (8) to (10) are:

a) $y \rightarrow \infty$, all x:

$u \rightarrow U_e$ and

$T \rightarrow T_e$

i.e., at a sufficiently large distance from the solid boundary the flow and temperature fields are unaffected by the presence of the body.

b) $y = 0$ (wall)

1. Assuming no slip: $u = 0$
2. Assuming no temperature jump: $T = T_w(x)$, specified. If the wall is adiabatic: $\left(\frac{\partial T}{\partial y}\right)_{y=0} = 0$

3. If the wall is impermeable $v = 0$. Otherwise, the normal velocity component may vary in any specified way as a function of x of the form $v = v_o(x)$.

The condition of no slip at the wall needs some more attention (R15). This condition has been verified experimentally, and it is a generally acceptable one for most cases as the following argument shows: However smooth a solid surface may appear it will always contain minute projections and cavities, where some fluid is trapped. Still, as far as fluid flow is concerned, such trapped fluid can be regarded as forming part of the surface and it does not move in the direction of the flow. If this fluid were moving, then one would expect extremely large values for the shear stress at the wall, but such values have never been observed.

There exist situations where the no-slip condition is violated, for example when a fluid flows over another with which it does not mix (because the "wall" is able to move when stresses are applied to it), and also when a gas is at a very low pressure. In the latter case however, the gas can no longer be treated as a continuum. Besides, the Navier-Stokes equations of motion are valid only if the characteristic length scale l of the flow is much larger than the mean free path z of the molecules, i.e., $z/l \ll 1$ (R10). For boundary layer flows the significant length scale is the thickness δ of the boundary layer, and to a first approximation

$\delta \propto Re^{\frac{1}{2}}$, whereas

$$z = \frac{\nu}{Cs} \left(\frac{n\gamma}{2} \right)^{\frac{1}{2}}$$

Cs = speed of sound

$\gamma = \frac{Cp}{Cv}$ (specific heat ratio)

Hence, one gets

$$M / Re^{\frac{1}{2}} \ll 1$$

where $M = U_e/Cs$ is the Mach number.

It can be shown (R10) that this condition is equivalent to the condition that the boundary layer assumptions hold.

2.5 BOUNDARY LAYER PARAMETERS

Important parameters in the calculation of the boundary layer are:

a) Displacement thickness, δ_1. It is defined by the relation

$$\delta_1 = \int_{y=0}^{\infty} \left(1 - \frac{u}{U_e}\right) dy \qquad (14)$$

The displacement thickness indicates the distance by which the external streamlines are shifted, owing to the formation of the boundary layer. It provides a more accurate measure of the boundary layer than the boundary layer thickness, whose definition is arbitrary.

b) Shear stress at the wall. It is defined as

$$\tau_w = \mu_w \left(\frac{\partial u}{\partial y}\right)_w \tag{15}$$

where μ is the viscosity of the fluid and the subscript w denotes wall values. The shear stress at the wall is an important design characteristic for most applications. It is for this reason that most results of calculation of boundary layers are reported in the literature in terms of the wall shear.

c) Heat flux at the wall. At the boundary between the fluid and the body, the transfer of heat takes place by conduction only (radiation effects are ignored). The heat flux is then

$$q_w = - K_w \left(\frac{\partial T}{\partial y}\right)_w \tag{16}$$

The heat flux at the wall is of analogous importance to design, as the shear stress.

2.6 PRESSURE GRADIENT AND SEPARATION

The pressure in the boundary layer has been assumed constant for all y. It is a function of the external velocity field only. The precise form of this relationship can be derived by replacing the velocity u by the main stream velocity U_e in the Navier-Stokes equation of motion in the x-direction. For constant properties, Equation (1) then reads

$$U_e \frac{\partial U_e}{\partial x} + V_e \frac{\partial U_e}{\partial y} = - \frac{1}{\rho} \frac{dp}{dx} + \mu \left(\frac{\partial^2 U_e}{\partial x^2} + \frac{\partial^2 U_e}{\partial y^2}\right) \tag{17}$$

Since we are concerned with the flow outside the boundary layer the derivatives of U_e with respect to y are negligible according to the fundamental boundary layer concept. Then (17) reduces to

$$U_e \frac{\partial U_e}{\partial x} = - \frac{1}{\rho} \frac{dp}{dx} + \frac{\partial^2 U_e}{\partial x^2} \tag{18}$$

For most boundary layer flows the term $\frac{\partial^2 U_e}{\partial x^2}$ may be neglected against the other terms in (18), so one finally obtains

$$- \frac{1}{\rho} \frac{dp}{dx} = U_e \frac{dU_e}{dx} \tag{19}$$

which is the required expression for the pressure gradient. If the external flow is one of constant velocity, i.e., $U_e = \text{const.}$, then obviously, there is no pressure gradient.

The phenomenon of separation of the boundary layer is connected with the existence of an adverse pressure gradient. It occurs primarily near blunt bodies such as cylinders and spheres. Behind such a body there exists a region of strongly decelerated flow. In this region the retarded fluid particles cannot penetrate far into the region of increased pressure owing to their small kinetic energy. The thickness of the boundary layer increases considerably in the downstream direction and the flow in the layer is reversed. This causes the decelerated particles to be forced outwards and the boundary layer separates from the wall. Separation is associated with the formation of vortices and large energy losses in the wake of the body. In general, the fluid particles behind the point of separation move in the direction of the pressure gradient, i.e., opposite to the external flow. Figure 2 represents the picture of the flow field near a point of separation.

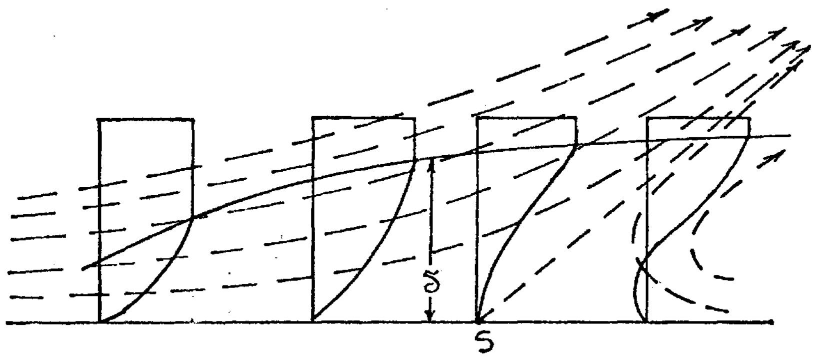

Figure 2. Flow in the Boundary Layer Near a Point of Separation, s.

The point of separation is defined as the point of the reversal of flow in the immediate neighbourhood of the wall. Mathematically, it is defined as the point on the wall where

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0 \qquad (20)$$

and obviously, the skin friction $\tau_w = 0$ there as well. Separation occurs invariably when an adverse pressure gradient ($\frac{dp}{dx} > 0$) exists.

2.7 VALIDITY OF THE BOUNDARY LAYER EQUATIONS

It has already been mentioned that the boundary layer equations are valid provided that the Reynolds number is sufficiently large, so that $\partial^2/\partial x^2$ terms can be neglected against the $\partial^2/\partial y^2$ terms. Clearly this assumption is not valid near the leading edge, where steep changes in the x-direction occur.

A rough estimate of the distance from the leading edge where the boundary layer equation may be expected to hold, may be obtained after some elementary algebra. For the case of the flat plate it was found (R16) that the thickness δ of the boundary layer varies as the square root of the distance x, from the leading edge

$$\delta \propto \sqrt{\frac{\nu x}{U_e}} \qquad (21)$$

One finds that if Re is $O(10^5)$ then ommision of the $\partial^2/\partial x^2$ terms will lead to an error of $O(10^{-2})$ or less from a dimensionless distance $x/L \gg 0.1$. If Re is $O(10^6)$ then for the same x the corresponding error would be reduced to $O(10^{-3})$.

It should be mentioned that laminar flow over a flat plate will prevail for $Re < 5 \times 10^5$ (R2). For values of $Re > 10^6$ the flow becomes turbulent. However, it has been proved that several techniques (for instance cooling of the wall or suction) can be used to keep the flow laminar at much higher Reynolds numbers (R17).

In the region close to the leading edge, where the Prandtl boundary layer equations do not apply, the problem of flow can be solved only by considering the full Navier-Stokes equations. In the past few years this problem has been delved into by several investigators who applied perturbation expansion techniques (R9, R18, R19).

2.8 STARTING PROFILES

Some remarks need to be made concerning starting (initial) profiles. The momentum and energy equations are partial differential equations of the parabolic type, and, in order to start the numerical integration of such equations, one needs, in addition to the boundary conditions, initial profiles, i.e., sets of values of u and T at all grid points along a line specified at a location $x = x_0$ in the x-y plane. The integration of the equations is achieved by use of a marching procedure. Knowing the u and T values along the line $x = x_0$ one may proceed to find the corresponding values on a line $x = x_1$ located one step further downstream, and so on.

The question of the choice of starting profiles has not been dealt with in detail in the literature. This is somewhat surprising, since it is generally known that differential equations of the parabolic type describe physical situations where downstream conditions are uniquely determined from given upstream and boundary conditions (R62, R63).

Stated differently, this means that there exists a solution of the equations which contains the starting profiles as interior, i.e., not starting, profiles (R9). Clearly, starting with such profiles at a location $x = x_0$ would be tantamount to assuming that particular solution that yields these profiles at the location x_o. The continuation of the solution further downstream would be uniquely determined by the boundary conditions of the problem in hand.

One then is faced with the problem of finding suitable profiles to start the numerical computations. One has to either obtain them experimentally or use profiles obtained by some other method. In the latter case, the problem actually solved is that of the continuation of a known solution.

However, problems arise when the flow situation presents a discontinuity ahead of the initial station x_o. In this case, if a starting profile is adopted that does not take this discontinuity into account, one would expect that the solution obtained farther downstream would be unaffected by the presence of the discontinuity. In other words, the solution would be identical to the solution of the problem without the discontinuity.

One way of getting around this difficulty is to carry out experiments and obtain the actual profiles as measured at some distance downstream from the discontinuity. Computation can then be started using these profiles. However, such experimental work is time consuming and in most cases quite tedious. Besides, one would have to perform such experiments for every particular type of flow.

The only other way to treat this problem is to start right from the leading edge utilizing the actual boundary conditions there, as starting profiles. Of course, it is well known that the Prandtl boundary layer equations are not valid in the immediate neighbourhood of the leading edge, because, as it has already been shown, there exist large gradients in the x-direction, which cannot be neglected as it is done in the derivation of the equations.

Wu (R42) has demonstrated, however, that use of such starting profiles leads to solutions at least as accurate as results obtained by methods using "similar" solutions, available at a particular location x_o, as starting profiles. Commenting on the use of the Wu-type profiles, Blotther and Fluegge-Lotz report (R43)

a) It is generally a serious problem to obtain starting profiles. This problem is minimized when similar solutions are sought, since similarity solutions are available in many cases.

b) Wu-type initial profiles lead to numerical solutions which are close to the exact solutions.

Wu's method (R42) is an explicit finite difference one, and in this case care must be taken that the proper mesh sizes are chosen. During the course of the present study, the possibility of using Wu-type initial profiles with the proposed implicit finite difference method has been investigated. It was found that the results were sufficiently accurate for both similar and non-similar flow situations at small distances from the leading edge. Therefore, Wu-type starting profiles have been adopted throughout the computation of solutions of the boundary layer equations to be reported in the last section.

It is clear from the foregoing discussion that use of such starting profiles has the added advantage of providing solutions to boundary layer problems where a discontinuity exists (for example, impulse suction or injection) near the leading edge. The accuracy of the results was verified in several flow cases examined during this study.

The following are referred to as Wu-type starting profiles:

a) The velocity u and temperature T are free-stream values at all the grid points across the layer except at the wall.

b) At the wall the velocity u is zero (no slip condition), and the temperature corresponds to the wall temperature.

c) The normal velocity component, v, is assumed zero at all the grid points across the boundary layer.

2.9 COMPATIBILITY CONDITIONS FOR STARTING PROFILES

In the case where other than the Wu-type initial profiles are used at a station x_o, in order to continue a known solution farther downstream, the starting profiles have to satisfy certain conditions known as "the compatibility conditions at the wall" (R2, R52).

The incompressible boundary layer equations are written as

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \tag{8b}$$

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = - \theta + \frac{\partial^2 u'}{\partial y^2} \tag{9b}$$

with boundary conditions

$$y' = 0 \qquad u', \; v' = 0$$

$$y' \rightarrow \infty \qquad u' \rightarrow 1$$

In these equations the following dimensionless variables have been introduced:

$$u' = \frac{u}{U_e}$$

$$v' = \frac{v}{U_e} \; Re$$

$$x' = \frac{x}{L}$$

$$y' = \frac{y}{L} \; Re$$

$$p' = \frac{p}{\rho \, U_e^2}$$

$$\theta = \frac{dp'}{dx'}$$

$$Re = \frac{U_e L}{\nu}$$

An expansion of the prescribed velocity profile u(x,y) in powers of y is assumed, the expansion being

$$u(x,y) = a_1 y + \frac{a_2}{2!}y^2 + \frac{a_3}{3!}y^3 + \ldots \qquad (22)$$

where the coefficients a_1, a_2, a_3 are functions of x. From (8b) and (19) one obtains a series expression for v:

$$-v = \frac{\dot{a}_1}{2!}y^2 + \frac{\dot{a}_2}{3!}y^3 + \frac{\dot{a}_3}{4!}y^4 + \ldots \qquad (23)$$

where a dot denotes differentiation with respect to x. Substituting (22) and (23) into (9b) one obtains the following conditions that the coefficients a_1, a_2, ... must satisfy:

a_1 free

$a_2 = 0$

$a_3 = 0$

$a_4 = a_1\dot{a}_1$, hence free

$a_5 = 2a_1\dot{\theta}$

$a_6 = 2\theta\dot{\theta}$

$a_7 = 4\ddot{a}_1{}^2 a - a_1\dot{a}_1{}^2$, hence free

$a_8 = 10a_1{}^2\ddot{\theta} - 13\dot{a}_1 a_1 + 9(a_1\ddot{a}_1 + \dot{a}_1{}^2)f$

$a_9 = 40a_1\theta\ddot{\theta} - 16a_1\dot{\theta}^2$

In sum, only the coefficients a_1, a_4, a_7, a_{10} etc., are free (they depend on the choice of a_1 only) while the remaining coefficients are connected with them through the compatibility conditions.

Furthermore, as it was shown by Prandtl, in order to start the numerical integration of the incompressible boundary layer equations (8a, 9a), specification of a starting profile for the velocity u, $u(x_o,y)$ at a location x_o, is enough. One does not need to specify a profile for the normal velocity v since, using the equation of continuity, one can eliminate $\partial u/\partial x$ from the equation of motion and thereby obtain v as a function of u and y. Nor is it necessary to specify the x-derivative of u, since this can be obtained from the continuity equation after v has been calculated. For the compressible flow problem specification of T at the same initial station is also required.

In conclusion, care must be taken in choosing the proper initial profiles, so that the proper variables are chosen, and that the compatibility conditions are satisfied. It has been shown (R56) that with numerical methods, a gross violation of the compatibility conditions may lead to an erratic sequence of velocity profiles.

3. LITERATURE SURVEY

Various methods have been adopted in the effort to solve the boundary layer equations since Prandtl's hypothesis was set forth. The existing literature in this area is quite extensive and one could only hope to give an outline of the techniques that have been used in the past.

3.1 SIMILARITY SOLUTIONS

Blasius (R16) obtained a solution for the incompressible laminar flow over a flat plate at zero incidence. To solve the equations (8a) and (9a), he introduced the similarity variable $n = y\sqrt{U_e/\nu x}$ and defined a stream function ψ by the equations

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

Upon introduction of a dimensionless stream function f(n) defined by

$$\psi = \sqrt{xU_e\nu}\, f(n)$$

equation (9a) is reduced to the following differential equation:

$$ff'' + 2f''' = 0 \qquad (24)$$

(where a prime denotes differentiation with respect to n) with boundary conditions

$$n = 0 \qquad f = 0 \qquad f' = 0$$

$$n \to \infty \qquad f' = 1$$

The solution of (24) obtained by Blasius has been verified by subsequent investigators and has been confirmed experimentally with measurements performed by Nikuradse (R20).

Blasius' solution is a "similar" solution in the sense that two velocity profiles u(x,y) located at different locations x differ only by a scale factor in u and y.

Similar solutions of the boundary layer equations attracted the interest of many early investigators (R21, R22). This is due to the fact that, in cases where similar solutions exist, it is possible to reduce the system of non-linear partial differential equations (8 - 10) to one involving ordinary differential equations. This is considered a definite advantage since methods of analytical or numerical solution of ordinary differential equations are more readily available and easier to apply than methods for the solution of partial differential equations (R23, R24).

Similar solutions exist, and the reduction to ordinary differential equations is possible if the variables in the undisturbed potential flow satisfy certain conditions (R2, R25, R26, R27).

3.2 APPROXIMATE METHODS

Approximate methods of solution have been applied with various degrees of success. One such method, widely used in the past, was suggested by Pollhausen (R28) and extended by Hollstein and Bohlen (R29). The method is based on Von Karman's momentum integral equation (R2). Details of this method may be found in (R2, R28, R29).

Von Karman and Millikan (R30) devised a different approximate method which yields better results than Pollhausen's method, particularly in the case of flows where separation occurs.

From the numerous other approximate methods that have been proposed, one should mention that of Smith (R31) and the one suggested by Wieghardt (R32) and simplified by Walz (R33). These methods seem to be rapid and produce satisfactory results.

For a detailed review of approximate methods, reference (R34) should be consulted.

3.3 ANALYTICAL SOLUTIONS

The applicability of analytical methods is limited to particular flow situations and in fact the majority of available analytical solutions concern mainly flow cases where similar solutions exist.

The failure of analytical methods to treat complicated problems of boundary layer flow, is basically due to the character of the partial differential equations involved. These are non-linear, and in most cases where similar solutions do exist, and a transformation to ordinary differential equations can be obtained, one usually has to turn to series expansion techniques or numerical methods in order to solve the resulting ordinary differential equations. The term, exact solutions of the boundary layer equations, adopted by workers in this field, should not be misleading, in so far as it refers to a solution of the boundary layer equations over the entire flow regime, irrespective of whether it has been obtained analytically or by numerical methods.

3.4 NUMERICAL METHODS

It is not surprising then that numerical methods were used as early as the 30's although computing equipment was scarce. With the advent of the fast computers the use of numerical methods was generalized and several finite-difference techniques were developed in an attempt to obtain accurate solutions of the boundary layer equations. Definite advantages of numerical methods over other existing methods of solution are that they can handle complicated flow problems and that their applicability is not limited to particular forms of variation of the fluid properties, (μ, K, Cp) contained in the equations. In addition, accuracy and the possibility of obtaining easily results for any prescribed changes of the boundary conditions, including discontinuities, and capability to handle a variety of boundary layer problems with a minimum of additional effort, are sufficient reasons to justify the increasing use of numerical methods.

At a first glance it might appear that there already exist many different numerical techniques that have been applied to boundary layer problems, and indeed, many numerical solutions are available in the literature. However, there exist but a few successful methods that have been proposed. Others appear as modifications or extensions of previously suggested techniques and much of the literature concerns investigation of a particular problem rather than development of a "general" method applicable to a variety of boundary layer flows. In what follows, the most successful methods that have been suggested at recent times will be reviewed briefly and their advantages and shortcomings will be discussed. This topic will be delved into again in this study after the procedure developed in this report has been described, so that a comparison can be made.

For several years Smith and Clutter's method (R36 - R40) has been widely used. Their procedure, as applied to incompressible flows, involved the definition of a stream function ψ through the relations

$$u = \frac{\partial \psi}{\partial y}$$

$$v = - \frac{\partial \psi}{\partial x}$$

By the introduction of a dimensionless "height" $n = y\sqrt{U_e/\nu x}$ and a dimensionless stream function $f(x,n) = \psi/\sqrt{U_e \nu x}$ the equation of momentum (9a) transforms to a non-linear partial differential equation in f of third order with respect to n and first order with respect to x. To solve this equation the x-derivatives are replaced by finite differences while the n-derivatives are left in differential form, an idea advanced by Hartree and Womersley (R41), aimed at reducing the partial differential equation to an ordinary one in n. Details of their method of solution can be found in (R36 - R40) and will not be discussed here.

Their method was used successfully in the computation of several boundary layer problems. Advantages of the method are said to be:

a) The method is inherently stable
b) It reduces the problem to the solution of ordinary differential equations
c) It produces highly accurate results
d) The transformed equations exhibit no singularity
e) There are no difficulties in starting the solution in the region near the leading edge since the transformed equation "shows a very good behaviour at the origin".

The most serious disadvantage of this method is the difficulty in satisfying the boundary conditions at the outer edge. This difficulty necessitates additional assumptions at the outer edge in order to make the integration possible, as an initial value problem starting at the wall. The computed values of the functions are compared with the desired values, the assumed conditions are corrected and the process repeated until the outer conditions are satisfied. Though this iteration method of satisfying boundary conditions is acceptable in cases when a single condition is imposed at the outer edge, it is quite cumbersome when the momentum and energy equations are coupled (compressible flow cases) and it can lead to large computing times.

Stability of the method is not ascertained. Smith and Clutter report (R39) that in their computation of a linearly retarded flow over a flat plate, an error of one in the sixth decimal point in the initial profile can produce a difference of one in the second decimal point near the separation point, located at a distance $x = 0.96$ from the origin. Such a sensitivity of the method is unacceptable, for the mere reason that it is extremely difficult to obtain (experimentally or otherwise) profiles accurate to the sixth decimal point.

It is also reported that there exists a rather severe limitation on the step size ΔX to be used. In their calculations $x/\Delta X$ has to be kept less than 25. For smaller values of X the solutions diverged, an evidence of instability. Besides, small step sizes ΔX are essential near separation points or in any region where changes in the outer flow take place rapidly.

To start the numerical integration their method requires specification of u and $\partial u/\partial x$ (and for compressible flows T and $\partial T/\partial x$) in contradiction of the compatibility conditions (R51).

There is no clear indication as to the computation of the normal velocity component, v, and its behaviour is not taken into account during the course of the calculations. This last remark applies to all numerical methods that use a transformation to eliminate v from the momentum and energy equations by automatically (i.e., upon substitution) satisfying the continuity equation, which is discarded. This may lead to violation of the compatibility conditions for the starting profiles and in some cases the results may become eroneous (R51).

For the reasons stated, the method cannot be considered a generally applicable one, though it produced good results in many cases.

Wu (R42) developed an explicit-finite difference method for the solution of the boundary layer problems. While results obtained by him are in good agreement with known solutions, one must be very careful with the choice of the proper step sizes. Simply reducing the step sizes does not always improve the results.

As is always the case with explicit finite-difference methods the step sizes are limited by stability considerations. In Wu's method the stability criterion is

$$\Delta X < \frac{u}{2\nu}(\Delta Y)^2$$

where u is the velocity at any point in the boundary layer. Since u approaches zero near the wall this criterion necessitates a small ΔX and, consequently, a large number of steps have to be taken in the x-direction. In problems with adverse pressure gradient or injection, when u tends to zero at some interior point, ΔX must approach zero and the scheme fails. A similar difficulty appears when ΔY has to be taken small in order to obtain accurate profiles further downstream, when the starting profile is complicated, as in the case of a boundary layer - jet interaction.

Wu was the first to recommend use of the exact boundary conditions at the leading edge as starting profiles as described previously. However, one would not recommend general use of an explicit finite difference method.

Baxter and Fluegge-Lotz (R44) modified the boundary layer equations using the Crocco transformation (R45). An explicit finite difference scheme was then used to solve the resulting equations. It was found that the method was not completely satisfactory due to the small step sizes required to ensure stability. Besides, the Crocco transformation exhibits a singularity at the outer edge of the boundary layer and this must be taken into account, when the numerical approximations are made. Another disadvantage of methods based on the Crocco transformation is that they cannot handle velocity profiles with overshoot. Such velocity profiles occur in certain cases of heated walls with favourable pressure gradient and this effect becomes more important for boundary layer flows with helium injection (R46). In these cases, the velocity at some points in the boundary layer exceeds the free stream velocity.

To overcome these difficulties Fluegge-Lotz and Yu (R47) investigated an explicit finite difference scheme to solve the boundary layer equations in the physical plane (x, y co-ordinates). However, the method was not found satisfactory and, particularly with heated walls and high Mach numbers, the step size requirements were so severe that it was impossible to obtain stable solutions.

An implicit finite difference scheme was developed by Fluegge-Lotz and Baxter (R48). This method appears to be sufficiently suitable for boundary layer problems. However, in order to avoid the extremely involved task of solving a great number of non-linear difference equations, the authors have used a linearization of the non-linear terms, such as $u\frac{\partial u}{\partial x}$ and $u\frac{\partial T}{\partial x}$. Such a linearization introduces an overall truncation error of $O(\Delta X)$ and therefore a small ΔX is required to ensure accuracy of the results.

In order to circumfere this difficulty, Davis and Fluegge-Lotz (R49) introduced a modified method where 3-point differences are used to replace the x-derivatives. In this case, the truncation error is $O(\Delta X^2)$ but one needs initial profiles at two stations x in order to start the calculations. Obtaining two sets of initial profiles is, however, quite a problem. As it has already been mentioned, use of a particular profile at a location $x = x_o$ restricts the applicability of the method to similar flows, unless that particular set of data has been obtained experimentally, or by use of some other method for the same flow being considered. It is obvious that requiring two sets of starting profiles complicates the problem, and this was in fact reported later (R50). Reference (R48) is indeed devoted to the treatment of "similar" flow problems. Besides, specification of profiles at two stations is equivalent to specifying the profiles at one station and also prescribing the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial T}{\partial x}$ at that station, a requirement inconsistent with the original problem.

The authors of (R43) and (R48) find it necessary to specify in addition to the longitudinal velocity profile, a profile of the normal velocity component, in order to start the computation. Ting (R51) shows clearly that this is not in agreement with the formulation of the problem. Indeed Prandtl (R1, R52) showed that once the longitudinal velocity profile is specified it is possible to obtain the normal velocity from the equation of continuity.

Krause (R53), based on Ting's remarks, reformulated the method of reference (R48) so as to eliminate the requirement of an initial profile for the normal velocity profile. His method is basically the same as that set forth by Fluegge-Lotz and Blottner, except for an iteration scheme for the computation of the normal velocity component. It appears that such an iteration, not only eliminates the difficulties at the start of the integration but it also improves the results. However, Krause's method still suffers from the shortcomings of the Fluegge-Lotz and Blottner method, namely that of the linearization of the non-linear terms and the requirement of an accurate starting profile for the longitudinal velocity component specified at some location $x = x_o$.

Pallone (R54) suggested a different approach to the numerical solution of the boundary layer equations. The boundary layer is divided into a number of strips (four or six) parallel to the wall, and the equations are integrated from the wall to the various strips, where the velocity and temperature profiles have been approximated by a polynomial in y. The advantage of the method is that it reduces the equations to a set of ordinary differential equations. However, since a polynomial is fitted through a large number of points in each strip, one would question the accuracy of the method, in so far as there is no guarantee that such a fit can always be made, or that the method would converge to the exact solution as the number of strips is increased. The polynomials can be greatly different from the exact profiles between the fitting points.

Recently, Cebeci, Smith and Wang (R55) presented a new method capable of handling both compressible and incompressible turbulent or laminar flows. A stream-function is introduced and the Probstein-Elliot/Levy-Lees transformation is used to reduce the system of the boundary layer equation to a set of ordinary differential equations. However, the transformation equations used in their method are a natural consequence of seeking similarity solutions. Linearization is necessary in this method as well, in order to avoid solution of non-linear difference equations, after the finite-difference substitutions are made.

The method appears as a modification and extension of the older Smith and Clutter methods. Special attention to the boundary conditions at the outer edge is necessary again. It is reported that double precision arithmetic was used in some cases in order to reduce the truncation error. The results presented (all of them in terms of skin friction and heat transfer coefficients) agree quite well with available data.

From the previous discussion one may conclude that there is still a need for a satisfactory finite-difference method that would avoid the shortcomings of the methods suggested up to now, and that would satisfy most of the following requirements:

1. The method should be simple, accurate and fast.
2. It should handle reliably many problems of boundary layer flow.
3. One should be able to start the integration from the leading edge using the exact boundary conditions there.
4. Stability of the method should be independent of the choice of the step sizes.
5. Linearizations that increase the truncation error should be avoided.
6. Boundary conditions should be easy to satisfy even in the presence of discontinuities.
7. The normal velocity component should be easy to calculate and should be used during the course of the computations in order to improve the accuracy of the results.
8. Starting profiles of u and T and at one station only should be required.
9. A minimum of additional effort should be required in order that the method handles a different boundary layer problem.

10. Any variation of fluid properties and wall or free stream conditions inclucing discontinuities should be easily accounted for.

In what follows an implicit finite difference technique will be described for the solution of the boundary layer equations. Several problems were investigated using this new technique, in an effort to examine the suitability of the method for use with the complicated non-similar boundary layer flows. It is believed that this procedure does not have the shortcomings of the methods discussed in this section.

4. THE FINITE DIFFERENCE SOLUTION OF THE BOUNDARY LAYER EQUATIONS

4.1 DERIVATION OF THE DIMENSIONLESS EQUATIONS

As a first step to the solution of the system of equations (8) to (11), the equations are made dimensionless. Handling of dimensionless equations is convenient from the point of view that all variables stay in scale during the computations, and direct comparison of results is easier.

For convenience, the system of the boundary layer equations is re-written here.

Continuity

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (8)$$

Motion

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = - \frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \qquad (9)$$

Energy

$$\rho Cp\, u\frac{\partial T}{\partial x} + \rho Cp\, v\frac{\partial T}{\partial y} = u\frac{dp}{dx} + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \qquad (10)$$

State (ideal gas):

$$p = \rho RT \qquad (11)$$

The following substitutions are made:

$$\overset{*}{u} = \frac{u}{U_e}$$

$$\overset{*}{v} = \frac{v}{U_e} \tag{26}$$

$$\overset{*}{x} = \frac{x}{L} \tag{27}$$

$$\overset{*}{y} = \frac{y}{L} \tag{28}$$

$$\overset{*}{p} = \frac{p}{\rho_e U_e^2} \tag{29}$$

$$\overset{*}{\rho} = \frac{\rho}{\rho_e} \tag{30}$$

$$\overset{*}{K} = \frac{K}{K_e} \tag{31}$$

$$\overset{*}{C}p = \frac{Cp}{Cp_e} \tag{32}$$

$$\overset{*}{\mu} = \frac{\mu}{\mu_e} \tag{33}$$

$$\overset{*}{T} = \frac{T - T_W}{T_e - T_W} \quad \text{or} \tag{34}$$

$$\overset{*}{T} = \frac{T - T_W}{\Delta T_o} \tag{35}$$

$$\Delta T_o = T_e - T_W \tag{36}$$

Stars denote dimensionless variables. L is a characteristic length of the body, the subscript e denotes main stream values and the subscript w, values at the wall. The temperature is made dimensionless so as to vary from zero at the wall to 1 at the outer edge. In this way direct comparison between the temperature and velocity profiles can be made. If the wall temperature is equal to the free stream temperature or if the heat flux and not the temperature at the wall is given, a convenient definition of a dimensionless temperature is

$\overset{*}{T} = T/T_e$. Subsequent discussion is based on the definition of equation (34). If (34) does not apply, the modification is obvious.

Substituting for u, v,...T, from (25) to (34) into equations (8) to (11) one obtains after some elementary algebra the following equivalent set:

Continuity

$$\frac{\partial(\overset{*}{\rho}\overset{*}{u})}{\partial \overset{*}{x}} + \frac{\partial(\overset{*}{\rho}\overset{*}{v})}{\partial \overset{*}{y}} = 0 \tag{37}$$

Motion

$$\overset{*}{\rho}\overset{*}{u}\frac{\partial \overset{*}{u}}{\partial \overset{*}{x}} + \overset{*}{\rho}\overset{*}{v}\frac{\overset{*}{u}}{\overset{*}{y}} = -\frac{d\overset{*}{p}}{d\overset{*}{x}} + \left\{\frac{\mu_e}{LU_e\rho_e}\right\}\frac{\partial}{\partial \overset{*}{y}}\left(\overset{*}{\mu}\frac{\partial \overset{*}{u}}{\partial \overset{*}{y}}\right) \tag{38}$$

Energy

$$\begin{aligned}\overset{*}{C}p\,\overset{*}{\rho}\,\overset{*}{u}\frac{\overset{*}{T}}{\overset{*}{x}} + \overset{*}{C}p\,\overset{*}{\rho}\,\overset{*}{v}\frac{\partial \overset{*}{T}}{\partial \overset{*}{y}} = &-\left\{\frac{U_e^{\,2}}{Cp_e\Delta T_o}\right\}\overset{*}{u}\frac{d\overset{*}{p}}{d\overset{*}{x}} \\ &+\left\{\frac{K_e}{\rho_e Cp_e U_e L}\right\}\frac{\partial}{\partial \overset{*}{y}}\left(K\frac{\partial \overset{*}{T}}{\partial \overset{*}{y}}\right) \\ &+\left\{\frac{\mu_e U_e}{L\,\rho_e Cp_e\Delta T_o}\right\}\overset{*}{\mu}\left(\frac{\partial \overset{*}{u}}{\partial \overset{*}{y}}\right)^2\end{aligned} \tag{39}$$

State

$$\overset{*}{\rho} = \frac{T_e}{T} \tag{40}$$

or

$$\overset{*}{\rho} = \frac{T_e}{\overset{*}{T}\Delta T_o + T_w} \tag{40a}$$

In deriving the dimensionless equation of state, use is made of the fact that in the boundary layer the pressure $p(x_1, y)$

remains constant on any line $(x = x_1)$. Then, since (11) applies throughout the boundary layer, one obtains from (11):

$$p = \rho RT = \rho_e R T_e = p_e$$

whence (40) is obtained.

The quantities in brackets in equation (38, 39) are evaluated as follows:

$$\frac{\mu_e}{L \rho_e U_e} = \frac{1}{Re} \tag{41}$$

$$\frac{U_e^2}{Cp_e \Delta T_o} = - E \tag{42}$$

$$\frac{K_e}{\rho_e Cp_e U_e L} = \frac{K_e}{Cp_e \mu_e} \frac{\mu_e}{\rho_e L U_e} = \frac{1}{Pr} \frac{1}{Re} \tag{43}$$

$$\frac{\mu_e U_e}{L \rho_e Cp_e \Delta T_o} = \frac{U_e^2}{Cp_e \Delta T_o} \frac{\mu_e}{U_e L \rho_e} = - \frac{E}{Re} \tag{44}$$

Re is the Reynolds number, E is the Eckert number and Pr is the Prandtl number, all based on free stream conditions. One may note the following concerning the physical significance of the dimensionless numbers Re, Pr, E. The Reynolds number represents the ratio of the inertia forces to the friction forces. The Eckert number relates the temperature increase through adiabatic compression in the boundary layer (this temperature increase is equal to $U_e^2/2Cp_e$ (R2)) to the temperature difference between the main stream and the body. Heat effects due to friction and compression are important if $E = O(1)$. It can be proved (R2) that the Eckert number is related to the Mach number $M = U_e/C_s$ through the equation

$$E = (\gamma - 1)M^2 \frac{T_e}{T_w - T_e} \tag{45}$$

where $\gamma = Cp/Cv$, the ratio of specific heats of the medium, and C_s is the speed of sound at the main stream conditions.

The Prandtl number depends only on the properties of the medium. For gases Pr is O(1) while for liquids it is much greater ($O(10^3)$ for oils). The thickness of the temperature boundary layer is proportional, to a first approximation, to $Pr^{-\frac{1}{2}}$.

Substituting from (41) to (44) into equations (38) and (39) the following dimensionless boundary layer equations are obtained:

Continuity

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{46}$$

Motion

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{1}{Re}\frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y}) \tag{47}$$

Energy

$$Cp\,\rho\,u\frac{\partial T}{\partial x} + Cp\,\rho\,v\frac{\partial T}{\partial y} = \frac{1}{RePr}\frac{\partial}{\partial y}(K\frac{\partial T}{\partial y})$$

$$- E\left\{\frac{\mu}{Re}(\frac{\partial u}{\partial y})^2 - u\frac{dp}{dx}\right\} \tag{48}$$

State

$$\rho = \frac{T_e}{T\Delta T_o + T_w} \tag{49}$$

Stars have been dropped but it will be understood that all quantities in equations (46) to (49) are dimensionless (except T_e, T_w, ΔT_o in (49)). All quantities referred to from now on will be dimensionless too, unless otherwise stated.

Expressions for the viscosity μ, thermal conductivity K, and specific heat Cp, as functions of temperature of the form

$$\mu = \mu(T) \tag{50}$$

$$K = K(T) \tag{51}$$

$$Cp = Cp(T) \tag{52}$$

complement the system of equations (46) to (49).

Boundary conditions usually imposed on equations (46) to (48) are:

at the wall, $y = 0$

$$u = 0$$

$$T = 0 \quad \text{or}$$

$$T = T_o(x) \quad \text{or}$$

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = q(x)$$

$$v = 0 \quad \text{or}$$

$$v = v_o(x)$$

at the outer edge, $y \to \infty$

$$u = 1$$

$$T = 1$$

4.2 THE NUMERICAL METHOD

4.2.a Finite Difference Approximations of the Momentum and Energy Equations

Paskonov (R58) has pointed out that it is possible to write both the equations of motion and energy in the form

$$a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(c\frac{\partial f}{\partial y}\right) + d + ef \tag{53}$$

where f stands for either the velocity u or the temperature T. The coefficients a, b, c, d, e, may depend on u or T and their derivatives. It is easy to bring equations (47) and (48) to the form of equation (53). The coefficients are respectively

Motion:
$$a_1\frac{\partial u}{\partial x} + b_1\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(c_1\frac{\partial u}{\partial y}\right) + d_1 + e_1 u \tag{54}$$

$$a_1 = \rho u \tag{55}$$

$$b_1 = \rho v \tag{56}$$

$$c_1 = \frac{\mu}{Re} \tag{57}$$

$$d_1 = -\frac{dp}{dx} \tag{58}$$

$$e_1 = 0 \tag{59}$$

Energy:
$$a_2\frac{\partial T}{\partial x} + b_2\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(c_2\frac{\partial T}{\partial y}\right) + d_2 + e_2 T \tag{60}$$

$$a_2 = Cp\,\rho u \tag{61}$$

$$b_2 = Cp\,\rho v \tag{62}$$

$$c_2 = \frac{K}{RePr} \tag{63}$$

$$d_2 = E\left\{u\frac{dp}{dx} - \frac{\mu}{Re}\left(\frac{\partial u}{\partial y}\right)^2\right\} \tag{64}$$

$$e_2 = 0 \tag{65}$$

Since both equations (54) and (60) are similar to the general equation (53), discussion will be focused on (53). The conclusions will concern both (54) and (60), taking into account the coefficients defined by (55) to (59) and (61) to (65).

Following Paskonov (R58), equation (49) is solved by an implicit finite difference scheme in the x and y plane. A rectangular grid is constructed

$$X = I\Delta X \tag{66}$$

$$Y = N\Delta Y \tag{67}$$

where $1 \leq I \leq IX$ and $1 \leq N \leq NY$, IX, NY being the number of points in the x- and y-directions respectively. An auxiliary net

$$X = (I + \tfrac{1}{2})\,\Delta X \tag{68}$$

$$Y = (N + \tfrac{1}{2})\,\Delta Y \tag{69}$$

is superimposed (Figure 3).

The coefficients a, b, c, d, e are computed at the auxiliary net, i.e., at the "central points of the basic grid.

The terms in (53) are replaced by their corresponding difference approximations as follows:

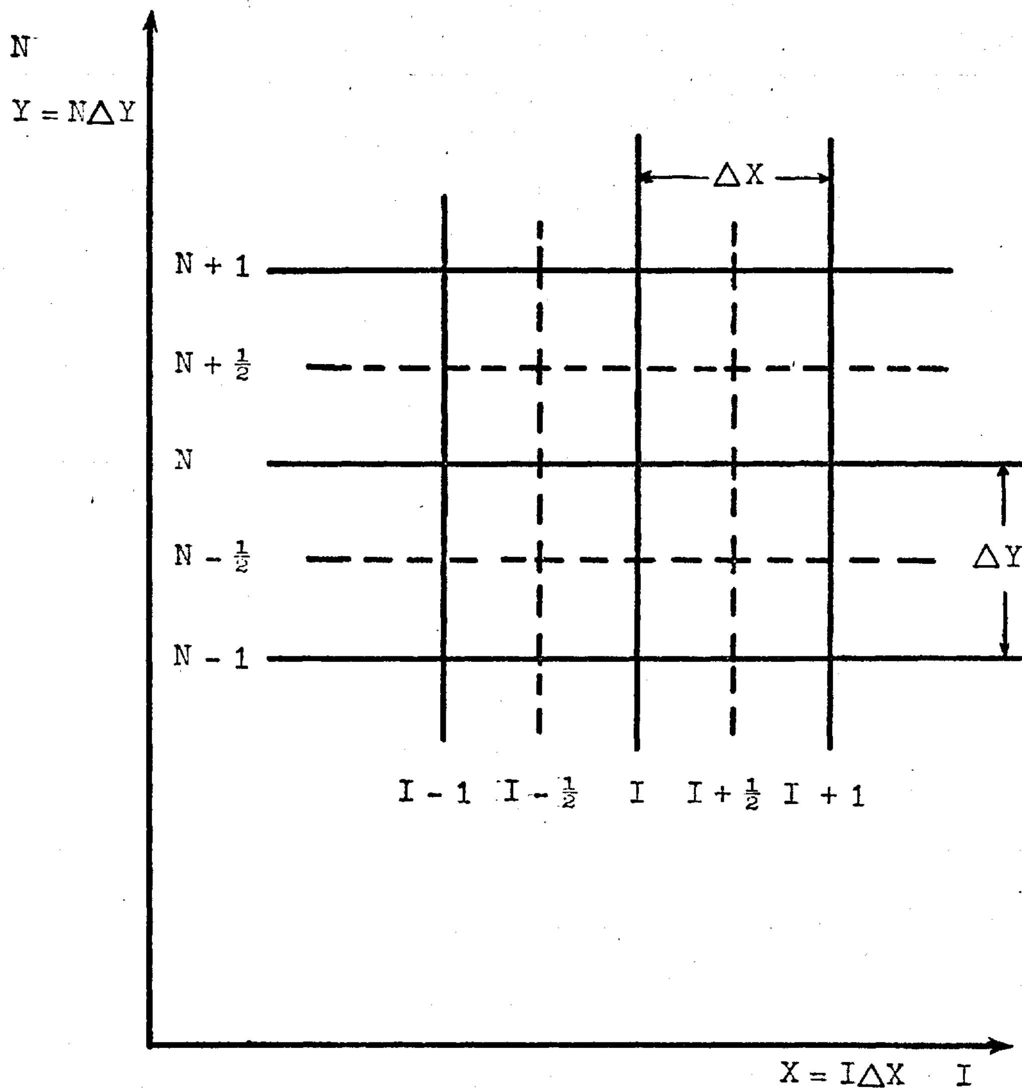

Figure 3: Rectangular grid for the finite difference method

$$a\frac{\partial f}{\partial x} = (a^N_{I-\frac{1}{2}}) \frac{f^N_I - f^N_{I-1}}{\Delta X} + \frac{1}{2}\Delta X \frac{\partial^2 f}{\partial x^2} \tag{70}$$

$$b\frac{\partial f}{\partial y} = \frac{b^N_{I-\frac{1}{2}}}{2\Delta Y}\left\{S(f^{N+1}_I - f^{N-1}_I) + (1-S)(f^{N+1}_{I-1} - f^{N-1}_{I-1})\right\} - \frac{1}{6}\Delta Y^2 \frac{\partial^3 f}{\partial y^3} \tag{71}$$

$$\frac{\partial}{\partial y}(c\frac{\partial f}{\partial y}) = \frac{1}{\Delta Y^2}\left\{S(f^{N+1}_I - 2f^N_I + f^{N-1}_I)\ c^N_{I-\frac{1}{2}} + (1-S)(f^{N+1}_{I-1} - 2f^N_{I-1} + f^N_{I-1})c^N_{I-\frac{1}{2}}\right\} + O(\Delta Y^2) \tag{72}$$

$$d = d^N_{I-\frac{1}{2}} \tag{73}$$

$$ef = e^N_{I-\frac{1}{2}}\ (1-S)\ f^N_{I-1} + Sf^N_I \tag{74}$$

In equations (71), (72) and (74), S is an averaging parameter. In order that no restriction is placed on the step sizes X and Y, S should be chosen in the range $\frac{1}{2} \leqslant S \leqslant 1$.

4.2.b <u>Solution of the Finite Difference Equations</u>

Substituting from (70) to (74) into (53) and rearranging terms, the following difference equation is obtained

$$\alpha^N f_I^{N-1} + \beta^N f_I^N + \gamma^N f_I^{N+1} = \delta^N \tag{75}$$

where

$$\alpha^N = -\frac{S}{2\Delta Y}\left\{b^N_{I-\frac{1}{2}} + \frac{1}{\Delta Y}\left(C^N_{I-\frac{1}{2}} + C^{N-1}_{I-\frac{1}{2}}\right)\right\} \tag{76}$$

$$\beta^N = \frac{a_{I-\frac{1}{2}}}{\Delta X} + \frac{S}{2(\Delta Y)^2}\left\{C^{N+1}_{I-\frac{1}{2}} + 2C^N_{I-\frac{1}{2}} + C^N_{I-\frac{1}{2}}\right\} \tag{77}$$

$$\gamma^N = \frac{S}{2\Delta Y}\left\{b^N_{I-\frac{1}{2}} - \frac{1}{\Delta Y}\left(C^{N+1}_{I-\frac{1}{2}} + C^N_{I-\frac{1}{2}}\right)\right\} \tag{78}$$

$$\begin{aligned}\delta^N = {} & \frac{(1-S)}{2\Delta Y}\left\{b^N_{I-\frac{1}{2}} + \frac{1}{\Delta Y}\left(C^N_{I-\frac{1}{2}} + C^{N-1}_{I-\frac{1}{2}}\right)\right\} f^{N-1}_{I-1} \\ & + \left\{\frac{a^N_{I-\frac{1}{2}}}{\Delta X} - \frac{(1-S)}{2(\Delta Y)^2}\left[\left(C^{N+1}_{I-\frac{1}{2}} + 2C^N_{I-\frac{1}{2}} + C^{N-1}_{I-\frac{1}{2}}\right) - e^N_{I-\frac{1}{2}}\right]\right\} f^N_{I-1} \\ & - \frac{(1-S)}{2\Delta Y}\left\{b^N_{I-\frac{1}{2}} - \frac{1}{\Delta Y}\left(C^{N+1}_{I-\frac{1}{2}} + C^N_{I-\frac{1}{2}}\right)\right\} f^{N+1}_{I-1} \\ & + d^N_{I-\frac{1}{2}}\end{aligned} \tag{79}$$

Equation (75) enables one to compute the values of f along the grid line I provided that the values of f on the previous grid line (I - 1) and the values of the coefficients a, b, c, d, e on the grid line $(I - \frac{1}{2})$ are known. Boundary conditions for f at the wall and at the outer edge should also be specified. In most cases

$$N = 1, \text{ Wall:} \qquad f_I^{N=1} = 0 \tag{80}$$

$$N = NY, \text{ Outer Edge: } f_I^{N=NY} = 1 \tag{81}$$

(The particular form of equations (80) and (81) can be determined from the prescribed boundary conditions. It is easy to see that (80) and (81) are the actual boundary conditions for the velocity u, and for T in the case where the wall temperature is specified.)

In this fashion one may proceed from grid line (I - 1) to line (I) and so on to integrate the equations. The values of f on the grid line (I = 1) are known, once a starting profile for f (i.e., u and T) has been specified. However, the values of the coefficients a, b, c, d, e on the grid line $(I - \frac{1}{2})$ are still unknown. Thus, one has to compute these coefficients as functions of unknown functions. This can be achieved by iteration in the usual way. In the first approximation the values of a function on the $(I - 1)^{th}$ line are assumed for the corresponding points on the I^{th} line.

It can be seen from (76) that $\alpha^N < 0$. Also from (77), $\gamma^N < 0$, in general, since b is O(v), (equation (56, 62)) while $c/\Delta Y$ is $O(1/\Delta Y)$. Obviously, $\beta^N > 0$ and moreover

$$\beta^N > -(\alpha^N + \gamma^N) \tag{82}$$

as it can be verified by direct substitution from (76) to (78) into (82).

This last remark suggests that direct use of the recurrence relation (75) should be avoided. This is because in order to compute a particular value of f_I^N one would have to subtract very nearly equal numbers, and so large errors may arise, especially at the start of the calculations, since both u and T increase almost exponentially with N.

Therefore a different procedure has been employed (R63, R64). It will be noted that all quantities stay in scale during the computations by this method.

We require two quantities A^N and B^N termed "forcing coefficients", such that for any f_I^N the relation

$$f_I^N = A^N f_I^{N+1} + B^N \tag{83}$$

holds. If (79) is to be true for any f_I^N one finds, taking (80) into account that

$$A^1 = 0 \tag{84}$$

$$B^1 = 0 \tag{85}$$

Applying (83) with $N-1$ instead of N there results

$$f_I^{N-1} = A^{N-1} f_I^N + B^{N-1} \tag{86}$$

Substituting for f_I^{N-1} from (86) into (75) one obtains

$$f_I^N = \frac{-\gamma^N}{\alpha^N A^{N-1} + \beta^N} f_I^{N+1} + \frac{\delta^N - \alpha^N B^{N-1}}{\alpha^N A^{N-1} + \beta^N} \tag{87}$$

Direct comparison of (83) and (87) shows that

$$A^N = \frac{-\gamma^N}{\alpha^N A^{N-1} + \beta^N} \tag{88}$$

$$B^N = \frac{\delta^N - \alpha^N B^{N-1}}{\alpha^N A^{N-1} + \beta^N} \tag{89}$$

Starting with the known values of A^1 and B^1 one may proceed to find A^N and B^N in the direction of increasing N by use of (88) and (89). Knowing A^N and B^N for all $1 \leqslant N \leqslant NY$, one can easily determine all f_I^N from (83) starting with the boundary condition at the outer edge(81).

Note that if $A^{N-1} < 1$, then

$$A^N \leqslant \frac{-\gamma^N}{\alpha^N + \beta^N} < \frac{-\gamma^N}{-\gamma^N} = 1$$

by use of (82). Since $A^1 = 0$, this establishes that $0 \leqslant A^N < 1$ for all N. Since f_I^N is generally less than 1 (or, at any rate bounded) it can be seen from (83) that B^N will also be of the same order of magnitude as A^N.

It is seen then that use of (83) instead of (75) is advantageous from the computational point of view. Moreover, this method for solving difference equations of the type of (75) is faster than conventional matrix methods. To illustrate the point, suppose that the coefficients α^N, β^N, γ^N, δ^N in (75) are constant, then use of (83) requires only three multiplications and two divisions per point per line, while matrix methods involve inversion of the matrix of the coefficients α^N, β^N, γ^N and post-multiplication of the inverse matrix by the vector of the "constants" δ^N. The last operation alone requires NY multiplications per point, per line.

4.2.c Finite Difference Approximation of the Continuity Equation

The finite difference expression for the terms in the continuity equation (42) are

$$\frac{\partial(\rho u)}{\partial x} = \frac{1}{2\Delta X}\left\{(\rho u)_I^N - (\rho u)_{I-1}^N + (\rho u)_I^{N+1} - (\rho u)_{I-1}^{N+1}\right\} \quad (90)$$

$$\frac{\partial(\rho v)}{\partial y} = \frac{1}{\Delta Y}\left\{(\rho v)_{I-\frac{1}{2}}^{N+1} - (\rho v)_{I-\frac{1}{2}}^{N}\right\} \quad (91)$$

The particular choice of expressions (90) and (91) lies in the fact that the truncation errors are small ($O(\Delta X^2)$ and $O(\Delta Y^2)$). Also the truncated terms involve only third order derivatives of (ρv) with respect to y. This is important since the second derivatives may assume large values near the wall.

Substituting (90) and (91) into (46) one obtains:

$$v_{I-\frac{1}{2}}^{N+1} = \frac{1}{\rho_{I-\frac{1}{2}}^{N+1}}\left\{(\rho v)_{I-\frac{1}{2}}^{N} - \frac{\Delta Y}{2\Delta X}\left((\rho u)_I^N - (\rho u)_{I-1}^N + (\rho u)_I^{N+1} - (\rho u)_{I-1}^{N+1}\right)\right\} \quad (92)$$

Having computed the values of u and T at all grid points on the lines (I), (I - ½), (I - 1), equation (92) is used to calculate the normal velocity component v starting from the known boundary condition at the wall (N = 1).

4.2.d <u>The Computation of the Boundary Layer Parameters</u>

The displacement thickness, wall shear and heat transfer are computed from finite-difference approximations of equations (14, 15, 16). In the calculation of the displacement thickness Simpson's rule is used for the evaluation of the integral. Thus:

$$\delta_1 = NY \times \Delta Y - \frac{\Delta Y}{3}\left(4\sum_{N=2}^{NY-1} u_N(\text{even}) + 2\sum_{N=3}^{NY-1} u_N(\text{odd}) + u_{NY}\right) + O(\Delta Y^5) \qquad (93)$$

Also

$$\tau_W = \frac{\mu_W}{6\Delta Y}\ (18u_2 - 9u_3 + 2u_4) + O(\Delta y^4) \qquad (94)$$

$$Q_W = -\frac{K_W}{6\Delta Y}\ (18T_2 - 9T_3 + 2T_4 - 11T_1) + O(\Delta Y^4) \qquad (95)$$

In (93) and (94), u_1, the velocity at the wall, is taken to equal zero and the quantities in equations (94) and (95) are dimensional.

A dimensionless skin friction may be defined as follows:

$$Cf = \frac{\tau_W}{\rho_e U_e^2} \qquad (96)$$

or, introducing the dimensionless u, and y, from equations (25, 28),

$$Cf = \frac{\mu_w}{\mu_e} \frac{1}{Re} \left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{97}$$

and Cf may be computed numerically from

$$Cf = \frac{\mu_w}{\mu_e} \frac{1}{Re} (18u_2 - 9u_3 + 2u_4) \tag{98}$$

A heat transfer coefficient λ may be defined, such that according to Newton's law of cooling, the quantity of heat exchanged between the solid and the fluid per unit area and time is:

$$Q = \lambda (T_e - T_w) = \lambda \Delta T_o \tag{99}$$

Introducing the dimensionless temperature $\overset{*}{T}$ and $\overset{*}{y}$ from equations (34) and (28), equation (16) assumes the form

$$Q = - \frac{K_w \Delta T_o}{L} \left(\frac{\partial \overset{*}{T}}{\partial \overset{*}{y}}\right)_w \tag{100}$$

Introducing the dimensionless Nusselt number

$$Nu = \frac{\lambda L}{K_w} \tag{101}$$

(100) becomes

$$Q = - \frac{\lambda \Delta T_o}{Nu} \left(\frac{\partial \overset{*}{T}}{\partial \overset{*}{y}}\right)_w \tag{102}$$

and taking into account that $Q = Q_w$, and equation (99) one obtains

$$Nu = - \left(\frac{\partial \overset{*}{T}}{\partial \overset{*}{y}}\right)_w \tag{103}$$

The Nusselt number may be thus used to characterize the heat transfer at the wall. It may be computed from the equation

$$Nu = -\frac{1}{6\Delta Y}\left(18T_2 - 9T_3 + 2T_4\right) \qquad (104)$$

where all the quantities are dimensionless and $T_1 = 0$ according to (34).

5. CONVERGENCE AND STABILITY

Whenever a differential equation is replaced by a finite difference approximation one must investigate the convergence and the stability of the difference scheme. Let D represent the exact solution of a differential equation, Δ the exact solution of the difference equation and A the numerical solution of the difference equation. Then, $(D - \Delta)$ is called the truncation error. It arises because of the finite distance between the points of the difference net. Convergence is concerned with the conditions under which $\Delta \rightarrow D$. $(\Delta - A)$ is called the numerical error. Though $(\Delta - A)$ may consist of several errors we may consider that it is basically due to round-off errors during the computations. Finding the conditions under which $(D - A)$ remains small throughout the entire region of integration is a problem of stability.

The problems of the convergence and stability of finite difference schemes used to solve non-linear partial differential equations has not been resolved completely up to now. More difficulties are encountered when a study of the stability of a system of such equations is attempted.

Convergence may be ascertained if the truncation error involved in replacing the differential by difference quotients goes to zero as the step sizes go to zero. That the finite difference approximations used in this study satisfy this requirement can be noted from equations (70) to (72).

The faster the truncation error goes to zero, the more accurate is the difference approximation and it would be expected that the numerical solution would approximate the exact solution more accurately.

However, convergence does not guarantee stability. To ensure stability of a numerical scheme, one must ensure that an error occurred in the computations will remain bounded. Von Neumann suggested that the stability of a finite difference method may be determined by considering the growth of the general error term. To do so one must perform a Fourier series expansion of a line of errors.

A lucid account of Von Neumann's theory may be found in (R59), (R61). However, only partial differential equations with constant coefficients can be treated using this method, and to the best of the author's knowledge, no general method of treating the stability of numerical schemes approximating non-linear partial differential equations is available up to the present time.

An approximate stability analysis for the case in hand can be performed if the following assumptions are made:

1. The momentum equation dominates in determining the stability of the finite difference method. With this assumption one needs to consider only the

stability requirements for the equation of motion, instead of the whole system of equations (46) to (48). Blottner and Fluegge-Lotz report, (R43), that this assumption can be made in so far as it has been justified repeatedly from previous experience.

2. The mesh sizes are sufficiently small so that the coefficients in the equation of motion (47) vary only slightly between adjacent grid points and can be considered constant at these points.

An approximate stability analysis of the finite difference scheme presented in this report based on these assumptions is given in the appendix. The result of the analysis indicates that the finite difference scheme is stable for any choice of the step sizes ΔX and ΔY provided the averaging parameter S is chosen so that

$$\tfrac{1}{2} \leqslant S \leqslant 1$$

6. THE INTEGRATION PROCEDURE

The basic algorithm used to accomplish the numerical integration of the boundary layer equations is given in Figure 4. A detailed algorithm of the computations is given in the appendix along with the computer listings.

Wu-type starting profiles have been used throughout the computations to be reported in the last section. It has been found that use of such profiles leads to quite accurate results at a short distance from the leading edge.

The iteration scheme presented in the algorithm of Figure 4, is one of several alternates that were examined during the course of this study. It was found that iteration for the normal velocity v is not necessary in general. In most cases, results obtained when iteration for v was included differ in the fourth decimal place from the results computed without such iteration.

A different scheme was also examined for the iteration for u and T. In this case the scheme involved an iteration until the profiles of the velocity components u and v were established along a particular line using only the equations of motion and continuity. These profiles were then used to compute T from the energy equation. However, it turned out that additional iterations for T were necessary, and the values of u and v had to be re-computed. Although in the end

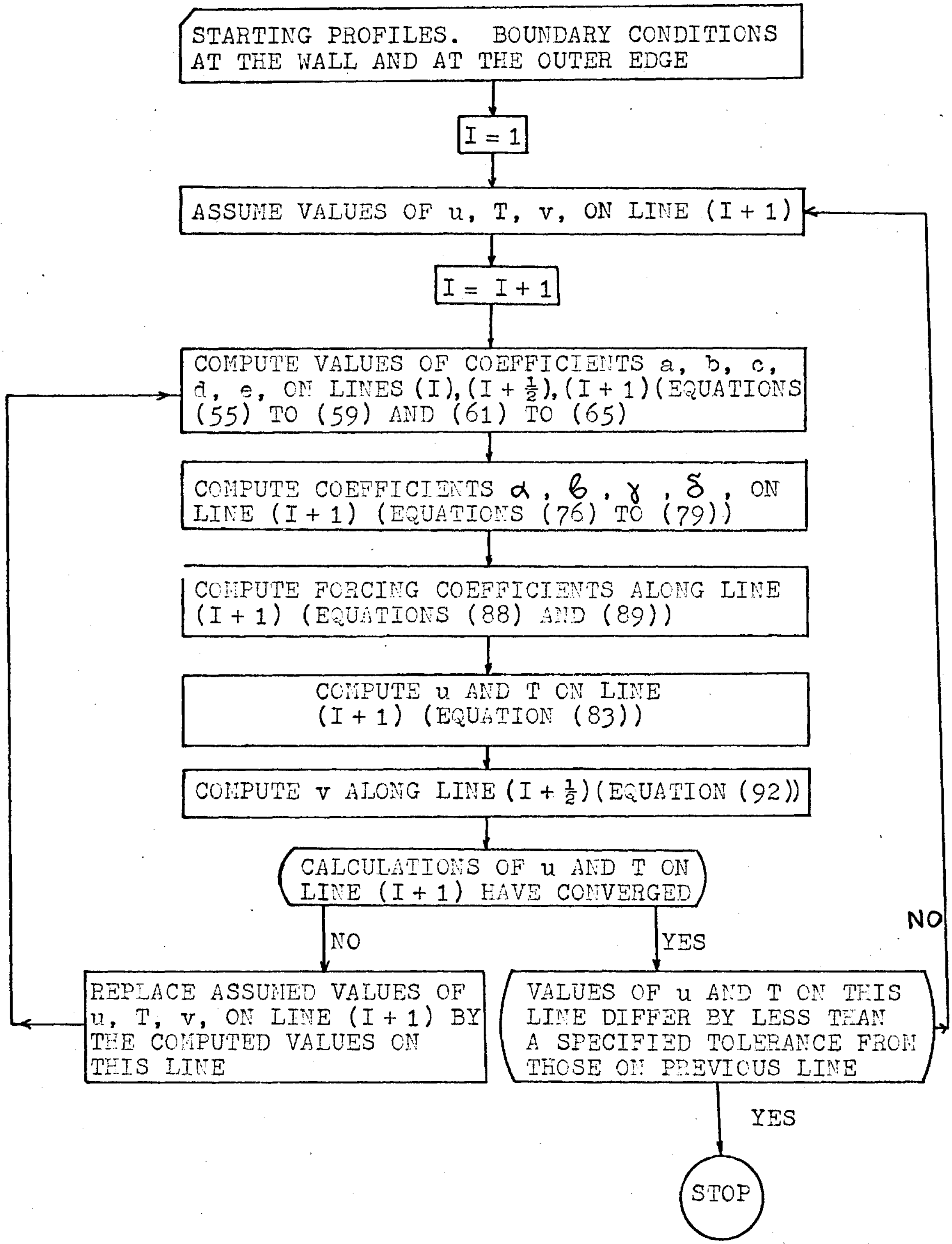

Figure 4. Basic Computer Algorithm

the results were identical to those obtained by the algorithm of Figure 4, almost twice as many iterations were necessary and the computer time was increased by approximately 40%. This was expected since the momentum and continuity equations involve the density ρ, and satisfactory profiles for u and v can be obtained only after ρ, i.e., T, has been calculated.

The number of iterations to establish profiles along a particular line was invariably small. With a tolerance of 0.001 for convergence of the velocity and temperature profiles along a line, an average of 10 iterations per line was required at the start of the calculations for the flow over a flat plate. Farther downstream, after the first 5 stations, the number of iterations never exceeded three.

6.1 DISCUSSION OF COMPUTER RUNS

Results obtained with the algorithm of Figure 4 at the tenth station downstream from the edge were accurate to the second decimal point. Near the twentieth station accuracy in the third decimal point was obtained and beyond the 50th station the profiles changed by less than 0.0001. The greatest changes occurred always near the wall.

As it has been mentioned already there is no limitation on the step sizes ΔX and ΔY to be used in the numerical integration, provided $1 \leqslant S \leqslant \frac{1}{2}$. Several computer runs were made in a effort to investigate the influence of the step sizes on the accuracy of the results and on the computer time.

The results of such a series of runs with different step size ratios are presented in the last section for the case of a flat plate at zero incidence. It may be noted here, that step size ratios in the region

$$\frac{1}{20} \leqslant \frac{\Delta X}{\Delta Y} \leqslant 100$$

have been found to produce accurate results in most cases. From equations (70) to (72) it is clear that use of small ΔX is recommended in order that the truncation error of the finite difference approximation for $\frac{\partial f}{\partial x}$ (equation (66)) remains small. However, use of small ΔX would require many steps in the x-direction in order to cover a sufficient distance in the downstream direction. It was found that a step size ratio

$$1 \leqslant \frac{\Delta X}{\Delta Y} \leqslant 20$$

produces results accurate to the 4^{th} decimal point and at the same time enables one to proceed with large steps in the downstream direction. For ratios

$$\frac{\Delta X}{\Delta Y} > 100$$

less accurate results were obtained and approximately twice as many iterations per line were necessary to ensure convergence of the profiles at a particular station, as compared to the number of iterations when $\frac{\Delta X}{\Delta Y} = 20$. Iteration for the normal velocity component **v** was found to improve the accuracy of the results in this case. It was also found that while the computation of the longitudinal velocity U is relatively insensitive to the use of large step ratios (error in the fourth decimal place in the case of a flat plate with $\frac{\Delta X}{\Delta Y} = 100$) the computation of the normal velocity component v is subject to severe errors when iteration for v is not used (error in the second decimal point for the same case). Iteration for v increases the computer time by approximately 10% and gives values of v accurate to 1% for the flat plate with $\frac{\Delta X}{\Delta Y} = 100$

The computer time is invariably small. For 100 points in the y-direction and 100 stations in the x-direction with outputs at 10 stations, the computer time on the CDC 6400 is 25 seconds for the incompressible boundary layer on a flat plate at zero incidence and 35 seconds for the compressible flow in the same case. Computer time for other case studies is of the same order of magnitude, i.e., approximately 0.25 seconds per station for the incompressible and 0.35 seconds per station for the compressible boundary layer.

7. CASE STUDIES

Several boundary layer flow problems were solved using the method developed in this report, in order to evaluate the accuracy and speed of the finite difference technique. The examples presented in this section have been chosen from the cases where accurate solutions have already been obtained by other methods, so that direct comparisons can be made.

7.1 BOUNDARY LAYER ON A FLAT PLATE, INCOMPRESSIBLE FLOW

This is the simplest example of the application of the boundary layer equations. Historically, this was the first boundary layer case treated by Blasius (R16) in his Doctor's thesis at Goettingen, in order to illustrate the application of Prandtl's boundary layer theory. Several investigators have since then confirmed Blasius' results. For an account of previous work on this problem reference (R2) may be consulted.

This flow problem has been solved using the present method, because it provides a good test of the accuracy of the numerical technique, since exact solutions are available.

The dimensionless boundary layer equations to be solved in this case are:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re}\ \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$y = 0 \qquad u = v = 0$$

$$y \rightarrow \infty \qquad u = 1$$

The results obtained are plotted in Figure 5 for the longitudinal velocity component, Figure 6 for the normal velocity component, and Figure 7 for the dimensionless local shear stress at the wall, and are compared to Howarth's (R64) results. The excellent agreement in all cases may be noted from these plots. Table 1 presents a study of the dependence of the longitudinal velocity u, on the ratio of the step sizes $\Delta X/\Delta Y$. It is seen from this table that the variation of the results is negligible (4^{th} decimal point). A similar study for the normal velocity component v is given in Figure 8. In this case results obtained with and without iteration for v are presented. It may be noted that for large step size ratios $\Delta X/\Delta Y$, iteration for v is necessary in order to obtain accurate results.

Figures 9 and 10 show the variation of computer time with the number of points in the y-direction and the number of stations in the x-direction. Total time and average time per station are shown. It is seen that the computer time is indeed small.

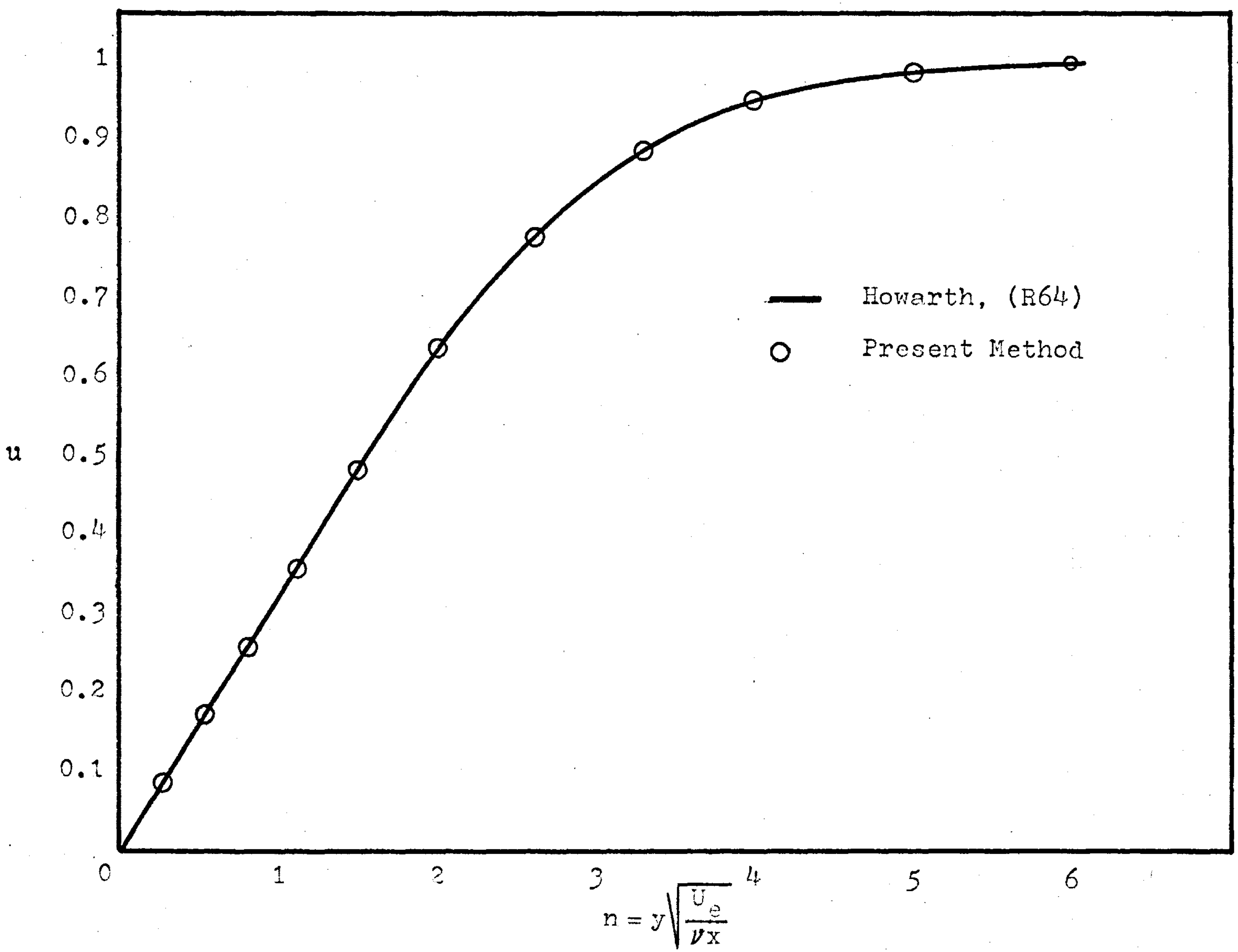

Figure 5: Incompressible boundary layer on a flat plate.

The longitudinal velocity component

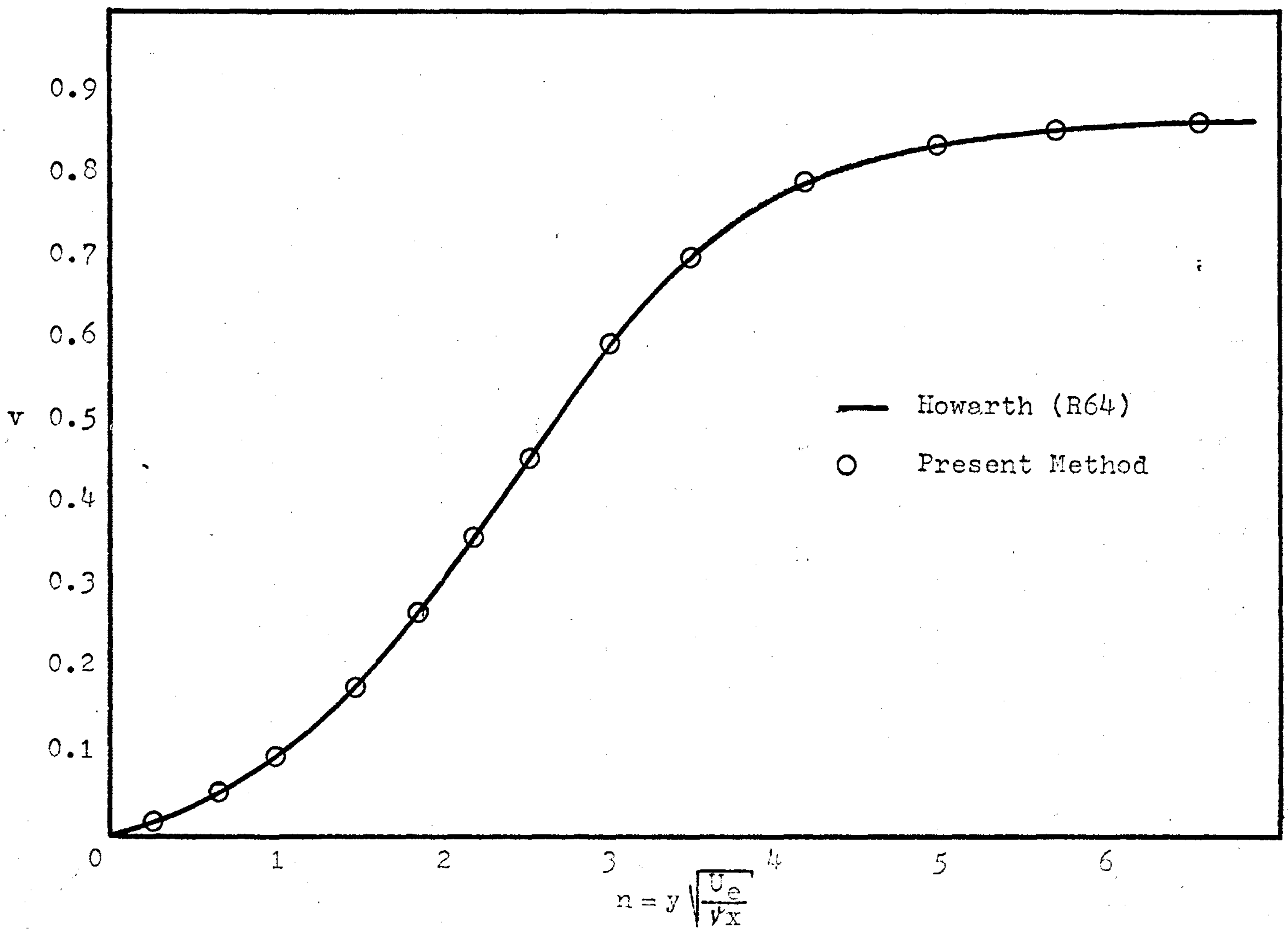

Figure 6: Incompressible boundary layer on a flat plate. The normal velocity component

Table 1

INCOMPRESSIBLE BOUNDARY LAYER ON A FLAT PLATE

THE LONGITUDINAL VELOCITY COMPONENT,u

	HOWARTH (R64)	Present Method 100 y-points			
$\Delta X/\Delta Y$		0.1	1	10	50
$n=y\sqrt{\frac{Re}{x}}$		u			
0	0	0	0	0	0
0.2	0.06641	0.06625	0.06601	0.06592	0.06576
0.4	0.13277	0.13250	0.13228	0.13214	0.13206
0.6	0.19894	0.19876	0.19854	0.19828	0.19813
0.8	0.26471	0.26465	0.26448	0.26437	0.26423
1.0	0.32979	0.32972	0.32963	0.32950	0.32938
1.2	0.39378	0.39375	0.39368	0.39363	0.39359
1.4	0.45627	0.45624	0.45616	0.45608	0.45604
1.6	0.51676	0.51666	0.51664	0.51657	0.51651
1.8	0.57477	0.57470	0.57467	0.57461	0.57449
2.0	0.62977	0.62973	0.62970	0.62966	0.62958
2.2	0.68132	0.68130	0.68129	0.68125	0.68122
2.4	0.72899	0.72900	0.72895	0.72893	0.72888
2.6	0.77246	0.77248	0.77244	0.77241	0.77236
2.8	0.81152	0.81154	0.81150	0.81147	0.81143
3.0	0.84605	0.84607	0.84601	0.84599	0.84595
3.2	0.87609	0.87612	0.87607	0.87602	0.87600
3.4	0.90177	0.90181	0.90175	0.90172	0.90170
3.6	0.92333	0.92336	0.92328	0.92325	0.92322
3.8	0.94112	0.94117	0.94111	0.94109	0.94105
4.0	0.95552	0.95559	0.95552	0.95550	0.95547
4.4	0.97587	0.97596	0.97591	0.97588	0.97586
4.8	0.98779	0.98791	0.98784	0.98780	0.98777
5.2	0.99425	0.99432	0.99426	0.99423	0.99419
5.6	0.99748	0.99760	0.99753	0.99749	0.99745
6.0	0.99898	0.99911	0.99902	0.99897	0.99895
6.4	0.99961	0.99976	0.99965	0.99961	0.99956
6.8	0.99987	1.00000	0.99994	0.99990	0.99988
7.2	0.99996	1.00000	1.00000	0.99995	0.99994
7.6	0.99999	1.00000	1.00000	1.00000	1.00000
8.0	1.00000	1.00000	1.00000	1.00000	1.00000

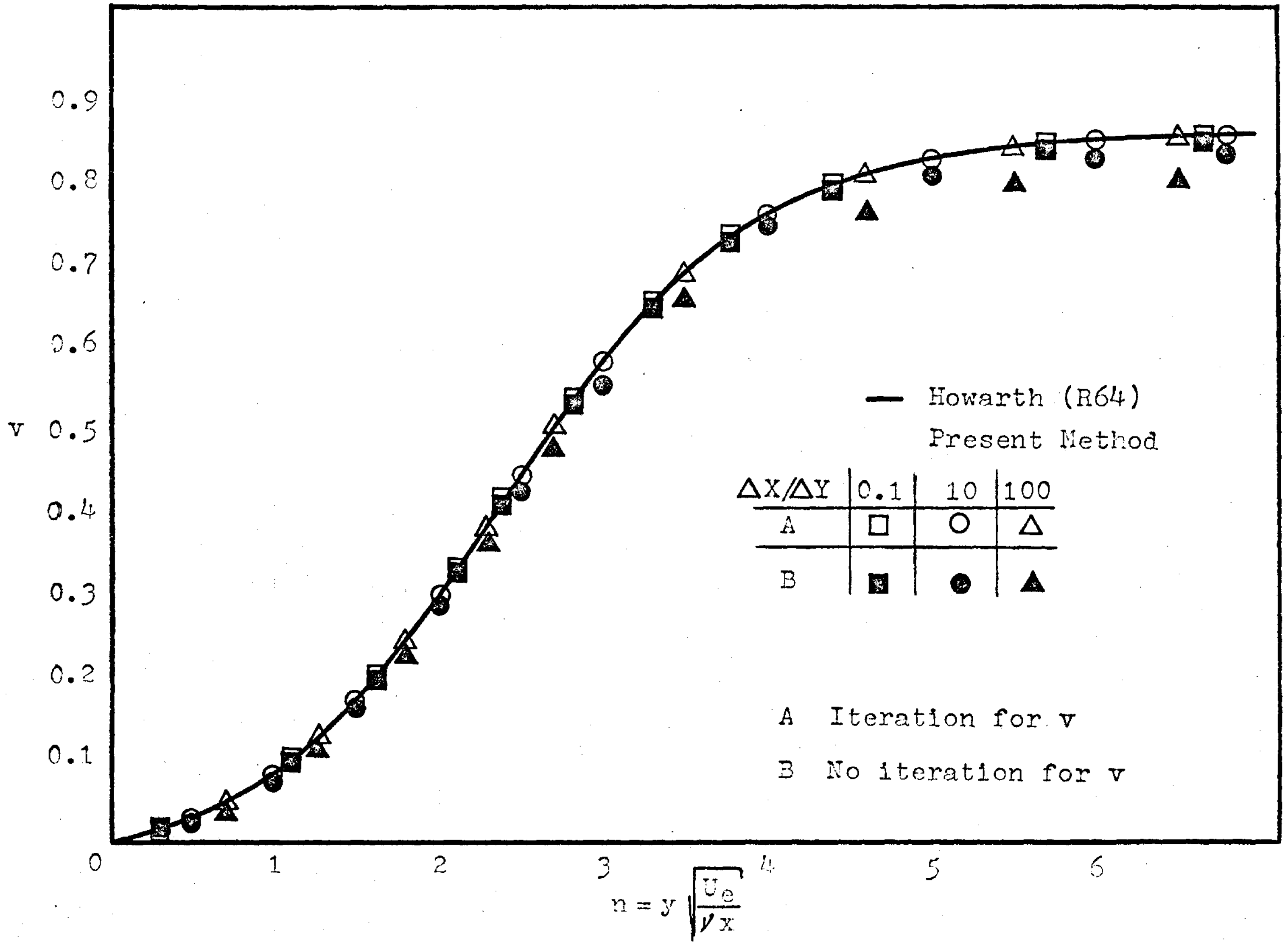

Figure 8: Variation of the normal velocity component with the step size ratio and with iteration for v

Figure 9

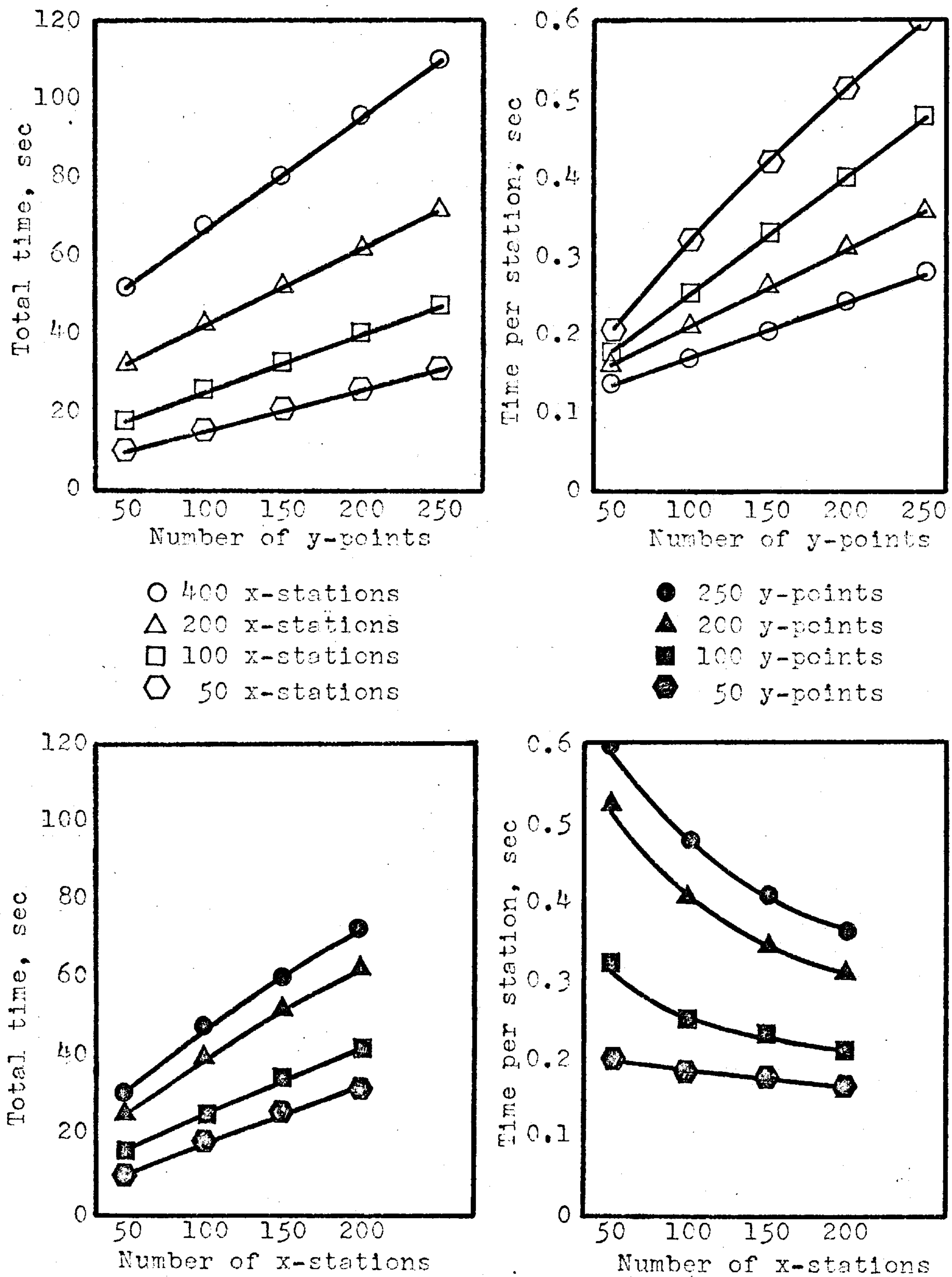

Figure 10

7.2 BOUNDARY LAYER ON A FLAT PLATE, COMPRESSIBLE FLOW

In this case, the system of equations to be solved is:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = \frac{1}{Re}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

$$Cp\,\rho u\frac{\partial T}{\partial x} + Cp\,\rho v\frac{\partial T}{\partial y} = \frac{1}{RePr}\frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) - \frac{E}{Re}\mu\left(\frac{\partial u}{\partial y}\right)^2$$

$$\rho = \frac{T_e}{T\Delta T_o + T_w}$$

$$\mu = f_1(T)$$

$$k = f_2(T)$$

$$y = 0 \qquad u = 0$$
$$v = 0$$
$$T = 0$$

$$y \to \infty \qquad u = 1$$
$$T = 1$$

This system has been solved using the present method. The algorithm of Figure 4 is employed for the numerical solution. Cohen and Reshotko (R65), Brown and Donoughe (R66) and Chapman and Rubesin (R67) have considered the same problem. The velocity and temperature profiles obtained by the present

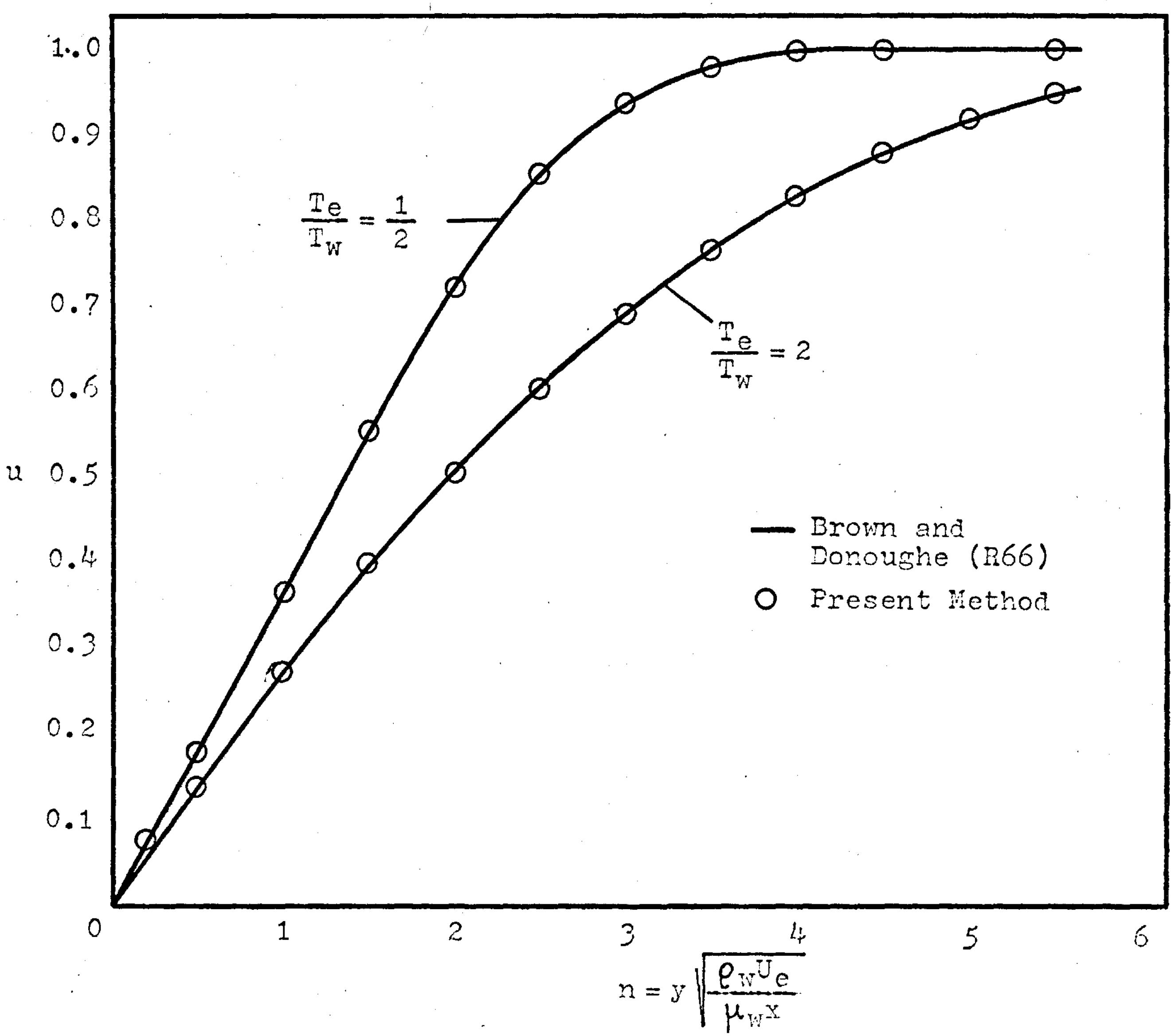

Figure 11: Compressible boundary layer on a flat plate. The longitudinal velocity component

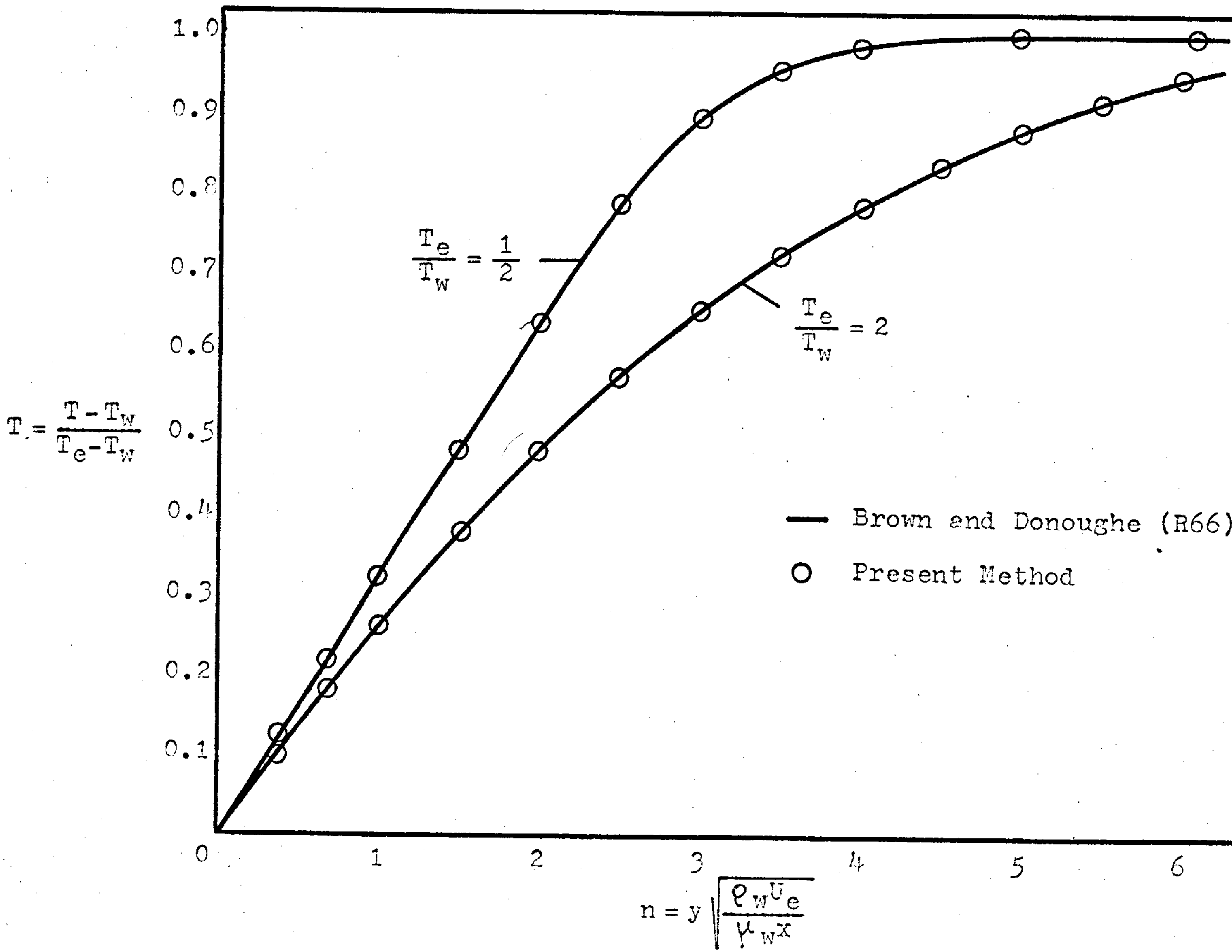

Figure 12: Compressible boundary layer on a flat plate. The temperature profile

method are compared with those of Brown and Donoughe in Figures 11 and 12. In order to make direct comparison possible, μ and K are taken as the following functions of T (T here is dimensional, ^{o}R):

$$\frac{\mu}{\mu_w} = \left(\frac{T-460}{T_w-460}\right)^{0.7}$$

$$\frac{K}{K_w} = \left(\frac{T-460}{T_w-460}\right)^{0.85}$$

$$T > 460$$

The agreement between the results can be noted from Figures 11 and 12.

7.3 INCOMPRESSIBLE BOUNDARY LAYER WITH ADVERSE PRESSURE GRADIENT

A number of investigators have studied the boundary layer flow on a flat plate with a linearly retarded velocity field

$$U_e(x) = 1 - ax$$

$$a = 0.125$$

Reasons for studying this flow is that it leads to separation, and it provides a good test of the numerical method, under conditions of adverse pressure gradients. Both analytical and numerical solutions are available in this case. Howarth (R64) was the first to obtain an analytical solution. Leigh (R68) presented an extensive study of the flow near the separation

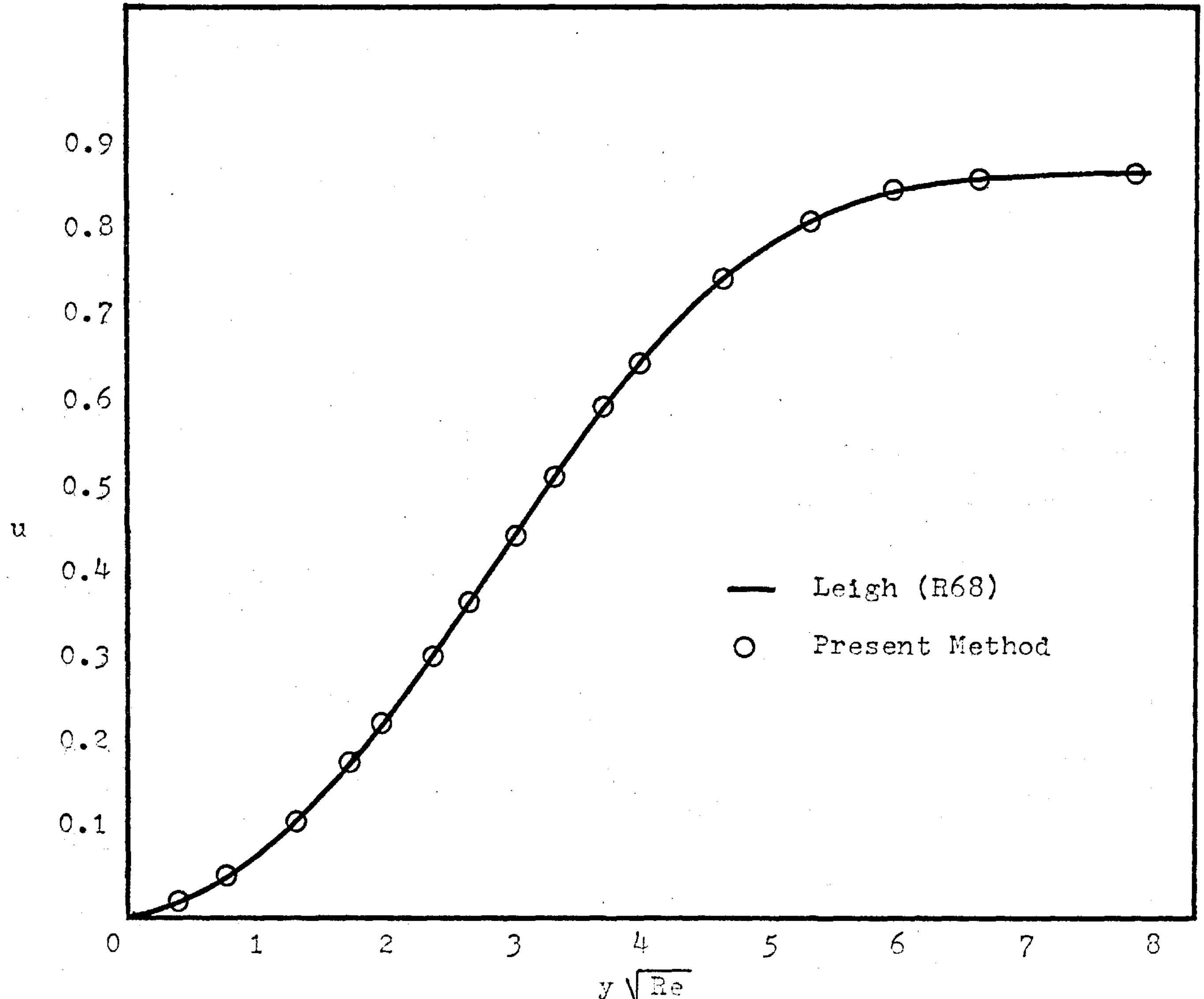

Figure 13: Flow with adverse pressure gradient: $U_e(x) = 1 - 0.125x$. The longitudinal velocity component near the separation point, $x = 0.9560$

Table 2

SHEAR STRESS AT THE WALL FOR THE FLOW

WITH ADVERSE PRESSURE GRADIENT

ax	$\tau_w (1/a)^{\frac{1}{2}}$				
	Howarth (R64)	Smith and Clutter (R35)	Krause (R53)	Cebeci et al (R55)	Present Method
0.0125	2.739		2.730	2.7400	2.7356
0.0250	1.772	1.7713	1.768	1.7721	1.7693
0.0375	1.309		1.306	1.3090	1.3072
0.0500	1.011	1.0106	1.009	1.0093	1.0079
0.0625	0.790		0.789	0.7898	0.7896
0.0750	0.613		0.610	0.6115	0.6083
0.0875	0.459		0.456	0.4573	0.4552
0.1000	0.315	0.3155	0.312	0.3137	0.3124
0.1100		0.195	0.195	0.1944	0.1929
0.1125	0.163		0.161	0.1611	0.1598
0.1150		0.128	0.126	0.1241	0.1242
0.1175			0.080	0.0791	0.0822
0.1195		0.034		0.0137	0.0305
0.11972					0.0194
0.11986		0.019			

point. Smith and Clutter (R35), Krause (R53) and Cebeci et al. (R55) have obtained solutions by numerical methods.

The separation point obtained by extrapolation by all previous investigators is at $ax = 0.120$. The present method computed the flow to a point very close to the separation point, up to $ax = 0.11972$. The velocity profile near the separation point is compared to that given by Leigh (R68) in Figure 13 while the shear stress at the wall is compared with results from previous investigators in Table 2.

7.4 INCOMPRESSIBLE BOUNDARY LAYER ON A FLAT PLATE WITH CONTINUOUS SUCTION

Fluid suction has favourable effects on the development of the boundary layer, in that it reduces the drag on the body and it stabilizes the boundary layer. Such stabilization is understood to account for two effects:

a) Prevention of separation, i.e., formation of a boundary layer which is capable of overcoming a greater adverse pressure gradient.

b) Laminar flow is maintained at much higher Reynolds numbers (approximately 100 times larger than if no suction is applied).

The effect of suction consists in the removal of decelerated particles from the boundary layer. The application of suction which was originally tried by Prandtl, was later widely used in the design of aircraft wings.

In order to ensure that a flow with suction satisfies the simplifying conditions of the boundary layer theory, it is necessary to limit the suction velocity at the wall v_o, to a magnitude of $O(Re^{-\frac{1}{2}})$ (R2). In this case the external flow may be assumed to remain unaffected by the presence of suction, since the quantity of fluid removed from the stream is so small that only fluid particles in the immediate neighbourhood of the wall are sucked away.

The equations to be solved in this case are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$

$$y = 0 \qquad u = 0$$

$$v = v_o = \text{const} < 0$$

$$y \rightarrow \infty \qquad u = 1$$

A particular solution for the fully developed boundary layer flow may be obtained in this case by putting $\frac{\partial u}{\partial x} = 0$. The equation of continuity then gives $v(x,y) = v_o =$ const. and the equation of motion becomes:

$$v_o\frac{\partial u}{\partial y} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$

which has the solution:

$$u(y) = 1 - \exp\,(v_o Re\, y)$$

with $$v(x,y) = v_o < 0$$

The local shear stress coefficient at the wall is given by

$$C_w = \frac{1}{Re} \left(\frac{\partial u}{\partial y}\right)_w = \frac{1}{Re} (-v_o Re) = - v_o$$

This silution may be regarded as an "asymptotic suction solution". It was in fact shown by Iglisch (R69) that this asymptotic case is reached after a length of about

$$x = \frac{4}{Re\, v_o^2}$$

Iglisch (R69) made a detailed study of the development of the boundary layer on a flat plate with continuous suction applied from the leading edge.

This problem was solved by the present method. The results obtained are compared to Iglisch's results in Figure 14 for the longitudinal velocity profile, and in Tables 3 and 4 for the asymptotic velocity profile and the shear stress at the wall respectively. The agreement of the results is quite good. The asymptotic velocity profile for $v = -10^{-3}$ and $Re = 10^6$ was found by the present method at $x = 4.0098$ in excellent agreement with Iglisch's prediction. At that station, the normal velocity component at all points across the boundary layer was equal to the value at the wall and the profile of the longitudinal velocity component differed by less than 10^{-5} from the values at the previous station. Since a considerable distance along the flat plate had to be covered, in order to reach the asymptotic state, a large step

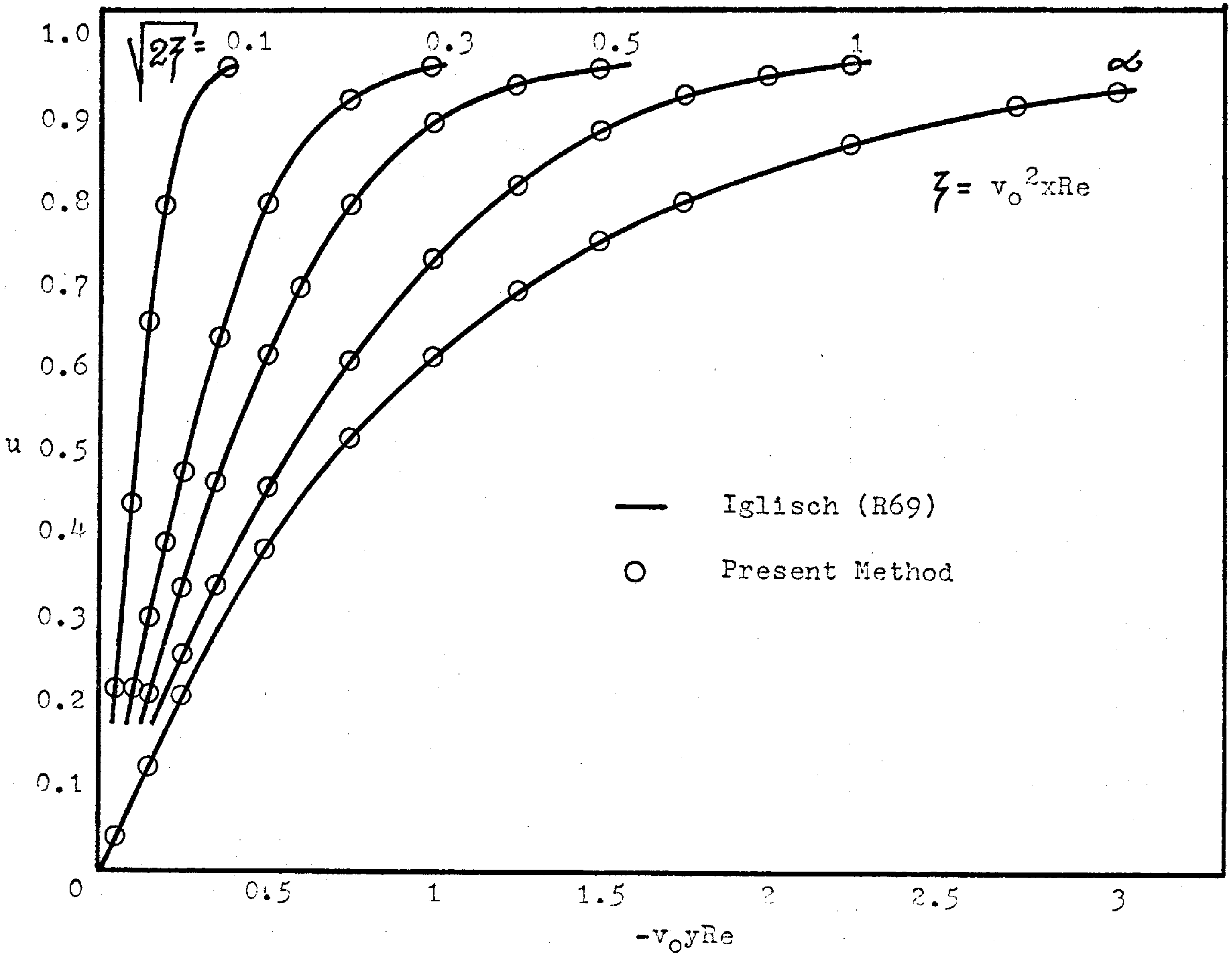

Figure 14: Development of the velocity profile
for the flow with continuous suction

Table 3

ASYMPTOTIC SUCTION PROFILE

	Iglisch (R69)	Present Method
$\overset{+}{y} = -v_o Rey$	u	u
0	0	0
0.2	0.1813	0.1820
0.4	0.3297	0.3304
0.6	0.4512	0.4517
0.8	0.5507	0.5512
1.0	0.6321	0.6329
1.2	0.6989	0.6994
1.4	0.7534	0.7539
1.6	0.7981	0.7985
1.8	0.8347	0.8351
2.0	0.8647	0.8650
2.2	0.8892	0.8894
2.4	0.9093	0.9096
2.6	0.9257	0.9258
2.8	0.9392	0.9393
3.0	0.9502	0.9504
3.2	0.9592	0.9593
3.4	0.9666	0.9666
3.6	0.9727	0.9728
3.8	0.9776	0.9775
4.0	0.9817	0.9817
4.2	0.9850	0.9849
4.4	0.9877	0.9876
4.6	0.9899	0.9900
4.8	0.9918	0.9917
5.0	0.9933	0.9932
5.5	0.9959	0.9960
6.0	0.9972	0.9972
6.5	0.9985	0.9985
7.0	0.9991	0.9992

Table 4

BOUNDARY LAYER ON A FLAT PLATE WITH CONTINUOUS SUCTION

	Iglisch (R69)	Present Method
$\zeta = -v_0 \sqrt{Re\,x}$	τ_w	τ_w
0.005	5.322	5.304
0.02	2.986	2.969
0.045	2.216	2.197
0.08	1.835	1.829
0.125	1.612	1.611
0.18	1.467	1.463
0.245	1.366	1.362
0.32	1.292	1.290
0.405	1.237	1.236
0.5	1.194	1.193
0.72	1.135	1.133
0.98	1.094	1.094
1.28	1.068	1.068
2.00	1.036	1.036
2.88	1.019	1.020
5.12	1.009	1.009

size ratio ($\Delta X/\Delta Y = 100$) was used with iteration for the normal velocity component. Computer time for 400 stations in the x-direction and 100 points in the y-direction was 75 seconds on the CDC 6400. The corresponding computer time for $\Delta X/\Delta Y = 10$ and without iteration for v was 320 seconds with results differing at the fourth decimal point.

7.5 BOUNDARY LAYER ON A FLAT PLATE WITH DISCONTINUOUS SUCTION VELOCITY

The problem of the boundary layer flow with discontinuous suction velocity has not received much attention in spite of its importance in practice. Rheinboldt (R70) in his Ph.D. thesis presented an accurate analytical solution based on series expansion techniques. Smith and Clutter (R39) attempted a numerical solution, using their method, which was discussed previously. They report that the calculated values of the wall shear stress differed in the second decimal point from those obtained by Rheinboldt, just aft of the suction region. This error is introduced by the very short steps ΔX which are necessary at the discontinuity in order to achieve convergence of the solution further downstream.

Two cases were treated using the present numerical method. The equations to be solved are again:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$

$$y = 0 \qquad u = 0$$

$$y \to \infty \qquad u = 1$$

In one of the case studies (Case A) the boundary condition for v at the wall was taken as

$$y = 0 \; : \; 0 \leq x \leq 1 \qquad v = 0$$

$$: \; 1 < x \qquad v = v_o = \text{const} < 0$$

i.e., the wall is assumed impermeable up to $x = 1$, and porous thereafter, with continuous suction applied in the porous region.

In the second case (Case B) the boundary condition for v was

$$y = 0 \qquad 0 \leq x < 1 \qquad v = 0$$

$$1 \leq x \leq 1.15 \qquad v = v_o = \text{const} < 0$$

$$1.15 < x \qquad v = 0$$

i.e., "impulse" suction is assumed in the region $1 \leq x \leq 1.15$, with constant suction velocity v_o, while the rest of the wall is impermeable. In Figures 15 through 18, the results obtained by the present method are compared to Rheinboldt's results, for CaseA, for the following values of v_o: $v_o\sqrt{Re} = -0.5$, -1.0, -1.5. Results for Case B are presented in Figures 19 through 21. In all cases the agreement is quite satisfactory.

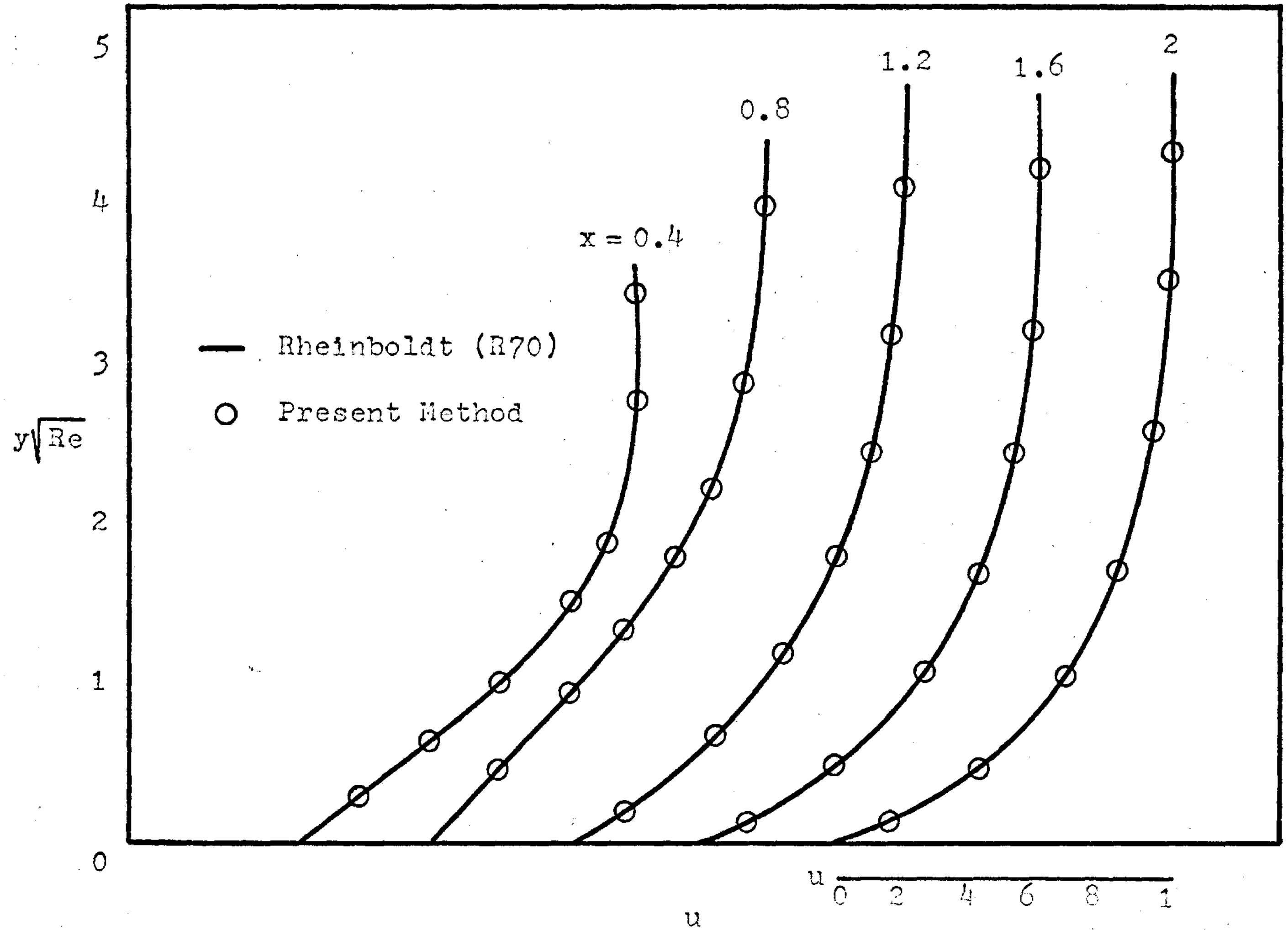

Figure 15: Velocity profiles for the flow with discontinuous suction. Case A. $v_o = -10^{-3}$

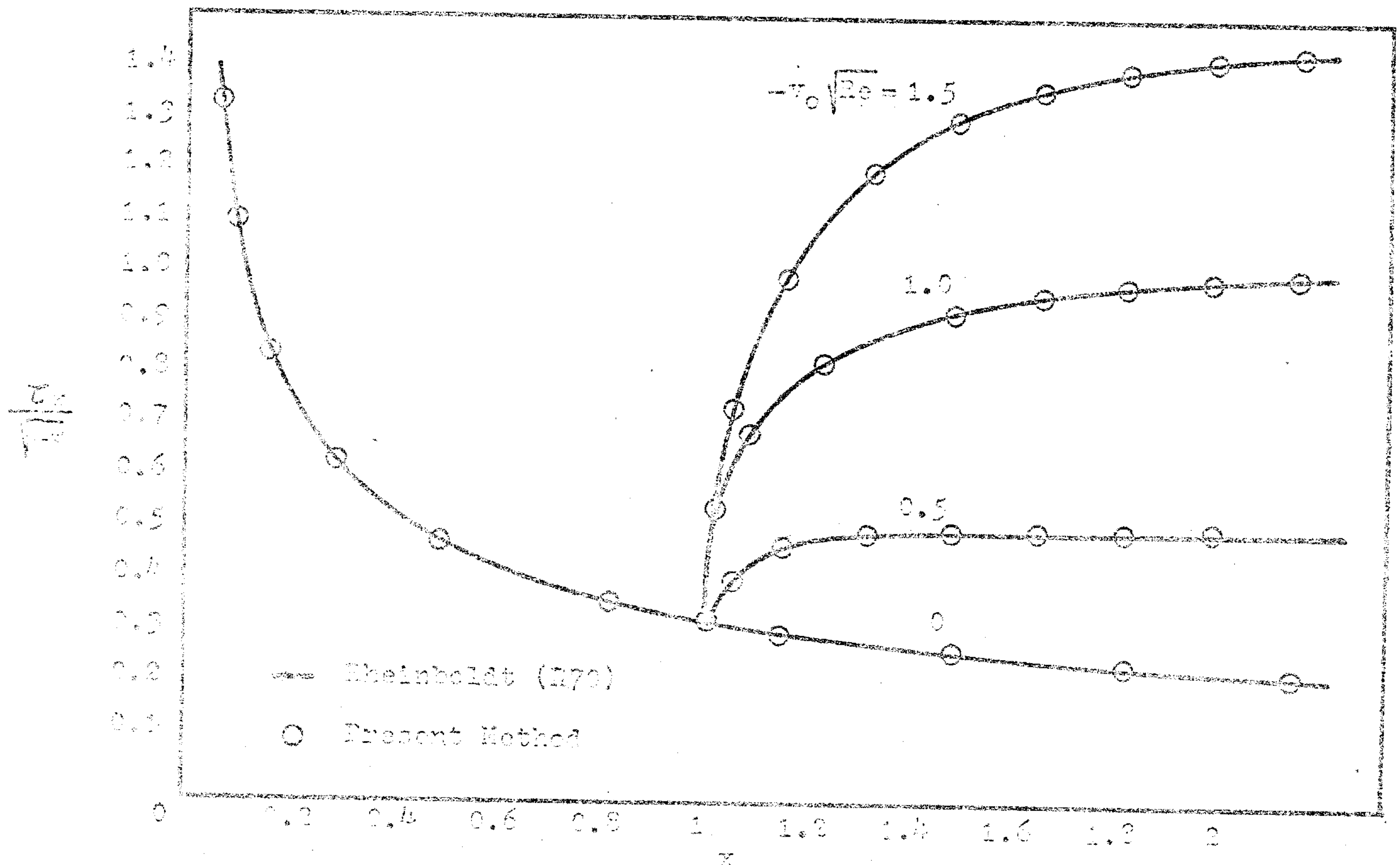

Figure 16: Wall shear stress for various suction velocities. Discontinuous suction Case A

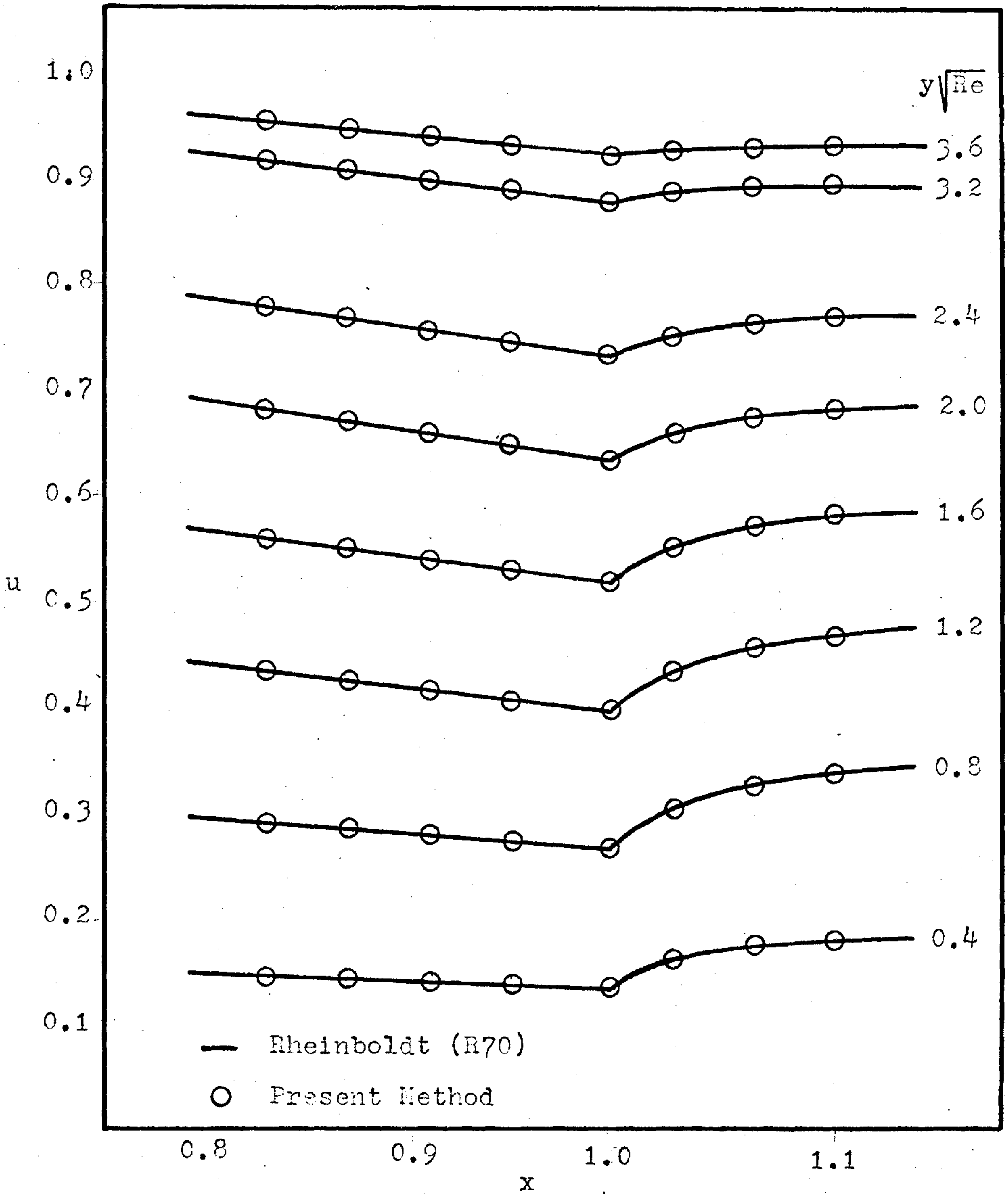

Figure 17: Velocity profiles at constant distance from the wall for the flat plate with discontinuous suction. Case A

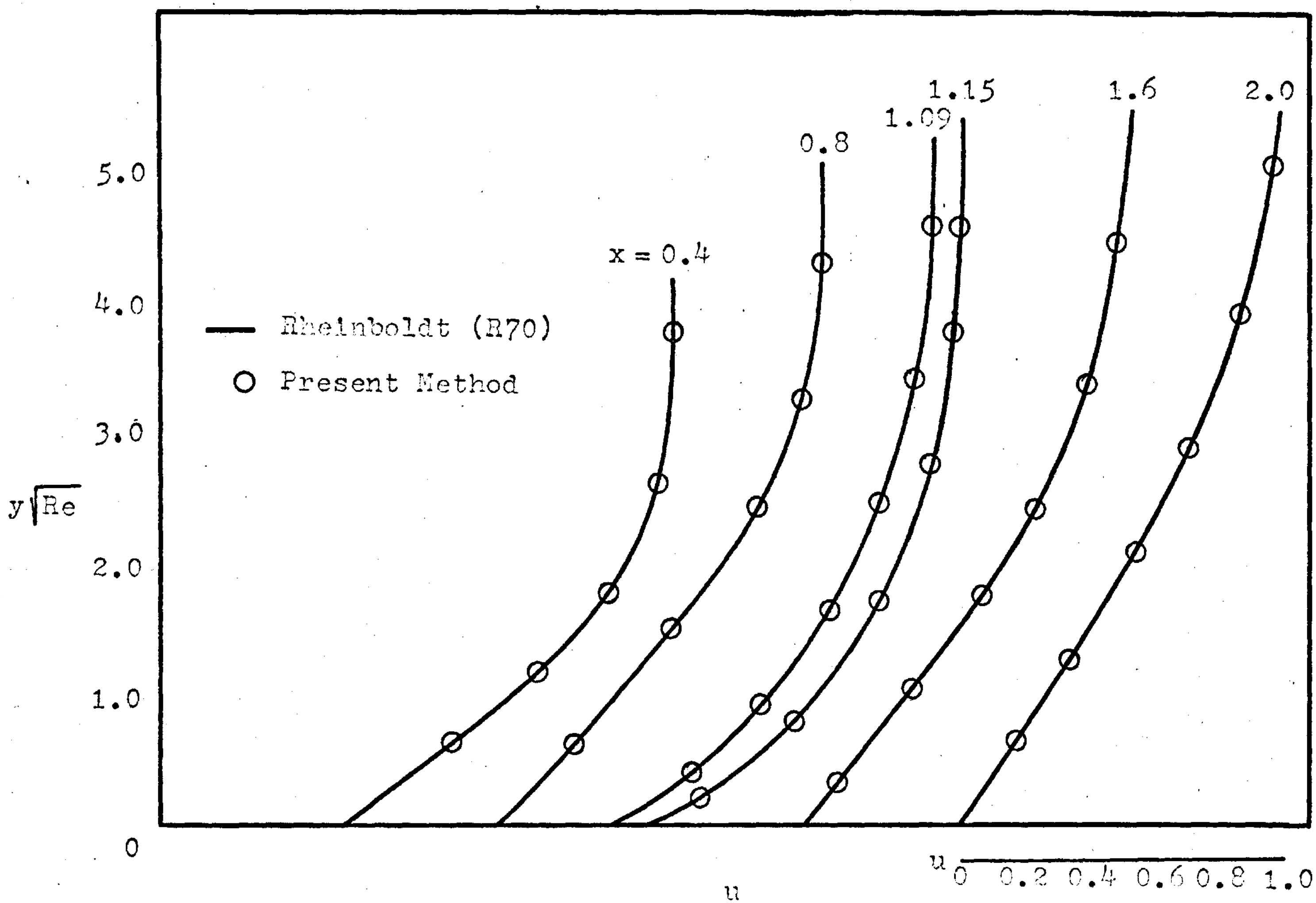

Figure 18: Velocity profiles for the flat plate with discontinuous suction. Case B. $v_o = -1.5\ 10^{-3}$

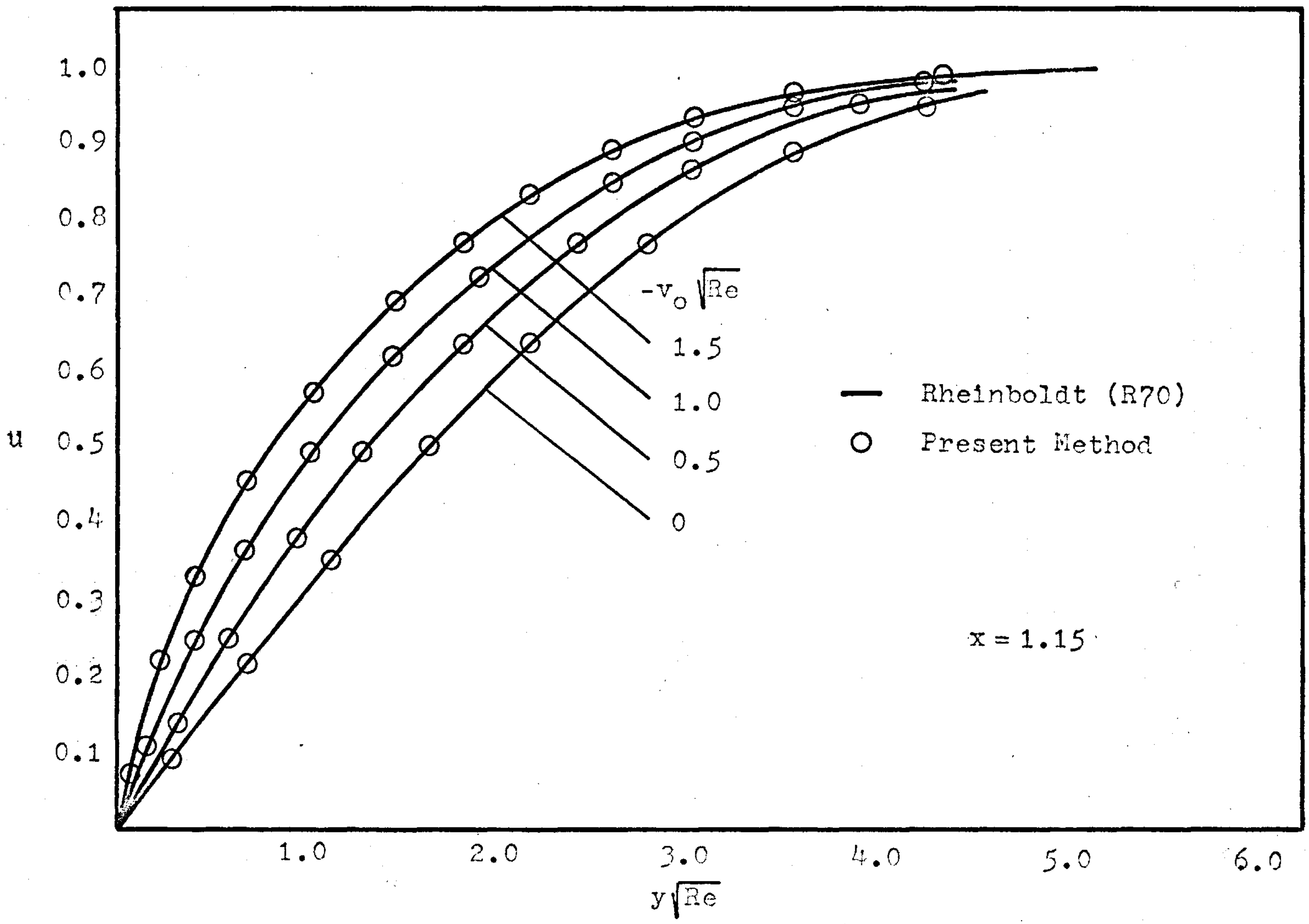

Figure 19: Velocity profiles for various suction velocities at the end of the suction region. Case B. x = 1.15

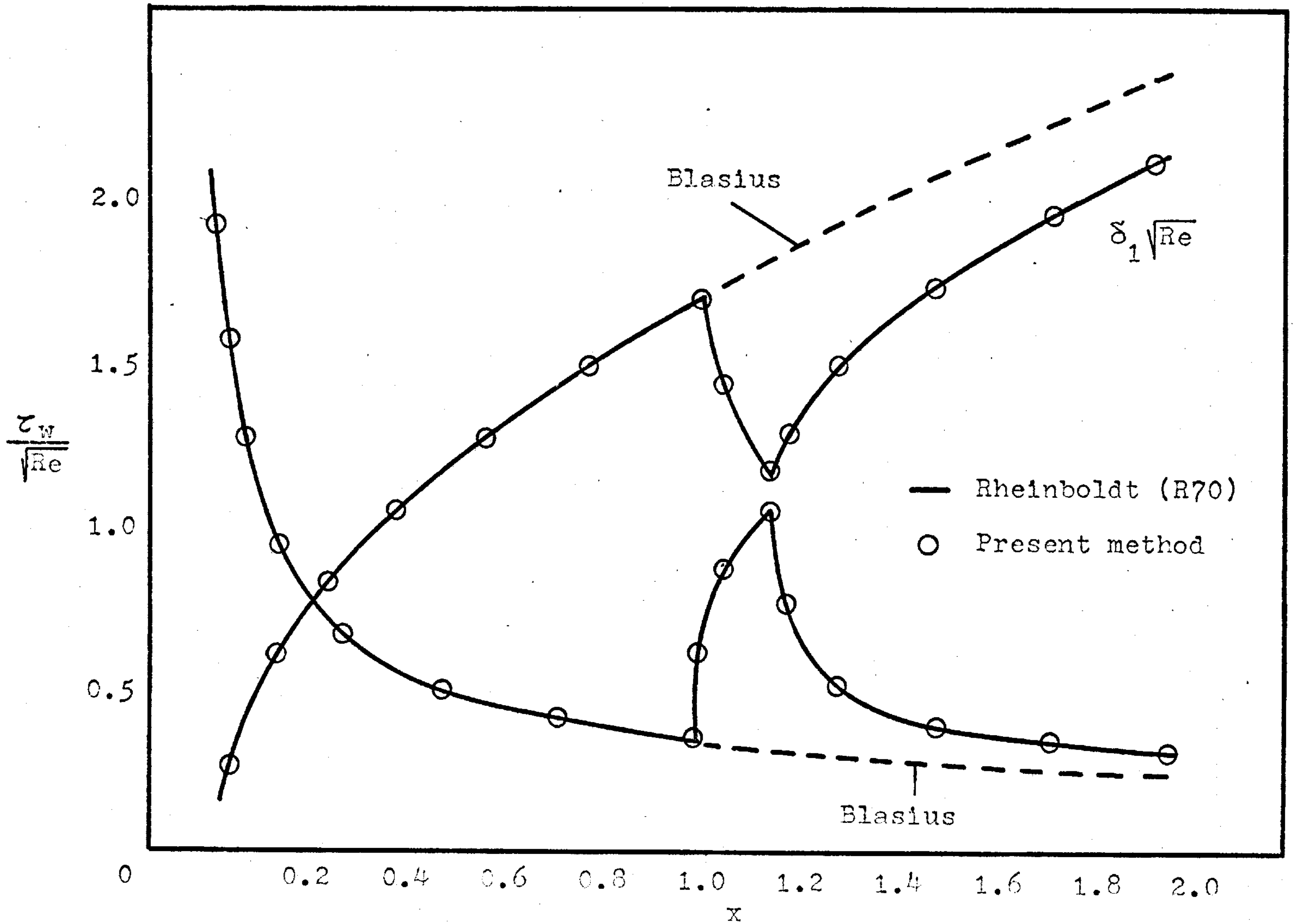

Figure 20: Shear at the wall and displacement thickness for the discontinuous suction case. Case B. $v_o = -1.5\ 10^{-3}$

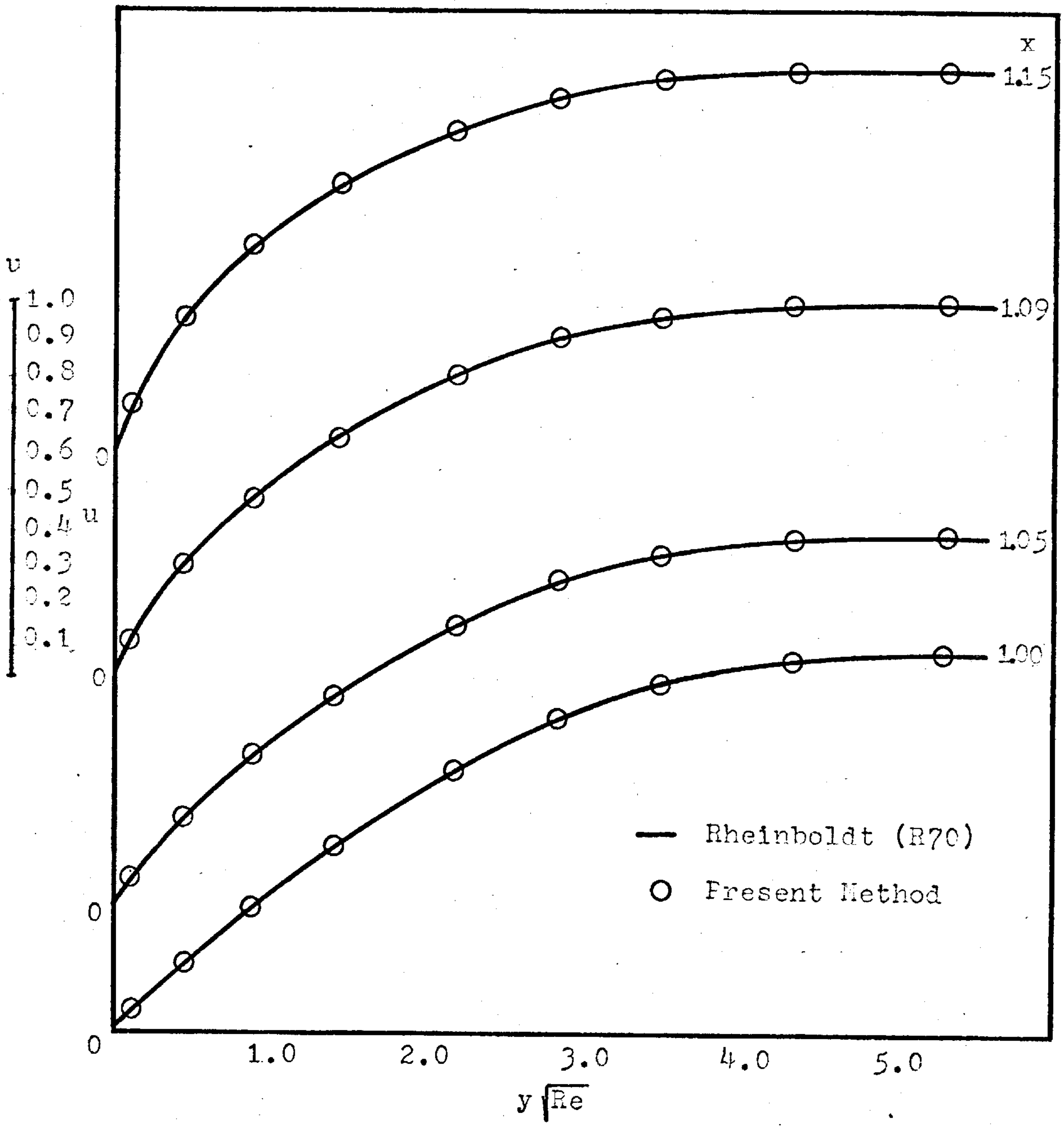

Figure 21: Velocity profiles within the region of suction. $v_o = -1.5\times10^{-3}$. Case B

7.6 INCOMPRESSIBLE BOUNDARY LAYER WITH UNIFORM INJECTION

While suction has a stabilizing effect on the boundary layer, uniform similar fluid injection destabilizes the boundary layer and eventually leads to separation. Moreover, the velocity profile near the separation point assumes the extreme form of what is known as "blow-off profile". In this case the boundary layer is blown away from the wall.

Catheral et al. (R71) made an extensive analytical and numerical study of this case, which presents particular difficulties due to the blow-off effect. They located the separation point at a distance $x = 0.7456$ from the leading edge, and they showed that near the separation point the velocity profile approaches the profile $u = 0$ at all grid points across the boundary layer, no matter how far from the plate the outer edge is located. This means that as x approaches the separation point x_s the distance between the wall and the edge of the boundary layer tends to infinity. Clearly this leads to the impossibility of meeting the outer edge boundary condition near the separation point.

Catheral et al. (R71) found that it was impossible to obtain a completely satisfactory analytical solution, and that a satisfactory agreement between the analytical and the numerical solution could not be reached.

An attempt was made to solve the boundary layer equations with fluid injection, using the present method. The equations to be solved in this case are:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re}\ \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$y \rightarrow \infty \qquad u = 1$

$y = 0 \qquad u = 0$

$v = v_o = \text{const} > 0$

The results obtained for the shear stress at the wall are presented in Figure 22, and agree quite well with Catheral's et al. up to $x = 0.70$. Further downstream difference between the results develops and at $x = 0.73$ there is a difference in the first significant digit. Separation is predicted at $x = 0.7432$.

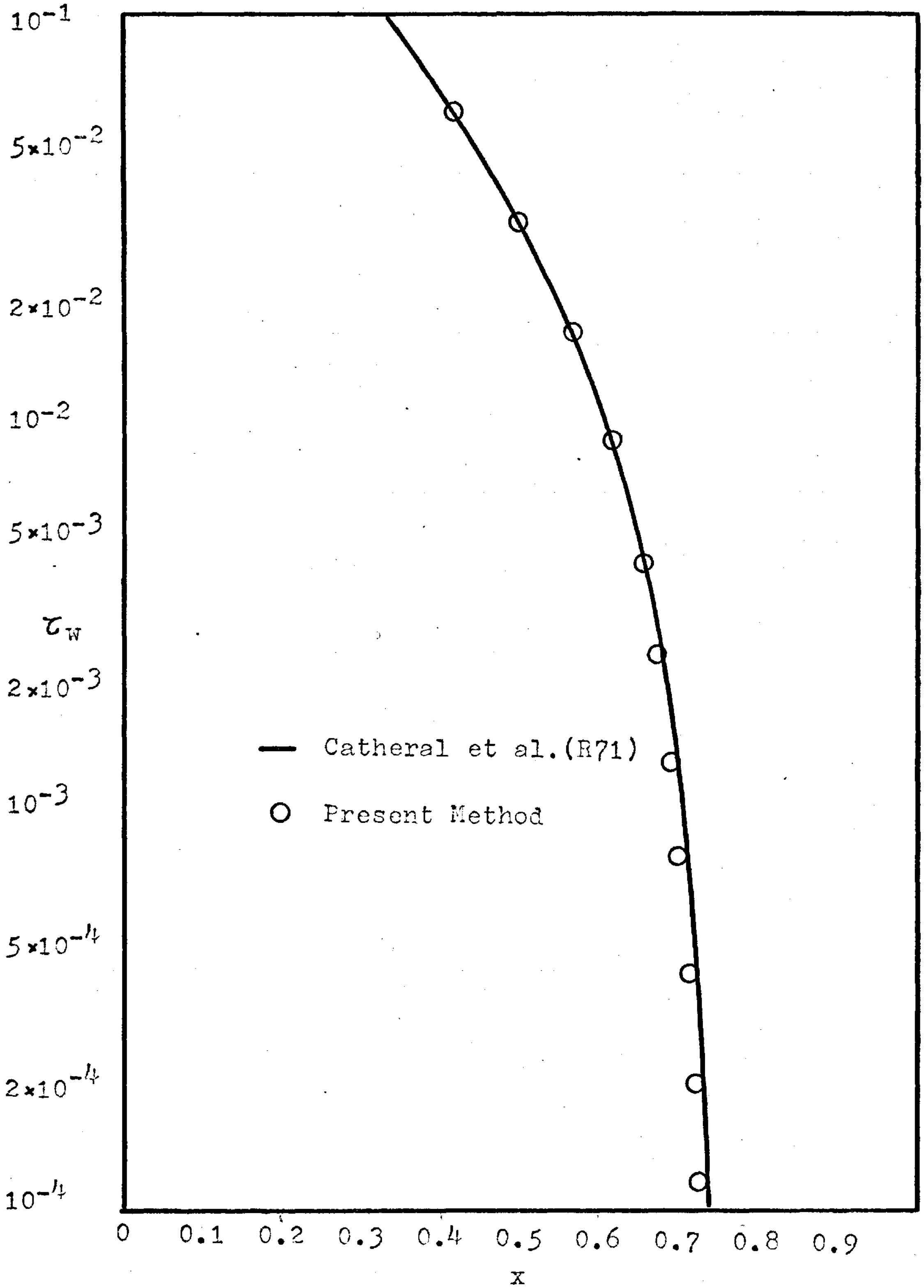

Figure 22: Shear stress at the wall near the separation point. Continuous injection

8. CONCLUSIONS

An implicit finite difference method has been described and its convergence, accuracy and speed have been evaluated. Various flows have been calculated by the present method, and comparisons with analytical and numerical solutions have been made. In all cases, it was found that the method is accurate for both incompressible and compressible flows. The computer time requirements are very small.

It is believed that the present method compares favourably with previously proposed numerical methods. The technique presented in this report may be extended easily to cover magnetohydronamic boundary layer flows, turbulent flows, non-Newtonian boundary layers and binary (or multicomponent) boundary layers.

NOTATION

A	:	Forcing coefficient, Equation (88)
a	:	Coefficient in reccurrence formula (53)
B	:	Forcing coefficient, Equation (89)
b:	:	Coefficient in reccurrence formula (53)
Cp	:	Specific heat, constant pressure
Cs	:	Speed of sound
Cv	:	Specific heat, constant volume
C	:	Coefficient in reccurrence formula (53)
d	:	Coefficient in reccurrence formula (53)
E	:	Eckert number
e	:	Coefficient in reccurrence formula (53)
f	:	Velocity u or temperature T
K	:	Thermal conductivity
L	:	Characteristic length
M	:	Mach number
Nu	:	Nusselt number
p	:	Pressure
Pr	:	Prandtl number
q	:	Heat flux
Re	:	Reynolds number
S	:	Averaging parameter
T	:	Temperature

u	:	Longitudinal velocity component
U_e	:	Main stream velocity
v	:	Normal velocity component
v_o	:	Suction (or injection) velocity
x	:	Distance in the direction of the flow
y	:	Distance perpendicular to the direction of the flow
α	:	Coefficient in equation (75)
β	:	Coefficient in equation (75)
γ	:	Coefficient in equation (75)
δ	:	Coefficient in equation (75)
ΔT_o	:	Temperature difference between the wall and the mean stream
ΔX	:	Step size in the x-direction
ΔY	:	Step size in the y-direction
λ	:	Heat transfer coefficient
μ	:	Viscosity
ν	:	Kinematic viscosity
ρ	:	Density
τ	:	Shear stress
ϕ	:	Dissipation function

Subscripts

e	:	Main stream variables
w	:	Wall variables
I	:	Denotes grid points in the direction of the flow

Superscripts

N : Denotes grid points in the direction normal to the flow

* : Denotes dimensionless variables

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APPENDIX A

STABILITY ANALYSIS OF THE FINITE DIFFERENCE SCHEME

If the finite difference approximations (70) to (72) are substituted into the momentum equation (47) with $f \equiv u$ and under the assumptions that the coefficients a_1, b_1, c_1 defined in equations (55) to (58) are constant (moreover, we assume, without loss of generality, that the pressure gradient is zero, i.e., $d_1 = 0$) the following difference equation is obtained:

$$a_1 \frac{a_I^N - u_{I-1}^N}{\Delta X} + \frac{b_1}{2\Delta Y}\left\{S(u_I^{N+1} - u_I^{N-1}) + (1-S)(u_{I-1}^{N+1} - u_{I-1}^{N-1})\right\}$$

$$= \frac{c_1}{\Delta Y^2}\left\{S(u_I^{N+1} - 2u_I^N + u_I^{N-1}) + (1-S)(u_{I-1}^{N+1} - 2u_{I-1}^N + u_{I-1}^{N-1})\right\}$$

Rearranging terms, and putting for simplification (See Note 1) $S = \frac{1}{2}$, one obtains:

$$u_I^{N+1}\left(\frac{b_1}{4\Delta Y} - \frac{c_1}{2\Delta Y^2}\right) + u_I^N\left(\frac{a_1}{\Delta X} + \frac{c_1}{\Delta Y^2}\right) + u_I^{N+1}\left(-\frac{b_1}{4\Delta Y} - \frac{c_1}{2\Delta Y^2}\right)$$

$$+ u_{I-1}^{N+1}\left(\frac{b_1}{4\Delta Y} - \frac{c_1}{2\Delta Y^2}\right) + u_{I-1}^N\left(-\frac{a_1}{\Delta X} + \frac{c_1}{\Delta Y^2}\right) + u_{I-1}^{N-1}\left(-\frac{b_1}{4\Delta Y} - \frac{c_1}{2\Delta Y^2}\right)$$

$$= 0 \qquad \text{(A1)}$$

Dividing through by $\frac{a_1}{\Delta X}$ (see Note 2) and introducing the parameters:

$$\frac{b_1 \Delta X}{4 a_1 \Delta Y} = G$$

$$\frac{c_1 \Delta X}{2 a_1 \Delta Y^2} = H$$

$$K_1 = G - H$$

$$K_2 = 1 + 2H$$

$$K_3 = - G - H$$

$$L_1 = G - H = K_1$$

$$L_2 = - 1 + 2H$$

$$L_3 = - G - H = K_3$$

(A1) can be written as

$$K_1 u_I^{N+1} + K_2 u_I^N + K_3 u_I^{N-1} + L_1 u_{I-1}^{N+1} + L_2 u_{I-1}^N + L_3 u_{I-1}^{N-1} = 0 \qquad \text{(A2)}$$

According to Von Neumann's theory the term u_I^N in the expression (A2) is substituted by

$$u_I^N = A e^{lx} e^{jmy} \qquad \text{(A3)}$$

or

$$u_I^N = A e^{lI \Delta X} e^{jmN \Delta Y} \qquad \text{(A4)}$$

where $A = 0$ and l, m are real constants, $j = \sqrt{-1}$. In order that an original error e^{jmY} will not increase as x increases, a necessary and sufficient condition is

$$\left| e^{l \Delta X} \right| \leqslant 1 \qquad \text{(A5)}$$

Setting $Z = e^{l \Delta X}$ and substituting from (A4) into (A2) there results

$$Z = -\frac{L_1 e^{jm(N+1)\Delta Y} + L_2 e^{jm\Delta Y} + L_3 e^{jm(N-1)\Delta Y}}{K_1 e^{jm(N+1)\Delta Y} + K_2 e^{jmY} + K_3 e^{jm(N-1)\Delta Y}}$$

or

$$Z = -\frac{(G - H)e^{jm\Delta Y} + (-1 + 2H) + (-G - H)e^{-jm\Delta Y}}{(G - H)e^{jm\Delta Y} + (1 + 2H) + (-G - H)e^{-jm\Delta Y}}$$

or

$$Z = -\frac{1 - 2Gj\sin(m\Delta Y) - 2H(1 - \cos(m\Delta Y))}{1 + 2Gj\sin(m\Delta Y) + 2H(1 - \cos(m\Delta Y))}$$

Putting

$$2G\sin(m\Delta Y) = g$$

$$2H(1 - \cos(m\Delta Y)) = h > 0$$

we obtain

$$Z = \frac{1 - h - gj}{1 + h + gj}$$

or

$$Z = \frac{1 - h^2 - g^2 - 2gj}{(1 + h)^2 + g^2} \qquad \text{(A6)}$$

Since, according to (A5)

$$|Z| \leqslant 1$$

or

$$|Z|^2 \leqslant 1$$

we obtain

$$|Z|^2 = \frac{(1 - h^2)^2 + 2g^2(1 + h^2) + g^4}{(1 + h)^2 + 2g^2(1 + h)^2 + g^4} \qquad \text{(A7)}$$

If $h = 0$, (A7) becomes

$$|z|^2 = 1$$

and (A5) is satisfied.

With $h \neq 0$, and since

$$(1 - h^2)^2 = (1 + h)^2(1 - h)^2 \leqslant (1 + h)^4$$

and

$$1 + h^2 < (1 + h)^2$$

it is clear that the numerator of (A7) is smaller than the denominator and so (A5) holds, for all G and H. Therefore, the implicit method is stable regardless of the choice of the step sizes.

Note 1

If the parameter S is left in the difference equation, the derivation of the analogue of (A7) is similar, but the algebra is somewhat more complicated. The analogue of (A7) arrived at by identical considerations is in this case:

$$2Sg_1^2h_1^2(-1-S^2+S) + h_1^2(1 - 6S) + 2h_1g_1^2(-1+S-2S^2) + g_1^2(1 - 2S)$$
$$+ Sh_1^3(1 - 6S^2) + Sg_1^4(S - 2) + S^2h_1^4(1 - 2S) - 2h_1 < 0 \qquad \text{(A7a)}$$

where

$$g_1 = 2g$$
$$h_1 = 2h$$

If $1 \leqslant S \leqslant \frac{1}{2}$, h_1, $g_1 \neq 0$, then the terms in parentheses in (A7a) are either negative or zero, since $h_1 > 0$ and (A5) is satisfied. The stability criterion concerning the case

$0 < S < \frac{1}{2}$ is difficult to obtain since it requires the solution of the inequality (A7a).

<u>Note 2</u>

If $a_1 = 0$, i.e., $u = 0$, ommission of the $\frac{a_1}{\Delta X}$ terms in (A1) leads to

$$|Z|^2 = 1$$

so that the stability condition (A5) is again fulfilled.

APPENDIX B

THE COMPUTER PROGRAM

B.1 COMPUTER PROGRAM NOMENCLATURE

Indices

N : 1, 2, 3,...NPY : Number of points in the y-direction

I : 1, 2, 3,...NPX : Number of stations in the x-direction

Non Subscripted Variables

DIFTMX : Maximum difference between the assumed and the calculated temperature profile on a line

DIFUMX : Maximum difference between the assumed and the calculated longitudinal velocity profile on a line

DIFVMX : Maximum difference between the assumed and the calculated normal velocity profile on a line

DIS : Displacement thickness

DTO : Difference between the wall temperature and the free stream temperature

DTPMA : Maximum difference between the calculated temperature profile on a line and that on the previous line

DUPMA : Maximum difference between the calculated longitudinal velocity profile on a line and that on the previous line

DX : Step size in the x-direction

DY : Step size in the y-direction

ECK : The Eckert number

EXCP : The specific heat

EXL	:	The characteristic length L
EXP	:	The pressure
EXT	:	The free stream temperature
EXU	:	The free stream velocity
KS	:	Control variable for the printout of the wall stress and displacement thickness
KWR	:	Control variable for the printout of the velocity and temperature profiles
LEND	:	Control variable to stop the program when calculations converge
NB	:	Number of points across the boundary layer
NPY	:	Number of points in the domain of integration in the y-direction
NPYN	:	NPY - 1
NPX	:	Number of stations in the x-direction
NSIM	:	NPY - 2
NITERT	:	Number of iterations to establish the temperature profile along a line
NITERU	:	Number of iterations to establish the longitudinal velocity profile along a line
NITERV	:	Number of iterations to establish the normal velocity profile on a line
PRA	:	The Prandtl number
REY	:	The Reynolds number
SHEAR	:	Shear stress at the wall
STEPRA	:	Step size ratio
S1	:	Averaging parameter in the finite difference approximation of the momentum equation

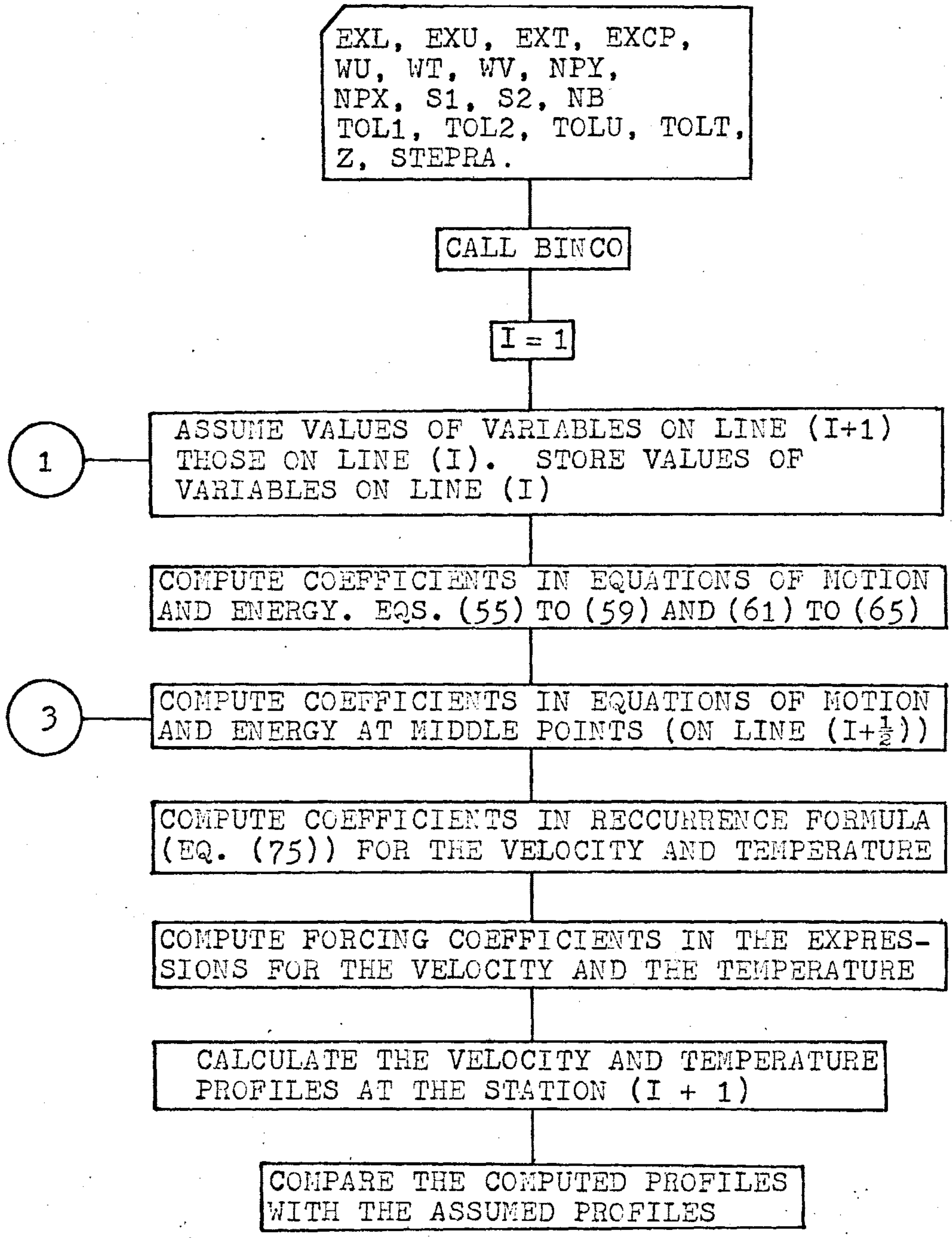
S2	:	Averaging parameter in the finite difference approximation of the energy equation
TOLT	:	Tolerance for the convergence of calculations for the temperature
TOLU	:	Tolerance for the convergence of calculations for the velocity
TOL1	:	Tolerance for the velocity profile along a line
TOL2	:	Tolerance for the temperature profile along a line
VELGR	:	The velocity gradient at the wall
WT	:	The wall temperature
WU	:	The wall longitudinal velocity
WV	:	The wall normal velocity
X	:	Distance along the body
Y	:	Distance perpendicular to the body
Z	:	Parameter used to achieve acceleration of convergence of the profiles along a line

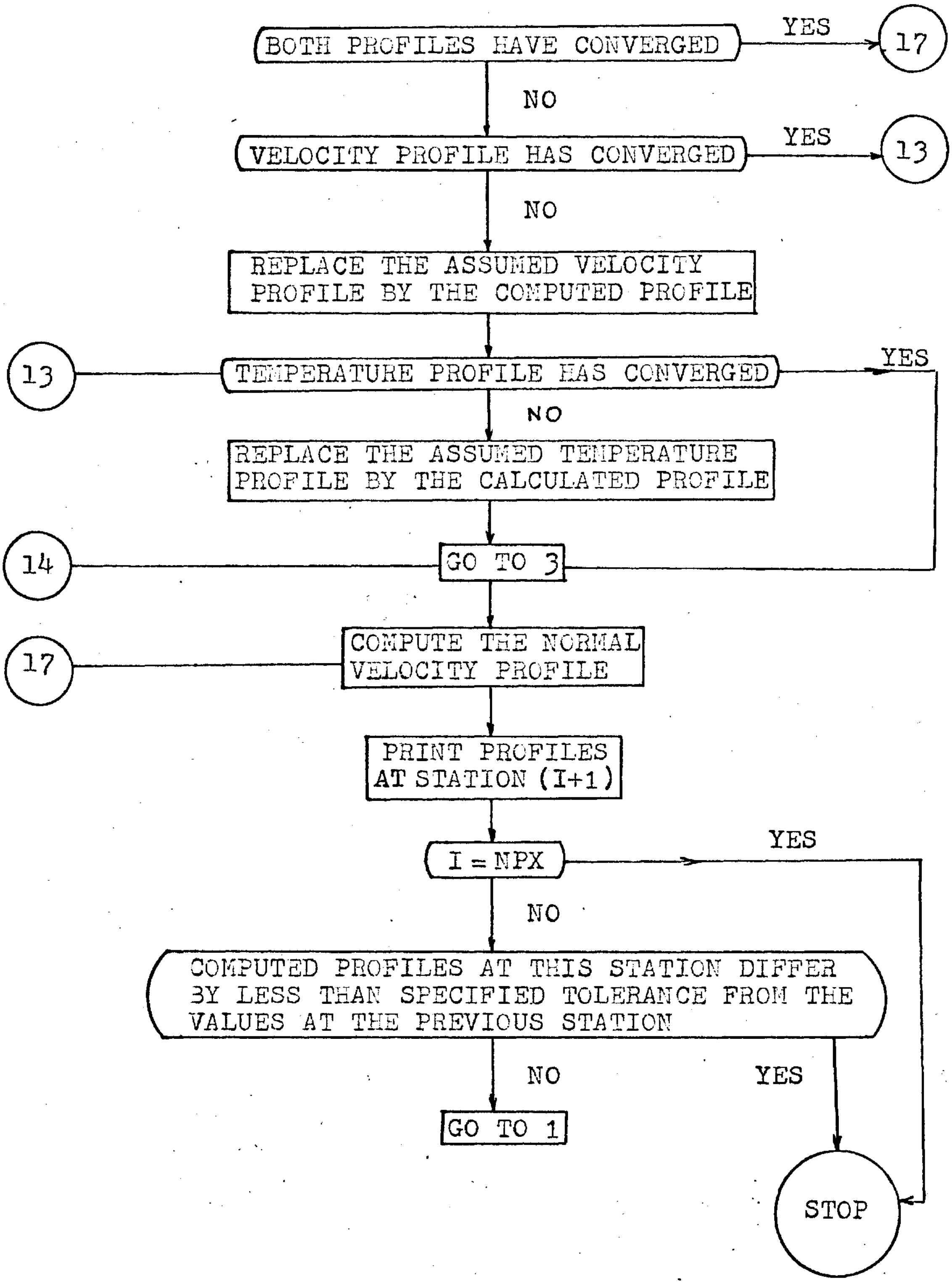
Subscripted Variables

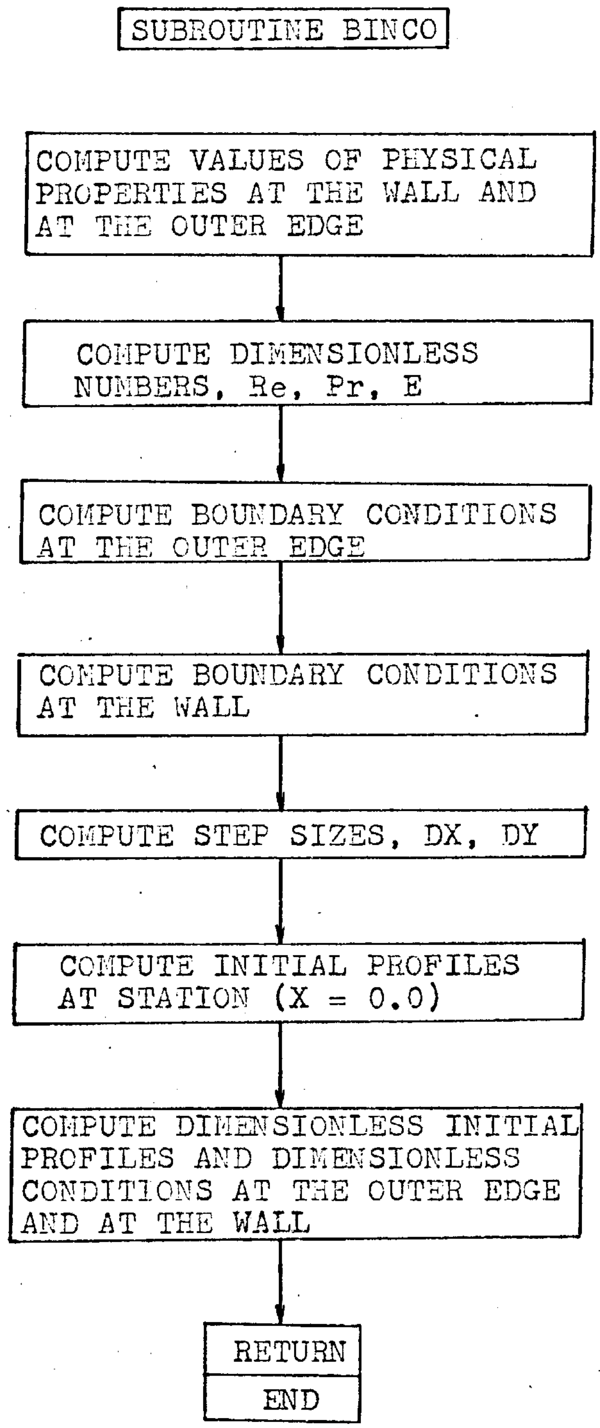
A, B, C, D	:	Coefficients in the equation of motion and energy
AL1, AL2, BE1, BE2, GA1, GA2, DE1, DE2		Coefficients in the recurrence relation for the velocity (suffix 1) and for the temperature (suffix 2)
FCA1, FCB1	:	Forcing coefficients in the reccurrence formula for the velocity
FCA2, FCB2	:	Forcing coefficients in the reccurrence formula for the temperature
R	:	The density
T	:	The temperature
U	:	The longitudinal velocity component

V	:	The normal velocity component
RAS	:	Assumed density for the points on the next line
TAS	:	Assumed temperature for the points on the next line
UAS	:	Assumed longitudinal velocity for the points on the next line
VAS	:	Assumed normal velocity component for the points on the next line
RM	:	Storage array for calculated values of density on a line
TM	:	Storage array for calculated values of temperature on a line
UM	:	Storage array for the calculated values of the longitudinal velocity component on a line
VM	:	Storage array for the calculated values of the normal velocity component on a line
RIED	:	Density along the outer edge
TIED	:	Temperature along the outer edge
UIED	:	Velocity along the outer edge
RIW	:	Density along the wall
TIW	:	Temperature along the wall
UIW	:	Longitudinal velocity component along the wall
VIW	:	Normal velocity component along the wall

ALGORITHM FOR THE COMPUTER PROGRAM




BOTH PROFILES HAVE CONVERGED
YES
17
NO
VELOCITY PROFILE HAS CONVERGED
YES
13
NO
REPLACE THE ASSUMED VELOCITY PROFILE BY THE COMPUTED PROFILE
13
TEMPERATURE PROFILE HAS CONVERGED
YES
NO
REPLACE THE ASSUMED TEMPERATURE PROFILE BY THE CALCULATED PROFILE
14
GO TO 3
17
COMPUTE THE NORMAL VELOCITY PROFILE
PRINT PROFILES AT STATION (I+1)
I = NPX
YES
NO
COMPUTED PROFILES AT THIS STATION DIFFER BY LESS THAN SPECIFIED TOLERANCE FROM THE VALUES AT THE PREVIOUS STATION
NO
YES
GO TO 1
STOP


SUBROUTINE BINCO
COMPUTE VALUES OF PHYSICAL PROPERTIES AT THE WALL AND AT THE OUTER EDGE
COMPUTE DIMENSIONLESS NUMBERS, Re, Pr, E
COMPUTE BOUNDARY CONDITIONS AT THE OUTER EDGE
COMPUTE BOUNDARY CONDITIONS AT THE WALL
COMPUTE STEP SIZES, DX, DY
COMPUTE INITIAL PROFILES AT STATION (X = 0.0)
COMPUTE DIMENSIONLESS INITIAL PROFILES AND DIMENSIONLESS CONDITIONS AT THE OUTER EDGE AND AT THE WALL
RETURN
END

```
C   PROGRAM FOR THE NUMERICAL SOLUTION OF THE BOUNDARY LAYER
C   EQUATIONS FOR COMPRESSIBLE FLOW WITH VARIABLE FLUID PROPERTIES.
C
C
C   THE EQUATIONS ARE SOLVED BY A FULLY IMPLICIT FINITE DIFFERENCE
C   TECHNIQUE IN THE X-Y PLANE.
C
C
C   INPUT DATA NEEDED ARE
C
C    EXL     CHARACTERISTIC LENGTH
C    EXU     MAIN STEAM VELOCITY
C    EXT     MAIN STREAM TEMPERATURE
C    EXCP    SPECIFIC HEAT
C    EXP     PRESSURE
C    WU      WALL VELOCITY  (ZERO, IF NO SLIP CONDITION APPLIES)
C    WV      NORMAL VELOCITY AT THE WALL
C    WT      WALL TEMPERATURE
C    NPY     NUMBER OF POINTS IN THE DOMAIN OF INTEGRATION
C    NPX     NUMBER OF STATIONS IN THE X-DIRECTION
C    S1      AVERAGING PARAMETER   (S1.LE.1 . AND S1.GE.1/2 )
C    S2      AVERAGING PARAMETER   (S2.LE.1 . AND S2.GE.1/2 )
C    NB      NUMBER OF POINTS ACCROSS THE BOUNDARY LAYER
C    TOL1    TOLERANCE FOR THE CONVERGENCE OF THE VELOCITY
C            PROFILE ALONG A LINE
C    TOL2    TOLERANCE FOR THE CONVERGENCE OF THE TEMPERATURE
C            PROFILE ALONG A LINE
C    TOLU    TOLERANCE FOR CONVERGENCE OF THE CALCULATIONS
C            OF THE VELOCITY PROFILE
C    TOLT    TOLERANCE FOR CONVERGENCE OF THE CALCULATIONS
C            OF THE TEMPERATURE PROFILE
C    Z       PARAMETER FOR THE ACCELERATION OF CONVERGENCE
C    STEPRA  STEP SIZE RATIO (DX/DY)
C    NEXTKR  CONTROLS   PRINTOUT STATIONS FOR THE PROFILES
C    NEXTKS  CONTROLS PRINTOUT STATIONS FOR THE WALL SHEAR
C
C
C   COMMON AND DIMENSION STATEMENTS
C
       COMMON  EXL,EXU,EXR,EXT,EXCP,EXK,EXM,DTO,  ECK,REY,PRA,
     $WU,WT,WR,WV,  NPY,NPX,DY,DX,  THICK, NPYN, NB, EXP, N,
     $ U(300), T(300), R(300), V(300), Y(300),
     $UIW(510),TIW(510),RIW(510),VIW(510),
     $ UIED(510),TIED(510),RIED(510)
       DIMENSION UAS(300), TAS(300), RAS(300), VAS(300),
     $  UM(300), TM(300), RM(300), VM(300),
     $  A(300), B(300), D(300), AA(300), BB(300), DD(300),
     $  AOLD(300), BOLD(300), DOLD(300), RNEW(300), C1(300),
     $ C2(300), COLD1(300), COLD2(300), CC1(300), CC2(300),
     $  FCA1(300), FCA2(300), FCB1(300), FCB2(300),
```

```
     $  AL1(300), BE1(300), GA1(300), DE1(300),
     $  AL2(300), BE2(300), GA2(300), DE2(300),
     $ DUP(300) ,DTP(300), HTA(300), VDIM(300),
     $ VOLD(300) ,DIFV(300) ,
     $ UU(300),TT(300), RR(300),
     $  DIFU(300), DIFT(300)
C
C
      WRITE(6,9011)
      READ(5,901) EXL,EXU,EXT,EXCP,EXP
      WRITE(6,9012) EXL,EXU,EXT,EXCP,EXP
      WRITE(6,9021)
      READ (5,902 )  WU,WV,WT
      WRITE (6,9022) WU,WV,WT
      WRITE(6,9031)
      READ(5,903)  NPY,NPX,S1,S2,NB
      WRITE(6,9032)NPY,NPX,S1,S2,NB
      READ (5,904) TOL1,TOL2,TOLU,TOLT
      WRITE (6,9041) TOL1,TOL2,TOLU,TOLT
      READ( 5,905)   Z,STEPRA
      WRITE (6,9051) Z,STEPRA
      READ (5,906)   NEXTKR,NEXTKS
      WRITE (6,9061) NEXTKR,NEXTKS
C
      NPYN =NPY-1
C
 8888 CALL BINCO
C
      I=1
      LEND = 1
      KS=2
      KWR = 12
      X=0.0
C
 1    DO 1010 N=1,NPY
C  ASSIGN VALUES AT I+1 THOSE AT I
      UAS(N) = U(N)
      TAS(N) = T(N)
      RAS(N) = R(N)
      VAS(N) = V(N)
C
C  STORE VALUES AT I
      UM(N) = U(N)
      TM(N) = T(N)
      RM(N) = R(N)
      VM(N) = V(N)
 1010 CONTINUE
```

```
C   COMPUTATIONS OF COEFFICIENTS IN EQUATIONS OF MOTION AND ENERGY.
 2      DO 1020 N=1,NPY
        A(N) = R(N)*U(N)
        B(N) = R(N)*V(N)
        C1(N) = VIS(T,N)/REY
        C2(N) = CON(T,N)/(REY*PRA)
        IF(N.GE.NPYN)    U(N+2) = U(NPY)
        D(N) = (ECK*((U(N+2)-U(N))/(2*DY))**2)*(1./REY)
 1020  CONTINUE
C   NITERU,NITERT,NITERV COUNT ITERATIONS FOR U,V,T AT THE SAME I.
        NITERU=1
        NITERT=1
        NITERV =1
C
C   COMPUTATION OF COEFFICIENTS IN EQUATIONS OF MOTION AND ENERGY
C   AT MIDDLE POINTS.
        I=I+1
        X=X+DX
 3      DO 1030 N=1,NPY
        AOLD(N) = UAS(N)*RAS(N)
        BOLD(N) = VAS(N)*RAS(N)
        COLD1(N) = VIS(TAS,N)/REY
        COLD2(N) = CON(TAS,N)/(REY*PRA)
        IF (N.GE.NPYN)  UAS(N+2) = UAS(NPY)
        DOLD(N) = (ECK*((UAS(N+2)-UAS(N))/(2*DY))**2)*(1./REY)
        AA(N) = (A(N)+AOLD(N))/2
        BB(N) = (B(N)+BOLD(N))/2
        DD(N) = (D(N)+DOLD(N))/2
        CC1(N) = (C1(N)+COLD1(N))/2.
        CC2(N) = (C2(N)+COLD2(N))/2.
        VOLD(N) = (VAS(N) + VM(N))/2.
 1030 CONTINUE
C
C
C   COMPUTATION OF COEFFICIENTS IN RECCURENCE FORMULAE
C   FOR VELOCITY (SUFFIX 1) AND TEMPERATURE (SUFFIX 2)
C
C
 4      DO 1040 N=2,NPY
        IF (N.EQ.NPY)   UM(N+1) = UM(N)
        IF (N.EQ.NPY)   TM(N+1) = TM(N)
        IF(N.EQ.NPY) CC1(N+1)= CC1(N)
        IF(N.EQ.NPY) CC2(N+1)= CC2(N)
        AL1(N) = -S1*(BB(N)+(CC1(N)+CC1(N-1))/DY)/(2*DY)
        BE1(N) = AA(N)/DX+(S1/2)*(CC1(N+1)+2*CC1(N)+CC1(N+1))/(DY**2)
        GA1(N) =S1*(BB(N)-(CC1(N+1)+CC1(N))/DY)/(2.*DY)
        DE1(N) = (1-S1)*(BB(N)+(CC1(N)+CC1(N-1))/DY)*UM(N-1)/(2*DY)
       $+(AA(N)/DX-(1-S1)*(CC1(N+1)+CC1(N)*2+CC1(N-1))/(2*(DY**2)))*UM(N
       $ -(1-S1)*(BB(N)-(CC1(N+1)+CC1(N))/DY) *UM(N+1)/(2*DY)
```

```
      AL2(N) = -S2*(BB(N)+(CC2(N)+CC2(N-1))/DY)/(2*DY)
      BE2(N) = AA(N)/DX+(S2/2)*(CC2(N+1)+2*CC2(N)+CC2(N+1))/(DY**2)
      GA2(N) =S2*(BB(N)-(CC2(N+1)+CC2(N))/DY)/(2.*DY)
      DE2(N) = (1-S2)*(BB(N)+(CC2(N)+CC2(N-1))/DY)*TM(N-1)/(2*DY)
     $+(AA(N)/DX-(1-S2)*(CC2(N+1)+CC2(N)*2+CC2(N-1))/(2*(DY**2)))*TM(N
     $ -(1-S2)*(BB(N)-(CC2(N+1)+CC2(N))/DY) *TM(N+1)/(2*DY)
     $ +DD(N)
 1040 CONTINUE
C
      IF (I.GT.2.OR.NITERU.GT.1) GO TO 41
C  WRITE STATEMENTS TO CHECK THE COEFFICIENTS IN THE RECCURENCE
C  FORMULAE
      WRITE (6,9100)
      WRITE (6,9101) (AL1(N),BE1(N), GA1(N),DE1(N),
     $ AL2(N),BE2(N),GA2(N),DE2(N),N=1,19)
      DO 1045 N=20,NPY,10
      WRITE (6,9101)  AL1(N),BE1(N), GA1(N),DE1(N),
     $ AL2(N),BE2(N),GA2(N),DE2(N)
 1045 CONTINUE
C
C  COMPUTE FORCING COEFFICIENTS  (SUFFIX 1 FOR VELOCITY
C  AND 2 FOR TEMPERATURE)
 41   FCA1(1)=0.0
      FCA2(1)=0.0
      FCB1(1)=0.0
      FCB2(1)=0.0
      IF (I.GT.2.OR.NITERU.GT.1) GO TO 5
      WRITE (6,9110) I
 5    DO 1050 N=2,NPY
      FCA1(N)= -GA1(N)/(AL1(N)*FCA1(N-1)+BE1(N))
      FCA2(N)= -GA2(N)/(AL2(N)*FCA2(N-1)+BE2(N))
      FCB1(N)= (DE1(N)-AL1(N)*FCB1(N-1))/(AL1(N)*FCA1(N-1)+BE1(N))
      FCB2(N)= (DE2(N)-AL2(N)*FCB2(N-1))/(AL2(N)*FCA2(N-1)+BE2(N))
 1050 CONTINUE
      IF (I.GT.2.OR.NITERU.GT.1) GO TO 51
C  WRITE STATEMENTS TO CHECK THE FORCING COEFFICIENTS
      WRITE (6,9111)(N,FCA1(N),FCA2(N),FCB1(N),FCB2(N) ,N=1,19)
      DO 1055 N= 20,NPY,10
      WRITE (6,9111) N,FCA1(N),FCA2(N),FCB1(N),FCB2(N)
 1055 CONTINUE
C
C  CALCULATION OF DEPENDENT VARIABLES AT NEXT STATION DOWNSTREAM
 51   U(NPY) = UIED(I)
      T(NPY) = TIED(I)
      R(NPY) = RIED(I)
 6    DO 1060 L=2,1000
      IF (L.GT.NPY) GO TO 7
      N= NPY+1-L
      U(N) = FCA1(N)*U(N+1) + FCB1(N)
      T(N) = FCA2(N)*T(N+1) + FCB2(N)
 1060 CONTINUE
```

```
C  COMPUTE DENSITY AT I
 7     DO 1070 N =1,NPY
       T(N) = T(N)*DTO + WT
       R(N) = EXT*EXR/T(N)
       R(N) = R(N)/EXR
 1070 CONTINUE
C
C  RE-ESTABLISH DIMENSIONLESS T(I,N).  NECCESSARY BECAUSE OF 7
 8     DO 1080 N=1,NPY
       T(N) = (T(N) - WT)/DTO
 1080 CONTINUE
C
C  CHECK RESULTS AT THE START OF THE CALCULATIONS
        IF (I.GT.3.OR. NITERU.GT.1) GO TO 81
       WRITE (6,9120) I
       WRITE (6,9121) ( N,U(N), T(N) ,R(N), V(N),N=1,19)
       DO  85  N=20,NPY,10
       WRITE (6,9121)   N,U(N), T(N) ,R(N),V(N)
 85    CONTINUE
C  TEST TO ESTABLISH PROFILES AT I
 81    DIFUMX =0.0
       DIFTMX =0.0
 9     DO 1090  N=2,NPY
       DIFU(N) = ABS( U(N) - UAS(N))
       DIFT(N) = ABS( T(N) - TAS(N))
C  LOCATE MAXIMUM DIFU DIFT, DIFV.
       IF (DIFU(N).GT.DIFUMX) DIFUMX =DIFU(N)
       IF (DIFT(N).GT.DIFTMX) DIFTMX =DIFT(N)
 1090 CONTINUE
C
C  COMPARE RESULTS WITH ASSUMED PROFILES
       IF (I.GT.10) GO TO 10
       WRITE(6,9130) DIFUMX,DIFTMX,I
C
 10    IF (DIFUMX.LT.TOL1  .AND. DIFTMX.LT.TOL2) GO TO 17
       IF (DIFUMX.LT.TOL1  ) GO TO 13
 11    DO 1110 N=1,NPY
       UU(N) = U(N) +Z*(U(N)-UAS(N))
       IF (UU(N) .LE.0.) GO TO 1125
 1110 CONTINUE
       DO 1120 N=1,NPY
       UAS(N) = UU(N)
 1120 CONTINUE
       GO TO 1128
 1125 DO 1127 N=1,NPY
       UAS(N) = U(N)
 1127 CONTINUE
 1128 NITERU = NITERU +1
       IF(I.GT. 10) GO TO 13
       WRITE (6,9140)
```

```
 13    IF (DIFTMX.LT.TOL2) GO TO 14
       DO 1135 N=1,NPY
       TT(N) = T(N) +Z*(T(N)-TAS(N))
       IF (TT(N) .LE.0.) GO TO 1145
 1135  CONTINUE
       DO 1140 N=1,NPY
       TAS(N) = TT(N)
       RAS(N) = R(N) +Z*(R(N)-RAS(N))
 1140  CONTINUE
       GO TO 1148
 1145  DO 1147 N=1,NPY
       TAS(N) = T(N)
       RAS(N) = R(N)
 1147  CONTINUE
 1148  NITERT = NITERT +1
       IF(I.GT.10)  GO TO 15
       WRITE (6,9150)
C
C
 14     WRITE (6,9160)I, NITERU, NITERT
 15    IF (NITERU.GT.20.OR.NITERT.GT.20) GO TO 16
 3333  GO TO 3
 16    WRITE (6,9170)I,NITERU,NITERT,DIFUMX,DIFTMX
C
C  COMPUTE DENSITY AT THE AUXILIARY GRID
 17    DO 1150 N=1,NPY
       RNEW(N) = (RM(N)+R(N))/2.
 1150   CONTINUE
C  COMPUTE NORMAL VELOCITY AT THE AUXILIARY GRID
 18    DO 1160 N=1,NPYN
       V(N+1) = ( 1./RNEW(N+1))*(V(N)*RNEW(N)
      $-(DY/(2.*DX))*(R(N)*U(N) - RM(N)*UM(N)
      $+R(N+1)*U(N+1)-RM(N+1)*UM(N+1)))
 1160  CONTINUE
C
 19    IF (I.NE.KWR.AND.LEND.NE.10) GO TO 21
        KWR = KWR+NEXTKR
 191   WRITE (6,9180) I
       DO 1170 N=1,NPY
       HTA(N) = Y(N)*SQRT(REY/X)*SQRT(WR/EXR)
       VDIM(N) = V(N)*SQRT(REY*X)
 1170  CONTINUE
 20    WRITE (6,9190)(N,U(N), T(N),R(N), V(N),Y(N),HTA(N),VDIM(N),N=1,19
       DO 1180 N=20,NPY,10
       WRITE (6,9190) N,U(N), T(N),R(N), V(N),Y(N),HTA(N),VDIM(N)
 1180  CONTINUE
       IF (I.NE.NPX.AND.LEND.NE.10) GO TO 21
       GO TO 555
 21    DUPMA =0.
```

```
      DTPMA =0.
      DO 1190 N=1,NPY
      DUP(N) = ABS(U(N) - UM(N))
      DTP(N) = ABS(T(N) - TM(N))
      IF (DUP(N) .GT.DUPMA) DUPMA=DUP(N)
      IF (DTP(N) .GT.DTPMA) DTPMA=DTP(N)
 1190 CONTINUE
      IF (DUPMA.GT.TOLU .AND.DTPMA.GT.TOLT ) GO TO 22
      LEND=10
      WRITE (6,9200) I,X
      GO TO 1701
 22   IF (I.EQ.NPX) GO TO 191
 23   DO 1200 N=1,NPY
      V(N) = 2.*V(N) -VAS(N)
 1200 CONTINUE
      IF (I.NE.KS) GO TO 1111
      KS = KS+ NEXTKS
      VELGR = (18*U(2)-9*U(3)+2*U(4))/(6*DY)
      SHEAR = VELGR/REY
      SUMIN = 0.0
      NSIM = NPY-2
      DO 1210 N=1,NSIM,2
      SUMIN = SUMIN + (U(N) +4*U(N+1) +U(N+2))
 1210 CONTINUE
      DIS = DY*(NPY-SUMIN/3.)
      WRITE (6,9210) I,X,SHEAR ,DIS
 1111 GO TO 1
 555  IF (LEND.EQ.10) GO TO 5555
      WRITE (6,9230)   X
 5555 STOP
C
C
C  FORMAT STATEMENTS
C
 9011 FORMAT( 15X,3HEXL,7X,3HEXU,7X,3HEXT,6X,4HEXCP,7X,3HEXP)
  901 FORMAT (     F10.6,F10.3,F10.3, F10.3,F10.3)
 9012 FORMAT(  12X,F10.6,F10.3,F10.3, F10.3,F10.3//)
 9021 FORMAT (24X,2HWU,18X,2HWV,18X,2HWT)
  902 FORMAT (    F10.4,10X,F10.4,10X,F10.4)
 9022 FORMAT (20X,F10.4,10X,F10.4,10X,F10.4//)
 9031 FORMAT( 24X,3HNPY,7X,3HNPX,7X,2HS1,8X,2HS2,8X,2HNB/)
 903  FORMAT(      2I10,2F10.3 ,I10)
 9032 FORMAT( 20X,2I10,2F10.3 I10//)
 904  FORMAT (4F10.0)
 9041 FORMAT (20X,4HTOL1,6X,4HTOL2,6X,4HTOLU,6X,4HTOLT/
     $ 19X,4(F6.4,5X))
```

```
 905   FORMAT(2F10.0)
 9051  FORMAT (20X,2HZ=,F5.2,10X,7HSTEPRA=,F6.3/)
 906   FORMAT (2I10)
 9061  FORMAT (20X,7HNEXTKR=,I4,7HNEXTKS=,I4/)
 9100  FORMAT(15X,3HAL1,10X,3HBE1,10X,3HGA1,10X,3HDE1,
      $         10X,3HAL2,10X,3HBE2,10X,3HGA2,10X,3HDE2/)
 9101  FORMAT(8(5X,E10.3))
 9110  FORMAT (20X,26HFORCING COEFFICIENTS AT I=,I2/
      $ 10X,1HN,9X,7HFCA1(N),9X,7HFCA2(N),9X,7HFCB1(N),9X,7HFCB2(N)/)
 9111  FORMAT (8X,I3,8X,E9.2,8X,E9.2,8X,E9.2,8X,E9.2)
 9120  FORMAT (10X, 27HRESULTS FROM LOOP SIX AT I=, I3/
      $ 18X,1HN,7X,1HU,14X,1HT,14X,1HR,14X,1HV/)
 9121  FORMAT (10X,I10,F10.6,5X,F10.6,5X,F10.6,5X,F10.6)
 9130  FORMAT (20X,6HDIFUMX,10X,6HDIFTMX,10X,I/19X,E8.2,9X,E8.2,7X,I3/)
 9140  FORMAT (10X, 22HDIFUMX IS NOT.LT. TOL1)
 9150  FORMAT (10X, 22HDIFTMX IS NOT.LT. TOL2)
 9160  FORMAT (20X,1HI,7X,6HNITERU,6X,6HNITERT/18X,I3,10X,I3,10X,I3/)
 9170  FORMAT( 10X,48HNUMBER OF ITERATIONS TO ESTABLISH PROFILES AT I=,
      $I3,3HARE/ 17X,6HNITERU,              4X,6HNITERT/ 10X,2I10//
      $10X,22HMAXIMUM DEVIATIONS ARE/ 17X,6HDIFUMX,4X,
      $6HDIFTMX/ 10X,2F10.6//)
 9180  FORMAT  (10X,22HPROFILES AT STATION I=,I3// 18X,1HN, 7X,1HU,
      $14X,1HT,14X,1HR,14X,1HV,14X,1HY,12X,3HHTA,11X,4HVDIM/)
 9190  FORMAT (10X,I10,7(F10.6,5X))
 9200  FORMAT (//10X,28HCALCULATIONS CONVERGED AT I=,I4,5X,2HX=,F10.5//)
 9210  FORMAT (10X,2HI=,I4,10X,2HX=,F8.4,10X,6HSHEAR=,E12.5,
      $ 10X,4HDIS=,E12.5)
 9230  FORMAT(10X,52HCOMPUTATIONS COMPLETED UP TO LAST STATION DOWNSTREA
      $//15X,2HX=,F10.5/)
C
       END



       SUBROUTINE BINCO
C
C  SUBROUTINE TO CALCULATE BOUNDARY CONDITIONS AND INITIAL PROFILES
C
       COMMON  EXL,EXU,EXR,EXT,EXCP,EXK,EXM,DTO,  ECK,REY,PRA,
      $WU,WT,WR,WV,  NPY,NPX,DY,DX,  THICK, NPYN, NB, EXP, N,
      $ U(300), T(300), R(300), V(300), Y(300),
      $UIW(510),TIW(510),RIW(510),VIW(510),
      $ UIED(510),TIED(510),RIED(510)
C
C  DENSITY ,VISCOSITY AND CONDUCTIVITY STATEMENT FUNCTIONS
       RHO(T) = 1.26*EXP/T
       VIM(T) = 2.270*(T**1.5)*(10.**(-8))/(T+198.6)
       CO(T) = 0.2161*(0.0133+0.00002*(T-460))
C
C
C  COMPUTE VALUES OF PROPERTIES AT THE WALL AND AT THE OUTER EDGE
       EXR = RHO(EXT)
       WR = RHO(WT)
```

```
      EXM = VIM(EXT)
      WM = VIM(WT)
      EXK = CO(EXT)
      WK = CO(WT)
      WRITE (6, 90) EXR,WR,EXM,WM,EXK,WK
C
C
C
C  COMPUTE DIMENSIONLESS PARAMETERS
      REY = (EXL*EXU*EXR)/EXM
      DTO=EXT-WT
      ECK = -(EXU**2)/(EXCP*DTO)
      PRA= (EXM*EXCP)/EXK
      WRITE( 6,91) REY,DTO,ECK,PRA
C  COMPUTE BOUNDARY CONDITIONS AT OUTER EDGE
      DO 110 I=2,NPX
      UIED(I) = EXU
      TIED(I) = EXT
      RIED(I) = EXR
 110  CONTINUE
C  COMPUTE BOUNDARY CONDITIONS AT WALL
      DO 120 I=2,NPX
      UIW(I) = WU
      TIW(I) = WT
      RIW(I) = WR
      VIW(I) = WV
 120  CONTINUE
      THICK= 5.*SQRT(EXM*EXL/(EXR*EXU))
      DY = THICK/NB
      DX = STEPRA*DY
      WRITE (6,92) THICK,DY,DX
C  COMPUTE INITIAL PROFILES AT STATION X=0.0
      WRITE (6,93)
      Y(1)=0.0
      DO 130 N=1,NPY
      Y(N+1) = Y(N) +DY
      IF (N.GT.1)   GO TO 135
      U(N)  = 0.0
      T(N)  = WT
      R(N)  =EXT*EXR/T(N)
      GO TO 130
 135  U(N) = EXU
      T(N) = EXT
      R(N) = EXR
 130  CONTINUE
      WRITE(6,94)(N,Y(N), U(N), T(N), R(N)   ,N=1,19)
      DO 131 N=20,NPY,10
      WRITE(6,94) N,Y(N), U(N), T(N), R(N)
 131  CONTINUE
```

```
C  COMPUTE DIMENSIONLESS INITIAL PROFILES AT X=0.0
      WRITE (6,95)
      DO 140 N=1,NPY
      Y(N)= Y(N)/EXL
      U(N) = U(N)/EXU
      T(N) = (T(N)-WT)/DTO
      R(N) = R(N)/EXR
      V(N) = 0.00
 140  CONTINUE
      WRITE (6, 96)( N,Y(N), U(N), T(N), R(N) ,N=1,19)
      DO 145 N=20,NPY,10
      WRITE (6, 96)  N,Y(N), U(N), T(N), R(N)
 145  CONTINUE
C  COMPUTE DIMENSIONLESS BOUNDARY CONDITIONS AT OUTER EDGE
      DO 150 I= 2,NPX
      UIED(I) = UIED(I)/EXU
      TIED(I) = (TIED(I)-WT)/DTO
      RIED(I) = RIED(I)/EXR
 150  CONTINUE
C  DIMENSIONLESS BOUNDARY CONDITIONS AT THE WALL
      DO 160 I= 2,NPX
      UIW(I) = UIW(I)/EXU
      TIW(I) = (TIW(I)-WT)/DTO
      RIW(I) = RIW(I)/EXR
      VIW(I) = VIW(I)/EXU
 160  CONTINUE
C  COMPUTE DIMENSIONLESS DY,DX
      DY = DY/EXL
      DX=DX/EXL
      WRITE (6, 97) DY,DX
C
C
C  FORMAT STATEMENTS
C
 90   FORMAT(10X,24HEXR,WR,EXM,WM,EXK,WK ARE//10X,6(E10.3,5X))
 91   FORMAT ( 10X,9HREYNOLDS=,F10.1,3X,7HEXT-WT=,F10.2,3X,7HECKERT=,
     $F10.5,3X,8HPRANDTL=,F10.4//)
 92   FORMAT (10X,30HTHICKNESS OF BOUNDARY LAYER IS,F10.6/ 30HSTEP SIZE
     $INTHE Y-DIRECTION IS,F12.9/30HSTEP SIZE IN THEX-DIRECTION IS,
     $F12.9//)
 93   FORMAT (10X,33HINITIAL PROFILES AT STATION X=0.0//18X,1HN,7X,1HY,
     $14X,1HU,14X,1HT,14X,1HR/)
 94   FORMAT (10X,I10,F10.5,5X,F10.5,5X,F10.5,5X,F10.5)
 95   FORMAT (10X,39HDIMENSIONLESS INITIAL PROFILES AT X=0.0//18X,1HN,
     $7X,1HY,14X,1HU,14X,1HT,14X,1HR14X,1HV/)
 96   FORMAT (10X,I10,F10.5,5X,F10.5,5X,F10.5,5X,F10.5)
 97   FORMAT (10X,28HDIMENSIONLESS STEP SIZES DY=,F12.8,3HDX=F12.8//
     $30HEND OF SUBROUTINE BINCO OUTPUT////)
C
      RETURN
      END
```

```
      FUNCTION VIS(T,N)
C
C   VIS IS THE VISCOSITY-TEMPERATURE RELATIONSHIP.
C
      COMMON NPY,EXM ,EXT,WT
      DIMENSION T(300)
C   CHANGE TO DIMENSIONAL TEMPERATURE.
      T(N) = T(N)*(EXT-WT)+WT
      VIS     = 2.270*(T(N)**1.5)*(10.**(-8))/ (T(N)+198.6)
      VIS = VIS/EXM
      RETURN
      END
```

```
      FUNCTION CON(T,N)
C
C   CON IS THE CONDUCTIVITY - TEMPERATURE RELATIONSHIP.
C
      COMMON NPY,EXK ,EXT,WT
      DIMENSION T(300)
C   CHANGE TO DIMENSIONAL TEMPERATURE.
      T(N) = T(N)*(EXT-WT)+WT
      CON       = 0.2161*(0.0133+0.00002*(T(N)-460))
      CON = CON/EXK
      RETURN
      END
```