THE JOINT DISTRIBUTION OF TWO LINEAR COMBINATIONS OF RANDOM VARIABLES UNIFORMLY DISTRIBUTED ON A SIMPLEX

THE JOINT DISTRIBUTION OF TWO LINEAR COMBINATIONS OF RANDOM VARIABLES UNIFORMLY DISTRIBUTED ON A SIMPLEX

By

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SCOPE AND CONTENTS :

This thesis deals with linear combinations of a set of random variables uniformly distributed on a simplex. The exact joint distribution of two general linear combinations with real constant coefficients is considered and the results found in the form of the joint probability density function. Application of the result is also illustrated.

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INTRODUCTION

Consider n random variables $u_1, u_2, \ldots, u_n$ uniformly distributed over the simplex of points defined by $\left\{ (u_1, \ldots, u_n) : u_i \geqslant 0, i=1,\ldots,n; \sum_{i=1}^{n} u_i \leq 1 \right\}$ so that their joint density function is given by

$$f(u_1, \ldots, u_n) = n! \qquad (u_i \geq 0, \ i=1,\ldots,n; \sum_{i=1}^{n} u_i \leq 1).$$

The aim of this manuscript is to generalize a formula for the joint density function of any two arbitrary linear combinations of $(u_1, u_2, \ldots, u_n)$, specifically the combinations

$$y_1 = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$$

$$y_2 = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n ,$$

where the coefficients a_{ij} $(i = 1,2 ; j = 1,2,\ldots,n)$ are real constants.

This result can be applied to find the joint density function of linear combinations of other variables. Consider n independent random variables $x_1, x_2, \ldots, x_n$ each uniformly distributed over the interval $[0 , 1]$ so that their joint density function is

$$f(x_1, x_2, \ldots, x_n) = 1 \qquad (0 \leqslant x_i \leqslant 1 , i = 1,2,\ldots,n).$$

The ordered variates $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ obtained by ordering $(x_1, x_2, \ldots, x_n)$ in ascending order of magnitude are known to have a joint density function of the form

$$f(x_{(1)}, x_{(2)}, \ldots, x_{(n)}) = n! \qquad 0 \leq x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(1)} \leq 1$$

since there are exactly $(n)!$ permutations of $(x_1, x_2, \ldots, x_n)$ corresponding to each $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$.

If one writes

$$u_1 = x_{(1)}$$

$$u_i = x_{(i)} - x_{(i-1)} \qquad (i = 2,3,\ldots,n+1)$$

with $\quad x_{(n+1)} = 1$

then one has

$$x_{(i)} = \sum_{j=1}^{i} u_j \qquad (i = 1,2,\ldots,n+1).$$

It is easy to see that

$$u_i \geq 0 \qquad (i = 1,2,\ldots,n+1)$$

and

$$\sum_{i=1}^{n+1} u_i = x_{(n+1)} = 1$$

or

$$\sum_{i=1}^{n} u_i \leq 1 .$$

The Jacobian $\partial(u_1,u_2,\ldots,u_n)/\partial(x_{(1)},x_{(2)},\ldots,x_{(n)}) = 1$, and so the joint density function of $(u_1,u_2,\ldots,u_n)$ is

$$f(u_1,u_2,\ldots,u_n) = n! \qquad (u_i \geq 0,\ i=1,2,\ldots,n\ ;\ \sum_{i=1}^{n} u_i \leq 1).$$

By this transformation, the joint distribution of $(x_{(1)},x_{(2)},\ldots,x_{(n)})$ will be that of $(u_1,u_1+u_2,\ldots,\sum_{j=1}^{i} u_j,\ldots,\sum_{j=1}^{n} u_j)$, and therefore the joint density function of the linear combinations

$$X_1 = c_{11}x_{(1)} + c_{12}x_{(2)} + \ldots + c_{1n}x_{(n)}$$

$$X_2 = c_{21}x_{(1)} + c_{22}x_{(2)} + \ldots + c_{2n}x_{(n)}$$

is the same as that of

$$y_1 = \sum_{i=1}^{n} a_{1i}u_i$$

$$y_2 = \sum_{i=1}^{n} a_{2i}u_i ,$$

where

$$a_{ij} = \sum_{k=j}^{n} c_{ik} \qquad (i = 1,2\ ;\ j = 1,2,\ldots,n)$$

For reasons apparent from this derivation, u_i can be referred to

as an "interval".

The joint density function of linear combinations of the ordered variates $u_{(1)} \leq u_{(2)} \leq \ldots \leq u_{(n)}$, obtained by ordering $\left\{u_i : u_i \geq 0 , i = 1,2,\ldots,n; \sum_{i=1}^{n} u_i \leq 1\right\}$ in ascending order of magnitude, can be found in a similar manner. One knows that the joint density function of $\left\{ u_{(i)} : 0 \leq u_{(1)} \leq \ldots \leq u_{(n)} , \sum_{i=1}^{n} u_{(i)} \leq 1 \right\}$ is

$$f(u_{(1)},u_{(2)},\ldots,u_{(n)}) = n!n! \quad (0 \leq u_{(1)} \leq u_{(2)} \leq \ldots \leq u_{(n)}; \sum_{i=1}^{n} u_{(i)} \leq 1)$$

since there are exactly $n!$ permutations of $(u_1,u_2,\ldots,u_n)$ corresponding to each $(u_{(1)} \leq u_{(2)} \leq \ldots \leq u_{(n)})$.

Let

$$v_1 = n\, u_{(1)}$$

$$v_j = (n+1-j)(u_{(j)} - u_{(j-1)}) \quad (j = 2,3,\ldots,n) .$$

Then $u_{(i)} = \sum_{j=1}^{i} v_j/(n+1-j)$ $(i=1,2,\ldots,n)$, $\partial(v_1,v_2\ldots,v_n)/\partial(u_{(1)},\ldots,u_{(n)}) = n!$

and the joint density function of $(v_1,v_2,\ldots,v_n)$ is given by

$$f(v_1,v_2,\ldots,v_n) = n! \qquad (v_i \geq 0, i = 1,\ldots,n ; \sum_{i=1}^{n} v_i \leq 1).$$

Thus the joint density function of

$$U_1 = d_{11}u_{(1)} + d_{12}u_{(2)} + \ldots + d_{1n}u_{(n)}$$

$$U_2 = d_{21}u_{(1)} + d_{22}u_{(2)} + \ldots + d_{2n}u_{(n)}$$

is the same as that of

$$y_1 = \sum_{i=1}^{n} a_{1i}u_i$$

$$y_2 = \sum_{i=1}^{n} a_{2i}u_i ,$$

where

$$a_{ij} = (\sum_{k=j}^{n} d_{ik})/(n+1-i) \qquad (i = 1,2 ; j = 1,2,\ldots,n).$$

The distributions of functions of $\left\{x_{(i)} : 0 \leq x_{(1)} \leq \ldots \leq x_{(n)} \leq 1\right\}$, $\left\{u_{(i)} : u_{(1)} \leq u_{(2)} \leq \ldots \leq u_{(n)} ; \sum_{i=1}^{n} u_{(i)} \leq 1\right\}$ or $\left\{u_i : u_i \geq 0 , i=1,\ldots,n; \sum_{i=1}^{n} u \leq 1\right\}$ are of interest in a wide class of statistical problems. In some problems it is required to test whether a given set of points on the unit interval could have resulted from independent selection of points from the unit interval. If the corresponding $\left\{u_i : u_i \geq 0 , i = 1,\ldots,n ; \sum_{i=1}^{n} u_i \leq 1\right\}$ are almost all equal, then it might be suspected that the selection was not independent. For example in the study of the Poisson process, if $z_1, z_2, \ldots, z_{n+1}$ are (n+1) independent random variables representing the intervals between successive occurences of the phenomena, each having the density function $\lambda e^{-\lambda z}$, $\lambda > 0$, $z > 0$, then the joint distribution of $\left\{ z_i / \sum_{j=1}^{n+1} z_j , i = 1,2,\ldots,n\right\}$ is the same as the joint distribution of $\left\{u_i : i = 1,\ldots,n , \sum_{i=1}^{n} u_i \leq 1\right\}$ and has been used in studying the Poisson process.

Also consider a set of n independent identically distributed random variables $w_1, w_2, \ldots, w_n$ with the common continuous cummulative distribution function $F(w)$. Then it is well known that if the corresponding order statistics are $w_{(1)} < w_{(2)} < \ldots < w_{(n)}$ and $w_{(0)} = -\infty$, $w_{(n+1)} = +\infty$, then the joint distribution of $\left\{F(w_{(i+1)}) - F(w_{(i)}) : i=0,\ldots, n-1\right\}$ is the same as that of $\left\{u_i : i = 1,\ldots,n ; \sum_{i=1}^{n} u_i \leq 1\right\}$. Thus tests for the randomness of a given set of points on the unit interval can be used to test the hypothesis that the w_i's came from a population whose cummulative distribution function is $F(w)$.

Although linear combinations are not the only functions of $\left\{u_i : u_i \geq 0, i=1,\ldots,n ; \sum_{i=1}^{n} u_i \leq 1\right\}$ that are useful, the discussion will

be restricted to these functions and their exact distributions.

In Chapter I, a brief survey is given of recent papers that have dealt with the exact distributions of the linear combinations described above. In Chapter II, the joint distribution of two linear combinations is considered and a stronger result than that available from the literature is found under weaker conditions. In Chapter III, a result is proved that often allows one to use the result of Chapter II even when these weaker conditions do not hold. This result is used to find certain useful distributions.

Chapter I: Survey of papers on the exact distributions of linear combinations of random variables uniformly distributed on a simplex

§ 1.1 Distribution of one linear combination

Although there are earlier papers that consider the problems of finding the distributions of linear combinations of $(u_1, u_2, \ldots, u_n)$, the 1951 paper of Mauldon is the first of a number of more recent papers that led to the complete and thorough solution for the distribution of one linear combination. In this paper he considered the distribution of the particular linear combination

$$u = \sum_{i=n+2-k}^{n+1} u_{(i)} \qquad \left(u_{(1)} \leq u_{(2)} \leq \ldots \leq u_{(n+1)}, \sum_{i=1}^{n+1} u_{(i)} = 1 \right),$$

that is, the distribution of the sum of the k largest of the intervals $\left\{ u_i : u_i \geq 0 \, , \, i = 1, 2, \ldots, n+1 \, ; \sum_{i=1}^{n+1} u_i = 1 \right\}$. He considered a moment generating function $M(t) = E\left\{(1 - ut)^{-(n+1)}\right\}$ and after obtaining the moments independently he found that

$$E\left\{(1-ut)^{-(n+1)}\right\} = \frac{(n+1)!}{k!} (1 - t)^{-k} \prod_{j=k+1}^{n+1} (j-kt)^{-1} \quad .$$

Then he used a geometrical interpretation to point out that the density function $f(u)$ is actually a polynomial in u of degree not greater than $(n-1)$ in the range $k/p < u < k/(p-1), (p=k+1, \ldots, n+1)$ and is zero otherwise . Thus he was able to write

$$E\left\{(1-ut)^{-(n+1)}\right\} = \int_{k/n+1}^{1} (1 - ut)^{-(n+1)} f(u) \, du$$

$$= \frac{(n+1)!}{k!} (1-t)^{-k} \prod_{j=k+1}^{n+1} (j-kt)^{-1}$$

or writing $f_p(u)$ for $f(u)$ in the range $k/p < u < k/(p-1)$, $(p=k+1,\ldots,n+1)$,

$$\sum_{p=k+1}^{n+1} \int_{k/p}^{k/(p-1)} (u-t)^{-(n+1)} f_p(u)\, du$$

$$= \frac{(n+1)!}{k!} (1-t)^{-k} \prod_{j=k+1}^{n+1} (k-jt)^{-1}$$

for all t except $t=k/j$ $(j=k,\ldots,n+1)$. The behaviour of each side of this equation as t tends to one of the values k/p was then used to evaluate $f(u)$ for $k/p < u < k/(p-1)$. The formula for the density function of u was found to be

$$f_p(u) = n \sum_{j=p}^{n+1} (-1)^{n+1-j} \frac{1}{k^{n-k}(j-k)^{k-1}} \cdot \frac{(n+1)!}{(n+1-j)!(j-k)!k!} (ju-k)^{n-1}$$

in the range $k/p < u < k/(p-1)$ $(p = k+1,\ldots,n+1)$.

For $k=1$, the distribution function, obtained by integrating the density function, agrees with the result found by Fisher (1929) in another connection which in fact was the problem of finding the distribution of the largest interval among $(u_1, u_2, \ldots, u_{n+1})$. Fisher's method was geometrical. Irwin (1955) also derived a similar formula using Whitworth's theorem. The theorem states that if there are N sets of letters and if out of r assigned letters $\alpha_1, \alpha_2, \ldots, \alpha_r$, each letter occurs in N_1 sets, each combination of two letters occurs in N_2 sets, and so on, and finally all the r letters occur in N_r sets, the number of sets free from all the letters will be

$$N + \sum_{i=1}^{r} \binom{r}{i} N_i (-1)^i .$$

He found by simple probability theory, that the probability density for any r intervals $u_1, u_2, \ldots, u_r$ is

$$n(n-1)\ldots(n+1-r)(1-u_1-u_2-\ldots-u_r)^{(n+1)-r-1}\, du_1 du_2 \ldots du_r .$$

This, when integrated for values of $u_1, u_2, \ldots, u_r$ all greater than some value u, gives

$$\Pr(u_1 > u, u_2 > u, \ldots, u_r > u) = (1-ru)^n \qquad \text{for } u \leq 1/r$$
$$= 0 \qquad \text{otherwise.}$$

Therefore by Whitworth's theorem, the probability that all $(u_1, \ldots, u_n)$ are less than u is

$$1 - \sum_{(i)} (-1)^{i-1} \binom{n+1}{i} (1 - iu)^n ,$$

where $\sum_{(i)}$ includes all terms for which $(1 - iu)^n$ is positive.

Therefore

$$\Pr(u_{(n+1)} \geq n) = 1 - (1 - \sum_{(i)} (-1)^{i-1} \binom{n+1}{i} (1 - iu)^n)$$
$$= \sum_{(i)} (-1)^{i-1} \binom{n+1}{i} (1 - iu)^n .$$

Darling's (1953) approach to the distribution of one linear combination was essentially an extension of the Dirichlet integral —— a formula for evaluating some particular class of multiple integrals. In his paper he gave a unified treatment of the distribution of various functions $G_n = \sum_{j=1}^{n+1} h_j(u_j)$ for quite arbitrary functions $h_j(u_j)$. For $h_j(u_j) = u_j$, he obtained the well-known result that $x_{(i)} = \sum_{j=1}^{i} u_j$ has a Beta distribution.

Four years after Mauldon considered the sums of the k largest intervals, Barton and David (1955) considered the slightly more general variables $u = \sum_{i=r}^{s} u_{(i)}$ and $x = \sum_{i=r}^{s} x_{(i)}/(s-r+1)$. The probability densities were obtained by straight-forward integration of the joint distribution of $\left\{ u_{(i)} : i = r, \ldots, s \right\}$ and the joint distribution of $\left\{ x_{(i)} : i = r, \ldots, s \right\}$ after making the appropriate transformations. The formulae obtained were

$$f(u) = \frac{(n+1)!n!}{(n+1-s)!(s-r)!(r-1)!(n+1-s)^{s-r-1}} \cdot$$

$$\cdot \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r}{j} \frac{(s-r-j)^{s-r} \cdot C}{(j+1)(n+1-s)-i(s-r-j)}$$

where

$$C = \left\{ H(1 - u\frac{(n-r+i+2)}{s-r+1}) \right\}^{n-1} - \left\{ H(1-u\frac{(n+1-r-j)}{s-r-j}) \right\}^{n-1},$$

$$H(x) = \frac{1}{2}\left\{ x + |x| \right\} \quad ,$$

and

$$f(x) = \frac{(s-r+1)}{(n-r)!(s-r-1)!(r-1)!} \sum_{j=0}^{s-r-1} \frac{(s-r-j)^{-r}(-1)^{s-r-j-1}}{(j+1)^{n-s}} \cdot$$

$$\cdot \binom{s-r-1}{j} \sum_{i=0}^{n-1} \frac{1}{(n-i)} \cdot A \quad ,$$

where

$$A = \left\{ \binom{n-r}{i} \left[H(x(s-r-j))\right]^{n-1} \left[H(j+1-x(s-r+1))\right]^{i} + \right.$$

$$\left. + \binom{r-1}{i} \left[H((j+1)(1-x))\right]^{n-i} \left[H(x(s-r+1)-(j+1))\right]^{i} \right\} \quad .$$

Then in 1956 in another paper, they showed it possible to find the distribution of $u = \sum_{i=s}^{t} r_i u_{(i)}$ $(s \leq t \leq n+1)$ where r_i is the positional rank of $u_{(i)}$, and r_i and $u_{(i)}$ are independent. They did this by summing the conditional distribution of u given a set of the $\{r_i\}$ over all possible permutations of the $\{r_i\}$ but they were not able to carry out the summation because of the complicated form of the conditional distribution. To obtain the moments of the random variables under consideration, they used the same moment generating function $M(t) = E\{(1 - ut)^{-(n+1)}\}$ for the variable u as Mauldon(1951).

Then in 1957, Mauldon was the first to find the distribution of the linear combinations $x = \sum_{i=1}^{n} c_i x_{(i)}$ and $u = \sum_{i=1}^{n+1} d_i u_{(i)}$ where the coefficients are arbitrary real numbers. That is, all the previous papers had only considered particular examples. He considered, for a random variable ν, the moment generating function

$$\phi_{(n+1)}(t) = E\left\{(t-\nu)^{-(n+1)}\right\}$$

$$= \int_{-\infty}^{\infty} (t-\nu)^{-(n+1)} \, dF(\nu)$$

and showed (by deriving the appropriate inversion formulae) that if

$$\phi_{(n+1)}(t) = \prod_{k=1}^{p} (t - g_k)^{-n_k}$$

$$= \left\{\psi(t)\right\}^{-1}$$

where $\{g_i : i = 1,2,\ldots,p\}$ are distinct real values and $\{n_i : i=1,\ldots,p\}$ are positive integers, then $\sum_{i=1}^{p} n_i = n+1$ and

$$\left.\begin{matrix} F(\nu) \\ 1 - F(\nu) \end{matrix}\right\} = \sum_{(k)} \left(\sum_{q=1}^{n_k} \binom{n}{q-1} a_{kq} (g_k - \nu)^{n+1-q} \right)$$

where $\sum_{(k)}$ is taken over the set of all values of k such that $g_k \lessgtr \nu$, using the sign $<$ for $F(\nu)$ and $>$ for $1-F(\nu)$, and

$$a_{kq} = (-1)^{q-1} b_k \, c_{kq}$$

where

$$b_k = \prod_{i \neq k} (g_k - g_i)^{-n_i}$$

$$= (n_k)! \left\{ \left(\frac{d}{dt}\right)^{n_k} \psi(g_k) \right\}^{-1}$$

$$c_{kq} = \sum_{(t)} \binom{n_i + t_i - 1}{n_i - 1} (g_k - g_i)^{-t_i}$$

and $\sum_{(t)}$ is taken over all sets of non-negative integers t_i $(i \neq k)$ such that $\sum t_i = q-1$. Then he pointed out that for the random variable u it could be derived independently that

$$\phi_{(n+1)}(t) = \prod_{k=1}^{n+1} \left(t - \sum_{i=1}^{k} d_i/k \right)^{-1} ,$$

and the preceeding result could then be applied directly once the values of $\left(\sum_{i=1}^{k} d_i\right)/k$ were known. Similarly for the distribution

of x, it could be derived that

$$\phi_{(n+1)}(t) = \prod_{k=1}^{n+1} \left(t - \sum_{i=1}^{k-1} c_i \right)^{-1} \qquad \left(k=1,\ldots,n+1; \ \sum_{i=1}^{k-1} c_i = 0 \text{ for } k=1 \right)$$

and hence the distribution also followed from the above result.

Dwass in his 1961 paper dealt with the general distribution of essentially the same linear combination $y = \sum_{i=1}^{n+1} a_i u_i$ for any real constant coefficients. Although Mauldon was interested in $\sum_{i=1}^{n} c_i x_{(i)}$ and $\sum_{i=1}^{n+1} d_i u_{(i)}$, it was pointed out in the introduction that they can be expressed as $\sum_{i=1}^{n} a_i u_i$ and in fact his derivation of $\phi_{(n+1)}(t)$ made use of this fact. However, although the problem considered by the two authors is really the same, the proofs differ. Dwass assumed that among the a_i's there were m distinct numbers, labelled them as $a_1', a_2', \ldots, a_m'$ and let each a_i' be repeated n_i times. Then it is well-known that

$$E(e^{a(z_1+z_2+\ldots+z_k)}) = (1-a/\lambda)^{-k}$$

for any real constant 'a' where $z_1, z_2, \ldots, z_{n+1}$ are independently and identically distributed with density $\lambda e^{-\lambda z}$, $\lambda > 0$, $z > 0$. Therefore

$$\begin{aligned} E(e^{\sum_{i=1}^{n+1} a_i z_i}) &= \prod_{i=1}^{n+1} (1-a_i/\lambda)^{-1} \\ &= \prod_{i=1}^{m} (1-a_i'/\lambda)^{-n_i} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n_i} b_{ij} (1 - a_i'/\lambda)^{-j} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n_i} b_{ij} E(e^{a_i'(z_1+z_2+\ldots+z_j)}) \end{aligned}$$

where b_{ij} $(i=1,\ldots,m; j=1,\ldots,n_i)$ are coefficients in the partial

fraction expansion. But

$$\sum_{i=1}^{m}\sum_{j=1}^{n_i} b_{ij}\int_0^\infty E(\, e^{a_i'(z_1+\cdots+z_j)} \,|\, S = t\,)\, f_\lambda(t)\, dt$$

where $S = \sum_{i=1}^{n+1} z_i$ and $f_\lambda(t)$ denotes the density function of $(z_1+z_2+\ldots+z_{n+1})$,

$$= \sum_{i=1}^{m}\sum_{j=1}^{n_i} b_{ij}\int_0^\infty E(\, e^{a_i' t(u_1+\cdots+u_j)})\, f_\lambda(t)\, dt$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n_i} b_{ij}\int_0^\infty \Big[\int_{-\infty}^{\infty} e^{yt} g_{a_i',j}(y)\, dy\Big]\, f_\lambda(t)\, dt \quad ,$$

$$= \int_0^\infty \Big[\int_{-\infty}^{\infty} e^{yt} \sum_{i=1}^{m}\sum_{j=1}^{n_i} b_{ij}\, g_{a_i',j}(y)\, dy\Big]\, f_\lambda(t)\, dt$$

where $g_{a_i',j}(y)$ denotes the density function of $a_i'(u_1+u_2+\ldots+u_j)$.

But
$$E(e^{\sum_{i=1}^{n+1} a_i z_i}) = \int_0^\infty E(e^{\sum_{i=1}^{n+1} a_i z_i} \,|\, S = t)\, f_\lambda(t)\, dt$$

$$= \int_0^\infty E\Big[\exp(t\sum_{i=1}^{n+1} a_i u_i)\Big]\, f_\lambda(t)\, dt$$

Therefore
$$E\Big[\exp(t\sum_{i=1}^{n+1} a_i u_i)\Big] = \int_{-\infty}^{\infty} e^{ty} \sum_{i=1}^{m}\sum_{j=1}^{n_i} b_{ij}\, g_{a_i',j}(y)\, dy$$

from which it follows that

$$f(y) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} b_{ij}\, g_{a_i',j}(y) \quad .$$

Dwass gave the explicit expression for $g_{a_i',j}(y)$ but not for the coefficients b_{ij}. This result can be used to find the density of $u = \sum_{i=1}^{n+1} d_i u_{(i)}$ as shown earlier.

In Ali's (1969) paper, one of the objectives was to find the distribution function of the linear combination $x = \sum_{i=1}^{n} c_i x_{(i)}$ for any set of real constant coefficients. He obtained the distribution function of x in the form

$$\left.\begin{aligned} F_n(x) &= \sum_{j=0}^{n} h(x,a_j,n) \Big/ \prod_{\substack{i=0 \\ \neq j}}^{n} (a_i - a_j) \\ \text{where} \quad & a_0 = 0, \; a_j = \sum_{i=j}^{n} c_i \;, \; j = 1,2,\ldots,n, \\ \text{and} \quad h(x,a,n) &= \left\{ \frac{1}{2}\left[(x-a) + |x-a|\right] \right\}^n \\ &= \begin{cases} 0 & \text{if } x \leq a \\ (x-a)^n & \text{if } x > 0 \;. \end{cases} \end{aligned}\right\} \qquad (1.1.1)$$

Two methods were used to obtain this result — (1) the method of induction and (2) the method of characteristic functions. Even though he pointed out how the distribution could be found in the case that some of the a_i's coincide, no precise formula was given, and the proofs were actually done for $a_0, a_1, \ldots, a_n$ distinct. In both derivations he considered the problem in terms of the $\left\{ u_i : u_i \geq 0, \; i = 1,2,\ldots,n \; ; \; \sum_{i=1}^{n} u_i \leq 1 \right\}$, so that

$$x = \sum_{i=1}^{n} a_i u_i \;,$$

where a_i's are as defined by Formula (1.1.1).

For the method of induction he noted that

$$F_n(x) = n! \int_{S_n \cap T_n} du_1 du_2 \ldots du_n \qquad (1.1.2),$$

where S_n is the simplex : $\left\{ (u_1,u_2,\ldots,u_n) : \sum_{i=1}^{n} u_i \leq 1, \; u_i \geq 0 \right\}$

and T_n is the simplex : $\left\{ (u_1,u_2,\ldots,u_n) : \sum_{i=1}^{n} a_i u_i \leq x \right\}$.

For $n = 1, 2,$ Formula (1.1.1) is easily verified by integrating (1.1.2) directly on the right-hand side. Then, assuming that

$$F_n(x) = \sum_{j=0}^{n} h(x,a_j,n)/\prod_{\substack{i=0\\ \neq j}}^{n}(a_i - a_j)$$

is true for n, consider

$$F_{(n+1)}(x) = (n+1)! \int_{S_{n+1}\cap T_{n+1}} du_1 \ldots du_{n+1} ,$$

where S_{n+1} is the simplex : $\left\{ (u_1,u_2,\ldots,u_{n+1}) : \sum_{i=1}^{n+1} u_i \leq 1,\ u_i \geqslant 0 \right\}$

and T_{n+1} is the simplex : $\left\{ (u_1,u_2,\ldots,u_{n+1}) : \sum_{i=1}^{n+1} a_i u_i \leq x \right\}$

with a_{n+1} distinct from $a_1,a_2,\ldots,a_n$. With the transformation

$$u_{n+1} = u \quad , \quad u_i = (1 - u)v_i \qquad (i = 1,2,\ldots,n)$$

the set $\left\{ v_i : i = 1,2,\ldots,n \right\}$ is defined in the region

$$S_n' : \left\{ (v_1,\ldots,v_n) : \sum_{i=1}^{n} v_i \leq 1,\ v_i \geqslant 0 \right\} ,$$

$$T_n' : \left\{ (v_1,\ldots,v_n) : \sum_{i=1}^{n} a_i v_i \leq (x - a_{n+1}u)/(1 - u) \right\} ,$$

and

$$F_{n+1}(x) = (n+1)\int_0^1 (1-u)^n \left[n! \int_{S_n' \cap T_n'} dv_1 \ldots dv_n \right] du$$

$$= (n+1)\int_0^1 (1-u)^n F_n((x-a_{n+1}u)/(1-u))\, du$$

$$= \sum_{j=0}^{n} (n+1) \int_0^1 h(x,a_j(1-u) + ua_{n+1},n)du/\prod_{\substack{i=0\\ \neq j}}^{n}(a_i-a_j),$$

which by evaluation of $\int_0^1 h(x,a_j(1-u) + ua_{n+1},n)\, du$, is

$$\sum_{(A_1)} (x - a_j)^{n+1} (a_{n+1} - a_j)^{-1} / \prod_{\substack{i=0\\ \neq j}}^{n}(a_i - a_j)$$

$$- \sum_{(A_2)} (x - a_{n+1})^{n+1} (a_{n+1} - a_j)^{-1} / \prod_{\substack{i=0\\ \neq j}}^{n}(a_i - a_j)$$

where $\sum_{(A_1)}$ is over the set of all j such that $x - a_j \geq 0$; and $\sum_{(A_2)}$ is over all j while $x - a_{n+1} \geq 0$.

Hence

$$F_{n+1}(x) = \sum_{j=0}^{n} h(x,a_j,n+1)/\prod_{\substack{i=0\\ \neq j}}^{n+1}(a_i - a_j)$$

$$- h(x,a_{n+1},n+1)\sum_{j=0}^{n}\Big(\prod_{\substack{i=0\\ \neq j}}^{n+1}(a_i - a_j)\Big)^{-1}$$

$$= \sum_{j=0}^{n+1} h(x,a_j,n+1)/\prod_{\substack{i=0\\ \neq j}}^{n+1}(a_i - a_j)$$

$$- h(x,a_{n+1},n+1)\sum_{j=0}^{n+1}\Big(\prod_{\substack{i=0\\ \neq j}}^{n+1}(a_i - a_j)\Big)^{-1}$$

$$= \sum_{j=0}^{n+1} h(x,a_j,n+1)/\prod_{\substack{i=0\\ \neq j}}^{n+1}(a_i - a_j)$$

since by a well known result of divided differences

$$\sum_{j=0}^{n+1}\Big(\prod_{\substack{i=0\\ \neq j}}^{n+1}(a_i - a_j)\Big)^{-1} = (-1)^n \Delta^{n+1}(1;a_0,a_1,\ldots,a_{n+1})$$

$$= 0$$

where $\Delta^n(g(x)\,;a_0,a_1,\ldots,a_n)$ represents the nth divided difference of $g(x)$ with respect to the arguments $a_0,a_1,\ldots,a_n$. Thus Formula(1.1.1) is also true for (n+1) and the induction proof is complete.

To use the method of characteristic functions, consider that

$$E(u_1^{r_1} u_2^{r_2}\ldots u_n^{r_n}) = r_1!r_2!\ldots r_n!n!/(n+r_1+\ldots+r_n)!$$

by the use of the Dirichlet distribution (Wilks p.177). Therefore for $x = \sum_{i=0}^{n} a_i u_i$ with $a_0 = 0$,

$$E(x^r) = n!r!\sum{}' a_0^{r_0} a_1^{r_1}\ldots a_n^{r_n}/(n+r)!$$

where $\sum'$ is over all the different products with positive integers $r_0, r_1, \ldots, r_n$, $r_0 + r_1 + \ldots + r_n = r$.

Then by a known result of divided differences,

$$E(x^r) = n!r!\Delta^n(z^{n+r}\,;\, a_0, a_1, \ldots, a_n)/(n+r)!$$

Since x has a finite range distribution, it is known that the characteristic function $\phi(t)$ of x is

$$\phi(t) = \sum_{r=0}^{\infty} (it)^r\, n!\Delta^n(z^{n+r}\,;\, a_0, a_1, \ldots, a_n)/(n+r)!$$

Put $m = n + r$; then by the known results that

$$\Delta^n(z^m\,;\, a_0, a_1, \ldots, a_n) = 0 \quad \text{for } 0 \leq m < n$$

and

$$\Delta^n(z^m;\, a_0, a_1, \ldots, a_n) = \sum_{j=0}^{n} a_j^m \Big/ \prod_{\substack{i=0\\ \neq j}}^{n} (a_j - a_i) \quad \text{for } m \geq n,$$

$$\phi(t) = \sum_{m=0}^{\infty} n!\ (it)^{m-n}\Delta^n(z^m;\, a_0, \ldots, a_n)/m!$$

$$= n!\ (it)^{-n} \sum_{m=0}^{\infty} \sum_{j=0}^{n} (ita_j)^m \Big/ \prod_{\substack{i=0\\ \neq j}}^{n} (a_j - a_i)m!$$

$$= n!\ (it)^{-n} \sum_{j=0}^{n} e^{ita_j} \Big/ \prod_{\substack{i=0\\ \neq j}}^{n} (a_j - a_i)$$

by interchanging the order of the summation and by expansion of e^{ita_j}. By the inversion formula for the characteristic function,

$$\frac{d}{dx} F_n(x) = n!(2\pi)^{-1} \int_{-\infty}^{\infty} \sum_{j=0}^{n} \frac{e^{it(a_j - x)}}{(it)^n \prod_{\substack{i=0\\ \neq j}}^{n} (a_j - a_i)}\, dt$$

$$= n \sum_{j=0}^{n} h(x, a_j, n-1) \Big/ \prod_{\substack{i=0\\ \neq j}}^{n} (a_i - a_j)$$

which upon integration, gives

$$F_n(x) = \sum_{j=0}^{n} h(x,a_j,n) / \prod_{\substack{i=0 \\ \neq j}}^{n} (a_i - a_j) \quad .$$

This method is almost the same as the method used by Mauldon (1957), the essential difference being that Mauldon used a different moment generating function than the characteristic function.

The main objective of Weisberg (1971) was to find the general distribution of $x = \sum_{i=1}^{n} c_i x_{(i)}$ ($c_i \geq 0$) or equivalently $x = \sum_{i=1}^{s} c_i x_{(l_i)}$ for any real coefficients $c_i > 0$ and $l_0 = 0 < l_1 < \ldots < l_s \leq n$.

He transformed the $\{x_{(i)} : 0 \leqslant x_{(1)} \leqslant \ldots \leqslant x_{(n)} \leqslant 1\}$ to the $\{u_i : u_i \geqslant 0 , i = 1,2,\ldots,n ; \sum_{i=1}^{n} u_i \leq 1\}$ and wrote a_i ($i = 1,2,\ldots,n+1$) as

$$a_{l_i} = a_{l_i+1} + c_i \qquad \text{for } i = 1,2,\ldots,s,$$

$$a_j = a_{j+1} \qquad \text{for } j \notin (l_1,l_2,\ldots,l_s)$$

and $\quad a_{n+1} = 0.$

Then he had

$$\sum_{i=1}^{s} c_i x_{(l_i)} = \sum_{i=1}^{n} a_i u_i \ ,$$

and for the special case $s = n$, when all $a_1,a_2,\ldots,a_n$ are distinct and strictly decreasing, Dempster and Kleyle (1968) had already shown by a completely geometrical approach that

$$\Pr(\sum_{i=1}^{n} c_i x_{(l_i)}) = \Pr(\sum_{i=1}^{n} a_i u_i \leq x)$$

$$= 1 - \sum_{j=1}^{r} \frac{(a_j - x)^n}{a_j \prod_{i \neq j} (a_j - a_i)} \qquad (1.1.3)$$

for $0 \leq x \leq a_1$, where r is the largest integer such that $x \leq a_r$.

For $s < n$, let

$$a_{l_{i-1}+1} = a_{l_{i-1}+2} = \cdots = a_{l_i} = a_{(i)} \qquad (i = 1,2,\ldots,s).$$

Then there are $r_i = (l_i - l_{i-1})$ $(i = 1,2,\ldots,s)$ of the $a_1, a_2, \ldots, a_n$ that take the value $a_{(i)}$ and $r_{s+1} = (n - l_s)$ take the value $a_{(s+1)} = 0$, and in this situation Formula (1.1.3) is not applicable. Weisberg's derivation to generalize Formula (1.1.3) to include repeated coefficients involved altering the coefficients to a new set of coefficients $b_i(\varepsilon)$ defined by

$$b_{l_{i-1}+j}(\varepsilon) = a_{(i)} + (r_i - j)\varepsilon\,, \quad \varepsilon > 0\,, \; j = 1,\ldots,r_i\,; \quad i = 1,\ldots,s$$

$$b_{l_s+j}(\varepsilon) = (r_{s+1} + 1 - j)\varepsilon\,, \quad j = 1,2,\ldots,r_{s+1}$$

and evaluating the limit of the distribution function for the b_i's as ε approaches zero.

If m is the largest integer such that $x \le a_m$, then by Formula (1.1.3)

$$\Pr\Big(\sum_{i=1}^{n} b_i(\varepsilon)\, u_i \le x\Big)$$

$$= 1 - \sum_{j=1}^{l_1} \frac{(b_j(\varepsilon) - x)^n}{b_j(\varepsilon) \prod_{i \ne j} (b_j(\varepsilon) - b_i(\varepsilon))} - \cdots$$

$$- \sum_{j=l_{m-1}+1}^{l_m} \frac{(b_j(\varepsilon) - x)^n}{b_j(\varepsilon) \prod_{i \ne j} (b_j(\varepsilon) - b_i(\varepsilon))}$$

$$= 1 - \sum_{i=1}^{m} \Big(\sum_{j=l_{i-1}+1}^{l_i} \frac{f_i(b_j(\varepsilon))}{\prod_{l_{i-1} < k \le l_i,\, k \ne j} (b_j(\varepsilon) - b_k(\varepsilon))} \Big),$$

where

$$f_i(a) = \frac{(a - x)^n}{a \prod_{j \le l_{i-1},\, j > l_i} (a - b_j(\varepsilon))}\,.$$

Thus
$$\Pr\left(\sum_{i=1}^{n} b_i(\varepsilon)\, u_i \le x\right)$$
$$= 1 - \sum_{i=1}^{m}\left(\sum_{\rho=1}^{r_i} \frac{f_i(a_{(i)} + (r_i - \rho)\varepsilon)}{\rho^{r_i-1}(r_i-\rho)!(\rho-1)!}(-1)^{\rho-1}\right)$$
$$= 1 - \sum_{i=1}^{m}\left(\sum_{j=0}^{r_i-1} \frac{f_i(a_{(i)} + j\varepsilon)}{(r_i-1)!}\binom{r_i-1}{j}(-1)^{r_i-1-j}\right)$$
by putting $j = r_i - \rho$.

This is equivalent to
$$1 - \sum_{i=1}^{m} \frac{\Delta^{r_i-1} f_i(a_{(i)})}{\varepsilon^{r_i-1}(r_i-1)!}$$
where Δ is the forward difference operator defined by
$$\Delta^k f(x) = \Delta^{k-1} f(x+h) - \Delta^{k-1} f(x), \quad k = 1,2,\ldots.$$
Then he proved that
$$\Pr\left(\sum_{i=1}^{s} c_i x_{(l_i)} \le x\right)$$
$$= \lim_{\varepsilon\to 0} \Pr\left(\sum_{i=1}^{n} b_i(\varepsilon)\, u_i \le x\right)$$
$$= 1 - \lim_{\varepsilon\to 0} \sum_{i=1}^{m} \frac{\Delta^{r_i-1} f_i(a_{(i)})}{\varepsilon^{r_i-1}(r_i-1)!}$$
Then since
$$\lim_{\varepsilon\to 0} \frac{\Delta^k f(x)}{\varepsilon^k} = f^{(k)}(x)$$
for any function $f(x)$ whose k^{th} derivative exists at x, and
$$\lim_{\varepsilon\to 0} b_{l_{j-1}+i} = a_{(j)} \qquad (i = 1,2,\ldots,r_j),$$
therefore
$$\lim_{\varepsilon\to 0} \Pr\left(\sum_{i=1}^{n} b_i(\varepsilon)\, u_i \le x\right)$$
$$= 1 - \lim_{\varepsilon\to 0} \sum_{i=1}^{m} \frac{f_i^{(r_i-1)}(a_{(i)})}{(r_i-1)!}$$

$$= 1 - \sum_{i=1}^{m} \frac{g_i^{(r_i-1)}(a_{(i)})}{(r_i - 1)!}$$

where

$$g_i(a) = \frac{(a-x)^n}{a \prod_{j \neq i} (a - a_{(j)})} .$$

Thus

$$\Pr\left(\sum_{i=1}^{s} c_i x_{(l_i)} \leq x \right) = 1 - \sum_{i=1}^{m} \frac{g_i^{(r_i-1)}(a_{(i)})}{(r_i - 1)!}$$

where m is the largest integer such that $x \leq a_{(m)}$.

As pointed out in the introduction, linear combinations of $(u_{(1)}, u_{(2)}, \ldots, u_{(n)})$ and $(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ can be transformed to linear combinations of $(u_1, u_2, \ldots, u_n)$, so the results in the various papers are basically the same, the only difference being in the conditions the coefficients must satisfy. Specifically, the papers by Mauldon(1957), Dwass(1961), Ali(1969) and Weisberg(1971) all contain results that either give directly the distribution of $y = \sum a_i u_i$, $x = \sum c_i x_{(i)}$ or $u = \sum d_i u_{(i)}$, for any values of the coefficients or at least can be used to find the distribution of any of these variates. The final results agree when identical combinations are considered.

§ 1.2 Distributions of two linear combinations

The joint distribution of more than one linear combination has not been investigated very thoroughly. For example, it was only pointed out by Dempster and Kleyle (1968) as a remark that in principle the joint distribution of several linear combinations could follow from the same geometrical approach given in that paper. As a particular case, Barton and David (1955) tried to find the joint

distribution of $U_1 = \sum_{i=r}^{s} u_{(i)}$ and $U_2 = \sum_{i=j}^{k} u_{(i)}$ and found the form was too complicated to be expressed by one single formula. Niven(1963) obtained the joint distribution for the sample mean $y_1 = (\sum_{i=1}^{n} x_{(i)})/n$ and the sample range $y_2 = x_{(n)} - x_{(1)}$ for $n = 4$. By a simple geometrical interpretation, she found separate formulae for the density in the different regions with non-zero probability for the mean and range; that is, no general formula was given. Ali and Mead (1969) investigated the joint distribution for several combinations

$$y_1 = a_{11}u_1 + a_{12}u_2 + \dots + a_{1n}u_n$$

$$y_2 = a_{21}u_1 + a_{22}u_2 + \dots + a_{2n}u_n$$

$$- \quad - \quad - \quad - \quad -$$

$$y_k = a_{k1}u_1 + a_{k2}u_2 + \dots + a_{kn}u_n.$$

The method was to invert the characteristic function in a straight forward manner. In the case of two linear combinations, they found that the joint density function of y_1 and y_2 was given by

$$f(y_1,y_2) = \frac{n!}{(n-2)!} \sum_{i_1=0}^{n} \sum_{\substack{i_2=0 \\ \neq i_1}}^{n} \lambda(y_1 - a_{1i_1}) \lambda \left[\frac{\begin{vmatrix} 1 & a_{1i_1} & a_{2i_1} \\ 1 & a_{1i_2} & a_{2i_2} \\ 1 & y_1 & y_2 \end{vmatrix}}{\begin{vmatrix} 1 & a_{1i_1} \\ 1 & a_{1i_2} \end{vmatrix}} \right] \cdot$$

$$\cdot \frac{\begin{vmatrix} 1 & a_{1i_1} & a_{2i_1} \\ 1 & a_{1i_2} & a_{2i_2} \\ 1 & y_1 & y_2 \end{vmatrix}^{n-2}}{\prod_{i_3 \neq i_1, i_2} \begin{vmatrix} 1 & a_{1i_1} & a_{2i_1} \\ 1 & a_{1i_2} & a_{2i_2} \\ 1 & a_{1i_3} & a_{2i_3} \end{vmatrix}} \qquad (1.2.1)$$

provided

$$\begin{vmatrix} 1 & a_{1i_1} & a_{2i_1} \\ 1 & a_{1i_2} & a_{2i_2} \\ 1 & a_{1i_3} & a_{2i_3} \end{vmatrix} \neq 0$$

and

$$\begin{vmatrix} 1 & a_{1i_1} \\ 1 & a_{1i_2} \end{vmatrix} \neq 0$$

for all distinct i_1, i_2, i_3.

The symbol $\lambda(x)$ is defined by

$$\lambda(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0, \end{cases}$$

and $a_{10} = a_{20} = 0$. They pointed out that even if

$$\begin{vmatrix} 1 & a_{1i_1} & a_{2i_1} \\ 1 & a_{1i_2} & a_{2i_2} \\ 1 & a_{1i_3} & a_{2i_3} \end{vmatrix} = 0$$

for certain i_1, i_2, i_3, the joint density $f(y_1,y_2)$ could still be obtained by applying the continuity theorem for characteristic functions. They illustrated this by a simple example and the general formula that would arise in this case was not given. Such a formula is found in the next chapter.

Chapter II : The Joint Distribution of Two Linear Combinations

§ 2.1 : Introduction

It was mentioned in § 1.2 that under certain conditions the joint density $f(y_1,y_2)$ is given by Formula (1.2.1). Since $u_i \leqslant 1$ ($i = 1,2,\ldots,n$), it is clear that, geometrically (y_1,y_2) must lie in the polygon with vertices (a_{10},a_{20}), $(a_{11},a_{21}),\ldots,(a_{1n},a_{2n})$ in the 2 - dimensional plane. Using this interpretation of $(y_1,y_2),(a_{1i},a_{2i})$, ($i = 0,1,\ldots,n$) as points in the 2 - dimensional Euclidean space E. R. Mead (1969) found that Formula (1.2.1) can be equivalently expressed as

$$f(y_1,y_2) = \frac{n!}{(n-2)!}\sum_{i_1=2}^{n}\sum_{i_2=1}^{i_1-1}\lambda_{i_1,i_2}\,\mathrm{sgn}\begin{vmatrix} a_{1i_1} & a_{2i_1} \\ a_{1i_2} & a_{2i_2} \end{vmatrix}\cdot\frac{\begin{vmatrix} 1 & a_{1i_1} & a_{2i_1} \\ 1 & a_{1i_2} & a_{2i_2} \\ 1 & y_1 & y_2 \end{vmatrix}^{n-2}}{\prod\limits_{\substack{i_3=0 \\ \neq i_1, \\ i_2}}^{n}\begin{vmatrix} 1 & a_{1i_1} & a_{2i_1} \\ 1 & a_{1i_2} & a_{2i_2} \\ 1 & a_{1i_3} & a_{2i_3} \end{vmatrix}} \tag{2.1.1}$$

provided none of the determinants in the denominator are zero, where $(a_{10},a_{20}) = (0,0)$ and

$$\lambda_{ij} = \begin{cases} 1 & \text{if } (y_1,y_2) \text{ falls in the triangle with vertices } (0,0),\ (a_{1i},a_{2i}) \text{ and } (a_{1j},a_{2j}) \\ 0 & \text{otherwise.} \end{cases}$$

Geometrically, when two or more points are coincident or three

or more points are colinear, the determinants in the denominator become indeterminate. In this chapter, a more general form of the joint density $f(y_1,y_2)$ is given which can always be used if there are no colinear coefficient points and which can often be used with modification for colinear cases. Unless specified otherwise, the coefficient points (a_{1i},a_{2i}), $i=0,1,\ldots,n$ are assumed to be non-colinear throughout this chapter.

§ 2.2 The joint density $f(y_1,y_2)$

Let

$(n+1)$ = total number of points (a_{1i},a_{2i}) $(i=0,1,\ldots,n)$

p = total number of distinct points

n_i = number of actual points coinciding at the ith distinct point $(i=1,2,\ldots,n)$.

Then without loss in generality it can be assumed that

$$(a_{10},a_{20}) = (a_{11},a_{21}) = \cdots = (a_{1\,n_1-1},\, a_{2\,n_1-1})$$

$$(a_{1n_1},a_{2n_1}) = (a_{1\,n_1+1},a_{2\,n_1+1}) = \cdots = (a_{1\,n_1+n_2-1},a_{2\,n_1+n_2-1})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(a_{1\,n_1+n_2+\cdots+n_{p-1}},a_{2\,n_1+n_2+\cdots+n_{p-1}}) = \cdots$$

$$= (a_{1\,n_1+n_2+\cdots+n_p-1},a_{2\,n_1+n_2+\cdots+n_p-1}).$$

Denote

$$(a_{1k_1},a_{2k_1}) = (a_{10},a_{20})$$

$$(a_{1k_2},a_{2k_2}) = (a_{1n_1},a_{2n_1})$$

$$\cdots\cdots\cdots\cdots$$

$$(a_{1k_i},a_{2k_i}) = (a_{1\,n_1+n_2+\cdots+n_{i-1}},a_{2\,n_1+n_2+\cdots+n_{i-1}}),$$

$$(i=2,3,\ldots,p)$$

Also define (i j y) and (i j k) to be

$$(i\ j\ y) = \begin{vmatrix} 1 & a_{1i} & a_{2i} \\ 1 & a_{1j} & a_{2j} \\ 1 & y_1 & y_2 \end{vmatrix}, \quad (i\ j\ k) = \begin{vmatrix} 1 & a_{1i} & a_{2i} \\ 1 & a_{1j} & a_{2j} \\ 1 & a_{1k} & a_{2k} \end{vmatrix}.$$

<u>Theorem 2.1</u>. If no three of the coefficient points (a_{1i}, a_{2i}), $(i = 0,1,\ldots,n)$ are colinear, then the joint density of

$$y_1 = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$$

$$y_2 = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n$$

is given by

$$f(y_1,y_2) = \frac{n!}{(n-2)!} \sum_{i=3}^{p} \sum_{j=2}^{i-1} \lambda_{k_i k_j} \operatorname{sgn} \begin{vmatrix} a_{1k_i} & a_{2k_i} \\ a_{1k_j} & a_{2k_j} \end{vmatrix}$$

$$\cdot \frac{1}{\prod_{q=1}^{p} (n_q - 1)! \begin{vmatrix} 1 & a_{2k_i} \\ 1 & a_{2k_j} \end{vmatrix}^{(n+1)-p}} \frac{\partial^{(n+1)-p}}{\partial a_{1k_1}^{n_1-1} \cdots \partial a_{1k_p}^{n_p-1}} \left\{ \frac{(k_i k_j y)^{n-2}}{\prod_{\substack{l=1 \\ \neq i,j}}^{p} (k_i k_j k_l)} \right\} \qquad (2.2.1)$$

The following two lemmas are used to prove the theorem.

For convenience assume that the coincident points have been adjusted in such a way that the determinants $(i_1 i_2 i_3) \neq 0$ $(i_1 \neq i_2 \neq i_3)$ and such that any given (y_1, y_2) in the triangle with vertices (0,0), (a_{1k_i}, a_{2k_i}), (a_{1k_j}, a_{2k_j}) will still be in all the triangles with vertices (0,0), (a_{1i_1}, a_{2i_1}), (a_{1i_2}, a_{2i_2}) $(i_1 = k_i, \ldots, k_i + n_i - 1, i_2 = k_j, \ldots, k_j + n_j - 1)$.

Denote

$$f'_{k_i k_j}(y_1,y_2) = \lambda_{k_i k_j} \sum_{i_1=k_i}^{k_i+n_i-1} \sum_{i_2=k_j}^{k_j+n_j-1} \frac{n!}{(n-2)!} \operatorname{sgn} \begin{vmatrix} a_{1i_1} & a_{2i_1} \\ a_{1i_2} & a_{2i_2} \end{vmatrix} \frac{(i_1 i_2 y)^{n-2}}{\prod_{\substack{i_3=0 \\ \neq i_2, i_1}}^{n} (i_1 i_2 i_3)} \qquad (2.1.1)'$$

Also let $f_{k_i k_j}(y_1,y_2)$ represent the limiting value of $f'_{k_i k_j}(y_1,y_2)$ obtained by letting the adjusted points approach their original values in an appropriate way.

<u>Lemma 2.2</u> If (y_1,y_2) falls in the triangle with vertices $(0,0)$, (a_{1k_i},a_{2k_i}) and (a_{1k_j},a_{2k_j}), $i\neq j$, and if $n_i = n_j = 1$ and n_l is any positive integer for $1 \leq l \leq p$, $l \neq i,j$, then

$$f_{k_ik_j}(y_1,y_2) = \frac{n!}{(n-2)!}\,\mathrm{sgn}\begin{vmatrix} a_{1k_i} & a_{2k_i} \\ a_{1k_j} & a_{2k_j} \end{vmatrix} \cdot \frac{1}{\prod_{q=1}^{p}(n_q-1)!} \cdot$$

$$\cdot \frac{\partial^{n+1-p}}{\partial a_{1k_1}^{n_1-1}\cdots\partial a_{1k_p}^{n_p-1}} \left\{ \frac{(k_i\ k_j\ y)^{n-2}}{\prod_{\substack{l=1\\ \neq i,j}}^{p}(k_i\ k_j\ k_l)} \right\} .$$

For other values of (y_1,y_2),

$$f_{k_ik_j}(y_1,y_2) = 0$$

<u>Proof:</u>

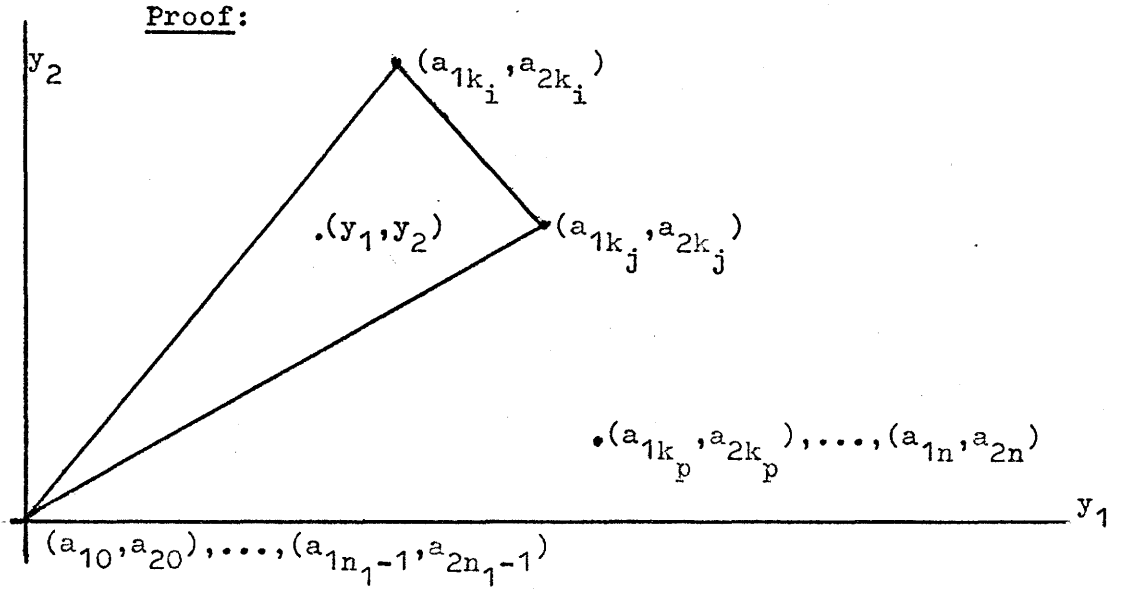

Figure I

Without loss in generality write $i=2$, $j=3$ so that $n_2=n_3=1$.

From Formula (2.1.1)', if (y_1,y_2) falls in the triangle with vertices $(0,0)$, (a_{1k_2},a_{2k_2}) and (a_{1k_3},a_{2k_3}), then

$$f'_{k_2k_3}(y_1,y_2) = C \cdot \frac{(k_2\ k_3\ y)^{n-2}}{\prod_{\substack{l=0\\ \neq k_2,k_3}}^{n} (k_2\ k_3\ l)}$$

and

$$f_{k_2k_3}(y_1,y_2) = C \cdot \frac{(k_2\ k_3\ y)^{n-2}}{\prod_{\substack{i=1\\ \neq 2,3}}^{p} (k_2\ k_3\ k_i)^{n_i}}$$

where

$$C = \frac{n!}{(n-2)!} \operatorname{sgn} \begin{vmatrix} a_{1k_2} & a_{2k_2} \\ a_{1k_3} & a_{2k_3} \end{vmatrix} .$$

For each n_i and k_i it is easy to see that

$$\frac{1}{(k_2\ k_3\ k_i)^{n_i}} = \frac{(-1)^{2(n_i-1)}}{(n_i-1)! \begin{vmatrix} 1 & a_{2k_2} \\ 1 & a_{2k_3} \end{vmatrix}^{n_i-1}} \frac{\partial^{n_i-1}}{\partial a_{1k_i}^{n_i-1}} \left\{ \frac{1}{(k_2\ k_3\ k_i)} \right\}$$

Notice that if $a_{2k_2} = a_{2k_3}$, then $a_{1k_2} \neq a_{1k_3}$ (otherwise $(a_{1k_2}, a_{2k_2}) \equiv (a_{1k_3}, a_{2k_3})$) and the derivative can be expressed in a similar way with respect to a_{2k_i}, that is,

$$\frac{1}{(k_2\ k_3\ k_i)^{n_i}} = \frac{(-1)^{2(n_i-1)}}{(n_i-1)! \begin{vmatrix} 1 & a_{1k_3} \\ 1 & a_{1k_2} \end{vmatrix}^{n_i-1}} \frac{\partial^{n_i-1}}{\partial a_{2k_i}^{n_i-1}} \left\{ \frac{1}{(k_2\ k_3\ k_i)} \right\} .$$

Therefore, for convenience, it will be assumed that $a_{2k_2} \neq a_{2k_3}$.

Thus

$$\frac{(k_2\ k_3\ y)^{n-2}}{\prod_{\substack{i=1\\ \neq 2,3}}^{p} (k_2\ k_3\ k_i)^{n_i}}$$

$$= \frac{1}{\prod_{\substack{i=1\\ \neq 2,3}}^{p} (n_i-1)! \begin{vmatrix} 1 & a_{2k_2} \\ 1 & a_{2k_3} \end{vmatrix}^{\sum_{i=1}^{p}{}' (n_i-1)}} \cdot \frac{\partial^{\sum_{i=1}^{p}{}' (n_i-1)}}{\partial a_{1k_1}^{n_1-1} \partial a_{1k_4}^{n_4-1} \cdots \partial a_{1k_p}^{n_p-1}} \left\{ \frac{(k_2\ k_3\ y)^{n-2}}{\prod_{\substack{l=1\\ \neq 2,3}}^{p} (k_2\ k_3\ k_l)} \right\}$$

$$= \frac{1}{\prod_{i=1}^{p} (n_i-1)! \begin{vmatrix} 1 & a_{2k_2} \\ 1 & a_{2k_3} \end{vmatrix}^{(n+1)-p}} \cdot \frac{\partial^{(n+1)-p}}{\partial a_{1k_1}^{n_1-1} \partial a_{1k_2}^{n_2-1} \cdots \partial a_{1k_p}^{n_p-1}} \left\{ \frac{(k_2\ k_3\ y)^{n-2}}{\prod_{\substack{l=1\\ \neq 2,3}}^{p} (k_2\ k_3\ k_l)} \right\}$$

where $i \neq 2,3$ in $\sum_{i=1}^{p}{}'$, since $(n_2-1) = (n_3-1) = 0$.

Finally, if (y_1,y_2) is not in the triangle with vertices $(0,0)$, (a_{1k_2},a_{2k_2}) and (a_{1k_3},a_{2k_3}), then it is obvious that

$$f_{k_2k_3}(y_1,y_2) = 0$$

since $\lambda_{k_2k_3} = 0$ by difinition.

Therefore for any $i \neq j$

$$f_{k_ik_j}(y_1,y_2) = \frac{n!}{(n-2)!} \operatorname{sgn} \begin{vmatrix} a_{1k_i} & a_{2k_i} \\ a_{1k_j} & a_{2k_j} \end{vmatrix} \cdot$$

$$\cdot \frac{1}{\prod_{q=1}^{p} (n_q-1)! \begin{vmatrix} 1 & a_{2k_i} \\ 1 & a_{2k_j} \end{vmatrix}^{(n+1)-p}} \cdot \frac{\partial^{(n+1)-p}}{\partial a_{1k_1}^{n_1-1} \cdots \partial a_{1k_p}^{n_p-1}} \left\{ \frac{(k_i\ k_j\ y)^{n-2}}{\prod_{\substack{l=1\\ \neq i,j}}^{p} (k_i\ k_j\ k_l)} \right\}$$

if (y_1,y_2) falls in triangle with vertices $(0,0)$, (a_{1k_i},a_{2k_i}) and (a_{1k_j},a_{2k_j}), and is zero otherwise. The lemma is thus proved.

<u>Lemma 2.3</u>. If n_i and n_j are any positive integers greater than zero, then

$$f_{k_i k_j}(y_1,y_2) = \frac{n!}{(n-2)!} \operatorname{sgn}\begin{vmatrix} a_{1k_i} & a_{2k_i} \\ a_{1k_j} & a_{2k_j} \end{vmatrix} \frac{1}{\prod_{q=1}^{p}(n_q-1)!} \cdot$$

$$\cdot \frac{1}{\begin{vmatrix} 1 & a_{2k_i} \\ 1 & a_{2k_j} \end{vmatrix}^{(n+1)-p}} \frac{\partial^{(n+1)-p}}{\partial a_{1k_1}^{n_1-1}\cdots\partial a_{1k_p}^{n_p-1}} \left\{ \frac{(k_i\ k_j\ y)^{n-2}}{\prod_{\substack{l=1\\ \neq i,j}}^{p} (k_i\ k_j\ k_l)} \right\}$$

if (y_1,y_2) falls in the triangle with vertices $(0,0)$, (a_{1k_i},a_{2k_i}) and (a_{1k_j},a_{2k_j}), and is zero otherwise.

<u>Proof</u>: Again assume for convenience that $i = 2$, $j = 3$. From Formula (2.1.1)', if (y_1,y_2) is in the triangle with vertices $(0,0)$, (a_{1k_2},a_{2k_2}) and (a_{1k_3},a_{2k_3}),

$$f'_{k_2k_3}(y_1,y_2) = C \cdot \sum_{i_1=k_2}^{k_2+(n_2-1)} \sum_{i_2=k_3}^{k_3+(n_3-1)} \frac{(i_1\ i_2\ y)^{n-2}}{\prod_{\substack{i_3=0\\ \neq i_1,i_2}}^{n} (i_1\ i_2\ i_3)} \qquad (2.2.2)$$

where

$$C = \frac{n!}{(n-2)!} \operatorname{sgn}\begin{vmatrix} a_{1k_2} & a_{2k_2} \\ a_{1k_3} & a_{2k_3} \end{vmatrix}$$

since

$$\operatorname{sgn}\begin{vmatrix} a_{1i_1} & a_{2i_1} \\ a_{1i_2} & a_{2i_2} \end{vmatrix} = \operatorname{sgn}\begin{vmatrix} a_{1k_2} & a_{2k_2} \\ a_{1k_3} & a_{2k_3} \end{vmatrix}$$

for all $i_1 = k_2,\ldots,k_2+(n_2-1)$, $i_2 = k_3,\ldots,k_3+(n_3-1)$. The zero determinants in the denominators can be avoided by replacing (a_{1i_1},a_{2i_1}) by

$$(a_{1i_1(\varepsilon)}, a_{2i_1(\varepsilon)}) = (a_{1k_2}+(i_1-k_2)\varepsilon, a_{2k_2}) \quad i_1=k_2,\ldots,k_2+(n_2-1)$$

and (a_{1i_2}, a_{2i_2}) by

$$(a_{1i_2(\varepsilon')}, a_{2i_2(\varepsilon')}) = (a_{1k_3}+(i_2-k_3)\varepsilon', a_{2k_3}) \quad i_2=k_3,\ldots,k_3+(n_3-1)$$

and taking the limit of (2.2.2) as ε, ε' approach zero. Therefore (2.2.2) becomes

$$f_{k_2k_3}(y_1,y_2) = \lim_{\varepsilon,\varepsilon'\to 0} C\cdot \sum_{i_1=k_2}^{k_2+(n_2-1)} \sum_{i_2=k_3}^{k_3+(n_3-1)} \frac{(i_1(\varepsilon)\ i_2(\varepsilon')\ y)^{n-2}}{\prod_{i_3=0}^{n_1-1}(i_1(\varepsilon)\ i_2(\varepsilon')\ i_3) \prod_{i_3=k_2,\, i_3\neq i_1}^{k_2+(n_2-1)}(i_1(\varepsilon)\ i_2(\varepsilon')\ i_3(\varepsilon))}$$

$$\cdot \frac{1}{\prod_{i_3=k_3,\, i_3\neq i_2}^{k_3+(n_3-1)}(i_1(\varepsilon)\ i_2(\varepsilon')\ i_3(\varepsilon')) \prod_{i_3=k_4}^{n}(i_1(\varepsilon)\ i_2(\varepsilon')\ i_3)} \qquad (2.2.3).$$

Expanding the determinants in the denominators

$$(i_1(\varepsilon)\ i_2(\varepsilon')\ i_3(\varepsilon))$$

$$= \begin{vmatrix} 1 & a_{1k_2}+(i_1-k_2)\varepsilon & a_{2k_2} \\ 1 & a_{1k_3}+(i_2-k_3)\varepsilon' & a_{2k_3} \\ 1 & a_{1k_2}+(i_3-k_2)\varepsilon & a_{2k_2} \end{vmatrix}$$

$$= \begin{vmatrix} a_{1k_3}+(i_2-k_3)\varepsilon' & a_{2k_3} \\ a_{1k_2}+(i_3-k_2)\varepsilon & a_{2k_2} \end{vmatrix} - [a_{1k_2}+(i_1-k_2)\varepsilon] \begin{vmatrix} 1 & a_{2k_3} \\ 1 & a_{2k_2} \end{vmatrix} +$$

$$+ a_{2k_2} \begin{vmatrix} 1 & a_{1k_3}+(i_2-k_3)\varepsilon' \\ 1 & a_{1k_2}+(i_3-k_2)\varepsilon \end{vmatrix}$$

$$= [a_{1k_3}+(i_2-k_3)\varepsilon']a_{2k_2} - a_{2k_3}[a_{1k_2}+(i_3-k_2)\varepsilon] -$$

$$- [a_{1k_2}+(i_1-k_2)\varepsilon](a_{2k_2}-a_{2k_3}) + a_{2k_2}[a_{1k_2}+(i_3-k_2)\varepsilon - a_{1k_3}-(i_2-k_3)\varepsilon']$$

$$= \varepsilon'(i_2-k_3)(a_{2k_2}-a_{2k_2}) + \varepsilon(i_3-k_2)(a_{2k_2}-a_{2k_3}) - (a_{2k_2}-a_{2k_3})(i_1-k_2)\varepsilon$$

$$= \varepsilon(a_{2k_2} - a_{2k_3})(i_3-i_1),$$

for $k_2 \le i_1 \ne i_3 \le k_2+(n_2-1)$ and $k_3 \le i_2 \le k_3+(n_3-1)$.

Similarly

$$\big(i_1(\varepsilon)\ i_2(\varepsilon')\ i_3(\varepsilon')\big) = \begin{vmatrix} 1 & a_{1k_2}+(i_1-k_2)\varepsilon & a_{2k_2} \\ 1 & a_{1k_3}+(i_2-k_3)\varepsilon' & a_{2k_3} \\ 1 & a_{1k_3}+(i_3-k_3)\varepsilon' & a_{2k_3} \end{vmatrix}$$

$$= [a_{1k_3}+(i_2-k_3)\varepsilon']a_{2k_3} - [a_{1k_3}+(i_3-k_3)\varepsilon']a_{2k_3} + $$
$$+ a_{2k_2}(i_3-i_2)\varepsilon'$$
$$= a_{2k_3}(i_2-i_3)\varepsilon' + a_{2k_2}(i_3-i_2)\varepsilon'$$
$$= \varepsilon'(i_3-i_2)(a_{2k_2}-a_{2k_3})$$

for $k_2 \le i_1 \le k_2+(n_2-1)$, $k_3 \le i_2 \ne i_3 \le k_3+(n_3-1)$.

Therefore for each $k_2 \le i_1 \le k_2 + (n_2-1)$, the products in the denominator become

$$\varepsilon^{n_2-1}(\varepsilon')^{n_3-1}(a_{2k_2}-a_{2k_3})^{n_2+n_3-2}\prod_{i_3=0}^{n_1-1}\big(i_1(\varepsilon)\ i_2(\varepsilon')\ i_3\big)\ \cdot$$

$$\cdot \prod_{\substack{i_3=k_2\\ \ne i_1}}^{k_2+(n_2-1)}(i_3-i_1)\prod_{\substack{i_3=k_3\\ \ne i_2}}^{k_3+(n_3-1)}(i_3-i_2)\prod_{i_3=k_4}^{n}\big(i_1(\varepsilon)\ i_2(\varepsilon')\ i_3\big)$$

and hence the summation with respect to i_2 for each fixed i_1 is

$$\sum_{i_2=k_3}^{k_3+(n_3-1)} \frac{1}{\varepsilon^{n_2-1}(\varepsilon')^{n_3-1}(a_{2k_2}-a_{2k_3})^{n_2+n_3-2}} \frac{(i_1(\varepsilon)\ i_2(\varepsilon')\ y)^{n-2}}{\prod_{\substack{i_3=k_2\\ \ne i_1}}^{k_2+(n_2-1)}(i_3-i_1)}\ \cdot$$

$$\cdot\left\{\prod_{\substack{i_3=k_3\\ \ne i_2}}^{k_3+(n_3-1)}(i_3-i_2)\prod_{i_3=0}^{n_1-1}\big(i_1(\varepsilon)\ i_2(\varepsilon')\ i_3\big)\prod_{i_3=k_4}^{n}\big(i_1(\varepsilon)\ i_2(\varepsilon')\ i_3\big)\right\}^{-1}$$

$$= \frac{1}{\varepsilon^{n_2-1}(\varepsilon')^{n_3-1}(a_{2k_2}-a_{2k_3})^{n_2+n_3-2}} \frac{1}{\prod\limits_{\substack{i_3=k_2\\ \neq i_1}}^{k_2+(n_2-1)} (i_3-i_1)} \cdot$$

$$\cdot \sum_{i_2=k_3}^{k_3+(n_3-1)} \frac{(i_1(\varepsilon)\ i_2(\varepsilon')\ y)^{n-2}}{\prod\limits_{\substack{i_3=k_3\\ \neq i_2}}^{k_3+(n_3-1)} (i_3-i_2) \prod\limits_{i_3=0}^{n_1-1} (i_1(\varepsilon)\ i_2(\varepsilon')\ i_3)} \cdot$$

$$\cdot \frac{1}{\prod\limits_{i_3=k_4}^{n} (i_1(\varepsilon)\ i_2(\varepsilon')\ i_3)} \cdot$$

$$= \frac{1}{\varepsilon^{n_2-1}(\varepsilon')^{n_3-1}(a_{2k_2}-a_{2k_3})^{n_2+n_3-2}(n_2-1)!(n_3-1)!} \left\{ (-1)^{i_1-k_2}\binom{n_2-1}{i_1-k_2} \cdot \right.$$

$$\left. \cdot \sum_{i_2=k_3}^{k_3+(n_3-1)} (-1)^{i_2-k_3}\binom{n_3-1}{i_2-k_3} \frac{(i_1(\varepsilon)\ i_2(\varepsilon')\ y)^{n-2}}{\prod\limits_{i_3=0}^{n_1-1}(i_1(\varepsilon)\ i_2(\varepsilon')\ i_3) \prod\limits_{i_3=k_4}^{n} (i_1(\varepsilon)\ i_2(\varepsilon')\ i_3)} \right\} \cdot$$

Substituting this into (2.2.3) gives

$$f_{k_2k_3}(y_1,y_2) = \lim_{\varepsilon,\varepsilon'\to 0} C \frac{1}{\varepsilon^{n_2-1}(\varepsilon')^{n_3-1}(a_{2k_2}-a_{2k_3})^{n_2+n_3-2}(n_2-1)!(n_3-1)!}$$

$$\sum_{i_1=k_2}^{k_2+(n_2-1)} (-1)^{i_1-k_2}\binom{n_2-1}{i_1-k_2} \left\{ \sum_{i_2=k_3}^{k_3+n_3-1} (-1)^{i_2-k_3}\binom{n_3-1}{i_2-k_3} \frac{(i_1(\varepsilon)\ i_2(\varepsilon')\ y)^{n-2}}{\prod\limits_{i_3=0}^{n_1-1}(i_1(\varepsilon)\ i_2(\varepsilon')\ i_3)} \right.$$

$$\left. \cdot \frac{1}{\prod\limits_{i_3=k_4}^{n} (i_1(\varepsilon)\ i_2(\varepsilon')\ i_3)} \right\} \qquad (2.2.4).$$

But if the $(m+n)^{th}$ derivative of a function $G(x,y)$ exists at the point (x,y) then

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} G(x,y) = \lim_{k,h\to 0} \frac{\Delta_x^m(\Delta_y^n G(x,y))}{k^m h^n}$$

where Δ_x^m is the forward difference operator defined by

$$\Delta_x^q G(x,y) = \Delta_x^{q-1} G(x+k,y) - \Delta_x^{q-1} G(x,y).$$

Therefore from (2.2.4)

$$f_{k_2k_3}(y_1,y_2) = \lim_{\varepsilon,\varepsilon'\to 0} \frac{C}{(n_2-1)!(n_3-1)! \begin{vmatrix} 1 & a_{2k_2} \\ 1 & a_{2k_3} \end{vmatrix}^{n_2+n_3-2}} \cdot$$

$$\cdot \Delta_{a_{1k_2}}^{n_2-1}\left[\Delta_{a_{1k_3}}^{n_3-1} \frac{(k_2\ k_3\ y)^{n-2}}{\prod_{i_3=0}^{n_1-1}(k_2\ k_3\ i_3)\prod_{i_3=k_4}^{n}(k_2\ k_3\ i_3)}\right] \Big/ \varepsilon^{n_2-1}(\varepsilon')^{n_3-1}$$

$$= \frac{C}{(n_2-1)!(n_3-1)! \begin{vmatrix} 1 & a_{2k_2} \\ 1 & a_{2k_3} \end{vmatrix}^{n_2+n_3-2}} \cdot \frac{\partial^{n_2+n_3-2}}{\partial a_{1k_2}^{n_2-1}\partial a_{1k_3}^{n_3-1}} \cdot$$

$$\cdot \frac{(k_2\ k_3\ y)^{n-2}}{\prod_{i_3=0}^{n_1-1}(k_2\ k_3\ i_3)\prod_{i_3=k_4}^{n}(k_2\ k_3\ i_3)}.$$

But from Lemma 2.2 ,

$$\frac{(k_2\ k_3\ y)^{n-2}}{\prod_{i_3=0}^{n_1-1}(k_2\ k_3\ i_3)\prod_{i_3=k_4}^{n}(k_2\ k_3\ i_3)} = \frac{(k_2\ k_3\ y)^{n-2}}{\prod_{\substack{i=1\\ \neq 2,3}}^{p}(k_2\ k_3\ k_i)^{n_i}}$$

$$= \frac{1}{\prod_{\substack{i=1\\ \neq 2,3}}^{p} (n_i-1)! \begin{vmatrix} 1 & a_{2k_2} \\ 1 & a_{2k_3} \end{vmatrix}^{\sum_{\substack{i=1\\ \neq 2,3}}^{p} (n_i-1)}} \frac{\partial^{\sum_{\substack{i=1\\ \neq 2,3}}^{p} (n_i-1)}}{\partial a_{1k_1}^{n_1-1} \partial a_{1k_4}^{n_4-1} \ldots \partial a_{1k_p}^{n_p-1}} \left\{ \frac{(k_2\ k_3\ y)^{n-2}}{\prod_{\substack{l=1\\ \neq 2,3}}^{p} (k_2\ k_3\ k_l)} \right\}.$$

Therefore

$$f_{k_2k_3}(y_1,y_2) = \frac{c}{\prod_{i=1}^{p} (n_i-1)! \begin{vmatrix} 1 & a_{2k_2} \\ 1 & a_{2k_3} \end{vmatrix}^{(n+1)}} \cdot \frac{\partial^{(n+1)-p}}{\partial a_{1k_1}^{n_1-1} \ldots \partial a_{1k_p}^{n_p-1}}$$

$$\left\{ \frac{(k_2\ k_3\ y)^{n-2}}{\prod_{\substack{l=1\\ \neq 2,3}}^{p} (k_2\ k_3\ k_l)} \right\}$$

and finally, if (y_1,y_2) is not in the triangle with vertices $(0,0)$, (a_{1k_2},a_{2k_2}) and (a_{1k_3},a_{2k_3}), then

$$f_{k_2k_3}(y_1\ ,y_2) = 0$$

for $\lambda_{k_2k_3} = 0$ by definition. Therefore for any given $i \neq j$,

$$f_{k_ik_j}(y_1,y_2) = \begin{cases} \dfrac{n!}{(n-2)!} \operatorname{sgn} \begin{vmatrix} a_{1k_i} & a_{2k_i} \\ a_{1k_j} & a_{2k_j} \end{vmatrix} \dfrac{1}{\prod_{q=1}^{p} (n_q-1)! \begin{vmatrix} 1 & a_{2k_i} \\ 1 & a_{2k_j} \end{vmatrix}^{n+1-p}} \cdot \\ \cdot \dfrac{\partial^{n+1-p}}{\partial a_{1k_1}^{n_1-1} \ldots \partial a_{1k_p}^{n_p-1}} \left\{ \dfrac{(k_i\ k_j\ y)^{n-2}}{\prod_{\substack{l=1\\ \neq i,j}}^{p} (k_i\ k_j\ k_l)} \right\} \\ \text{if } (y_1,y_2) \text{ is in the triangle with vertices} \\ (0,0),\ (a_{1k_i},a_{2k_i}) \text{ and } (a_{1k_j},a_{2k_j}) \\ 0 \text{ otherwise.} \end{cases}$$

Proof of Theorem 2.1 :

It follows from (2.1.1), (2.1.1)', Lemma 2.2 and Lemma 2.3 that the joint density of (y_1,y_2) is given by

$$f(y_1,y_2) = \sum_{i=3}^{p} \sum_{j=2}^{i-1} f_{k_i k_j}(y_1,y_2)$$

$$= \frac{n!}{(n-2)!} \sum_{i=3}^{p} \sum_{j=2}^{i-1} \lambda_{k_i k_j} \operatorname{sgn} \begin{vmatrix} a_{1k_i} & a_{2k_i} \\ a_{1k_j} & a_{2k_j} \end{vmatrix}$$

$$\cdot \frac{1}{\prod_{q=1}^{p} (n_q-1)! \begin{vmatrix} 1 & a_{2k_i} \\ 1 & a_{2k_j} \end{vmatrix}^{n+1-p}} \cdot \frac{\partial^{n+1-p}}{\partial a_{1k_1}^{n_1-1} \cdots \partial a_{1k_p}^{n_p-1}} \left\{ \frac{(k_i\ k_j\ y)^{n-2}}{\prod_{\substack{l=1 \\ \neq i,j}}^{p} (k_i\ k_j\ k_l)} \right\}$$

and the theorem is proved.

§ 2.3 A simple example of a coincident case

The application of the result (2.2.1) is illustrate by the following simple example.

Consider the problem of finding the joint density of

$$y_1 = x_{(2)}$$

$$y_2 = x_{(3)}$$

or equivalently

$$y_1 = u_1 + u_2$$

$$y_2 = u_1 + u_2 + u_3 \quad .$$

Here n=3, p=3, and the points

$$(a_{1k_1}, a_{2k_1}) \equiv (a_{10}, a_{20}) = (0,0)$$

$$(a_{1k_2}, a_{2k_2}) \equiv (a_{11}, a_{21}) = (a_{12}, a_{22}) = (1,1)$$

$$(a_{1k_3}, a_{2k_3}) \equiv (a_{13}, a_{23}) = (0,1)$$

are as shown in Figure II

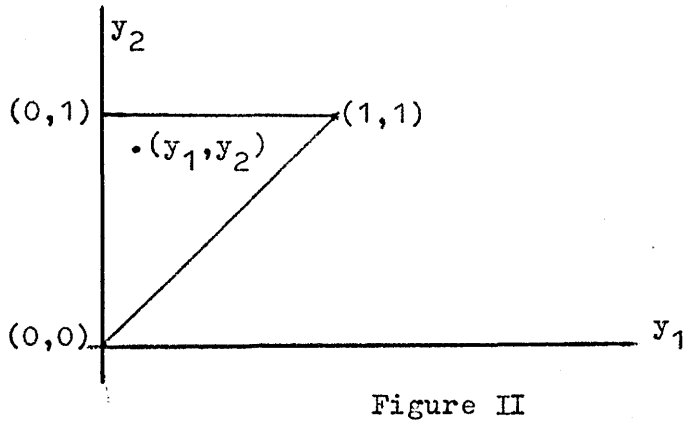

Figure II

From formula (2.2.1) $f(y_1,y_2)$ is positive only if (y_1,y_2) lies in the triangle formed by (0,0), (0,1) and (1,1) for $\lambda_{k_i k_j} = 0$ when (y_1,y_2) is not in the corresponding triangle. Therefore one only needs to consider (y_1,y_2) in this triangle. By (2.2.1),

$$f(y_1,y_2) = \frac{3!}{(3-2)!}\ \mathrm{sgn}\begin{vmatrix} a_{1k_3} & a_{2k_3} \\ a_{1k_2} & a_{2k_2} \end{vmatrix} \frac{1}{1!0!\begin{vmatrix} 1 & a_{1k_2} \\ 1 & a_{1k_3} \end{vmatrix}^1} \cdot \frac{\partial}{\partial a_{2k_2}} \left\{ \frac{(k_3\ k_2\ y)}{(k_3\ k_2\ k_1)} \right\}$$

$$= 3!\ \mathrm{sgn}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}} \cdot \frac{\partial}{\partial a_{21}} \left\{ \frac{(3\ 1\ 0)}{(3\ 1\ 0)} \right\}$$

$$= 3!\ \frac{(3\ 1\ 0)\begin{vmatrix} 1 & y_1 \\ 1 & a_{13} \end{vmatrix} - (3\ 1\ y)\begin{vmatrix} 1 & a_{10} \\ 1 & a_{13} \end{vmatrix}}{(3\ 1\ 0)^2}.$$

Substituting the values of a_{ij} into the above expression, gives the well known result

$$f(y_1,y_2) = \begin{cases} 3!y_1 & \text{if } 0 < y_1 < y_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Chapter III Distributions with colinear points

§ 3.1 Introduction

When some of the real coefficient points (a_{1i}, a_{2i}) are colinear, the density in Theorem 2.1 may still be in an indeterminate form. However, there are cases in which the indeterminate terms can be eliminated in such a manner that the joint density can still be expressed in the derivative form considered previously. In §3.2, a formula is developed that can be used in the elimination procedure. This is illustrated in §3.3 by an example involving colinear points. Also, there are some cases with colinear points in which there are no difficulties and Formula (2.2.1) can be used directly. This case is demonstrated by an example in §3.4.

§ 3.2 Some results related to divided differences

Since divided differences will be encountered, some known results in this connection will be discussed first.

Let f(x) be a function of the real variable x and let a_0, a_1, $\ldots, a_n$ be (n+1) distinct real numbers. Then the nth divided difference of f(x) with respect to the arguments $a_0, a_1, \ldots, a_n$ is defined by

$$\Delta^n[f(x);\ a_0,\ldots,a_n] = \frac{\Delta^{n-1}[f(x);\ a_0,\ldots,a_{n-1}] - \Delta^{n-1}[f(x); a_1,\ldots,a_n]}{a_0 - a_n}$$

$n = 1,2,\ldots$

It is well known and it can be proved by induction that

$$\Delta^n[f(x);\ a_0,\ldots,a_n] = \sum_{v=0}^{n} \frac{f(a_v)}{\prod_{\substack{j=0\\ \neq v}}^{n} (a_v - a_j)}. \qquad (3.2.1)$$

It can also be proved that another expression for $\Delta^n[f(x);\ a_0,\ldots,a_n]$ is given by

$$\Delta^n[f(x);\ a_0,\ldots,a_n] = \theta_n\theta_{n-1}\cdots\theta_1 f(a_0) \qquad (3.2.2)$$

where the operator θ_p which applies to $f(a_0)$ alone is defined as

$$\theta_p f(a_0) = \frac{f(a_0) - f(a_p)}{a_0 - a_p} \ . \qquad (3.2.3)$$

If $f(x)$ is a polynomial of degree n in x, then the rth divided difference is a polynomial of degree (n-r) in a_0. The operator θ_p reduces the degree by unity, as is seen from (3.2.3) where the numerator must contain the factor $(a_0 - a_p)$, as it vanishes for $a_0 = a_p$, and therefore by (3.2.2), $\Delta^r[f(x);\ a_0,\ldots,a_n]$ is a polynomial of degree (n-r) in a_0. Thus if $f(x)$ is a polynomial of degree n,

$$\Delta^{n+r}[f(x);\ a_0,\ldots,a_{n+r}] = \begin{cases} \text{constant, for } r = 0 \\ 0, \text{ for r a positive integer} \\ \qquad \text{greater than zero.} \end{cases} \qquad (3.2.4)$$

It has been pointed out in §3.1 that some colinear cases can be eliminated so that the joint density $f(y_1,y_2)$ can still be expressed in the derivative form of Chapter II. The elimination can be done with the help of the following lemma and theorem.

<u>Lemma 3.1.</u> For any fixed $i = 0,1,\ldots,n$ such that $(i\ j\ k) \neq 0$ for $j,k = 0,1,\ldots,i-1,i+1,\ldots,n$ and $j \neq k$,

$$\sum_{\substack{j=0\\ \neq i}}^{n} \frac{(i\ j\ y)^{n-2}}{\prod\limits_{\substack{k=0\\ \neq i,j}}^{n}(i\ j\ k)} = 0 \ . \qquad (3.2.5)$$

<u>Proof:</u> For a fixed value of i, the condition $(i\ j\ k) \neq 0$ for $i \neq j \neq k$ implies that (a_{1i},a_{2i}), (a_{1j},a_{2j}) and (a_{1k},a_{2k}) are distinct and not colinear.

Fix $i = 1$ for convenience. If one lets

$$(a_{11}, a_{21}) = (a,b)$$

where a, b are real numbers, and writes

$$(a_{1j}, a_{2j}) = (a+\alpha_j, b+\alpha_j m_j) \quad j = 0,2,\ldots,n;\ \alpha_j \neq 0$$

where m_j denotes the slope of the line through (a,b) and (a_{1j}, a_{2j}), then m_j $(j = 0,2,\ldots,n)$ must all be distinct. Now

$$(1\ j\ y)^{n-2} = \begin{vmatrix} 1 & a & b \\ 1 & a+\alpha_j & b+\alpha_j m_j \\ 1 & y_1 & y_2 \end{vmatrix}^{n-2}$$

$$= [ay_2 + \alpha_j y_2 - by_1 - m_j y_1 \alpha_j - ay_2 + ab + m_j a \alpha_j + by_1 - ab - \alpha_j b]^{n-2}$$

$$= \alpha_j^{n-2} [(y_2 - b) + m_j(a - y_1)]^{n-2}$$

$j = 0,2,\ldots,n$, and

$$(1\ j\ k) = \begin{vmatrix} 1 & a & b \\ 1 & a+\alpha_j & b+m_j\alpha_j \\ 1 & a+\alpha_k & b+m_k\alpha_k \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha_j & m_j\alpha_j \\ 1 & \alpha_k & m_k\alpha_k \end{vmatrix}$$

$$= \alpha_j \alpha_k (m_k - m_j)$$

$j \neq k;\ j,\ k = 0,2,\ldots,n.$

Substituting these values into the left hand side of (3.2.5), one obtains

$$\sum_{\substack{j=0 \\ \neq 1}}^{n} \frac{(1\ j\ y)^{n-2}}{\prod\limits_{\substack{k=0 \\ \neq 1,j}}^{n} (1\ j\ k)} = \sum_{\substack{j=0 \\ \neq 1}}^{n} \frac{\alpha_j^{n-2}[(y_2 - b) + m_j(a - y_1)]^{n-2}}{\alpha_j^{n-1} \prod\limits_{\substack{k=0 \\ \neq 1,j}}^{n} \alpha_k (m_k - m_j)}$$

$$= \frac{1}{\prod_{\substack{i=0\\ \neq 1}}^{n} \alpha_i} \sum_{\substack{j=0\\ \neq 1}}^{n} \frac{\left[(y_2 - b) + m_j(a - y_1)\right]^{n-2}}{\prod_{\substack{k=0\\ \neq 1,j}}^{n} (m_k - m_j)}$$

$$= \frac{(-1)^{n-1}}{\prod_{\substack{i=0\\ \neq 1}}^{n} \alpha_i} \sum_{\substack{j=0\\ \neq 1}}^{n} \frac{\left[(y_2 - b) + m_j(a - y_1)\right]^{n-2}}{\prod_{\substack{k=0\\ \neq 1,j}}^{n} (m_j - m_k)}.$$

The numerator is a polynomial of degree (n-2) in m_j, and m_j $(j = 0,2,\ldots,n)$ are distinct. Therefore by (3.2.1) and (3.2.4), the expression on the left-hand-side of (3.2.5) is a divided difference of order (n-1) of an $(n-2)^{th}$ degree polynomial in m_j and therefore vanishes.

If there is an m_j undefined (that is, the line joining (a_{1j},a_{2j}) and (a_{11},a_{21}) is vertical), then assume for convenience that it is for $j = n$. Then let $(a_{1n},a_{2n}) = (a,b+\alpha_n)$ and

$$\sum_{\substack{j=0\\ \neq 1}}^{n} \frac{(1\ j\ y)^{n-2}}{\prod_{\substack{k=0\\ \neq 1,j}}^{n} (1\ j\ k)} = \frac{1}{\prod_{\substack{k=0\\ \neq 1}}^{n} \alpha_k} \left\{ \sum_{\substack{j=0\\ \neq 1}}^{n-1} \frac{\left[(y_2 - b)+m_j(a-y_1)\right]^{n-2}}{\prod_{\substack{k=0\\ \neq 1,j}}^{n-1} (m_k - m_j)} + (-1)^{n-1}(a-y_1)^{n-2} \right\}$$

$$= \frac{1}{\prod_{\substack{k=0\\ \neq 1}}^{n} \alpha_k} \left\{ (-1)^{n-2} \Delta^{n-2}\left[((y_2-b)+m_j(a-y_1))^{n-2}; m_0, m_2, \ldots, m_{n-1}\right] + (-1)^{n-1}(a - y_1)^{n-2} \right\}$$

$$= \frac{1}{\prod_{\substack{k=0\\ \neq 1}}^{n} \alpha_k} \left[(-1)^{n-2}(a - y_1)^{n-2} + (-1)^{n-1}(a - y_1)^{n-2}\right]$$

$$= 0.$$

Thus the lemma is proved for any fixed $i = 0,1,\ldots,n$ if $(i\ j\ k) \neq 0$, $j,\ k = 0,1,\ldots,i-1,i+1,\ldots,n$ and $j \neq k$.

If the coefficient points (a_{1i}, a_{2i}) of the linear combinations y_1 and y_2 are not all distinct, suppose (as in Chapter II, §2.2) that there are p distinct such points, and that n_i points are coincident at the ith distinct point represented by (a_{1k_i}, a_{2k_i}) $(i = 1,2,\ldots,p)$ as shown in Figure III below

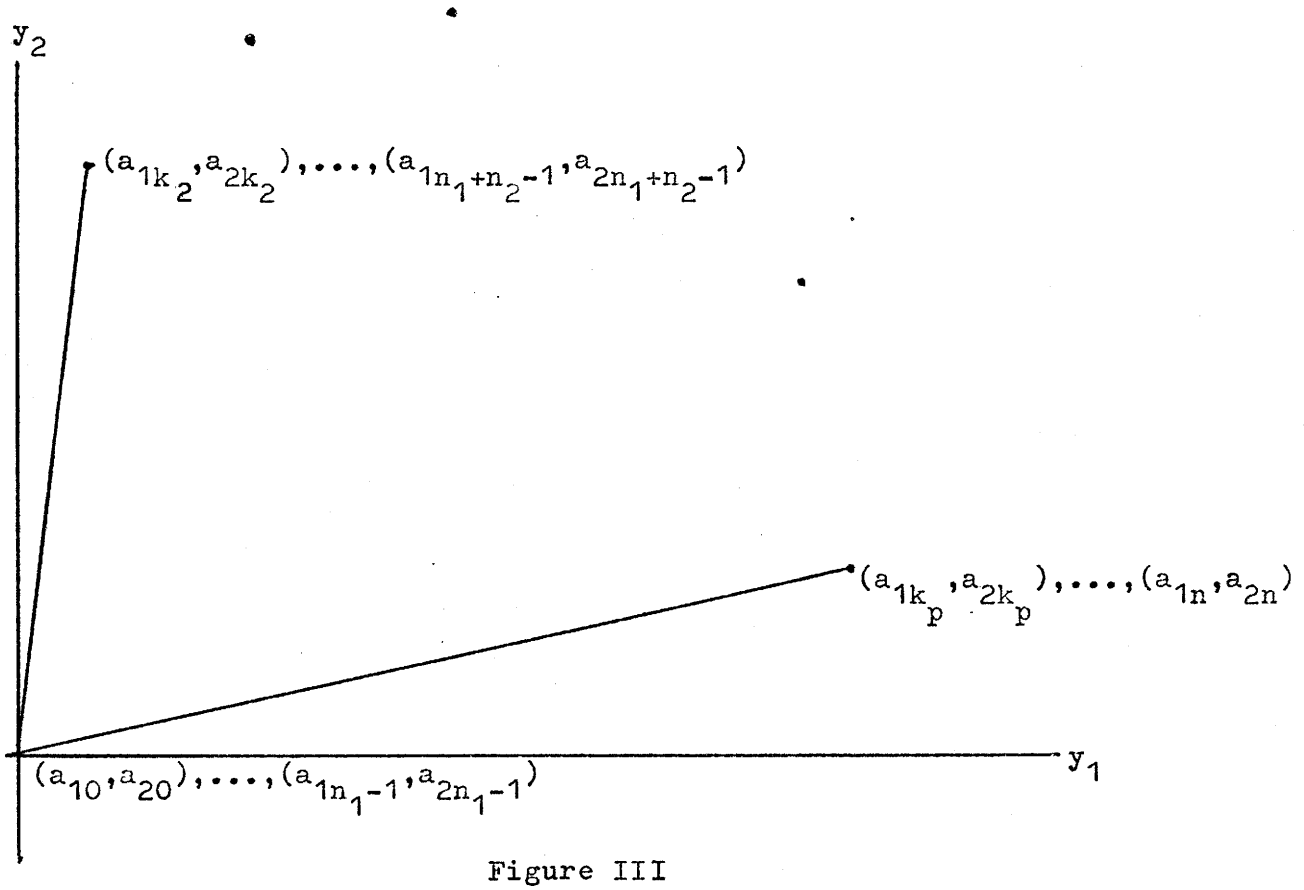

Figure III

<u>Theorem 3.2</u>. If for any fixed $r = 1,2,\ldots,p$, there exist no distinct j and l, $j,l = 0,1,\ldots,r-1,r+1,\ldots,p$, such that

$$(k_r\ k_j\ k_l) = 0,$$

then

$$\sum_{\substack{j=1\\ \neq r}}^{p} \frac{1}{\prod_{i=1}^{p}(n_i-1)!\begin{vmatrix}1 & a_{2k_r}\\ 1 & a_{2k_j}\end{vmatrix}^{n+1-p}} \frac{\partial^{n+1-p}}{\partial a_{1k_1}^{n_1-1}\cdots\partial a_{1k_p}^{n_p-1}} \left\{\frac{(k_r\ k_j\ y)^{n-2}}{\prod_{\substack{l=1\\ \neq r,j}}^{p}(k_r\ k_j\ k_l)}\right\} = 0. \tag{3.2.6}$$

<u>Proof</u>: Consider a fixed $r = 1,2,\ldots,p$ satisfying $(k_r\ k_j\ k_l) \neq 0$ for $j,l = 0,1,\ldots,r-1,r+1,\ldots,p;\ j \neq l$. Adjust the (n_j-1) coincident points $(j = 1,2,\ldots,p)$ (by introducing small $\varepsilon>0$) in such a way that they are distinct from (a_{1k_j}, a_{2k_j}) and the conditions in Lemma 3.1 are satisfied for all $i = k_r,\ldots,(k_r+n_r-1)$.

Therefore by Lemma 3.1, for each $i = k_r,\ldots,(k_r+n_r-1)$,

$$\sum_{\substack{j=0\\ \neq i}}^{n} \frac{(i\ j\ y)^{n-2}}{\prod\limits_{\substack{k=0\\ \neq i,j}}^{n} (i\ j\ k)} = 0$$

which implies that

$$\sum_{i=k_r}^{k_r+n_r-1} \left(\sum_{\substack{j=0\\ \neq i}}^{n} \frac{(i\ j\ y)^{n-2}}{\prod\limits_{\substack{k=0\\ \neq i,j}}^{n} (i\ j\ k)} \right) = 0 . \qquad (3.2.7)$$

But

$$\frac{(i\ j\ y)^{n-2}}{\prod\limits_{\substack{k=0\\ \neq i,j}}^{n} (i\ j\ k)} = - \frac{(j\ i\ y)^{n-2}}{\prod\limits_{\substack{k=0\\ \neq i,j}}^{n} (j\ i\ k)}$$

implies that

$$\sum_{i=k_r}^{k_r+n_r-1} \left[\sum_{\substack{j=k_r\\ \neq i}}^{k_r+n_r-1} \frac{(i\ j\ y)^{n-2}}{\prod\limits_{\substack{k=0\\ \neq i,j}}^{n} (i\ j\ k)} \right] = 0.$$

Therefore (3.2.7) becomes

$$\sum_{i=k_r}^{k_r+n_r-1} \left(\sum_{\substack{j=0\\ j\notin(k_r,\ldots,k_r+n_r-1)}}^{n} \frac{(i\ j\ y)^{n-2}}{\prod\limits_{\substack{k=0\\ \neq i,j}}^{n} (i\ j\ k)} \right) = 0 .$$

Then using the same method as in the proof of Lemma 2.3 and with ε understood in the notation

$$\lim_{\varepsilon \to 0} \sum_{i=k_r}^{k_r+n_r-1} \left(\sum_{\substack{j=0 \\ j \notin (k_r,\ldots,k_r+n_r-1)}}^{n} \frac{(i\ j\ y)^{n-2}}{\prod_{\substack{k=0 \\ \neq i,j}}^{n} (i\ j\ k)} \right)$$

$$= \sum_{\substack{j=1 \\ \neq r}}^{p} \frac{1}{\prod_{i=1}^{p} (n_i-1)! \begin{vmatrix} 1 & a_{2k_r} \\ 1 & a_{2k_j} \end{vmatrix}^{n+1-p}} \frac{\partial^{n+1-p}}{\partial a_{1k_1}^{n_1-1} \cdots \partial a_{1k_p}^{n_p-1}} \left\{ \frac{(k_r\ k_j\ y)^{n-2}}{\prod_{\substack{l=1 \\ \neq r,j}}^{p} (k_r\ k_j\ k_l)} \right\}.$$

Therefore

$$\sum_{\substack{j=1 \\ \neq r}}^{p} \frac{1}{\prod_{i=1}^{p} (n_i-1)! \begin{vmatrix} 1 & a_{2k_r} \\ 1 & a_{2k_j} \end{vmatrix}^{n+1-p}} \frac{\partial^{n+1-p}}{\partial a_{1k_1}^{n_1-1} \cdots \partial a_{1k_p}^{n_p-1}} \left\{ \frac{(k_r\ k_j\ y)^{n-2}}{\prod_{\substack{l=1 \\ \neq r,j}}^{p} (k_r\ k_j\ k_l)} \right\} = 0$$

for r satisfying the condition stated in the theorem.

Notice that in the proof of the theorem, we assume that $a_{2k_r} \neq a_{2k_j}$ for all $1 \leq j \leq p$ and $j \neq r$. If there is one such j with $a_{2k_j} = a_{2k_r}$, then as was discussed in the proof of Lemma 2.2, $a_{1k_j} \neq a_{1k_r}$ and the corresponding term in the summation of (3.2.6) would be

$$\frac{1}{\prod_{i=1}^{p} (n_i-1)! \begin{vmatrix} 1 & a_{1k_j} \\ 1 & a_{1k_r} \end{vmatrix}^{n+1-p}} \frac{\partial^{n+1-p}}{\partial a_{2k_1}^{n_1-1} \cdots \partial a_{2k_p}^{n_p-1}} \left\{ \frac{(k_r\ k_j\ y)^{n-2}}{\prod_{\substack{l=1 \\ \neq r,j}}^{p} (k_r\ k_j\ k_l)} \right\}.$$

§ 3.3 Example of colinear points

In this section, the use of Theorem 3.2 to eliminate troublesome terms caused by colinear points is illustrated by the following example.

Consider the problem of finding the joint density of

$$y_1 = x_{(r+1)} + x_{(r+2)} + \cdots + x_{(n-r)}$$

$$y_2 = x_{(n-r)} - x_{(r+1)}$$

where r is a non-negative integer less than (n-1)/2, that is,

$$y_1 = (n-2r)u_1 + (n-2r)u_2 + \dots + (n-2r)u_{r+1} + (n-2r-1)u_{r+2} +$$

$$+ (n-2r-2)u_{r+3} + \dots + 2u_{n-r-1} + u_{n-r}$$

$$y_2 = u_{r+2} + u_{r+3} + \dots + u_{n-r}.$$

Therefore among the (n+1) coefficient points including the origin (0,0), there are (r+1) points coinciding at (0,0), (r+1) points coinciding at (n-2r,0), and the remaining (n-2r-1) points (n-2r-1,1) (n-2r-2,1),...,(2,1),(1,1) are colinear.

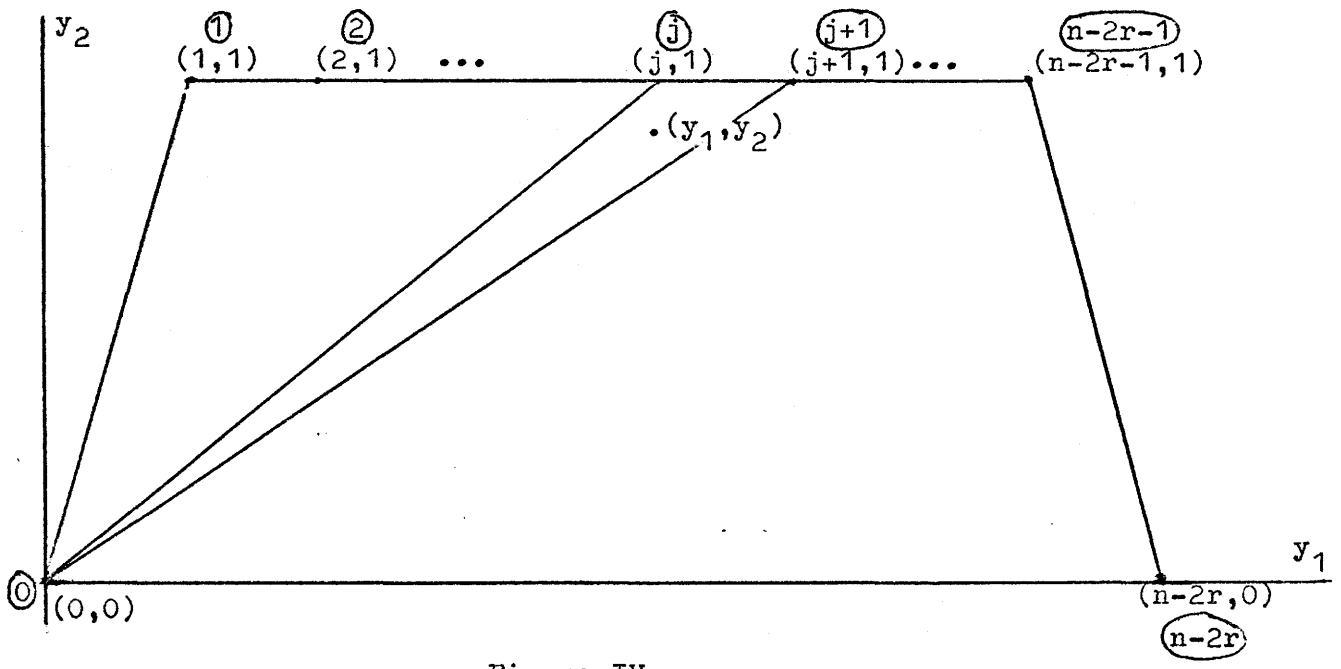

Figure IV

The distinct points are labelled 0,1,...,n-2r. Obviously the density is non-zero only if (y_1,y_2) falls in the trapezium with vertices (0,0), (1,1), (n-2r-1,1) and (n-2r,0). If (y_1,y_2) is in the triangle with vertices (0,0), (j,1) and (j+1,1) with the corresponding term of the density denoted by

$$[j,j+1] = \frac{1}{r!r!\begin{vmatrix} 1 & j+1 \\ 1 & j \end{vmatrix}^{2r}} \cdot \frac{\partial^{2r}}{\partial a_{20}^{r}\, \partial a_{2n-2r}^{r}} \left\{ \frac{(j\ j+1\ y)^{n-2}}{\prod\limits_{\substack{i=0 \\ \neq j,j+1}}^{n-2r} (j\ j+1\ i)} \right\},$$

then (y_1,y_2) is also in the triangle with terms $[k,l]$, $k=1,\ldots,j$; $l=j+1,\ldots,n-2r-1$. That is, for $j=1,\ldots,n-2r-2$, the terms of the density including the colinear points can be written as

$$T = \frac{n!}{(n-2)!}\sum_{j=1}^{n-2r-2}\lambda_{jj+1}\,\mathrm{sgn}\begin{vmatrix} j & 1 \\ j+1 & 1\end{vmatrix}\left\{\sum_{k=1}^{j}\sum_{l=j+1}^{n-2r-1}[k,l]\right\}.$$

But for k fixed, by Theorem 3.2

$$\sum_{\substack{l=0\\ \neq k}}^{n-2r}[k,l] = 0.$$

Therefore

$$\sum_{l=j+1}^{n-2r-1}[k,l] = -\sum_{\substack{l=0\\ \neq k}}^{j}[k,l] - \sum_{l=n-2r}^{n-2r}[k,l],$$

and

$$T = \frac{n!}{(n-2)!}\sum_{j=1}^{n-2r-2}\lambda_{jj+1}(-1)\left\{\sum_{k=1}^{j}\left(-\sum_{\substack{l=0\\ \neq k}}^{j}[k,l] - \sum_{l=n-2r}^{n-2r}[k,l]\right)\right\}$$

$$= \frac{n!}{(n-2)!}\sum_{j=1}^{n-2r-2}\lambda_{jj+1}\left\{\sum_{k=1}^{j}\sum_{\substack{l=1\\ \neq k}}^{j}[k,l] + \sum_{k=1}^{j}[k,0] + \sum_{k=1}^{j}[k,n-2r]\right\}.$$

Obviously $\sum_{k=1}^{j}\sum_{\substack{l=1\\ \neq k}}^{j}[k,l]$ is zero as the terms in the double summation cancel in pairs since $[k,l] = -[l,k]$. Thus

$$T = \frac{n!}{(n-2)!}\sum_{j=1}^{n-2r-2}\lambda_{jj+1}\left\{\sum_{k=1}^{j}\left([k,0] + [k,n-2r]\right)\right\}$$

$$= \frac{n!}{(n-2)!}\sum_{j=1}^{n-2r-2}\lambda_{jj+1}\left[\sum_{k=1}^{j}\frac{1}{r!r!\begin{vmatrix}1 & 0\\ 1 & k\end{vmatrix}^{2r}}\cdot\frac{\partial^{2r}}{\partial a_{20}^{r}\,\partial a_{2n-2r}^{r}}\left\{\frac{(k\ 0\ y)^{n-2}}{\prod_{\substack{i=1\\ \neq k}}^{n-2r}(k\ 0\ i)}\right\}\right.$$

$$\left. + \sum_{k=1}^{j}\frac{1}{r!r!\begin{vmatrix}1 & n-2r\\ 1 & k\end{vmatrix}^{2r}}\cdot\frac{\partial^{2r}}{\partial a_{20}^{r}\,\partial a_{2n-2r}^{r}}\left\{\frac{(k\ n-2r\ y)^{n-2}}{\prod_{\substack{i=0\\ \neq k}}^{n-2r-1}(k\ n-2r\ i)}\right\}\right.$$

This expression does not contain any indeterminate terms and therefore the colinearity problems have been eliminated. The joint density of y_1 and y_2 is thus

$$f(y_1, y_2) = T + \frac{n!}{(n-2)!} \sum_{j=1}^{n-2r-1} \lambda_{jn-2r} \operatorname{sgn}\begin{vmatrix} j & 1 \\ n-2r & 0 \end{vmatrix} \frac{1}{r!r! \begin{vmatrix} 1 & n-2r \\ 1 & j \end{vmatrix}^{2r}} \cdot$$

$$\cdot \frac{\partial^{2r}}{\partial a_{20}^r \, \partial a_{2n-2r}^r} \left\{ \frac{(j \; n-2r \; y)^{n-2}}{\prod_{\substack{i=0 \\ \neq j}}^{n-2r-1} (j \; n-2r \; i)} \right\}$$

$$= \frac{n!}{(n-2)!} \frac{1}{r!r!} \cdot \frac{\partial^{2r}}{\partial a_{20}^r \, \partial a_{2n-2r}^r} \left\{ \sum_{j=1}^{n-2r-2} \sum_{k=1}^{j} \lambda_{jj+1} \cdot \right.$$

$$\cdot \left[\frac{1}{\begin{vmatrix} 1 & 0 \\ 1 & k \end{vmatrix}^{2r}} \cdot \frac{(k \; 0 \; y)^{n-2}}{\prod_{\substack{i=0 \\ \neq k}}^{n-2r} (k \; 0 \; i)} + \frac{1}{\begin{vmatrix} 1 & n-2r \\ 1 & k \end{vmatrix}^{2r}} \cdot \frac{(k \; n-2r \; y)^{n-2}}{\prod_{\substack{i=0 \\ \neq k}}^{n-2r-1} (k \; n-2r \; i)} \right]$$

$$\left. - \sum_{j=1}^{n-2r-1} \lambda_{jn-2r} \frac{1}{\begin{vmatrix} 1 & n-2r \\ 1 & j \end{vmatrix}^{2r}} \cdot \frac{(j \; n-2r \; y)^{n-2}}{\prod_{\substack{i=0 \\ \neq j}}^{n-2r-1} (j \; n-2r \; i)} \right\}.$$

The distribution of the more general case of

$$y_1 = (r+1)x_{(r+1)} + x_{(r+2)} + \cdots + x_{(n-r-1)} + (r+1)x_{(n-r)}$$

$$y_2 = x_{(n-r)} - x_{(r+1)}$$

follows in an identical manner, the only difference being that the coefficient points are (0,0) repeated (r+1) times, (n,0) repeated (r+1) times, and (n-1,1), (n-2,1),..., (r+2,1), (r+1,1) which are colinear. The expression

$$y_1 = n^{-1}\left[(r+1)x_{(r+1)} + x_{(r+2)} + \cdots + x_{(n-r-1)} + (r+1)x_{(n-r)}\right]$$

is called a Winsorized mean, whereas

$$y_1 = (n-2r)^{-1} \sum_{i=r+1}^{n-r} x_{(i)}$$

is called a trimmed mean.

§ 3.4 The joint distribution of the smallest and largest intervals

In this section the problem of finding the joint distribution of $u_{(1)}$ and $u_{(n+1)}$ is considered. This will show that Theorem 2.1 can sometimes be applied directly even when there are colinear points. It will also demonstrate the use of Theorem 2.1 in the case of linear combinations of the form $\sum_{i=1}^{n+1} a_i u_i$ (compared to the form $\sum_{i=1}^{n} a_i u_i$). Since

$$u_{(n+1)} = 1 - \sum_{i=1}^{n} u_{(i)} ,$$

one may find the joint density of

$$y_1 = u_{(1)}$$
$$y_2 = \sum_{i=1}^{n} u_{(i)}$$

using Theorem 2.1 and then the joint density of $u_{(1)}$ and $u_{(n+1)}$ can easily be found from this. Thus Theorem 2.1 can be used even if the linear combinations involve all of the

$$\left(u_{(1)} \leq u_{(2)} \leq \ldots \leq u_{(n+1)} , \quad \sum_{i=1}^{n+1} u_{(i)} = 1 \right) .$$

As has been discussed before, the distribution of $(u_{(1)},\ldots,u_{(n+1)})$ is the same as the distribution of $(\frac{u_1}{n+1}, \frac{u_1}{n+1}+\frac{u_2}{n}, \ldots, \frac{u_1}{n+1}+\frac{u_2}{n}+\ldots+\frac{u_n}{2}+u_{n+1})$. Expressing y_1 and y_2 in terms of $(u_1,\ldots,u_n)$, one obtains

$$y_1 = \frac{u_1}{(n+1)}$$

$$y_2 = \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{u_j}{n+2-j}$$

$$= \sum_{i=1}^{n} \frac{n+1-i}{n+2-i} u_i$$

$$= \frac{n}{n+1} u_1 + \frac{n-1}{n} u_2 + \dots + \frac{2}{3} u_{n-1} + \frac{u_n}{2} \quad .$$

As can be seen in Figure V below, the coefficients points $\left\{(0, \frac{j}{j+1}),\ j = 0,1,\dots,n-1\right\}$ of the linear combinations are colinear along the y_2-axis. However, since the triangles formed by $(0,0)$, $(0, \frac{i}{i+1})$ and $(0, \frac{j}{j+1})$ $(i = 0,\dots,j-1;\ j = 1,\dots,n-1)$ are straight lines, $\lambda_{ij} = 0$. Thus one only needs to consider the triangles formed by $(0,0)$, $(0, \frac{j}{j+1})$ and $(\frac{1}{n+1}, \frac{n}{n+1})$ $(j = 1,\dots,n-1)$ and Formula (2.2.1) can be applied directly.

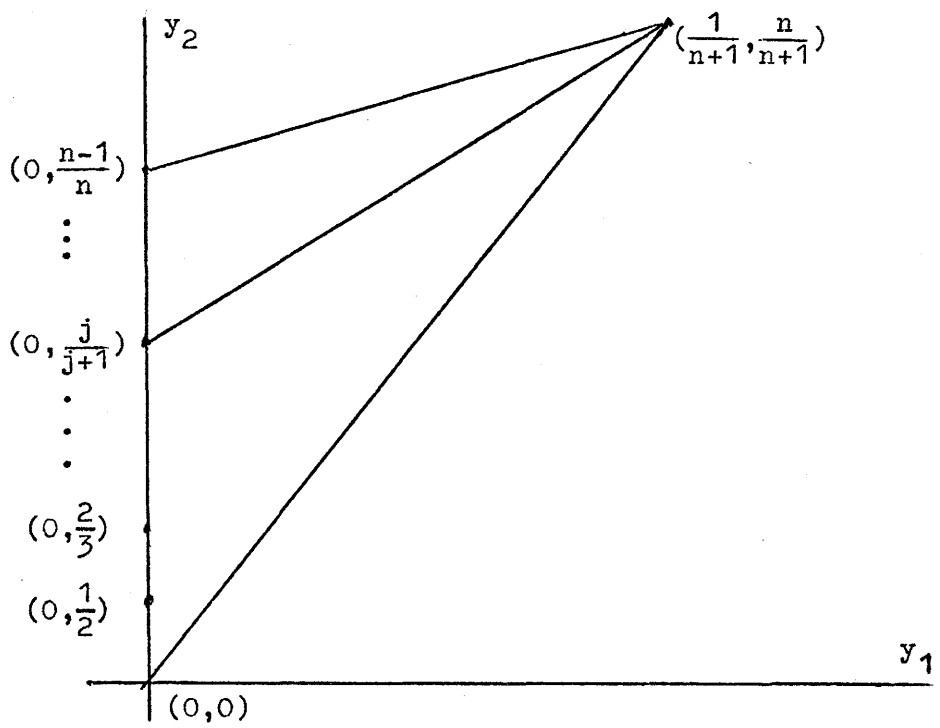

Figure V

<u>Lemma 3.3</u>. The joint density function of

$$y_1 = \frac{u_1}{n+1}$$

$$y = \frac{n}{n+1} u_1 + \frac{n-1}{n} u_2 + \dots + \frac{2}{3} u_{n-1} + \frac{1}{2} u_n$$

is given by

$$f(y_1,y_2) = \frac{n!}{(n-2)!} \sum_{j=0}^{n-1} \lambda'_j \frac{(-1)^j (n+1)! \left[-j-(n-j)y_1+(j+1)y_2\right]^{n-2}}{j!(n-1-j)!}$$

where

$$\lambda'_j = \begin{cases} 1 & \text{if } (-j-(n-j)y_1+(j+1)y_2) > 0 \text{ and } 0 < y_1 < \frac{1}{n+1} \\ 0 & \text{otherwise} \end{cases} .$$

<u>Proof</u>: By Formula (2.2.1), $f(y_1,y_2)$ is non-zero only for (y_1,y_2) in the triangle with vertices $(0,0)$, $(0, \frac{n-1}{n})$ and $(\frac{1}{n+1}, \frac{n}{n+1})$, and

$$f(y_1,y_2) = \frac{n!}{(n-2)!} \sum_{j=1}^{n-1} \lambda_j \operatorname{sgn} \begin{vmatrix} 0 & \frac{j}{j+1} \\ \frac{1}{n+1} & \frac{n}{n+1} \end{vmatrix} \cdot \frac{(j\ n\ y)^{n-2}}{\prod_{\substack{k=0 \\ \neq j}}^{n-1} (j\ n\ k)}$$

where

$$\lambda_j = \begin{cases} 1 & \text{if } (y_1,y_2) \text{ is in the triangle formed by } (0,0), \\ & (0, \frac{j}{j+1}) \text{ and } (\frac{1}{n+1}, \frac{n}{n+1}) \\ 0 & \text{otherwise} \end{cases} .$$

But

$$(j\ n\ y)^{n-2} = \begin{vmatrix} 1 & 0 & \frac{j}{j+1} \\ 1 & \frac{1}{n+1} & \frac{n}{n+1} \\ 1 & y_1 & y_2 \end{vmatrix}^{n-2}$$

$$= \left[\frac{y_2}{n+1} - \frac{ny_1}{n+1} + \frac{jy_1}{j+1} - \frac{j}{(n+1)(j+1)}\right]^{n-2}$$

$$= \left[\frac{1}{(n+1)(j+1)}\right]^{n-2} \left[-j-(n-j)y_1 + (j+1)y_2\right]^{n-2}$$

and for $k = 0,\ldots,n-1;\ k \neq j$

$$(j\ n\ k) = \begin{vmatrix} 1 & 0 & \frac{j}{j+1} \\ 1 & \frac{1}{n+1} & \frac{n}{n+1} \\ 1 & 0 & \frac{k}{k+1} \end{vmatrix}$$

$$= \frac{k}{(n+1)(k+1)} - \frac{j}{(j+1)(n+1)}$$

$$= \frac{k-j}{(n+1)(k+1)(j+1)} ,$$

and therefore

$$\frac{(j\ n\ y)^{n-2}}{\prod\limits_{\substack{k=0\\ \neq j}}^{n-1} (j\ n\ k)} = \frac{\left\{\left[(n+1)(j+1)\right]^{-1}\left[-j-(n-j)y_1+(j+1)y_2\right]\right\}^{n-2}}{(-j)^j j!(n-1-j)!(j+1) \ / \left[(n+1)(j+1)\right]^{n-1} n!}$$

$$= \frac{(n+1)!\left[-j-(n-j)y_1+(j+1)y_2\right]^{n-2}}{(-1)^j\ j!(n-1-j)!} .$$

Thus

$$f(y_1,y_2) = \frac{n!}{(n-2)!} \sum_{j=1}^{n-1} \lambda_j \frac{(-1)^{j-1}(n+1)!\left[-j-(n-j)y_1+(j+1)y_2\right]^{n-2}}{j!\ (n-1-j)!}$$

where

$$\lambda_j = \begin{cases} 1 & \text{if } (y_1,y_2) \text{ is in the triangle with vertices} \\ & (0,0),\ (0,\ \frac{j}{j+1}) \text{ and } (\frac{1}{n+1},\ \frac{n}{n+1}), \\ 0 & \text{otherwise.} \end{cases}$$

or equivalently for this case

$$\lambda_j = \begin{cases} 1 & \text{if } [-j-(n-j)y_1+(j+1)y_2] < 0 \text{ and } 0<y_1<\frac{1}{n+1} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f(y_1,y_2) = 0$ if $[-j-(n-j)y_1+(j+1)y_2] < 0$ for all $j = 0,\ldots,n-1$. Thus

$$f(y_1,y_2) = \frac{n!}{(n-2)!}\sum_{j=k}^{n-1} \frac{(-1)^j(n+1)![-j-(n-j)y_1+(j+1)y_2]^{n-2}}{j!\ (n-1-j)!}$$

for some integer $k \leq n-1$ and $0 < y_1 < \frac{1}{n+1}$.

Putting $i = n$ in Lemma 3.1 gives

$$\frac{n!}{(n-2)!}\sum_{j=0}^{n-1} \frac{(-1)^{j-1}(n+1)![-j-(n-j)y_1+(j+1)y_2]^{n-2}}{j!\ (n-1-j)!} = 0 .$$

Therefore

$$f(y_1,y_2) = -\sum_{j=0}^{k-1} \frac{(-1)^{j-1}(n+1)![-j-(n-j)y_1+(j+1)y_2]^{n-2}}{j!\ (n-1-j)!}$$

or equivalently

$$= \frac{n!}{(n-2)!}\sum_{j=0}^{n-1} \lambda'_j \frac{(-1)^j(n+1)![-j-(n-j)y_1+(j+1)y_2]^{n-2}}{j!\ (n-1-j)!}$$

where λ'_j is as defined in the statement of this lemma.

<u>Theorem 3.4</u>. The joint density function of $u_{(1)}$ and $u_{(n+1)}$ is given by

$$f(u_{(1)},u_{(n+1)}) = \frac{n!}{(n-2)!}\sum_{j=0}^{n-1} \lambda'_j \frac{(-1)^j(n+1)![1-(n-j)u_{(1)}-(j+1)u_{(n+1)}]^{n-2}}{j!\ (n-1-j)!}$$

where

$$\lambda'_j = \begin{cases} 1 & \text{if } (1-(n-j)u_{(1)}-(j+1)u_{(n+1)}) > 0 \\ 0 & \text{otherwise} . \end{cases}$$

Proof: The theorem follows from Lemma 3.2 by noting that

$$u_{(1)} = \frac{u_1}{n+1} = y_1$$

and

$$u_{(n+1)} = 1 - \sum_{i=1}^{n} u_{(i)} = 1 - y_2 \quad .$$

Now consider the problem of finding $\Pr(u_{(1)} > a, u_{(n+1)} < b)$. For this, the following lemma is required.

Lemma 3.5 . For any positive integer n

$$\sum_{j=1}^{n-1} (-1)^{j-1} \binom{n}{j} (n-j)^{n-1} = n^{n-1} \quad .$$

Proof: This equality is equivalent to

$$\sum_{j=0}^{n} (-1)^{j-1} \binom{n}{j} (n-j)^{n-1} = 0 \quad .$$

But the left-hand side is equal to

$$n! \sum_{j=0}^{n} \left[(-1)^{j-1} / j!(n-j)!\right](n-j)^{n-1}$$

$$= n! \sum_{j=0}^{n} (-1)^{j-1}(n-j)^{n-1}(-1)^{n-j} / \prod_{\substack{i=0 \\ \neq j}}^{n} (j-i)$$

$$= n!(-1)^{n-1} \sum_{j=0}^{n} (n-j)^{n-1} / \prod_{\substack{i=0 \\ \neq j}}^{n} (j-i)$$

$$= n!(-1)^{n-1} \Delta^n (n-x)^{n-1} \quad ,$$

x assuming the values $0,1,\ldots,n$.

But the function $(n-x)^{n-1}$ is a polynomial of degree $n-1$ in x, and by Formula (3.2.4),

$$\Delta^n (n - x)^{n-1} = 0 \quad .$$

Therefore

$$\sum_{j=0}^{n} (-1)^{j-1} \binom{n}{j} (n - j)^{n-1} = 0$$

or

$$\sum_{j=0}^{n-1} (-1)^{j-1} \binom{n}{j} (n - j)^{n-1} = n^{n-1} \quad .$$

<u>Theorem 3.6</u>. $\Pr(u_{(1)} > a,\ u_{(n+1)} < b) =$

$$= \sum_{j=0}^{n-1} \lambda''_j (-1)^j \binom{n+1}{j} (1-(n+1-j)a - jb)^n$$

for $0 < a < \frac{1}{n+1}$, $0 < 1 - b < \frac{n}{n+1}$, where

$$\lambda''_j = \begin{cases} 1 & \text{if } (1-(n+1-j)a - jb) > 0 \text{ and } 0 < a < \frac{1}{n+1} \\ 0 & \text{otherwise} \end{cases} \quad .$$

<u>Proof:</u> $\Pr(u_{(1)} > a,\ u_{(n+1)} < b)$

$$= \Pr(u_{(1)} > a,\ 1 - \sum_{i=1}^{n} u_{(i)} < b)$$

$$= \Pr(u_{(1)} > a,\ \sum_{i=1}^{n} u_{(i)} > 1 - b)$$

$$= \Pr(y_1 > a,\ y_2 > 1 - b) \quad .$$

One only needs to consider the region $0 < a < \frac{1}{n+1}$ and $0 < 1 - b < \frac{n}{n+1}$ since for any other values $(c, 1 - d)$

$$\Pr(y_1 > c,\ y_2 > 1 - d) = \Pr(y_1 > a, y_2 > 1 - b)$$

for some a and $(1 - b)$ in the region. This is easy to see because the joint density $f(y_1, y_2)$ as given by Formula(2.2.1)

is zero outside the triangle with vertices (0,0), $(0,\frac{n-1}{n})$ and $(\frac{1}{n+1},\frac{n}{n+1})$.

Diagramatically, it is easy to see that if (a,1-c) is below the line from (0,0) to $(\frac{1}{n+1},\frac{1}{n+1})$ and $0<a<\frac{1}{n+1}$ as shown in Figure VI,

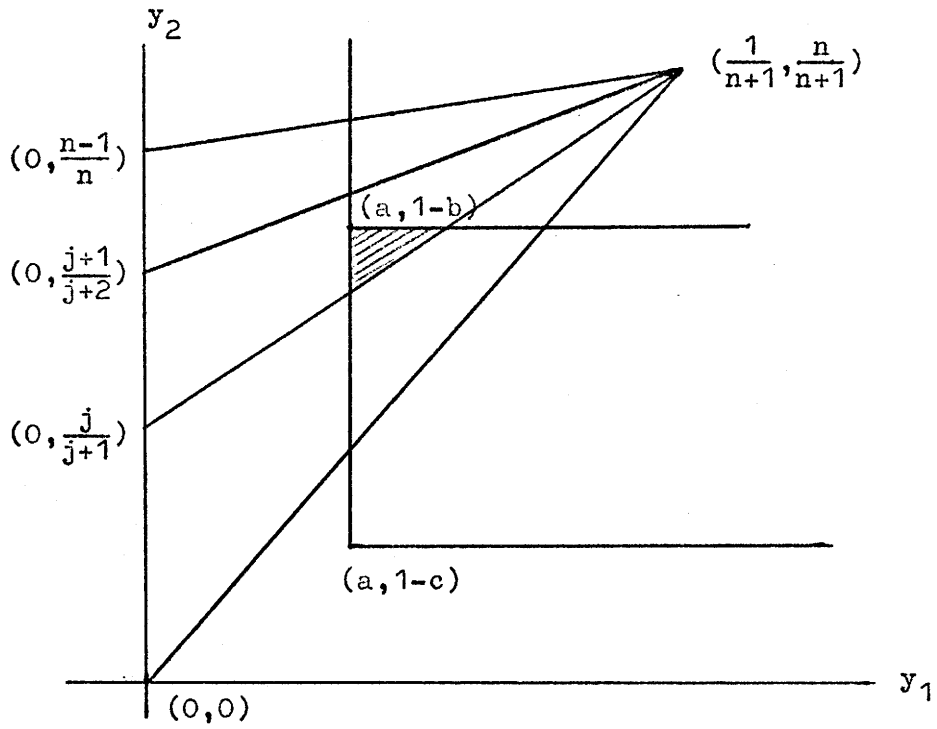

Figure VI

then integrating the density in Lemma 3.3 gives

$$\Pr(y_1>a,\ y_2>1-c)$$

$$=\frac{n!}{(n-2)!}\sum_{j=0}^{n-2}\frac{(-1)^j(n+1)!}{j!(n-1-j)!}\int_{y_1=a}^{\frac{1}{n+1}}\int_{y_2=\frac{j+(n-j)y_1}{j+1}}^{\frac{n-1+y_1}{n}}\left[-j-(n-j)y_1+(j+1)y_2\right]^{n-2}dy_2dy_1$$

$$=\frac{n!}{(n-2)!}\sum_{j=0}^{n-2}\frac{(-1)^j(n+1)!}{j!(n-1-j)!}\int_{y_1=a}^{\frac{1}{n+1}}\frac{\left[-j-(n-j)y_1+(j+1)\frac{(n-1)+y_1}{n}\right]^{n-1}}{(n-1)(j+1)}dy_1$$

$$=\sum_{j=0}^{n-2}\frac{(-1)^{j+1}(n+1)!\left[\frac{n-j-1}{n}(1-(1+n)a)\right]^n}{j!(n-j-1)!(j+1)(\frac{j+1}{n}-(n-j))}$$

$$= \sum_{j=0}^{n-2} \frac{(-1)^{j+1}\binom{n}{j+1}\left[\frac{n-j-1}{n}\right]^n (1-(1+n)a)^n}{\frac{j+1-n}{n}}$$

$$= (1-(1+n)a)^n \sum_{j=0}^{n-2} (-1)^j \binom{n}{j+1}\left(\frac{n-j-1}{n}\right)^{n-1}$$

$$= (1-(1+n)a)^n \frac{1}{n^{n-1}} \sum_{j=1}^{n-1} (-1)^{j-1} \binom{n}{j} (n-j)^{n-1}$$

$$= (1-(1+n)a)^n \quad \text{by using Lemma 3.5} .$$

However if $(a,1-b)$ is in the convex hull of $\left((0,0), (0,\frac{n-1}{n}), (\frac{1}{n+1}, \frac{n}{n+1})\right)$, above the lines from $(0,\frac{j}{j+1})$ to $(\frac{1}{n+1}, \frac{n}{n+1})$, for $j = 0,\ldots,k$ and below for $j = k+1,\ldots,n-1$, then $\Pr(y_1 > a, y_2 > 1-c)$ exceeds $\Pr(y_1 > a, y_2 > 1-b)$ by

$$\sum_{j=0}^{k} \frac{n!(-1)^j(n+1)!}{(n-2)!j!(n-1-j)!} \int_{y_1=a}^{\frac{(j+1)(1-b)-j}{n-j}} \int_{y_2=\frac{j+(n-j)y_1}{j+1}}^{1-b} [-j-(n-j)y_1+(j+1)y_2]^{n-2}\, dy_2 dy_1$$

$$= \sum_{j=0}^{k} \frac{n!(-1)^j(n+1)!}{(n-2)!j!(n-1-j)!} \int_{y_1=a}^{\frac{(j+1)(1-b)-j}{n-j}} \frac{[-j-(n-j)y_1+(j+1)(1-b)]^{n-1}}{(n-1)(j+1)}\, dy_1$$

$$= \sum_{j=0}^{k} (-1)^j \binom{n+1}{j+1}[1-(n-j)a-(j+1)b]^n .$$

Therefore

$$\Pr(y_1 > a, y_2 > 1-b)$$

$$= (1-(n+1)a)^n - \sum_{j=0}^{k} (-1)^j \binom{n+1}{j+1} [1-(n-j)a-(j+1)b]^n$$

$$= (1-(1+n)a)^n - \sum_{j=1}^{k+1} (-1)^{j+1} \binom{n+1}{j} [1-(n+1-j)a-jb]^n$$

$$= \sum_{j=0}^{k+1} (-1)^j \binom{n+1}{j} (1 - (n+1-j)a - jb)^n$$

$$= \sum_{j=0}^{n-1} \lambda''_j (-1)^j \binom{n+1}{j} (1 - (n+1-j)a - jb)^n$$

where λ''_j is as defined in the statement of the theorem.

If $(a, 1 - b)$ is above the line from $(0, \frac{n-1}{n})$ to $(\frac{1}{n+1}, \frac{n}{n+1})$, the above derivation still holds. That is, even though the ranges of integration would be different and an adjustment should be made, it happens that $\lambda'_j = 1$ for all $j=0,1,\ldots,n-1$ and the integrand for the part above the line joining $(0, \frac{n-1}{n})$ to $(\frac{1}{n+1}, \frac{n}{n+1})$ is

$$\frac{n!}{(n-2)!} \sum_{j=0}^{n-1} (-1)^j \frac{(n+1)!}{j!(n-1-j)!} [-j-(n-j)y_1 + (j+1)y_2]^{n-2} = 0$$

by Lemma 3.1 . Thus the adjustment would be zero anyway. Therefore

$$\Pr(u_{(1)} > a, u_{(n+1)} < b)$$

$$= \sum_{j=0}^{n-1} \lambda''_j (-1)^j \binom{n+1}{j} (1 - (n+1-j)a - jb)^n$$

for $0 < a < \frac{1}{n+1}$, $0 < 1 - b < \frac{n}{n+1}$, where λ''_j is as defined in the statement of the theorem.

This result was obtained by Darling (1953) in a different way. His method was to invert the characteristic function of the random variable $N_{n+1}(a,b)$ which was defined to be the number of the $\left\{u_i : u_i \geq 0,\ i = 1,\ldots,n+1;\ \sum_{i=1}^{n+1} u_i = 1\right\}$ satisfying $a < u_i < b$, and the required probability was found by setting $N_{n+1}(a,b) = n+1$.

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