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RELATIVE ADJOINTNESS AND PRESERVATION OF NON-EXISTING LIMITS

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ABSTRACT

Triples and the categories of triple algebras are relativized by a full faithful functors. The Tripleability Theorem in [1] is correspondingly relativized. The concept of the rank of a triple becomes intrinsic in this setting.

Preservation of non-existing limits is interpreted in terms of limit-colimit commutation property. This is used to account for the usual description of the category of algebras as the category of all product preserving set-valued functors on the opposite category of free algebras.

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I dedicate this work to my parents.

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INTRODUCTION

There are two known categorical approaches to the study of algebras. The first is an approach, in which the operations of algebras play the pivotal role. Not only just the generic operations - e.g. multiplication and identity in the case of monoid - but also all derived operations are considered. This approach was initiated notably by W. Lawvere [7] among others.

The second approach, which is referred to as the triple algebraic approach in the following, is that which is based on the adjointness situation between the categories of algebras and the category of sets. It was observed that an adjointness gives rise to a triple and conversely, a triple determines two, the largest and the smallest adjointness situations, called the Eilenberg-Moore Situation and the Kleisli Situation, respectively. They represent the category of all algebras and the category of all free algebras, respectively. (see [1])

The main difference between these two approaches lies in the consideration of the rank, i.e. the smallest regular cardinal greater than the arities of the operations of the algebras of the type under consideration. In the first approach a consideration of the rank is intrinsically included; in the second such is conspicuously ignored. Consequently, the category of compact Hausdorff spaces, for instance, is "algebraic" in the second sense but obviously not so in the first sense.

This work proposes one way that would somewhat reconcile the difference by refining the triple algebraic approach. This work is done by considering relative adjointness situations instead of adjointness situations. Moreover, it is noteworthy that the proposed way is not only a refinement but also a generalization of the triple algebraic theory, in so far as the relative adjointness is a generalization of the adjointness.

After having the above reconciliation between the two different approaches, the description of the category of algebras in the first approach, namely, as the category of all product preserving set-valued functors on the opposite category of free algebras, is justified in the triple algebraic sense.

Since in the arbitrary setting as is studied in this work existence of limit or colimit is not known, an appropriate modification of limit preservance of functors for non-existing limits is studied.

A word on the way the chapters and sections are referred is in order. The number preceding a colon refers to the number of the chapter, whereas the number immediately following the colon or the first number when there is no colon refers to the section number. Therefore 3:1.3 means the third statement in the Section 1 of the Chapter 3, while 2.5 means the fifth statement in the Section 2 of the same chapter.

Chapter 0

PRELIMINARIES

In this chapter we will review the basic definitions and some consequences that are needed in the later chapters.

Section 1: Categories of Functors and Yoneda Embeddings.

1.1 Preliminary Remarks.

1.1.0 In general, the collection of all functors from a category to another does not form a category. It fails to be a category only because the collection of all natural transformations from a functor to another may not be a set. As a foundation of the legitimate formulation of functor categories three possibilities exist:

1.1.1 One uses the set theory of von Neumann-Bernays-Gödel as a basis. The fundamental concept here is that of a "class." Sets are those classes which are elements of classes.

Small categories are those categories whose object classes are sets, equivalently those categories whose morphism classes are sets.

In this situation only those functor categories with small domain categories are legitimate. The set theory of von Neumann-Bernays-Gödel as a foundation does not permit to consider functor categories with arbitrary domain categories.

1.1.2 Instead of an axiomatic theory of sets as a basis, we could use an axiomatic theory of the category of categories which encompasses set theory as the theory of categories which encompasses set theory as the theory of discrete categories.

The formulation of functor categories is given as exponentiation [7]. For our purpose this approach is unnecessarily sophiscated.

1.1.3 One expands the set theory of Zermalo-Fraenkel by introducing universes as suggested by Grothendieck; i.e. one admits suitable inaccessible cardinals. Accounts of this approach could be found in [3]. We shall do no more than point out a few facts which will suffice for a formulation of functor categories.

1.2 <u>Universes</u>.

1.2.1 A universe is a non-empty set $\mathcal{U}$ subject to the following conditions:

(1) If $A \in \mathcal{U}$ and $B \in A$ then $B \in \mathcal{U}$.

(2) If $A, B \in \mathcal{U}$, then $\{A, B\} \in \mathcal{U}$.

(3) If $A \in \mathcal{U}$, then the power set $\mathcal{P}(A) \in \mathcal{U}$.

(4) If $\{A_i \mid i \in I \in \mathcal{U}\}$ is a family of elements of $\mathcal{U}$. then $\bigcup_{i \in I} A_i \in \mathcal{U}$.

1.2.2 From these axioms one can easily deduce the following properties:

-If $A \in \mathcal{U}$, then $\{A\} \in \mathcal{U}$.

-If $A \subset B \in \mathcal{U}$, then $A \in \mathcal{U}$.

-If $A, B \in \mathcal{U}$, the couple $(A, B) = \{\{A, B\}, A\}$ is an element of $\mathcal{U}$.

-If $A, B \in \mathcal{U}$, the union $A \cup B$ and the product $A \times B$ are elements of $\mathcal{U}$.

-If $\{A_i \mid i\in I\in\mathcal{U}\}$ is a family of elements of $\mathcal{U}$ then the product $\prod_{i\in I} A_i$ is an element of $\mathcal{U}$.

-If $A\in\mathcal{U}$, then card(A) < card($\mathcal{U}$). In particular the relation $\mathcal{U}\in\mathcal{U}$ can not be true.

In short: $\mathcal{U}$ is closed under the usual constructions of set theory carried out on the elements of $\mathcal{U}$.

1.2.3 An example of a universe is the set of all symbols of type $\{\{\phi\},\{\{\phi\},\phi\},$ etc.$\}$ where every element of this universe is a finite set and this universe is countable.

1.2.4 We require as an axiom that every set is an element of a universe. Thus in particular every universe is an element of a higher universe.

1.3 <u>$\mathcal{U}$-categories</u>.

1.3.1 In the following we fix a universe $\mathcal{U}$ containing an element of infinite cardinality, for instance the set $\mathbb{N}$ of natural numbers (and therefore also containing $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$). We make use of universe, but we choose a language which would allow us to a large extent to use the language of the set theory of von Nermann-Bernays-Gödel.

1.3.2 A <u>$\mathcal{U}$-small set</u> is a set belonging to $\mathcal{U}$. Subsets of $\mathcal{U}$ are called <u>$\mathcal{U}$-classes</u>. Whenever there is no risk of confusion, we usually drop the prefix $\mathcal{U}$.

1.3.3 A <u>category</u> (more precisely a <u>$\mathcal{U}$-category</u>) consists of $\mathcal{U}$-class Mor $\mathbb{A}$, and a composition rule which is a partially defined associative binary operation with left and right identities for each element. In particular the composition determines the class of identity morphism of $\mathbb{A}$, denoted by Ob $\mathbb{A}$, and the partitioning of Mor $\mathbb{A}$ into the classes $Hom_{\mathbb{A}}(A,B)$ of all elements of Mor $\mathbb{A}$ with A as the right identity and B as the left identity, which are required to be $\mathfrak{U}$-sets.

$Hom_{\mathbb{A}}(A,B)$ is often abbreviated as $[A,B]$ when there is no risk of confusion of the category under consideration.

A category is <u>$\mathfrak{U}$-small</u> if Ob $\mathbb{A}$ is a $\mathfrak{U}$-set.

1.3.4 Let $\mathbb{A}$ be a $\mathfrak{U}$-category. Let $\mathfrak{V}$ be a universe containing $\mathfrak{U}$. Then in particular $\mathbb{A}$ is a small $\mathfrak{V}$-category. For any $\mathfrak{V}$-category $\mathbb{B}$, the functors $\mathbb{A} \to \mathbb{B}$ and the natural transformations between them form a $\mathfrak{V}$-category. The composition of the morphisms is that of the natural transformations. This category is denoted by $[\mathbb{A}, \mathbb{B}]$. If $\mathbb{B}$ is also $\mathfrak{V}$-small, then $[\mathbb{A}, \mathbb{B}]$ is $\mathfrak{V}$-small.

In particular if $\mathbb{A}$ is a $\mathfrak{U}$-small category, $[\mathbb{A}, \mathbb{B}]$ is a $\mathfrak{U}$-category for every $\mathfrak{U}$-category $\mathbb{B}$. In general for any two categories we could then legetimately consider the functor category in an appropriately chosen universe.

1.4 <u>The Yoneda Embedding</u>

1.4.1 We write Ens for the category of all ($\mathfrak{U}$-) small sets. In view of 0.3.4 we have the category of all contravariant functors from a category $\mathbb{A}$ to Ens. We denote this category by $\widehat{\mathbb{A}}$, which is often called the <u>category of presheaves of sets on $\mathbb{A}$</u>. When $\mathbb{A}$ is a small $\mathcal{U}$-category, the category $\widehat{\mathbb{A}}$ is a $\mathcal{U}$-category. When $\mathbb{A}$ is a $\mathcal{U}$-category the category $\widehat{\mathbb{A}}$ is not in general a $\mathcal{U}$-category, but a $\mathcal{V}$-category for a universe $\mathcal{V}$ containing $\mathcal{U}$. In either case by choosing the universe appropriately, we could legetimately consider the categories of presheaves of sets.

1.4.2 For a $\mathcal{U}$-category $\mathbb{A}$, and for every $A \in \mathrm{Ob}\,\mathbb{A}$, we define a contravariant functor $[-,A] : \mathbb{A}^{\circ} \longrightarrow \mathrm{Ens}$ by the rule $B \rightsquigarrow [B,A] = \mathrm{Hom}_{\mathbb{A}}(B,A)$ and for $g: B \longrightarrow B'$, $[g,A]$ is given by composing with g on the right.

Given $f: A \longrightarrow A'$ in $\mathbb{A}$, we have a natural transformation $[-,f] : [-,A] \longrightarrow [-,A']$ where for $B \in \mathrm{Ob}\,\mathbb{A}$, $[B,f]$ is given by composing with f on the left.

For $A \in \mathrm{Ob}\,\mathbb{A}$, $f \in \mathrm{Mor}\,\mathbb{A}$, the rule $A \rightsquigarrow [-,A]$ and $f \rightsquigarrow [-,f]$ defines a full embedding $h_{*} : \mathbb{A} \longrightarrow \widehat{\mathbb{A}}$ called the <u>Yoneda Embedding</u>, or often denoted simply by h.

Therefore every category may be regarded as a full subcategory of a category of presheaves of sets.

1.4.3 If a $\mathcal{U}$-category $\mathbb{A}$ is not a small $\mathcal{U}$-category, $\widehat{\mathbb{A}}$ is no longer a $\mathcal{U}$-category.. Therefore in general we could not formulate the Yoneda Embedding within the universe $\mathcal{U}$. Noting that the Yoneda Embedding is a representation of objects of a $\mathcal{U}$-categoy $\mathbb{A}$ as set valued functors, we restrict ourselves to a subcategory of

$\hat{\mathbb{A}}$, consisting of all proper presheaves of sets on $\mathbb{A}$.

Let $\mathbb{A}$ be a $\mathfrak{U}$-category. A set valued functor $T: \mathbb{A} \longrightarrow$ Ens is called <u>proper</u> (more precisely <u>$\mathfrak{U}$-proper</u>), if there exists a $\mathfrak{U}$-small set $\mathfrak{D}$ of objects with the following property:

For every $A \in \mathrm{Ob}\,\mathbb{A}$, and for every $a \in T(A)$, there exists a suitable D in $\mathfrak{D}$ and $d \in T(D)$ together with a morphism $f: D \longrightarrow A$ such that $T(f)(D)=a$.

Such a set $\mathfrak{D}$ is called a <u>dominating set</u> for T.

It is then straightforward to see that the category of all proper presheaves of sets on a $\mathfrak{U}$-category $\mathbb{A}$ is again a $\mathfrak{U}$-category.

<u>Section 2</u>: <u>Adjoint Situations and Triples</u>.

In this section we recall the definitions and fundamental properties of triple algebras, thereby providing the ground work of Chapter 2.

2.1. let $J: \mathbb{C} \longrightarrow \mathbb{A}$, $U: \mathbb{X} \longrightarrow \mathbb{A}$ and $F: \mathbb{C} \longrightarrow \mathbb{X}$ be three functors. F is said to be <u>relative left adjoint to U with respect to J</u> or simply <u>J-relative left adjoint</u>, if there exists a natural equivalence α, which is called the <u>adjunction isomorphism</u>,

$$\alpha_{C,X}: [FC, X] \xrightarrow{\sim} [JC, UX]$$

where $C \in \mathrm{Ob}\,\mathbb{C}$ and $X \in \mathrm{Ob}\,\mathbb{X}$.

We sometimes write this situation in symbols as

$$F \dashv U \bmod J$$

If in particular $J=Id_{\mathbb{A}}$, then we simply say that F is a <u>left adjoint to</u> U, which case is written in symbol $F \dashv U$.

2.2 Let F be a left adjoint to U with the adjunction isomorphism α. We let $\eta_A = \alpha_{A,FA}(1_{FA})$ and $\varepsilon_X = \alpha^{-1}_{UX,X}(1_{UX})$ for every $A \in Ob\,\mathbb{A}$ and $X \in Ob\,\mathbb{X}$, respectively. Then $\eta = \{\eta_A\}$, $\varepsilon = \{\varepsilon_X\}$ define natural transformations; $\eta : Id_{\mathbb{A}} \longrightarrow UF$ and $\varepsilon : FU \longrightarrow Id_{\mathbb{X}}$ respectively called the <u>front</u> and the <u>back</u> <u>adjunctions</u>.

Let $T=UF$ and $\mu=U\varepsilon F$. Then the 3-tuple(T,η,μ) satisfies

(1) $\mu(\eta T) = \mu(T\eta) = 1_T$, and

(2) $\mu(\mu T) = \mu(T\mu)$.

2.3 Let $\mathbb{A}$ be a category. A <u>triple</u> (<u>monad</u>, or <u>triad</u>) $\mathbb{T} = (T,\eta,\mu)$ over $\mathbb{A}$ is an endofunctor $T : \mathbb{A} \longrightarrow \mathbb{A}$ with natural transformations $\eta : 1 \longrightarrow T$ and $\mu : TT \longrightarrow T$ which satisfies (1) and (2) of (2.2).

2.4 Let $\mathbb{A}$ be a category and $\mathbb{T} = (T,\eta,\mu)$ a triple over $\mathbb{A}$. A <u>$\mathbb{T}$-algebra</u> is a pair (A,a) with $A \in Ob\,\mathbb{A}$ and $a : TA \longrightarrow A$ such that

(1) $a\eta_A = 1_A$, and

(2) $a\mu_A = aT(a)$.

A is called the <u>carrier</u> or the <u>underlying object</u> in of the algebra (A,a), and a is called the <u>structure map</u>.

A $\mathbb{T}$-homomorphism $(A,a) \longrightarrow (B,b)$ is a 3-tuple (a, f,b) where f is an $\mathbb{A}$-morphism $A \longrightarrow B$, making the following square commute:

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ a \downarrow & & \downarrow b \\ A & \xrightarrow[f]{} & B \end{array}$$

f is called the underlying $\mathbb{A}$-morphism of the homomorphism (a,f,b). We usually drop the reference to the structure maps.

We have the category $\mathbb{A}^{\mathbb{T}}$ of $\mathbb{T}$-algebras and $\mathbb{T}$-homomorphisms.

2.5 Let $\mathbb{T}=(T,\eta,\mu)$ be a triple over the category . Then there is the forgetful functor $U^{\mathbb{T}}: \mathbb{A}^{\mathbb{T}} \longrightarrow \mathbb{A}$ which assigns to each algebra its carrier and to each homomorphism the underlying $\mathbb{A}$-morphism.

The forgetful functor $U^{\mathbb{T}}$ is faithful.

We define the free functor $F^{\mathbb{T}}: \mathbb{A} \longrightarrow \mathbb{A}^{\mathbb{T}}$ by the rule $A \leadsto (TA, \mu_A)$ and $f \leadsto T(f)$ where $f: A \longrightarrow B$ in $\mathbb{A}$.

We note that $F^{\mathbb{T}}$ is a left adjoint to $U^{\mathbb{T}}$ and $U^{\mathbb{T}}F^{\mathbb{T}} = T$. Moreover the triple induced by the adjointness situation $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ is the same as the triple we started with.

We call the category $\mathbb{A}^{\mathbb{T}}$ the Eilenberg-Moore category corresponding to the triple $\mathbb{T}$, and the adjointness situation $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ the Eilenberg-Moore situation corresponding to the triple $\mathbb{T}$.

2.6 Let $F: \mathbb{A} \rightarrow \mathbb{X}$ be a functor. The full image of F is the category $\mathbb{X}_F$ whose objects are those of $\mathbb{A}$ and whose morphism sets $\mathrm{Hom}_{\mathbb{X}_F}(A,B)$ are precisely $\mathrm{Hom}_{\mathbb{X}}(FA,FB)$.

There exist functors $\mathrm{clF}: \mathbb{A} \longrightarrow \mathbb{X}_F$, $\mathrm{fimF}: \mathbb{X}_F \longrightarrow \mathbb{X}$ so that the following holds:

(1) $\mathrm{fimF}\cdot\mathrm{ClF} = F$,

(2) clF is bijective on objects,

(3) fimF is full faithful.

2.7 In (2.6), we have a factorization of a functor in the category Cat of categories. The factorization is not an epi-mono factorization, but comes very close to it.

More precisely, fimF satisfies the following: Given any two functors $G,\ H: \mathbb{Y} \longrightarrow \mathbb{X}_F$, if $(\mathrm{fimF})\cdot G$ is naturally equivalent to $(\mathrm{fimF})\cdot H$, then G is naturally equivalent to H. And clF satisfies the following: Given any two functors $L,\ M: \mathbb{X}_F \longrightarrow \mathbb{Z}$, if $L\cdot(\mathrm{clF})$ is naturally equivalent to $M\cdot(\mathrm{clF})$ then L is naturally equivalent to M.

In other words they satisfy the definitions of monomorphism and epimorphism in Cat in which the equality is replaced by the natural equivalence. This observation constitutes the ground for calling them 2-monomorphism and 2-epimorphism respectively in the 2-category Cat. (See for instance [4] for the concept of 2-ness)

2.8 Let $\mathbb{T} = (T, \eta, \mu)$ be a triple over $\mathbb{A}$, and $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ the Eilenberg-Moore situation corresponding to the triple $\mathbb{T}$. Let $F_{\mathbb{T}} = \mathrm{clF}^{\mathbb{T}}$ and $U_{\mathbb{T}} = U^{\mathbb{T}} \cdot \mathrm{fimF}^{\mathbb{T}}$. We remark that there is an adjointness situation $F_{\mathbb{T}} \dashv U_{\mathbb{T}}$, which gives rise to the same triple $\mathbb{T}$.

We call the category $(\mathbb{A}^{\mathbb{T}})_{F^{\mathbb{T}}}$ (see 2.6), for short $\mathbb{A}_{\mathbb{T}}$, the Kleisli category corresponding to the triple $\mathbb{T}$; and the adjointness situation $F_{\mathbb{T}} \dashv U_{\mathbb{T}}$ the Kleisli situation corresponding to the triple $\mathbb{T}$.

2.9 The Kleisli category $\mathbb{A}_{\mathbb{T}}$ corresponding to a triple $\mathbb{T}=(T,\eta,\mu)$ could be described in a more direct manner:

The objects are the same as those of the category $\mathbb{A}$. For every A and A' in $\mathbb{A}$, $\mathrm{Hom}_{\mathbb{A}_{\mathbb{T}}}(A,A')$ consists of all $\mathbb{A}$-morphisms $f: TA \longrightarrow TA'$ such that

$$\begin{array}{ccc} TTA & \xrightarrow{\quad Tf \quad} & TTA' \\ \mu_A \big\downarrow & & \big\downarrow \mu_B \\ TA & \xrightarrow{\quad f \quad} & TA' \end{array}$$

commutes. The composition is the one induced from the category $\mathbb{A}$.

In this description, $U_{\mathbb{T}}$ is determined by the rule $A \rightsquigarrow TA$ and $f \rightsquigarrow f$. $F_{\mathbb{T}}$ is determined by $A \rightsquigarrow A$ and $h \rightsquigarrow Th$.

2.10 Let $F \dashv U$ be an adjointness situation with front adjunction η and back adjunction ε , where $F: \mathbb{A} \longrightarrow \mathbb{X}$, and $U: \mathbb{X} \longrightarrow \mathbb{A}$. This adjointness situation gives rise to a triple $\mathbb{T}=(T,\eta,\mu)$ where $T=UF$ and $\mu=U\varepsilon F$ (see 2.2), which in turn gives rise to two adjointness situations, namely the Eilenberg-Moore situation and the Kleisli situation.

There exist two functors $N: \mathbb{A}_{\mathbb{T}} \to \mathbb{X}$ and $K: \mathbb{X} \to \mathbb{A}^{\mathbb{T}}$, both of which are called <u>comparison functors</u>.

N is defined by the rule $A \rightsquigarrow FA$ and $f: FA \longrightarrow FB$ $\rightsquigarrow FA \xrightarrow[F\eta_A]{} FUFA \xrightarrow[Ff]{} FUFB \xrightarrow[\varepsilon_{FB}]{} FB$.

K is defined by the rule $X \rightsquigarrow (UX, U\varepsilon_X)$ and $g: X \longrightarrow Y$ $\rightsquigarrow Ug$.

2.11 We need some notions about functors relative to diagrams. Let $U: \mathbb{X} \longrightarrow \mathbb{A}$ be a functor. Let $\mathcal{Q}$ be a categorical property of diagrams (e.g. monomorphism, limits, etc.). Assume that with every diagram D in $\mathbb{X}$, i.e. a functor D into the category $\mathbb{X}$ with a (small) domain category, with property $\mathcal{Q}$, the dagram $U \cdot D$ in $\mathbb{A}$ also has the property $\mathcal{Q}$. In this case one says that U <u>preserves the property $\mathcal{Q}$</u>. Assume that each diagram D in $\mathbb{X}$ for which the diagram $U \cdot D$ in $\mathbb{A}$ has the property $\mathcal{Q}$ has itself the property $\mathcal{Q}$, then we say that U <u>reflects the property $\mathcal{Q}$</u>.

We say U <u>creates colimits for a diagram $D: \mathbb{I} \longrightarrow \mathbb{X}$</u>, if the followings are satisfied:

(1) there exists a colimit of $U \cdot D$ in $\mathbb{A}$, say $\lambda_i: UD(i) \longrightarrow A$,

(2) there exists exactly one pair (X, σ) consisting of an object X of $\mathbb{X}$ and a cone $\sigma_i: D(i) \longrightarrow X$ such that $UX = A$ and $U\sigma_i = \lambda_i$, and lastly

(3) this cone $\sigma_i: D(i) \longrightarrow X$ is itself a colimit cone in $\mathbb{X}$.

We could similarly define the <u>limit-creation property</u>.

U is said to <u>create isomorphism</u>, if given any isomorphism $f: U(X) \longrightarrow B$ in $\mathbb{A}$ for arbitray X of $\mathbb{X}$, there exists exactly one morphism v with domain X such that v is an isomorphism and $U(v)=f$.

2.12 Let $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ be the Eilenberg-Moore situation corresponding to the triple $\mathbb{T}=(T,\eta,\mu)$ over the category and $D: \mathbb{I} \longrightarrow \mathbb{A}^{\mathbb{T}}$ a digram. The following is always true:

(1) $U^{\mathbb{T}}$ creats limits, in particular isomorphisms,

(2) If $U^{\mathbb{T}} \cdot D$ has a colimit which is preserved by T and by TT, then $U^{\mathbb{T}}$ creates colimits of D.

2.13 A <u>fork</u> is a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{r} C$$

with rf=rg. A fork <u>splits</u> if there are morphisms

$$A \xleftarrow{j} B \xleftarrow{i} C$$

such that $ri=1_C$, $fi=1_B$, and $gj=ir$.

Let $U: \mathbb{X} \longrightarrow \mathbb{A}$ be a functor and $u, v: X \longrightarrow Y$ a pair of morphisma in $\mathbb{X}$. We say that the pair is <u>split by U</u>, if U(u) and U(v) can be completed to a split fork.

2.14 As an answer to the question when the comparison functor is equivalence or isomorphism, we have the theorem due to J. Beck [1].

Let the adjoint situation $F \dashv U$ generate the triple $\mathbb{T}=(T,\eta,\mu)$ where $U: \mathbb{X} \longrightarrow \mathbb{A}$, and let $N: \mathbb{A}_{\mathbb{T}} \longrightarrow \mathbb{X}$, and $K: \mathbb{X}$ -

$\to \mathbb{A}^{\mathbb{T}}$ be the comparison functors. We consider the pairs of morphisms in $\mathbb{X}$ which are split by U. Then the following holds:

(1) N is dense if and only if K is full faithful if, and only if U reflects coequalizers of these pairs of morphisms.

(2) K is an equivalence of categories if and only if there are coequalizers of the pairs of morpnisms considered in $\mathbb{X}$ and if U preserves and reflects these coequalizers.

(3) K is isomorphism of categories if and only if U creates coequalizers of these pairs of morphisms.

2.15 Lastly we give a characterization of the Eilenberg-Moore categories as functor categories. [8]

Let $\mathbb{T}=(T,\eta,\mu)$ be a triple over a category $\mathbb{A}$. The following square is a pullback square in the category of categories (within the framework of a suitably chosen universe).

$$\begin{array}{ccc} \mathbb{A}^{\mathbb{T}} & \xrightarrow{\quad P \quad} & [\mathbb{A}_{\mathbb{T}}^{o}, Ens] \\ {\scriptstyle U^{\mathbb{T}}}\downarrow & & \downarrow{\scriptstyle [F_{\mathbb{T}}^{o},\ Ens]} \\ \mathbb{A} & \xrightarrow[\quad h \quad]{} & [\mathbb{A}^{o}, Ens] \end{array}$$

where for the definitions of $U^{\mathbb{T}}$ and $F_{\mathbb{T}}$ see (2.5) and (2.8) respectively; h is the Yoneda Embedding; and P is the associated functor (more precisely a left Kan extension) to $\text{fim}F^{\mathbb{T}}: \mathbb{A}_{\mathbb{T}} \longrightarrow \mathbb{A}^{\mathbb{T}}$, see (2.8).

Section 3: Kan Extensions

In this section we review some rudimentary properties of the Kan Extension which are extensively used in the sequel. For details see for instance [2].

3.1 Let $J:\mathbb{C} \longrightarrow \mathbb{A}$ and $F:\mathbb{C} \longrightarrow \mathbb{B}$ be functors. A functor $\mathrm{Lan}_J(F):\mathbb{A} \longrightarrow \mathbb{B}$ together with a natural transformation η_F: $F \longrightarrow \mathrm{Lan}_J(F)\cdot J$ is called <u>left Kan Extension of F along J</u>, if for each functor $T:\mathbb{A} \longrightarrow \mathbb{B}$ the map

$$[\mathrm{Lan}_J(F),\ T] \longrightarrow [F,\ TJ],\ \psi \rightsquigarrow (\psi J)\cdot\eta_F$$

is bijective.

The pair $(\mathrm{Lan}_J(F), \eta_F)$ is up to isomorphism uniquely determined. It is obvious that the left Kan extension $\mathrm{Lan}_{J'\cdot J}(F)$ exists, if $\mathrm{Lan}_{J'}(\mathrm{Lan}_J(F))$ exists and they are isomorphic.

3.2 Let $J:\mathbb{C} \longrightarrow \mathbb{A}$ be a functor. For any category $\mathbb{B}$,

$$[J,\mathbb{B}] : [\mathbb{A},\mathbb{B}] \longrightarrow [\mathbb{C},\mathbb{B}]$$

is a functor. Suppose $[J,\mathbb{B}]$ has a left adjoint which we designate by Lan_J. Then the left Kan extension $(\mathrm{Lan}_J(F), \eta_F)$ is precisely the front adjunction at F in the category $[\mathbb{C},\mathbb{B}]$.

3.3 We give two very important examples:

(1) Let $J:\mathbb{C} \longrightarrow \mathbb{A}$ a functor, and $C \in \mathrm{Ob}\,\mathbb{C}$. In view

of the Yoneda Lemma, we have

$$[[JC,-],T] \xrightarrow{\sim} TJC \xrightarrow{\sim} [[C,-],TJ]$$

for every functor $T: \mathbb{A} \longrightarrow \mathrm{Ens}$. Hence $\mathrm{Lan}_J[C,-] = [JC,-]$ and $\eta_{[C,-]}: [C,-] \longrightarrow [JC,J-]$ is the map induced by the functor J.

(2) Let $J: \mathbb{C} \longrightarrow \mathbb{A}$ a functor, and let $h: \mathbb{C} \longrightarrow \hat{\mathbb{C}}$ be the Yoneda Embedding. We claim $\mathrm{Lan}_J(h)(A) = [J-,A]$ for every $A \in \mathrm{Ob}\,\mathbb{A}$, and $\eta_h: h \longrightarrow \mathrm{Lan}_J(h) \cdot J$ is given at $C \in \mathrm{Ob}\,\mathbb{C}$, by $\eta_{h,C}: [-,C] \longrightarrow [J-,JC]$ which is induced by the functor J.

3.4 The Kan construction gives us a case when left Kan extensions exist.

Let $J: \mathbb{C} \longrightarrow \mathbb{A}$ and $F: \mathbb{C} \longrightarrow \mathbb{B}$ be functors. For any object A in $\mathbb{A}$, we define the (Lawvere) comma category associated to A: objects are pairs (C, ξ) where $C \in \mathrm{Ob}\,\mathbb{C}$ and $\xi: JC \longrightarrow A$ in $\mathbb{A}$, and morphisms are $\mathbb{C}$-morphisms $f: (C,\xi) \longrightarrow (C',\xi')$, where $f: C \longrightarrow C'$ in $\mathbb{C}$ satisfies $\xi' \cdot Jf = \xi$. We denote this category by J/A. We define $J_A: J/A \longrightarrow \mathbb{C}$ by $(C,\xi) \rightsquigarrow C$ and $f \rightsquigarrow f$.

If $\varinjlim FJ_A$ exists in $\mathbb{B}$ for every $A \in \mathrm{Ob}\,\mathbb{A}$, then there exists $\mathrm{Lan}_J(F)$ and $\mathrm{Lan}_J(F)(A) = \varinjlim FJ_A$.

3.5 We have another interpretation of left Kan extensions. Let $J: \mathbb{C} \longrightarrow \mathbb{A}$ and $F: \mathbb{C} \longrightarrow \mathbb{B}$ be functors. As seen in (3.3) we always have $\mathrm{Lan}_J(h)$ and $\mathrm{Lan}_F(h)$ where $h: \mathbb{C} \longrightarrow \hat{\mathbb{C}}$ is the Yoneda Embedding. Then $\mathrm{Lan}_J(F)$ is precisely a relative left adjoint to $\mathrm{Lan}_F(h)$ with respect to $\mathrm{Lan}_J(h)$.

3.6 We recall a very useful proposition concerning relative adjointness:

Let $P: \mathbb{M}' \longrightarrow \mathbb{M}$, $S: \mathbb{M}' \longrightarrow \mathbb{N}$ and $R: \mathbb{N} \longrightarrow \mathbb{M}$ be functors such that S is P-relative left adjoint to R (see 2.1). Let $D: \mathbb{I} \longrightarrow \mathbb{M}'$ be any diagram with a colimit $\delta_i: D(i) \longrightarrow L$. If P preserves the colimit $\{\delta_i\}$, then so does S.

In other words, S preserves all colimits that are preserved by P.

We could claim more:

In the same situation as above. Let $\alpha_i: D(i) \longrightarrow A$ be a cone. If P transforms $\{\alpha_i\}$ into a colimit cone, then so does S.

As an application of the above, we consider 3.5, thereby concluding that $Lan_J(F)$ preserves all colimits that are preserved by $Lan_J(h)$.

3.7 Let $G: \mathbb{B} \longrightarrow \mathbb{X}$ and $I: \mathbb{C} \longrightarrow \mathbb{X}$ be two additional functors in the same situation as in 3.1. Suppose that $Lan_J(I)$ and $Lan_J(F)$ exist. If F is I-relative left adjoint to G, then $Lan_J(F)$ is $Lan_J(I)$-relative left adjoint to G.

Section 4: Path Categories

A diagram scheme Σ consists of two sets Vx and Ar and two maps $o, e: Ar \longrightarrow Vx$. The elements of Vx are called vertices and those of Ar arrows; for $a \in Ar$, $o(a)$ is called the origin and $e(a)$ the end of a. We say that a is an arrow from $o(a)$ to $e(a)$.

If $\mathbb{C}$ is a small category, we obviously have the underlying diagram scheme of the category $\mathbb{C}$ by forgetting the composition of $\mathbb{C}$.

4.2 A <u>diagram</u> D in a category $\mathbb{C}$ <u>of type</u> Σ consists of two maps $\mathit{Vt} \longrightarrow \mathrm{Ob}\,\mathbb{C}$, and $\mathit{Ar} \longrightarrow \mathrm{Mor}\,\mathbb{C}$, both of which are written by D, such that for any $a \in \mathit{Ar}$, D(o(a)) is the domain of D(a) and D(e(a)) is the codomain of D(a).

A natural transformation between diagrams of type Σ in $\mathbb{C}$ is defined by tramsforming the definition of the natural transformation between functors in the obvious way. One obtains a category $[\Sigma, \mathbb{C}]$ which is analoguous to a functor category.

4.3 A <u>path</u> w in a diagram scheme Σ is a finite sequence of arrows $a_1, a_2, \ldots, a_n$ such that $e(a_i)=o(a_{i+1})$ for $i=1,2,\ldots,n$ (n 1) is called the <u>length</u> of w. For such a path we write $w=a_n a_{n-1}\cdots a_2 a_1$ and define $o(w)=o(a_1)$ as the <u>origin</u> and $e(w)=e(a_n)$ as the <u>end</u>.

4.4 There is an obvious <u>composition of paths</u>. If $w=a_n a_{n-1}\cdots a_1$ and $v=b_m b_{m-1}\cdots b_1$ are two paths with $e(w)=o(v)$, then $b_m b_{m-1}\cdots b_1 a_n a_{n-1}\cdots a_1$ is again a path which we denote by vw and read v following w. Obviously this composition of paths is associative.

4.5 If Σ is a diagram scheme, we construct its trivial extension Σ_0 by adding to every vertex i of Σ an identity arrow 1_i whose origin and end are both i itself.

The trivial extension D_0 of a diagram D:

is obtained by defining $D_o(1_i)=1_{D(i)}$, $D_o|\Sigma=D$.

4.6 A <u>commutativity condition</u> for the diagram scheme Σ is a pair of paths (v,w) in the trivial extension Σ_o of Σ, where v and w have the same origin and the same end.

A diagram $D:\Sigma\longrightarrow\mathbb{C}$ <u>satisfies the commutativity condition (v,w)</u>, if for the trivial extension D_o of D, $D_o(v)=D_o(w)$ holds.

4.7 Let Σ be a diagram scheme and K a set of commutativity conditions for Σ. A diagram is said to be <u>of type</u> Σ/K, if it is of type Σ and satisfies all commutativity conditions of K.

If $\mathbb{C}$ is a category, then the diagrams of type Σ/K in $\mathbb{C}$ together with their natural transformations form a category, which we denote by $[\Sigma/K,\mathbb{C}]$. It is a full subcategory of $[\Sigma,\mathbb{C}]$.

4.8 Let Σ be a diagram scheme and let K be a set of commutativity conditions for Σ. We define the category $\mathcal{G}(\Sigma/K)$ as follows: Thé objects are the vertices of Σ. For any two paths u_1 and u_2 in the trivial extension Σ_o of Σ, we say that u_1 and u_2 are K-related, if there exist subpaths v_i of u_i (i=1,2) such that $(v_1,v_2)\in K$. Define $\mathrm{Hom}_{\mathcal{G}(\Sigma/K)}(i_1,i_2)$ as the set of all equivalence classes of paths in Σ_o with origin i_1 and end i_2 with respect to the equivalence relation generated by K-relatedness. The composition of paths in Σ_o induces a composition of the equivalence classes.

There exists a diagram $\Delta:\Sigma \longrightarrow \mathcal{S}(\Sigma/K)$ with the following universal property:

For any category $\mathbb{C}$,

(1) For any diagram $D:\Sigma \longrightarrow \mathbb{C}$ of type Σ/K, there exists exactly one functor $T_D: \mathcal{S}(\Sigma/K) \longrightarrow \mathbb{C}$ such that $D=T_D\cdot\Delta$.

(2) There is an isomorphism of categories

$$[\Sigma/K, \mathbb{C}] \xrightarrow{\sim} [\mathcal{S}(\Sigma/K), \mathbb{C}],$$

where the map for objects is given by the rule $D \rightsquigarrow T_D$ in (1).

Chapter 1

DENSITY

The notion of density was studied notably in [3], [5], [6], and [10]. Density presupposes a rule with respect to which density can be asserted. The rule is either limit or colimit operation. By considering a certain class of colimits we refine the concept of density.

Section 1: Dense Functors.

In this section we provide a new perspective to density in terms of a cancellation property.

1.1 Definition A functor J from a category $\mathbb{C}$ to a category $\mathbb{A}$ is said to be dense, if each object A of $\mathbb{A}$ is a colimit of $J \cdot J_A$ where J_A is the canonical functor from the comma category J/A of objects $(C, \xi : JC \longrightarrow A)$ into assigning C to (C, ξ).

1.2 Proposition Given a functor $J : \mathbb{C} \longrightarrow \mathbb{A}$. The followings are equivalent:

(1) J is dense.

(2) The left Kan extension $Lan_J(h)$ of the Yoneda Embedding $h : \mathbb{C} \longrightarrow \hat{\mathbb{C}}$ along J is full faithful.

(3) The left Kan extension $Lan_J(J)$ of J along J exists and is equivalent to $Id_{\mathbb{A}}$.

The above is a standard fact and the proof is therefore omitted.

1.3 Theorem Let $\mathbb{C}$ be a small category and $J:\mathbb{C} \longrightarrow \mathbb{A}$ a full faithful dense functor. Let G be a functor from $\mathbb{A}$ to any cocomplete category $\mathbb{X}$. Then G is a left Kan extension of GJ along J, if, and only if G preserves all colimits which are preserved by $Lan_J(h)$.

Proof: The necessity is included in 0:3.6. Sufficiency follows since, for every $A \in Ob\,\mathbb{A}$, $Lan_J(h)(\varinjlim_{J/A} JJ_A) \xrightarrow{\sim}$

$\varinjlim_{J/A} Lan_J(h)JJ_A$, $G(A) \simeq G(\varinjlim JJ_A) \simeq \varinjlim GJJ_A \xrightarrow{\sim}$

$Lan_J(GJ)(A)$, where the middle isomorphism is guaranteed by the assumption.

1.3.1 Remark We write $[\mathbb{A},\mathbb{X}]_{J'\text{-left}}$ as the category of all J'-relative left adjoint functors where $J' = Lan_J(h)$; and $Cont_{[J']}[\mathbb{A},\mathbb{X}]$ as the category of all functors which preserve all colimits preserved by J'. Then there exists an isomorphism

$$[\mathbb{A},\mathbb{X}]_{J'\text{-left}} \xrightarrow{\sim} Cont_{[J']}[\mathbb{A},\mathbb{X}]$$

1.4 Corollary Let $\mathbb{C}$ be a small category, $\mathbb{X}$ a cocomplete category and $J:\mathbb{C} \longrightarrow \mathbb{A}$ a dense functor. The functor

$$[J,\mathbb{X}] : [\mathbb{A},\mathbb{X}] \longrightarrow [\mathbb{C},\mathbb{X}]$$

induces the maps

$$[G,G'] \longrightarrow [GJ,G'J]$$

for any pair G, G' in $[\mathbb{A},\mathbb{X}]$. If G preserves colimits, then the above maps are bijective.

In particular, for two cocontinuous functors G and G', G and G' are equivalent, if, and only if GJ and G'J are equivalent.

1.5 <u>Remark</u> The import of 1.4 is that equivalence of two cocontinuous functors is completely determined by the equivalence of respective restrictions on a dense subcategory. This fact indeed characterizes density.

1.6 <u>Theorem</u> Let $\mathbb{C}$ be a small category. The followings are equivalent for a full faithful functor $J:\mathbb{C} \longrightarrow \mathbb{A}$.

(1) J is dense.

(2) Let Cocont $[\mathbb{A},Ens^\circ]$ be the full subcategory of $[\mathbb{A}, Ens^\circ]$ consisting of cocontinuous functors. Then the functor

$$[J,Ens^\circ]:\mathrm{Cocont}\,[\mathbb{A},Ens^\circ] \longrightarrow [\mathbb{C},Ens^\circ]$$

is full faithful.

(3) For any cocomplete category $\mathbb{X}$, the functor

$$[J,\mathbb{X}]:\mathrm{Cocont}\,[\mathbb{A},\mathbb{X}] \longrightarrow [\mathbb{C},\mathbb{X}]$$

is full faithful.

<u>Proof</u>: (1) $\Longrightarrow$ (3) follows from the following commutative diagram:

$$\begin{array}{ccc} [G,G'] & \xrightarrow{\sim} & [Lan_J(GJ),G'] \\ \downarrow & \swarrow\sim & \\ [GJ,G'J] & & \end{array}$$

(3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) For any pair of objects A, B of $\mathbb{A}$, $[-,A]$ and $[-,B]$ are cocontinuous functors from $\mathbb{A}$ into Ens^{o}. By the assumption,

$$[[-,A],[-,B]] \longrightarrow [[J-,A],[J-,B]]$$

is bijective. Since $[J-,A] \simeq \varinjlim_{J/A} hJ_A$, we have $[A,B] \simeq [[-,A],[-,B]] \simeq [[J-,A],[J-,B]] \simeq \varprojlim_{J/A} [hJ_A,[J-,B]]$ $\varprojlim_{J/A} [JJ_A,B]$. Therefore $A \simeq \varinjlim_{J/A} JJ_A$.

Section 2: Δ-Dense Functors.

In this section we study a refined notion of dense functors. We observe that properties concerning dense functors are analoguously carried over. We use saturated classes in [2] as our means of refinement.

2.1 Definition Let Δ be a class of samll categories. Δ is said to be saturated if it satisfies the following:

(1) The final category $\mathbb{1}$ belongs to Δ.

(2) For any cofinal functor $H:\mathbb{I} \longrightarrow \mathbb{J}$, if $\mathbb{I}$ belongs to Δ, so does $\mathbb{J}$.

(3) Let $H:\mathbb{X} \longrightarrow$ Cat be a functor, where Cat is the category of all small categories. If $\mathbb{X} \in \Delta$ and for each $X \in \mathbb{X}$, $H(X) \in \Delta$ then $\varinjlim H(X)$ also belongs to Δ.

2.2 For a saturated class Δ and for any category $\mathbb{C}$, the Δ-cocompletion $K_\Delta(\mathbb{C})$ of $\mathbb{C}$ is the full subcategory of $\hat{\mathbb{C}}$ consisting of functors which are Δ-colimits of representable functors in $\hat{\mathbb{C}}$, where Δ-colimits are colimits of diagrams with domain categories in Δ.

We call the canonical embedding $\mathbb{C} \longrightarrow K_\Delta(\mathbb{C})$ h_Δ.

2.2.1 Remark For a given universe $\mathcal{U}$, let Δ be a saturated class of $\mathcal{U}$-small categories, and $\mathbb{C}$ a $\mathcal{U}$-category. Although the functor category $\hat{\mathbb{C}}$ may not be a $\mathcal{U}$-category, $K_\Delta(\mathbb{C})$ is always a $\mathcal{U}$-category. Indeed let F, G be any pair of objects of $K_\Delta(\mathbb{C})$. Then there exists $\mathbb{I}, \mathbb{J}$ in Δ together with two functors $\mathbb{I} \longrightarrow \mathbb{C}$, $\mathbb{J} \longrightarrow \mathbb{C}$ such that

$\varinjlim (\mathbb{I} \longrightarrow \mathbb{C} \longrightarrow \hat{\mathbb{C}}) \simeq F$ and $\varinjlim (\mathbb{J} \longrightarrow \mathbb{C} \longrightarrow \hat{\mathbb{C}})$ $\simeq$ G. Write the corresponding functors $\mathbb{I} \longrightarrow \mathbb{C}$ (resp. $\mathbb{J} \longrightarrow \mathbb{C}$) by F (resp. G) again. Then $[F,G] \simeq [\varinjlim hF(i), \varinjlim hG(j)] \simeq \varprojlim_i [hF(i), \varinjlim_j hG(j)] \simeq \varprojlim_i \varinjlim_j [hF(i), hG(j)]$ $\simeq \varprojlim_i \varinjlim_j [F(i), G(j)]$. Hence $[F,G]$ is isomorphic to a $\mathcal{U}$-small set, which means that $K_\Delta(\mathbb{C})$ is a $\mathcal{U}$-category.

In particular, when Δ is the class of all small categories, which is certainly a saturated class, $K_\Delta(\mathbb{C})$ is precisely the category of all proper presheaves on $\mathbb{C}$. It is this reason why proper functors are sometimes called essentially small functors [3].

2.3 <u>Definition</u> Let $J: \mathbb{C} \longrightarrow \mathbb{A}$ be a functor and Δ a saturated class. J is said to be <u>Δ-dense</u>, if J is dense and the left Kan extension $Lan_J(h)$ of the Yoneda Embedding $h: \mathbb{C} \longrightarrow \hat{\mathbb{C}}$ along J factorizes over $K_\Delta(\mathbb{C})$.

2.3.1 <u>Remark</u> (1) In the Definition 2.3, the condition that J is dense is redundant.

(2) In the Definition 2.3, the first factor $J': \mathbb{A} \longrightarrow K_\Delta(\mathbb{C})$ is precisely $Lan_J(h_\Delta)$. Indeed let $I_\Delta : K_\Delta(\mathbb{C}) \longrightarrow \hat{\mathbb{C}}$ be the canonical embedding. For any $G: \mathbb{A} \longrightarrow K_\Delta(\mathbb{C})$ the following chain of isomorphisms holds: $[J', G] \simeq [I_\Delta J', I_\Delta G]$ $\simeq [Lan_J(h), I_\Delta G] \simeq [h, I_\Delta GJ] \simeq [h_\Delta, GJ]$.

(3) In view of (2) we can rewrite 2.3 as follows: $Lan_J(h_\Delta)$ exists and is full faithful.

2.4 <u>Remark</u> Let $\mathbb{B}$ be any Δ-cocomplete category, and F a functor $\mathbb{C} \longrightarrow \mathbb{B}$. Then $Lan_h(F)$ is a I_Δ-relative right adjoint functor.

<u>Proof</u>: Define a functor L from $K_\Delta(\mathbb{C})$ into $\mathbb{B}$ as follows: For any object X of $K_\Delta(\mathbb{C})$, define $L(X) = \varinjlim_i FX(i)$ where the functor $X:\mathbb{I} \longrightarrow \mathbb{C}$ represents the object X.

For morphisms, L is defined via:

$$\begin{array}{ccc} [X,Y] & & [LX,LY] \\ \wr| & & |\wr \\ \varprojlim_i \varinjlim_j [X(i),Y(j)] & \longrightarrow & \varprojlim_i \varinjlim_j [FX(i),FY(j)] \end{array}$$

where the bottom row is the canonical map induced by the functor F. The following string of isomorphisms completes the proof: $[LX,\mathcal{B}] \simeq \varprojlim_i [FX(i),\mathcal{B}] \simeq \varprojlim_i [hX(i), [F-,\mathcal{B}]] \simeq [\varinjlim_i hX(i), Lan_h(F)(\mathcal{B})] \simeq [I_\Delta(X), Lan_h(F)(\mathcal{B})]$, where $\mathcal{B}$ is an object of $\mathbb{B}$.

2.5 <u>Proposition</u> Let $J:\mathbb{C} \longrightarrow \mathbb{A}$ be Δ-dense for a saturated class Δ, and $\mathbb{X}$ a Δ-cocomplete category and $F:\mathbb{C} \longrightarrow \mathbb{X}$ a functor. Then $Lan_J(F)$ always exists.

<u>Proof</u>: In view of the Remark 2.3.1, $Lan_J(h) = I\ Lan_J(h_\Delta)$ and since left Kan extension is preserved by the relative left adjoint functor L of 2.4, $Lan_J(F)$ is given as the composite functor $L\ Lan_J(h_\Delta)$.

2.6 <u>Proposition</u> Let $J: \mathbb{C} \longrightarrow \mathbb{A}$ be a functor. Suppose J is Δ-dense. Then for any $G: \mathbb{A} \longrightarrow \mathbb{X}$ with Δ-cocomplete $\mathbb{X}$, if G is Δ-cocontinuous, then G is the left Kan extension of GJ along J.

Proof: Since J is Δ-dense, for $A \in Ob\mathbb{A}$, there exists a cofinal functor $H: \mathbb{I} \longrightarrow J/A$ with $\mathbb{I}$ belonging to Δ. We have: $G(A) = G(\underset{J/A}{\underline{\lim}} JJ_A) = G(\underset{I}{\underline{\lim}} JJ_A H(i)) = \underset{i}{\underline{\lim}}$ $GJJ_A H(i) = \underset{J/A}{\underline{\lim}} GJJ_A = Lan_J(GJ)(A)$.

2.6.1 Corollary Let $J: \mathbb{C} \to \mathbb{A}$ be a Δ-dense functor. For any Δ-cocomplete category $\mathbb{X}$ the functor

$$[J, \mathbb{X}] : Cocont_{\Delta}[\mathbb{A}, \mathbb{X}] \longrightarrow [\mathbb{C}, \mathbb{X}]$$

is full faithful, where $Cocont_{\Delta}[\mathbb{A}, \mathbb{X}]$ is the full subcategory of $[\mathbb{A}, \mathbb{X}]$, consisting of all Δ-cocontinuous functors.

2.7 Proposition Let $\mathbb{A}$ be a category. The following are equivalent:

(1) $\mathbb{A}$ is a Δ-retract, i.e. there exists a small category $\mathbb{C}$ together with an adjointness situation

$$\mathbb{A} \underset{R}{\overset{L}{\rightleftarrows}} K_{\Delta}(\mathbb{C}), \quad L \dashv R$$

with full faithful R.

(2) $\mathbb{A}$ is Δ-cocomplete and has a small Δ-dense subcategory.

Proof: (1) $\Longrightarrow$ (2) It is enough to show that Lh_{Δ} is Δ-dense. This will follow if $R = Lan_{Lh_{\Delta}}(h_{\Delta})$. Indeed $Lan_{Lh_{\Delta}}(h_{\Delta}) \simeq Lan_L(Lan_{h_{\Delta}}(h_{\Delta})) \simeq Lan_L(Id) \simeq R$.

(2) $\Longrightarrow$ (1) First observe that J is h_{Δ}-relative left adjoint to $Lan_J(h_{\Delta})$. Since $\mathbb{A}$ is Δ-cocomplete, $Lan_{h_{\Delta}}(J)$

exists. The sought adjointness situation is then

$$\mathrm{Lan}_{h_\Delta}(J) \longrightarrow\!\!| \ \mathrm{Lan}_J(h_\Delta).$$

2.8 Theorem Let $\mathbb{A}$ be a category. The following are equivalent:

(1) $\mathbb{A}$ is Δ-cocomplete, and has a small Δ-dense subcategory of objects C for which $[C,-]$ preserves Δ-colimits.

(2) $\mathbb{A}$ is equvalent to $K_\Delta(\mathbb{C})$ for a small category $\mathbb{C}$.

Proof: The nontrivial part is (1) $\Longrightarrow$ (2). In view of Proposition 2.7, it is enough to show that $\mathrm{Lan}_{h_\Delta}(J)$ and $\mathrm{Lan}_J(h_\Delta)$ provide an equivalence, in other words, the front and the back adjunctions are equivalences. Since $\mathrm{Lan}_J(h_\Delta)$ is full faithful, the back adjunction is an equivalence. The front adjunction being an equivalence follows from Δ-cocontinuity of $\mathrm{Lan}_J(h_\Delta)$. For if $\mathrm{Lan}_J(h_\Delta)$ is Δ-cocontinuous, in view of Corollary 2.6.1, for two Δ-cocontinuous functors Id and $\mathrm{Lan}_J(h_\Delta)\cdot\mathrm{Lan}_{h_\Delta}(J)$, the front adjunction, being an equivalence when restricted by h_Δ, is itself an equivalence. The Δ-cocontinuity of $\mathrm{Lan}_J(h_\Delta)$ follows from 0:3.6.

2.9 Let $J: \mathbb{C} \longrightarrow \mathbb{A}$ be a functor, and $\mathbb{X}$ a cocomplete category. In 0:3.4, we have seen that for every functor $F: \mathbb{C} \longrightarrow \mathbb{X}$, $\mathrm{Lan}_J(F)$ exists. In the following Theorem, we establish that this fact completely determines the cocompleness of the category.

2.9.1 Theorem Let $\mathbb{X}$ be a category satisfying the following:

For any Δ-dense functor $J: \mathbb{C} \longrightarrow \mathbb{A}$ such that for every $C \in \mathrm{Ob}\,\mathbb{C}$, $[JC,-]$ preserves all Δ-colimits, and for any $F: \mathbb{C} \longrightarrow \mathbb{X}$, $\mathrm{Lan}_J(F)$ exists.

Then $\mathbb{X}$ is Δ-cocomplete.

Proof: For any $\mathbb{I}$ in Δ, and any $H: \mathbb{I} \longrightarrow \mathbb{X}$, we claim that $\varinjlim_i H(i) \simeq \mathrm{Lan}_{h_\Delta}(H)(\varinjlim_i h(i))$, where $h_\Delta: \mathbb{I} \longrightarrow K_\Delta(\mathbb{I})$. By the assumption $\mathrm{Lan}_{h_\Delta}(H)$ preserves Δ-colimits. Therefore $\mathrm{Lan}_{h_\Delta}(H)(\varinjlim_i h_\Delta(i)) \simeq \varinjlim_i \mathrm{Lan}_{h_\Delta}(H)(h_\Delta(i)) \simeq \varinjlim_i H(i)$. The latter isomorphism is due to the full faithfulness of h_Δ.

Section 3: Density with respect to a Functor $V: \mathbb{B} \longrightarrow \mathrm{Ens}$.

We introduce a notion of density which generalizes the density in Section 1. This provides an interpretation of (algebraic) structured objects in concrete categories.

3.1 Definition Let $V: \mathbb{B} \longrightarrow \mathrm{Ens}$ be a functor. A functor $J: \mathbb{C} \longrightarrow \mathbb{A}$ is said to be V-dense, if $\mathrm{Lan}_J(h)$ is factorized over $[\mathbb{C}^\circ, V]: [\mathbb{C}^\circ, \mathbb{B}] \longrightarrow [\mathbb{C}^\circ, \mathrm{Ens}]$ with the first factor being full faithful.

3.2 Proposition Let V have a left adjoint F and $\mathbb{B}$ be cocomplete. Then the followings are equivalent:

(1) $\mathbb{A}$ is a retract of $[\mathbb{C}^\circ, \mathbb{B}]$ for some small category $\mathbb{C}$.

(2) $\mathbb{A}$ is cocomplete and admits a small V-dense functor.

Proof: (1) $\Rightarrow$ (2) Let

$$\mathbb{A} \underset{T}{\overset{S}{\rightleftarrows}} [\mathbb{C}^\circ, \mathbb{B}], \quad S \dashv T,$$

with full faithful T be the adjointness situation of the assumption. $\mathbb{A}$ is obviously cocomplete. Consider $S \cdot [\mathbb{C}^\circ, F] \cdot h: \mathbb{C} \longrightarrow \mathbb{A}$. We claim that $S \cdot [\mathbb{C}^\circ, F] \cdot h$ is the V-dense functor. It is enough to show that $[\mathbb{C}^\circ, V] \cdot T = \mathrm{Lan}_{S \cdot [\mathbb{C}^\circ, F] \cdot h}(h)$. But $\mathrm{Lan}_{S \cdot [\mathbb{C}^\circ, F] \cdot h}(h) \simeq \mathrm{Lan}_{S \cdot [\mathbb{C}^\circ, F]}(\mathrm{Lan}_h(h)) \simeq \mathrm{Lan}_{S \cdot [\mathbb{C}^\circ, F]}(\mathrm{Id}) \simeq [\mathbb{C}^\circ, V] \cdot T$. The last isomorphism is due to the fact that $S \cdot [\mathbb{C}^\circ, F]$ is a left adjoint to $[\mathbb{C}^\circ, V] \cdot T$.

(2) $\Longrightarrow$ (1) Since $\mathbb{A}$ is cocomplete, $\mathrm{Lan}_h(J)$ exists. Put $S = \mathrm{Lan}_{[\mathbb{C}^\circ, F]}(\mathrm{Lan}_h(J)) \xrightarrow{\sim} \mathrm{Lan}_{[\mathbb{C}^\circ, F]\cdot h}(J)$. It is now enough to show that S is a left adjoint to T. By the definition of S, it is equivalent to show that J is $([\mathbb{C}^\circ, F]\cdot h)$-relative left adjoint to T. Indeed $\left[[\mathbb{C}^\circ, F]\cdot h(C), T(A)\right] \xrightarrow{\sim} \left[h(C), [\mathbb{C}^\circ, V]\cdot T(A)\right] \xrightarrow{\sim} \left[h(C), \mathrm{Lan}_J(h)(A)\right] \xrightarrow{\sim} [JC, A]$, where $C \in \mathrm{Ob}\,\mathbb{C}$ and $A \in \mathrm{Ob}\,\mathbb{A}$.

3.3 Examples (1) The canonical functor from the category of all finitely generated free monoids into the category of rings, assigning the monoid rings, is dense with respect to the forgetful functor on the category of Abelian groups.

(2) The embedding of finitely generated free algebras as discrete topological algebras into the category of topological algebras is dense with respect to the forgetful functor on the category of topological spaces.

Chapter 2

TRIPLES ASSOCIATED WITH RELATIVE ADJOINTNESS SITUATIONS

An adjointness situation is known to give rise to a triple on a category. (see [1], [9]) In this chapter we study triples associated with relative adjointness situations.

Section 1: Triples Generated by Relative Adjointness Situations.

1.0 Let $J:\mathbb{C} \longrightarrow \mathbb{A}$ be a full faithful dense functor and $s:\mathbb{C} \longrightarrow \mathbb{X}$, and $r:\mathbb{X} \longrightarrow \mathbb{A}$ functors such that s is J-relative left adjoint to r, where the adjunction transformation for $C \in \mathrm{Ob}\,\mathbb{C}$ and $X \in \mathrm{Ob}\,\mathbb{X}$ is

$$\alpha_{C,X}: [sC, X] \xrightarrow{\sim} [JC, rX].$$

Let for every $C \in \mathrm{Ob}\,\mathbb{C}$, $\eta_C = \alpha_{C,sC}(1_{sC})$ and $T = rs$. Then $\{\eta_C\}$ define a natural transformation $\eta: J \longrightarrow T$, which is often called the front adjunction of the relative adjointness.

Put $\hat{T} = \mathrm{Lan}_h(J' \cdot T)$ where $J' = \mathrm{Lan}_J(h)$ and h is the Yoneda Embedding $\mathbb{C} \longrightarrow \hat{\mathbb{C}}$. We define a triple structure on the endofunctor $\hat{T}$.

1.0.1 For every $H:\mathbb{C}^{\bullet} \longrightarrow \mathrm{Ens}$, define $\hat{\eta}_H$ to be the unique natural natural transformation making the following diagram commute:

$$\begin{array}{ccccc}
H & & \overset{\hat{\eta}_H}{\dashrightarrow} & & \hat{T}(H) \\
\| & & & & \| \\
\underset{h/H}{\varinjlim}\, hh_H & & & & \underset{h/H}{\varinjlim}\, J'Th_H \\
\xi \uparrow & & & & \uparrow \sigma_{(C,\xi)} \\
hh_H(C,\xi) & & & & J'Th_H(C,\xi) \\
\| & & & & \| \\
[-,C] & \xrightarrow{s_{-,C}} & [s-,sC] & \xrightarrow{\alpha_{-,sC}} & [J-,TC]
\end{array}$$

Where $\sigma_{(C,\xi)}$ are the colimit morphisms. Indeed such an $\hat{\eta}_H$ exists uniquely, since for every $\varphi:(C,\xi) \longrightarrow (D,\zeta)$ in the category h/H, $\sigma_{(D,\zeta)} \cdot \alpha_{-,sD} \cdot s_{-,D} \cdot [-,\varphi] = \sigma_{(D,\zeta)} \cdot \alpha_{-,sD} \cdot [s-,s\varphi] \cdot s_{-,C} = \sigma_{(D,\zeta)} \cdot [J-,T] \cdot \alpha_{-,sC} \cdot s_{-,C} = \sigma_{(C,\xi)} \cdot \alpha_{-,sC} \cdot s_{-,C}$, which means that $\{\sigma_{(C,\xi)} \cdot \alpha_{-,sC} \cdot s_{-,C}\}$ is natural in $(C,\xi) \in Ob(h/H)$.

1.0.2 Let $\lambda_{(D,\zeta)}: J'Th_{\hat{T}(H)}(D,\zeta) \longrightarrow \underset{h/\hat{T}(H)}{\varinjlim} J'Th_{\hat{T}(H)}$ be colimit maps, where $\zeta: [-,D] \longrightarrow \hat{T}(H)$. Since $[h(D),-]$ preserves all colimits in $\hat{C}$, there exists $(C,\xi) \in Ob(h/H)$ and $\zeta': [-,D] \longrightarrow J'Th_H(C,\xi) = [J-,TC]$ such that the following diagram commutes:

$$\begin{array}{ccc}
[-,D] & \xrightarrow{\zeta} & \hat{T}(H) \\
 & \underset{\zeta'}{\searrow} & \uparrow \sigma_{(C,\xi)} \\
 & & [J-,TC]
\end{array}$$

Via the sequence of isomorphisms

$$[[-,D],[J-,TC]] \xrightarrow{\sim} [JD,TC] \xleftarrow{\alpha_{D,sC}} [sD,sC]$$

ζ' corresponds to $\zeta_0 = \alpha^{-1}_{D,sC}(\zeta'_D(1_D))$.

1.0.3 <u>Lemma</u> Given $\zeta: [-,D] \longrightarrow \hat{T}(H)$, $\sigma_{(C,\xi)}\cdot[J-, r\zeta_0]$ does not depend on the choice of (C,ξ) and ζ' in 1.0.2.

Furthermore, $\{\sigma_{(C,\xi)}\cdot[J-,r\zeta_0]\}$ is natural in (D,ζ) over the category $h/\hat{T}(H)$.

<u>Proof</u>: Suppose $(\bar{C},\bar{\xi})$ and $\bar{\zeta}'$ be another such as those in 1.0.2. i.e. $\sigma_{(\bar{C},\bar{\xi})}\cdot\bar{\zeta}' = \zeta$. In view of the isomorphism $[[-,D], \varinjlim J'Th_H] \xrightarrow{\sim} \varinjlim [[-,D], J'Th_H]$, ζ' and $\bar{\zeta}'$ are equivalent in $\varinjlim [[-,D], J'Th_H]$. From the way colimits in Ens are constructed, it is enough to show that whenever there is $\varphi:(C,\xi) \longrightarrow (\bar{C},\bar{\xi})$ in h/H with $\bar{\zeta}' = [J-, T\varphi]\cdot\zeta'$, $\sigma_{(\bar{C},\bar{\xi})}\cdot[J-,r\bar{\zeta}_0] = \sigma_{(C,\xi)}\cdot[J-,r\zeta_0]$, where $\bar{\zeta}_0 = \alpha^{-1}_{D,s\bar{C}}(\bar{\zeta}'_D(1_D))$. By the naturality of the adjunction transformation α, we have $\alpha^{-1}_{D,s\bar{C}}(T\varphi\cdot\zeta'_D(1_D)) = s\varphi\cdot\alpha^{-1}_{D,sC}(\zeta'_D(1_D))$. Since $\{\sigma_{(C,\xi)}\}$ is natural, $\sigma_{(\bar{C},\bar{\xi})}\cdot[J-,r\bar{\zeta}_0] = \sigma_{(\bar{C},\bar{\xi})}\cdot[J-, rs\varphi\cdot r\alpha^{-1}_{D,sC}(\zeta'_D(1_D))] = \sigma_{(\bar{C},\bar{\xi})}\cdot[J-,T\varphi]\cdot[J-,r\zeta_0] = \sigma_{(C,\xi)}\cdot [J-,r\zeta_0]$. This proves the claim.

Let $f:(D,\zeta) \longrightarrow (D,\bar{\zeta})$ be a morphism in $h/\hat{T}(H)$. Suppose $\bar{\zeta} = \sigma_{(C,\xi)}\cdot\bar{\zeta}'$ as in 1.0.2. In view of what has been established in the above, it is enough to show that $\sigma_{(C,\xi)}\cdot[J-,r\zeta_0] = \sigma_{(C,\xi)}\cdot [J-,r\alpha^{-1}_{D,sC}(\bar{\zeta}'_D\cdot[D,f](1_D))]$. But

$\alpha^{-1}_{D,sC}(\varsigma'_D\cdot[D,f](1_D)) = \alpha^{-1}_{D,sC}(\varsigma'_D(f)) = \alpha^{-1}_{D,sC}(\varsigma_D(1_D))$. This completes the proof.

1.0.4 For every $H:\mathbb{C}^{\circ} \longrightarrow \mathrm{Ens}$, define $\hat{\mu}_H$ to be the unique morphism making the following diagram commute:

$$\begin{array}{ccc}
\hat{T}\hat{T}(H) & \overset{\mu_H}{\dashrightarrow} & \hat{T}(H)\\
\| & & \|\\
\varinjlim_{h/\hat{T}(H)} J'Th_{\hat{T}(H)} & & \varinjlim_{h/H} J'Th_H\\
\uparrow \lambda_{(D,\varsigma)} & & \uparrow \sigma_{(C,\xi)}\\
J'Th_{\hat{T}(H)}(D,\varsigma) & & J'Th_H(C,\xi)\\
\| & & \|\\
[J-,TD] & \xrightarrow{[J-,r\varsigma_0]} & [J-,TC]
\end{array}$$

where (C,ξ) and ς_0 are as in 1.0.2. The existence of $\hat{\mu}_H$ is guaranteed in view of Lemma 1.0.3.

1.0.5 <u>Remark</u> (1) $\hat{\eta}_{h(C)} = \alpha_{-,sC}\cdot s_{-,C}$ for every $C\in \mathrm{Ob}\,\mathbb{C}$.

(2) Since $\hat{T}(h(C)) \simeq [J-,TC]$, $\hat{\mu}_{h(C)}\cdot\lambda_{(D,\varsigma)} = [J-,r\varsigma_0]$ for every $(D,\varsigma)\in\mathrm{Ob}(h/\hat{T}h(C))$, where $\varsigma_0 = \alpha^{-1}_{D,sC}(\varsigma)$. In particular, for $(C,\bar{\xi})\in\mathrm{Ob}(h/\hat{T}h(C))$, where $\bar{\xi}$ corresponds to η_C via the Yoneda Embedding h, $\hat{\mu}_{h(C)}\cdot\lambda_{(C,\bar{\xi})} = [J-,T(1_C)]$.

1.0.6 <u>Lemma</u> $\{\hat{\eta}_H\}$, and $\{\hat{\mu}_H\}$ are extended to natural transformations in $H\in\mathrm{Ob}\,\hat{\mathbb{C}}$.

<u>Proof</u>: Given $\theta:H \longrightarrow K$ in $\hat{\mathcal{C}}$, for any $(C,\xi)\in\mathrm{Ob}(h/H)$, $\hat{T}(\theta)\cdot\hat{\eta}_H\cdot\xi = \hat{T}(\theta)\cdot\sigma_{(C,\xi)}\cdot\alpha_{-,sC}\cdot s_{-,C} = \sigma_{(C,\theta\xi)}\cdot\alpha_{-,sC}\cdot s_{-,C} =$

$\hat{\eta}_K\cdot\theta\cdot\xi$. Hence $\hat{T}(\theta)\hat{\eta}_H=\hat{\eta}_K\cdot\theta$.

For each $(D,\zeta)\in Ob(h/\hat{T}(H))$, $(D,\hat{T}(\theta)\cdot\zeta)\in Ob(h/\hat{T}(K))$. Suppose $\zeta=\sigma_{(C,\xi)}\cdot\zeta'$. Then we know $\hat{T}(\theta)\cdot\zeta=\sigma^K_{(C,\theta\xi)}\cdot\zeta'$, since $\hat{T}(\theta)\cdot\zeta=\hat{T}(\theta)\cdot\sigma_{(C,\xi)}\cdot\zeta'=\sigma^K_{(C,\theta\xi)}\cdot\zeta'$, where $\{\sigma^K_{(C,\xi)}:[-,C]\longrightarrow\hat{T}(K)\}$ are the colimit maps. Therefore $\hat{T}(\theta)\cdot\hat{\mu}_H\cdot\lambda_{(D,\zeta)}=\hat{T}(\theta)\cdot\sigma_{(C,\xi)}\cdot[J-,r\zeta_0]=\sigma_{(C,\theta\xi)}\cdot[J-,r\zeta_0]=\hat{\mu}_K\cdot\lambda^K_{(D,\hat{T}(\theta)\cdot\zeta)}=\hat{\mu}_K\cdot\hat{T}\hat{T}(\theta)\cdot\lambda_{(D,\zeta)}$, where $\lambda^K_{(D,\zeta)}:[J-,TD]\longrightarrow\hat{T}\hat{T}(K)$ are the colimit morphisms. Since $\{\lambda_{(D,\zeta)}\}$ are colimit morphisms, $\hat{T}(\theta)\cdot\hat{\mu}_H=\hat{\mu}_K\cdot\hat{T}\hat{T}(\theta)$. This completes the proof.

1.0.7 <u>Proposition</u> $(\hat{T},\hat{\eta},\hat{\mu})$ is a triple on $\hat{\mathcal{C}}$.

<u>Proof</u>: (1) We show $\hat{\mu}_H\cdot\hat{\eta}_{\hat{T}(H)}=1_{\hat{T}(H)}$ for every $H\in Ob\,\hat{\mathcal{C}}$. For every $(D,\zeta)\in Ob(h/\hat{T}(H))$, the following sequence of equalities holds:

$$\hat{\mu}_H\cdot\hat{\eta}_{\hat{T}(H)}\cdot\zeta\overset{①}{=}\hat{\mu}_H\cdot\lambda_{(D,\zeta)}\cdot\alpha_{-,sC}\cdot s_{-,C}\overset{②}{=}\sigma_{(C,\xi)}\cdot[J-,r\zeta_0]\cdot\alpha_{-,sC}\cdot s_{-,C}\overset{③}{=}\sigma_{(C,\xi)}\cdot\alpha_{-,sC}\cdot[s-,\zeta_0]\cdot s_{-,C},$$

where ① and ② follow from 1.0.1 and 1.0.4 respectively, and ③ from the naturality of α. In view of Lemma 1.0.3, it is enough to show $\alpha_{-,sC}\cdot[s-,\zeta_0]\cdot s_{-,C}=\zeta'$, which follows from the definition of ζ_0 from ζ' in 1.0.2.

(2) $\hat{\mu}_H\cdot\hat{T}\hat{\eta}_H=1_{\hat{T}(H)}$ for every $H\in Ob\,\hat{\mathcal{C}}$. For every $(C,\xi)\in Ob(h/H)$, $\hat{\mu}_H\cdot\hat{T}\hat{\eta}_H\cdot\sigma_{(C,\xi)}\overset{①}{=}\hat{\mu}_H\cdot\lambda_{(C,\hat{\eta}_H\cdot\xi)}\overset{②}{=}\sigma_{(C,\xi)}\cdot$

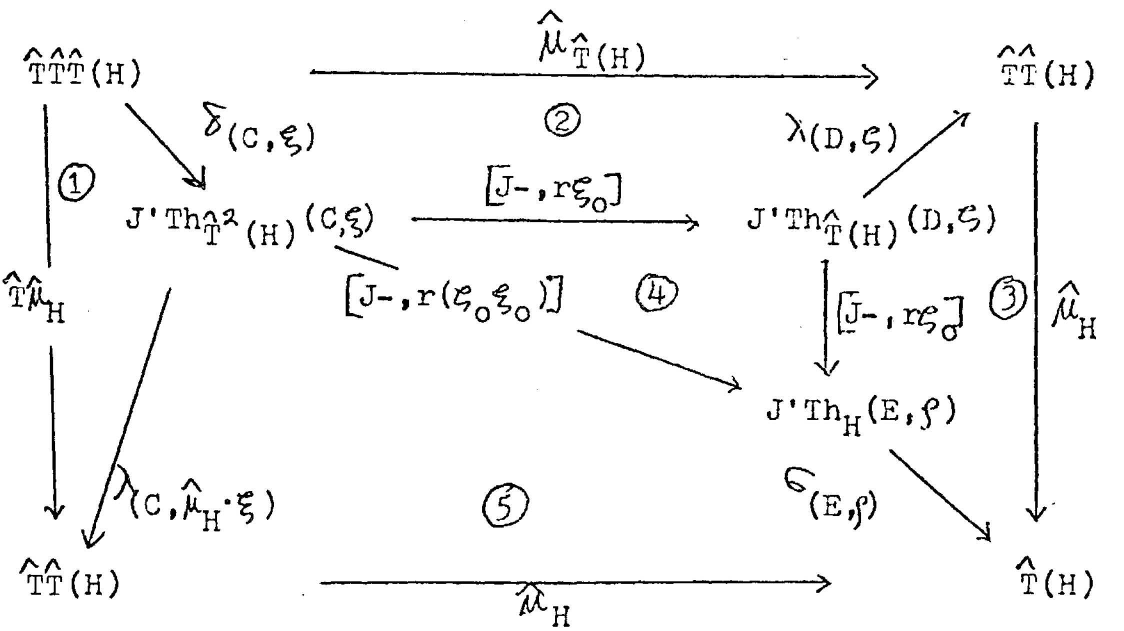
$[J-, r\rho_o]$, where ρ_o corresponds to $\hat{\eta}_H \cdot \xi$ as in 1.0.2, and where ① follows from the definition of $\hat{T}$ and ② from 1.0.4. But in view of 1.0.2, ρ_o is ipso facto 1_{sC}.

(3) $\hat{\mu}_H \cdot \hat{\mu}_{\hat{T}(H)} = \hat{\mu}_H \cdot \hat{T}\hat{\mu}_H$ for every $H \in Ob\,\hat{C}$. Consider the following diagram, where $\delta_{(C,\xi)} : J'Th_{\hat{T}^2(H)}(C,\xi) \longrightarrow \hat{T}^3(H)$ are colimit maps, and ξ_o and ζ_o are defined analoguously as in 1.0.2:

$$\lambda_{(D,\zeta)} \cdot \xi' = \xi \text{ and } \sigma_{(E,\rho)} \cdot \zeta' = \zeta;$$

$$[h(C), [J-, TD]] \simeq [JC, TD] \simeq [sC, sD],\ \xi' \rightsquigarrow \xi_o, \text{ and}$$

$$[h(D), [J-, TE]] \simeq [JD, TE] \simeq [sD, sE],\ \zeta' \rightsquigarrow \zeta_o.$$



The diagrams ② and ③ commute respectively by the definition of $\hat{\mu}$. Diagram ① commutes by the definition of $\hat{T}$, and commutativity of ④ is obvious. It is then enough to show the commutativity of ⑤, which means in other words $\sigma_{(E,\rho)} \cdot \overline{\alpha_{C,sE}(\varepsilon_0 \xi_0)} = \hat{\mu}_H \cdot \xi$, where $\overline{\alpha_{C,sE}(\varepsilon_0 \xi_0)}$ is the corresponding natural transformation to $\varepsilon_0 \xi_0$ via $[h(C),[J-,TE]] \xrightarrow{\sim} [JC,TE]$. $\hat{\mu}_H \cdot \xi = \hat{\mu}_H \cdot \lambda_{(D,\sigma)} \cdot \xi'$ $= \sigma_{(E,\rho)} \cdot [J-, r\varepsilon_0] \cdot \xi'$. But $\overline{\alpha_{C,sE}(\varepsilon_0 \xi_0)} = [J-, r\varepsilon_0] \cdot \xi'$ by the naturality of α and the Yoneda Lemma. This completes the proof of the proposition.

1.1 A given relative adjointness situation has been shown to give rise to a triple. We now show that this triple is always cocontinuous, i.e. preserves all colimits.

1.1.1 <u>Lemma</u> Let $J: \mathbb{C} \longrightarrow \mathbb{E}$ be a functor, and $I: \mathbb{I} \longrightarrow \mathbb{E}$ a diagram with $\varinjlim I = (L, \lambda)$, where λ is the colimit cone. If for each $C \in Ob\mathbb{C}$, $[JC,-]$ preserves $\varinjlim I$, then $(J/L, J/\lambda) = \underset{i}{\varinjlim} J/I(i)$.

<u>Proof</u>: $\{J/\lambda(i)\}$ is certainly natural in $i \in \mathbb{I}$. Given a natural cone $\{\sigma_i : J/I(i) \longrightarrow \mathbb{X}\}$, define $\bar{\sigma}: J/L \longrightarrow \mathbb{X}$ as follows: For each $(C,\xi) \in Ob(J/L)$, since $[JC, \varinjlim I] \xrightarrow{\sim} \varinjlim [JC, I(i)]$, there exists $i \in \mathbb{I}$ and $\xi_i : JC \longrightarrow I(i)$ such that $\xi = \lambda_i \cdot \xi_i$. We need to show $\sigma_i(C, \xi_i)$ is independent of the choice of ξ_i. Suppose $\xi = \lambda_j \xi_j$ is another such factorization. In view of an analoguous reason to that in the proof of 1.0.3, we could without loss of generality assume that there exists $\varphi: i \longrightarrow j$ in $\mathbb{I}$ such that $I(\varphi) \cdot \xi_i = \xi_j$. Since $\{\sigma_i\}$ is natural in $i \in Ob\mathbb{I}$, $\sigma_i = \sigma_j \cdot$

$J/I(\varphi)$. Therefore $\sigma_i(C,\xi_i) = \sigma_j(C,I(\varphi)\cdot\xi_i) = \sigma_j(C,\xi_j)$. Hence we could now define $\bar{\sigma}(C,\xi) = \sigma_i(C,\xi_i)$.

For $f:(C,\xi) \longrightarrow (C',\xi')$ in J/L, suppose $\xi' = \lambda_j\cdot\xi_j$. Then $f:(C,\xi_j\cdot J(f)) \longrightarrow (C',\xi_j)$ is also a morphism in $J/I(j)$. Define $\bar{\sigma}(\varphi) = \sigma_j(\varphi)$. The independence of this definition of the choice of j can be shown analoguously to the above. That this $\bar{\sigma}$ defines a functor is obvious. The uniqueness easily follows from that $[JC,-]$ preserves $\varinjlim$ I.

1.1.2 Proposition In the same situation as in 1.0, $\hat{T}$ is cocontinuous.

Proof: Let $I:\mathbb{I} \longrightarrow \hat{\mathbb{C}}$ be any diagram. Consider the following sequence of isomorphisms: $\hat{T}(\varinjlim I) =$

$$\mathrm{Lan}_h(J'T)(\varinjlim I) = \varinjlim_{h/\varinjlim I} J'Th_{\varinjlim I} \overset{①}{\xrightarrow{\sim}} \varinjlim_{\varinjlim_i h/I(i)} J'Th_{\varinjlim I}$$

$$\overset{②}{\xrightarrow{\sim}} \varinjlim_i \varinjlim_{h/I(i)} J'Th_{I(i)} \xrightarrow{\sim} \varinjlim_i \mathrm{Lan}_h(J'T)(I(i)) =$$

$\varinjlim_i \hat{T}(I(i))$, Where ① follows from Lemma 1.1.1 and ② is not difficult to see.

1.2 Let $J:\mathbb{C} \to \mathbb{A}$ be a full faithful dense functor, and $\hat{\mathbb{T}} = (\hat{T},\hat{\eta},\hat{\mu})$ a triple on $\hat{\mathbb{C}}$. Let $F^{\hat{\mathbb{T}}} \dashv U^{\hat{\mathbb{T}}}$ be the Eilenberg-Moore situation for $\hat{\mathbb{T}}$. Consider the following pullback, where $J' = \mathrm{Lan}_h(J)$:

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{W} & \hat{\mathcal{C}}^{\hat{\mathbb{T}}} \\ R\downarrow & & \downarrow U^{\hat{\mathbb{T}}} \\ \mathbb{A} & \xrightarrow{J'} & \hat{\mathbb{C}} \end{array}$$

1.2.1 <u>Lemma</u> In the situation of 1.2, if $\hat{\mathbb{T}}$ is induced by a J-relative adjoint situation as in 1.0 and Proposition 1.0.7, then there exists a functor $S:\mathbb{C} \longrightarrow \mathbb{P}$ such that S is J-relative left adjoint to R and

$$F^{\hat{\mathbb{T}}}\cdot J'\cdot J = W\cdot S.$$

<u>Proof</u>: Since $J'T(C) = \hat{T}\cdot h(C)$ for every $C\in \mathrm{Ob}\,\mathbb{C}$, there exists a functor $S:\mathbb{C} \longrightarrow \mathbb{P}$ such that $T = RS$ and $F^{\hat{\mathbb{T}}}\cdot h = W\cdot S$. For every $C\in \mathrm{Ob}\,\mathbb{C}$, $P\in \mathrm{Ob}\,\mathbb{P}$, $[S(C),P] \xrightarrow{\sim} [WS(C),W(P)] = [F^{\hat{\mathbb{T}}}h(C),W(P)] \simeq [h(C),U^{\hat{\mathbb{T}}}W(P)] = [h(C),J'R(P)] \xrightarrow{\sim} [J(C),R(P)]$.

1.2.2 <u>Definition</u> Let $J:\mathbb{C} \longrightarrow \mathbb{A}$ be a full faithful dense functor, and $\hat{\mathbb{T}} = (\hat{T},\hat{\eta},\hat{\mu})$ a triple on $\hat{\mathbb{C}}$. The <u>J-relative Eilenberg-Moore situation for $\hat{\mathbb{T}}$</u> is a 3-tuple $(J^{\hat{\mathbb{T}}},S^{\hat{\mathbb{T}}},R^{\hat{\mathbb{T}}})$ consisting of the following data: $J^{\hat{\mathbb{T}}}$ is a category, $S^{\hat{\mathbb{T}}}:\mathbb{C} \longrightarrow J^{\hat{\mathbb{T}}}$ and $R^{\hat{\mathbb{T}}}:J^{\hat{\mathbb{T}}} \longrightarrow \mathbb{A}$ are functors; and satisfying the following:

There exists a functor $W:J^{\hat{\mathbb{T}}} \longrightarrow \hat{\mathbb{C}}^{\hat{\mathbb{T}}}$ such that $F^{\hat{\mathbb{T}}}\cdot J'\cdot J = F^{\hat{\mathbb{T}}}\cdot h = W\cdot S^{\hat{\mathbb{T}}}$, and

$$\begin{array}{ccc} J^{\hat{\mathbb{T}}} & \xrightarrow{W} & \hat{\mathbb{C}}^{\hat{\mathbb{T}}} \\ R^{\hat{\mathbb{T}}}\downarrow & & \downarrow U^{\hat{\mathbb{T}}} \\ \mathbb{A} & \xrightarrow{J'} & \hat{\mathbb{C}} \end{array}$$

is a pullback square.

The J-relative Kleisli Situation for $\hat{\mathbb{T}}$ is a 3-tuple $(J_{\hat{\mathbb{T}}}, S_{\hat{\mathbb{T}}}, R_{\hat{\mathbb{T}}})$, where $J_{\hat{\mathbb{T}}}$ is a category, $S_{\hat{\mathbb{T}}}: \mathbb{C} \longrightarrow J_{\hat{\mathbb{T}}}$ and $R_{\hat{\mathbb{T}}}: J_{\hat{\mathbb{T}}} \longrightarrow \mathbb{A}$ are functors, satisfying the following:

There exists a functor $Q: J_{\hat{\mathbb{T}}} \longrightarrow \hat{\mathbb{C}}^{\hat{\mathbb{T}}}$ such that $Q \cdot S_{\hat{\mathbb{T}}}$ is the full image factorization of $F^{\hat{\mathbb{T}}} \cdot h$, and $J' R_{\hat{\mathbb{T}}} = U^{\hat{\mathbb{T}}} \cdot Q$.

1.2.2.1 Remark (1) From the pullback property of the square 1.2.2.1 there exists a unique functor $M: J_{\hat{\mathbb{T}}} \longrightarrow J^{\hat{\mathbb{T}}}$ such that $R^{\hat{\mathbb{T}}} \cdot M = P_{\hat{\mathbb{T}}}$ and $Q = W \cdot P$.

(2) $S^{\hat{\mathbb{T}}}$ (resp. $S_{\hat{\mathbb{T}}}$) is a J-relative left adjoint to $R^{\hat{\mathbb{T}}}$ (resp. $R_{\hat{\mathbb{T}}}$).

1.2.3 Notation When $\hat{\mathbb{T}}$ is induced by a J-relative adjoint situation as in Proposition 1.0.7, we write (J^T, S^T, R^T) (resp. (J_T, S_T, R_T)) for J-relative Eilenberg-Moore Situation (resp. Kleisli Situation) for $\hat{\mathbb{T}}$.

1.2.4 Remark (1) In the situation of 1.2.3, J_T is defined as follows: the objects are the same as those of $\mathbb{C}$; J_T-morphisms from C into D for $C, D \in \mathrm{Ob} J_T$ are morphisms $\varphi: TC \longrightarrow TD$ in $\mathbb{A}$ such that $\hat{\mu}_{h(D)} \cdot \hat{T} J' \varphi = J' \varphi \cdot \hat{\mu}_{h(C)}$. Then S_T is given as $C \rightsquigarrow C$ and $f \rightsquigarrow Tf$; and R_T is $C \rightsquigarrow TC$ and $\varphi \rightsquigarrow \varphi$.

(2) $T = R_T S_T = R^R S^T$ holds in the notation of (1).

1.3 Proposition Let $\mathbb{T} = (T, \eta, \mu)$ be a triple on $\mathbb{A}$. Put $\hat{T} = \mathrm{Lan}_h(h \cdot T)$. Then the Id -relative Eilenberg-Moore Situation (resp. Kleisli Situation) for $\hat{\mathbb{T}}$ is precisely the Eilenberg-Moore (resp. Kleisli) Situation for $\mathbb{T}$.

<u>Proof</u>: First we observe that $\hat{T}\cdot h(A) = Lan_h(h\cdot T)((h(A)) = h\cdot T(A)$ for every $A \in Ob\,\mathbb{A}$. By Remark 1.0.5, $\hat{\eta}_{h(A)} = h(\eta_A)$. Since $(TA, 1_{TA}) \in Ob(Id_{\mathbb{A}}/TA)$ is the final object, again by Remark 1.0.5, $\hat{\mu}_{h(A)} = \hat{\mu}_{h(A)}\cdot\lambda_{(TA,1_{TA})} = [-,\mu_A]$ hlods. Hence $\hat{\mu}_{h(A)} = h(\mu_A)$.

Define a functor $W: \mathbb{A}^{\mathbb{T}} \longrightarrow \hat{\mathbb{A}}^{\hat{\mathbb{T}}}$ by $(A,a) \rightsquigarrow (h(A), h(a))$ and $\varphi \rightsquigarrow h(\varphi)$. This functor is well defined in view of the above observations. Consider now the following diagram:

$$\begin{array}{ccc} \mathbb{A}^{\mathbb{T}} & \xrightarrow{\quad W \quad} & \hat{\mathbb{A}}^{\hat{\mathbb{T}}} \\ {\scriptstyle U^{\mathbb{T}}}\downarrow & & \downarrow{\scriptstyle U^{\hat{\mathbb{T}}}} \\ \mathbb{A} & \xrightarrow[\quad h \quad]{} & \hat{\mathbb{A}} \end{array}$$

This diagram commutes by the definition of W. For every $(X, x: \hat{T}X \longrightarrow X)$ in $\hat{\mathbb{A}}^{\hat{\mathbb{T}}}$ with $X = h(A)$, by the Yoneda Embedding there exists a unique $a: TA \longrightarrow A$ such that $h(a) = x$. From the fact that x is a structure map, it follows that a is also a structure map for $\mathbb{T}$. This proves the square is indeed pullback. Finally $W\cdot F^{\mathbb{T}} = F^{\hat{\mathbb{T}}}\cdot h$ follows from $\hat{\mu}_{h(A)} = h(\mu_A)$.

1.4 <u>Definition</u> let $J: \mathbb{C} \longrightarrow \mathbb{A}$ be a full faithful functor. <u>J-absolute</u> colimits are those colimits in $\mathbb{A}$ which are preserved by $J' = Lan_h(J)$.

1.4.1 <u>Remark</u> In view of 0:3.6 we could easily conclude that colimits are J-absolute, if, and only if they are preserved by all those functors which are the left Kan extensions of their restrictions on $\mathbb{C}$.

1.5 <u>Lemma</u> Let

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{W} & \mathbb{Y} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle V} \\ \mathbb{A} & \xrightarrow[H]{} & \mathbb{B} \end{array}$$

be a pullback diagram.

(1) If H has one of the properties: faithful, full, injective on objects, surjective on objects, then W has the same property.

(2) If isomorphisms are lifted (uniquely) or, resp. if they are created by V, then the same is true for for U.

(3) Let $D:\mathbb{I} \longrightarrow \mathbb{X}$ be a diagram for which UD has a limit which is preserved by H. If V creates limits of WD, then U creates limits of D. Corresponding statements hold for colimits.

1.5 <u>Proposition</u> Let $\mathbb{T} = (T,\eta,\mu)$ be a triple over $\mathbb{A}$, and $J:\mathbb{C} \longrightarrow \mathbb{A}$ a full faithful dense functor. Then the following are equivalent:

(1) $(\mathbb{A}^{\mathbb{T}}, F^{\mathbb{T}}\cdot J, U^{\mathbb{T}})$ is a J-relative Eilenberg-Moore

Situation.

(2) $\mathbb{T}$ preserves all J-absolute colimits.

<u>Proof</u>: Consider the following diagram:

$$\begin{array}{ccccc} & \overset{F^{\mathbb{T}}\cdot J}{\nearrow} & \mathbb{A}^{\mathbb{T}} & \xrightarrow{\;W\;} & \hat{\mathbb{C}}^{\hat{\mathbb{T}}} \\ & & \Big\downarrow U^{\mathbb{T}} & & \Big\downarrow U^{\hat{\mathbb{T}}} \\ \mathbb{C} & \xrightarrow[J]{} & \mathbb{A} & \xrightarrow[J']{} & \hat{\mathbb{C}} \end{array}$$

(1) $\Rightarrow$ (2) is included in Lemma 1.5.

(2) $\Rightarrow$ (1) First observe that for every $A \in \mathrm{Ob}\,\mathbb{A}$, $\hat{T}J'(A) = \mathrm{Lan}_h(J'U^{\mathbb{T}}F^{\mathbb{T}}J)(J'(A)) = \varinjlim_{J/A} J'U^{\mathbb{T}}F^{\mathbb{T}}JJ_A \overset{*}{\simeq} J'U^{\mathbb{T}} \cdot F^{\mathbb{T}}(\varinjlim JJ_A) = J'U^{\mathbb{T}}F^{\mathbb{T}}(A) = [J-, T(A)]$, where * follows from (2). For each $(A,a) \in \mathrm{Ob}\,\mathbb{A}^{\mathbb{T}}$, define $W(A,a) = (J'(A), [J-,a])$ and $W(\varphi) = [J-,\varphi]$. Then W is a well defined functor. For every $(X,x) \in \mathrm{Ob}\,\hat{\mathbb{C}}^{\hat{\mathbb{T}}}$, with $X = [J-,A]$, since J' is full faithful, x defines a unique $a: TA \longrightarrow A$. That this a is a structure map for $\mathbb{T}$ follows from x's being a structure map for $\hat{\mathbb{T}}$. This together with the full faithfulness of W proves the pullback of the right end square in the above diagram.

1.7 <u>Theorem</u> Given a J-relative adjoint situation $\mathbb{C} \xrightarrow{\;s\;} \mathbb{X} \xrightarrow{\;r\;} \mathbb{A}$ where for every $C \in \mathrm{Ob}\,\mathbb{C}$, $X \in \mathrm{Ob}\,\mathbb{X}$,

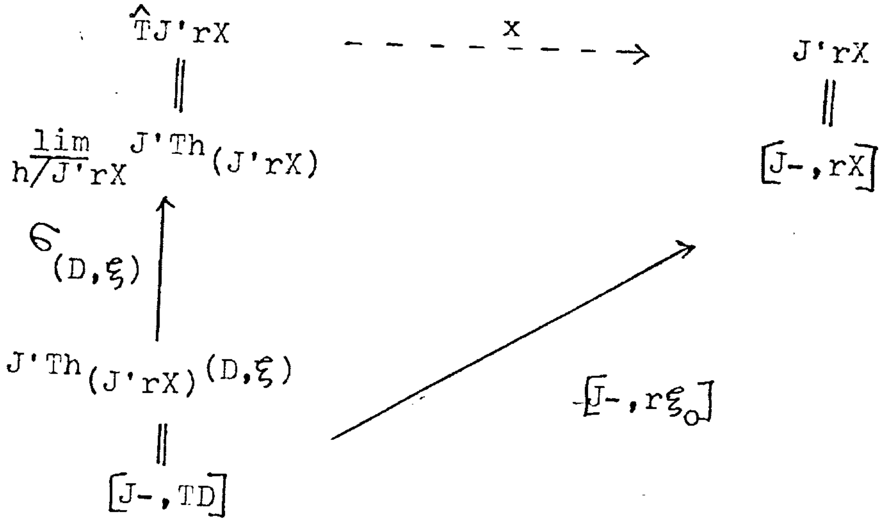
$$\alpha_{C,X} : [sC, X] \xrightarrow{\;\sim\;} [JC, rX]$$

is the adjunction transformation, there exists a unique functor $N: J_T \longrightarrow \mathbb{X}$ (resp. $K: \mathbb{X} \longrightarrow J^T$) such that

$N\cdot S_T = s$ (resp. $R^T\cdot K = r$).

Moreover $rN = R_T$ and $M = KN$. (see 1.2.2.1)

<u>Proof</u>: Define x to be the unique morphism making the following diagram commute for every $X \in Ob\mathcal{X}$:



where $\xi : h(D) \longrightarrow J'rX$ and $\xi_0 = \alpha^{-1}_{D,X}(\xi_D(1_D))$. This is possible since $\{[J-, r\xi_0]\}$ is natural in $(D,\xi) \in Ob(h/J'rX)$. Indeed for any $\varphi : (D,\xi) \longrightarrow (D',\xi')$ in $h/J'rX$, since $\alpha_{D,X}\cdot[s\varphi, X] = [J\varphi, X]\cdot\alpha_{D',X}$, $[J-, r\xi'_0]\cdot[J-, T\varphi] = [J-, r(\xi'_0\cdot s\varphi)] = [J-, r(\alpha^{-1}_{D',X}(\xi'_{D'}(1_{D'}))\cdot s\varphi) = [J-, r(\alpha^{-1}_{D',X}(\xi'_{D'}(1_{D'})\cdot J\varphi) = [J-, r(\alpha^{-1}_{D,X}(\xi_D(1_D)))] = [J-, r\xi_0]$.

We claim that $(J'rX, x) \in Ob\, \hat{\mathcal{C}}^{\hat{\mathbb{T}}}$.

Firstly, we show $x\cdot\hat{\eta}_{J'rX} = 1_{J'rX}$. Indeed for every $(D,\xi) \in Ob(h/J'rX)$, by the definition of $\hat{\eta}$ (see 1.0.1), $x\cdot\hat{\eta}_{J'rX}\cdot\xi = x\cdot\sigma_{(D,\xi)}\cdot\alpha_{-,sD}\cdot s_{-,D} = [J-, r\xi_0]\cdot\alpha_{-,sD}\cdot s_{-,D} = \alpha_{-,X}\cdot[s-,\xi_0]\cdot s_{-,D}$. But $\alpha_{D,X}\cdot[sD,\xi_0]\cdot s_{D,D}(1_D) = \alpha_{D,X}(\xi_0) =$

$\xi_D(1_D)$. By the Yoneda Lemma, $x\cdot\hat{\eta}_{J'rX}\cdot\xi = \xi$. Now since $\{\xi\}$ are colimit maps $x\cdot\hat{\eta}_{J'rX} = 1_{J'rX}$. Secondly, we show $x\cdot\hat{\mu}_{J'rX} = x\cdot\hat{T}(x)$. Indeed for every $(D,\varsigma)\in Ob(h/J'rX)$, by the definition of $\hat{\mu}$ (see 1.0.4), $x\cdot\hat{\mu}_{J'rX}\cdot\lambda_{(D,\varsigma)} = x\cdot\sigma_{(C,\xi)}$ $[J-,r\varsigma_0] = [J-,r(\xi_0\varsigma_0)]$, where ς' is such that $\sigma_{(C,\xi)}\cdot\varsigma'$ $=\varsigma$ and $\varsigma_0 = \alpha^{-1}_{D,sC}(\varsigma'_D(1_D))$, and $\xi_0 = \alpha^{-1}_{D,X}(\xi_D(1_D))$. On the other hand, $x\cdot\hat{T}(x)\cdot\lambda_{(D,\varsigma)} = x\cdot\sigma_{(D,x\cdot\varsigma)}$. Therefore by the definition of x, it is enough to show $\alpha^{-1}_{D,X}((x\cdot\varsigma)_D(1_D)) =$ $\xi_0\varsigma_0$. But $x\cdot\varsigma = x\cdot\sigma_{(C,\xi)}\cdot\varsigma' = [J-,r\xi_0]\cdot\varsigma'$. Since $\xi_0:sC$ $\longrightarrow X$ and $\alpha_{D,X}\cdot[sD,\xi_0] = [JD,r\xi_0]\cdot\alpha_{D,sC}$, $\alpha_{D,C}(\xi_0\varsigma_0) =$ $r\xi_0\cdot\varsigma'_D(1_D)$. Hence $(x\varsigma)_D(1_D) = [JD,r\xi_0]\cdot\varsigma'_D(1_D) = r\xi_0\cdot\varsigma'_D(1_D)$ $=\alpha_{D,X}(\xi_0\varsigma_0)$. Therefore $x\cdot\hat{\mu}_{J'rX}\cdot\lambda_{(D,\varsigma)} = x\cdot\hat{T}(x)\cdot\lambda_{(D,\varsigma)}$ and $x\cdot\hat{\mu}_{J'rX} = x\cdot\hat{T}(x)$ follows.

For any morphism $\varphi: X \longrightarrow Y$ in $\mathcal{X}$, we claim that $J'r\varphi:(J'rX,x) \longrightarrow (J'rY,y)$ is a homomorphism, where y is defined analoguously. For every $(D,\xi)\in Ob(h/J'rX)$, $J'r\varphi\cdot x\cdot\sigma_{(D,\xi)} = J'r\varphi\cdot[J-,r\xi_0] = [J-,r(\varphi\cdot\xi_0)]$. On the other hand, $y\cdot\hat{T}J'r\varphi\cdot\sigma_{(D,\xi)} = y\cdot\sigma_{(D,J'r\varphi\cdot\xi)}$. By the definition of y, it is enough to show $(J'r\varphi\cdot\xi)_D(1_D) =$ $\alpha_{D,Y}(\varphi\cdot\xi_0)$. Since $\alpha_{D,Y}[sD,\varphi] = [JD,r\varphi]\cdot\alpha_{D,X}$, we have $\alpha_{D,Y}(\varphi\,\xi_0) = r\varphi\cdot\xi_D(1_D)$. And $(J'r\varphi\cdot\xi)_D(1_D) = [JD,r\varphi]\cdot$ $\xi_D(1_D) = r\varphi\cdot\xi_D(1_D) = \alpha_{D,Y}(\varphi\cdot\xi_0)$. It follows then from the colimit property of $\hat{T}J'rX$, that $J'r\varphi\cdot x = y\cdot\hat{T}J'r\varphi$, which proves that $J'r\varphi$ is a $\hat{\mathbb{T}}$-homomorphism.

Define $H(X) = (J'rX, x)$ and $H(\varphi) = J'r\varphi$. H is a well defined functor. Moreover $J'r = U^{\hat{\mathbb{T}}}\cdot H$. Since the square in 1.2.2 is a pullback, there exists a unique functor $K: \mathbb{X} \longrightarrow J^T$ such that $r = R^T\cdot K$ and $W\cdot K = H$.

Define $N: J_T \longrightarrow \mathbb{X}$ as follows: the object map is assigning sC to $C \in \mathrm{Ob} J_T$ and for any $\varphi: C \longrightarrow D$ in J_T, i.e. $\varphi: TC \longrightarrow TD$ satisfying a certain condition, define $N(\varphi) = \alpha^{-1}_{C,sD}(\varphi\cdot\eta_C)$ where $\eta_C = \alpha_{C,sC}(1_{sC})$. We show that this rule is extended to a functor. For any $\varphi: C \longrightarrow D$, and $\psi: D \longrightarrow E$ in J_T, $N(\psi)\ N(\varphi) = \alpha^{-1}_{D,sE}(\psi\cdot\eta_D)\cdot\alpha^{-1}_{C,sD}(\varphi\cdot\eta_C)$ and $N(\psi\varphi) = \alpha^{-1}_{C,sE}(\psi\cdot\varphi\cdot\eta_C)$. Since $\alpha^{-1}_{C,sE}\cdot[JC,\psi] = [sC, \alpha^{-1}_{D,sC}(\psi\cdot\eta_D)]\cdot\alpha_{C,sD}$, and $r(\alpha^{-1}_{D,sE}(\psi\cdot\eta_D)) = \psi$, we have $\alpha^{-1}_{D,sE}(\psi\cdot\eta_D)\cdot\alpha^{-1}_{C,sD}(\varphi\cdot\eta_C) = \alpha^{-1}_{C,sD}(\psi\cdot\varphi\cdot\eta_C)$.

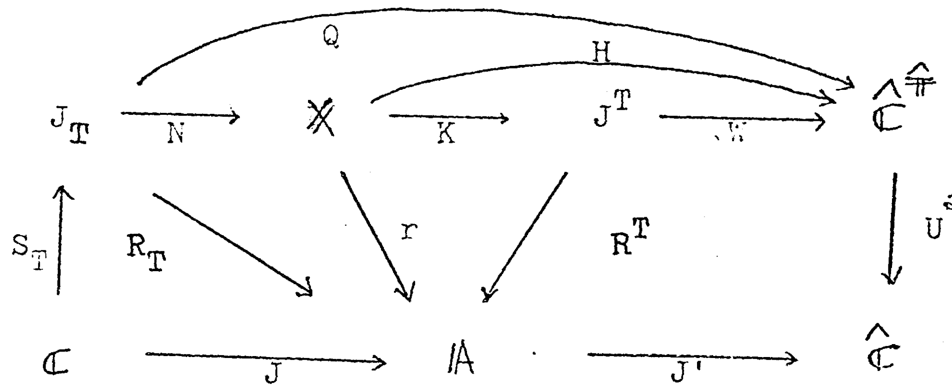
Finally we show $N\cdot S_T = s$. Indeed $N\cdot S_T(C) = sC$, and for any J_T-morphism , $NS_T(\varphi) = N(rs\varphi) = \alpha^{-1}_{C,sD}(rs\varphi\cdot\eta_C)$. But since $\alpha_{C,sD}(s\varphi) = rs\varphi\cdot\eta_C$, $NS_T(\varphi) = s\varphi$.

The last statement follows from the commutativity conditions and full faithfulness of J'.

1.7.1 <u>Definition</u> The functors N and K in 1.7 are called <u>comparison functors</u>.

Section 2: Characterization of Relative Eilenberg-Moore Situations.

2.1 Proposition Given a J-relative adjointness situation $\mathbb{C} \xrightarrow{s} \mathbb{X} \xrightarrow{r} \mathbb{A}$ with relative adjunction α, in view of the Definition 1.2.2 and the Theorem 1.7, we have the following diagram:



Then the following hold:

(1) $r = Lan_N(R_T)$, $R^T = Lan_K(r) = Lan_{KN}(R_T)$, and $U = Lan_Q(J'R_T)$.

(2) $Lan_N(J'R_T) = J'\ Lan_N(R_T)$ and $Lan_{KN}(J'R_T) = J'\ Lan_{KN}(R_T)$.

(3) $W = Lan_{KN}(Q)$, $H = Lan_N(Q)$ and $W = Lan_K(H)$.

Proof: (1) (a) We show that $r = Lan_N(R_T)$. For any $X \in Ob\mathbb{X}$, any $(C, \xi : sC \longrightarrow X) \in Ob(N/X)$, $\{r\xi : TC \longrightarrow rX\}$ is a natural cone in (C,ξ) $Ob(N/X)$. We show $\{r\xi\}$ are colimit maps. Let $\{\varsigma_{(C,\xi)} : TC \longrightarrow A\}$ be any natural cone. Since J is dense, and since for any $(D,d) \in Ob(J/rX)$, $(D, \alpha^{-1}_{D,X}(d)) \in Ob(N/X)$, and since $r\alpha^{-1}_{D,X}(d) \cdot \eta_D = d$ and

$\{\sigma_{(D,\bar{\alpha}^{1}_{D,X}(d))}\cdot\eta_D\}$ is natural in $(D,d)\in Ob(J/rX)$, there exists a unique morphism $f:rX \longrightarrow A$ such that $f\cdot r\bar{\alpha}^{1}_{D,X}(d)\eta_D = \sigma_{(D,\bar{\alpha}^{1}_{D,X}(d))}$. We need to show that for every (C,ξ) in N/X, $f\cdot r\xi = \sigma_{(C,\xi)}$. Since J is dense, it is enough to show that for any $h:JD \longrightarrow TC$, $f\cdot r\xi\cdot h = \sigma_{(C,\xi)}\cdot h$. Then for any $h:JD \longrightarrow TC$, $(D,\xi\cdot\bar{\alpha}^{1}_{D,sC}(h))\in Ob(N/X)$ and $r\bar{\alpha}^{1}_{D,sC}(h):$ $(D,\xi\cdot\bar{\alpha}^{1}_{D,sC}(h)) \longrightarrow (C,\xi)$ is an (N/X)-morphism. By the naturality of $\{\sigma_{(C,\xi)}\}$, $\sigma_{(C,\xi)}\cdot r\bar{\alpha}^{1}_{D,sC}(h) = \sigma_{(D,\xi\cdot\bar{\alpha}^{1}_{D,sC}(h))}$.

Now $f\cdot r\xi\cdot h \overset{①}{=} f\cdot r\bar{\alpha}^{1}_{D,X}(r\xi\cdot h)\cdot\eta_D \overset{②}{=} \sigma_{(D,\bar{\alpha}^{1}_{D,X}(r\xi\cdot h))}\cdot\eta_D \overset{③}{=}$

$\sigma_{(D,\xi\cdot\bar{\alpha}^{1}_{D,sC}(h))}\cdot\eta_D = \sigma_{(C,\xi)}\, r\bar{\alpha}^{1}_{D,sC}(h)\cdot\eta_D = \sigma_{(C,\xi)}\cdot h$,

where ① follows from $\alpha_{D,X} = [\eta_D, rX]\cdot r_{sD,X}$; ② follows from the definition of f; and ③ from the naturality of α. We show the uniqueness of such f. Suppose $g:rX \longrightarrow A$ with $g\cdot r\xi = \sigma_{(C,\xi)}$. Since J is dense, it is enough to see that $g\cdot d = f\cdot d$ for any $d:JD \longrightarrow rX$. The proof then follows from $g\cdot d = g\cdot r\bar{\alpha}^{1}_{D,X}(d)\cdot\eta_D = \sigma_{(D,\bar{\alpha}^{1}_{D,X}(d))}\cdot\eta D = f\cdot d$.

(b) We show $R^T = Lan_{KN}(R_T)$. Since $R^T S^T = T$, the comparison functor $J_T \longrightarrow T^T$ is precisely KN, hence by (a), the claim follows.

(c) $R^T = Lan_{KN}(R_T) = Lan_K(Lan_N(R_T)) = Lan_K(r)$.

(d) Since $J'P_T = \hat{U}^{\#}\cdot Q$, and Q is full faithful; and $\hat{U}^{\#}$ is cocontinuous $\hat{U}^{\#} = \mathrm{Lan}_Q(J'P_T)$.

(2) For any $X \in \mathrm{Ob}\mathcal{X}$, and for every $(C,\xi) \in \mathrm{Ob}(N/X)$, $\{[J-,r\xi] : [J-,TC] \longrightarrow [J-,rX]\}$ is natural in $(C,\xi)\in \mathrm{Ob}(N/X)$. We show that $\{[J-,r\xi]\}$ are colimit maps in $\hat{\mathcal{C}}$, Let $\{\delta_{(C,\xi)} : [J-,TC] \longrightarrow L$ be a natural cone in $(C,\xi)\in\mathrm{Ob}(N/X)$. Define $\bar{\delta} : [J-,rX] \longrightarrow L$ as follows: for every $D\in\mathrm{Ob}\mathcal{C}$, for every $h\in[JD,rX]$ define $\bar{\delta}_D(h) = \delta_{(D,\bar{\alpha}^{-1}_{D,X}(h)),D}(\eta_D)$. Then $\{\bar{\delta}_D\}$ is a natural transformation. Indeed for any $\varphi : E \longrightarrow D$ in $\mathcal{C}$, we first claim that $rs\varphi : (E,\bar{\alpha}^{-1}_{E,X}(h\cdot J\varphi)) \longrightarrow (D,\bar{\alpha}^{-1}_{D,X}(h))$ is an (N/X)-morphism. In other words, $\bar{\alpha}^{-1}_{D,X}(h)\cdot\bar{\alpha}^{-1}_{E,sD}(rs\varphi\cdot\eta_E) =$
$\bar{\alpha}^{-1}_{E,X}(h\cdot J\varphi)\cdot\bar{\alpha}^{-1}_{D,X}(h)\cdot\bar{\alpha}^{-1}_{E,sD}(rs\varphi\cdot\eta_E) = \bar{\alpha}^{-1}_{D,X}(h)\cdot\bar{\alpha}^{-1}_{E,sD}(\eta_D\cdot J\varphi)$
$\overset{①}{=} \bar{\alpha}^{-1}_{D,X}(h)\cdot s\varphi \overset{②}{=} \bar{\alpha}^{-1}_{E,X}(h\cdot J\varphi)$, where ① follows from $[J\varphi,rsD]\cdot\alpha_{D,sD} = \alpha_{E,sD}\,[s\varphi,sD]$ and ② from $[J\varphi,X]\cdot\alpha_{D,X} = \alpha_{E,X}\cdot[s\varphi,X]$.
Now by the naturality of $\{\delta_{(C,\xi)}\}$ in $(C,\xi)\in\mathrm{Ob}(N/X)$,

$\delta_{(E,\bar{\alpha}^{-1}_{E,X}(h\cdot J\varphi))} = \delta_{(D,\bar{\alpha}^{-1}_{D,X}(h))}\cdot[J-,T]$. Therefore for every $\varphi : E \longrightarrow D$ in $\mathcal{C}$ and for every $h : JD \longrightarrow rX$, $\bar{\delta}_E\cdot[J\varphi,rX]$
$(h) = \bar{\delta}_E(h\cdot J\varphi) = \delta_{(E,\bar{\alpha}^{-1}_{E,X}(h\cdot J\varphi)),E}(\eta_E) = \delta_{(D,\bar{\alpha}^{-1}_{D,X}(h)),E}$
$[JE,T\varphi](\eta_E) = \delta_{(D,\bar{\alpha}^{-1}_{D,X}(h)),E}(T\varphi\cdot\eta_E) = \delta_{(D,\bar{\alpha}^{-1}_{D,X}(h)),E}(\eta_D\cdot J\varphi)$
$= \delta_{(D,\bar{\alpha}^{-1}_{D,X}(h))}\,[J\varphi,TD]\,(\eta_D) \overset{*}{=} L(\varphi)\cdot\delta_{(D,\bar{\alpha}^{-1}_{D,X}(h)),D}(\eta_D) =$

$L(\varphi)\cdot\bar{\delta}_D(h)$, where (*) follows from the naturality of $\{\delta_{(D,\bar{\alpha}^{-1}_{D,X}(h)),D}\}$ in D. Therfore $\{\bar{\delta}_D\}$ is a natural transformation. We next show that for every $(C,\xi)\in Ob(N/X)$, $\bar{\delta}\cdot[J-,r\xi] = \delta_{(C,\xi)}$. Let $k:JD \longrightarrow TC$ be an $\mathbb{A}$-morphism. Since $r\bar{\alpha}^{-1}_{D,sC}(k):(D,\bar{\alpha}^{-1}_{D,X}(r\xi\cdot k)) \longrightarrow (C,\xi)$ is an (N/X)-morphism and since $\delta_{(C,\xi),D}\,[JD,r\bar{\alpha}^{-1}_{D,sC}(k)] = \delta_{(D,\bar{\alpha}^{-1}_{D,X}(r\xi\cdot k)),D}$, we have $\bar{\delta}_D(r\xi\cdot k) = \delta_{(D,\bar{\alpha}^{-1}_{D,X}(r\xi\cdot k)),D}(\eta_D) = \delta_{(C,\xi),D}(r\bar{\alpha}^{-1}_{D,sC}(k)\cdot\eta_D) = \delta_{(C,\xi),D}(k)$. For the uniqueness of such $\bar{\delta}$, let τ be another such that $\delta_{(C,\xi)} = \tau\cdot[J-,r\xi]$. For every $D\in Ob\,\mathbb{C}$, and for every $h:JD \longrightarrow rX$, $\tau_D(h) = \tau_D(r\bar{\alpha}^{-1}_{D,X}(h)\cdot\eta_D) = \tau_D\cdot[JD,r\bar{\alpha}^{-1}_{D,X}(h)](\eta_D) = \delta_{(D,\bar{\alpha}^{-1}_{D,X}(h))}(\eta_D) = \bar{\delta}_D(h)$. Consequently we have shown that $Lan_N(J'R_T) = J'r$.

$Lan_{KN}(J'R_T) = J'Lan_{KN}(R_T)$ follows from $Lan_N(J'R_T) = J\,Lan_N(R_T)$ for precisely the same reason as $R^T = Lan_{KN}(R_T)$ follows from $r = Lan_N(R_T)$.

(3) (a) Since $U^{\hat{\#}}\,Lan_{KN}(Q) \overset{①}{=} Lan_{KN}(U^{\hat{\#}}\cdot Q) = Lan_{KN}(J'R_T) \overset{②}{=} J'\cdot Lan_{KN}(R_T) \overset{③}{=} J'R^T = U^{\hat{\#}}\cdot W$, and $U^{\hat{\#}}$ creates all colimits, where ① follows from the cocontinuity of $U^{\hat{\#}}$, ② from (2) and ③ from (1), we conclude that $Lan_{KN}(Q) = W$.

(b) could be proved analoguously to (a), since $U^{\hat{\#}}\cdot Lan_N(Q) = Lan_N(U^{\hat{\#}}\cdot Q) = Lan_N(J'R_T) = J'\,Lan_N(R_T) = J'r =$

$U^{\hat{\mathbb{T}}} \cdot H$, we conclude $H = \mathrm{Lan}_N(Q)$.

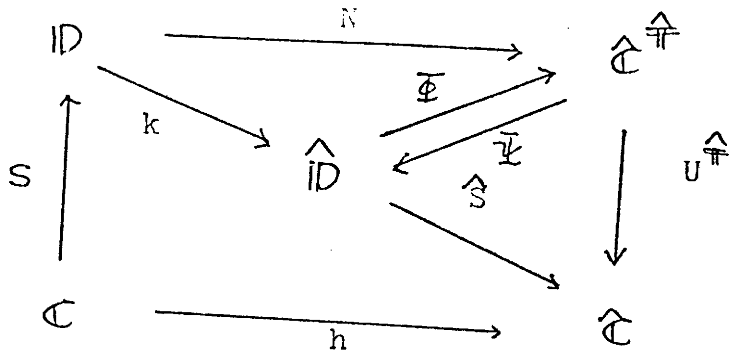
(c) Since $W = \mathrm{Lan}_{KN}(Q) = \mathrm{Lan}_K(\mathrm{Lan}_N(Q)) = \mathrm{Lan}_K(H)$, the claim follows.

2.2 <u>Corollary</u> In the same situation as in Proposition 2.1, if r reflects J-absolute colimits, then N is dense.

<u>Proof</u>: For any $X \in \mathrm{Ob}\,\mathbb{X}$, we have a natural cone $\{\xi : NN_X(C,\xi) \longrightarrow X\}$. $\{r\xi : rNN_X(C,\xi) \longrightarrow rX\}$ is a colimit cone by Proposition 2.1 (1), and by (2) the above colimit is preserved by J'. Since r reflects all such colimits, $\{\xi : NN_X(C,\xi) \longrightarrow X\}$ is a colimit cone.

2.3 <u>Theorem</u> Let $\mathbb{C}$ be a small category, and $\hat{\mathbb{T}} = (\hat{T}, \hat{\eta}, \hat{\mu})$ a triple over $\hat{\mathbb{C}}$. Assume that $\hat{T}$ is cocontinuous. Then $\hat{\mathbb{C}}^{\hat{\mathbb{T}}} \simeq \hat{\mathbb{D}}$, where $\mathbb{D}$ is the full image of $\mathbb{C}$ in $\hat{\mathbb{C}}^{\hat{\mathbb{T}}}$.

<u>Proof</u>: Let $\mathbb{C} \xrightarrow{S} \mathbb{D} \xrightarrow{N} \hat{\mathbb{C}}^{\hat{\mathbb{T}}}$ be the full image factorization of $F^{\hat{\mathbb{T}}} \cdot h$. It is well known that if $\hat{T}$ is cocontinuous, $U^{\hat{\mathbb{T}}}$ creates all colimits and therefore $\hat{\mathbb{C}}^{\hat{\mathbb{T}}}$ is cocomplete. (see 0:2.12) By the Corollary 2.2, N is dense. For each $D \in \mathrm{Ob}\,\mathbb{D}$, there exists a unique $C \in \mathrm{Ob}\,\mathbb{C}$ such that $s(C) = D$. Hence $[N(D),-] = [Ns(C),-] = [F^{\hat{\mathbb{T}}}h(C),-] \simeq [h(C), U^{\hat{\mathbb{T}}}-] = [h(C),-\ U^{\hat{\mathbb{T}}}(-)]$ is cocontinuous. Consider the following diagram, where $\Phi = \mathrm{Lan}_K(N)$ and $\Psi = \mathrm{Lan}_N(k)$, and k is the Yoneda Embedding. In view of Proposition 1.6, Proposition 2.1 is applicable to this situation.



In particular, $\hat{S}\cdot k = U^{\hat{\#}}\cdot N$ and $\hat{S} = U^{\hat{\#}}\cdot \Phi$. For every $X \in Ob\,\hat{\mathbb{C}}^{\#}$, $U^{\hat{\#}}(X) = U^{\hat{\#}}(\varinjlim_{N/X} NN_X) \overset{(1)}{=} \varinjlim_{N/X} U^{\hat{\#}} NN_X = \varinjlim_{N/X} \hat{S}\cdot kN_X \overset{(2)}{=} \hat{S}\cdot \varinjlim_{N/X} kN_X$ $\hat{S}\cdot\Psi(X)$, where (1) and (2) follow from the colimit creation properties of $U^{\hat{\#}}$ and $\hat{S}$ respectively. Therefore $U^{\hat{\#}} = \hat{S}\cdot\Psi$. The Characterization Theorem of presheaf categories [10] now show Φ snd Ψ are equivalences, and that they are isomorphisms of categories follows from the fact that $U^{\hat{\#}}$ creates all colimits.

2.4 <u>Corollary</u> Given a J-relative adjointness situation $s \dashv r$ mod J as in the proposition 2.1, $\hat{\mathbb{C}}^{\#}$ is isomorphic to J_T and Q can be identified with the Yoneda Embedding.

2.5 <u>Theorem</u> Let $\mathbb{C} \xrightarrow{s} \mathbb{X} \xrightarrow{r} \mathbb{A}$ be a J-relative adjointness situation as in Proposition2.1. Then the followings are equivalent:

(1) N is dense.

(2) K is full faithful.

(3) r reflects J-absolute colimits.

<u>Proof</u>: In view of Corollary 2.4, we identify $\hat{\mathbb{C}}^{\#}$ and

$\hat{J}_T$ in Proposition 2.1.

(1) $\Longrightarrow$ (2) By the Proposition 2.1, N is dense, if, and only if $H = \mathrm{Lan}_N(Q)$ is full faithful, if, and only if K is full faithful, since W is always full faithful.

(2) $\Longrightarrow$ (3) Let $\{\xi_i : X_i \longrightarrow X\}$ be any natural cone in $\mathbb{X}$ such that $\{r\xi_i : rX_i \longrightarrow rX\}$ is a colimit cone in $\mathbb{A}$, and $\{J'r\xi_i : J'rX_i \longrightarrow J'rX\}$ is also a colimit cone. Since $U^{\hat{\#}}$ creates all colimits, and since $U^{\hat{\#}} \cdot H = J'r$, $\{H\xi_i : HX_i \longrightarrow HX\}$ is a colimit cone. Since H is full faithful, H reflects colimits. Hence $\{X_i \longrightarrow X\}$ is a colimit cone in $\mathbb{X}$,

(3) $\Longrightarrow$ (1) is proved in the Corollary 2.2.

2.6 <u>Corollary</u> In the same situation as the Theorem 2.5, the followings are equivalent:

(1) $\mathbb{C} \xrightarrow{s} \mathbb{X} \xrightarrow{r} \mathbb{A}$ is a J-relative Kleisli situation.

(2) s is bijective on objects and r reflects all J-absolute colimits.

2.7 <u>Theorem</u> Let $\mathbb{C} \xrightarrow{s} \mathbb{X} \xrightarrow{r} \mathbb{A}$ be a J-relative adjointness situation. Let $\mathbb{C} \xrightarrow{t} \mathbb{Y} \xrightarrow{N} \mathbb{X}$ be the full image factorization of s. Consider

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{N'} & \hat{\mathbb{Y}} \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \hat{t}} \\ \mathbb{A} & \xrightarrow{J'} & \hat{\mathbb{C}} \end{array}$$

where $N' = Lan_N(k)$, and $k : \mathbb{Y} \longrightarrow \hat{\mathbb{Y}}$ the Yoneda Embedding, and $\hat{t} = [t^\circ, Ens]$. The followings are equivalent:

(1) The above square is a pullback square.

(2) r creates all J-absolute colimits of diagrams in $\mathbb{Y}$.

Proof: (1) $\Longrightarrow$ (2) is obvious in view of Lemma 1.5.

(2) $\Longrightarrow$ (1) First observe that rN reflects all J-absolute colimits, since N is full faithful. By the Corollary 2.6, $\mathbb{C} \xrightarrow{t} \mathbb{Y} \xrightarrow{rN} \mathbb{A}$ is a J-relative Kleisli situation. Since r creates all J-absolute colimits of diagrams in $\mathbb{Y}$, an investigation of the proof of the Theorem 2.5 produces that N is dense.

Now let $\Phi : \mathbb{Z} \longrightarrow \hat{\mathbb{Y}}$, and $\overline{\Psi} : \mathbb{Z} \longrightarrow \mathbb{A}$ be functors with $\hat{t}\Phi = J'\overline{\Psi}$. We show that for every $Z \in Ob\mathbb{Z}$, the colimit of the diagram $k/\Phi(Z) \longrightarrow \mathbb{Y} \longrightarrow \mathbb{X} \longrightarrow \mathbb{A}$ is $\overline{\Psi}(Z)$. Indeed for every $\xi : k(Y) \longrightarrow \Phi(Z)$, observe that $[\hat{t}\cdot k(Y), \hat{t}.\Phi(Z)] = [J'rN(Y), J'\overline{\Psi}(Z)]$ and for every ξ in $[k(Y), \Phi(Z)]$ put $\bar{\xi} = J'^{-1}_{rN(Y),\overline{\Psi}(Z)}(\hat{t}_{k(Y),\Phi(Z)}(\xi))$, which is an element of $[rN(Y), \overline{\Psi}(Z)]$. Then $\bar{\xi} : rN(Y) \longrightarrow \overline{\Psi}(Z)$ is a natural cone in $(Y, \xi) \in Ob(k/\Phi(Z))$. For given any $\varphi : Y \longrightarrow Y'$ in $\mathbb{Y}$ with $\xi'.k(\varphi) = \xi$, we need to see $\bar{\xi}'\cdot rN\varphi = \bar{\xi}$. This follows from a diagram chasing of the diagram in the next page.

Since $J'(\bar{\xi}) = \hat{t}(\xi)$, and since $\hat{t}$ creates all colimits and J' is full faithful, it follows that $\{\bar{\xi}\}$ is a colimit

$$
\begin{array}{ccccccc}
[k(Y),\bar{\Phi}(Z)] & \xrightarrow{\hat{t}} & [\hat{t}k(Y),\hat{t}\bar{\Phi}(Z)] & = & [J'rN(Y),J'\bar{\Psi}(Z)] & \xleftarrow{J'} & [rN(Y),\bar{\Psi}(Z)] \\
\uparrow{\scriptstyle [k(\varphi),\bar{\Phi}(Z)]} & & \uparrow{\scriptstyle [\hat{t}k(\varphi),\hat{t}\bar{\Phi}(Z)]} & & \uparrow{\scriptstyle [J'rN(\varphi),J'\bar{\Psi}(Z)]} & & \uparrow{\scriptstyle [rN(\varphi),\bar{\Psi}(Z)]} \\
[k(Y'),\bar{\Phi}(Z)] & \xrightarrow[\hat{t}]{} & [\hat{t}k(Y),\hat{t}\bar{\Phi}(Z)] & = & [J'rN(Y'),J'\bar{\Psi}(Z)] & \xleftarrow[J']{} & [rN(Y'),\bar{\Psi}(Z)]
\end{array}
$$

cone in $\mathbb{A}$, and is preserved by J'. By (2), there exists a unique $P(Z) \in \mathrm{Ob}\,\mathbb{X}$, with $\sigma_{(Y,\xi)}: N(Y) \longrightarrow P(Z)$ such that $\{\sigma_{(Y,\xi)}\}$ is a colimit; and $r\sigma_{(Y,\xi)} = \bar{\xi}$ and $rP(Z) = \tilde{\Phi}(Z)$. Furthermore $N'P(Z) = \bar{\Phi}(Z)$ follows from the colimit creation property of $\hat{t}$.

2.8 <u>Corollary</u> Let $\mathbb{C} \xrightarrow{s} \mathbb{X} \xrightarrow{r} \mathbb{A}$ be a J-relative adjointness situation. $(\mathbb{X}, s, r)$ is a J-relative Eilenberg-Moore Situation if and only if r creates all J-absolute colimits.

Chapter 3

LIMIT PRESERVING FUNCTORS

The concept of limit preservance of a functor is only meaningful when the domain category has limits. In this chapter we discuss a concept of limit preservance, which does not presuppose the existence of limits in the domain category, nor in the codomain category, and which therefore allows us to study the limit preserving functors even when the existence of limits is not known.

Section 1: Limit Preserving Functors.

In this section we define a concept of limit preservance of a functor and study the basic properties.

What forms the basis for limit preserving functors are the representable functors. Every set-valued functor admit a colimit representation of representable functors. The concept of limit preservance of functors can then be formulated as a commutativity of the limit under consideration and a certain colimit in the category of sets.

1.1 Definition Let $T:\mathbb{C}^{\circ} \longrightarrow Ens$ be an essentially small functor, where $\mathbb{C}$ is a category (not necessarily small). Let $\mathbb{X}$ be a small category, and $H:\mathbb{X} \longrightarrow \mathbb{C}$ a functor. The functor T is said to be H-continuous, if the canonical map

$$\underset{h/T}{\underrightarrow{\lim}}\ \underset{\mathbb{X}}{\underleftarrow{\lim}} \left[H(X), h_T(C,\xi)\right] \longrightarrow \underset{\mathbb{X}}{\underleftarrow{\lim}}\ \underset{h/T}{\underrightarrow{\lim}} \left[H(X), h_T(C,\xi)\right]$$

is an isomorphism, where h is the Yoneda Embedding into the category of all essentially small functors (see 0.1.4), and h/T is the comma category associated with T. h/T may not be small but by the essential smallness condition of T, it admits a cofinal functor from a small category, therefore the colimit indexed by h/T exists in Ens.

T is said to be *$\mathbb{X}$-continuous*, for a small category $\mathbb{X}$, if T is H-continuous for every $H; \mathbb{X} \longrightarrow \mathbb{C}$.

Let Δ be a class of small categories. T is said to be *Δ-continuous*, if T is $\mathbb{X}$-continuous for every $\mathbb{X}$ in Δ.

It is often convenient to say for T to be *continuous with respect to H (resp., $\mathbb{X}$, or Δ)* meaning H-(resp., $\mathbb{X}$-, or Δ-) continuity of T.

T is said to be *continuous* if T is continuous with respect to all small categories.

Let $T: \mathbb{C} \longrightarrow \mathbb{A}$ be a functor. We assume for convenience the smallness of $\mathbb{C}$. T is said to be *H-cocontinuous* for a small diagram $H: \mathbb{X} \longrightarrow \mathbb{C}$, if for every $A \in Ob\,\mathbb{A}$,

$$[T-, A] : \mathbb{C}^{\circ} \longrightarrow Ens$$

is H-continuous. The functor T is said to be *H-continuous*, if the dual functor

$$\mathbb{C}^{\circ} \longrightarrow \mathbb{C} \longrightarrow \mathbb{A} \longrightarrow \mathbb{A}^{\circ}$$

is H-cocontinuous.

Similarly we define $\mathbb{X}$- and Δ- continuity.

The definition indeed is reduced to the known concept of limit preservance in the case when limits exist.

1.2 <u>Proposition</u> Let $T: \mathbb{C}^{o} \longrightarrow Ens$ be a an essentially small functor. Let $H: \mathbb{X} \longrightarrow \mathbb{C}$ be a small diagram in $\mathbb{C}$. Suppose (L, λ) be a colimit of H in $\mathbb{C}$. Then T is H-continuous, if, and only if $(TL, T\lambda)$ is a limit of TH.

<u>Proof</u>: The proof is an easy consequence of the following commutative diagram:

$$\begin{array}{ccc}
\varinjlim_{h/T} \varprojlim_{\mathbb{X}} [H(X), h_T(C,\xi)] & \longrightarrow & \varprojlim_{\mathbb{X}} \varinjlim_{h/T} [H(X), h_T(C,\xi)] \\
\wr\vert & & \wr\vert \\
\varinjlim_{h/T} [L, h_T(C,\xi)] & & \varprojlim_{\mathbb{X}} \varinjlim_{h/T} [hH(X), hh_T(C,\xi)] \\
\wr\vert & & \wr\vert \\
\varinjlim_{h/T} [h(L), hh_T(C,\xi)] & & \varprojlim_{\mathbb{X}} [hH(X), T] \\
\wr\vert & & \downarrow Pr_X \\
[h(L), T] & & [hH(X), T] \\
\wr\vert & & \wr\vert \\
T(L) & \xrightarrow{T\lambda_X} & TH(X)
\end{array}$$

where Pr_X is the X-th projection.

1.3 <u>Corollary</u> Let $T: \mathbb{C} \longrightarrow \mathbb{A}$ be a functor, and $\mathbb{C}$ a small category. Let $H: \mathbb{X} \longrightarrow \mathbb{C}$ be a small diagram with with a colimit (L, λ) in $\mathbb{C}$. Then T is H-cocontinuous, if, and only if $(TL, T\lambda)$ is a colimit of TH.

This corollary is due to the proposition 1.2 and the fact that a cone μ is a colimit cone in $\mathbb{A}$, if, and only if for every $A \in Ob\mathbb{A}$, $[\mu, A]$ is a limit cone in Ens.

Before we prove some basic properties of our new continuous functors, we need a technical lemma:

1.4 Lemma Let $T:\mathbb{C}\longrightarrow\mathbb{A}$ and $S:\mathbb{A}\longrightarrow\mathbb{B}$ be two functors and let $\mathbb{C}$ be a small category. For every $B\in \mathrm{Ob}\,\mathbb{B}$, we have the canonical isomorphism

$$h/[ST-,B] \xleftarrow{\sim} \varinjlim_{(A,a)} h/[T-,A]$$

where the colimit runs over $(A,a)\in \mathrm{Ob}(k/[S-,B])$, and where h and k are the Yoneda Embeddings.

Proof: Consider for every $(A,a)\in \mathrm{Ob}(k/[S-,B])$,

$$h/[T-,A] \longrightarrow h/[ST-,B]$$

$$(c,\ \xi:TC\to A) \rightsquigarrow (C, STC \xrightarrow[S\xi]{} SA \xrightarrow[a]{} B)$$

This assignment defines a functor $\sigma_{(A,a)}$ and the family $\{\sigma_{(A,a)}\}$ is natural in $(A,a)\in \mathrm{Ob}(k/[S-,B])$. Let $\{\delta_{(A,a)}: h/[T-,A]\longrightarrow \mathbb{P}\}$ be a natural family. We define $\overline{\delta}$: $h/[ST-,B]\longrightarrow\mathbb{P}$ as follows: For $(C,\ \rho:STC\longrightarrow B)\in \mathrm{Ob}(h/[ST-,B])$, since (TC,ρ) $\mathrm{Ob}(k/[S-,B])$, we define $\overline{\delta}(C,\rho)=\delta_{(TC,\rho)}(C,Id_{TC})$. This takes care of the object part. For morphisms, let $g:(C,\rho)\longrightarrow(C',\rho')$ be a morphism in $h/[ST-,B]$. Then since $Tg:(TC,\rho)\longrightarrow(TC',\rho')$ is a morphism in $k/[S-,B]$, we have $\delta_{(TC',\rho')}\cdot h/[T-,Tg]=\delta_{(TC,\rho)}$. Hence $\delta_{(TC,\rho)}(C,Id_C)=\delta_{(TC',\rho')}(C,Tg)$. Therefore we define $\overline{\delta}(g)=\delta_{(TC',\rho')}(g)$ where $g:(C,Tg)\longrightarrow(C',Id_{TC'})$ in $h/\ T-,TC'$. It is obvious that the above defines a functor $\overline{\delta}$. Moreover $\overline{\delta}\cdot\sigma_{(A,a)}=\delta_{(A,a)}$. Indeed for $(C,\xi)\in \mathrm{Ob}(h/[T-,A])$, $\overline{\delta}\cdot\sigma_{(A,a)}(C,\xi)=\overline{\delta}(C,a\cdot S\xi)$ $=\delta_{(TC,a\cdot S\xi)}(C,Id_{TC})=\delta_{(A,a)}(C,\xi)$, where the last equality follows from the naturality of $\{\delta_{(A,a)}\}$ with respect to a morphism $\xi:(TC,a\cdot S\xi) \longrightarrow (A,a)$ in $h/[S-,B]$. Now let $\overline{\overline{\delta}}$ be another functor such that $\overline{\overline{\delta}}\cdot \sigma_{(A,a)} = \delta_{(A,a)}$. For any $(C,\rho)\in Ob(h/[ST-,B])$, $(TC,\rho)\in Ob(h/[S-,B])$. Hence $\overline{\overline{\delta}}(C,\rho) = \overline{\overline{\delta}}\cdot \sigma_{(TC,\rho)}(C,Id_{TC}) = \delta_{(TC,\rho)}(C,Id_{TC}) = \overline{\delta}(C,\rho)$, where the middle equality follows from the assumption. This completes the proof.

1.5 <u>Proposition</u> Let $\mathbb{C}$ be a small category. Let T, $T':\mathbb{C} \longrightarrow \mathbb{A}$, $S:\mathbb{A} \longrightarrow \mathbb{B}$ be functors.

(1) If T and T' are equivalent and T is H-continuous for a diagram $H:\mathbb{D} \longrightarrow \mathbb{C}$, then so is T'.

(2) If T and S are $\mathbb{X}$-cocontinuous for a small category $\mathbb{X}$, then ST is also $\mathbb{X}$-cocontinuous.

(3) If ST is H-cocontinuous for a small diagram $H:\mathbb{D} \longrightarrow \mathbb{C}$ and S is full faithful, T is H-cocontinuous.

<u>Proof</u>: (1) is obvious.

(2) We first observe that $h_{[ST-,B]}\cdot \sigma_{(A,a)} = h_{[T-,A]}$, where $h_{[ST-,B]}$ and $h_{[T-,A]}$ are the canonical functors. The following sequence of isomorphisms proves the claim: For any small diagram $H:\mathbb{X} \longrightarrow \mathbb{C}$,

$$\underset{h/[ST-,B]}{\varinjlim}\ \underset{X\in\mathbb{X}}{\varprojlim}\ [H(X), h_{[ST-,B]}] \overset{①}{\simeq} \underset{(A,a)}{\ } \underset{h/[S-,B]}{\varinjlim}$$

$$\underset{h/[T-,h_{[S-,B]}(A,a)]}{\varinjlim}\ \underset{X\in\mathbb{X}}{\varprojlim}[H(X), h_{[T-,h_{[S-,B]}(A,a)]}] \overset{②}{\simeq}$$

$$\underset{(A,a)}{\varinjlim}\ \underset{X}{\varprojlim}\ \varinjlim[H(X), h_{[T-,h_{[S-,B]}(A,a)]}] \simeq \underset{(A,a)}{\varinjlim}\ \underset{X}{\varprojlim}[TH(X),$$

$h[S-,B](A,a)] \overset{③}{\xrightarrow{\sim}} \varprojlim_X \varinjlim_{(A,a)} [TH(X), h[S-,B](A,a)] \overset{④}{\xrightarrow{\sim}}$

$\varprojlim_X \varinjlim_{h/[ST-,B]} [H(X), h[ST-,B]]$, where the isomorphisms ① and ④ follow from Lemma 1.4; ② from the $\mathbb{X}$-cocontinuity of T and ③ from $\mathbb{X}$-cocontinuity of S.

(3) It is enough to see for every $A \in Ob\,\mathbb{A}$, $[T-,A]$ is H-continuous. Since S is full faithful, $[T-,A] \simeq [ST-, SA]$ is H-continuous by assumption.

1.6 Corollary Let $\mathbb{C}$ be a small category and $T: \mathbb{C} \longrightarrow \mathbb{A}$ a functor. Let $J: \mathbb{B} \longrightarrow \mathbb{A}$ be a codense functor. For any small diagram $H: \mathbb{D} \longrightarrow \mathbb{C}$, T is H-cocontinuous, if, and only if for any $B \in Ob\,\mathbb{B}$, $[T-,JB]$ is H-continuous.

Proof: Since J is codense, the associated functor $J': \mathbb{A} \longrightarrow [\mathbb{B}, Ens]^{op}$, $A \rightsquigarrow [A, J-]$ is full faithful. The claim then follows from the proposition 1.5, and the pointwise construction of limits in a functor category.

1.7 Proposition Let $\mathbb{C}$ be a small category. For any $C \in Ob\,\mathbb{C}$, $[-,C]: \mathbb{C}^{\circ} \longrightarrow Ens$ is continuous.

This is obvious since the category $h/[-,C]$ has a terminal object.

1.8 Corollary: Every right adjoint functor is continuous, or, equivalently, every left adjoint functor is cocontinuous.

We have a slightly more general claim:

1.9 Proposition: Let $\mathbb{C}$ be a small category and $J: \mathbb{C} \longrightarrow \mathbb{A}$, $t: \mathbb{C} \longrightarrow \mathbb{Y}$ and $r: \mathbb{Y} \longrightarrow \mathbb{A}$ functors where t is J-relative left adjoint to r. Let $H:\mathbb{D} \longrightarrow \mathbb{C}$ be a small diagram such that J is H-cocontinuous. Then t is also H-cocontinuous. (Compare with 0.3.6)

<u>Proof</u>: For any $Y \in \mathrm{Ob}\,\mathcal{Y}$, we need to show the canonical map

$$\underset{t/Y}{\underrightarrow{\lim}}\ \underset{D}{\underleftarrow{\lim}}[H(D),t_Y] \longrightarrow \underset{D}{\underleftarrow{\lim}}\ \underset{t/Y}{\underrightarrow{\lim}}[H(D),t_Y]$$

is an isomorphism. From the relative adjointness, we conclude $t/Y \simeq J/rY$ and $t_Y \simeq J_{rY}$, and the isomorphism follows from the assumption J being H-cocontinuous.

Section 2: Limit and Colimit Commutation.

In this section we generalize the concept of cofilteredness of a category, obtaining a concept which is slightly more general than the corresponding generalization in [2], but still retaining the property of limit-colimit commutativity.

2.1 Definition Let $\mathbb{X}$ and $\mathbb{D}$ be small categories. $\mathbb{X}$ is said to be $\mathbb{D}$ -cofiltered, if for any $H: \mathbb{D} \longrightarrow \mathbb{X}$, the canonical map

$$\varinjlim_{X} \varprojlim_{D} [H(D), X] \longrightarrow \varprojlim_{D} \varinjlim_{X} [H(D), X]$$

is an isomorphism.

Let Δ be a class of small categories, $\mathbb{X}$ is said to be Δ-cofiltered, if $\mathbb{X}$ is $\mathbb{D}$-cofiltered for every category $\mathbb{D}$ in Δ.

2.2 Remark (1) Observe in 2.1 that $\varprojlim_{D} \varinjlim_{X} [H(D), X]$ is always a singleton set. Therefore in a $\mathbb{D}$-cofiltered category $\mathbb{X}$ the category of all cones from H is connected.

In particular there exists a cone from H to an object X of $\mathbb{X}$. In other words, the existence of a commutative completion of the diagram of H in $\mathbb{X}$.

(2) If $\mathbb{X}$ is $\mathbb{D}$-cofiltered and $G: \mathbb{X} \longrightarrow \mathbb{Y}$ a cofinal functor, then $\mathbb{Y}$ is also $\mathbb{D}$-cofiltered.

2.3 Example (1) Let Δ_α be the class of all α-small categories. Then Δ_α-cofiltered categories are precisely α-cofiltered categories. (See [2])

(2) The (Cat)-cofiltered categories are precisely the coabsolute categories. (See 4.4)

2.4 <u>Theorem</u> Let $\mathbb{C}$ be a small category, and $T:\mathbb{C}^{o} \longrightarrow \mathrm{Ens}$ a functor. Let Δ be a class of small categories and $h:\mathbb{C} \longrightarrow \hat{\mathbb{C}}$ the Yoneda Embedding. Then the followings are equivalent:

(1) T is Δ-continuous,

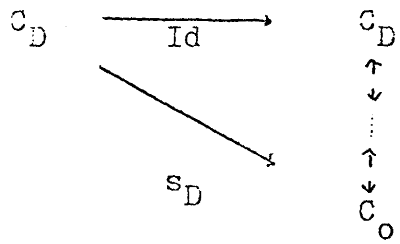
(2) h/T is Δ-cofiltered.

<u>Proof</u>: (1) $\Rightarrow$ (2) For every $\mathbb{D} \in \Delta$ and every $H:\mathbb{D}$ h/T, $D \rightsquigarrow (C_D, \xi_D:[-,C_D] \to T)$, consider the following commutative diagram:

$$\begin{array}{ccc} \varinjlim_{(C,\xi)} \varprojlim_{D} [H(D),(C,\xi)] & \longrightarrow & \varprojlim_{D} \varinjlim_{(C,\xi)} [H(D),(C,\xi)] \\ \varphi \downarrow & & \downarrow \\ \varinjlim_{(C,\xi)} \varprojlim_{D} [h_T H(D),h_T(C,\xi)] & \xrightarrow{\sim} & \varprojlim_{D} \varinjlim_{(C,\xi)} [h_T H(D),h_T(C,\xi)] \end{array}$$

We note that the one element of $\varprojlim \varinjlim [H(D),(C,\xi)]$ is mapped into $\{[\mathrm{Id}_{C_D},(C_D,\xi_D)]\}$ in $\varprojlim \varinjlim [h_T H(D),h_T(C,\xi)]$, where $[\mathrm{Id}_{C_D},(C_D,\xi_D)]$ is the equivalence class containing the image of the element Id_{C_D} of $[h_T H(D),h_T(C_D,\xi_D)]$.
We first claim that φ is one-one. Let $[\{r_D\},(C_o,\xi_o)]$, $[\{r'_D\},(C'_o,\xi'_o)]$ be two elements of $\varinjlim \varprojlim [H(D),(C,\xi)]$ such that their images under φ coincide. In view of the construction of colimits in Ens, it is enough to show: if for any (h/T)-morphism $u:(C_o,\xi_o) \longrightarrow (C'_o,\xi'_o)$ such that

$C_D \xrightarrow{r_D} C_o \xrightarrow{u} C'_o = C_D \xrightarrow{r'_D} C'_o$ in $\mathbb{C}$, then $r'_D = u \cdot r_D$ is also valid in h/T, which is obvious. We now show the ontoness. Since $\{[Id_{C_D}, (C_D, \xi_D)]\}$ belongs to $\varinjlim \varprojlim$ $[h_T H(D), h_T(C, \xi)]$, there exists $(C_o, \xi_o) \in Ob(h/T)$ and $s_D : h_T H(D) \longrightarrow h_T(C_o, \xi_o)$ such that $[Id_{C_D}, (C_D, \xi_D)] = [s_D, (C_o, \xi_o)]$. Then by the construction of colimit in Ens, there exists a finite sequence of (h/T)-morphisms between (C_D, ξ_D) and (C_o, ξ_o) making the following diagram commute:



But the sequence being in h/T, we could conclude that $\xi_o \cdot [-, s_D] = \xi_D$, claiming that s_D is indeed an (h/T)-morphism.

(2) $\Longrightarrow$ (1) Let $\mathbb{D} \in \Delta$ and $H : \mathbb{D} \longrightarrow \mathbb{C}$. We need to see the isomorphism of the canonical map

$$\varinjlim_{(C,\xi)\in h/T} \varprojlim_{D\in\mathbb{D}} [H(D), h_T(C, \xi)] \longrightarrow \varprojlim \varinjlim [H(D), h_T(C, \xi)]$$

First we show it onto. Since $\varprojlim \varinjlim [H(D), h_T(C, \xi)] \xrightarrow{\sim} \varprojlim_D [hH(D), T]$, let $x = \{\xi_D : [-, H(D)] \longrightarrow T\}$ be an element of $\varprojlim \varinjlim [H(D), h_T(C, \xi)]$. Consider the assignment $D \rightsquigarrow (H(D), \xi_D : [-, H(D)] \longrightarrow T)$. That this assignment can be extended to a functor $H_x : \mathbb{D} \longrightarrow h/T$ is obvious. Since

h/T is Δ-cofiltered, there exists a $(C_o, \xi_o) \in Ob(h/T)$ and $r_D: H(D) \longrightarrow C_o$ such that $\xi_D = \xi_o \cdot [-, r_D]$. We claim that $[\{r_D\}, (C_o, \xi_o)]$ is a preimage to $\{\xi_D\} = x$, i.e. for every $D \in Ob\,\mathbb{D}$, $[\{r_D\}, (C_o, \xi_o)] = \xi_D$ in $\varinjlim_{h/T} [H(D), h_T(C, \xi)]$. But this means precisely that $\xi_D = \xi_o \cdot [-, r_D]$, which is always the case.

We now show the one-oneness. Let $[\{r_D\}, (C_o, \xi_o)]$, $[\{r'_D\}, (C'_o, \xi'_o)]$ be two elements of $\varinjlim \varprojlim [H(D), h_T(C, \xi)]$ such that for every $D \in Ob\,\mathbb{D}$, $[r_D, (C_o, \xi_o)] = [r'_D, (C'_o, \xi'_o)] = \xi_D$. Let $x = \{\xi_D\}$ and define H_x as above. By Δ-cofilteredness of h/T, we could find $(\bar{C}_o, \bar{\xi}_o)$ and $\bar{r}_D: H(D) \longrightarrow \bar{C}_o$ such that $\xi_o \cdot [-, r_D] = \xi'_o \cdot [-, r'_D] = \bar{\xi}_D \cdot [-, \bar{r}_D] = \xi_D$. It is then easy to see that both $[\{r_D\}, (C_o, \xi_o)]$ and $[\{r'_D\}, (C'_o, \xi'_o)]$ are the image of the unique element of $\varinjlim \varprojlim [H_x(D), (C, \xi)]$. This concludes the proof.

Cofiltered categories as defined in 2.1 give rise to a commutativity condition, and are characterized by it:

2.5 <u>Theorem</u> Let $\mathbb{X}$ and $\mathbb{D}$ be small categories. The followings are equivalent:

(1) $\mathbb{X}$ is $\mathbb{D}$-cofiltered,

(2) For every category $\mathbb{C}$, every functor $G: \mathbb{X} \longrightarrow \mathbb{C}$, and $H: \mathbb{D} \longrightarrow \mathbb{C}$, the canonical map

$$\varinjlim_X \varprojlim_D [H(D), G(X)] \longrightarrow \varprojlim_D \varinjlim_X [H(D), G(X)]$$

is an isomorphism.

<u>Proof</u>: (1) $\Rightarrow$ (2) Let $T = \varinjlim_X hG(X)$ in $\hat{\mathbb{C}}$, where

h is the Yoneda Embedding. Then by 2.2, h/T is $\mathbb{D}$-cofiltered. By Theorem 2.4, T is then $\mathbb{D}$-continuous. The proof then follows from the sequence of isomorphisms: $\varprojlim_{D} \varinjlim_{X} [H(D), G(X)] \xrightarrow{\sim} \varprojlim_{D} \varinjlim_{X} [hH(D), hG(X)] \xrightarrow{\sim} \varprojlim_{D} [hH(D), T] \xrightarrow{\sim} \varprojlim_{D} \varinjlim_{h/T} [H(D), h_T] \xrightarrow{\sim} \varinjlim_{h/T} \varprojlim_{D} [H(D), h_T] \xrightarrow{\sim} \varinjlim_{X} \varprojlim_{D} [H(D), G(X)]$.

(2) $\Rightarrow$ (1) Consider any $H: \mathbb{D} \longrightarrow \mathbb{X}$ and $Id_{\mathbb{X}} : \mathbb{X} \longrightarrow \mathbb{X}$. Then using (2), we have

$$\varinjlim_{X} \varprojlim_{D} [H(D), X] \xrightarrow{\sim} \varprojlim_{D} \varinjlim_{X} [H(D), X]$$

which is what is required.

Section 3: The Category of Continuous Functors.

In this section we apply our results of the previous sections in the situation arising in the Sec. 2 of the Chap. 2.

3.1 Definition Let $\mathbb{C}$ be a small category and $J: \mathbb{C} \longrightarrow \mathbb{A}$ a functor. Denote by $[J]$ the class of all small categories $\mathbb{D}$ such that for any $H: \mathbb{D} \longrightarrow \mathbb{C}$, J is H-co-continuous. Denote by $\mathrm{Cont}_J(\mathbb{C})$ the full subcategory of $\hat{\mathbb{C}}$ determined by all those $T: \mathbb{C}^{\circ} \longrightarrow \mathrm{Ens}$ where T is $[J]$-continuous. (see 1.1)

3.2 Proposition Let $\mathbb{C}$ be a small category and $J: \mathbb{C} \longrightarrow \mathbb{A}$ a functor. Then the following holds:

(1) Every representable functor is $[J]$-continuous, hence there exists an embedding $h_J: \mathbb{C} \longrightarrow \mathrm{Cont}_J(\mathbb{C})$.

(2) There exists a functor $\overline{J}: \mathbb{A} \longrightarrow \mathrm{Cont}_J(\mathbb{C})$ making the following diagram commute:

$$\begin{array}{ccccc} \mathbb{C} & \xrightarrow{h_J} & \mathrm{Cont}_J(\mathbb{C}) & \longrightarrow & \hat{\mathbb{C}} \\ {\scriptstyle J}\downarrow & \overline{J}\nearrow & & \nearrow & \mathrm{Lan}_J(h) \\ \mathbb{A} & & & & \end{array}$$

(3) h_J is $[J]$-cocontinuous.

(4) $\mathrm{Cont}_J(\mathbb{C})$ is $[J]$-cofiltered cocomplete, i.e. for any $[J]$-cofiltered category $\mathbb{X}$ and for any $P: \mathbb{X} \longrightarrow \mathrm{Cont}_J(\mathbb{C})$, there exists a colimit of P in $\mathrm{Cont}_J(\mathbb{C})$.

Proof: (1) is obvious.

(2) follows from the fact that $Lan_J(h)(A)=[J-,A]$ is continuous for all those diagrams for which J is co-continuous. (see 0:3.2)

(3) Given any $[J]$-continuous $T:\mathbb{C}^\circ \longrightarrow \mathcal{E}ns$, we need to show the $[J]$-continuity of $[h_J-,T]$. The result then follows from the observation that $[h_J-,T] \xrightarrow{\sim} T$.

(4) Let $\mathbb{X}$ be a $[J]$-cofiltered and $P:\mathbb{X} \longrightarrow Cont_J(\mathbb{C})$ a diagram. Let $T = \varinjlim P(X)$ in $\hat{\mathbb{C}}$. We need to see T is $[J]$-continuous. Let $\mathbb{D}\in[J]$ and $H:\mathbb{D} \longrightarrow \mathbb{C}$ be a functor. The following sequence of isomorphisms establishes $[J]$-continuity of T:

$$\varinjlim_{h/T} \varprojlim_{D} [H(D),h_T] \overset{①}{\xrightarrow{\sim}} \varinjlim_{X} \varinjlim_{h/P(X)} \varprojlim_{D} [H(D),h_{P(X)}] \overset{②}{\xrightarrow{\sim}} \varinjlim_{X} \varprojlim_{D} [hH(D),P(X)] \overset{③}{\xrightarrow{\sim}} \varprojlim_{D} \varinjlim_{X} [hH(D),P(X)] \xrightarrow{\sim} \varprojlim_{D} \varinjlim_{X} [H(D),h_T],$$

where ① follows from $T = \varinjlim P(X)$, ② from $[J]$-continuity of $P(X)$ and ③ from Theorem 2.5.

3.3 <u>Remark</u> From the proof of 3.2, we conclude that the canonical embedding $i_J:Cont_J(\mathbb{C}) \longrightarrow \hat{\mathbb{C}}$ is $[J]$-cofiltered cocontinuous.

3.4 <u>Theorem</u> Let $\mathbb{C}$ be a small category and $J:\mathbb{C} \longrightarrow \mathbb{A}$ a functor. Then $Cont_J(\mathbb{C})$ is $[J]$-cofiltered cocompletion of $\mathbb{C}$ in the following sense:

(1) $Cont_J(\mathbb{C})$ is $[J]$-cofiltered cocomplete,

(2) For any $[J]$-cofiltered cocomplete category $\mathbb{B}$ and a functor $K:\mathbb{C} \longrightarrow \mathbb{B}$, there exists a $[J]$-

cofiltered cocontinuous functor $\overline{K}:\mathrm{Cont}_J(\mathbb{C}) \longrightarrow \mathbb{B}$ such that $\overline{K}\cdot h_J = K$.

Proof: (1) is Proposition 3.2 (4).

(2) We define $\overline{K}$ to be $\mathrm{Lan}_{h_J}(K)$, which exists in view of Theorem 2.5 and $[J]$-cofiltered cocompleteness of $\mathbb{B}$. (see 0:3.4) That $\mathrm{Lan}_{h_J}(K)=\overline{K}$ is $[J]$-cofiltered cocontinuous follows from the fact that K is i_J-relative left adjoint to $\mathrm{Lan}_K(h)$ and Proposition 1.9.

3.5 Remark Theorem 3.4 is also valid, even when $\mathbb{C}$ is not small. In this case we use the standard procedure by redefining $\mathrm{Cont}_J(\mathbb{C})$ as the category of essentially small $[J]$-continuous functors.

3.6 Theorem Let $\mathbb{C}$ be a small category and $J:\mathbb{C} \longrightarrow \mathbb{A}$ a functor. Consider $h_J:\mathbb{C} \longrightarrow \mathrm{Cont}_J(\mathbb{C})$ as in 3.2. Let $t: \mathbb{C} \longrightarrow \mathbb{Y}$ and $r: \mathbb{Y} \longrightarrow \mathrm{Cont}_J(\mathbb{C})$ be functors. Let $\mathbb{P}$ be the pullback of the diagram:

$$\begin{array}{ccc} \mathbb{P} & \longrightarrow & \hat{\mathbb{Y}} \\ \big\downarrow & & \big\downarrow \hat{t} \\ \mathrm{Cont}_J(\mathbb{C}) & \longrightarrow & \hat{\mathbb{C}} \end{array}$$

Then $\mathbb{P}$ is precisely the full subcategory of $\hat{\mathbb{Y}}$ consisting of all functors $R:\mathbb{Y}^{\circ} \longrightarrow \mathrm{Ens}$ such that R is tH-continuous for all $H:\mathbb{D} \longrightarrow \mathbb{C}$, and $\mathbb{D} \in [J]$.

Conversely, this property determines the pullback $\mathbb{P}$.

Proof: Let $R:\mathbb{Y}^{\circ} \longrightarrow \mathrm{Ens}$ be tH-continuous for all $H:\mathbb{D} \longrightarrow \mathbb{C}$, and $\mathbb{D} \in [J]$. We claim that $R\cdot t^{\circ}$ is $[J]$-continuous. For any $\mathbb{D} \in [J]$, any $H: \mathbb{D} \longrightarrow \mathbb{C}$, we need to show $\underset{h/R\cdot t^\circ}{\underrightarrow{\lim}}\ \underset{D}{\underleftarrow{\lim}}[H(D), h_{R\cdot t^\circ}] \xrightarrow{\sim} \underset{D}{\underleftarrow{\lim}}\ \underset{h/R\cdot t^\circ}{\underrightarrow{\lim}}[H(D), h_{R\cdot t^\circ}]$. We observe that $\underset{k/R}{\underrightarrow{\lim}}\ i_J r k_R = R\cdot t^\circ$, where $k: \mathcal{Y} \longrightarrow \hat{\mathcal{Y}}$ is the Yoneda Embedding and k_R is the diagram associated with R. Consider the following sequence of isomorphisms: $\underset{D}{\underleftarrow{\lim}}\ \underset{h/R t}{\underrightarrow{\lim}}[H(D), h_{R\cdot t^\circ}] \xrightarrow{\sim} \underset{D}{\underleftarrow{\lim}}$ $\underset{h/R\cdot t^\circ}{\underrightarrow{\lim}}[hH(D), hh_{R\cdot t^\circ}] \xrightarrow{\sim} \underset{D}{\underleftarrow{\lim}}[hH(D), \underset{k/R}{\underrightarrow{\lim}}\ i_J r k_R] \xrightarrow{\sim} \underset{D}{\underleftarrow{\lim}}$ $\underset{k/R}{\underrightarrow{\lim}}[hH(D), i_J r k_R] \xrightarrow{\sim} \underset{D}{\underleftarrow{\lim}}\ \underset{k/R}{\underrightarrow{\lim}}[h_J H(D), r k_R] \overset{①}{\xrightarrow{\sim}} \underset{D}{\underleftarrow{\lim}}$ $\underset{k/R}{\underrightarrow{\lim}}[tH(D), k_R] \overset{②}{\xrightarrow{\sim}} \underset{k/R}{\underrightarrow{\lim}}\ \underset{D}{\underleftarrow{\lim}}[tH(D), k_R] \overset{③}{\xrightarrow{\sim}} \underset{h/R\cdot t^\circ}{\underrightarrow{\lim}}\ \underset{D}{\underleftarrow{\lim}}$ $[H(D), h_{R\cdot t^\circ}]$, where ① follows from relative adjointness, ② from R being tH-continuous, and ③ follows from the following consideration: First observe that $h/R\cdot t^\circ \simeq \underset{(Y,\rho)}{\underrightarrow{\lim}}$ $h/rk_R(Y,\rho)$. Then consider the following diagram.

$$
\begin{array}{ccc}
\underset{h/R\cdot t^\circ}{\underrightarrow{\lim}}\ \underset{D}{\underleftarrow{\lim}}[H(D), h_{R\cdot t^\circ}] & \xrightarrow{\quad \Phi \quad} & \underset{k/R}{\underrightarrow{\lim}}\ \underset{D}{\underleftarrow{\lim}}[tH(D), k_R] \\
\wr\Big| & & \Big\uparrow \delta_{(Y,\rho)} \\
\underset{(Y,\rho)\in k/R}{\underrightarrow{\lim}}\ \underset{h/rk_R(Y,\rho)}{\underrightarrow{\lim}}\ \underset{D}{\underleftarrow{\lim}}[H(D), h_{rk_r(Y,\rho)}] & & \underset{D}{\underleftarrow{\lim}}[tH(D), Y] \\
\wr\Big| \sigma_{(Y,\rho)} & & \\
\underset{h/rY}{\underrightarrow{\lim}}\ \underset{D}{\underleftarrow{\lim}}[H(D), h_{rY}] & & \wr\Big| \ ② \\
\wr\Big| \ ① & & \\
\underset{D}{\underleftarrow{\lim}}\ \underset{h/rY}{\underrightarrow{\lim}}[H(D), h_{rY}] & \overset{\sim}{\text{———}} & \underset{D}{\underleftarrow{\lim}}[hH(D), rY]
\end{array}
$$

where $\varsigma_{(Y,\rho)}$ and $\delta_{(Y,\rho)}$ are colimit maps respectively; and ① follows from H-continuity of rY and ② from relative adjointness. Therefore we conclude an isomorphism Φ making the diagram commute. Therefore we have shown that R·t is J -continuous, fom which the tH-continuity of R follows. This then concludes the proof.

We write the category defined in 3.6 by $\mathrm{Cont}_{J,t}(\mathcal{Y})$.

3.7 Theorem In the same situation as in 3.6, the following holds:

(1) Given any $K:\mathbb{C} \longrightarrow \mathbb{B}$, and any [J]-cofiltered cocomplete category $\mathbb{B}$, if K is [J]-cocontinuous, then $\mathrm{Lan}_{h_J}(K)$ is concontinuous.

(2) Given any $F:\mathcal{Y} \longrightarrow \mathbb{Q}$, with cocomplete category $\mathbb{Q}$, if F is tH-cocontinuous for all $H:\mathbb{D} \longrightarrow \mathbb{C}$ where J is H-cocontinuous, then $\mathrm{Lan}_N(F)$ is cocontinuous, where N is the canonical embedding $\mathcal{Y} \longrightarrow \mathrm{Cont}_{J,t}(\mathcal{Y})$.

Proof: (1) $\mathrm{Lan}_{h_J}(K)$ is a left adjoint functor with $\mathrm{Lan}_K(h_J)$ as a right adjoint. This follows from that $\mathrm{Lan}_K(h_J)(B) = [K-,B] = [-,B]\cdot K$ and Proposition 1.5. The proof of (2) is analoguous to that of (1).

Section 4: Absoluteness.

In this section we study special cofiltered categories in the sense of 2.1, namely (Cat)-cofiltered categories. These categories arise naturally from absolute colimits [9] - those colimits which are preserved by every functor.

4.1 Definition Let H be a small diagram $\mathbb{X} \longrightarrow \mathbb{C}$. H is said to be coabsolute if every functor $F: \mathbb{C} \longrightarrow \mathbb{A}$ is H-cocontinuous.

Dually, we could define the absoluteness of a diagram.

4.2 Proposition Let $H: \mathbb{X} \longrightarrow \mathbb{C}$ be a small diagram. Let h be the Yoneda Embedding on $\mathbb{C}$. The followings are equivalent:

(1) H is a coabsolute diagram.

(2) For every $C \in \mathrm{Ob}\,\mathbb{C}$, $[C,-]$ is H-cocontinuous.

(3) Every $T: \mathbb{C}^{o} \longrightarrow \mathrm{Ens}$ is H-continuous.

(4) h is H-continuous.

The proof is trivial. One way to show is $(1) \Longrightarrow (2) \Longrightarrow (4) \Longrightarrow (3) \Longrightarrow (1)$.

4.3 Definition Let $\mathbb{X}$ be a small category. $\mathbb{X}$ is said to be coabsolute, if every functor $H: \mathbb{X} \longrightarrow \mathbb{C}$ for any $\mathbb{C}$ is coabsolute.

Dually, we define absolute categories with respect to limits.

4.4 <u>Theorem</u> Let $\mathbb{X}$ be a small category. The followings are equivalent:

(1) $\mathbb{X}$ is coabsolute.

(2) There exists a family of morphisms $\mu_X : X \longrightarrow X_o$ for all $X \in \mathrm{Ob}\,\mathbb{X}$ which is a cone from $\mathrm{Id}_{\mathbb{X}}$ to X_o.

(3) $\mathbb{X}$ is (Cat)-cofiltered.

<u>Proof</u>: (1) $\Longrightarrow$ (3) Let $\mathbb{D}$ be any small category, and $H : \mathbb{D} \longrightarrow \mathbb{X}$ a functor. Consider the following functor $H' : \mathbb{X} \longrightarrow \hat{\mathbb{D}}$, $X \rightsquigarrow [H-, X]$ and $\varprojlim_D : \hat{\mathbb{D}} \longrightarrow \mathrm{Ens}$, $T \rightsquigarrow \varprojlim T$. Since $\hat{\mathbb{D}}$ and Ens are cocomplete and $\mathbb{X}$ is coabsolute, we have the following sequence of isomorphisms: $\varprojlim_D \varinjlim_X [H(D), X] \xrightarrow{\sim} \varprojlim_D (\varinjlim_X H'(X))(D) \xrightarrow{\sim} \varinjlim_X (\varprojlim_D H'(X))(D) \xrightarrow{\sim} \varinjlim_X \varprojlim_D [H(D), X]$. Hence $\mathbb{X}$ is (Cat)-cofiltered.

(3) $\Longrightarrow$ (2) Since $\mathbb{X}$ is (Cat)-cofiltered, for $\mathrm{Id}_{\mathbb{X}} : \mathbb{X} \longrightarrow \mathbb{X}$, $\varinjlim_{X'} \varprojlim_X [X, X'] \simeq \varprojlim_X \varinjlim_{X'} [X, X'] \simeq$ one point set. Hence there exist an $X_o \in \mathrm{Ob}\,\mathbb{X}$, and a cone from $\mathrm{Id}_{\mathbb{X}}$ to X_o.

(2) $\Longrightarrow$ (1) Let $\{\mu_X : X \rightarrow X_o\}$ be a cone from $\mathrm{Id}_{\mathbb{X}}$ to X_o in the category $\mathbb{X}$. Let $H : \mathbb{X} \longrightarrow \mathbb{C}$ and $T : \mathbb{C}^o \longrightarrow \mathrm{Ens}$ be two functors. We wish to show T is H-continuous. i.e. the canonical map

$$\varinjlim_{h/T} \varprojlim_X [H(X), h_T] \xrightarrow{\varphi} \varprojlim_X \varinjlim_{h/T} [H(X), h_T]$$

is an isomorphism. Given any $\{[r_X : H(X) \longrightarrow h_T(C_X, \xi_X),$

$(C_X, \xi_X)]\}$ in $\varprojlim_X \varinjlim_{h/T} [H(X), h_T]$ we define a map ψ by assigning to it $[\{r_{X_0} \cdot H(\mu_X)\}, (C_{X_0}, \xi_{X_0})]$. We claim that ψ is a two-sided inverse to the canonical map φ . Given any $f = \{[r_X, (C_X, \xi_X)]\}$ in $\varprojlim_X \varinjlim_{h/T} [H(X), h_T]$, $\varphi\psi(f)$ $= \{[r_{X_0} \cdot H(\mu_X), (C_{X_0}, \xi_{X_0})]\}$. We need to show $[r_X, (C_X, \xi_X)] = [r_{X_0} \cdot H(\mu_X), (C_{X_0}, \xi_{X_0})]$, which is equivalent to $\xi_{X_0} \cdot hr_{X_0} \cdot hH(\mu_X) = \xi_X \cdot hr_X$. But since $\{\xi_X \cdot hr_X\} \in \varprojlim_X [hH(X), T]$, the result follows. Conversely, let $g = [\{s_X\}, (\overline{C}, \overline{\xi})]$ in the set $\varinjlim_{h/T} \varprojlim_X [H(X), h_T(C, \xi)]$, $\psi\varphi(g) = [\{s_{X_0} \cdot H(\mu_X)\}, (\overline{C}, \overline{\xi})]$. We need to see for all $X \in Ob\mathcal{X}$, $s_{X_0} \cdot H(\mu_X) = s_X$, which follows from the naturality of $\{s_X\}$. This completes the proof.

4.5 <u>Lemma</u> Let $\mathbb{D}$ be a small category and $H: \mathbb{D} \longrightarrow \mathbb{C}$ a functor. Let $(T, \lambda) = \varinjlim hH(D)$, where h is the Yoneda Embedding. Then there exists a cofinal functor $P: \mathbb{D} \longrightarrow h/T$ $D \rightsquigarrow (H(D), \lambda_D)$ such that $h_T \cdot D = H$.

<u>Proof</u>: For any $(C, \xi) \in Ob(h/T)$, $\varinjlim_D [(C, \xi), (H(D), \lambda_D)] \simeq \varinjlim_D [([-,C], \xi), ([-,H(D)], \lambda_D)] \simeq [([-,C], \xi), \varinjlim_D ([-,H(D)], \lambda_D)] \simeq [([-,C], \xi), (T, id_T)]$ $\simeq$ singleton set, where the last three hom sets are taken

in $\hat{\mathbb{C}}/T$, and the last isomorphism follows since (T, id_T) is a terminal object in $\hat{\mathbb{C}}/T$.

4.6 Theorem In the same situation as in 4.5, the followings are equivalent:

(1) h/T is coabsolute.

(2) T is continuous.

(3) H is coabsolute.

Proof: The equivalence of (1) and (2) follows from the Theorem 4.4 and the Theorem 2.4. (1) $\Longrightarrow$ (3) is trivial. It remains to prove (3) $\Longrightarrow$ (2). For any small diagram $K: \mathbb{X} \longrightarrow \mathbb{C}$, consider the following sequence of isomorphism:

$$\varprojlim_{X} \varinjlim_{h/T} [K(X), h_T] \overset{①}{\xrightarrow{\sim}} \varprojlim_{X} \varinjlim_{D} [K(X), H(D)] \overset{②}{\xrightarrow{\sim}} \varprojlim_{X} [hK(X), \varinjlim_{D} hH(D)] \overset{③}{\xrightarrow{\sim}} \varinjlim_{D} \varprojlim_{X} [hK(X), hH(D)] \xrightarrow{\sim} \varinjlim_{D} \varprojlim_{X} [K(X), H(D)] \overset{④}{\xrightarrow{\sim}} \varinjlim_{h/T} \varprojlim_{X} [K(X), h_T],$$

where the isomorphisms ① and ④ follow from Lemma 4.5; ② from the pointwise construction of colimits in $\hat{\mathbb{C}}$; and ③ from the H-cocontinuity of the functor $\varprojlim_{X} [hK(X), h-]$ and Proposition 4.2.

4.7 Remark From the Theorem 4.6, we conclude that a coabsolute diagram can be factorized through a diagram from a coabsolute index category. This coabsolute index category may not be small. In the following we find a small coabsolute category for a coabsolute diagram so that the colimit of the coabsolute diagram could be represented as a colimit of a diagram indexed by this coabsolute category.

4.8 Theorem Let $\mathbb{D}$ be a small category and $H: \mathbb{D} \to \mathbb{C}$ a coabsolute diagram. Suppose $(L, \lambda) = \varinjlim H(D)$ in $\mathbb{C}$. Then there exists a coabsolute category $\mathbb{X}$ and a commuting diagram

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{Q} & \mathbb{X} \\ & {\scriptstyle H}\searrow \quad \swarrow{\scriptstyle K} & \\ & \mathbb{C} & \end{array}$$

such that $\varinjlim K(X) = (L, \bar{\lambda})$ and $\bar{\lambda}_{Q(D)} = \lambda_D$.

Proof: We recall the characterization theorem of absolute colimits in [9], from which we have: There exists $D_o \in \mathrm{Ob}\,\mathbb{D}$ and $d_o: L \longrightarrow H(D_o)$ such that

(1) for all $D \in \mathrm{Ob}\,\mathbb{D}$, $(D_o, d_o \cdot \lambda_D)$ and $(D, id_{H(D)})$ are connected in $H(D)/H$, and

(2) $\lambda_{D_o} \cdot d_o = id_L$.

Let Σ be a set bijective with the $\mathrm{Ob}\,\mathbb{D}$. Let $b: \Sigma \xleftarrow{\sim} \mathrm{Ob}\,\mathbb{D}$ be the bijection. Let $\mathbb{D}'$ be the underlying diagram scheme of the category $\mathbb{D}$, and $\mathbb{D}''$ be the diagram scheme obtained from $\mathbb{D}'$ by adding a set Σ of arrows where for $D \in \mathrm{Ob}\,\mathbb{D}$, the origin of $b(D) = D$ and the end of $b(D) = D_o$. For $\mathbb{D}''$ we consider a set Φ of commutativity condition, which consists of all those coming from the category $\mathbb{D}$, and all pairs $(b(D')\cdot f, b(D))$ for all $f: D \longrightarrow D'$. We set $\mathbb{X} = \mathcal{P}(\mathbb{D}''/\Phi)$, the path category as in (0:4.8). We note that $\mathbb{D}$ and $\mathbb{X}$ have the same objects. For $f \in \mathrm{Mor}\,\mathbb{D}$, we define $Q(f)$ to be the equivalence class of the path $\{f\}$ of length one. $K: \mathbb{X} \longrightarrow \mathbb{C}$ is defined as follows: $K(D) = H(D)$. We consider an assignment $\mathbb{D}'' \longrightarrow \mathbb{C}$, $b(D) \rightsquigarrow d_o \cdot \lambda_D$, and $f \rightsquigarrow H(f)$ for all other f in $\mathrm{Mor}\,\mathbb{D}$. This assignment transforms the commutativity condition into identities in $\mathbb{C}$, since $d_o \cdot \lambda_{D'} \cdot H(f) = d_o \cdot \lambda_D$. By (0:4.8) we have a functor $K: \mathbb{X} \longrightarrow \mathbb{C}$. We observe that $\mathbb{X}$ is coabsolute with $[b(D)]: D \longrightarrow D_o$ as a cone from $Id_{\mathbb{X}}$ into D_o, where $[b(D)]$ is the equivalence class of the path $\{b(D)\}$ of length 1. In order to show that $\varinjlim K(X) \xrightarrow{\sim} (L, \lambda)$, it is enough to show: for every $C \in \mathrm{Ob}\,\mathbb{C}$, the canonical map

$$\varprojlim_X [K(X), C] \longrightarrow \varprojlim_D [H(D), C]$$

is an isomorphism. Let $\{r_D : D \in \mathrm{Ob}\,\mathbb{D}\}$ be an element of $\varprojlim [H(D), C]$. We need to see that it is still natural with respect to the category $\mathbb{X}$. It suffices to show that $r_D = r_{D_o} \cdot d_o \cdot \lambda_D$ for all $D \in \mathrm{Ob}\,\mathbb{D}$. But since $\lambda_{D_o} \cdot d_o = 1_L$; $r_D = r \cdot \lambda_D = r(\lambda_{D_o} \cdot d_o) \cdot \lambda_D = r_{D_o} \cdot d_o \cdot \lambda_D$. This completes the proof.

4.9 <u>Example</u> We consider a coequalizer diagram

$$A \underset{v}{\overset{u}{\rightrightarrows}} B \xrightarrow{e} C$$

Suppose this is coabsolute, we have then

$$A \xleftarrow{s} B \xleftarrow{r} C$$

such that $er = 1_C$, $us = 1_B$ and $vs = rs$. Obviously $vs = re = f$ is an idempotent. We now consider the coabsolute category

$$A \underset{v}{\overset{u}{\rightrightarrows}} B \circlearrowleft f.$$

Clearly every morphism $k: B \longrightarrow X$ with $ku = kv$ also satisfies $kf = k$. Therefore C is also a colimit of the diagram indexed by the coabsolute category.

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