VARIETIES OF DISTRIBUTIVE PSEUDO-COMPLEMENTED LATTICES

THE LATTICE OF VARIETIES

OF

DISTRIBUTIVE PSEUDO-COMPLEMENTED LATTICES

By

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SCOPE AND CONTENTS: The lattice of varieties of distributive pseudo-complemented lattices is completely described, viz. a chain of type $\omega + 1$. Moreover, each variety is determined by a single equation in addition to those equations which define distributive pseudo-complemented lattices. Characterizations of distributive pseudo-complemented lattices satisfying a certain equation are given which turn out to be generalizations of L. Nachbin's result for Boolean algebras and the results for Stone algebras obtained by G. Grätzer-E. T. Schmidt and J. C. Varlet.

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INTRODUCTION

The concept of variety (i.e. equational class) of algebras of a given type was first introduced in 1935 by Garrett Birkhoff [3]. In his paper, he has shown that a class of algebras of a given type forms a variety if and only if it is closed under the formation of homomorphic images, subalgebras and direct products. In the same paper, he has also shown that, if one ignores the foundation problems which in any case can be easily circumvented, the varieties of algebras of a given type form a complete lattice under class inclusion.

Recently there has been much interest in investigating the properties of this lattice for algebras of a given type, and, if possible, giving a complete description of this lattice.

The former problem has been dealt with more successfully than the latter. For example, the lattice of varieties of lattices has been extensively studied by several authors. Results in this area may be found in K. A. Baker [1], G. Grätzer [14], B. Jónsson [18], [19] and R. McKenzie [21],[22].

However, up to the present, the only complete descriptions of (non-trivial) lattices of varieties of algebras of a given type are the lattice of varieties of algebras with one unary operation, the lattice of varieties of idempotent semigroups, and the lattice of varieties of commutative monoids given by E. Jacob-R. Schwabauer [17], J. A. Gerhard [10], [11] and T. J. Head [16], respectively.

In this thesis, we investigate the lattice of varieties of distributive pseudo-complemented lattices and give a complete description of this lattice which turns out to be a chain of type $\omega + 1$. Further, each variety in the lattice is completely determined by an equation in addition to those equations characterizing distributive pseudo-complemented lattices. An outline of the thesis, by sections, follows:

Section 1: We show that the class of all distributive pseudo-complemented lattices is equational (i.e. a variety).

Section 2: The equation (E_n) is introduced and a characterization of the class $(E_n)^*$ of distributive pseudo-complemented lattices which satisfy the equation (E_n) is given. This characterization is a generalization of L. Nachbin's result for Boolean algebras (L. Nachbin [20]) and the results for Stone algebras obtained by G. Grätzer-E. T. Schmidt [12] and J. C. Varlet [24].

Section 3: In this section, we construct a subdirectly irreducible DP-algebra (i.e. distributive pseudo-complemented lattice regarded as an algebra) $\overline{B}$ from a given Boolean algebra B. The classes $\mathcal{B}_n$, $\mathcal{R}_n$, $\mathcal{G}_n$ are introduced and it is shown that $(E_n)^* = \mathcal{B}_n = \mathcal{R}_n = \mathcal{G}_n$.

Section 4: We show that the class of all DP-algebras is generated by its finite members. Moreover, a complete description of the lattice is given. We also show, as a corollary of the above result, that $\overline{B}_n (n \geq 0)$ are exactly the finite subdirectly irreducible DP-algebras.

Section 5: Two characterizations of generalizations of relative Stone algebras are given.

Section 1

Distributive Pseudo-complemented Lattices

In this section, we shall show that the class of all distributive pseudo-complemented lattices is equational.

A pseudo-complemented lattice is a lattice L with zero element 0 such that for each element $a \in L$ there exists an element $a^* \in L$ so that, for all $x \in L$, $a \wedge x = 0$ if and only if $x \leq a^*$. It is evident that for each element a of a pseudo-complemented lattice L, the element a^* is uniquely determined by $a \in L$. Thus * can be regarded as a unary operation on L.

Pseudo-complemented lattices form an extensively studied class of lattices and have been explored in detail by J. C. Varlet [25]. However, in his paper, the most interesting results require at least the assumption of modularity, sometimes distributivity.

Examples of distributive pseudo-complemented lattices are Boolean lattices, the lattice of all open subsets of a topological space, the lattice of all ideals of a distributive lattice with zero, the lattice of all congruence relations of an arbitrary lattice and the Lindenbaum algebra of intuitivistic logic.

It is obvious that every pseudo-complemented lattice contains the unit, viz. 0^*. It therefore follows that every pseudo-complemented lattice L can be regarded as an algebra $(L; (\vee, \wedge, *, 0, 1))$ of the type (2, 2, 1, 0, 0). In this paper, we are interested only in distributive pseudo-complemented lattices. For simplicity, we call such a lattice, regarded as an algebra, a DP-algebra. Thus, a DP-algebra is an algebra $(L; (\vee, \wedge, *, 0, 1))$ of the type (2, 2, 1, 0, 0) such that $(L; (\vee, \wedge, 0, 1))$ is a distributive lattice with zero (the smallest element) 0 and unit (the largest element) 1, and * is the pseudo-complementation.

The following proposition lists some fundamental properties of pseudo-complemented lattices:

Proposition 1. Let $(L; (\vee, \wedge, *, 0, 1))$ be a pseudo-complemented lattice. Then, for all $a, b \in L$, we have

(i) $0^* = 1$

(ii) $a \wedge a^* = 0$

(iii) $a \leq b \Rightarrow b^* \leq a^*$

(iv) $a \leq a^{**}$, i.e. $a \vee a^{**} = a^{**}$

(v) $a^{***} = a^*$

(vi) $a \wedge b = 0 \Leftrightarrow a \wedge b^{**} = 0$

(vii) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

(viii) $(a \vee b)^* = a^* \wedge b^*$.

Proof: (i)—(iv) follow immediately from the definition.

(v). By (iv), $a \leq a^{**}$, hence $a^{***} \leq a^*$ by (iii). Also, $a^* \leq a^{***}$ by (iv). Thus $a^{***} = a^*$.

(vi). Clearly, $a \wedge b^{**} = 0 \Rightarrow a \wedge b = 0$. Assume $a \wedge b = 0$.

Then $a \leq b^*$ and hence $b^{**} \leq a^*$, i.e. $a \wedge b^{**} = 0$.

(vii). Clearly, $(a \wedge b)^{**} \leq a^{**} \wedge b^{**}$. By applying (vi) repeatedly, we have $a \wedge b \wedge (a \wedge b)^* = 0 \Rightarrow a^{**} \wedge b^{**} \wedge (a \wedge b)^* = 0$ $\Rightarrow a^{**} \wedge b^{**} \leq (a \wedge b)^{**}$. Consequently, $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

(viii). It is obvious that $(a \vee b)^* \leq a^* \wedge b^*$. It remains to show that $a^* \wedge b^* \leq (a \vee b)^*$. But this follows from the following observation:

$$(a \vee b)^* \geq a^* \wedge b^* \Leftrightarrow (a \vee b)^{**} \leq (a^* \wedge b^*)^*$$

$$\Leftrightarrow a \vee b \leq (a^* \wedge b^*)^*$$

$$\Leftrightarrow a,\ b \leq (a^* \wedge b^*)^*$$

$$\Leftrightarrow a \wedge (a^* \wedge b^*) = 0 = b \wedge (a^* \wedge b^*).$$

<u>Remark</u>: The dual of (viii) is not true in general. For example, let R be the real line (with usual topology) and $\mathcal{L}$ the lattice of all open subsets of R. Consider $A = \{x \in R \mid x < 0\}$ and $B = \{x \in R \mid x > 0\}$. Then

$$(A \wedge B)^* = \emptyset^* = IC\,\emptyset = R$$

$$A^* \vee B^* = ICA \vee ICB = B \vee A = R - \{0\}$$

and hence

$$(A \wedge B)^* \neq A^* \vee B^*.$$

The following theorem shows that the class of all DP-algebras is equational.

THEOREM 1. An algebra $(A; (\vee, \wedge, *, 0, 1))$ of the type $(2, 2, 1, 0, 0)$ is a DP-algebra iff $(A; (\vee, \wedge, 0, 1))$ is a distributive lattice with zero 0 and unit 1 and satisfies the following equations:

(i) $a \wedge a^* = 0$

(ii) $a \vee a^{**} = a^{**}$

(iii) $(a \vee b)^* = a^* \wedge b^*$

(iv) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

(v) $0^* = 1$.

In particular, the class of all DP-algebras is equational.

Proof: Proposition 1 shows that DP-algebras satisfy the conditions. Assume conversely that (i) - (v) are satisfied in a distributive lattice A with zero 0 and unit 1. We have to show that, for all $a, x \in A$, $a \wedge x = 0$ iff $x \leq a^*$. Clearly, by (i), $x \leq a^*$ implies $a \wedge x = 0$. Assume now that $a \wedge x = 0$, then we have

$$
\begin{aligned}
x &\leq x^{**} && \text{(by (ii))} \\
&= x^{**} \wedge 1 \\
&= x^{**} \wedge 0^* && \text{(by (v))} \\
&= x^{**} \wedge (a^* \wedge a^{**})^* && \text{(by (i))} \\
&= x^{**} \wedge (a \vee a^*)^{**} && \text{(by (iii))} \\
&= (x \wedge (a \vee a^*))^{**} && \text{(by (iv))} \\
&= ((x \wedge a) \vee (x \wedge a^*))^{**} && \text{(by distributivity)}
\end{aligned}
$$

$= (x \wedge a^*)^{**}$ (since $a \wedge x = 0$)

$= x^{**} \wedge a^{***}$ (by (iv))

$= x^{**} \wedge a^*$ (since $a^* = a^{***}$ by (ii) and (iii))

$\leqslant a^*$.

Remark: R. Balbes and A. Horn [2] have shown that an algebra $(A; (\wedge, *, 0))$ of the type (2, 1, 0) is a pseudo-complemented semi-lattice iff it satisfies the following equations:

(i) $a \wedge b = b \wedge a$

(ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(iii) $a \wedge a = a$

(iv) $0 \wedge a = 0$

(v) $a \wedge (a \wedge b)^* = a \wedge b^*$

(vi) $a \wedge 0^* = a$

(vii) $0^{**} = 0$.

In particular, the class of all DP-algebras is equational.

Section 2

A Characterization of the classes $(E_n)^*$

For DP-algebras we consider the following equations ($n \geq 1$):

$$(E_n) \qquad (x_1 \wedge \ldots \wedge x_n)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge \ldots \wedge x_i^* \wedge \ldots \wedge x_n)^* = 1.$$

It is evident that for n = 1, the equation (E_n) becomes

$$(E_1) \qquad x^* \vee x^{**} = 1.$$

The problem of characterizing the class of DP-algebras satisfying the equation (E_1) was first raised by M. H. Stone; since then several solutions have been offered - the first was given by G. Grätzer-E.T. Schmidt [12] who named this class of DP-algebras Stone algebras. Later solutions were given by J. C. Varlet [24], O. Frink [9], G. Grätzer [13] and G. Bruns [5]. Other results concerning Stone algebras may be found in R. Balbes-A. Horn [2], C. C. Chen-G. Grätzer [6], [7], T. P. Speed [23] and J. C. Varlet [25].

We see immediately that DP-algebras satisfying the equations (E_n) ($n \geq 1$) are generalizations of Stone algebras. For each $n \geq 1$, we denote by $(E_n)^*$ the class of all DP-algebras which satisfy the equation (E_n). In the following theorem, a characterization of the variety $(E_n)^*$ is given which turns out to be a generalization of L. Nachbin's result for Boolean algebras (L. Nachbin [20]) and the results for Stone algebras obtained by G. Grätzer-E. T. Schmidt [12] and J. C. Varlet [24].

THEOREM 2. For a DP-algebra A, the following two conditions are equivalent ($n \geqslant 1$):

(1) $A \in (E_n)^*$.

(2) every prime filter in A is contained in at most n distinct maximal (proper) filters.

To prove theorem 2, we need the following:

LEMMA 1. (M. H. Stone). Let L be a distributive lattice, F a filter and I an ideal in L such that $F \cap I = \emptyset$. Then there exists a prime filter $P \supseteq F$ such that $P \cap I = \emptyset$.

Proof: Let $\mathcal{A} = \{ Q \mid F \subseteq Q,\ Q \cap I = \emptyset$ and Q a filter in $L \}$. Then $\mathcal{A}$ is inductive and hence there is a maximal element $P \in \mathcal{A}$.

Evidently, $P \supseteq F$ and $P \cap I = \emptyset$. It remains to show that P is prime, i.e. $a \vee b \in P \Longrightarrow a \in P$ or $b \in P$. In fact, if $a \notin P$ and $b \notin P$, then $P \subset P \vee [a, \rightarrow]$ and $P \subset P \vee [b, \rightarrow]$ ($[a, \rightarrow] = \{ x \in L \mid a \leqslant x \}$). By maximality of P, we have $(P \vee [a, \rightarrow]) \cap I \neq \emptyset$, $(P \vee [b, \rightarrow]) \cap I \neq \emptyset$. We claim that there exists $p_1 \in P$ such that $p_1 \wedge a \in I$. Indeed, if $p \wedge a \notin I$ for all $p \in P$, then, since $x \geqslant p \wedge a$ for all $x \in P \vee [a, \rightarrow]$, where $p \in P$, we have $x \notin I$. Consequently $(P \vee [a, \rightarrow]) \cap I = \emptyset$, a contradiction. Similarly, there exists $p_2 \in P$ such that $p_2 \wedge b \in I$. Put $p = p_1 \wedge p_2 \in P$, then $a \wedge p \in I$ and $b \wedge p \in I$ and hence $(a \vee b) \wedge p = (a \wedge p) \vee (b \wedge p) \in P \cap I$, contradicting the fact that $P \cap I = \emptyset$.

<u>Proof of theorem 2</u>: (1) $\Longrightarrow$ (2). Assume that (2) is not true. Then there would exist a prime filter P and n + 1 distinct maximal (proper) filters $M_1, \dots, M_{n+1}$ containing P. By distributivity and maximality, we have, for i = 1, 2, ..., n + 1, $\bigcap_{j \neq i} M_j \nsubseteq M_i$. Take $a_i \in \bigcap_{j \neq i} M_j - M_i$ (i = 1, 2, ..., n). Then $a_i \in M_j$ (i = 1, 2, ..., n; j = 1, 2, ..., n + 1; $i \neq j$). We claim that $a_i^* \in M_i$. Indeed, $a_i \notin M_i$ and hence $M_i \vee [a_i, 1] = A$ by the maximality of M_i. Thus $0 = x \wedge a_i$ for some $x \in M_i$, it follows then that $x \leqslant a_i^*$, i.e. $a_i^* \in M_i$. Now $a_1 \wedge \dots \wedge a_n \in M_{n+1}$, $a_1 \wedge \dots \wedge a_i^* \wedge \dots \wedge a_n \in M_i$ (i = 1, 2, ..., n), hence $(a_1 \wedge \dots \wedge a_n)^* \notin P$ and $(a_1 \wedge \dots \wedge a_i^* \wedge \dots \wedge a_n)^* \notin P$ (i = 1, 2, ..., n). Since P is prime, it follows that $\left(\bigwedge_{i=1}^{n} a_i\right)^* \vee \bigvee_{i=1}^{n} (a_1 \wedge \dots \wedge a_i^* \wedge \dots \wedge a_n)^* \notin P$. But $1 \in P$, thus the equation (E_n) is not satisfied.

(2) $\Longrightarrow$ (1). Assume that the DP-algebra A does not satisfy the equation (E_n). Then there would exist $a_1, \dots, a_n \in A$ such that $c = \left(\bigwedge_{i=1}^{n} a_i\right)^* \vee \bigvee_{i=1}^{n} (a_1 \wedge \dots \wedge a_i^* \wedge \dots \wedge a_n)^* < 1$. By Stone's lemma, there exists a prime filter P such that $c \notin P$. Put

$$b_{n+1} = \bigwedge_{i=1}^{n} a_i$$

$$b_i = a_1 \wedge \dots \wedge a_i^* \wedge \dots \wedge a_n \quad (i = 1, 2, \dots, n)$$

and consider the filters $F_j = P \vee [b_j, 1]$ $(j = 1, 2, \ldots, n + 1)$. For $i \neq j$ $(i, j = 1, 2, \ldots, n + 1)$, we have $b_i \notin F_j$, for otherwise we would have $0 = b_i \wedge b_j \in F_j$ and hence there would exist $p \in P$ such that $p \wedge b_j = 0$, i.e. $p \leqslant b_j^*$, thus $c \in P$, a contradiction. It follows that $F_i (i = 1, 2, \ldots, n + 1)$ are proper filters. Moreover, we have $F_i \vee F_j = A$ $(i \neq j;\ i, j = 1, 2, \ldots, n + 1)$ by the definition of F_i. Let M_i be a maximal (proper) filter containing F_i $(i = 1,2,\ldots,n+1)$. Then $M_1, \ldots, M_{n+1}$ are $n + 1$ distinct maximal (proper) filters containing P, thus (2) is not satisfied.

Section 3

The classes $\mathcal{B}_n$, $\mathcal{R}_n$ and $\mathcal{J}_n$ and the relationship between them

In this section, we construct a subdirectly irreducible DP-algebra $\overline{B}$ from a given Boolean algebra B by adjoining a new unit 1 to B. The classes $\mathcal{B}_n$, $\mathcal{R}_n$ and $\mathcal{J}_n$ are introduced and we show that

$$(E_n)^* = \mathcal{B}_n = \mathcal{R}_n = \mathcal{J}_n \ (n \geqslant 1).$$

We recall that a Boolean algebra ia an algebra $(B; (\vee, \wedge, ', 0, e))$ of the type (2, 2, 1, 0, 0) such that $(B; (\vee, \wedge, 0, e))$ is a distributive lattice with zero element 0 and unit e, and ' is the complementation, i.e. for each $a \in B$, we have $a \wedge a' = 0$, $a \vee a' = e$. Put $\overline{B} = B \cup \{1\}$, where $x < 1$ for all $x \in B$, and define

$$x^* = \begin{cases} x', & \text{if } 0 \neq x \in B; \\ 1, & \text{if } x = 0; \\ 0, & \text{if } x = 1. \end{cases}$$

We have the following:

Proposition 2. $(\overline{B}; (\vee, \wedge, *, 0, 1))$ is a subdirectly irreducible DP-algebra and is called the DP-algebra obtained from the Boolean algebra B by adjoining a new unit 1.

Proof: It is obvious that $(\overline{B}; (\vee, \wedge, 0, 1))$ is a distributive lattice with zero 0 and unit 1.

We claim that * is the pseudo-complementation on $\overline{B}$, i.e. for all $a, x \in \overline{B}$, $a \wedge x = 0 \Longleftrightarrow x \leq a^*$. It is trivial if $a = 0$ or $a = 1$. Assume that $0 < a \leq e$, then $a \wedge x = 0 \Longleftrightarrow x \leq a' = a^*$. Consequently, $(\overline{B}; (\vee, \wedge, *, 0, 1))$ is a DP-algebra.

It remains to show that $(\overline{B}; (\vee, \wedge, *, 0, 1))$ is subdirectly irreducible. We shall prove this by showing that there is a least congruence relation $\Theta > \Delta$, where $\Delta = \{(x, x) \mid x \in \overline{B}\}$. To do this, let us consider the binary relation $\Theta_0 = \Delta \cup \{(1, e), (e, 1)\}$ on $\overline{B}$, where $\Delta = \{(x, x) \mid x \in \overline{B}\}$ and e is the unit of the Boolean algebra B. It is evident that Θ_0 is a (DP-algebra) congruence relation. We claim that $\Theta_0 \subseteq \Theta$ for all congruences $\Theta \neq \Delta$ on $\overline{B}$. Indeed, let Θ be a congruence relation on $\overline{B}$ such that $\Theta > \Delta$. Then $x \Theta y$ for some $x, y \in \overline{B}$ with $x \neq y$. To show that $\Theta_0 \subseteq \Theta$, it suffices to show that $e \Theta 1$. It is trivial if either x or y is 1. If neither x nor y is 1, then $x, y \in B$ and hence we have $x \vee y^* \Theta e$ and $x^* \vee y \Theta e$. We assert that either $x \vee y^*$ or $x^* \vee y$ is not e, for otherwise we would have $x = x \wedge e = x \wedge (x^* \vee y) = x \wedge y$ and $y = y \wedge e = y \wedge (x \vee y^*) = y \wedge x$ and hence $x = y$, a contradiction. Consequently, $a \Theta e$ for some $a \in B$ with $a \neq e$. It then follows that $a^{**} \Theta e^{**}$, i.e. $a \,\Theta\, 1$ and thus $e \,\Theta\, 1$. (Note that $a^{**} = a$ for all $a \in B$ with $0 \leq a < e$, and that $e^{**} = 1$).

Let B_n $(n \geqslant 0)$ be the 2^n-element Boolean algebras, and $\overline{B}_n$ the DP-algebras obtained from B_n by adjoining a new unit 1. Then $\overline{B}_n$ are all finite subdirectly irreducible DP-algebras. They play a very significant role in characterizing the varieties $(E_n)^*$ as well as in the description of the lattice of varieties of DP-algebras. Some of the diagrams of $\overline{B}_n$ are given in **Figure 1.**

To give a characterization of the variety $(E_n)^*$ in terms of $\overline{B}_n$ we need the following:

LEMMA 2. Let A be a DP-algebra, P a prime filter in A, $M_1, \ldots, M_n$ $(n \geqslant 0)$ all distinct maximal (proper) filters properly containing P, and let $a_1, \ldots, a_n$ be the atoms of B_n $(n \geqslant 0)$. Define the mapping $\varphi : A \longrightarrow \overline{B}_n$ by

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in P; \\ \bigvee \{ a_i \mid x \in M_i \}, & \text{if } x \notin P. \end{cases}$$

Then φ is a DP-algebra homomorphism (i.e. φ preserves all the operations on A) of A onto $\overline{B}_n$.

Proof: (1) $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$.

It is trivial if $\varphi(x \vee y) = 1$. If $\varphi(x \vee y) \leqslant e$, then $x \vee y \notin P$ and hence

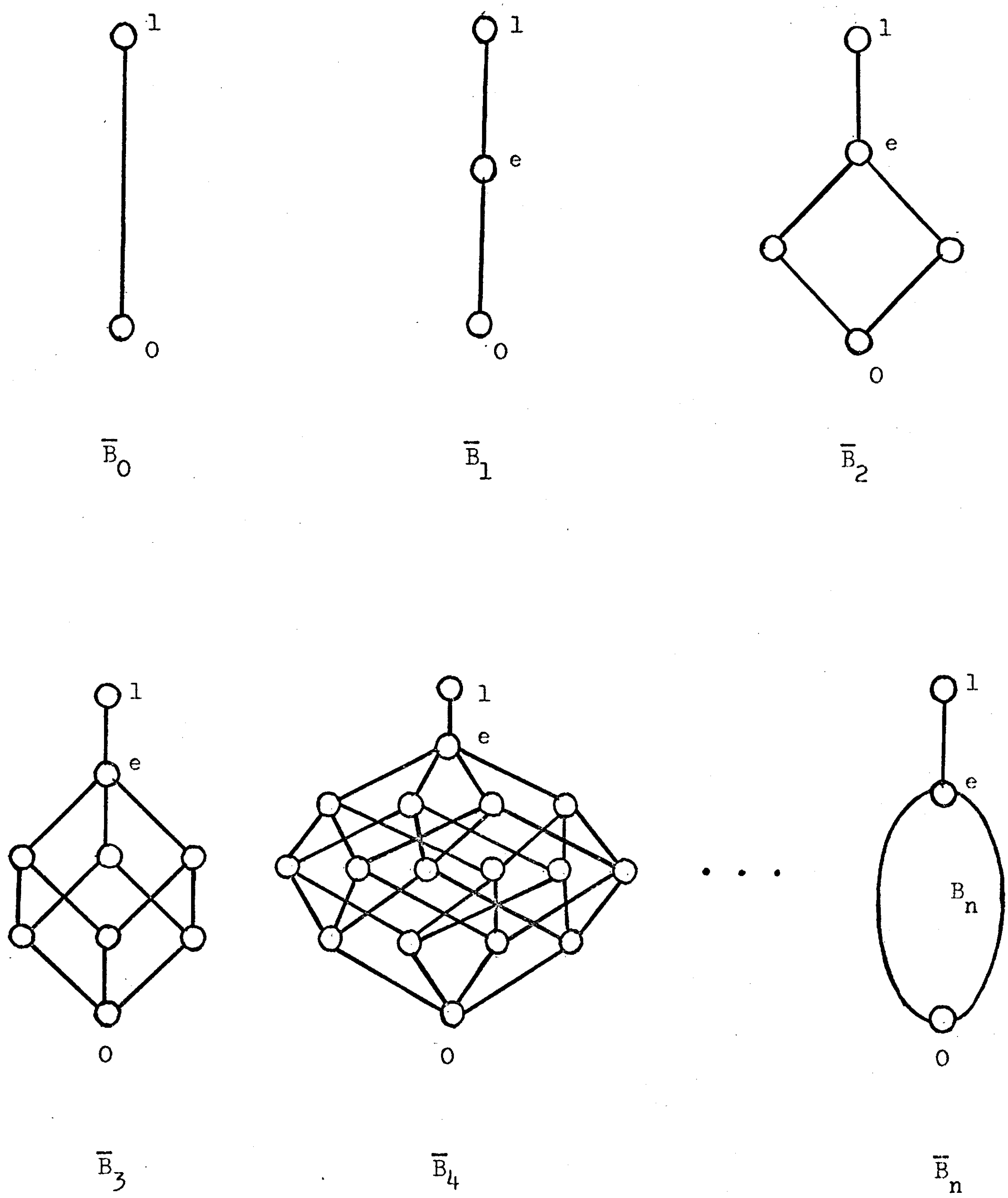

Figure 1: The $\bar{B}_n$ $(n \geqslant 0)$.

$$\varphi(x) \vee \varphi(y) = (\bigvee \{a_i \mid x \in M_i\}) \vee (\bigvee \{a_j \mid y \in M_j\})$$

$$= \bigvee \{a_i \mid x \in M_i \text{ or } y \in M_i\}$$

$$= \bigvee \{a_i \mid x \vee y \in M_i\}$$

$$= \varphi(x \vee y).$$

(2) $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$.

If one of the x, y is an element of P, say $x \in P \subset M_i$, then

$$\varphi(x) \wedge \varphi(y) = \varphi(y) = \bigvee \{a_i \mid y \in M_i\}$$

$$= \bigvee \{a_i \mid x \wedge y \in M_i\}$$

$$= \varphi(x \wedge y).$$

If neither x nor y is an element of P, then

$$\varphi(x) \wedge \varphi(y) = (\bigvee_i \{a_i \mid x \in M_i\}) \wedge (\bigvee_j \{a_j \mid y \in M_j\})$$

$$= \bigvee_{i,j} \{a_i \wedge a_j \mid x \in M_i \text{ and } y \in M_j\}$$

$$= \bigvee_i \{a_i \mid x \wedge y \in M_i\}$$

$$= \varphi(x \wedge y).$$

(3) $\varphi(0) = 0$ and $\varphi(1) = 1$ by definition of φ.

(4) $\varphi(x^*) = \varphi(x)^*$.

If $x \in P$, then $x^* \notin M_i$ for all $i = 1, 2, \dots, n$, and hence $\varphi(x^*) = 0 = \varphi(x)^*$. If $x \notin M_i$ for all $i = 1, 2, \dots, n$, then $x^* \in P$ for otherwise we would have $p \nleq x^*$ for all $p \in P$, i.e. $p \wedge x \neq 0$ for all $p \in P$, and hence the filter $P \vee [x, 1]$ would be proper. But then every maximal (proper) filter $M \supseteq P \vee [x, 1]$ would be different from all M_i $(i = 1, 2, \dots, n)$, a contradiction. It follows that $\varphi(x^*) = 1 = \varphi(x)^*$. Finally, assume that $x \in M_i - P$ for some i, $1 \leqslant i \leqslant n$. Since the pseudo-complementation in $\overline{B}_n$ of an element y satisfying $0 < y \leqslant e$ is the complement of y in the Boolean algebra $B_n = [0, e]$, and since this is the join of all atoms not contained in y, we have

$$\begin{aligned}\varphi(x)^* &= (\bigvee\{a_i \mid x \in M_i\})^* \\ &= \bigvee\{a_i \mid x \notin M_i\} \\ &= \bigvee\{a_i \mid x^* \in M_i\} \\ &= \varphi(x^*).\end{aligned}$$

(5) φ is onto.

If $n = 0$, then P is a maximal (proper) filter in A and φ is, in fact, a DP-algebra homomorphism of A onto $\overline{B}_0$.

Assume $n > 0$. We have to show that for each $a \in \overline{B}_n$, there exists an element $x \in A$ such that $\varphi(x) = a$. It is trivial if $a = 0$ or $a = 1$. If $a = e$, then pick $x_i \in M_i - P$ $(i = 1, 2, \ldots, n)$ and put $x = \bigvee_{i=1}^{n} x_i$. Clearly, $x \notin P$ and $x \in M_i$ for all $i = 1, 2, \ldots, n$. It follows that $\varphi(x) = \bigvee \{ a_i \mid x \in M_i \} = e$. Finally, if $0 < a < e$, then there exist a_i, a_j $(i, j = 1, 2, \ldots, n;\ i \neq j)$ such that $a_j \leq a$ and $a_i \nleq a$. For all i such that $a_i \nleq a$, we have evidently $\bigcap \{ M_j \mid a_j \leq a \} \nsubseteq M_i$. Pick $y \in \bigcap \{ M_j \mid a_j \leq a \} - M_i$ for all i such that $a_i \nleq a$, and put $x = \bigwedge \{ y_i \mid a_i \nleq a \}$. Then

$$\varphi(x) = \bigvee \{ a_j \mid x \in M_j \} = \bigvee \{ a_j \mid a_j \leq a \} = a.$$

The following theorem gives another characterization of the variety $(E_n)^*$.

<u>THEOREM 3</u>. Let A be a DP-algebra. Then the following two conditions are equivalent $(n \geq 1)$:

(1) $A \in (E_n)^*$

(2) A is isomorphic with a subdirect product of copies of $\overline{B}_0, \overline{B}_1, \ldots, \overline{B}_n$.

<u>Proof</u>: (1) $\Longrightarrow$ (2). Let $a, b \in A$ with $a \neq b$. We have to show that there exists a DP-algebra homomorphism φ of A onto $\overline{B}_k$ $(0 \leq k \leq n)$ such that $\varphi(a) \neq \varphi(b)$. We can assume, without loss of generality, that $a \nleq b$. By Stone's lemma, there exists a

prime filter P such that $a \in P$ and $b \notin P$. By theorem 2, there exist at most n distinct maximal filters containing P. Let $M_1, \ldots, M_k$ $(0 \leqslant k \leqslant n)$ be all distinct maximal filters properly containing P. Lemma 2 then implies that there exists a DP-algebra homomorphism φ of A onto $\overline{B}_k$ such that $\varphi(a) \neq \varphi(b)$.

(2) $\Longrightarrow$ (1). It is trivial that $\overline{B}_0$ satisfies the equation (E_n) $(n \geqslant 1)$. Moreover, since each $\overline{B}_n$ $(n \geqslant 1)$ has exactly n distinct maximal filters, viz. the principal filters generated by atoms of $\overline{B}_n$, we see immediately, by theorem 2, that $\overline{B}_1, \ldots, \overline{B}_n$ all satisfy the equation (E_n). Consequently, $A \in (E_n)^*$.

Notations:

$\mathcal{B}_{-1}$ = the class of all one-element DP-algebras;

$\mathcal{B}_n$ = HSP $(\overline{B}_n)$ = the variety of DP-algebras generated by $\overline{B}_n$ $(n \geqslant 0)$;

$\mathcal{B}_F$ = the class of all finite DP-algebras;

$\mathcal{B}_\infty$ = the variety of all DP-algebras;

$\mathcal{K}_n$ = the class of all DP-algebras A such that every prime filter in A is contained in at most n distinct maximal (proper) filters $(n \geqslant 1)$;

$\mathcal{J}_n$ = the class of all DP-algebras A such that every (proper) prime ideal in A contains at most n distinct minimal prime ideals $(n \geqslant 1)$.

We see immediately that $\mathcal{B}_0$ is the variety of all Boolean algebras while $\mathcal{B}_1$ is the variety of all Stone algebras.

As an immediate consequence of theorems 2 and 3, we have the following:

<u>Corollary 1</u>. For $n \geq 1$, we have

$$(E_n)^* = \mathcal{R}_n = \mathcal{B}_n.$$

<u>Proof</u>: $(E_n)^* = \mathcal{R}_n$ is the content of theorem 2.

Since $\overline{B}_0, \overline{B}_1, \ldots, \overline{B}_n$ are all subalgebras of $\overline{B}_n$, it follows from theorem 3 that $(E_n)^* \subseteq \mathcal{B}_n$. On the other hand, $\mathcal{B}_n \subseteq (E_n)^*$ by virtue of the fact that $\overline{B}_n \in (E_n)^*$. Thus $(E_n)^* = \mathcal{B}_n$.

The following theorem gives another characterization of the variety $(E_n)^*$.

<u>THEOREM 4</u>. Let L be a distributive lattice with 0 and 1. Then the following three conditions are equivalent ($n \geq 1$):

(1) every prime filter in L is contained in at most n distinct maximal (proper) filters.

(2) every (proper) prime ideal in L contains at most n distinct minimal prime ideals.

(3) L is the lattice-theoretical join of any n + 1 distinct minimal prime ideals in L.

<u>Proof</u>: (1)$\Longleftrightarrow$(2). Trivial.

(2) $\Rightarrow$ (3). If not, then there would exist n + 1 distinct minimal prime ideals $Q_1, \ldots, Q_{n+1}$ such that $\bigvee_{i=1}^{n+1} Q_i \subset L$. It follows from Stone's lemma that there exists a prime filter P disjoint from $\bigvee_{i=1}^{n+1} Q_i$. Clearly, L-P is a (proper) prime ideal containing Q_1, $Q_2, \ldots, Q_{n+1}$, a contradiction.

(3) $\Rightarrow$ (2). If not, then there would exist a (proper) prime ideal I containing n + 1 distinct minimal prime ideals $Q_1, Q_2, \ldots, Q_{n+1}$. But then we have $\bigvee_{i=1}^{n+1} Q_i \subseteq I \subset L$, contradicting (3).

Combining the results of theorems 2, 3, and 4, and corollary 1, we have the following:

<u>THEOREM 5</u>. Let A be a DP-algebra. Then the following conditions are equivalent ($n \geq 1$):

(1) $A \in (E_n)^*$

(2) $A \in \mathcal{B}_n$

(3) $A \in \mathcal{R}_n$

(4) $A \in \mathcal{J}_n$

(5) A is the lattice-theoretical join of any n + 1 distinct minimal prime ideals in A.

Section 4

The lattice

We shall show in this section that the variety $\mathcal{B}_\omega$ of all DP-algebras is generated by its finite members. Furthermore, a complete description of the lattice of varieties of DP-algebras is given, viz. a chain of type $\omega + 1$. As a first result in this respect, we have the following:

THEOREM 6. $\mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n \subset \mathcal{B}_{n+1} \subset \cdots \subset \mathcal{B}_\omega$.
("$\subset$" means proper inclusion).

Proof: It is evident that $\mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1$. For $n \geq 1$, we have, clearly, $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$. It remains to show that $\mathcal{B}_n \neq \mathcal{B}_{n+1}$. But this follows immediately from the fact that $\overline{B}_{n+1} \in \mathcal{B}_{n+1}$ contains a prime filter $\{1\}$ which is contained in exactly $n + 1$ distinct maximal filters.

LEMMA 3. Let A be a DP-algebra, $e_1, \ldots, e_m$ $(m \geq 1)$ elements of A satisfying the conditions:

$$e_i \wedge e_j = 0 \ (i, j = 1, 2, \ldots, m;\ i \neq j);$$

$$e_i^{**} = e_i \ (i = 1, 2, \ldots, m).$$

Put $S = \left\{ \bigwedge_{\alpha \in J} e_\alpha^* \mid J \subseteq \{1, 2, \ldots, m\}\right\} \cup \left\{ (\bigwedge_{\alpha \in J} e_\alpha^*)^* \mid J \subseteq \{1, 2, \ldots, m\}\right\}$

and $T = \{ x_1 \vee \dots \vee x_n \mid x_i \in S,\ 1 \leq i \leq n \}$. Then T is the subalgebra of A generated by $\{ e_1, \dots, e_m \}$.

In particular, T is finite.

Proof: It is clear that S is closed under *.

We claim that S is closed under $\wedge$. To do this, it suffices to show that

$$\Big(\bigwedge_{\alpha \in J_1} e_\alpha{}^* \Big) \wedge \Big(\bigwedge_{\beta \in J_2} e_\beta{}^* \Big)^* \in S \text{ and}$$

$$\Big(\bigwedge_{\alpha \in J_1} e_\alpha{}^* \Big)^* \wedge \Big(\bigwedge_{\beta \in J_2} e_\beta{}^* \Big)^* \in S$$

where J_1, J_2 are arbitrary subsets of $\{1, 2, \dots, m\}$. Indeed, we have

$$\begin{aligned}\Big(\bigwedge_{\alpha \in J_1} e_\alpha{}^* \Big) \wedge \Big(\bigwedge_{\beta \in J_2} e_\beta{}^* \Big)^* &= \Big(\bigvee_{\alpha \in J_1} e_\alpha \Big)^* \wedge \Big(\bigvee_{\beta \in J_2} e_\beta \Big)^{**} \\ &= \Big(\bigvee_{\alpha \in J_1} e_\alpha \Big)^{***} \wedge \Big(\bigvee_{\beta \in J_2} e_\beta \Big)^{**} \\ &= \Big(\Big(\bigvee_{\alpha \in J_1} e_\alpha \Big)^* \wedge \bigvee_{\beta \in J_2} e_\beta \Big)^{**}\end{aligned}$$

$$= \left(\left(\bigwedge_{\alpha \in J_1} e_\alpha{}^* \right) \wedge \bigvee_{\beta \in J_2} e_\beta \right)^{**}$$

$$= \left(\bigvee_{\beta \in J_2} \left(e_\beta \wedge \bigwedge_{\alpha \in J_1} e_\alpha{}^* \right) \right)^{**}$$

$$= \left(\bigvee_{\beta \in J_2 - J_1} e_\beta \right)^{**}$$

$$= \left(\bigwedge_{\beta \in J_2 - J_1} e_\beta{}^* \right)^{*}$$

and

$$\left(\bigwedge_{\alpha \in J_1} e_\alpha{}^* \right)^* \wedge \left(\bigwedge_{\beta \in J_2} e_\beta{}^* \right)^* = \left(\bigvee_{\alpha \in J_1} e_\alpha \right)^{**} \wedge \left(\bigvee_{\beta \in J_2} e_\beta \right)^{**}$$

$$= \left(\left(\bigvee_{\alpha \in J_1} e_\alpha \right) \wedge \left(\bigvee_{\beta \in J_2} e_\beta \right) \right)^{**}$$

$$= \left(\bigvee \left\{ e_\alpha \wedge e_\beta \mid \alpha \in J_1,\ \beta \in J_2 \right\} \right)^{**}$$

$$= \left(\bigvee \left\{ e_\alpha \mid \alpha \in J_1 \cap J_2 \right\} \right)^{**}$$

$$= \left(\bigwedge_{\alpha \in J_1 \cap J_2} e_\alpha{}^* \right)^{*}.$$

Evidently, 0, $1 \in T$ and $\{e_1, \ldots, e_m\} \subseteq S \subseteq T$. Moreover, T is closed under $\bigvee$ by the definition of T, and T is closed under $\bigwedge$ by distributivity. Finally, T is closed under * since

$$(x_1 \vee \ldots \vee x_n)^* = x_1^* \wedge \ldots \wedge x_n^* \in S \subseteq T.$$

Hence T is a subalgebra of A containing $\{e_1, \ldots, e_m\}$, and T is evidently the subalgebra generated by $\{e_1, \ldots, e_m\}$.

With the aid of lemma 3, we are now in a position to prove the following:

THEOREM 7. Let $\mathcal{K}$ be a variety (equational class) of DP-algebras. If $\mathcal{K} \nsubseteq \mathcal{B}_n$ $(n \geq -1)$, then $\mathcal{B}_{n+1} \subseteq \mathcal{K}$.

Proof: If $n = -1$, then $\mathcal{K}$ contains a non-trivial DP-algebra A by virtue of the fact that $\mathcal{K} \nsubseteq \mathcal{B}_{-1}$. But then A contains $\overline{B}_0$ as a subalgebra and hence $\mathcal{B}_0 \subseteq \mathcal{K}$.

If $\mathcal{K} \nsubseteq \mathcal{B}_0$, then there exists a DP-algebra $A \in \mathcal{K}$ which is not Boolean. Hence there exists an element $a \in A$ such that $a \vee a^* < 1$. It then follows that $\{0, a \vee a^*, 1\}$ is a subalgebra of A isomorphic with $\overline{B}_1$. Hence $\mathcal{B}_1 \subseteq \mathcal{K}$.

Now assume that $n \geq 1$ and take $A \in \mathcal{K} - \mathcal{B}_n$. By corollary 1, there exist $a_1, \ldots, a_n \in A$ such that

$$\left(\bigwedge_{i=1}^{n} a_i\right)^* \vee \bigvee_{i=1}^{n} (a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n)^* < 1.$$

Put

$$e_i = (a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n)^{**} = a_1^{**} \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n^{**}$$

$$(i = 1, 2, \ldots, n)$$

$$e_{n+1} = \left(\bigwedge_{i=1}^{n} a_i \right)^{**} = \bigwedge_{i=1}^{n} a_i^{**}.$$

Clearly, $e_i \wedge e_j = 0$ $(i, j = 1, 2, \ldots, n + 1;\ i \neq j)$ and $e_i^{**} = e_i$ $(i = 1, 2, \ldots, n + 1)$. By lemma 3, the subalgebra B generated by $\{ e_1, \ldots, e_{n+1} \}$ is finite. For $1 \leqslant i \leqslant n$, we have $e_i^* = (a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n)^* \geqslant a_i^{**}$ so that $(e_1^* \wedge \ldots \wedge e_n^*)^* \leqslant \left(\bigwedge_{i=1}^{n} a_i^{**} \right)^* = \left(\bigwedge_{i=1}^{n} a_i \right)^{***} = e_{n+1}^*$. Moreover, $(e_1^* \wedge \ldots \wedge e_i \wedge \ldots \wedge e_n^*)^* = e_i^*$. We claim that $B \notin (E_n)^*$. In fact, put $x_i = e_i^*$ $(i = 1, 2, \ldots, n)$, we have

$$\left(\bigwedge_{i=1}^{n} x_i \right)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge \ldots \wedge x_i^* \wedge \ldots \wedge x_n)^* = \left(\bigwedge_{i=1}^{n} e_i^* \right)^* \vee \bigvee_{i=1}^{n} (e_1^* \wedge \ldots \wedge e_i \wedge \ldots \wedge e_n^*)^*$$

$$\leqslant e_{n+1}^* \vee \bigvee_{i=1}^{n} e_i^*$$

$$= \left(\bigwedge_{i=1}^{n} a_i \right)^* \vee \bigvee_{i=1}^{n} (a_1 \wedge \ldots \wedge a_i^* \wedge \ldots \wedge a_n)^*$$

$$< 1.$$

By theorem 2, there exists a natural number $k \geqslant n + 1$ and a prime filter P in B which is properly contained in exactly k distinct maximal (proper) filters in B. Hence $\bar{B}_k \in \mathcal{K}$ by lemma 2, and therefore $\mathcal{B}_{n+1} \subseteq \mathcal{B}_k \subseteq \mathcal{K}$ by theorem 6.

The following theorem shows that the class of all DP-algebras is generated by its finite members.

<u>THEOREM 8.</u> $\mathcal{B}_\infty = H\ S\ P(\mathcal{B}_F)$.

<u>Proof</u>: We have to show that every equation which does not hold in some DP-algebra A does not hold in some finite DP-algebra B. Let $\mathcal{F}_\tau(V)$ be an algebra of the type τ of DP-algebras, absolutely freely generated by some countable set V, let p, q $\in \mathcal{F}_\tau(V)$ and assume that the equation (p, q) does not hold in some DP-algebra A, i.e. there is a homomorphism $\varphi : \mathcal{F}_\tau(V) \longrightarrow A$ such that $\varphi(p) \neq \varphi(q)$. There exists a finite sequence of finite sets $F_0 \subseteq F_1 \subseteq \dots \subseteq F_n$ such that $F_0 \subseteq V$, p, q $\in F_n$ and for each i = 1, 2, ..., n and each $a \in F_i$, one of the following holds:

(a) there exist b, c $\in F_{i-1}$ such that $a = b \vee c$ or $a = b \wedge c$;

(b) there exists $b \in F_{i-1}$ such that $a = b^*$

(c) a = 0 or a = 1.

Define $M = \varphi(F_n) \cup \{\varphi(a^*) \mid a \in F_n\} \cup \{0_A, 1_A\}$. Let B be the sublattice (not sub-DP-algebra) of A generated by M. Then B as a finite distributive lattice is pseudo-complemented, and hence can be regarded as a DP-algebra. Furthermore, since the pseudo-complement of every element $x \in \varphi(F_n)$ belongs to M, the pseudo-complement of every element $x \in \varphi(F_n)$ is the same in both A and B. Now let

$\psi : \mathcal{F}_\tau(V) \longrightarrow B$ be a homomorphism extending $\varphi | F_0$. We shall show by induction on i that $\psi | F_i = \varphi | F_i$ for all $i = 0, 1, \ldots, n$. It is trivial for $i = 0$. Assume that it is true for $i - 1$ $(i \geqslant 1)$. Take $a \in F_i$. We have to show that $\psi(a) = \varphi(a)$. It is trivial if $a = 0$ or $a = 1$. If $a = b \vee c$, where $b, c \in F_{i-1}$, then

$$\varphi(a) = \varphi(b \vee c) = \varphi(b) \vee_A \varphi(c) = \psi(b) \vee_A \psi(c) = \psi(b) \vee_B \psi(c) = \psi(b \vee c) = \psi(a)$$

Similarly, $\varphi(a) = \psi(a)$ for $a = b \wedge c$, where $b, c \in F_{i-1}$. Finally, if $a = b^*$ for some $b \in F_{i-1}$, then $\varphi(a) = \varphi(b^*) = \varphi(b)^* = \psi(b)^* = \psi(b^*) = \psi(a)$. It follows, in particular, that $\psi(p) = \varphi(p) \neq \varphi(q) = \psi(q)$, i.e. the equation (p, q) does not hold in B.

LEMMA 4. $\mathcal{B}_\infty = \bigvee \{\mathcal{B}_n | n = -1, 0, 1, \ldots\}$. ($\bigvee$ is the join in the lattice of varieties of DP-algebras).

Proof: Let A be a finite DP-algebra. We are going to show that $A \in \mathcal{B}_n$ for some integer $n \geqslant -1$. It then follows, by theorem 8, that $\mathcal{B}_\infty = \bigvee \{\mathcal{B}_n | n = -1, 0, 1, \ldots\}$. It is trivial if $A \in \mathcal{B}_{-1}$ or $A \in \mathcal{B}_0$. Assume that A is a non-trivial finite DP-algebra which is not Boolean. Then A has finitely many maximal (proper) filters and hence $A \in \mathcal{R}_n = \mathcal{B}_n$ for some $n \geqslant 1$ by theorems 2 and 5. This completes the proof of the lemma.

Now we are in a position to prove the following:

THEOREM 9. The chain of theorem 6 is the whole lattice of varieties of DP-algebras.

Proof: Let $\mathcal{K}$ be an arbitrary variety of DP-algebras, we are going to show that either $\mathcal{K} = \mathcal{B}_\infty$ or $\mathcal{K} = \mathcal{B}_n$ for some integer $n \geqslant -1$. In fact, we have either $\mathcal{K} \supseteq \mathcal{B}_n$ for all $n = -1, 0, 1, \ldots$, in which case $\mathcal{K} \supseteq \bigvee \{\mathcal{B}_n \mid n = -1, 0, 1, \ldots\} = \mathcal{B}_\infty$, i.e. $\mathcal{K} = \mathcal{B}_\infty$, or else there exists a largest n such that $\mathcal{B}_n \subseteq \mathcal{K}$. But then we have $\mathcal{B}_n = \mathcal{K}$, for otherwise we would have $\mathcal{K} \nsubseteq \mathcal{B}_n$ and hence, by theorem 7, $\mathcal{B}_{n+1} \subseteq \mathcal{K}$. This contradicts the choice of n.

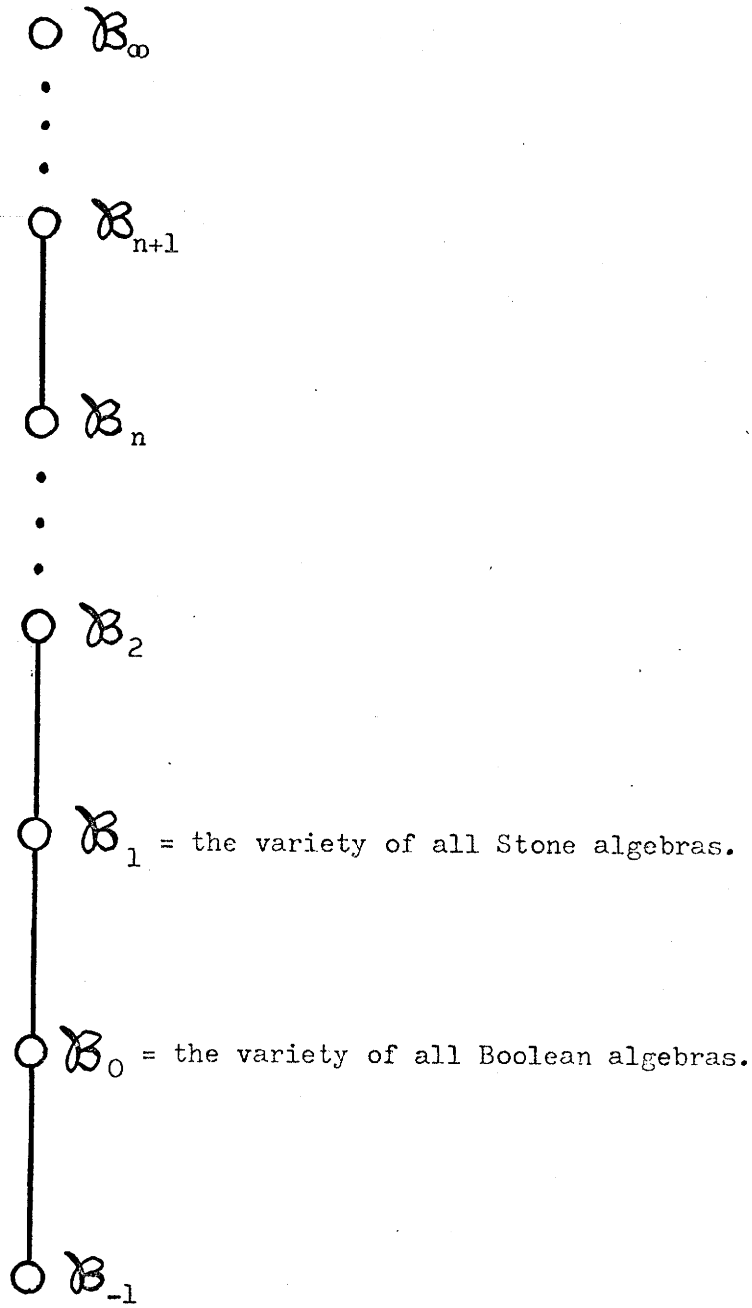

Figure 2: The lattice.

The following corollary shows that the DP-algebras $\overline{B}_n$ $(n \geq 0)$ are exactly the finite subdirectly irreducible DP-algebras.

Corollary 2. The DP-algebras $\overline{B}_n$ $(n \geq 0)$ are exactly the finite subdirectly irreducible members of $\mathcal{B}_\infty$.

Proof: By proposition 2, $\overline{B}_n$ $(n \geq 0)$ are all subdirectly irreducible. It remains to show that every finite subdirectly irreducible DP-algebra L is isomorphic with some $\overline{B}_n$ $(n \geq 0)$. In fact, there is a natural number n such that $|\overline{B}_n| \leq |L| < |\overline{B}_{n+1}|$ ($|A|$ = the cardinality of A). Put $\mathcal{L} = H\ S\ P\ (L)$. If $\mathcal{L} \nsubseteq \mathcal{B}_n$, then $\mathcal{B}_{n+1} \subseteq \mathcal{L}$ by theorem 7. In particular, $\overline{B}_{n+1} \in H\ S\ P\ (L)$. By corollary 3.4, B. Jónsson [18], $\overline{B}_{n+1} \in H\ S(L)$. This is impossible. Thus we have $\mathcal{L} \subseteq \mathcal{B}_n$, and hence $L \in H\ S(\overline{B}_n)$. Consequently, $L \cong \overline{B}_n$.

Remark: H. Lakser has shown that every subdirectly irreducible DP-algebra is of the form $\overline{B}$, where B is a Boolean algebra. Hence $\overline{B}$ are exactly the subdirectly irreducible members of $\mathcal{B}_\infty$.

Section 5

Generalizations of relative Stone algebras

A lattice L is said to be a relative Stone algebra if every closed interval in L is a Stone algebra. (G. Grätzer-E. T. Schmidt [12]). In their paper [12] , G. Grätzer and E. T. Schmidt have shown that a distributive lattice L in which every closed interval (as a sublattice) is pseudo-complemented is a relative Stone algebra iff one of the following two equivalent conditions holds:

(1) for any pair of incomparable prime ideals P and Q in L, $P \vee Q = L$.

(2) $\overline{B}_2$ is not a lattice homomorphic image of L.

In this section, we shall prove two theorems which turn out to be generalizations of the result mentioned above.

THEOREM 10. Let L be a distributive lattice in which every closed interval (as a sublattice) is pseudo-complemented. Then the following two conditions are equivalent:

(1) every closed interval [a, b] in L satisfies the equation (E_n), i.e. $[a, b] \in (E_n)^*$.

(2) $L = \bigvee_{i=1}^{n+1} Q_i$ for any n + 1 pairwise incomparable prime ideals $Q_1, \ldots, Q_{n+1}$ in L.

Proof: (1) $\Longrightarrow$ (2). If not, there would exist n + 1 pairwise incomparable prime ideals $Q_1, Q_2, \ldots, Q_{n+1}$ such that $\bigvee_{i=1}^{n+1} Q_i \subset L$. For each i, $1 \leq i \leq n+1$, we have

$$\bigwedge_{\substack{j\neq i\\ j=1}}^{n+1} Q_j - Q_i \neq \emptyset$$

for otherwise we would have $Q_i \supseteq \bigwedge_{\substack{j\neq i\\ j=1}}^{n+1} Q_j$ for some i $(1 \leq i \leq n+1)$ and hence $Q_i = Q_i \vee \bigwedge_{\substack{j\neq i\\ j=1}}^{n+1} Q_j = \bigwedge_{\substack{j\neq i\\ j=1}}^{n+1} (Q_i \vee Q_j)$. Since Q_i, Q_j are incomparable, $Q_i \vee Q_j > Q_i$. Hence Q_i is $\wedge$-reducible, contradicting the fact that Q_i is prime. Take $b_i \in \bigwedge_{\substack{j\neq i\\ j=1}}^{n+1} Q_j - Q_i$ $(i = 1, 2, \ldots, n+1)$, and put $c_i = b_1 \vee \ldots \vee b_{i-1} \vee b_{i+1} \vee \ldots \vee b_{n+1}$ $(i = 1, 2, \ldots, n+1)$.

Now consider the closed interval $I = \left[\bigwedge_{i=1}^{n+1} c_i,\ a \vee \bigvee_{i=1}^{n+1} c_i \right]$, where $a \in L - \bigvee_{i=1}^{n+1} Q_i$. Clearly, $c_i \in I$ for all $i = 1, 2, \ldots, n+1$, and hence $c_i^* \in I$ exist $(i = 1, 2, \ldots, n+1)$ by hypothesis. It is evident that $c_j \notin Q_i$ for all $j \neq i$, for otherwise we would have $b_i \leq c_j \in Q_i$ and hence $b_i \in Q_i$, contradicting the choice of b_i. Since Q_i is prime, we have $\bigwedge_{j\neq i} c_j \notin Q_i$. In particular, $\bigwedge_{i=1}^{n} c_i \notin Q_{n+1}$.

But then $\bigwedge_{j\neq i} c_j \notin Q_i$ implies $c_i^* \notin Q_i$, and hence $c_1 \wedge \ldots \wedge c_i^* \wedge \ldots \wedge c_n \notin Q_i$.

Consequently, $\left(\bigwedge_{i=1}^{n} c_i\right)^* \in Q_{n+1}$, $(c_1 \wedge \ldots \wedge c_i^* \wedge \ldots \wedge c_n)^* \in Q_i$ $(i = 1, 2, \ldots, n)$ (since for a prime ideal Q, $x \notin Q \Longrightarrow x \wedge x^* = 0 \in Q \Longrightarrow x^* \in Q$). We have therefore

$$a \vee \bigvee_{i=1}^{n+1} c_i = \left(\bigwedge_{i=1}^{n} c_i\right)^* \vee \bigvee_{i=1}^{n} (c_1 \wedge \ldots \wedge c_i^* \wedge \ldots \wedge c_n)^* \in \bigvee_{i=1}^{n+1} Q_i$$

by (1). Hence $a \in \bigvee_{i=1}^{n+1} Q_i$ which contradicts the choice of a.

(2) $\Longrightarrow$ (1). If not, there would exist a closed interval $[a, b]$ in L such that there are n + 1 distinct minimal prime ideals Q_i' $(i = 1, 2, \ldots, n + 1)$ in $[a, b]$ with $\bigvee_{i=1}^{n+1} Q_i' < [a, b]$ by theorem 4.

The mapping $\varphi : L \longrightarrow [a, b]$ defined by $\varphi(x) = (x \vee a) \wedge b$ is clearly an epimorphism. Put $Q_i = \varphi^{-1}[Q_i']$ $(i = 1, 2, \ldots, n + 1)$. Then each Q_i is clearly a prime ideal in L. Moreover, $Q_1, \ldots, Q_{n+1}$ are pairwise incomparable, because $Q_i \subseteq Q_j$ $(i \neq j)$ would imply $Q_i' = \varphi[Q_i] \subseteq \varphi[Q_j] = Q_j'$, and hence $Q_i' = Q_j'$, a contradiction.

By (2), we have $L = \bigvee_{i=1}^{n+1} Q_i$. It follows immediately that $[a, b] = \varphi[L] = \varphi\left[\bigvee_{i=1}^{n+1} Q_i\right] = \bigvee_{i=1}^{n+1} \varphi[Q_i] = \bigvee_{i=1}^{n+1} Q_i'$. This is a contradiction.

<u>THEOREM 11</u>. Let L be a distributive lattice in which every closed interval (as a sublattice) is pseudo-complemented. Then the following two conditions are equivalent:

(1) every closed interval $[a, b]$ in L satisfies the equation (E_n), i.e. $[a, b] \in (E_n)^*$.

(2) $\overline{B}_{n+1}$ is not a lattice homomorphic image of L.

Proof: (1) $\Rightarrow$ (2). If not, then $\overline{B}_{n+1}$ would be a lattice homomorphic image of L. Let $\varphi : L \longrightarrow \overline{B}_{n+1}$ be the (lattice) epimorphism and $a_1, \ldots, a_{n+1}$ the atoms of $\overline{B}_{n+1}$. Now consider the principal ideals $Q_i' = [0, a_i^*]$ $(i = 1, 2, \ldots, n + 1)$, then they are pairwise incomparable prime ideals of $\overline{B}_{n+1}$. Put $Q_i = \varphi^{-1}[Q_i']$, then $Q_1, \ldots, Q_{n+1}$ are n + 1 pairwise incomparable prime ideals in L. By theorem 10, we have $L = \bigvee_{i=1}^{n+1} Q_i$. This implies $\overline{B}_{n+1} = \varphi[L] =$

$\bigvee_{i=1}^{n+1} \varphi[Q_i] = \bigvee_{i=1}^{n+1} Q_i'$. But this is impossible since $\bigvee_{i=1}^{n+1} Q_i' =$

$\overline{B}_{n+1} - \{1\}$.

(2) $\Rightarrow$ (1). If not, then there would exist n + 1 pairwise incomparable prime ideals $Q_1, \ldots, Q_{n+1}$ such that $\bigvee_{i=1}^{n+1} Q_i < L$. By Stone's lemma, there would exist a prime ideal $R \supseteq \bigvee_{i=1}^{n+1} Q_i$. Consider

the family $\mathcal{F}$ consisting of the following subsets of L:

$$L - R,\ R - \bigcup_{i=1}^{n+1} Q_i,\ Q_i - \bigcup_{j \neq i} Q_j\ (i = 1, 2, \ldots, n + 1),$$

$$(Q_i \cap Q_j) - \bigcup_{\substack{k \neq i \\ k \neq j}} Q_k,\ \ldots,\ \bigcap_{j=1}^{\ell} Q_{i_j} - \bigcup_{k \neq i_j} Q_k,\ (1 \leq i_j \leq n + 1;\ i_j \text{ all distinct})$$

$$, \ldots, \bigcap_{i=1}^{n+1} Q_i.$$

$\mathcal{F}$ is clearly a partition of L and hence induces an equivalence relation θ on L. We claim that θ is a lattice congruence relation on L. This follows from the following observations:

(i) $x \in L - R,\ y \in R - \bigcup_{i=1}^{n+1} Q_i \Rightarrow x \vee y \in L - R,\ x \wedge y \in R - \bigcup_{i=1}^{n+1} Q_i.$

(ii) $x \in L - R,\ y \in \bigcap_{\alpha \in J} Q_\alpha - \bigcup_{\beta \in I-J} Q_\beta,\ \emptyset \neq J \subseteq I = \{1, \ldots, n+1\}$

$$\Rightarrow x \vee y \in L - R,\ x \wedge y \in \bigcap_{\alpha \in J} Q_\alpha - \bigcup_{\beta \in I-J} Q_\beta$$

(iii) $x \in R - \bigcup_{i=1}^{n+1} Q_i,\ y \in \bigcap_{\alpha \in J} Q_\alpha - \bigcup_{\beta \in I-J} Q_\beta,\ \emptyset \neq J \subseteq I \Rightarrow x \vee y \in R - \bigcup_{i=1}^{n+1} Q_i,$

$$x \wedge y \in \bigcap_{\alpha \in J} Q_\alpha - \bigcup_{\beta \in I-J} Q_\beta.$$

(iv) $x \in \bigcap_{\alpha \in J_1} Q_\alpha - \bigcup_{\beta \in I-J_1} Q_\beta,\ y \in \bigcap_{\gamma \in J_2} Q_\gamma - \bigcup_{\delta \in I-J_2} Q_\delta,\ \emptyset \neq J_1, J_2 \subseteq I,$

$$\Rightarrow x \vee y \in \begin{cases} \bigcap_{\alpha \in J_1 \cap J_2} Q_\alpha - \bigcup_{\beta \in I-J_1 \cap J_2} Q_\beta, & \text{if } J_1 \cap J_2 \neq \emptyset; \\ R - \bigcup_{i=1}^{n+1} Q_i, & \text{if } J_1 \cap J_2 = \emptyset. \end{cases}$$

$$x \wedge y \in \bigcap_{\alpha \in J_1 \cup J_2} Q_\alpha - \bigcup_{\beta \in I-J_1 \cup J_2} Q_\beta$$

(v) $x \in \bigcap_{i=1}^{n+1} Q_i$, and $y \in A \in \mathcal{F} \Rightarrow x \vee y \in A,\ x \wedge y \in \bigcap_{i=1}^{n+1} Q_i.$

Evidently, $L/\Theta \cong \overline{B}_{n+1}$, hence $\overline{B}_{n+1}$ is a lattice homomorphic image of L, a contradiction.

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