ON SEMI-DEFINITE FORMS IN ANALYSIS

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by

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SCOPE AND CONTENT: Using the representation theory of positive definite sequences some propositions in additive number theory are obtained and H. Bohr's approximation theorem is deduced. A unified approach to theorems by S. Bochner, S. N. Bernstein and H. Hamburger is discussed and some operator versions of numerical moment problems are studied. The Appendix contains comments to J. von Neumann's spectral theorem for self-adjoint operators in Hilbert space.

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INTRODUCTION

In the spectral resolution of self-adjoint operators in Hilbert space it has proved very advantageous to employ the integral representation of the resolvent. When this approach is taken certain theorems of classical analysis play a prominent role. In this connection we mention the following theorems and their authors:

<u>I (S. Bochner)</u>: Let k denote a continuous function on $(-A,A)$, $0 < A \leqq \infty$. In order that the representation

$$k(x) = \int_{-\infty}^{\infty} e^{ixt} \, d\sigma(t) \qquad (-A < x < A)$$

holds, where σ denotes a nondecreasing function of bounded variation on the interval $(-\infty,\infty)$, it is necessary and sufficient that k be a positive definite function, i. e., for any numbers $0 \leqq x_1 < \ldots < x_n < A$ $(n < \infty)$ and any complex numbers $\lambda_1, \ldots, \lambda_n$ we have

$$\sum_{j,m=1}^{n} k(x_j - x_m)\, \lambda_j \overline{\lambda_m} \geqq 0.$$

<u>II (F. Riesz & G. Herglotz)</u>: In order that a finite function f defined on the disk $|z| < 1$ have the representation

$$f(z) = i \operatorname{Im} f(0) + (2\pi)^{-1} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \, d\rho(t),$$

where ρ is a non-decreasing function on $(-\pi, \pi)$, it is necessary and sufficient that f be holomorphic in $|z| < 1$ and its values have non-negative real part for $|z| < 1$.
It is clear that ρ has bounded variation for f(0) is finite.

III (R. Nevanlinna): In order that a finite function g defined on $\operatorname{Im} z > 0$ have the representation

$$g(z) = \mu z + \nu + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} \, d\tau(t),$$

where $\mu \geqq 0$ and ν are two real constants and τ is a non-decreasing function on the entire numerical line, it is necessary and sufficient that g be holomorphic in the half-plane $\operatorname{Im} z > 0$ and its values on this half-plane have non-negative imaginary part. From the finiteness of g(i) follows that the function τ is of bounded variation on $(-\infty, \infty)$.

In this thesis there will be occasion to exhibit instances where the relationship between spectral resolution of self-adjoint operators on the one hand and classical analysis theorems on the other hand is to a certain degree mutual.
A detailed description of the efforts of the author of this thesis will be found further on in the INTRODUCTION.

In the course of proving the foregoing theorem II G. Herglotz established a proposition concerning the Fourier - Stieltjes integral representation of positive definite sequences. For a wording of the proposition in question the interested reader is refered to page 23; the notion of a positive definite sequence is defined on page 1.

A famous theorem due to C. Caratheodory and O. Toeplitz has contributed substantially to ensure a place of distinction in analysis for positive definite sequences. This theorem offers another solution to the problem underlying theorem II and thereby gives a characterization of an important class of functions occuring in the interpolation theory of analytic functions as well; the theorem of Caratheodory and Toeplitz states that the function f holomorphic in $|z| < 1$,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

maps $|z| < 1$ to $\operatorname{Re} w \geqq 0$ if and only if the sequence with terms c_n $(n = \ldots,-1,0,1,\ldots)$, where $c_0 = a_0 + \overline{a_0}$, $c_m = a_m$, $c_{-m} = \overline{a_m}$ for $m = 1,2,\ldots$, is positive definite.

The author of this thesis uses the Fourier - Stieltjes integral representation theorem for positive definite sequences in chapter II to offer a new proof of a well-known theorem in the theory of almost periodic functions of Harald Bohr.

The theorem in question is stated as proposition I on page 41. The method of proof yields also another proposition, stated on pp. 53-54; the author believes that this proposition is new.

As a point of interest the author wishes to mention that he avoided the customary practice of using limit periodic functions in deducing Bohr's approximation theorem from the theorem concerning almost periods and instead made suitable estimates (see pp. 67-71).

In the author's quest for self-contained presentation of these matters he was obliged to line up a lot of material in chapters I and II which belongs to the past of the subject.

The main result aimed at in chapter III is a unified approach to theorems by S. Bochner, S. N. Bernstein and H. Hamburger. The theorem of Bochner we have stated already further above; we shall quote now the remaining two theorems we have in mind.

<u>Theorem of S. N. Bernstein</u>: Let k be a continuous function on the interval (A_1, A_2), where $-\infty \leqq A_1 < A_2 \leqq \infty$ and $A_1 < 0 < A_2$. In order that

$$k(x) = \int_{-\infty}^{\infty} e^{xt} \, d\,\sigma(t) \qquad (A_1 < x < A_2)$$

hold, where σ is a certain non-decreasing bounded function on $(-\infty, \infty)$, it is necessary and sufficient that for $x_1, \ldots, x_n$, where $A_1 < x_1 < \ldots < x_n < A_2$ $(n < \infty)$ and any complex numbers $\lambda_1, \ldots, \lambda_n$ the relation

$$\sum_{j,m=1}^{n} k(x_j + x_m)\, \lambda_j \overline{\lambda_m} \geqq 0$$

hold.

<u>Theorem of H. Hamburger</u>: Let $(a_m)_{m=0}^{\infty}$ be a sequence of real numbers. A necessary and sufficient condition that

$$a_m = \int_{-\infty}^{\infty} t^m \, d\sigma(t) \qquad (m = 0,1,2,\ldots)$$

where σ denotes some non-decreasing bounded function on $(-\infty, \infty)$ is the requirement

$$\sum_{j,k=0}^{n} a_{j+k}\, \lambda_j \overline{\lambda_k} \geqq 0.$$

The author would like to describe briefly how he treats the problem at hand.

Let V denote a vector space consisting of complex-valued continuous functions on $(-\infty, \infty)$; the algebraic operations are defined as usual. Let W denote the linear hull of the set of functions of the form $f\,\bar{g}$, where $f, g \in V$. Let Ψ be a positive linear functional acting on W, i. e., Ψ is an additive homogeneous mapping such that $\Psi(f\,\bar{f}) \geqq 0$. The problem posed now is to get a sufficient condition that a given positive linear functional Ψ on W have an integral representation of the form

$$\Psi(\phi) = \int_{-\infty}^{\infty} \phi(t)\, d\sigma(t) \qquad (\phi \in W)$$

where σ is some non-decreasing bounded function on $(-\infty, \infty)$.

If one has Hamburger's theorem in mind the set of all polynomials must be taken as domain of definition of the functional Ψ.

To use the spectral theory of operators on our problem we need a Hilbert space in terms of our problem. But a suitable Hilbert space is available. It can be seen that

$$\langle f, g \rangle = \Psi(f\,\bar{g})$$

is an inner product on V if we identify in V any two vectors f and g for which $\langle f-g, f-g \rangle = 0$. Completion of V with respect to the norm induced by this inner product gives a Hilbert space $\mathfrak{H}$ in which V is dense.

Given $g \in V$, let $\hat{g}$ be the function defined by

$$\hat{g}(s) = sg(s) \qquad\qquad -\infty < s < \infty .$$

We impose on V the additional requirement that the set F of all $g \in V$ for which $\hat{g} \in V$, be dense in V with the norm

$$\| f \| = (\Psi(f\,\bar{f}))^{\frac{1}{2}} \qquad\qquad (f \in V).$$

Under these circumstances we can introduce in $\mathcal{G}$ the operator of multiplication by the independent variable s, having defined it initially on the set F. This operator is symmetric and can be closed. This closure we denote by the operator D. It can be seen that D is well-defined by observing that $\langle f,f \rangle = 0$ implies $\langle \hat{f},\hat{f} \rangle = 0$.

Suppose for a minute that our problem was solved and that for any $f,g \in V$ we had

$$\langle f,g \rangle = \int_{-\infty}^{\infty} f(t)\, \overline{g(t)}\, d\sigma(t).$$

The functions f and g appear here in two roles: firstly, as elements of the space $\mathcal{G}$ and, secondly, as elements of L^2_σ. If we suppose for example that V consists of all polynomials and if by abuse of symbolism we signify by f(t) that $f \in L^2_\sigma$ and by f(s) that $f \in V$ and if $u = u(s)$ denotes the unit polynomial (identically equal to 1) we can see that

$$f(s) - f(t)u = (D-tE)g_t(s),$$

where E is the identity operator in $\mathcal{G}$, D is the operator of multiplication by the independent variable introduced above and $g_t(s)$ is an element of $\mathcal{G}$ and depends on the parameter t; evidently we have simply used the familiar elementary fact that s-t factors $s^m - t^m$ for $m = 1,2,\ldots$.

The foregoing is a heuristic guide to the formulation of our theorem (see p. 118):

Let D be a closed symmetric operator in some Hilbert space $\mathcal{G}$, and let E_t be a spectral function of D. Suppose that given any elements $f,h \in \mathcal{G}$ there exists a function $g:(-\infty,\infty) \to \mathcal{D}_D$ and a differentiable function φ on $(-\infty,\infty)$ such that

$$f - \varphi(t)h = (D-tE)g(t).$$

Then

$$f = \int_{-\infty}^{\infty} \varphi(t)\, d\, E_t h \quad \text{and} \quad \langle f,f\rangle = \int_{-\infty}^{\infty} |\varphi(t)|^2\, d\langle E_t h,h\rangle.$$

Since σ is assumed to be only some non-decreasing bounded function on $(-\infty,\infty)$ the foregoing theorem is strong enough to get us within range of the solution of our problem:

A sufficient condition that a given positive linear functional Ψ defined on W has an integral representation of the form

$$\Psi(\phi) = \int_{-\infty}^{\infty} \phi(t)\, d\,\sigma(t) \qquad (\phi \in W)$$

with non-decreasing bounded function σ on $(-\infty,\infty)$ is that the following threefold requirement be satisfied: The function 1 belong to V, the set F of all functions $g \in V$ for which $\hat{g} \in V$ be dense in V with the norm

$$\| f \| = (\Psi (f \bar{f}))^{\frac{1}{2}} \qquad (f \in V)$$

and from $f \in V$ it follow that for any real t the function of s

$$(f(s)-f(t))/(s-t)$$

belong to V.

The theorems of Bochner, Bernstein and Hamburger are deduced in terms of this result.

The proof of the theorem quoted on page VIII is made dependent on certain facts known from operator theory. Central among these is a theorem of J. von Neumann (see p. 95). The author gives his own proof for this theorem and in this connection introduces the notion of fractional-linear transformation of a linear operator and establishes several proposition (for detail see pp. 86-91). As additional evidence for the usefulness of this notion the author employs it in proving a theorem of M. H. Stone K.O. Friedrichs and H. Freudenthal (see p. 130). On the basis of the latter theorem the author offers an operator version of a classical analysis theorem of T. J. Stieltjes on page 145. Another operator-valued theorem of this sort is given on page 141.

Since the spectral theorem for self-adjoint operators in Hilbert space enters on numerous occasions in chapter III the author felt a need to offer a proof in the appendix. This proof is based on the theory of vector lattices. The author makes the observation that every strongly closed ring

$\mathfrak{A}$ of bounded self-adjoint operators is a complete vector lattice in the usual operator-theoretic sense of partial ordering and that if $\mathfrak{A}$ contains the unit operator, one can take it as unit of the vector lattice and that the basis of the vector lattice in this case consists of all projection operators contained in $\mathfrak{A}$. This observation allows the use of H. Freudenthal's integral representation theorem of elements of a complete vector lattice with unit; the spectral resolution of a bounded self-adjoint operator is then a direct consequence of Freudenthal's theorem. To obtain the spectral resolution of an unbounded self-adjoint operator, the author uses the vector lattice theoretic union of complete vector lattices of bounded self-adjoint operators. This approach succeeds with only a modest amount of information from operator theory. The author refrained from writing out proofs for the vector lattice theoretic part as he has prepared a set of notes which contain all the necessary detail some time ago.

Chapter 1

POSITIVE DEFINITE SEQUENCES

1.1 Definition and Basic Relations

A sequence (finite or not) whose terms are the complex numbers c_n ($n = \dots -1, 0, 1, \dots$) is said to be p o s i t i v e d e f i n i t e, if for any finite set of complex numbers $\lambda_1, \dots, \lambda_N$ the inequality

$$\sum_{k,j=1}^{N} c_{k-j} \lambda_k \overline{\lambda_j} \geqq 0 \qquad (1)$$

holds; $\overline{\lambda_j}$ denotes the complex conjugate of λ_j.

From the above definition we deduce at once the following basic relations for a positive definite sequence:

(i) $c_0 \geqq 0$

(ii) $c_{-n} = \overline{c_n}$

(iii) $|c_n| \leqq c_0$

(iv) $|c_n - c_{n+m}|^2 \leqq 2c_0(c_0 - \mathrm{Re}\{c_m\})$.

Here $\mathrm{Re}\{c_m\}$ is to signify the real part of c_m.

Indeed, substituting $N = 1$, $\lambda_1 = 1$ into formula (1), we get inequality (i).

To obtain equality (ii), we let $N = n+1$, $\lambda_1 = 1$, $\lambda_2 = \ldots$ $\ldots = \lambda_n = 0$, $\lambda_{n+1} = \lambda$ in formula (1). Evidently

$$0 \leqq \sum_{k=1}^{n+1} \sum_{j=1}^{n+1} c_{k-j} \lambda_k \overline{\lambda_j} = c_0 + c_{-n}\overline{\lambda} + c_n\lambda + c_0\lambda\overline{\lambda}.$$

Since $c_0 \geqq 0$ by inequality (i), we have that

$$c_{-n}\overline{\lambda} + c_n\lambda$$

is real for any choice of the complex number λ. Taking $\lambda = 1$ and $\lambda = i$, successively, we see: If $c_{-n}\overline{\lambda} + c_n\lambda$ is real for any complex number λ, then $c_{-n} = \overline{c_n}$ is true. The converse of the latter statement obviously holds.

Next we verify inequality (iii); from it will follow in particular that $c_0 = 0$ implies $c_n = 0$ for all n. Suppose now that $c_0 = 0$. Then from

$$c_0 + c_{-n}\overline{\lambda} + c_n\lambda + c_0\lambda\overline{\lambda} \geqq 0$$

we get, putting $\lambda = c_{-n}$, that $2c_n\overline{c_n} = 0$ or $c_n = 0$.

Suppose, on the other hand, that $c_0 > 0$, then substituting $\lambda \;\; -\overline{c_n} / c_0$ into

$$c_0 + c_{-n}\overline{\lambda} + c_n\lambda + c_0\lambda\overline{\lambda} \geqq 0,$$

we obtain by equality (ii) that

$$c_0^{\,2} - c_n \overline{c_n} \geqq 0.$$

Hence $|c_n| \leqq c_0$ holds in both cases.

Finally, we show that inequality (iv) is true. For this purpose we take $N = n+m+1$, $\lambda_1 = 1$, $\lambda_2 = \dots = \lambda_n = 0$, $\lambda_{n+1} = \lambda$, $\lambda_{n+2} = \dots = \lambda_{n+m} = 0$, $\lambda_{n+m+1} = -\lambda$.

We obtain

$$0 \leqq \sum_{k=1}^{n+m+1} \sum_{j=1}^{n+m+1} c_{k-j} \lambda_k \overline{\lambda_j} =$$

$$= c_0 + c_{-n}\overline{\lambda} - c_{-n-m}\overline{\lambda} + c_n\lambda + c_0\lambda\overline{\lambda} - c_{-m}\lambda\overline{\lambda}$$

$$-c_{n+m}\lambda - c_m\lambda\overline{\lambda} + c_0\lambda\overline{\lambda} =$$

$$= c_0 + 2\,\mathrm{Re}\left\{(c_n - c_{n+m})\lambda\right\} + 2(c_0 - \mathrm{Re}\left\{c_m\right\})|\lambda|^2$$

which holds for any complex number λ. If we let

$$\lambda = \frac{-\overline{(c_n - c_{n+m})}}{2(c_0 - \mathrm{Re}\left\{c_m\right\})}$$

then we obtain $0 \leqq 2c_0(c_0 - \mathrm{Re}\left\{c_m\right\}) - |c_n - c_{n+m}|^2$, which gives inequality (iv).

1.2 Representation Theory

In this section we discuss two types of realization of positive definite sequences. The first type is geometric in nature. It is based on a proposition of E. H. Moore concerning Hermitian positive semi-definite matrices and reduces the study of positive definite sequences to the consideration of stationary sequences in complex Hilbert space. The second type of realization is analytic in nature. It is due to G. Herglotz and gives the representation of positive definite sequences in terms of Fourier-Stieltjes integrals. We commence with the geometric representation theory, stating first Kolmogoroff's definition of the notion of stationary sequence.

A sequence whose terms u_n $(n = 1, 2, \ldots)$ _belong to a Hilbert space H is said to be s t a t i o n a r y, if the inner product_ $\langle u_h, u_k \rangle$ _depends only on the difference_ h-k.

PROPOSITION (E. H. Moore & A. Kolmogoroff): _A necessary and sufficient condition for the existence of a complex Hilbert space and a set of elements_ $u_1, u_2, \ldots$ _in this space which satisfy the conditions_

$$\langle u_h, u_k \rangle = c_{hk} \qquad (h,k = 1,2, \ldots)$$

is that the matrix (c_{hk}) _be Hermitian positive semi-definite, that is_

$$\sum_{h,k=1}^{n} c_{hk} \lambda_h \overline{\lambda_k} \geqq 0 \tag{2}$$

<u>for any finite set of complex numbers</u> $\lambda_1, \ldots, \lambda_n$.

Proof: It is easy to see that the condition (2) is necessary because

$$\sum_{h,k=1}^{n} \langle u_h, u_k \rangle \lambda_h \overline{\lambda_k} =$$

$$= \| \lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_n u_n \|^2 \geqq 0.$$

Next we show that condition (2) is sufficient as well. We define a complex linear space L as follows. Each element of L consists of a sequence of complex coordinates, of which only a finite number of terms are different from zero. Multiplication by a complex number of an element of L and addition of elements of L we define as usual, namely coordinate-wise.

We define a function Φ on L × L by

$$\Phi(x,y) = \sum_{h,k=1}^{\infty} c_{hk} x_h \overline{y_k},$$

where $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$.

It is easy to see by (2) that

$$\left| \Phi(x,y) \right|^2 \leqq \Phi(x,x)\, \Phi(y,y) \tag{3}$$

and consequently

$$\begin{aligned} \left| \sqrt{\Phi(x,x)} - \sqrt{\Phi(y,y)} \right| &\leqq \sqrt{\Phi(x \pm y, x \pm y)} \leqq \\ &\leqq \sqrt{\Phi(x,x)} + \sqrt{\Phi(y,y)} \end{aligned} \tag{3'}$$

hold. We also have that

$$\sqrt{\Phi(\alpha x, \alpha y)} = |\alpha| \sqrt{\Phi(x,y)}$$

for any complex number α . The set of elements for which

$$\Phi(x,x) = 0$$

is a linear subspace M. If the element x belongs to M, then it does not necessarily follow however that all its coordinates are zero. Therefore we consider the quotient space

$$\widetilde{L} = L / M.$$

The elements of $\widetilde{L}$ are the subsets $X \subset L$ with the property that with $\hat{x} \in X$ all other elements x of X have the form

$$x = \hat{x} + z$$

with an arbitrary element z from M. The multiplication of an element of $\tilde{L}$ by a complex number α is defined so that the product αX is that element Y of $\tilde{L}$ which as subset of the linear space L contains the element αx $(x \in X)$. Analogously we define the addition. Furthermore we define

$$\Phi(X,Y) = \Phi(x,y) \qquad (x \in X,\ y \in Y).$$

This definition is independent of the choice of the representatives x and y because of the inequality (3'). It is therefore well-defined. It can be seen that it acts as an inner product for the space $\tilde{L}$ by either verifying directly the relations an inner product has to satisfy, or by simply showing that the "parallelogram law" holds, for it is clear that $\tilde{L}$ is a normed linear space with respect to the norm

$$\| \quad \| = \sqrt{\Phi(\ , \ )}.$$

Completion of $\tilde{L}$ with respect to the above norm gives us the required Hilbert space.

Finally, if u_n is that element of the Hilbert space whose n-th coordinate is 1 and all others are 0, we see that

$$\Phi(u_h,u_k) = \langle u_h,u_k \rangle = c_{hk}.$$

The proof is complete.

By comparison of formula (1) in the definition of positive definite sequences and formula (2) in the foregoing proposition, we obtain the following <u>Geometric Representation Theorem for Positive Definite Sequences</u>:

THEOREM: <u>In order that a sequence whose terms are the complex numbers</u> c_n, $n = 0, \pm 1, \pm 2, \ldots$, <u>be positive definite, it is necessary and sufficient that there exist a stationary sequence whose terms</u> u_n, $n = 1, 2, \ldots$, <u>belong to some complex Hilbert space H such that</u>

$$\langle u_h, u_k \rangle = c_{h-k} \qquad (h,k = 1, 2, \ldots).$$

At this point we wish to make a brief digression and consider the "parallelogram law" mentioned further above in the construction of a suitable Hilbert space; we are refering to the Jordan - v. Neumann theorem concerning the characterization of an inner product space. After proving this theorem in a manner which is free from the usual limit argument, we shall return to the representation theory for positive definite sequences.

THEOREM (P. Jordan & J. von Neumann): *Let X be a normed linear space. If the norm on X satisfies the "parallelogram law"*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2),$$

then there exists an inner product on X, $\langle x,y\rangle \quad (x,y \in X)$ *such that* $\sqrt{\langle x,x\rangle} = \|x\|$ *for every* $x \in X$. *If* $\langle x,y\rangle$ *can be so defined, this can be done in only one way.*

Proof: Define

$$\langle x,y\rangle = (1/4)\{(\|x+y\|^2 - \|x-y\|^2) + i(\|x+iy\|^2 - \|x-iy\|^2)\}.$$

First we show the following equalities: $\langle ix,y\rangle = i\langle x,y\rangle$, $\langle x,iy\rangle = -i\langle x,y\rangle$, $\sqrt{\langle x,x\rangle} = \|x\|$, and $\langle y,x\rangle = \overline{\langle x,y\rangle}$.

$$\langle ix,y\rangle =$$

$$= (1/4)\{(\|ix+y\|^2 - \|ix-y\|^2) + i(\|ix+iy\|^2 - \|ix-iy\|^2)\}$$

$$= (1/4)\{(\|i(x-iy)\|^2 - \|i(x+iy)\|^2) + i(\|i(x+y)\|^2 - \|i(x-y)\|^2)\}$$

$$= (1/4)\{i(\|x+y\|^2 - \|x-y\|^2) - (\|x+iy\|^2 - \|x-iy\|^2)\} = i\langle x,y\rangle.$$

$$\langle x,iy\rangle =$$

$$= (1/4)\{(\|x+iy\|^2 - \|x-iy\|^2) + i(\|x-y\|^2 - \|x+y\|^2)\}$$

$$= (1/4)\{-i(\|x+y\|^2 - \|x-y\|^2) + (\|x+iy\|^2 - \|x-iy\|^2)\} = -i\langle x,y\rangle.$$

$$\langle x,x\rangle = (1/4)\{4\|x\|^2 + i(|i+1|^2 - |1-i|^2)\|x\|^2\} = \|x\|^2.$$

$$\begin{aligned}\langle y,x\rangle &= \\ &= (1/4)\{(\|y+x\|^2-\|y-x\|^2)+i(\|y+ix\|^2-\|y-ix\|^2)\} = \\ &= (1/4)\{(\|x+y\|^2-\|x-y\|^2)-i(\|x+iy\|^2-\|x-iy\|^2)\} = \overline{\langle x,y\rangle}.\end{aligned}$$

Next we show that for any complex number λ

$$\langle \lambda x,y\rangle + \langle x,\lambda y\rangle = (\lambda+\bar{\lambda})\langle x,y\rangle$$

holds.

$$\begin{aligned}&\langle \lambda x,y\rangle + \langle x,\lambda y\rangle = \\ &= (1/4)\{(\|\lambda x+y\|^2-\|\lambda x-y\|^2)+i(\|\lambda x+iy\|^2-\|\lambda x-iy\|^2)\} + \\ &+ (1/4)\{(\|x+\lambda y\|^2-\|x-\lambda y\|^2)+i(\|x+i\lambda y\|^2-\|x-i\lambda y\|^2)\}.\end{aligned}$$

By the "parallelogram law":

$$\|\lambda x+y\|^2+\|x+\lambda y\|^2 = (1/2)\left[\|(\lambda+1)x+(\lambda+1)y\|^2 + \|(\lambda-1)x-(\lambda-1)y\|^2\right]$$

$$\|\lambda x-y\|^2+\|x-\lambda y\|^2 = (1/2)\left[\|(\lambda+1)x-(\lambda+1)y\|^2 + \|(\lambda-1)x+(\lambda-1)y\|^2\right]$$

$$\|\lambda x+iy\|^2+\|x+i\lambda y\|^2 = (1/2)\left[\|(\lambda+1)x+i(\lambda+1)y\|^2 + \|(\lambda-1)x-i(\lambda-1)y\|^2\right]$$

$$\|\lambda x-iy\|^2+\|x-i\lambda y\|^2 = (1/2)\left[\|(\lambda+1)x-i(\lambda+1)y\|^2 + \|(\lambda-1)x+i(\lambda-1)y\|^2\right].$$

We therefore obtain

$$\langle \lambda x,y\rangle + \langle x,\lambda y\rangle =$$

$$= (1/4)(1/2)\Big[\|(\lambda+1)x+(\lambda+1)y\|^2 - \|(\lambda+1)x-(\lambda+1)y\|^2$$

$$+ i\|(\lambda+1)x+i(\lambda+1)y\|^2 - i\|(\lambda+1)x - i(\lambda+1)y\|^2$$

$$+ \|(\lambda-1)x - (\lambda-1)y\|^2 - \|(\lambda-1)x+(\lambda-1)y\|^2$$

$$+ i\|(\lambda-1)x - i(\lambda-1)y\|^2 - i\|(\lambda-1)x+i(\lambda-1)y\|^2\Big]$$

$$= (1/8)\Big[(|\lambda+1|^2 - |\lambda-1|^2)(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)\Big].$$

But

$$|\lambda+1|^2 - |\lambda-1|^2 = 2(\lambda+\overline{\lambda})$$

holds for any complex number λ.

The relation $\langle \lambda x,y\rangle + \langle x,\lambda y\rangle = (\lambda+\overline{\lambda})\langle x,y\rangle$ gives, putting $\lambda = ir$, where r is any real number,

$$\langle rx,y\rangle = \langle x,ry\rangle .$$

Thus

$$2\langle rx,y\rangle = \langle rx,y\rangle + \langle x,ry\rangle = 2r\langle x,y\rangle$$

and hence

$$\langle rx,y\rangle = r\langle x,y\rangle$$

for every real number r.

It is clear that

$$(1/2)\langle x+y,2z\rangle = \langle x+y,z\rangle .$$

We now show that

$$\langle x,z\rangle + \langle y,z\rangle = (1/2)\langle x+y,2z\rangle .$$

$$\langle x,z\rangle + \langle y,z\rangle =$$

$$= (1/4)\Big\{(\|x+z\|^2 - \|x-z\|^2) + i(\|x+iz\|^2 - \|x-iz\|^2$$

$$+ (\|y+z\|^2 - \|y-z\|^2) + i(\|y+iz\|^2 - \|y-iz\|^2)\Big\}.$$

By the "parallelogram law":

$$\|x+z\|^2 + \|y+z\|^2 = (1/2)\|x+y+2z\|^2 + (1/2)\|x-y\|^2$$

$$\|x-z\|^2 + \|y-z\|^2 = (1/2)\|x+y-2z\|^2 + (1/2)\|x-y\|^2$$

$$\|x+iz\|^2 + \|y+iz\|^2 = (1/2)\|x+y+i2z\|^2 + (1/2)\|x-y\|^2$$

$$\|x-iz\|^2 + \|y-iz\|^2 = (1/2)\|x+y-i2z\|^2 + (1/2)\|x-y\|^2 .$$

Therefore $\langle x,z\rangle + \langle y,z\rangle =$

$$= (1/8)\Big[\|x+y+2z\|^2 - \|x+y-2z\|^2 + i(\|x+y+i2z\|^2 - \|x+y-i2z\|^2)\Big]$$

$$= (1/2)\langle x+y,2z\rangle .$$

This establishes the additive condition

$$\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle .$$

Finally, the additive condition combined with $\langle ix,y\rangle = i\langle x,y\rangle$ and $\langle rx,y\rangle = r\langle x,y\rangle$, for all real r, yields

$$\langle \lambda x,y\rangle = \lambda\langle x,y\rangle$$

for any complex number λ.

Consequently, the "polarization identity"

$$\langle x,y\rangle = (1/4)\{\|x+y\|^2 - \|x-y\|^2 + i(\|x+iy\|^2 - \|x-iy\|^2)$$

defines an inner product on the space X. Since any inner product (x,y) satisfies the "polarization identity" (for replace in the identity $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + (x,y) + (y,x)$ the term y by -y, iy, and -iy), it is clear that $\langle x,y\rangle$ is unique. This completes the proof.

We now return to the representation theory of positive definite sequences; by way of preparation for the proof of the analytic representation theorem, we take up certain propositions first.

THEOREM (E. Helly): <u>Let f be a continuous function on [a,b]. Suppose that h_n, n = 1,2, ... , are the terms of a sequence of functions which converges to a finite function h at each point of [a,b]. If the total variation satisfies</u>

$$\bigvee_a^b (h_n) \leqq K < \infty$$

<u>for all n, then</u>

$$\lim_{n\to\infty} \int_a^b f(x)\, dh_n(x) = \int_a^b f(x)\, dh(x).$$

Proof: If we split the interval $[a,b]$ in an arbitrary way, we have for $n = 1, 2, \ldots$

$$\sum_{k=0}^{m-1} \left| h_n(x_{k+1}) - h_n(x_k) \right| \leqq K.$$

Passing to the limit as $n \to \infty$, we get

$$\sum_{k=0}^{m-1} \left| h(x_{k+1}) - h(x_k) \right| \leqq K.$$

Since the splitting of $[a,b]$ was arbitrary, it follows that

$$\bigvee_a^b (h) \leqq K.$$

This shows that the limit function is of finite variation also.

We now pick an arbitrary $\varepsilon > 0$ and split $[a,b]$ by means of the points x_k, $k = 0, 1, \ldots, m$, into subintervals $[x_k, x_{k+1}]$ in such a manner that the oscillation of f is less than $\varepsilon/3K$ on each subinterval $[x_k, x_{k+1}]$. We see that

$$\int_a^b f(x)\, dh(x) = \sum_{k=0}^{m-1} \int_{x_k}^{x_{k+1}} f(x)\, dh(x) =$$

$$\sum_{k=0}^{m-1} \int_{x_k}^{x_{k+1}} (f(x) - f(x_k))\, dh(x) + \sum_{k=0}^{m-1} f(x_k) \int_{x_k}^{x_{k+1}} dh(x).$$

But

$$\int_{x_k}^{x_{k+1}} dh(x) = h(x_{k+1}) - h(x_k).$$

Since

$$|f(x) - f(x_k)| < \varepsilon/3K$$

for any $x \in [x_k, x_{k+1}]$, we have

$$\left| \int_{x_k}^{x_{k+1}} (f(x) - f(x_k))dh(x) \right| \leqq (\varepsilon/3K) \bigvee_{x_k}^{x_{k+1}} (h).$$

Thus

$$\left| \sum_{k=0}^{m-1} \int_{x_k}^{x_{k+1}} (f(x)-f(x_k))dh(x) \right| \leqq (\varepsilon/3K) \bigvee_a^b (h) \leqq \varepsilon/3.$$

Therefore

$$\int_a^b f(x)dh(x) = \sum_{k=0}^{m-1} f(x_k)(h(x_{k+1})-h(x_k)) + \vartheta(\varepsilon/3)$$

where $|\vartheta| \leqq 1$. Similarly we get

$$\int_a^b f(x)dh_n(x) = \sum_{k=0}^{m-1} f(x_k)(h_n(x_{k+1})-h_n(x_k)) + \vartheta_n(\varepsilon/3)$$

where $|\vartheta_n| \leqq 1$.

Since

$$\lim_{n \to \infty} h_n(x) = h(x)$$

foor all $x \in [a,b]$, there is a natural number n_0 such that

$$\left| \sum_{k=0}^{m-1} f(x_k)(h_n(x_{k+1})-h_n(x_k)) - \sum_{k=0}^{m-1} f(x_k)(h(x_{k+1})-h(x_k)) \right|$$

is less than $\varepsilon/3$ for all $n > n_0$. Hence we have that

$$\left| \int_a^b f(x)dh_n(x) - \int_a^b f(x)dh(x) \right| < \varepsilon$$

for all $n > n_0$ and the theorem is established.

THEOREM (D. Hilbert): Let $(g_n)_{n=1}^{\infty}$ be a sequence of real-valued functions defined on some set E such that

$$\sup_{x \in E} |g_n(x)| \leqq K \quad \text{for all } n.$$

Then given any countable subset D of E there exists a subsequence $(g_{n_i})_{i=1}^{\infty}$ which converges at every point of D.

Proof: Let $D = \{x_1, x_2, \ldots\}$. Then $(g_n(x_1))_{n=1}^{\infty}$ is a bounded sequence of real numbers; hence, by the Bolzano-Weierstrass theorem, we may choose a subsequence $(g_n^{(1)})_{n=1}^{\infty}$ of $(g_n)_{n=1}^{\infty}$ such that the sequence $(g_n^{(1)}(x_1))_{n=1}^{\infty}$ converges, say,

$$\lim_{n \to \infty} g_n^{(1)}(x_1) = A_1.$$

Next we consider the sequence $(g_n^{(1)}(x_2))_{n=1}^{\infty}$. Again by the Bolzano-Weierstrass theorem we can select a subsequence $(g_n^{(2)})_{n=1}^{\infty}$ of $(g_n^{(1)})_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} g_n^{(2)}(x_2) = A_2$$

exists.

Continuing this process indefinitely we obtain a sequence of subsequences $(g_n^{(m)})_{n=1}^{\infty}$, $m = 1,2,\ldots$, of $(g_n)_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} g_n^{(m)}(x_m) = A_m$$

exists, $m = 1,2,\ldots$. We then consider the diagonal sequence

$(g_n^{(n)})_{n=1}^{\infty}$. For fixed k, $(g_n^{(n)}(x_k))_{n \geqq k}$ is a subsequence of $(g_n^{(k)}(x_k))_{n \geqq k}$, and hence converges to A_k. Therefore $(g_n^{(n)})_{n=1}^{\infty}$ converges at every point of D. QED

Let X be a Banach space and let X^* denote its adjoint space. If X is separable, then every closed sphere is sequentially compact in the weak* topology of X^*, that is from any sequence of linear functionals $(g_n)_{n=1}^{\infty}$ with bounded norm one can select a subsequence which converges weakly to a functional g_0. We consider the space $V[a,b]$ of functions of bounded variation on $[a,b]$, with norm defined by

$$\| g \| = \bigvee_a^b (g).$$

By the F. Riesz representation theorem, the space $V[a,b]$ may be identified with the adjoint space of the separable Banach space $C[a,b]$ of continuous functions on $[a,b]$ in the following way. For each $\varphi \in (C[a,b])^*$ there exists a unique $g \in V[a,b]$ such that

$$\varphi(f) = \int_a^b f(t)dg(t)$$

for all $f \in C[a,b]$. Moreover, $\| \varphi \| = \| g \|$. Making use of these observations, it is easy to prove the following

THEOREM (E. Helly): Let $(g_n)_{n=1}^{\infty}$ be a sequence of functions of bounded variation defined on $[a,b]$, and suppose that

$$\sup_{a \leqq x \leqq b} |g_n(x)| \leqq K < \infty , \text{ and } \bigvee_a^b (g_n) \leqq K < \infty$$

for all n. Then there exists a subsequence $(g_{n_i})_{i=1}^{\infty}$ which converges pointwise in $[a,b]$ to a function of bounded variation.

Proof: Without loss of generality we can take $[a,b] = [0,1]$, and since every function of bounded variation is the difference of two non-decreasing functions we may assume that every g_n is non-decreasing. Let φ_n be the linear functional on $C[0,1]$ associated with g_n, that is

$$\varphi_n(f) = \int_0^1 f(t)dg_n(t)$$

for every continuous f on $[0,1]$. Since $\bigvee_0^1 (g_n) \leqq K$, the sequence $(\varphi_n)_{n=1}^{\infty}$ is bounded in norm, hence we can select a weakly convergent subsequence of the φ_n's, which we still call $(\varphi_n)_{n=1}^{\infty}$. Let φ be the weak limit, g the function of bounded variation corresponding to φ. The function g has at most countably many points of discontinuity $x_1, x_2, \ldots$. Let x be a point of continuity of g, and define functions h_m, $m = 1, 2, \ldots$, by

$$h_m(t) = \begin{cases} 1; & 0 \leqq t \leqq x \\ -m(t-x)+1; & x \leqq t \leqq x+(1/m) \\ 0; & x+(1/m) \leqq t \leqq 1 \end{cases}$$

Since h_m is continuous the weak convergence $\varphi_n \longrightarrow \varphi$ implies

$$\int_0^1 h_m(t)dg_n(t) \longrightarrow \int_0^1 h_m(t)dg(t), \quad n \longrightarrow \infty .$$

From the continuity of g at t = x and the boundedness of h_m it follows that

$$\int_x^{x+\frac{1}{m}} h_m(t)dg(t) \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty .$$

Consequently

$$\int_0^1 h_m(t)dg(t) = \int_0^x h_m(t)dg(t) + \int_x^{x+\frac{1}{m}} h_m(t)dg(t) \longrightarrow$$

$$\longrightarrow g(x) - g(0).$$

But g_n is non-decreasing and $h_m(t) \geqq 0$ so that

$$\int_0^1 h_m(t)dg_n(t) \geqq \int_0^x h_m(t)dg_n(t) = g_n(x) - g_n(0).$$

We therefore get

$$\limsup_n \, (g_n(x)-g_n(0) \leqq g(x) - g(0).$$

On the other hand, since h_m is non-increasing,

$$\int_0^1 h_m(t+(1/m))dg_n(t) = \int_0^x h_m(t+(1/m))dg_n(t) \leqq$$

$$\leqq \int_0^x h_m(t)dg_n(t) = g_n(x) - g_n(0),$$

whence

$$\int_0^1 h_m(t+(1/m))dg(t) \leqq \liminf_n (g_n(x) - g_n(0)).$$

By the continuity of g at x:

$$\int_0^1 h_m(t+(1/m))dg(t) = \int_0^{x-\frac{1}{m}} h_m(t+(1/m))dg(t) +$$

$$+ \int_{x-\frac{1}{m}}^x h_m(t+(1/m))dg(t) =$$

$$= g(x-(1/m)) - g(0) + \int_{x-\frac{1}{m}}^x h_m(t+(1/m))dg(t) \longrightarrow g(x)-g(0).$$

Hence

$$g(x) - g(0) \leqq \liminf_n (g_n(x)-g_n(0)).$$

We can therefore conclude that

$$\lim_n (g_n(x)-g_n(0)) = g(x)-g(0).$$

By assumption $|g_n(0)| \leqq K$, $|g_n(x_k)| \leqq K$ for all n and k.

Hence by the theorem on page 16 there exists a subsequence $(g_{n_i})_{i=1}^{\infty}$ such that for $i \to \infty$

$$g_{n_i}(0) \longrightarrow a_0 \quad \text{and} \quad g_{n_i}(x_k) \longrightarrow a_k.$$

From this and the relation

$$\lim_n \; (g_n(x)-g_n(0)) = g(x)-g(0),$$

established further above, we get convergence for every $x \neq x_k$, $k = 1,2, \ldots$:

$$g_{n_i}(x) \longrightarrow g(x) - g(0) + a_0 \quad \text{as } i \to \infty$$

The function

$$\tilde{g}(x) = \begin{cases} a_k & \text{when } x = x_k \\ g(x)-g(0)+a_0 & \text{when } x \neq x_k \end{cases}$$

for $k = 1,2, \ldots$ is of bounded variation, where for each $x \in [0,1]$ we obtain

$$g_{n_i}(x) \longrightarrow \tilde{g}(x)$$

and the proof is complete.

After these preparations we are in a position to establish the <u>Analytic Representation Theorem for Positive Definite Sequences</u>.

THEOREM (G. Herglotz): <u>A necessary and sufficient condition for a sequence to be positive definite, is that its terms $(c_n)_{n=-\infty}^{\infty}$ have the representation</u>

$$c_n = \int_{-\pi}^{\pi} e^{inx}\, dg(x) \qquad (n = 0, \pm 1, \pm 2, \dots),$$

<u>where</u> g <u>denotes a non-decreasing function of bounded variation</u>.

Proof: The condition is sufficient. Indeed

$$\sum_{k=1}^{N} \sum_{j=1}^{N} c_{k-j} \lambda_k \overline{\lambda}_j =$$

$$= \sum_{k=1}^{N} \sum_{j=1}^{N} \left\{ \int_{-\pi}^{\pi} e^{i(k-j)x}\, dg(x) \right\} \lambda_k \overline{\lambda}_j =$$

$$= \int_{-\pi}^{\pi} \left(\sum_{k=1}^{N} e^{ikx} \lambda_k \right) \left(\sum_{j=1}^{N} e^{-ijx} \overline{\lambda}_j \right) dg(x) =$$

$$= \int_{-\pi}^{\pi} \left| \sum_{k=1}^{N} e^{ikx} \lambda_k \right|^2 dg(x) \geqq 0.$$

The condition is necessary. Since $(c_n)_{n=-\infty}^{\infty}$ is positive definite, we have in particular that

$$\sum_{k=1}^{N} \sum_{j=1}^{N} c_{k-j}\, e^{-i(k-j)x} \geqq 0.$$

We observe that

$$\sum_{k=1}^{N} \sum_{j=1}^{N} c_{k-j} \, e^{-i(k-j)x} = N \sum_{m=-N+1}^{N-1} \left(1 - \frac{|m|}{N}\right) c_n \, e^{-inx}.$$

We let

$$Q_N(x) = \sum_{m=-N+1}^{N-1} \left(1 - \frac{|m|}{N}\right) c_n \, e^{-inx}.$$

Since

$$\int_{-\pi}^{\pi} e^{-i(n-m)x} \, dx = \begin{cases} 0 & \text{for } n \neq m \\ 2\pi & \text{for } n = m \end{cases}$$

we see that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} Q_N(x) \, e^{inx} \, dx = \left(1 - \frac{|m|}{N}\right) c_n.$$

We put

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} Q_N(x) \, e^{inx} \, dx = \int_{-\pi}^{\pi} e^{inx} \, dg_N(x).$$

Then

$$g_N(x) = \int_{-\pi}^{x} Q_N(y) \, dy;$$

g_N is a non-decreasing function with total variation

$$\int_{-\pi}^{\pi} dg_N(x) = c_0.$$

By the theorem on page 19 we can find a subsequence $(g_{N_k})_{k=1}^{\infty}$ which converges pointwise to a limit function g which is again non-decreasing and of bounded variation.
By the theorem on page 13 we finally obtain

$$\lim_{k \to \infty} \int_{-\pi}^{\pi} e^{inx} \, dg_{N_k}(x) = \int_{-\pi}^{\pi} e^{inx} \, dg(x)$$

and the theorem is proved.

Chapter 2.

ALMOST PERIODIC FUNCTIONS

2.1 Basic Definitions and Properties

A complex-valued continuous function f defined on the entire real line R is said to be u n i f o r m l y a l m o s t p e r i o d i c (UAP for short), if to every $\varepsilon > 0$ there corresponds a number $L = L(\varepsilon) > 0$ such that in each interval on R of length L is situated an ε - a l m o s t p e r i o d of f, that is to say a number $\tau = \tau(\varepsilon)$ so that for all $x \in R$ we have:

$$|f(x+\tau) - f(x)| < \varepsilon$$

It is customary to use the symbol $E\{\varepsilon;f\}$ to denote the set of ε-almost periods of the function f and to use the symbol $\overline{E}\{\varepsilon;f\}$ to signify the set of all integers of $E\{\varepsilon;f\}$.

It is well-established usage to refer to a set E of real numbers as r e l a t i v e l y d e n s e, if there exists a number $T > 0$ such that any interval of length T contains at least one element of E.

In terms of these conventions the definition of a UAP function reads as follows: A complex-valued continuous function f defined on the entire real line R is a UAP function if for any $\varepsilon > 0$ the set $E\{\varepsilon;f\}$ is relatively dense.

UAP functions were invented and thoroughly studied by Harald Bohr. In this section we shall look at some basic propositions in the theory of UAP functions.

I. <u>A UAP function is bounded on the entire real line R.</u>

Proof: Let f be a UAP function and take $\varepsilon=1$. Since f is continuous, the function $|f|$ has a maximum M on the closed interval $[0,L(1)]$, where L(1) is obtained through the definition above. Suppose that x_0 is an arbitrary real number. We select an almost period $\tau = \tau(1)$ in the interval $-x_0 \leqq x \leqq -x_0+L(1)$. Thus $0 \leqq x_0+\tau \leqq L(1)$ and we have

$$|f(x_0)| \leqq |f(x_0)-f(x_0+\tau)| + |f(x_0+\tau)| \leqq 1+M$$

and the proposition is proved.

II. <u>A UAP function is uniformly continuous on the entire real line R.</u>

Proof: Denote by ε an arbitrary positive number and take a number $L = L(\varepsilon/3)$. The UAP function f is uniformly continuous on the closed interval $[-1,1+L]$. Hence we can find a positive number $\delta < 1$ such that for any y_1 and y_2 in $[-1,1+L]$, for which $|y_2-y_1| < \delta$ holds, the inequality

$$|f(y_2) - f(y_1)| < \varepsilon/3$$

is satisfied. Suppose that x_1 and x_2 is an arbitrary pair of real numbers for which $|x_2-x_1| < \delta$. Denote by τ an $\varepsilon/3$-almost period of f included in the interval $[-x_1,-x_1+L]$. Since $|x_2-x_1| < \delta$ and $0 \leqq x_1 + \tau \leqq L$, we easily see that $-1 < x_2 + \tau < 1+L$. Therefore

$$|f(x_2)-f(x_1)| \leqq |f(x_2)-f(x_2+\tau)| + |f(x_2+\tau)-f(x_1+\tau)| +$$
$$+ |f(x_1+\tau)-f(x_1)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

and since we have chosen ε arbitrarily, the proof is finished.

III. <u>Let f be a UAP function. For any positive number ε it is possible to select numbers $L = L(\varepsilon)$ and $\delta = \delta(\varepsilon)$ so that in each interval on the real line R of length L we can find a subinterval of length δ all points of which are ε-almost periods of f.</u>

Proof: Denote by $T = T(\varepsilon/2)$ a positive number possessing the property that in each interval on the real line R of length T there is at least one $\varepsilon/2$-almost period of f. Furthermore denote by $\tilde{\delta} = \tilde{\delta}(\varepsilon/2)$ a positive number such that

$$\sup_{-\infty < x < \infty} |f(x+h) - f(x)| < \varepsilon/2$$

if $|h| < \tilde{\delta}$. The number $\tilde{\delta}$ exists by proposition II. Let τ be an $(\varepsilon/2)$-almost period of f contained in the interval $(\alpha + \tilde{\delta}, \alpha + T + \tilde{\delta})$. If $|h| < \tilde{\delta}$, then the number $\tau + h$ is in the interval $(\alpha, \alpha + T + 2\tilde{\delta})$ and is an ε-almost period of f; the latter follows from the estimate

$$|f(x+\tau+h) - f(x)| \leqq |f(x+h+\tau) - f(x+h)| + \\ + |f(x+h) - f(x)| < \varepsilon .$$

Hence the numbers $L = T + 2\tilde{\delta}$ and $\delta = 2\tilde{\delta}$ satisfy the claim made above.

IV. Let f and g be two UAP functions. Then for any $\varepsilon > 0$ there exists a number $M(\varepsilon)$ such that every interval on the real line R of length $M(\varepsilon)$ contains at least one number $\gamma(\varepsilon)$ which is a common ε-almost period of both functions f and g.

Proof: For $\varepsilon > 0$ let δ be as in proposition III. Let the positive integer M be so large that every interval of length $L = M\delta$ contains an $(\varepsilon/2)$-almost period of f as well as an $(\varepsilon/2)$-almost period of g. Divide the real line into intervals $I_n = [(n-1)L, nL)$. Then for each integer n, I_n contains a $\sigma_n \in E\{\varepsilon/2;f\}$ and a $\tau_n \in E\{\varepsilon/2;g\}$. Divide the interval $[-L,L)$ into 2M intervals J_k of length δ. Since $\sigma_n, \tau_n \in I_n$ we have $|\sigma_n - \tau_n| < L$, hence $\sigma_n - \tau_n \in J_k$ for some k. This k may be called the index of n. Since only finitely many indices are available, there exists a positive integer N such that every possible index is attained as n runs from -N to N. Hence there exists an m_n, $-N < m_n < N$, such that n and m_n have the same index k, i.e., $\sigma_n - \tau_n \in J_k$ and $\sigma_{m_n} - \tau_{m_n} \in J_k$.

Let $\overline{\sigma}_n = \sigma_n - \sigma_{m_n}$, $\overline{\tau}_n = \tau_n - \tau_{m_n}$.

Clearly $\overline{\sigma}_n \in E\{\varepsilon;f\}$, $\overline{\tau}_n \in E\{\varepsilon;g\}$.

Also,

$$|\overline{\sigma}_n - \overline{\tau}_n| = |(\sigma_n - \tau_n) - (\sigma_{m_n} - \tau_{m_n})| < \delta .$$

Since $\overline{\sigma}_n \in E\{\varepsilon;f\}$ it follows from proposition III that $\overline{\tau}_n \in E\{\varepsilon;f\}$, i.e., $\overline{\tau}_n$ is a common ε-almost period of f and g. It is easy to see that the set $\{\overline{\sigma}_n : n = 0, \pm 1, \ldots\}$ is relatively dense: for any n

$$|\overline{\sigma}_{n+1} - \overline{\sigma}_n| = |(\sigma_{n+1} - \sigma_{m_{n+1}}) - (\sigma_n - \sigma_{m_n})| \leqq$$

$$\leqq |\sigma_{n+1} - \sigma_n| + |\sigma_{m_{n+1}} - \sigma_{m_n}| < 2L + (L + 2NL).$$

Since each $\overline{\tau}_n$ differs from the corresponding $\overline{\sigma}_n$ by less than δ, the set $\{\overline{\tau}_n : n = 0, \pm 1, \ldots\}$ is likewise relatively dense.

Remark: We see that proposition IV holds for any finite system of UAP functions.

V. The set of integral ε-almost periods of a UAP function f is relatively dense.

Proof: Let $g(x) = \sin 2\pi x$. Taking $\varepsilon_1 < \varepsilon$, by proposition III there is a $\delta > 0$ such that all numbers whose distances from $E\{\varepsilon_1;f\}$ are less than δ belong to $E\{\varepsilon;f\}$. Take an $\varepsilon_2 > 0$ such that the elements of the set $E\{\varepsilon_2;g\}$ differ from integers by less than δ. Pick $\varepsilon_3 = \min(\varepsilon_1, \varepsilon_2)$. Then the ε_3-almost periods common to f and g, $E\{\varepsilon_3;f,g\}$, is by IV a relatively dense set and the elements of $E\{\varepsilon_3;f,g\}$

differ from integers by less than δ because g is periodic with period 1. Let J denote the set of integers nearest to each number of $E\{\varepsilon_3;f,g\}$. Then J is relatively dense. But $E\{\varepsilon_3;f,g\} \subset E\{\varepsilon_1;f\}$ and therefore the distance of each number of J from the set $E\{\varepsilon_1;f\}$ is less than δ and consequently each number of J belongs to $E\{\varepsilon;f\}$, and clearly to $\overline{E}\{\varepsilon;f\}$. Thus J being relatively dense, $\overline{E}\{\varepsilon;f\}$ is also relatively dense. The proof is finished.

Consider the complete metric space M_∞ of bounded complex-valued functions on $(-\infty, \infty)$, where distance between two elements f and g is defined by

$$\rho(f,g) = \sup_{-\infty < x < \infty} |f(x)-g(x)| .$$

By proposition I each UAP function belongs to M_∞.

We shall call a set Q in a metric space X sequentially compact if every infinite subset of Q contains a convergent sequence with limit in X (but not necessarily belonging to the set Q).

VI. <u>A set S of UAP functions is sequentially compact in</u> M_∞ <u>if and only if</u>: 1) <u>the functions of the set S are uniformly bounded and equi-continuous and</u> 2) <u>the functions of the set S are equi-almost periodic, that is for every</u> $\eta > 0$ <u>there exists an</u> $L = L(\eta)$ <u>such that each interval of length L contains a number p which is a</u> η <u>-almost period for all functions of the set S</u>.

Proof: Necessity. Consider condition 1) first. Let S be sequentially compact. The uniform boundedness of the function

in S follows the fact that every sequentially compact set S of a metric space is totally bounded. We show that S is an equi-continuous set of functions. For given $\varepsilon > 0$ we construct a finite $\varepsilon/3$-net $u_1, \dots, u_n$ for the set S. Since the u_k's of the $\varepsilon/3$ - net can be taken as belonging to S, it is clear from proposition II that the u_k's are uniformly continuous on $(-\infty, \infty)$. For each u_k we select a δ_k such that

$|u_k(x_1)-u_k(x_2)| < \varepsilon/3$ holds for $|x_1-x_2| < \delta_k$. Let

$$\delta = \min_{1 \leqq k \leqq n} \delta_k.$$

If $|x_1-x_2| < \delta$, then for each function $u \in S$ we have

$$|u(x_1)-u(x_2)| \leqq \sup_{-\infty < x_1 < \infty} |u(x_1)-u_k(x_1)| + |u_k(x_1)-u_k(x_2)|$$

$$+ \sup_{-\infty < x_2 < \infty} |u_k(x_2)-u(x_2)| < 2\rho(u,u_k) + \varepsilon/3.$$

If we pick from the net that function u_k for which

$$\rho(u_k,u) < \varepsilon/3,$$

then

$$|u(x_1)-u(x_2)| < \varepsilon .$$

Since $\varepsilon > 0$ was arbitrarily chosen and since the above estimate does not depend on the position of the points x_1 and x_2 nor on the choice of the function u of S, we get that the set S of functions is equi-continuous.

Now we consider condition 2). Since S is sequentially compact, there exists for each $\eta > 0$ a finite $\eta/3$ - net for the set S consisting of elements $v_1, \dots, v_n$; all these functions we can consider as belonging to the set S. By the remark following proposition IV there exists a number $L > 0$ with the property that each interval $(\alpha, \alpha + L)$ contains a number t which is for all v_i $(i = 1, \dots, n)$ a common $\eta/3$-almost period:

$$|v_i(x+t)-v_i(x)| < \eta/3, \quad i = 1, \dots, n; \quad -\infty < x < \infty .$$

On the other hand $(v_i)_{i=1}^{n}$ constitutes an $\eta/3$ - net. Thus for each function $v \in S$ there exists some v_i for which

$$|v(x)-v_i(x)| < \eta/3; \quad -\infty < x < \infty .$$

From the last two inequalities follows that

$$|v(x+t)-v(x)| \leqq |v(x+t)-v_i(x+t)| + |v_i(x+t)-v_i(x)|$$

$$+ |v_i(x)-v(x)| < 3\eta/3 = \eta \quad \text{for} \quad -\infty < x < \infty .$$

Hence t is an η-almost period for all $v \in S$ and the necessity of condition 2) is shown.

Sufficiency. We assume that a set S of UAP functions satisfies conditions 1) and 2) and we select an $\eta > 0$. Furthermore let $L = L(\eta)$ be so determined that each interval of length L has an η-almost period for all $w \in S$. For each function $w \in S$ we define a function $\overline{w}$ by

$$\overline{w}(x) = \begin{cases} w(x) & \text{for } -L \leqq x \leqq L \\ w(x-r_n) \quad \text{for} & \begin{array}{ll} nL < x \leqq (n+1)L & n = 1,2,\ldots \\ nL \leqq x < (n+1)L & n = -2,-3,\ldots \end{array} \end{cases}$$

Here r_n is an η-almost period for all $w \in S$ which is in the interval $(nL,(n+1)L)$.

We denote the set of all functions $\overline{w}$ by S_η. The functions $\overline{w}$ in S_η satisfy on the interval $[-L,L]$ the conditions of Ascoli's theorem, namely: If Q is compact, then a set in C(Q) is sequentially compact if and only if it is bounded and equi-continuous. Thus S_η is sequentially compact in the sense of uniform convergence on this interval. Since $x-r_n \in [-L,L]$, by the definition of $\overline{w}$, a sequence of these functions, which converges uniformly on the interval $[-L,L]$, also converges uniformly on the entire real line. Hence the set S_η is sequentially compact in the sense of uniform convergence on the entire real line, that is in the sense of the metric of M_∞. For arbitrary $w \in S$ and the corresponding $\overline{w} \in S_\eta$ we have

$$w(x) - \overline{w}(x) = 0 \quad \text{for} \quad -L \leqq x \leqq L$$

and

$$w(x) - \overline{w}(x) = w(x) - w(x-r_n)$$

for $\begin{cases} nL < x \leqq (n+1)L & n = 1,2,3, \ldots \\ nL \leqq x < (n+1)L & n = -2,-3, \ldots \, . \end{cases}$

Since r_n is an η-almost period of w we have for arbitrary x:

$$|w(x) - \overline{w}(x)| < \eta .$$

Thus the sequentially compact set S_η forms an η- net for S in the space M_∞. Hence S is sequentially compact and it is verified that conditions 1) and 2) are in fact sufficient. This ends the proof of the proposition.

For $\lambda \in R$ the translate f_λ of the function f is defined by

$$f_\lambda(x) = f(x+\lambda).$$

VII. <u>A continuous function is UAP if and only if the set of its translates is sequentially compact in</u> M_∞.

Proof: Let f be UAP. Evidently the set of translates satisfies both conditions of proposition VI.
Conversely suppose that the set $\{f_\lambda : \lambda \in R\}$ is sequentially compact. Then it contains a finite ε - net $f_{\lambda_1}, \ldots, f_{\lambda_n}$.
We order the f_{λ_i} according to rising index $\lambda_1 < \lambda_2 < \ldots < \lambda_n$.
For each f_λ there exists an f_{λ_i} such that

$$\rho(f_\lambda, f_{\lambda_i}) = \sup_{-\infty < x < \infty} |f_\lambda(x) - f_{\lambda_i}(x)| < \varepsilon$$

or

$$|f(x+\lambda)-f(x+\lambda_i)| < \varepsilon \quad \text{for all } x.$$

If we put $x+\lambda_i = x'$, we get

$$|f(x'+\lambda-\lambda_i)-f(x')| < \varepsilon, \quad -\infty < x' < \infty .$$

Thus for an arbitrary real λ one of the numbers $\lambda-\lambda_i$, $i=1,2, \ldots ,n$, is an ε-almost period. It follows that each interval $(a,a+\lambda_n-\lambda_1)$ contains an ε-almost period. For if we put $a+\lambda_n=\lambda$, we obtain as ε-almost period one of the numbers $a+\lambda_n-\lambda_i$, $i=1,2, \ldots ,n$. Since $\lambda_1 \leqq \lambda_i \leqq \lambda_n$ we have $a+\lambda_n-\lambda_i \in [a,a+\lambda_n-\lambda_1]$

and the proof is complete.

<u>Remark:</u> The proposition VII can serve as an alternate definition for UAP functions; this was done by S. Bochner. Using this definition of UAP function we easily obtain that the <u>sum of two UAP functions is again a UAP function</u>. Indeed, if f and g are two UAP functions, then by proposition VII any sequence $(f+g)_{\lambda_i} = f_{\lambda_i} + g_{\lambda_i}$ has $\mu_n = \lambda_{i_n}$

such that (f_{μ_n}) and (g_{μ_n}) are both uniformly convergent.

Thus $((f+g)_{\mu_n}) = (f_{\mu_n}+g_{\mu_n})$ is also uniformly

convergent and by VII $f+g$ is a UAP function as well.

It is trivial that with f, the functions af, f_c and $|f|$, where a is a complex and c is a real number, are UAP also. Since

$$|f^2(x+t)-f^2(x)| = |f(x+t)+f(x)| \; |f(x+t)-f(x)|$$

and

$$f(x)g(x) = (1/4)\{(f(x)+g(x))^2 - (f(x)-g(x))^2\}$$

we see that the product of two UAP functions is UAP as well.

VIII. <u>The uniform limit f of a sequence $(f_n)_{n=1}^{\infty}$ of UAP functions is again a UAP function.</u>

Proof: Given any $\varepsilon > 0$, pick $N = N(\varepsilon)$ such that

$$\sup_{-\infty < x < \infty} |f(x)-f_N(x)| < \varepsilon/3.$$

Let t be an $\varepsilon/3$-almost period of f_N. We have

$$|f(x+t)-f(x)| \leqq |f(x+t)-f_N(x+t)| + |f_N(x+t)-f_N(x)|$$
$$+ |f_N(x)-f(x)| < 3\,\varepsilon/3$$

proving the proposition because $E\{\varepsilon/3; f_N\}$ is relatively dense, and because of the above inequality each $\varepsilon/3$-almost period of f_N is an ε-almost period of f, $E\{\varepsilon; f\}$ is relatively dense.

<u>Remark</u>: We may summarize some of the above results in the following statement:

<u>The set of UAP functions forms a complex Banach space under the norm</u>

$$\| f \| = \sup_{-\infty < x < \infty} | f(x) | .$$

An important consequence of proposition VIII is the following. Consider the set of all exponential polynomials

$$s_n(x) = \sum_{k=1}^{m} a_k e^{i \lambda_k x} ,$$

where λ_k are real numbers and a_k are complex numbers for $k = 1, \ldots ,n$. Each summand in the above expression is a periodic function with period $2\pi / |\lambda_k|$ if $\lambda_k \neq 0$, or constant, and therefore a UAP function. Thus the sum s_n is a UAP function as well. Looking at the class of all possible uniform limits of exponential polynomials, we get by force of proposition VIII that all the functions thus resulting are UAP also.

In the remainder of this chapter we occupy ourselves with establishing the converse result, namely that each UAP function is the uniform limit of exponential polynomials. This result gives a deep characterization of the space of

UAP functions and is the content of H. Bohr's

APPROXIMATION THEOREM: <u>Every UAP function f can be approximated uniformly for</u> $-\infty < x < \infty$ <u>by finite sums of the form</u>

$$s(x) = \sum_{k=1}^{m} a_k e^{i \lambda_k x},$$

<u>that is, for each</u> $\varepsilon > 0$ <u>there exists a sum s such that</u>

$$|f(x)-s(x)| \leqq \varepsilon \quad \underline{\text{for all } x}.$$

In conclusion of this section we mention the following:
If

$$\sum_{k=1}^{\infty} |a_k| < \infty \qquad \text{(the } a_k\text{'s are complex numbers)}$$

and $(\lambda_k)_{k=1}^{\infty}$ is a set of real numbers, then

$$\sum_{k=1}^{\infty} a_k e^{i \lambda_k x}$$

is a UAP function. The foregoing statement is an immediate consequence of proposition VIII and will be used in the next section.

2.2 On a Theorem in Additive Number Theory

In this section we employ the representation theory of positive definite sequences to derive some propositions of number theoretic character. We shall call a set E of positive integers r e l a t i v e l y d e n s e, if there exists a number $T > 0$ such that any interval of length T on the positive part of the real line contains at least one element of the set E. We prove first the following proposition:

I. For any relatively dense set of positive integers E it is possible to exhibit real numbers $\lambda_1, \ldots, \lambda_m$ such that all integers n, for which the numbers

$$\frac{\lambda_k n}{2\pi} \qquad (k = 1, \ldots, m)$$

differ from integers by not more than 1/4, are representable in the form

$$n = n_p + n_q - n_r - n_s,$$

where n_p, n_q, n_r, and n_s belong to the set E.

The foregoing proposition will be used in the next section to show that the almost periods of a UAP function coincide with the solutions of a system of inequalities of the form

$$|\lambda_k t| < \delta \pmod{2\pi}; \quad k = 1, \dots, m.$$

These inequalities signify that there exist integers n_k for which the ordinary inequalities $|\lambda_k t - 2\pi n_k| < \delta$, $k = 1, \dots, m$, are satisfied.

We now turn to the proof of proposition I, stated on page 41.

Suppose that E is a relatively dense set of positive integers and let χ_E signify the characteristic function of the set E. We denote by

$$A_N = \sum_{(0 < n < N)} \chi_E(n)$$

and for all integers n we let

$$\varphi_N(n) = \frac{1}{A_N} \sum_{(0 < n_1 < N)} \chi_E(n+n_1)\, \chi_E(n_1).$$

In the sequel we consider only those natural numbers N' for which the foregoing function φ_N is defined, that is for which $A_N \neq 0$.

We note that

$$0 \leqq \varphi_{N'}(n) \leqq 1.$$

By the theorem on page 16 we see that from the sequence $(\varphi_{N'})$ we can select a subsequence $(\varphi_{N''})$ which converges pointwise, that is for each integer n, to some limit function φ_ω.

Moreover we have:

$$0 \leqq \varphi_\omega(n) \leqq 1.$$

We observe: If $\varphi_\omega(n) > 0$ for some n, then this n can be represented as difference of two elements of the set E. Indeed, if $\varphi_\omega(n) > 0$ for some n, then there exists an N for which $\varphi_N(n) > 0$. Thus for one of the numbers $n_1 = 1, \dots, N-1$ we have $\chi_E(n+n_1)\,\chi_E(n_1) > 0$ and consequently $n+n_1 = n_2 \in E$ and $n_1 \in E$; but $n = n_2 - n_1$.

We now verify that $\varphi_\omega(n)$ is a positive definite sequence for $n = 0, \pm 1, \pm 2, \dots$; we show that for any complex numbers $\rho_0, \rho_1, \dots, \rho_m$ $(m < \infty)$ we have that

$$H = \sum_{\substack{0 \leqq n_1 \leqq m \\ 0 \leqq n_2 \leqq m}} \varphi_\omega(n_1 - n_2)\, \rho_{n_1} \overline{\rho_{n_2}} \geqq 0.$$

Consider an approximating sequence $(\varphi_{N'})$ for φ_ω. Then

$$H = \lim_{N' \to \infty} \sum_{\substack{0 \leqq n_1 \leqq m \\ 0 \leqq n_2 \leqq m}} \varphi_{N'}(n_1 - n_2)\, \rho_{n_1} \overline{\rho_{n_2}} .$$

But

$$\sum_{\substack{0 \leqq n_1 \leqq m \\ 0 \leqq n_2 \leqq m}} \varphi_{N'}(n_1 - n_2)\, \rho_{n_1} \overline{\rho_{n_2}} =$$

$$= \frac{1}{A_{N'}} \sum_{\substack{-n_2 < n_3 < N'-n_2 \\ 0 \leqq n_1 \leqq m \\ 0 \leqq n_2 \leqq m}} \chi_E(n_1+n_2)\,\chi_E(n_2+n_3)\,\rho_{n_1}\overline{\rho_{n_2}}\,.$$

Hence

$$\sum_{\substack{0 \leqq n_1 \leqq m \\ 0 \leqq n_2 \leqq m}} \varphi_{N'}(n_1-n_2)\,\rho_{n_1}\overline{\rho_{n_2}} =$$

$$= \frac{1}{A_{N'}} \sum_{(0 < n_3 < N')} \Big| \sum_{(0 \leqq n \leqq m)} \chi_E(n+n_3)\,\rho_n \Big|^2 + \mathcal{R}_{N'}$$

where

$$\mathcal{R}_{N'} = \frac{1}{A_{N'}} \sum_{\substack{-n_2 < n_3 \leqq 0 \\ 0 \leqq n_1 \leqq m \\ 0 \leqq n_2 \leqq m}} \chi_E(n_1+n_2)\,\chi_E(n_2+n_3)\,\rho_{n_1}\overline{\rho_{n_2}}$$

$$- \frac{1}{A_{N'}} \sum_{\substack{N'-n_2 \leqq n_3 < N' \\ 0 \leqq n_1 \leqq m \\ 0 \leqq n_2 \leqq m}} \chi_E(n_1+n_2)\,\chi_E(n_2+n_3)\,\rho_{n_1}\overline{\rho_{n_2}}\,.$$

Since $A_{N'} \longrightarrow \infty$ as $N' \longrightarrow \infty$ we have that $\mathcal{R}_{N'} \longrightarrow 0$ as $N' \longrightarrow \infty$. Therefore: $H \geqq 0$.

In view of the analytic representation theorem for positive definite sequences (see page 23) we can write

$$\varphi_\omega(n) = \int_{-\pi}^{\pi} e^{int}\, dg(t), \qquad (n = 0, \pm 1, \pm 2, \dots)$$

where g is a non-decreasing function whose total variation is

$$\int_{-\pi}^{\pi} dg(t) = \varphi_\omega(0) \leqq 1.$$

Decomposing the function g into its monotone increasing jump-function g_d and its monotone increasing continuous part g_c, we define

$$\Psi(n) = \int_{-\pi}^{\pi} e^{int}\, dg_d(t) \quad \text{and}$$

$$\Phi(n) = \int_{-\pi}^{\pi} e^{int}\, dg_c(t).$$

We note that $\Psi(n)$ represents an absolutely convergent series with non-negative coefficients:

$$\Psi(n) = \sum c_k e^{i\lambda_k n}.$$

We also see that $\Phi(n)$ possesses the following asymptotic behaviour:

$$\Phi(n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

The latter assertion is a consequence of the Riemann-Lebesgue theorem; $\Phi(n)$ represents Fourier coefficients of a summable function.

We now show that $\Psi(n)$ is non-negative.
The proof conveniently splits into two steps; firstly, we show that the assumption $\Psi(n_0) < 0$ for some n_0 leads to a contradiction and secondly, we show that the case $\mathrm{Im}(\Psi(n_0)) \neq 0$ for some n_0 is impossible as well.

Assume that for a certain n_0 we have $\Psi(n_0) < 0$. Then we can find a real number $\delta > 0$ such that the set

$$\mathcal{E}_\delta = \{ n: \Psi(n) < -\delta \}$$

is not empty. Since $\sum |c_k| < \infty$, the function Ψ is UAP by what was said at the end of the last section (see page 40). Put $\varepsilon = -(\Psi(n_0) + \delta)$. The set $\mathcal{E}$ of integral ε-almost periods of Ψ is relatively dense by proposition IV on page 31. Take any $m \in \mathcal{E}$. Then $|\Psi(n_0 + m) - \Psi(n_0)| < \varepsilon$ so that $\Psi(n_0 + m) < \Psi(n_0) + \varepsilon = -\delta$. Hence $n_0 + m \in \mathcal{E}_\delta$, and since $\mathcal{E}$ is relatively dense, so is $\mathcal{E}_\delta$, We can find a number $\tilde{n} \in \mathcal{E}_\delta$ such that $|\Phi(\tilde{n})| < \delta/2$ on account of the asymptotic behaviour of $\Phi(n)$ as $n \longrightarrow \infty$. Therefore we have $\varphi_\omega(\tilde{n}) \leqq \Psi(\tilde{n}) + |\Phi(\tilde{n})| < -\delta/2 < 0$. But this contradicts the fact that $0 \leqq \varphi_\omega(\tilde{n}) \leqq 1$ (see page 43).

In a similar manner we convince ourselves that $\Psi(n)$ in not complex. For, if there was a number n_0 such that $\Psi(n_0)$ was a proper complex number, then we could find a disk in the complex plane with center at the point $\Psi(n_0)$ such that all points of this disk would be at a distance, say $\delta > 0$ or more, from the real axis. The set of numbers n for which $\Psi(n)$ are located inside the mentioned disk would constitute a non-empty set which would in fact be relatively dense. In this relatively dense set we could pick an element $\tilde{n}$ such that $|\Phi(\tilde{n})| < \delta/2$ holds. Therefore the sum $\Psi(\tilde{n}) + \Phi(\tilde{n}) = \mathcal{P}_\omega(\tilde{n})$ would have to be a proper complex number and we once again have reached a contradiction.

Next we want to show that the average

$$c_0 = \lim_{K \to \infty} \frac{1}{K} \sum_{(0 < n < K)} \Psi(n)$$

is strictly positive.

From the asymptotic behaviour of $\Phi(n)$ we have

$$\frac{1}{N} \sum_{(0 < n < N)} \Phi(n) \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty.$$

Thus

$$c_0 = \lim_{K \to \infty} \frac{1}{K} \sum_{(0 < n < K)} \mathcal{P}_\omega(n).$$

On the other hand

$$\frac{1}{K}\sum_{(0<n<K)}\varphi_N(n)=\frac{1}{K\,A_N}\sum_{\substack{0<n_1<N\\0<n<K}}\chi_E(n+n_1)\,\chi_E(n_1)=$$

$$=\frac{1}{A_N}\sum_{(0<n_1<N)}\chi_E(n_1)\left\{\frac{1}{K}\sum_{(n_1<n_2<n_1+K)}\chi_E(n_2)\right\}.$$

Since E is a relatively dense set of positive integers, one can find numbers $a>0$ and $K_0>0$ such that the number of elements of the set E situated in any interval of length K, where $K \geqq K_0$, will be larger than aK.

Consequently,

$$\frac{1}{K}\sum_{(n_1<n_2<n_1+K)}\chi_E(n_2)\geqq a$$

and therefore

$$\frac{1}{K}\sum_{(0<n<K)}\varphi_N(n)\geqq a.$$

Passage to the limit as $N\longrightarrow\infty$ gives

$$\frac{1}{K}\sum_{(0<n<K)}\varphi_\omega(n)\geqq a$$

so that we get the desired result, namely: $C_0 \geqq a > 0$.

We consider the convolution

$$\Lambda_N(n) = \frac{1}{N} \sum_{(0 < n_1 < N)} \mathcal{S}_\omega(n+n_1)\, \mathcal{S}_\omega(n_1).$$

Since $\mathcal{S}_\omega(n) = \Psi(n) + \Phi(n)$ and making use of the asymptotic behaviour of $\Phi(n)$ as $n \to \infty$, it is seen that the following limit exists:

$$\Lambda(n) = \lim_{N \to \infty} \Lambda_N(n) = \lim_{N \to \infty} \frac{1}{N} \sum_{(0 < n_1 < N)} \Psi(n+n_1)\, \Psi(n_1);$$

we obtain

$$\Lambda(n) = c_0^2 + \sum_{\lambda_k \neq 0} c_k^2\, e^{i \lambda_k n} .$$

If for some n_0 we have the inequality $\Lambda(n_0) > 0$, then we can find a number n_1 such that

$$\mathcal{S}_\omega(n_0 + n_1) > 0 \quad \text{and} \quad \mathcal{S}_\omega(n_1) > 0$$

holds. In this case however the numbers $n_0 + n_1$ and n_1 can be represented as difference of two elements of the set E (see page 43). Thus every n, for which $\Lambda(n) > 0$ is satisfied, has a representation of the type

$$n = n_p + n_q - n_r - n_s,$$

where n_p, n_q, n_r, and n_s belong to the set E.

The series

$$\Lambda(n) = c_0^2 + \sum_{\lambda_k \neq 0} c_k^2\, e^{i \lambda_k n}$$

converges absolutely; we also know that $c_0 > 0$, $c_k \geqq 0$ and $\Lambda(n) \geqq 0$ because $\Psi(n) \geqq 0$. We write

$$\Lambda(n) = c_0^2 + \sum_{k=1}^{m} c_k^2 \cos \lambda_k n + \sum_{k=m+1}^{\infty} c_k^2 \cos \lambda_k n,$$

where it can be assumed that

$$\left| \sum_{k=m+1}^{\infty} c_k^2 \cos \lambda_k n \right| < c_0^2/2$$

upon suitable choice of the number m. This then means that

$$\Lambda(n) > c_0^2/2 + \sum_{k=1}^{m} c_k^2 \cos \lambda_k n.$$

The quantity on the right side of the last inequality is larger than zero provided

$$\sum_{k=1}^{m} c_k^2 \cos \lambda_k n \geqq 0.$$

This will be the case when $\cos \lambda_k n \geqq 0$; the latter condition amounts to the requirement that

$$|\lambda_k n| \leqq \pi/2 \pmod{2\pi}.$$

The foregoing therefore answers the question, when is $\Lambda(n) > 0$, considering the series expansion for $\Lambda(n)$.

Evidently we are now done with the proof of the proposition stated on page 41.

In the foregoing proof we established that $\varphi_\omega(n) > 0$ implies that n permits representation as difference of two elements of the relatively dense set E of positive integers. We wish to find out next what analogous claim can be made relative to the expression $\Psi(n)$. We commence with a definition.

<u>If S is any set of positive integers, let</u> $\pi_n(S)$ <u>denote the number of elements of S less than the natural number n. We say that a relatively dense set E of positive integers satisfies a certain property P for n e a r l y a l l elements of E, if the subset</u> $E_1 \subset E$ <u>of elements not satisfying property P is negligible in the sense that the ratio</u> $\pi_n(E_1) / \pi_n(E)$ <u>tends to zero as</u> $n \longrightarrow \infty$.

We show that if $\delta > 0$ is a sufficiently small fixed number, then nearly all n for which $\Psi(n) > \delta$ can be represented as difference of two elements of the relatively dense set E of positive integers.

Consider the set of positive integers

$$\mathfrak{A}_\delta = \{ n : \Psi(n) > \delta \}.$$

That the set $\mathfrak{A}_\delta$ is infinite for some $\delta > 0$ follows from the fact that

$$\lim_{K \to \infty} \frac{1}{K} \sum_{0 < n < K} \Psi(n) = c_0 > 0.$$

Let $\mathfrak{A}'_\delta$ denote the subset of $\mathfrak{A}_\delta$ whose elements are not representable as difference of numbers in E, then by what we know about φ_ω,

$$\varphi_\omega(n) = 0 \text{ if and only if } n \in \mathfrak{A}'_\delta .$$

Whence

$$\sum_{\substack{0 < n < N \\ n \in \mathfrak{A}'_\delta}} \Phi(n) = - \sum_{\substack{0 < n < N \\ n \in \mathfrak{A}'_\delta}} \Psi(n) < - \delta \Pi_N(\mathfrak{A}'_\delta)$$

and therefore

$$\sum_{\substack{0 < n < N \\ n \in \mathfrak{A}_\delta}} |\Phi(n)| \geqq \sum_{\substack{0 < n < N \\ n \in \mathfrak{A}'_\delta}} |\Phi(n)| > \delta \Pi_N(\mathfrak{A}'_\delta).$$

Thus we get

$$\delta \frac{\Pi_N(\mathfrak{A}'_\delta)}{\Pi_N(\mathfrak{A}_\delta)} \leqq \frac{1}{\Pi_N(\mathfrak{A}_\delta)} \sum_{\substack{0 < m < N \\ m \in \mathfrak{A}_\delta}} |\Phi(m)|.$$

But for any sequence of numbers converging to zero, the sequence of consecutive arithmetic means converges to zero. Since $\mathfrak{A}_\delta$ is infinite, the asymptotic behaviour of $\Phi(n)$ as $n \to \infty$ implies

$$\frac{1}{\Pi_N(\mathfrak{A}_\delta)} \sum_{\substack{0 < m < N \\ m \in \mathfrak{A}_\delta}} |\Phi(m)| \longrightarrow 0 \text{ as } N \to \infty$$

and therefore

$$\frac{\Pi_N(\mathfrak{M}_\delta')}{\Pi_N(\mathfrak{M}_\delta)} \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty .$$

But this is what we set out to do.

By an argument completely analogous to the one given on page 50, we observe that the consideration of the series

$$\Psi(n) = c_0 + \sum_{\lambda_k \neq 0} c_k e^{i \lambda_k n}$$

leads to the following statement: It is possible to exhibit real numbers $\lambda_1, \ldots, \lambda_m$, $\eta > 0$, $\delta > 0$ such that $\Psi(n) > \delta$ for any positive integer n, for which all numbers

$$\frac{\lambda_k n}{2\pi} \qquad (k = 1, 2, \ldots, m)$$

differ from integers by not more than η.

It is now easy to see the validity of the following proposition:

<u>For any relatively dense set E of positive integers we can find real numbers $\lambda_1, \ldots, \lambda_m$ and $\eta > 0$ such that nearly all integers n, for which the numbers</u>

$$\frac{\lambda_k n}{2\pi} \qquad (k = 1, \ldots , m)$$

<u>differ from integers by not more than</u> η , <u>are representable as difference of two elements of the given set E</u>.

2.3 Theorem Concerning Almost Periods

Using the results of the last section we prove a theorem concerning the almost periods of a UAP function; this theorem gives a deep characterization of almost periods. In the next section we shall derive from it Bohr's approximation theorem without the use of limit periodic functions.

THEOREM CONCERNING ALMOST PERIODS: <u>If f is a UAP function, then for any given $\varepsilon > 0$ we can find a $\delta > 0$, and reals $\lambda_0, \lambda_1, \dots, \lambda_m$ such that all solutions t of the system of inequalities</u>

$$|\lambda_k t| < \delta \pmod{2\pi}; \quad k = 0, 1, \dots, m$$

<u>are ε-almost periods of the function f.</u>

The proof of the foregoing theorem conveniently decomposes into two lemmas.

LEMMA 1: <u>Let $\{t_1, t_2, \dots\}$ be a relatively dense set of positive reals. Moreover, suppose that there is a positive real α such that for any distinct indices n_1 and n_2</u>

$$|t_{n_1} - t_{n_2}| > \alpha > 0; \quad n_1, n_2 = 1, 2, \dots$$

<u>holds. Then for any given $\beta > 0$ we can find a $\delta > 0$, and</u>

reals $\lambda_0, \lambda_1, \dots , \lambda_m$ so that all solutions t of the system of inequalities

$$|\lambda_k t| < \delta \pmod{2\pi}; \quad k = 1, 2, \dots , m$$

satisfy as well an inequality of the form

$$|t - (t_p + t_q - t_r - t_s)| < \beta$$

with suitable elements t_p, t_q, t_r, and t_s, dependent on t and belonging to the relatively dense set $\{t_1, t_2, \dots\}$.

Proof of the lemma: We first determine a natural number M so large that

$$(1/M) < \alpha \quad \text{and} \quad (1/M) < (\beta/5)$$

holds. Next we select natural numbers n_i by the following rule:

$$n_i = [M\, t_i], \qquad i = 1, 2, \dots ,$$

where [ ] signifies the integral part of the number so enclosed. We now observe that for $i \neq k$ and $t_i > t_k$ we have

$$n_i - n_k = [M\, t_i] - [M\, t_k] \geqq [M(t_i - t_k)] \geqq [\tfrac{1}{\alpha}\alpha] = 1.$$

Thus the numbers n_i form a relatively dense set of distinct positive integers and we can apply the proposition I of the last section (see page 41); there are reals $\lambda_1, \dots , \lambda_m$ so that all numbers n which solve

$$|\lambda_k n| < \pi/2 \pmod{2\pi}; \quad k = 1, 2, \dots , m$$

are of the form $n = n_p + n_q - n_r - n_s$. Since we operate with residue classes mod 2π, we may assume that the reals λ_1, ... , λ_m lie in the interval $[0,2\pi)$. We put

$$\lambda_0 = 2\pi M \quad \text{and} \quad \delta = \pi/(4M)$$

and show that the numbers λ_0, λ_1, ... , λ_m and δ are such as lemma 1 asserts. Let t be a solution of

$$|\lambda_k t| < \delta \pmod{2\pi}; \quad k = 0, 1, \dots , m.$$

Then there exists a number n such that

$$|2\pi Mt - 2\pi n| < \pi/(4M)$$

or

$$\left| t - \frac{m}{M} \right| < 1/(8M^2) < \beta/5.$$

In addition, there exist integers n_1, ... , n_m such that

$$\left|\lambda_k \frac{m}{M} + 2\pi n_k\right| \leqq |\lambda_k t + 2\pi n_k| + \lambda_k \left|t - \frac{m}{M}\right| <$$

$$< \pi/(4M) + 2\pi/(8M^2) \leqq \pi/(2M)$$

holds for $k = 1, 2, \dots , m$. Therefore the number n satisfies for $k = 1, 2, \dots , m$ the inequalities

$$|\lambda_k n| < \pi/2 \pmod{2\pi}.$$

But $n = n_p + n_q - n_r - n_s$ and we have

$$\left| t - (t_p + t_q - t_r - t_s) \right| \leqq$$

$$\leqq \left| t - \frac{m}{M} \right| + \left| \frac{m}{M} - \left(\frac{m_p}{M} + \frac{m_q}{M} - \frac{m_r}{M} - \frac{m_s}{M} \right) \right| +$$

$$+ \left| \frac{m_p}{M} - t_p \right| + \left| \frac{m_q}{M} - t_q \right| + \left| \frac{m_r}{M} - t_r \right| + \left| \frac{m_s}{M} - t_s \right| <$$

$$< \beta/5 + 0 + \beta/5 + \beta/5 + \beta/5 + \beta/5.$$

This means that we have what we set out to do.

Remark: In lemma 1 we can delete the condition: "suppose that there is a positive real α such that for any distinct indices n_1 and n_2

$$\left| t_{n_1} - t_{n_2} \right| > \alpha > 0; \quad n_1, n_2 = 1, 2, \ldots \quad \text{holds}".$$

This condition is a bonus of the fact that the set $\{t_1, t_2, \ldots\}$ is relatively dense. Indeed, if $\{t_1, t_2, \ldots\}$ is relatively dense, then we can select a $T > 0$ such that every interval of length $T/2$ contains an element of the set $\{t_1, t_2, \ldots\}$. If we pick from each interval $\left((i-1/2)T, iT\right)$, $i = 1, 2, \ldots$, an element t_{n_i}, then

$$\left| t_{n_i} - t_{n_k} \right| > T/2 > 0 \qquad i \neq k.$$

In view of the preceding remark we have:

LEMMA 2: <u>Let</u> $E = \{t_1, t_2, \dots\}$ <u>be a relatively dense set of positive reals. Then for any given</u> $\beta > 0$ <u>we can find a</u> $\delta > 0$ <u>and reals</u> $\lambda_0, \dots, \lambda_m$ <u>so that all solutions t of</u>

$$|\lambda_k t| < \delta \pmod{2\pi}; \quad k = 0, 1, \dots, m$$

<u>satisfy as well an inequality of the form</u>

$$|t - (t_p + t_q - t_r - t_s)| < \beta$$

<u>with suitable elements</u> t_p, t_q, t_r, <u>and</u> t_s <u>dependent on t and belonging to the set E</u>.

Before we start with the verification of the theorem concerning almost periods stated on page 55 we insert the following two observations. Firstly, it is not necessary to get into a separate discussion of positive and negative almost periods because if t is an ε-almost period of a UAP function f, then so is -t. Secondly, if t_1 and t_2 are ε_1- and ε_2-almost periods of a UAP function f, respectively, then $t_1 \pm t_2$ are $(\varepsilon_1 + \varepsilon_2)$-almost periods of f.

We select some relatively dense set $E = \{t_1, t_2, \dots\}$ of $\varepsilon/8$-almost periods of the UAP function f and apply to it lemma 2. First of all we pick β to be arbitrary. We get that the solutions t of the system of inequalities

$$|\lambda_k t| < \delta \pmod{2\pi};\ k = 0, 1, \dots, m$$

with the numbers $\lambda_0, \dots, \lambda_m$ and δ dependent on β also satisfy an inequality of the form

$$\left| t - (t_p + t_q - t_r - t_s) \right| < \beta.$$

Since the elements of the set E are $\varepsilon/8$-almost periods of f, we have that

$$t_p + t_q - t_r - t_s = t(\frac{\varepsilon}{2})$$

is an $\varepsilon/2$-almost period of f. The number $t(\frac{\varepsilon}{2})$ differs from t by at most β . The function f is uniformly continuous on the entire real line by proposition II on page 27. Thus we can take β so small that every number t which differs from an almost period $t(\frac{\varepsilon}{2})$ of f by less than β ;

$$\left| t - t(\frac{\varepsilon}{2}) \right| < \beta$$

is an ε -almost period of the function f. Thus, if $\varepsilon > 0$ is given, we select $\beta > 0$ so small as was just explained. Then by lemma 2 we choose the numbers δ , $\lambda_0, \lambda_1, \dots, \lambda_m$ and note that these numbers are precisely those whose existence is claimed in the theorem concerning almost periods. This finishes the proof.

2.4 Proof of Bohr's Approximation Theorem

Keeping in mind that our problem is essentially the construction of Fourier series for UAP functions, we are obliged to look first at a theorem which is the key to all these considerations.

THEOREM CONCERNING MEAN VALUE: <u>For each UAP function f there exists the m e a n v a l u e</u>

$$M\left\{f(x)\right\} = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x)\, dx.$$

<u>Moreover, the limit</u>

$$\lim_{T \to \infty} \frac{1}{T} \int_a^{a+T} f(x)\, dx = M\left\{f(x+a)\right\}$$

<u>exists uniformly for a</u>.

Proof: Let $\varepsilon > 0$ be given. We put

$$L = L(\varepsilon/2) \quad \text{and} \quad A = \sup_{-\infty < x < \infty} |f(x)| .$$

Denote by α an arbitrary real number and by t an $\varepsilon/2$-almost period of f situated in the interval $(\alpha, \alpha + L)$. Then

$$\left| \frac{1}{T} \int_0^T f(x)dx - \frac{1}{T} \int_\alpha^{\alpha+T} f(x)dx \right| \leqq$$

$$\leqq \left| \frac{1}{T}\int_0^T f(x)dx - \frac{1}{T}\int_t^{t+T} f(x)dx \right| +$$

$$+ \left| \frac{1}{T}\int_\alpha^t f(x)dx \right| + \left| \frac{1}{T}\int_{t+T}^{\alpha+T} f(x)dx \right| \leqq$$

$$\leqq \frac{1}{T}\int_0^T |f(x)-f(x+t)|\ dx + \frac{1}{T}\int_\alpha^t |f(x)|\ dx + \frac{1}{T}\int_{t+T}^{\alpha+T} |f(x)|\, dx$$

$$< \varepsilon/2 + (2AL)/T \qquad (1)$$

because

$$\int_\alpha^{\alpha+T} = \int_\alpha^t + \int_t^{t+T} + \int_{t+T}^{\alpha+T}$$

and

$$\int_t^{t+T} f(y)dy = \int_0^T f(x+t)dx$$

when we set $y = x+t$.

Considering the arithmetic average of the n differences

$$\frac{1}{T}\int_0^T f(x)dx - \frac{1}{T}\int_{(m-1)T}^{mT} f(x)dx; \qquad m = 1, 2, \ldots, n$$

we get from (1):

$$\left| \frac{1}{T}\int_0^T f(x)dx - \frac{1}{nT}\int_0^{mT} f(x)dx \right| < \varepsilon/2 + (2AL)/T. \qquad (2)$$

Let T_1 and T_2 be positive numbers such that $m_1T_1 = m_2T_2$, where m_1 and m_2 are integers. From (2) it follows that

$$\left| \frac{1}{T_1} \int_0^{T_1} f(x)dx - \frac{1}{T_2} \int_0^{T_2} f(x)dx \right| < \varepsilon + 2AL\left(\frac{1}{T_1} + \frac{1}{T_2}\right). \quad (3)$$

The last inequality carries over to arbitrary positive numbers T_1 and T_2 by continuity consideration. If T_1 and T_2 are strictly greater than $(4AL)/\varepsilon$, then we see from (3) that

$$\left| \frac{1}{T_1} \int_0^{T_1} f(x)dx - \frac{1}{T_2} \int_0^{T_2} f(x)dx \right| < 2\varepsilon$$

which proves the existence of the limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(x)dx = M\{f(x)\}.$$

Taking $n \to \infty$ in inequality (2) we get:

$$\left| \frac{1}{T} \int_0^T f(x)dx - M\{f(x)\} \right| \leqq \varepsilon/2 + (2AL)/T. \quad (4)$$

To get the second assertion of the theorem, we note first that for a constant a:

$$M\{f(x+a)\} = M\{f(x)\}$$

because

$$\frac{1}{T}\int_0^T f(x+a)dx = \frac{1}{T}\int_a^{a+T} f(x)dx =$$

$$= \frac{1}{T}\int_a^0 f(x)dx + \frac{1}{T}\int_0^T f(x)dx + \frac{1}{T}\int_T^{a+T} f(x)dx,$$

where for $T \to \infty$ the second term on the right hand side of the last equation tends to $M\{f(x)\}$, whereas the other two terms tend to zero because the absolute value of each is $\leqq |a| A/T$.

It remains to show that for each $\varepsilon > 0$ there exists a number $T_0 = T_0(\varepsilon)$ independent of the number a such that for $T > T_0$ the inequality

$$\left| \frac{1}{T}\int_0^T f(x+a)dx - M\{f(x+a)\} \right| < \varepsilon$$

holds. But this follows directly from inequality (4) because the numbers A and L are independent of the number a. Taking in particular $a = -T$ we get

$$M\{f(x)\} = \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{0} f(x)dx = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)dx.$$

The proof of the theorem is complete.

In the interval $-\infty < x < \infty$ the system $\{e^{i\lambda x} : \lambda \text{ real}\}$ is an orthonormal system in the sense:

$$M\left\{e^{i\lambda_1 x}\, e^{i\lambda_2 x}\right\} = \begin{cases} 0 & \text{if } \lambda_1 \neq \lambda_2 \\ 1 & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

We let

$$a(\lambda) = M\left\{f(x)\, e^{-i\lambda x}\right\}.$$

The non-zero $a(\lambda)$'s are at most countable and are called the Fourier exponents of the function f.

Indeed, let $\lambda_1, \ldots, \lambda_N$ be distinct real numbers and $c_1, \ldots$ $\ldots, c_N$ arbitrary complex numbers. Then

$$M\left\{\left|f(x) - \sum_{n=1}^{N} c_n e^{i\lambda_n x}\right|^2\right\} =$$

$$= M\left\{|f(x)|^2\right\} - \sum_{n=1}^{N} |a(\lambda_n)|^2 + \sum_{n=1}^{N} |c_n - a(\lambda_n)|^2.$$

First of all it is clear that the mean values appearing in the foregoing equality exist because the functions involved are UAP. The equality is verified as follows:

$$M\left\{\left|f(x) - \sum_{n=1}^{N} c_n e^{i\lambda_n x}\right|^2\right\} =$$

$$M\left\{\left(f(x) - \sum_{n=1}^{N} c_n e^{i\lambda_n x}\right)\left(\overline{f(x)} - \sum_{n=1}^{N} \overline{c_n}\, e^{-i\lambda_n x}\right)\right\} =$$

$$M\left\{f(x)\overline{f(x)}\right\} - \sum_{n=1}^{N} \overline{c}_n \, M\left\{f(x)\, e^{-i\lambda_n x}\right\}$$

$$- \sum_{n=1}^{N} c_n \, M\left\{\overline{f(x)}\, e^{i\lambda_n x}\right\}$$

$$+ \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} c_{n_1}\overline{c}_{n_2} \, M\left\{e^{i\lambda_{n_1} x}\, e^{-i\lambda_{n_2} x}\right\} =$$

$$= M\left\{|f(x)|^2\right\} - \sum_{n=1}^{N} c_n a(\lambda_n) - \sum_{n=1}^{N} c_n\overline{a(\lambda_n)} + \sum_{n=1}^{N} c_n\overline{c}_n =$$

$$= M\left\{|f(x)|^2\right\} - \sum_{n=1}^{N} (c_n - a(\lambda_n))\overline{(c_n - a(\lambda_n))} - \sum_{n=1}^{N} a(\lambda_n)\overline{a(\lambda_n)} =$$

$$= M\left\{|f(x)|^2\right. - \sum_{n=1}^{N} |a(\lambda_n)|^2 + \sum_{n=1}^{N} |c_n - a(\lambda_n)|^2 .$$

Taking in particular $c_n = a(\lambda_n)$, $n = 1, \dots, N$, and since $M\left\{|f(x) - \sum_{n=1}^{N} c_n e^{i\lambda_n x}|^2\right\} \geqq 0$, we get Bessel's inequality

$$\sum_{n=1}^{N} |a(\lambda_n)|^2 \leqq M\left\{|f(x)|^2\right\}.$$

From it we see that the number of λ for which $|a(\lambda)| > d$ is less than $M\left\{|f(x)|^2\right\}/d^2$. Taking $d_n = 1/n$ $(n = 1, 2, \dots)$ we consider the sets $B_1 = \left\{\lambda : |a(\lambda)| > 1\right\}$ and $B_n = \left\{\lambda : d_n \geqq a(\lambda) > d_{n+1}\right\}$, $n = 2, 3, \dots$, we get that the set of λ for which $a(\lambda) \neq 0$ is at most countable.

We now turn to the proof of Bohr's approximation theorem. Let f be a UAP function. By M_f we genote the vector space over the rationals generated by $\lambda_1, \lambda_2, \dots$, the Fourier exponents of f. M_f has a basis $\beta_1, \beta_2, \dots$, and each β_k may be chosen from the set $\lambda_1, \lambda_2, \dots$. Since the β_k's form a basis there exist positive integers $m = m(\lambda_1,\dots,\lambda_n)$ and $q = q(\lambda_1,\dots,\lambda_n)$ such that $\lambda_j = (s_{j1}\beta_1 + \dots + s_{jm}\beta_m)/q$, $j = 1,\dots,n$, where s_{jk} is an integer, $j = 1,\dots,n$; $k = 1,\dots,m$. We put

$$\delta = 1/(4m \max_{\substack{1 \leqq j \leqq n \\ 1 \leqq k \leqq m}} |s_{jk}|).$$

Then it follows that every t which satisfies the inequalities

$$|\beta_k t/q| < \delta \pmod{2\pi}; \quad k = 1,\dots,m \tag{5}$$

also satisfies the inequalities

$$|\lambda_j t| < \pi/2 \pmod{2\pi}; \quad j = 1,\dots,n. \tag{6}$$

Let N be an arbitrary natural number. We consider the Bochner-Fejér kernel defined by

$$K_N(\alpha_k t) = \frac{1}{N}\left[(\sin(N\alpha_k t)/2)/(\sin(\alpha_k t)/2)\right]^2 =$$

$$\sum_{|\nu| < N} (1 - \frac{|\nu|}{N})\, e^{-i\nu\alpha_k t},$$

where $\alpha_k = \beta_k/q$, $k = 1,\ldots,m$. If $|\alpha_k t| \geqq \delta \pmod{2\pi}$ $(\delta < \pi)$, then $|\sin(\alpha_k t/2)| \geqq |\sin(\delta/2)|$. Thus for such t

$$K_N(\alpha_k t) < (N \sin^2(\delta/2))^{-1} \tag{7}$$

We note two properties of the Bochner - Fejér kernel: It is never negative and its mean value equals 1 because it is equal to the constant term of K_N.

We consider the composite Bochner - Fejer kernel:

$$K^N(t) = K_N(\alpha_1 t)\, K_N(\alpha_2 t) \ldots K_N(\alpha_m t).$$

It is again seen that it never is negative and its mean value is

$$M\left\{K^N(t)\right\} = 1. \tag{8}$$

Let

$$E = \left\{ t\colon |\alpha_k t| < \delta \pmod{2\pi},\ k = 1,\ldots,m \right\}$$

$$E_k = \left\{ t\colon |\alpha_k t| < \delta \pmod{2\pi} \right\};\ k = 1, \ldots, m,$$

where $\alpha_k = \beta_k/q$, $k = 1,\ldots,m$. Then $E = E_1 \cap \ldots \cap E_m$.

For $T > 0$, put $E^{(T)} = E \cap (-T,T)$, $E_k^{(T)} = E_k \cap (-T,T)$, $k = 1,\ldots,m$.

Let g be any non-negative UAP function and $A = \sup_t g(t)$. Then by (7):

$$\frac{1}{2T}\int_{-T}^{T} g(t)\, K^N(t)dt =$$

$$= \frac{1}{2T}\int_{E_1^{(T)}} g(t)\, K^N(t)dt +$$

$$+ \frac{1}{2T}\int_{(-T,T)-E_1^{(T)}} g(t)K_N(\alpha_1 t)K_N(\alpha_2 t)\dots K_N(\alpha_m t)dt \leqq$$

$$\leqq \frac{1}{2T}\int_{E_1^{(T)}} g(t)\, K^N(t)dt +$$

$$+ \frac{1}{N\sin^2(\delta/2)}\int_{(-T,T)-E_1^{(T)}} g(t)K_N(\alpha_2 t)\dots K_N(\alpha_m t)dt \leqq$$

$$\leqq \frac{1}{2T}\int_{E_1^{(T)}} g(t)\, K^N(t)dt +$$

$$\frac{1}{N\sin^2(\delta/2)}\int_{-T}^{T} g(t)\,\frac{K^N(t)}{K_N(\alpha_1 t)}\,dt.$$

Separating the points of the set $E_2^{(T)}$ from the set $E_1^{(T)}$ we get

$$\frac{1}{2T}\int_{-T}^{T} g(t)\, K^N(t)\, dt \leqq \frac{1}{2T}\int_{E_1^{(T)}\cap E_2^{(T)}} g(t)\, K^N(t)\, dt\ +$$

$$+\ \frac{A}{N\sin^2(\delta/2)}\ \frac{1}{2T}\int_{-T}^{T}\left(\frac{K^N(t)}{K(\alpha_1 t)}+\frac{K^N(t)}{K_N(\alpha_2 t)}\right)dt.$$

Continuing this process and passing to the limit as $T \to \infty$, we obtain, using the fact that $E = E_1 \cap \dots \cap E_m$,

$$M\{g(t)K^N(t)\} \leqq \lim_{T\to\infty}\frac{1}{2T}\int_{E^{(T)}} g(t)\, K^N(t)\, dt\ +\ \frac{mA}{N\sin^2(\delta/2)} \tag{9}$$

Now we invoke the theorem concerning almost periods (see page 55): Every $t \in E$ is an $\varepsilon/2$-almost period of f;

$$|f(x+t) - f(x)| < \varepsilon/2. \tag{10}$$

We consider the Bochner - Fejér polynomial

$$P_N(x) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\, K^N(t)\, dt.$$

Applying (9) to the UAP function $g(t) = |f(x+t)-f(x)|$ we get

$$|P_N(x)-f(x)| \leqq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(x+t)-f(x)|\, K^N(t)\, dt$$

$$\leqq \lim_{T\to\infty} \frac{1}{2T} \int_{E(T)} |f(x+t)-f(x)|\, K^N(t)\, dt +$$

$$+ \frac{2Cm}{N \sin^2(\delta/2)}$$

where $C = \sup_t |f(t)|$. Hence by (10)

$$|P_N(x)-f(x)| \leqq \frac{\varepsilon}{2} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} K^N(t)dt +$$

$$+ \frac{2Cm}{N \sin^2(\delta/2)}$$

$$= (\varepsilon/2) + (2Cm)/(N \sin^2(\delta/2)). \tag{11}$$

For fixed m and ε , N can be taken so large that the inequality

$$\frac{2Cm}{N \sin^2(\delta/2)} < \frac{\varepsilon}{2} \tag{12}$$

is satisfied. We see therefore that the estimate (11) holds uniformly in x and Bohr's approximation theorem is proved.

The Bochner-Fejér polynomial P_N is seen to be of the form

$$\sum_{\substack{|\nu_1| < N \\ \cdots\cdots \\ |\nu_m| < N}} B_{1,\ldots,m}\, a(\nu_1\alpha_1 + \ldots + \nu_m\alpha_m)\, e^{i(\nu_1\alpha_1 + \ldots + \nu_m\alpha_m)x}$$

where

$$B_{1,\ldots,m} = \left(1 - \frac{|\nu_1|}{N}\right) \ldots \left(1 - \frac{|\nu_m|}{N}\right)$$

and $a(\alpha) = M\left\{f(t)\, e^{-i\alpha t}\right\}$.

From page 39 we recall that the class of UAP functions forms a complex Banach space under the supremum norm. Using convolution multiplication we can define a product of two UAP functions:

$$(f * g)(x) = M\left\{f(x-t)g(t)\right\}$$

(here the mean value is evaluated with respect to t); we note that we again get a UAP function. The Banach space of UAP functions turns out to be a commutative Banach algebra under convolution multiplication. Bohr's approximation theorem can be interpreted as follows: Every closed ideal in the Banach algebra of UAP functions is the intersection of the regular maximal ideals containing it.

Chapter 3

HILBERT SPACES WITH POSITIVE DEFINITE KERNELS

3.1 Extension of a Symmetric Operator

A linear operator A in a Hilbert space $\mathfrak{H}$ with domain of definition $\mathfrak{D}_A$ and range $\mathfrak{R}_A$ is called H e r m i t i a n if

$$\langle Af,g\rangle = \langle f,Ag\rangle \qquad (f,g \in \mathfrak{D}_A)$$

and is called s y m m e t r i c if it is Hermitian and $\mathfrak{D}_A$ is dense in $\mathfrak{H}$.

If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are Hilbert spaces and A is an operator from the space $\mathfrak{H}_1$ into the space $\mathfrak{H}_2$, where $\mathfrak{D}_A$ is dense in $\mathfrak{H}_1$, it can happen that for certain $h \in \mathfrak{H}_2$ a representation of the form

$$\langle Af,g\rangle = \langle f,h\rangle$$

holds for all $f \in \mathfrak{D}_A$. By a theorem of F. Riesz (see Neumark's book on normed algebras, § 5, section 3) this is the case if and only if

$$\langle Af,g\rangle = F_g(f)$$

is a bounded linear form in $\mathfrak{D}_A$. Let $\mathfrak{D}^*$ be the collection of all such g. By

$$A^* g = h$$

we define an operator A^* from $\mathfrak{H}_2$ into $\mathfrak{H}_1$ whose domain of definition $\mathcal{D}_{A^*}$ is $\mathcal{D}^*$. The element h is uniquely determined by g and A^* is called the <u>a d j o i n t   o p e r a t o r</u> of A.

<u>An operator A in a Hilbert space</u> $\mathfrak{H}$ <u>is said to be s e l f - - a d j o i n t if it is symmetric and</u> $A = A^*$.

<u>An operator U in a Hilbert space</u> $\mathfrak{H}$ <u>is called u n i t a r y if it is isometric, that is</u>

$$\langle Uf, Uf \rangle = \langle f, g \rangle \qquad (f, g \in \mathcal{D}_U),$$

<u>and if</u> $\mathcal{D}_U = \mathcal{R}_U = \mathfrak{H}$.

<u>An operator</u> $\tilde{A}$ <u>is said to be an e x t e n s i o n of an operator A in a Hilbert space</u> $\mathfrak{H}$ <u>if</u> $\mathcal{D}_{\tilde{A}} \supset \mathcal{D}_A$ <u>and</u> $\tilde{A}f = Af$ <u>for all</u> $f \in \mathcal{D}_A$; <u>we shall sometimes write</u> $A \subset \tilde{A}$ <u>to indicate that</u> $\tilde{A}$ <u>is an extension of A</u>.

<u>An operator A in a Hilbert space</u> $\mathfrak{H}$ <u>is called c l o s e d if</u> $f_n \in \mathcal{D}_A$, $\lim_{n \to \infty} f_n = f$, $\lim_{n \to \infty} Af_n = g$ <u>implies that</u> $f \in \mathcal{D}_A$ <u>and</u> $Af = g$.

If there exist closed extensions of the operator A, then there is a unique minimal closed extension $\bar{A}$ of the operator A which we shall call the <u>c l o s u r e</u> of A; all other closed extensions of A are extensions of $\overline{A}$. We note that symmetric operators always have a closure and the closure is always a Hermitian operator; if A is symmetric, then $A \subseteq A^*$, but A^* is closed.

By the <u>o r t h o g o n a l   s u m</u> $\mathfrak{H}$ of two Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$:

$$\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$$

we mean the set of all ordered pairs (f_1,f_2), $f_1 \in \mathfrak{H}_1$, $f_2 \in \mathfrak{H}_2$ for which the algebraic operations and the inner product are defined as follows:

$$\alpha(f_1,f_2) = (\alpha f_1, \alpha f_2)$$

$$(f_1,f_2) + (g_1,g_2) = (f_1+g_1, f_2+g_2)$$

$$\langle (f_1,f_2),(g_1,g_2)\rangle = \langle f_1,g_1\rangle + \langle f_2,g_2\rangle .$$

$\mathfrak{H}$ is then a Hilbert space and $\mathfrak{H}_1$ and $\mathfrak{H}_2$ can be viewed as mutually orthogonal subspaces of $\mathfrak{H}$, provided one identifies them with $\mathfrak{H}_1 \oplus \{0\}$ and $\{0\} \oplus \mathfrak{H}_2$, respectively.

Let A be an operator from $\mathfrak{H}_1$ to $\mathfrak{H}_2$. The set $\{(f,Af);\ f \in \mathfrak{D}_A\}$ in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ is called the <u>g r a p h</u> of A. The operator A is closed if and only if its graph in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ is a closed set.

<u>By the o p e n i n g of two linear manifolds in a Hilbert space $\mathfrak{H}$ we mean the norm of the difference of the projection operators which map $\mathfrak{H}$ onto the closure of these linear manifolds.</u>

Denoting the opening of the linear manifolds M_1 and M_2 by $Q(M_1,M_2)$ we therefore have

$$Q(M_1,M_2) = \| P_1-P_2 \| ,$$

where P_1 and P_2 are the projection operators which map $\mathfrak{H}$ onto the subspaces $\overline{M_1}$ and $\overline{M_2}$, respectively. By definition

$$Q(M_1,M_2) = Q(\overline{M_1},\overline{M_2}) = Q(\mathfrak{H} \ominus M_1, \mathfrak{H} \ominus M_2).$$

Let E denote the identity operator. For any $h \in \mathfrak{H}$ we have

$$(P_2-P_1)h = P_2(E-P_1)h - (E-P_2)P_1h.$$

Since the vectors $P_2(E-P_1)h$ and $(E-P_2)P_1h$ are orthogonal, we see that

$$\| (P_2-P_1)h\|^2 = \| P_2(E-P_1)h\|^2 + \| (E-P_2)P_1h\|^2 \leqq$$
$$\leqq \| (E-P_1)h\|^2 + \| P_1h \|^2 = \| h\|^2 \qquad (1)$$

holds. The inequality (1) shows that

$$0 \leqq Q(M_1,M_2) \leqq 1.$$

The opening of two linear manifolds is actually equal to 1 if one of these manifolds contains a non-zero vector which is orthogonal to the other manifold.

We consider some propositions.

I. <u>If the opening of two linear manifolds</u> M_1 <u>and</u> M_2 <u>is less then</u> 1, <u>then</u>

$$\dim M_1 = \dim M_2.$$

Proof: By the remark made above it suffices to show that the inequality $\dim M_2 > \dim M_1$ implies the existence of a non-zero vector in $\overline{M_2}$ which is orthogonal to $\overline{M_1}$. To see that this is actually the case we project $\overline{M_1}$ onto $\overline{M_2}$. We obtain the sub--space $G = P_2 \overline{M_1}$ whose dimension of course does not exceed the dimension of $\overline{M_1}$ and therefore is less than the dimension of $\overline{M_2}$. Hence there is in $\overline{M_2} \ominus G$ a non-zero vector, which means that in $\overline{M_2}$ there is a non-zero vector that is orthogonal to G. This vector will also be orthogonal to $\overline{M_1}$ because $\overline{M_1} \ominus G$ is orthogonal to $\overline{M_2}$. This proves the proposition.

The next object for consideration is the formula

$$Q(M_1,M_2) = \max\Big\{ \sup_{f \in \overline{M_2},\ \|f\|=1} \|(E-P_1)f\|, \quad \sup_{g \in \overline{M_1},\ \|g\|=1} \|(E-P_2)g\| \Big\} \qquad (2).$$

We note first of all that the quantity

$$\|(E-P_1)f\| = \operatorname{dist}\left[f,\overline{M_1}\right]$$

represents the distance between the element f and $\overline{M_1}$:

$$\operatorname{dist}\left[f,\overline{M_1}\right] = \inf_{g \in \overline{M_1}} \|f-g\|$$

and therefore we can express (2) in the form

$$Q(M_1,M_2) = \max\left\{ \sup_{f\in \overline{M_2},\ \|f\|=1} \operatorname{dist}\left[f,\overline{M_1}\right],\ \sup_{g\in \overline{M_1},\ \|g\|=1} \operatorname{dist}\left[g,\overline{M_2}\right]\right\}.$$

We now turn to establishing equation (3).

$$Q(M_1,M_2) = \sup_{h\in \mathcal{G}} \frac{\|(P_2-P_1)h\|}{\|h\|} =$$

$$= \sup_{h\in \mathcal{G}} \frac{\sqrt{\|P_2(E-P_1)h\|^2 + \|(E-P_2)P_1h\|^2}}{\|h\|} \tag{3}$$

Therefore

$$Q(M_1,M_2) \geqq \sup_{h\in \overline{M_1}} \frac{\sqrt{\|P_2(E-P_1)h\|^2 + \|(E-P_2)P_1h\|^2}}{\|h\|} =$$

$$= \sup_{h\in \overline{M_1}} \frac{\|(E-P_2)h\|}{\|h\|} = r_2$$

In the same manner we obtain

$$Q(M_1,M_2) = \sup_{h\in \overline{M_2}} \frac{\|(E-P_1)h\|}{\|h\|} = r_1.$$

Consequently

$$Q(M_1,M_2) \geqq \max\left\{r_1,r_2\right\}.$$

We show next that the inequality sign in the last relation can be inverted.

By the definition of the number r_2 we have

$$\|(E-P_2)P_1h\|^2 \leqq r_2^{\,2} \|P_1h\|^2. \tag{4}$$

On the other hand

$$\|P_2(E-P_1)h\|^2 = \langle P_2(E-P_1)h, P_2(E-P_1)h\rangle = \langle P_2(E-P_1)h; (E-P_1)h\rangle =$$

$$= \langle P_2(E-P_1)h, (E-P_1)^2h\rangle = \langle (E-P_1)P_2(E-P_1)h, (E-P_1)h\rangle \leqq$$

$$\leqq \|(E-P_1)P_2(E-P_1)h\| \cdot \|(E-P_1)h\|$$

and hence by the definition of r_1

$$\|P_2(E-P_1)h\|^2 \leqq r_1^{\,2} \|P_2(E-P_1)h\| \cdot \|(E-P_1)h\|.$$

Therefore

$$\|P_2(E-P_1)h\| \leqq r_1 \|(E-P_1)h\|. \tag{5}$$

Using (4) and (5) we get

$$\|(E-P_2)P_1h\|^2 + \|P_2(E-P_1)h\|^2 \leqq r_2^{\,2} \|P_1h\|^2 + r_1^{\,2} \|(E-P_1)h\|^2 \leqq$$

$$\leqq \max\{r_1^{\,2}, r_2^{\,2}\}\left[\|P_1h\|^2 + \|(E-P_1)h\|^2\right] = \|h\|^2 \max\{r_1^{\,2}, r_2^{\,2}\}$$

so that by (3):

$$Q(M_1, M_2) \leqq \max\{r_1, r_2\}$$

and we have what we set out to verify.

Let A be a linear operator in a complex Hilbert space $\mathfrak{H}$ and suppose that $\mathfrak{D}_A$ is dense in $\mathfrak{H}$; let E denote the identity operator in $\mathfrak{H}$.

A point λ of the complex plane will be called a p o i n t o f r e g u l a r t y p e for the operator A if there is a positive real k_λ such that

$$\|(A-\lambda E)f\| \geqq k_\lambda \|f\| \qquad (f \in \mathfrak{D}_A). \tag{6}$$

The points of regular type for an operator A form an open set. Indeed, if λ_o is of regular type for the operator A, then for $|\lambda - \lambda_o| < k_{\lambda_o}$ we have

$$\|(A-\lambda E)f\| \geqq \|(A-\lambda_o E)f\| - |\lambda - \lambda_o| \, \|f\| \geqq k_\lambda \|f\| ,$$

where $f \in \mathfrak{D}_A$ and $k_\lambda = k_{\lambda_o} - |\lambda - \lambda_o|$.

Any point λ of regular type will be called a r e g u l a r p o i n t for the operator A if the set $\mathfrak{R}_{(A-\lambda E)}$ coincides with the entire space $\mathfrak{H}$.

REMARK: If A is a self-adjoint operator in $\mathfrak{H}$, then $\mathfrak{R}_{(A \pm iE)} = \mathfrak{H}$. If A is a symmetric operator in $\mathfrak{H}$ and $\mathfrak{R}_A = \mathfrak{H}$, then A is self-adjoint.

We verify the first assertion as follows. Let A be self-adjoint in $\mathfrak{H}$, then for $f \in \mathcal{D}_A$ we have

$$\|Af \pm if\|^2 = \langle Af, Af \rangle \mp i \langle Af, f \rangle \pm i \langle f, Af \rangle + \langle f, f \rangle =$$

$$= \|Af\|^2 + \|f\|^2 . \tag{7}$$

Thus $Af \pm if = 0$ only for $f = 0$. $\mathcal{R}_{(A \pm iE)}$ is dense in $\mathfrak{H}$. If g is orthogonal to $\mathcal{R}_{(A+iE)}$, then $0 = \langle g, Af + if \rangle =$ $= \langle g, Af \rangle - \langle ig, f \rangle$, thus $g \in \mathcal{D}_{A^*} = \mathcal{D}_A$ and $Ag = ig$. But as we have just seen this can only be if g 0. To see that $\mathcal{R}_{(A+iE)} = \mathfrak{H}$ we take an $h \in \mathfrak{H}$. Since $\mathcal{R}_{(A+iE)}$ is dense in $\mathfrak{H}$, there is a sequence (h_n), $n = 1, 2, \ldots$ such that

$$h_n = Af_n + if_n \longrightarrow h. \tag{8}$$

By (7)

$$\|h_n - h_m\|^2 = \|A(f_n - f_m) + i(f_n - f_m)\|^2 = \|A(f_n - f_m)\|^2 + \|f_n - f_m\|^2 .$$

Thus (f_n) and (Af_n), $n = 1, 2, \ldots$, converge to certain vectors f and g. Since A is closed, $f \in \mathcal{D}_A$ and $g = Af$. By (8) $h = Af + if \in \mathcal{R}_{(A+iE)}$. Thus $\mathcal{R}_{(A+iE)} = \mathfrak{H}$. In the same way we obtain that $\mathcal{R}_{(A-iE)} = \mathfrak{H}$.

To verify the second assertion we only have to show that $\mathcal{D}_{A^*} \subset \mathcal{D}_A$. If $h \in \mathcal{D}_{A^*}$ and $g = A^* h$, then, using the assumption that $\mathcal{R}_A = \mathcal{G}$, we have a vector $h' \in \mathcal{D}_A$ for which $g = Ah'$. From this we get for arbitrary $f \in \mathcal{D}_A$

$$\langle Af,h\rangle = \langle f,A^*h\rangle = \langle f,Ah'\rangle = \langle Af,h'\rangle$$

and $\langle Af,h-h'\rangle = 0$. Since $\mathcal{R}_A = \mathcal{G}$, we have $h-h' = 0$, that is $h = h' \in \mathcal{D}_A$.

We recall from page 80 that points of regular type form an open set in the complex plane; we shall refer to this set of points as the <u>d o m a i n   o f   r e g u l a r i t y</u> of the operator.

II. <u>Let D denote a connected region of the complex plane consisting of points of regular type for the operator A. Then the orthogonal complement</u> $\mathfrak{N}_\lambda$ <u>of</u> $\mathcal{R}_{(A-\lambda E)}$ <u>in</u> $\mathcal{G}$ <u>will have the same dimension for all</u> $\lambda \in D$.

Proof: We shall show that for each point $\lambda_0 \in D$ we can find a neighborhood W such that the dimension of $\mathfrak{N}_\lambda$ and the dimension of $\mathfrak{N}_{\lambda_0}$ will be equal for all $\lambda \in W$; from this the proposition will follow by the Heine-Borel theorem. Let W be a neighborhood of the point λ_0 with radius $(1/3)k_{\lambda_0}$. Then by (6) for W we get

$$\|(A-\lambda E)f\| \geqq \|(A-\lambda_0 E)f\| - |\lambda-\lambda_0|\,\|f\| > (2/3)k_{\lambda_0}\|f\|$$

for all $f \in \mathcal{D}_A$ and

$$\|(A-\lambda E)f - (A-\lambda_0 E)f\| = |\lambda-\lambda_0|\,\|f\| < (1/2)\,\|(A-\lambda E)f\|,$$

$$\|(A-\lambda E)f - (A-\lambda_0 E)f\| < (1/3)\,\|(A-\lambda_0 E)f\|.$$

This shows that the opening of the subspaces $\overline{\mathcal{R}_{(A-\lambda_0 E)}}$ and $\overline{\mathcal{R}_{(A-\lambda E)}}$ is not larger than 1/2 for all $\lambda \in W$ by formula (2) (see page 77). Hence by proposition I (see page 77) the dimension of $\mathfrak{N}_\lambda$ ($\lambda \in W$) equals the dimension of the subspace $\mathfrak{N}_{\lambda_0}$. This completes the proof of the proposition.

The foregoing proposition lends meaning to the following definition:

<u>The dimension of the subspace $\mathfrak{N}_\lambda$ for $\lambda \in D$ will be called the *deficiency number* of the operator A in the connected region</u> D.

Let A be a symmetric operator and $\lambda = a+ib$ ($b \neq 0$), then

$$\|(A-\lambda E)f\|^2 = \|(A-aE)f\|^2 + b^2\|f\|^2 \geq b^2\|f\|^2$$

for $f \in \mathcal{D}_A$. We see therefore that the upper and the lower half-plane of the complex plane are connected regions for the domain of regularity of the operator A. If one of the points of the real axis in the complex plane is of regular type for the operator A, then the deficiency numbers of A are equal. Thus symmetric operators can have no more than two distinct deficiency numbers.

<u>The pair</u> (m,n), <u>where m is the deficiency number of the symmetric operator A in the upper half-plane and n is the deficiency number in the lower half-plane of the complex plane, is called the d e f i c i e n c y i n d e x of the operator A</u>.

From the remark on page 80 it is seen that the deficiency index of a symmetric operator A is (0,0) if and only if A is self-adjoint.

The domain of regularity of an isometric operator U with dense domain of definition also contains two connected regions: The region inside and the region outside the unit circle in the complex plane because for $|\lambda| < 1$

$$\|(U-\lambda E)f\| \geqq \|Uf\| - |\lambda|\,\|f\| = (1-|\lambda|)\,\|f\|$$

and for $|\lambda| > 1$

$$\|(U-\lambda E)f\| \geqq |\lambda|\,\|f\| - \|Uf\| = (|\lambda|-1)\,\|f\| .$$

Let V denote an isometric operator which maps the entire Hilbert space $\mathfrak{H}$ onto a proper part of $\mathfrak{H}$. As we have just seen, all points situated inside and outside the unit circle of the complex plane are points of regular type for V. Suppose now that the element $f_0 \in \mathfrak{H}$ $(\|f_0\| = 1)$ is orthogonal to $\mathcal{R}_{(V-\lambda E)}$. Then

$$|\lambda| = \|\lambda f_0\| = \min_{g \in \mathcal{R}_{(V-\lambda E)}} \|\lambda f_0 - g\| \leqq \|\lambda f_0 - (\lambda f_0 - Vf_0)\| = 1$$

Hence for $|\lambda| > 1$ the set $\mathcal{R}_{(V-\lambda E)}$ coincides with the entire space $\mathfrak{H}$. For $\lambda = 0$ by assumtion $\mathcal{R}_V$ does not coincide with $\mathfrak{H}$; therefore by proposition II above for all λ, $|\lambda| < 1$, the set $\mathcal{R}_{(V-\lambda E)}$ is a proper subspace of $\mathfrak{H}$. The orthogonal complements $\mathfrak{N}_\lambda$ have dimension n equal to the dimension of $\mathfrak{N}_0$, the orthogonal complement to $\mathcal{R}_V$. We see therefore that the operator V has two deficiency numbers, namely 0 and n. Since the deficiency numbers of the operator V cannot be equal, V cannot have points of regular type on the unit circle.

<u>We call an operator V which maps the entire Hilbert space $\mathfrak{H}$ onto a proper part of $\mathfrak{H}$ isometrically a s e m i - - u n i t a r y operator</u>.

The deficiency index for an isometric operator with dense domain of definition is given in terms of the region outside and the region inside the unit circle in the complex plane. A unitary operator has deficiency index (0,0).

We wish to obtain some information about the Cayley transformation of a linear operator. Following a suggestion in M. H. Stone's book on linear transformations in Hilbert space (see ch. VIII, § 2), we commence by studying a more general situation which is analogous to the fractional-linear transformation in complex analysis, namely

$$w = (az+b)/(cz+d), \quad ad-bc \neq 0.$$

<u>Suppose that</u> A <u>is a linear operator in a Hilbert space</u> $\mathfrak{H}$. <u>Assume that for some complex numbers</u> c <u>and</u> d <u>the operator</u> $cA+dE$ <u>is one-one (i.e.,</u> $h \in \mathfrak{D}_A$ <u>and</u> $cAh+dh=0$ <u>implies</u> $h=0$). <u>For any complex numbers</u> a <u>and</u> b <u>such that</u> $ad-bc \neq 0$ <u>we define a linear operator</u> B <u>on</u> $\mathfrak{R}_{cA+dE}$ <u>by</u>

$$Bf = aAg+bg \tag{9}$$

<u>where</u>

$$f = cAg+dg \qquad (g \in \mathfrak{D}_A). \tag{10}$$

<u>That is,</u>

$$B = (aA+bE)(cA+dE)^{-1}, \tag{11}$$

<u>and we shall call</u> B <u>a *fractional-linear transformation* of</u> A.

Without loss of generality we can assume that $ad-bc=1$; we shall do this.

We now solve the system of equations

$$cAg + dg = f$$

$$aAg + bg = Bf$$

for Ag and g. We get

$$Ag = \frac{\begin{vmatrix} f & d \\ Bf & b \end{vmatrix}}{\begin{vmatrix} c & d \\ a & b \end{vmatrix}} = dBf - bf$$

$$g = \frac{\begin{vmatrix} c & f \\ a & Bf \end{vmatrix}}{\begin{vmatrix} c & d \\ a & b \end{vmatrix}} = -cBf + af \tag{12}$$

for $g \in \mathcal{D}_A$ and $f \in \mathcal{D}_B$. Moreover, the operator $-cB + aE$ is one-one, because if $g = 0$, then $0 = Ag = dBf - bf$ and $0 = g = -cBf + af$; hence $0 = (ad-bc)f = f$. We sum this up in proposition

III. <u>If the operator</u> B <u>is expressed in terms of the operator</u> A <u>as indicated in formula</u> (11), <u>then the operator</u> A <u>is given by the formula</u>

$$A = (dB - bE)(-cB + aE)^{-1}. \tag{13}$$

IV. <u>If the operators</u> A <u>and</u> B <u>are fractional-linear transformations of each other, then from the fact that one is a closed operator follows that the other is a closed operator as well.</u>

Proof: Suppose that the operator A is closed. If the sequences $f_n \in \mathcal{D}_B$ and Bf_n, $n = 1,2, \ldots$, converge to f_0 and h_0, respectively, then the sequences

$$g_n = -cBf_n + af_n$$

$$Ag_n = dBf_n - bf_n$$

converge to g_0 and k_0, respectively, and $g_n \in \mathcal{D}_A$. Since the operator A is closed, $g_0 \in \mathcal{D}_A$ and $k_0 = Ag_0$. By (10) and (9):

$$f_n = cAg_n + dg_n$$

$$Bf_n = aAg_n + bg_n.$$

Passing to the limit as $n \to \infty$ we get

$$f_0 = ck_0 + dg_0 = cAg_0 + dg_0$$

$$h_0 = ak_0 + bg_0 = aAg_0 + bg_0$$

and hence $f_0 \in \mathcal{D}_B$ and $h_0 = Bf_0$. But this means that the operator B is closed. The proof is finished.

From proposition IV it follows that if A is a closed operator and λ is a point of regular type for A, then $\mathcal{R}_{A-\lambda E}$ is a subspace. In fact, the operator $(A-\lambda E)^{-1}$, acting on $\mathcal{R}_{A-\lambda E}$ is bounded and by proposition IV closed; therefore it is defined on a closed manifold.

V. <u>If the numbers</u> a, b, c, <u>and</u> d <u>are real and the operator A is Hermitian (resp. self-adjoint), then the operator B will be Hermitian (resp. self-adjoint) also</u>.

Proof: If A is Hermitian, then for any $g_1, g_2 \in \mathcal{D}_A$:

$$\langle aAg_1 + bg_1, cAg_2 + dg_2 \rangle = ac\langle Ag_1, Ag_2\rangle + ad\langle Ag_1, g_2\rangle +$$
$$+ bc\langle g_1, Ag_2\rangle + bd\langle g_1, g_2\rangle =$$
$$= ac\langle Ag_1, Ag_2\rangle + ad\langle g_1, Ag_2\rangle + bc\langle Ag_1, g_2\rangle + bd\langle g_1, g_2\rangle =$$
$$= \langle cAg_1 + dg_1, aAg_2 + bg_2\rangle$$

so that by (10) and (9) for $f_1, f_2 \in \mathcal{D}_B$

$$\langle Bf_1, f_2\rangle = \langle f_1, Bf_2\rangle .$$

Next, let the operator A be self-adjoint. We verify that for any λ which is not real the set $\mathcal{R}_{B-\lambda E}$ coincides with the space $\mathcal{G}$ and therefore the Hermitian operator B is self-adjoint.

By (10) and (9) the set $\mathcal{R}_{B-\lambda E}$ is made up of vectors of the form:

$$aAg + bg - \lambda(cAg + dg) = (a - \lambda c)Ag + (b - \lambda d)g \qquad (g \in \mathcal{D}_A).$$

Since a, b, c, and d are real, then for non-real λ :

$$a - \lambda c \neq 0 \quad \text{and} \quad \mu = -(b - \lambda d)/(a - \lambda c)$$

are non-real as well and therefore

$$\mathcal{R}_{B-\lambda E} = \mathcal{R}_{A-\mu E} = \mathcal{G} .$$

In particular, if the operator A^{-1} exists, then it is self-adjoint.

VI. <u>If</u> $c = \bar{a}$, $d = \bar{b}$ <u>and the operator A is Hermitian (resp. self-adjoint), then the operator B is isometric (resp. unitary)</u>.

Proof: If A is Hermitian, then for any $g_1, g_2 \in \mathcal{D}_A$:

$$\langle aAg_1 + bg_1, aAg_2 + bg_2\rangle = a\bar{a}\langle Ag_1, Ag_2\rangle + a\bar{b}\langle Ag_1, g_2\rangle +$$
$$+ \bar{a}b\langle g_1, Ag_2\rangle + b\bar{b}\langle g_1, g_2\rangle =$$
$$= c\bar{c}\langle Ag_1, Ag_2\rangle + d\bar{c}\langle g_1, Ag_2\rangle + c\bar{d}\langle Ag_1, g_2\rangle + d\bar{d}\langle g_1, g_2\rangle =$$
$$= \langle cAg_1 + dg_1, cAg_2 + dg_2\rangle$$

which means

$$\langle Bf_1, Bf_2\rangle = \langle f_1, f_2\rangle \qquad (f_1, f_2 \in \mathcal{D}_B);$$

therefore the operator B is isometric.

Let the operator A be self-adjoint. Then the domain of definition of B, $\mathcal{D}_B = \mathcal{R}_{(\bar{a}A + \bar{b}E)}$, and the range of B, $\mathcal{R}_B = \mathcal{R}_{(aA + bE)}$ coincide with the space $\mathfrak{H}$ and the isometric operator B is unitary. This proves the proposition.

Analogously one shows the converse statement:

VII. <u>If</u> $c = \overline{a}$, $d = \overline{b}$ <u>and the operator B is isometric (resp. unitary), then the operator A is Hermitian (resp. self-adjoint).</u>

From the definition of fractional-linear transformation we get directly:

VIII. <u>Suppose that the operator B is a fractional-linear transformation of A:</u>

$$B = (aA + bE)(cA + dE)^{-1}.$$

<u>If</u> $\widetilde{A}$ <u>is an extension of the operator A and if the operator</u> $c\widetilde{A} + dE$ <u>is one-one (i.e.,</u> $c\widetilde{A}f + df = 0$ <u>implies</u> $f = 0$ <u>for</u> $f \in \mathcal{D}_{\widetilde{A}}$<u>), then the fractional-linear transformation of the operator</u> $\widetilde{A}$:

$$\widetilde{B} = (a\widetilde{A} + bE)(c\widetilde{A} + dE)^{-1}$$

<u>is an extension of the operator B.</u>

<u>A fractional-linear transformation of a Hermitian operator A for which</u> $c = \overline{a}$, $d = \overline{b}$ <u>holds we call a</u> <u>C a y l e y - t r a n s f o r m a t i o n</u> <u>of A; if we put</u> $\lambda = -d/c$ <u>we get it in the form:</u>

$$U_\lambda = (A - \overline{\lambda}E)(A - \lambda E)^{-1} \qquad (\operatorname{Im}\lambda \neq 0). \qquad (14)$$

In view of the propositions VI, VII, and VIII the Cayley-transformation reduces the problem of finding self-adjoint extensions of a symmetric operator A to the problem of finding unitary extensions of the operator

$$U_\lambda = (A - \overline{\lambda} E)(A - \lambda E)^{-1}.$$

Let A be closed, then the operator U_λ has as domain of definition the subspace

$$\mathcal{D}_{U_\lambda} = \mathcal{R}_{(A-\lambda E)} = L_\lambda$$

and as range the subspace

$$\mathcal{R}_{U_\lambda} = \mathcal{R}_{(A-\overline{\lambda}E)} = L_{\overline{\lambda}} .$$

The orthogonal complements to L_λ and $L_{\overline{\lambda}}$ we denote by $\mathfrak{N}_\lambda$ and $\mathfrak{N}_{\overline{\lambda}}$, respectively. To assume that A is closed, is no essential restriction because a symmetric operator A has a closed symmetric extension $A^{**} = (A^*)^*$.

Keeping in mind that isometric operators map orthogonal elements into orthogonal elements, we can see that any closed isometric extension $\widetilde{U_\lambda}$ of the operator U_λ can be obtained in the following manner.

Choose in $\mathfrak{N}_\lambda$ and $\mathfrak{N}_{\overline{\lambda}}$ orthonormal systems $(\varphi_\nu)_{\nu \in N}$ and $(\psi_\nu)_{\nu \in N}$ of the same cardinality and on the subspace $\widetilde{L_\lambda}$, being the orthogonal sum of L_λ and the closed linear hull of the system $(\varphi_\nu)_{\nu \in N}$, let for $f \in L_\lambda$

$$\widetilde{U}_\lambda (f + \sum_{\nu \in N} \xi_\nu \varphi_\nu) = U_\lambda f + \sum_{\nu \in N} \xi_\nu \psi_\nu, \tag{15}$$

where ξ_ν $(\nu \in N)$ are arbitrary numbers satisfying

$$\sum_{\nu \in N} |\xi_\nu|^2 < \infty .$$

In this way we get an isometric operator which will map $\widetilde{L}_\lambda$ onto $\widetilde{L}_{\overline{\lambda}}$, the orthogonal sum of the subspace $L_{\overline{\lambda}}$ and the closed linear hull of the system $(\psi_\nu)_{\nu \in N}$.

If the subspaces $\mathfrak{N}_\lambda$ and $\mathfrak{N}_{\overline{\lambda}}$ have distinct dimensions (this corresponds to the case when the symmetric operator

$$A = (\lambda U_\lambda - \overline{\lambda} E)(U_\lambda - E)^{-1}$$

has distinct deficiency numbers), then we select in the subspace of smaller dimension a dense orthonormal basis and in the second subspace a basis of the same cardinality and we extend the isometric operator U_λ up to semi-unitary. Such an extension is usually called maximal isometric extension because it does not admit further isometric extension.

If the dimensions of $\mathfrak{N}_\lambda$ and $\mathfrak{N}_{\overline{\lambda}}$ are equal (this corresponds to the case when the symmetric operator has equal deficiency numbers), the isometric operator U_λ can be extended up to unitary which will be a maximal extension.

If the dimensions of $\mathfrak{N}_\lambda$ and $\mathfrak{N}_{\overline{\lambda}}$ are equal and finite, then, besides the unitary, there is no other maximal isometric extension of the operator U_λ.

If the dimensions of $\mathfrak{N}_\lambda$ and $\mathfrak{N}_{\overline{\lambda}}$ are equal and infinite, then, besides the unitary, there can be constructed other maximal isometric extensions. For example, to get a semi-unitary extension $\widetilde{U}_\lambda$ of the operator U_λ, select in $\mathfrak{N}_{\overline{\lambda}}$ an incomplete orthonormal system (ψ_ν) whose cardinality equals the dimension of $\mathfrak{N}_{\overline{\lambda}}$ and select in $\mathfrak{N}_\lambda$ a complete orthonormal system (φ_ν) and define the operator $\widetilde{U}_\lambda$ by the formula (15) given on the last page.

To be able to proceed from the isometric extension $\widetilde{U}_\lambda$, by inverting the fractional-linear transformation, to the construction of the symmetric extension $\widetilde{A}$ of the operator A

$$\widetilde{A} = (\lambda \widetilde{U}_\lambda - \overline{\lambda} E)(\widetilde{U}_\lambda - E)^{-1}$$

it is necessary and sufficient according to proposition VIII that

$$\widetilde{U}_\lambda \varphi - \varphi = 0 \text{ imply } \varphi = 0 \text{ for } \varphi \in \mathfrak{D}_{\widetilde{U}_\lambda} .$$

Let A be a symmetric operator. We show now that for any isometric extension $\widetilde{U}_\lambda$ of the operator

$$U_\lambda = (A - \overline{\lambda} E)(A - \lambda E)^{-1}$$

the condition of proposition VIII is satisfied. Let $\widetilde{U}_\lambda$ be an isometric extension of the operator U_λ and

$$\langle \tilde{U}_\lambda \varphi - \varphi, h \rangle = 0 \qquad (h \in \mathfrak{H}).$$

In particular, then, we have for all $h = \tilde{U}_\lambda g = U_\lambda g$, where $g \in \mathfrak{D}_{U_\lambda}$,

$$\langle \tilde{U}_\lambda \varphi - \varphi, \tilde{U}_\lambda g \rangle = \langle \varphi, g - U_\lambda g \rangle = 0;$$

hence by (13), $\langle \varphi, f \rangle = 0$ $(f \in \mathfrak{D}_A)$. But $\mathfrak{D}_A$ is dense in $\mathfrak{H}$ and we get $\varphi = 0$.

In view of what was said about isometric extensions of the operator U_λ and recalling the content of propositions VII and VIII, we have arrived at a theorem of J. von Neumann:

THEOREM: <u>A symmetric operator has self-adjoint extensions if and only if its deficiency numbers are equal</u>.

If a symmetric operator A has deficiency numbers which are not equal, then - as was just shown - extension of A in the space $\mathfrak{H}$ to a self-adjoint operator is impossible. However, it is possible to extend a symmetric operator A to a self-adjoint operator by passage to an enlarged Hilbert space. The following theorem is due to M. A. Neumark:

THEOREM: <u>Any symmetric operator A defined in a Hilbert space $\mathfrak{H}$ and with arbitrary deficiency index</u> (m,n) <u>can be extended to a self-adjoint operator A^+ which is defined in a larger Hilbert space</u> $\mathfrak{H}^+ \supset \mathfrak{H}$.

Proof: Given two Hilbert spaces $\mathfrak{H}$, $\mathfrak{H}'$ we form

$$\mathfrak{H} \oplus \mathfrak{H}'$$

in the manner as was explained on page 75. Suppose that the operator A acts in $\mathfrak{H}$ and the operator B acts in $\mathfrak{H}'$. Using these two operators we define a new operator C, denoting it by $A \oplus B$, acting in $\mathfrak{H} \oplus \mathfrak{H}'$ as follows:

$$C(f,f') = (Af,Bf')$$

where (f,f') is an element of $\mathfrak{H} \oplus \mathfrak{H}'$, $f \in \mathfrak{D}_A$, $f' \in \mathfrak{D}_B$.

We observe the following properties:

1: If the operators A and B are symmetric, then the operator C is likewise symmetric.

2. If the operator A has deficiency index (m,n) and the operator B has deficiency index (m',n'), then the operator C has deficiency index (m+m',n+n').

Property 2 is seen as follows. Since for $f \in \mathfrak{D}_A$, $f' \in \mathfrak{D}_B$

$$(C-\lambda E)(f,f') = ((A-\lambda E)f,(B-\lambda E)f')$$

holds, we get

$$\mathfrak{R}_{C-\lambda E} = \mathfrak{R}_{A-\lambda E} \oplus \mathfrak{R}_{B-\lambda E}$$

and consequently

$$\mathfrak{H} \oplus \mathfrak{H}' = \mathfrak{R}_{C-\lambda E} \oplus \mathfrak{N}_\lambda \oplus \mathfrak{N}'_\lambda \cdot$$

where

$$\mathfrak{N}_\lambda = \mathfrak{H} \ominus \mathfrak{R}_{A-\lambda E}$$

and

$$\mathfrak{N}'_\lambda = \mathfrak{H}' \ominus \mathfrak{R}_{B-\lambda E} .$$

Suppose that the symmetric operator A, acting in $\mathfrak{H}$, has deficiency index (m,n), where $m \neq n$. Then we can find a symmetric operator B, acting in some Hilbert space $\mathfrak{H}'$, with deficiency index (n,m). In fact the following choice will do: let $B = -A$ and $\mathfrak{H}' = \mathfrak{H}$. Then the operator $C = A \oplus B$, acting in $\mathfrak{H} \oplus \mathfrak{H}'$ will be symmetric and its deficiency numbers will be equal. By von Neumann's theorem the operator C has a self-adjoint extension. But any self-adjoint extension of C is also a self-adjoint extension of the symmetric operator A originally given and it goes outside the given space $\mathfrak{H}$ and into the space $\mathfrak{H} \oplus \mathfrak{H}'$. This completes the proof of the theorem.

If B is a symmetric extension of a symmetric operator A, then $A \subseteqq B$ and $B \subseteqq B^*$. It follows therefore that $B \subseteqq B^* \subseteqq A^*$; this means that any symmetric extension of a symmetric operator A is contained in the adjoint operator A^*. If A is self-adjoint, then $A = A^*$ and A cannot have a symmetric extension.

We say that a subspace M in a Hilbert space <u>reduces</u> an operator A if for $f \in \mathfrak{D}_A$ we also have $Pf \in \mathfrak{D}_A$ and $APf = PAf$, where P is the projection operator onto M.

If a symmetric extension A^+ in $\mathfrak{H}^+$ of an operator A in $\mathfrak{H}$, where $\mathfrak{H}^+ \supseteqq \mathfrak{H}$, is reduced by a subspace $G^+ \subset \mathfrak{H}^+ \ominus \mathfrak{H}$, then we shall exclude this subspace G^+ from $\mathfrak{H}^+$ (i. e. we substitute for the space $\mathfrak{H}^+$ the space $\mathfrak{H}^+ \ominus G^+$ and the operator A^+ is replaced by its component in $\mathfrak{H}^+ \ominus G^+$.

If A^+ is any symmetric extension of a symmetric operator A, then $\mathfrak{D}_A \subseteqq \mathfrak{D}_{A^+} \cap \mathfrak{H} \subseteqq \mathfrak{D}_{A^+}$ obtains. This permits a classification of proper symmetric extensions into:

Type 1: $\mathfrak{D}_A \neq \mathfrak{D}_{A^+} \cap \mathfrak{H} = \mathfrak{D}_{A^+}$

Type 2: $\mathfrak{D}_A = \mathfrak{D}_{A^+} \cap \mathfrak{H} \neq \mathfrak{D}_{A^+}$

Type 3: $\mathfrak{D}_A \neq \mathfrak{D}_{A^+} \cap \mathfrak{H} \neq \mathfrak{D}_{A^+}$.

According to this classification, symmetric extensions not leading outside the space are of Type 1 and maximal symmetric operators have only symmetric extensions of type 2.

3.2 Extension of a Symmetric Operator (Continued)

By an orthogonal resolution of the identity we mean a one-parameter family $(E_t)_{t \in [a,b]}$ of projection operators on a Hilbert space $\mathfrak{H}$, where $[a,b]$ is a finite or infinite interval and

(a): $E_a = 0, \quad E_b = E,$

where 0 and E denote the zero and the identity operator, respectively;

(b): $E_{t-0} = E_t \qquad (a < t \leqq b);$

(c): $E_u E_v = E_s \qquad (s = \min\{u,v\}).$

If the interval $[a,b]$ is infinite, we put

$$E_{-\infty} = \lim_{t \to -\infty} E_t \quad \text{and} \quad E_{\infty} = \lim_{t \to \infty} E_t.$$

From the foregoing definition we have that for an arbitrary element f in $\mathfrak{H}$ the inner product

$$\langle E_t f, f \rangle = \sigma_f(t)$$

is a continuous function from the left; in addition this function is non-decreasing and of bounded variation. Indeed, $\sigma_f(a) = 0$, $\sigma_f(b) = \langle f,f\rangle$ and for $s < t$ we have

$$\langle E_s f,f\rangle = \|E_s f\|^2 = \|E_s E_t f\|^2 \leqq \|E_t f\|^2 = \langle E_t f,f\rangle.$$

We denote by E_Δ the difference $E_{t''} - E_{t'}$, where $\Delta = [t',t''] \subset [a,b]$. For any two intervals Δ_1, Δ_2 we get by condition (c):

$$E_{\Delta_1} E_{\Delta_2} = E_{\Delta_1 \cap \Delta_2}.$$

If Δ_1 and Δ_2 are disjoint intervals, then

$$E_{\Delta_1} E_{\Delta_2} = 0;$$

this means that the subspaces of the projection operators E_{Δ_1} and E_{Δ_2} are orthogonal.

SPECTRAL THEOREM for self-adjoint operators:
<u>For each self-adjoint operator B in a Hilbert space $\mathfrak{H}$ there is one and only one operator function</u> E_t $(-\infty < t < \infty)$ <u>such that</u>

1) E_t <u>is a projection operator;</u>

2) $E_t E_s = E_t$ <u>for</u> $t \leqq s$;

3) E_t <u>permutes with each bounded operator which permutes with B</u>;

4) $\lim_{t \to -\infty} E_t f = 0, \quad \lim_{t \to \infty} E_t f = f$ <u>for all</u> $f \in \mathcal{G}$;

5) $E_t f$ <u>is a left-continuous function for all</u> $f \in \mathcal{G}$;

6) $f \in \mathcal{D}_B$ <u>if and only if</u>

$$\int_{-\infty}^{\infty} t^2 \, d\langle E_t f, f \rangle < \infty$$

<u>In this case</u>

$$Bf = \int_{-\infty}^{\infty} t \, dE_t f$$

<u>and</u>

$$\|Bf\|^2 = \int_{-\infty}^{\infty} t^2 \, d\langle E_t f, f \rangle .$$

The operator function E_t satisfying 1) to 6) is called the <u>s p e c t r a l f u n c t i o n</u> of B and the second last formula the <u>s p e c t r a l r e s o l u t i o n</u> of B.

For convenience we recall that a linear operator A is said to p e r m u t e with a bounded linear operator C if CA = AC on the domain of definition of the product CA, i.e. on $\mathcal{D}_{CA}$.

We note in passing that in the APPENDIX we shall consider a method of proof for the foregoing Spectral Theorem.

<u>By a resolution of the identity we mean a one-parameter family of self-adjoint operators</u> F_t <u>which satisfies:</u>

(A): <u>For</u> $t_2 > t_1$ <u>the difference</u> $F_{t_2} - F_{t_1}$ <u>is a bounded positive operator, that is for every</u> $f \in \mathfrak{H}$ <u>we have</u>

$$\langle F_{t_2} f, f \rangle \geqq \langle F_{t_1} f, f \rangle ;$$

(B): $F_{t-0} = F_t$;

(C): $F_{-\infty} = 0, \quad F_{\infty} = E.$

In contrast to the orthogonal resolution of the identity it is no longer required that the F_t's be projection operators. The corresponding requirement of orthogonality (condition (c)) is dropped because from condition (c) and condition (A) above it would follow that F_t is a projection operator.

We agree to denote by F_Δ the difference $F_{t_2} - F_{t_1}$, where $\Delta = [t_1, t_2]$. We permit the case that $t_1 = t_2$.

Let S be an arbitrary set, Φ a complex-valued function on $S \times S$. If

$$\Phi(f,g) = \overline{\Phi(g,f)}$$

for all $f,g \in S$ and

$$\sum_{j,k=1}^{n} \Phi(f_j,f_k)\ \lambda_j\ \overline{\lambda_k} \geqq 0$$

for $f_1, \dots, f_n$ ($n < \infty$) belonging to S and any complex numbers $\lambda_1, \dots, \lambda_n$, then we call Φ a <u>Hermitian-positive function</u>.

Given a Hermitian-positive function on an arbitrary set S, it is possible to imbed S into a Hilbert space $\mathfrak{H}^+$ in such a manner that for any two elements f and g of S the inner product is defined by

$$\langle f,g\rangle = \Phi(f,g).$$

The details of the construction for the separable case were carried out on pages 5 to 7. It remains to consider here the case when S is an uncountable set. For each $s \in S$ let C_s be a copy of the set of complex numbers and let L be the (weak) direct sum of the family of vector spaces $(C_s)_{s \in S}$. For $f,g \in L$ define

$$\langle f,g\rangle = \sum_{s,t \in S} \Phi(s,t)\ f(s)\ \overline{g(t)}.$$

As on pp. 5-7, we put $M = \{f \in L: \langle f,f\rangle = 0\}$, and denote the completion of L/M by $\mathfrak{H}^+$. S can be imbedded in $\mathfrak{H}^+$ by mapping $s \in S$ to $\tilde{s} \in L/M$, where $\tilde{s}(t) = \delta_{st}$ (Kroenecker delta) for all $t \in S$. Then

$$\langle \tilde{s},\tilde{t}\rangle = \sum_{s',t' \in S} \Phi(s',t')\,\tilde{s}(s')\,\tilde{t}(t') = \Phi(s,t).$$

For convenience we will always identify $\tilde{s}$ and s.

<u>We say that the Hilbert space</u> $\mathfrak{H}^+$ <u>has positive definite kernel</u> Φ.

The following theorem is due to M. A. Neumark:

THEOREM: <u>Let</u> F_t <u>be a resolution of the identity of the space</u> $\mathfrak{H}$. <u>Then there exists a Hilbert space</u> $\mathfrak{H}^+$ <u>which contains</u> $\mathfrak{H}$ <u>as subspace and there is an orthogonal resolution of the identity</u> E_t^+ <u>of the space</u> $\mathfrak{H}^+$ <u>such that for each</u> $f \in \mathfrak{H}$ <u>the relation</u>

$$F_t f = P^+ E_t^+ f$$

<u>holds. Here</u> P^+ <u>denotes the projection operator of</u> $\mathfrak{H}^+$ <u>onto</u> $\mathfrak{H}$.

Proof: We introduce the set S of all pairs ω of the form

$$\omega = (\Delta, f)$$

where Δ denotes a subinterval of $I = [-\infty, \infty]$ and f is in $\mathcal{G}$. On $S \times S$ we define a function Φ by

$$\Phi(\omega_1, \omega_2) = \langle F_{\Delta_1 \cap \Delta_2} f_1, f_2 \rangle ,$$

where $\omega_j = (\Delta_j, f_j)$, $j = 1,2$.

We show that the function Φ is Hermitian-positive:

$$\Phi(\omega_1, \omega_2) = \langle F_{\Delta_1 \cap \Delta_2} f_1, f_2 \rangle = \langle f_1, F_{\Delta_1 \cap \Delta_2} f_2 \rangle =$$

$$= \overline{\langle F_{\Delta_1 \cap \Delta_2} f_2, f_1 \rangle} = \overline{\Phi(\omega_2, \omega_1)} .$$

Moreover

$$\sum_{j,k=1}^{n} \Phi(\omega_j, \omega_k) \lambda_j \overline{\lambda_k} =$$

$$\sum_{j,k=1}^{n} \langle F_{\Delta_j \cap \Delta_k} f_j, f_k \rangle \lambda_j \overline{\lambda_k} \geqq 0 . \qquad (1)$$

If the intervals Δ_j $(j = 1, \ldots , n)$ are pairwise disjoint, then

$$\sum_{j,k=1}^{n} \langle F_{\Delta_j \cap \Delta_k} f_j, f_k \rangle \lambda_j \overline{\lambda_k} =$$

$$= \sum_{j=1}^{n} \langle F_{\Delta_j} f_j, f_j \rangle |\lambda_j|^2 \geqq 0. \qquad (2)$$

If the intervals Δ_j $(j = 2,3, \ldots ,n)$ are pairwise disjoint and the intervals Δ_1 and Δ_2 coincide, then the sum on the right hand side of equation (1) splits into two summands: The first summand, containing the indices from 3 to n, is of the form (2); the second summand, containing the indices 1 and 2, is of the form

$$\sum_{j,k=1}^{2} \langle F_{\Delta_j \cap \Delta_k} f_j, f_k \rangle \lambda_j \overline{\lambda_k} =$$

$$= \sum_{j,k=1}^{2} \langle F_{\Delta_1} f_j, f_k \rangle \lambda_j \overline{\lambda_k} =$$

$$= \langle F_{\Delta_1} \sum_{j=1}^{2} \lambda_j f_j, \sum_{k=1}^{2} \lambda_k f_k \rangle \geqq 0.$$

Now we note that regardless of the manner in which the intervals Δ_j $(j = 1,2, \ldots ,n)$ are situated, we can reduce the consideration to the cases already looked at by an additional splitting of the given set of intervals into a system of disjoint or coincident intervals and by using additivity: If $\Delta_1 \cap \Delta_2 = \Box$, then

$$\langle F_{(\Delta_1 \cup \Delta_2) \cap \Delta_3} f, g \rangle = \langle F_{(\Delta_1 \cap \Delta_3) \cup (\Delta_2 \cap \Delta_3)} f, g \rangle =$$

$$= \langle F_{\Delta_1 \cap \Delta_3} f, g \rangle + \langle F_{\Delta_2 \cap \Delta_3} f, g \rangle.$$

This shows that Φ is a Hermitian-positive function on $S \times S$.

Now we imbed S into a Hilbert space $\mathcal{G}^+$ as described before. If we denote the inner product in the space $\mathcal{G}^+$ with the subscript $+$, then we have

$$\langle \omega_1, \omega_2 \rangle_+ = \Phi(\omega_1, \omega_2).$$

Clearly the subspace $\{(I,f): f \in \mathcal{G}\}$ of $\mathcal{G}^+$ is isomorphic to $\mathcal{G}$ (as a vector space) under the correspondence $f \longleftrightarrow (I,f)$. Since

$$\langle (I,f),(I,g) \rangle_+ = \langle F_I f,g \rangle = \langle f,g \rangle$$

this isomorphism preserves inner products and hence is an isometric embedding of $\mathcal{G}$ in $\mathcal{G}^+$. Thus $\mathcal{G}$ may be regarded as a subspace of $\mathcal{G}^+$.

The problem is to find the projection of the element (Δ,f) of $\mathcal{G}^+$ onto the subspace $\mathcal{G}$. If this projection is denoted by (I,g), then for each h in $\mathcal{G}$ we must have

$$\langle (\Delta,f) - (I,g),(I,h) \rangle_+ = 0$$

or

$$\langle (\Delta,f),(I,h) \rangle_+ - \langle (I,g),(I,h) \rangle_+ =$$

$$= \langle F_\Delta f,h \rangle - \langle g,h \rangle = \langle F_\Delta f - g,h \rangle = 0.$$

From this it follows that

$$g = F_\Delta f$$

which means that

$$P^{+}(\Delta,f) = (I,F_{\Delta}f). \tag{3}$$

Now we define the operator E_{Δ}^{+} by

$$E_{\Delta}^{+}(\Delta',f) = (\Delta \cap \Delta',f), \quad (\Delta',f) \in \mathcal{G}^{+}. \tag{4}$$

Then for any $f \in \mathcal{G}$

$$P^{+}E_{\Delta}^{+}f = P^{+}E_{\Delta}^{+}(I,f) = P^{+}(\Delta \cap I,f) = P^{+}(\Delta,f) = (I,F_{\Delta}f) =$$

$$= F_{\Delta}f.$$

Hence the proof of the theorem is complete if we verify that the operators E_{Δ}^{+} form an orthogonal resolution of the identity of the space $\mathcal{G}^{+}$.

But E is an additive operator function on intervals. From

$$(E_{\Delta}^{+})^{2}(\Delta',f) = E_{\Delta}^{+}(\Delta \cap \Delta',f) = (\Delta \cap \Delta \cap \Delta',f) = E_{\Delta}^{+}(\Delta',f)$$

and

$$\langle E_{\Delta}^{+}(\Delta',f),(\Delta'',g)\rangle_{+} = \langle(\Delta \cap \Delta',f),(\Delta'',g)\rangle_{+} =$$

$$= \langle F_{\Delta \cap \Delta' \cap \Delta''}f,g\rangle = \langle F_{\Delta' \cap \Delta \cap \Delta''}f,g\rangle = \langle(\Delta',f),E_{\Delta}^{+}(\Delta'',g)\rangle_{+}$$

it follows that E_{Δ}^{+} is a projection operator. It is also clear that

$$E_{I}^{+}(\Delta',f) = (\Delta',f)$$

holds. Since the set of elements of the form (Δ',f) is dense in $\mathcal{G}^{+}$, each E_{Δ}^{+} can be extended to a continuous operator on the entire space $\mathcal{G}^{+}$. In view of all the properties we have established, the extended operators E^+_Δ form an orthogonal resolution of the identity of the space $\mathfrak{H}^+$ and the proof is finished.

Let A be a symmetric operator in a Hilbert space $\mathfrak{H}$. Suppose that A has been extended to a self-adjoint operator A^+ (by passage from the space $\mathfrak{H}$ to an enlarged space $\mathfrak{H}^+$). Let E^+_Δ be the spectral function of the operator A^+ and P^+ denote the projection operator of $\mathfrak{H}^+$ onto $\mathfrak{H}$. Let us put

$$F_\Delta = P^+ E^+ .$$

Then for any two elements $f \in \mathfrak{D}_{A^+}$ and $g \in \mathfrak{H}^+$ we have

$$\langle A^+ f, g \rangle = \int_{-\infty}^{\infty} t \, d\langle E^+_t f, g \rangle$$

and

$$\| A^+ f \|^2 = \int_{-\infty}^{\infty} t^2 \, d\langle E^+_t f, f \rangle .$$

If $f \in \mathfrak{D}_A$ and $g \in \mathfrak{H}$, then these two formulas can be written in the following way

$$\langle Af, g \rangle = \int_{-\infty}^{\infty} t \, d\langle F_t f, g \rangle \tag{5}$$

$$\| Af \|^2 = \int_{-\infty}^{\infty} t^2 \, d\langle F_t f, f \rangle . \tag{6}$$

By comparing formulas (5) and (6) with the corresponding formulas (see page 101) appearing in the spectral theorem for self-adjoint operators, it is natural to make the following definition:

If A is a symmetric operator and F_t a resolution of the identity such that for each $f \in \mathfrak{D}_A$ and each $g \in \mathfrak{H}$ the formulas (5) and (6) hold, then we refer to F_t as a s p e c t r a l f u n c t i o n of the symmetric operator A.

THEOREM: Each spectral function of a symmetric operator A defined in a Hilbert space $\mathfrak{H}$ is of the form

$$F_t = P^+ E_t^+ .$$

In this formula E_t^+ denotes the spectral function of a self-adjoint extension A^+ of the operator A which one obtains by passage from the space $\mathfrak{H}$ to the space $\mathfrak{H}^+ \supseteqq \mathfrak{H}$ and P^+ stands for the projection operator from $\mathfrak{H}^+$ onto $\mathfrak{H}$.

Proof: Using the theorem stated on page 104 we form the space $\mathfrak{H}^+$ and determine in it an orthogonal resolution of the identity E_t^+ such that

$$F_t = P^+ E_t^+ \tag{7}$$

holds. We show that the operator A^+, whose domain of definition consists of all $f \in \mathcal{G}^+$ for which

$$\int_{-\infty}^{\infty} t^2 \, d\langle E_t f, f\rangle < \infty$$

and which is defined by the formula

$$A^+ f = \int_{-\infty}^{\infty} t \, d\, E_t^+ f ,$$

is actually a self-adjoint extension of the operator A.

First of all the operator A^+ is self-adjoint; this is seen from the spectral theorem for self-adjoint operators. Next, if $f \in \mathcal{D}_A$, then $f \in \mathcal{D}_{A^+}$ because

$$\int_{-\infty}^{\infty} t^2 \, d\langle E_t^+ f, f\rangle = \int_{-\infty}^{\infty} t^2 \, d\langle F_t f, f\rangle = \| Af \|^2 < \infty .$$

Moreover, for $f \in \mathcal{D}_A$ and $g \in \mathcal{G}$ we have

$$\langle Af, g\rangle = \int_{-\infty}^{\infty} t \, d\langle F_t f, g\rangle = \int_{-\infty}^{\infty} t \, d\langle E_t^+ f, g\rangle = \langle A^+ f, g\rangle$$

and therefore

$$Af = P^+ A^+ f \qquad (f \in \mathcal{D}_A). \tag{8}$$

On the other hand, if $f \in \mathcal{D}_A$, then

$$\| Af \|^2 = \int_{-\infty}^{\infty} t^2 \, d\langle F_t f, f\rangle = \int_{-\infty}^{\infty} t^2 \, d\langle E_t^+ f, f\rangle = \| A^+ f \|^2. \tag{9}$$

From (8) and (9) we get that

$$Af = A^{+}f \qquad (f \in \mathfrak{D}_A).$$

In the foregoing we can assume that A^{+} is not reduced by any subspace of $\mathfrak{H}^{+} \ominus \mathfrak{H}$. For, if there was a subspace G of $\mathfrak{H}^{+} \ominus \mathfrak{H}$ which reduced the operator A^{+}, then it would likewise reduce the orthogonal resolution of the identity E_t^{+}. In that case the deletion of G from $\mathfrak{H}^{+}$ would force the deletion of the component in G of the operator E_t^{+}. That however does not affect formula (7) and the proof is complete.

If a spectral function F_t of a symmetric operator is represented in the form (7), where E_t^{+} is the spectral function of a self-adjoint extension A^{+} of the operator A, then we may say that the spectral function F_t is generated by the self-adjoint extension A^{+}. In this manner then every self-adjoint extension of a symmetric operator A generates a spectral function of this operator and, conversely, every spectral function of the operator A is generated by its self-adjoint extension.

The spectral function associated with a self-adjoint operator in the spectral theorem for self-adjoint operators is in the sense of our present terminology the only spectral function; a self-adjoint operator has no self-adjoint extensions.

By the spectral theorem for self-adjoint operators every orthogonal resolution of the identity is the spectral function of a self-adjoint operator. This kind of claim cannot be made for symmetric operators and non-orthogonal resolutions of the identity.

For non-orthogonal resolutions of the identity the passage from the "weak" representation (5) to the "strong" representation

$$Af = \int_{-\infty}^{\infty} t \, d \, F_t f \tag{10}$$

is impossible; the equation (10) is only a symbolic way of writing formulas (5) and (6). The representation (10) holds in the "strong" sense for self-adjoint operators.

From the orthogonality property of the spectral function E_t of a self-adjoint operator B it follows that for each finite interval Δ of the real line the vector $E_\Delta f$ $(f \in \mathfrak{H})$ belongs to $\mathfrak{D}_B$ and that for each $g \in \mathfrak{H}$ we have

$$\langle B \, E_\Delta f, g \rangle = \int_\Delta t \, d \langle E_t f, g \rangle .$$

This statement does not carry over in its entirety to integral representations of symmetric operators. However the following is true:

IX. <u>If A is a symmetric operator,</u> F_t <u>its spectral function,</u> Δ <u>a finite interval on the real line, and h an arbitrary vector of</u> $\mathfrak{H}$, <u>then for every</u> $g \in \mathfrak{H}$ <u>we have</u>

$$F_\Delta g \in \mathfrak{D}_{A^*}$$

<u>and</u>

$$\langle A^* F_\Delta g, h \rangle = \int_\Delta t \, d \langle F_t g, h \rangle .$$

Proof: Let A^+ be the extension which generates F_t and P^+ be the projection operator of $\mathfrak{H}^+$ onto $\mathfrak{H}$. Then for each $f \in \mathfrak{D}_A$ we have

$$\langle Af, F_\Delta g \rangle = \langle A^+ f, P^+ E_\Delta^+ g \rangle = \langle A^+ f, E_\Delta^+ g \rangle = \int_\Delta t \, d \langle E_t^+ f, g \rangle =$$

$$= \int_\Delta t \, d \langle E_t^+ f, P^+ g \rangle = \int_\Delta t \, d \langle F_t f, g \rangle . \qquad (11)$$

The integral

$$\int_\Delta t \, d \langle F_t h, g \rangle$$

is a bilinear functional of h and g in $\mathfrak{H}$ and has therefore the representation $\langle h, Dg \rangle$ with a bounded operator D. Thus

$$\langle Af, F_\Delta g \rangle = \langle f, g^* \rangle ,$$

and therefore

$$F_\Delta g \in \mathfrak{D}_{A^*} .$$

Hence by (11) we get the representation

$$\langle f, A^* F_\Delta g \rangle = \int_\Delta t \, d\langle F_t f, g \rangle .$$

But $\mathfrak{D}_A$ is dense in $\mathfrak{H}$ and therefore one can put instead of $f \in \mathfrak{D}_A$ an arbitrary vector $h \in \mathfrak{H}$. This ends the proof.

The content of the foregoing proposition is sometimes used to define a spectral function of a symmetric operator A; by a spectral function of a symmetric operator A one then means any resolution of the identity F_t satisfying the following condition: for any $g \in \mathfrak{H}$ and any finite interval $\Delta = [t', t'']$ the vector $F_\Delta g = (F_{t''} - F_{t'})g$ belongs to $\mathfrak{D}_{A^*}$ and

$$\langle A^* F_\Delta g, h \rangle = \int_\Delta t \, d\langle F_t g, h \rangle ,$$

where h is an arbitrary element of $\mathfrak{H}$.

X. <u>Let A be a self-adjoint operator in a Hilbert space</u> $\mathfrak{H}$ <u>and</u> E_t <u>be its spectral function. Suppose that given any elements</u> $f,h \in \mathfrak{H}$ <u>there exists an interval</u> (a,b), <u>a function</u> $g:(a,b) \longrightarrow \mathcal{D}_A$, <u>and a differentiable function</u> φ <u>on</u> (a,b) <u>such that</u>

$$f-\varphi(t)h = (A-tE)g(t) \tag{12}$$

<u>for every</u> $t \in (a,b)$. <u>Then</u>

$$(E_{t''}-E_{t'})f = \int_{t'}^{t''} \varphi(t)\, d\, E_t h \tag{13}$$

<u>is valid in any interval</u> $(t',t'') \subset (a,b)$.

Proof: We consider the vector-valued function

$$w(s) = \int_s^{t''} d\, E_t f \;-\; \int_s^{t''} \varphi(t)\, d\, E_t h \qquad (t' \leqq s \leqq t'')$$

which is zero for $s = t''$. We verify that for any $s \in (t',t'')$ the following two properties hold: (a) w is continuous relative to the norm and (b) the relation

$$\lim_{\delta \to 0} (1/\delta)\, \| w(s \pm \delta) - w(s) \| = 0$$

is satisfied for any $\delta > 0$. From this will follow that w has at each point $s \in (t',t'')$ a strong derivative, equal to zero, and thus w is seen to be independent of s.

By the definition of the function w and by (11) we obtain

$$w(s) - w(s+\varepsilon) = \int_s^{s+\varepsilon} d\, E_t f \;-\; \int_s^{s+\varepsilon} \varphi(t)\, d\, E_t h =$$

$$= \int_s^{s+\varepsilon} d_t\, E_t(A-sE)g(s) - \int_s^{s+\varepsilon} (\varphi(t)-\varphi(s))\, d\, E_t h =$$

$$= \int_s^{s+\varepsilon} (t-s)\, d_t\, E_t g(s) - \int_s^{s+\varepsilon} (\varphi(t)-\varphi(s))\, d\, E_t \dot{h}.$$

Therefore

$$\begin{aligned} \| w(s)-w(s+\varepsilon)\| \leqq |\varepsilon|\, \|(E_{s+\varepsilon}-E_s)g(s)\| &+ \\ +|\varepsilon|\, |\varphi'(s+\Theta\varepsilon)|\, \|(E_{s+\varepsilon}-E_s)h\| & \end{aligned} \tag{14}$$

where $0 < \Theta < 1$. The statement (a) follows directly from the foregoing inequality. Moreover, taking $\varepsilon = -\delta < 0$, we write (14) in the form

$$(1/\delta)\, \| w(s-\delta)-w(s)\| \leqq \|(E_s - E_{s-\delta})g(s)\| +$$

$$+ |\varphi'(s-\Theta\delta)|\, \|(E_s - E_{s-\delta})h\|$$

from which, because of $E_{t-0} = E_t$, follows the second relation in statement (b). For the verification of the first relation in statement (b) we use the continuity of the function w and we write (14), taking $\varepsilon = \delta > 0$, in the form

$$(1/\delta)\, \| (w(s+\delta)-w(s)\| \leqq \|(E_{s+\delta} - E_{s+0})g(s)\| +$$

$$+ |\varphi'(s+\Theta\delta)|\, \|(E_{s+\delta} - E_{s+0})h\|.$$

The right side of this inequality tends to zero with δ.

In view of what has been established, we have that $w = 0$ and the proposition is proved.

The foregoing proposition can be extended to the following

THEOREM: <u>Let D be a closed symmetric operator in a Hilbert space</u> $\mathfrak{H}$, <u>and let</u> E_t <u>be a spectral function of D. Suppose that given any elements</u> $f, h \in \mathfrak{H}$ <u>there exists a function</u> $g : (-\infty, \infty) \longrightarrow \mathfrak{D}_D$ <u>and a differentiable function</u> φ <u>on</u> $(-\infty, \infty)$ <u>such that</u>

$$f - \varphi(t)h = (D - tE)g(t).$$

Then

$$f = \int_{-\infty}^{\infty} \varphi(t)\, d\, E_t h \quad \underline{\text{and}} \quad \langle f,f\rangle = \int_{-\infty}^{\infty} |\varphi(t)|^2\, d\langle E_t h, h\rangle .$$

Proof: If D was self-adjoint, the theorem would at once follow from proposition X by taking $a = -\infty$ and $b = \infty$. In (13) we would let $t' \rightarrow -\infty$ and $t'' \rightarrow \infty$. Since the E_t's are orthogonal when D is self-adjoint, it would be immediate that $\langle f,f\rangle$ is of the form claimed in the theorem.

We assume that D is a closed symmetric operator. In this case D has a self-adjoint extension of type 2 (see page 98). By the theorem on page 110 for every spectral function E_t of D we can construct a self-adjoint operator D^+ in some enlarged Hilbert space $\mathfrak{H}^+ \supset \mathfrak{H}$ such that

$$D^+ f = Df \quad (f \in \mathfrak{D}_D) \quad \text{and} \quad E_t = P^+ E_t^+ \quad (-\infty < t < \infty),$$

where E_t^+ is the spectral function of D^+ and P^+ denotes the projection operator of $\mathfrak{H}^+$ onto $\mathfrak{H}$.

By what has been proved already, we have

$$f = \int_{-\infty}^{\infty} \varphi(t)\, d\, E_t^+ h \quad \text{and} \quad \langle f,f\rangle = \int_{-\infty}^{\infty} |\varphi(t)|^2\, d\langle E_t^+ h,h\rangle .$$

Applying the operator P^+ we can write these two equations in the form

$$f = P^+ f = \int_{-\infty}^{\infty} \varphi(t)\, d\, P^+ E_t^+ h = \int_{-\infty}^{\infty} \varphi(t)\, d\, E_t h$$

and

$$\langle f,f\rangle = \int_{-\infty}^{\infty} |\varphi(t)|^2\, d\langle E_t h,h\rangle$$

because

$$\langle E_t^+ h,h\rangle = \langle E_t^+ h,P^+ h\rangle = \langle P^+ E_t^+ h,h\rangle = \langle E_t h,h\rangle .$$

This completes the proof of the theorem.

3.3 Extension of a Positive Symmetric Operator

We commence with a lemma on norm-invariance.

LEMMA I: <u>Any bounded Hermitian operator A such that $\mathcal{D}_A \subset \mathfrak{H}$ has at least one self-adjoint extension $\widetilde{A}$ whose norm equals the norm of A</u>.

Proof: Since A is bounded, we may assume without loss of generality that $\|A\| = 1$. For any $f \in \mathfrak{H}$ and $g \in \mathcal{D}_A$ we have

$$|\langle Ag,f\rangle| \leqq \|A\| \, \|g\| \, \|f\| = \|g\| \, \|f\|$$

and therefore $\langle Ag,f\rangle$ is a linear continuous functional on $\mathcal{D}_A$. By a theorem of F. Riesz (see § 5, section 3 of Neumark's book on normed algebras) to any $f \in \mathfrak{H}$ there corresponds uniquely a certain element $h \in \mathcal{D}_A$ such that $\langle Ag,f\rangle = \langle g,h\rangle$ for $g \in \mathcal{D}_A$.

For $f \in \mathfrak{H}$, $h \in \mathcal{D}_A$ we put

$$h = A^{o}f.$$

The operator A^{o} is linear and $\|A^{o}f\| \leqq \|f\|$ for all $f \in \mathfrak{H}$ because

$$\|A^{o}f\|^{2} = \langle A^{o}f,A^{o}f\rangle = \langle AA^{o}f,f\rangle \leqq \|A^{o}f\| \, \|f\|$$

for all $f \in \mathfrak{H}$.

We denote by P the projection operator onto the closure of $\mathcal{D}_A$. Then

$$\langle A^{o}g,h\rangle = \langle g,Ah\rangle = \langle Ag,h\rangle = \langle PAg,h\rangle$$

for $g,h \in \mathcal{D}_A$ and

$$A^{o}g = PAg \quad \text{for} \quad g \in \mathcal{D}_A . \tag{1}$$

We introduce in $\mathcal{G}$ a new inner product:

$$\langle g,f\rangle_1 = \langle g,f\rangle - \langle A^{o}g,A^{o}f\rangle \quad \text{for} \quad g,f \in \mathcal{G} .$$

The corresponding norm is then given by

$$\| f \|_1^2 = \| f \|^2 - \| A^{o}f \|^2 .$$

The new inner product satisfies all reqirement associated with an inner product except perhaps that from $\| f \|_1 = 0$ it may not follow that $f = 0$. We say that the elements $g,f \in \mathcal{G}$ are e q u i v a l e n t, $g \sim f$ in notation, if $\| g-f \|_1 = 0$. The transitivity of the relation $\sim$ follows from the inequality

$$\| g-h \|_1 \leqq \| g-f \|_1 + \| f-h \|_1 .$$

It is easy to see that we are dealing with an equivalence relation; the elements of $\mathcal{G}$ are split into equivalence classes. We denote by $\hat{g}$ the class of elements equivalent to the element $g \in \mathcal{G}$. The set of classes of equivalent elements forms an incomplete Hilbert space if we define in it

$\alpha \hat{g} + \beta \hat{f} = \hat{h}$ $(h = \alpha g + \beta f)$ and give the inner product as

$$\langle \hat{g}, \hat{f} \rangle = \langle g, f \rangle_1 .$$

We observe that $\|\hat{g}\| = \|g\|_1$. The completion of the set of all classes of equivalent elements relative to the metric induced by the introduced inner product we shall denote by $\hat{\mathfrak{H}}$. The completion of the set of classes $\hat{g}$ for all $g \in \mathfrak{D}_A$ forms a subspace $\hat{\mathfrak{D}} \subset \hat{\mathfrak{H}}$.

On $\mathfrak{D}_A$ we define the operator $B = A - A^o$; hence by (1) we have $Bg \in \mathfrak{N}$ for $g \in \mathfrak{D}_A$, where $\mathfrak{N} = \mathfrak{H} \ominus \mathfrak{D}_A$. Since

$\|Bg\|^2 = \|Ag\|^2 - \|A^o g\|^2 \leqq \|g\|^2 - \|A^o g\|^2$

we get for $g \in \mathfrak{D}_A$:

$$\|Bg\| \leqq \|g\|_1 . \tag{2}$$

By the foregoing inequality we have that $g \sim h$ $(g, h \in \mathfrak{D}_A)$ implies $Bg = Bh$. Thus we can define on a dense set in $\hat{\mathfrak{D}}$ an operator B':

$$B' \hat{g} = Bg \quad (g \in \mathfrak{D}_A,\ B' \hat{g} \in \mathfrak{N})$$

for which inequality (2) assumes the form

$$\|B' \hat{g}\| \leqq \|\hat{g}\| \quad (g \in \mathfrak{D}_A). \tag{3}$$

From (3) we have that the sequence

$$B' \hat{g}_n = Bg_n \qquad (g_n \in \mathfrak{D}_A,\ n = 1,2, \dots)$$

converges in $\mathfrak{H}$ to a certain element from $\mathfrak{M} \subset \mathfrak{H}$ as the sequence $\hat{g}_n$ converges in $\hat{\mathfrak{H}}$ to a certain element from $\hat{\mathfrak{D}}$. Closing the operator B' we get the operator $\hat{B}$, defined on all of $\hat{\mathfrak{D}}$ ($\subset \hat{\mathfrak{H}}$) with values belonging to $\mathfrak{M}$ ($\subset \mathfrak{H}$); hence by (3) we will have

$$\| \hat{B} \hat{g} \| \leqq \| \hat{g} \| \qquad (\hat{g} \in \hat{\mathfrak{D}}). \tag{4}$$

Let $\hat{P}$ denote the orthogonal projection operator onto $\hat{\mathfrak{D}}$ in the space $\hat{\mathfrak{H}}$. Define on the entire space $\mathfrak{H}$ the operator

$$B^o f = \hat{B} \hat{P} \hat{f} \qquad (f \in \mathfrak{H}).$$

Then inequality (4) gives

$$\| B^o f \| = \| \hat{B} \hat{P} \hat{f} \| \leqq \| \hat{P} \hat{f} \| \leqq \| \hat{f} \| = \| f \|_1 .$$

which means that for $f \in \mathfrak{H}$:

$$\| B^o f \|^2 \leqq \| f \|^2 - \| A^o f \|^2 . \tag{5}$$

We now consider the operator $A_1 = A^o + B^o$. From (5) we get that $\| A_1 \| \leqq 1$ and therefore $\| A_1^* \| \leqq 1$.

Since $\hat{P} \hat{g} = \hat{g}$ for $g \in \mathfrak{D}_A$, we have that $B^o g = Bg$ $(g \in \mathfrak{D}_A)$ and from the definition of the operator B it follows that

$$A_1 g = Ag \qquad (g \in \mathfrak{D}_A).$$

For all $f \in \mathcal{H}$, $g \in \mathcal{D}_A$ we have

$$\langle A_1^* g, f\rangle = \langle g, A_1 f\rangle = \langle g, A^0 f + B^0 f\rangle.$$

Since $B^0 f \in \mathfrak{N}$ and $\langle g, B^0 f\rangle = 0$, we obtain

$$\langle A_1^* g, f\rangle = \langle g, A^0 f\rangle = \langle Ag, f\rangle.$$

We therefore conclude that

$$A_1^* g = Ag \qquad (g \in \mathcal{D}_A).$$

We have shown that the operators A_1 and A_1^* are extensions of the operator A which preserve norm. Hence the self-adjoint operator

$$\tilde{A} = (A_1 + A_1^*)/2$$

is an extension of A and $\| \tilde{A} \| = 1$. The lemma is proved.

<u>A Hermitian operator H is called **positive**</u> ($H > 0$ <u>in notation) if</u> $\langle Hf, f\rangle \geqq 0$ $(f \in \mathcal{D}_H)$ <u>and if for at least one</u> $g \in \mathcal{D}_H$ <u>we have</u> $\langle Hg, g\rangle > 0$.

Let S and T be two bounded self-adjoint operators; we will write $S > T$ and say that S is <u>larger</u> than T if the operator $(S-T)$ is positive.

Let H denote a bounded self-adjoint positive operator. Then there exists one and only one bounded self-adjoint positive operator B such that $B^2 = H$. We denote B by $H^{\frac{1}{2}}$.
Indeed, if E_t is the spectral function of H, then $E_t = 0$ for $t < 0$ because H is positive. We denote by $\mathfrak{D}_B$ the set of all vectors f for which

$$\int_0^\infty t \, d\|E_t f\|^2 < \infty$$

and put for such f

$$H^{\frac{1}{2}} f = Bf = \int_0^\infty \sqrt{t} \; d\, E_t f \,.$$

The operator B is positive, self-adjoint and satisfies $B^2 = H$. The uniqueness of B follows from the uniqueness of the spectral function E_t. From the construction of B it is also clear that it commutes with all those bounded operators with which H commutes. In the Appendix (see page 179) we consider another method of proof which makes no use of the spectral theorem for self-adjoint operators.

Let A denote a bounded Hermitian operator with $\|A\| \leqq 1$ and closed domain of definition $\mathfrak{D}_A \neq \mathfrak{H}$. We denote by $\mathfrak{L}(A)$ the set of self-adjoint extensions of the operator A whose norm does not exceed 1. By Lemma I the set $\mathfrak{L}(A)$ is not empty.
The difference C of any two self-adjoint extensions of the operator A is a self-adjoint operator vanishing on $\mathfrak{D}_A$.

The range of the operator C will be contained in the orthogonal complement $\mathfrak{N}$ of $\mathfrak{D}_A$ because $\langle Cg, f\rangle = 0$ ($g \in \mathfrak{D}_A$, $f \in \mathfrak{H}$) implies that $\langle g, Cf\rangle = 0$ and the latter means that $Cf \in \mathfrak{N}$. Thus, if $\tilde{A}$ is one of the operators of the set $\mathcal{L}(A)$, then all operators of the set will have the form $\tilde{A} + C$, where C has values in $\mathfrak{N}$ and the condition

$$\left|\langle \tilde{A} f + Cf, f\rangle\right| \leqq \langle f, f\rangle \qquad (f \in \mathfrak{H})$$

holds. The foregoing condition is equivalent with

$$-\langle (E + \tilde{A})f, f\rangle \leqq \langle Cf, f\rangle \leqq \langle (E - \tilde{A})f, f\rangle \qquad (f \in \mathfrak{H})$$

or

$$-(E + \tilde{A}) \leqq C \leqq E - \tilde{A}. \tag{6}$$

We note that the operators $E - \tilde{A}$ and $E + \tilde{A}$ are positive.

LEMMA II: <u>Let $\mathfrak{N}$ be some subspace of $\mathfrak{H}$ and H a positive operator. Then the set $\mathfrak{M}$ of self-adjoint operators C satisfying the condition $C \leqq H$ and $\mathfrak{R}_C \subset \mathfrak{N}$ has a largest element $H_{\mathfrak{N}}$, that is an operator larger than any other operator C of the set.</u>

Proof: The set $\mathfrak{M}$ is not empty since at least the operators 0 and $P_{\mathfrak{N}}$ belong to it; $P_{\mathfrak{N}}$ denotes the orthogonal projection operator onto $\mathfrak{N}$.

We denote by $\mathfrak{D}$ the orthogonal complement of $\mathfrak{N}$ in $\mathfrak{H}$.

Then $\langle Cg,g\rangle = 0$ ($g \in \mathcal{D}$) and $\langle Cf,g\rangle = 0$ ($f \in \mathcal{G}$, $g \in \mathcal{D}$) and therefore $\langle Cf,f\rangle = \langle C(f-g),f-g\rangle$. Since the operator C is not larger than the operator H, we get

$$\langle Cf,f\rangle \leqq \langle H(f-g),f-g\rangle = \| H^{\frac{1}{2}}(f-g)\|^2 \quad (f \in \mathcal{G}\,;\, g \in \mathcal{D})$$

and

$$\langle Cf,f\rangle = \inf_{g \in \mathcal{D}} \| H^{\frac{1}{2}}f - H^{\frac{1}{2}}g \|^2 . \tag{7}$$

We denote by $\mathcal{L}$ the set of all elements h which are orthogonal to $H^{\frac{1}{2}}\mathcal{D}$ and by $P_{\mathcal{L}}$ the orthogonal projection operator onto $\mathcal{L}$. From (7) we get for $f \in \mathcal{G}$:

$$\langle Cf,f\rangle \leqq \| P_{\mathcal{L}} H^{\frac{1}{2}}f \|^2 = \langle H^{\frac{1}{2}}P_{\mathcal{L}} H^{\frac{1}{2}}f,f\rangle. \tag{8}$$

However $h \in \mathcal{L}$ if and only if $\langle h,H^{\frac{1}{2}}g\rangle = 0$ ($g \in \mathcal{D}$), that is $\langle H^{\frac{1}{2}}h,g\rangle = 0$, and therefore $H^{\frac{1}{2}}h \in \mathfrak{N}$. It follows that

$$H_{\mathfrak{N}} = H^{\frac{1}{2}} P_{\mathcal{L}} H^{\frac{1}{2}}$$

belongs to the set $\mathfrak{M}$ and is the sought for largest operator. The lemma is proved.

Using the notation of Lemma II, from relation (6) on the last page we see that

$$-(E + \tilde{A})_{\mathfrak{N}} \leqq C \leqq (E - \tilde{A})_{\mathfrak{N}} \quad (\mathfrak{N} = \mathcal{G} \ominus \mathcal{D}_A) \tag{10}$$

and that all self-adjoint extensions of the operator A which belong to the set $\mathcal{L}(A)$ contain the smallest extension, A_m, and the largest extension, A_M, where

$$A_m = \tilde{A} - (E + \tilde{A})_{\mathfrak{N}}$$

$$A_M = \tilde{A} + (E - \tilde{A})_{\mathfrak{N}} \quad . \tag{11}$$

We have therefore shown the necessity of the condition in the following

LEMMA III: <u>In order that a self-adjoint operator</u> A_1 <u>belong to the set</u> $\mathcal{L}(A)$ <u>it is necessary and sufficient that the following condition be satisfied</u>:

$$A_m \leqq A_1 \leqq A_M \; . \tag{12}$$

Proof: It remains to establish the sufficiency of the condition (12).

From (12) we get that $\| A_1 \| \leqq \max (\|A_m\|, \|A_M\|) \leqq 1$ and that the operator $(A_1 - A_m)$ is not larger than the operator $(A_M - A_m)$. Thus for $g \in \mathfrak{D}_A$:

$$0 \leqq \langle (A_1 - A_m)g, g \rangle \leqq \langle (A_M - A_m)g, g \rangle = 0$$

and therefore

$$\langle (A_1 - A_m)g, g \rangle = \| (A_1 - A_m)^{\frac{1}{2}} g \|^2 = 0$$

and

$$A_1 g = A_m g = Ag \qquad (g \in \mathfrak{D}_A).$$

The lemma is established.

From what has been discussed and considering relation (10) it can be seen that some self-adjoint extension $\tilde{A}$ of a bounded Hermitian operator A is the unique self-adjoint extension with norm $\leqq 1$ if and only if $(E + \tilde{A})_{\mathfrak{N}} = (E - \tilde{A})_{\mathfrak{N}} = 0$.

LEMMA IV: Let an operator A be a fractional-linear transformation of a symmetric operator R:

$$A = (aR + bE)(cR + dE)^{-1}$$

where a, b, c, d are real numbers and ad-bc $\neq 0$. Then given any self-adjoint extension $\tilde{A}$ of A there is a self-adjoint extension $\tilde{R}$ of R given by

$$\tilde{R} = (d\tilde{A} - bE)(-c\tilde{A} + aE)^{-1} .$$

Proof: By propositions III (see page 87), V (page 89), and VIII (page 91) it is enough to show that $c\tilde{A}f - af = 0$ implies $f = 0$. Now let $c\tilde{A}f - af = 0$, then $\langle c\tilde{A}f-af,h\rangle = 0$ for $h \in \mathfrak{H}$. In particular we have $\langle c\tilde{A}f-af,w\rangle = 0$, where $w \in \mathfrak{D}_A$. Therefore $\langle f,c\tilde{A}w-aw\rangle = \langle f,cAw-aw\rangle = 0$ which means that $\langle f,g\rangle = 0$ for $g \in \mathfrak{D}_R$. But $\mathfrak{D}_R$ is dense in $\mathfrak{H}$ as R is a symmetric operator. It follows that $f = 0$ and the lemma is proved.

Let a Hermitian operator S satisfy the condition: $S > 0$. Then for any $a > 0$ we have

$$\| Sf + aE \|^2 = \| Sf \|^2 + 2a \langle Sf, f \rangle + a^2 \| f \|^2 \geqq a^2 \| f \|^2 .$$

Thus for positive symmetric operators the set of points of regular type is connected (containing all non-real and positive numbers); it therefore follows that the deficiency numbers are equal. In section 1 of this chapter we have seen that a symmetric operator has self-adjoint extensions if and only if its deficiency numbers are equal.

Our aim is to establish an interesting theorem concerning the extension of positive symmetric operators due to M. H. Stone (see his book, Thm. 9.21), K. O. Friedrichs and H. Freudenthal which reads:

THEOREM: <u>Each positive symmetric operator S has at least one positive self-adjoint extension $\widetilde{S}$</u>.

Proof: Consider the fractional-linear transformation of the operator S:

$$A = (E - S)(E + S)^{-1}$$

$$S = (E - A)(E + A)^{-1}.$$

The operator A exists because $f + Sf = 0$ implies $f = 0$ in view of the fact that $\langle f,f \rangle \leqq \langle f,f \rangle + \langle Sf,f \rangle = \langle f + Sf, f \rangle$ holds for $f \in \mathcal{D}_S$. By proposition V on page 89 the operator A is Hermitian.

Using formulas (10) and (9) given on page 86 we obtain

$$g = f + Sf, \quad Ag = f - Sf \qquad (f \in \mathfrak{D}_S,\ g \in \mathfrak{D}_A).$$

Since S is positive we have

$$\langle Ag, Ag\rangle = \langle f - Sf, f - Sf\rangle = \|f\|^2 - 2\langle Sf, f\rangle + \|Sf\|^2 \leqq$$

$$\leqq \|f\|^2 + 2\langle Sf, f\rangle + \|Sf\|^2 = \langle f + Sf, f + Sf\rangle = \langle g, g\rangle$$

and therefore the operator A is bounded: $\|A\| \leqq 1$.

By Lemma I (see page 120) the operator A has at least one self-adjoint extension $\widetilde{A}$ with norm $\leqq 1$. By Lemma IV the self-adjoint operator

$$\widetilde{S} = (E - \widetilde{A})(E + \widetilde{A})^{-1}$$

will be a self-adjoint extension of the operator S. Since $\|\widetilde{A}\| \leqq 1$, the operator $\widetilde{S}$ is positive; if $f \in \mathfrak{D}_{\widetilde{S}}$ we find an element $g \in \mathfrak{D}_{\widetilde{A}}$ such that

$$f = g + \widetilde{A}g \quad \text{and} \quad \widetilde{S}f = g - \widetilde{A}g$$

and therefore

$$\langle \widetilde{S}f, f\rangle = \langle g - \widetilde{A}g, g + \widetilde{A}g\rangle = \|g\|^2 - \|\widetilde{A}g\|^2 \geqq 0.$$

This proves the theorem.

3.4 Examples of Kernel Spaces and Applications

Let V denote a vector space whose elements are certain complex-valued continuous functions on $(-\infty, \infty)$; the algebraic operations in V are defined as usual. Let W denote the linear hull of the set of all functions $f\,\overline{g}$, where $f, g \in V$.

We shall call a linear (i.e., additive and homogeneous) functional Ψ acting on W positive if for $f \in V$ we have that $\Psi(f\,\overline{f}) \geqq 0$.

We observe that the form $\langle f,g\rangle = \Psi(f\,\overline{g})$ acts like an inner product, except that $\langle f,f\rangle = 0$ does not imply that $f = 0$. To see this we only have to verify that $\langle f,g\rangle = \overline{\langle g,f\rangle}$ holds for $f, g \in V$. Taking any complex number λ and expanding

$$\Psi((f+\lambda g)\,\overline{(f+\lambda g)}) \geqq 0$$

we obtain that the expression

$$\lambda\,\Psi(g\,\overline{f}) + \overline{\lambda}\,\Psi(f\,\overline{g})$$

is a real number. Choosing $\lambda = 1$ and $\lambda = i$, successively, we get that

$$\Psi(g\,\overline{f}) = \overline{\Psi(f\,\overline{g})} \quad \text{or} \quad \Psi(f\,\overline{g}) = \overline{\Psi(g\,\overline{f})} .$$

Since

$$0 \leqq \langle f-\lambda g, f-\lambda g\rangle = \langle f,f\rangle - \lambda\langle g,f\rangle - \overline{\lambda}\langle f,g\rangle + |\lambda|^2\langle g,g\rangle$$

we arrive at the inequality of Schwartz:

$$|\langle f,g\rangle|^2 \leqq \langle f,f\rangle\langle g,g\rangle ;$$

indeed, if $\langle g,f\rangle = 0$, then the Schwartz inequality holds trivially and if $\langle g,f\rangle \neq 0$, then we may take $\lambda = \langle f,f\rangle / \langle g,f\rangle$.

In view of the foregoing observation we are permitted to define in V the inner product

$$\langle f,g\rangle = \Psi(f\,\overline{g});$$

we identify in V any two vectors f and g for which

$$\langle f-g, f-g\rangle = 0.$$

Completion of V with respect to the norm induced by this inner product gives a Hilbert space $\mathfrak{H}$ in which V is dense.

Given $g \in V$ let $\hat{g}$ be the function defined by

$$\hat{g}(s) = sg(s), \quad -\infty < s < \infty .$$

We impose on V the following <u>restriction</u>: We require that the set F of all functions $g \in V$ for which $\hat{g}$ belongs to V be dense in V and therefore dense in $\mathfrak{H}$.

Under these conditions we can introduce in $\mathfrak{H}$ the operator of multiplication by the independent variable s, having defined it initially on the set F. This operator is symmetric since the set F is dense in $\mathfrak{H}$ and

$$\langle f, \hat{g} \rangle = \Psi(f \overline{\hat{g}}) = \langle \hat{f}, g \rangle$$

for any $f, g \in F$. Therefore we can close the operator in question; this closure we shall denote by the operator D. To see that the operator D is well defined we have to show that $\langle f, f \rangle = 0$ implies $\langle \hat{f}, \hat{g} \rangle = 0$. It is sufficient to show this for $f \in F$. However $\langle f, f \rangle = 0$ implies that $\langle f, g \rangle = 0$ for any $g \in V$ and in particular for any $g = \hat{h}$, $h \in F$. Thus for any $h \in F$ we have

$$\langle \hat{f}, h \rangle = \langle f, \hat{h} \rangle = 0.$$

Hence, by virtue of the fact that F is dense in V, the required equation $\langle \hat{f}, \hat{f} \rangle = 0$ follows.

LEMMA: A sufficient condition that a given positive functional Ψ defined on W permit an integral representation of the form

$$\Psi(\phi) = \int_{-\infty}^{\infty} \phi(t)\, d\,\sigma(t) \qquad (\phi \in W)$$

with non-decreasing bounded function σ on $(-\infty, \infty)$ is that the following threefold requirement be satisfied: The function 1 belong to V, the set F of all functions $g \in V$ for which $\hat{g} \in V$ be dense in V with the norm

$$\|f\| = (\Psi(f\,\bar{f}))^{\frac{1}{2}} \qquad (f \in V),$$

and from $f \in V$ it follow that for any real t the function of s

$$(f(s)-f(t))/(s-t)$$

belong to V.

Proof: We construct the Hilbert space $\mathfrak{H}$ as was done further above and in it we introduce the closed symmetric operator D of multiplication by the independent variable s.
We note that the function f appears in two roles. On the one hand f is an element of the space $\mathfrak{H}$ and on the other hand f is an element of the space L^2_σ of square-integrable functions with weight function σ.
We wish to apply the theorem stated on page 118. To facilitate the presentation we rewrite the condition $f-\varphi(t) = (D-tE)g(t)$ appearing in the mentioned theorem as follows:

$$f - \varphi(t)h = (D-tE)g_t.$$

We note that $g_t \in \mathfrak{D}_D$. We take as element h the function $1 \in V$ and as value of the function φ at t the value of $f \in V$ at t. The requirement $f-\varphi(t)h = (D-tE)g_t$ then assumes the form

$$f(s)-f(t)1 = (s-t)g_t(s)$$

and is satisfied because both functions of s

$$g_t(s) = (f(s)-f(t))/(s-t), \qquad sg_t(s) = tg_t(s) + f(s)-f(t)1$$

belong to V. In the foregoing we chose to indicate by writing $f(s)$ that $f \in \mathfrak{H}$. This completes the proof of the lemma.

THEOREM (S. Bochner): Let k denote a continuous function on $(-A,A)$, $0 < A \leqq \infty$. In order that the representation

$$k(x) = \int_{-\infty}^{\infty} e^{ixt}\, d\sigma(t) \qquad (-A < x < A)$$

holds, it is necessary and sufficient that k be a positive definite function, i.e., for any numbers $0 \leqq x_1 < \dots < x_n < A$ $(n < \infty)$ and any complex numbers $\lambda_1, \dots, \lambda_n$ we have

$$\sum_{j,m=1}^{n} k(x_j - x_m)\, \lambda_j \overline{\lambda_m} \geqq 0.$$

Proof: To see the necessity of the condition we observe that

$$\sum_{j,m=1}^{n} k(x_j - x_m)\, \lambda_j \overline{\lambda_m} = \int_{-\infty}^{\infty} \Big| \sum_{j=1}^{n} \lambda_j e^{ix_j t} \Big|^2 d\sigma(t) \geqq 0.$$

To verify the sufficiency of the condition we make use of the foregoing lemma. For V we take the set of all functions of the form

$$f(t) = \int_{\alpha}^{\beta} e^{itx}\, d\rho(x),$$

where $[\alpha, \beta] \subset (-A,A)$ and ρ is an arbitrary, in general complex function of bounded variation. We then construct W and for a start define the functional Ψ by

$$\Psi(e^{ix_1 t}\, e^{-ix_2 t}) = k(x_1 - x_2) \qquad (x_1, x_2 \in (-A,A)).$$

We verify that the conditions stated in the lemma are fulfilled. It is clear that $1 \in V$. As the set F we take the family of functions of the form

$$v(t) = \int_{\alpha}^{\beta} e^{itx} g(x)\, dx,$$

where g is continuous. To show that the set F is dense in V with norm $\|f\| = (\Psi(f\bar{f}))^{\frac{1}{2}}$, it suffices to establish that the function $f_0(t) = e^{ict}$, for $c \in (-A,A)$, can be approximated by a function $v \in F$. For this purpose we take

$$v_0(t) = (c_2-c_1)^{-1} \int_{c_1}^{c_2} e^{itx}\, dx \in F,$$

where $c \in (c_1,c_2)$ and c_2-c_1 is as small as we please. We get

$$\Psi((f_0-v_0)\overline{(f_0-v_0)}) =$$

$$= (c_2-c_1)^{-2} \int_{c_1}^{c_2} \int_{c_1}^{c_2} \Psi((e^{ict}-e^{itx})(e^{-ict}-e^{-ity}))\, dx\, dy =$$

$$= (c_2-c_1)^{-2} \int_{c_1}^{c_2} \int_{c_1}^{c_2} (k(0)-k(c-y)-k(x-c)+k(x-y))\, dx\, dy$$

and it remains to take into account the continuity of the function k. To verify the last part of the requirement in the lemma it suffices also to do this for the function e^{ict} for any $c \in (-A,A)$. But we have

$$(e^{ict}-e^{ics})/(s-t) = i \int_{0}^{c} e^{ixt} e^{is(c-x)}\, dx \in V.$$

Remark:

In the foregoing we based the proof of the representation theorem concerning positive definite functions on the lemma stated on page 134. In a completely analogous manner one can deduce from the same lemma a theorem of S. N. Bernstein concerning the representation of exponentially convex functions. The theorem in question states:

Let k be a continuous function on the interval (A_1, A_2), *where* $-\infty \leqq A_1 < A_2 \leqq \infty$ *and* $A_1 < 0 < A_2$. *In order that*

$$k(x) = \int_{-\infty}^{\infty} e^{xt}\, d\sigma(t) \qquad (A_1 < x < A_2)$$

hold, where σ *is a certain non-decreasing bounded function on* $(-\infty, \infty)$, *it is necessary and sufficient that for* $x_1, \ldots, x_n$, *where* $A_1 < x_1 < \ldots < x_n < A_n$, $(n < \infty)$ *and any complex numbers* $\lambda_1, \ldots, \lambda_n$ *the relation*

$$\sum_{j,m=1}^{n} k(x_j + x_m)\, \lambda_j \overline{\lambda_m} \geqq 0$$

hold.

We give one more application of the lemma.

THEOREM (H. Hamburger): <u>Let $(a_m)_{m=0}^{\infty}$ be a sequence of real numbers. A necessary and sufficient condition that</u>

$$a_m = \int_{-\infty}^{\infty} t^m \, d\sigma(t) \qquad (m = 0,1,2,\dots)$$

<u>where σ denotes a certain non-decreasing bounded function is the requirement</u>:

$$\sum_{j,k=0}^{n} a_{j+k} \lambda_j \overline{\lambda_k} \geqq 0$$

Proof: We show the sufficiency of the condition. Suppose that V consists of all polynomials in the indeterminate t, then the unit function belongs to V. We apply the lemma given on page 134. We observe that the set W coincides with V and the conditions of the mentioned lemma are fulfilled. Hence there exists a non-decreasing bounded function σ on $(-\infty, \infty)$ so that

$$\Psi(t^m) = \int_{-\infty}^{\infty} t^m \, d\sigma(t) \qquad (m = 0,1,2,\dots)$$

However, any sequence of reals $(a_m)_{m=0}^{\infty}$ defines on V a linear functional Ψ when we let $\Psi(t^m) = a_m$ $(m = 0,1,2,\dots)$ and this linear functional will be positive on $W = V$ if for $n < \infty$:

$$\sum_{j,k=0}^{n} a_{j+k} \lambda_j \overline{\lambda_k} = \Psi\left(\Big| \sum_{j=0}^{n} \lambda_j t^j \Big|^2 \right) \geqq 0.$$

It is easy to verify the necessity of the condition.

Let $\mathcal{E}$ be some Hilbert space in which the inner product of two vectors u and v is denoted by $\langle u,v \rangle$. Let with each point t of some closed interval $[a,b]$ (finite or infinite) be associated a certain bounded self-adjoint operator F_t, acting in $\mathcal{E}$. We call F_t a non-decreasing operator function if

$$F_{t_1} \leqq F_{t_2} \quad \text{for } t_1 < t_2 \quad (t_1, t_2 \in [a,b]).$$

For a non-decreasing operator function F_t the limit operators F_{t-0} and F_{t+0} exist in the sense that for any $u \in \mathcal{E}$ we have

$$F_{t-0}u = \lim_{s \uparrow t} F_s u \,, \qquad F_{t+0}u = \lim_{s \downarrow t} F_s u \,.$$

We shall consider integrals of the form

$$J = \int_a^b g(t) \, d\, F_t \,,$$

where g is some continuous function on $[a,b]$, and assume that $F_a = 0$ and $F_t = F_{t-0}$ for $a < t < b$. We shall understand the above integral in the sense that

$$\langle Ju,v \rangle = \int_a^b g(t) \, d \langle F_t u,v \rangle \qquad (u,v \in \mathcal{E}).$$

As illustration of the analytical significance of the lemma on page 120 and the theorem on page 130 we shall prove two theorems.

THEOREM: <u>In order that the sequence</u> $S_0, S_1, \ldots, S_{2n}$ <u>of bounded self-adjoint operators in a Hilbert space</u> $\mathcal{E}$ <u>satisfy the representation</u>

$$S_k = \int_{-1}^{1} t^k \, d F_t \qquad (k = 0,1,\ldots,2n) \tag{1}$$

<u>it is necessary and sufficient that for any</u> $x_i \in \mathcal{E}$ $(i = 0,1,\ldots,2n)$ <u>the conditions</u>

$$\sum_{j,k=0}^{n} \langle S_{j+k} x_j, x_k \rangle \geqq 0 \tag{I}$$

$$\sum_{j,k=0}^{n-1} \langle (S_{j+k} - S_{j+k+2}) x_j, x_k \rangle \geqq 0 \tag{II}$$

<u>hold</u>.

Proof: From the integral representation we obtain

$$\sum_{j,k=0}^{n} \langle S_{j+k} x_j, x_k \rangle = \int_{-1}^{1} \sum_{j,k=0}^{n} t^{j+k} \, d \langle F_t x_j, x_k \rangle =$$

$$= \lim_{N \to \infty} \sum_{m=1}^{N} \langle \Delta_m F_t \, y_m , y_m \rangle ,$$

where

$$\Delta_m F_t = F_{t_m} - F_{t_{m-1}}, \quad y_m = \sum_{j=0}^{n} t_m^j x_j ,$$

$$t_m = -1 + \frac{2m}{N} \qquad (m = 1,2,\ldots,N).$$

Hence we get the necessity of condition I.

The necessity of condition II follows from a similar consideration:

$$\sum_{j,k=0}^{n-1} \left\langle (S_{j+k} - S_{j+k+2})x_j, x_k \right\rangle =$$

$$= \lim_{N \to \infty} \sum_{m=1}^{N} (1 - t_m^2) \left\langle \Delta_m F_t\, y_m , y_m \right\rangle .$$

Next we show the sufficiency of the conditions I and II. Consider the linear set $\mathcal{P}$ of polynomials of the form

$$P(t) = \sum_{j=0}^{n} t^j x_j \qquad (x_j \in \mathcal{E} \ ; \ j = 0,1,\ldots,n).$$

Let

$$Q(t) = \sum_{k=0}^{n} t^k y_k \qquad (y_k \in \mathcal{E} \ ; \ k = 0,1,\ldots,n)$$

and define on $\mathcal{P}$ a bilinear Hermitian functional (P,Q) by

$$(P,Q) = \sum_{j,k=0}^{n} \langle S_{j+k} x_j , y_k \rangle .$$

By condition I we have for any P

$$(P,P) \geqq 0.$$

Assume first that formula I is strictly positive, i.e., if not all coefficients x_j are zero, then $(P,P) > 0$.
In this case the bilinear functional (P,Q) can be taken as inner product in $\mathcal{P}$. Completion of $\mathcal{P}$ under the norm induced by this inner product produces a certain Hilbert space $\mathcal{G}$.

Denote by $\mathcal{R}$ the set of all polynomials $Q \in \mathcal{P}$ of degree not larger than n-1;

$$Q(t) = \sum_{j=0}^{n-1} t^j y_j \qquad (y_j \in \mathcal{E} \ ; \ j = 0,1,\ldots,n-1).$$

On $\mathcal{R}$ we define an operator A by

$$AQ(t) = tQ(t) = \sum_{j=0}^{n-1} t^{j+1} y_j .$$

But for any $Q_1, Q_2 \in \mathcal{R}$ we have $(tQ_1,Q_2) = (Q_1,tQ_2)$; hence the operator A is Hermitian. Moreover, by condition II,

$$(AQ,AQ) = (tQ,tQ) \leqq (Q,Q) \qquad (Q \in \mathcal{R}). \tag{2}$$

It follows that $\| A \| \leqq 1$.

We now apply the lemma stated on page 120. Let E_t be the spectral function of $\tilde{A}$:

$$\tilde{A} = \int_{-1}^{1} t \, d \, E_t \; .$$

Then for any $x, y \in \mathcal{E}$ we will have

$$\langle S_{j+k} x, y \rangle = (t^j x, t^k y) = (\tilde{A}^j x, \tilde{A}^k y) = (\tilde{A}^{j+k} x, y) =$$
$$= \int_{-1}^{1} t^{j+k} \, d \, (E_t x, y) \qquad (j+k = 0, 1, \ldots, 2n). \tag{3}$$

We note that the operators E_t, $-1 < t < 1$, act in $\mathfrak{H}$ and not in $\mathcal{E}$; since

$$| (E_t x, x) | \leqq (x, x) = \langle S_0 x, x \rangle \leqq \| S_0 \| \langle x, x \rangle ,$$

we have that the bilinear Hermitian functional $(E_t x, y)$ is continuous. For any $x \in \mathcal{E}$ the functional $(E_t x, x)$ is non-decreasing. Using a theorem of F. Riesz (see for example § 5, section 3 of Neumark's book on normed algebras) we get that to each E_t there corresponds a bounded self-adjoint operator F_t, acting in $\mathcal{E}$, such that

$$(E_t x, y) = \langle F_t x, y \rangle \qquad (x, y \in \mathcal{E}); \tag{4}$$

F_t, $-1 < t < 1$, is a nondecreasing operator function. Since (3) and (4) imply (1) the theorem is proved for the case when formula I is strictly positive.

If formula I is not strictly positive, we regard the polynomials $P,Q \in \mathcal{P}$ as equivalent whenever

$$(P-Q,P-Q) = 0$$

is satisfied. By (2) the operator A transforms equivalent polynomials into equivalent polynomials. If we identify equivalent polynomials, then we have reduced the case under consideration to the case considered further above. This completes the proof of the theorem.

THEOREM: <u>In order that the sequence</u> $S_0, S_1, S_2, \ldots$ <u>of bounded self-adjoint operators in a Hilbert space</u> $\mathcal{E}$ <u>have a representation of the form</u>

$$S_k = \int_0^\infty t^k \, d\, F_t \qquad (k = 0,1,2,\ldots), \tag{5}$$

<u>where</u> F_t $(0 \leqq t < \infty)$ <u>is some non-decreasing operator-function, it is necessary and sufficient that for any</u> $u_j \in \mathcal{E}$ $(j = 0,1,2,\ldots)$ <u>the following two conditions hold</u>:

$$\sum_{j,k=0}^{n} \langle S_{j+k} u_j , u_k \rangle \geqq 0 \qquad (n = 0,1,\ldots), \tag{I}$$

$$\sum_{j,k=0}^{n} \langle S_{j+k+1} u_j , u_k \rangle \geqq 0 \qquad (n = 0,1,\ldots), \tag{II}$$

Proof: From the integral representation (5) we see that conditions I and II are necessary:

$$\sum_{j,k=0}^{n} \langle S_{j+k}u_j, u_k \rangle = \lim_{N\to\infty} \lim_{p\to\infty} \sum_{m=1}^{p} \langle \Delta_m F_t v_m, v_m \rangle,$$

$$\sum_{j,k=0}^{n} \langle S_{j+k+1}u_j, u_k \rangle = \lim_{N\to\infty} \lim_{p\to\infty} \sum_{m=1}^{p} t_m \langle \Delta_m F_t v_m, v_m \rangle,$$

where

$$t_m = (mN^2)/p, \quad v_m = \sum_{j=0}^{n} t^j u_j, \quad \Delta_m F_t = F_{t_{m+1}} - F_{t_m}$$

for $m = 1,2,\ldots,p$.

Next we show that conditions I and II are sufficient. We assume first that formula I is strictly positive, i.e.,

$$\sum_{j,k=0}^{n} \langle S_{j+k}u_j, u_k \rangle = 0 \text{ implies } u_j = 0 \text{ for } j = 0,1,\ldots,n.$$

We denote by $\mathcal{R}$ the set of all polynomials of the form

$$P(t) = \sum_{j=0}^{n} t^j u_j \quad (u_j \in \mathcal{E}\ ;\ n = 0,1,\ldots).$$

Let

$$Q(t) = \sum_{k=0}^{m} t^k v_k \quad (v_k \in \mathcal{E}\ ;\ m = 0,1,\ldots).$$

We define

$$(P,Q) = \sum_{j=0}^{n} \sum_{k=0}^{m} \langle S_{j+k} u_j, v_k \rangle. \tag{6}$$

It is straightforward to verify that (P,Q) has all properties of an inner product (here we use that the operators S_j are Hermitian and that formula I is strictly positive). Let $\mathfrak{H}$ be the completion of $\mathfrak{R}$ relative to the norm induced by this inner product. We define on $\mathfrak{R}$ the operator H by

$$HP(t) = tP(t) = \sum_{j=0}^{n} t^{j+1} u_j .$$

Since the operators S_j are Hermitian and since $\mathfrak{R}$ is dense in $\mathfrak{H}$ we have that H is a symmetric operator in $\mathfrak{H}$. From condition II we get

$$(HP,P) = (tP,P) = \sum_{j,k=0}^{n} \langle S_{j+k+1} u_j, u_k \rangle \geqq 0;$$

this means that H is a positive operator.

By virtue of the theorem on page 130 the operator H has a positive self-adjoint extension $\tilde{H}$. Let E_t be the spectral function of the operator $\tilde{H}$:

$$\tilde{H} = \int_0^{\infty} t \, d E_t .$$

Then for any $u,v \in \mathcal{E}$ we have

$$\langle S_k u,v\rangle = (\widetilde{H}^k u , v) = \int_0^\infty t^k \, d\,(E_t u,v) \tag{7}$$

$(k = 0,1,\ldots)$. We note that the operators E_t act in $\mathcal{G}$ but not in $\mathcal{E}$; however $(E_t u,v)$ is a Hermitian bilinear functional in $\mathcal{E}$ and it is continuous:

$$0 \leqq (E_t u,u) \leqq (u,u) = \langle S_0 u,u\rangle \leqq \| S_0 \| \langle u,u\rangle .$$

Thus there corresponds to it a bounded self-adjoint operator F_t, acting in $\mathcal{E}$ and such that

$$(E_t u,v) = \langle F_t u,v\rangle \qquad (u,v \in \mathcal{E}) \tag{8}$$

F_t will be a non-decreasing operator-function. Hence (7) and (8) imply (5).

In case the form I is not strictly positive, we shall say that the polynomials $P,Q \in \mathcal{R}$ are equivalent if and only if

$$(P-Q,P-Q) = 0,$$

where (P,Q) is the bilinear functional defined by (6). The operator H (multiplying by t) maps equivalent polynomials into equivalent polynomials. To see this it suffices to verify that $(P,P) = 0$ implies that $(tP,tP) = 0$. But this is clear from $(tP,tP)^2 = (P,t^2P)^2 \leqq (P,P)(t^2P,t^2P)$. By identification of equivalent polynomials we have reduced the case in which the form I is not strictly positive to the case in which the form I was assumed to be strictly positive and the proof of the theorem is finished.

APPENDIX

Spectral Theorem for Self-adjoint Operators:

For any self-adjoint operator H in a Hilbert space $\mathfrak{H}$ there exists one and only one operator function E_t $(-\infty < t < \infty)$ satisfying

1) E_t is a projection operator;
2) $E_t E_s = E_t$ for $t \leqq s$;
3) E_t permutes with every bounded operator A which permutes with H;
4) $\lim_{t \to -\infty} E_t f = 0, \quad \lim_{t \to \infty} E_t f = f$ for all $f \in \mathfrak{H}$;
5) $E_t f$ is left-continuous for all $f \in \mathfrak{H}$;
6) $f \in \mathfrak{D}_H$ if and only if

$$\int_{-\infty}^{\infty} t^2 \, d\|E_t f\|^2 < \infty .$$

In this case

$$Hf = \int_{-\infty}^{\infty} t \, d\, E_t f .$$

The convergence involved in the integral representation is the strong operator convergence.

The foregoing version of the spectral theorem is due to v. Neumann; before him D. Hilbert had cast the foundation for spectral theory but Hilbert's work in connection with linear integral equations was essentially restricted to the study of completely continuous linear operators (taking bounded sets into sequentially compact sets).

Following ideas of F. Riesz a modernisation of spectral theory was given by H. Freudenthal (Teilweise geordnete Moduln, Proc. Acad. Wet. Amsterdam, 39, 641-651 (1936)) and S. W. P. Steen (An introduction to the theory of operators, Proc. London Math. Soc. 41, 361-392 (1936)). Commencing in 1941, H. Nakano published a long series of papers in which he went substantially beyond his predecessors both in terms of results and in methods. A systematic account of these matters can be found in Nakano's treatises listed in the Bibliography. Here we want to consider the Freudenthal - Steen - Nakano theory in relation to J. von Neumann's version of the spectral theorem for self-adjoint operators. To this end we shall first survey some of the basic information concerning vector lattices and partially ordered rings which is relevant to abstract spectral theory.

1. Let E denote a vector space over the field of real numbers. Suppose that in E there is given a certain set of elements of which it is asserted that they are "larger

than zero"; this is signified by writing $x > \Theta$, where Θ denotes the zero vector in E. We call all elements which are larger than zero and the element Θ positive and denote by E^+ the collection of all positive elements of E. We put $x > y$ if $x-y > \Theta$. As usual, $x \geqq y$ means that either $x > y$ or $x = y$.

The vector space E is said to be a <u>vector lattice</u> if the ordering introduced above satisfies the requirements:

A) if $x > \Theta$, then $x \neq \Theta$;

B) if $x > \Theta$ and $y > \Theta$, then $x+y > \Theta$;

C) for any two elements $x, y \in E$ the supremum, $x \vee y$, exists;

D) if $x > \Theta$ and α is a real larger than 0, then $\alpha x > \Theta$.

Another definition, equivalent to the foregoing one, is the following: A vector space E over the real numbers is called a vector lattice if E is a lattice and the relation $x > y$ implies that (a): $x+z > y+z$ for any $z \in E$ and (b): for any real α larger than 0 we have $\alpha x > \alpha y$.

2. E is called a <u>complete vector lattice (resp. σ-complete vector lattice</u>) if every bounded (resp. bounded and countable) subset of the vector lattice E has a supremum and an infimum.

3. Let E be a vector lattice. If $x_t \in E$ $(t \in T)$ and if $y = \sup(-x_t)$ exists, then $-y = \inf x_t$. In particular we have

$$x \wedge y = -\left[(-x) \vee (-y)\right]. \tag{1}$$

Moreover, in a vector lattice the following identities always hold:

$$(x \vee y) + z = (x + z) \vee (y + z) \tag{2}$$

$$(x \wedge y) + z = (x + z) \wedge (y + z). \tag{3}$$

A special case of formula (2) is:

$$\left[(x - y) \vee \Theta\right] + y = x \vee y. \tag{4}$$

4. Let E be a vector lattice, $x_t \in E$ $(t \in T)$ and $\sup x_t$ exist. Then $y + \sup x_t = \sup (y + x_t)$ for any $y \in E$ and

$$\alpha \sup x_t = \begin{cases} \sup (\alpha x_t) & \text{for any real } \alpha \geqq 0, \\ \inf (\alpha x_t) & \text{for any real } \alpha \leqq 0. \end{cases}$$

Using formula (4), the first part of the foregoing statement and formula (1), we get

$x \vee y = \left[(x-y) \vee \Theta\right] + y = x + y + \left[(-y) \vee (-x)\right] = x + y - x \wedge y.$

Hence we obtain

$$(x \vee y) + (x \wedge y) = x + y. \tag{5}$$

5. Let E be a vector lattice and $x \in E$. We define the following elements:

$x^{+} = x \vee \Theta$ (positive part of x);

$x^{-} = (-x) \vee \Theta$ (negative part of x);

$|x| = x^{+} + x^{-}$ (modulus of x).

6. If E is a vector lattice and $x \in E$, then $x = x^{+} - x^{-}$. If $x = y - z$, where $y, z \in E^{+}$, then $x^{+} \leqq y$ and $x^{-} \leqq z$.

7. The following relationships hold in a vector lattice E:

$$(x+y)^{\pm} = x^{\pm} + y^{\pm};$$

$$(\alpha x)^{+} = \begin{cases} \alpha x^{+} & \text{if } \alpha \geqq 0 \\ -\alpha x^{-} & \text{if } \alpha \leqq 0; \end{cases} \qquad (\alpha x)^{-} = \begin{cases} \alpha x^{-} & \text{if } \alpha \geqq 0 \\ -\alpha x^{+} & \text{if } \alpha \leqq 0; \end{cases}$$

$-|x| \leqq x \leqq |x|$; $|x+y| \leqq |x| + |y|$; $|\alpha x| = |\alpha||x|$; $|x| = \Theta$ only when $x = \Theta$. If $x \leqq y$, then $x^{+} \leqq y^{+}$ and $x^{-} \geqq y^{-}$.

8. Let E be a vector lattice and $x_t \in E$ $(t \in T)$. If $x = \sup x_t$ exists, then

$$x^{+} = \sup x_t^{+}, \quad x^{-} = \inf x_t^{-}. \tag{6}$$

9. In any vector lattice E the infinite distributive laws $x \wedge \sup y_t = \sup (x \wedge y_t)$ and $x \vee \inf y_t = \inf (x \vee y_t)$ are satisfied.

10. The elements x and y of a vector lattice E are said to be <u>disjunct</u> (in symbols xdy) if $|x| \wedge |y| = \Theta$. Two subsets of E are called disjunct if every element of the first subset is disjunct from every element of the second subset. It is clear that xdx only if $x = \Theta$; two disjunct subsets of E can therefore have only one common element, namely Θ.

11. Let E be a vector lattice. For any $x \in E$ we have $x^+ d\, x^-$. If $x = y - z$, where $y, z \in E^+$ and $y d z$, then $y = x^+$ and $z = x^-$. For any $x \in E$ we have that $|x| = x \vee (-x)$. If $-y \leqq x \leqq y$, then $|x| \leqq y$. If $z \leqq x$ and $y \leqq u$, then $|x-y| \leqq u-z$.

12. Let E be a vector lattice. Suppose that in E for some element u we have $\theta \leqq u \leqq x+y$, where $x, y \in E^+$. Then u can be represented in the form $u = v + w$, where $\theta \leqq v \leqq x$ and $\theta \leqq w \leqq y$.

13. If the sets E_1 and E_2 in a vector lattice E are disjunct, then so are their linear hulls.

14. If E is a vector lattice, $x_j \in E^+$ $(j = 1, 2, \ldots, n)$ and $x_j d x_k$ for $j \neq k$, then

$$x_1 + x_2 + \ldots + x_n = x_1 \vee x_2 \vee \ldots \vee x_n. \qquad (8)$$

15. If E is a vector lattice, $x, y \in E$ and $x d y$, then we have

$$(x+y)^{\pm} = x^{\pm} + y^{\pm} \quad \text{and} \quad |x+y| \leqq |x| + |y|.$$

16. If $x = x_1 + x_2 + \ldots + x_n$, where x_j belongs to the vector lattice E for $j = 1, 2, \ldots, n$ and $x_j d x_k$ for $j \neq k$ and $x \in E^+$, then $x_j \in E^+$ for $j = 1, 2, \ldots, n$. If $x_1 + x_2 + \ldots + x_n = \theta$ and $x_j d x_k$ for $j \neq k$, then $x_j = \theta$

for $j = 1,2, \dots ,n$. If $x_1,x_2, \dots ,x_n \neq \theta$ and $x_j d x_k$, then $x_1, x_2, \dots , x_n$ are linearly independent.

17. Let E be a vector lattice. If $x_t \in E$ $(t \in T)$, $y \in E$ and $x_t d y$ for all $t \in T$ and if $x = \sup x_t$ (or $x = \inf x_t$) exists, then xdy.

18. Let L be a lattice. A sequence (x_n), $n = 1,2, \dots$, of elements of L is said to be <u>order-convergent</u> ((o)-convergent for short) to the limit $x \in L$, if there exist two monotone sequences of elements in L, where (y_n) is decreasing and (z_n) is increasing, such that $x = \inf y_n = \sup z_n$ and $z_n \leqq x_n \leqq y_n$ for $n = 1,2, \dots$.

The (o)-limit is unique, provided it exists. The algebraic and lattice theoretic operations in a vector lattice E are (o)-continuous and from this follow the statements:

(a): $x_n \xrightarrow{(o)} x$ implies $x_n^+ \xrightarrow{(o)} x^+$, $x_n^- \xrightarrow{(o)} x^-$ and $|x_n| \xrightarrow{(o)} |x|$;

(b): $|x_n| \xrightarrow{(o)} \theta$ implies $x_n \xrightarrow{(o)} \theta$;

(c): $x_n \xrightarrow{(o)} x$ is equivalent with $|x_n - x| \xrightarrow{(o)} \theta$;

(d): $x_n \xrightarrow{(o)} x$, $x_n' \xrightarrow{(o)} x'$ and $x_n d x_n'$ for all n implies xdx';

(e): $x_n \xrightarrow{(o)} x$ if and only if there exists a monotone sequence $v_n \downarrow \theta$ in the sense of (o)-convergence such that $|x_n - x| \leqq v_n$ for all n.

19. A subset E_1 of a vector lattice is said to be <u>normal</u>, if it is stable under the formation of linear combinations of its elements and if $x \in E_1$, $y \in E$ and $|y| \leqq |x|$ imply that $y \in E_1$.

A normal subset of a vector lattice is itself a vector lattice.

20. Let E be a vector lattice and E_1 be normal in E. If $x_t \in E_1$ $(t \in T)$ and $x = \sup x_t$ $(x = \inf x_t)$ exists in E_1, then x is the supremum (infimum) of the set $\{x_t : t \in T\}$ in E.

21. Let E be a vector lattice and E_1 be normal in E. Then $x_n \xrightarrow{(o)} x$ in E_1, where $x_n \in E_1$ for all n, if and only if $x_n \xrightarrow{(o)} x$ in E and $\{x_n : n = 1, 2, \ldots\}$ is bounded in E_1.

22. A vector lattice E is called <u>Archimedean</u>, if from $x \in E^+$ and the boundedness of the set $\{nx : n = 1, 2, \ldots\}$ it follows that $x = \theta$.

If E is an Archimedean vector lattice, then

(a): if $nx \leqq y$ for all $n = 1, 2, \ldots$, then $x \leqq \theta$;

(b): if $\alpha = \sup \alpha_t$ ($\alpha = \inf \alpha_t$), where α_t are real numbers, and $x \in E^+$, then $\alpha x = \sup \alpha_t x$ ($\alpha x = \inf \alpha_t x$);

(c): if $|x_n| \leqq y$ $(n = 1,2, \ldots)$ and $\alpha_n \longrightarrow 0$, then $\alpha_n x_n \xrightarrow{(o)} \theta$;

(d): if $x_n \xrightarrow{(o)} x$ and $\alpha_n \longrightarrow \alpha$, then $\alpha_n x_n \xrightarrow{(o)} \alpha x$.

23. A positive element of a vector lattice E is called <u>unit</u> and is denoted by $\mathbb{1}$, if $x \wedge \mathbb{1} > \theta$ for any $x > \theta$ and $x \in E$; in other words, if $x \in E$ and $x d \mathbb{1}$, then $x = \theta$.

If $\mathbb{1}$ is a unit of a vector lattice E and if E contains elements different from θ, then $\mathbb{1} > \theta$; if E consists of θ only, then θ is the unit element.

24. Let E be a vector lattice with unit $\mathbb{1}$. An element $e \in E$ is called <u>unitary</u>, if $e \wedge (\mathbb{1} - e) = \theta$. The set of all unitary elements is called <u>basis</u> of the vector lattice E and is denoted by $\mathcal{L}(E)$.

From the definition of unitary element it follows that $e \geqq \theta$ and $\mathbb{1} - e \geqq \theta$; this means that $\theta \leqq e \leqq \mathbb{1}$. The elements θ and $\mathbb{1}$ belong to the basis. If $e \in \mathcal{L}(E)$, then $\mathbb{1} - e \in \mathcal{L}(E)$ and conversely.

25. The basis $\mathcal{L}(E)$ of a vector lattice E is a sublattice of E. If for any set of unitary elements e_t $(t \in T)$

$e = \sup e_t$ or $e = \inf e_t$ exists, then $e \in \mathcal{L}(E)$. The basis, under the ordering already existing in it, represents a Boolean algebra and for any $e \in \mathcal{L}(E)$ the element $e' = 1 - e$ is the complement of e.

26. Let $e_1, e_2 \in \mathcal{L}(E)$. If $e_1 d e_2$, then $e_1 + e_2 \in \mathcal{L}(E)$. If $e_1 \geqq e_2$, then $e_1 - e_2 \in \mathcal{L}(E)$. The element e_2 is disjunct from the element $e_1 - (e_1 \wedge e_2)$.

27. If E is a vector lattice with unit 1, then every element $x \in E$ for which $|x| \leqq \delta 1$ for a certain real δ (dependent on x) is called <u>bounded</u>. A vector lattice with unit, all elements of which are bounded, is called a <u>vector lattice of bounded elements</u>. All bounded elements of a vector lattice with unit form a normal set in the vector lattice. The notion of bounded element depends on the choice of unit in a given vector lattice.

28. For any Boolean algebra $\mathcal{L}$ there exists an Archimedean vector lattice of bounded elements, the basis of which is isomorphic to the algebra $\mathcal{L}$.

29. If in a vector lattice E each non-empty bounded set of positive elements has a supremum, then E is a complete vector lattice.

30. A normal subset E_1 of a complete vector lattice E is a complete vector lattice also.

31. If in a vector lattice E for every countable, bounded set of positive elements there exists a supremum, then E is a σ-complete vector lattice.

32. A normal subset of a σ-complete vector lattice is a σ-complete vector lattice.

33. Any σ-complete vector lattice is Archimedean.

34. The basis $\mathcal{L}(E)$ of a complete (σ-complete) vector lattice E with unit $\mathbb{1}$ is a complete (σ-complete) Boolean algebra.

35. If E is a σ-complete vector lattice with unit $\mathbb{1}$, then for any $x \in E^+$ and $n = 1,2, \ldots$

$$\sup_n (n\mathbb{1} \wedge x) = x. \tag{9}$$

36. A normal subset E_1 of a σ-complete vector lattice E is called a <u>component</u> of E provided that the following condition is satisfied: if $X \subset E_1$ and $\sup X$ ($\inf X$) exists in E, then $\sup X \in E_1$ ($\inf X \in E_1$).

37. If X is an arbitrary subset of a σ - complete vector lattice E, then the set X^d consisting of all $x \in E$ which are disjunct from the set X is called the <u>disjunct complement</u> of X.

38. The disjunct complement of any $X \in E$ is a component of the σ-complete vector lattice E.

39. Let E_1 be a component of a σ-complete vector lattice E and $x \in E^+$. If in the set of all elements $y \in E_1$ satisfying the inequality $\theta \leqq y \leqq x$ there exist a largest element, then this element is called the <u>projection</u> of the element x onto the component E_1 and is denoted by $pr_{E_1} x$. For an arbitrary element $x \in E$ we define the projection of x onto E_1 by the formula

$$pr_{E_1} x = pr_{E_1} x^+ - pr_{E_1} x^-$$

provided that $pr_{E_1} x^+$ and $pr_{E_1} x^-$ exist. By definition, if $x \in E^+$ and $pr_{E_1} x$ exists, then $\theta \leqq pr_{E_1} x \leqq x$. For an arbitrary $x \in E$, if the projection exists, we have $\theta \leqq pr_{E_1} x^{\pm} \leqq x^{\pm}$; therefore

$$|pr_{E_1} x| = pr_{E_1} x^+ + pr_{E_1} x^- \leqq x^+ + x^- = |x|. \qquad (10)$$

40. Let E_1 be a component of a σ-complete vector lattice E and E_1^d its disjunt complement. In order that $pr_{E_1} x$ exist for

an element $x \in E$, it is necessary and sufficient that x be representable in the form $x = y + z$, where $y \in E_1$ and $z \notin E_1^d$. If such a representation exists, then it is unique and we have

$$x = pr_{E_1} x + pr_{E_1^d} x. \tag{11}$$

Therefore, if the projection of the element x onto E_1 exists, then its projection onto the disjunct complement E_1^d exists as well.

41. If E is a complete vector lattice, then for any $x \in E$ the projection onto any component E_1 of E exists.

42. The component E_1 of a σ-complete vector lattice E is called <u>essential component</u> if E_1 is a σ-complete vector lattice with unit.

43. In a σ-complete vector lattice E the projection of any $x \in E$ onto any essential component exists. If u is a unit of E_1 and $x \in E^+$, then for $n = 1, 2, \ldots$:

$$pr_{E_1} x = \sup_n (x \wedge nu). \tag{12}$$

44. Suppose that X is an arbitrary subset of a σ-complete vector lattice E. It is clear that $X^{dd} = (X^d)^d$ is a component and $X \subset X^{dd}$ holds. The component X^{dd} is called the <u>component generated by the set</u> X. If a component is generated by a

single element u, we write E_u $(=\{u\}^{dd})$.

45. X^{dd} is the smallest component of the complete vector lattice E containing the set X; if X itself is a component, then $X = X^{dd}$.

46. If the sets $X_1, X_2 \subset E$ and $X_1 d X_2$, where E is a complete vector lattice, then the components generated by X_1 and X_2 are disjunct. If $|v| \leqq |u|$, then $E_v \subset E_u$.

47. For a σ-complete vector lattice E the notion of essential component and the notion of component generated by an element are equivalent; in the component E_u the element $|u|$ plays the role of unit.

48. For the projection onto an essential component the following notation is customary: if E_u is the component generated by the element u, one writes (u)x for $pr_{E_u} x$. It is clear that $(\mathbb{1})x = x$ for all $x \in E$. By formula (12) the projection onto the component E_u for $x \in E^+$ becomes

$$(u)x = \sup_n (x \wedge nu). \tag{13}$$

49. In a complete vector lattice E with unit $\mathbb{1}$ all components are essential.

50. Let E be σ-complete vector lattice and E_1 a component such that for every $x \in E$ the projection $pr_{E_1} x$ exists. If E is a complete vector lattice, then E_1 can be any of its components. The projection mapping satisfies the properties:

(a) $pr_{E_1}(\alpha x + \beta y) = \alpha\, pr_{E_1} x + \beta\, pr_{E_1} y$ (linearity);

(b) $(pr_{E_1} x)^{\pm} = pr_{E_1} x^{\pm}$, $|pr_{E_1} x| = pr_{E_1} |x|$;

(c) $pr_{E_1} x = \begin{cases} x & \text{if and only if } x \in E_1 \\ \theta & \text{if and only if } x d E_1; \end{cases}$

(d) if $x \geqq y$, then $pr_{E_1} x \geqq pr_{E_1} y$;

(e) if $x_n \xrightarrow{(o)} x$, then $pr_{E_1} x_n \xrightarrow{(o)} pr_{E_1} x$.

51. If E is a vector lattice, if the elements $x_t \in E$ $(t \in T)$ are mutually disjunct and if $\sup x_t^+$ and $\sup x_t^-$ exist in E, then the element $\underset{t}{S}\, x_t$ defined by

$$\underset{t}{S}\, x_t = \sup x_t^+ - \sup x_t^-$$

is called the <u>union</u> of the elements x_t $(t \in T)$.

54. The union of a finite number of elements x_k $(k = 1, \ldots, n)$ of a vector lattice E exists. Moreover,

$$\overset{n}{\underset{k=1}{S}}\, x_k = \sum_{k=1}^{n} x_k$$

holds. If E is a complete vector lattice, the union of any bounded set of pairwise disjunct elements x_t $(t \in T)$ exists because in this case we have the existence of sup x_t^+ and sup x_t^-. In a σ-complete vector lattice the union of any bounded, countable set of pairwise disjunct elements exists.

53. Let E be a vector lattice and $x_t \in E$ $(t \in T)$ be such that their union exists. Since $x_t d x_s$ for $t \neq s$, then $x_t^+ d\, x_s^-$ for any $t, s \in T$; therefore by Number 17: $\sup x_t^+ \; d \; \sup x_t^-$ and by Number 11: $(\underset{t}{S}\, x_t)^+ = \sup x_t^+$, $(\underset{t}{S}\, x_t)^- = \sup x_t^-$. Hence we get

$$|\underset{t}{S}\, x_t| = \sup x_t^+ \vee \sup x_t^- = \sup (x_t^+ \vee x_t^-) = \sup |x_t| .$$

54. If $\underset{t}{S}\, x_t$ exists and $x_t dy$ for all x_t $(t \in T)$ in a vector lattice E, then $(\underset{t}{S}\, x_t)dy$.

55. The associative law holds for unions; in a complete vector lattice E the associative law reads: If the set $\{x_t : t \in T\}$ is bounded and consists of pairwise disjunct elements and the set T of indices t is representable in terms of disjoint subsets T_a $(a \in A)$ $(T = \bigcup_{a \in A} T_a)$, then

$$\underset{t \in T}{S} x_t = \underset{a \in A}{S} (\underset{t \in T_a}{S} x_t). \tag{14}$$

56. The set of elements $\{x_t \in E: t \in T\}$ is called <u>complete</u> in a vector lattice E, if in E there is no element, except θ, which is disjunct from x_t for all $t \in T$.

In Number 36 the notion of component of a σ-complete vector lattice was defined; in the same manner one defines the concept of component of a vector lattice. Replacing in the above definition the elements x_t by components E_t of E, we can define completeness of the set of components $E_t \subset E$ $(t \in T)$.

57. We say that the set of components E_t $(t \in T)$ of a σ-complete vector lattice E form a <u>decomposition</u> of E, if the following is satisfied: (1) these components are pairwise disjunct, (2) the set $\{E_t: t \in T\}$ is complete in E and (3) for any $x \in E$ the projection onto each of the components E_t $(t \in T)$ exists. An example of a system of two components forming a decomposition of a σ-complete vector lattice is any essential component and its disjunct complement.

58. If two components E_1 and E_2 form a decomposition of a σ-complete vector lattice E, then $E_1 = E_2^d$.

59. If in a σ-complete vector lattice E there is given a complete set of pairwise disjunct elements x_t $(t \in T)$, then the components E_t, generated by these elements $(E_t = E_{x_t})$,

form a decomposition of E.

60. If the components E_t $(t \in T)$ form a decomposition of a complete vector lattice E, then any element $x \in E$ is representable uniquely in the form of the union

$$x = \underset{t \in T}{S} x_t, \tag{15}$$

where $x_t \in E_t$ and $x_t = pr_{E_t} x$.

61. Given a family $(E_t)_{t \in T}$ of complete vector lattices, let E be the cartesian product of the sets E_t; define addition and scalar multiplication in E componentwise and let $E^+ = \{(x_t)_{t \in T} : x_t \in E_t^+ \text{ for all } t \in T\}$. With these definitions E is a complete vector lattice, called the <u>union</u> of the vector lattices E_t, and denoted by $\underset{t \in T}{S} E_t$. Note that if $x = (x_t)_{t \in T} \in E$, then $|x| = (|x_t|)_{t \in T}$.

If $t_o \in T$ is fixed, then the set Z_{t_o} of all $x \in E$ for which $x_{t_o} \in E_{t_o}$, and $x_t = \theta$ for $t \neq t_o$, will be a component in E. This component is algebraically and lattice theoretically isomorphic to E_{t_o}. Moreover, if each element $x_{t_o} \in E_{t_o}$ is identified with the family $(x_t)_{t \in T}$, where $x_t = \delta_{tt_o} x_{t_o}$ (Kroenecker delta), then E_{t_o} is identified with Z_{t_o}. Therefore the E_t's form a decomposition of E and $pr_{E_t} x = x_t$ for $x = (x_t)_{t \in T}$. We can write the elements of E as $x = \underset{t \in T}{S} x_t$ and call x the union of the elements x_t.

62. The union $E = S\, E_t$ of the complete vector lattices E_t with units $\mathbb{1}_t$ is a complete vector lattice with unit; as unit in E we can take $S\, \mathbb{1}_t$.

63. If the complete vector lattice E is decomposed into components E_t, then E is algebraically and lattice theoretically isomorphic to a certain normal sublattice of the complete vector lattice $Y = S\, E_t$.

64. Let E be a σ-complete vector lattice with unit $\mathbb{1}$ and $\mathcal{L}(E)$ be its basis. Then

(a) If $\mathbb{1} = S\, e_t$ ($e_t \in \mathcal{L}(E)$; $t \in T$), then the set $(e_t)_{t \in T}$ is complete.

(b) For any $e \in \mathcal{L}(E)$, $E_e^{\,d} = E_{\mathbb{1}-e}$.

(c) If $e = S\, e_t$ ($e_t \in \mathcal{L}(E)$, $t \in T$), then for any $x \in E$ we have that

$$(e)x = \underset{t \in T}{S} (e_t)x.$$

(d) If $e, e' \in \mathcal{L}(E)$, then for any $x \in E$

$$(e')\left[(e)x\right] = (e \wedge e')x. \tag{16}$$

(e) If $e_n \downarrow \theta$ ($e_n \in \mathcal{L}(E)$), then $(e_n)x \xrightarrow{(o)} \theta$ for any $x \in E$.

65. Let E be a σ-complete vector lattice with unit $\mathbb{1}$ and $\mathfrak{L}$(E) its basis. Then every unitary element is a projection of the unit onto a certain essential component. Conversely, the projection of the unit element onto any essential component is a unitary element, acting in this component as unit element.

66. The basis $\mathfrak{L}$(E) is isomorphic with the set of all essential components of the σ-complete vector lattice E with unit $\mathbb{1}$, ordered by inclusion. If E is a complete vector lattice, the basis $\mathfrak{L}$(E) is isomorphic with the set of all its components.

67. Let E be a σ-complete vector lattice with unit $\mathbb{1}$ and let $\mathfrak{L}$(E) denote its basis. For any $e, e' \in \mathfrak{L}(E)$ we have

$$(e')e = e \wedge e'. \tag{17}$$

68. Let E be a σ-complete vector lattice with unit $\mathbb{1}$ and let $\mathfrak{L}$(E) denote its basis. The projection of the unit onto the essential component E_x, generated by the element $x \in E$, is called the <u>trace</u> of the element x and is denoted by e_x; thus

$$e_x = (x)\mathbb{1} = \sup_n (\mathbb{1} \wedge n|x|).$$

By Number 65 the trace $e_x \in \mathfrak{L}(E)$, for any $x \in E$, is a unit in the component E_x, that is $E_{e_x} = E_x$. Thus, if $x \neq \theta$, then $e_x \neq \theta$. Moreover, since $E_x = E_{|x|}$, we have $e_x = e_{|x|}$.

69. The trace satisfies the following properties:

(a) $x d y$ if and only if $e_x d e_y$;

(b) $|x| \leqq |y|$ implies $e_x \leqq e_y$;

(c) $(e_x)x = x$;

(d) $(e_{x^+})x = x^+$, $(1 - e_{x^+})x = -x^-$, $(e_{x^-})x = -x^-$,

$(1 - e_{x^-})x = x^+$;

(e) $e_x = e_{x^+} + e_{x^-}$;

(f) if $x = \sup x_t$, where $x_t \in E^+$ $(t \in T)$, then $e_x = \sup e_{x_t}$;

if $x = \inf x_t$, where $x_t \in E^+$ $(t \in T)$, then for $t \in T$

we have $e_x \leqq e_{x_t}$.

70. Let E be a σ-complete vector lattice with unit 1.
For each $x \in E$ and each real number λ we put

$$e_\lambda^x = e_{(\lambda 1 - x)^+};$$

e_λ^x is the trace of the positive part of the element $\lambda 1 - x$;
$e_\lambda^x \in \mathcal{L}(E)$. For fixed $x \in E$ the system of elements (e_λ^x),

where λ runs through $(-\infty, \infty)$ is called the <u>resolution</u> of x.

From the definition of resolution it follows that

$$(e_\lambda^x)x \leqq \lambda e_\lambda^x \quad \text{and} \tag{18}$$

$$(1 - e_\lambda^x)x \geqq \lambda(1 - e_\lambda^x). \tag{19}$$

71. Let E be a σ-complete vector lattice with unit 1. For each $x \in E$ the resolution of the element x has the properties:

(a) $e_\mu^x \geqq e_\lambda^x$ for $\mu \geqq \lambda$;

(b) $\sup\limits_\lambda e_\lambda^x = 1$;

(c) $\inf\limits_\lambda e_\lambda^x = \Theta$;

(d) for each λ the resolution is left-continuous:

$$\sup_{\mu < \lambda} e_\mu^x = e_\lambda^x ;$$

(e) if $\mu_1 \geqq \mu_2 \geqq \lambda_1 \geqq \lambda_2$, then

$$(e_{\mu_1}^x - e_{\mu_2}^x) \; d \; (e_{\lambda_1}^x - e_{\lambda_2}^x).$$

72. Let E be a σ-complete vector lattice with unit 1. For any $\mu \geqq \lambda$ we have the formulas:

$$e^x_\mu \wedge (1 - e^x_\lambda) = e^x_\mu - e^x_\lambda \tag{20}$$

$$\lambda(e^x_\mu - e^x_\lambda) \leqq (e^x_\mu - e^x_\lambda)x \leqq \mu(e^x_\mu - e^x_\lambda). \tag{21}$$

73. In a σ-complete vector lattice E one can introduce the notion of infinite series $\sum_{n=0}^{\infty} x_n$ ($x_n \in E$) with the help of (o)-convergence, where $\underset{p \to \infty}{\text{(o)-lim}} (\sum_{n=0}^{p} x_n)$ is taken to be the sum of the series under consideration. The (o)-convergence of the series under consideration implies that $x_n \xrightarrow{(o)} \theta$ as $n \to \infty$. For the (o)-convergence of a positive series (where all $x_n \in E^+$) it is necessary and sufficient that the sequence of its partial sums be bounded; in this case the sum of the series coincides with the supremum of its partial sums. If $\theta \leqq x_n \leqq y_n$ for all $n > n_0$, then from the (o)-convergence of the series $\sum y_n$ follows the (o)-convergence of $\sum x_n$. If the series $\sum |x_n|$ is (o)-convergent, then $\sum x_n$ is likewise (o)-convergent and the latter series is said to be absolutely convergent.

Let all terms of the series $\sum x_n$ be pairwise disjunct and suppose that the set $\{x_n : n = 1, 2, \ldots\}$ is bounded. In this case the union $\underset{n}{S}\, x_n$ exists and

$$\overset{\infty}{\underset{n=1}{S}}\, x_n = \sum_{n=1}^{\infty} x_n$$

holds.

The sum of the two-sided infinite series $\sum_{-\infty}^{\infty} x_n$ is defined as the (o)-limit of the partial sums $\sum_{n=p}^{q} x_n$ for $p \to -\infty$ and $q \to \infty$.

74. The <u>integral representation theorem of H. Freudenthal:</u> Let E be a σ-complete vector lattice with unit $\mathbb{1}$. Then each element $x \in E$ is representable in the form

$$x = \int_{-\infty}^{\infty} \lambda \, d\, e_{\lambda}^{x} \quad ,$$

where the integral sign signifies the (o)-limit of the integral sums

$$J = \sum_{-\infty}^{\infty} t_n (e_{\lambda_n}^{x} - e_{\lambda_{n-1}}^{x})$$

formed for an arbitrary partition of $(-\infty, \infty)$ by the points

$$-\infty < \ldots < \lambda_{-1} < \lambda_0 < \lambda_1 < \ldots < \infty$$

where $\lambda_{n-1} \leqq t_n \leqq \lambda_n$ for all n and the (o)-limit is taken under the condition that $\varepsilon = \sup_n (\lambda_n - \lambda_{n-1}) \longrightarrow 0$.

75. A σ-complete vector lattice E with a binary multiplication is called a <u>partially ordered ring</u> provided the following conditions hold:

(A) $(xy)z = x(yz)$;

(B) $x(y+z) = xy+xz, \quad (y+z)x = yx+zx$;

(C) $(\alpha x)y = x(\alpha y) = \alpha(xy)$ for every real number α ;

(D) there is a multiplicative unit 1 ($> \Theta$) such that for all $x \in E$ we have $x1 = 1x = x$ (hence there is only one unit);

(E) $x \geqq \Theta$ and $y \geqq \Theta$ implies $xy \geqq \Theta$;

(F) $x \wedge 1 = \Theta$ implies $x = \Theta$.

Remark: If $x \geqq y$ and $z \geqq \Theta$, then $xz \geqq yz$ and $zx \geqq zy$.

76. Let $e, e' \in \mathcal{L}(E)$, the basis of the partially ordered ring E. Then we have

$$(e')e = (e)e' = ee' = e'e.$$

77. If E is a partially ordered ring, $x \in E^+$ and $y \in E$, then

$$|xy| \leqq x|y| \quad \text{and} \quad |yx| \leqq |y|x.$$

78. If x and y belong to a partially ordered ring E and $\theta \leqq x \leqq \beta 1$, then the products xy and yx, for fixed x, are continuous in the sense of the (o)-limit.

79. From the Numbers 74 & 78 follows that for any unitary element e in a partially ordered ring E we have the formula

$$(e)x = ex = xe \tag{24}$$

for any $x \in E$.

This completes our survey of basic information concerning vector lattices and partially ordered rings.

We now return to the consideration of linear operators in Hilbert space.

Let $\mathcal{B}$ denote the set of all bounded linear operators in a Hilbert space $\mathcal{G}$. If $f_1,\ldots,f_n \in \mathcal{G}$ and $\varepsilon > 0$, then $\mathcal{U}(A_o;f_1,\ldots,f_n;\varepsilon)$ is to denote the set of operators A in $\mathcal{B}$ satisfying the system of inequalities

$$\|(A - A_o)f_j\| < \varepsilon \qquad (j = 1,2,\ldots,n)$$

for a given A_o in $\mathcal{B}$. The set $\mathcal{U}(A_o;f_1,\ldots,f_n;\varepsilon)$ is called a strong neighborhood of the operator A_o. The set of strong neighborhoods generates a topology of $\mathcal{B}$. The convergence of a sequence (A_n), $n = 1,2,\ldots$, to an operator A in this topology means that for every $f \in \mathcal{G}$ $\|A_nf - Af\| \longrightarrow 0$ as $n \longrightarrow \infty$; A is called the <u>strong</u> limit of the sequence (A_n).

The operations αA, $A + B$, AB (in the product one factor is kept fixed) are continuous relative to the strong operator convergence. Passage from A to A^* is not necessarily continuous with respect to the strong operator convergence.

We recall that an arbitrary linear operator A in a Hilbert space $\mathcal{G}$ is said to <u>permute</u> with a bounded linear operator

B in $\mathfrak{H}$ if $BA \subset AB$ holds, that is if $BA = AB$ on $\mathfrak{D}_{BA}$. If both A and B are bounded linear operators, then we say that they permute if they commute. We shall use the symbol AvB to signify that the operator A permutes with the operator B.

A self-adjoint operator A is said to be positive, if $\langle Af,f \rangle \geqq 0$ holds for all $f \in \mathfrak{D}_A$; we signify this by writing $A \geqq 0$. If A is a bounded self-adjoint operator, then A^2 is a self-adjoint positive operator.

We now take up several elementary propositions which will turn out to be usefull at a later stage of the discussion.

I: If the bounded linear operators A_1 and A_2 are self-adjoint, if $A_2 \geqq 0$ and $A_1 v A_2$, then the self-adjoint operator $A_1{}^2A_2$ is positive.

Proof: $\langle A_1{}^2A_2f,f \rangle = \langle A_1A_2f,A_1f \rangle = \langle A_2A_1f,A_1f \rangle \geqq 0$.

II: If the bounded linear self-adjoint operators A and B are positive and permutable, then the self-adjoint operator $AB = BA$ is also positive.

Proof: We may assume that $A \neq 0$ and hence $\| A \| > 0$. We define a sequence (A_n), $n = 1,2,\ldots$, of self-adjoint operators by putting

$$A_1 = \frac{A}{\|A\|}\ , \quad A_{n+1} = A_n - A_n^{\ 2} \quad (n = 1,2,\ldots)$$

and we show that $0 \leqq A_n \leqq E$ for every n; here E denotes the identity operator. For $n = 1$ we have

$$0 \leqq \langle A_1 f, f\rangle = \langle Af, f\rangle \,/\, \|A\| \leqq \langle f,f\rangle$$

and therefore $0 \leqq A_1 \leqq E$. Since

$$E - A_{n+1} = (E - A_n) + A_n^{\ 2}$$

and

$$A_{n+1} = A_n - A_n^{\ 2} = A_n^{\ 2} - A_n^{\ 3} + A_n - 2A_n^{\ 2} + A_n^{\ 3}$$

we have that

$$A_{n+1} = A_n^{\ 2}(E - A_n) + A_n(E - A_n)^2.$$

By the inductive assumption $A_n \geqq 0$ and $E - A_n \geqq 0$ so that by proposition I above we obtain $E - A_{n+1} \geqq 0$ and $A_{n+1} \geqq 0$ and therefore $0 \leqq A_{n+1} \leqq E$. From

$$A_1 = A_1^{\ 2} + A_2^{\ 2} + \ \ldots\ + A_n^{\ 2} + A_{n+1} \quad \text{and} \quad A_{n+1} \geqq 0$$

it follows that

$$\sum_{j=1}^{m} \langle A_j f, A_j f\rangle = \sum_{j=1}^{m} \langle A_j^{\ 2} f, f\rangle = \langle A_1 f, f\rangle - \langle A_{n+1} f, f\rangle \leqq$$

$$\leqq \langle A_1 f, f\rangle$$

holds; this means that the series $\sum \| A_j f \|^2$ converges and therefore $\lim \| A_j f \| = 0$. It follows that

$$\lim \left(\sum_{j=1}^{n} A_j^2 \right) f = \lim (A_1 f - A_{n+1} f) = A_1 f.$$

Since BvA by assumption, we see that BvA_j for every j; hence

$$\langle BAf, f \rangle / \| A \| = \langle BA_1 f, f \rangle = \lim \sum_{j=1}^{n} \langle BA_j^2 f, f \rangle =$$

$$= \lim \sum_{j=1}^{n} \langle A_j B A_j f, f \rangle = \lim \sum_{j=1}^{n} \langle BA_j f, A_j f \rangle \geqq 0$$

and this shows that BA is positive.

III: If all operators of an increasing sequence (A_n) of bounded linear self-adjoint operators are mutually permutable and if there exists a self-adjoint bounded linear operator B such that $A_n vB$ and $A_n \leqq B$ for all n, then the sequence (A_n) converges to a self-adjoint operator A satisfying $A_n vA$ and $A_n \leqq A \leqq B$. A similar statement holds for decreasing sequences.

Proof: We consider the decreasing sequence of positive operators $H_n = B - A_n$. They are mutually permutable; for $m > n$ the operators $H_m(H_n - H_m)$ and $(H_n - H_m)H_n$ are positive by proposition II above. Hence, for every $f \in \mathcal{G}$:

$$\langle H_m^2 f, f \rangle \leqq \langle H_m H_n f, f \rangle \leqq \langle H_n^2 f, f \rangle ,$$

from which we infer that the non-increasing sequence of non-negative numbers $\langle H_m^2 f, f \rangle = \| H_m f \|^2$ has a limit, dependent on f, and that $\langle H_m H_n f, f \rangle$ has the same limit for $m, n \longrightarrow \infty$. Thus we have

$$\lim \| (H_m - H_n) f \|^2 = \lim \langle (H_m - H_n)^2 f, f \rangle =$$

$$= \lim \left[\langle H_m^2 f, f \rangle - 2 \langle H_m H_n f, f \rangle + \langle H_n^2 f, f \rangle \right] = 0.$$

By the completeness of the space $\mathcal{G}$ the sequence $H_n f$ converges for every $f \in \mathcal{G}$ so that the same is true of the sequence $A_n f$. We define $Af = \lim A_n f$ and note that A is linear and self-adjoint. The relations $A_n \leqq A_{n+1}$ and $A_n \leqq B$, for all n, imply $A_n \leqq A \leqq B$ and the relations $A_n v A_j$ and $\lim A_j = A$ imply $A_n v A$.

IV: If A is a bounded linear self-adjoint positive operator, then there exists a unique bounded self-adjoint positive operator B such that $B^2 = A$.

Proof: Evidently we may assume that $A \leqq E$, since we may replace A by $A / \| A \|$. We define a sequence of self-adjoint operators B_n as follows

$$B_0 = 0, \quad B_{n+1} = B_n + (1/2)(A - B_n^2) \qquad (n = 0, 1, 2, \ldots).$$

It is seen by induction that if a bounded self-adjoint operator C permutes with A, then it permutes with all B_n;

hence we have AvB_n (because AvA). Next, $B_m vA$, for all m, implies $B_m vB_n$ for all m and n.

Since

$$E-B_{n+1} = E-B_n-(1/2)(A-B_n{}^2) = (1/2)(E-B_n)^2 + (1/2)(E-A)$$

we have that $E-B_n \geqq 0$ for all n. Since $B_n vB_{n-1}$ we have

$$B_{n+1}-B_n = B_n + (1/2)(A-B_n{}^2)-B_{n-1}-(1/2)(A-B_{n-1}{}^2) =$$

$$= B_n-B_{n-1}-(1/2)(B_n{}^2-B_{n-1}{}^2) = (B_n-B_{n-1})(E-(1/2)(B_n+B_{n+1})) =$$

$$= (1/2)\ \left[(E-B_n)+(E-B_{n-1})\right]\ (B_n-B_{n-1})$$

which means that $B_{n+1} \geqq B_n$. Indeed, for $n = 0$ we have

$$B_1 = (1/2)A \geqq 0 = B_0.$$

By the inductive assumption $B_n-B_{n-1} \geqq 0$, so that by proposition II above $B_{n+1}-B_n \geqq 0$ holds as well because $B_{n+1} - B_n$ is the product of the permutable and positive operators $(1/2)\ \left[(E-B_n)+(E-B_{n-1})\right]$ and $B_n - B_{n-1}$.

The sequence (B_n), satisfying therefore $0 = B_0 \leqq B_1 \leqq \dots \leqq E$, converges by force of proposition III to a positive operator B. Hence, taking $n \longrightarrow \infty$ in $B_{n+1} = B_n + (1/2)(A - B_n{}^2)$, we get that

$$B = B + (1/2)(A-B^2) \quad \text{or} \quad B^2 = A.$$

To see that B is uniquely determined, we observe first that every bounded self-adjoint operator C, permutable with A, is also permutable with B since CvA implies CvB_n, and this in turn implies CvB. If therefore C is bounded and positive, and $C^2 = A$, we get that CvB holds because $AC = CA = C^3$. Let $B^{\frac{1}{2}}$ and $C^{\frac{1}{2}}$ be two positive operators satisfying $(B^{\frac{1}{2}})^2 = B$ and $(C^{\frac{1}{2}})^2 = C$ (their existence is assured by the first part of the proof), let $f \in \mathfrak{H}$ be arbitrary and $g = (B-C)f$. Then we get, since CvB,

$$\| B^{\frac{1}{2}}g \|^2 + \| C^{\frac{1}{2}}g \|^2 = \langle Bg, g \rangle + \langle Cg, g \rangle =$$

$$= \langle (B+C)(B-C)f, g \rangle = \langle (B^2 - C^2)f, g \rangle = 0;$$

Hence $B^{\frac{1}{2}}g = C^{\frac{1}{2}}g = 0$ and therefore $Bg = 0$ and $Cg = 0$. But this implies

$$\| (B-C)f \|^2 = \langle (B-C)^2 f, f \rangle = \langle (B-C)g, f \rangle = 0$$

or $Bf = Cf$. Since f is arbitrary, we have that $B = C$.

V. Given an unbounded self-adjoint operator A in a Hilbert space $\mathfrak{H}$. Then the positive bounded self-adjoint operator $B = (E + A^2)^{-1}$ exists, A permutes with B, and $\|B\| \leqq 1$.

Proof: Let T denote a linear operator whose domain of definition is dense in $\mathfrak{H}$ (a self-adjoint operator satisfies this condition). We show that the operator $B = (E + T^*T)^{-1}$ and the operator $C = T(E + T^*T)^{-1}$ are defined everywhere and $\|B\| \leqq 1$ and $\|C\| \leqq 1$; moreover B is symmetric and positive.

The graph of T, denoted by G_T, is the set of all pairs (f,Tf), where f runs through the elements of $\mathcal{D}_T$, in the Hilbert space $\mathfrak{H} \oplus \mathfrak{H}$. In $\mathfrak{H} \oplus \mathfrak{H}$ we consider the following two mappings

$$U(f,g) = (g,f), \quad V(f,g) = (g,-f).$$

We note that these mappings are unitary operators and that

$$UV = -VU \text{ and } U^2 = -V^2 = I,$$

where I stands for the identity mapping on $\mathfrak{H} \oplus \mathfrak{H}$. With this notation the equation $\langle Tf,g \rangle = \langle f,g^* \rangle$, which defines the adjoint operator $T^* g = g^*$, can be put into the form

$$\langle V(f,Tf),(g,g^*) \rangle = 0.$$

This means that G_{T^*} is made up of those elements of $\mathfrak{H} \oplus \mathfrak{H}$ which are orthogonal to VG_T. G_{T^*} is therefore a subspace of $\mathfrak{H} \oplus \mathfrak{H}$, namely the orthogonal complement of the closure of the set VG_T (that is $\overline{VG_T}$). Since $\overline{VG_T} = V\,\overline{G_T}$, we can write

$$G_{T^*} = (\mathfrak{H} \oplus \mathfrak{H}) \ominus V\,\overline{G_T}.$$

The graph of a linear operator is closed if and only if the operator is closed. Thus, if T is a closed linear operator with dense domain of definition in $\mathfrak{H}$, then G_T and G_{T^*} are orthogonal complements of each other in $\mathfrak{H} \oplus \mathfrak{H}$. Let h be an arbitrary element of $\mathfrak{H}$. One then can decompose the element $(h,0)$ of $\mathfrak{H} \oplus \mathfrak{H}$ into the sum of an element of G_T and an element of VG_{T^*} in a unique manner:

$$(h,0) = (f, Tf) + (T^* g, -g).$$

By passing to the components, we can write the system of equations

$$h = f + T^* g \quad \text{and} \quad 0 = Tf - g$$

with unique solution f in $\mathfrak{D}_T$ and g in $\mathfrak{D}_{T^*}$. By putting

$$f = Bh \quad \text{and} \quad g = Ch$$

we define two mappings on $\mathfrak{H}$ into itself which are linear. We can write the system of equations in the form

$$E = B + T^* C \quad \text{and} \quad 0 = TB - C,$$

whence $C = TB$ and $E = B + T^* TB = (E + T^* T)B$. From the decomposition of $(h,0)$ we get that

$$\|h\|^2 = \|(h,0)\|^2 = \|(f,Tf)\|^2 + \|(T^* g,-g)\|^2 =$$

$$= \|f\|^2 + \|Tf\|^2 + \|T^* g\|^2 + \|g\|^2$$

so that

$$\|Bh\|^2 + \|Ch\|^2 = \|f\|^2 + \|g\|^2 \leqq \|h\|^2 .$$

Thus $\|B\| \leqq 1$ and $\|C\| \leqq 1$.

For an element u in the domain of definition of $T^* T$ we have

$$\langle (E + T^* T)u, u \rangle = \langle u,u \rangle + \langle Tu, Tu \rangle \geqq \langle u,u \rangle$$

and therefore $(E + T^*T)u = 0$ implies $u = 0$. Consequently, the inverse $(E + T^*T)^{-1}$ exists. It is clear that $(E + T^*T)^{-1}$ is defined everywhere and equals B because of the equation $E = B + T^*TB = (E + T^*T)B$ which appeared further above. Since

$$\langle Bu,v\rangle = \langle Bu,(E + T^*T)Bv\rangle = \langle Bu,Bv\rangle + \langle Bu,T^*TBv\rangle =$$

$$= \langle Bu,Bv\rangle + \langle T^*TBu,Bv\rangle = \langle (E + T^*T)Bu,Bv\rangle = \langle u,Bv\rangle$$

and

$$\langle Bu,u\rangle = \langle Bu,(E + T^*T)Bu\rangle = \langle Bu,Bu\rangle + \langle TBu,TBu\rangle \geqq 0$$

it follows that B is symmetric and positive.

Finally, we consider the self-adjoint operator A and let $B = (E + A^2)^{-1}$ and $C = AB = A(E + A^2)^{-1}$. By what has been proved already, B is symmetric and bounded; in fact $0 \leqq B \leqq E$. Multiplying both members of the equation $A(E + A^2) = (E + A^2)A$ by B on the left and on the right and keeping in mind that $(E + A^2)B = E$ and $B(E + A^2) \subset E$ we get at once that $BA \subset AB$; this means that A permutes with B.

VI: Let $M_1, M_2, \ldots, M_j, \ldots$ be a sequence of subspaces of a Hilbert space $\mathfrak{H}$ which are mutually orthogonal and whose direct sum is the entire space $\mathfrak{H}$. For an arbitrary element f in $\mathfrak{H}$ let f_j denote the projection of f onto M_j. Let $A_1, A_2, \ldots, A_j, \ldots$ be a sequence of given linear operators such that the component of A_j in M_j is a bounded self-adjoint operator mapping M_j into itself. Under these conditions there exists one and only one self-adjoint operator A on $\mathcal{H}$, in general not bounded, whose component on M_j is A_j for $j = 1,2, \ldots$. $\mathcal{D}_A$ consists of all elements f for which the series

$$\sum_{j=1}^{\infty} \| A_j f_j \|^2$$

converges, and for these f we have

$$Af = \sum_{j=1}^{\infty} A_j f_j .$$

Proof: First we note that the mapping A defined above is linear. $\mathcal{D}_A$ is dense in $\mathcal{H}$ because it contains all elements of the form $f_1 + f_2 + \ldots + f_n$. In addition, A is symmetric since for all elements $f, g \in \mathcal{D}_A$ we have

$$\langle Af, g \rangle = \sum \langle A_j f_j, g_j \rangle = \sum \langle f_j, A_j g_j \rangle = \langle f, Ag \rangle .$$

Let $g \in \mathcal{D}_{A^*}$, then for all $f \in \mathcal{D}_A$ we get $\langle Af, g \rangle = \langle f, A^* g \rangle$ and therefore $\sum_{j=1}^{\infty} \langle A_j f_j, g_j \rangle = \sum_{j=1}^{\infty} \langle f_j, (A^* g)_j \rangle$.

Choosing as f an arbitrary element of M_i, then $f_j = 0$ for $j \neq i$

and

$$\langle A_i f_i, g_i \rangle = \langle f_i, (A^* g)_i \rangle .$$

When A_i is assumed to be self-adjoint in M_i, then

$$(A^* g)_i = A_i g_i .$$

We obtain

$$\sum_{i=1}^{\infty} \| A_i g_i \|^2 = \sum_{i=1}^{\infty} \| (A^* g)_i \|^2 = \| A^* g \|^2 ;$$

thus g also belongs to $\mathcal{D}_A$ and we have

$$Ag = \sum_{i=1}^{\infty} A_i g_i = \sum_{i=1}^{\infty} (A^* g)_i = A^* g .$$

This shows that $A^* \subseteqq A$. But A is symmetric, hence $A^* = A$.

It remains to verify that A is unique. Let A' be any self-adjoint operator which has component A_j in M_j. Since A' is closed, it is necessarily defined for all elements f for which the series $\sum_{j=1}^{\infty} A' f_j$ converges; the sum of this series will also be equal to A'f. Since $A' f_j = A_j f_j$, and since the convergence of a series of orthogonal elements is equivalent with the convergence of the series of squares of the norms, the set of these elements f coincides with $\mathcal{D}_A$ and for these f one has $A'f = Af$; thus $A' \supseteqq A$. But A is selfadjoint and therefore maximally symmetric; hence $A' = A$.

Let $\mathfrak{V}$ denote the set of all linear bounded self-adjoint operators in a Hilbert space $\mathfrak{H}$. If $A, B \in \mathfrak{V}$, then the product $AB \in \mathfrak{V}$ if and only if $AB = BA$. Let $A \in \mathfrak{V}$ and α be a scalar, then $\alpha A \in \mathfrak{V}$ if and only if α is a real number. Motivated by these observations, we call a set $\mathfrak{A}$ of bounded linear operators in $\mathfrak{H}$ a <u>ring</u>, if for any $A, B \in \mathfrak{A}$ and any real number α the operators $A + B$, AB, and αA also belong to $\mathfrak{A}$.

In the set $\mathfrak{V}$ we can introduce a partial order by writing $A \geqq 0$ if and only if A is positive; $A > 0$ means that $\langle Af, f \rangle \geqq 0$ for all $f \in \mathfrak{H}$ and $A \neq 0$. R. V. Kadison (Order properties of bounded self-adjoint operators, Proc. Amer. Math. Soc. 2. 505-510 (1951)) showed that under this ordering the set $\mathfrak{V}$ does not form a lattice (since $A \vee B$ exists in $\mathfrak{V}$ if and only if $A \leqq B$ or $B \leqq A$).

We give some definitions before going on with the considerations.

A partially ordered set R is said to be <u>directed upward</u> (downward), if for any two elements $a, b \in R$ there exists an element $c \in R$ such that $c \geqq a$ and $c \geqq b$ ($c \leqq a$ and $c \leqq b$). An upward directed set R is called a <u>path</u> if for any $a \in R$ there is an element $b \in R$ such that $b > a$ (i.e., R has no greatest element). By a path of operators will be meant a family of operators $(A_t)_{t \in T}$, where T is a path. A path of

self-adjoint operators will be called increasing (decreasing) if and only if $s \leqq t$ implies $A_s \leqq A_t$ ($A_s \geqq A_t$). A path $(A_t)_{t \in T}$ of bounded operators is said to <u>converge</u> to the operator A in the sense of the strong operator topology, if given any neighborhood U of A in the topological space $\mathcal{B}$ (see page 175), then A_t is eventually in U, that is, if there exists an index t_o such that $A_t \in U$ for each $t \geqq t_o$.
A subset of the set $\mathcal{T}$ is said to be strongly closed, if it is closed in the sense of the strong operator topology, that is the limit of any strongly convergent path of operators of the subset also belongs to the subset.
If $A_t \longrightarrow A$ in the sense of the strong operator topology, A_t and A belong to $\mathcal{T}$ and if A_t permutes with $B \in \mathcal{T}$ for each $t \in T$, then A permutes with B as well. Indeed, for any $f \in \mathcal{D}_B$ we have $BA_tf = A_tBf \longrightarrow ABf$. But $A_tf \longrightarrow Af$ and, since B is closed, $Af \in \mathcal{D}_B$ and $BAf = ABf$.

The proof of the following proposition is completely analogous to the proof of proposition III further above.

VII: Let $(A_t)_{t \in T}$ be an increasing path of pairwise permuting self-adjoint operators such that $A_t \leqq B$ for all $t \in T$ and $B \in \mathcal{T}$. Then $\sup A_t = A$ exists in $\mathcal{T}$ and A is the strong operator limit of $(A_t)_{t \in T}$.

Proof: Let $H_t = B_1 - A_t$ $(t \in T)$, where $B_1 = \|B\| E$ and E is the identity operator on $\mathcal{G}$. It is clear that $H_t \geqq 0$ and that the operators H_t form a decreasing path. If $s \geqq t$ for $s,t \in T$, then $H_s \leqq H_t$ and therefore $(H_t-H_s)H_t$ and $H_s(H_t-H_s)$ are positive by proposition II above. Just as in the proof of proposition III we obtain

$$0 \leqq \langle H_s^2 f,f\rangle \leqq \langle H_s H_t f,f\rangle \leqq \langle H_t^2 f,f\rangle \text{ for any } f \in \mathcal{G} .$$

Thus $\lim \langle H_t^2 f,f\rangle = \lim \langle H_s H_t f,f\rangle$. Therefore

$$\lim \| H_t f-H_s f \|^2 = \lim \langle H_t f - H_s f, H_t f - H_s f\rangle =$$

$$= \lim \left[\langle H_t^2 f,f\rangle -2\langle H_s H_t f,f\rangle + \langle H_s^2 f,f\rangle\right] = 0;$$

this means that for each $f \in \mathcal{G}$ the strong limit of $H_t f$ exists and therefore the strong limit of $A_t f$ exists. In other words, there exists a bounded linear operator A such that $A_t f \longrightarrow Af$ for all $f \in \mathcal{G}$. The operator A is symmetric and therefore $A \in \mathcal{T}$. Since the path $(A_t)_{t\in T}$ is increasing, for any $f \in \mathcal{G}$ we have $\langle A_t f,f\rangle \leqq \langle Af,f\rangle \leqq \langle Bf,f\rangle$. Hence $A_t \leqq A$ for all $t \in T$ and $A \leqq B$. Since B can be taken as any upper bound of the set $\{A_t\colon t\in T\}$ we have that $A = \sup A_t$.

VIII: Let $\mathfrak{A}$ be any strongly closed ring of bounded self-adjoint operators, then $C \vee 0$ exists for any $C \in \mathfrak{A}$.

Proof: Let $C \in \mathfrak{A}$. Since $\mathfrak{A}$ is a ring, $C^2 \in \mathfrak{A}$. It is clear that $C^2 \geqq 0$. From the proof of proposition IV above concerning the existence and uniqueness of the square root of a positive bounded self-adjoint linear operator we know that this square root can be represented as the strong limit of a certain sequence of polynomials in terms of the "radical" operator without free term. We recall that the sequence of approximating polynomials for the square root of the operator A was of the form:

$$B_0 = 0, \quad B_{n+1} = B_n + (1/2)(A - B_n^2),$$

where we assumed that $\| A \| \leqq 1$. Since $C^2 \in \mathfrak{A}$, then any polynomial in terms of C^2 without free term is also contained in $\mathfrak{A}$, and since $\mathfrak{A}$ is strongly closed, $B = (C^2)^{\frac{1}{2}} \in \mathfrak{A}$. We put

$$A = (1/2)(B + C) \qquad (A \in \mathfrak{A})$$

and show that $A = C \vee 0$.

Let $D = A-C = (1/2)(B-C) \in \mathfrak{A}$. Let $\mathfrak{N}_1$ denote the null-space of the operator D, that is the subspace consisting of all $f \in \mathfrak{H}$ such that $Df = 0$. Let $\mathfrak{N}_2$ denote the orthogonal complement of $\mathfrak{N}_1$ in $\mathfrak{H}$. It is easy to see that $\mathfrak{N}_1$ is invariant for any operator $Q \in \mathfrak{A}$; indeed, if $f \in \mathfrak{N}_1$, then

$$D(Qf) = QDf = Q0 = 0$$

(because $\mathfrak{A}$ is a commutative ring) and therefore $Qf \in \mathfrak{N}_1$. From the invariance of $\mathfrak{N}_1$ also follows the invariance of $\mathfrak{N}_2$ for any operator $Q \in \mathfrak{A}$. From the definition of $\mathfrak{N}_1$

we have that $C = B = A$ on $\mathfrak{M}_1$. Moreover,

$$DA = (1/4)(B-C)(B+C) = (1/4)(B^2 - C^2) = 0.$$

Thus $DAf = 0$ for any $f \in \mathfrak{H}$ and therefore $Af \in \mathfrak{M}_1$ for all $f \in \mathfrak{H}$. On the other hand, if $f \in \mathfrak{M}_2$, then $Af \in \mathfrak{M}_2$ because of the invariance of the subspace $\mathfrak{M}_2$. Thus $A = 0$ on $\mathfrak{M}_2$. But $D = B-A$ and therefore $B = D$ on $\mathfrak{M}_2$.

Any $f \in \mathfrak{H}$ can be represented in the form $f = f_1 + f_2$, where $f_k \in \mathfrak{M}_k$ $(k = 1,2)$. By the invariance of the subspace $\mathfrak{M}_k$ $(k = 1,2)$ we have for any operator $Q \in \mathfrak{A}$:

$$\langle Qf,f\rangle = \langle Qf_1,f_1\rangle + \langle Qf_2,f_2\rangle . \qquad (25)$$

But $Af_2 = Df_1 = 0$. We therefore get

$$\langle Af,f\rangle = \langle Af_1,f_1\rangle = \langle Bf_1,f_1\rangle \qquad (26)$$

and

$$\langle Df,f\rangle = \langle Df_2,f_2\rangle = \langle Bf_2,f_2\rangle.$$

By the definition of square root $B \geqq 0$; thus $A \geqq 0$ and $D \geqq 0$ implying $A \geqq C$.

Suppose now that H is an arbitrary operator of $\mathfrak{A}$ satisfying $H \geqq 0$ and $H \geqq C$. Then for any $f \in \mathfrak{H}$ we have by virtue of the fact that $C = B$ on $\mathfrak{M}_1$ and by force of the equations (25) and (26) that $\langle Af,f\rangle = \langle Bf_1,f_1\rangle = \langle Cf_1,f_1\rangle \leqq$ $\leqq \langle Hf_1,f_1\rangle \leqq \langle Hf_1,f_1\rangle + \langle Hf_2,f_2\rangle = \langle Hf,f\rangle$. This means that $A \leqq H$ and consequently $A = C \vee 0$.

Remark: It is easy to see that $\mathfrak{A}$ is a vector lattice; we only need to verify that $A, B \in \mathfrak{A}$ implies the existence of $A \vee B$. The foregoing proposition VIII gives that $(A-B) \vee 0$ exists. But it can be seen that $\left[(A-B) \vee 0\right] + B = A \vee B$.

IX: Any strongly closed ring $\mathfrak{A}$ of bounded self-adjoint linear operators on a Hilbert space forms a complete vector lattice.

Proof: By the foregoing remark it remains to show that an arbitrary subset $\mathcal{C}$ of $\mathfrak{A}$ which is bounded from above has a supremum. We adjoin to $\mathcal{C}$ the suprema of all its finite subsets. We can then regard $\mathcal{C}$ as an upward directed set. If in $\mathcal{C}$ there is a largest operator, then it will be the supremum of the set $\mathcal{C}$. If however there is no such operator in $\mathcal{C}$, then the operators making up $\mathcal{C}$ form an increasing path and by proposition VII above this path has a strong limit A, where $A = \sup \mathcal{C}$ in $\mathfrak{S}$. Since the ring $\mathfrak{A}$ is strongly closed, $A \in \mathfrak{A}$, and $A = \sup \mathcal{C}$ in the lattice $\mathfrak{A}$ as well. This completes the proof.

Let $C^{+} = C \vee 0$, $C^{-} = (-C) \vee 0$ and $|C| = C^{+} + C^{-}$. In the proof of proposition VIII we showed that $C^{+} = A = (1/2)(B+C)$. Thus $C^{-} = C^{+} - C = (1/2)(B-C) = D$, $|C| = C^{+} + C^{-} = B$,

$|C| = (C^2)^{\frac{1}{2}}$, $C^- = 0$ on $\mathfrak{M}_1$ and $C^+ = 0$ on $\mathfrak{M}_2$; the space $\mathfrak{H}$ decomposes into the orthogonal subspaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Each operator of $\mathfrak{A}$ is invariant on $\mathfrak{M}_1$ and on $\mathfrak{M}_2$. This means that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ reduce $\mathfrak{A}$. $C = |C| = C^+$ on $\mathfrak{M}_1$ and $C = -|C| = -C^-$ on $\mathfrak{M}_2$. Therefore for any $f \in \mathfrak{H}$ we get $Cf = \overset{+}{C} f_1 - \overset{-}{C} f_2$, where $f_k \in \mathfrak{M}_k$ for $k = 1,2$ ($f_k = \mathrm{pr}_{\mathfrak{M}_k} f$; $k = 1,2$) and $\overset{+}{C} f_1 \in \mathfrak{M}_1$ and $\overset{-}{C} f_2 \in \mathfrak{M}_2$.

On the other hand $|C|\, f = \overset{+}{C} f_1 + \overset{-}{C} f_2$ and therefore for any $f \in \mathfrak{H}$

$$\|\, |C|\, f \,\| = \| Cf \| . \tag{27}$$

By (27) we get that the null-spaces of the operators A and $|A|$ coincide and, moreover, that for any $A \in \mathfrak{A}$

$$\| A \| = \|\, |A| \,\| . \tag{28}$$

X: Let $A_j \in \mathfrak{A}$ $(j = 1,2,\ldots,n)$ and $A = A_1 \vee A_2 \vee \ldots \vee A_n$. Then $\mathfrak{H}$ can be decomposed into mutually orthogonal subspaces $\mathfrak{H}_j$ reducing $\mathfrak{A}$, such that $A = A_j$ on $\mathfrak{H}_j$ for each j.

Proof: First we verify the statement for $n = 2$. Then

$$A = A_2 + (A_1 - A_2)^+ = A_2 + (1/2)\left[(A_1 - A_2) + |A_1 - A_2|\right] =$$
$$= (1/2)\left[(A_1 + A_2) + |A_1 - A_2|\right].$$

By what has been said above, $\mathfrak{H}$ decomposes into the orthogonal subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, reducing $\mathfrak{A}$, such that $|A_1 - A_2| = A_1 - A_2$ on $\mathfrak{H}_1$ and $|A_1 - A_2| = A_2 - A_1$ on $\mathfrak{H}_2$. Then $A = A_1$ on $\mathfrak{H}_1$ and $A = A_2$ on $\mathfrak{H}_2$.

We now proceed by induction and suppose that the proposition is true for n operators. Let $A = A_1 \vee \dots \vee A_n \vee A_{n+1}$ ($A_j \in \mathfrak{A}$). We put $B = A_1 \vee \dots \vee A_n$ so that $A = B \vee A_{n+1}$. As in the case of two operators, we decompose $\mathfrak{H}$ into two subspaces $\mathcal{L}$ and $\mathfrak{M}$, reducing $\mathfrak{A}$, such that

$$A = \begin{cases} B \text{ on } \mathcal{L} \\ A_{n+1} \text{ on } \mathfrak{M}. \end{cases}$$

For each operator $Q \in \mathfrak{A}$, we signify by Q' its component in the subspace $\mathcal{L}$. The operators Q' also form a strongly closed ring which we denote by $\mathfrak{A}'$ and by proposition IX $\mathfrak{A}'$ forms a complete vector lattice. The modulus of each operator Q' of $\mathfrak{A}'$ is defined by $((Q')^2)^{\frac{1}{2}}$, that is the component of the operator $|Q|$ in the subspace $\mathcal{L}$. Then $(Q')^+$ and $(Q')^-$ are the components of Q^+ and Q^-, respectively. We see that the analogous statement for bounds of finite sets of operators of $\mathfrak{A}'$ holds as well and in $\mathfrak{A}'$ we have the equality $B' = A_1' \vee \ldots \vee A_n'$. By the inductive assumption there exists a decomposition of the subspace $\mathcal{L}$ into mutually orthogonal subspaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$, reducing $\mathfrak{A}'$, and therefore also reducing $\mathfrak{A}$, such that $B' = A_j$ on $\mathfrak{H}_j$. Thus we have that $B = A_j$ on $\mathfrak{H}_j$. If we put $\mathfrak{H}_{n+1} = \mathfrak{M}$, then we see that the subspaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ and $\mathfrak{H}_{n+1}$ form the required decomposition.

Remark: A similar proposition holds for infimum.

XI: In order that the operators $A, B \in \mathfrak{A}$ be disjunct as elements of the complete vector lattice $\mathfrak{A}$, it is necessary and sufficient that the space $\mathfrak{H}$ decomposes into orthogonal subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ such that $A = 0$ on $\mathfrak{H}_1$ and $B = 0$ on $\mathfrak{H}_2$.

Proof: Since the null-space of any operator A of $\mathfrak{A}$ coincides with the null-space of the modulus $|A|$, it is enough to consider the case when $A, B \geqq 0$.

If $A \wedge B = 0$, then by the above remark the condition is seen to be necessary.

Conversely, suppose that the condition holds. We put $C = A \wedge B$. We observe that $U, V \in \mathfrak{A}$ and $0 \leqq U \leqq V$ imply $\|Uf\| \leqq \|Vf\|$ for any $f \in \mathfrak{H}$. Using this fact, we see that from $0 \leqq C \leqq A$ we get at once $C = 0$ on $\mathfrak{H}_1$. In the same manner we show that $C = 0$ on $\mathfrak{H}_2$. Thus $Cf = 0$ for all $f \in \mathfrak{H}$ and therefore $A \wedge B = 0$.

XII: If the strongly closed ring $\mathfrak{A}$ contains the identity operator E of $\mathfrak{H}$, then one can take E as unit of the complete vector lattice $\mathfrak{A}$. In this case the basis $\mathfrak{L}(\mathfrak{A})$ of the complete vector lattice will consist of all projection operators, contained in $\mathfrak{A}$, and $\mathfrak{A}$ itself will be a complete vector lattice of bounded elements.

Proof: First we verify that E acts as unit in $\mathfrak{A}$. Let $A \in \mathfrak{A}$, $A \geqq 0$ and $A \wedge E = 0$. By proposition XI above there exist mutually orthocomplemented subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ such that $A = 0$ on $\mathfrak{H}_1$ and $E = 0$ on $\mathfrak{H}_2$. But the latter means that $\mathfrak{H}_2$ contains the zero element of $\mathfrak{H}$ only. Hence $\mathfrak{H}_1 = \mathfrak{H}$ and $Af = 0$ on all of $\mathfrak{H}$.

Let E be taken as unit in the complete vector lattice $\mathfrak{A}$. If A is an arbitrary operator in $\mathfrak{A}$ and $m = \inf_{\|f\|=1} \langle Af,f \rangle$ and $M = \sup_{\|f\|=1} \langle Af,f \rangle$, then for any $f \in \mathfrak{H}$ we have

$$\langle mEf,f \rangle = m\langle f,f \rangle \leqq \langle Af,f \rangle \leqq M\langle f,f \rangle = \langle MEf,f \rangle .$$

Thus $mE \leqq A \leqq ME$ which means that A is a bounded element of the complete vector lattice $\mathfrak{A}$.

Let the operator $P \in \mathfrak{A}$ project the space $\mathfrak{H}$ onto the subspace $\mathfrak{L}$. Then $E-P$ projects $\mathfrak{H}$ onto the subspace

$\mathfrak{M} = \mathfrak{H} \ominus \mathfrak{L}$. Hence, by proposition XI above, we get that $P \wedge (E-P) = 0$; this means that P belongs to the basis $\mathfrak{B}(\mathfrak{A})$.

Conversely, let P be an arbitrary unitary element of $\mathfrak{A}$, $\mathfrak{H}_1$ the null-space of P and $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1$. Since $P \wedge (E-P) = 0$, then by proposition XI above we get that $E-P = 0$ on $\mathfrak{H}_2$, or $P = E$ on $\mathfrak{H}_2$, and $P = 0$ on $\mathfrak{H}_1$. This means that P is the projection operator of $\mathfrak{H}$ onto $\mathfrak{H}_2$. The proof of the proposition is finished.

Let A denote an arbitrary (not necessarily bounded) self-adjoint linear operator in a Hilbert space $\mathfrak{H}$. Let $\mathfrak{E}$ be the set of all bounded self-adjoint linear operators in $\mathfrak{H}$ which permute with A. The set $\mathfrak{E}$ is strongly closed, but might not be a ring because the operators permutable with A might not commute with each other. Select from $\mathfrak{E}$ the subset $\mathfrak{A}$, consisting of all operators contained in $\mathfrak{E}$ and permuting with any operator from $\mathfrak{E}$. It is clear that $\mathfrak{A}$ is not empty because it contains the identity operator E. $\mathfrak{A}$ is a ring. In fact, it is enough to verify that multiplication does not lead outside of $\mathfrak{A}$. If $U, V \in \mathfrak{A}$, then $U, V \in \mathfrak{E}$ and therefore $UV = VU$ and this product belongs to $\mathfrak{A}$. $\mathfrak{A}$ is strongly closed. By proposition IX $\mathfrak{A}$ is a complete vector lattice. We take E as unit in this vector lattice (see proposition XII).

If the operator A is bounded, then $A \in \mathfrak{A}$ and by the integral representation theorem of H. Freudenthal we have

$$A = \int_{-\infty}^{\infty} t\, d\, E_t ,$$

where $E_t = e_t^A$, that is (E_t), $-\infty < t < \infty$, is the resolution of the operator A. By proposition XII the resolution (E_t) consists of projection operators. From Number 71 (see page 170) we get the basic properties of the resolution. In particular, the left-continuity of the resolution means that

$$E_t = \lim_{s \to t-0} E_s \tag{29}$$

is in the sense of strong convergence. Indeed, by proposition VII we have that $\lim\limits_{s < t} E_s$ in $\mathcal{V}$ is the strong limit of the path (E_s), and since all $E_s \in \mathfrak{A}$, this limit also belongs to $\mathfrak{A}$, because the ring $\mathfrak{A}$ is strongly closed. This limit has to be the supremum of the set $\{E_s : s < t\}$ in $\mathfrak{A}$; by Number 71 it coincides with E_t and we have formula (29). The resulting representation of the operator A forms the content of the spectral theorem for bounded self-adjoint operators.

We now suppose that A is unbounded. By proposition V the positive bounded self-adjoint linear operator $B = (E + A^2)^{-1}$

exists, A permutes with B, and $\|B\| \leqq 1$. We show that $B \in \mathfrak{U}$. Let C be any operator in $\mathcal{E}$. Then $CA^2 \subset A^2C$ and therefore $C(E+A^2) \subset (E+A^2)C$. From the latter we obtain $BC(E+A^2)B \subset$ $\subset B(E+A^2)CB$. But $(E+A^2)B = E$ and $B(E+A^2) \subset E$. Thus $BC \subset CB$. Since $\mathcal{D}_{BC} = \mathfrak{H}$, we get $BC = CB$. Thus $B \in \mathcal{E}$ and B commutes with any $C \in \mathcal{E}$; this means that $B \in \mathfrak{U}$.

Let (E_t) be the resolution of the operator B. Since $0 \leqq B \leqq E$, we have that $E_t = 0$ for $t \leqq 0$ and $E_t = E$ for $t > 1$. We verify that the resolution (E_t) is continuous at $t = 0$ and to the right. Let $E_0 = \inf\limits_{t > 0} E_t$. Then E_0 is a projection operator and by formula (18) (see page170) it follows that $0 \leqq (E_0)B \leqq tE_0$ for $t > 0$; but this is true for any $t > 0$ and therefore $(E_0)B = 0$. By Number 79 (see page 174) the projection onto a component is equivalent with multiplication by the unitary element generating this component. Thus $E_0B = 0$, that is $B = 0$ on the subspace $\mathcal{L}$ onto which the operator E_0 projects $\mathfrak{H}$. Since, however, the operator B has an inverse, $\mathcal{L}$ has to consist of the zero element of $\mathfrak{H}$ only and therefore $E_0 = 0$.

Let $P_0 = E-E_1$ and $P_k = E_{1/k} - E_{1/(k+1)}$ for $k = 1,2,\ldots$.

By the continuity of the resolution (E_t) at $t = 0$ we have

$$\underset{k=0}{\overset{\infty}{S}} P_k = \sum_{k=0}^{\infty} P_k = E. \tag{30}$$

We denote by $\mathfrak{A}_k$ (k 0,1,2,...) the component of the complete vector lattice $\mathfrak{A}$ generated by the operator P_k and by $\mathfrak{H}_k$ the subspace of the Hilbert space $\mathfrak{H}$, on which the projection operator P_k is realized. By properties of the resolution, the operators P_k $(k = 0,1,2,\ldots)$ are pairwise disjunct; from proposition XI it follows that the subspaces $\mathfrak{H}_k$ $(k = 0,1,2,\ldots)$ are pairwise orthogonal. Formula (30) implies that the system ($\mathfrak{H}_k$) is complete in $\mathfrak{H}$ and that the components $\mathfrak{A}_k$ generate a decomposition of the complete vector lattice $\mathfrak{A}$ (see Number 59, pages 165 - 166). For, if we suppose that the system of subspaces $\mathfrak{H}_k$, generating the subspace $\mathfrak{L}$, were different from $\mathfrak{H}$ and P be the projection operator onto $\mathfrak{L}$, then by proposition VII it would follow that $P = \sum_{k=0}^{\infty} P_k$ in the sense of strong convergence and thus $P \in \mathfrak{A}$ and $P = \sup P_k$ in $\mathfrak{A}$. But this would contradict formula (30).

Since $P_k \in \mathfrak{A}$, all subspaces $\mathfrak{H}_k$ are invariant relative to the operator A and relative to any operator of $\mathfrak{A}$.

The range of the operator B coincides with the domain of definition of the operator A^2 and $\mathfrak{D}_{A^2} \subset \mathfrak{D}_A$. We have $BP_k = (P_k)E \geqq (1/(k+1))P_k$; therefore the operator BP_k has on $\mathfrak{H}_k$ a positive infimum and thus maps $\mathfrak{H}_k$ onto $\mathfrak{H}_k$ which means that $\mathcal{H}_k \subset \mathcal{D}_A$. Thus the operator A is symmetric on $\mathcal{H}_k$, mapping all of $\mathcal{H}_k$ into itself. Hence the operator A is bounded and self-adjoint on $\mathcal{H}_k$ and the operator $A_k = AP_k$ is bounded and self-adjoint on all of $\mathcal{H}$.

It is clear that all $A_k \in \mathcal{E}$. We verify that $A_k \in \mathfrak{A}$. Let $C \in \mathcal{E}$, then taking into account that $P_k \in \mathfrak{A}$, we have $CA_k = CAP_k \subset ACP_k = AP_kC = A_kC$. This means that A_k commutes with any $C \in \mathcal{E}$ and therefore $A_k \in \mathfrak{A}$.

The operators A_k are pairwise disjunct as elements of the complete vector lattice $\mathfrak{A}$, and A_k is disjunct from P_j for $k \neq j$. From the latter we get that $A_k \in \mathfrak{A}_k$ for each k.

We form the union X of the complete vector lattices $\mathfrak{A}_k$. Each element $x \in X$ has the form $x = \underset{k=0}{\overset{\infty}{S}} Q_k$, where $Q_k \in \mathfrak{A}_k$. From the set of bounded self-adjoint operators Q_k we construct the self-adjoint operator Q in the space $\mathcal{H}$, coinciding on each $\mathcal{H}_k$ with the operator Q_k (see proposition VI). Let Y be the set of all self-adjoint operators obtained by this method. We have $\mathfrak{A} \subset Y$. Moreover, if $x = S\, A_k$, then the corresponding operator coincides with A and therefore $A \in Y$.

By this construction one establishes a one-to-one correspondence between the elements of the complete vector lattice X and the operators of Y, and therefore, with the usual definitions for the algebraic operations and ordering, Y is turned into a complete vector lattice. By Number 62 (see page 167) we note that E can be taken as unit in Y and then by Number 63 (see page 167) we get that the two bases $\mathcal{L}(\mathfrak{A})$ and $\mathcal{L}(Y)$ coincide.

Thus we have an imbedding of the operator A into a certain complete vector lattice with unit, the basis of which consists of projections. Then the spectral resolution of the operator A is gotten in the same way as for bounded self-adjoint operators further above. From the very method for getting the spectral resolution it directly follows that the spectral family of a self-adjoint operator A consists of projection operators E_t, permuting with any operator $C \in \mathfrak{Y}$ which permutes with A.

References

Ahieser, N. I. & M. Krein: Some questions in the theory of moments. vol. 2, Trans. of math. monographs, Amer. Math. Soc., Providence 1962

Bernstein, S. N. Sur les fonctions absolument monotones. Acta Math. 52(1928), 1-66

Birkhoff, G. Lattice theory. Amer. Math. Soc. Colloquium Pub. 25(1940)

Bochner, S. Beitraege zur Theorie der fastperiodischen Funktionen. Math. Ann. 96(1927), 119-147 and 383-409

Bochner, S. Vorlesungen ueber Fouriersche Integrale. Akad. Verlagsgesellschaft, Leipzig 1932

Bohr, H. Collected mathematical works. Ed. by E. Folner & B. Jessen. 3 vols. Kopenhagen 1952

Dunford, N. & J. T. Schwartz: Linear operators. Part I: General theory, Part II: Spectral theory. Interscience Publishers, New York 1958, 1963

Edwards, R. E. Functional analysis, theory and applications. Holt, Rinehart and Winston, 1965

Freudenthal, H. Teilweise geordnete Moduln. Neder. Akad. Wetensch. Proc. 39(1936), 641-651

Freudenthal, H. Ueber die Friedrichssche Fortsetzung halbbeschraenkter Hermitescher Operatoren. Ned. Akad. Wetensch. Proc. 39(1936), 832-833

Friedrichs, K. O. Spektraltheorie halbbeschraenkter Operatoren und ihre Anwendung auf die Spektralzerlegung von Differentialoperatoren. Math. Ann. 109(1934), 465-487

Gelfand, I. M. & D. A. Raikow & G. E. Schilow: Kommutative normierte Algebren. VEB Deutscher Verlag der Wissenschaften, Berlin 1964

Hamburger, H. Ueber eine Erweiterung des Stieltjes'schen Momemtenproblems. Math. Ann. 81(1920), 235-319 and 82(1921), 120-164, 168-187

Helly, E. Ueber lineare Funktionaloperationen. S.-B. Akad. Wiss. Wien 121(1912), 265-297

Herglotz, G. Ueber Potenzreihen mit positivem, reellen Teil im Einheitskreis. S.-B. Saechs. Akad. Wiss. 63(1911), 501-511

Jordan, P. & J. v. Neumann: On inner products in linear, metric spaces. Ann. of Math. (2) 36(1935), 719-723

Kadison, R. V. Order properties of bounded self-adjoint operators. Proc. Amer. Math. Soc. 2(1951), 505-510

Kantorovic, L. V. & B. Z. Vulih & A. G. Pinsker: Partially ordered groups and partially ordered linear spaces. Trans. Amer. Math. Soc., vol. 27; pp. 51-124

Kolmogoroff, A. Stationary sequences in Hilbert space. Boll. Mos. Gos. Univ. Mat. 2(1940); Trans. into Spanish in Trabajos de Estadistica, Madrid, 4, 55-73 and 243-270

Koethe, G. Topologische lineare Raeume I. Die Grundlehren d. math. Wiss., vol. 107, Springer, 1960

Maak, W. Fastperiodische Funktionen. Die Grundlehren d. math. Wiss., vol. 61, Springer, 1950

Meschkowski, H. Hilbert Raeume mit Kernfunktionen. Die Grundlehren d. math. Wiss., vol. 113, Springer, 1962

Moore, E. H. General analysis, I, II. Mem. Amer. Philos. Soc., Philadelphia 1935, 1939

Nakano, H. Modulared semi-ordered linear spaces. Tokyo Math. Book Series, vol. I. Maruzen Co. Ltd., Tokyo 1950

Nakano, H. Modern spectral theory. Tokyo Math. Book Series, vol. II. Maruzen Co. Ltd., Tokyo 1950

Nakano, H. Semi-ordered linear spaces. Japan Society for the Promotion of Science. Tokyo 1955

Neumann, v. J. Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren. Math. Ann. 102(1929), 49-131

Neumark, M. A. Normierte Algebren. VEB Deutscher Verlag der Wissenschaften, Berlin 1960

Neumark, M. A. Lineare Differentialoperatoren. Akademie-Verlag, Berlin 1960

Ogasawara, T. Theory of vector lattices. J. Sci. Hiroshima Univ., Ser. A, 12(1942), 37-100 and 13(1944), 41-161; Trans. into English by S. Kakutani, Yale University

Riesz, F. & B. v. Sz.-Nagy: Lecons d'analyse fonctionnelle. Akademiai Kiado, Budapest 1955

Smirnow, W. A. Lehrgang der hoeheren Mathematik, Teil V. VEB Deutscher Verlag der Wissenschaften, Berlin 1962

Steen, S. W. P. An introduction to the theory of operators I. Proc. London Math. Soc. 41(1936), 361-392

Stone, M. H. Linear transformations in Hilbert space and their application to analysis. Colloquium Publications of the Amer. Math. Soc. 1932

Sz.-Nagy, v. B. Perturbations des transformations auto-adjointes dans l'espace de Hilbert. Commentarii Math. Helv. 19(1946/47), 347-366

Widder, D. V. The Laplace transform. Princeton Univ. Press, Princeton 1946

Wilansky, A. Functional analysis. Blaisdell Publishing Co. 1964

Yosida, K. Functional analysis. Die Grunglehren d. math. Wiss. vol. 123, Academic Press; Springer, 1965

Zaanen, A. C. Linear analysis. Interscience Publishers Inc. New York 1953