AN ALGEBRA OF

BOUNDED ANALYTIC FUNCTIONS

A FILTER DESCRIPTION FOR THE HOMOMORPHISMS OF THE ALGEBRA OF BOUNDED ANALYTIC FUNCTIONS ON THE UNIT DISC

By

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SCOPE AND CONTENTS: For any filter F defined on the unit disc D, F* is the filter generated by ϵ-neighbourhoods of the sets of F, using hyperbolic distance. Any complex homomorphism Φ of $\mathfrak{B}$, the algebra of bounded analytic functions on D, is given by $\Phi(g) = \lim g(F^*)$ for some maximal closed filter F. The homomorphisms can be classified according to the direction of approach to the boundary of the corresponding filters. For those which are obtained by oricycle or non-tangential approach, the *-filters are in 1-1 correspondence with the homomorphisms; and into these subsets, one can analytically embed discs. On the Silov boundary of $\mathfrak{B}$, the above correspondence fails to be 1-1, and smaller filters are considered.

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INTRODUCTION

Let $\mathcal{B}$ denote the algebra of all bounded analytic functions on the open unit disc D. We shall attempt to characterize the maximal ideal space $\mathfrak{M}$ of this algebra in terms of filters on D. A characterization of this type can be given for the algebra $\mathcal{A}$ of all analytic functions on D; in this case, as in the similar case of the ring of all entire functions, maximal ideals are in one-one correspondence with ultrafilters of zero sets. In these examples, the zero sets are the discrete closed subsets of D or of the complex plane respectively, and each maximal ideal corresponds to the ultrafilter generated by all zero sets of functions in the ideal. See Banaschewski [1]. However when we consider the algebra of bounded analytic functions on D, this description fails in several respects. For one thing, it is no longer true that every function without zeros is invertible. One might hope to avert this difficulty by using the same filters, but a different correspondence between filters and ideals. A situation of this type holds for the maximal ideals in the algebras of all continuous and of all bounded continuous functions on a completely regular space. This is discussed fully in [10] for real valued functions. The maximal ideals of both algebras are in one-one correspondence with maximal filters in the lattice of zero sets, but the correspondence differs in the two cases. For the algebra of all continuous functions it is as given above, but for bounded functions a maximal filter corresponds to the ideal of all functions approaching zero along it.

Although we adopt this latter description in the case of bounded analytic functions, we find that the filters describing the maximal ideals of $\mathcal{A}$ still fail for several reasons to work for the subalgebra $\mathcal{B}$. In the first place, the zero sets differ in the two algebras. Any discrete closed subset of D, that is any sequence without cluster points in D, will serve as a zero set for an analytic function, but a sequence (z_n) of points in D will be the zero set of a bounded analytic function only if it approaches the boundary rapidly enough to satisfy the condition

$$\Sigma\,(1 - |z_n|) < \infty.$$

A second difficulty arises from the following classical theorem of Pick. Suppose (a_n) is a sequence of points in D, which approaches the boundary, and f is a function from B which tends to the value α along this sequence. For any sequence (b_n) in D, where the hyperbolic distances $h(a_n,b_n)$ approach zero as n increases, the sequence $(f(b_n))$ also tends to α. One can use this to construct different ultrafilters along which every function in $\mathcal{B}$ has the same limit, and which therefore correspond to the same maximal ideal.

Finally, $\mathcal{B}$ contains maximal ideals which cannot correspond to any filter containing the zero set of a function in $\mathcal{B}$. These maximal ideals (from the Silov boundary) are generated by functions which have no zeros, and which are therefore units in $\mathcal{A}$. The filters which are required to describe these contain only sets which are much larger than the zero set of an analytic function.

As an attempt to overcome these difficulties, we shall proceed as follows. It has recently been shown that the compact space $\mathfrak{M}$ is

an extension space of D. This means that there is a continuous map over D from βD onto $\mathfrak{M}$ where βD is the Stone-Cech compactification of D. The points of βD can be described using maximal closed filters on D, and this description also applies to the points of $\mathfrak{M}$. However the maximal closed filters are too large, and are in a many-one correspondence with the maximal ideals of B. In the last section of Chapter 1, we use the hyperbolic metric to make identifications to form smaller filters called *-filters. In chapters 2 and 3, we find subsets, $\mathfrak{N}$ and $\mathfrak{D}$, of the non-trivial portion of $\mathfrak{M}$ which are in 1-1 correspondence with *-filters. These subsets correspond to filters whose approach to the boundary of D, in terms of the hyperbolic geometry, is simple. The sets $\mathfrak{N}$ and $\mathfrak{D}$ correspond to filters which approach the boundary along hypercycles and along oricycles respectively.

The 1-1 correspondence just described will quite possibly hold for a larger part of $\mathfrak{M}$; this is not decided because the proofs given depend on the special mode of approach to the boundary. However we see in Chapter 4 that the correspondence fails radically to be 1-1 on the Silov boundary X of $\mathcal{B}$. The space X is known to be a Stone space, and the points in it can be put in correspondence with filters on the unit circle. These in turn can be used to construct filters on D which must correspond to the points in the Silov boundary. It is then shown that each of these filters is contained in many different *-filters.

The notion of an analytic embedding of the unit disc in $\mathfrak{M}$ is defined in Hoffman [14] pp 166-169, and an example is given. By means of filters this example, which maps D into $\mathfrak{D}$, is made more explicit

in Chapter 5, and another example which maps into $\mathcal{N}$ is given. The chapter begins by constructing an analytic mapping of D (not necessarily an embedding) whose image contains any preassigned point of $\mathcal{M}$.

In the last chapter, the topology of $\mathcal{M}$ is discussed, and an attempt is made to arrange the points in $\mathcal{M}$ according to the geometric position in D of their filters. Some cardinality results are given; both $\mathcal{M}$ and X contain $2^{2^{\aleph_0}}$ points.

CHAPTER 0

<u>Introduction</u> We collect various introductory material together in this chapter. In §1, we discuss commutative Banach algebras with identity, giving the Gelfand representation. The existence of the Silov boundary is proved, and examples are given of algebras used subsequently. The material in §2 is from Hoffman [14]; several classical theorems about analytic functions on the unit disc are needed. Among these are the Fatou theorem, the theorem of F. and M. Riesz, and the decomposition theory for functions in H^1. Section §3 discusses the Wallman description of the Stone-Cech compactification of a normal space. The last section deals with the hyperbolic metric in the unit disc D; a further reference for this is Carathéodory [6].

§1 BANACH ALGEBRAS

A Banach algebra $\mathfrak{A}$ is a Banach space, that is a complete normed linear space, with a multiplication satisfying

$$||X.Y|| \leq ||X||.||Y|| \quad \text{for all} \quad X, Y \in \mathfrak{A}.$$

We shall restrict ourselves from the outset to commutative algebras with identity. As it is no less general, we assume that the identity of $\mathfrak{A}$, denoted by 1, has norm unity. We always take the complex field C for the field of scalars.

A fundamental property of Banach algebras is the 1-1 correspondence between maximal ideals and complex homomorphisms. This is a consequence of the following theorem.

<u>THEOREM 1 (Gelfand-Mazur).</u> Every complex Banach algebra which is a field is isomorphic to the complex field C.

Before the proof, we give several preliminary results.

<u>Lemma 1</u>. An element $x \in \mathfrak{A}$ is regular if $||x-1|| < 1$. Its inverse is given by the formula

$$x^{-1} = 1 + \Sigma_1^{\infty} (1 - x)^n.$$

Proof. Since $||(1-x)^n|| \leq ||1-x||^n$, the partial sums of $\Sigma_1^{\infty} (1-x^n)$ form a Cauchy sequence, and we use the infinite sum to denote the limit of this (necessarily convergent) sequence. Defining $y = 1 + \Sigma_1^{\infty} (1-x)^n$, we obtain

$y - xy = (1-x)y = (1-x) + \Sigma_2^{\infty} (1-x)^n = y - 1$

or $xy = 1$ and $y = x^{-1}$.

<u>Lemma 2</u>. The set G of all regular elements of $\mathfrak{A}$ is open, and the

mapping $x \longrightarrow x^{-1}$ for $x \in G$ is a homeomorphism of G onto itself.

<u>Proof.</u> Take any $y \in G$. We have $||y^{-1}|| > 0$, and can define a neighbourhood N of y

$$N = \{x \in \mathfrak{A} : \ ||x-y|| < 1/||y^{-1}||\}$$

But for any $x \in N$, we have

$$||y^{-1}x - 1|| = ||y^{-1}(x-y)|| \leq ||y^{-1}|| \cdot ||x-y|| < 1.$$

By lemma 1 this means $y^{-1}x$ is invertible, and therefore so is $x = y(y^{-1}x)$. In other words, the entire neighbourhood N of y is contained in G.

To show that the mapping is a homeomorphism onto, it is sufficient to establish continuity. This is done by finding for each $y \in G$ a neighbourhood $N_y = \{x \in \mathfrak{A} : \ ||x-y|| < 1/2||y^{-1}||\}$, contained in G, on which

$$(1) \qquad ||x^{-1} - y^{-1}|| < C_y \cdot ||x-y||$$

where the constant C_y depends on y, but not on x. This immediately gives the required continuity, for it shows that x^{-1} will lie inside a given neighbourhood of y^{-1} whenever x lies in a small enough neighbourhood of y.

If $x \in N_y$, we have $y^{-1}x \in G$ since

$$(2) \qquad ||y^{-1}x - 1|| \leq ||y^{-1}|| \cdot ||x-y|| < \frac{1}{2},$$

and we can expand $x^{-1}y = (y^{-1}x)^{-1} = 1 + \sum_1^\infty (1 - y^{-1}x)^n$. Hence

$$||x^{-1} - y^{-1}|| \leq ||y^{-1}|| \cdot ||x^{-1}y - 1|| = ||y^{-1}|| \cdot ||\sum_1^\infty (1 - y^{-1}x)^n||$$

$$\leq ||y^{-1}|| \cdot ||1 - y^{-1}x|| \left(\sum_0^\infty ||1 - y^{-1}x||^n \right)$$

$$\leq 2\,||y^{-1}||\,.\,(||1 - y^{-1}x||).$$

Applying the first part of (2) to the bracketed part, we obtain (1) as required, where $C_y = 2\,||y^{-1}||^2$. This completes the proof of Lemma 2. The following important Lemma 3 will lead quickly to a proof of the theorem. It depends on the notion of the spectrum of an element in the algebra.

We define the <u>spectrum</u> of $x \in \mathfrak{A}$ to be the set $\sigma(x) = \{\lambda : x-\lambda$ is singular$\}$ of scalars. (We shall write constant multiples $\lambda 1$ of the identity as λ.) We see readily that the spectrum of x is contained in the circle about the origin of radius $||x||$. In other words

(3) $\qquad |\lambda| > ||x|| \implies x-\lambda$ is invertible.

This follows from lemma 1, for $1 - \frac{1}{\lambda}x = \frac{-1}{\lambda}(x-\lambda)$ is invertible since $||\frac{1}{\lambda}x|| = \frac{1}{|\lambda|}\,||x|| < 1$. Hence $x-\lambda$ is also invertible.

We wish to show that $\sigma(x)$ is never empty. That it is closed follows because $x-\lambda$ is a continuous function of λ for x fixed, and the spectrum is the inverse image under this mapping of the set of singular elements in $\mathfrak{A}$, a closed set. We consider the function $(x-\lambda)^{-1}$ of λ, defined on the complement $\rho(x)$ of the spectrum, and use the theory of analytic functions to obtain a contradiction if $\rho(x)$ is the entire complex plane.

<u>Lemma 3</u>. $\sigma(x)$ is non-empty for each $x \in \mathfrak{A}$.

<u>Proof.</u> Take the function

$$x(\lambda) = (x-\lambda)^{-1}$$

defined and continuous on $\rho(x)$, an open set. If both λ and μ are in $\rho(x)$, then

$$\begin{aligned} x(\lambda) &= x(\lambda)\ (x-\mu)\ x(\mu) \\ &= x(\lambda)\ ((x-\lambda) - (\lambda-\mu))\ x(\mu) \\ &= (1 + (\lambda-\mu)\ x(\lambda))\ x(\mu) \end{aligned}$$

or $x(\lambda) - x(\mu) = (\lambda-\mu)\ x(\lambda)\ x(\mu)$.

Take any bounded linear functional F on $\mathfrak{A}$, and define the function $F(\lambda) = F((x-\lambda)^{-1})$ for $\lambda \in \rho(x)$. We find that f is analytic on $\rho(x)$.

$$\begin{aligned} f'(\mu) &= \lim_{\lambda \to \mu} \frac{f(\lambda) - f(\mu)}{\lambda-\mu} \\ &= \lim_{\lambda \to \mu} F\left(\frac{x(\lambda) - x(\mu)}{\lambda-\mu}\right) \\ &= \lim_{\lambda \to \mu} F\ (x(\lambda).x(\mu)) \\ &= F(x(\mu)^2). \end{aligned}$$

Furthermore, if we allow λ to tend to ∞, we have

$$\begin{aligned} |f(\lambda)| &\leq ||F||\ .\ ||(x-\lambda)^{-1}|| \\ &\leq \frac{||F||}{|\lambda|} \cdot \frac{1}{||1 - \frac{x}{\lambda}||} \longrightarrow 0 \end{aligned}$$

If $\sigma(x)$ is empty, we have entire functions f for each functional F, and by Liouville's theorem they must be constant. Because the constant can only be zero, $(x-\lambda)^{-1}$ is annihilated by every functional F, and must be zero.. This is impossible.

Proof of Theorem 1. Our Banach algebra is a field. If we choose any element $x \in \mathfrak{A}$, there must be a scalar $\lambda \in \sigma(x)$. The element $x - \lambda$, being singular in a field, is necessarily zero. Thus every element of $\mathfrak{A}$ is a scalar multiple of the identity, and the algebra is isomorphic to the complex field.

THEOREM 2. Every maximal ideal M in $\mathfrak{A}$ is closed and is the kernel of a continuous homomorphism onto the field of complex numbers.

Proof. The identity is in the open set G of units, none of which can belong to a maximal ideal M. The ideal cannot be dense in $\mathfrak{A}$; and its closure $\overline{M}$, being an ideal, must be the same as M. Because of this $\mathfrak{A}/M$ becomes a Banach space under the usual quotient norm

$$||x + M|| = \inf \{||x + y|| \mid y \in M\}.$$

It is also clear that

$$||1 + M|| = 1$$

$$||(x + M)(y + M)|| \leq ||x + M|| \cdot ||y + M||$$

in the quotient space, where we can treat $\mathfrak{A}/M$ as an algebra since M is an ideal. $\mathfrak{A}/M$ is therefore a Banach algebra; however it is also a field because M is maximal. By the Gelfand-Mazur theorem we have an identification of $\mathfrak{A}/M$ with C, and this gives rise to a homomorphism ϕ of $\mathfrak{A}$ onto C having kernel M. It is clear that $\phi(1) = 1$. Also

$$|\phi(x)| \leq ||x||$$

follows from $|\phi(x)| = ||x + M|| = \inf_{y \in M} ||x + y|| \leq ||x||$.

Together these two last remarks tell us that, as a linear functional, ϕ has norm exactly one. This of course applied to every complex homomorphism of $\mathfrak{A}$, since each one has for kernel some maximal ideal M.

As an example of a Banach algebra, we consider the complex

algebra $C(X)$ of all bounded continuous complex valued functions on a compact Hausdorff space X. Multiplication and addition are defined pointwise and the norm is the supremum over X of a given function. We can readily see that all the homomorphisms are "fixed", in other words are evaluations of the functions of $C(X)$ at some point $x_0 \in X$. For if a proper ideal I has a function f_x with $f_x(x) \neq 0$ for each $x \in X$, then by compactness one can construct $f = \sum_{i=1}^{n} f_{x_i} \bar{f}_{x_i} \in I$ for a suitable finite set $x_1, x_2, \ldots, x_n$ of points of X such that $f \geq \delta > 0$ on the space X. But this means that the ideal I contains an invertible element f, which is a contradiction.

If we denote by $\mathfrak{M}$ the set of maximal ideals of $C(X)$, and give it the hull kernel topology, then we remark that X is homeomorphic with $\mathfrak{M}$ under the mapping

$$x \longrightarrow M_x = \{f \in C(X) : f(x) = 0\}.$$

For if $B \subseteq X$, and we write

$$I_B = \{f \in C(X) : f|B = 0\}$$

then $f \in I_B$ implies $f(x) = 0$ for any $x \in \bar{B}$ by continuity. This means that $x \in \bar{B}$ implies $M_x \in \text{hull}\ (I_B)$. If on the other hand $x \notin \bar{B}$, we can by complete regularity of X find $f \in I_B$ with $f(x) \neq 0$, and this means $M_x \notin \text{hull}\ (I_B)$. Together these two observations show that the bijection $x \longrightarrow M_x$ carries the closure operator on X into the closure operator on $\mathfrak{M}$.

Suppose we again consider the Banach algebra $\mathfrak{A}$, and denote the space of non zero complex homomorphisms by $\mathfrak{M}$ (often called the maximal ideal space). We define for each element of $\mathfrak{A}$ a function on $\mathfrak{M}$ by the relation

(4) $\hat{f}(\phi) = \phi(f)$ for $f \in \mathfrak{A}$, $\phi \in \mathfrak{M}$.

$\mathfrak{M}$ is then given the weakest topology such that each such function $\hat{f}$ will be continuous. In other words a basic neighbourhood of $\phi_0 \in \mathfrak{M}$ has the form

(5) $\{\phi \in \mathfrak{M} \mid |\hat{f}_i(\phi) - \hat{f}_i(\phi_0)| < \epsilon \; ; \; f_1, f_2, \quad f_n \in \mathfrak{A} \; ; \; \epsilon > 0\}$.

Under this topology, the weak topology, $\mathfrak{M}$ will be a compact Hausdorff space. Observe that the topology we have defined, if we treat our complex homomorphisms as linear functionals on $\mathfrak{A}$, is the restriction to $\mathfrak{M}$ of the weak - * topology. Since the closed unit ball of the conjugate of a Banach space is always compact Hausdorff in the weak - * topology, and since $\mathfrak{M}$ is a subset of the unit sphere of $\mathfrak{A}^*$, we need only show that $\mathfrak{M}$ is a closed subset of $\mathfrak{A}^*$. The following set $\mathfrak{N}$ consists of $\mathfrak{M}$ united with the zero functional

$$\mathfrak{N} = \bigcap_{f,g \in \mathfrak{A}} \{\phi \in \mathfrak{A}^* : \phi(fg) = \phi(f)\,\phi(g)\}.$$

$$= \bigcap_{f,g \in \mathfrak{A}} \{\phi \in \mathfrak{A}^* : (\widehat{fg} - \hat{f}\hat{g})(\phi) = 0\}$$

where we extend the definition (4) for $\hat{f}$ to all functionals $\phi \in \mathfrak{A}^*$. $\mathfrak{N}$ is evidently closed in the weak-* topology, since each $\widehat{fg} - \hat{f}\hat{g}$ is continuous; and because $\mathfrak{M}$ is separated from the zero functional by $\hat{1}$, it is likewise closed.

In the formula (5), it is always possible to replace $f_1, \ldots, f_n$ by other functions of $\mathfrak{A}$, namely $f_1 - \phi_0(f_1), \ldots, f_n - \phi_0(f_n)$, which are zero at ϕ_0. Thus

$$\{\phi \in \mathfrak{M} \mid |\hat{f}_i(\phi)| < \epsilon, \quad f_1, f_2, \ldots, f_n \in \mathfrak{A} \; ; \; \epsilon > 0\}$$

is a general basic neighbourhood of ϕ_0, where $\hat{f}_i(\phi_0) = 0$, $i = 1,2, \ldots, n$.

To generate the topology of $\mathfrak{M}$ which we are using, it is sufficient to take in (5) functions $f \in \mathfrak{A}$ which belong to some set of generators for $\mathfrak{A}$. By a set of generators G for a Banach algebra $\mathfrak{A}$, we mean a collection of functions which, with the identity, generate a subalgebra of $\mathfrak{A}$ that is dense in $\mathfrak{A}$. In other words each element $g \in \mathfrak{A}$ is the limit of a sequence of polynomials P_K in a finite number of variables from G (and with constant terms). Suppose we generate a topology on $\mathfrak{M}$ using sets of the form (5) with $f_1, \ldots, f_n \in G$. If g is arbitrary in $\mathfrak{A}$, we can find a sequence (P_K) of polynomials as above with $\lim_{K \to \infty} ||P_K - g|| = 0$. The P_K are all continuous in the new topology. But $\hat{g}(\phi) = \lim_{K \to \infty} \hat{P}_K(\phi)$ uniformly for $\phi \in \mathfrak{M}$, and hence $\hat{g}$ is the uniform limit of continuous functions. Thus $\hat{g}$ is continuous for each $g \in \mathfrak{A}$, and the new topology coincides with the old.

Under the Gelfand representation, described in the following theorem, a general Banach algebra is mapped homomorphically into $C(\mathfrak{M})$.

<u>THEOREM 3.</u> The mapping of $\mathfrak{A}$ into $C(\mathfrak{M})$ given by $f \longrightarrow \hat{f}$ is a norm decreasing algebra homomorphism. Its kernel is the intersection of all maximal ideals of $\mathfrak{A}$. The image $\hat{\mathfrak{A}}$ is a subalgebra of $C(\mathfrak{M})$ which contains the identity and separates the points of $\mathfrak{M}$. An element $f \in \mathfrak{A}$ is regular if and only if $\hat{f}$ has no zeros on $\mathfrak{M}$.

<u>Proof.</u> That the mapping is a homomorphism follows readily from the formula $\hat{f}(\phi) = \phi(f)$ defining $\hat{f}$. For $\hat{f}$ to be identically zero, it is necessary and sufficient that $\phi(f) = 0$ for all $\phi \in \mathfrak{M}$, in other words that f belong to the radical of $\mathfrak{A}$, the intersection of all maximal ideals. For any $\phi \in \mathfrak{M}$, $|\hat{f}(\phi)| = |\phi(f)| \leq ||f||$ holds

because ϕ is a functional of norm one. Hence

$$||\hat{f}|| = \sup_{\phi \in \mathfrak{M}} |\hat{f}(\phi)| < ||f||.$$

The image of the identity of $\mathfrak{A}$ is a function which is identically one for all $\phi \in \mathfrak{M}$, in other words $\hat{\mathfrak{A}}$ contains the identity of $C(\mathfrak{M})$. Two homomorphisms ϕ and ψ of $\mathfrak{M}$, if they are different, must differ for some $f \in \mathfrak{A}$, which says $\hat{f}(\phi) \neq \hat{f}(\psi)$. $\hat{\mathfrak{A}}$ must therefore separate points in $\mathfrak{M}$. $\hat{\mathfrak{A}}$ is the image of an algebra, and is therefore itself an algebra. Finally, an element $f \in \mathfrak{A}$ fails to be regular if and only if it belongs to a proper ideal, in which case it belongs to a maximal ideal M and is annihilated by the homomorphism having kernel M.

In the Gelfand representation, the norm, in certain cases, is actually decreased. If, for example, $\mathfrak{A}$ has a non-trivial ideal as radical, then any f in this ideal will have $||f|| > 0$ and $||\hat{f}|| = ||\hat{0}|| = 0$. However even a semi-simple Banach algebra (one with zero radical) can have a Gelfand representation which actually decreases the norm. The situation is clarified by defining the <u>spectral norm</u> or spectral radius of elements in $\mathfrak{A}$.

$$r(t) = \sup\{|\lambda| \ : \ \lambda \in \sigma(f)\}.$$

The spectral norm $r(f)$ of any $f \in \mathfrak{A}$ is the maximum distance from the origin to a point in the spectrum. By (3), $r(f) \leq ||f||$. We will show that the spectral norm of f is the same as the norm of $\hat{f}$ in $C(\mathfrak{M})$. Then the Gelfand representation will not only be faithful but also norm preserving if the original norm of $\mathfrak{A}$ is identical with the spectral norm.

THEOREM 4. For each $f \in \mathfrak{A}$

$$r(f) = ||\hat{f}||$$

where $r(f)$ is the spectral norm, and $||\hat{f}||$ refers to the supremum norm on $\mathfrak{M}$.

Proof. If a constant $\lambda \in \sigma(f)$, then $f - \lambda$ is singular, and $\widehat{(f-\lambda)}(\phi) = 0$ for some $\phi \in \mathfrak{M}$. For this homomorphism ϕ, we have therefore $\hat{f}(\phi) = \lambda$. This argument is reversible, and hence the spectrum of f is precisely the set of zeros of $\hat{f}$. This gives the required result.

A Banach algebra in which the Gelfand representation is an isometric isomorphism is called a function algebra. Such an algebra is always a subalgebra of $C(\mathfrak{M})$ for some compact space $\mathfrak{M}$. An important notion in the study of function algebras is that of the Silov boundary.

THEOREM 5. The maximal ideal space $\mathfrak{M}$ of a function algebra $\mathfrak{A}$ contains a uniquely determined closed subset X, the smallest closed set in $\mathfrak{M}$ on which each $\hat{f} \in \hat{\mathfrak{A}}$ attains its maximum modulus. X is called the Silov boundary of $\mathfrak{A}$.

Proof. If we take any chain of closed subsets of $\mathfrak{M}$ on which each $\hat{f} \in \hat{\mathfrak{A}}$ attains its maximum modulus, it is evident that the intersection of the chain has the same property. For the set on which $\hat{f}_0$ attains its maximum modulus is compact for each $f_0 \in \mathfrak{A}$ and if it meets each of the compact sets of the chain it must also meet the intersection. Hence Zorn's lemma applies, and we obtain a minimal closed set X of the required type.

To see that X is in fact the smallest such set, suppose Y is another and $X \not\subseteq Y$. We choose $\phi_0 \in Y \setminus X$ and a neighbourhood N of ϕ_0 with $N \cap X = \emptyset$. N has the form

$$N = \{\phi \in \mathfrak{M} : |\hat{f}_i(\phi) - \hat{f}_i(\phi_0)| < \epsilon \ , \quad f_1, \ldots, f_n \in \mathfrak{A}\}$$

We can add suitable constants to these functions to make each $\hat{f}_i(\phi_0) = 0$. Suppose we also insist, by making N smaller if necessary, that $|\hat{f}_i| \leq 1$ for each i. Since Y is minimal, there must exist $f_0 \in \mathfrak{A}$ which does not attain its maximum modulus on $Y \setminus N$. We may take $||\hat{f}_0|| = 1$, and by replacing f_0 by a sufficiently high power $f_0^{\,n}$, we may also assume $|\hat{f}_0| < \epsilon$ on the compact set $Y \setminus N$. Then the inequality $|\hat{f}_i . \hat{f}_0| < \epsilon$ holds on Y for each i, and hence holds on all of $\mathfrak{M}$. If we take a point $\phi_1 \in X$ with $|\hat{f}_0(\phi_1)| = 1$, we have for $i = 1,2, \ldots, n$ that $|\hat{f}_i(\phi_1)| < \epsilon$, which implies $\phi_1 \in N$. Since $X \cap N$ is empty this is a contradiction and the theorem follows.

The following theorem does not require that the two algebras be Banach algebras; it is valid for arbitrary complex commutative algebras $\mathfrak{A}_i$, $i = 1,2$, with identity if we introduce the weak topology into the sets $\mathfrak{M}_i$ of non-zero complex homomorphisms.

<u>THEOREM 6.</u> Suppose τ is a unitary algebra homomorphism of $\mathfrak{A}_1$ into $\mathfrak{A}_2$. Then there is a continuous map τ' of $\mathfrak{M}_2$ into $\mathfrak{M}_1$ given by

$$\tau'\phi = \phi \circ \tau \qquad \phi \in \mathfrak{M}_2$$

If $\mathfrak{A}_1$ is a subalgebra of $\mathfrak{A}_2$ and τ is the injection mapping, then $\tau'\mathfrak{M}_2$ consists of all homomorphisms of $\mathfrak{A}_1$ which can be extended to $\mathfrak{M}_2$. If τ is onto $\mathfrak{A}_2$ and has kernel K, then τ' is a homeomorphism of $\mathfrak{M}_2$ onto the hull of K.

<u>Proof.</u> An arbitrary sub-basic open set in $\mathfrak{M}_1$ has the form

$V = \{\phi \in \mathfrak{M}_1 : |\hat{f}(\phi)| < \epsilon\}$. Under τ', the open set $U = \{\psi \in \mathfrak{M}_2 : |\tau f(\psi)| < \epsilon\}$ in $\mathfrak{M}_2$ maps into V, because

(6) $$\hat{f}(\tau'\psi) = (\psi \circ \tau)(f) = \psi(\tau f) = \widehat{\tau f}(\psi)$$

for $\psi \in \mathfrak{M}_2$. Thus τ' is continuous. It is a restriction in the case where τ is an injection, and so a homomorphism of $\mathfrak{M}_1$ is in the image of τ' in this case if and only if it is equal to the restriction of some homomorphism of the larger algebra. Finally, assuming τ is onto, it is clear that τ' maps into $h(K)$, the hull of K. For if $f \in K$ and $\phi \in \mathfrak{M}_2$ are arbitrary,

$$\tau'\phi(f) = (\phi \circ \tau)f = \phi(\tau f) = \widehat{\tau f}(\phi) = 0.$$

If τ' were not 1-1, we would have $\tau'\phi = \tau'\psi$ for two different homomorphisms ϕ and ψ of $\mathfrak{M}_2$. Suppose the function which separates ϕ and ψ is written $\widehat{\tau f}$ for $f \in \mathfrak{A}_1$. Then $\widehat{\tau f}(\phi) \neq \widehat{\tau f}(\psi)$, which by (6) implies $\hat{f}(\tau'\phi) \neq \hat{f}(\tau'\psi)$ which contradicts our assumption that $\tau'\phi = \tau'\psi$. Hence τ' is 1-1. Next take any $\psi \in h(K)$. We define a function ϕ on $\mathfrak{A}_2$ by the following formula. If an element $f \in \mathfrak{A}_2$ is written $f = \tau g$, then

(7) $$\phi(f) = \psi(g).$$

This definition does not depend on the choice of g; if $f = \tau g'$ also, then $\tau(g-g') = 0$ which means $g - g' \in K$ and $\psi(g) = \psi(g')$. By the relation $f = \tau g$ we can verify in a routine fashion that ϕ is a homomorphism in $\mathfrak{M}_2$. We see that $\tau'\phi = \psi$ by rewriting (7).

$$\psi(g) = \phi(f) = \phi(\tau g) = \phi \circ \tau(g) = \tau'\phi(g) \qquad g \in \mathfrak{A}_1$$

Finally we must show that the inverse of τ' is continuous on $h(K)$. The most general sub-basic open set of $\mathfrak{M}_2$ is $U = \{\psi \in \mathfrak{M}_2 : |\hat{g}(\psi)| < \epsilon\}$. If we write $g = \tau f$ for some $f \in \mathfrak{A}_1$, we find that the

set $V = \{\phi \in h(K) : |\hat{f}(\phi)| < \epsilon\}$ is mapped by the inverse of τ' into U (actually onto). This again is a consequence of (6).

The idea of the Stone-Cech compactification of a completely regular space X can be introduced into this context. We first mention two properties of weak topologies. The weak topology generated by the restrictions to a subset Y of functions defined on a set X coincides with the relative topology on Y obtained from the weak topology on X. Also if we take the set of functions in $C(X)$ and use these to generate a weak topology on X, then this will co-incide with the old topology if X is completely regular.

<u>THEOREM 7</u>. Suppose X is completely regular and $\mathfrak{M}$ is the space of complex homomorphisms of $C(X)$. Then the mapping $x \longrightarrow \phi_x$, where $\phi_x(f) = f(x)$ for $f \in C(X)$, is a homeomorphism. If we suppose that X is identified with its image in $\mathfrak{M}$, we have :

(1) X is dense in $\mathfrak{M}$.

(2) Every function in $C(X)$ has a unique extension to a function in $C(\mathfrak{M})$

(3) Any compact Hausdorff space $\mathfrak{M}'$ with properties (1) and (2) is homeomorphic to $\mathfrak{M}$ over X.

$\mathfrak{M}$ is called the Stone-Cech compactification of X and is written βX.

<u>Proof.</u> That the obviously 1-1 mapping $x \longrightarrow \phi_x$ is a homeomorphism is a consequence of the two introductory remarks. We are using the weak topology on $\mathfrak{M}$, and X is completely regular. The Gelfand representation is an isometric isomorphism, since

$$||f|| = \sup_X |f(x)| = \sup_X |\hat{f}(\phi_x)| \leq \sup_{\mathfrak{M}} |\hat{f}(\phi)| = ||\hat{f}|| \leq ||f||,$$

and we may identify the functions f and $\hat{f}$. This representation is onto by the Stone-Weierstrass theorem, because the image is uniformly closed, unitary, point separating, and contains the complex conjugate of each function. $C(X)$ is therefore the set of restrictions to X of functions in $C(\mathfrak{M})$. From the isomorphism of $C(X)$ onto $C(\mathfrak{M})$ we deduce that each function in $C(X)$ has a unique extension to $\mathfrak{M}$, and that X must be dense in $\mathfrak{M}$. The property (3) remains to be shown. Suppose $\mathfrak{M}'$ has properties (1) and (2). Then we have an isomorphism of $C(X)$ onto $C(\mathfrak{M}')$. If we compose this with the isomorphism of $C(\mathfrak{M})$ onto $C(X)$ we obtain an algebra isomorphism between $C(\mathfrak{M})$ and $C(\mathfrak{M}')$. We will treat element of $\mathfrak{M}$ and $\mathfrak{M}'$ as maximal ideals. This algebra isomorphism determines a 1-1 correspondence between the maximal ideals of the two, and this is seen to be a homeomorphism since the two topologies, being the hull-kernel in each case, are determined by the algebraic structure. Thus $\mathfrak{M}$ and $\mathfrak{M}'$ are homeomorphic. Also the homeomorphism leaves the fixed maximal ideals invariant, which implies that the homeomorphism is over X.

Another example of a Banach algebra is the following. Take the collection of all bounded measurable complex-valued functions defined on the unit circle Γ. Identifications are made of functions differing only on a set of measure zero. The addition and multiplication of functions is pointwise, and the norm is the essential supremum. This can be defined as the infimum of all K for which $\{x \mid |f(x)| \geq K\}$ has zero measure, and is just denoted by $||f||$. The algebra is called L^∞.

We consider the Gelfand representation of L^∞. Let X be the maximal ideal space.

<u>THEOREM 8</u>. L^∞ is isometrically isomorphic with $C(X)$. The maximal ideal space X is totally disconnected. Let $\tilde{B}$ be the lattice of open-closed sets in X, and T be the space of maximal filters in $\tilde{B}$ provided with the hull kernel topology. Then the mapping which maps each point of X to the filter of open-closed sets containing it, is a homeomorphism of X onto T.

<u>Proof.</u> We first show that the Gelfand homomorphism is an isometry. It is known that $||\hat{f}|| \leq ||f||$. From the definition of essential supremum, there will exist a complex constant λ with $|\lambda| = ||f||$, such that the set $\{x \in \Gamma \mid |f(x) - \lambda| < \epsilon\}$ has positive measure for arbitrary positive ϵ. The function $f - \lambda$ will not be invertible in L^∞, since $1/f - \lambda$ is not essentially bounded. Therefore $f - \lambda$ belongs to a proper ideal in L^∞, and must be annihilated by some homomorphism $\phi \in X$. Hence $||\hat{f}|| \geq |\hat{f}(\phi)| = |\lambda| = ||f||$. This proves the representation is faithful and norm preserving. Its image is uniformly closed in $C(X)$. Moreover the image contains the identity, and is closed under complex conjugation. By the Stone-Weierstrass theorem the image is all of $C(X)$.

We next investigate the compact Hausdorff space X. Because the simple functions are dense in L^∞, the collection of characteristic functions χ_E of measurable subsets E of Γ form a set of generators of the Banach algebra L^∞. We note also that $\hat{\chi}_E$ can only take on the values 0 and 1, because $\chi_E{}^2 = \chi_E$, which implies $(\phi(\chi_E))^2 = \phi(\chi_E)$ for $\phi \in X$. Therefore the topology of X has a basis of sets of the

form

$$\left\{\phi \in X \;\middle|\; |\hat{\chi}_{E_i}(\phi)| < \epsilon \;, \quad E_i \text{ measurable}, \; i = 1, \ldots, n\right\}$$

and if $\epsilon < 1$, this is the same as

$$\left\{\phi \in X \;\middle|\; \hat{\chi}_{E_i}(\phi) = 0 \;, \quad E_i \text{ measurable}, \; i = 1, \ldots, n.\right\}$$

The latter can be written

$$(8) \qquad \left\{\phi \in X \;:\; \hat{\chi}_E(\phi) = 0\right\}$$

where $E = \bigcup_{i=1}^{n} E_i \subseteq \Gamma$. This is closed as well as open. X is therefore totally disconnected.

We next observe that every open-closed set in X is of the form (8) where E is a measurable subset of Γ. If K is open and closed in X, then its characteristic function is continuous, and belongs to $C(X)$. It must be $\hat{f}$ for some idempotent in L^∞. But every idempotent in L^∞ has the form χ_E for some measurable set $E \subseteq \Gamma$. Hence $K = \left\{\phi \;:\; \hat{\chi}_E(\phi) = 1\right\}$ which gives (8) when E is replaced by its complement in Γ.

Each neighbourhood filter of a point in X determines a maximal filter in $\tilde{B}$ when it is restricted to $\tilde{B}$. This restricted filter is of course a basis for the neighbourhood filter since X is totally disconnected. By the compactness of X, every maximal filter in $\tilde{B}$ is obtained in this way. Hence the mapping defined in the theorem is 1-1 onto.

We will show that T has a basis of open closed sets of the form

$$(9) \qquad T_F = \left\{\mathfrak{a} \in T \;\middle|\; F \in \overset{\mathfrak{a}}{\mathfrak{X}}\right\} \qquad F \in \tilde{B}.$$

This will mean that the 1-1 mapping carries a basis of one topology onto

a basis of the other, and must be a homeomorphism. To show that the hull-kernel topology on T gives basic open closed sets of the form (9), we will generate a topology with these sets, and then verify that it is the hull-kernel.

The collection of all T_F, for F open closed in Γ, is closed under finite intersection, as

$$T_F \cap T_G = T_{F \cap G}, \qquad F,G \in \tilde{B}.$$

They generate a topology in which a basis for the neighbourhood system of $\mathfrak{a} \in T$ is $\{T_F \mid F \in \mathfrak{a}\}$.

In the hull-kernel topology, the closure operation Cl is defined by the following, where $S \subseteq T$,

$$Cl(S) = \{\mathfrak{a} \in T \mid \mathfrak{a} \supseteq \bigcap_{\mathfrak{b} \in S} \mathfrak{b}\}$$

We must therefore show that every closed set $C \subseteq T$ has the form

(10) $$C = \{\mathfrak{a} \in T \mid \mathfrak{a} \supseteq \bigcap_{\mathfrak{b} \in C} \mathfrak{b}\}$$

It is first of all necessary to check that sets C satisfying (10) are closed. If we take $\mathfrak{a}_0$ in the closure of C, then $T_F \cap C$ is non-empty for each $F \in \mathfrak{a}_0$. Let us also define the filter

(11) $$\mathfrak{F} = \bigcap_{\mathfrak{b} \in C} \mathfrak{b} .$$

We know that, for each $F \in \mathfrak{a}_0$, there is $\mathfrak{a} \in T_F \cap C$, which means $F \in \mathfrak{a}$ and $\mathfrak{a} \supseteq \mathfrak{F}$. Hence F is compatible with $\mathfrak{F}$, where F is an arbitrary set in $\mathfrak{a}_0$. Therefore $\mathfrak{a}_0$ is compatible with $\mathfrak{F}$, which means $\mathfrak{a}_0 \supseteq \mathfrak{F}$ by maximality. In other words $\mathfrak{a}_0 \in C$, so that C is closed.

Suppose now that C is closed. To show that (10) holds, we must show $C = T_{\mathfrak{F}} = \{\mathfrak{a} \in T \mid \mathfrak{a} \supseteq \mathfrak{F}\}$ where $\mathfrak{F}$ is defined by (11).

Certainly $C \subseteq T_{\mathfrak{F}}$ is trivial, and to show the opposite inequality we start with a filter $\mathfrak{A}_0$ of T which is not in the closed set C. This means that for some $F \in \mathfrak{A}_0$ we have

(12) $$T_F \cap C = \emptyset.$$

We can take the open closed set G which is the complement of F. We have $F \cup G = X$ belonging to each filter in C, and F belonging to no filter of C by (12). By the maximality of the filters in C, this means $G \in \mathfrak{A}$ for each $\mathfrak{A} \in C$, and therefore $G \in \mathfrak{F}$. The filter $\mathfrak{A}_0$, which was an arbitrary member of T that did not lie in C, is now seen to lie outside $T_{\mathfrak{F}}$. For if $\mathfrak{A}_0 \in T_{\mathfrak{F}}$, then it must contain both F and G, a contradiction. Hence $T_{\mathfrak{F}} \subseteq C$ and the theorem follows.

§2 H^p SPACES

We denote by L^p, $p \geq 1$, the linear space of all complex-valued measurable functions F, defined on the unit circle Γ, for which $\int_{-\pi}^{\pi} |F(e^{i\theta})|^p \, d\theta$ is bounded. Then L^p is a Banach space under the norm

$$||F||_p = \left(\int_{-\pi}^{\pi} |F(e^{i\theta})|^p \, d\theta \right)^{1/p} \qquad F \in L^p.$$

In the previous section, we discussed the Banach space L^∞, consisting of bounded measurable functions on Γ, where the norm, denoted by $||F||$ or by $||F||_\infty$ for $F \in L^\infty$, is the essential supremum. Another description of L^∞ is the collection of functions belonging to L^p for each $p \geq 1$, and having $||F|| = \lim_{p \to \infty} ||F||_p$. In all L^p spaces, $1 \leq p \leq \infty$, identifications are made of functions differing only on a set of measure zero.

It is frequently useful to express a function F from L^p in terms of θ rather than $e^{i\theta}$, so we define $\tilde{F}$ to be the composition of F with the mapping $\theta \longrightarrow e^{i\theta}$. In other words

$$\tilde{F}(\theta) = F(e^{i\theta}) \qquad -\pi \leq \theta \leq \pi .$$

The H^p spaces are quite simply the subspaces of the L^p spaces, $1 \leq p \leq \infty$, for which the negative Fourier coefficients vanish.

$$H^p = \left\{ F \mid F \in L^p, \int_{-\pi}^{\pi} F(e^{i\theta}) e^{in\theta} \, d\theta = 0, \quad n = 1,2,3, \ldots \right\}$$
$$1 \leq p \leq \infty.$$

These are closed, and hence are Banach spaces.

We will see, in Theorem 2, that the H^p spaces are identifiable with spaces of functions analytic in the open unit disc.

We shall use Γ for the unit circle and D for the open unit

disc,

$$\Gamma = \{z \in C \mid |z| = 1\}$$
$$D = \{z \in C \mid |z| < 1\}$$

where C is the complex field.

If we take a function F in L^1, the largest of these spaces, and define a function $f(z)$ for $z \in D$ by

$$(1) \qquad f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{F}(t)\, P_r(\theta - t)\, dt$$

where $P_r(\theta)$ is the Poisson kernel

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r \operatorname{Cos} \theta + r^2} = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

we may treat this formula as a convolution $f_r = \tilde{F} * P_r$ where

$$f_r(\theta) = f(re^{i\theta})$$

belonging to L^1. If F has the sequence $(c_n)_{-\infty}^{\infty}$ of Fourier coefficients we have a Fourier expansion

$$f(re^{i\theta}) = f_r(\theta) = \sum_{n=-\infty}^{\infty} c_n\, r^{|n|}\, e^{in\theta}$$

for f_r, since the coefficients of two functions simply multiply when the functions are convolved. The function $f(z)$, already known to be harmonic, will be analytic if and only if

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{F}(\theta)\, e^{in\theta}\, d\theta = 0 \qquad n = 1,2,3, \ldots$$

The following classical theorem tells us that an analytic function f on D, if it has a Poisson representation (1) in terms of an integrable function F on the boundary Γ, will recover the values of F almost everywhere on Γ when a radial limit is taken. The result is needed in a somewhat more general form in terms of measures. It applies to harmonic functions whether or not they are also analytic.

THEOREM 1 (FATOU). Let μ be a finite complex Borel measure on the unit circle, and let f be the harmonic function in the open unit disc defined by

$$f(re^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta-t)\, d\mu(t).$$

Let θ_0 be any point where μ is differentiable with respect to Lebesgue measure. Then

$$\lim_{r\to 1} f(re^{i\theta_0}) = 2\pi \frac{d\mu}{d\theta}(\theta_0).$$

Proof. The finite measure $d\mu$ must be dF for some complex valued function of bounded variation on the interval $[-\pi,\pi]$. Suppose $F'(\theta_0)$ exists. One can assume $F(-\pi) = F(\pi)$; this is the same as assuming $\mu(\Gamma) = 0$, which in turn can be obtained by subtracting a suitable multiple of Lebesgue measure (for which the Theorem is trivial) from $d\mu$. We now apply integration by parts.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, dF(t) = \frac{1}{2\pi} P_r(\theta-t)F(t)\Big|_{-\pi}^{\pi} + \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r'(\theta-t)F(t)dt.$$

Because the integrated part is zero, this gives

$$\begin{aligned}
\frac{1}{2\pi} f(re^{i\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r'(\theta-t)F(t)dt \\
&= \frac{1}{2\pi}\left(\int_0^{\pi} + \int_{-\pi}^{0}\right)(P_r'(t))F(\theta-t)dt \\
&= \frac{1}{2\pi}\int_0^{\pi} P_r'(t)[F(\theta-t) - F(\theta+t)]dt \\
&= \frac{1}{2\pi}\int_0^{\pi} [-\operatorname{Sin} t\, P_r'(t)]\left[\frac{F(\theta+t) - F(\theta-t)}{\operatorname{Sin} t}\right]dt.
\end{aligned}$$

Because $P_r'(t)$ is an odd function, we can write the formula as

$$\frac{1}{2\pi} f(re^{i\theta}) = \frac{r}{2\pi}\int_{-\pi}^{\pi} K_r(t)\left[\frac{F(\theta+t) - F(\theta-t)}{2\operatorname{Sin} t}\right]dt$$

by introducing the kernel

$$K_r(t) = -\frac{1}{r}\,\mathrm{Sin}\,t\,P_r'(t) = \frac{2(1-r^2)}{(1-2r\,\mathrm{Cos}\,t+r^2)^2}\ .$$

The collection of functions $\{K_r : 0 < r < 1\}$ satisfy the following three conditions:

(i) K_r is a continuous and non-negative function of t for $-\pi \leq t \leq \pi$

(ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_r(t)dt = 1$

(iii) If $0 < \delta < \pi$, then

$$\lim_{r\to 1}\ \sup_{|\theta| \geq \delta} |K_r(\theta)| = 0$$

If we set $G(t) = \dfrac{F(\theta_0+t) - F(\theta_0-t)}{2\,\mathrm{Sin}\,t}$, we have a function continuous at $t = 0$ with value $F'(\theta_0)$, and applying (iii) to the equation

$$\frac{1}{2\pi} f(re^{i\theta_0}) - F'(\theta_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_r(t)\,[G(t) - F'(\theta_0)]dt$$

$$= \frac{1}{2\pi}\int_{|t|\geq\delta} K_r(t)[G(t) - F'(\theta_0)]dt$$

$$+ \frac{1}{2\pi}\int_{-\delta}^{\delta} K_r(t)\,[G(t) - F'(\theta_0)]dt.$$

We obtain

$$\lim_{r\to 1}\ \frac{1}{2\pi} f(re^{i\theta_0}) = F'(\theta_0)$$

which proves the theorem. Note that F' exists almost everywhere on Γ, because F has bounded variation.

A collection of functions satisfying conditions (i), (ii), and (iii) is called an approximate identity; these will appear later along with quite similar arguments.

The correspondence between functions of H^p and the analytic functions will certainly be 1-1 by Fatou's theorem, and the Poisson representation guarantees linearity of the correspondence. We shall actually see that the correspondences are Banach space isomorphisms.

THEOREM 2. The mapping of the functions $F \in H^p$, $1 \leq p \leq \infty$, to their Poisson integrals f

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{F}(t)\, P_r(\theta - t)$$

is an isometric isomorphism of H^p onto the Banach space of all analytic functions f for which the functions f_r are bounded in L^p. The norm for this space is

$$||f|| = \lim_{r \to 1} ||f_r||_p.$$

Recall that $f_r(\theta) = f(re^{i\theta})$ and that $|| \quad ||_p$ refers to the norm in L^p. The proof of Theorem 2 is given in three parts for the cases (a) $1 < p < \infty$, (b) $p = \infty$, and (c) $p = 1$. Several other theorems will be required for a complete proof.

Proof of Theorem 2(a). Suppose we take any $F \in L^p$, $1 < p < \infty$. We first show that f_r converges to F in L^p as $r \to 1$; in the Theorem this is only applied to the special case $F \in H^p$. It will give

$$||f|| = \lim_{r \to 1} ||f||_p = ||F||.$$

for our mapping $F \to f$.

Define q by the equation $\frac{1}{p} + \frac{1}{q} = 1$, and let $g \in L^q$ be arbitrary. We can extend functions periodically, and so obtain

$$f_r(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{F}(t)\, P_r(\theta - t)dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{F}(\theta - t)\, P_r(t)dt.$$

$\{P_r : 0 \leq r < 1\}$ is an approximate identity, and by property (i)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f_r(\theta)-\tilde{F}(\theta))g(\theta)d\theta = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}[\tilde{F}(\theta-t)-\tilde{F}(\theta)]g(\theta)\,P_r(t)d\theta dt$$

Hence

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f_r(\theta)-\tilde{F}(\theta))g(\theta)d\theta\right| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(\tilde{F}(\theta-t)-\tilde{F}(\theta))g(\theta)d\theta\right|P_r(t)dt$$

If we define $\tilde{F}_t(\theta) = \tilde{F}(\theta-t)$, the inside integral is majorized by $||g||_q \cdot ||F_t-F||_p$. Hence

$$(2)\qquad \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f_r(\theta)-\tilde{F}(\theta))g(\theta)d\theta\right| \le ||g||_q \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi}||F_t-F||_p\,P_r(t)dt.$$

Now g is arbitrary in L^q, which is the conjugate space of L^p. For any function $h \in L^p$, there exists a functional ϕ in the conjugate space with $\phi(h) = ||h||_p$, $||\phi|| = 1$. In other words there exists $g \in L^q$ with $\int hg = ||h||_p$ and $||g||_q = 1$. If we take $h = f_r - F$, and use the corresponding g in (2), we have

$$||f_r-F||_p \le \int_{-\pi}^{\pi}||F_t-F||_p\,P_r(t)dt$$

$||F_t-F||_p$ can be made small by making $|t| \le \delta$. Then

$$||f_r-F||_p \le \int_{-\delta}^{\delta}||F_t-F||_p\,P_r(t)dt + \int_{|t|\ge\delta}||F_t-F||_p\,P_r(t)dt$$

$$\le 2\pi\cdot\sup_{|t|<\delta}||F_t-F||_p + 4\pi\,||F||_p\sup_{|t|\ge\delta}P_r(t).$$

This means that, as $r \longrightarrow 1$,

$$||f_r-F||_p \longrightarrow 0.$$

Next we take any harmonic function f defined in the unit disc with the property that $\lim_{r\to 1}||f_r||_p$ is finite. We wish to show f is represented by the Poisson integral of some function $F \in L^p$. Our hypothesis says that the functions f_r lie in a bounded set in L^p.

Because L^p is reflexive (with conjugate space L^q), the weak and weak-* topologies agree, and hence any bounded set is weakly compact. If we pick a sequence $1 < r_1 < r_2 < r_3 \ldots < 1$ of numbers r_k tending monotonely to 1, the sequence f_{r_k} must have a weak cluster point F in L^p. This means $\int f_{r_k} g\, dx$ lies inside an arbitrary neighbourhood of $\int F g\, dx$ for an infinity of k where $g \in L^q$ is arbitrary. If we let $g = e^{imx}$, $m = 0, \pm 1, \pm 2, \ldots$, which belongs to L^q, we find that the Fourier coefficients of F are cluster points of the Fourier coefficients of the f_{r_k}. The Fourier expansion of f_r must be of the form

$$(3) \qquad f_r(\theta) = \sum_{-\infty}^{\infty} c_n r^{|n|} e^{in\theta},$$

however. If f is real harmonic then $f = \frac{h + \bar{h}}{2}$ for some analytic function $h(z) = \sum_0^\infty a_n z^n$. This means $f(z) = \frac{1}{2}(\sum_0^\infty a_n z^n + \sum_0^\infty \bar{a}_n \bar{z}^n)$ or $f(z) = \Sigma c_n r^{|n|} e^{in\theta}$ where $c_0 = 2\,\mathrm{Re}(a_0)$, $c_n = a_n$ for $n > 0$, and $c_n = \bar{a}_{-n}$ for $n < 0$. A similar argument will apply to a purely imaginary harmonic function, and together these give formula (3). From (3) it is evident that $f_r(\theta) = f(re^{i\theta})$ is obtained as the Poisson integral of the function with Fourier coefficients c_n. This function is precisely F, which is in the space L^p, because the Fourier coefficients of f_r are $c_n r^{|n|}$, which approach c_n as r increases to 1. Also, having now a Poisson representation, we can apply Fatou's theorem to see that $f_r(\theta) \longrightarrow F(e^{i\theta})$ for almost all θ.

<u>Proof of Theorem 2(b)</u>. For the case $p = \infty$, the argument is not very different. If $F \in L^\infty$ is arbitrary, then the Poisson representation (1) gives a bounded harmonic function f in D. The functions f_r converge to the function $F \in L^\infty$; not necessarily in the norm topology, but in the weak-* topology on L^∞. However this is sufficient to give

$\lim_{r\to 1} ||f_r||_\infty = ||F||_\infty$. To prove the existence of F we argue as follows. For any $g \in L^1$

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f_r(\theta) - \tilde{F}(\theta))g(\theta)d\theta \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\frac{1}{2\pi} \int_{-\pi}^{\pi} (\tilde{F}(\theta-t) - F(\theta))g(\theta)d\theta\right| P_r(t)dt,$$

using an argument the same as the one preceding (2) in the proof of 2(a). We can majorize the integral as before

$$\left|\frac{1}{2\pi} \int_{-\pi}^{\pi} (f_r(\theta) - F(\theta))g(\theta)d\theta\right| \leq \frac{1}{2\pi} \int_{-\delta}^{\delta} \left|\frac{1}{2\pi} \int_{-\pi}^{\pi} (\tilde{F}(\theta-t) - \tilde{F}(\theta))g(\theta)d\theta\right| P_r(t)dt$$

$$+ 2||f||_\infty \cdot ||g||_1 \sup_{|t| \geq \delta} P_r(t).$$

The second term, for any δ, will approach zero as $r \longrightarrow 1$. The first term is less than

$$\sup_{|t| < \delta} \left|\frac{1}{2\pi} \int_{-\pi}^{\pi} (\tilde{F}(\theta-t) - \tilde{F}(\theta))g(\theta)d\theta\right| = \sup_{|t| < \delta} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} F(y)(g(y) - g(y-t))dy\right|$$

and this is arbitrarily small for a sufficiently small δ, since $||g-g_t||_1 \longrightarrow 0$ as $t \longrightarrow 0$. $(g_t(x) = g(x-t))$. This implies that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f_r(\theta)g(\theta)d\theta \longrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{F}(\theta)g(\theta)d\theta \quad \text{for each } g \in L^1$$

which means that the functions f_r approach F in the weak-* topology on L^∞.

The second part is proved exactly as in 2(a). Given a bounded harmonic function f on D, the functions f_r will be contained in a bounded set in L^∞, and will have a weak-* cluster point $F \in L^\infty$. As before one shows that f is obtained from F by the Poisson integral.

Again we only apply this argument to L^∞ functions which are also in H^∞. Thus we identify H^∞ with the space of all bounded analytic functions on D.

<u>Proof of Theorem 2(c)</u>. In proving parts (a) and (b), we used 1-1 corres-

pondences between collections of harmonic functions f for which $\{||f_r||_p \mid 0 < r < 1\}$ was bounded and the spaces L^p $1 < p \leqslant \infty$. However this correspondence fails for the space L^1. There exist harmonic functions f for which $\{||f_r||_1 \mid 0 < r < 1\}$ is bounded, but which have no Poisson representation in terms of a function F. There is a 1-1 correspondence, however, when we allow the use of measures on Γ in the Poisson representation. In the special case of analytic functions f, a theorem of F. and M. Riesz says that any measure which gives an analytic function must be absolutely continuous. This means that H^1 can be identified with the set of all analytic functions f such that $\{||f_r||_1 \mid 0 < r < 1\}$ is bounded. The proof of the classical F. and M. Riesz theorem is given later; we now discuss the correspondence between measures on Γ and harmonic functions on D.

Given any finite complex Borel measure μ defined on Γ, we can obtain a harmonic function f by the Poisson formula. We claim that as $r \longrightarrow 1$, the measures $\frac{1}{2\pi} f_r(\theta)d\theta$ approach the measure μ in the space of measures on Γ in the weak-* topology. This means that for each $g \in C(\Gamma)$ we have

$$(4) \qquad \int_{-\pi}^{\pi} g(\theta) \frac{1}{2\pi} f_r(\theta)d\theta \longrightarrow \int g(\theta)d\mu(\theta) \quad \text{as} \quad r \longrightarrow 1.$$

This is a consequence of the Riesz-Kakutani Theorem, which identifies the space of linear functionals on $C(\Gamma)$ with the set of measures on Γ. Then the definition (4) of convergence is just weak-* convergence in the conjugate space of measures.

Take now the collection $\{\frac{1}{2\pi} f_r(\theta)d\theta \mid 0 < r < 1\}$ of measures, and let $g \in C(\Gamma)$ be arbitrary.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)f_r(\theta)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\int P_r(\theta-t)d\mu(t)d\theta$$

$$= \int d\mu(t)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)P_r(\theta-t)d\theta\right)$$

$$= \int g_r(t)d\mu(t)$$

where $g_r(t) = g(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)P_r(t-\theta)d\theta$

is the harmonic function with continuous boundary values $g(t)$. As $r \longrightarrow 1$, it is well known that $g_r(t) \longrightarrow g(t)$ uniformly, and this is sufficient to interchange limit and integral.

$$\lim_{r\to 1}\int_{-\pi}^{\pi} g(\theta)\frac{1}{2\pi}f_r(\theta)d\theta = \lim_{r\to 1}\int g_r(t)d\mu(t) = \int g(t)d\mu(t).$$

This shows that μ is the weak-* limit of the collection of measures as $r \longrightarrow 1$.

One can readily show in this case that the collection $\{||f_r||_1 \mid 0 < r < 1\}$ is bounded. We have

$$f_r(\theta) = \int P_r(\theta-t)d\mu(t)$$

$$\int_{-\pi}^{\pi} |f_r(\theta)|d\theta \leq \int d|\mu(t)| \int_{-\pi}^{\pi} P_n(\theta-t)d\theta \leq 2\pi|\mu|(\Gamma)$$

where $|\mu|$ is the total variation of μ; $|\mu|(\Gamma)$ is finite and therefore we have a uniform bound for all $||f_r||_1$.

We argue conversely just as before; suppose f is harmonic and $\{||f_r||_1 \mid 0 < r < 1\}$ is bounded. This means that the measures $\frac{1}{2\pi}f_r(\theta)d\theta$ lie in a bounded set, and along some sequence $0 < r_1 < r_2 < \ldots < 1$ converging to 1 will have a weak-* cluster point μ. Since $e^{im\theta}$ is continuous for $m = 0, \pm 1, \pm 2, \ldots,$ we can use the definition (4) with $g(\theta) = e^{im\theta}$. This tells us that the Fourier coefficients $c_m = \int e^{-im\theta}\, d\mu(\theta)$ of μ are cluster points of the corres-

ponding Fourier coefficients of f_r as $r \longrightarrow 1$. However, f is harmonic in D, and has the form

$$f_r(\theta) = f(re^{i\theta}) = \sum_{-\infty}^{\infty} d_n r^{|n|} e^{in\theta}$$

for some coefficients d_n. Evidently $c_n = d_n$ for all n, since $d_n r^{|n|}$ is the Fourier coefficient of f_r and as $r \longrightarrow 1$ this must have c_n as its only cluster point. This series representation also tells us that the measure μ under the Poisson integral gives the function f back again. This establishes the required correspondence except for the Riesz theorem to follow later.

In the cases where f_r converges to a function in L^1 when $r \longrightarrow 1$, then the corresponding measure is absolutely continuous, and the Poisson formula is of the form (1).

Theorem 2 follows.

The following Theorem applies to any function in H^1. However we only give a proof for a bounded analytic function.

<u>Theorem 3.</u> Suppose f is a bounded analytic function on D, and f has a radial limit λ at some point α on the boundary. Suppose any two chords through α are drawn on either side of the radius at α. Then f approaches λ as z approaches α on any path between the two chords.

<u>Proof</u>. We translate D for simplicity; let the centre be mapped to i, and the point α to the origin. Take any pair of chords from the origin, and a pair of circles, say $|z| = 1$ and $|z| = \frac{1}{2}$. Call the closed region bounded by the two chords and the two circles R. Define a sequence of analytic functions in R by the formula

$$f_n(z) = f(\frac{1}{2^n} z).$$

This means that the values of f_n as z ranges over R correspond to the values of f in a similar region which is smaller and closer to α. If we pick the point $z_0 = 3i/4$, and evaluate $f_n(z_0)$, we see that this approaches λ as n approaches infinity. By Vitali's theorem, this suffices to prove that f_n converges uniformly to an analytic function F in K. However this function will be constantly equal to λ for any value $z = ki$, $\frac{1}{2} < k < 1$, because for any such z, we know $f_n(z) \longrightarrow \lambda$ as $n \longrightarrow \infty$. Therefore F is the constant function λ. This means that f must tend to λ along any approach to the origin lying between the two chords.

Note also that this convergence is uniform, in the sense that all points within a certain distance $\delta(\epsilon)$ from the origin and between the two chords will give a functional value within ϵ of λ where ϵ is arbitrary.

We next consider the space H^1 of analytic functions, the largest of the spaces considered. Let A be the algebra of analytic functions on D which are continuous on $D \cup \Gamma$, and let A_0 be the ideal in A of functions vanishing at the origin. The following result and the sequence of lemmas used to prove it, will give the Riesz theorem and several other propositions.

<u>THEOREM 4</u> (Szego; Kolmogoroff-Krein). Let μ be a finite positive Borel measure on the unit circle and let h be the derivative of μ with respect to normalized Lebesgue measure. Then

$$(5) \qquad \inf_{f \in A_0} \int |1-f|^2 \, d\mu = \exp\left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \log h(\theta) d\theta\right]$$

If we let $L^2(d\mu)$ be the space of functions on Γ which are

integrable with respect to $d\mu$, this is a Hilbert space, and the left side is the square of the distance in $L^2(d\mu)$ from 1 to the closed subspace generated by A_0.

<u>Lemma 1</u>. If μ is a finite real measure on the circle such that $\int f\, d\mu = 0$ for each $f \in A_0$, then μ is a multiple of Lebesgue measure.

<u>Proof</u>. Fejer's theorem says that the real parts of the functions in A are uniformly dense in the space of all real valued continuous functions on Γ. This means that any real measure annihilating A must annihilate the set $\{\mathrm{Re}(f) \mid f \in A\}$, and therefore must be zero. Suppose now μ annihilates A_0. Write $\lambda = \int d\mu$ and $d\mu_1 = d\mu - (\lambda/2\pi)d\theta$. Then μ_1 is a real measure annihilating A, for if $f \in A$,

$$\int f\, d\mu_1 = \int (f - f(0))d\mu_1 + \int f(0)d\mu_1 = 0.$$

This means $d\mu_1 = 0$ and

$$d\mu = (\lambda/2\pi)d\theta.$$

In the sequel, μ refers to the measure of Theorem 4.

<u>Lemma 2</u>. Suppose 1 is not in the closed span of A_0, taken in $L^2(d\mu)$. Let F be the orthogonal projection of 1 on this closed subspace. Then

$$|1-F|^2\, d\mu$$

is a constant multiple of Lebesgue measure.

<u>Proof</u>. Let S be the closed subspace of $L^2(d\mu)$ spanned by A_0. The function $1-F$ is orthogonal to S. If $f \in A_0$ is arbitrary, then $f(1-F)$ is in S, because

$$f(1-F) = \lim_{n\to\infty} f(1-f_n)$$

where $f_n \in A_0$ and $f_n \longrightarrow F$. Thus $1-F$ is orthogonal to $f(1-F)$

$$\int f|1-F|^2\, d\mu = 0 \qquad f \in A_0.$$

By Lemma 1, $|1-F|^2\, d\mu$ must be a constant multiple of Lebesgue measure. We let μ_a and μ_s be the absolutely continuous and the singular parts of μ; with h as defined in the Theorem, we have $d\mu_a = \frac{h}{2\pi}\, d\theta$.

<u>Lemma 3</u>. The function $(1-F)^{-1}$ is in H^2, and the function $(1-F)h$ is in

$$L^2 = L^2(\frac{1}{2\pi}\, d\theta)$$

if we assume that 1 is not in the closed span of A_0 in $L^2(d\mu)$.

<u>Proof</u>. By the previous lemma $|1-F^2|d\mu = kd\theta$, $k \neq 0$. This means that $d\mu_a = |1-F|^{-2}\, kd\theta$, which proves that $(1-F)^{-1}$ is in L^2. If f is arbitrary in A_0,

$$\int (1-F)^{-1} f\, d\theta = \int \frac{(1-\overline{F})f}{|1-F|^2}\, d\theta = \frac{1}{k} \int (1-\overline{F})f\, d\mu = 0 \quad ,$$

because $1-F$ is orthogonal to $f \in A_0$. This holds for the particular choices $f(\theta) = e^{in\theta}$, $n = 1,2,3, \ldots,$ and therefore $(1-F)^{-1}$ is in H^2.

We have

$$(6) \qquad |1-F|^2\, d\mu_a + |1-F|^2\, d\mu_s = kd\theta.$$

The measure $|1-F|^2\, d\mu_a$ equals $\frac{1}{2\pi}\, |1-F|^2\, h\, d\theta$ and the measure $|1-F|^2\, d\mu_s$ must be zero since it is singular with respect to Lebesgue measure, but by (6) it is a multiple of $d\theta$. Hence $|1-F|^2\, h$ equals a constant ρ. This means $|1-F|h = \rho|1-F|^{-1}$ and since $(1-F)^{-1}$ is in L^2, so is $(1-F)h$.

Observe that $|1-F|$ must be zero almost everywhere with respect to $d\mu_s$, because $|1-F|^2\, d\mu_s$ is the zero measure.

<u>Lemma 4</u>. If μ is a positive measure on the circle with absolutely continuous part μ_a, then

$$\inf_{f \in A_0} \int |1-f|^2 \, d\mu = \inf_{f \in A_0} \int |1-f|^2 \, d\mu_a .$$

In particular, if μ is singular, the function 1 is in the $L^2(d\mu)$ closure of A_0.

Proof. The square of the distance from 1 to A_0 in $L^2(d\mu)$ is $\int |1-F|^2 \, d\mu$. This quantity is the same as $\int |1-F|^2 \, d\mu_a$, by the observation that $|1-F|^2 \, d\mu_s$ is the zero measure, where this new integral represents the square of the distance between 1 and F in $L^2(d\mu_a)$. The function F is in the closure of A_0 in $L^2(d\mu_a)$ however, and $1-F$ is still orthogonal to A_0 as is easily seen from previous arguments if we note that $|1-F|^2 \, d\mu_s = 0$. Hence the $L^2(d\mu_a)$ distance between 1 and the closure of A_0 is still obtained from the function $1-F$. The Lemma follows.

<u>Lemma 5</u>. If μ is a finite complex Borel measure on the circle which is orthogonal to A_0, then both the absolutely continuous and the singular parts of μ are orthogonal to A_0.

<u>Proof</u>. The measure ρ defined by

$$d\rho = \frac{1}{2\pi} (1 + |h|) d\theta + d|\mu_s|$$

is positive, and satisfies the properties

(i) μ is absolutely continuous with respect to ρ, and $\frac{d\mu}{d\rho}$ is bounded.

(ii) $\frac{d\rho}{d\theta} \geq \frac{1}{2\pi}$

If $f \in A_0$ is arbitrary, we have by (ii)

$$\int |1-f|^2 \, d\rho \geq \frac{1}{2\pi} \int (1-f)(1-\bar{f}) d\theta \geq 1.$$

We let F_1 be the orthogonal projection of 1 into the closed subspace

of $L^2(d\rho)$ spanned by A_0. Then

$$\int |1-F_1|^2 \, d\rho \geq 1,$$

and we can apply Lemma 3 to obtain $(1-F_1)^{-1} \in H^2$ and $(1-F_1)(1+|h|) \in L^2$. The latter implies $(1-F_1)h \in L^2$.

We can choose a sequence (f_n) of functions from A_0 converging in $L^2(d\rho)$ to F_1. Then for g arbitrary in A_0,

$$\int (1-F_1)g \, d\mu = \lim_{n\to\infty} \int (1-f_n)g \, d\mu = 0.$$

We can interchange limit and integral because $f_n \longrightarrow F_1$ in $L^2(d\rho)$ and $\frac{d\mu}{d\rho}$ is bounded. The function $(1-f_n)g \in A_0$, and hence is annihilated by μ. Again $1-F_1$ vanishes almost everywhere with respect to $d\rho_s$, and hence also with respect to $d\mu_s$. We get

$$(7) \qquad 0 = \int (1-F_1)g \, d\mu = \int (1-F_1)g \, h \, d\theta \qquad g \in A_0 .$$

We can select a sequence (g_n) of functions from A which converge to $(1-F_1)^{-1}$, since A is dense in H^2. Then we use $g = g_n f$ in (7) for f arbitrary in A_0.

$$\int g_n f \, (1-F)h \, d\theta = 0 \qquad f \in A_0$$

Letting n tend to infinity, we have $(1-F)h \in L^2$, and also that g_n converges in L^2 to $(1-F)^{-1}$, we obtain

$$\int f \, h \, d\theta = 0 \qquad \text{which means}$$

$$\int f \, d\mu_a = 0 \qquad \text{for } f \in A_0.$$

Thus the absolutely continuous part is orthogonal to A_0. The singular part must also be orthogonal, and lemma 5 follows.

<u>Lemma 6</u>. Given a non-negative, integrable function h on Γ,

$$\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log h \, d\theta\right] = \inf_{f \in A_0} \frac{1}{2\pi}\int_{-\pi}^{\pi} h \, e^{\mathrm{Re}(f)} \, d\theta \quad ,$$

Since $\log h$ is bounded above by h, an integrable function, we have $\log h$ non-integrable if and only if $\int_{-\pi}^{\pi} \log h \, d\theta = -\infty$, and in this case the left-hand side is taken to be zero.

<u>Proof.</u> We first show that the left side is not greater than the right, by making use of the inequality

$$\exp[\frac{1}{2\pi} \int \log h \, d\theta] \leq \frac{1}{2\pi} \int h \, d\theta,$$

which is a generalization of the familiar relation between arithmetic and geometric means. We apply this to the function he^g, where g is real valued and integrable. We also assume $\int g \, d\theta = 0$ which means that g does not appear on the left side

$$\exp[\frac{1}{2\pi} \int \log h \, d\theta] \leq \frac{1}{2\pi} \int he^g \, d\theta.$$

The set G of all such functions is contained in the set R where

$$G = \{g \in L^1 \mid g = \bar{g}, \int g \, d\theta = 0\}$$

$$R = \{g \mid g = \mathrm{Re}(f) \; f \in A_0\}.$$

Hence

$$(8) \qquad \exp[\frac{1}{2\pi} \int \log h \, d\theta] \leq \inf_{g \in G} \frac{1}{2\pi} \int he^g \, d\theta \leq \inf_{f \in A_0} \frac{1}{2\pi} \int he^{\mathrm{Re}(f)} \, d\theta.$$

To prove the reverse inequality, we first show that the two infima of (8) are equal. If g is a real function of L^1 for which $\int g \, d\theta = 0$ we can find a sequence (g_n) of functions in L^∞ which tend monotonely to g (in the sense that the positive and negative parts of g_n increase monotonely to the positive and negative parts of g). We can apply the monotone convergence theorem to the integrals $\int he^{g_n} \, d\theta$ to show that the infimum is the same for the class of real L^∞ functions as it is for the class of real L^1 functions. For each

real $g \in L^\infty$, one can find a bounded sequence of functions of the form $\mathrm{Re}(f)$, $f \in A$, which converges almost everywhere to g. This time we can use the dominated convergence theorem to show that the infimum is the same for real L^∞ functions as it is for the class of functions $\mathrm{Re}(f)$, $f \in A$.

Finally, we choose $g \in G$ such that the first inequality of (8) becomes equality. If $\log h$ is integrable, define $\lambda = \frac{1}{2\pi}\int \log h \, d\theta$, $g = \lambda - \log h$. Then g is real, in L^1, and $\int g \, d\theta = 0$. Also

$$\frac{1}{2\pi}\int he^g \, d\theta = \frac{1}{2\pi}\int e^\lambda \, d\theta = \exp[\frac{1}{2\pi}\int \log h \, d\theta].$$

If $\log h$ is non-integrable, then for any $\epsilon > 0$, the function $\log(h+\epsilon)$ can be used with the previous equality

$$\exp[\frac{1}{2\pi}\int \log(h+\epsilon)d\theta) = \inf_{f\in A_0} \frac{1}{2\pi}\int (h+\epsilon)e^{\mathrm{Re}(f)}d\theta \geq \inf_{f\in A_0} \frac{1}{2\pi}\int he^{\mathrm{Re}(f)}d\theta.$$

Then as ϵ tends to zero we find that the limit of the left side is not as big as the right side

$$\exp[\frac{1}{2\pi}\int \log h \, d\theta] \geq \inf_{f\in A_0} \frac{1}{2\pi}\int he^{\mathrm{Re}(f)}d\theta$$

as required.

<u>Proof of Theorem 4</u>. Given the positive finite Borel measure μ on Γ, we must show that

$$\inf_{f\in A_0} \int |1-f|^2 \, d\mu = \exp[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log h \, d\theta] \tag{9}$$

where h is the derivative of μ with respect to $d\theta/2\pi$. By Lemma 6, the right side of (9) is equal to

$$\inf_{f\in A_0} \frac{1}{2\pi}\int he^{\mathrm{Re}(f)}d\theta = \inf_{g\in A_0} \frac{1}{2\pi}\int he^{2\,\mathrm{Re}(g)}d\theta.$$

Whenever $g \in A_0$, $e^{2\,\mathrm{Re}(g)} = |e^g|^2 = |1-f|^2$ for some $f \in A_0$. Hence

$$\inf_{g\in A_0} \frac{1}{2\pi} \int h e^{2\,\mathrm{Re}(g)} d\theta \geq \inf_{f\in A_0} \frac{1}{2\pi} \int |1-f|^2 h\, d\theta = \inf_{f\in A_0} \int |1-f|^2 d\mu$$

where the equality follows from Lemma 4. This proves (9) in one direction; the right-side is at least as great as the left. The inequality in the other direction will follow if we can show

$$(10) \qquad \exp[\frac{1}{2\pi} \int \log h\, d\theta] \leq \inf_{f\in A_0} \frac{1}{2\pi} \int |1-f|^2 h\, d\theta.$$

We define $h = |1-q|^2$ where q is any element in A_0, and substitute in formula (10) with the opposite inequality; in this direction the inequality has been established. This gives

$$\exp[\frac{1}{2\pi} \int \log h\, d\theta] \geq \inf_{f\in A_0} \frac{1}{2\pi} \int |1-f|^2 |1-q|^2 d\theta.$$

The integrals on the right side are of the form

$$\frac{1}{2\pi} \int |1-\phi|^2 d\theta$$

where $\phi = f+g - fg \in A_0$, and all have at least 1 for their value.

$$\frac{1}{2\pi} \int (1-\phi)(1-\bar{\phi})\, d\theta = 1 + \frac{1}{2\pi} \int |\phi|^2 d\theta \geq 1,$$

as we have seen before.

Hence $\exp[\frac{1}{2\pi} \int \log |1-q|^2 d\theta] \geq 1$, which means that $\log |1-q|^2$ is is Lebesgue integrable and

$$\int \log |1-q|^2 d\theta \geq 0 \qquad \text{for } g \in A_0.$$

Hence $|1-q|^2 = ke^p$ where p is a real L^1 function with $\int p d\theta = 0$, and where $k \geq 1$.

Returning to the original h, we see that

$$\frac{1}{2\pi}\int |1-q|^2\, h\, d\theta = k\,\frac{1}{2\pi}\int he^{p}\, d\theta \;\geq\; \inf_{f\in A_0} \frac{1}{2\pi}\int he^{Re(f)}\, d\theta$$

which gives (10) when we take the infimum over all $g \in A_0$, since the right side is just $\exp[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log h\, d\theta]$.

We now give several important theorems on the basis of these results.

THEOREM 5. (F. and M. Riesz). Let μ be a finite complex Borel measure on the unit circle such that

$$\int e^{in\theta}\, d\mu(\theta) = 0 \qquad n = 1,2,3, \ldots .$$

Then μ is absolutely continuous with respect to Lebesgue measure.

Proof. We must show μ_s is the zero measure. Since μ is orthogonal to A_0, μ_s is also, in virtue of Lemma 5. If we consider the space $L^2(d|\mu_s|)$, we are using a singular measure, and according to Lemma 4, the function 1 is in the closure of A_0 in this space. Let (f_n), $f_n \in A_0$, be a sequence of functions converging to 1 in $L^2(d|\mu_s|)$. Then

$$\int d\mu_s = \lim_{n\to\infty} \int f_n\, d\mu_s = 0,$$

because each f_n is in A_0, and must be annihilated by μ_s. This means that μ_s is also orthogonal to 1. Consider now the measure $e^{-i\theta}\, d\mu_s$. This annihilates $e^{in\theta}$ for $n = 1,2,3, \ldots$ and so is orthogonal to A_0. It is also singular and hence we can repeat the above argument to show that it is orthogonal to 1. If we repeat this process, we find that

$$\int e^{in\theta}\, d\mu(\theta) = 0 \qquad n = 0,-1,-2,-3, \ldots$$

which, when taken with the hypothesis of the theorem, is enough to ensure that μ_s is the zero measure.

THEOREM 6. Suppose f is any non zero function in H^1. Then $\log |f(\theta)|$ is integrable and if $f(0) \neq 0$

(11) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log |f(e^{i\theta})| d\theta \geq \log |f(0)| \qquad \text{(Jensen's Inequality)}$$

Proof. We assume first that $f \in H^2$ where $H^2 \subset H^1$, and that $f(0) \neq 0$. We apply Theorem 5 to the measure $\frac{1}{2\pi}|f|^2 d\theta$.

$$\exp[\frac{1}{2\pi}\int \log |f|^2 d\theta] = \inf_{g \in A_0} \frac{1}{2\pi}\int |1-g|^2 |f|^2 d\theta$$

For any $g \in A_0$, the integral on the right-hand side is

$$\frac{1}{2\pi}\int |f-fg|^2 d\theta = \frac{1}{2\pi}\int |f(0) - \phi|^2 d\theta \geq |f(0)|^2$$

where $\phi = (f(0) - f) + fg \in A_0$, so that we can apply the same argument as at the conclusion of the proof of Theorem 4. The Jensen inequality follows. For $f \in H^1$ but $f \notin H^2$, it is possible to choose a sequence (f_n) of functions in H^2 which converges to f. We also need $f_n(0) = f(0)$ for each n. This could be obtained, for example, by letting f_n be the n'th Cesaro mean of the Fourier series for f. This gives for each n that $\frac{1}{2\pi}\int \log |f_n| d\theta \geq \log |f_n(0)| = \log |f(0)|$, and from this we obtain (11) by proceeding to the limit and applying Fatou's lemma.

If $f(0) = 0$, we can write $f(z) = z^n g(z)$ where $g(0) \neq 0$ and obtain a lower bound for $\int \log |f| d\theta$ as follows:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log |f(e^{i\theta})| d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log |g(e^{i\theta}) d\theta \geq \log |g(0)| > -\infty.$$

Of course if f were to vanish on a set of positive measure on Γ, then $\int \log |f| d\theta = -\infty$. We have the result.

COROLLARY. If $f \in H^1$ and $f = 0$ on a set of positive measure on the

unit circle, then f is identically zero.

THEOREM 7. Every function in H^1 is the product of two functions in H^2.

Proof. We can assume $f(0) \neq 0$. The function $\log |f|$ is integrable by the last theorem. We consider $L^2(d\mu)$ using the measure

$$d\mu = \frac{1}{2\pi} |f| \, d\theta, \qquad h = |f|$$

whose derivative with respect to normalized Lebesgue measure is $h = |f|$. In Szego's theorem, the right hand side, $\exp[\frac{1}{2\pi} \int_{-\pi}^{\pi} \log h \, d\theta]$, is strictly greater than zero, which means that 1 is not in the $L^2(d\mu)$ closure of A_0. We now appeal to Lemma 3, in which we see that $|1-F|^2 h$ equals a constant ρ almost everywhere. (F is the projection of 1 on the closure of A_0 in $L^2(d\mu)$.) Also $(1-F)^{-1} \in H^2$ and $(1-F)f \in L^2$ are given in Lemma 3. But $(1-F)f$ is not only in L^2, but also in H^2. This gives the factorization

$$f = (1-F)^{-1} \cdot (1-F)f.$$

DEFINITION. An outer function is an analytic function q on the unit disc which is of the form

$$(12) \qquad q(z) = \lambda \exp[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log h(\theta) d\theta] \qquad |\lambda| = 1$$

where $h(\theta)$ is non-negative, and $\log h(\theta)$ is integrable. We will always assume that $h(\theta)$ is also Lebesgue integrable.

We show that, for almost all θ, $\lim_{r \to 1} |q(re^{i\theta})| = h(\theta)$, or in other words $h(\theta) = |Q(\theta)|$ where $Q \in H^1$ is the boundary function corresponding to q. We have $|q| = e^u$, where u is the Poisson integral of $\log h$, because $\mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)$ is just the Poisson kernel. The inequality

$\exp[\int g d\mu] \leq \int e^g \, d\mu$ holds for positive measures μ of mass 1. Such a measure is $\frac{1}{2\pi} Pr(\theta-t)dt$, and we have

$$\exp[u(re^{i\theta})] = \exp[\frac{1}{2\pi} \int_{-\pi}^{\pi} Pr(\theta-t) \log h(t)dt]$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} Pr(\theta-t) \, h(t)dt.$$

Now integrating the equation $|q| = e^u$, we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |q(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[u(re^{i\theta})]d\theta$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, \frac{1}{2\pi} \int_{-\pi}^{\pi} Pr(\theta-t) \, h(t)dt$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} h(t)dt$$

by interchanging the two integrations.

Letting $r \longrightarrow 1$, we see that $q \in H^1$. Also we have, for almost all θ that

$$\lim_{r \to 1} |q(re^{i\theta})| = \lim_{r \to 1} \exp[u(re^{i\theta})] = h(\theta)$$

because u is represented by a Poisson integral, and by Theorem 2 it will have boundary limits almost everywhere equal to $\log h(\theta)$.

We observe that, if h is bounded, then $|q|$ has the same bound in D.

We can construct an H^1 function with given modulus h, whenever h and $\log h$ are integrable. The converse is known from Theorem 6. Together we have

<u>THEOREM 8</u>. Let h be a non-negative integrable function on the circle. Then h is the modulus of a non-zero H^1 function if and only if $\log h$ is integrable. If h is non-negative and in L^∞, then h is the

modulus of a non zero H^∞ function if and only if $\log h$ is integrable.

Suppose we select any analytic function f with boundary function $F \in H^1$. Since both $|F|$ and $\log |F|$ are integrable on the unit circle, we can form an outer function q with $h = |F|$.

$$(13) \qquad q(z) = \exp[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log |F(e^{i\theta})| d\theta]$$

Of course q has a boundary function Q in H^1 and $|Q| = |F|$ almost everywhere. Taking moduli in (13) we have

$$(14) \qquad \log |q(re^{i\phi})| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(\theta)| \, Pr(\phi-\theta) d\theta.$$

Since q is never zero, we can obtain the inequality

$$(15) \qquad \log |f(re^{i\phi})| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(\theta)| \, Pr(\phi-\theta) d\theta$$

as a more general form of Jensen's inequality where we use the measure $dm = \frac{1}{2\pi} Pr(\phi-\theta) d\theta$ in place of ordinary Lebesgue measure. The point $re^{i\phi}$ then plays the rôle of the origin.

Combining (14) and (15), we obtain the inequality $\left| \frac{f(re^{i\phi})}{q(re^{i\phi})} \right| \leq 1$. As $r \longrightarrow 1$ however, we have for almost all ϕ that $|f| \longrightarrow |F| = |Q|$. The function f/q is analytic since q is non-zero, and it is bounded above by 1. Furthermore it has a boundary function which is almost everywhere of modulus 1. Such functions are called inner functions.

DEFINITION. An inner function is an analytic function g on the unit disc such that $|g(z)| < 1$, and $|G(e^{i\theta})| = 1$ almost everywhere.

We obtain the following unique factorization for H^1 functions.

<u>THEOREM 9</u>. Every non-zero function f in H^1 can be written as a product $f = g.q$ where g is an inner function and q is an outer

function. This factorization is unique up to a constant of modulus 1.

Proof. We have given the construction for q, and g is defined to be the inner function f/q. For the uniqueness we need suppose $f = g_1 \cdot q_1$ is a second factorization. On the boundary we have $|Q| = |F| = |Q_1|$ almost everywhere, which implies by the definition of outer functions that $Q_1 = \lambda Q$ where $|\lambda| = 1$. This means $g = \lambda g_1$ and the two factorizations are the same up to the constant λ.

It is easily seen that outer functions are characterized by the following maximal property. A function q in H^1 is outer if and only if $|f(z)| \leq |q(z)|$, $z \in D$, holds for every $f \in H^1$ with $|F| \leq |Q|$ almost everywhere on Γ.

We consider a break down of inner functions next. If $f = g.q$ is the factorization of some $f \in H^1$, then all the zeros of f must appear in g. We will discuss that part of the inner function which reproduces all the zeros of the function. Since inner functions are bounded by 1, we can restrict ourselves to bounded analytic functions.

THEOREM 10. If f is bounded analytic in D and has zeros (α_n) where multiple zeros are repeated, then

$$\Sigma\,(1 - |\alpha_n|) < \infty. \tag{16}$$

Proof. We omit the zeros at the origin, and assume $f(0) \neq 0$. This will not affect (16); since now all α_n are non zero, we need only show $\prod_n |\alpha_n|$ converges to a non-zero value. We may assume that $|f| \leq 1$ on D. Let us define functions

$$B_n(z) = \prod_{k=1}^{n} \frac{\bar{\alpha}_k}{|\alpha_k|} \cdot \frac{\alpha_k - z}{1 - \bar{\alpha}_k z} \qquad n = 1,2,3, \ldots$$

which are analytic in the closed disc, and have modulus 1 on the boundary. Since

$$|F(\theta)/B_n(e^{i\theta})| = |F(\theta)| \leq 1$$

almost everywhere on the boundary, f/B_n is also bounded and analytic on D with modulus less than or equal to unity.

We apply the inequality $|f(z)| \leq |B_n(z)|$ at $z = 0$ to obtain

$$0 < |f(0)| \leq |B_n(0)| = \prod_{k=1}^{n} |\alpha_k|.$$

This gives convergence of $\prod_{k=1}^{\infty} |\alpha_k|$ since each $|\alpha_k| < 1$.

We next show that there exists a bounded analytic function with zeros (α_n) for any sequence satisfying (16).

<u>THEOREM 11</u>. Let (α_n) be a sequence in D. The product

$$b(z) = \prod_{n=1}^{\infty} \frac{\bar{\alpha}_n}{|\alpha_n|} \cdot \frac{\alpha_n - z}{1 - \bar{\alpha}_n z} = \prod_{n=1}^{\infty} b_n(z)$$

will converge uniformly on compact sets in D to an inner function with zeros $\alpha_1, \alpha_2, \alpha_3, \ldots$ if and only if

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty .$$

Such a function is called a <u>Blaschke product</u>.

<u>Proof</u>. Theorem 10 immediately gives one implication. Assume now that $\Sigma (1 - |\alpha_n|)$ converges.

If we consider the (continuous) restrictions of the functions B_n to the unit circle, we will show that these form a Cauchy sequence in H^2. Bearing in mind that $|B_n| = 1$ on Γ,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |B_m - B_n|^2 \, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\,|B_m|^2 + |B_n|^2 + 2\,\mathrm{Re}(B_n\bar{B}_m)]d\theta$$

$$= 2[1 - \mathrm{Re}\,\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{B_n}{B_m}\, d\theta].$$

We can assume $n > m$, in which case B_n/B_m is analytic in the closed disc, and the integral is just evaluation at the origin.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{B_n}{B_m}\, d\theta = \frac{B_n}{B_m}(0) = \prod_{k=m+1}^{n} |\alpha_k|.$$

By our hypothesis the latter product can be made arbitrarily close to 1 for m sufficiently large.

Thus (B_n) converges to some function B in L^2. This implies that (B_n) converges to B in L^1, and from the Poisson formula, for $z = re^{i\phi}$ restricted to $\{z : |z| \leq R < 1\}$

$$|B_n(z) - b(z)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |B_n(\theta) - B(\theta)|\, \mathrm{Pr}(\theta-\phi)\, d\theta$$

$$\leq ||B_n - B||_1 \cdot K(R)$$

where $K(R)$ is a constant. This gives uniform convergence of B_n to b on compact subsets of D.

The mean convergence of B_n to B implies that a subsequence converges almost everywhere. Thus $|B| = 1$ almost everywhere and b is an inner function.

Another method of proof gives information about the behaviour of $b(z)$ outside the unit disc.

<u>THEOREM 12</u>. If $\Sigma\,(1 - |\alpha_n|) < \infty$ where $0 < |\alpha_n| < 1$ for each n, then the Blaschke product with zeros $\alpha_1, \alpha_2, \ldots$, converges at all points z in the complex plane except those in the compact set

$$K = \text{closure} \left\{ 1/\alpha_n \;\middle|\; n = 1,2,3, \ldots \right\}$$

The convergence is uniform in any closed set in the plane which is disjoint from K, and $b(z)$ is therefore analytic off K.

The set K just consists of the inverses with respect to the unit circle of the points α_n, along with their points of accumulation, all of which will lie on the unit circle. They are just the same as the accumulation points of the α_n.

<u>Proof</u>. It is sufficient to prove the uniform convergence on any closed set disjoint from K. For any such closed set C, there exists a minimum distance $r > 0$ between points of C and of the compact set K. We let $\rho > 0$ be the smallest value of $|\alpha_n|$. We remove all the zeros α_n which lie inside the set C. There can only be a finite number of these. To show that the product $b_n(z)$ of the remaining factors converges uniformly to a function with no zeros on C, we show that $\Sigma\, |1 - b_n(z)|$ is uniformly summable.

$$b_n(z) = \frac{\alpha_n}{|\alpha_n|} \; \frac{\alpha_n - z}{1 - \bar{\alpha}_n z}$$

$$1 - b_n(z) = \frac{1}{|\alpha_n|} \left[1 - \frac{|\alpha_n|^2 - \bar{\alpha}_n z}{1 - \bar{\alpha}_n z}\right] + 1 - \frac{1}{|\alpha_n|}$$

$$= \frac{1 - |\alpha_n|}{|\alpha_n|} \left[\frac{1 + |\alpha_n|}{1 - \bar{\alpha}_n z} - 1\right].$$

Then for all n

$$|1 - b_n(z)| \leq \frac{1 - |\alpha_n|}{\rho} \left[\frac{2}{\rho r} - 1\right] \qquad \text{for } z \in C,$$

since $|1 - \bar{\alpha}_n z| = |\alpha_n| \left|\frac{1}{\bar{\alpha}_n} - z\right| \geq \rho r.$

Summing, we have $\Sigma\,|1 - b_n(z)|$ is uniformly bounded for $z \in C$, since $\Sigma\,(1 - |\alpha_n|) < \infty$. Furthermore, having deleted all factors $b_n(z)$ with zeros for $z \in C$, this argument gives an analytic function with no zeros in C for the deleted product $\prod b_n(z)$. This means $b(z)$ is analytic in C, and has precisely the α_n lying in C for its zeros there.

We next investigate a second type of inner function. This one, which has no zeros in D, is called a singular function. Then we show that every inner function factors uniquely into a product of the two types.

<u>THEOREM 13</u>. Suppose g is an inner function with no zeros. If we assume $g(0) > 0$, there is a unique singular positive measure μ on the unit circle such that

$$g(z) = \exp\left[-\int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)\right]. \tag{17}$$

<u>Proof</u>. Because g has no zeros in D, we may choose an analytic function h such that $g = e^{-h}$. If we write $h = u + iv$, the harmonic function u must be non-negative, because $|g| \leq 1$. We express u with the Poisson formula

$$u(re^{i\theta}) = \int Pr(\theta - t)\, d\mu(t). \tag{18}$$

The measure μ is unique; moreover it must be a positive measure since u is non-negative. We also have

$$h(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)$$

because the integral gives an analytic function with u as real part,

and with imaginary part vanishing at the origin. Now we impose the condition that g is an inner function. This means $|G(\theta)| = 1$ for almost all θ, or that the radial limits of u vanish almost everywhere. But from Theorem 1, since we have a Poisson representation (18), these limits are almost everywhere equal to $\frac{1}{2\pi}\frac{d\mu}{d\theta}$. Hence μ is a singular measure.

We should discuss the existence of the representation (18) having positive measure for the non-negative harmonic function u. We argue in the context of Theorem 2(c). We are given a harmonic function $u \geq 0$. We evaluate the norm in L^1 of the function u_r where $u_r(\theta) = u(re^{i\theta})$ and obtain

$$||u_r||_1 = \int_{-\pi}^{\pi} |u_r(\theta)|d\theta = \int_{-\pi}^{\pi} u(re^{i\theta})d\theta = 2\pi\, u(0).$$

Thus the collection $\{u_r : 0 < r < 1\}$ is contained in a bounded set in L^1, and as $r \longrightarrow 1$, the measures $\frac{1}{2\pi} u_r(\theta)d\theta$ converge (weak-*) to a measure μ which represents u. It is evident that μ is non-negative.

Suppose g is a singular function with singular measure μ. The closed support K of μ is the complement of the union of all open sets on Γ which have μ measure zero. Just as with the Blaschke product, the singular function has an analytic continuation outside the open unit disc D under certain conditions.

<u>THEOREM 14</u>. The singular function g is analytic everywhere in the complex plane except at points in the closed support K of its singular measure. If g is bounded away from zero in some neighbourhood (restricted to D) of a point $e^{i\theta}$, then this point does not belong to K. In particular, g cannot be continuously extended to any point in K.

Proof. If $z \notin K$ then using the representation (17) for g, the function h is analytic at z with derivative

$$h'(z) = \int \frac{2e^{i\theta}}{(e^{i\theta} - z)^2}\, d\mu(\theta)$$

and hence $g = e^{-h}$ is analytic.

Suppose now that $|g| \geq \rho > 1$ in the neighbourhood of $e^{i\theta_0}$. This incidentally holds if g has a continuous extension to $e^{i\theta_0}$, since the value will necessarily be of modulus one. Then we have for r close enough to 1 and $|\theta-\theta_0| \leq \delta$ for some constant δ, that $\operatorname{Re} h(re^{i\theta})$ is bounded above. This quantity, which we call $u(re^{i\theta})$ is known to be bounded below by zero, since $|g| = e^{-u} \leq 1$. We have

$$u_r(\theta) = u(re^{i\theta}) = \int Pr(\theta-t)\, d\mu(t). \tag{19}$$

We restrict the functions u_r to the interval $|\theta-\theta_0| \leq \delta$, and call these restrictions ϕ_r. These are a bounded family of continuous functions defined on an interval of length 2δ, and they will have some weak-star cluster point, a bounded measurable function ϕ. Take now any function $F(\theta)$ which is continuous on the circle, but vanishes outside $|\theta-\theta_0| \leq \delta$. We consider $I = \lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)\, u_r(\theta)\, d\theta$ twice; first substituting from (19) and interchanging the two integrations, we get

$$I = \int d\mu(t) \lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)\, Pr(\theta-t)\, d\theta = \int F\, d\mu$$

since the inner Poisson integral reproduces the continuous boundary values $F(\theta)$. We next note that, as $r \longrightarrow 1$, the integrals

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)\, u_r(\theta)\, d\theta = \frac{1}{2\pi}\int_{\theta_0-\delta}^{\theta_0+\delta} F(\theta)\, u_r(\theta)\, d\theta$$

cluster at

$$\frac{1}{2\pi}\int_{\theta_0-\delta}^{\theta_0+\delta} F(\theta)\,\phi(\theta)\,d\theta.$$

This must be equal to I. Hence

$$\frac{1}{2\pi}\int_{\theta_0-\delta}^{\theta_0+\delta} F(\theta)\,\phi(\theta)\,d\theta = \int_{\theta_0-\delta}^{\theta_0+\delta} F(\theta)\,d\mu(\theta)$$

for all continuous functions F (which vanish at the end points, no serious restriction) on the interval. Hence μ is absolutely continuous for $|\theta-\theta_0| < \delta$, and $\frac{d\mu}{d\theta} = \frac{1}{2\pi}\phi$.

But μ is singular, and therefore $\phi = 0$. Hence $\frac{d\mu}{d\theta} = 0$ in a neighbourhood of $e^{i\theta_0}$, which shows that this point is not in K. In particular no point on Γ to which g can be continuously extended is in K. This completes the proof.

We state the factorization theorem for inner functions. This can of course be used in combination with Theorem 9 to give a factorization for any H^1 function into a Blaschke product, a singular function, and an outer function in H^1.

<u>THEOREM 14</u>. Every bounded analytic function f can be written in the form $f = bh$ where b is a Blaschke product and h is bounded analytic without zeros. Every inner function f can be written in the form

$$f = bg$$

where b is a Blaschke product and g is singular. The factorizations are unique.

<u>Proof</u>. In either case, if f has zeros $\alpha_1, \alpha_2, \ldots$, we can form a convergent Blaschke product from these. By an argument previously given,

f/b is bounded; its modulus has the same supremum as f. If f is inner, then $f/b = g$ is also inner, and since it has no zeros, it is by definition a singular function. The uniqueness holds because the zeros (α_n) determine b.

<u>THEOREM 15</u>. Every inner function is the uniform limit of Blaschke products.

<u>Proof</u>. Let $g \in \mathcal{B}$ be an inner function. We may assume $g(0) \neq 0$. Define the function $\phi(r,\alpha) = \frac{1}{2\pi r} \int_{|z|=r} \log \left|\frac{g(z) - \alpha}{1 - \bar{\alpha}\, g(z)}\right| |dz|$ for $|\alpha| < 1$. By Jensen's formula, we have, if $\alpha \neq g(0)$, that

$$\phi(r,\alpha) = \log\left(\frac{r^n}{|z_1|\ |z_2| \ \dots\ |z_n|} \cdot \left|\frac{g(0) - \alpha}{1 - \bar{\alpha}\, g(0)}\right|\right)$$

where $z_1, z_2, \dots z_n$ are the zeros of the function $g - \alpha$ in the circle $|z| \leq r < 1$. We observe that $\phi(r,\alpha) \leq 0$, and moreover that $\phi(r,\alpha)$ is non-decreasing in r, and is uniformly bounded if we impose the condition on α that $|\alpha - g(0)| \geq \delta$. Call $\phi(\alpha) = \lim_{r\to 1} \phi(r,\alpha)$; we have $\phi \leq 0$. Take any $\rho < |g(0)|$ and consider $\int_0^{2\pi} \phi(\rho e^{i\theta})d\theta$. By the uniform boundedness of $\phi(r,\alpha)$ along $|\alpha| = \rho$, this is just

$$\lim_{r\to 1} \int_0^{2\pi} \phi(r, \rho e^{i\theta})d\theta = \lim_{r\to 1} \frac{1}{2\pi r} \int_0^{2\pi} \int_{|z|=1} \log \left|\frac{g(z) - \rho e^{i\theta}}{1 - \rho e^{-i\theta} g(z)}\right| |dz| d\theta$$

$$= \lim_{r\to 1} \int_{|z|=r} \frac{1}{2\pi} \int_0^{2\pi} \log \left|\frac{g(z) - \rho e^{i\theta}}{1 - \rho e^{-i\theta} g(z)}\right| d\theta\ |dz|.$$

We use Jensen's formula on the inner integral.

$$\frac{1}{2\pi} \int_0^{2\pi} \log|g(z) - \rho e^{i\theta}| - \frac{1}{2\pi} \int_0^{2\pi} \log|e^{i\theta} - \rho g(z)|\ d\theta.$$

The second part is zero, and the first gives $\log |g(z)|$ if $\rho < |g(z)|$. However if $\rho \geq |g(z)|$ the Jensen formula then gives $\log |g(z)| + \log \left|\frac{\rho}{g(z)}\right|$. Thus the inner integral is

$$\max(\log |g(z)|, \log \rho)$$

We call this $F(z)$. Because g is an inner function, $|g(re^{i\theta})| \longrightarrow 1$ as $r \longrightarrow 1$ for almost all θ. This means

$$0 = \lim_{r \to 1} \int_{|z|=r} F(z)\, |dz| = \int_0^{2\pi} \phi(\rho e^{i\theta})\, d\theta.$$

Since $\phi(\alpha) \leq 0$, it follows that $\phi(\rho e^{i\theta}) = 0$ for almost all θ. Let $\alpha = \rho e^{i\theta}$ be chosen so that $\phi(\alpha) = 0$. Consider $\frac{g(z) - \alpha}{1 - \bar{\alpha}\, g(z)}$, and let $B(z)$ be the Blaschke product with the same zeros. The function

$$U(z) = \frac{g(z) - \alpha}{1 - \bar{\alpha}\, g(z)} \quad \frac{1}{B(z)}$$

is an inner function. We have

$$0 = \phi(\alpha) = \lim_{r \to 1} \phi(r,\alpha) = \lim_{r \to 1} \log \left(\frac{r^n}{|z_1|\ |z_2| \ \dots\ |z_n|} \quad \frac{g(0) - \alpha}{|1 - \bar{\alpha}\, g(0)} \right),$$

and letting $r \longrightarrow 1$ in the latter

$$0 = \log \left(\left| \frac{g(0) - \alpha}{1 - \bar{\alpha}\, g(0)} \right| \Big/ \pi |z_1| \right) = \log \left(\left| \frac{g(0) - \alpha}{1 - \bar{\alpha}\, g(0)} \right| \Big/ |B(0)| \right)$$

$$= \log |U(0)|.$$

This means that $|U(0)| = 1$, so that U is constant, and we choose t so that $V(z) = e^{it}$. We now let $B_1(z)$ be the Blaschke product $e^{it} B(z)$. We have $\frac{g(z) - \alpha}{1 - \bar{\alpha}\, g(z)} = B_1(z)$. This Blaschke product will be arbitrarily close in norm to g, since ρ is arbitrarily small, and we have

$$|g(z) - B_1(z)| = |\alpha| \left| \frac{1 - g(z)}{1 - \bar{\alpha}\, g(z)} \right| \leq \frac{2\rho}{1-\rho}.$$

§3 THE WALLMAN COMPACTIFICATION

We require the following:

THEOREM. If X is a normal space, then each complex homomorphism of $C(X)$ is of the form

$$f \longrightarrow \lim f(\mathfrak{M}) \qquad f \in C(X)$$

for some uniquely determined maximal closed filter $\mathfrak{M}$ on X.

The homomorphisms of $C(X)$ are just the points of βX by §1, Theorem 6. To prove the theorem we take the set T of maximal closed filters on X, and introduce a topology which makes this into an extension of X (with suitable identifications) having properties (1) and (2) of §1, Theorem 6. Then T is homeomorphic to βX over X. The construction of a compactification using maximal closed filters is due to Wallman; it gives the Stone-Cech compactification whenever X is normal. The following sequence of lemmas will describe this approach. We will deal with maximal filters in the lattice of closed subsets of X rather than maximal closed filters on X. These are in an obvious 1-1 correspondence.

Let X be a normal topological space and denote by $\mathfrak{F}$ the lattice of closed subsets of X. Let T be the set of all maximal filters in $\mathfrak{F}$, with a topology generated by the basic closed sets of the form

$$\Omega_F = \{\mathfrak{M} \in T \; ; \; F \in \mathfrak{M}\} \qquad \text{for } F \in \mathfrak{F} .$$

Because the filters are maximal,

$$F \cup G \in \mathfrak{M} \Longrightarrow F \in \mathfrak{M} \text{ or } G \in \mathfrak{M}, \text{ which gives}$$

$$\Omega_F \cup \Omega_G = \Omega_{F \cup G} .$$

Therefore the collection of Ω_F is closed under finite unions, which

means that it qualifies as the basis (for closed sets) of a topology. We also have

$$\Omega_F \cap \Omega_G = \Omega_{F \cap G}.$$

An arbitrary closed set $C \subseteq T$ is of the form

$$C = \bigcap_{F \in \mathcal{J}} \Omega_F \qquad ,$$

and it can be assumed that $\mathcal{J}$ is a filter.

<u>Lemma 1</u>. T is compact Hausdorff in this topology.

<u>Proof</u>. Take different filters $\mathfrak{M}$ and $\mathfrak{N}$ in T. There must exist sets $C \in \mathfrak{M}$, $D \in \mathfrak{N}$ with $C \cap D = \emptyset$. By the normality of X there must exist $E, F \in \mathfrak{F}$ with $E \cup F = X$ and $D \cap E = \emptyset = C \cap F$. ($E$ and F are just complements in X of the two open sets whose existence is guaranteed by the definition of normality.) The open sets $O = T \setminus \Omega_E$, $U = T \setminus \Omega_F$ are neighbourhoods in T of $\mathfrak{M}$ and $\mathfrak{N}$ respectively. Furthermore

$$O \cap U = T \setminus (\Omega_E \cup \Omega_F) = T \setminus \Omega_{E \cup F} = \emptyset,$$

and therefore T is Hausdorff.

Now take a filter $\mathcal{A}$ of closed sets in T. A typical closed set is of the form

$$C = \bigcap_{F \in \mathcal{G}} \Omega_F \qquad \mathcal{G} \text{ a filter, } \mathcal{G} \subseteq \mathfrak{F} \quad .$$

Because $\mathcal{A}$ is assumed proper, each of the filters $\mathcal{G}$ is also proper. Because the sets C make up a filter, the collection of all $\mathcal{G}$ is a directed set, and its union is a proper filter $\mathcal{H}$. If we now choose any maximal filter $\mathfrak{M}$ above $\mathcal{H}$, we see that $\mathfrak{M}$ belongs to each closed set $C \in T$. For $\mathcal{G} \subseteq \mathfrak{M}$, and if $F \in \mathcal{G}$ we have $F \in \mathfrak{M}$. Hence for each F in any particular $\mathcal{G}$, we have $\mathfrak{M} \in \Omega_F$, so that

$\mathfrak{N} \in \bigcap_{F \in \mathcal{G}} \Omega_F = C$. This proves that an arbitrary proper filter A contains a point of adherance $\mathfrak{N}$, and therefore the space T is compact.

The injection mapping X into T is, of course, ω: $x \longrightarrow [x]$ where $[x] = \{F \in \mathcal{F} : x \in F\}$ is the maximum filter of all closed sets containing the one point x.

<u>Lemma 2</u>. ω is a homeomorphism of X into T.

<u>Proof</u>. Given any closes set $C = \bigcap_{F \in \mathcal{G}} \Omega_F$ in T, we define a closed set $F_0 = \bigcap_{F \in \mathcal{G}} F$ in X. The inclusion $\Omega_{F_0} \subseteq C$ always holds. We show that the traces of these two sets on $\omega(X)$ are the same.

$$C \cap \omega(X) = \Omega_{F_0} \cap \omega(X).$$

This is a consequence of the equivalent statements $[y] \in C \Longleftrightarrow F \in [y]$ for each $F \in \mathcal{G} \Longleftrightarrow y \in F$ for each $F \in \mathcal{G} \Longleftrightarrow y \in F_0 \Longleftrightarrow [y] \in \Omega_{F_0}$.

Hence in dealing with the restriction to $\omega(X)$ of the topology of T, one need only consider closed sets of the form Ω_F. But this immediately shows that ω induces a 1-1 correspondence between the closed sets of X and $\omega(X)$.

<u>Lemma 3</u>. $\omega(X)$ is dense in T. If we identify X and $\omega(X)$ then T is the Stone-Cech compactification of X.

<u>Proof</u>. To show the first part, it suffices to find, for any basic closed set $\Omega_F \subset T$, an element $x \in X$ with $[x] \notin \Omega_F$. Any element $x \in X \setminus F$ will suffice, and such an element exists because if $F = X$ then $\Omega_F = T$.

Assuming $X \subseteq T$, we see immediately that any two disjoint closed sets in X must have disjoint closures in T, because if $F \cap G = \emptyset$

we have $\Omega_F \cap \Omega_G = \emptyset$. This is however a necessary and sufficient condition for the property (2) to hold when (1) holds in §1, Theorem 7.

We prove the implication in the direction required. Assume that (2) fails to hold. There is a function f which we may take to be real valued, which is bounded continuous on X, but which has no extension to T as a bounded continuous function. Pick any $t \in T \setminus X$. The range of f is bounded, and if we assume there is neighbourhood U_y of each point y in the closure of this range such that $t \notin \Gamma_T f^{-1}(U_y)$, we obtain a contradiction of the continuity of f by a compactness argument. Γ_T denotes the closure operator on T. Hence there is some point y_0 in the closure of the range of f such that the inverse image of each neighbourhood of y_0 contains t in its closure. Suppose now that t is a point to which f has no continuous extension. This means there is a second point y_1 with the same property that y_0 has. If we now take disjoint closed neighbourhoods C_1 and C_0 of y_1 and y_0, the sets $f^{-1}(C_1)$ and $f^{-1}(C_0)$ are disjoint closed sets in X, but their closures in T intersect.

We close the discussion of the Wallman compactification with the following remark. Suppose f is bounded and continuous on X, and we denote by $\hat{f}$ its extension to T. Given any maximal closed filter $\mathfrak{A}_0$ on X, we will denote by $\mathfrak{A}$ the filter in T obtained by restricting the sets of $\mathfrak{A}_0$ to the lattice $\mathfrak{F}$ of closed subsets of X.

<u>Lemma 4</u>. For each maximal closed filter $\mathfrak{A}_0$ on the normal space X, $\lim f(\mathfrak{A}_0)$ exists and equals $\hat{f}(\mathfrak{A})$ for each $f \in C(X)$.

<u>Proof</u>. We can assume $f(\mathfrak{A}) = 0$, and by continuity, find a neighbourhood N_ϵ for any $\epsilon > 0$ with

$$|\hat{f}(\mathfrak{B})| < \epsilon \quad \text{for} \quad \mathfrak{B} \in N_\epsilon .$$

By the regularity of the space T, we can find a closed set C such that $\mathfrak{A} \in C \subseteq N_\epsilon$. By compactness we can find, if $C = \bigcap_{\mathfrak{B}} \Omega_F$, a single Ω_F with $\mathfrak{A} \in \Omega_F \subseteq N_\epsilon$. Hence $|\hat{f}(\mathfrak{B}) < \epsilon$ holds for $\mathfrak{B} \in \Omega_F$, and in particular $|f(x)| = |\hat{f}([x])| < \epsilon$ for $x \in F$. Also $\mathfrak{A} \in \Omega_F$ gives $F \in \mathfrak{A}$. This proves that $\lim f(\mathfrak{A})$ exists and equals $\hat{f}(\mathfrak{A})$; however the former is the same as $\lim f(\mathfrak{A}_0)$ since $\mathfrak{A}_0$ is generated by $\mathfrak{A}$.

§4 HYPERBOLIC METRIC

We shall discuss the introduction of the hyperbolic metric into the open unit disc D. Another metric giving the same topology and called the pseudo-hyperbolic metric will also be used frequently. The latter is defined by the formula

$$(1) \qquad \psi(a,b) = \left|\frac{a-b}{1-\bar{a}b}\right| \qquad a,b \in D$$

The most general bilinear transformation S which maps D onto itself is given by

$$(2) \qquad \omega = S(z) = e^{i\phi} \cdot \frac{c-z}{1-\bar{c}z} \qquad |c| < 1, \ \phi \text{ real.}$$

By direct substitution we obtain

$$(3) \qquad \psi(S(a), S(b)) = \psi(a,b)$$

for any transformation S of the type (2).

We show that ψ is a metric on D. It satisfies the condition $\psi(a,b) \geq 0$, and is only zero if $a = b$. Observe also that $\psi(a,b) < 1$ for $a,b \in D$. Clearly $\psi(a,b)$ is the same as $\psi(b,a)$. To obtain the triangle inequality for any three points, we simplify matters by using a transformation S of type (2) to send one point a_0 to the origin, and a second b_0 to a point $b > 0$ on the positive reals. The third, c_0, will be mapped to an unspecified point c. Then the triangle inequality

$$\psi(a_0,c_0) \leq \psi(a_0,b_0) + \psi(b_0,c_0)$$

is just the same as

(4) $$|c| \leq b + \left|\frac{b - c}{1 - bc}\right|$$

The latter obviously holds for $|c| \leq b$, and so we assume $|z| = \rho > b$. On the circle $|z| = \rho$ we have

$$1 - \left|\frac{b - z}{1 - bz}\right|^2 = \frac{(1-bz)(1-\overline{bz}) - (b-z)(\overline{b}-\overline{z})}{|1 - bz|^2}$$

$$= \frac{(1-b^2)(1-\rho^2)}{|1 - bz|^2}$$

which attains its maximum value when $z = \rho$, by inspection of the denominator of the right side of the equation. This means that the minimum value of $\left|\frac{b - z}{1 - bz}\right|$ occurs on $|z| = \rho$ when $z = \rho$. It is sufficient therefore to check (4) in the case $z = \rho$.

$$\rho < b + \frac{\rho - b}{1 - b\rho} .$$

But this is trivial, as $b + \frac{\rho - b}{1 - b\rho} = \frac{1 - b^2}{1 - b\rho} > \rho$ since $\rho > b$. Hence (4) holds in general; observe that it always holds with strict inequality when the three points are all different.

Hence ψ is a metric on D. We list some of its properties:

(a) An ϵ-neighbourhood of a point b consists of all points inside a circle containing b, but with centre on the line segment joining b to the origin. For example, if $0 \leq b < 1$ and $0 < \epsilon < 1$, the circle $\psi(b,z) = \epsilon$ meets the real axis at $r_1 = \frac{b - \epsilon}{1 - b\epsilon}$ and $r_2 = \frac{b + \epsilon}{1 + b\epsilon}$ where $r_1 < a < r_2$. The centre of the circle is $r = \frac{b(1-\epsilon^2)}{1 - b^2\epsilon^2} \leq b$.

(b) $\psi(|a|, |b|) \leq \psi(a,b)$ $\qquad a,b \in D$.

(c) Suppose $1 > a > b > 0$. The closest point on the circle

$|z| = a$ to the point $z = b$ is the point $z = a$. This follows from (b).

The straight lines of D, when it is given the metric ψ, are the circles orthogonal to Γ, or rather their restrictions to D. We investigate the general rigid motion (2) of D in this metric. Three cases arise, and these can be distinguished by the values of the constant $\lambda = \mathrm{Sin}^2(\phi/2)/(1 - |c|^2)$. For a development of the following equations, see Caratheodory [6].

The case $\lambda < 1$ is of little interest for us. The transformation (2) in this case rotates the points of D about a fixed $z_0 \in D$. The points move on circles $\psi(z,z_0) = \text{const.}$ which, as we have seen, contain z_0 but have their centres on the line segment joining z_0 to the origin.

If $\lambda = 1$, the rigid motion is called a limiting rotation. It can be thought of as a rotation about an ideal point not in the geometry, actually a point α on the unit circle. The points in D move along individual members of the family of circles touching Γ at α, but otherwise lying inside D. Such circles are called oricycles. The equation for a limiting rotation about α has the form

$$\omega = \frac{(1+iu)z - \alpha}{\bar{\alpha}z - (1-iu)} \qquad |\alpha| = 1$$

where u is an arbitrary non-zero real number. The same equation may be written

$$\omega = \frac{z - i(z-\alpha)}{1 - i(\bar{\alpha}z-1)}$$

if we replace u by $1/\rho$ and multiply numerator and denominator by -i. Again the constant ρ is any non-zero real number. The limiting rotation is illustrated in the following diagram.

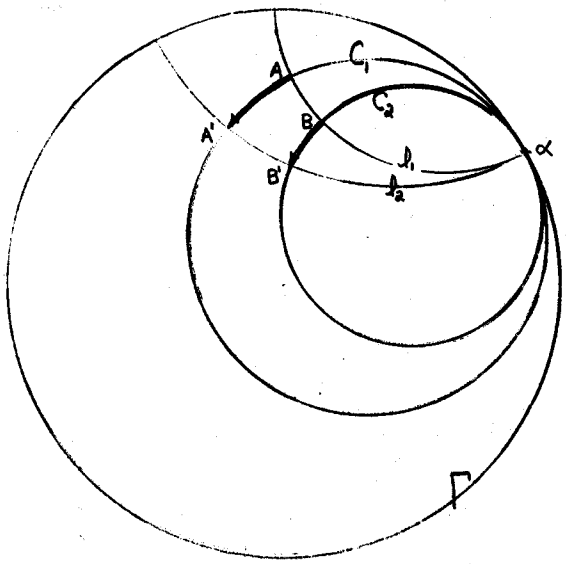

Figure 1

This transformation moves all points in D, and in fact has only α as a fixed point. It moves points along the oricycles through α, of which C_1 and C_2 are examples. Points on the circle ℓ_1 are mapped into points on the circle ℓ_2; note that these are non-Euclidean lines. If A and B have images A' and B', we know that $\psi(A',B') = \psi(A,B)$ because the transformation is a rigid motion. The oricycles are curves at a constant distance from each other, and this applies to any pair of oricycles through α.

The third case is a translation, and it corresponds to $\lambda > 1$. If α and β are two points on the unit circle, then the transformation given by

$$\omega = \frac{(\alpha+\beta)\sigma + (\alpha-\beta)z - 2\sigma\alpha\beta}{2\sigma z - (\alpha+\beta)\sigma - (\alpha-\beta)} \qquad -1 < \sigma < 1$$

translates points along the family of circles through α and β. If σ is positive, the points move from α towards β, and σ represents

the pseudo-hyperbolic distance travelled by points on the non-Euclidean line C_0 joining α and β. The circles through α and β are called hypercycles, and we again consider in a diagram the family of non-Euclidean lines perpendicular to the curves along which the points move.

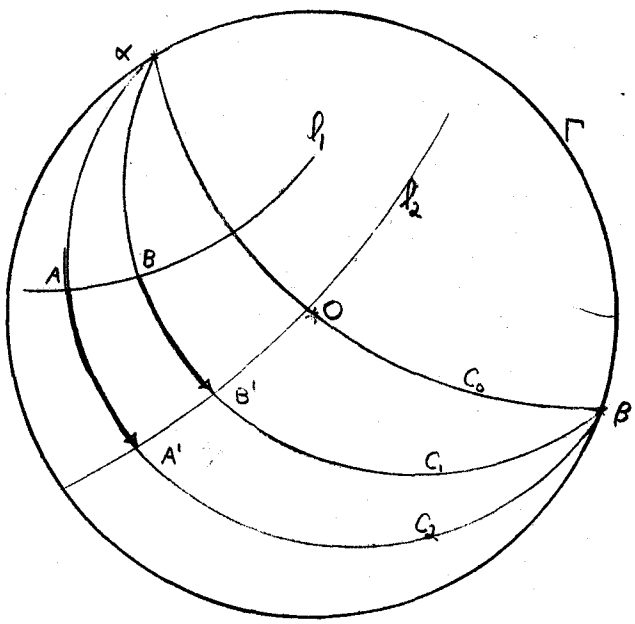

Figure 2

Points on one non-Euclidean line ℓ_1 map into points on another. Again $\psi(A',B') = \psi(A,B)$, and in general two hypercycles of the same family are curves at a constant hyperbolic distance from each other. The hyperbolic metric h is equivalent to ψ and it has the advantage of being additive for three points which are situated on a non-Euclidean line. This does not hold for ψ ; we have seen that the triangle inequality is strict for any three distinct points. The hyperbolic metric $h(a,b)$ can be obtained from $\psi(a,b)$ by assuming that, for some function g, $h(a,b) = g(\psi(a,b))$, and imposing the condition that h will be additive on the real line.

$$h(0,r+k) = h(0,r) + h(r,r+k)$$

for $r,k \geq 0$ and $r+k = 1$. In terms of g this gives

$$g(r+k) = g(r) + g\left(\frac{k}{1 - (r+k)r}\right).$$

We set $v = \dfrac{k}{1 - (r+k)r}$ which gives $k = v(1-r^2)/(1+rv)$. Thus

$$\frac{g(r+k) - g(r)}{k} = \frac{g(0)}{k} = \frac{g(v)}{v} \cdot \frac{1 + vr}{1 - r^2}.$$

We let $k \longrightarrow 0$, which means $v \longrightarrow 0$.

$$g^1(r) = g^1(0)/(1-r^2)$$

which gives, assuming $g(0) = 0$, that $g(r) = \tanh^{-1}(r/c)$ where $c = g^1(0)$ will be given the value 1. Hence the equation defining the hyperbolic metric is

$$\text{(5)} \qquad \tanh\ (h(a,b)) = \left|\frac{a - b}{1 - \bar{a}b}\right|$$

Each ϵ - neighbourhood of a point in the ψ-metric is just a δ - neighbourhood of the same point in the h-metric with $\tanh(\delta) = \epsilon$. The function $y = \tanh x$ is monotone increasing; as x increases from 0 to ∞, y increases from 0 to 1.

The differential of arc length in the hyperbolic metric is given by $|dz|/1-|z|^2$, and from this it is evident that the hyperbolic distance between points is greater than the Euclidean distance. In the neighbourhood of the origin the two are close to each other, but the hyperbolic distance is magnified when the points approach the boundary. It is this property, namely

$$|z_1-z_2| \leq h(z_1,z_2) \quad , \quad z_1,z_2 \in D \quad ,$$

which will make the following Theorem more useful to us when stated in

terms of the hyperbolic metric. In other cases however, we shall use the pseudo-hyperbolic distance.

The content of Schwarz's Lemma was first stated in the following invariant form by G. Pick.

THEOREM. Suppose $f: D \longrightarrow D$ is a holomorphic function where D is the unit disc. Then for arbitrary z_1 and z_2 in D

$$h(f(z_1), f(z_2)) \leq h(z_1, z_2).$$

Proof. The transformations

$$t = \frac{z - z_1}{1 - \bar{z}_1 z} \quad \text{and} \quad \omega = \frac{f(z) - f(z_1)}{1 - \overline{f(z_1)}\, f(z)}$$

are holomorphic and map D into itself. The first is a conformal map of D onto itself; we can invert it (and combine with the second) to obtain a function

$$\omega = g(t)$$

where $g: D \longrightarrow D$ is holomorphic and $g(0) = 0$. Applying Schwarz's Lemma to g we obtain

$$|g(t)| \leq |t|$$

or

$$\left| \frac{f(z) - f(z_1)}{1 - \overline{f(z_1)}\, f(z)} \right| \leq \left| \frac{z - z_1}{1 - \bar{z}_1 z} \right| .$$

Setting $z = z_2$, this equation gives

$$\psi(f(z_1), f(z_2)) \leq \psi(z_1, z_2) \quad \text{and hence}$$

$$h(f(z_1), f(z_2)) \leq h(z_1, z_2)$$

We include the following lemma at this point since its proof runs

along the same lines as the proof of Pick's theorem.

<u>Lemma</u>. If f is analytic in the open unit disc D, and if $|f(z)| \leq 1$ for $z \in D$, then

$$|f^1(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2} \qquad \text{for } z \in D.$$

<u>Proof</u>. As in the previous theorem we construct $\omega = g(t)$. Noting that $|g(t)| \leq t$ implies $|g^1(0)| \leq 1$, we write the latter inequality in terms of f.

$$\left| \frac{d}{dt}\left(\frac{f(z) - f(z_1)}{1 - f(z_1)\, f(z)}\right) \right|_{t=0} \leq 1.$$

This gives

$$\frac{|f^1(z)|\,(1 - |f(z_1)|^2)(1 - |z_1|^2)}{|1 + z_1 t|^2} \leq 1$$

which is the required formula if we set $t = 0$, making $z = z_1$.

CHAPTER I

BOUNDED ANALYTIC FUNCTIONS ON THE UNIT DISC

Introduction. The chapter begins with a few of the known results about the algebra $\mathcal{B}$ of bounded analytic functions on the unit disc. A more comprehensive account of this is found in the last chapter of Hoffman [14]. In the next two sections, we deal with Carleson's proof of the Corona conjecture [8], and his theorem on interpolating sequences [7]. In section 4 we begin the attempt to describe the homomorphisms of $\mathcal{B}$ with filters.

§1 THE ALGEBRA $\mathcal{B}$

The set of all bounded analytic functions on the unit disc, to be denoted by $\mathcal{B}$, is an algebra under pointwise addition and multiplication. Under the norm $||f|| = \sup_{z\in D}|f(z)|$, it is readily seen to be a Banach algebra, commutative with identity. Henceforth $\mathcal{M}$ will be used to denote the space of maximal ideals of $\mathcal{B}$; $\mathcal{M}$ is compact Hausdorff under the weakest topology which makes all the functions $\hat{f}$, for $f \in \mathcal{B}$, continuous. If $\hat{\mathcal{B}} = \{\hat{f} : f \in \mathcal{B}\}$, the Gelfand homomorphism of $\mathcal{B}$ onto $\hat{\mathcal{B}} \subset C(\mathcal{M})$ is an isometric isomorphism. This follows upon consideration of the subset Δ of $\mathcal{M}$ consisting of fixed homomorphisms or evaluations. For each $z \in D$ the mapping $\Phi_z : f \longrightarrow f(z)$, for $f \in \mathcal{B}$, is a homomorphism in $\mathcal{M}$. The set $\{\Phi_z : z \in D\} = \Delta$ is obviously in 1-1 correspondence with D; we shall shortly show that this is a homeomorphism. Take now any $f \in \mathcal{B}$. The norm of $\hat{f}$, which we will simply write as $||\hat{f}||$, will satisfy

$$||\hat{f}|| = \sup_{\Phi\in\mathcal{M}}|\hat{f}(\Phi)| \leq \sup_{\Phi\in\Delta}|\hat{f}(\Phi)| = \sup_{z\in D}|f(z)| = ||f||.$$

Since the inequality $||\hat{f}|| \leq ||f||$ is always valid for a Gelfand representation, we have

$$||\hat{f}|| = ||f|| \qquad \text{for all } f \in \mathcal{B}.$$

This condition is sufficient to make the Gelfand mapping into an isometric isomorphism. $\mathcal{B}$ is therefore a function algebra.

Suppose we select the function z from the algebra $\mathcal{B}$. For any homomorphism $\Phi \in \mathcal{M}$, we have $||\Phi|| = 1$, so that $|\Phi(f)| \leq ||f||$ holds for functions f in $\mathcal{B}$. The function z evidently satisfies $||z|| = 1$, so that

$$|z(\Phi)| = |\Phi(z)| \leq ||z|| = 1. \qquad \Phi \in \mathcal{M}.$$

In other words $\hat{z}$ maps all of $\mathfrak{M}$ into the closed unit disc $D \cup \Gamma$. Also $\hat{z}$ maps onto D; indeed, for any $a \in D$, the homomorphism Φ_a maps z into a, so that $\hat{z}(\Phi_a) = a$. But D is dense in $D \cup \Gamma$, $\hat{z}$ is continuous, and $\mathfrak{M}$ is compact. From these properties we see that $\hat{z}$ maps onto $D \cup \Gamma$.

We can also establish that $\hat{z}$ takes only the homomorphisms of Δ into points of D, that the set $\mathfrak{M} \setminus \Delta$ is entirely mapped onto Γ. Suppose that $\hat{z}(\Phi) = \lambda$ for some λ with modulus less than 1. If f is any function of $\mathcal{B}$ vanishing at λ, then it can be written $f(z) = (z-\lambda)g(z)$, and

$$\Phi(f) = (\Phi(z) - \lambda)\, \Phi(g) = 0$$

Therefore Φ annihilates the ideal in $\mathcal{B}$ of all functions that are zero at λ. Because this is a maximal ideal it must be the kernel of Φ, and $\Phi = \Phi_\lambda$. Therefore $|\hat{z}(\Phi)| = 1$ for $\Phi \in \mathfrak{M} \setminus \Delta$.

Our mapping $\hat{z}$ is known to be 1-1 and continuous from Δ onto D. It must actually be a homeomorphism, for the topology on Δ is the weakest for which each $\hat{f}$ is continuous, whereas all the analytic functions f in $\mathcal{B}$ are necessarily continuous on D. Hence the subset Δ of $\mathfrak{M}$ is a homeomorphic copy of the unit disc.

We have seen that among the remaining homomorphisms in $\mathfrak{M} \setminus \Delta$, there exist ones mapped by $\hat{z}$ to each of the points of the unit circle. It is useful to classify the set $\mathfrak{M} \setminus \Delta$ in this way. For $|\alpha| = 1$, define the fibre above α to be the set

$$\mathfrak{M}_\alpha = \{\Phi \in \mathfrak{M} \mid \hat{z}(\Phi) = \alpha\}.$$

These are non-empty compact subsets of $\mathfrak{M}$. They are complicated topological spaces, and a study of the structure of $\mathfrak{M}$ depends to a large

extent on a knowledge of an individual fibre. That all fibres are homeomorphic is not difficult to see.

Any rotation of the complex plane about the origin will bring about a conformal mapping of D onto itself. This causes an algebra automorphism of $\mathcal{B}$, which in turn gives a homeomorphism of $\mathcal{M}$ onto itself. The original rotation τ will operate on the unit circle, and from the definition of the fibres we see that the homeomorphism takes $\mathcal{M}_\alpha$ onto $\mathcal{M}_{\tau\alpha}$. Therefore it is frequently enough to give an argument for a particular fibre, often the fibre $\mathcal{M}_1$.

In the following Theorem, we see that the value of $\Phi(f)$ depends on the behaviour of f in the vicinity of the point α, where $\Phi \in \mathcal{M}_\alpha$, $|\alpha| = 1$, and $f \in \mathcal{B}$.

<u>THEOREM 1.</u> For $f \in \mathcal{B}$ and $|\alpha| = 1$, the range of $\hat{f}$ on $\mathcal{M}_\alpha$ consists of all complex numbers ζ for which there is a sequence of points λ_n in D with $\lim \lambda_n = \alpha$ and $\lim f(\lambda_n) = \zeta$. The function f is continuously extendable to $D \cup \{\alpha\}$ if and only if $\hat{f}$ is constant on $\mathcal{M}_\alpha$.

Proof. Suppose $\lim \lambda_n = \alpha$ and $\lim f(\lambda_n) = \zeta$. The ideal $J = \{g \in \mathcal{B}: \lim g(\lambda_n) = 0\}$ is non-trivial. It contains the functions $f - \zeta$ and $z - \alpha$; however, it does not contain the identity 1. Any maximal ideal containing J is the kernel of a homomorphism Φ with

$$\Phi(f - \zeta) = 0 \quad \text{and} \quad \Phi(z - \alpha) = 0$$

which means

$$\Phi(f) = \zeta \quad \text{and} \quad \Phi \in \mathcal{M}_\alpha.$$

This gives the first claim in one direction, and leads immediately to one implication in the second claim. If $\hat{f}$ is constant on $\mathcal{M}_\alpha$ with value ζ, then $(f(\lambda_n))$ cannot cluster at any different value η for

any sequence (λ_n) converging to α. If it did, f would converge to η along a subsequence, and this would give $\Phi(f) = \eta$ for some $\Phi \in \mathfrak{M}_\alpha$, a contradiction. Hence f can in this case be continuously extended to α.

For the converse of this last claim, we make use of the function

(1) $\quad h(\lambda) = \frac{1}{2}(1 + \bar{\alpha}\lambda)$

which is continuous on $D \cup \Gamma$ and satisfies $|h| < 1$ there, except at the one point α where $h(\alpha) = 1$. We assume f has continuous extension to α with value zero. Form the sequence $((1 - h^n)f)$ $n = 1, 2, \ldots$ of functions in $\mathcal{B}$. In the topology of $\mathcal{B}$, which is the uniform topology, this sequence converges to f. For to make $||h^n f|| < \epsilon$ we can choose a neighbourhood (in D) of α on which $|f| < \epsilon$, and then since $|h| \leq \rho < 1$ on the remainder of D, a suitable value of N will give $|h^n| < \epsilon/||f||$. for $n \geq N$.

If we take now any $\Phi \in \mathfrak{M}_\alpha$, we have

$$\Phi(h) = \tfrac{1}{2}(1 + \bar{\alpha}\,\Phi(\lambda)) = \tfrac{1}{2}(1 + \bar{\alpha}\,\alpha) = 1$$

and, because Φ is continuous

$$\Phi(f) = \lim_{n \to \infty} \Phi((1 - h^n)f) = 0$$

Hence $\hat{f}\,|\mathfrak{M}_\alpha = 0$ and the claim follows.

Finally we must find, assuming $\Phi(f) = \zeta$ and $\Phi \in \mathfrak{M}_\alpha$, a sequence (λ_n) converging to α with $\lim f(\lambda_n) = \zeta$. We again can let $\zeta = 0$. If no such sequence exists, then for some neighbourhood N of α

$$|f(z)| \geq \delta > 0 \qquad \text{for } z \in D \cap N$$

If $f = b.g.q$ is the decomposition of f into Blaschke product, singular function, and outer function, we show that both b and g are bounded away from zero in the neighbourhood $D \cap N$. We appeal to two theorems

on inner functions from section 2 on H^p spaces in Chapter 0. The point α is not a cluster point of the zeros of b; by Theorem 12, b is analytic at α. In like fashion, Theorem 14 tells us that g is analytic at α since it is bounded away from zero in a neighbourhood of α. Thus we have $|b(\alpha)| = |g(\alpha)| = 1$. For the outer function

$$q(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|\tilde{F}(\theta)|d\theta\right]$$

where the boundary function F of f satisfies

$$|\tilde{F}(\theta)| \geq \delta > 0 \qquad \text{for } e^{i\theta} \in N \cap \Gamma$$

On this set $\log|\tilde{F}|$ is bounded and hence

$$h(z) = \exp\left[\frac{1}{2\pi}\int_{N\cap\Gamma} \frac{e^{i\theta}+z}{e^{i\theta}-z}(-\log|\tilde{F}(\theta)|)d\theta\right]$$

is bounded analytic. Now the function

$$q(z)\,h(z) = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} k(\theta)d\theta\right]$$

since $k(\theta)$ is identically zero on $N\cap\Gamma$, will be analytic at α with value zero. Hence $|(qh)(\alpha)| = 1$. Together we have that $fh = b\cdot g\cdot q\cdot h$ is analytic at α, with a value of modulus one. This means that $\widehat{fh}$ has a constant value of modulus one on $\mathcal{M}_\alpha$. Hence for any Φ in $\mathcal{M}_\alpha$

$$|\Phi(f)||\Phi(h)| = |\Phi(fh)| = |(\widehat{fh})(\Phi)| = 1$$

so that $\Phi(f) \neq 0$, a contradiction. This completes the proof.

We next consider the Corona conjecture. This is the conjecture that Δ is a dense subset of the compact topological space $\mathcal{M}$. The next section is devoted to Lennart Carleson's proof of this conjecture. What he actually proves is the following interpolation theorem.

<u>THEOREM.</u> If $f_1, f_2, \ldots, f_n$ are bounded analytic functions such that

(2) $\quad |f_1(\lambda)| + |f_2(\lambda)| + \ldots + |f_n(\lambda)| \geq \delta > 0 \,, \qquad |\lambda| < 1,$

then there exist bounded analytic functions $p_1, p_2, \ldots, p_n$ with

(3) $\quad p_1 f_1 + p_2 f_2 + \ldots + p_n f_n = 1.$

We conclude this section by showing that the assertion in this theorem is equivalent to the Corona conjecture.

Suppose the above theorem fails for the n functions $f_1, f_2, \ldots, f_n$. Then these functions generate a proper ideal. Let Φ_o be any homomorphism whose kernel contains this ideal. The open neighbourhood of Φ_o given by

$$\left\{\Phi \in \mathfrak{M} : |\Phi(f_i)| < \delta/n \,,\ i = 1,2,\ldots,n\right\}$$

cannot meet Δ since $|\hat{f}_1(\Phi_a)| + \ldots + |\hat{f}_n(\Phi_a)| = |f_1(a)| + \ldots + |f_n(a)| \geq \delta$ for all $a \in D$. Hence Δ is not dense in $\mathfrak{M}$.

Suppose conversely that some $\Phi_o \in \mathfrak{M}$ has a neighbourhood which fails to meet Δ. This neighbourhood must contain a basic open neighbourhood of Φ_o of the form

$$N = \left\{\Phi \in \mathfrak{M} : |\Phi(f_i)| < \delta \,,\ i = 1,2,\ldots,n\right\}$$

for some functions $f_1, f_2, \ldots, f_n$ in $\mathcal{B}$. Because N does not meet Δ, for each $a \in D$, $|\Phi_a(f_i)| < \delta$ must fail for some i. This means that

$$|\hat{f}_i| + \ldots + |\hat{f}_n| \geq \delta \quad \text{on } \Delta$$

which is the same as (2). However, (3) cannot possibly hold, since all the functions $f_1, f_2, \ldots, f_n$ are annihilated by Φ_o.

§2 THE CORONA PROBLEM

In this section we give Carleson's proof of the Corona conjecture. Theorems 1, 2, and 3 are required for this proof. These results are sufficient to give Theorem 4, and interpolation result, suggested by D. J. Newman, which leads to a proof of the corona conjecture given in Theorem 5. An additional interpolation result, relevant to the next section, is then proved.

THEOREM 1. Let $\mu(z)$ be a non-negative measure in D and assume that

(1.1) $\mu(S) \leq C\ell$

holds for all sets S of the form

$$S = \left\{re^{i\theta} : r \geq 1-\ell,\ \theta_0 \leq \theta \leq \theta_0 + \ell\right\} \qquad \ell \leq 1$$

Then there is an absolute constant A so that

(1.2) $\int |G(z)|^p \, d\mu(z) \leq A.C \, ||G||_p^p$

for all $G \in H^p$, $p \geq 1$. Conversely, if an inequality (1.2) holds for a certain constant C_1 in place of $A.C$, $\mu(S)$ satisfies (1.1) with C independent of S.

The following Lemma 2 is a weakened form of the key result used in Carleson's first paper [7] to obtain the condition for a sequence to be an interpolating sequence (see next section). The theorem is then shown to follow from this result. Lemma 1 is required in the proof of Lemma 2.

Lemma 1. Let $c \geq 2$ be a positive integer. Suppose (a_n) is a sequence satisfying the conditions

$$\sum_{v=n}^{\infty} a_v \leq c\, a_n \qquad n = 0, 1, 2, \ldots$$

$$a_0 \geq a_1 \geq a_2 \geq \cdots \geq 0.$$

Then

(1.3) $$a_n \leq 4(1 - (\tfrac{1}{c}))^n a_o \qquad \text{for all } n.$$

Proof. Fix any value $N > c$, and suppose (a_n), with the restriction $a_o = 1$, maximizes a_N under the above restrictions. It is evident that $a_n = 0$ for $n > N$. Taking the value a_n for $h = N - c + 1$ as determined, the restriction

$$a_n + \dots + a_N \leq c\, a_n$$

obtained from the above formula will give, for a_N maximal, $a_n = a_N$. This will hold for $N - c + 1 \leq n \leq N$. The values of a_n for $n < N - c$ are obtained by recursion to be

$$a_n = (1 - (\tfrac{1}{c}))^n.$$

Hence

$$a_N \leq a_{N-} = (1 - (\tfrac{1}{c}))^N (1 - (\tfrac{1}{c}))^{-c} < 4(1 - (\tfrac{1}{c}))^N$$

Lemma 2. Let (ω_i) be a sequence of open subintervals of $(0,1)$. Denote by m_1 the length of σ_i. We suppose that each interval ω_i has the form $(\frac{k}{2^n}, \frac{k+1}{2^n})$ for suitable integers k and n. If a constant C exists such that

(1.4) $$\sum_{\omega_j \leq \omega_i} m_j \leq C.m_i \qquad \text{for all } i,$$

then a constant A can be found such that

(1.5) $$\sum_{i=1}^{\infty} (\tfrac{1}{m_i}) \left(\int_{\omega_i} f(x)dx \right)^2 \leq A.C \int_0^1 f(x)^2\, dx$$

for all square integrable functions f.

Proof. Let $X_i(x)$ be the characteristic function of ω_i, and define

$$K(x,y) = \sum_1^{\infty} X_i(x)\, X_i(y) / m_i$$

We can restrict to functions f that are non-negative.

$$(1.6)\quad \sum_1^\infty \left(\frac{1}{m_i}\right)\left(\int_{\omega_i} f(x)dx\right)^2 = \int_0^1 f(x))dx \left(\int_0^1 K(x,y)\, f(y)dy\right)$$

$$\leq \left\{\int_0^1 f(x)^2dx\right\}^{\frac{1}{2}} \cdot \left\{\int_0^1 \left(\int_0^1 K(x,y)\, f(y)dy\right)^2 dx\right\}^{\frac{1}{2}}$$

by Schwarz's inequality. We express the last factor as the square root of a quadratic form

$$(1.7)\qquad Q = \sum_{i,j} \left(\frac{m_{ij}}{\sqrt{m_i}\,\sqrt{m_j}}\right) x_i\, x_j$$

where

$$x_i = \left(1/\sqrt{m_i}\right) \int_{\omega_i} f(x)dx$$

$$m_{ij} = m(\omega_i \cap \omega_j)$$

If we can prove that Q has a bound $A \subset \sum_1^\infty x_i^2$, the lemma follows, since (1.6) becomes

$$\Sigma\, x_i^2 \leq ||f|| \cdot \sqrt{AC\Sigma\, x_i^2}$$

The summation of 1.7 is carried out in a special way. We say that the index i belongs to the first generation G_1 if ω_i is not included in any larger interval ω_j. (If several intervals are the same, choose any one as being a member of the first generation.) The second generation G_2 is defined as the collection of first generations corresponding to the intervals associated with G_1. We proceed in this manner; every interval will have its index in one of these sets.

If ω_i has index i belonging to G_k, we denote by G_{iv} the members of G_{k+v} corresponding to the subintervals of ω_i, $v = 0,1,2,\ldots$. Let a_v be the total length of the intervals corresponding to G_{iv} for some i. $(a_0 = m_i)$ We then obtain

$$\sum_{v=n}^{\infty} a_v \leq c\, a_n \qquad n = 0,1,2,\ldots$$

from the hypothesis of the lemma, and also

$$a_o \geq a_1 \geq a_2 \geq \cdots \geq 0,$$

which means Lemma 1 will apply. The quadratic form Q has symmetric coefficients, and

$$Q = \sum_{i=1}^{\infty} \frac{x_i}{\sqrt{m_i}} \sum_{j=1}^{\infty} \frac{m_{ij}}{\sqrt{m_j}} x_j \leq 2 \sum_{i=1}^{\infty} \frac{x_i}{\sqrt{m_i}} \sum_{v=o}^{\infty} \sum_{j \in G_{iv}} \sqrt{m_j}\, x_j$$

taking only pairs (i,j) with $j \in$ some G_{iv}. For these $m_{ij} = m_j$. If $j \notin$ some G_{jv}, then $i \in$ some G_{jv}, or else $m_{ij} = 0$; thus the inequality follows. Hence

$$Q^2 \leq 4 \left[\sum_{i=1}^{\infty} \left(\frac{1}{\sqrt{m_i}} \sum_{v=o}^{\infty} \sum_{j \in G_{iv}} \sqrt{m_j}\, x_j \right) x_i \right]^2$$

$$\leq 4 \sum_{i=1}^{\infty} \frac{1}{m_i} \left(\sum_{v=o}^{\infty} \sum_{j\, G_{iv}} \sqrt{m_j}\, x_j \right)^2 \sum_{i=1}^{\infty} x_i^2$$

We define $k^2 = 1 - (\frac{1}{c})$, and find for the first factor the estimate

$$\sum_{i=1}^{\infty} \left(\frac{1}{m_i}\right) \sum_{v=o}^{\infty} k^{-v} \left(\sum_{j \in G_{iv}} \sqrt{m_j} \sum_{j \in G_{iv}} x_j \right)^2 . \left(\sum k^v \right)$$

by merely applying $1 \leq k^{-v} \sum_{o}^{\infty} k^v \quad v = 0,1,2,\ldots$. This gives a quantity

$$\leq \sum_{i=1}^{\infty} \left(\frac{1}{m_i}\right) \sum_{v=o}^{\infty} k^{-v} \sum_{j \in G_{iv}} m_j \sum_{j \in G_{iv}} x_j^2 \left(\frac{1}{1-k}\right)$$

We next apply Lemma 1;

$$\sum_{j \in G_{iv}} m_j = a_v \leq 4\left(1 - \left(\frac{1}{c}\right)\right)^v a_o = 4\, k^{2v} m_i$$

and the estimate is

$$\leq \frac{4}{1-k} \sum_{\mu=1}^{\infty} \sum_{i \in G_\mu} \sum_{v=o}^{\infty} k^v \sum_{j \in G_{iv}} x_j^2$$

$$\leq \frac{4}{1-k} \sum_{\mu=1}^{\infty} \sum_{v=o}^{\infty} k^{v} \sum_{j \in G_{\mu+v}} x_j^{2}$$

$$\leq \frac{4}{1-k} \sum_{j=1}^{\infty} x_j^{2} \sum_{v=o}^{\infty} k^{v} = \frac{4}{(1-k)^2} \Sigma x_i^{2} .$$

This gives the required estimate for Q.

$$Q^2 \leqslant \frac{16}{(1-k)^2} (\Sigma x_i^{2})^2$$

because we have, since $0 < k = \sqrt{1 - \frac{1}{c}} < 1$

$$\frac{16}{(1-k)^2} = \frac{16(1+k)^2}{(1-k^2)^2} \leq \frac{64}{(\frac{1}{c})^2} = Ac^2$$

for the constant $A = 64$. This completes the proof of Lemma 2.

<u>Proof of Theorem 1</u>. We use the notation

$$\Upsilon_{vn} = \left\{ z \mid \frac{1}{2^{n+1}} < 1 - |z| \leq \frac{1}{2^n} , \ \frac{v.2\pi}{2^{n+1}} \leq \arg z < \frac{(v+1)2\pi}{2^{n+1}} \right\}$$

ω_{vn} is the range of $\arg z$ for $z \in \Upsilon_{vn}$

$$z_{vn} = (1 - \frac{1}{2^n}) e^{(v+\frac{1}{2})2\pi i/2^{n+1}}$$

$$n = 0,1,2,\ldots, ; \quad v = 0,1,\ldots, 2^{n+1} - 1.$$

Since any zeros of $G \in H^p$ can be divided out by means of a Blaschke product, whose boundary value has modulus one almost everywhere, we need only prove the Theorem for functions without zeros. Then, a proof for $p=2$ will suffice, since for $G \in H^p$, $G(z) \neq 0$, the function $G^{2/p}$ is defined and in H^2. The result for H^2 applied to $G^{2/p}$ gives immediately the required result for G. For the case $p=2$, if $G = u+iv$, since $|G|^2 = u^2 + v^2$ we find that it is enough to prove

$$(1.8) \qquad \int u^2(z) \, d\mu(z) \leq A_1 C \int_0^{2\pi} f^2(\theta) d\theta$$

for an arbitrary positive harmonic function u, which has boundary

value $f(\theta) \in L^2(0,2\pi)$. A_1 must be a constant, and we use the notation $A_2, A_3, \ldots$ in the sequel for numerical constants.

By Harnack's inequality

$$(1.9) \qquad \int (u(z))^2 d\mu(z) \leq A_2 \Sigma_{v,n} \mu(r_{vn})(u(z_{vn}))^2.$$

For a point Z_{vn} let w_j^o be the arc $w_{ij}, j \leq n$, to which $\arg z_{vn}$ belongs, and let w_j^1 denote $w_{i-1,j}$ and w_j^2 denote $w_{i+1,j}$. Then simple estimates in the Poisson kernel show that

$$u(z_{vn})^2 \leq A_3 \, 2^{2n} \, \Sigma_{k=o}^{2} \, \Sigma_{j=o}^{n} \, 8^{j-n} \left(\int_{w_j^k} f(\theta) d\theta \right)^2$$

We insert this in (1.9) and find

$$(1.10) \qquad \int (u(z))^2 d\mu(z) \leq A_4 \, \Sigma_{i,j} \, \lambda_{ij} \left(\frac{2^{j+1}}{2\pi} \int_{w_{ij}} f(\theta) d\theta \right)^2$$

where $\lambda_{ij} = \sum_{v,n \geq j} 2^{j-n} \mu(r_{vn})$, the summation being extended over those v for which $\arg z_{vn} \in w_{i-1,j} \cup w_{ij} \cup w_{i+1,j}$. The hypothesis (1.1) of the theorem implies that

$$(1.11) \qquad \sum_{\omega_{ij} \leq \omega_{km}} \lambda_{ij} \leq A_5 C \, 2^{-m}.$$

At this point we will make changes which depend on the individual function f, so we take a fixed f. We change the constants λ_{ij} systematically into new constants λ_{ij}^* to which lemma 2 applies. Start with λ_{i1}. If $\lambda_{i1} \geq \frac{1}{2}$, we leave it alone, but if $\lambda_{i1} < \frac{1}{2}$, we add it to $\lambda_{2i,2}$ or $\lambda_{2i+1,2}$ and replace λ_{i1} by zero. This can be done so that the right side of 1.10 increases, since the corresponding two integrals for $j=2$ have the integral for $j=1$ as their mean value. When this process has been carried out for all the λ_{i1}, we call the new coefficients $\lambda_{ij}^{(1)}$. Next we carry out exactly the same operation with all the numbers $\lambda_{i2}^{(1)}$. We add $\lambda_{i2}^{(1)}$ to either $\lambda_{2i,3}^{(1)}$ or to

$\lambda^{(1)}_{2i+1,3}$ if $\lambda^{(1)}_{i2} < \frac{1}{2^2}$ and replace it by zero. If $\lambda^{(1)}_{i2} \geq \frac{1}{2^2}$ we make no change. We proceed in this fashion.

The finals constants, called λ^*_{ij}, will still satisfy (1.11) if we replace A_5C by $(A_5C + 2) < a_6C$, assuming $C \geq 1$. We have either $\lambda^*_{ij} = 0$ or $\lambda^*_{ij} \geq \frac{1}{2^j}$. Thus

$$\Sigma_{\omega_{ij} \subseteq \omega_{km}} \lambda_{ij} \leq \frac{A_6C}{2^m} \leq A_6C \ \lambda^*_{km}$$

This means we can apply Lemma 2, which gives the required formula (1.8).

Take the finite Blaschke product of the points (a_v) $v = 1,2,\ldots,s$.

$$A(z) = \prod_1^s \frac{a_v - z}{1 - z\bar{a}_v} \cdot \frac{\bar{a}_v}{|a_v|}$$

It is necessary to construct a system of curves γ around the zeros of A, on which A is neither too small nor too large; in addition the measure μ defined as arc length along γ must satisfy the hypothesis of Theorem 1. We define, for $0 < \varepsilon < 1$, the sets

$$a(\varepsilon) = \{z;\ |z| < 1,\ |A(z)| \leq \varepsilon\}$$
$$b(\varepsilon) = \{z;\ |z| \leq 1,\ |A(z)| \geq \varepsilon\}$$

<u>THEOREM 2</u>. There exist absolute constants K and A, such that for any finite Blaschke product $A(z)$ and any $\varepsilon \leq \frac{1}{4}$, we can find a finite number of disjoint regions $\Omega_1, \Omega_2, \ldots, \Omega_p$ with boundary $\Gamma = \bigcup_j \partial\Omega_j$ having properties:

(i) $\Gamma \subseteq a(\varepsilon^K) - a(\varepsilon)$

(ii) The sets $\partial\Omega_j$ are rectifiable curves, and the set function μ, defined as arc length of Γ', satisfies the condition

(2.1) $$\int_S d\mu(z) \leq A\ell/\varepsilon^2$$

for all sets S of Theorem 1.

The proof of this theorem is divided into parts (A) - (E).

(A) We use the sets Υ_{vn} defined previously, but sub-divide each of these sets further into 2^{2N} subsets $\Upsilon_{vn}(i)$, $i = 1,2,\ldots,2^{2N}$, by dividing the ranges of $|z|$ and $\arg z$ into 2^N equal parts for $z \in \Upsilon_{vn}$. The integer N, which will shortly be chosen small enough to make the variation of A have a modulus less than ε, must depend on the choice of A and ε. We include all points on the boundary of each set $\Upsilon_{vn}(i)$ as well as on Υ_{vn} itself for this discussion. In fact, the system Γ of curves consists of certain radial lines and arcs from the boundaries of sets $\Upsilon_{vn}(i)$.

It is known by the lemma in §4, Chapter 0, that

$$|A'| \leq \frac{1}{1 - |z|^2}$$

since A is a Blaschke product and $||A|| = 1$. Furthermore, it is possible to join any two points, λ and μ, in $\Upsilon_{vn}(i)$ by an arc of length less than $\frac{A_7}{2^{n+1}.2^N}$ for some constant A_7. Hence

$$|A(\lambda) - A(\mu)| \leq \frac{A_7}{2^N}$$

We pick any integral value of N which will make this less than ε. In other words

(2.2) $$2^{-N} < (A_7)^{-1}\varepsilon .$$

We define

$$\alpha = \bigcup\{\Upsilon_{vn}(i) \ : \ \Upsilon_{vn}(i) \cap a(\varepsilon) \neq \emptyset\}$$

where the union is over all permissible values of v, n, and i,

which make the above intersection non-empty. If N is chosen as described, we have

$$a(\varepsilon) \subseteq \alpha \subseteq a(2\varepsilon).$$

A second set β, disjoint from α, is defined by

$$\beta = b(\varepsilon^K)$$

where K is discussed in (E), and will be sufficiently small to ensure that $\alpha \cap \beta = \emptyset$. (Actually we will choose $K = \frac{1}{25} e^{-24\pi^2}$.)

(B) We now construct a system P of line segments and arcs, all boundaries of the $\gamma_{vn}(i)$, which will separate the sets α and β. In (C), Γ is obtained as a subset of P. In (E), it is shown that Γ satisfies (2.1) by proving that the larger set P does.

P is first assumed to contain $|z| = 1$ and all dividing lines of $\gamma_{vo}(i)$. We next describe a construction which is applied to the four sets $\gamma_{o1}(i)$, $\gamma_{11}(i)$, $\gamma_4(i)$, $\gamma_{31}(i)$. This same argument will again be used later, and we shall refer to it as the construction (U); in the description, we shall deal with $\gamma_{o1}(i)$.

Suppose γ_{o1} meets β. This case is called type I. We place the arc $|z| = \frac{1}{2}$, $0 \leq \arg z \leq \pi/2$ and the segments $\frac{1}{2} \leq |z| \leq 1$, $\arg z = 0, \pi/2$ in P. For every $\gamma = \gamma_{o1}(i)$ which intersects α, assuming γ is defined by $C_1 \leq |z| \leq d_1$ and $u_1 \leq \arg z \leq v_1$, we let the arc $|z| = C_1$, $u_1 \leq \arg z \leq v_1$, and the two segments $c_1 \leq |z| \leq 1$, $\arg z = u_1, v_1$ belong to P. We denote by $h(\gamma)$ the rectangular-shaped domain determined by these three sides along with a portion of $|z| = 1$. We also include in P all boundary lines of all $\gamma_{vn}(i) \subseteq h(\gamma)$ for which $n \leq N + 1$.

Next we take all those sets $\gamma_{v2}(i)$ lying within the angle

$0 \leq \arg z \leq \pi/2$ which we are considering, which meet α, but which do not lie in any of the $h(\gamma)$ previously considered. For each such $\gamma_{v2}(i) = \gamma$, we do a similar construction, including the outside boundaries of $h(\gamma)$, along with the boundary lines of all those $\gamma_{vn}(i) \leq h(\gamma)$ for which, this time, $h \leq N + 2$.

This construction is continued; the next step will deal with all $\gamma_{v3}(i)$ in $0 \leq \arg z \leq \pi/2$ which intersect α but do not lie in either of the preceding sets $h(\gamma)$. Eventually, since $|A|$ tends uniformly to 1 as z approaches the boundary, there will be no sets which meet α, and the first stage is completed.

If γ_{o1} fails to meet β, the case called type II, we omit the boundary lines of the sets $\gamma_{o1}(i)$, and proceed to the two next sets, γ_{o2} and γ_{12}. If possible these are dealt with according to the method of type I just described. If either of these fails to meet β, we take the two next sets γ_{v3} outside this one, and continue in this way until sets γ_{vn} have been found meeting β for the entire arc $0 \leq \arg z \leq \pi/2$.

It is now necessary to deal with the outer portions of all the sets $h(\gamma)$ which have been encountered in this construction. These are called s-sets of the first generation. These sets s all have the following form. If the set γ corresponds to the angle $2\pi/_2 k$ at the origin, then

$$(2.3) \qquad s = \left\{ z \mid 1 - \frac{1}{2^k} \leq |z| \leq 1;\ \frac{2\pi t}{2^k} \leq \arg z \leq \frac{2\pi(t+1)}{2^k} \right\}$$

for some integer t (note that $k \geq N + 1$ will always hold). All the bounding lines of sets $\gamma_{vn}(i)$ lying inside $h(\gamma)\backslash s$ have been included in P. We now treat the s-sets according to the

construction (U). The construction is applied to the two sets $a_{2t,k}$ and $a_{2t+1,k}$ for an s-set of the form (2.3).

For each of these new applications of (U) to the first generation s-sets, we will obtain new s-sets, all of which form the second generation. This sequence of new generations (applying inside the other three quadrants as well) will eventually terminate, since for τ large enough, $\{z;\ \tau \leq |z| \leq 1\}$ lies entirely inside β. This completes the description of P.

The system P of segments separates the sets α and β in the sense that a continuous curve joining points from the two sets must intersect P. The sets s are separated from the rest of D. Hence any point in γ_{01} is separated from any s-set. If γ_{01} meets β, then γ_{01} is separated from points z with $\pi/2 \leq \arg z \leq 2\pi$ which meet α but which are not in s sets have been encircled by lines of P. If γ_{01} fails to meet β, but, for example γ_{02} does, the same sort of argument applies, and this means that P separates any two points of α and β which are not in the same s-set. However, the divisions inside each s-set are precisely of the same type as those obtained from γ_{01} by the construction (U), and so this case presents no difficulty.

(C) The system Γ is a subsystem of P. We first form P* from P by deleting all lines which consist of interior points of α or of β. P* will still separate α and β; for suppose that $\gamma : z = z(t)$, $z(0) \in \alpha$, $z(1) \in \beta$ is continuous and does not meet P*. We let t_0 be the maximum value of t such that $z(t) \in \alpha$, and t_1 the minimum value of $t > t_0$ with $z(t) \in \beta$. Then

the curve $z(t)$, $t_o \leq t \leq t_1$ joins α and β, but cannot cross P, this giving a contradiction.

For any $p \in \alpha$ we let $\Omega(p)$ be the open subset of D consisting of points which can be reached from p by a continuous curve not meeting P^*. $\Omega(p)$ contains, along with p, the interior of the set $\gamma_{vn}(i)$ to which p belongs. If $q \in \Omega(p)$, then $\Omega(p) = \Omega(q)$ and if $q \notin \Omega(p)$ then $\Omega(p) \cap \Omega(q) = \emptyset$. Because α is compact we will have a finite number of these disjoint regions $\Omega_1, \Omega_2, \ldots, \Omega_p$ covering α but not meeting β. The boundary system Γ for these satisfies $\Gamma \subseteq P^* \subseteq P$. Since (i) of Theorem 2 follows, we proceed to the proof of (ii). The proof is given in (E) for P, following a preparatory lemma.

(D) The following Lemma appears in [12].

<u>Lemma</u> Let E be a closed subset of the half plane $H = \{\zeta : \zeta = \xi + i\eta, \eta > 0\}$ and define $E^* = \{\zeta : \eta = 0, \xi = |\zeta'|$ for some $\zeta' \in E\}$. Let $w(\zeta)$ be the harmonic measure of E with respect to $H \backslash E$, and let $w^*(\zeta)$ be the harmonic measure of E^* with respect to H. Then

$$w(\xi + i\eta) \geq (2/3)\, w^*(-|\xi| + i\eta)$$

We modify the lemma as follows. Let $R \subseteq D$ be the ring $R = \{z \in D: \rho < |z| < 1\}$. Suppose E_1 is a compact subset of R, and define E_1^* on $|z| = 1$. by $E_1^* = \{e^{i\theta} : re^{i\theta} \in E_1$ for some $r\}$. Suppose we restrict attention to an angle $|\theta| < \lambda|\log\rho|$ for some constant λ, and denote by M_1^* the subset of E_1^* lying within this angle. Let m_1^* be the measure on $|z| = 1$ of the set M_1^*. Suppose also that $w_1(z)$ and $w_1^*(z)$ are the harmonic

measures of E_1 and $E_1{}^*$ with respect to $R\backslash E_1$ and R. The inequality we need is the following

$$(2.4) \qquad m_1{}^* \leq (3)(e^{3\pi\lambda})|\log\rho| \cdot w_1(\sqrt{\rho}).$$

To obtain this relation, we consider the conformal mapping

$$(2.5) \qquad \zeta = \xi + i\eta = z^{i\pi/\log\rho} = r^{i\pi/\log\rho}, e^{-\pi\theta/\log\rho}$$

of the universal covering surface R^∞ of R onto the upper half plane H. Under this mapping, points $e^{i\theta}$ for $\theta \geq 0$ map onto the real axis for $\xi \geq 1$, and as $\theta \longrightarrow \infty$ the image point moves to the right along the reals. For $\theta \leq 0$ and $\theta \longrightarrow -\infty$ however, the point moves to the left from $\zeta = 1$ towards $\zeta = 0$. If we take the points $\rho e^{i\theta}$, we have a similar situation on the negative reals, and points $re^{i\theta}$ for $\rho < r < 1$ map on to radial lines through $\zeta = 0$. In particular, points $\sqrt{\rho}e^{i\theta}$ map onto the positive imaginary axis, and $\sqrt{\rho}$ maps to i.

If we take the set E_1 and repeat it in each sheet of R^∞, call E its image under the mapping (2.5), and apply the above lemma, we obtain

$$w(\zeta) \geq (\tfrac{2}{3})w^*(\zeta) \qquad \text{for} \qquad \xi \leq 0 \qquad \text{and} \qquad \eta > 0$$

for the harmonic measures defines in the lemma. The image functions under the transformation taking this back to R^∞ will also satisfy a relation of this type. Also, these harmonic functions must by symmetry be the same at corresponding points in each sheet, and so these functions are the same as w_1 and $w_1{}^*$ on R. Note also that the condition $\xi \leq 0$ becomes $|z| \leq \sqrt{\rho}$. Thus

$$(2.6) \qquad w_1(z) \geq (\tfrac{2}{3})\, w_1{}^*(z) \qquad \text{for} \quad |z| \leq \sqrt{\rho}$$

Considering next the set M_1^* on $|z| = 1$ lying in the angle $|\theta| < \lambda|\log \rho|$, we map this onto M^* on the positive ξ axis (taking it in just the one sheet). The measure m^* of M^* will satisfy

$$(2.7) \qquad m^* = \int_{M^*} d\xi = \int_{M_1^*} - d \exp\left(\frac{-\pi\theta}{\log \rho}\right) \geq \left|\frac{\pi}{\log \rho}\right| e^{-\pi\lambda} m_1^*,$$

noting that $r = 1$ for points of M_1^*.

Since points of M^* are centred around $\zeta = 1$, and in fact all lie in the interval $(0, e^{\lambda\pi})$, we have that w^* at the point $\zeta = i$ satisfies

$$(2.8) \qquad w^*(i) = \frac{1}{\pi} \int_{E^*} \frac{d\xi}{1+\xi^2} \geq \frac{1}{\pi} \int_{M^*} \frac{d\xi}{1+\xi^2}$$

$$\geq \frac{m^*}{\pi(1 + e^{2\pi\lambda})}$$

Combining (2.7) and (2.8)

$$m_1^* \leq \frac{m^* e^{\pi\lambda} |\log \rho|}{\pi} \leq (e^{\pi\lambda} + e^{3\pi\lambda}) \, |\log \rho| \, w^*(i).$$

We apply (2.6) to $w_1^*(\sqrt{\rho})$ which is the same as $w^*(i)$, and obtain from the above

$$m_1^* \leq \frac{3}{2} (e^{\pi\lambda} + e^{3\pi\lambda}). \, |\log \rho|. \, w_1(\sqrt{\rho})$$

which proves 2.4. This inequality will hold also for a compact set E_1 in $\rho \leq |z| \leq 1$.

(E) The proof of Theorem 2 is completed when we show now that our set P will satisfy the property (2.1) for μ defined to be arc length along curves of P, rather than along the smaller set Γ.

We prove this property for the s-sets of the form 2.3, and this is enough to give the result for sets S of the form described in Theorem 1.

Suppose we have a set of the form $h(\gamma)$; this will contribute arcs and line segments to P of total length less than $A_8.N.2^N$ times the arc length of γ. for some absolute constant A_8. We measure arc length on $|z| = 1$. Recall, (formula (2.2)), that N was chosen large enough to give $2^{-N} < (A_7)^{-1}\varepsilon$. We can simultaneously demand that N not be chosen too much larger than this value. We can suppose, for example, that $(\frac{1}{4})(A_7)^{-1}\varepsilon \leq 2^{-N}$. We have

$$A_8.N.2^N < A_8\, 2^{2N} \leq 16A_8.A_7^2/\varepsilon^2 = A_9/\varepsilon^2$$

This means that, if we set $C_1 = A_9/\varepsilon^2$, the total length added to P from arcs and segments of $h(\gamma)$ (we do not include later generations) is less than C_1 times the arc length of γ.

We start with an s-set s_o, and carry out one generation of the construction (U) within it. Suppose we have type I. We add lines to P for sets h of the form $h(\gamma)$. These sets give disjoint arcs on $|z| = 1$ and for each of these the contribution to P is less than C_1 times the arc length of $h(\gamma)$. If we add we obtain

$$(2.9) \qquad L' \leq C_1 \ell_o$$

where L' is the contribution to P from this one generation and ℓ_o is the total arc length of s_o. We have omitted the three interior bounding lines of s_o which are also added to P, but these are less than some constant multiple of ℓ, and we will assume that this constant is absorbed in C_1. This can be somewhat generalized. If s' is an s-set contained in s_o, having arc length ℓ', then the contribution L' to that part of P in-

side s' due to the one generation of the construction must satisfy $L' \leq C_1 \ell'$.

In the case where type I does not apply in s_o, we can move towards $|z| = 1$ until our sets meet β, and there apply the construction for type I. No lines are added to P when this is done. Then (2.9) can be obtained by addition of the result for each application of the type I construction.

We have obtained an estimate for the total length of P from one generation; we will obtain a similar estimate which will include all generations. If L is the length of P inside s_o, we will show

$$L \leq 2 C_1 \ell_o.$$

To do this, we need to use the results about harmonic measure. We will show that whenever we can use the construction of type I on s_o, in other words whenever a point of β appears in the inner ring of s_o, we have an improved inequality,

$$\sum_{j=1} \ell_j \leq \frac{1}{2} \ell_o. \tag{2.10}$$

Here, $\ell_1, \ell_2, \ldots, \ell_r$ are the arc lengths on $|z| = 1$ of the sets $\Gamma_{vn}(i)$ which determine the $h(r)$ sets when (U) is applied to s_o. The inequality $\sum_{j=1} \ell_j \leq \ell_o$ is trivially satisfied, but in this special case it can be improved to give (2.10).

Recall that a constant K was used in (A) to determine a set $\beta = b(\varepsilon^K)$ disjoint from α. The constant $K = \frac{1}{25} e^{-24\pi^2}$ will be used. We have $a(\varepsilon) \subseteq \alpha \subseteq a(2\ )$. If $w(z,a(2\varepsilon))$ and $w(z,\alpha)$ are the harmonic measures with respect to D of $a(2\varepsilon)$ and α respectively, and if $z \in \beta$, we have

$$(2.11) \qquad w(z,\alpha) \leq w(z,a(2\varepsilon)) = \frac{\log |A(z)|}{\log 2\varepsilon} \leq \frac{K \log \varepsilon}{\log 2\varepsilon} \leq 2K$$

recalling that $\varepsilon \leq \frac{1}{4}$.

Next assume that a construction of type I is carried out in s_o. There must exist a point $z_o \in s_o \cap \beta$ which we write $z_o = \sqrt{\rho}\, e^{i\theta}$. for some constant $\rho < 1$ which must satisfy

$$\frac{1}{2^{k+1}} \leq 1 - \sqrt{\rho} \leq \frac{1}{2^k}$$

If E_1' denotes the sets $\gamma_{vn}(i)$ in s_o which are used in the construction of $h(r)$ sets, each of these must intersect the ring $R = \{z: \rho \leq |z| \leq 1\}$. We define $E_1 = E_1' \cap R$ and use the result of the previous section (D), We take $\lambda = 8\pi$, so that the sector $|\theta - \theta_o| < \lambda\, |\log \rho|$ will contain s_o. This gives, noting also that $|\log \rho| < \frac{1}{2^{k-1}}$,

$$(2.12) \qquad m_1^* \leq 3\ (e^{24\pi^2})\ (\frac{1}{2^{k-1}})\ w_1(z_o)$$

Here of course w_1 is the harmonic measure of E_1 with respect to R. If we let w_2 be the harmonic measure of E_1 with respect to D, we have

$$(2.13) \qquad w_1(z_o) \leq w_2(z_o) \leq w(z_o,\alpha)$$

From (2.11) and (2.13)

$$w_1(z_o) \leq 2K = \frac{3}{25}\, e^{-24\pi^2}$$

which we substitute in (2.12)

$$m_1^* \leq \frac{12}{25}\frac{1}{2^k} < \frac{1}{2} \cdot \frac{1}{2^k} = \frac{\ell_o}{2}$$

However $m_1^* = \sum_{j=1} \ell_j$, the total lengths of the arcs on $|z| = 1$ corresponding the the $h(r)$ sets required. This proves (2.10) for the special case where s_o can be divided by a construction of

type I. However, the type II is obtained from dividing and applying type I, and the proof applies there also.

The final argument is an induction on the number of generations required. Assume that we have for every s which contains at most m generations of sets the equation

(2.14) $$L \leq 2\, C_1 \ell .$$

Suppose s_o is a set which contains $m + 1$ generations. We can suppose that a construction of Type I is carried out on s_o. The contribution L' to P from the first generation will satisfy

$$L' \leq C_1 \ell_o$$

This is the formula 2.9 (we observe that this will prove our result (2.14) for $m = 1$). We have $h(r)$ sets of length $\ell_1, \ell_2, \ldots, \ell_\sigma$ each of which gives rise to at most m generations. By the induction hypothesis, the contribution P of these is at most $\sum_{j=1}^{\sigma} 2\, C_1 \ell_j$. In view of (2.10) this gives a total contribution of at most

$$2\, C_1 \sum_{j=1}^{\sigma} \ell_j \leq C_1 \ell_o$$

which, when added to the first generation contribution L', will give the required equation

$$L \leq 2\, C_1 \ell_o .$$

Theorem 2 follows.

<u>THEOREM 3</u>. Let m_∞ be the least possible maximum of an analytic function f in D satisfying the condition

(3.1) $$f(z_v) = w_v \qquad v = 1, 2, \ldots, s.$$

for a given set of complex constants $w_1, w_2, \ldots, w_n$, where $z_1, z_2, \ldots, z_n$ are different points in D. Then

(3.2) $$m_\infty = \sup_{||G||_1 = 1} \left| \sum_1^s \frac{G(z_v)w_v}{\pi'(z_v)} \right|$$

where

(3.3) $$\pi(z) = \prod_1^s \left(\frac{z-z_v}{1-z\,\bar{z}_v}\right)$$

<u>Proof</u>. We consider the extremal problem

(3.4) $$\inf_{f \in H^q} ||f||_q = m_q$$

under the condition that f satisfy (3.1). H^q is a Banach space, and we see that the lower bound m_q is assumed for some function f. This function, necessarily unique, we shall call f_q. Any other function which satisfies (3.1) and is in H^q must be of the form $f = f_q + \eta \pi_n \cdot q$ for $g \in H^q$ and η some complex constant which we use to obtain the condition for $||f||_q$ to be minimum when $f = f_q$. The condition is

(3.5) $$\int_{|z|=1} |f_q|^{q-2} \cdot \bar{f}_q \pi \cdot g d\theta = 0 \qquad g \in H^q$$

If we consider the measure $d\mu = |f|^{q-2} \bar{f}_q \pi \cdot d\theta$ on the unit circle, and let $g(z) = z^p$, $p = 0,1,2,\ldots$ we see that $d\mu$ is analytic in the sense that $\int z^p \, d\mu = 0$, $p = 0,1,2,\ldots$ By the Theorem of F. and M. Riesz, Chapter 0, §2, Theorem 5, this gives a function in $H_0^1 = \{f \in H^1; f(0) = 0\}$ with boundary values almost everywhere equal to $|f_q|^{q-2} \bar{f}_q \pi$. Hence we can find $F_q \in H^1$ with boundary values

(3.6) $$F_q = \frac{|f_q|^{q-2} \bar{f}_q \pi \cdot m_q^{1-q}}{z} \qquad \text{a.e. on } |z| = 1$$

We evaluate the norm in H^1

(3.7) $$||F_q||_1 \leq 1$$

Cauchy's integral formula gives

(3.8) $$m_q = m_q^{1-q} \cdot m_q = \frac{1}{2\pi} m_q^{1-q} \int_{|z|=1} |f_q|^q \, d\theta = \frac{1}{2\pi i} \int_{|z|=1} \left(\frac{F_q \cdot f_q}{\pi}\right) dz$$

$$= \sum_{v=1}^{s} \frac{F_q(z_v).w_v}{\pi'(z_v)}$$

We now let $q \longrightarrow \infty$ through integral values, and select a subsequence along which f_q and F_q converge to limit functions f and F. This is done as follows:

Taking $q = 2,3,4,\ldots$ we have two sequences of functions $f_2, f_3, f_4, \ldots$

$$f_q \in H^q \subseteq H^{q-1} \subseteq \ldots \subseteq H^2$$

$$||f_q||_q = m_q \geq m_{q-1} \geq \ldots \geq m_2$$

$F_2, F_3, F_4, \ldots \quad F_q \in H^1 \qquad ||F_q||_1 \leq 1$

Since the Banach spaces H^p for $p \geq 1$ are reflexive, any bounded subset is sequentially compact in the weak topology. The second sequence is contained in the unit sphere in H^1, and in the first sequence, for any q, all the functions from f_q onward are bounded in H^q by the value m_q. It is possible to choose a sequence $2 \leq q_1 < q_2 < q_3 < \ldots$ of integers so that $F_{q_1}, F_{q_2}, F_{q_3}, \ldots$ converges weakly in H^1, and so that $f_{q_1}, f_{q_2}, f_{q_3}, \ldots$ has the following property. For any $q > 1$, if we omit the functions f_{q_i} with $q_i < q$, we are left with a sequence of functions converging weakly to some function $f \in H^q$. Since $f \in H^q$ for all q, we have $f \in \mathfrak{B}$. Clearly f satisfies the interpolation, and since each f_q gives the best interpolation in H^q, we must have $||f|| = m_\infty$. If we let $q \rightarrow \infty$ through the values of our subsequence (3.8) becomes

$$(3.9) \qquad m_\infty = \sum_{v=1}^{s} \frac{F(z_v)\, w_v}{\pi'(z_v)} \qquad F \in H^1.$$

However, we have for any $G \in H^1$ with $||G||_1 \leq 1$

$$\left|\sum_1^s \frac{G(z_v)w_v}{\pi'(z_v)}\right| = \left|\frac{1}{2\pi i}\int_{|z|=1} Gf\pi^{-1}\,dz\right| \le m_\infty\,||G||_1 \le m_\infty$$

which means that (3.9) can be rewritten as

$$m_\infty = \sup_G \left|\sum_1^s \frac{G(z_v)\cdot w_v}{\pi'(z_v)}\right| \qquad G\in H^1,\quad ||G||_1 \le 1$$

which is the required equation.

<u>THEOREM 4</u>. Let $A(z)$ be the finite Blaschke product

$$A(z) = \prod_1^s \frac{a_v - z}{1 - z\bar{a}_v}\cdot\frac{a_v}{|a_v|}$$

and assume that the set $\{z : |A(z)| < \delta\}$, $\delta < \frac{1}{2}$, has the (simply connected) components $D_1, D_2, \ldots, D_q$. Let $F_i(z)$ be holomorphic in D_i, and assume $|F_i(z)| < 1$ there. Then the interpolation problem

$$(4.1) \qquad f(a_v) = F_i(a_v) \qquad a_v\in D_i,\ f\in \mathfrak{B},$$

has a solution f with $||f|| < \frac{1}{\delta^{A_{11}}}$

<u>Proof</u>. We choose ε so that $\varepsilon^K = \delta$ for the constant K of Theorem 2. Clearly $\varepsilon \le \frac{1}{4}$. We can construct the system Γ of curves around the zeros of A. We set $F(z) = F_i(z)$ for $z\in D_i$. According to Theorem 3, a function f_o satisfying the required interpolation can be found with norm

$$||f_o|| = \sup_G \left|\sum_{v=1}^s \frac{F(a_v)G(a_v)}{A'(a_v)}\right| \qquad G\in H^1,\ ||G||_1 = 1$$

We express each term of this sum by Cauchy's integral formula along the curve Γ. Γ lies inside the region $\bigcup_{i=1}^q D_i$ on which F is homomorphic, and it winds around each point a_v. We have

$$||f_o|| = \sup_G \left|\frac{1}{2\pi i}\int_\Gamma \frac{F(z)G(z)}{A(z)}\,dz\right|$$

We have $|F| \leq 1$, and on Γ, $|A| \geq \varepsilon$. This gives

$$||f_o|| \leq \frac{1}{2\pi\varepsilon} \int_\Gamma |G(z)||dz|$$

Because Γ satisfies the conditions of Theorem 1, with a constant $C = C_1 = A_9/\varepsilon^2$, and because $\delta = \varepsilon^K$, this means that an absolute constant A_{11} exists with

$$||f_o|| \leq \frac{1}{A_{11}}$$

The above interpolation theorem was suggested by D. J. Newman, who showed that it leads to a proof of the following, the corona conjecture.

<u>THEOREM 5.</u> Let $f_1(z), f_2(z), \ldots, f_n(z)$ be functions of $\mathcal{B}$ such that

(5.1) $$|f_1(z)| + |f_2(z)| + \ldots + |f_n(z)| \geq \delta > 0 \qquad z \in D$$

for some δ. Then these functions generate all of $\mathcal{B}$. Furthermore, if $||f_v|| \leq 1$, $v = 1,2,\ldots,n$, and $\delta \leq \frac{1}{2}$, there exists $p_v(z) \in \mathcal{B}$ such that

(5.2) $$\sum_{v=1}^{n} p_v f_v = 1 \quad \text{and} \quad ||p_v|| \leq n! \left(\frac{2}{\delta}\right)^{nA_{12}}$$

for some absolute constant A_{12}.

<u>Proof.</u> We make an induction on n. For $n = 1$ the condition (5.1) gives a single function f_1 in $\mathcal{B}$ which is bounded away from zero and hence invertible. Its inverse p_1 will satisfy the inequality of (5.2) as long as $A_{12} \geq 1$.

Assume the theorem holds for $(n-1)$ functions. We will use the invariance of this assertion under conformal mappings; (5.2) will hold for $(n-1)$ functions in any simply connected domain.

Assume first that f_n is a finite Blaschke product $B(z)$ with simple zeros $b_1, b_2, \ldots, b_s$. The set $\{z \in D: |B(z)| < \delta/2\}$ has com-

ponents $D_1, D_2, \ldots, D_q$. In each D_i, there exist functions $P_{iv}(z)$ such that

$$\sum_{v=1}^{n-1} P_{iv}(z)\, f_v(z) = 1 \qquad ||P_{iv}|| \leq (n-1)!\, \left(\tfrac{2}{\delta}\right)^{(n-1)A_{12}}$$

By Theorem 3 applied to the holomorphic functions P_{iv} in D_i for $i = 1,2,\ldots,q$, we obtain functions $p_v \in B$ for each value $v = 1,2,\ldots,n-1$ with $p_v(b_j) = P_{iv}(b_j)$ b_j D_i

The functions p_v will satisfy the inequality

$$||p_v|| \leq (n-1)!\, \left(\tfrac{2}{\delta}\right)^{(n-1)A_{12}+A_{11}}$$

since we use $\frac{\delta}{2}$ instead of δ in Theorem 4, and the functions P_{iv} used are not majorized by 1, but by $(n-1)(\frac{2}{\delta})^{(n-1)A_{12}}$. We next define $p_n(z)$ by the formula

$$p_n(z) = (1 - \textstyle\sum_1^{n-1} p_v(z)\, f_v(z))B(\bar{z})^{-1}$$

The zeros of the numerator will cancel the finite Blaschke product of the denominator, leaving a function in $\mathcal{B}$ which satisfies (5.2) with exponent $(n-1)A_{12} + A_{11}$. This means we have, for $v = 1,2,\ldots,n$,

$$||p_v|| \leq n!\, \left(\tfrac{2}{\delta}\right)^{(n-1)A_{12}+A_{11}}$$

We will shortly set $A_{12} = A_{11}+1$, which will give (5.2) for the case in which f_n is a finite Blaschke product.

To prove (5.2) in general, we choose $\rho < 1$ and replace $f_v(z)$ by $g_v(z) = f_v(\rho z)$. Note that $g_v(z)$ is analytic in the closed disc $|z| \leq 1$. If we can prove (5.2) for an infinite sequence $\rho_\mu \to 1$, we have proved it generally. We choose ρ so that $g_n(z) \neq 0$ on $|z| = 1$, and choose $G_n(z)$ analytic and non-zero in D such that

$$|G_n(e^{i\theta})| = \mathrm{Min}\left\{\frac{1}{|g_n(e^{i\theta})|}, \frac{2}{\delta}\right\}$$

The functions $g_1, g_2, \ldots, G_n g_n$ again satisfy (5.1), for $|G_n| \geq 1$ on $|z| = 1$, and hence also on $|z| < 1$ by the minimum principle, since it is non-zero. We can choose a sequence $\{B_k(z)\}$ of finite Blaschke products which converges uniformly to $G_n(z)g_n(z)$ outside any neighbourhood of the set on $|z| = 1$ where $|g_n| \leq \delta/2$. (See [9], p.195.) We apply the result from above to the n functions $g_1, g_2, \ldots, g_{n-1}, B_k$. If we let $k \longrightarrow \infty$ we have

$$\lim_{k \to \infty} \left\{ \inf_{|z|<1} (|g_1| + |g_2| + \cdots + |g_{n-1}| + |B_k|) \right\} \geq \delta/2.$$

We can obtain functions $P_1^{(k)}, P_2^{(k)}, \ldots, P_n^{(k)} \in B$ for all k sufficiently large which will give

$$P_1^{(k)} g_1 + P_2^{(k)} g_2 + \cdots P_{n-1}^{(k)} g_{n-1} + P_n^{(k)} B_k = 1$$

Because the $P_v^{(k)}$ are uniformly bounded, we can use normality of the families to extract convergent sequences $P_v^{(k_\ell)}$. These can be taken to have the same indices (k_ℓ) for each v. If p_v are the limits of these subsequences, we have

$$p_1 g_1 + \cdots + p_{n-1} g_{n-1} + p_n G_n g_n = 1$$

$$||p_v|| \leq n! \left(\frac{2}{\delta}\right)^{(n-1)A_{12} + A_{11}} \qquad v = 1, 2, \ldots, n.$$

We have $||G_n|| \leq \frac{2}{\delta}$ which was

$$||p_n G_n|| \leq n! \left(\frac{2}{\delta}\right)^{(n-1)A_{12} + A_{11} + 1} \leq n! \left(\frac{2}{\delta}\right)^{nA_{13}}$$

where $p_n G_n$ is the coefficient of g_n in the identity $p_1 g_1 + \cdots + p_{n-1} g_{n-1} + (p_n G_n) g_n = 1$; if we rename this p_n, then we have satisfied (5.2) because the functions $p_1, p_2, \ldots, p_{n-1}$ also satisfy $||p_v|| \leq n! \left(\frac{2}{\delta}\right)^{(n-1)A_{12} + A_{11}} \leq (n!)\left(\frac{2}{\delta}\right)^{nA_{13}}$. Thus the corona conjecture is verified.

Another consequence of Theorems 1,2, and 3, is the following Interpolation result.

<u>THEOREM 6.</u> Let $\{b_v\}_1^\infty$ and $\{c_v\}_1^\infty$ be two sequences of complex numbers in $|z| < 1$ such that $\sum(1 - |b_v|) < \infty$ and $\sum(1 - |c_v|) < \infty$. Let $B(z)$ and $C(z)$ be the corresponding Blaschke products. Then the interpolation

(6.1) $f(b_v) = 0, \quad f(c_v) = 1, \quad v = 1,2,\ldots$

is possible with $f \in B$ if and only if $\delta > 0$ exists such that

(6.2) $|B(z)| + |C(z)| \geq \delta > 0 \quad |z| < 1$

If (6.2) holds, (6.1) can be solved with $||f|| < 1/\delta^{A_{13}}$, $\delta < \frac{1}{2}$, for a suitable constant A_{13}.

<u>Proof.</u> We assume first that (6.1) holds; we have

$$f = Bg, \; f - 1 = Ch \quad ||f|| = ||g|| \quad ||h|| \leq ||f|| + 1,$$

factoring out the zeros of f and $f - 1$. If we subtract,

$1 = Bg - Ch$ which leads to (6.2) directly, since

$$1 \leq |B(z)|\cdot||g|| + |C(z)|\cdot||h|| \leq (|B| + |C|)(||f|| + 1)$$

will give the value $\delta = (||f|| + 1)^{-1}$

For the converse, we appeal to Theorem 3 to draw a system of curves around the zeros of

$$C_s(z) = \prod_1^s \frac{c_v - z}{1 - z\bar{c}_v} \cdot \frac{c_v}{|c_v|}$$

using an ε chosen sothat $\varepsilon^K = \delta/2$. We define $B_s(z)$ in an analogous way, and apply Theorem 3 to evaluate $||f_s||$ for the function of least norm (in $\mathcal{B}$) which satisfies (6.1) for $v \leq s$. The Blaschke product is $B_s \cdot C_s$ and the w's are zeros and ones.

We obtain

$$||f_s|| = \sup_G \left|\sum_{v=1}^s \frac{G(c_v)}{B_s(c_v)C_s'(c_v)}\right| \qquad G \in H^1, \; ||G||_1 = 1,$$

and Cauchy's integral formula leads to

$$(6.3) \qquad ||f_s|| = \sup_G \left| \frac{1}{2\pi i} \int_\Gamma \frac{G(z)}{B_s(z)C_s(z)}\, dz \right| \qquad ||G||_1 = 1$$

On the curve Γ we have $|C_s(z)| \geq \varepsilon = (\frac{\delta}{2})^{1/K}$, and also from $|C_s(z)| \leq \varepsilon^K = \delta/2$ and the hypothesis (6.2), we obtain $|B_s(z)| \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}$. Thus from (6.3), $||f_s|| \leq \frac{1}{2\pi} (\frac{2}{\delta})^{1/K+1} \int_\Gamma |G(z)||dz| \leq (\frac{1}{\delta})^{A_{13}}$ where we make use of the second property of the curves Γ to estimate the integral in terms of δ and an absolute constant. This completes the proof.

§3 INTERPOLATING SEQUENCES

The following question was posed by R.C.Buck in connection with the attempt to settle the corona problem. What condition must be satisfied by a sequence (z_n) of points in D in order that the interpolation

(1) $\quad f(z_n) = w_n \qquad n = 1,2,3,\ldots$

can be satisfied by a function f in $\mathcal{B}$ for any bounded sequence (w_n) of complex numbers? Such a sequence is called an interpolating sequence. Obvious necessary conditions are that the z_n be distinct, and that $\sum(1 - |z_n|) < \infty$. This question was completely settled by Carleson, some time before the corona problem, in [7]. Newman [18] and Hayman [13] obtained results independently on the same question. Carleson demonstrated that the condition

(C) $$\prod_{n \neq m} \left| \frac{z_n - z_m}{1 - z_n \bar{z}_m} \right| \geq \delta > 0 \qquad m = 1,2,3,\ldots$$

is necessary and sufficient for (z_n) to be an interpolating sequence. We give the proof in this section. The necessity is easier and is given in Lemma 1. For the sufficiency we give an argument from Carleson's corona paper which is shorter than his original discussion, but which uses the powerful results of this paper.

<u>Lemma 1</u>. If (z_n) is an interpolating sequence, then, for some constant $\delta > 0$, the condition (C) holds.

<u>Proof.</u> Denote by ℓ^∞ the Banach space of bounded sequences of complex numbers under the supremum norm, and define the homomorphism $R:\mathcal{B} \to \ell^\infty$

$$Rf = (f(z_1), f(z_2), \ldots).$$

If I is the closed ideal of functions in $\mathcal{B}$ vanishing on the set $\{z_1, z_2, \ldots\}$, then R induces an isomorphism from $\mathcal{B}/I$ onto ℓ^∞. This isomorphism is norm decreasing, where $\mathcal{B}/I$ has the quotient norm, and by the closed graph theorem has a bounded inverse T. For some

constant M, we have

$$||Tw|| < M \sup_n |w_n|$$

for any non-zero sequence $w = (w_1, w_2, \ldots)$ in ℓ^∞. Suppose we choose $f \in \mathcal{B}$ such that $Rf = w$. Then $||Tw|| = \inf_{h \in I} ||f + h||$ where the latter norm is in $\mathcal{B}$. We can always choose h such that $g = f+h \in \mathcal{B}$ gives

(2) $$||g|| \leq M \sup_n |w_n| \qquad Rg = w$$

Suppose we interpolate the sequence $w_n = \delta_{mn}$, $n = 1,2,\ldots$ with a function g_m satisfying (2). We obtain

$$||g_m|| \leq M, \quad g_m(z_n) = \delta_{mn} \qquad n = 1,2,\ldots .$$

If B_m is the Blaschke product of the points (z_n) with z_m omitted, we have

$$g_m = B_m G_m \qquad ||G_m|| \leq M$$

which gives, evaluating at z_m

$$1 = |g_m(z_m)| = |B_m(z_m)| \cdot |G_m(z_m)| \leq M\, |B_m(z_m)|.$$

We now set $\delta = M^{-1}$, and obtain

$$|B_m(z_m)| \geq \delta$$

which is precisely (C).

The following lemma, more general than is immediately needed, will be used on several later occasions.

<u>Lemma 2.</u> If $B(z)$ is the Blaschke product of a sequence (z_n) satisfying the condition (C), and if $\varepsilon > 0$ is given, there exists a constant δ_0 depending only on δ and ε such that

$$|B(z)| \geq \delta_0 \text{ whenever } \psi(z,z_n) \geq \varepsilon \quad \text{for all } n.$$

<u>Proof.</u> We first prove the following property of D, where all distances $\psi(z,z')$ referred to are pseudo-hyperbolic. A constant N

exists, depending only on ε, such that from any infinite sequence of points of D distant at least ε from the origin, one can extract a finite subset with at most N members, such that every other point of the sequence is at least as close to one of these as to the origin. To do this it is sufficient to find an absolute constant $\theta_o > 0$ such that for any point A in D with modulus a, we have $\psi(P,A) \leq \psi(P,0)$ for any point P lying outside the circle

(3) $$x^2 + y^2 = a^2$$

and satisfying $|\arg(P) - \arg(A)| \leq \theta_o$. For we can start with a point A_1 of minimal modulus, and use it to eliminate all points from the sequence within an angle $2\theta_o$ about OA_1. We repeat for a point A_2 of minimal modulus outside this angle, and continue until all points of the given sequence have been eliminated. This can be done in at most N steps for some suitable integer N.

To show the existence of θ_o, we assume A is the point (a,0) and draw the neighbourhood $\{P' \in D: \psi(P',A) \leq a\}$, a circle with centre $(\frac{a}{1+a^2}, 0)$ and radius $\frac{a}{1+a^2}$. We intersect this circle whose equation is

(4) $$x^2 - \frac{2ax}{1+a^2} + y^2 = 0$$

with (3) to obtain points R and Q. The co-ordinates (x_o, y_o) of R are $x_o = \frac{a}{2}(1+a^2)$, $y_o = \frac{a}{2}\sqrt{(5+a^2)(3-a^2)}$, and we have

$$\frac{y_o}{x_o} = \frac{\sqrt{(5+a^2)(3-a^2)}}{\sqrt{(1+a^2)(1+a^2)}}$$

which is bounded away from zero when a is allowed to range. Hence the angle $ROQ \geq 2\theta_o$ for some constant θ_o.

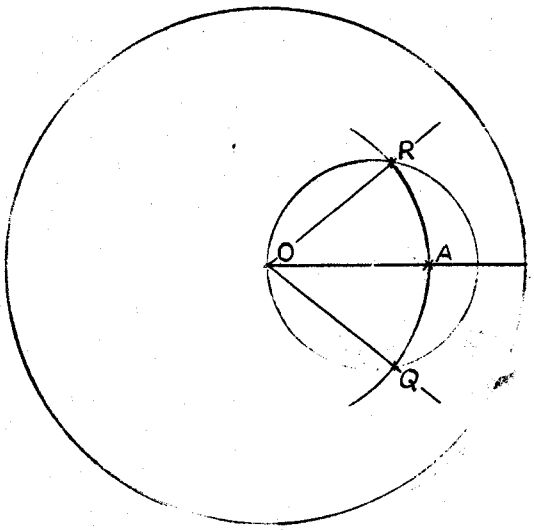

Figure 3

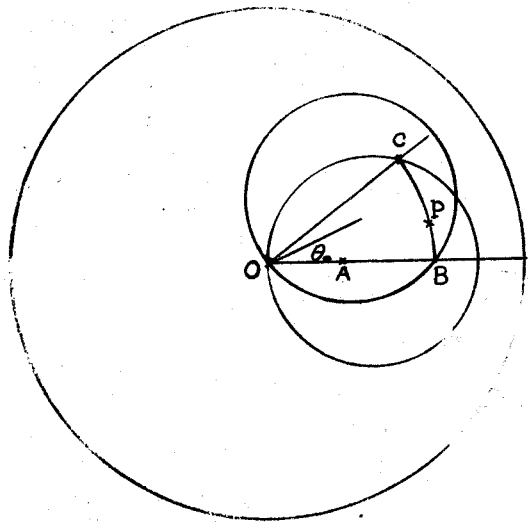

Figure 4

To prove θ_o has the required property, take any point P outside (3) and within the angle $2\theta_o$ about OA. See Fig. 4. The circle through P, centred at the origin has the equation $x^2 + y^2 = b^2$. Let B be the point (b,o) and let C be a point where the circle $x^2 + y^2 = b^2$ meets the circle $\{P' \in D: \psi(P',B) = b\}$. To show $\psi(P,A) \leq \psi(P,O)$ it is sufficient to show $\psi(C,A) \leq \psi(C,O)$. However, $b = \psi(C,O) = \psi(C,B) \geq \psi(C,A)$ where the concluding inequality is evident when we draw the neighbourhood $\{P' \in D;\ \psi(P',C) \leq b\}$ and observe that A lies inside it.

We next take any point $z_o \in D$ with $\psi(z_o,z_n) \geq \varepsilon$ $n = 1,2,3,\ldots,$ and by a suitable non-Euclidean rigid motion T, move z_o to the origin. Denote by z_n' the point Tz_n for $n = 1,2,\ldots$ and let B' be the Blaschke product on the points z_n'. Since $B(z) = B'(Tz)$ holds in

general, we must show $|B'(0)| \geq \delta_o$. Because T is a rigid motion we have that z_n' is ψ-distant from the origin by at least ε for each n, and also that (C) is satisfied by the sequence (z_n')

$$\prod_{n \neq m} \left| \frac{z_m' - z_n'}{1 - \bar{z}_n' z_m'} \right| = \prod_{n \neq m} \psi(z_m' \, z_n') \geq \delta$$

If the sequence is rearranged so that every member is as close to one of $z_1', z_2', \ldots, z_N'$ as to the origin, then we can make a partition of the set of integers $\{N + 1, N + 2, N + 3, \ldots\}$ into sets $A_1, A_2, \ldots, A_N$ satisfying the condition: $n \in A_i$ implies $\psi(z_n', z_i') \leq \psi(z_n', 0)$. We obtain

$$|B'(0)| \geq \varepsilon^N \prod_{n \in A_1} \psi(z_n', z_1') \cdots \prod_{n \in A_N} \psi(z_n', z_N')$$
$$\geq (\varepsilon\delta)^N = \delta_o.$$

Note that δ_o does not depend on the sequence (z_n) of points given, but only on the value of δ used in (C).

<u>THEOREM</u> For a sequence (z_n) of complex numbers in $|z| < 1$, the following two conditions are equivalent:

(I) For any bounded sequence (w_n) of complex numbers, a function $f \in \mathcal{B}$ can be found such that

$$f(z_n) = w_n \qquad n = 1,2,3,\ldots$$

$$\text{(C)} \qquad \prod_{n \neq m} \left| \frac{z_n - z_m}{1 - z_n \bar{z}_m} \right| \geq \delta \qquad m = 1,2,3,\ldots$$

for some constant $\delta > 0$ which does not depend on m.

<u>Proof.</u> We have proved that (I) implies (C) in Lemma 1. Assume that (C) holds, and choose any integer s along with any decomposition of $\{z_1, z_2, \ldots, z_s\}$ into two disjoint non-empty sets. If $B(z)$ and $C(z)$

are the corresponding Blaschke products, we demonstrate the existence of δ_o, depending only on δ, such that

$$|B(z)| + |C(z)| \geq \delta_o \qquad z \in D.$$

Indeed, $\psi(z_m, z_n) \geq \delta$ whenever $m \neq n$, and hence for any $z \in D$, $\psi(z, z_n) \geq \delta/2$ for either the zeros of B or the zeros of C. If we suppose it is B apply Lemma 2 with $\varepsilon = \delta/2$ and get

$$|B(z)| \geq \delta_o$$

using the fact that (C) still applies when we restrict to the zeros of B.

This condition enables us to apply the interpolation formula of Theorem 6 in the previous section. We can interpolate zeros and ones arbitrarily at z_n, $n \leq s$, and with a function whose norm is bounded by constant $C = (1/\delta_o)^{A_{13}}$ independent of s.

We suppose without loss of generality that $|w_n| \leq 1$ for all n, and rearrange $(z_n)_1^s$ so that

$$u_1 \leq u_2 \leq \cdots \leq u_s \qquad z_n = u_n + iv_n$$

We obtain a function f_n for each n, $0 \leq n < s$, such that $f_n(z_i) = 1$ $i \leq n$ and $f_n(z_i) = 0$ $n < i \leq s$.
If we set $u_o = 0$, and define

$$g(z) = \sum_1^s (u_n - u_{n-1}) f_n(z)$$

we have $g(z_n) = u_n$, along with the estimate

$$||g|| \leq \sum_1^s (u_n - u_{n-1})||f_n|| \leq C \sum_1^s (u_n - u_{n-1}) \leq 2C.$$

A similar argument will give a function $h \in \mathcal{B}$ with

$$h(z_n) = v_n, \qquad ||h|| \leq 2C.$$

The function $g + ih$ gives the required interpolation for $n \leq s$, and has norm at most $4C$. If we call this function f_s, we have a uniformly

bounded sequence of functions (f_s). We can extract a convergent subsequence to obtain a function $f \in \mathcal{B}$ satisfying the condition $f(z_n) = w_n$ for all n.

A sequence of points (z_n) in D is said to approach the boundary exponentially if

$$\frac{1 - |z_{n+1}|}{1 - |z_n|} \leq c \qquad n = 1,2,3,\ldots \tag{5}$$

For some constant $c < 1$.

<u>COROLLARY.</u> If the sequence (z_n) approaches the boundary exponentially, then it is an interpolating sequence. Conversely, if $0 \leq z_1 \leq z_2 \leq z_3 \leq \cdots$ where (z_n) is an interpolating sequence, then the sequence approaches the boundary exponentially.

<u>Proof.</u> We have

$$\prod_{n \neq m} \left|\frac{z_n - z_m}{1 - z_n \bar{z}_m}\right| \geq \prod_{n<m} \frac{|z_m| - |z_n|}{1 - |z_n||z_m|} \cdot \prod_{n>m} \frac{|z_n| - |z_m|}{1 - |z_n||z_m|},$$

and assuming (5) for $n > m$

$$1 - |z_n| \leq c^{n-m}(1 - |z_m|)$$

$$|z_n| - |z_m| = (1 - |z_m|) - (1 - |z_n|) \geq (1 - c^{n-m})(1 - |z_m|).$$

For the denominator we have

$$|z_n||z_m| \geq (1 + c^{n-m}(1 - |z_m|)).\ |z_m|$$

$$1 - |z_n||z_m| \leq (1 - |z_m|)(1 + |z_m|c^{n-m}) \leq (1 + c^{n-m})(1 - |z_m|)$$

Thus the second factor gives

$$\prod_{n>m} \frac{|z_m| - |z_n|}{1 - |z_n|.|z_m|} \geq \prod_{n=m+1}^{\infty} \frac{1 - c^{n-m}}{1 + c^{n-m}} = \prod_{k=1}^{\infty} \left(\frac{1 - c^k}{1 + c^k}\right)$$

For the case $n < m$ we use a similar approach

$$1 - |z_m| \leq c^{m-n}(1 - |z_n|)$$

$$|z_m| - |z_n| \geq (1 - c^{m-n})(1 - |z_n|)$$

$$1 - |z_n||z_m| \leq (1 + c^{m-n})(1 - |z_n|)$$

which gives

$$\prod_{n<m} \frac{|z_m| - |z_n|}{1 - |z_n||z_m|} \geq \prod_{n=1}^{m-1} \frac{(1 - c^{m-n})}{(1 + c^{m-n})} \geq \prod_{k=1}^{\infty} \left(\frac{1 - c^k}{1 + c^k}\right)$$

Hence (z_n) satisfies condition (C).

Conversely, if (C) holds for a monotone increasing sequence (z_n) of non-negative numbers, we have

$$\delta \leq \frac{z_{n+1} - z_n}{1 - z_n z_{n+1}} \qquad n = 1,2,3,\ldots$$

But the inequalities

$$\frac{z_{n+1} - z_n}{1 - z_n z_{n+1}} \leq \frac{z_{n+1} - z_n}{1 - z_n} = 1 - \frac{1 - z_{n+1}}{1 - z_n}$$

give the condition for exponential approach to the boundary since

$$\frac{1 - z_{n+1}}{1 - z_n} \leq 1 - \delta < 1.$$

This converse fails to hold in general. There exist interpolating sequences which do not tend to the boundary exponentially (see Hoffman [14] page 204). We note also that every sequence of points in D with a cluster point on the boundary contains an interpolating subsequence, for it is always possible to extract a subsequence tending exponentially to the cluster point on the boundary.

§4 A FILTER DESCRIPTION OF $\mathfrak{M}$

We may consider $\mathfrak{B}$ as a subalgebra of the algebra $C(D)$ of all continuous complex valued functions on D. Both algebras having the uniform norm, it is apparent that $\mathfrak{B}$ is a sub-Banach algebra of $C(D)$. The latter algebra has maximal ideal space βD, and is in fact nothing other than $C(\beta D)$. By §1, Theorem 6 of Chapter 0, the injection of $\mathfrak{B}$ in $C(\beta D)$ induces a continuous map

$$\theta : \beta D \longrightarrow \mathfrak{M}$$

which is clearly a homeomorphism onto Δ when restricted to D. The remaining points of βD, that is, the points of $\beta D \setminus D$ are mapped into the various fibres $\mathfrak{M}_\alpha$. If a filter $\mathfrak{N}$ in βD, which means a maximal closed filter on D, converges to a point α on the unit circle, then θ maps this filter into some homomorphism of $\mathfrak{M}\alpha$.

We know that Δ is dense in $\mathfrak{M}$. The continuous map θ maps βD into a compact subset of $\mathfrak{M}$ which contains Δ ; in other words, θ maps βD onto $\mathfrak{M}$.

In view of this, we see that every homomorphism in $\mathfrak{M}$ is the image under θ of some homomorphism of $C(\beta D)$, but these latter are all obtained by taking the limit along some maximal closed filter of the various functions in $C(\beta D)$. Furthermore, $\theta(\phi)$ is known to be simply the restriction of ϕ to the smaller algebra for any homomorphism ϕ of the larger. Hence every homomorphism of $\mathfrak{B}$ has the form

$$(1) \qquad f \longrightarrow \lim f(\mathfrak{N}) \qquad f \in \mathfrak{B}$$

where $\mathfrak{N}$ is some maximal closed filter on D. For a fixed homomorphism ϕ_a of $\mathfrak{B}$, the maximal closed filter $\mathfrak{N}$ will be the filter of all closed sets containing a. Moreover each homomorphism ϕ_a of Δ is determined by just this one filter. However, this fails for homomorphisms

of $\mathcal{M} \setminus \Delta$.

R.C.Buck [5] considers this mapping from βD to $\mathcal{M}$, and remarks that one can use Pick's theorem to show that it is not 1-1. In the context of filters, this suggests that two different filters of βD, if they are sufficiently 'close' in the hyperbolic metric, will give the same homomorphism on $\mathcal{B}$. We use this idea to make an identification of certain classes of filters in βD all of which give the same homomorphism. More properly stated, we take smaller filters on D which are contained in each maximal closed filter of the above class, and which determine, in the sense of (1), the same homomorphisms as the members of the class.

For any maximal closed filter $\mathfrak{N}$ on D, we define a filter $\mathfrak{N}^*$ by taking as basis all hyperbolic ε-neighbourhoods of the sets in $\mathfrak{N}$. If we define for any subset A of D

$$N(A,\varepsilon) = \{z \in D \mid \psi(z,w) < \varepsilon \text{ for some } w \in A\}$$

then $\mathfrak{N}^*$ is the filter generated by the collection

$$\{N(A,\varepsilon) \mid A \in \mathfrak{N},\ \varepsilon > 0\}$$

We will only apply the notion of *-filter to maximal closed filters $\mathfrak{N}$, although the definition is meaningful for any filter on D.

We see from the next proposition that the *-filters do in fact give all the homomorphisms of $\mathcal{B}$ in the sense of (1)

<u>PROPOSITION 1.</u> For each maximal closed filter $\mathfrak{N}$ on D, and each $f \in \mathcal{B}$, $\lim f(\mathfrak{N}^*)$ exists.

<u>Proof.</u> We can restrict the proof to functions f with $||f|| \leq 1$, since every function in $\mathcal{B}$ is a scalar multiple of one such. Let $\lim f(\mathfrak{N}) = \alpha$, and take any positive number ε. Choose a set $A \in \mathfrak{N}$ with $|f(z) - \alpha| < \varepsilon/2$ for $z \in A$. For any point $w \in N(A,\varepsilon/2)$,

there exists a point $z \in A$ with $h(w,z) < \varepsilon/2$. Now

$$|f(w) - \alpha| \leq |f(w) - f(z)| + |f(z) - \alpha| < \varepsilon$$

since, by Pick's theorem applied to f,

$$|f(w) - f(z)| \leq h(f(w),f(z)) \leq h(w,z) < \varepsilon/2.$$

It therefore follows that $\lim f(\mathfrak{N}^*)$ exists and equals $\lim f(\mathfrak{N})$.

<u>COROLLARY.</u> The filter $\mathfrak{N}^*$ determines a homomorphism of $\mathcal{B}$ given by $f \longrightarrow f(\mathfrak{N}^*)$ for $f \in \mathcal{B}$. Every homomorphism of $\mathcal{B}$ can be represented in this way.

The question immediately arises whether or not this identification is sufficient to give a 1-1 correspondence between filters and homomorphisms. Does each homomorphism correspond to a unique *-filter? This question will be considered at length in subsequent chapters. At this point we make a few remarks which will be useful in this investigation.

For each maximal closed filter $\mathfrak{N}$, it is evident that $\mathfrak{N}^* \subseteq \mathfrak{N}$. If $\mathfrak{N}^* \subseteq \mathcal{H}$ for any maximal closed filter $\mathcal{H}$, we will see that $\mathfrak{N}^* = \mathcal{H}^*$. We first show $\mathfrak{N}^* \subseteq \mathcal{H}^*$. Take any set B from the former; it must contain a set of the form $N(A,\varepsilon)$ with $A \in \mathfrak{N}$. Moreover, we must have $N(A,\frac{\varepsilon}{2}) \in \mathfrak{N}^*$ because $A \in \mathfrak{N}$. Hence, $N(A,\frac{\varepsilon}{2}) \in \mathcal{H}$, and the set $N(N(A,\frac{\varepsilon}{2}),\frac{\varepsilon}{2})$ belongs to $\mathcal{H}^*$. But this latter set is contained in $N(x,\varepsilon) \subseteq B$. B must therefore belong to $\mathcal{H}^*$; thus $\mathfrak{N}^* \subseteq \mathcal{H}^*$. The opposite containment follows at once if $\mathcal{H}^* \subseteq \mathfrak{N}$, simply by repeating the same argument. But the case $\mathcal{H}^* \nsubseteq \mathfrak{N}$ leads immediately to a contradiction. If a set $B \in \mathcal{H}^* \setminus \mathfrak{N}$, then we can take a basic set $N(A,\varepsilon)$ for with $A \in \mathcal{H}$ having the same property. Indeed, the smaller closed set $\overline{N(A,\frac{\varepsilon}{2})}$ is also in $\mathcal{H}^* \setminus \mathfrak{N}$. By the maximality of $\mathfrak{N}$, we can find $C \in \mathfrak{N}$ with $\overline{N(A,\frac{\varepsilon}{2})} \cap C = \emptyset$. The disjointness of these last sets

easily implies $N(A,\frac{\varepsilon}{8}) \cap N(C,\frac{\varepsilon}{8}) = \emptyset$. The first of these is in $\mathcal{X}^*$, and the second is in $\mathcal{N}^*$, a contradiction.

<u>PROPOSITION 2.</u> If $\mathcal{N}^*$ and $\mathcal{X}^*$ are distinct *-filters, then there exists a positive constant ε, and sets $A \in \mathcal{N}$, $B \in \mathcal{X}$ with

$$\psi(a,b) \geq \varepsilon \quad \text{whenever} \quad a \in A, \quad b \in B.$$

<u>Proof.</u> We have only to show that distinct *-filters $\mathcal{N}^*$ and $\mathcal{X}^*$ are incompatible, in other words sets C and D can be found, $C \in \mathcal{N}^*$ and $D \in \mathcal{X}^*$, with $C \cap D = \emptyset$. For then we can find $A \in \mathcal{N}$, and $\delta_1 > 0$ such that $N(A,\delta_1) \subseteq C$, along with $B \in \mathcal{X}$ and $\delta_2 > 0$ such that $N(B,\delta_2) \subseteq D$. From the disjointness property

$$N(A,\delta_1) \cap N(B,\delta_2) = \emptyset$$

the proposition will certainly follow with $\varepsilon = \min(\delta_1,\delta_2)$.

We suppose now that the two *-filters are compatible, and from this obtain a contradiction. Two filters which are compatible always generate a proper filter, so that we can choose an ultrafilter $\mathcal{F}$ above both $\mathcal{N}^*$ and $\mathcal{X}^*$. The closed filter $\mathcal{G}$ obtained by taking as a basis the closures of all sets in $\mathcal{F}$ will retain the property that it lies above both *-filters. This is a consequence of the following property of *-filters: Every basic generating set $N(A,\varepsilon)$ contains a closed set $\overline{N(A,\frac{\varepsilon}{2})}$ in the filter. We now take, if necessary, a maximal closed filter $\mathcal{M}$ above $\mathcal{F}$. Evidently $\mathcal{N}^* = \mathcal{M}^* = \mathcal{X}^*$, a contradiction. This special property of the generating sets of *filters makes the following slightly stronger statement valid.

<u>COROLLARY.</u> If $\mathcal{N}^*$ and $\mathcal{X}^*$ are distinct *-filters, then there exists a positive constant ε, and sets $A \in \mathcal{N}^*$, $B \in \mathcal{X}^*$, with

$$\psi(a,b) \geq \varepsilon \quad \text{whenever} \quad a \in A, \quad b \in B.$$

A few remarks on terminology: For any maximal closed filter $\mathcal{N}$, we say that the homomorphism

$$\phi_{\mathcal{N}} : f \longrightarrow \lim f(\mathcal{N}) \qquad f \in \mathcal{B}$$

is <u>determined by</u> $\mathcal{N}$, or <u>corresponds to</u> $\mathcal{N}$. We frequently use the fact that $\phi_{\mathcal{N}}(f)$ must be in the closure of $f(A)$, if A is a set in $\mathcal{N}$. We say that we can <u>separate</u> two *-filters $\mathcal{N}^*$ and $\mathcal{K}^*$ if we can find $f \in \mathcal{B}$ with $\phi_{\mathcal{N}}(f) \neq \phi_{\mathcal{K}}(f)$. Of course, the two filters determine different homomorphisms if and only if such a function f exists.

Let $\mathcal{M}^*$ be the set of all *-filters. Noting that any *-filter has a basis of closed sets, as we have just seen, we can introduce in $\mathcal{M}^*$ a topology in the same fashion as the topology was introduced in the set of all maximal closed filters (Chapter 0, §3). This gives a compact Hausdorff extension space of the unit disc D. It is the quotient topology of βD, and therefore induces a continuous map

$$\mathcal{M}^* \longrightarrow \mathcal{M}$$

We shall show that this mapping is 1-1 on certain subsets of $\mathcal{M}$, other than D, but fails to be 1-1 everywhere.

The space $\mathcal{M}^*$ can be characterized as the largest compact extension space of D in which the embedding $D \longrightarrow \mathcal{M}^*$ is uniformly continuous in the hyperbolic metric. For a discussion of such matters, see [2].

CHAPTER II

NON-TANGENTIAL HOMOMORPHISMS

<u>Introduction</u> Each non-fixed homomorphism of $\mathcal{B}$ is assigned an angle which is determined by the direction of approach to the boundary of the corresponding filter. By means of this angle of approach to a given α on the unit circle, an open subset $\mathcal{N}_\alpha$ of the fibre $\mathcal{M}_\alpha$ is defined, the set of non-tangential homomorphisms. It is shown that each homomorphism in $\mathcal{N}_\alpha$ is determined by a unique *-filter.

§1 ANGLE OF APPROACH

Recall that a hypercycle is the restriction to D of some circle which meets the unit circle in exactly two points.

DEFINITION A filter on D converging to α with $|\alpha| = 1$, is called <u>non-tangential</u> if and only if two hypercycles through α and $-\alpha$ can be chosen along with a set from the filter which lies between them.

DEFINITION A homomorphism $\Phi \in \mathfrak{M}_\alpha$ for $|\alpha| = 1$, is called <u>non-tangential</u> if and only if $\Phi = \Phi_{\mathfrak{N}}$ for some non-tangential maximal closed filter on D which converges to α.

We denote by $\mathfrak{N}_\alpha$ the subset of $\mathfrak{M}_\alpha$ consisting of all these non-tangential homomorphisms, and let $\mathfrak{N} = \bigcup_{\alpha \in \Gamma} \mathfrak{N}_\alpha$. It is sufficient to prove results for a particular value of α, say $|\alpha|=1$. Let $\mathcal{C}$ be the family of hypercycles through -1 and 1. The terms 'above' and 'below' are used in the obvious geometric sense.

<u>PROPOSITION 1</u>. If the filter $\mathfrak{N}^*$ is non-tangential and converges to $\alpha=1$, there exists $C_0 \in \mathcal{C}$ with this property: For completely arbitrary $C_1 \in \mathcal{C}$ above C_0, and $C_2 \in \mathcal{C}$ below C_0, some set from $\mathfrak{N}^*$ lies between C_1 and C_2.

<u>Proof</u>. $\mathfrak{N}$ is a maximal closed filter. Take $F \in \mathfrak{N}$ and $C \in \mathcal{C}$. Let F_1 be the closed set of those points of F lying above or on C, and F_2 the closed set of those points below or on C. $F = F_1 \cup F_2$, and by the maximality of the filter, either $F_1 \in \mathfrak{N}$ or $F_2 \in \mathfrak{N}$. In the former case, we say the filter lies above, in the latter case below, the circle C. One of these must hold. If a filter lies above [below] any member of $\mathcal{C}$, then it is in the same relation to any lower [higher] member of .

Since $\mathfrak{N}^*$ is non-tangential, it and therefore $\mathfrak{N}$ will lie below some member of $\mathcal{C}$, and above some member of $\mathcal{C}$. From these properties we infer the existence of some $C_o \epsilon \mathcal{C}$ such that $\mathfrak{N}$ lies below any circle which is above C_o, and above any circle which lies below C_o. Because of the relation between the family $\mathcal{C}$ and the hyperbolic metric, we can make the same assertion about $\mathfrak{N}^*$. For if C_1 is above C_o, let d be the constant pseudo-hyperbolic distance between the two, and choose $F \epsilon \mathfrak{N}$ lying below that member of $\mathcal{C}$ which lies exactly half way between C_o and C_1 (using the pseudo-hyperbolic metric). The set $N(F,d/2)$ belongs to $\mathfrak{N}^*$ and lies below C_1. Thus $\mathfrak{N}^*$ lies below C_1 and must likewise lie above any C_2 below C_o. A set from $\mathfrak{N}^*$ can therefore be found, which lies between C_1 and C_2; it is clear that this set can be taken to lie strictly between C_1 and C_2, if necessary.

We see in the above proof that a filter $\mathfrak{N}$ is non-tangential if and only if $\mathfrak{N}^*$ is non-tangential.

With each circle $C \epsilon \mathcal{C}$, we associate the positive angle k between the upward vertical and the inward directed tangent to C, both taken at $\alpha=1$. We have $0 < k < \pi$. For each non-tangential $\mathfrak{N}^*$ converging to $\alpha=1$, we let the <u>angle of approach</u> to $\alpha=1$ be the k of the circle C_o determined by $\mathfrak{N}^*$ (see Proposition 1). The remaining *-filters converging to $\alpha=1$ must either lie above every member of $\mathcal{C}$, or below every member of $\mathcal{C}$; we assign the values $k=0$ and $k=\pi$ to the two cases respectively.

<u>PROPOSITION 2</u>. If $\mathfrak{N}^*$ and $\mathfrak{X}^*$ have different angles of approach to $\alpha=1$, then $\Phi_{\mathfrak{N}} \neq \Phi_{\mathfrak{X}}$.

<u>Proof</u>. The function $\frac{\pi}{2} - \arg w$ is harmonic in the right half W-plane

given by $-\pi/2 < \arg w < \pi/2$; it is the imaginary part of $\frac{\pi i}{2} - \log w$. On radial lines through the origin, this function is constant.

The conformal map $z = (w - 1)/(w + 1)$ maps this half plane onto D, taking the radial lines into the members of $\mathcal{C}$. The given function, now defined on D, remains harmonic; we call it u. The constant value of u on each $C \in \mathcal{C}$ is just the k associated with the circle C.

Let v be the harmonic conjugate of u, and define $f = \exp\ (u + iv)$. Then $f \in \mathcal{B}$, since it is analytic, and since $|f| = e^u$ is bounded. We observe that, on C, $|f| = e^k$.

Assume $\mathcal{N}^*$ is non-tangential with circle of approach C_0. $\Phi_{\mathcal{N}}(f)$ must belong to $\overline{f(A)}$ for each $A \in \mathcal{N}^*$, and these sets A can be confined between two circles of $\mathcal{C}$ as close as we like to C_0. It is impossible for $|\Phi_{\mathcal{N}}(f)|$ to be other than e^k. If $\mathcal{N}^*$ has $k = 0$ for angle of approach to $\alpha = 1$, then for each $\epsilon > 0$, we can choose $A \in \mathcal{N}$ above a high enough circle of $\mathcal{C}$, such that $u(z) < \epsilon$ for all $z \in A$. Thus $|f| = e^u$ can be made arbitrarily close to 1 and $\Phi_{\mathcal{N}}(f)$ must have modulus 1. Similarly if the angle $k = \pi$, we must have $|\Phi_{\mathcal{N}}(f)| = e^{\pi}$.

In all cases $|\Phi_{\mathcal{N}}(f)| = e^k$. It is evident that the proposition follows.

From this proposition, it is obvious that non-tangential homomorphisms are determined <u>only</u> by maximal closed filters which are non-tangential. We also have

<u>COROLLARY</u> $\mathcal{N}_\alpha$ is an open subset of $\mathfrak{M}_\alpha$.

<u>Proof</u>. For the function f used in the proposition, since f is continuous,

$$\mathcal{N}_\alpha = \{\Phi \in \mathfrak{M}_\alpha : 1 < |\hat{f}(\Phi)| < e^{\pi}\}$$

must be an open subset of $\mathfrak{M}_\alpha$.

We consider now the case of two different *-filters, $\mathfrak{N}^*$ and $\mathfrak{X}^*$, both of which are non-tangential and approach $\alpha=1$ along the same hypercycle C_o. It is known by the corollary of Proposition 2 in Chapter 1, §4, that sets $A\epsilon\ \mathfrak{N}^*$, $B\epsilon\ \mathfrak{X}^*$ exist, along with a positive constant which we take to be 8ϵ, such that $\psi(a,b) \geq 8\epsilon$ whenever $a\epsilon\ A$, $b\epsilon\ B$. Take two circles C_1 and C_2 from the family $\mathcal{C}$ which are equal pseudo-hyperbolic distances above and below C_o; this distance is assumed to be less than 2ϵ. A and B can be made to lie between C_1 and C_2. We also restrict A and B to the right half of D.

It is next shown that the points of A and B can be separated into a sequence of blocks, which consist alternately of points of A and B, tending to $\alpha=1$. See Figure 5, in which this is illustrated for the special case where C_o is the real axis. Take the family $\mathcal{B}$ of circles orthogonal to the circles of $\mathcal{C}$. In other words $\mathcal{B}$ is the family of circles with centre on the real axis, and orthogonal to the unit circle Γ. These are straight lines in the hyperbolic geometry when we restrict to D, and are orthogonal to the members of $\mathcal{C}$. Each point of A lies on a circle from $\mathcal{B}$ which determines some point of intersection on C_o. The set A_o of these points of intersection is called the trace of A on C_o. A similar set B_o is defined, and we readily see that the two sets are separated in the pseudo-hyperbolic metric.

$$\psi(\alpha,\beta) \geq 4\epsilon \quad \text{for } \alpha\epsilon\ A_o\ ,\ \beta\epsilon\ B_o\ .$$

This follows from the existence of $a\epsilon\ A$ with $\psi(a,\alpha) < 2\epsilon$ and of $b\epsilon\ B$ with $\psi(b,\beta) < 2\epsilon$, for then $\psi(\alpha,\beta) \geq \psi(a,b) - \psi(a,\alpha) - \psi(b,\beta) > 8\epsilon-2\epsilon-2\epsilon = 4\epsilon$.

We introduce an order relation on C_o; a is greater than b if it is closer to $\alpha=1$. In the diagram, this is the natural ordering of the reals.

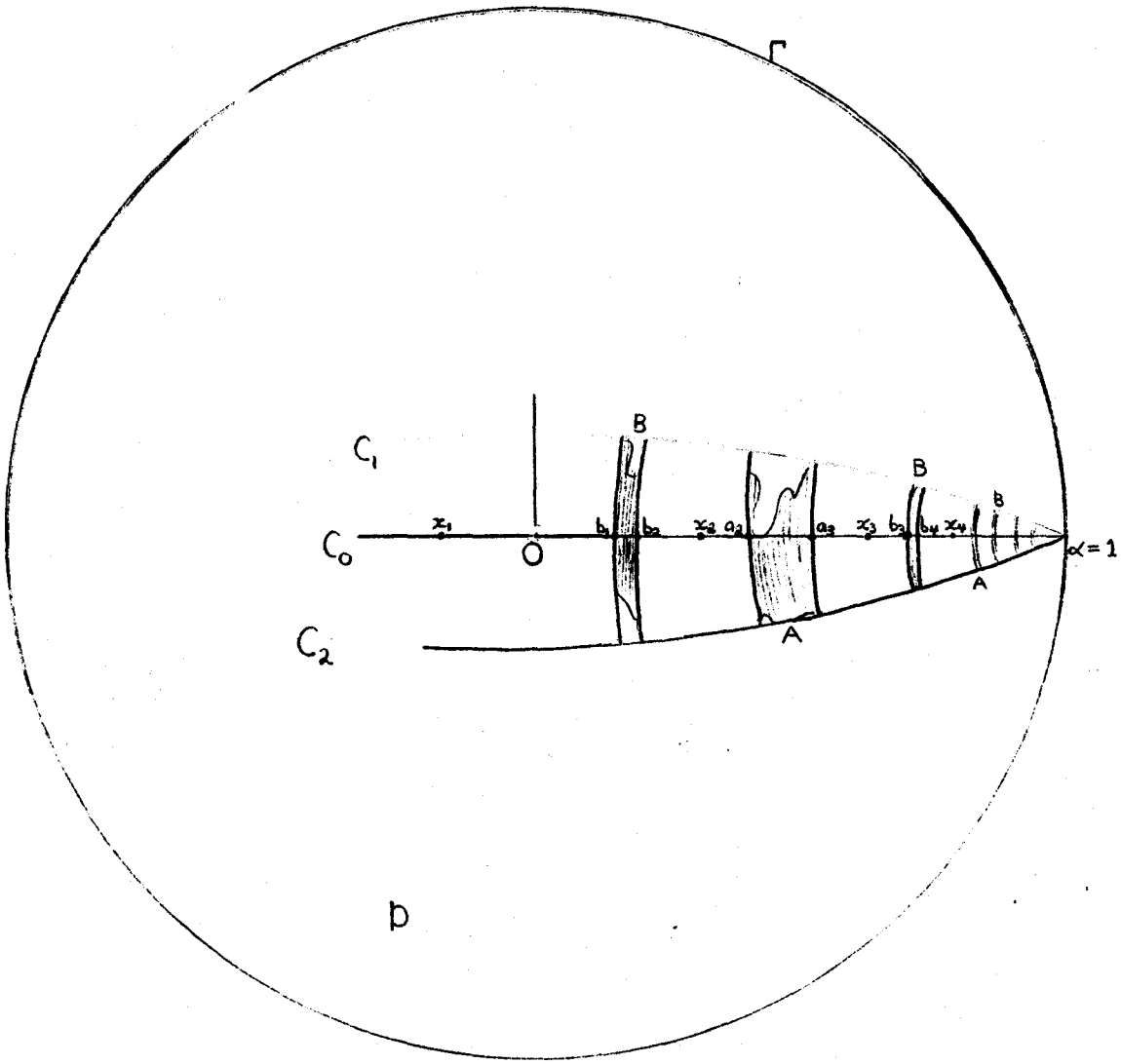

Figure 5

We use the appropriate notation: $[\alpha,\beta]$ denotes the closed subarc of C_0 joining α and β, and we take infima and suprema of subsets of C_0 using the given ordering. Start from the left and move towards $\alpha = 1$. Suppose without loss of generality that $b_1 = \inf\{\beta : \beta \in B_0\}$ is smaller than any $\alpha \in A_0$. If b_2 is the supremum of the set of points in B_0 which are less than each point of A_0, then $b_1 \leqslant b_2$. Let $a_2 = \inf\{\alpha : \alpha \in A_0\}$. Then $b_2 \leqslant a_2$. In fact we can choose points $\beta \in B_0$ and $\alpha \in A_0$ arbitrarily close to b_2 and a_2, which means that $\psi(b_2, a_2) \geqslant 4\epsilon$ and $b_2 < a_2$. Define a_3 to be the supremum of those points of A_0 less than each point in B_0, excluding those in $[b_1, b_2]$. We continue in this way; b_3 is the infimum of B_0 after $[b_1, b_2]$ has been removed, and b_4 is the supremum of those $\beta \in B_0$ which are less than any $\alpha \in A_0$ with $\alpha > b_3$. We obtain the following :

$$b_1 \leqslant b_2 < a_2 \leqslant a_3 < b_3 \leqslant b_4 < a_4 \ \ldots.$$

$$\psi(a_n, b_n) > 4\epsilon \text{ for } n = 2, 3, 4, \ldots.$$

$$B_0 \subseteq [b_1, b_2] \cup [b_3, b_4] \cup [b_5, b_6] \cup \ldots.$$

$$A_0 \subseteq [a_2, a_3] \cup [a_4, a_5] \cup [a_6, a_7] \cup \ldots.$$

We shall make use of this decomposition to find a function in $\mathcal{B}$ which separates the filters $\mathfrak{A}^*$ and $\mathfrak{B}^*$. This is done in the next section for the special case, shown in Figure 5, where C_0 is the line segment joining -1 and +1. Then the result is generalized in section §3.

§2 RADIAL APPROACH TO THE BOUNDARY

We assume throughout this section that $\mathcal{V}^*$ and $\mathcal{X}^*$ have angle of approach $k = \pi/2$ to the point $\alpha = 1$. Taking the sequences (a_n) and (b_n) of points obtained at the end of the previous section, we define now a sequence (x_n) where x_n bisects the gap between a_n and b_n. Thus

$$x_n \geqslant 0, \quad \psi(a_n, x_n) = \psi(b_n, x_n), \quad n = 2, 3, 4, \ldots.$$

We have the inequalities

$$\psi(x_n, a_n) \geqslant 2\epsilon, \ \psi(x_n, b_n) \geqslant 2\epsilon \quad n = 2, 3, 4, \ldots.$$

Since $4\epsilon < \psi(a_n, b_n) \leqslant \psi(a_n, x_n) + \psi(b_n, x_n) = 2\psi(a_n, x_n) = 2\psi(b_n, x_n)$

Define in addition $x_1 < b_1$ with

$$\psi(x_1, b_1) = 2\epsilon$$

We start with circles C_1 and C_2, and sets $A \epsilon \mathcal{V}^*$, $B \epsilon \mathcal{X}^*$ lying between the two circles. At various stages, restrictions will be placed on C_1 and C_2 which bring them closer to C_0. When this happens we restrict the initial A and B to the region between the new circles. The restrictions, which we still denote by A and B, must also belong to the filters $\mathcal{V}^*$ and $\mathcal{X}^*$ respectively. Finally sets A and B are obtained which are separated by the function $F \epsilon \mathcal{B}$.

$$F(z) = \prod_{n=1}^{\infty} \left(\frac{x_n - z}{1 - x_n z} \right)$$

This function takes on real values on the real axis, and in fact is positive on A_0, negative on B_0. It will be shown that F is separated away from zero on these sets, and indeed that it separates the larger sets A and B.

Take an individual block enclosed by C_1, C_2 and the circles of $\mathfrak{B}$ through a_{n-1} and a_n (n odd). See Figure 6. We draw a circle centred at the origin, and passing through the right hand corner points of this block. Let a_n^+ be the point where this circle meets the real line. In like fashion we obtain b_n^+ for n even.

<u>Lemma</u> 1 Circles C_1 and C_2 can be chosen such that

$$\psi(a_n, a_n^+) < \epsilon \qquad n = 3, 5, 7, \ldots.$$

$$\psi(b_n, b_n^+) < \epsilon \qquad n = 2, 4, 6, \ldots.$$

<u>Proof</u> See Figure 7. Suppose C_1 and C_2 are given. Corresponding to each λ, $0 < \lambda < 1$, is a point $\lambda^+ > \lambda$ obtained as a_n^+ is obtained from a_n. We show that for suitable C_1 and C_2, $\psi(\lambda, \lambda^+)$ can be made less than an arbitrary positive number δ independent of λ.

The tangent to C_1 at $\alpha = 1$ intersects the circle through λ at a point from which a perpendicular is dropped to the real axis. Let the distance from the foot of the perpendiculars to λ and to $\alpha = 1$ be p and q respectively, and let θ be the angle between the real line and the tangent at $\alpha = 1$.

As $\lambda \to 1$, the radius of the circle through λ^+ approaches 1, whereas that of the circle through λ approaches 0. At the point where the two radii are equal, it is clear that $p > (\lambda^+ - \lambda)/2$; and this inequality holds for all larger values of λ. Thus there exists λ_0 such that

$$\lambda^+ - \lambda < 2p \text{ for } \lambda \geqslant \lambda_0 .$$

Since the angle at the top vertex of the large triangle is always less than $\pi/2$, the angle Φ (marked on the diagram) must be less than θ.

$$p = r \tan \Phi = q \tan \theta \tan \Phi < q \tan^2 \theta$$

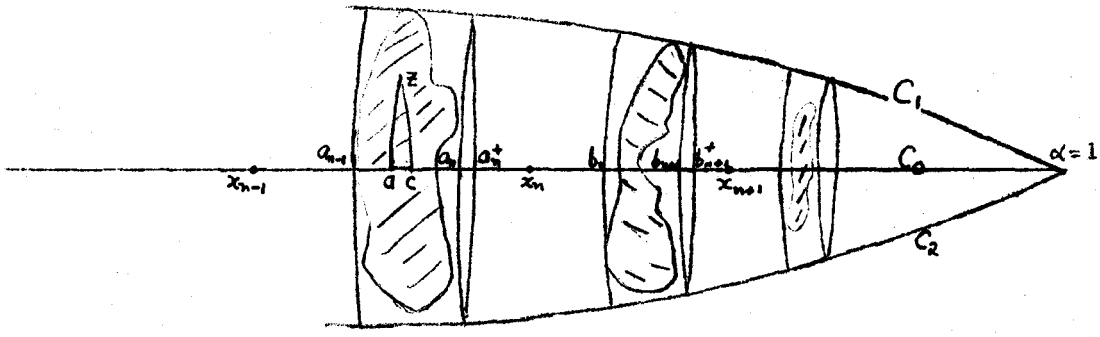

Figure 6

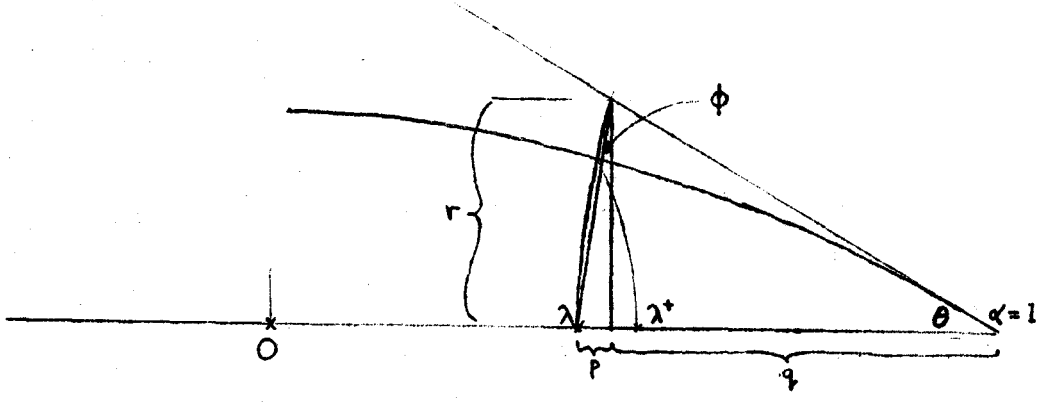

Figure 7

$$\lambda^+ - \lambda < 2q \tan^2 \theta \qquad \text{for } \lambda \geqslant \lambda_0 .$$

Hence

$$\psi(\lambda,\lambda^+) = \frac{\lambda^+ - \lambda}{1 - \lambda\lambda^+} < \frac{2q \tan^2 \theta}{1 - \lambda} < 2 \tan^2 \theta' \qquad \text{for } \lambda \geqslant \lambda_0 .$$

This means that, by choosing C_1 and C_2 to give sufficiently small θ, we can make $\psi(\lambda,\lambda^+) < \delta$ for $\lambda \geqslant \lambda_0$. On the remaining segment, $0 \leqslant \lambda \leqslant \lambda_0$, the ratio of pseudo-hyperbolic to Euclidean metric is bounded, and since $\lambda^+ - \lambda$ can obviously be made uniformly small on the interval, we can obtain here also the inequality $\psi(\lambda,\lambda^+) < \delta$.

The lemma follows for the value $\delta = \epsilon$, and we assume henceforth this restriction on C_1 and C_2.

In Figure 6, we have enclosed certain points of A in a closed block bounded by two circles of $\mathcal{C}$, one circle of $\mathcal{L}$, and one circle centred at the origin. We denote by $\tilde{A}$ the union of all such blocks. In addition we define a set

$$A_1 = [a_2 a_3^+] \cup [a_4 a_5^+] \cup [a_6 a_7^+] \quad \ldots .$$

Similar sets $\tilde{B}$ and B_1 are defined.

<u>Corollary 1</u>. For each $z \in \tilde{A} \cup \tilde{B}$, $\psi(z,x_n) \geqslant \epsilon$ $n = 1, 2, 3, \ldots$.

<u>Proof</u> We observe in Figure 6 that $\psi(x_n,a_n^+) \geqslant \epsilon$, since $\psi(a_n,a_n^+) < \epsilon$ and $\psi(x_n,a_n) \geqslant 2\epsilon$. On the circle through a_n^+, the closest point to x_n using ψ-distance is a_n^+ itself. Any point of that part of $\tilde{A}$ shown must therefore be a distance of at least ϵ from x_n. On the left, $\psi(x_{n-1},a_{n-1}) \geqslant 2\epsilon$ and the points of $\tilde{A}$ shown are at least as far from x_{n-1}. This shows that any $z \in \tilde{A}$ is ψ-distant from any x_n by at least ϵ, and the same argument holds for $\tilde{B}$.

Corollary 2 Given $\eta > 0$, circles C_1 and C_2 can be chosen such that to any $z \epsilon \tilde{A}$ ($z \epsilon \tilde{B}$) corresponds $c \epsilon A_1$ ($c \epsilon B_1$) with $|z| = c$ and $\psi(z,c) < \eta$.

Proof. Choose C_1 and C_2 to satisfy two conditions. First, their ψ-distance from C_0 has a value less than $\eta/2$. Second, they are chosen such that $\psi(\lambda,\lambda^+) < \delta$ for $0 \leqslant \lambda < 1$, where $\delta = \eta/2$. Points a and c are the intersection with C_0 of circles through z orthogonal to C_1 and C_2, and centred at the origin respectively. See Figure 6. By the first conditions $\psi(z,a) < \eta/2$, and by the second $\psi(a,c) < \eta/2$. Hence $\psi(z,c) < \eta$. It is obvious that $c = |z|$ and that $c \epsilon A_1$. The case of $z \epsilon \tilde{B}$ is the same.

Lemma 2 The sequence (x_n) is an interpolating sequence.

Proof. For each n, $\psi(x_n, x_{n+1}) \geqslant \epsilon > 0$. We show in general that any increasing sequence of reals with this property approaches the boundary exponentially.

$$\psi(x_n, x_{n+1}) = \frac{x_{n+1} - x_n}{1 - x_{n+1}x_n} \geqslant \epsilon$$

$$x_{n+1}(1 + \epsilon x_n) \geqslant \epsilon + x_n$$

$$1 - x_{n+1} < 1 - \frac{\epsilon + x_n}{1 + x_n\epsilon} = \frac{(1 - \epsilon)(1 - x_n)}{1 + x_n\epsilon}$$

Hence $\dfrac{1 - x_{n+1}}{1 - x_n} \leqslant \dfrac{1 - \epsilon}{1 + x_n \epsilon} < 1 - \epsilon < 1$

Because $(1 - x_{n+1})/(1 - x_n) < 1 - \epsilon$ for all n, the sequence is said to approach the boundary exponentially. Such a sequence is necessarily an interpolating sequence by the Corollary to the Theorem of Chapter 1, §3. This guarantees that F is a convergent Blaschke product, and hence defines a function in $\mathcal{B}$.

In addition, this implies, by the Theorem just mentioned, the existence

of a positive constant δ, such that

$$\prod_{m \neq n} \left|\frac{x_n - x_m}{1 - x_n x_m}\right| = \left|F^{\hat{n}}(x_n)\right| \geqslant \delta \quad \text{for all } n.$$

<u>Lemma</u> 3 There exists a positive constant ρ such that

$$|F(z)| \geqslant \rho \quad \text{for } z \in \tilde{A} \cup \tilde{B}$$

Take without loss of generality $z \in \tilde{A}$. Suppose $x_{n-1} < |z| < x_n$ See Figure 6.

$$|F(z)| = \prod_{m=1}^{n-2} \left|\frac{x_m - z}{1 - x_m z}\right| \cdot \left|\frac{x_n - z}{1 - x_{n-1} z}\right| \cdot \left|\frac{x_n - z}{1 - x_n z}\right| \cdot \prod_{m=n+1}^{\infty} \left|\frac{x_m - z}{1 - x_m z}\right|$$

For $m \leqslant n - 2$ $\left|\frac{x_m - z}{1 - x_m z}\right| \geqslant \left|\frac{x_m - x_{n-1}}{1 - x_m x_{n-1}}\right|$

$$\prod_{m=1}^{n-2} \left|\frac{x_m - z}{1 - x_m z}\right| > \prod_{m=1}^{n-2} \left|\frac{x_m - x_{n-1}}{1 - x_m x_{n-1}}\right| \geqslant \prod_{m \neq n-1} \left|\frac{x_m - x_{n-1}}{1 - x_m x_{n-1}}\right| \geqslant \delta$$

For $m = n - 1$ and $m = n$

$$\left|\frac{x_m - z}{1 - x_m z}\right| = \psi(x_m, z) \geqslant \epsilon$$

For $m \geqslant n + 1$ $\left|\frac{x_m - z}{1 - x_m z}\right| \geqslant \left|\frac{x_m - x_n}{1 - x_m x_n}\right|$

$$\prod_{m=n+1}^{\infty} \left|\frac{x_m - z}{1 - x_m z}\right| \geqslant \prod_{m=n+1}^{\infty} \left|\frac{x_m - x_n}{1 - x_m x_n}\right| \geqslant \delta$$

Hence $|F(z)| \geqslant \delta . \epsilon . \epsilon . \delta = \delta^2 \epsilon^2 = \rho$

This could also be obtained as a consequence of Lemma 2 of chapter 1, §3.

<u>Lemma</u> 4 $\{F(x)\} \geqslant \rho$ for $x \in A_1$

$\{F(x)\} \leqslant -\rho$ for $x \in B_1$

<u>Proof</u>. We have already remarked that F takes on real values on the real axis and that these values are positive on A_0, negative on B_0. The same applies to A_1 and B_1. But $|F(x)| \geqslant \rho$ for $x \in A_1$ or for $x \in B_1$ by Lemma 3

since $A_1 \subseteq \tilde{A}$, $B_1 \subseteq \tilde{B}$.

<u>Lemma</u> 5 Circles C_1 and C_2 in $\mathbb{C}$ can be so chosen that to every z in $\tilde{A}$ or $\tilde{B}$ corresponds to c in A_1 or B_1 respectively with

$$|F(z) - F(c)| < \rho/2$$

<u>Proof</u> Set $\eta = \rho/16$. We give the proof for $\tilde{A}$. Circles C_1 and C_2 corresponding to this value of η can be selected, according to Lemma 1, Corollary 2, to give for each $z \in \tilde{A}$, a point $c \in A_1$ with

$$\psi(z,c) < \eta \quad \text{and} \quad |z| = c.$$

Integrating along the straight line L from c to z.

$$F(z) - F(c) = \int_L F'(\xi)d\xi.$$

We can apply the lemma at the end of Chapter 0, §4, to F, since any Blaschke product has modulus less than one inside D. This gives

$$|F'(\xi)| \leqslant \frac{1}{1 - |\xi|} \leqslant \frac{1}{1 - c}$$

for points ξ on the line segment L.

Since the line segment L has length $|z - c|$,

$$|F(z) - F(c)| \leqslant \frac{1}{1 - c} \cdot |z - c|$$

Because we have $\psi(z,c) = \left|\frac{z - c}{1 - cz}\right| < \eta$, it is sufficient to show

(1) $\quad |1 - cz| < 8(1 - c)$

for then

$$|F(z) - F(c)| < \frac{1}{1 - c} \cdot \eta \cdot |1 - cz| < \frac{\rho}{2}$$

We see (1) from Figure 8. Take the base P of the perpendicular from the point cz to the real axis, and the intersection Q with the real axis of the circle, centred at 0, which passes through cz. The angle Φ is less than θ, and as long as $\theta < \pi/3$, we have Sec Φ < Sec θ < 2.

PR < 2QR always holds, and in fact as $z \to 1$, $PR/QR \to 1$. Hence

$$|1 - zc| = PR \operatorname{Sec} \Phi < 2.QR.\operatorname{Sec} \theta < 2(1 - c^2)\cdot 2 < 8(1 - c)$$

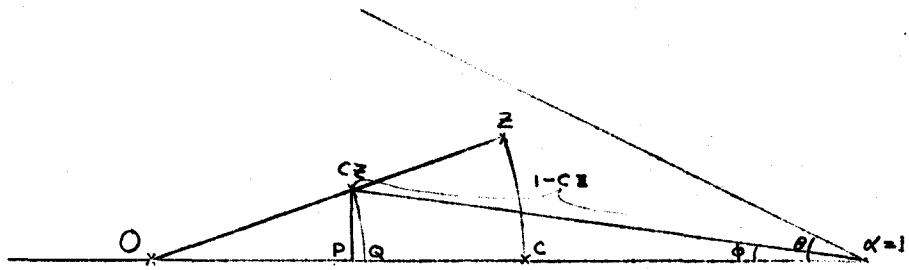

Figure 8

<u>PROPOSITION 3</u> Two different $*$-filters with angle of approach $k = \pi/2$ to the point $\alpha = 1$ give different homomorphisms.

<u>Proof</u>. We take $\mathfrak{A}^*$ and $\mathfrak{X}^*$ along with the sets $A \in \mathfrak{A}^*$, $B \in \mathfrak{X}^*$ restricted to the region between two circles C_1 and C_2 which satisfy the conditions of the previous lemmas. We have obtained $F \in \mathfrak{B}$ whose value at any $z \in A$, since $A \subseteq \tilde{A}$, is within $\rho/2$ of the value of F at some point of A_1. But on A_1, F is real with a value at least ρ. Hence $\operatorname{Re}(F(z)) \geqslant \rho/2$ for each $z \in A$. Likewise $\operatorname{Re}(F(z)) \leqslant -\rho/2$ whenever $z \in B$. The sets F(A) and F(B) in the complex plane are separated by an infinite strip of width ρ. Hence $\phi_{\mathfrak{A}}(F) \neq \phi_{\mathfrak{X}}(F)$ and $\phi_{\mathfrak{A}} \neq \phi_{\mathfrak{X}}$.

§3 GENERAL NON-TANGENTIAL APPROACH

In this section we extend the result of Proposition 3 to the case of arbitrary non-tangential approach to the point $\alpha = 1$. When this, proved in Proposition 4, is taken along with Proposition 2, we see that any non-tangential homomorphism in $\mathfrak{M}_1$ is determined by exactly one *-filter. Since any fibre $\mathfrak{M}_\alpha$ evidently has the same property, we have

<u>THEOREM</u> Each non-tangential homomorphism is determined by a unique *-filter.

As mentioned, this Theorem is a consequence of the following:

<u>PROPOSITION 4</u> Two different *-filters approaching $\alpha = 1$ along the same circle of approach C_0 give different homomorphisms.

<u>Proof</u>. In the discussion at the end of section 1, two such filter $\mathfrak{A}^*$ and $\mathfrak{B}^*$ are taken. Sets $A \epsilon \mathfrak{A}^*$ and $B \epsilon \mathfrak{B}^*$ are obtained, and traces on C_0 of these sets, which give a sequence of

$$b_1 \leqslant b_2 < a_2 \leqslant a_3 < b_3 \ldots.$$

of points on C_0 satisfying $\psi(a_n, b_n) \geqslant 4\epsilon$, for $n = 2,3 \ldots.$, for some ϵ. Recall that the sets A and B can be cut down to lie between arbitrary members of $\mathcal{C}$ above and below C_0.

Consider the conformal map L:

$$z' = L(z) = \frac{-z + ci}{ciz - 1} \qquad -1 < c < 1 .$$

where ci is the intercept of C_0 on the imaginary axis. L leaves - 1 and +1 invariant, and maps C_0 to the line segment between these points. Furthermore individual points move along circles of the family $\mathcal{L}$, and this means that the sets A and B map into sets A' and B' similar to the sets

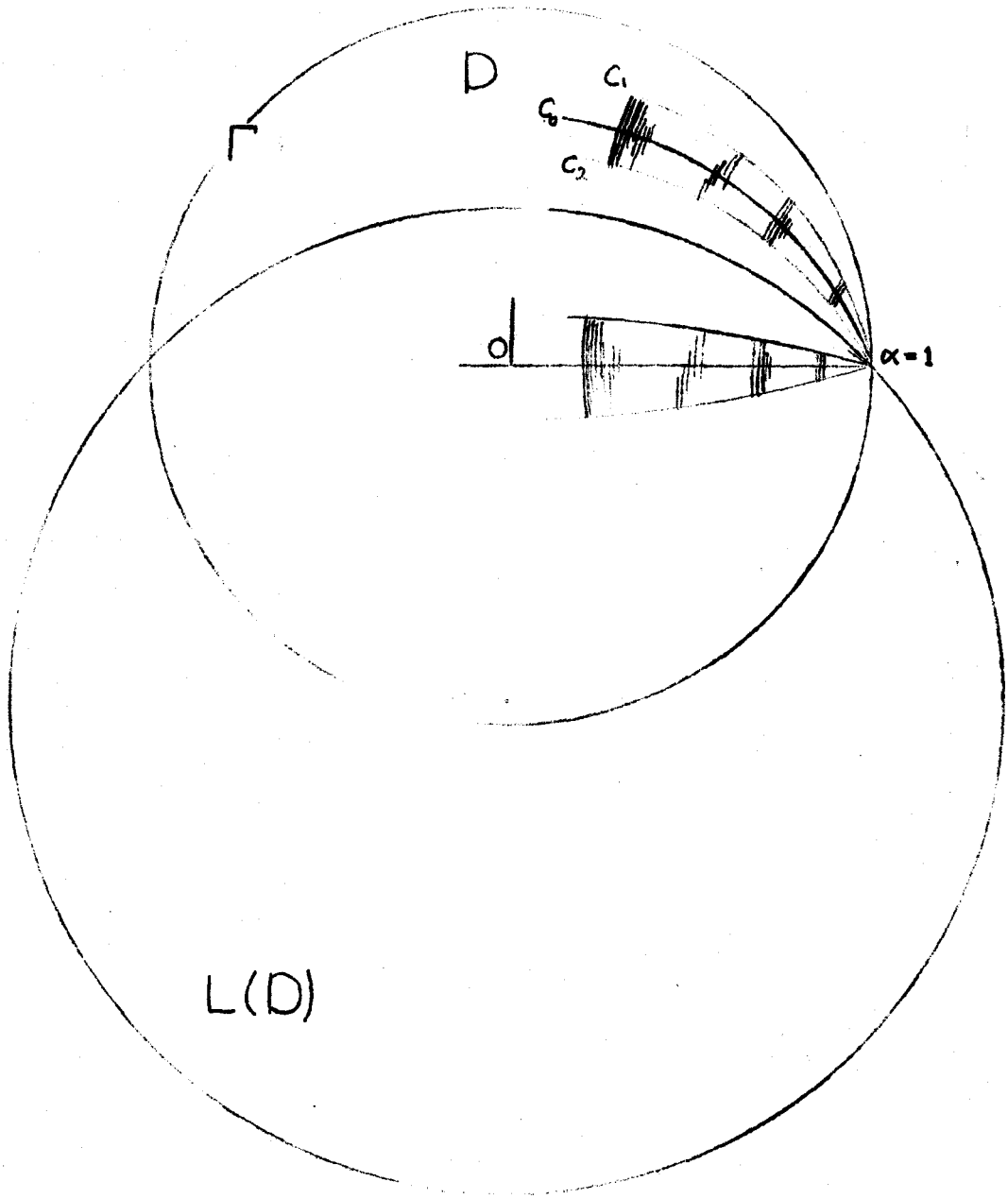

Figure 9

which are separated in Proposition 3. See Figure 9. We have a sequence of points

$$b_1' \leqslant b_2' < a_2' \leqslant a_3' < b_3' \ \ldots,$$

the images of the points in the previously mentioned sequence. We are able to use the argument of Proposition 3, defining a sequence (x_n) of points and a function F which separates A' and B'. The argument requires that A' and B' be restricted to the region between two circles of $\mathcal{C}$ above and below the real axis, but this only means an appropriate selection of C_1 and C_2 around C_0. This is because every circle through - 1 and + 1 is carried into another with the same property.

To make this method of proof valid, two points must be verified. The function F cannot properly be defined unless the points of the sequence satisfy

$$\psi(a_n b_n) \geqslant \epsilon' \quad n = 2, 3, 4, \ldots,$$

for some positive number ϵ'. This is shown in Lemma 6, and gives a Blaschke product F which separates the sets A' and B'. However we are using the composite function FoL to separate A and B, and we show in Lemma 7 that F is bounded and analytic on the set L(D). L(D) is the interior of a finite circle passing through - 1 and +1.

<u>Lemma 6</u> Let C_0 be the portion within the open unit disc of some circle passing z = -1 and z = 1. If L is the bilinear function leaving these two points fixed, and mapping C_0 to the real axis, then

$$\psi(L(z_1), L(z_2)) \geqslant k\ \psi(z_1, z_2) \quad \text{for arbitrary } z_1, z_2 \in C_0,$$

where k is a constant depending on C_0.

<u>Proof</u>. We have only to substitute in the formulae. If ci is the intersection of C_0 with the imaginary axis, we have

$$L(z) = \frac{-z + ci}{ciz - 1}$$

For $z_1, z_2 \in C_o$, the numbers $w_1 = L(z_1)$ and $w_2 = L(z_2)$ are real and

$$z_n = \frac{w_n + ci}{ciw_n + 1} \qquad n = 1,2.$$

$$\psi(z_1,z_2) = \frac{\dfrac{w_2+ci}{ciw_2+1} - \dfrac{w_1+ci}{ciw_1+1}}{1 - \dfrac{w_2-ci}{-ciw_2+1}\cdot\dfrac{w_1+ci}{ciw_1+1}}$$

$$= \left(\frac{1 + c^2}{1 - c^2}\right)\cdot\left|\frac{w_2 - w_1}{1-w_1w_2 + \dfrac{2ci}{1-c^2}(w_1-w_2)}\right|$$

$$< \left(\frac{1 + c^2}{1 - c^2}\right)\cdot\left|\frac{w_2 - w_1}{1 - w_1w_2}\right| = \frac{1}{k}\,\psi(w_1,w_2)\ ,$$

where $k = \dfrac{1 - c^2}{1 + c^2}$.

In particular, for the given sequences, we have on C_o that

$$\psi(a_n,b_n) > 4\epsilon \quad \text{when } n=2,3,4, \ldots$$

Setting $\epsilon' = 4\epsilon k$, we obtain

$$\psi(a_n',b_n') > \epsilon' \quad \text{when } n=2,3,4, \ldots$$

<u>Lemma 7</u> Given a Blaschke product

$$F(z) = \prod_{n=1}^{\infty} \frac{x_n - z}{1 - x_n z}$$

whose zeros form an increasing sequence of reals which gives an interpolating sequence. Then F is bounded and analytic in the interior of any circle passing through -1 and +1.

<u>Proof</u>. The function F is defined and analytic everywhere in the whole plane except for poles at points $1/x_n$ $n=1,2,3, ..$ and for an essential singularity at the limit point of these poles, namely at $\alpha = 1$. This

follows from the Schwarz reflection principle, since F is analytic with $|F| = 1$ everywhere on the unit circle except at $\alpha = 1$. This also follows from Theorem 12 of Chapter 0, §2.

Given a circle C through -1 and +1, (see Figure 10). The function is clearly bounded inside C except possibly in the neighbourhood of $\alpha = 1$. We draw some neighbourhood N of this point. The boundedness of F is in doubt only in that part of the interior of C which lies outside $D \cup \Gamma$ which we call R.

Let R' be the reflection of this region in the circle Γ. The values of $|F|$ inside R are just the values of $|1/F|$ in R', and hence we must prove that F is bounded away from zero on the set R'.

However this is an immediate consequence of Chapter 1, §3, Lemma 2. The pseudo-hyperbolic distances from points in R' to points of the sequence (x_n) is certainly bounded away from zero. If we draw the hypercycle H through - 1 and +1 which is tangent to C' and $\alpha = 1$, then the above distances are at least as great as the constant pseudo-hyperbolic distance between H and the real axis.

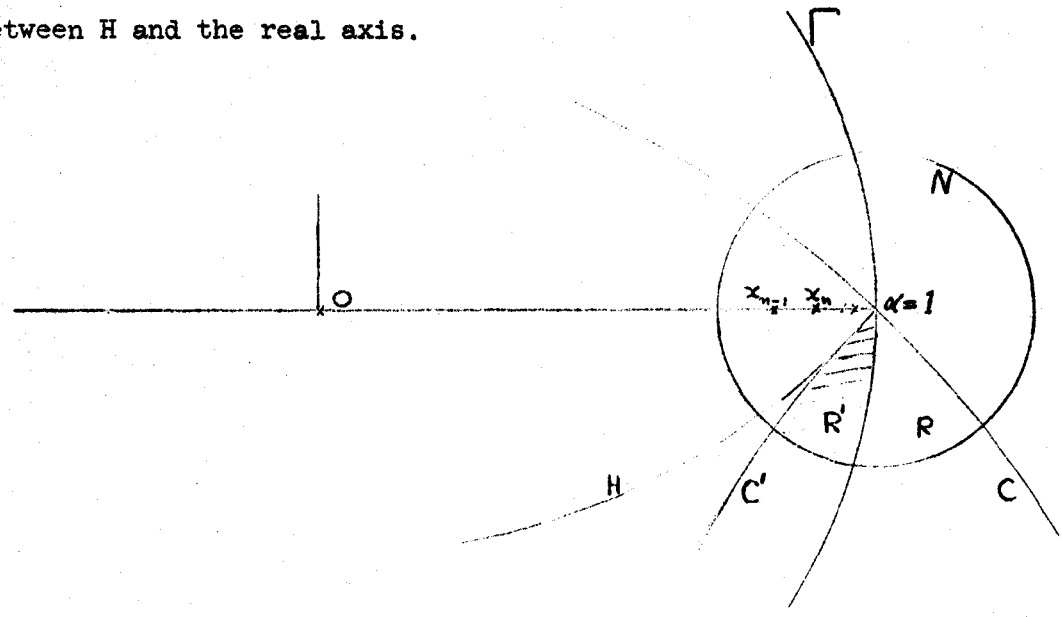

Figure 10

CHAPTER III

ORICYCLE HOMOMORPHISMS

Introduction. A second subset of $\mathfrak{M}_\alpha$ is here considered, consisting of those homomorphisms whose filters approach the point α in a fashion similar to the oricycles at α. This open subset is called $\mathfrak{D}_\alpha$. The arguments are similar to those of the last chapter, with oricycles replacing hypercycles, and it is shown that each homomorphism in $\mathfrak{D}_\alpha$ is determined by a unique *-filter.

§1 APPROACH TO THE BOUNDARY ALONG ORICYCLES

DEFINITION. A filter on D converging to some α with $|\alpha| = 1$ is called an oricycle filter if some member of it lies between two oricycles at α.

DEFINITION. A homomorphism $\phi \in \mathfrak{M}_\alpha$, $|\alpha| = 1$, is called an oricycle homomorphism if there exists a maximal closed filter $\mathfrak{N}$ on D, which is an oricycle filter, such that $\phi = \phi_{\mathfrak{N}}$. Such homomorphisms form a subset of $\mathfrak{M}_\alpha$ which we call $\mathcal{D}_\alpha$. Let $\mathcal{D} = \bigcup_{\alpha \in \Gamma} \mathcal{D}_\alpha$.

We restrict our attention to the fibre at $\alpha=1$. Any maximal closed oricycle filter must have either a set in the upper half of D, or a set in the lower half of D; this follows from the maximality of the filter. We say that $\mathfrak{N}$ approaches the point $\alpha = 1$ from above or from below respectively. In either case, we can argue with the family $\mathcal{E}$ of oricycles at $\alpha = 1$ exactly as we did with the family $\mathcal{C}$. A particular circle C_o of $\mathcal{E}$ must exist, such that for arbitrary C_1 and C_2 of $\mathcal{E}$ inside and outside C_o, some set of $\mathfrak{N}$ will lie between C_1 and C_2. Because the circles of $\mathcal{E}$ are separated by constant hyperbolic distance, we can argue, as before, that the filter $\mathfrak{N}^*$ must also have this property. If we assume that $\mathfrak{N}$ approaches $\alpha = 1$ from above, this set can be assumed to lie in the upper half of D. The filter is said to approach $\alpha = 1$ along C_o.

Any oricycle filter approaching $\alpha = 1$ from above, the case we will consider without loss of generality, will lie above any circle from the family $\mathcal{C}$. One can always restrict to a small enough neighbourhood of $\alpha = 1$ to force the oricycles enclosing some set of the filter to lie above any given member of $\mathcal{C}$ within this neighbourhood. Since the set

in the filter can be restricted to this neighbourhood, this means that the filter lies above the arbitrary circle of $\mathcal{C}$. In like fashion, an oricycle filter approaching $\alpha = 1$ from below lies below every member of $\mathcal{C}$.

<u>PROPOSITION 1</u>. If $\mathfrak{N}$ is a maximal closed filter approaching $\alpha = 1$ from above along any oricycle C_o, then any maximal closed filter $\mathfrak{X}$ with $\phi_{\mathfrak{N}} = \phi_{\mathfrak{X}}$ also has this property. The same holds if $\mathfrak{N}$ approaches $\alpha = 1$ from below.

<u>Proof</u>. The function $g = \exp(\frac{z+1}{z-1})$ is defined and analytic everywhere in $D \cup \Gamma$ except at $\alpha = 1$; $||g|| = 1$. On Γ, except at $\alpha = 1$, we have $|g| = 1$. The modulus of $f(z)$ is constant for z ranging over any fixed member of $\mathcal{E}$, with value less than 1. This value tends monotonely to zero as the oricycle becomes smaller. In particular, on the circle C_o, $|g| = k$ for some constant k with $0 < k < 1$. This means, by the same argument that was used in Chapter II, Prop. 2, that $|\phi_{\mathfrak{N}}(g)| = k$.

Suppose $\mathfrak{X}$ is another maximal closed filter converging to $\alpha = 1$ with $\phi_{\mathfrak{N}} = \phi_{\mathfrak{X}}$. If $\mathfrak{X}$ is an oricycle filter, then it is associated with some oricycle C, and for the above function g, $|\phi_{\mathfrak{X}}(g)|$ is equal to the constant value of $|g|$ on C. This means that C and C_o must be the same oricycle. To see that $\mathfrak{X}$ also approaches $\alpha = 1$ from above (from below), we can use the function f of Chapter II, Prop. 2. This is the function used to separate non-tangential filters having different angles of approach to $\alpha = 1$. A filter which lies above every member of $\mathcal{C}$ has angle of approach $k = 0$, and its homomorphism ϕ gives $|\phi(f)| = 1$. On the other hand, a filter lying below every member of $\mathcal{C}$ will give an angle of approach $k = \pi$, and the corresponding homomorphism ϕ gives $|\phi(f)| = e^{\pi}$. Since $\mathfrak{N}$ and $\mathfrak{X}$ give the same homomorphism, the two

filters must both approach $\alpha = 1$ from above (or below).

If $\mathcal{X}$ is not an oricycle filter, then two possibilities arise. $\mathcal{X}$ may lie outside every member of $\mathcal{E}$; in other words, for any oricycle from $\mathcal{E}$, some set of $\mathcal{X}$ lies outside this oricycle. In this case, $|\phi_{\mathcal{X}}(g)| = 1 \neq |\phi_{\mathcal{N}}(g)|$. Or $\mathcal{X}$ may lie inside every member of $\mathcal{E}$, and in this case $|\phi_{\mathcal{X}}(g)| = 0 \neq |\phi_{\mathcal{N}}(g)|$. Both of these contradict the hypothesis that $\phi_{\mathcal{N}} = \phi_{\mathcal{X}}$. This completes the proof.

<u>COROLLARY 1</u>. $\mathcal{D}_\alpha$ is an open subset of $\mathcal{M}_\alpha$.

Proof. We see from the above proof that

$$\mathcal{D}_1 = \{\phi \in \mathcal{M}_1 : 0 < |\hat{g}(\phi)| < 1\}$$

where $g \in \mathcal{B}$, and $\hat{g}$ is continuous on $\mathcal{M}$.

We can split each $\mathcal{D}_\alpha$ into two parts $\mathcal{D}_\alpha^+$ and $\mathcal{D}_\alpha^-$ according to whether the filter approaches α along the part of an oricycle through $\arg z > \arg \alpha$ or $\arg z < \arg \alpha$ respectively. Thus $\mathcal{D}_1^+$ ($\mathcal{D}_1^-$) consists of those homomorphisms of $\mathcal{D}_1$ whose filters approach $\alpha = 1$ from above (below). The function f used in the above Proposition completely separates these two sets whose union is $\mathcal{D}_1$. Hence we can say in general that both $\mathcal{D}_\alpha^+$ and $\mathcal{D}_\alpha^-$ are open in $\mathcal{M}_\alpha$.

<u>COROLLARY 2</u>. If $\mathcal{N}^*$ approaches $\alpha = 1$ from above (or below) on some oricycle, and $\mathcal{X}^*$ does not, then $\phi_{\mathcal{N}} \neq \phi_{\mathcal{X}}$.

This means that the main problem is to separate two filters $\mathcal{N}$ and $\mathcal{X}$ with $\mathcal{N}^* \neq \mathcal{X}^*$, both of which approach $\alpha = 1$ along the same oricycle C_o, from, let us say, above. We proceed in exactly the same fashion as before. For some $\epsilon > 0$, we can select $A \in \mathcal{N}^*$, $B \in \mathcal{X}^*$ such that $\psi(a,b) \geqslant 8\epsilon$ whenever $a \in A$ and $b \in B$. We restrict to the portion between two oricycles C_1 and C_2, lying an equal pseudo-hyperbol-

ic distance outside and inside C_o. This distance is taken to be less than 2ϵ, but the choice of C_1 and C_2 will be further restricted later. We can restrict to the first quadrant. The family of non-Euclidean straight lines perpendicular to the members of $\mathcal{E}$ consists of all circles passing through $\alpha = 1$ at right angles to Γ. Call this family $\mathcal{F}$. The families $\mathcal{E}$ and $\mathcal{F}$ are in the same relation with respect to the hyperbolic metric as were $\mathcal{C}$ and $\mathcal{B}$, and we proceed with a construction analogous to that in the previous chapter. The points of C_o can be ordered according to their nearness to $\alpha = 1$, and we can obtain traces of the sets A and B on C_o, called A_o and B_o. A sequence of points

$$b_1 \leqslant b_2 < a_2 \leqslant a_3 < b_3 \leqslant b_4 < a_4 \ \ldots\ldots$$

on C_o with

$$\psi(a_n, b_n) \geqslant 4\epsilon \qquad \text{for } n = 2,3,4, \ \ldots ,$$

is obtained. The trace sets A_o and B_o again satisfy

$$B_o \subseteq [b_1,b_2] \cup [b_3,b_4] \cup [b_5,b_6] \cup \ldots\ldots$$

$$A_o \subseteq [a_2,a_3] \cup [a_4,a_5] \cup [a_6,a_7] \cup \ldots\ldots$$

In order to construct a function which will separate the sets A and B, we map all of D conformally into the right half plane, and make use of the Poisson formula in the half plane.

§2 ORICYCLE FILTERS MAPPED INTO THE RIGHT HALF PLANE

The conformal mapping which sends the right half of the W-plane onto D has the equation $z = \frac{w - 1}{w + 1}$. The mapping is shown in Figure 11.

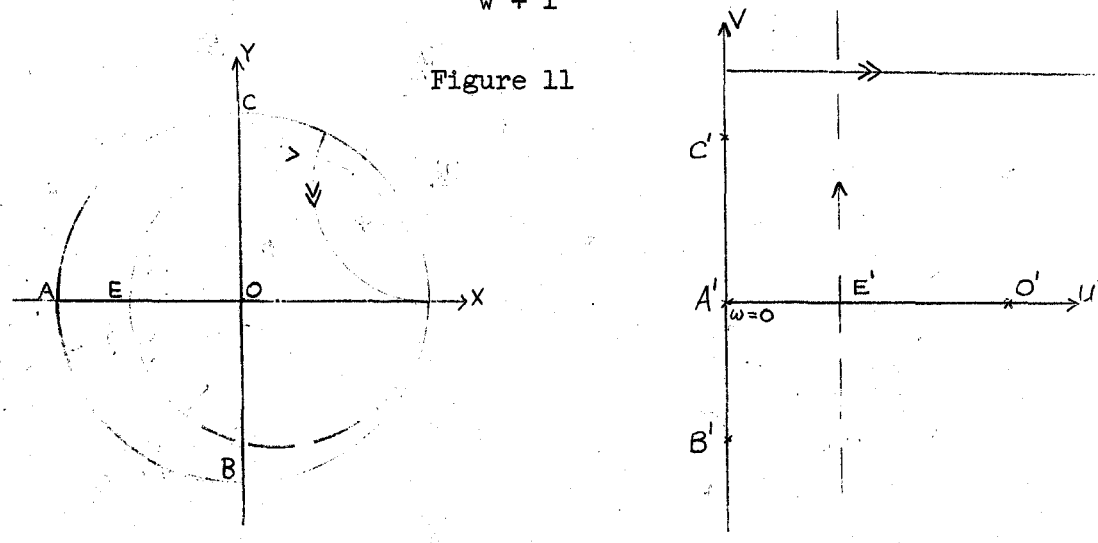

Figure 11

The boundary point $z = 1$ corresponds to the point at infinity in the W-plane. The oricycles of the family $\mathcal{E}$ are the images of vertical lines, and the circles of the family $\mathcal{F}$ are the images of horizontal lines in the right half of the W-plane.

If two points z_1 and z_2 in D have distance $\psi(z_1, z_2) = \rho$, then the corresponding points w_1 and w_2 are related by the equation

$$\left| \frac{w_1 - w_2}{\overline{w}_1 + w_2} \right| = \rho .$$

For we have

$$\rho = \left| \frac{z_1 - z_2}{1 - \overline{z}_1 z_2} \right| = \left| \frac{\frac{w_1 - 1}{w_1 + 1} - \frac{w_2 - 1}{w_2 + 1}}{1 - \left(\frac{w_1 - 1}{w_1 + 1}\right)\left(\frac{w_2 - 1}{w_2 + 1}\right)} \right| .$$

If the points z_1 and z_2 both lie on one member of $\mathcal{E}$, we have $w_1 = u + iv_1$ and $w_2 = u + iv_2$ since w_1 and w_2 must lie on the same vertical line. Hence

$$\rho = \left| \frac{i(v_1 - v_2)}{2u + i(v_1 - v_2)} \right| = \frac{| v_1 - v_2 |}{\sqrt{4u^2 + (v_1 - v_2)^2}}$$

and $\quad |v_1 - v_2| = 2u\rho / \sqrt{1 - \rho^2}$.

Observe that this is independant of the position of z_1 and z_2 on the oricycle.

We now investigate the conformal image of the sets in the two filters $\mathfrak{N}^*$ and $\mathfrak{X}^*$. Points on the trace sets A_o and B_o will all map into some vertical line C_o' with equation $u = u_o$, $u_o > 0$, in the W-plane. The sequence $b_1, b_2, a_2, a_3, b_3, b_4, a_4, a_5, \ldots$ also maps into this line. If we call the image sequence $b_1', b_2', a_2', a_3', \ldots$, then

$$\mathrm{Im}(b_1') \leqslant \mathrm{Im}(b_2') < \mathrm{Im}(a_2') \leqslant \mathrm{Im}(a_3') < \mathrm{Im}(b_3') \leqslant \ldots$$

In fact, for each $n \geqslant 2$,

$$|a_n' - b_n'| \geqslant \frac{2.u_o.4\epsilon}{\sqrt{1 - (4\epsilon)^2}} = \text{constant.}$$

If we let x_n bisect the interval between a_n' and b_n' for $n \geqslant 2$, we obtain a sequence $x_2, x_3, x_4, \ldots$ on C_o' with

$$|x_n - a_n'| \geqslant d \ , \quad |x_n - b_n'| \geqslant d \qquad n = 2,3,4, \ldots$$

where d is some positive constant, namely half the one obtained earlier. We can also insert a point x_1 d units below b_1 on C_o'.

Since the family of oricycles through $\alpha = 1$ is mapped into the family of vertical lines in the W-plane, this means that A' and B', the images of A and B respectively, can be made to lie between any two vertical lines C_1' and C_2' on either side of C_o'. This means that for an arbitrary neighbourhood N of u_o, we can choose A', for example, such that whenever $u + iv \in A'$, we have $u \in N$ and $u_o + iv \in A_o'$.

We shall construct in Proposition 2 a bounded harmonic function in the half plane $\mathrm{Re}(w) > 0$, which is positive and bounded away from zero on A' , and is negative on B' . Under the conformal map, this becomes a bounded harmonic function p on D, and if q is its conjugate, $f = \exp(p + iq) \in \mathscr{B}$ will separate A and B. To do this we use the Poisson formula in the half plane. Given a real measurable function $H(t)$ $-\infty < t < \infty$, where t is measured from the origin in the direction of the V-axis. Suppose H is integrable with respect to the measure $(1 + t^2)^{-1}dt$. Then

$$h(u + iv) = \frac{1}{\pi}\int_{-\infty}^{\infty} H(t)\frac{u}{u^2 + (t - v)^2}\,dt$$

is harmonic for $u > 0$. In our case, H is chosen so that $|H| \leqslant 1$, implying $|h| \leqslant 1$.

When dealing with any point $u + iv$, we make a change of variable in the integral by setting $s = t - v$. This means that we measure distances along the V-axis from a point P at the same height as $u + iv$. Observe that when H is constant, the integral gives rise to the inverse tangent. Suppose $H = 1$ on the interval from X to Y.

$$h(u + iv) = \frac{1}{\pi}\int_X^Y \frac{u\,ds}{u^2 + s^2} = \frac{1}{\pi}\,\mathrm{arcTan}(Y/u) - \frac{1}{\pi}\,\mathrm{arcTan}(X/u)$$

$$= T(Y/u) - T(X/u).$$

where we define, for convenience, $T(x) = \frac{1}{\pi}\,\mathrm{arcTan}(x)$.

<u>PROPOSITION 2</u>. If $\mathfrak{N}$ and $\mathfrak{X}$ are maximal closed filters approaching $\alpha = 1$ from above along the same oricycle, and if $\mathfrak{N}^* \neq \mathfrak{X}^*$, then $\phi_{\mathfrak{N}} \neq \phi_{\mathfrak{X}}$.

<u>Proof</u>. Take a sequence of points (ξ_n) on the V-axis with ξ_n and x_n at

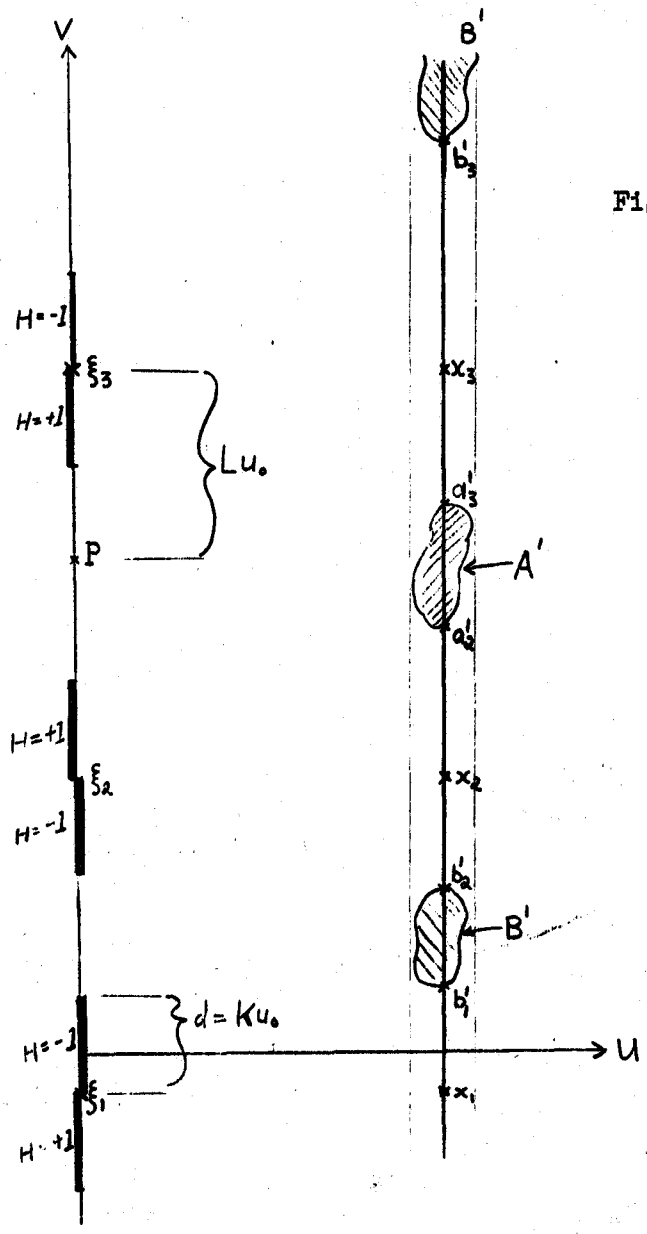

Figure 12

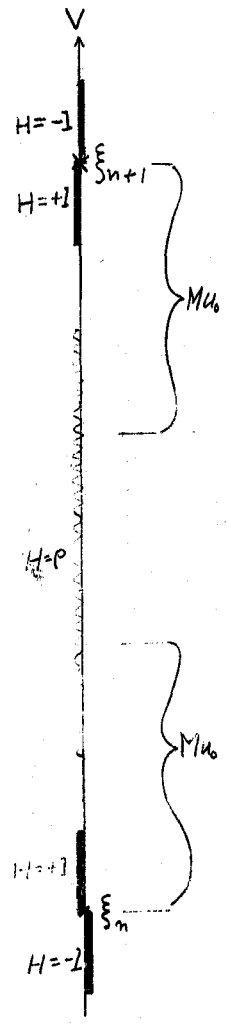

Figure 13
(n even)

the same height for each n. Define H as follows:

$$
\begin{aligned}
H(t) &= (-1)^n && \text{for } \xi_n < t < \xi_n + d\,, && n \text{ even.}\\
&= -(-1)^n && \text{for } \xi_n - d < t < \xi_n\,, && n \text{ odd.}\\
&= 0 && \text{otherwise.}
\end{aligned}
$$

This is illustrated in Figure 12. The function h is defined as the Poisson integral of H.

Take a point $u + iv$ for which $u_o + iv \in A_o'$. There corresponds at the same height on the V-axis a point P, which is above $\xi_n + d$ and below $\xi_{n+1} - d$ for some even value of n. It is convenient to write all distances on the V-axis as the product of a suitable number with u_o. Thus $d = Ku_o$ for suitable K. Suppose the points ξ_{n+1} and ξ_{n+2} are at a distance of Lu_o and L_1u_o from P respectively. We consider the contribution to h due to the values of H around these two points. This gives

$$
\left[\frac{1}{\pi}\int_{(L-K)u_o}^{Lu_o} (+u)/(u^2+s^2)\,ds \;+\; \frac{1}{\pi}\int_{Lu_o}^{(L+K)u_o} (-u)/(u^2+s^2)\,ds \right]
$$

$$
+ \left[\frac{1}{\pi}\int_{(L_1-K)u_o}^{L_1u_o} (-u)/(u^2+s^2)\,ds + \frac{1}{\pi}\int_{L_1u_o}^{(L_1+K)u_o} (+u)/(u^2+s^2)\,ds \right]
$$

$$
= \left[2T(Lu_o/u) - T\big((L-K)u_o/u\big) - T\big((L+K)u_o/u\big) \right]
$$

$$
+ \left[2T(L_1u_o/u) - T\big((L_1-K)u_o/u\big) - T\big((L_1+K)u_o/u\big) \right]
$$

$$
= 2r - 2r_1\,.
$$

where r and r_1 are shown on the following diagram.

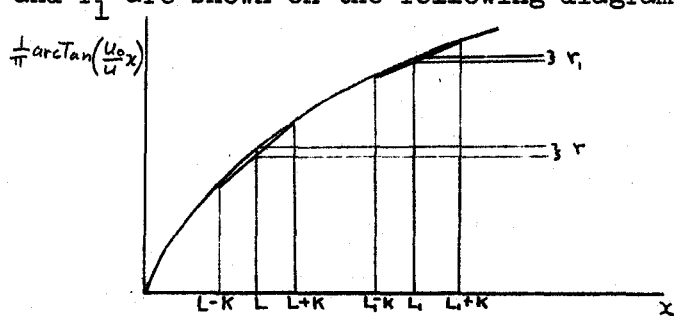

Figure 14

The quantity $2r - 2r_1$ is positive since $L_1 > L$; note also that it is a continuous function of u for $u > 0$, when the other quantities are left fixed.

All the points of the sequence (ξ_n) can be paired just as these two, including the even number lying below P. The contribution due to each pair is positive and so $h(u + iv) > 0$. (In the case of B_o', a similar pairing into negative terms is obtained; there is one term left over at ξ_1 , but the contribution from this single term is also negative.)

It is necessary to have h not only positive, but bounded away from zero on A' . This can be achieved with the present function h if we assume that the sequence $(|x_{n+1}-x_n|)$ n=1,2,3, ... is bounded; in other words if we suppose that the succeeding points of the sequence (x_n) do not become arbitrarily far apart. This means that our point $u + iv$ gives us a point P on the V-axis whose distance from ξ_{n+1} is bounded. This means, for some constant M, that $L \leqslant M$. We also have $L_1 - L \geqslant 2K$. These two conditions give, for any fixed value of u, a minimum (positive) value for $2r - 2r_1$. This is the actual value of $2r - 2r_1$ if L=M and $L_1 - L = 2K$,the worst possible case. For $u=u_o$, take this minimum value, and let ρ be any smaller positive number. This means that any point $u_o + iv \in A_o'$ gives rise to a quantity $2r - 2r_1$ such that $h(u_o + iv_o) > 2r - 2r_1 > \rho$. If we also invoke the property that $2r - 2r_1$ is a continuous function of u at $u=u_o$, we have the inequality $2r - 2r_1 > \rho$ when u is allowed to range over some neighbourhood N of u_o. When $u \in N$ and $u_o + iv \in A_o'$, we have $h(u + iv) > \rho > 0$. Since A' can be chosen so that all points in it satisfy these restrictions, this means h is bounded away from zero on A' .

If we discard the restriction that there is an upper bound for the numbers $|x_{n+1} - x_n|$ $n=1,2,3, \ldots$, the function H which we have defined on the boundary must be supplemented by inserting values at places where the distances between ξ_{n+1} and ξ_n become too large. This must be done without disturbing the inequalities already proved.

We continue in the previous setting. The point $u + iv$ gives P between ξ_n and ξ_{n+1} for n even. For arbitrary M, we can find a constant $\rho > 0$, and can deal with all points P which are below ξ_{n+1} by an amount at most Mu_o. Similarly, the previous arguments will give the required conclusion for any points P at most Mu_o above ξ_n.

Choose M such that $T(M/2) > 3/8$. This is possible since $T(x) \longrightarrow 1/2$ as $x \longrightarrow \infty$. It is possible to find a neighbourhood N_1 of u_o with $T(Mu_o/2u) > 3/8$ by the continuity of T at $u=u_o$. Corresponding to this value of M, there will exist $\rho > 0$, as in the first part of the proof. For any pair ξ_n and ξ_{n+1} separated by more than $2Mu_o$, we introduce an additional amount $(-1)^n\rho$ into the function H between $\xi_n + Mu_o/2$ and $\xi_{n+1} - Mu_o/2$. See Figure 13.

For P in the interval between ξ_n and ξ_{n+1} , the total additional contribution of negative values must come from points outside the interval, and is not more in absolute value than

$$\frac{2}{\pi} \int_{Mu_o/2}^{\infty} {}^{(\rho u)}/_{(u^2 + s^2)}\, ds = 2\rho\left[\frac{1}{2} - T(Mu_o/2) \right] < \rho/4$$

If P lies within a distance Mu_o of ξ_n or ξ_{n+1} , this means h must still be more than $\rho - \frac{\rho}{4} = \frac{3\rho}{4}$. For the remaining points P between ξ_n and ξ_{n+1} , if any exist, the additional values extend at least a distance of $Mu_o/2$ in either direction. This gives at least

$$\frac{2}{\pi}\int_0^{Mu_o/2} (\rho u)/(u^2+s^2)ds \quad = \quad 2\rho\, T(Mu_o/2) > \frac{3\rho}{4} \quad .$$

Thus the values on this part for h are at least $3\rho/4 - \rho/4 = \rho/2$, and in all cases h is at least $\rho/2$.

On the set $A' = \{u + iv : u \in N \cap N_1 \ , \ u_o + iv \in A_o'\}$, h is everywhere positive and greater than $\rho/2$. A similar argument will show that on the set $B' = \{u + iv : u \in N \cap N_1 \ , \ u_o + iv \in B_o'\}$, the function h is negative and less than $-\rho/2$, although it suffices to have h negative on B'.

As mentioned, the Proposition follows from the construction of this bounded harmonic function h which separates A' and B', because A' and B' are the images of sets A and B in D which belong respectively to $\mathfrak{N}^*$ and $\mathfrak{X}^*$.

The results of Proposition 1 and its corollaries, along with Proposition 2, give for $\mathfrak{D}_\alpha$ the result which was proved in the previous chapter for $\mathfrak{N}_\alpha$.

<u>THEOREM</u>. Any oricycle homomorphism of $\mathfrak{B}$ is determined by a unique *-filter.

CHAPTER IV

THE SILOV BOUNDARY

Introduction. The Silov boundary of $\mathcal{B}$ was identified in the paper of I. J. Schark [23], and this material is given in Propositions 1 and 2. The remaining discussion treats the homomorphisms of the Silov boundary in terms of filters. It is shown that a single such homomorphism is determined by many different *-filters. Some topological properties of the Silov boundary as a subspace are given in the third section.

§1 ESSENTIALLY BOUNDED MEASURABLE FUNCTIONS ON Γ

We have seen in Theorem 2 of Chapter 0, §2, that $\mathscr{B}$ and H^∞ are isomorphic as Banach spaces. The isomorphism in one direction is obtained by mapping the functions in H^∞ into their Poisson integrals. In the opposite direction, one maps a bounded analytic function into its limit function on the boundary. This exists almost everywhere in virtue of the Fatou Theorem.

<u>PROPOSITION 1</u>. The mapping τ which assigns to each $f \in \mathscr{B}$ the function F defined for almost all θ in the interval $-\pi \leqslant \theta \leqslant \pi$ by

$$F(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

is a Banach algebra isomorphism of $\mathscr{B}$ into L^∞.

<u>Proof</u>. Given $f, g \in \mathscr{B}$, we have seen from the Poisson formula that

$$(f \pm g)(re^{i\theta}) \longrightarrow (F \pm G)(e^{i\theta}),$$

except on a set of measure zero. Since $F = \tau f$ and $G = \tau g$, this means $\tau(f \pm g) = \tau f + \tau g$. An identical argument gives $\tau(f.g) = \tau f.\tau g$ because, for almost all θ,

$$(f.g)(re^{i\theta}) \longrightarrow (F.G)(e^{i\theta}).$$

In other words, τ is an algebra homomorphism. As we remarked in Chapter 0, it is also norm preserving. This can be deduced directly from the Poisson representation. We have, first of all

$$|f(re^{i\phi})| = \left|\frac{1}{\pi} \int_{-\pi}^{\pi} F(e^{i\theta}) P_r(\theta - \phi) d\theta\right| \leqslant \frac{1}{2\pi} . ||F|| . 2\pi = ||F|| .$$

where $||F||$ is the essential supremum on Γ. Thus $||f|| \leqslant ||F||$. Take now any $\epsilon > 0$. For almost all points of the boundary, $F(e^{i\theta})$ is defined, and for each of these we can find $z \in D$ with $|f(z)| \geqslant$

$|F(e^{i\theta})| - \epsilon$. From this we see that $|F|$ is almost everywhere less than $||f|| + \epsilon$. Hence $||F|| \leqslant ||f|| + \epsilon$ for arbitrary ϵ; and $||F|| \leqslant ||f||$. Thus $||F|| = ||f||$.

Recall that the space X of homomorphisms of L^∞ is a totally disconnected space, and that L^∞ is isomorphic and isometric with the ring of all continuous functions on X. We denote by $\hat{F}$ the function on X corresponding to any f under the isomorphism. See the discussion in Chapter 1, §1.

<u>PROPOSITION 2</u>. The injection τ of $\mathscr{B}$ in L^∞ induces a homeomorphism $\tau' : X \longrightarrow \mathfrak{M}$ which has as its image set the Silov boundary of $\mathscr{B}$.

<u>Proof</u>. Because $\tau : \mathscr{B} \longrightarrow L^\infty$ is an algebra homomorphism, it induces a continuous mapping τ' from X into $\mathfrak{M}$ defined for any $x \in X$ by

$$\tau'(x)(f) = x(\tau f) \qquad f \in \mathscr{B}.$$

Take two points x_1 , $x_2 \in X$ and let U be a basic open-closed neighbourhood of x_1 which does not contain x_2 . This set, by Chapter 0, §1, Theorem 8, has the form

$$U = X_E = \{x \in X : \hat{\chi}_E(x) = 1\}$$

where $E \subseteq \Gamma$ is a set with positive measure.

If u is the harmonic extension of χ_E to the disc, v its harmonic conjugate, and $f = \exp(u + iv)$, then $f \in \mathscr{B}$, and $|F| = \exp(\chi_E)$ almost everywhere on Γ , where $F = \tau f$. Hence

(1) $$|\hat{F}| = e \quad \text{on } U.$$
$$= 1 \quad \text{on } X \setminus U.$$

We have for any $x \in X$ and $f \in \mathscr{B}$ that

(2) $$\hat{f}(\tau' x) = \tau'(x)(f) = x(\tau f) = \hat{F}(x) .$$

Hence for the above defined f, we see that

$$\hat{f}(\tau' x_1) = e \ , \qquad \hat{f}(\tau' x_2) = 1 \ .$$

Since x_1 and x_2 were arbitrary, this shows that τ' is 1-1. By the compactness of X, this is sufficient to show that τ' is a homeomorphism.

Every function in $\mathcal{B}$ attains its maximum on $\tau'(X)$. We see this from (2). Choose $x \in X$ on which the function $|\hat{F}|$ attains its maximum value, where $F = \tau f$. This maximum value is of course $||F||$. Then

$$|\hat{f}(\tau' x)| = |\hat{F}(x)| = ||F|| = ||f|| \ ,$$

and $\hat{f}$ attains its maximum at $\tau' x$.

Suppose we now take any proper closed subset of $\tau'(X)$, which we write as $\tau'(C)$ where C is a closed subset of X. We can choose an open-closed set $U \neq \emptyset$ in the set $X \setminus C$. By the construction of the first part, for any open-closed set U there exists $f \in \mathcal{B}$ for which $F = \tau f$ satisfies (1). The modulus of the function $\hat{f}$ has constant value 1 on $\tau'(C)$, but on $\tau'(U)$ it has the value e. Thus $\hat{f}$ does not attain its maximum value on $\tau'(C)$. It is evident from these arguments that $\tau'(X)$ is the Silov boundary of $\mathcal{B}$ — the smallest closed subset of $\mathfrak{M}$ on which each $\hat{f}$, for $f \in \mathcal{B}$, attains its maximum modulus.

By the maximum principle, Δ does not meet the Silov boundary of $\mathcal{B}$. Hence τ' maps into the set $\bigcup_{|\alpha|=1} \mathfrak{M}_\alpha$. If we fibre X according to the formula

$$X_\alpha = \{x \in X : \hat{z}(x) = \alpha\} \quad \text{for } |\alpha| = 1 \ ,$$

it is clear that τ' maps X_α into $\mathfrak{M}_\alpha$.

If we have $F \in L^\infty$ which is defined and continuous at $\alpha \in \Gamma$ with value ζ, then $\hat{F}$ is constant with value ζ on X_α. For any complex number η fails to be in the range of $\hat{F}$ on X_α if and only if $F-\eta$ and $z-\alpha$ do not both lie in any proper ideal of L^∞. In other words if and

only if the functions $g,h \in L^{\infty}$ can be chosen such that

$$(F - \eta)g + (z - \alpha)h = 1$$

This is possible if and only if $\eta \neq \zeta$. Hence the range of $\hat{F}$ on X_{α} consists of the one value ζ.

Because X is a Stone space, its points can be considered as maximal filters in the Boolean lattice of open-closed sets of X. This was discussed in Chapter 0, §2. The open-closed sets of X, which generate the topology, are all of the form

$$X_E = \{x \in X : \hat{\chi}_E(x) = 1\} ,$$

where $E \subseteq \Gamma$ has positive measure. The open-closed sets making up any of the maximal filters also provide a neighbourhood basis for the homomorphism which corresponds to the filter.

The Boolean lattice in question is isomorphic to the lattice $\mathcal{M}'$ of all measurable subsets of Γ modulo sets of measure zero. Hence X is homeomorphic to the space $\mathcal{X}'$ of all maximal filters on $\mathcal{M}'$. Recall that $\mathcal{X}'$ has the hull kernel topology; the closure of any collection of maximal filters in $\mathcal{X}'$ consists of all maximal filters in $\mathcal{X}'$ which contain their intersection.

Let $\mathcal{M}$ be the collection of all measurable subsets of Γ, and θ be the natural homomorphism

$$\theta : \mathcal{M} \longrightarrow \mathcal{M}'$$

which factors out the ideal $\mathcal{Z}$ of sets of zero Lebesgue measure. It is useful to deal with the inverse images under θ of the maximal filters on $\mathcal{M}'$. This gives a set $\mathcal{X}$ of filters on $\mathcal{M}$ which consists of all maximal filters on $\mathcal{M}$ that do not meet $\mathcal{Z}$. The topology of $\mathcal{X}$ is still hull kernel. If a set E belongs to such a filter, then so do all sets F

differing from E only on a set of measure zero. This enables us to discard, for any $F \in L^\infty$, a set of measure zero on which the function is not defined, when we examine the limit of F along a filter from $\mathcal{H}$.

Let us denote by M_x the filter in $\mathcal{H}$ corresponding to $x \in X$ under the homeomorphism induced by the lattice isomorphism between the two lattices of open-closed sets. We will show that $\hat{F}(x) = \lim F(M_x)$. The correspondence between open-closed sets is given by $\{x : \hat{\chi}_E(x)=1\} \longrightarrow \{M_x : E \in M_x\}$. Hence

$$(3) \qquad \{x : E \in M_x\} = \{x : \hat{\chi}_E(x)=1 = \{x : \hat{\chi}_{\Gamma \setminus E}(x) = 0\} .$$

Any filter $M \in \mathcal{H}$ must converge to some point $\alpha \in \Gamma$. The open sets are all measurable, so a maximal filter of measurable subsets on the compact space Γ must contain a neighbourhood filter. We note that some member of $\mathcal{H}$ must lie in a half neighbourhood of α.

<u>PROPOSITION 3</u>. Given $F \in L^\infty$, and a set $E \subseteq \Gamma$ of positive measure. Then ζ is an essential value of F on E if and only if, for some $x \in X$ with $E \in M_x$, we have $\hat{F}(x) = \zeta$.

<u>Proof</u>. We first give a proof of the proposition for the case $E = \Gamma$. In other words: F has ζ for an essential value if and only if $\hat{F}(x) = \zeta$ for some $x \in X$. Take $\zeta = 0$ without loss of generality. $\hat{F}$ is never zero on X if and only if F has an inverse in L^∞, and this happens if and only if F is essentially bounded away from zero on Γ.

Suppose now $E \subset \Gamma$, and let F have essential value $\zeta=0$ on E. Denote by $\complement E$ the complement of E in Γ, and let $k = ||F||$. Suppose that on the set $\{x : E \in M_x\}$, $\hat{F}$ is never zero. By (3), this is the set $\{x : \hat{\chi}_{\complement E}(x)=0\}$. On the complement of this set, the function $\hat{\chi}_{\complement E}$ is

equal to 1. The following function must be non-zero on all of X:

$$2k\hat{\chi}_{\complement E} + \hat{F} .$$

This means that $2k\chi_{\complement E} + F$ is invertible in L^∞. But on the set E, $2k\chi_{\complement E} + F = F$ has an essential zero, which is a contradiction.

Conversely, suppose that $\hat{F}(x) = \zeta$ for some x with $E \in M_x$. The function $2k\hat{\chi}_{\complement E} + \hat{F}$ equals ζ at x, and therefore $2k\chi_{\complement E} + F$ must have ζ for an essential value. But this function is never equal to ζ on $\complement E$, and on E it is the same as F. This means F has ζ for essential value on E.

<u>PROPOSITION 4</u>. For each $y \in X$,

$$\hat{F}(y) = \lim F(M_y) \qquad \text{for any } F \in L^\infty.$$

<u>Proof</u>. Let $y \in X$ be arbitrary and take any $F \in L^\infty$. We give a proof for the case $\hat{F}(y)=0$, and the proposition for general $F \in L^\infty$ follows easily. Since $\hat{F}$ is continuous on X, we can, for arbitrary $\epsilon > 0$, find a neighbourhood of y on which $|\hat{F}| < \epsilon$. We let U be a basic open-closed neighbourhood of y lying inside this.

$$U = \{x \in X : \hat{\chi}_E(x) = 1\}$$

for some measurable set $E \subseteq \Gamma$. From (3) we know that

$$U = \{x \in X : E \in M_x\} .$$

Hence for each x with $E \in M_x$, we have $|\hat{F}(x)| \leqslant \epsilon$. By Proposition 2, the essential values of F on E are all of modulus at most ϵ. In other words

$$|\hat{F}(t)| \leqslant \epsilon \quad \text{for almost all } t \in E.$$

We can remove the set of measure zero on which $|F| > \epsilon$ from E, and then we have $|F| \leqslant \epsilon$ on some set $E_1 \in M_x$. Since ϵ was arbitrary, it follows that $\lim F(M_x) = 0$.

<u>COROLLARY</u>. A point $y \in X_\alpha$ if and only if M_y converges to α.

<u>Proof</u>. Suppose that, in the above Proposition $y \in X_\alpha$. Then every one of the sets E chosen above must meet every neighbourhood in Γ of α in a set of positive measure. Otherwise χ_E can be made constantly equal to zero on a neighbourhood of α by changing χ_E on a set of measure zero. In L^∞ this gives an equivalent function. This would mean that $\hat{\chi}_E$ is identically zero on X_α, contrary to the assumption that $y \in U$. In other words, if $y \in X_\alpha$, then M_y converges to α.

Conversely, if M_y converges to α, we can take any $\beta \neq \alpha$, and prove that $y \notin X_\beta$. Since $\beta \neq \alpha$, we can pick an open neighbourhood E of α in Γ such that $\beta \in \bar{E}$. The set $U = \{x \in X : \hat{\chi}_E(x) = 1\}$ contains y, but it does not meet X_β since χ_E is continuous with value zero at β, so that $\hat{\chi}_E | X_\beta = 0$.

§2 SILOV BOUNDARY FILTERS

We agree to identify X and $\tau'(X)$, where τ' is the homeomorphism embedding X in $\mathfrak{M}$. Thus X will henceforth be the Silov boundary of $\mathcal{B}$, a subset of $\mathfrak{M}$. We note that, with this agreement,

$$X_\alpha = \mathfrak{M}_\alpha \cap X \quad .$$

To each $x \in X$ corresponds a filter M_x on Γ. We use these M_x to construct filters on D which correspond to the homomorphisms of X.

We take the case $\alpha=1$. Recall that $\mathcal{E}$ represents the family of oricycles through $\alpha=1$, and that $\mathcal{F}$ represents the family of circles orthogonal to the members of $\mathcal{E}$, and passing through $\alpha=1$. For any M_x, with $x \in X_1$, we know by the maximality of the filter that some set in the filter will belong to the upper half of Γ, or to the lower half of Γ. We assume the former. Then M_x has a basis consisting of sets in the upper half of Γ. Take any such set E. We form a set $\tilde{E} \subseteq D$ as follows (see Figure 15) : For each $\alpha \in E$, take the member of $\mathcal{F}$ through α, and take some open arc of this circle in D, which terminates at α. The set $\tilde{E}$ will be the closure of the union of these arcs, each of arbitrary non-zero length, corresponding to each $\alpha \in E$. We take the collection of all E corresponding to each $E \in M_x$, and then let E range over all M_x. This gives a filter basis for a filter which we call $\mathfrak{n}_x$. Although the description only applies to homomorphisms in X_1, it is obvious that we can obtain, for any $x \in X$, a similar filter $\mathfrak{n}_x$.

<u>PROPOSITION 5</u>. Each $f \in \mathcal{B}$ has a limit along $\mathfrak{n}_x$ for each $x \in X$. If $F \in L^\infty$ is the boundary function for f, then

$$\lim f(\mathfrak{n}_x) = \lim F(M_x) = \hat{f}(x) \quad .$$

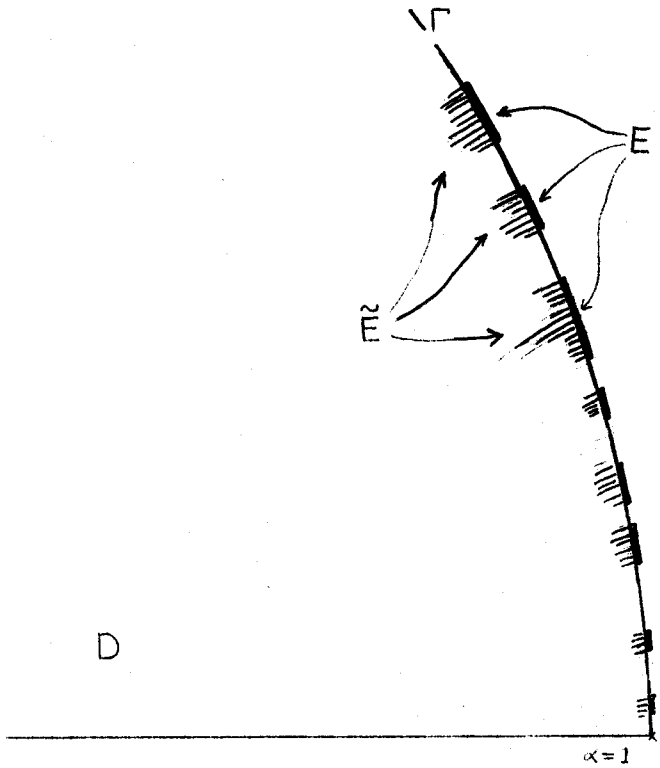

Figure 15

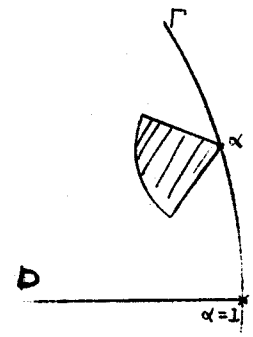

Figure 16

<u>Proof</u>. We take the case $x \in X_1$. Suppose $\hat{f}(x)=0$. The function $\hat{F}$ defined on X is just $\hat{f}|X$, since we have identified X with the Silov boundary of $\mathcal{B}$. Hence $\hat{F}(x)=0$ and F approaches zero along M_x. For any $\epsilon > 0$, we can find $E \in M_x$ such that F is defined on E, and

$$|F(t)| < \epsilon/2 \quad \text{for} \quad t \in E \quad .$$

The function f has a limit, given by F, as we approach any point of E along a radial line. But we know (Theorem 3, Chapter 0, §2) that this implies a limit along any non-tangential path leading to this point. In particular, on the member of $\mathcal{F}$ leading to $\alpha \in E$, we can choose a suitable open arc I, terminating at α, with

$$|f(z) - F(\alpha)| < \epsilon/2 \quad \text{for} \quad z \in I \ .$$

Hence we have $|f(z)| < \epsilon$ on the union of all the arcs I leading to

points of E. The closure of this union gives a set $\tilde{E}$ in $\mathfrak{N}_x$, and we have

$$|f(z)| \leq \epsilon \quad \text{for all} \quad z \in \tilde{E} \ .$$

This means that $\lim f(\mathfrak{N}_x)$ exists and equals zero, proving the Proposition in the special case $\hat{f}(x)=0$. The general case reduces to this when we define a new function $g = f-\hat{f}(x)$.

Remark: We see that the sets of the filter are rather large. In the Theorem which follows, we show that many different *-filters contain any single $\mathfrak{N}_x$ .We first observe, however, that we could further increase the size of the sets of the filter $\mathfrak{N}_x$, to obtain a filter which still satisfies Proposition 5. We made use of a result which states, more precisely, that whenever a bounded analytic function f has a radial limit at $\alpha\epsilon\Gamma$, (which occurs for almost all α), the function tends uniformly to this limit inside any two chords of the circle Γ, terminating at α. Instead of taking arcs of circles terminating at α, we could equally well have taken sectors of circles, as shown in Figure 16, symmetric about the radial line to α. A set in the new filter would consist of the closure of the union of such sectors, where α ranges over a set $E\epsilon M_x$. And f must still have a limit along such a filter, which would be equal to $\lim F(M_x)$.

<u>THEOREM</u>. Each homomorphism of the Silov boundary of $\mathcal{B}$ is determined by at least $2^{2^{\aleph_0}}$ different *-filters.

<u>Proof</u>. Take $x\epsilon X$, which we assume to belong to X_1. Pick any oricycle O_1 at $\alpha=1$, and then form the sequence $O_1, O_2, O_3, \ldots$ of oricycles at $\alpha=1$, increasing in size, and with a fixed pseudo-hyperbolic distance, say d, between O_n and O_{n+1} for $n=1,2,3, \ldots$.

Take any ultrafilter $\mathcal{U}$ on the set N of natural numbers. If

$\theta : N \longrightarrow (O_n)$ is the natural 1-1 correspondence of N onto the sequence of oricycles, then θ maps $\mathcal{U}$ into a filter on D. Choose an arbitrary set $\tilde{E}$ from the filter $\mathfrak{a}_x$, and an arbitrary set $U \in \mathcal{U}$. The set $\theta(U) \cap \tilde{E}$ is a closed set; it is the trace on $\tilde{E}$ of a certain infinite collection of oricycles. If we let $\tilde{E}$ vary over $\mathfrak{a}_x$, and U vary over $\mathcal{U}$, we obtain a closed filter on D. This filter contains $\mathfrak{a}_x$, and any maximal closed filter above it will determine the homomorphism x.

Suppose a different filter $\mathcal{V}$ on N is chosen. Since sets $U \in \mathcal{U}$, and $V \in \mathcal{V}$ can be chosen which are disjoint, the points of $\theta(U)$ and $\theta(V)$ are separated in the pseudo-hyperbolic metric. If $\lambda \in O(U)$ and $\mu \in O(V)$, then $\psi(\lambda, \mu) \geqslant d$. This leads us to conclude that the maximal closed filter obtained as above from $\mathcal{V}$, and the maximal closed filter we have already obtained from $\mathcal{U}$, must also have this separation property. This in turn means that their *-filters cannot be the same. Hence each ultrafilter on N gives rise to a different *-filter associated with x, and the Theorem follows, because the number of distinct ultrafilters on N is $2^{2^{\aleph_0}}$ (see [3]).

§3 TOPOLOGICAL DISCUSSION OF X

The topology of $\mathfrak{M}$ is considered in Chapter VI. At this point, however, we make some remarks about the topology of $\mathfrak{M}$ which have special reference to the Silov boundary.

We can split each fibre X_α into two parts X_α^+ and X_α^-, just as was done with $\mathfrak{D}_\alpha$ in Chapter III. Take the case $\alpha=1$. If we denote by $\mathfrak{X}_\alpha$ the filters in $\mathfrak{X}$ converging to α, we have for each $x \in X_1$ a filter $M_x \in \mathfrak{X}_1$. If this filter contains a set lying entirely in the upper half of Γ, then we say that $x \in X_1^+$. If, on the other hand, some set of M_x lies entirely in the lower half of Γ, we say $x \in X_1^-$. By the maximality of M_x, on of these cases will hold, and it is surely impossible for both to hold. We might observe that if $x \in X_1^+$, then $\mathfrak{a}_x$ lies above any hypercycle from the family $\mathcal{C}$, and indeed it lies above (outside) any oricycle from the family $\mathcal{E}$. Likewise any $x \in X_1^-$ lies below the hypercycles of $\mathcal{C}$ and also below the oricycles of $\mathcal{E}$.

The set X is closed. Since each $\mathfrak{M}_\alpha$ is closed, each fibre X_α is closed. In addition, we can separate the sets X_α^+ and X_α^- just as we did the sets $\mathfrak{D}_\alpha^+$ and $\mathfrak{D}_\alpha^-$ in Chapter III, so that both X_α^+ and X_α^- are closed.

Suppose we take a set E of positive measure in Γ, and consider the two sets

$$X_E = \{x \in X : \hat{\chi}_E(x) = 1\}$$
$$X_{E,\alpha} = \{x \in X_\alpha : \hat{\chi}_E(x) = 1\}$$

We show that the closure of the set $X_E \setminus X_{E,\alpha}$ contains $X_{E,\alpha}$; it is in fact X_E. This is perhaps best seen in terms of filters in $\mathfrak{X}$. We let

$$\mathcal{H}_E = \{M_x \in \mathcal{H} : E \in M_x\}$$
$$\mathcal{H}_{E;\alpha} = \{M_x : x \in X_\alpha , E \in M_x\}$$

be the corresponding sets in the homeomorphic space $\mathcal{H}$. We wish to show that the set $\mathcal{H}_E \setminus \mathcal{H}_{E,\alpha}$ has for its closure $\mathcal{H}_E$, where the topology is the hull kernel topology. It is sufficient to show that both have the same kernel, since $\mathcal{H}_E$ is clearly closed. This common kernel is the filter in $\mathcal{M}$ generated by E along with all sets differing from E on a set of measure xero. The kernel of $_E \quad _{E,\alpha}$ is no bigger than this, for if $F \subset E$ for a measurable set F with $mF < mE$, there always exists a maximal filter M_x , which converges to some point different from α, containing E, but not F.

We apply the above for the special sets $\Gamma, \Gamma^+ = \{\alpha \in \Gamma : \mathrm{Im}(\alpha) > 0\}$ $\Gamma^- = \alpha \in \Gamma : \mathrm{Im}(\alpha) < 0\}$, and obtain the following:

$$X_1 \subseteq (\overline{\bigcup_{\alpha=1} X_\alpha}) , \quad X_1^+ \subseteq (\overline{\bigcup_{\mathrm{Im}(\alpha)>0} X_\alpha}) , \quad X_1^- \subseteq (\overline{\bigcup_{\mathrm{Im}(\alpha)<0} X_\alpha})$$

Consider now the closed set $X_\alpha \subseteq \mathfrak{M}_\alpha$. We show

<u>PROPOSITION 6</u>. X_α has no interior in $\mathfrak{M}_\alpha$.

<u>Proof</u>. Assume $\alpha=1$. Take $\phi_o \in X_1$, and an arbitrary open set U containing ϕ_o . U has the form

$$U = \{\phi \in \mathfrak{M}_1 : |\hat{f}_i(\phi) - \hat{f}_i(\phi_o)| < \epsilon , i=1,2, \quad ,m\}$$

We can assume $f_1, \quad ,f_m$ are chosen such that

$$f_i(\phi_o) = 0 \qquad i=1,2, \quad ,m.$$

By considering the filter M_{ϕ_o} on Γ which corresponds to ϕ_o , we can obtain a sequence $E_1, E_2, \ldots$ of subsets of Γ, each of positive measure, such that

$$|F_i(\alpha)| < 1/n \quad \text{for} \quad \alpha \in E_n \qquad i=1,2, \quad ,m.$$

F_i is the boundary function for f_i.

This can be done simultaneously for the m functions, and we can

discard the set of measure zero where they are not all defined. For each n, we can choose $\alpha_n \epsilon E_n$, and a point λ_n on the radial line to α_n such that $|f_i(\lambda_n)| < 1/n$ i=1, ,m . We also take $|\lambda_n|$ close enough to 1 so that $\sum_{n=1}^{\infty} (1-|\lambda_n|) < \infty$.

If we take any maximal closed filter $\mathfrak{M}$ on D which contains the set $\{\lambda_1, \lambda_2, \ldots\}$, it will determine a homomorphism which is not in the Silov boundary. This is because the Blaschke product B defined on the sequence (λ_n) is annihilated by this homomorphism ϕ. Since $f_i(\lambda_n) \longrightarrow 0$ as $n \longrightarrow \infty$, we have $\hat{f}_i(\phi) = 0$ for i=1, ,m , so that $\phi \epsilon U$.

Hence any open set U containing ϕ_o contains homomorphisms that are not in the Silov boundary.

§4 A CHARACTERIZATION OF THE SILOV BOUNDARY

We insert at this point a result of D. J. Newman [19]; it is also discussed in [14]. Every homomorphism of $\mathscr{B}$ which is not in the Silov boundary will annihilate a Blaschke product. We give this in Proposition 9. Since $\phi(b)$ always has modulus unity, if $\phi \in X$ and b is a Blaschke product, it is evident that the converse holds also, and this gives a characterization of the Silov boundary.

<u>PROPOSITION 7</u>. If a homomorphism $\phi \in \mathfrak{M}$ is non-zero for every Blaschke product, then for each inner function g, we have

$$|\phi(g)| = 1 .$$

<u>Proof</u>. Every inner function is the uniform limit of Blaschke products, by Theorem 15, Chapter 0, §2. Hence we need only give a proof for these special inner functions. The claim is obvious for any finite Blaschke product; we know $\phi \notin \Delta$, and therefore ϕ is in some fibre, which means $|\phi(z)|=1$. This means $|\phi(\ (\alpha-z)/(1-\bar{\alpha}z)\ )| = 1$ for any $\alpha \in D$. Take now an infinite Blaschke product.

$$b(z) = \prod_{n=1}^{\infty} \left(\frac{\bar{\alpha}_n}{|\alpha_n|}\right)\left(\frac{\alpha_n - z}{1-\bar{\alpha}_n z}\right) = \prod_{n=1}^{\infty} b_n(z) \quad , \quad |\alpha_n| < 1 \ , \Sigma(1-|\alpha_n|) < \infty$$

Choose a non-decreasing sequence of integers (p_n) with $\lim_{n\to\infty} p_n = \infty$, and $\Sigma p_n(1-|\alpha_n|) < \infty$.

For any positive integer N we may define

$$\tilde{b}(z) = \prod_{n=1}^{\infty} [b_n(z)]^{p_n} \qquad P_N(z) = \prod_{n=1}^{N} b_n(z)^{p_n}$$

$$Q_N(z) = \prod_{n=N+1}^{\infty} [b_n(z)]^{p_n - p_N}$$

Then we have $\tilde{b} = b_1^{p_N} \cdot P_N \cdot Q_N$.

Because $|\phi(P_N)| = 1$ and $|\phi(Q_N)| \leq 1$, we have $|\phi(\tilde{b})| \leq |\phi(b_1)|^{p_N}$. The left side is strictly greater than zero, and is fixed. The right side, which is the same as $|\phi(b)|^{p_N}$, must have the value 1, since p_N can be increased indefinitely. Hence $|\phi(b)| = |\phi(b_1)| = 1$.

<u>PROPOSITION 8</u>. Each $\phi \in \mathfrak{M}$ has a unique Hahn-Banach extension to a linear functional Ψ in $(L^\infty)^*$, given by

$$\Psi(f) = \ell(\mathrm{Re}(f)) + i\ell(\mathrm{Im}(f)) \quad ,$$

where ℓ is a bounded non-negative linear functional on the space of real valued L^∞ functions. It is defined by

$$\ell(u) = \log|\phi(F_u)| \ , \ F_u(z) = \exp[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} u(e^{i\theta})d\theta] \ ,$$

for any real valued u in L^∞. F_u is in $\mathcal{B}$ and is a unit.

<u>Proof</u>. The linearity of ℓ is immediate; its boundedness follows from the inequality, if $|u| \leq 1$, $1/e \leq |F_u| \leq e$, which gives $1/e \leq |\phi(F_u)| \leq e$. The first part of this inequality depends on the fact that F_u is invertible ($F_u^{-1} = F_{-u}$). Being the outer function of an essentially bounded u, it is evidently in $\mathcal{B}$, so that $\phi(F_u)$ is well-defined. The last inequality shows, by taking logarithms, that $-1 \leq \ell(u) \leq 1$, so that $||\ell|| \leq 1$. However $||\ell||$ is actually equal to 1 in view of the fact that $\ell(1) = 1$. To see that ℓ is non-negative, take $u \geq 0$. This means $|F_u| \geq 1$, and therefore $|\phi(F_u)| \geq 1$. Conclusion, $\ell(u) \geq 0$.

Take now $f=u+iv \in H^\infty$, and assume, without loss of generality, that $f(0)=0$. Then

$$F_u = e^f \qquad\qquad F_v = e^{-if}$$

Hence $\Psi(f) = \ell(u) + i\ell(v)$

$$= \log|\phi(e^{f})| + i.\log|\phi(e^{-if})|$$

$$= \log|e^{\phi(f)}| + i.\log|e^{-i\phi(f)}|$$

$$= \mathrm{Re}(\phi(f)) + i.\mathrm{Re}(-i\phi(f))$$

$$= \phi(f)$$

The functional Ψ is indeed an extension of ϕ. Suppose Ψ' is another (positive) Hahn-Banach extension of ϕ. We have for any u that

$$|\phi(F_u)| \leq \Psi(e^{u}) \qquad |\phi(F_u^{-1})| \leq \Psi'(e^{-u})$$

Hence $1 \leq \Psi(e^{u}).\Psi'(e^{-u})$. This gives, if we replace u by tu, for t an arbitrary real number, that

$$\Psi(e^{tu}) \geq \frac{1}{\Psi'(e^{-tu})}$$

$$1 + t\Psi(u) + \frac{t^2}{2}\Psi^2(u) + \ldots \quad \geq \quad 1 + t\Psi'(u) + \ldots\ldots$$

which leads directly to the relation $\Psi(u) \geq \Psi'(u)$ for positive u. A similar argument gives $\Psi'(u) \geq \Psi(u)$, and this leads directly to the equality of Ψ and Ψ' everywhere in L^{∞}.

<u>PROPOSITION 9</u>. Any homomorphism of $\mathcal{B}$ which fails to be in the Silov boundary will annihilate a Blaschke product.

<u>Proof</u>. Take any $\phi_o \in \mathfrak{M}$ which satisfies the condition $\phi_o(b) \neq 0$ for each Blaschke product b. We wish to show ϕ_o is in X, or, which is the same thing, to show that its extension to a linear functional ϕ_o on L^{∞} is a homomorphism of L^{∞}. All such homomorphisms are evaluations at some point of X. The extension Ψ of a homomorphism $\phi \in \mathfrak{M}$ is written in the form $\Psi(f) = \ell(\mathrm{Re}(f)) + i\ell(\mathrm{Im}(f))$, where ℓ is a bounded positive functional on the space of real valued continuous functions on X. We will show that ℓ_o is just evaluation at some point in X, where ℓ_o is

is associated with ψ_o and ϕ_o.

Suppose ℓ_o is not an evaluation. Because the set of simple functions is dense in the space of real essentially bounded functions, we can find an idempotent function e for each $x \epsilon X$, with $\hat{e}(x)=1$ and $\ell_o(e) \neq 1$. Since $\ell_o(e)$ must be either 0 or 1, we have

$$\hat{e}(x) = 1 \quad , \quad \ell_o(e) = 0 \quad .$$

A similar operation for each $x \epsilon X$ and a compactness argument gives a finite collection $e_1, e_2, \quad , e_n$ of idempotent functions (characteristic functions) with $\ell_o(e_1) = \ell_o(e_2) = \ldots = \ell_o(e_n) = 0$. If we let $v = e_1 + e_2 + \ldots + e_n$, we have

$$\ell_o(v) = 0 \quad , \quad \hat{v} \geq 1 \text{ on } X \quad .$$

Recall the definition of ℓ. For any real valued function in L^∞, we define $F_u \epsilon \mathscr{B}$ by $F_u(z) = \exp[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u(e^{i\theta}) d\theta]$, and then let the functional ℓ be determined by

$$\ell(u) = \log|\phi(F_u)| \quad .$$

If we choose the particular function v, we have a function $F_v \epsilon \mathscr{B}$ which satisfies, for $\phi \epsilon X$, the equation $\log|\hat{F}_v(\phi)| = \hat{v}(\phi) \geq 1$, which means $\hat{F}_v \geq e$ on X. For the particular homomorphism ϕ_o , we have $\phi_o(F_v) = \exp[\ell_o(\phi_o)] = 1$. Thus we have a function F_v such that the value $\phi_o(F_v)$ does not appear in the range of $\hat{F}_v$ on X. If we let $f = F_v - 1$, then f is a function with $\phi_o(f) = 0$, but $0 \epsilon \hat{f}(X)$. if we decompose $f = g.q$ into inner function g and outer function q, we have $0 = \phi_o(f) = \phi_o(g).\phi_o(q)$. Since g is an inner function, we have $|\phi_o(g)| = 1$ by hypothesis. Moreover $\hat{f}$ is bounded away from zero on the compact set X, so that the boundary function of f is essentially bounded away from zero on Γ. Its outer part, q, is thus invertible in L^∞, and $\phi_o(g) \neq 0$, a contradiction.

CHAPTER V

EMBEDDING OF DISCS

Introduction. This chapter is centred around a result appearing in the paper of I.J. Schark [23], in which is considered an 'analytic embedding' of the unit disc in an individual fibre. This is a mapping $\Psi : D \to \mathfrak{M}_\alpha$ which is a homeomorphism, and has the additional property of being analytic – in other words $\hat{f} o \Psi$ is analytic on D for each $f \in \mathcal{B}$. From this mapping is inferred the existence of a homeomorphic copy of $\mathfrak{M}$ in $\mathfrak{M}_\alpha$.

By using filters, more general analytic mappings of the unit disc into $\mathfrak{M}_\alpha$ are discussed in this chapter. An analytic mapping Ψ can be found which takes any given point $z_o \in D$ into any homomorphism $\phi_o \in \mathfrak{M}_\alpha$. In section 2, the 'parts' of $\mathfrak{M}$ are discussed. Special cases of mappings Ψ which are embeddings into $\mathcal{D}_\alpha$ (Schark's case) and into $\mathcal{N}_\alpha$ are treated in section 3.

§1 MAPPINGS FROM THE UNIT DISC INTO $\mathfrak{M}_\alpha$

For any given α with $|\alpha| = 1$, a non-Euclidean rigid motion leaving α invariant will be called an α-motion. For $\alpha \in \Gamma$ and $a, z \in D$, there exists exactly one α-motion S with $S(z)=a$. This is obtained by drawing the circle through a, z, and α; if this is an oricycle touching Γ at the point α, then S is that limiting rotation carrying z to a; if this circle cuts Γ in a second point β, then S is a non-Euclidean translation carrying points of D along the members of the family of circles through α and β, and carrying z to a.

Fix $z_o \in D$ and $\alpha \in \Gamma$. Corresponding to each $z \in D$, we define a mapping $F_z : D \longrightarrow D$ as follows: First define for $a_o \in D$ the α-motion S_{a_o} which maps z_o into a_o. The point $a = F_z(a_o)$ is the image of z under S_{a_o}.

$$F_z(a_o) = S_{a_o}(z) \qquad \text{for all } a_o \in D .$$

If $F_z(a_o) = F_z(b_o)$, then S_{a_o} and S_{b_o} are two α-motions taking z into the same point. Hence they are equal, and $a_o = b_o$. Thus F_z is 1-1. F_z is onto as well. For any $a \in D$, let S be the α-motion mapping z to a, and let $a_o = S(z_o)$. Then $S = S_{a_o}$, and $a = F_z(a_o)$. Finally we show that F_z is a homeomorphism. It is sufficient to prove continuity, since by the above remarks, F_z^{-1} is the same type of function as F_z; it would be F_{z_o} if we had started with z and a rather than z_o and a_o.

That $F_z(a_o)$ is a continuous function of a_o for z fixed is evident when we explicitly write down the bilinear function S_{a_o}.

$$F_z(a_o) = S_{a_o}(z) = \frac{(z_o-\alpha)(1-\bar{a}_o\alpha)(1-\bar{z}_o z)a_o-(1-\bar{z}_o\alpha)(z_o-z)(a_o-\alpha)}{(z_o-\alpha)(1-\bar{a}_o\alpha)(1-\bar{z}_o z)-(1-\bar{z}_o\alpha)(z_o-z)(a_o-\alpha)\bar{a}_o} .$$

Although F_z is a homeomorphism of D onto itself, it is not necessarily a non-Euclidean rigid motion. However it has the following property: The hyperbolic distance between a_o and $a = F_z(a_o)$ is a constant independant of a_o, depending only on z and z_o. For,

$$h(a_o,a) = h(S_{a_o}(z_o), S_{a_o}(z)) = h(z_o,z) \quad ,$$

which is a constant.

For any $A \subseteq D$, and for any filter $\mathfrak{A}$ of subsets of D, we define

$$A_z = F_z(A) \quad , \quad \mathfrak{A}_z = \{A_z : A \in \mathfrak{A}\} .$$

$\mathfrak{A}_z$ is clearly a filter. We have

$$A_{z_o} = A \quad , \quad \mathfrak{A}_{z_o} = \mathfrak{A} \quad .$$

If $\mathfrak{N}$ is any maximal closed filter on D, $\mathfrak{N}_z$ must likewise be maximal closed, because F_z is a homeomorphism on D. If $\mathfrak{N}$ converges to some $\alpha \in \Gamma$, so also will $\mathfrak{N}_z$; this follows from the property that each point of any $A_z \in \mathfrak{N}_z$ lies at a fixed hyperbolic distance from the corresponding point of $A \in \mathfrak{N}$.

If a second point z' is chosen, this gives a second homeomorphism $F_{z'}$ on D with the same properties as F_z. The mapping $F = F_{z'} \circ F_z^{-1}$ which maps a to $a' = F_{z'}(a_o)$ is necessarily of this same type; in particular $h(a,a') = h(z,z')$ is a constant depending only on z and z'.

<u>THEOREM 1</u>. Take any homomorphism $\phi_o \in \mathfrak{M}_\alpha$ for $|\alpha|=1$, and any maximal closed filter $\mathfrak{N}$ on D with $\phi_o = \phi_{\mathfrak{N}}$. Let $z_o \in D$ be arbitrary. The mapping $\Psi : D \longrightarrow \mathfrak{M}_\alpha$, $\Psi(z) = \phi_{\mathfrak{N}_z}$, has the following properties:

(i) $\Psi(z_o) = \phi_o$.

(ii) Ψ is continuous.

(iii) For each $f \in \mathcal{B}$, $\hat{f} \circ \Psi$ is analytic on D.

<u>Proof</u>. Ψ clearly has property (i). We write ϕ_z for $\phi_{\mathfrak{N}_z}$. To show continuity for any $z \in D$, take the arbitrary sub-basic neighbourhood N of ϕ_z.

$$N = \{\phi \in \mathfrak{M} : \quad |\hat{f}(\phi_z) - \hat{f}(\phi)| < \epsilon\} \quad ,$$

where f is any function in $\mathcal{B}$, but can without loss of generality satisfy $||f|| \leq 1$. Since $\hat{f}(\phi_z) = \lim f(\mathfrak{N}_z)$, there exists $A \in \mathfrak{N}$ with

(1) $$|\hat{f}(\phi_z) - f(a)| < \epsilon/4 \quad \text{for all } a \in A_z \ .$$

Take the hyperbolic neighbourhood M of z

$$M = \{z' \in D : h(z,z') < \epsilon/4\} \ .$$

For any $z' \in M$, and any $a' \in A_{z'}$, there must exist $a \in A_z$ with $h(a,a') < \epsilon/4$. In fact $A_{z'}$ and A_z are in 1-1 correspondence under a mapping which carries each point a distance $h(z,z')$ to its image.

We apply Pick's Theorem to the function f.

$$h(f(a) - f(a')) \leq h(a,a') \ .$$

(2) $$|f(a) - f(a')| < \epsilon/4 \ .$$

Combining (1) and (2), we have $|\hat{f}(\phi_z) - f(a')| < \epsilon/2$ for all $a' \in A_{z'}$, and since $A_{z'} \in \mathfrak{N}_{z'}$, this gives

(3) $$|\hat{f}(\phi_z) - \hat{f}(\phi_{z'})| \leq \epsilon/2 < \epsilon$$

Thus $\phi_{z'} = \Psi(z') \in N$ whenever $z' \in M$; but M is also a neighbourhood of z in the ordinary topology. The function Ψ maps a suitable neighbourhood of z into each sub-basic neighbourhood of $\Psi(z)$, and must be continuous.

We next show (iii). Let K be any compact subset of D. We construct for $f \in \mathcal{B}$ a sequence of functions Ψ_n in $\mathcal{B}$ such that $\hat{f} \circ \Psi$ is the uniform limit of the sequence $\hat{f} \circ \Psi_n$ on K. We again suppose $||f|| \leq 1$, and appeal to the formula of the first part of the proof. Let n be a positive integer. For each $z \in K$, there must exist a neighbourhood M of z, and $A \in \mathfrak{N}$, with

(4) $\quad |\hat{f}(\phi_{z'}) - f(a')| < 1/n$ for all $a' \in A_{z'}$ whenever $z' \in M$.

This follows from (1), (2), and (3), if ϵ is taken to be less than $1/n$.

K can be covered by a finite number of these neighbourhoods, $M_1, M_2, \ldots, M_k$, of the points $z_1, z_2, \ldots, z_k$. Let $A_1, A_2, \ldots, A_k$ be the members of $\mathfrak{N}$ which are obtained with these neighbourhoods. Their intersection $\tilde{A} = A_1 \cap A_2 \cap \ldots \cap A_k$ must belong to $\mathfrak{N}$. We use this $\tilde{A}$ in (4), along with the fact that any $z' \in K$ must lie in one of the $M_1, M_2, \ldots M_k$, to obtain

(5) $\quad |\hat{f}(\phi_{z'}) - f(a')| < 1/n$ for all $a' \in A_{z'}$ whenever $z' \in K$.

We now pick some fixed element $a_o \in A$, and define Ψ_n to be that non-Euclidean rigid motion leaving α fixed and with $\Psi_n(z_o) = a_o$. In other words $\Psi_n = S_{a_o}$. For any $z' \in K$, This means that the point in $A_{z'}$ corresponding to a_o in $\tilde{A}$ must be $a' = \Psi_n(z')$, and we use this a' in the formula (5). Since $\phi_{z'} = \Psi(z')$, this gives

$$|\hat{f}(\Psi(z')) - f(\Psi_n(z'))| < 1/n \text{ whenever } z' \in K .$$

Hence on the arbitrary compact subset of D, $\hat{f} \circ \Psi$ is the limit of a uniformly convergent sequence of analytic functions. $\hat{f} \circ \Psi$ must be analytic.

Remark: The trivial map which sends all of D into the single homomorphism ϕ_o will satisfy (i), (ii), and (iii). We see in the next section that Ψ is this trivial mapping whenever ϕ_o belongs to the Silov boundary. Suppose $\phi_o \in \mathcal{N}_\alpha$. Then any maximal closed filter $\mathfrak{N}$ with $\phi_o = \phi_{\mathfrak{N}}$ has an angle of approach k_o to the point α. If we let z vary over D, the filters $\mathfrak{N}_z$ will all be non-tangential, but will have different angles of approach to α. In fact, by a suitable choice of z, any angle k with $0 < k < \pi$ is possible. Now if two maximal closed filters have a

different angle of approach to α, they must give different homomorphisms in $\mathcal{N}_\alpha$ (Proposition 2, Chapter II). Therefore ψ is non-trivial; it also maps all of D into $\mathcal{N}_\alpha$. A similar remark can be made for $\mathcal{D}_\alpha$.

§2 PARTS OF $\mathfrak{M}$

Assume that X is some compact space, and that A is a unitary point-separating subalgebra of C(X), closed in the uniform topology. A is called a function algebra. Gleason [11] introduces the following equivalence relation on M, the space of homomorphisms of A. For $\lambda,\mu \in M$, we say $\lambda \sim \mu$ if and only if $||\lambda-\mu|| < 2$, where $\lambda-\mu$, a linear functional on A, has its norm calculated in the dual space of A. since $||\lambda|| = ||\mu|| = 1$, the inequality $||\lambda-\mu|| \leq 2$ always holds; if this is a strict inequality, then $\lambda \sim \mu$. In other words

$$\lambda \sim \mu \text{ if and only if } \sup_{f \in A, ||f|| \leq 1} |\lambda(f) - \mu(f)| < 2$$

Gleason states that this gives an equivalence relation on M, and and calls the equivalence classes the <u>parts</u> of the space of homomorphisms. He suggests that these parts are related to the analytic structure of M.

A is called a <u>Dirichlet algebra</u> if every real valued continuous function on X is the uniform limit of functions of the form Re(f) for $f \in A$. Wermer [27] proves the striking result that in a Dirichlet algebra, any part that is not a single point is the continuous 1-1 image of the unit disc under some mapping ψ. Furthermore, for any $f \in A$, denoting by $\hat{f}$ its extension to M, the composition $\hat{f} \circ \psi$ is analytic on D.

$\mathcal{B}$ is not a Dirichlet algebra; see [14].

The next two propositions apply to any function algebra.

<u>PROPOSITION 1</u>. The relation $\sim$ is an equivalence relation on M.

<u>Proof</u>. Suppose $\lambda \sim \mu$ and $\mu \sim \nu$ for homomorphisms $\lambda,\mu,\nu \in M$. Let $K < 2$ be larger than each of the numbers $||\lambda-\mu||$ and $|||\mu-\nu||$, and

set $N = (2-K)/2$. Let U and V be open neighbourhoods of radius N about the points -1 and 1 in the complex plane. Choose bilinear transformations S and T, each leaving -1 and 1 invariant, as follows. T must map the set $\{z : z \in \bar{D}, \mathrm{Re}(z) \geq 0\}$ into V. Note that, by continuity, we can find a neighbourhood U_1 of -1, with $T(U_1) \subseteq U$. S must map the set $\{z : z \in \bar{D}, \mathrm{Re}(z) \leq 0\}$ into U. Let V_1 be a neighbourhood of +1 such that $S(V_1) \subseteq V$.

Assume that $||\lambda-\nu|| = 2$; in other words we assume the proposition false. We can find $f \in A$ with $||f|| \leq 1$ and with $|\lambda(f)-\nu(f)| = |\hat{f}(\lambda)-\hat{f}(\nu)|$ arbitrarily close to 2. Since we can take $\hat{f}(\nu)$ positive, we obviously can so choose f to have $\hat{f}(\lambda) \in U_1$, and $\hat{f}(\nu) \in V_1$.

Suppose $\mathrm{Re}(\hat{f}(\mu)) \geq 0$. If T is chosen as above, and is required to be a non-Euclidean rigid motion on D, then the function $T \circ f$ has the form

$$g = T \circ f = e^{i\theta}\left(\frac{a-f}{1-\bar{a}f}\right) \quad \text{where } |a| < 1.$$

Since $f \in A$; $a-f \in A$, and $1-\bar{a}f \in A$. The latter, being bounded away from zero, is invertible. Hence $g \in A$. Also $\hat{g} = T \circ \hat{f}$, by the continuity of T in the closed unit disc. With the function g, we obtain a contradiction.

$$\hat{g}(\lambda) = T(\hat{f}(\lambda)) \in U \quad , \quad \hat{g}(\mu) = T(\hat{f}(\mu)) \in V.$$

Hence $|\hat{g}(\lambda) - \hat{g}(\mu)| > 2-2N = K$. But since $||g|| \leq 1$, we have $|\hat{g}(\lambda) - \hat{g}(\mu)| \leq ||\lambda-\mu|| < K$.

A similar contradiction is obtained using the mapping S if $\mathrm{Re}(\hat{f}(\mu)) < 0$. Hence $||\lambda-\nu|| < 2$, and $\lambda \sim \nu$.

<u>PROPOSITION 2</u>. Take $\lambda, \mu \in M$. These are in different parts of M whenever there exists $f \in A$, $||f||=1$, with

$$|\hat{f}(\lambda)| = 1 \quad , \quad |\hat{f}(\mu)| < 1 .$$

Proof. Take without loss of generality $\hat{f}(\lambda)=1$, and choose a non-trivial Euclidean rigid motion S_n, with -1 and 1 as invariant points, such that

$$|1 + S_n(\hat{f}(\mu))| < 1/n \quad , \quad n \text{ a positive integer.}$$

The function $g = S_n \circ f$ belongs to A, and $\hat{g} = S_n \circ \hat{f}$. Note that $\hat{g}(\lambda) = S_n(\hat{f}(\lambda)) = S_n(1) = 1$. We now have

$$|\hat{g}(\lambda) - \hat{g}(\mu)| = |1 - S_n(\hat{f}(\mu))| \geq 2 - \frac{1}{n}$$

Hence $||\lambda-\mu|| = \sup_{h\in A, ||h||\leq 1} |\hat{h}(\lambda) - \hat{h}(\mu)| = 2$,

and λ is not in the relation $\sim$ with μ.

We return now to the particular function algebra $\mathcal{B}$. For α with $|\alpha|=1$, and arbitrary $\phi \in \mathcal{M}_\alpha$, it is clear that the entire part containing ϕ lies within $\mathcal{M}_\alpha$. We can apply Proposition 2, since there exists a function $f\in \mathcal{B}$, continuous on $\bar{D}$, such that

$$f(\alpha) = 1 \quad , \quad |f(\beta)| < 1 \quad , \quad \text{for } \beta \in \bar{D} \ , \ \beta\neq\alpha \ .$$

For homomorphisms of the Silov boundary X, we have

PROPOSITION 3. For each $\phi \in X$, $\{\phi\}$ is a part of $\mathcal{M}$.

Proof. First let $\phi' \in X$ be different from ϕ. Assume $\mathfrak{a}$ and $\mathfrak{a}'$ are the filters on Γ which determine ϕ and ϕ' respectively. We choose $A\in \mathfrak{a}$, $A'\in \mathfrak{a}'$; two sets of positive measure on Γ with $A\cap A' = \emptyset$. There exists an outer function $f\in \mathcal{B}$ with $|f|=1$ on A, and $|f|=\frac{1}{2}$ on A'. This means $|\phi(f)| = 1$ and $|\phi'(f)| = \frac{1}{2}$, and by Proposition 2, ϕ' is not in the relation $\sim$ with ϕ.

Suppose now $\phi' \notin X$. Then there exists a Blaschke product $b \in \mathcal{B}$ with $\hat{b}(\phi') = 0$, as proved in Chapter IV, §4. But $|\hat{b}|$ is identically one on X, and hence $|\hat{b}(\phi)| = 1$. Again by Proposition 2, this means that ϕ and ϕ' cannot belong to the same part.

PROPOSITION 4. For each $M > 0$, there exists a constant K, $0 < K < 2$, with the following property: For arbitrary $\lambda, \mu \in \mathcal{M}$ satisfying $h(\lambda,\mu) \leq M$, and arbitrary $f \in \mathcal{B}$ with $||f|| \leq 1$,

$$|\hat{f}(\lambda) - \hat{f}(\mu)| \leq K .$$

Proof. Take any such λ, μ, and f, and let T be any non-Euclidean rigid motion on D with $T(f(\mu)) = 0$. We can apply Pick's Theorem to the function $g = T \circ f \in \mathcal{B}$, since $||g|| \leq 1$. Because $g(\mu)=0$,

$$|T(f(\lambda))| = |g(\lambda)| = \left| \frac{g(\lambda) - g(\mu)}{1 - \bar{g}(\lambda)g(\mu)} \right| \leq \left|\frac{\lambda-\mu}{1-\bar{\lambda}\mu}\right| = \tanh M < 1$$

In other words, the values $|T(f(\lambda))|$ are bounded away from 1. This statement is contradicted, however, if we assume the proposition false. Suppose $|f(\lambda) - f(\mu)|$ can be made arbitrarily close to 2. We can without loss of generality make $f(\mu)$ negative, and for arbitrary $\epsilon < \frac{1}{2}$ can choose f so that $|1-f(\lambda)| < \epsilon$, $1 + f(\mu) < \epsilon$. The non-Euclidean rigid motion T, leaving -1 and 1 invariant, and mapping $f(\mu)$ to 0, must take $f(\lambda)$ closer yet to $z=1$. This gives

$$|1 - T(f(\lambda))| < \epsilon ,$$

for a bilinear transformation T satisfying all the conditions stipulated above, where ϵ is arbitrarily small. This contradiction assures the existence of a constant K satisfying the conditions of the proposition.

As an immediate consequence we have

COROLLARY 1. Δ is one part of $\mathcal{M}$.

Proof. For any two fixed points λ and μ of D, letting ϕ_λ and ϕ_μ be the associated homomorphisms of Δ, we have $|\hat{f}(\phi_\lambda) - \hat{f}(\phi_\mu)| = |f(\lambda)-f(\mu)| \leq K < 2$ for some constant K, where f is any member of $\mathcal{B}$ of norm at most one. Hence $||\phi_\lambda - \phi_\mu|| \leq K < 2$ and the two are in the same part.

For $\lambda \in D$, and $\mu \in \mathfrak{M}_\alpha$, where $|\alpha|=1$, there exists $f \in \mathcal{B}$, with $\hat{f}$ constant and equal to 1 on $\mathfrak{M}_\alpha$, but with $|f(\lambda)| < 1$. Take for example the rotation about the origin taking α into 1; this is continuous on $\bar{D}$, and hence is constant on each fibre. Applying Proposition 2, we see that μ is not in the same part as λ. Hence Δ is a complete $\sim$ equivalence class. By the same argument, we have for the mappings Ψ obtained in Theorem 1 the following:

<u>COROLLARY 2</u>. Each of the mappings Ψ of D into $\mathfrak{M}_\alpha$ maps into one part of $\mathfrak{M}$.

<u>Proof</u>. This follows from the fact that, for arbitrary $f \in \mathcal{B}$, $\hat{f} \circ \Psi$ is analytic on D.

Recall that in Theorem 1, when $\mathfrak{N}$ is a maximum closed filter on D, the sets in any pair $\mathfrak{N}_z$ and $\mathfrak{N}_{z'}$ of filters have this property: To each $A \in \mathfrak{N}_z$ corresponds a set $A' \in \mathfrak{N}_{z'}$ under a 1-1 mapping F such that $h(a,F(a))$ is some constant depending only on z and z'. We now show that a much weaker assumption on two filters is still sufficient to place the corresponding homomorphisms in the same part of $\mathfrak{M}$.

<u>PROPOSITION 5</u>. Let $\mathfrak{N}$ and $\mathfrak{N}'$ be maximal closed filters on D. Suppose there exists a constant M such that to arbitrary $A \in \mathfrak{N}$, and $A' \in \mathfrak{N}'$, there belong points a and a' respectively, with $h(a,a') \leq M$. Then $\phi = \phi_{\mathfrak{N}}$ and $\phi' = \phi_{\mathfrak{N}'}$ are in the same part of $\mathfrak{M}$.

<u>Proof</u>. Let $f \in \mathcal{B}$ with $||f|| \leq 1$ be given. By Proposition 4 there exists $K < 2$, independant of f, such that

(1) $$|f(a) - f(a')| \leq K \quad \text{whenever} \quad h(a,a') \leq M.$$

Set $\epsilon = (2-K)/4$. Choose $A \in \mathfrak{N}$ such that

(2) $$|\hat{f}(\phi) - f(a)| < \epsilon \quad \text{whenever} \quad a \in A,$$

and likewise choose $A' \in \mathfrak{A}'$ with

(3) $$|f(a') - \hat{f}(\phi')| < \epsilon \quad \text{whenever } a' \in A'.$$

By hypothesis, we can select a and a' such that $h(a,a') \leq M$, and then (1), (2), and (3) all hold. This gives

$$|\hat{f}(\phi) - \hat{f}(\phi')| \leq \epsilon + K + \epsilon = (2+K)/2 < 2$$

If we take the supremum over $f \in \mathcal{B}$ with $||f|| \leq 1$, we have $||\phi - \phi'|| < 2$ as required.

§3 ANALYTIC EMBEDDINGS OF THE UNIT DISC IN $\mathfrak{M}_\alpha$

In this section we investigate special mappings Ψ from D into $\mathfrak{M}_\alpha$ which are homomorphisms; Two types are given – one mapping into $\mathfrak{D}_\alpha$, the subset of $\mathfrak{M}_\alpha$ consisting of oricycle homomorphisms; the other mapping into the set $\mathfrak{N}_\alpha$ of non-tangential homomorphisms of $\mathfrak{M}_\alpha$. We treat the oricycle case first. The example given by I.J. Schark is of this type. In order to show that the mapping Ψ from D is a homeomorphism, Schark uses a function $h \in \mathcal{B}$ with the property that $\hat{h}|\Psi(D)$ is precisely the inverse of Ψ. We follow this method here.

Without loss of generality we set $\alpha=1$. Take a limiting rotation L on D, with $\alpha=1$ as the invariant point on Γ. The most general such transformation has the form

$$L(z) = \frac{z - \rho i(z-1)}{1 - \rho i(z-1)} \qquad -\infty < \rho < \infty \quad .$$

We denote by $L^{(n)}$ the n'th composition of L with itself. Fix any $z_o \in D$, and define the set

$$A = \left\{ z_o, L(z_o), L^{(2)}(z_o), L^{(4)}(z_o), \ldots, L^{(2^n)}(z_o), \ldots \right\}$$

Let $\mathfrak{U}$ be an arbitrary free ultrafilter on D which contains the set A. The pair z_o and $\mathfrak{U}$ define an analytic mapping Ψ of D into $\mathfrak{M}_1$. Since all the points of A lie on the oricycle through z_o, $\Psi(z_o) = \phi_{\mathfrak{U}} \in \mathfrak{D}_1$. This means that all of D is mapped into the set $\mathfrak{D}_1$, as was observed at the end of §1.

By induction, we show that

$$L^{(n)}(z) = \frac{z - n\rho i(z-1)}{1 - n\rho i(z-1)} \quad .$$

Assuming the above formula for a given n, we obtain

$$L^{(n+1)}(z) = L(L^{(n)}(z))$$

$$= \frac{\frac{z-n\rho i(z-1)}{1-n\rho i(z-1)} - \rho i\,\frac{z-n\rho i(z-1)}{1-n\rho i(z-1)} - 1}{1 - \rho i\,\frac{z-n\rho i(z-1)}{1-n\rho i(z-1)} - 1}$$

$$= \frac{z-n\rho i(z-1)-\rho i(z-1)}{1-n\rho i(z-1)-\rho i(z-1)}$$

$$= \frac{z - (n+1)\rho i(z-1)}{1 - (n+1)\rho i(z-1)} \quad .$$

This proves the formula for n=1,2,3, ... By an inversion, we have

$$L^{(-1)}(z) = \frac{z + \rho i(z-1)}{1 + \rho i(z-1)} \quad ,$$

and the above formula is proved for negative integers by the same induction as above, if we replace n by -n.

For any $z \in D$ we have $\lim_{|n| \to \infty} L^{(n)}(z) = 1$.

In fact, if we restrict z to a compact subset of D, we show that

$$(1) \qquad |L^{(n)}(z) - 1| \leq \frac{K}{|n|} \qquad n \neq 0 \quad .$$

We note that, as z ranges over the compact set, $|z-1|$ is bounded away from zero and hence its reciprocal is bounded. We have

$$|L^{(n)}(z) - 1| = \left|\frac{z - 1}{1 - n\rho i(z-1)}\right| = \frac{1}{|n\rho i - {}^{1}/_{(z-1)}|} \leq \frac{K_o}{|n|}$$

for large enough n, where K_o is a suitable constant. The required constant K can now be obtained.

Define a function $h(z) = \prod_{k=0}^{\infty} L^{(-2^k)}(z)$. This product converges uniformly on compact subsets of D, since on any compact set

$$| L^{(-2^k)} - 1| \leq \frac{K}{2^k}$$

Hence h is analytic on D. It must be bounded with $||h|| \leq 1$, since

$|L^{(n)}| \leq 1$ on D for all n. Thus $h \in \mathcal{B}$. We wish to show that, for any $z \in D$, $\hat{h}(\Psi(z)) = z$. The homomorphism $\Psi(z)$ is determined by the ultrafilter $\mathcal{N}_z$. Since $A \in \mathcal{N}$, this filter must contain the set

$$A_z = \{z, L(z), L^{(2)}(z), L^{(4)}(z), \ldots, L^{(2^n)}(z), \ldots\} .$$

Any function which has a limit on the sequence of points defining A_z, must necessarily have the same limit along $\mathcal{N}_z$. We show that $\lim_{n\to\infty} h(L^{(2^n)}(z)) = z$, which means that $z = \lim h(\mathcal{N}_z) = \hat{h}(\Psi(z))$.

We work within a compact subset of D.

$$\begin{aligned}
|h(L^{(2^n)}(z)) - z| &= \left| \prod_{k=0}^{n-1} L^{(-2^k+2^n)}(z).z. \prod_{k=n+1}^{\infty} L^{(-2^k+2^n)}(z) - z \right| \\
&= |z| \left| \prod_{k=0}^{n-1} L^{(2^n-2^k)}(z). \prod_{k=n+1}^{\infty} L^{(-2^k+2^n)}(z) - 1 \right| \\
&\leq |z| \left[\sum_{k=0}^{n-1} |L^{(2^n-2^k)}(z)-1| + \sum_{k=n+1}^{\infty} |L^{(-2^k+2^n)}(z)-1| \right],
\end{aligned}$$

applying the inequality, for $t_k \in D$, $\left|\prod_{k=1}^{\infty} t_k - 1\right| \leq \sum_{k=1}^{\infty} |t_k - 1|$.

We now apply the inequality (1), valid inside any compact set.

$$\sum_{k=0}^{n-1} |L^{(2^n-2^k)}(z) - 1| \leq \sum_{k=0}^{n-1} \frac{K}{2^n-2^k} \leq \frac{nK}{2^{n-1}}$$

$$\sum_{k=n+1}^{\infty} |L^{(-2^k+2^n)}(z) - 1| \leq \sum_{k=n+1}^{\infty} \frac{K}{2^k-2^n} \leq \frac{K}{2^{n-1}}$$

Hence $|h(L^{(2^n)}(z)) - z| \leq \dfrac{|z|.K.(n+1)}{2^{n-1}}$

which tends to zero as $n \to \infty$.

This proves that $\hat{h} \circ \Psi$ is the identity function on D. Since $\hat{h}$ is continuous, Ψ is necessarily a homeomorphism.

We next describe a very similar embedding of the unit disc, this time into $\mathcal{M}_\alpha$. A slight variation is necessary in the approach. For a limiting rotation L at $\alpha=1$, we have that both $L^{(n)}(z)$ and $L^{(-n)}(z)$ approach 1 for large integral values of n. This is not the case for a non-Euclidean translation; If the translation L maps points towards 1 along circles through -1 and 1, then $L^{(-n)}(z)$ approaches -1 for n large.

We will map the disc into the fibre $\mathcal{M}_{-1}$. L will be an arbitrary non-Euclidean translation towards -1 along the family of circles through -1 and 1. The set A will this time consist only of the alternate members of the sequence $(L^{(2^n)}(z_o))$.

$$A = \{z_o, L^{(4)}(z_o), L^{(16)}(z_o), \ldots, L^{(2^{2n})}(z_o), \ldots\}$$

An arbitrary ultrafilter containing A will give the mapping Ψ, as before. It is clear that the ultrafilter is non-tangential; this means that $\Psi(D) \subseteq \mathcal{M}_1$. The mapping

$$L(z) = \frac{\sigma + z}{1 + \sigma z} \qquad \sigma \leq 0,$$

is that translation of the type described which moves points on the real axis a pseudo-hyperbolic distance σ to the right. The origin, for example, maps to the point σ, which is Ψ-distant exactly σ from it. If σ is replaced by $-\sigma$, the translation then moves the same amount to the left. It is useful to use the hyperbolic distance so that $\sigma = \tanh(s)$. We take an arbitrary translation towards -1.

$$L(z) = \frac{-\tanh(s) + z}{1 - \tanh(s)z} \qquad s > 0.$$

We again obtain by induction a formula for $L^{(n)}$, $n = 0,1,-1,2,-2,..$

$$L^{(n)}(z) = \frac{-\tanh(ns) + z}{1 - \tanh(ns)z}.$$

For, assuming that this holds for some n,

$$L^{(n+1)}(z) = L(L^{(n)}(z))$$

$$= \frac{-\tanh(s) + \left(\frac{-\tanh(ns) + z}{1 - \tanh(ns)z}\right)}{1 - \tanh(s)\left(\frac{-\tanh(ns) + z}{1 - \tanh(ns)z}\right)}$$

$$= \frac{-(\tanh(s)+\tanh(ns)) + z(1+\tanh(s)\tanh(ns))}{(1+\tanh(s)\tanh(ns)) - z(\tanh(s)+\tanh(ns))}$$

$$= \frac{-\tanh(n+1)s + z}{1 - \tanh(n+1)s.z}$$

Hence it holds for positive n. As before, the above formula can be proved for negative n by a similar induction.

For large values of n, $L^{(n)}(\lambda)$ approaches -1. Indeed if λ is restricted to some compact subset of D, we show that

$$(2) \qquad |L^{(n)}(\lambda) + 1| = \left|\frac{(\lambda+1)(1-\tanh(ns))}{1 - \tanh(ns).\lambda}\right| \leq \frac{K}{n} \quad , n=1,2,..$$

for some constant K. We have $|\lambda+1| \leq 2$, and because λ cannot approach -1, the denominator $|1 - \tanh(ns).\lambda|$ is bounded away from zero. Also

$$1 - \tanh(ns) = \frac{2e^{-ns}}{e^{ns} + e^{-ns}} \leq 2e^{-ns} ,$$

which tends to zero faster than $1/n$. We need in addition to obtain a similar inequality for n tending to $-\infty$; in this case, $L^{(n)}(\lambda) \longrightarrow 1$. Taking the same compact set,

$$(3) \qquad |L^{(n)}(\lambda) - 1| = \left|\frac{(\lambda-1)(1+\tanh(ns))}{1 - \tanh(ns).\lambda}\right| \leq \frac{K'}{|n|} \quad , n=-1,-2,..$$

where K' is suitably large. Here $|\lambda-1| \leq 2$ and the denominator is this time bounded away from zero because λ cannot come arbitrarily close to -1. The other term, for $n \to -\infty$, tends to zero faster than $1/|n|$, since

$$1 + \tanh(ns) = \frac{2e^{ns}}{e^{ns} + e^{-ns}} \leq 2e^{ns} .$$

The function h, defined as before, is $\prod_{k=0}^{\infty} L^{(-2^k)}$.

The product expansion for h again converges uniformly on compact subsets of D, in virtue of (3); this gives $h \in \mathcal{B}$, and $||h|| \leq 1$. When we evaluate

$$h(L^{(2^{2n})}(z)) = \prod_{k=0}^{2n-1} L^{(-2^k+2^{2n})}(z) \cdot z \cdot \prod_{k=2n+1}^{\infty} L^{(-2^k+2^{2n})}(z)$$

for large n, we see that the terms in the last product are close to 1 because of (3). But those of the first product are close to -1 by (2); There are, however, an even number, and the product is sufficiently close to 1. We use the identity $ab-1 = (a+1)(b+1) - (a+1) - (b+1)$ to obtain, inside the compact set, that

$$\begin{aligned} |L^{(m)}(z).L^{(n)}(z) - 1| &\leq |L^{(m)}(z)+1||L^{(n)}(z)+1| + |L^{(m)}(z)+1| \\ &\qquad + |L^{(n)}(z)+1| \\ &\leq \frac{K''}{n} \end{aligned}$$

where m and n are arbitrary positive integers of which n is the smaller, and K'' is another constant. Take now, for any z in the compact set,

$$\begin{aligned} |h(L^{(2^{2n})}(z) - z| &= |z|.\left| \prod_{k=0}^{2n-1} L^{(-2^k+2^{2n})}(z). \prod_{k=2n+1}^{\infty} L^{(-2^k+2^{2n})}(z) - 1 \right| \\ &\leq |z| \left[\sum_{\ell=0}^{n-1} |L^{(-2^{2\ell}+2^{2n})}(z).L^{(-2^{2\ell+1}+2^{2n})}(z) - 1| \right. \\ &\qquad \left. + \sum_{k=2n+1}^{\infty} |L^{(-2^k+2^{2n})}(z) - 1| \right] \\ &\leq |z| \left[\sum_{\ell=0}^{n-1} \left(\frac{K''}{2^{2n}-2^{2\ell+1}}\right) + \sum_{k=2n+1}^{\infty} \left(\frac{K}{|-2^k+2^{2n}|}\right) \right] \\ &\leq |z|.\left(\frac{nK'' + K}{2^{2n-1}} \right) \end{aligned}$$

This is arbitrarily small for all n large enough, which means that

$$\hat{h}(\Psi(z)) = z$$

As described by Schark [23], we find a copy of $\mathfrak{M}$ inside the fibre $\mathfrak{M}_\alpha$ for the cases where Ψ has a continuous inverse $\hat{h}$. If we consider the homomorphism of $\mathcal{B}$ into itself defined by $\Theta: f \longrightarrow \hat{f} \circ \Psi$, we see in this special case that it is an onto map. The pre-image of any $g \in \mathcal{B}$ is the function $g \circ h$, for it maps into $(g \circ h)\hat{} \circ \Psi = (g \circ \hat{h}) \circ \Psi = g$. The collection $\{\hat{f} \circ \Psi \mid f \in \mathcal{B}\}$ is the algebra of bounded analytic functions on the disc D. If we transfer the metric structure of D into its homeomorphic copy $\Psi(D)$, we can consider the collection of all $\hat{f}|\Psi(D)$ on the set $\Psi(D)$. This is just an isomorphis copy of $\mathcal{B}$, and it will have a maximal ideal space $\mathfrak{M}'$ homeomorphic to $\mathfrak{M}$. This space can be identified as a subset of $\mathfrak{M}$, however, because we can consider the space $\mathfrak{M}'$ as the hull in $\mathfrak{M}$ of the kernel of the homomorphism Θ (see Theorem 6, Chapter 0, §1) .That is

$$\mathfrak{M}' = \{\phi \in \mathfrak{M} : \hat{f}(\phi) = 0 \text{ whenever } \hat{f}|\Psi(D) = 0\} \quad .$$

This is obviously a subset of $\mathfrak{M}_\alpha$.

CHAPTER VI

SOME PROPERTIES OF $\mathfrak{M}$

<u>Introduction.</u> A general arrangement is made of the homomorphisms of $\mathfrak{M}_\alpha$ according to the order of approach to the boundary. It is shown that an extensive part of $\mathfrak{M}_\alpha$ lies outside the subsets $\mathfrak{N}_\alpha$, $\mathfrak{D}_\alpha$, and X_α, but little is known of these. In particular the question of whether a 1-1 correspondence exists between *-filters and homomorphisms is unsettled. In addition, the question of which homomorphisms in $\mathfrak{M}_\alpha$ can be approached by points in other fibres is broached. Finally the space $\mathfrak{M}$ as a whole is discussed, and some remarks are made about cardinality.

§1 ORDER OF APPROACH TO THE BOUNDARY

Suppose we take the fibre $\mathfrak{M}_\alpha$ and consider what homomorphisms can exist besides those that have been studied, that is, besides the homomorphisms of $\mathfrak{N}_\alpha$, $\mathfrak{D}_\alpha$, and X_α. As usual we will deal without loss of generality only with $\alpha = 1$, and in fact only with those homomorphisms whose filters lie above all hypercycles from the family $\mathcal{C}$. These are called homomorphisms approaching $\alpha = 1$ from above, and aside from $\mathfrak{N}_1$, every homomorphism in $\mathfrak{M}_1$ must approach $\alpha = 1$ from above or from below.

Given an arbitrary hypercycle from $\mathcal{C}$, and an arbitrary oricycle from $\mathcal{E}$, there always exists a non empty closed set lying above the hypercycle, and inside the oricycle. This is because, for a small enough neighbourhood of $\alpha = 1$, the oricycle must lie outside the hypercycle. If we consider all such closed sets, we obtain a filter basis for a proper closed filter, and the homomorphisms determined by maximal closed filters above this filter lie, in a sense, between $\mathfrak{N}_1$ and $\mathfrak{D}_1^+$. In like fashion we can take closed sets in D which lie outside and above any oricycle from $\mathcal{E}$, and generate a proper closed filter with these. This permits us to obtain a non empty set of homomorphisms in $\mathfrak{M}_1$ lying 'outside' those of $\mathfrak{D}_1^+$, and one can see that among these will be some not in the Silov boundary. Take a sequence (O_n) of oricycles with radius tending to unity, and a sequence (z_n) of points in D with z_n above O_n for each n, and with $\Sigma\,(1 - |z_n|) < \infty$. Any maximal closed filter containing the set $\{z_1, z_2, \ldots\}$ will lie above our closed filter, but it determines a

homomorphism which annihilates a Blaschke product, and hence does not lie in X_1.

In order to investigate such homomorphisms, we consider a large number of families of curves having different order of contact with the circle Γ at $\alpha = 1$. It is convenient to study these curves in a half plane, rather than in D. At the outset we look at $\mathfrak{M}_1{}^*$, the space of *-filters converging to $\alpha = 1$. We continue to deal only with filters lying above every hypercycle from the family $\mathcal{C}$.

We map the point $\alpha = 1$ to the origin, and the region D conformally onto the upper half plane under the mapping

$$\omega = i\left(\frac{1 - z}{1 + z}\right) .$$

This means $z = \frac{i - \omega}{i + \omega}$, and from this we obtain a pseudo-hyperbolic metric, also denoted by Ψ, in the upper half plane. If $\Psi(z_1,z_2) = d$ and ω_i is the image of z_i, $i = 1,2$, we have

$$\begin{aligned} d &= \left|\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right| \\ &= \left|\frac{\dfrac{i - \omega_2}{i + \omega_2} - \dfrac{i - \omega_1}{i + \omega_1}}{1 - \left(\overline{\dfrac{i - \omega_1}{i + \omega_1}}\right)\left(\dfrac{i - \omega_2}{i + \omega_2}\right)}\right| \\ &= \left|\frac{\omega_1 - \omega_2}{\bar{\omega}_1 - \omega_2}\right| \end{aligned}$$

Under this mapping, hypercycles from the set $\mathcal{C}$ are sent into radial lines through the origin. The filters we are dealing with are

mapped into filters which lie to the right of each of these radial lines and converge to the origin. Maximal closed filters preserve this property since the mapping is a homeomorphism, and if we use the above metric to form the *-filter of the image of a maximal closed filter $\mathfrak{M}$, this will be just the image of $\mathfrak{M}^*$.

For each $\gamma \geq 1$, consider the family of curves $v = ku^\gamma$, $k \geq 0$ arbitrary. We can restrict to values $0 \leq u \leq 1$, because every filter with which we deal contains a set satisfying this restriction. Observe that $\gamma = 1$ gives the family of radial lines mentioned above.

By arguments used several times already, if we fix γ and choose a maximal closed filter $\mathfrak{M}$, exactly one of the following three possibilities will hold.

(1) Some set from $\mathfrak{M}$ can be found lying above any curve of the form $v = ku^\gamma$.

(2) Some set from $\mathfrak{M}$ can be found lying below any curve of the form $v = ku^\gamma$.

(3) There exists a particular curve $v = k_0u^\gamma$, and some set from $\mathfrak{M}$ can be found lying between any two curves $v = k_1u^\gamma$ and $v = k_2u^\gamma$ above and below this curve.

It is possible to construct filters for any γ and any k_0 which satisfy (3). One has only to take a maximal closed filter of sets lying in the curve $v = k_0u^\gamma$. We must show, however, that the *-filter $\mathfrak{M}^*$ also satisfies exactly that one of the above conditions satisfied by $\mathfrak{M}$. To do this it is sufficient to prove the following:

<u>Lemma 1</u>. For any two different curves $v = k_1u^\gamma$ and $v = k_2u^\gamma$ of the same family, the Ψ-distance between points on the two curves is

bounded away from zero in some neighbourhood of the origin.

Proof. We suppose $k_1 > k_2$.

Take a point A (see Fig.17) on the curve $v = k_2u^\gamma$, at which the angle of inclination of the tangent is less than $\pi/3$ and suppose the horizontal line from A intersects the curve $v = k_1u^\gamma$ at $u = u_0$. We restrict ourselves to values $u \leq u_0$ for points on the upper curve. Let ω_1 be such a point, and let ω_2 be the point on the lower curve for which the distance

$$d = \left|\frac{\omega_1 - \omega_2}{\bar{\omega}_1 - \omega_2}\right|$$

is the least. The point ω_2 must lie somewhere between points B and D. We have

$$|\omega_1-\omega_2| > c = a \operatorname{Sin} \theta > a/2$$

since $\theta > \frac{\pi}{6}$. But $a = k_1u^\gamma - k_2u^\gamma$ which gives

$$|\omega_1-\omega_2| > (k_1-k_2)u^\gamma/2.$$

The denominator $|\bar{\omega}_1-\omega_2|$ in the above expression must be less than $4u^\gamma$. Hence

$$d = \left|\frac{\omega_1 - \omega_2}{\bar{\omega}_1 - \omega_2}\right| > \frac{k_1 - k_2}{8}$$

is bounded away from zero.

From this lemma, we see that if a maximal closed filter is associated with a curve $v = k_0u^\gamma$ from a given family in the sense of (3), then $\mathfrak{N}^*$ also has this property. Furthermore if $\mathfrak{N}$ lies above (below) every member of some family, so also will $\mathfrak{N}^*$.

Note that we do not assert that each *-filter is associated with

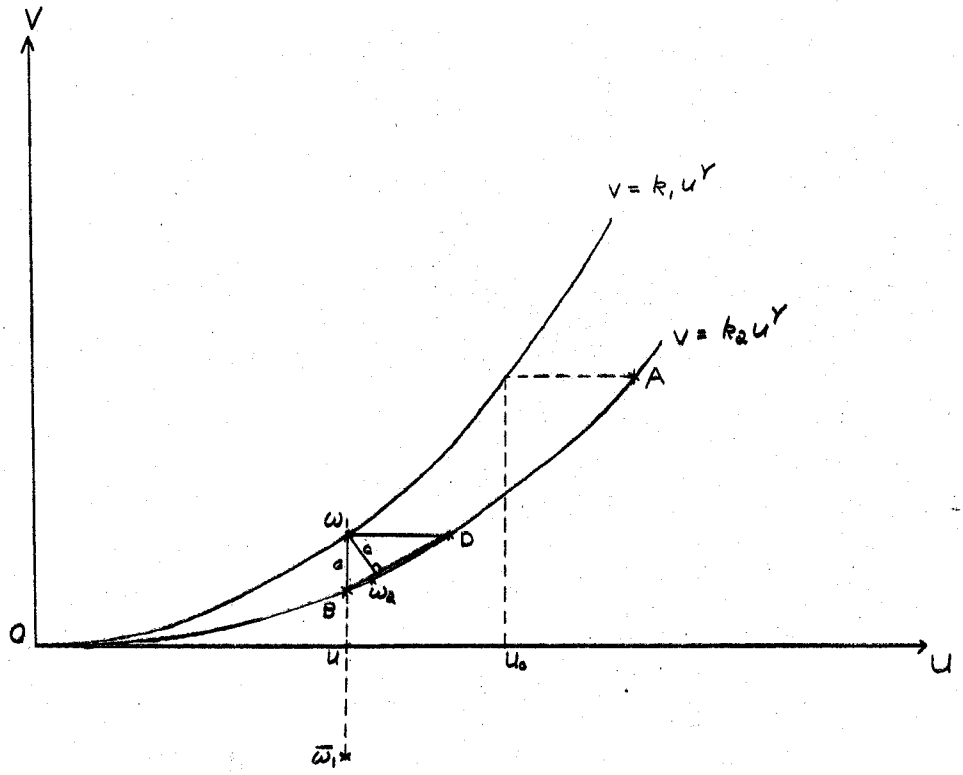

Figure 17

a member of some family for a particular value of γ. We have seen that for each γ there are *-filters associated with each member of the family, but one can construct *-filters not belonging to any family. Even these, however, will have a fixed position relative to the different families, in the sense that any such filter must lie above or below all the members of any given family.

As an example of a filter not belonging to any family, take any value $\gamma_0 \geq 1$, and define the following sequences: Let (k_n) and (ℓ_n) be sequences of positive constants tending to infinity and zero respectively, and let (γ_n) be a sequence of numbers $\gamma_n > \gamma_0$ tending to γ_0. For each value of n, we can choose a point ω_n lying above

$v = k_n u^{\gamma_0}$ but below $v = \ell_n u^{\gamma_n}$. The sequence (ω_n) must tend to 0. Then, if $\mathfrak{N}$ is a maximal closed filter containing the set $\{\omega_1, \omega_2, \ldots\}$, $\mathfrak{N}^*$ will not be associated with any member of any family. We see, however, that $\mathfrak{N}^*$ lies above every member of the family of curves $v = ku^{\gamma_0}$, and that it lies below every member of any family $v = \ell u^{\gamma}$ for $\gamma > \gamma_0$.

We might observe that oricycle filters giving homomorphisms of $\mathcal{D}_1^+$ will all map into filters associated with the family of curves

$$v = ku^2 \qquad k \geq 0 \text{ arbitrary.}$$

Indeed, oricycles map into circles of the form

$$u^2 + (v-r)^2 = r^2$$

which reduces to

$$v = r\left(1 - \sqrt{1 - \left(\frac{u}{r}\right)^2}\right) = \frac{1}{2r} u^2 + \frac{1}{8r^3} u^4 + \ldots$$

which can be confined between two curves of the form $v = ku^2$ and $v = \ell u^2$ in a small enough neighbourhood of the origin.

If we now map back into D, we have an arrangement of those *-filters of $\mathfrak{M}_1^*$ lying above every hypercycle of the family $\mathcal{C}$. We can similarly deal with *-filters lying below every hypercycle, and these two collections, together with non tangential *-filters give the entire space $\mathfrak{M}_1^*$.

A similar arrangement will of course exist for each $\mathfrak{M}_\alpha^*$.

If a *-filter lying above every hypercycle from $\mathcal{C}$ has an image in the upper half plane associated with the family of curve $v = ku^{\gamma}$, then its approach to $\alpha = 1$ from above is said to be of order γ. If the filter has an image which lies above (below) the family of curves

$v = ku^{\gamma}$, then it is said to have order less (greater) than γ. We use this terminology loosely to apply even when the given *-filter is not itself associated with any fixed γ_0 in the sense of (3) above. Similar remarks apply to approach to $\alpha = 1$ from below.

Certain *-filters have order of approach greater than any γ. The construction of filters for Silov boundary homomorphisms given in Chapter IV will evidently give such an example.

§2 STRUCTURE OF THE FIBRES

When we turn our attention from $\mathfrak{M}_\alpha^*$ to $\mathfrak{M}_\alpha$ itself, we see that the mapping of the former onto the latter determines an order of approach to α for each $\phi \in \mathfrak{M}_\alpha$. This mapping has not been shown 1-1 except in special cases; consequently it may be possible for a homomorphism to have different orders of approach. The members of $\mathfrak{N}_\alpha$ and $\mathfrak{D}_\alpha$ have orders $\gamma = 1$ and $\gamma = 2$ of approach respectively, and only these orders. It is worth pointing out that for any value of $\gamma \geq 1$, there are a considerable number of homomorphisms with order γ and with no other order. This follows from the next proposition.

<u>PROPOSITION 1</u>. If a maximal closed filter $\mathfrak{N}$ in D contains a set $A = \{z_1, z_2, z_3, \ldots\}$ where (z_n) is an interpolating sequence, then $\phi_{\mathfrak{N}}$ is determined by exactly one *-filter $\mathfrak{N}^*$.

Proof. We construct the Blaschke product $F \in \mathfrak{B}$ with zeros z_n $n = 1, 2, \ldots$. Take any $\epsilon > 0$ and construct hyperbolic ϵ-neighbourhoods $N_1, N_2, N_3, \ldots$ about the points $z_1, z_2, \ldots$. We know from Lemma 2 of Chapter 0, §4 that F is bounded away from zero on the set $D \setminus \bigcup_{i=1}^{\infty} N_1$. This will be sufficient to prove the proposition, for if $\mathfrak{X}^*$ is a *-filter different from $\mathfrak{N}^*$, we can, for some ϵ, find sets $A' \in \mathfrak{N}$ and $B \in \mathfrak{X}$ with $\psi(a,b) \geq \epsilon$ whenever $a \in A'$ and $b \in B$. We can, by intersection with A if necessary, have $A' \subseteq A$. The function F will separate the filters; we know $\Phi_{\mathfrak{N}}(F) = 0$ and we have $\Phi_{\mathfrak{X}}(F) \neq 0$ because F is bounded away from zero on the set $B \in \mathfrak{X}$.

If we take the image in D of any curve $v = ku^\gamma$ where $k > 0$

and $\gamma \geq 1$ are arbitrary, we can place a sequence (z_n) of points along this curve which approaches the boundary exponentially. If a maximal closed filter $\mathfrak{N}$ contains the set $\{z_1, z_2, z_3, \ldots\}$, it homomorphism $\phi_{\mathfrak{N}}$ must approach $\alpha = 1$ from above with order γ. By the above proposition it can have no other order of approach. It is of course possible to approach $\alpha = 1$ from below with any given order in exactly the same manner, and everything as well applies to arbitrary α with $|\alpha| = 1$.

There are actually a large number of these homomorphisms for each γ, because of the large number of ultrafilters containing the set $\{z_1, z_2, z_3, \ldots\}$. This is discussed in the next section.

The following theorem is proved by W. Seidel in [24], pages 3 and 4.

Consider a Jordan arc C all of whose points with the exception of $z = 1$ are in D. Joining the end points of C by another Jordan arc C^1 which lies in D, and has no points in common with C except its two end points, we obtain a simply connected region G bounded by two curves C and C^1. If we set

$$\limsup_{\substack{z \to 1 \\ z \in C}} |f(z)| = A$$

we have:

<u>THEOREM</u>. Let z_i be an arbitrary sequence of points lying in G and converging toward $z = 1$, for which

$$\lim_{n \to \infty} |f(z_n)| = \alpha \text{ exists. Then } \alpha \leq A.$$

From this result, we can immediately obtain a sort of maximum principle in $\mathfrak{M}_\alpha$. When we mention homomorphisms with order of approach γ to the point α, we refer both to approach from the positive and from

the negative side of α.

<u>PROPOSITION 2</u>. If $|\hat{f}| \leq M$ for all homomorphisms of $\mathfrak{M}_\alpha$ with order of approach γ to the point α, then $|\hat{f}| \leq M$ for all homomorphisms having smaller order of approach, where f is an arbitrary function in $\mathfrak{B}$.

It seems possible that any homomorphism with bounded order of approach to α will fail to be in $\overline{\bigcup_{\beta \neq \alpha} \mathfrak{M}_\beta}$. This is easily shown for order of approach $\gamma \leq 2$. We take $\alpha = 1$ and again make use of the function $f \in \mathfrak{B}$ given by $f = \exp\left(\frac{z+1}{z-1}\right)$ which has constant modulus $|f| < 1$ on each oricycle of the family $\mathcal{E}$. It is known that $|\phi(f)| < 1$ for oricycle homomorphisms ϕ, and it is clear that if the homomorphism has order of contact less than 2 we have $|\phi(f)| = 0$. In either case ϕ is contained in an open set which does not meet $\overline{\bigcup_{\beta \neq 1} \mathfrak{M}_\beta}$, since $|\hat{f}| = 1$ identically on $\mathfrak{M}_\beta$ whenever $\beta \neq 1$. Note that this shows that any *-filter which determines a homomorphism of the Silov boundary must have a basis which lies outside any oricycle of $\mathcal{E}$. For we know that any $\phi \in X_1$ lies in the closure of $\bigcup_{\beta \neq 1} X_\beta$. Again it seems probable that every *-filter determining a Silov boundary homomorphism must have order of approach larger than any γ, just like the filters described in Chapter IV.

We do not have any example of a homomorphism in $\mathfrak{M}_\alpha$, not in X_α, which is in the closure of $\bigcup_{\beta \neq \alpha} \mathfrak{M}_\beta$. One can readily construct homomorphisms which have unbounded order of approach to α, but which

fail to be in this closure.

We take $\alpha = 1$ and approach from above. Take sequences $1 \leq \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 \ldots$ and $0 < k_1 < k_2 < k_3$ both tending to infinity. Then let (z_n) be a sequence of points of D tending to α and satisfying $\Sigma\,(1 - |z_n|) < \infty$. In addition we insist that for each n, the image of z_n in the upper half plane lies below the curve $v = k_n u^{\gamma_n}$. Then we take any homomorphism $\phi_{\mathcal{N}}$ where $\mathcal{N}$ is a maximal closed filter containing the set $\{z_1, z_2, \ldots\}$. The Blaschke product F with zeros z_n, $n = 1,2, \ldots$ gives $|\hat{F}| = 1$ identically on $\mathfrak{M}_\beta$ for $\beta \neq 1$, but we have $\hat{F}(\phi_{\mathcal{N}}) = 0$. Hence we can construct an open set, say $\{\phi \in \mathfrak{M} : |\hat{F}(\phi)| < \frac{1}{2}\}$, which does not meet $\bigcup_{\beta \neq 1} \mathfrak{M}_\beta$.

Another observation; the homeomorphic copy $\mathfrak{M}'$ of $\mathfrak{M}$ lying inside $\mathfrak{M}_\alpha$ which was constructed in the last chapter for the embedding Ψ of the unit disc into $\mathfrak{M}_\alpha$, consisted of the following homomorphisms.

$$\mathfrak{M}' = \{\phi \in \mathfrak{M}_\alpha : \hat{f}(\phi) = 0 \text{ whenever } \hat{f}|\Psi(D) = 0.\}$$

If we take $\alpha = 1$, consider the non-tangential embedding, and take the function $f = \exp\left(\frac{z+1}{z-1}\right)$ discussed above, we see that $\mathfrak{M}'$ consists entirely of homomorphisms with order of approach less than 2.

§3 THE SPACE $\mathfrak{M}$

The homomorphisms of Δ all have countable neighbourhood bases, since Δ is homeomorphic with the unit disc. We see that this fact characterizes Δ as a subset of $\mathfrak{M}$.

<u>PROPOSITION 3</u>. A homomorphism $\phi \in \mathfrak{M}$ has a countable neighbourhood basis if and only if $\phi \in \Delta$. In fact any G_δ in $\mathfrak{M} \setminus \Delta$ has at least the cardinality of the continuum.

<u>Proof</u>. Take a G_δ set $A = \bigcap_{n=1}^{\infty} G_n$ which contains a homomorphism ϕ in $\mathfrak{M} \setminus \Delta$. The sets G_n are all open, and hence must meet Δ; we choose, for each n, $\phi_n \in G_n \cap \Delta$. We can without loss of generality take (z_n) to be an interpolating sequence, where z_n is just the point in D corresponding to ϕ_n (that is $\phi_n = \phi_{z_n}$). This is because there exists a subsequence of (z_n) which is an interpolating sequence, and then one can discard the G_n for n not in the subsequence and renumber the sequence. Suppose we order the rationals in the unit interval in a sequence (ω_n). There exists in $\mathfrak{B}$ a function f with $f(z_n) = \omega_n$ $n = 1,2, \ldots$. Now if λ is an arbitrary real number in the unit interval, there exists a subsequence of (ω_n) which converges to λ. Hence there is a subsequence (z_{n_k}) of (z_n) on which f approaches λ. If we take an ultrafilter on D containing the set $\{z_{n_1}, z_{n_2}, z_{n_3}, \ldots\}$, this will determine a homomorphism ϕ_λ in $\mathfrak{M} \setminus \Delta$. This homomorphism has the property first that $\phi_\lambda(f) = \lambda$, and secondly that $\phi_\lambda \in A$. Therefore A contains continuously many homomorphisms.

The first part of the Proposition follows from the second. If the sets G_n form a neighbourhood system of ϕ, then ϕ cannot be outside Δ by the argument just completed. On the other hand if $\phi \in \Delta$, then it is known to have a countable neighbourhood base.

We can obtain an estimate of the cardinality of $\mathfrak{M}$ as follows.

<u>PROPOSITION 4</u>. There are $2^{2^{\aleph_0}}$ homomorphisms in $\mathfrak{M}$.

<u>Proof</u>. We can choose a countable dense subset N in Δ, and this will also be dense in $\mathfrak{M}$. This means that there must be an ultrafilter containing N which converges to any member of $\mathfrak{M}$. This means there are at most $2^{2^{\aleph_0}}$ homomorphisms, the number of ultrafilters on N.

That this many homomorphisms do exist follows if we take a sequence (x_n), $x_n \geq 0$ of points tending exponentially to the boundary. Any ultrafilter containing the set $A = \{x_1, x_2, \ldots\}$ will give a homomorphism, and we see that different ultrafilters give different ones. Indeed two different ones will contain subsets of A which are disjoint, and this means that they contain sets separated by a positive hyperbolic distance, for we know $\psi(x_n, x_m) \geq \delta > 0$ for $m \neq n$ because (x_n) is an interpolating sequence. Hence the two ultrafilters give different *-filters, and these in turn give different homomorphisms by Proposition 1. Thus we already have $2^{2^{\aleph_0}}$ homomorphisms in $\mathfrak{M}$ by this construction.

The second part of the above proof can be readily generalized to give the following.

<u>PROPOSITION 5</u>. There are $2^{2^{\aleph_0}}$ homomorphisms in any $\mathfrak{M}_\alpha$ with any given order of approach γ to the point α where $\gamma \geq 1$.

<u>Proof</u>. We take a curve approaching α with the required order of approach, and take any sequence of points on this curve which approaches

the boundary exponentially. By the above argument this will give a collection of homomorphisms with the required cardinality, all with order of approach γ. This applies in particular, of course, to the special cases $\gamma = 1$ and $\gamma \neq 2$.

<u>COROLLARY</u>. For any α, $|\alpha| = 1$, there are $2^{2^{\aleph_0}}$ homomorphisms in $\mathcal{N}_\alpha$ and $\mathcal{D}_\alpha$.

We conclude with a similar cardinality statement about the Silov boundary.

<u>PROPOSITION 6</u>. There are $2^{2^{\aleph_0}}$ homomorphisms in the Silov boundary X.

<u>Proof</u>. We show that this many homomorphisms exist in X_α for any α. This is sufficient in view of Proposition 4.

Take an infinite sequence $E_1, E_2, \ldots$ of open intervals on Γ which converge to some α. We may assume that these intervals have disjoint closures. We can take any ultrafilters $\mathcal{U}$ on the integers N and from it construct a filter on Γ consisting of all sets of the form $\bigcup_{n \in U} E_n$ for $U \in \mathcal{U}$. This is a filter of sets of positive measure, and it determines a homomorphism in X_α. A different ultrafilter $\mathcal{V}$ on N will determine a different homomorphism of X_α because there will exist sets $U \in \mathcal{U}$, $V \in \mathcal{V}$ with $U \cap V = \emptyset$. From this it follows that X_α has the stated cardinality.

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