PHASE SHIFT APPROXIMATION TO REACTION MATRIX ELEMENTS IN AN OSCILLATOR REPRESENTATION

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Бу

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Phase Shift Approximation to Reaction Matrix Elements in an Oscillator Representation

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This thesis presents a derivation of a method to obtain two-body, diagonal and non-diagonal, reaction matrix elements for central and tensor forces respectively directly from nucleon-nucleon scattering phase shifts. This procedure eliminates the necessity for constructing a nuclear potential.

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PREFACE

In recent years, many attempts have been made to calculate the structure of real nuclei starting from first principles, using two-body forces which accurately describe the interaction of two free nucleons. Among these, T. T. S. Kuo and G. E. Brown (1), C. W. Wong (2), H. S. Kohler (3), A. D. Mackellar and R. L. Beckar (4), A. K. Kerman et al (5), and M. K. Pal and A. P. Stamp (6) have used methods derived from the theory of nuclear matter. The procedure consists in the determination and evaluation of the G-matrix. This G-matrix takes the two-body interaction into account to all orders of perturbation theory in the presence of other particles (Pauli and dispersive Effects). Of course, all the theories referred to above, make some approximation in their evaluation of the G-matrix, and the crux of the matter is to decide which of these approximations is the best.

On the other hand, there is also considerable ambiguity in the theory of the potential V(r). Starting with the Gammel-Thaler potential, which was fitted to low energy data and to Stapp's phase shifts at 310 MeV, there has been a steady increase in the accuracy with which phenomenological potentials reproduce observed scattering

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There has also been a steady increase in the amount data. and quality of the data to be fitted. Presumably the modern phenomenological potentials such as Hamada-Johnston (7), Yale (8), Reid (9), and Bressel-Kerman (10) are of comparable validity. The best phenomenological potentials fit nearly one thousand pieces of n-p and p-p scattering data with a χ^2 ratio approaching 2.0 or slightly more. This figure, of course, depends somewhat on the selection of data employed. A potential fitted to one data selection would not seem very good when tested against another set of data. P. Signell, for example, has computed χ^2 for a large number of the more recent potentials and he might be considered an unbiased umpire in this field. Without going into the validity of particular models, we believe that there is not a great deal to choose between them on the basis of "goodness of fit". At the present time, the main progress is in the area of n-p triple scattering experiments (as evidenced, for example, in the Florida Conference report (24)) and in careful analysis of low energy experiments which now seem to yield definite results even for the P-waves in p-p scattering in the ten to fifty MEV range (1.1).

Finally there is the question of whether the short range, repulsive two-body force is better described as an infinite hard core, a finite square core, a Yukawa core,

Private communication from Professor Sprung.

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*

a separable non-local core (12), or an energy independent boundary condition (13). The scattering data by itself is unable to resolve this ambiguity.

To try to avoid the preceding uncertainties, in the force and in the method of calculation, Elliott, Sanderson, and Mavromatis (14) have proposed an elegant alternative to the construction of the nuclear potential V(r) for use in nuclear structure calculations. The required matrix elements of G between finite nucleus, single particle eigenstates would be evaluated directly as a weighted sum of integrals over the two-body scattering phase shifts. These phase shifts can be taken from various fits to the data, for example, those of G. Breit <u>et al</u> (15) or R. A. Arndt and M. H. MacGregor (16).

Actually, Elliott's method is based on the following two assumptions. The first is that the wave functions for the finite nucleus are taken to be those of the harmonic oscillator. More general ones, such as Woods-Saxon, could be expanded in oscillator states however. Secondly, Elliott's formulation relies to some extent on the assumption that the basic two-body interaction can be represented by a weak, finite, and local interaction V(r). In this case, the present derivation, like Elliott's is based on a Born approximation. D. S. Koltun (18), however, has been able to derive a very similar formula on the

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opposite assumption that the nuclear force is very strong and short ranged. This indicates, possibly, that the method has a wider range of validity than the Born approximation would suggest. However, only V(r) and not G is considered.

A different way of looking at this theory has been discussed, for example, by D. M. Brink and R. E. Peierls (26). We assume some two-body potential exists which may have any degree of complexity in strength and non locality. According to the Brueckner theory, one should solve for the finite nucleus reaction matrix G^N by the following equation:

$$G^{N} = V - V \frac{Q}{e} G^{N}$$

Here the operator Q is the Pauli operator which prevents scattering of particles into occupied states. The quantity e is the energy denominator which, in a many-body system, includes both kinetic and potential energy effects. It is possible to relate G^N to the free two-nucleon reaction matrix t^F :

 $t^{F} = V - V \frac{P}{e_{o}} t^{F}$; $P \rightarrow Principal value$ $e_{o} \rightarrow Kinetic energies only$

by the following relation.

$$G^{N} = t^{F} - t^{F} \left(\frac{Q}{e} - \frac{P}{e}\right) G^{N}$$

This relation has been given by Bethe, Brandow and Petschek (29)

who used it to relate the nuclear matter G-matrix to the reference matrix G^R. This approach to approximate the nuclear reaction matrix by the free two nucleon reaction matrix has, in fact, been advocated for several years by Kahana and co-workers. Provided the effects of $\frac{Q}{e}$ approximately cancel the effects of $\frac{P}{e_{a}}$, the method will have great merit. In binding energy calculations, one certainly cannot justify this. But for calculation of nuclear spectra, one might expect reasonable results for those nucleons near the top of the Fermi sea because they are in a low density region. One may go further and include correction terms to t^{F} ; this is being studied. To the extent that t^F approximates G^N , the nuclear problem reduces to evaluating matrix elements of t^F between harmonic oscillator states. Since diagonal matrix elements of t^F between plane wave states are related to the tangents of the phase shifts, Elliott's method can be applied and will then express the desired G^N matrix elements as integrals over the phase shifts. There is, however, one remaining difficulty. V or t^F must be local (26) for Elliott's formulation to work. Undoubtedly t^F is a very non local operator in coordinate space. However, it may be possible to approximate t^F by some local \hat{t}^F making errors comparable to those in the previous assumptions. For example, in the work of Bethe and Siemens*, it has been shown that the G-matrix in

> Private communication from Dr. H.A. Bethe and Mr. P. Siemens. (viii)

nuclear matter can be reasonably well approximated by a local, effective operator G. Equivalent suggestions have been made by R. K. Bhaduri and C. S. Warke (28), and by I. J. Donnelly (30). It is reasonable to hope that t^F can be similarly approximated, and thus the use of Elliott's method would be justified.

For simplicity in the following discussion, we will write the equations as though applying Elliott's method for a weak potential V(r); however, this other viewpoint of regarding V(r) as a local free two nucleon reaction matrix should be kept in mind.

We now illustrate the application of Elliott's method for reduced matrix elements in conjunction with Moshinsky-Talmi (17) techniques. When the relative matrix elements are known, the two-body matrix element can be computed as shown below (27).

 $< (l_{1} l_{2}) j_{1} (l_{2} l_{2}) j_{2} JMTT_{z} | V_{eff} | (l_{3} l_{2}) j_{3} (l_{4} l_{2}) j_{4} JMTT_{z} >$ $= \frac{1}{(1 + \delta_{n_{1}n_{2}} \delta_{l_{1}l_{2}} \delta_{j_{1}j_{2}})^{l_{2}}} \cdot \frac{1}{(1 + \delta_{n_{3}n_{4}} \delta_{l_{3}l_{4}} \delta_{j_{3}j_{4}})^{l_{2}}}$ $\times \sum_{\substack{\lambda \\ \lambda \lambda' J'nn'} l' k NL} (-)^{\lambda + \lambda'} (2J' + 1) (2\lambda + 1) (2\lambda' + 1) (2S + 1)$

 $:: \left[(2j_{1}+1) (2j_{2}+1) (2j_{3}+1) (2j_{4}+1) \right]^{\frac{1}{2}} < n_{1} \ell_{1} n_{2} \ell_{2} \lambda | n \ell NL \lambda >$ $: < n_{3} \ell_{3} n_{4} \ell_{4} \lambda' | n' \ell' NL \lambda' > \begin{cases} \ell_{1} \ell_{2} \lambda \\ \frac{1}{2} \ell_{2} S \\ j_{1} j_{2} J \end{cases} \begin{cases} \ell_{3} \ell_{4} \lambda \\ \frac{1}{2} \ell_{2} S \\ j_{3} j_{4} J \end{cases}$

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 $x \begin{cases} L \ell' \lambda' \\ S J J' \end{cases} \begin{cases} L \ell \lambda \\ S J J' \end{cases} \begin{bmatrix} L \ell \lambda \\ S J J' \end{bmatrix} \begin{bmatrix} 1 - (-)^{\ell+S+T} \end{bmatrix}$ $x < n\ell SJ' | V_{eff} | n'\ell' SJ' >$

The above is obtained by transforming the two particle state from j-j coupling scheme to the L-S representation with inclusion of the Moshinsky transformation. The quantity we are interested in is the reduced matrix element $\langle n \ell S J' | V_{eff} | n' \ell' S J' \rangle$; all other factors are essentially geometric in origin and contain no physics about the interaction. The usual method for obtaining the reduced matrix element will now be described.

Since the oscillator wave functions are partly comprised of Laguerre functions which are polynomials in the square of the relative coordinate r, any relative matrix element evaluated in an oscillator representation is expressed essentially as a weighted sum of terms. These terms are integrals over various even powers of r times the interaction potential and a qaussian in r. Hence, according to Talmi (17), a two-body relative matrix element can be written as

$$\langle R_{n\ell} | V(r) | R_{n'\ell'} \rangle = \Sigma B(n,\ell,n',\ell',p) I_{p}$$

where

$$R_{n\ell}(r) = \left[\frac{2n!}{\Gamma(n+\ell+3/2)}\right]^{\frac{1}{2}} r^{\ell} e^{-r^{2}/2} L_{n}^{\ell+1/2}(r^{2})$$

is the radial part of the general oscillator eigenstate. The $L_n^{\ell+1/2}(r^2)$ are Laguerre polynomials as defined by Erdelyi <u>et al</u> (25) in the relative coordinate r. The $B(n,\ell,n',\ell',p)$ factors are Talmi coefficients and I. p are Talmi integrals as defined as follows:

$$Ip = \frac{2}{\Gamma(p+3/2)} \int_{0}^{\infty} r^{2p} e^{-r^{2}} V(r) r^{2} dr$$

The summation index p is always integral valued since the following inequality must hold

$$\frac{1}{2}(l+l')$$

and where l = l' for diagonal matrix elements and $l = l' \pm 2$ for off diagonal matrix elements.

In the present derivation, we continue to use the above formulation of Talmi, but follow the suggestion of Elliott for evaluating Talmi integrals not as in their definition but as in their relation to integrals over the phase shifts. For l=l'=1 and n=n'=0, only one Talmi integral is needed. Elliott et al have given for this case

$$I_{1} = \frac{2}{3} \left(\frac{n\omega}{\pi} \right) \int e^{-E} (1 + 4E - 4E^{2}) \tan \delta_{\ell} (E) dE$$

where δ_{ℓ} are the known phase shifts appropriate to this value of ℓ and E is the laboratory energy of the nucleon in units of the oscillator energy $\hbar\omega$. Since B(0,1,0,1,1) = 1,

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it follows that the relative matrix element is equal to I1.

 $<0,1|V_{1}(r)|0,1> = I_{1}$

The purpose of this thesis is threefold:

- To generalize Elliott's method to all n and l.
 To find a convenient way to evaluate the Talmi integrals -- a recursive method is derived.
- 3) To generalize to the case of a tensor force where matrix elements which are non-diagonal in the orbital angular momentum & occur.

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CHAPTER I

MATHEMATICAL FORMULATION

Since most functions found in physics can be expressed in hypergeometric expansions, it would be desirable to obtain an integral over hypergeometric functions whose value is also expressed as a hypergeometric series. This expression was obtained from Slater (18) and is given in equation (1.1). This is useful since we desire an integral over spherical Bessel functions to be equated to various powers of r^2 . These powers of r^2 will then yield a relation between Talmi integrals and integrals over the phase shifts. The general hypergeometric function ${}_{A}F_{B}$ with A arguments of type (a) and B arguments of type (b) is defined as follows:

$$= \sum_{\nu=0}^{\infty} \frac{(a_1, a_2, \dots, a_A)}{(b_1, b_2, \dots, b_B)} ; x$$

Here,

$$(a)_{1} = a(a+1)(a+2) \dots (a-1+v)$$

When A < B, the above hypergeometric function is absolutely

convergent for all values of the factors a_i of type (a) and b_i of type (b) provided that x is finite and none of the b_i is zero or a negative integer. We begin with the following expression:

$$\int_{0}^{\infty} e^{-k't} t^{d-1} A^{F}_{B} [(a); (b); kt] I^{F}_{1} [a'; b'; k't] dt$$

$$= (k')^{-d} \frac{\Gamma(d) \Gamma(b') \Gamma(b'-a'-d)}{\Gamma(b'-a') \Gamma(b'-d)}$$
(1.1)

$$x_{A+2}F_{B+1} [(a), d, 1+d-b'; (b), 1+d+a'-b'; \frac{k}{k'}]$$

In equation (1.1), we now make the following substitutions which are consistent with the requirements for its validity:

$$k' = q^{2}$$
; $k = -r^{2}$
 $d = l+3/2$; $t = k^{2}$
 $a' = -l-l-u$; $b' = 1/2-k$

Here μ , ℓ are positive integers or zero. The form of the hypergeometric function is chosen to have the following construction:

$${}_{A}F_{B} \left[(a); (b); kt \right] = {}_{1}F_{2} \left[\ell+1; \ell+3/2, 2\ell+2; -k^{2}r^{2} \right]$$

Equation (1.1), with the new variables, appears below.
$${}_{2} \int_{0}^{\infty} e^{-k^{2}q^{2}} k^{2\ell} {}_{1}F_{2}(\ell+1; \ell+3/2, 2\ell+2; -k^{2}r^{2})$$
$${}_{x} {}_{1}F_{1}(-\ell-1-\mu; 1/2-\ell; k^{2}q^{2})k^{2}dk$$
$$= \frac{1}{(q)^{2\ell+3}} \frac{\Gamma(\ell+3/2)}{\Gamma(\mu+3/2)} \frac{\Gamma(1/2-\ell)}{\Gamma(-2\ell-1)} \frac{\Gamma(\mu-\ell)}{\Gamma(-2\ell-1)}$$

 $x_{3}F_{3}$ [l+1, l+3/2, 2l+2: l+3/2, 2l+2, $l+1-\mu$; $-r^{2}/q_{2}$] (1.2) Since we will require the hypergeometric function ${}_1F_2$ in equation (1.2) to eventually have the form of the square of the spherical Bessel function, the choice of a' and b' in $_{1}F_{1}$ (a'; b'; k't), appears to be arbitrary. However, an improper choice for these variables will lead to an infinite series for the hypergeometric function $A+2^{F}_{B+1}$ which would present a great inconvenience from a numerical point of view. But for the selection of a' and b' that has been made, the hypergeometric function $A+2^{F}_{B+1}$ is reduced to ${}_{1}F_{1}$ as well as becoming a polynomial of low order. The polynomial ₁F₁ (a', b', k't) with the new variables is actually a member of the family of Laguerre polynomials but is identical to neither of the common ones associated with the wavefunctions of the hydrogen atom nor the spherical harmonic oscillator. It is immediately apparent that the hypergeometric function ${}_{3}F_{3}$ reduces in Eq.(1.2) to 1^{F_1} since the two factors which are common in both the numerator and denominator will cancel out in all terms of the series. One must now investigate the behaviour of

$$\frac{\Gamma(\mu-\ell)}{\Gamma(-2\ell-1)} {}_{1}F_{1}(\ell+1; \ell+1-\mu; \frac{-r^{2}}{q^{2}})$$

which appears in right side of equation (1.2). Since the gamma function of zero or a negative integer is infinite and the hypergeometric function contains a factor $\ell+1-\mu$ in

the denominator, a zero will appear in some term and all succeeding ones. We now study two cases.

Case I: where ℓ-µ≥0

Now the term $l+l-\mu$ can never be zero or a negative integer so that the hypergeometric function is well behaved. The factor $\frac{\Gamma(\mu-l)}{\Gamma(-2l-1)}$ outside consists of the ratio of two gamma functions of negative integral arguments. Abramowitz and Stegun (19) give for gamma function of negative integers:

$$\lim_{z \to n} \Gamma(-z) = \frac{(-)^n}{n!} \lim_{z \to n} (\frac{1}{z-n})$$

where n is zero or a positive integer. Therefore

$$\frac{\Gamma(-(\ell-\mu))}{\Gamma(-(2\ell+1))} = (-)^{\ell+\mu+1} \frac{(2\ell+1)!}{(\ell-\mu)!}$$

and is no longer indeterminate.

Case II: 1-µ<o

Now the hypergeometric function containes a term $l+l-\mu$ which will yield zeros in the denominator but $\Gamma(\mu-l)$ is now well behaved. Here one must take the ratio of the limit of each term as one of its factors approaches zero with the limit of the gamma function $\Gamma(-2l-1)$. It will be found that those terms in the hypergeometric series which contained no singularities are now zero and those which did are now finite.

We now desire the expansion of a product of two

ordinary Bessel functions in terms of a hypergeometric series. From Rainville (20), we obtain n+m

$$J_{n}(x) J_{m}(x) = \frac{\left(\frac{x}{2}\right)}{\Gamma(n+1) \Gamma(m+1)}$$

 $x_{2}F_{3}\left[\frac{n+m+1}{2}, \frac{n+m+2}{2}; n+1, m+1, n+m+1, -x^{2}\right].$ (1.3)

The spherical Bessel functions, which are the eigenfunctions of a free particle in spherical coordinates, are related, according to Schiff (21), as follows

$$j_{\ell}(x) = \sqrt{\frac{\pi}{2\chi}} J_{\ell+\frac{1}{2\chi}}(x)$$

Hence for $n=m=l+\frac{1}{2}$

$$j_{\ell}^{2}(kr) = \frac{\left(\frac{\pi}{2}\right)}{2^{\ell+1}} \frac{(kr)^{2\ell}}{\left[\Gamma(\ell+3/2)\right]^{2}}$$

× $_{1}F_{2}$ [$\ell+1$; $\ell+3/2$, $2\ell+2$; $-k^{2}r^{2}$] . (1.4)

The hypergeometric function ${}_{1}F_{2}$ in equation (1.4) is exactly that obtained under the integral in equation (1.2) and is thus replaced by the spherical Bessel function squared to yield

$$\frac{2}{\pi} \int_{0}^{\infty} e^{-k^{2}q^{2}} 2^{2\ell+2} \frac{j_{\ell}^{2}(kr)}{r^{2\ell}} \left[\Gamma(\ell+3/2) \right]^{2}$$

$$\times {}_{1}F_{1} \left[-\ell - 1 - \mu; 1/2 - \ell; k^{2}q^{2} \right] k^{2}dk$$

$$= \frac{1}{q^{2\ell+3}} \frac{\Gamma(\ell+3/2)}{\Gamma(\mu+3/2)} \frac{\Gamma(1/2 - \ell) \Gamma(\mu-\ell)}{\Gamma(-2\ell-1)} {}_{1}F_{1} \left[\ell+1; \ell+1 - \mu; -\frac{r^{2}}{q^{2}} \right]$$
(1.5)

It is now desirable to convert the hypergeometric function on the right side of equation (1.5) into a new hypergeometric function times an exponential which when multiplied by $r^{2\ell}$ begins to appear like a product of oscillator wavefunctions. From Slater (18), we have that

$$e^{-x} {}_{1}F_{1} [a; b; x] = {}_{1}F_{1} [b-a; b; -x].$$

It can also be shown that, for $l-\mu>0$

$$\frac{\Gamma(1/2-\ell)}{\Gamma(\mu+3/2)} \frac{\Gamma(\mu-\ell)}{\Gamma(-2\ell-1)} = -(-)^{\mu} \frac{2^{2\ell+2+\mu}}{(2\mu+1)!!} (\ell+1/2) \frac{\ell!}{(\ell-\mu)!}$$

where

 $(2\mu+1)!! = (2\mu+1)(2\mu-1)(2\mu-3) \dots 5.3.1$

The resulting equation takes the following form:

$$\int_{0}^{\pi} e^{-k^{2}q^{2}} j_{\ell}^{2} (kr) {}_{1}F_{1}(-\ell-1-\mu; {}_{2}^{\ell}-\ell; k^{2}q^{2})k^{2}dk$$

$$= \frac{(-\frac{\pi}{2})}{q^{3}} \frac{(-)^{\mu} 2^{\mu}}{\Gamma(\ell+{}_{2}^{\ell})(2\mu+1)!!} \frac{(\frac{r^{2}}{2})}{q^{2}} e^{-r^{2}/q^{2}} \frac{\ell!}{(\ell-\mu)!}$$

$$\times {}_{1}F_{1}(-\mu; \ell+1-\mu; \frac{r^{2}}{q^{2}}) \qquad (1.6)$$

The general procedure to be followed in the succeeding chapters will now be explained. Both sides of equation (1.6) are multiplied by appropriate nuclear potential and integrations performed over r. The integral over the square of the spherical Bessel function is replaced by a phase shift and the integrals over the finite number of terms of the hypergeometric function become Talmi integrals. The number of integrals obtained depends on the value of μ , as well as ℓ . To get the appropriate Talmi integrals, for $\mu = \mu_0$, one needs to know the Talmi integrals corresponding to $\mu = \mu_0$ -1. Hence one always begins at $\mu = 0$, in which case the single Talmi integral may be evaluated numerically, and procedes by recursion to obtain all Talmi integrals required for the maximum value of μ . When multiplied by Talmi coefficients and summed, these Talmi integrals yield the two-body relative matrix element. What makes one matrix element different from another is the number of terms which are summed since this depends on μ and where $0 \leq \mu \leq n+n'$.

CHAPTER II

DIAGONAL MATRIX ELEMENTS

We now consider the factor

$$\frac{\ell!}{(\ell-\mu)!} {}_{1}F_{1}(-\mu; \ell+1-\mu; r^{2}/q^{2})$$

in equation (1.6). Since μ is a positive integer, the hypergeometric function is a polynomial of degree μ and if $\ell-\mu>0$ the expression is well behaved. When $\ell-\mu<0$ then $\frac{\ell!}{(\ell-\mu)!}$ vanishes alone but when multiplied by the hypergeometric function, cancellations occur which yield a non zero result. Specifically,

$$\frac{\ell!}{(\ell-\mu)!} {}_{1}F_{1}(-\mu, \ell+1-\mu, r^{2}/q^{2}) = \frac{\ell!}{(\ell-\mu)!} \left[1 + \frac{(-\mu)}{(\ell-\mu+1)} - \frac{q}{1!} + \frac{(-\mu)(-\mu+1)}{(\ell-\mu+2)(\ell-\mu+1)} - \frac{q}{2!} + \dots \right]$$

$$= \ell! \left[\frac{1}{(\ell-\mu)!} + \frac{(-\mu)}{(\ell-\mu+1)!} - \frac{q}{1!} + \frac{(-\mu)(-\mu+1)}{(\ell-\mu+1)!} - \frac{q}{2!} + \dots \right]$$

$$= \frac{\mu}{\nu} \left[(-\mu)^{\nu} + \frac{\mu!\ell!}{(\mu-\nu)!} + \frac{q}{(\ell-\mu+\nu)!} - \frac{q}{\nu!} + \frac{(-\mu)(-\mu+1)}{(\ell-\mu+1)!} - \frac{q}{2!} + \dots \right]$$

where

$$\alpha = \max \{0, \mu - \ell\}$$

We now make the substitution $v' = \mu - v$ to obtain

$$(-)^{\mu} \sum_{\nu=0}^{\beta} \frac{(-)^{\nu} \mu! \ell! q^{2}}{(\mu-\nu)! (\ell-\nu)! \nu!}$$

where $\beta = \min \{\mu, \ell\}$. (2.1)

The new form of equation (1.6) is as follows

$$\int_{0}^{\infty} e^{-k^{2}q^{2}} j_{\ell}^{2} (kr) {}_{1}F_{1}(-\ell-1-\mu; {}^{l}_{2}-\ell; k^{2}q^{2}) k^{2}dk$$

$$= \frac{(-\frac{\pi}{2})}{q^{3}} \frac{2^{\mu} \frac{(r^{2})}{q^{2}} e^{-r^{2}/q^{2}}}{F(\ell+\frac{1}{2})(2\mu+1)!!} \sum_{\nu=0}^{\beta} \frac{(-)^{\nu}\mu!\ell! q^{2}}{(\mu-\nu)!(\ell-\nu)! \nu!}$$
(2.2)

where the factor $(-)^{2\mu}$ drops out since it is always positive. We wish to make the terms in the sum in equation (2.2) into Talmi integrals. The Talmi integrals will now be defined in a slightly more general manner as given in the introduction. We consider an uncoupled partial wave with a definite l, S, J, and set

$$I(\ell, \mu) = \frac{2}{\Gamma(\ell+\mu+3/2)} \int_{0}^{\infty} (\frac{r^{2}}{q^{2}})^{\ell+\mu} e^{-r^{2}/q^{2}} V_{\ell}(r) \frac{r^{2}}{q^{2}} \frac{dr}{q} (2.3)$$

where $V_{\ell}(r)$ is the effective potential appropriate to the value of ℓ and a relative matrix element is now expressed for $\ell = \ell'$

$$\langle n\ell | V_{\ell}(\mathbf{r}) | n'\ell \rangle = \Sigma B(n, \ell, n', \ell, \ell+\mu) I(\ell, \mu)$$

 μ

and the former P is written as

$$P = \frac{1}{2}(\ell + \ell') + \mu$$

so that in the summation, we have

$$0 < \mu < n+n!$$

This just reflects the fact that each oscillator wavefunction contains a factor r^{ℓ} . An explicit calculation of a diagonal

matrix element l=l'=l and $\mu=l$ will be performed below. In this case, the reduced and rearranged form of equation (2.2) is

$$\frac{(-2)}{\pi} \Gamma(\ell + \frac{1}{2}) q^{3} \int_{0}^{\infty} e^{-k^{2}q^{2}} j_{\ell}^{2}(kr) \Gamma^{F_{1}}(-\ell - 1 - \mu; \frac{1}{2} - \ell; k^{2}q^{2}) k^{2} dk$$

$$= \frac{2^{\mu}}{(2\mu + 1)!!} (\frac{r^{2}}{q^{2}})^{\ell} e^{-r^{2}/q^{2}} \left[(\frac{r^{2}}{q^{2}})^{\mu} - \frac{\mu\ell}{1!} (\frac{r^{2}}{q^{2}})^{\mu - 1} \right]$$

$$(2.4)$$

where the explicit ℓ and μ dependence is shown. Equation (2.4) is multiplied by

$$\frac{2}{\Gamma(\ell+\mu+3/2)}\int_{0}^{\infty}V_{\ell}(r)\frac{r^{2}}{q^{2}}\frac{dr}{q}$$

and the order of integration over k and r is interchanged to obtain

$$\frac{(-2)}{\pi} \Gamma(\ell+1/2) \frac{2}{\Gamma(\ell+\mu+3/2)} \int_{0}^{\infty} e^{-k^{2}q^{2}} \left[\int_{0}^{\infty} j_{\ell}^{2}(kr) V_{\ell}(r)r^{2}dr \right]$$

$$\times {}_{1}F_{1} \left[-\ell - 1 - \mu; 1/2 - \ell; k^{2}q^{2} \right] k^{2}dk$$

$$= \frac{2^{\mu}}{(2\mu+1)!!} 2 \int_{0}^{\infty} (\frac{r^{2}}{q^{2}})^{\ell} e^{-r^{2}/q^{2}} \left[\frac{(\frac{r^{2}}{2})^{\mu}}{\Gamma(\ell+\mu+3/2)} - \frac{(\frac{r^{2}}{q^{2}})^{\ell}}{(\ell+\mu+1/2)} \right] V_{\ell}(r) \frac{r^{2}}{q^{2}} \frac{dr}{q}$$

$$(2.5)$$

Inside the integral over k of the left side of equation (2.5), an expression appears which is identical to the first order Born approximation for the phase shift which is given

(2.4)

in Schiff (21).

$$-\frac{\hbar^{2} \delta_{\ell}(k^{2})}{Mk} = \int_{\ell}^{\infty} j_{\ell}^{2}(kr) V_{\ell}(r)r^{2}dr \qquad (2.6)$$

where M is the nucleon mass, $\hbar k$ is the momentum of one nucleon in the center of mass frame, and $\delta_{\ell}(k^2)$ are the phase shifts appropriate to the value of ℓ for various values of the energy which is proportional to the square of k. The energy of the nucleon projectile in the laboratory frame is

$$E_{LAB} = \frac{2\hbar^2 k^2}{M}$$

and the oscillator energy is $\frac{2\hbar^2}{mq^2} = \hbar\omega$. The factor 2 in the preceding equation occurs because the usual size parameter b is related to the oscillator energy by $\hbar\omega = \frac{\hbar^2}{mb^2}$. The relative coordinate is expressed for this definition of b as $\bar{r} = \bar{r}_1 - \bar{r}_2$. However, in Moshinsky's notion $\bar{r}_m = \frac{1}{\sqrt{2}}(\bar{r}_1 - \bar{r}_2)$ so that the size parameter squared is $q^2 = 2b^2$. An additional parameter E is introduced which is equal to the laboratory energy in units of the oscillator energy $\hbar\omega$.

$$E = \frac{E_{LAB}}{\hbar\omega} = k^2 q^2$$

The factor

$$\frac{\Gamma(l+1/2)}{\Gamma(l+\mu+3/2)}$$

can be shown to be equal to

$$\frac{2^{\mu+1}}{(2\ell+1)} \frac{(2\ell+1)!!}{(2\ell+2\mu+1)!!}$$

Thus the left side of equation (2.5) becomes

$$\frac{-(2)^{\mu+1}}{(2\ell+1)} \frac{(2\ell+1)!!}{(2\ell+2\mu+1)!!} \frac{4}{\pi} \int_{0}^{\infty} e^{-k^{2}q^{2}} \left[-\frac{\hbar^{2}}{M} \delta_{\ell}(\epsilon) \right]$$

$$\times {}_{1}F_{1}(-\ell-1-\mu, \frac{1}{2}-\ell, k^{2}q^{2}) kdk$$

We observe that the right side of equation (2.5) is just a combination of two Talmi integrals and that

$$\frac{dE}{2q^2} = kdk$$

The final expression for the case $l = l' = l = \mu$ is

$$I(\ell, \mu) - \frac{\mu \ell}{(\ell + \mu + \frac{1}{2})} I(\ell, \mu - 1)$$

$$= \frac{2(2\mu + 1)!!(2\ell + 1)!!}{(2\ell + 1)(2\ell + 2\mu + 1)!!} (\frac{\pi}{\pi}) \int_{0}^{\infty} e^{-E} \delta_{\ell}(E)$$

$$\times {}_{1}F_{1}(-\ell - 1 - \mu; \frac{1}{2} - \ell; E) dE$$
(2.7)

By substituting $\mu = 0$ up to $\mu = n+n'$ in equation (2.7), one obtains the required Talmi integrals which when multiplied by the appropriate Talmi coefficients and summed provide the relative matrix element. It may be noted that for $\mu = 0$ in equation (2.7) one obtains the same formula as given by Elliott et al for $\ell = \ell' = 1$ and n+n' = 0. The general expression for arbitrary μ and ℓ follows $\sum_{\nu=0}^{\beta} (-)^{\nu} \frac{\mu! \ell! 2^{\nu}}{(\mu-\nu)!} \frac{(2\ell+2\mu+1-2\nu)!!}{(\ell-\nu)! \nu! (2\ell+2\mu+1)!!} I(\ell, \mu-\nu)$ (2.8) $= \frac{2(2\mu+1)!!}{(2\ell+1)!} \cdot \frac{(2\ell+2\mu+1)!!}{(2\ell+2\mu+1)!!} (\frac{\hbar\omega}{\pi}) \int_{0}^{\infty} e^{-E} \delta_{\ell}(E) \frac{1}{2} F_{1}(-\ell-1-\mu; \frac{1}{2}-\ell, E) dE$. The method presented here applies equally well to the diagonal matrix elements (l=l') in the case of coupled states of angular momentum J and parity $(-)^{J+1}$. The off-diagonal matrix elements with $l=l'\pm 2$ are considered next.

CHAPTER III

OFF-DIAGONAL MATRIX ELEMENTS

In this chapter, we apply the techniques for obtaining diagonal matrix elements to off-diagonal matrix elements where $|\ell - \ell'| = 2$. Coupled states due to the tensor force consist of two triplet states with the same total angular momentum but whose orbital angular momenta differ by 2. What we have to work with are the part of the total phase shift that would be produced if the tensor force alone acted for each member of the coupled pair as well as a coupling parameter which describes the strength of the interaction. One might have expected that only the coupling parameter would be required, but this is not the case.

To get a useful formula for this case, it will become apparent that one must use equation (2.2) with the value of ℓ being numerically identical to the total angular momentum J of the coupled state. Thus for the ${}^{3}S_{1} - {}^{3}D_{1}$ coupled state equation (2.2) with $\ell=1$ was used and the notation changed from ℓ to J to obtain

$$-\left(\frac{2}{\pi}\right) \Gamma\left(J+\frac{1}{2}\right)q^{3} \int_{0}^{\infty} e^{-k^{2}q^{2}} j_{J}^{2}(kr) {}_{1}F_{1}\left[-J-1-\mu; \frac{1}{2}-J; k^{2}q^{2}\right] k^{2}dk$$

$$= \frac{2^{\mu}}{(2\mu+1)!!} \left(\frac{r^{2}}{q^{2}}\right)^{J} e^{-r^{2}/q^{2}} \left[\left(\frac{r^{2}}{q^{2}}\right)^{\mu} - \frac{\mu}{1!} \frac{J}{q^{2}}\left(\frac{r^{2}}{q^{2}}\right)^{\mu-1}\right] \qquad (3.1)$$

where the μ and J dependence is maintained but is valid only for J=1. (General J is considered below). We wish to convert the square of spherical Eessel function j_J into a product of Bessel functions $j_{l=J-1}(kr)$ and $j_{l=J+1}(kr)$ which are related to the component states of the coupled one. To achieve this, we operate on equation (3.1) with $r \frac{\partial}{\partial r}$ observing that the only r-dependence occurs in the Bessel function and the terms on the right hand side. Since

$$r \frac{\partial}{\partial r}(j_J^2(kr)) = 2kr j_J(kr) \frac{\partial}{\partial (kr)}[j_J(kr)]$$

and from Schiff we obtain the derivative of the spherical Bessel function to achieve

$$\frac{\partial}{\partial r} \left[j_J^2(kr) \right]$$

 $= \frac{2k^{2}r^{2}}{(2J+1)^{2}} \left[J j_{J-1}^{2}(kr) - (J+1) j_{J+1}^{2}(kr) - j_{J+1}(kr) j_{J-1}(kr) \right]$ (3.2)

Using this expression and carrying out the differentiation of the right side, the new form of the equation is:

$$-\left(\frac{2}{\pi}\right) \Gamma\left(J+\frac{1}{2}\right)q^{3} \int_{0}^{\infty} e^{-k^{2}q^{2}} \frac{2k^{2}r^{2}}{\left(2J+1\right)^{2}} \left[J j_{J-1}^{2}(kr) - \left(J+1\right)j_{J+1}^{2}(kr)\right]$$

$$- j_{J-1}(kr) j_{J+1}(kr)]_{1}F_{1} [-J-1-\mu; \frac{1}{2}-J; k^{2}q^{2}] k^{2}dk$$

$$= \frac{2^{\mu+1}}{(2\mu+1)!!} (\frac{r^{2}}{q^{2}})^{J} e^{-r^{2}/q^{2}} [-(\frac{r^{2}}{q^{2}})^{\mu} + (\mu J+J+\mu) (\frac{r^{2}}{q^{2}})^{\mu-1}$$

$$- \mu J (J+\mu-1) (\frac{r^{2}}{q^{2}})^{\mu-2}] (\frac{r^{2}}{q^{2}}) . (3.3)$$

In a completely analogous manner to that in Chapter II, we now write down the Born approximation to the phase shifts τ_{ℓ} and $\tau_{\ell'=\ell+2}$ due to the tensor force for both partial waves and the coupling parameter ε^{J} as follows:

$$-\frac{\hbar^{2}}{Mk}\tau_{J-1}(k^{2}) = \int_{0}^{\infty} j_{J-1}^{2}(kr) V_{T}(r) r^{2} dr$$

$$-\frac{\hbar^{2}}{Mk}\tau_{J+1}(k^{2}) = \int_{0}^{\infty} j_{J+1}^{2}(kr) V_{T}(r) r^{2} dr$$
(3.4)

and

$$\frac{-h^{2} \varepsilon^{J}}{Mk} (k^{2}) = \frac{6 \sqrt{J} (J+1)}{(2J+1)} \int_{0}^{\infty} j_{J-1} (kr) j_{J+1} (kr) V_{T}(r) r^{2} dr$$

 $V_{\rm T}(r)$ is the radial part of the tensor interaction and the J-dependent factor is the matrix element of S_{12} . Again we wish to make Talmi integrals out of the terms on the right side of equation (3.3), so we multiply equation (3.3) by

$$\frac{1}{\Gamma(J+\mu+3/2)}\int_{0}^{\infty}V_{T}(r)\frac{dr}{q}$$

and interchange the order of integration over k and r on the left side. Hence

$$\frac{(2J+1)!! 2^{\mu+1}}{(2J+2\mu+1)!! (2J+1)^3} \left(\frac{h\omega}{\pi}\right) \int_{0}^{\infty} e^{-E} \left[J \tau_{J-1}(E) - (J+1) \tau_{J+1}(E)\right]$$
(3.5)

$$= \frac{(2J+1)}{6\sqrt{J(J+1)}} e^{J(E)} \frac{1}{1} r_{1}(-J-1-\mu; \frac{1}{2}-J; E) EdE$$

$$= \frac{2^{\mu}}{(2\mu+1)!!} 2 \int_{0}^{\infty} e^{-r^{2}/q^{2}} \left[\frac{-(\frac{r}{2})}{r(J+\mu+3/2)} + \frac{(\mu J+J+\mu)}{(J+\mu+1/2)} \frac{q}{r(J+\mu+1/2)} - \frac{(\frac{r}{2})}{r(J+\mu+1/2)} + \frac{(\frac{r}{2})}{r(J+\mu+1/2)} \right] V_{T}(r) \frac{r^{2}}{q^{2}} \frac{dr}{q}$$

where we have replaced $k^{3}dk$ by $\frac{EdE}{2q^{4}}$. The Talmi integral for the tensor potential is defined analogously to that in Chapter II.

$$VT(J, \mu) = \frac{2}{\Gamma(J+\mu+3/2)} \int_{0}^{\infty} e^{-r^{2}/q^{2}} \left(\frac{r^{2}}{q^{2}}\right)^{J+\mu} V_{T}(r) \frac{r^{2}}{q^{2}} \frac{dr}{q}$$

The final form of equation (3.5) is

$$VT(J, \mu) - \frac{2(1+2\mu)}{(2\mu+3)} VT(J, \mu-1) + \frac{4\mu^2}{(2\mu+3)(2\mu+1)} VT(J, \mu-2)$$

$$= \frac{2}{(2\mu+3)} \frac{(\frac{h\omega}{\pi})}{(2J+1)^2} \int_{0}^{\infty} e^{-E} \left[2\tau_{J+1}(E) + \frac{(2J+1)}{6\sqrt{J(J+1)}} \epsilon^{J}(E) \right]$$

 $\times {}_{1}F_{1}(-J-1-\mu; \frac{1}{2}-J, E)$ EdE

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Ordinarily the term J $\tau_{J-1}(E)$ would appear in the preceding equation but for the ${}^{3}S_{1} - {}^{3}D_{1}$ coupled state $\tau_{J-1}(E)$ is identically zero since the tensor force does not connect the S-state to itself. However, equation (3.6) is not useful unless one knows what VT(J, -1) is, in order to start the recursion. In the case of diagonal matrix elements, the corresponding term vanished since it was multiplied by $\mu = 0$. We can supply this deficiency by an alternative deduction from equation (3.1). Setting $\mu=0$, we obtain

$$\left(\frac{r^2}{q^2}\right)^{J} e^{-r^2/q^2}$$

 $= -(\frac{2}{\pi}) \Gamma(J+\frac{1}{2}) \int_{0}^{\infty} e^{-k^{2}q^{2}} j_{J}^{2}(kr) {}_{1}F_{1}\left[-J-1; \frac{1}{2}-J; k^{2}q^{2}\right] k^{2}dk$

and from Schiff (21) we make the substitution

$$j_{J}^{2}(kr) = \frac{k^{2}r^{2}}{(2J+1)^{2}} (j_{J-1}(kr) + j_{J+1}(kr))^{2}$$

which also provides the convenient factor k^2r^2 . Thus

$$(\frac{r^2}{q^2})^{-1} e^{-r^2/q^2} (\frac{r^2}{q^2})$$

 $= -(\frac{2}{\pi}) \frac{\Gamma(J+\frac{1}{2})}{(2J+1)^2} \int_{0}^{\infty} e^{-k^2 q^2} \left[j_{J-1}^{2}(kr) + j_{J+1}^{2}(kr) \right]$

+ $2j_{J-1}(kr) j_{J+1}(kr)]_{1}F_{1}[-J-1, k^{2}-J, k^{2}q^{2}] r^{2}k^{4} dk$. (3.7) Equation (3.7) is multiplied by

$$\frac{2}{\Gamma(J-1+3/2)}\int_{0}^{0}V_{T}(r)\frac{dr}{q}$$

and the analogous substitutions for k^2q^2 , k^3dk , and the Bessel functions were made to obtain

$$VT(J, -1) = \frac{\left(\frac{h\omega}{\pi}\right)}{(2J+1)^2} \int_{0}^{\infty} e^{-E} {}_{1}F_{1}\left[-J-1; {}_{2}^{J}-J, E\right]$$

$$\left[\tau_{J-1}(E) + \tau_{J+1}(E) + \frac{2(2J+1)}{6\sqrt{J(J+1)}} \epsilon^{J}(E)\right] EdE$$
(3.8)

Again τ_{J-1} is identically zero for the ${}^{3}S_{1} - {}^{3}D_{1}$ coupled state only. Equation (3.8) is correct for all J. Now an expression will be given for the general recursion formula of the off-diagonal Talmi integrals.

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$$\begin{array}{c} \nabla T(J, \mu) \\ \beta+1 \\ \Sigma \\ \nu=1 \end{array} (-)^{\nu} 2^{\nu} \frac{\mu!}{(\mu+1-\nu)!} \frac{J!}{(J+1-\nu)!} \frac{1}{\nu!} \frac{(2J+2\mu+1-2\nu)!!}{(2J+2\mu+1)!!} \end{array}$$

$$x \left[v (J+\mu+l-\nu) + \theta_{\nu,\beta+1} (\mu+l-\nu) (J+l-\nu) \right] VT (J, \mu-\nu)$$

$$= \frac{2(2\mu+1)!!}{(2J+2\mu+1)!!} \frac{(2J+1)!!}{(2J+1)^3} \left(\frac{h\omega}{\pi}\right) \int_{0}^{\infty} e^{-E} \left[-J \tau_{J-1}(E) + (J+1) \tau_{J+1}(E) \right]$$

$$+ \frac{(2J+1)}{6\sqrt{J(J+1)}} e^{J}(E) \int_{1} F_{1}(-J-l-\mu; \frac{1}{2}-J; E) EdE$$

$$93.9)$$

$$where \beta = \min \{\mu, J\} and$$

$$\theta_{\gamma,\delta} = 0$$
 if $\gamma = \delta$

= 1 otherwise

Now we explicitly illustrate how the phases $\tau_{l}(E)$ are obtained from the total nucleon-nucleon scattering phase shift. The general nuclear potential, which consists of central, spin-orbit, and tensor interactions, can be written as

$$V(r) = V_{c}(r) + L.S V_{LS}(r) + \hat{S}_{12} V_{T}(r)$$

When matrix elements of the above potential are taken, one obtains, in Born approximation, phase shifts which were shown (22) to be in proportion to the respective matrix elements. The coefficients of the respective matrix elements are the expectation values of the spin-orbit (L.S) and tensor (S_{12}) operators for each value of j in the triplet state. Thus

$$\Delta^{3} \ell_{J=\ell-1} = \delta_{C} - (\ell+1) \delta_{LS} - \frac{2(\ell+1)}{(2\ell-1)} \delta_{T}$$

$$\Delta^{3} \ell_{J=\ell} = \delta_{C} - \delta_{LS} + 2 \delta_{T}$$

$$\Delta^{3} \ell_{J=\ell+1} = \delta_{C} + \ell \delta_{LS} - \frac{2\ell}{(2\ell+3)} \delta_{T}$$

where $\Delta^3 \iota_J$ are the measured nuclear phase shifts. In order to solve for the "component phase shifts" one requires the following three conditions:

1)
$$\frac{1}{3(2\ell+1)} \sum_{\substack{J=\ell-1 \\ J=\ell-1}}^{\ell+1} (2J+1) \Delta^{3}\ell_{J} = \delta_{c}$$
2)
$$\sum_{\substack{J=\ell-1 \\ J=\ell-1}}^{\ell+1} A_{J} \Delta^{3}\ell_{J} = \delta_{LS}$$
3)
$$\sum_{\substack{J=\ell-1 \\ J=\ell-1}}^{\ell+1} B_{J} \Delta^{3}\ell_{J} = \delta_{T}$$

where A_J , B_J are undetermined coefficients. We are interested only in δ_T (which were called τ_l previously) so that we need calculate only B_J . By applying condition 3) above, one obtains

$$B_{\ell-1} + B_{\ell} + B_{\ell+1} = 0$$

$$-(\ell+1)B_{\ell-1} - B_{\ell} + \ell B_{\ell+1} = 0$$

$$-\frac{2(\ell+1)}{(2\ell-1)}B_{\ell-1} + 2B_{\ell} - \frac{2\ell}{(2\ell+3)}B_{\ell+1} = 1$$

By writing the first two equations in symmetric form, one sees the contribution by each component of the triplet to the tensor phase shift.

$$\frac{B_{\ell}-1}{\ell+1} = \frac{B_{\ell}}{-(2\ell+1)} = \frac{B_{\ell}+1}{\ell} = k_{\ell}$$

where k_{ℓ} is a common factor depending on ℓ . With the last of the three equations, it can easily be shown that

$$k_{\ell} = -\frac{1}{12} \frac{(2\ell-1)(2\ell+3)}{\ell(\ell+1)(2\ell+1)}$$

As an example, for *l*=1

$$B_0 = -\frac{10}{72}$$
; $B_1 = \frac{15}{72}$; $B_2 = -\frac{5}{72}$

so that

$$\tau_{l=1} = -\frac{10}{72} \Delta^{3} P_{0} + \frac{15}{72} \Delta^{3} P_{1} - \frac{5}{72} \Delta^{3} P_{2}$$

CHAPTER IV

NUMERICAL PROCEDURE

All calculations were made at the computation centre of McMaster University on an IBM 7040 originally and then on a CDC 6400. Diagonal matrix elements were calculated for values of l=0 up to l=5 for various combinations of n and n' such that

$0 \leq n (or n') \leq 3$

Off-diagonal matrix elements were calculated up to the ${}^{3}F_{4} - {}^{3}H_{4}$ coupled state for similar values of n and n'. Advantage was taken of the fact that a symmetric interchange of n! with n'!' in the Talmi coefficients leaves them unchanged. Thus, for the diagonal case only, matrix elements were calculated only for n' > n. The T=0 phase shifts of G. Breit et al (15) were used in the calculations. Interpolation in the phase shifts and coupling parameters was achieved by using a ten-point Lagrange formula evaluated according to the Aitken iteration method. This produces a ninth order polynomial.

In the hypergeometric function $_{1}F_{1}(-\ell-1-\mu; \frac{1}{2}-\ell, x)$, the order of the polynomial obtained is identical to the negative value of the first argument. In our case, the

maximum value of t was five and μ equal to six so that a twelfth order expansion must be evaluated. A novel procedure was employed to achieve this since by simple addition of each successive term, one obtains numbers of unusually great magnitude with a resultant loss of significance because each term is generally added with alternate sign. Hence, the polynomial was calculated in the "nested" way in which one begins with the coefficient of the highest power and procedes to the lower powers.

We now must integrate a polynomial of maximum order twenty-one for the diagonal matrix elements and of twenty-second order for off-diagonal matrix elements where an additional factor of the variable of integration occurs. For this, a fifteen-point Gauss-Laguerre guadrature formula was used which in principle will yield an exact solution for a polynomial of twenty-ninth degree or less. This proved more than adequate. Apart from the interpolation in the phase shifts, the preceding calculations were made using approximately fourteen significant figures for both the reduction of round-off and for better accuracy in the numerical integration. The Talmi coefficients were also evaluated with a similar degree of accuracy since for $\mu \ge 4$ the magnitude of the coefficients are of the order (n+n' 1) and for large n and n' can be quite large. 10

These coefficients are then multiplied by the Talmi integrals which are generally less than one and usually much less. Since the Talmi coefficients appear with alternate sign in the summation of Talmi integrals one may incur large round-off errors particularly when μ is large.

Elliott et al have compared the reduced matrix elements obtained by their method with those obtained by actual integration of the Tabakin (12) potential for the three states of the triplet P partial wave. They have found that the disagreement varies from within a few percent to over one hundred percent. The method thus appears to be moderately inaccurate. One source of this discrepancy lies in the Born approximation of the phase shift. It is known that one can expect the Born formula to be accurate provided the wavefunction of the scattered nucleon closely approximates the free particle wavefunction. This is generally true for high values of the relative kinetic energy of the two nucleon system. The requirement that the phase shift be small is insufficient to quarantee a good Born approximation since, at low energies the phase shift is small but the nucleon wave function is highly distorted compared to the free nucleon eigenstate. When one compares the phase shifts calculated exactly and in first Born approximation on the Reid (9) potential, it is found that for ${}^{1}S_{n}$ waves, the disagreement at low energies

can be as large as five hundred percent. At energies greater than one hundred MeV (laboratory frame) for the nucleon, agreement is obtained to within five or ten percent. For the ¹D₂ wave, similar accuracy was observed for energies greater than fifty MeV. The Born approximation becomes better for larger values of the orbital angular momentum since, in these cases, the scattered nucleon only sees the tail of the scattering potential. The wavefunction is therefore less distorted. Another consideration of Elliott's method is necessary since, in calculating nuclear spectra due to interactions of nucleons near the fermi surface, the relative kinetic energy of two nucleons is small. For two nucleons in the lowest relative S-state, the relative kinetic energy is of order no, which for light nuclei is approximately ten MeV. It was seen above that it is in this low energy region where the Born approximation is least accurate.

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