NUCLEAR POTENTIAL AND NUCLEAR MATTER

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By

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SCOPE AND CONTENTS:

The one-boson-exchange potential model is studied in some detail. The parameters of a velocity-dependent potential based on such a model are fitted to reproduce two-body scattering data. This potential is then used in nuclear matter calculations by means of the reference spectrum method. It is shown that the average binding energy of a nucleon in nuclear matter and the saturation density are in reasonable agreement with the experimental values.

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INTRODUCTION

One of the basic aims of nuclear physics is to predict various nuclear properties from a knowledge of nucleon-nucleon interactions. This at once presents two problems. First of all there is the need to determine the interaction itself, and secondly one is confronted with a many-body problem. In recent years there have been remarkable advances in both aspects.

To describe the nucleon-nucleon interactions Taketani (1951) advocated the adoption of a sort of semi-phenomenological approach. In his region I, the classical region, when the inter-nucleon distance is $r \gtrsim 2F$, the nucleon-nucleon interaction is dominated by the one-pion-exchange mechanism. In region II, the dynamical region ($1F \lesssim r < 2F$), qualitative behaviour can still be understood from a field-theoretic point of view. In region III, $r \lesssim 1F$, however, so many complicated effects take place that Taketani called it the phenomenological region. Recently, there have been two very successful phenomenological potential models. Both the Hamada-Johnston (1962) and the Yale (Breit et al, 1960; Hull et al, 1961) potentials are capable of correlating a large amount of two-body scattering data up to laboratory energy of about 300 Mev. Both have the well-established one-pion-exchange tail and both have a hard

core, that is, an infinite repulsion at an inter-nucleon distance of about 0.5 F. This is to be compared with the value of 0.4 F for the older Gammel-Thaler (1957) potential.

Another approach to the treatment of the nucleon-nucleon scattering is through dispersion relations. In this approach, one does not start with a Lagrangian or a Hamiltonian as in the convential field theory, nor are the results expressed in terms of potentials. The theme of this approach is that one should only deal with physically observable quantities and rely on a few general principles such as analyticity, unitarity and crossing symmetry of the scattering matrix.

As for the nuclear many-body problem, Brueckner (see for example, Brueckner and Gammel 1958) and Goldstone (1957) initiated a powerful formalism which pays special attention to the treatment of the hard core in the two-body interactions. Within this formalism there are two notable modifications which not only greatly simplify the calculation but also permit deeper physical insight into the problem. These are the separation method propounded by Moszkowski and Scott (1960), and the reference spectrum method developed by Bethe, Brandow and Petshek (1963).

The simplest test ground for a many-body theory is a hypothetical, uniform nucleus of infinite dimensions with equal numbers of electrically neutral protons and neutrons. Such a medium is called nuclear matter. It has the simplicity of having no Coulomb and surface effects to worry about. The

closest realization to such a hypothetical medium is perhaps the interior of a heavy nucleus. Nuclear matter is characterized by the average energy $\bar{E}$ per particle in the medium, and by its density ρ or equivalently, by the Fermi momentum k_F since the two quantities are simply related for a uniform gas:

$$\rho = \frac{N}{\Omega} = \frac{3}{4\pi r_o^3} = \frac{2}{3\pi^2} k_F^3.$$

Here r_o denotes the radius of the volume occupied by each nucleon. Empirically $\bar{E}$ is deduced by interpreting it to be the volume energy per particle in the Bethe-Weizsäcker semi-empirical mass formula, and r_o is obtained from high energy electron-nucleus scattering experiments. The appropriate values for nuclear matter, as recommended by Brandow (1964), are

$$\bar{E} = -15.8 \text{ Mev per particle}$$

and $$r_o = 1.12F \text{ corresponding to } k_F = 1.36F^{-1}.$$

More recently, however, Professor Bethe* seems to be in favour of a higher density $k_F \approx 1.43F^{-1}$. Using the reference spectrum method, Razavy (1963) obtained $\bar{E} = -7.8$ Mev per particle and $r_o = 1.35F$ with the Hamada-Johnston potential, whereas Brueckner and Masterson (1962) obtained $\bar{E} = -8.3$ Mev per particle and $r_o = 1.28F$ for the Yale potential.

The discrepancy with the empirical values may be due to

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*Private communication from Dr. D. W. L. Sprung.

several reasons. Perhaps one should sum over more subsets of diagrams in the Brueckner-Goldstone theory. Indeed, Bethe (1965) has summed the three-body cluster diagrams to all orders. Or it may also be the case that one cannot use a potential in nuclear matter calculation. Nuclear forces are derived from analysis of scattering experiments. Now in nucleon-nucleon scattering the energy and momentum of the centre of mass system before and after the collision are always the same. Formally this is expressed by saying that scattering experiments only shed information on the diagonal elements of the scattering matrix or one is always on the energy shell. In nuclear matter calculation, however, the situation may be a long way off the energy shell. This off-energy-shell effect in nuclear matter will be discussed in greater detail in Chapter III. For the present, suffice to say that when we wish to express the many-body composite scattering amplitude in terms of the two-body scattering amplitude, intermediate states in which energy is not conserved will occur. The extrapolation from an on-shell knowledge to the off-shell behaviour is usually made by assuming that the forces are given by a potential. This of course may be incorrect.

Without being so drastic, we shall assume the potential concept to be valid. Then a third possibility that may help to correct the discrepancy is to modify the nucleon-nucleon potential so as to give correct nuclear binding and density while maintaining the fit to two-body scattering data. That this can

be done is clear from the work of Gel'fand and Levitan (1951), and Jost and Kohn (1952). They have shown that a local potential in a given angular momentum state is uniquely determined if

(a) the phase-shifts in that angular momentum state are known for all positive energies,

(b) the energies of the two-body bound-states are known, and

(c) the residues of the S-matrix at the poles corresponding to the bound states are known.

Since in practice we cannot even fulfil condition (a), it is clear one does not have a unique potential. Thus we can construct a set of equivalent potentials which all will give the same two-body scattering data up to say 300 Mev laboratory energy. Bressel et al (1965) have made up a modified Hamada-Johnston potential with a finite core (core radius $r_c \approx .69F$). Nuclear matter calculations done by Sprung and Bhargava (1966) using the Bressel potential appear to be very encouraging. C. Wong (1964, 1965) has also made some calculation using a static, soft-core potential, and he has found the binding situation in nuclear matter is indeed improved. He employs a three-term Yukawa potential, and the interaction is assumed to act only in the singlet S-state. Although this potential is made up purely phenomenologically, one may assume that the three terms are due to the exchange of a pion, a scalar meson

and a vector meson. Now, meson-field-theoretical consideration does in fact favour a soft core over a hard core. Thus the idea of a soft core is not only attractive but also seems to be justified. Effects of soft-core potentials in nuclear matter have also been investigated recently by Köhler and Waghmare (1965).

It has also been fashionable to use a velocity-dependent potential instead of a static, hard-core one. From field theory, velocity dependence can also be expected. Notice that the Gel'fand-Levitan uniqueness theorem applies only to static potentials, so that even if phase shifts are known accurately there are many velocity-dependent potentials which are equivalent on the energy-shell. Baker (1962) has shown the extent of the equivalence between hard core and velocity-dependent potentials: they produce the same phase shifts in two-body scattering problem but are not equivalent for any system of more particles. Razavy et al (1962) have considered a phenomenological velocity-dependent potential of the type

$$V_o(r) + \underset{\sim}{p}\cdot w(r)\ \underset{\sim}{p},$$

and Green (1962) has used the form

$$V(r) + \underset{\sim}{p}^2 w(r) + w(r)\underset{\sim}{p}^2$$

in both scattering and nuclear matter calculation. The two types of velocity-dependence can be related by

$$pfp - \frac{(p^2 f + fp^2)}{2} = \frac{[p,\ [f,p]]}{2}\ .$$

In Green's potential, w(r) is taken to be a Gaussian. Green has found that his potential does not give saturation in the proper density region for nuclear matter. Bhaduri and Preston (1964) have studied this saturation problem in considerable detail.

The objective of the present investigation is to study the use in nuclear matter calculations of a soft-core, velocity-dependent potential model which has some contact with "fundamental" meson field theory. This potential is obtained from field theoretic basis.

In Chapter I we begin with a general survey and review of the nuclear force problem. Discussion is then centered around the one-boson-exchange potentials. In Chapter II we describe a simplified version of D. Wong's original model (1964). In this simplified version, the potential is assumed to arise from the exchange of three mesons, and it contains explicit velocity dependence. The Schrödinger equation is then set up for such a potential. Phase shift analysis is discussed at some length because, although simple in principle and treated in all quantum mechanics text books, nevertheless in practice one encounters difficulties which appear trivial only after one struggles through them. Therefore a concrete consideration outlining practical steps of phase shift analysis is not thought to be useless. The result of the potential parameter search is given at the end of this chapter. In Chapter III a review of the theory of nuclear matter precedes the calculation with the

velocity-dependent potential given in Chapter II. Conclusions and remarks form Chapter IV.

CHAPTER I

NUCLEON-NUCLEON POTENTIAL

The description of the nuclear force by a potential has its origin in analogy with the gravitational and electromagnetic forces. Although the concept of potential is essentially a non-relativistic one and thus it has only restricted validity, it is nevertheless extremely useful in describing nuclear phenomena at not too high energies. There are many comprehensive reviews on the nuclear potential (e.g. Moravcsik 1963). We shall merely discuss some relevant, general features before we proceed to D. Wong's potential.

In a loose manner, the difference between the total Hamiltonian and the Hamiltonian of free particles of a system is usually referred to as the potential. Thus, in the two-nucleon Schrödinger equation, the potential V is given by

$$V\Psi = (H-H_o)\Psi = (E-H_o)\Psi$$

where H_o is the Hamiltonian of the free particles. Phenomenologically, one can construct a potential with parameters adjusted to fit experimental data when the Schrödinger equation is solved. The potential cannot be, however, entirely arbitrary. The following requirements must be satisfied:

(1) hermiticity,

(2) translational and Galilean invariance; these require, respectively, that the potential be a function of the relative coordinate $\vec{r}=\vec{r}_1-\vec{r}_2$ and the relative momentum $\vec{p}=\frac{(\vec{p}_1-\vec{p}_2)}{2}$,

(3) rotation and space reflection invariance,

(4) particle exchange invariance, due to the indistinguishbility of the particles,

(5) time reversal invariance.

We now examine the possible "form" of a potential. For a two-particle system, we have (a) two-particle L-space, (b) two-particle S-space, and (c) two-particle J-space. Conservation of the total angular momentum $\underset{\sim}{J}$ means that the interaction is a zeroth-rank tensor in the J-space. Therefore the most general form of the interaction is

$$H_{int} \equiv T^{(o)}(J) \equiv \sum_k \alpha_k T^{(k)}(\Omega)\cdot T^{(k)}(S).$$

Where $T^{(k)}(S)$ is a tensor of rank k in the spin space, and $T^{(k)}(\Omega)$ is a tensor depending on angular variables. In the spin space, we can construct

$$(1)^{(o)}$$

$$(\underset{\sim}{S}\cdot\underset{\sim}{S})^{(o)}$$

$$(\underset{\sim}{S})^{(1)}$$

$$(\underset{\sim}{S}x\underset{\sim}{S})^{(2)}.$$

We cannot get tensors of rank higher than two because $\underset{\sim}{S}$ itself is a tensor of rank one only. The various combinations of

$T^{(k)}(\Omega)\cdot T^{(k)}(S)$ are

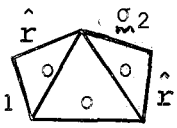
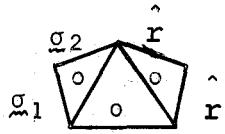
$(1)^{(o)} \leftrightarrow (1)^{(o)}$	central and spin-independent interaction,
$(\underset{\sim}{S}\cdot\underset{\sim}{S})^{(o)} \leftrightarrow (1)^{(o)}$	central but spin-dependent,
$(\underset{\sim}{S})^{(1)} \leftrightarrow (\underset{\sim}{L})^{(1)}$	spin-orbit interaction,
$(\underset{\sim}{S}x\underset{\sim}{S})^{(2)} \leftrightarrow (\underset{\sim}{r}x\underset{\sim}{r})^{(2)}$	tensor force,
$(\underset{\sim}{S}x\underset{\sim}{S})^{(2)} \leftrightarrow (\underset{\sim}{L}x\underset{\sim}{L})^{(2)}$	quadratic spin-orbit force.

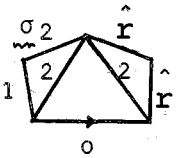
So far we have neglected the isotopic space. If we assume the interaction to be charge-symmetrical and also charge-independent, then only $(1)^{(o)}$ and $(\underset{\sim}{T}_1\cdot\underset{\sim}{T}_2)^{(o)}$ can occur. Taking all the above into account, we write down the most general form of the potential as

$$\begin{aligned} V = \{ & V_c(r,p) + V_\sigma(r,p)\ \underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2 + V_T(r,p)\ S_{12} \\ & + V_L(r,p)\ \underset{\sim}{L}\cdot\underset{\sim}{S} + V_{\sigma p}(r,p)\ \underset{\sim}{\sigma}_1\cdot\underset{\sim}{p}\ \underset{\sim}{\sigma}_2\cdot\underset{\sim}{p}\ \} \\ + \{ & \tilde{V}_c(r,p) + \tilde{V}_\sigma(r,p)\ \underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2 + \tilde{V}_T(r,p)\ S_{12} \\ & + \tilde{V}_L(r,p)\ \underset{\sim}{L}\cdot\underset{\sim}{S} + \tilde{V}_{\sigma p}(r,p)\ \underset{\sim}{\sigma}_1\cdot\underset{\sim}{p}\ \underset{\sim}{\sigma}_2\cdot\underset{\sim}{p}\ \}\ \underset{\sim}{T}_1\cdot\underset{\sim}{T}_2, \end{aligned} \tag{I-1}$$

where

$$S_{12} \equiv \frac{\underset{\sim}{\sigma}_1\cdot\underset{\sim}{r}\ \underset{\sim}{\sigma}_2\cdot\underset{\sim}{r}}{r^2} - \frac{\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2}{3}$$

$\equiv 3$  $-$ 

$\equiv \sqrt{5}$  $\sim\ T^{(2)}(S)\cdot T^{(2)}(L)$, the tensor force.

Instead of the $\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{p} \underset{\sim}{\sigma}_2 \cdot \underset{\sim}{p}$ quadratic term we could have written down the quadratic spin-orbit term $\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{L} \underset{\sim}{\sigma}_2 \cdot \underset{\sim}{L}$ since these are not entirely independent (Okubo and Marshak 1958).

Experimentally it is found that all the terms in equation (I-1) are necessary or desirable. The spin-dependent $\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2$ nature of nuclear force was recognized long ago by Wigner, as he noted that otherwise the measured zero-energy cross-section for n-p scattering and the deuteron binding energy were incompatible. The tensor force S_{12} is called for by the non-vanishing quadrupole moment of the deuteron, and by analysis of nucleon-nucleon differential cross section. There is no real proof of the spin-orbit $\underset{\sim}{L} \cdot \underset{\sim}{S}$ force, but there are many indications that one should include it in nuclear force. The ordering of the ^3P phase shifts at high energy ($\gtrsim$ 100 Mev laboratory energy)

$$\delta\ (^3P_2) > \delta(^3P_0) > \delta\ (^3P_1)$$

can be explained easily by the existence of a spin-orbit $\underset{\sim}{L} \cdot \underset{\sim}{S}$ interaction. The presence of such a force also improves fitting at 310 Mev. The indication of the quadratic spin-orbit term comes from the fact that otherwise it would be impossible to fit simultaneously the S- and D-state proton-proton scattering phase shifts as a function of energy.

We now discuss the radial dependence $V_i(r,p)$ of the potential. To explain the mysterious isotropy of the p-p differential corss-section even at 400 Mev, Jastrow (1951) proposed

the presence of a hard (i.e. infinite) repulsive core at about .5F. A qualitative argument may be given as follows. At low energy, only the S-wave gets inside .5F and feels the core. As energy increases, more and more of the S-state wave function is packed into smaller distance and so the repulsive core is felt much more appreciably. Eventually, at about 250 Mev,the S-phase becomes negative. The D-phase is still positive. Therefore the S-D interference term in the differential cross section is negative, and this together with the positive pure D-phase contribution, may make the resulting differential cross section isotropic.

A typical phenomenological potential is that of Hamada-Johnston (1962). Using the above most general form of the potential with a hard core of radius .48F, they manage to get a very good over-all fit to p-p and n-p data up to about 300 Mev by means of about thirty parameters.

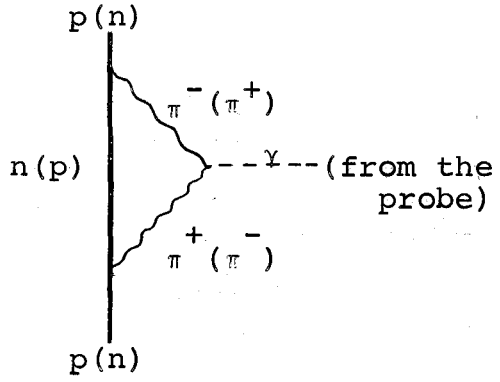
Another approach to the potential problem is to start from field theory. The various necessary invariance requirements are built in an interaction Lagrangian, which is chosen to be invariant under improper and proper (i.e. the full Lorentz group) Lorentz transformation. The physical idea of the meson-field-theoretic view point of the potential can easily be understood. Just as the photon is the field quantum of the electromagnetic field, the original Yukawa theory maintained that the pion is the field quantum of the nuclear field. A physical nucleon is surrounded by a cloud of pions, and nuclear force between two nucleons arises from an exchange of pions

between them. The potential generated by such one-pion exchange process is the Yukawa potential $\frac{e^{-\mu r}}{r}$, where μ is the reciprocal pion Compton wavelength. This potential is found to be true at least for two nucleons interacting at large distance ($\gtrsim$ 2F), and is now known as the OPEP (one-pion-exchange-potential) tail. The exchange of several pions simultaneously is a very complicated mechanism, and it is mathematically difficult to examine such processes. However, the well-known argument by Wick (1938) shows that the exchange of n pions contributes to the potential only for short distance $\left\{\lesssim \frac{1}{n\mu}\right\}$ of separation between two nucleons.

We have already noted that nucleon-nucleon interaction has a very strong, short-range repulsion, and a spin-orbit $\underset{\sim}{L}\cdot\underset{\sim}{S}$ term which is also short-range since low energy phenomena do not manifest its presence. The pion-exchange model has great difficulty in accounting for these two features. A strong, short-range repulsion does emerge when extended source technique is used, but only in even angular momentum states. It actually becomes attractive in odd angular momentum states. Relativistic corrections can give a spin-orbit $\underset{\sim}{L}\cdot\underset{\sim}{S}$ term, but it is far too weak. In general, pion-exchange calculation (two-pion-exchange) always gives very small spin-orbit force.

It was observed (Breit 1937, 1938, 1960; Nambu 1957; Sakurai 1960) that the exchange of a single vector meson could explain quite naturally both of these two short-range features. The vector meson would have a mass of several times the pion mass, as it is responsible for short-range phenomena. There

were also other evidences leading to the conjecture of vector mesons. In the Yukawa pion theory, the electromagnetic structure of the nucleon is explained by the virtual dissociation $p \leftrightarrow n + \pi^+$ and $n \leftrightarrow p + \pi^-$. Assuming the interaction strength to be the same in both cases, one would expect that the mean square charge radius of the proton and the neutron to be roughly equal in magnitude, though the sign is opposite.



Experimentally, however,

$$< r^2 >_p^{1/2} \sim 0.8F$$

and

$$< r^2 >_n^{1/2} \sim 0.0F.$$

Nambu (1957) suggested that an isoscalar vector meson could explain this observed data. This meson is now called the ω-meson. The coupling of the ω-meson to nucleons gives contribution not only of equal magnitude in the neutron and the proton case, but it is also of equal sign (positive). Therefore in the neutron case the negative contribution of the π^--cloud could possibly be cancelled out by the ω-meson contribution. In addition, an isovector vector meson was also called for by Frazer and Fulco (1959) to explain quantitatively the isovector part of the anomalous electromagnetic form factor. This meson is now called the ρ-meson. The ω-meson was discovered experi-

mentally by Maglic et al (1961), and the ρ-meson by Erwin et al (1961). Very recently, Cocho (1965) fitted the electromagnetic form factors of the nucleon with a pole model. Using a phenomenological cut-off and the physical masses of the ρ, ω and ϕ, he obtained excellent agreement with experiments. Table I-1 shows some of the mesons, based on the table compiled by Roos (1963).

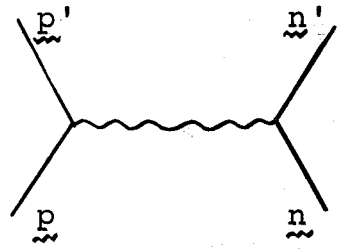
The potential generated by the exchange of various types of mesons is referred to as the OBEP (one-boson-exchange-potential) model. Many authors have discussed the OBEP model. Hoshizaki et al (1961) have given the exact expression for $V(\underset{\sim}{k},\underset{\sim}{k}')$, the potential in momentum space, as well as the expansion of $V(\underset{\sim}{k},\underset{\sim}{k}')$ with respect to k/M where M is the nucleon mass. The latter is done in order to get the potential in coordinate space. OBEP in the non-relativistic limit has also been given by Bryan et al (1963), among a number of other authors. In their work, Bryan et al considered the exchange of one scalar, one pseudoscalar, and one vector meson. The pseudoscalar meson is the pion. The scalar one is assumed to be isoscalar, and the vector meson is taken to be a mixture of the ρ and the ω. The one-boson-exchange potential in momentum space is defined by $V=T^{(2)}$, where $T^{(2)}$ denotes the scattering amplitude due to single meson exchange. The potential in the configuration space is the Fourier transform of the above V, and Bryan et al wrote down their V(r) by going to the non-relativistic limit. The potential was set equal to zero within .54F. They obtained good fits to the $^1D_2$-, $^3P_{0,1,2}$ -state

phase shifts, but they had rather poor result for the ^1S_o state. In a latter paper, Bryan and Scott(1964) used six mesons (ρ, ω, π, η, one isoscalar scalar and one isovector scalar meson) in their one-boson-exchange model. Note that there are two trivial misprints in their expressions for R_c and R_{12}. The potential was set equal to zero within .6F in all states. Probably because of this zero cut-off, the potential in the singlet odd state, for example, is much stronger than that in the corresponding state of hard core type cut-off potentials. With this potential Bryan and Scott were now able to fit a much larger number of phase parameters, although the ^1S_o and $^3S_1$ states still remain only qualitatively correct, the former being too positive and the latter too negative throughout the interesting energy region. In view of the large zero cut-off radius, this potential will cause virtual excitation mainly to the region just outside the Fermi sea. Therefore in nuclear matter calculation the Pauli principle will be most important. This is in contrast to the spirit of the reference spectrum method, in which the emphasis is on the spectral term rather than the Pauli term. Consequently it may be more appropriate to use the Bryan-Scott potential in conjunction with the Bethe-Goldstone equation (1957) in nuclear matter calculation.

Another OBEP model is due to D. Wong (1964) which we shall describe in some details. The mesons exchanged are one scalar meson (ω_{ABC}), two pseudoscalar mesons (π and η), and three vector mesons (ρ, ω and ϕ). To get the salient features

of his model, it may be sufficient to consider only three representative mesons and, in the vector meson case, we shall only consider vector coupling for simplicity. The S-matrix for the nucleon-nucleon scattering in the centre of mass system is related to the transition amplitude T by

$$(p'n'|S|pn) = \delta^{(3)}(\underset{\sim}{p}'-\underset{\sim}{p})\,\delta^{(3)}(\underset{\sim}{n}'-\underset{\sim}{n}) - 2\pi i\delta^{(4)}(p'+n'-p-n)\,\frac{M^2}{E_k E_{k'}}\,(\underset{\sim}{k}'|T|\underset{\sim}{k}), \qquad \text{(I-2)}$$



where

$$\underset{\sim}{k} = \frac{(\underset{\sim}{p}-\underset{\sim}{n})}{2}, \qquad \underset{\sim}{k}' = \frac{(\underset{\sim}{p}'-\underset{\sim}{n}')}{2}$$

suppose $T^{(2)}$ denotes the single boson exchange amplitude. Then D. Wong defines his potential in momentum space to be

$$V(\underset{\sim}{k},\underset{\sim}{k}') = \sqrt{\frac{M^2}{E_k E_{k'}}}\; T^{(2)}(\underset{\sim}{k},\underset{\sim}{k}'), \qquad \text{(I-3)}$$

This potential is to be used in the non-relativistic Schrödinger equation, and the Born amplitude(for the ℓ-th partial wave) is defined as

$$f_\ell(\underset{\sim}{k}') = \frac{m|\underset{\sim}{k}|}{2}\; V_\ell(\underset{\sim}{k},\underset{\sim}{k}')$$

The potential in the configuration space is defined as the Fourier transform of $V(\underset{\sim}{k},\underset{\sim}{k}')$. If $V(\underset{\sim}{k},\underset{\sim}{k}')$ depended only on the momentum transfer $(\underset{\sim}{k}-\underset{\sim}{k}')$, it would yield a local velocity-dependent potential. Our $V(\underset{\sim}{k},\underset{\sim}{k}')$, however, depends on $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ in a more complicated way so that it gives rise to a non-

local velocity-dependent potential. In doing Fourier transformation one puts $E_k = E_{k'} \equiv E$ (i.e., $|k^2| = |k'^2|$) because otherwise the integrals seem prohibitive. Then one can obtain the potential in the form of a product of a non-local factor N and an explicitly velocity-dependent part U

$$V_i(r,r') = U_i(r,p^2)\ N(|r-r'|). \qquad \text{(I-4)}$$

U is a polynomial in p^2. The factor N is the Fourier transform of $\frac{M}{E}$, and the nonlocality is confined to the order of one nucleon Compton wave length. If one approximates $\frac{M}{E}$ by $1 - \frac{k^2}{2M^2}$, then a purely local potential results. Writing the potential in polynomials in k^2, however, really means one is to stop at the term k^2 itself, for otherwise one would get a differential equation of higher than the second order. Moreover, if we use the potential expanded to $\frac{k^2}{M^2}$, we should somehow estimate the effect of the higher order terms.

We shall use only three representative mesons. The interaction Lagrangian is

$$L_{int} = L^{s}_{int} + L^{ps}_{int} + L^{v}_{int}$$

$$= \sqrt{4\pi}\ (g_s \bar{\psi}\psi\phi^{(s)} + g_{ps}\bar{\psi}\gamma_5\psi\phi^{(ps)} + g_v\bar{\psi}\gamma^{\mu}\psi\phi_{\mu}{}^{(v)}). \qquad \text{(I-5)}$$

Applying the Feynman-Dyson technique (e.g., see Schweber 1961), we have

$$T_s^{(2)}\ (\underset{\sim}{k},\underset{\sim}{k}') = -\frac{1}{2\pi^2}\, g_s^2\, \bar{U}(n')U(n)\ \frac{1}{\underset{\sim}{q}^2+\mu_s^2}\ \bar{U}(p')U(p),$$

$$T_{ps}^{(2)}(\underset{\sim}{k},\underset{\sim}{k}') = -\frac{1}{2\pi^2} g_{ps}^2 \bar{U}(n')\gamma_5 U(n) \frac{1}{\underset{\sim}{q}^2+\mu_{ps}^2} \bar{U}(p')\gamma_5 U(p),$$

and $$T_{v}^{(2)}(\underset{\sim}{k},\underset{\sim}{k}') = \frac{1}{2\pi^2} g_{v}^2 \bar{U}(n')\gamma^{\mu} U(n) \frac{g_{\mu\nu}}{\underset{\sim}{q}^2+\mu_{v}^2} \bar{U}(p')\gamma^{\nu} U(p), \tag{I-6}$$

where $\underset{\sim}{q}^2 \equiv (\underset{\sim}{k}'-\underset{\sim}{k})^2$. We adhere to Schweber's convention,

$$\bar{U} = U^{+}\gamma_o, \quad \bar{U}U = 1,$$

$$U = \sqrt{\frac{E+M}{2M}} \begin{pmatrix} V \\ \frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{p}}{E+M} V \end{pmatrix}, \tag{I-7}$$

$$V = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{or} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\gamma^o = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad k=1,2,3$$

and $g_{oo} = -g_{11} = -g_{22} = -g_{33} = +1$, $g_{\mu\nu} = 0$ for $\mu \neq \nu$.

Using equations (I-3), (I-5), (I-6) and (I-7), we get

$$V_s(\underset{\sim}{k},\underset{\sim}{k}') = -g_s^2 \frac{(E+M)(E'+M)}{4ME} \left[1 - \frac{\underset{\sim}{\sigma}_1\cdot\underset{\sim}{p}' \; \underset{\sim}{\sigma}_1\cdot\underset{\sim}{p}}{(E'+M)(E+M)}\right] \left[1 - \frac{\underset{\sim}{\sigma}_2\cdot\underset{\sim}{n}' \; \underset{\sim}{\sigma}_2\cdot\underset{\sim}{n}}{(E'+M)(E+M)}\right] \tag{I-8a}$$

$$\times \frac{1}{2\pi^2} \quad \frac{1}{\underset{\sim}{q}^2+\mu_s^2}$$

$$= -g_s^2 \frac{(E+M)^2}{4ME} \left[1 - \frac{\underset{\sim}{\sigma}_1\cdot\underset{\sim}{k}' \; \underset{\sim}{\sigma}_1\cdot\underset{\sim}{k}}{(E+M)^2}\right] \left[1 - \frac{\underset{\sim}{\sigma}_2\cdot\underset{\sim}{k}' \; \underset{\sim}{\sigma}_2\cdot\underset{\sim}{k}}{(E+M)^2}\right] \frac{1/2\pi^2}{\underset{\sim}{q}^2+\mu_s^2},$$

$$V_{ps}(\underset{\sim}{k},\underset{\sim}{k}') = -\frac{g_{ps}^2}{4ME} \frac{(\underset{\sim}{\sigma}_1\cdot\underset{\sim}{q})(\underset{\sim}{\sigma}_2\cdot\underset{\sim}{q})}{2\pi^2(\underset{\sim}{q}^2+\mu_{ps}^2)}, \tag{I-8b}$$

$$V_v(\underset{\sim}{k},\underset{\sim}{k}') = \frac{g_v^2}{4ME} \frac{1}{2\pi^2(\underset{\sim}{q}^2+\mu_v^2)} \left\{ \left(E+M+\frac{k^2-q^2/2}{E+M}\right)^2 + 4\underset{\sim}{k}^2 - \underset{\sim}{q}^2 \right.$$

$$+ \underset{\sim}{\sigma}_1\cdot\underset{\sim}{q}\underset{\sim}{\sigma}_1\cdot\underset{\sim}{q} - \underset{\sim}{q}^2\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2 + i\left[3+\frac{\underset{\sim}{q}^2}{(E+M)^2}\right]\left[(\underset{\sim}{\sigma}_1+\underset{\sim}{\sigma}_2)\cdot(\underset{\sim}{k}'\times\underset{\sim}{k})\right]$$

$$\left. - \frac{1}{(E+M)^2}\,\underset{\sim}{\sigma}_1\cdot(\underset{\sim}{k}'\times\underset{\sim}{k})\underset{\sim}{\sigma}_2\cdot(\underset{\sim}{k}'\times\underset{\sim}{k}) \right\}. \qquad \text{(I-8c)}$$

The potential in configuration space is given by

$$V(\underset{\sim}{r},\underset{\sim}{r}') = \frac{1}{(2\pi)^3}\int e^{i\underset{\sim}{k}'\cdot\underset{\sim}{r}'}\, V(\underset{\sim}{k},\underset{\sim}{k}')\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, d^3k'd^3k. \qquad \text{(I-9)}$$

Since $V(\underset{\sim}{k},\underset{\sim}{k}')$ is not symmetrical to start with, neither will $V(\underset{\sim}{r},\underset{\sim}{r}')$ be, and it has to be symmetrized. The resulting potential is

$$V(\underset{\sim}{r},\underset{\sim}{r}') = \left\{ g_s^2 Y(S)\left[-\frac{1}{2}\Phi_o(|\underset{\sim}{r}-\underset{\sim}{r}'|)+\frac{1}{2}\mu_s^2\Phi_1 - \nabla^2\Phi_1 - \frac{\mu_s^4}{8}\Phi_2 \right.\right.$$

$$\left. + \frac{\mu_s^2}{2}\nabla^2\Phi_2 - \frac{1}{2}\nabla^4\Phi_2\right] + g_v^2 Y(v)\left[\frac{1}{2}\Phi_o - 3\nabla^2\Phi_1 + \mu_v^2\Phi_1 + \frac{1}{2}\nabla^4\Phi_2\right.$$

$$\left.\left. - \frac{\mu_v^2}{2}\nabla^2\Phi_2 + \frac{1}{8}\mu_v^4\Phi_2\right]\right\} + \left\{ g_s^2 Y(s)\left[\frac{1}{3}\mu_s^2\nabla^2\Phi_2 - \frac{1}{8}\mu_s^4\Phi_2\right]\right.$$

$$+ g_{ps}^2 Y(ps)\frac{\mu_{ps}^2}{6}\Phi_1\,\underset{\sim}{T}_1\cdot\underset{\sim}{T}_2 + g_v^2 Y(v)\left[\frac{\mu_v^2}{3}\Phi_1 - \frac{\mu_v^2}{2}\nabla^2\Phi_2\right.$$

$$\left.\left. + \frac{\mu_v^4}{8}\Phi_2\right]\right\}\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2 + \left\{ -\frac{g_s^2 Y(s)}{2}\left(\frac{1}{r^2}+\frac{\mu_s}{r}+\frac{\mu_s^2}{3}\right)\nabla^2\Phi_2 \right.$$

$$+ g_{ps}^2 Y(ps) \left(\frac{1}{r^2} + \frac{\mu_{ps}}{r} + \frac{\mu_{ps}^2}{3}\right) \Phi_1 \underset{\sim}{T}_1 \cdot \underset{\sim}{T}_2 - \frac{1}{2} g_v^2 Y(v) \left(\frac{1}{r^2}\right.$$

$$\left. + \frac{\mu_v}{r} + \frac{\mu_v^2}{3}\right) \Phi_1 + \frac{g_{vY(v)}^2}{2} \left(\frac{1}{r^2} + \frac{\mu_v}{r} + \frac{\mu_v^2}{3}\right) \nabla^2 \Phi_2 \Bigg\} S_{12}$$

$$+ \left\{ g_s^2 \left[\frac{1}{r}\left(\frac{d}{dr} Y(s)\right) \Phi_1 + \frac{1}{r} \frac{d}{dr} Y(s) \nabla^2 \Phi_2 - \frac{\mu_s^2}{2r} \frac{d}{dr} Y(s) \Phi_2\right]\right.$$

$$\left. + g_v^2 \left[\frac{3}{r} \frac{d}{dr} Y(v) \Phi_1 - \frac{1}{r} \frac{d}{dr} Y(v) \nabla^2 \Phi_2 + \frac{\mu_v^2}{2} \frac{1}{r} \frac{dY(v)}{dr} \Phi_2\right]\right\} \underset{\sim}{L} \cdot \underset{\sim}{S}$$

$$+ \left\{ \frac{g_s^2 \mu_s^2}{2} \left(\underset{\sim}{\sigma}_2 \cdot \underset{\sim}{p} \; Y(s) \; \Phi_2 \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{p}' + \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{p} Y(s) \; \Phi_2 \underset{\sim}{\sigma}_2 \cdot \underset{\sim}{p}' \right) \right.$$

$$\left. - \frac{g_v^2 \mu_v^2}{2} \left(\underset{\sim}{\sigma}_2 \cdot \underset{\sim}{p} \; Y(v) \; \Phi_2 \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{p}' + \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{p} Y(v) \; \Phi_2 \underset{\sim}{\sigma}_2 \cdot \underset{\sim}{p}' \right) \right\}$$

$$+ \text{Terms with } \underset{\sim}{r} \leftrightarrow \underset{\sim}{r}', \text{ etc.} \qquad \text{(I-10)}$$

The notations in equation (I-9) are

$$Y(\alpha) = \frac{e^{-\mu_\alpha r}}{r} ,$$

$$\Phi_o = \frac{1}{(2\pi)^3} \int e^{i \underset{\sim}{k} \cdot (\underset{\sim}{r}' - \underset{\sim}{r})} \frac{(E+M)}{4ME} \; d^3k ,$$

$$\Phi_1 = \frac{1}{(2\pi)^3} \int e^{i\underset{\sim}{k}\cdot(\underset{\sim}{r}'-\underset{\sim}{r})} \frac{1}{4ME} d^3k ,$$

and $$\Phi_2 = \frac{1}{(2\pi)^3} \int e^{i\underset{\sim}{k}\cdot(\underset{\sim}{r}'-\underset{\sim}{r})} \frac{1}{4ME(E+M)^2} d^3k. \qquad \text{(I-11)}$$

We have in mind the representative scalar and pseudoscalar mesons to be the ω_{ABC} and the pion, respectively, and therefore we have assumed them to be isoscalar and isovector respectively. The representative vector meson is supposed to be a mixture of the ρ and the ω, so although we have written it as isoscalar in equation (I-10), we shall allow different coupling constants for the T=0 and T=1 states. In equation (I-10), we have also omitted certain contact terms.

SOME MESONS

Table I-1

	Symbol	J^p	Charge	T	Mass (Mev)
scalar singlet	ω_{ABC}	0^+	0	0	317
pseudoscalar octet	π	0^-	1	1	140
	π	0^-	0	1	135
	π	0^-	-1	1	140
	κ	0^-	1	1/2	494
	κ	0^-	-1	1/2	494
	κ	0^-	0	1/2	498
	$\bar{\kappa}$	0^-	0	1/2	498
	η	0^-	0	0	549
vector octet	ρ	1^-	1	1	757
	ρ	1^-	0	1	754
	ρ	1^-	-1	1	757
	κ^*	1^-	1	1/2	890
	κ^*	1^-	-1	1/2	890
	κ^*	1^-	0	1/2	890
	$\bar{\kappa}^*$	1^-	0	1/2	890
	ϕ	1^-	0	0	1020
vector singlet	ω	1^-	0	0	783

CHAPTER II

D. WONG'S POTENTIAL AND SCHRÖDINGER EQUATION

A. The Integral-differential Equation: David Wong's potential as given by equation (I-10) is a non-local quantity and therefore when used in the Schrödinger equation, an integro-differential equation will result. On the other hand, if one uses the potential by completely expanding it in powers of k^2/M^2, then one gets a local though explicitly velocity-dependent potential. In order to have a differential equation of the second order, one in fact only keeps terms up to k^2/M^2. It is difficult, however, to examine the neglected terms if we start with the expanded potential at the very beginning. In this Chapter we proceed to show that equation (I-10) can be written in such a way that the main terms do not contain integrals and are identical to those obtained in Wong's paper by expanding the potential to the k^2/M^2 terms, and that the remaining terms are small and may be treated as corrections. The advantage of using the integro-differential equation is thus obvious: corrections could be taken into account if so desired, although it greatly increases the mathematical complexity. Having shown that the corrections are not too large, one may then proceed with the differential equation alone.

For an arbitrary angular momentum state L, even the

formalism becomes very complicated, to say nothing of the numerical part. We therefore consider a particular case, the ^1S_o, T=1 state. The Schrödinger equation is of the form

$$(\nabla^2 + E)\ \psi\ (\underset{\sim}{r}) = \int V\ (\underset{\sim}{r},\underset{\sim}{r}')\ \psi\ (\underset{\sim}{r}')\ d\underset{\sim}{r}'. \qquad \text{(II-1)}$$

In our system of units, $\hbar=c=M=1$. To avoid confusion we do not yet put the energy E, a c-number, as k^2. We recall that $V(\underset{\sim}{r},\underset{\sim}{r}')$ consists of terms like $f_i(r)\hat{O}_i\Phi_i$ and $f_i(r')\hat{O}_i\Phi_i$. Let

$$\psi\ (\underset{\sim}{r}) = \sum_{S''L''J''} \frac{u_{S''L''J''}(r)}{r}\ \mathcal{Y}^{M}_{S''L''J''}\ . \qquad \text{(II-2)}$$

Then equation (II-1) becomes

$$\frac{d^2u_{ooo}}{dr^2} + Eu_{ooo} = r \sum_i d\underset{\sim}{r}'d\Omega_r \sum_{S'L'} \left\langle \mathcal{Y}^{o}_{ooo}(\Omega_r) \left| f_i \hat{O}_i \Phi_i \frac{u_{S'L'O}(r')}{r'} \right| \right.$$

$$\left. \mathcal{Y}^{o}_{S'L'O}(\Omega_{r'}) \right\rangle \ . \qquad \text{(II-3)}$$

where the scalar product notation is temporarily confined to spin-space. We shall neglect the quadratic $\underset{\sim}{\sigma}_1\cdot\underset{\sim}{p}\underset{\sim}{\sigma}_2\cdot\underset{\sim}{p}$ interaction, and we are here only concerned with $\hat{O}_i=1$ and $\underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2$, for which $\mathcal{Y}_{S'L'1}$ is an eigenfunction with eigenvalues λ_i (say). So

$$\left\langle \mathcal{Y}^{o}_{ooo}(\Omega_r) \left| f_i \hat{O}_i \Phi_i \frac{u_{S'L'O}(r')}{r'} \right| \mathcal{Y}^{o}_{S'L'O}(\Omega_{r'}) \right\rangle$$

$$= \lambda_i f_i \frac{u_{S'L'O}(r')}{r'} \left\langle \mathcal{Y}^{o}_{ooo}(\Omega_r) \left| \Phi \right| \mathcal{Y}^{o}_{S'L'O}(\Omega_{r'}) \right\rangle \ . \qquad \text{(II-4)}$$

Now consider

$$\Phi_i = \frac{1}{(2\pi)^3} \int e^{i\underset{\sim}{k}\cdot(\underset{\sim}{r}'-\underset{\sim}{r})} F_i(k^2)\, d^3k. \qquad \text{(II-5)}$$

We expand

$$e^{i\underset{\sim}{k}\cdot(\underset{\sim}{r}'-\underset{\sim}{r})} = 4\pi \sum_{\substack{\ell,m \\ \ell',m'}} (-i)^{\ell} j_{\ell}(kr) Y_{\ell}^{m}(\Omega_k) Y_{\ell}^{m*}(\Omega_r) 4\pi (i)^{\ell'} j_{\ell'}(kr') Y_{\ell'}^{m'*}(\Omega_k) Y_{\ell'}^{m'}(\Omega_{r'})$$

$$= (4\pi)^2 \sum_{\substack{\ell,m \\ \ell',m'}} (-)^{\ell} i^{\ell+\ell'} j_{\ell}(kr) j_{\ell'}(kr') Y_{\ell}^{m}(\Omega_k) Y_{\ell'}^{m'*}(\Omega_k) Y_{\ell}^{m*}(\Omega_r) Y_{\ell'}^{m'}(\Omega_{r'}). \qquad \text{(II-6)}$$

Therefore

$$\Phi_i = \sum_{\ell,m} \frac{(4\pi)^2}{(2\pi)^3} (-)^{\ell} \int k^2 dk j_{\ell}(kr) j_{\ell}(kr') Y_{\ell}^{m*}(\Omega_r) Y_{\ell}^{m}(\Omega_{r'}) F_i(k^2)$$

$$\equiv \sum_{\ell,m} G_i^{(\ell)}(r,r') Y_{\ell}^{m*}(\Omega_r) Y_{\ell}^{m}(\Omega_{r'}), \qquad \text{(II-7)}$$

where we denote

$$G_i^{(\ell)}(r,r') \equiv (-)^{\ell} \frac{(4\pi)^2}{(2\pi)^3} \int k^2 dk j_{\ell}(kr) j_{\ell}(kr') F_i(k^2). \qquad \text{(II-8)}$$

Hence

$$\left\langle \mathcal{Y}^{o}_{ooo}(\Omega_r) \middle| \Phi_i \middle| \mathcal{Y}^{o}_{S'L'O}(\Omega_{r'}) \right\rangle$$

$$= \sum_{\ell,m} G_i^{(\ell)}(r,r') \left\langle \chi^{o}_{o} Y^{o}_{o} \middle| Y_{\ell}^{-m}(\Omega_r) (-)^{m} Y_{\ell}^{m}(\Omega_i) \middle| \mathcal{Y}^{o}_{S'L'O}(\Omega_i) \right\rangle. \qquad \text{(II-9)}$$

Inserting equations (II-4) and (II-9) into (II-3), and integrating over $d\Omega_r$ and $d\Omega_{r'}$, we get

$$\frac{d^2u_{ooo}}{dr^2} + Eu_{ooo} = r\sum_i \lambda_i f_i(r) \int r'dr' G_i^{(o)}(r,r') u_{ooo}(r'). \tag{II-10}$$

The treatment for $f(r')\hat{O}_i\Phi_i$ is similar (f_i will be inside the integral now), and hence equation (II-10) indicates at least the structure of our equation.

Using the potential (I-10) the actual equation for the T=1, ^1S_o-state is explicitly (we now write u for u_{ooo})

$$\begin{aligned}
u''+Eu= &- \frac{g_s^2}{2} \int rr'dr' \mathcal{Y}(s) G_o^{(o)}(r,r') u(r') \\
&+ \frac{g_s^2}{2} \mu_s \int rr'dr' \mathcal{Y}(s) G_1^{(o)}(r,r') u(r') \\
&+ g_s^2 \int rr'dr' \mathcal{Y}(s) \mathcal{G}_1^{(o)} u(r') \\
&+ \frac{g_s^2 \mu_s^2}{2} \int rr'dr' \mathcal{Y}(s) \mathcal{G}_2^{(o)} u(r') \\
&+ \frac{g_s^2 \mu_s^4}{4} \int rr'dr' \mathcal{Y}(s) G_2^{(o)} u(r') \\
&- \frac{g_s^2}{2} \int rr'dr' \mathcal{Y}(s) H_2^{(o)} u(r') \\
&- \frac{g_\pi^2 \mu_\pi^2}{2} \int rr'dr' \mathcal{Y}(\pi) G_1^{(o)} u(r')
\end{aligned}$$

$$+ \frac{g_v^2}{2} \int rr'dr' \mathcal{Y}_{(v)} G_o^{(o)} u(r')$$

$$+ 3g_v^2 \int rr'dr' \mathcal{Y}_{(v)} \mathcal{G}_1^{(o)} u(r')$$

$$+ \frac{g_v^2}{2} \int rr'dr' \mathcal{Y}_{(v)} H_2^{(o)} u(r')$$

$$- \frac{g_v^2 \mu_v^2}{2} \int rr'dr' \mathcal{Y}_{(v)} \mathcal{G}_2^{(o)} u(r')$$

$$- \frac{g_v^2 \mu_v^4}{4} \int rr'dr' \mathcal{Y}_{(v)} G_2^{(o)} u(r') \qquad \text{(II-11)}$$

where

$$\mathcal{Y}_{(i)} = \frac{e^{-\mu_i r}}{r} + \frac{e^{-\mu_i r'}}{r'} , \qquad \text{(II-12)}$$

$$G_o^{(o)}(r,r') = \frac{2}{\pi} \int k^2 dk j_o(kr) j_o(kr') \frac{(\sqrt{k^2+1}+1)^2}{4\sqrt{k^2+1}} \qquad \text{(II-13a)}$$

$$G_1^{(o)}(r,r') = \frac{2}{\pi} \int k^2 dk j_o(kr) j_o(kr') \frac{1}{4\sqrt{k^2+1}} , \qquad \text{(II-13b)}$$

$$G_2^{(o)}(r,r') = \frac{2}{\pi} \int k^2 dk j_o(kr) j_o(kr') \frac{1}{4\sqrt{k^2+1}\,(\sqrt{k^2+1}+1)^2} ,$$

(II-13c)

$$\mathcal{G}_1^{(o)}(r,r') = \frac{2}{\pi} \int k^2 dk j_o(kr) j_o(kr') \frac{k^2}{4\sqrt{k^2+1}} , \qquad \text{(II-13d)}$$

$$\mathcal{G}_2^{(o)}(r,r') = \frac{2}{\pi} \int k^2 dk j_o(kr) j_o(kr') \frac{k^2}{4\sqrt{k^2+1}\,(\sqrt{k^2+1}+1)^2} ,$$

(II-13e)

and $$H_2^{(o)}(r,r') = \frac{2}{\pi}\int k^2 dk j_o(kr) j_o(kr') \frac{k^4}{4\sqrt{k^2+1}\,(\sqrt{k^2+1}+1)^2}.$$

(II-13f)

We also define

$$B_1^{(o)}(r,r') = \frac{2}{\pi}\int k^2 dk j_o(kr) j_o(kr') \frac{1}{\sqrt{k^2+1}\,(\sqrt{k^2+1}+1)}.$$

(II-13g)

We observe that we can write

$$\mathcal{G}_1^{(o)} = G_o^{(o)} - 2G_1^{(o)} - \frac{1}{2rr'}\delta(r-r'), \qquad \text{(II-14a)}$$

$$\mathcal{G}_2^{(o)} = G_1^{(o)} + 1/2\ B_1^{(o)}, \qquad \text{(II-14b)}$$

and

$$H_2^{(o)} = 2G_1^{(o)} + \mathcal{G}_1^{(o)} - \frac{1}{2rr'}\delta(r-r'). \qquad \text{(II-14c)}$$

Therefore we only have to consider $G_1^{(o)}$, $G_2^{(o)}$ and $B_1^{(o)}$. These are all of the type

$$I_o(\alpha) = \frac{2}{\pi}\int_o^\infty \frac{k^2 dk j_o(kr) j_o(kr')}{\sqrt{k^2+1}\,(\sqrt{k^2+1}+1)^\alpha}, \qquad \text{(II-15)}$$

with α=0, 1, and 2. This integral can be manipulated and re-written as

$$I_o(\alpha) = \frac{1}{\pi rr'}\int_1^\infty \frac{dy}{\sqrt{y^2-1}}\frac{1}{y^{2\alpha}}\,\mathrm{Re}(1+i\sqrt{y^2-1})^\alpha \left\{e^{-y|r-r'|} - e^{-y(r+r')}\right\}. \qquad \text{(II-16)}$$

In term of this integral I_o we have

$$G_1^{(o)} \equiv \frac{I_o(0)}{4} = \frac{1}{4\pi rr'} \int_1^\infty \frac{dy}{\sqrt{y^2-1}} \left\{ e^{-y|r-r'|} - e^{-y(r+r')} \right\}, \tag{II-17a}$$

$$G_2^{(o)} \equiv \frac{I_o(2)}{4} = \frac{1}{4\pi rr'} \int_1^\infty \frac{dy}{\sqrt{y^2-1}} \frac{1}{y^4} (2-y^2) \left\{ e^{-y|r-r'|} - e^{-y(r+r')} \right\} \tag{II-17b}$$

and

$$B_1^{(o)} \equiv I_o(1) = \frac{1}{\pi rr'} \int_1^\infty \frac{dy}{\sqrt{y^2-1}} \frac{1}{y^2} \left\{ e^{-y|r-r'|} - e^{-y(r+r')} \right\}. \tag{II-17c}$$

We now define a $\mathcal{K}$-function by

$$\mathcal{K}_N(x) \equiv C_N \int_1^\infty \frac{dy}{\sqrt{y^2-1}} \frac{e^{-xy}}{y^N} \tag{II-18}$$

with

$$C_o = \frac{1}{\pi}, \quad C_1 = C_2 = \frac{2}{\pi}$$

$$C_3 = C_4 = \frac{8}{3\pi}, \quad C_5 = C_6 = \frac{16}{5\pi} \tag{II-19}$$

Some properties and evaluation of these functions are discussed in Appendix A. For our present purpose, suffice it to say that the $\mathcal{K}$-functions are rapidly decaying functions. A few of these functions are sketched in Fig. II-1.

Returning to the Green's functions in our integro-differential equation, we see that now they can be expressed

in terms of the $\mathcal{K}$-functions. For example,

$$G_1^{(o)} = \frac{1}{4\pi rr'} \int_1^\infty \frac{dy}{\sqrt{y^2-1}} \left\{ e^{-y|r-r'|} - e^{-y(r+r')} \right\}$$

$$= \frac{1}{4rr'} \left[\mathcal{K}_o(|r-r'|) - \mathcal{K}_o(r+r') \right] . \qquad \text{(II-20)}$$

Thus the integrals in equation (II-11) are of the form

$$\int_o^\infty \frac{dr'}{(rr')^p} \mathcal{Y} u(r') \left[\pm \mathcal{K}_N(r+r') + \mathcal{K}_N(|r-r'|) \right], N \text{ even} \qquad \text{(II-21a)}$$

and

$$\int_o^\infty \frac{dr'}{(rr')^p} \mathcal{Y} u(r') \left[-(r+r') \mathcal{K}_N(r+r') + |r-r'| \mathcal{K}_N(|r-r'|) \right], \quad N \text{ odd.} \qquad \text{(II-21b)}$$

Let us consider equation (II-21a). With an obvious change of variables, equation (II-21a) can be written as

$$\pm \int_r^\infty dx \, \mathcal{K}_N(x) \frac{1}{r^{2p}\left(\frac{x}{r}-1\right)^p} u(x-r) \left[\frac{e^{-\mu r}}{r} + \frac{e^{\mu(r-x)}}{r\left(\frac{x}{r}-1\right)} \right]$$

$$+ \int_o^r dx \, \mathcal{K}_N(x) \frac{1}{r^{2p}\left(1-\frac{x}{r}\right)^p} u(r-x) \left[\frac{e^{-\mu r}}{r} + \frac{e^{\mu(x-r)}}{r\left(1-\frac{x}{r}\right)} \right]$$

$$+ \int_o^\infty dx \, \mathcal{K}_N(x) \frac{1}{r^{2p}\left(1+\frac{x}{r}\right)^p} u(r+x) \left[\frac{e^{-\mu r}}{r} + \frac{e^{-\mu(x+r)}}{r\left(1+\frac{x}{r}\right)} \right]$$

$$= \frac{e^{-\mu r}}{r}\frac{1}{r^{2p}}\int_0^\infty dx\ \mathcal{K}_N(x)\left\{\frac{1}{\left(1+\frac{x}{r}\right)^p}u(r+x)\left(1+\frac{e^{-\mu x}}{1+\frac{x}{r}}\right)\right.$$

$$\left.+\frac{[2S(r-x)]^{p-1}u(|r-x|)}{\left(1-\frac{x}{r}\right)^{p+o}}\left(1+\frac{2S(r-x)\theta(r-x)e^{\mu x}}{1-\frac{x}{r}}\right)\right\}, \qquad \text{(II-22)}$$

where

$$S(z) \equiv \begin{cases} -1/2 \text{ if } z \leq o \\ +1/2 \text{ if } z > o \end{cases} \qquad \text{(II-23)}$$

and

$$\theta(z) \equiv S(z)(1-e^{2\mu z}) + 1/2(1+e^{2\mu z}). \qquad \text{(II-24)}$$

We also define the following quantities:

$$u_{\pm}^{(p)}(r,x) \equiv \frac{1}{\left(1+\frac{x}{r}\right)^p}u(r+x)\left(1+\frac{e^{-\mu x}}{1+\frac{x}{r}}\right)$$

$$+\frac{[2S(r-x)]^{p-1}u(|r-x|)}{(1-\frac{x}{r})^{p+o}}\left(1+\frac{2S(r-x)\theta(r-x)e^{\mu x}}{1-\frac{x}{r}}\right), \qquad \text{(II-25)}$$

$$u'^{(p)}_{\pm}(r,x) \equiv u^{(p)}_{\pm}(r,x) - 4u(r), \qquad \text{(II-26)}$$

$$u''^{(p)}_{\pm}(r,x) \equiv u'^{(p)}_{\pm} - x^2\left[Y^{-1}\frac{d^2}{dr^2}(Yu) + \frac{d^2u}{dr^2}\right]. \qquad \text{(II-27)}$$

where as before Y stands for the Yukawa function, and

$$\left.\begin{aligned} \mathcal{J}^{(p)}_{N\pm}(r) &\equiv \int_0^\infty \mathcal{K}_N(x)\, u^{(p)}_{\pm}(r,x)\, dx \\ \mathcal{J}'^{(p)}_{N\pm}(r) &\equiv \int_0^\infty \mathcal{K}_N(x)\, u'^{(p)}_{\pm}(r,x)\, dx \\ \mathcal{J}''^{(p)}_{N\pm}(r) &\equiv \int_0^\infty \mathcal{K}_N(x)\, u''^{(p)}_{\pm}(r,x)\, dx \end{aligned}\right\} \text{ N even} \qquad \text{(II-28)}$$

For N odd, $x\,\mathcal{K}_N$ replaces $\mathcal{K}_N$ in equation (II-28). Using the properties of the $\mathcal{K}$-functions as well as the definitions above, we can write down

$$\mathcal{J}^{(p)}_{N\pm}(r) = \mathcal{J}'^{(p)}_{\pm}(r) + 2u(r). \qquad \text{(II-29)}$$

$$\mathcal{J}'^{(p)}_{N\pm}(r) = \mathcal{J}''^{(p)}_{N\pm}(r) + \frac{N+1}{N+2}\left[u'' + Y^{-1}\frac{d^2}{dr^2}(yu)\right], \text{ N even}$$

$$= \mathcal{J}''^{(p)}_{N\pm}(r) + \frac{3(N+2)}{(N+3)}\left[u''+Y^{-1}\frac{d^2}{dr^2}(Yu)\right], \text{ N odd.} \qquad \text{(II-30)}$$

We are now in a position to separate the integrals in the right-hand side of equation (II-11) into two parts: one without integrals and one with integrals which are relatively small. Thus for instance,

$$\int rr'dr'\, \mathcal{Y}u(r')G_1^{(o)} = 1/4 \int dr'\, \mathcal{Y}u(r')\left[\mathcal{K}_o(|r-r'|) - \mathcal{K}_o(r+r')\right]$$

$$= 1/4\, \frac{e^{-\mu r}}{r}\, \mathcal{J}^{(o)}_{o-}(r)$$

$$= 1/4\ \frac{e^{-\mu r}}{r}\ [2u+\ \mathcal{J}'^{(o)}_{o-}(r)]$$

$$= 1/4\ \frac{e^{-\mu r}}{r}\ [2u+\ \mathcal{J}''^{(o)}_{o-}+1/2[u''+Y^{-1}(Yu)'']$$

$$= 1/2\left[(Y+\frac{Y''}{4})u+\frac{Y'u'}{2}+\frac{Yu''}{2}\right]+1/4\ Y\ \mathcal{J}''^{(o)}_{o-} \qquad \text{(II-31)}$$

Similarly all the other terms in the right-hand side of equation (II-11) can be written in this way. One remark is in order at this point. There are some inherent ambiguities due to the fact that we set $\underset{\sim}{k}^2=\underset{\sim}{k}'^2$. This means for some $F_i(k^2)$ we can write it alternately as $k'^2F_i(k^2)$, and the result will of course be different. So far we have not explored the latter alternative. This situation occurs for $\mathcal{J}_1^{(o)}$ and $H_2^{(o)}$, but the latter can be expressed in terms of the former. Therefore for the original term which contains k^4 in $F_i(k^2)$, we take the average of the two terms with $k'^2F_i(k^2)$ and $F(k^2)$ respectively. For integrals involving these quantities, we also take the local limit, since otherwise they lead to highly singular terms near the origin. The resulting equation can then be written as

$$u''+k^2u=\left\{-g_s^2\left(1-\frac{x_s^2}{4}-\frac{x_s^4}{32}\right)Y(s)-\frac{g_\pi^2x_\pi^2}{4}Y(\pi)+g_v^2\left(1-\frac{x_v^4}{32}\right)Y(v)\right\}u$$

$$-\left\{\frac{g_s^2}{4}\left(1-\frac{x_s^2}{8}-\frac{x_s^4}{16}\right)(Y(s)u)''+\frac{g_\pi^2x_\pi^2}{16}(Y(\pi)u)''+\frac{g_v^2}{4}\left(3-\frac{x_v^2}{8}+\frac{x_v^4}{16}\right)\right.$$

$$\left.(Y(v)u)''\right\}$$

$$-\left\{\frac{g_s^2}{4}\left(1-\frac{x_s^2}{8}-\frac{x_s^4}{16}\right)Y(s)+\frac{g_\pi^2x_\pi^2}{16}Y(\pi)+\frac{g_v^2}{4}\left(3-\frac{x_v^2}{8}+\frac{x_v^4}{16}\right)Y(v)\right\}u''$$

$$+\frac{g_s^2x_s^2}{4}Y(s)\,\mathcal{J}''^{(o)}_{o-}(s)-\frac{g_s^2x_s^2}{8}Y(s)\,\mathcal{J}''^{(o)}_{2-}(s)+\frac{g_s^2x_s^4}{32}Y(s)$$

$$\left[3/2\;\mathcal{J}''^{(o)}_{4-}(s)-\mathcal{J}''^{(o)}_{2-}(s)\right]$$

$$-\frac{g_v^2x_v^2}{8}Y(v)\,\mathcal{J}''^{(o)}_{o-}(v)+\frac{g_v^2x_v^2}{8}Y(v)\,\mathcal{J}''^{(o)}_{2-}(v)$$

$$-\frac{g_v^2}{32}x_v^4\,Y(v)\left[3/2\;\mathcal{J}''^{(o)}_{4-}(v)-\mathcal{J}''^{(o)}_{2-}(v)\right]-\frac{g_\pi^2x_\pi^2}{8}Y(\pi)\,\mathcal{J}''^{(o)}_{o-}(\pi).$$

(II-32)

Here x_i has the numerical value of μ_i. The $\mathcal{J}$'s are those integrals defined in equation (II-28). Without these, equation (II-32) is the Schrödinger equation with a velocity-dependent potential of the usual type $V = v_o(r)+ p^2w(r)+w(r)p^2$, here $w(r)$ being sums of Yukawa functions. The integral part is relatively small compared with the rest. In fact, if we take $g_s^2 = g_v^2$, and $x_s= x_v$, they cancel out to a large extent, with only the terms

$$\frac{g_s^2x_s^2}{8}Y(s)\;\mathcal{J}''^{(o)}_{o-}(s)-\frac{g_\pi^2x_\pi^2}{8}Y(\pi)\;\mathcal{J}''^{(o)}_{o-}(\pi)$$

left. The first term has its origin in the central part of the original, complete potential. It is interesting to compare this result with the recent work of Green and Sharma (1965). In their velocity-dependent potential, by setting the coupling constants and masses of the scalar and vector mesons to be equal, they find that only the central part contributes to velocity-dependence. In our case we have seen

that only the central part contributes to the non-locality if we take $g_s^2 = g_v^2$ and $x_s = x_v$. The magnitude of the integral part in equation (II-32) of course depends on r: these terms are expected to become larger at smaller r. With a given set of potential parameters, we can get the zeroth order wave function by ignoring the integrals in equation (II-32). This wave function is then used to evaluate the integrals. These rough estimates show that they are one-seventh of those terms above the dotted line in equation (II-32) at a distance of one nucleon Compton wavelength and at an energy E_{cm}=27 Mev. They drop to about one-twentieth at about 2F.

B. The Differential Equation: In the last section we have shown how to write the ^1S_o-state Schrödinger equation consisting of derivatives and integral parts. The whole formalism can be extended to states of arbitrary angular momentum L. Since we are doing a three-meson model calculation only, we do not think it warrants such a tedious mathematical complication. Instead, we shall use David Wong's potential in the form expanded in $\frac{k^2}{M^2}$, i.e. we use the local but velocity-dependent form. While we have seen that the higher order correction is not particularly small at small distance (less than a nucleon Compton wavelength) of separation between two nucleons, we adopt a sort of phenomenological view point by fitting the local and velocity-dependent potential to experimental data. This velocity-dependent potential will nevertheless be of interest as it differs from the usual A. M. Green

type velocity-dependent potential in that the radial functions are Yukawas, and that the tensor and spin-orbit interactions are also velocity-dependent.

Let us write the total potential in the form

$$V_{total}=V+V_T S_{12}+V_{LS}\underset{\sim}{L}\cdot\underset{\sim}{S}+p^2[W+W_T S_{12}+W_{LS}\underset{\sim}{L}\cdot\underset{\sim}{S}]$$

$$+[W+W_T S_{12}+W_{LS}\underset{\sim}{L}\cdot\underset{\sim}{S}]p^2, \qquad \text{(II-33)}$$

where

$$V=V_c+V_\sigma \underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2,$$

and

$$W=W_c+W_\sigma \underset{\sim}{\sigma}_1\cdot\underset{\sim}{\sigma}_2.$$

We shall also write $V_1=V_L-3V_\sigma$, $V_3=V_L+V_\sigma$ and similarly for W_1 and W_3. We have not put in the $T_1\cdot T_2$ explicitly, but we shall write down separate potentials (with $\underset{\sim}{T}_1\cdot\underset{\sim}{T}_2$ properly taken into account) for the T=0 and the T=1 states. Let us consider the triplet case, as the singlet case can also be readily obtained from the former. We expand the total wave function

$$\psi=\sum_L \frac{u_{JL}(r)}{r}\,\mathcal{Y}^M_{JLS=1}.$$

The two-body Schrodinger equation is then (in units of $\hbar=M=C=1$)

$$\left(-\frac{1}{r}\frac{d^2}{dr^2}r+\frac{\hat{L}^2}{r^2}\right)\sum_L \frac{u_{JL}}{r}\mathcal{Y}^M_{JL1}+V_3\sum_L \frac{u_{JL}}{r}\mathcal{Y}^M_{JL1}$$

$$+V_T\sum_L S_{12}\frac{u_{JL}}{r}\mathcal{Y}^M_{JL1}+V_{LS}\sum_L \underset{\sim}{L}\cdot\underset{\sim}{S}\,\frac{u_{JL}}{r}\mathcal{Y}^M_{JL1}$$

$$+\left(-\frac{1}{r}\frac{d^2}{dr^2}r+\frac{\hat{L}^2}{r^2}\right)W_3\sum_L\frac{u_{JL}}{r}\mathcal{Y}^M_{JL1}$$

$$+\left(-\frac{1}{r}\frac{d^2}{dr^2}r+\frac{\hat{L}^2}{r^2}\right)W_T\sum_L S_{12}\frac{u_{JL}}{r}\mathcal{Y}^M_{JL1}$$

$$+\left(-\frac{1}{r}\frac{d^2}{dr^2}r+\frac{\hat{L}^2}{r^2}\right)W_{LS}\sum_L \underset{\sim}{L}\cdot\underset{\sim}{S}\frac{u_{JL}}{r}\mathcal{Y}^M_{JL1}$$

$$+W_3\left(-\frac{1}{r}\frac{d^2}{dr^2}r+\frac{\hat{L}^2}{r^2}\right)\sum_L\frac{u_{JL}}{r}\mathcal{Y}^M_{JL1}$$

$$+W_T S_{12}\left(-\frac{1}{r}\frac{d^2}{dr^2}r+\frac{\hat{L}^2}{r^2}\right)\sum_L\frac{u_{JL}}{r}\mathcal{Y}^M_{JL1}$$

$$+W_{LS}\underset{\sim}{L}\cdot\underset{\sim}{S}\left(-\frac{1}{r}\frac{d^2}{dr^2}r+\frac{\hat{L}^2}{r^2}\right)\sum_L\frac{u_{JL}}{r}\mathcal{Y}^M_{JL1}$$

$$=E\sum_L\frac{u_{JL}}{r}\mathcal{Y}^M_{JL1} \qquad \text{(II-34)}$$

Premultiplying by $\mathcal{Y}^{*M}_{JL1}$ on both sides and integrating over the solid angle $d\Omega$,

$$-\frac{1}{r}\frac{d^2}{dr^2}u_{JL}+\frac{L(L+1)}{r^2}\frac{u_{JL}}{r}+V_3\frac{u_{JL}}{r}+V_T\sum_{L'=J-1}^{J+1}\int \mathcal{Y}^{*M}_{JL1}S_{12}\mathcal{Y}^M_{JL'1}\,d\Omega\,\frac{u_{JL'}}{r}$$

$$+V_{LS}\sum_{L'}\int \mathcal{Y}^{*M}_{JL1}\underset{\sim}{L}\cdot\underset{\sim}{S}\,\mathcal{Y}^M_{JL'1}\,d\Omega\,\frac{u_{JL'}}{r}-\frac{1}{r}\frac{d^2}{r^2}(Wu_{JL})$$

$$+ 2\,\frac{L(L+1)}{r^2}\,W_3\,\frac{u_{JL}}{r} - \frac{W}{r}\,\frac{d^2}{dr^2}\,u_{JL} - \frac{1}{r}\,\frac{d^2}{dr^2}\,rW_T \sum_{L'} \int \mathcal{Y}^{M*}_{JL1}\, S_{12}\, \mathcal{Y}^{M}_{JL'1}\, d\Omega\, \frac{u_{JL'}}{r}$$

$$+ \frac{1}{r^2}\,W_T L(L+1) \sum_{L'} \int \mathcal{Y}^{M*}_{JL1}\, S_{12}\, \mathcal{Y}^{M}_{JL'1}\, d\Omega\, \frac{u_{JL'}}{r}$$

$$- \frac{1}{r}\,\frac{d^2}{dr^2}\,rW_{LS} \sum_{L'} \int \mathcal{Y}^{M*}_{JL1}\, \underset{\sim}{L}\cdot\underset{\sim}{S}\, \mathcal{Y}^{M}_{JL'1}\, d\Omega\, \frac{u_{JL'}}{r}$$

$$+ \frac{1}{r^2}\,W_{LS} L(L+1) \sum_{L'} \int \mathcal{Y}^{M*}_{JL1}\, \underset{\sim}{L}\cdot\underset{\sim}{S}\, \mathcal{Y}^{M}_{JL'1}\, d\Omega\, \frac{u_{JL'}}{r}$$

$$+ W_T \int d\Omega\, \mathcal{Y}^{M*}_{JL1}\, S_{12} \left(- \frac{1}{r}\,\frac{d^2}{dr^2}\, r + \frac{\hat{L}^2}{r^2} \right) \sum_{L'} \frac{1}{r}\, u_{JL'}\, \mathcal{Y}^{M}_{J'L1}$$

$$+ W_{LS} \int d\Omega\, \mathcal{Y}^{M*}_{JL1}\, \underset{\sim}{L}\cdot\underset{\sim}{S} \left(- \frac{1}{r}\,\frac{d^2}{dr^2}\, r + \frac{\hat{L}^2}{r^2} \right) \sum_{L'} \frac{1}{r}\, u_{JL'}\, \mathcal{Y}^{M}_{J'L1}$$

$$= E\,\frac{u_{JL}}{r}\,. \qquad \text{(II-35)}$$

We now make use of the matrix elements

$$\left(\mathcal{Y}^{M}_{J,J,1} \middle| S_{12} \middle| \mathcal{Y}^{M}_{J,J,1} \right) = 2$$

$$\left(\mathcal{Y}^{M}_{J,J+1,1} \middle| S_{12} \middle| \mathcal{Y}^{M}_{J,J+1,1} \right) = -\frac{2(J+2)}{(2J+1)}$$

$$\left(\mathcal{Y}^{M}_{J,J-1,1}\middle|S_{12}\middle|\mathcal{Y}^{M}_{J,J-1,1}\right) = -\frac{2(J-1)}{(2J+1)}$$

$$\left(\mathcal{Y}^{M}_{J,J+1,1}\middle|S_{12}\middle|\mathcal{Y}^{M}_{J,J-1,1}\right) = \frac{6\sqrt{J(J+1)}}{(2J+1)}$$

$$\left(\mathcal{Y}^{M}_{JL1}\middle|\underset{\sim}{L}\cdot\underset{\sim}{S}\middle|\mathcal{Y}^{M}_{JL1}\right) = \frac{1}{2}\left[J(J+1) - L(L+1) - 2\right].$$

and write equation (II-35) for L=J-1, L=J±1 and L=J separately. For L=J-1, we get

$$u''_{J,J-1}\left[-(1+2W_3) + \frac{4(J-1)}{(2J+1)}W_T - 2(J-1)W_{LS}\right]$$

$$+ u'_{J,J-1}\left[-2W'_3 + \frac{4(J-1)}{(2J+1)}W'_T - 2(J-1)W'_{LS}\right]$$

$$+ u_{J,J-1}\left[\frac{J(J-1)(1+2W_3)}{r^2} + V_3 - \frac{2(J-1)}{(2J+1)}V_T + (J-1)V_{LS} - W''_3\right.$$

$$\left. - \frac{4(J-1)^2 J}{(2J+1)}\frac{W_T}{r^2} + 2J(J-1)^2\frac{W_{LS}}{r^2} + \frac{2(J-1)}{(2J+1)}W''_T - (J-1)W''_{LS}\right]$$

$$+ u''_{J,J+1}\left[-\frac{12\sqrt{J(J+1)}}{(2J+1)}W_T\right] + u'_{J,J+1}\left[-\frac{12\sqrt{J(J+1)}}{2J+1}W'_T\right]$$

$$+ u_{J,J+1}\left[\frac{6\sqrt{J(J+1)}}{(2J+1)}V_T - \frac{6\sqrt{J(J+1)}}{(2J+1)}W''_T + \frac{12(J^2+J+1)\sqrt{J(J+1)}}{(2J+1)}\frac{W_T}{r^2}\right]$$

$$= k^2\, u_{J,J-1}. \qquad \text{(II-36)}$$

where we have now written $E=k^2$. The Schrödinger equation for $L=J+1$ is

$$u''_{J,J+1}\left[-(1+2W_3)+\frac{4(J+2)}{(2J+1)}W_T+2(J+2)W_{LS}\right]$$

$$+u'_{J,J+1}\left[-2W'_s+\frac{4(J+2)}{(2J+1)}W'_T+2(J+2)W'_{LS}\right]$$

$$+u_{J,J+1}\left[\frac{(J+1)(J+2)(1+2W)}{r^2}+V_3-\frac{2(J+2)V_T}{(2J+1)}-(J+2)V_{LS}-W''_3\right.$$

$$\left.-\frac{4(J+1)(J+2)^2W_T}{(2J+1)\,r^2}-\frac{2(J+1)(J+2)^2}{r^2}W_{LS}+\frac{2(J+2)}{(2J+1)}W''_T+(J+2)W''_{LS}\right]$$

$$+u''_{J,J-1}\left[-\frac{12\sqrt{J(J+1)}}{(2J+1)}W_T\right]+u'_{J,J-1}\left[-\frac{12\sqrt{J(J+1)}}{(2J+1)}W'_T\right]$$

$$+u_{J,J-1}\left[\frac{6\sqrt{J(J+1)}}{(2J+1)}V_T-\frac{6\sqrt{J(J+1)}}{(2J+1)}W''_T+\frac{12(J^2+J+1)\sqrt{J(J+1)}}{(2J+1)}\frac{W_T}{r^2}\right]$$

$$=k^2\,u_{J,J+1}. \qquad \text{(II-37)}$$

We note that equations (II-36) and (II-37) are coupled because of the tensor force. For $L=J$, the equation is uncoupled:

$$u''_{J,J}\,[-(1+2W_3)-4W_T+2W_{LS}]+u'_{J,J}\,[-2W'_3-4W'_T+2W'_{LS}]$$

$$+u_{J,J}\left[\frac{J(J+1)(1+2W_3)}{r^2}+V_3+2V_T-V_{LS}-W''_3-2W''_T\right.$$

$$\left.+4J(J+1)\frac{W_T}{r^2}+W''_{LS}-\frac{2J(J+1)}{2}W_{LS}\right]$$

$$=k^2\,u_{J,J}. \qquad \text{(II-38)}$$

To get the equation for the singlet case, we only have to change V_3 into V_1, W_3 into W_1, and set all quantities with subscripts T and LS to be zero in equation (II-38).

David Wong's potential is, using three representative mesons,

S=0, T=1

$$V_1 = -g_s^2 Y(s)\left(1 - \frac{x_s^2}{4} - \frac{x_s^4}{32}\right) - \frac{g_\pi^2 x_\pi^2}{4} Y(\pi) + g_v^2 Y(v)\left(1 - \frac{x_v^4}{32}\right),$$

$$W_1 = \frac{g_s^2 Y(v)}{4}\left(1 - \frac{x_s^2}{8} - \frac{x_s^4}{16}\right) + \frac{g_\pi^2 x_\pi^2}{16} Y(\pi) + \frac{g_v^2 Y(v)}{4}\left(3 - \frac{x_v^2}{8} + \frac{x_v^4}{16}\right),$$

All other V's and W's = 0 (II-39)

S=0, T=0

$$V_1 = -g_s^2 Y(s)\left(1 - \frac{x_s^2}{4} - \frac{x_s^4}{32}\right) + \frac{3g_\pi^2 x_\pi^2}{16} Y(\pi) + g_v^2 Y(v)\left(1 - \frac{x_v^4}{32}\right),$$

$$W_1 = \frac{g_s^2 Y(v)}{4}\left(1 - \frac{x_s^2}{8} - \frac{x_s^4}{16}\right) - \frac{3g_\pi^2 x_\pi^2}{16} Y(\pi) + \frac{g_v^2 Y(v)}{4}\left(3 - \frac{x_v^2}{8} + \frac{x_v^4}{16}\right),$$

All other V's and W's = 0 (II-40)

S=1, T=0'

$$V_3 = -g_s^2 Y(s)\left(1 - \frac{x_s^2}{4} + \frac{x_s^4}{32}\right) - \frac{g_\pi^2 x_\pi^2}{4} Y(\pi) + g_v^2 Y(v)\left(1 + \frac{2}{3} x_v^2 + \frac{x_v^4}{32}\right),$$

$$V_T = -\frac{g_\pi^2 Y(\pi)}{4}\left(\frac{3}{r^2} + \frac{3x_\pi}{r} + x_\pi^2\right) - \frac{g_v^2 Y(v)}{12}\left(\frac{3}{r^2} + \frac{3x_v}{r} + x_v^2\right),$$

$$V_{LS} = -g_s^2 Y(s)\left(\frac{1}{r^2} + \frac{x_s}{r}\right)\left(\frac{1}{2} - \frac{x_s^2}{16}\right) - g_v^2 Y(v)\left(\frac{1}{r^2} + \frac{x_v}{r}\right)\left(\frac{3}{2} + \frac{x_v^2}{16}\right),$$

$$W_3 = \frac{g_s^2 Y(s)}{4}\left(1 - \frac{11x_s^2}{24} + \frac{x_s^4}{16}\right) + \frac{g_\pi^2 x_\pi^2}{16} Y(\pi) + \frac{g_v^2 Y(v)}{4}\left(3 - \frac{11x_v^2}{24} - \frac{x_v^4}{16}\right)$$

$$W_T = \frac{g_s^2 Y(s)}{96}\left(\frac{3}{r^2} + \frac{3x_s}{r} + x_s^2\right) + \frac{g_\pi^2 Y(\pi)}{16}\left(\frac{3}{r^2} + \frac{3x_\pi}{r} + x_\pi^2\right)$$

$$+ \frac{g_v^2 Y(v)}{96}\left(\frac{3}{r^2} + \frac{3x_v}{r} + x_v^2\right), \qquad \text{(II-41)}$$

$$W_{LS} = \frac{g_s^2 Y(s)}{16}\left(\frac{1}{r^2} + \frac{x_s}{r}\right)\left(3 - \frac{x_s^2}{2}\right) + \frac{g_v^2 Y(v)}{16}\left(\frac{1}{r^2} + \frac{x_v}{r}\right)\left(5 + \frac{x_v^2}{2}\right).$$

<u>S=1, T=1</u>

$$V_3 = -g_s^2 Y(s)\left(1 - \frac{x_s^2}{4} + \frac{x_s^4}{32}\right) + \frac{g_\pi^2 x_\pi^2 Y(\pi)}{12} + g_v^2 Y(v)\left(1 + \frac{2}{3}x_v^2 + \frac{x_v^4}{32}\right),$$

$$V_T = \frac{g_\pi^2 Y(\pi)}{12}\left(\frac{3}{r^2} + \frac{3x_\pi}{r} + x_\pi^2\right) - \frac{g_v^2 Y(v)}{12}\left(\frac{3}{r^2} + \frac{3x_v}{r} + x_v^2\right),$$

$$V_{LS} = -g_s^2 Y(s)\left(\frac{1}{r^2} + \frac{x_s}{r}\right)\left(\frac{1}{2} - \frac{x_s^2}{16}\right) - g_v^2 Y(v)\left(\frac{1}{r^2} + \frac{x_v}{r}\right)\left(\frac{3}{2} + \frac{x_v^2}{16}\right),$$

$$W_3 = \frac{g_s^2 Y(s)}{4}\left(1 - \frac{11x_s^2}{24} + \frac{x_s^4}{16}\right) - \frac{g_\pi^2 x_\pi^2}{48} Y(\pi) + \frac{g_v^2 Y(v)}{4}\left(3 - \frac{11x_v^2}{24} - \frac{x_v^4}{16}\right),$$

$$W_T = \frac{g_s^2 Y(s)}{96}\left(\frac{3}{r^2} + \frac{3x_s}{r} + x_s^2\right) - \frac{g_\pi^2 Y(\pi)}{48}\left(\frac{3}{r^2} + \frac{3x_\pi}{r} + x_\pi^2\right)$$

$$+ \frac{g_v^2 Y(v)}{96}\left(\frac{3}{r^2} + \frac{3x_v}{r} + x_v^2\right),$$

$$W_{LS} = \frac{g_s^2 Y(s)}{16}\left(\frac{1}{r^2} + \frac{x_s}{r}\right)\left(3 - \frac{x_s^2}{2}\right) + \frac{g_v^2 Y(v)}{16}\left(\frac{1}{r^2} + \frac{x_v}{r}\right)\left(5 + \frac{x_v^2}{2}\right).$$

(II-42)

Here as before, $Y(i)= \frac{e^{-\mu_i r}}{r}$ and $x_i = \frac{\mu_i}{M} = \mu_i$ numerically. The pion is of course isovector. We have in mind the scalar meson to be something like the ω_{ABC} and therefore it is assumed to be isoscalar. For the model vector meson, it is really a combination of the ω, ρ and ϕ and so it does not have definite quantum number T. We have written it as if it has T=0, but if necessary, we shall allow it to have different coupling constants for different isospin states.

C. Phase Shift Analysis: In the singlet case, the Schrödinger equation is just a single equation and the phase shift is determined by the asymptotic behaviour of the wave function. In particular,

$$\tan \delta_\ell = \frac{kj'_\ell(ka) - w_\ell j_\ell(ka)}{kn'_\ell(ka) - w_\ell n_\ell(ka)} \qquad \text{(II-43)}$$

where differentiation is carried out with respect to the argument kr, and a is some point in the asymptotic region. w_ℓ is the logarithmic derivatives of the radial wave function at point a and is obtained numerically.

The triplet case is more complicated. We shall describe this in some details and indicate how the various formulae are obtained. For a given J there are now three states, namely, those with $L=J\pm1$ and $L=J$. The $L=J$ state is uncoupled, so equation (II-43) can still be used to find the phase shift. The two states $L=J\pm1$ are coupled. In the asymptotic region, however, the potential is negligible and

they become uncoupled, with the asymptotic form

$$u_{J,J-1} \to A_1 e^{-i[kr - 1/2(J-1)\pi]} - B_1 e^{i[kr - 1/2(J-1)\pi]},$$

$$u_{J,J+1} \to A_2 e^{-i[kr - 1/2(J+1)\pi]} - B_2 e^{i[kr - 1/2(J+1)\pi]}. \qquad \text{(II-44)}$$

The scattering matrix S is defined as

$$B = SA \qquad \text{(II-45)}$$

where

$$B \equiv \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \qquad A \equiv \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

and S is a 2x2 matrix which is unitary (conservation of probability) and symmetric (reciprocity). S can be diagonalized by a unitary transformation U which, applied to equation (II-45) from the left, yields

$$UB = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} UA \qquad \text{(II-46)}$$

where

$$USU^{-1} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \qquad \text{(II-47)}$$

and UA are the eigenvectors. The eigen-phase shifts δ_α and δ_β are defined by the eigenvalues of the S-matrix

$$e^{2i\delta_\alpha} \equiv S_1, \qquad e^{2i\delta_\beta} \equiv S_2$$

and we now denote the corresponding eigenvectors by

$$UA_\alpha = -\frac{e^{-i\delta_\alpha}}{2i}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad UA_\beta = -\frac{e^{-i\delta_\beta}}{2i}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{(II-48)}$$

Since the 2x2 S-matrix is unitary, it follows that δ_α and δ_β are real. In addition, the S-matrix is symmetric, and therefore it can be completely characterized by three independent parameters. We already have δ_α and δ_β as two of these, so U can now be expressed in terms of only one real number. Let us take

$$U = \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix} \qquad \text{(II-49)}$$

ϵ is called the 'mixing parameter'. From equations (II-48) and (II-49),

$$A_\alpha \equiv \begin{pmatrix} A_{1\alpha} \\ A_{2\alpha} \end{pmatrix} = -\frac{e^{-i\delta_\alpha}}{2i} \begin{pmatrix} \cos\epsilon \\ \sin\epsilon \end{pmatrix}, \quad A_\beta \equiv \begin{pmatrix} A_{1\beta} \\ A_{2\beta} \end{pmatrix} = -\frac{e^{-i\delta_\beta}}{2i} \begin{pmatrix} -\sin\epsilon \\ \cos\epsilon \end{pmatrix}, \qquad \text{(II-50)}$$

and from equations (II-46) and (II-50),

$$B_\alpha \equiv \begin{pmatrix} B_{1\alpha} \\ B_{2\alpha} \end{pmatrix} = -\frac{e^{i\delta_\alpha}}{2i} \begin{pmatrix} \cos\epsilon \\ \sin\epsilon \end{pmatrix}, \quad B_\beta \equiv \begin{pmatrix} B_{1\beta} \\ B_{2\beta} \end{pmatrix} = -\frac{e^{i\delta_\beta}}{2i} \begin{pmatrix} -\sin\epsilon \\ \cos\epsilon \end{pmatrix}. \qquad \text{(II-51)}$$

Thus there are two sets of independent solutions of the form of equation (II-44), one set with $A_1=A_{1\alpha}$, $B_1=B_{1\alpha}$, $A_2=A_{2\alpha}$ and $B_2=B_{2\alpha}$ which we shall call the α-solution, and the other set being the β-solution. Using the explicit expressions (II-50) and (II-51) for A and B in equation (II-44), we get

$$({}^{u}J,J-1)_\alpha \to \cos\epsilon \sin\left[kr - \frac{(J-1)\pi}{2} + \delta_\alpha\right], \qquad \text{(II-52a)}$$

$$({}^{u}J,J+1)_\alpha \to \sin\epsilon \sin\left[kr - \frac{(J+1)\pi}{2} + \delta_\alpha\right], \qquad \text{(II-52b)}$$

$$({}^{u}J,J-1)_{\beta} \to -\sin\epsilon \quad \sin\left[kr - \frac{(J-1)\pi}{2} + \delta_{\beta}\right], \qquad \text{(II-53a)}$$

$$({}^{u}J,J+1)_{\beta} \to \cos\epsilon \quad \sin\left[kr - \frac{(J+1)\pi}{2} + \delta_{\beta}\right]. \qquad \text{(II-53b)}$$

Also, by means of (II-50) and (II-51), it can easily be verified that

$$\tan\delta_{\alpha} = \frac{\mathrm{ReA}_1 + \mathrm{ReA}_2\tan\epsilon}{\mathrm{ImA}_1 + \mathrm{ImA}_2\tan\epsilon} \qquad \text{(II-54a)}$$

$$\tan\delta_{\beta} = \frac{\mathrm{ReA}_2 - \mathrm{ReA}_1\tan\epsilon}{\mathrm{ImA}_2 - \mathrm{ImA}_1\tan\epsilon} \qquad \text{(II-54b)}$$

It may be remarked that, although there appears to be four complex constants in equation (II-44), expressions (II-50) and (II-51) show that $\mathrm{ReA}_1 = -\mathrm{ReB}_1$ and $\mathrm{ImA}_1 = \mathrm{ImB}_1$, so that there are really only two independent complex numbers or four real constants. For practical purpose it is more convenient to use real constants and write the asymptotic wave function as

$${}^{u}J,J-1 \to r[Aj_{J-1}(kr) + Bn_{J-1}(kr)], \qquad \text{(II-55a)}$$

$${}^{u}J,J+1 \to r[A'j_{J+1}(kr) + B'n_{J+1}(kr)] \qquad \text{(II-55b)}$$

where A, B, A' and B' are real. Using the asymptotic forms for the spherical Bessel and Neumann functions and comparing equation (II-44) with (II-55), we find

$$A_1 = \frac{iA-B}{2k}, \qquad \text{(II-56a)}$$

and

$$A_2 = \frac{iA'-B'}{2k} \qquad \text{(II-56b)}$$

Hence we may rewrite equation (II-54) as

$$\tan \delta_\alpha = - \frac{B+B'\tan\epsilon}{A+A'\tan\epsilon} \qquad \text{(II-57a)}$$

$$\tan \delta_\beta = - \frac{B'+B\tan\epsilon}{A'-A\tan\epsilon} \qquad \text{(II-57b)}$$

The practical procedure of finding the phase shifts is as follows. We choose an arbitrary set of appropriate initial conditions and solve the two coupled differential equations numerically. By taking two different points r_1 and r_2 in the asymptotic region, we can determine the A, B, A' and B' in equation (II-55) for the particular set of initial values we have used. Now, we choose another set of initial conditions, distinct from the previous set, and again solve the two coupled equations numerically, thus similarly getting another set of $\widetilde{A}$, $\widetilde{B}$, $\widetilde{A}'$ and $\widetilde{B}'$. In view of equation (II-57), we have

$$\frac{\widetilde{B}+\widetilde{B}'\tan\epsilon}{\widetilde{A}+\widetilde{A}'\tan\epsilon} = \frac{B+B'\tan\epsilon}{A+A'\tan\epsilon} \qquad \text{(II-58)}$$

Since A, B, A', B', $\widetilde{A}$, $\widetilde{B}$, $\widetilde{A}'$ and $\widetilde{B}'$ are all known, we can find $\tan\epsilon$. Equation (II-57) can then be used to determine $\tan\delta_\alpha$ and $\tan\delta_\beta$.

It may appear that equation (II-58) gives two values for $\tan\epsilon$, and one wonders which one to use in equation (II-57). We can write equation (II-58) as

$$a\tan^2\epsilon + b\tan\epsilon + c=0 \qquad \text{(II-59)}$$

with

$$a \equiv A'\widetilde{B}' - \widetilde{A}'B',$$

$$b \equiv A'\widetilde{B} - \widetilde{A}'B + A\widetilde{B}' - \widetilde{A}B',$$

$$c \equiv A\widetilde{B} - \widetilde{A}B. \qquad \text{(II-60)}$$

Now, instead of equation (II-58) which is obtained by using equation (II-57a), we can also make use of equation (II-57b) and get

$$-\frac{\tilde{B}'+\tilde{B}\tan\epsilon}{\tilde{A}'-\tilde{A}\tan\epsilon} = \frac{-B'+B\tan\epsilon}{A'-A\tan\epsilon}$$

which would yield

$$c\tan^2\epsilon - b\tan\epsilon + a = 0. \qquad \text{(II-61)}$$

In order for equation (II-59) and (II-61) to be compatible, we must have c=-a. Then the roots of the quadratic equation (II-59) satisfy

$$(\tan\epsilon_1)(\tan\epsilon_2) = \frac{c}{a} = -1 \qquad \text{(II-62a)}$$

Or we have

$$\epsilon_1 = \epsilon_2 + \frac{\pi}{2} \quad \text{if} \quad \epsilon_2 < 0$$

$$= \epsilon_2 - \pi/2 \quad \text{if} \quad \epsilon_2 > 0 \qquad \text{(II-62b)}$$

restricting to the principal values of the inverse tangent. Incidentally, we note that equation (II-62a) also serves as a useful check in numerical work. Therefore although there are two values of $\tan\epsilon$, there is only <u>one</u> set of $\tan\delta_\alpha$ and $\tan\delta_\beta$, as equation (II-57) is invariant under the exchange of $\epsilon \to \epsilon \pm \frac{\pi}{2}$ which merely amounts to relabelling α and β . The ambiguity of calling which one α and which one β may be resolved by the following convention. Although in general the states L=J±1 are not eigenstates of the scattering matrix, in the limit of zero bombarding energy the difference between the centrifugal barrier effects for these two states is so pronounced that they become eigenstates. Equations (II-52) and

(II-53) imply that ϵ becomes either zero or $\frac{\pi}{2}$. These two values are also consistent with equation (II-52). We choose

$$\lim_{E\to 0} \epsilon \longrightarrow 0 \qquad \text{(II-63)}$$

Thus, in the zero-energy limit, the α-wave is essentially L=J-1, and the β-wave is essentially L=J+1. It is incorrect, however, to think of $\delta_\alpha^{(J)}$ and $\delta_\beta^{(J)}$ as the phase shifts for the state L=J-1 and L=J+1, respectively. Nevertheless it is a common practice to denote, for example, $\delta_\alpha^{(2)}$ and $\delta_\beta^{(2)}$ respectively as $\delta(^3P_2)$ and $\delta(^3F_2)$.

Our S-matrix so far is characterized by the so-called Blatt-Biedenharn phase shifts δ_α, δ_β and mixing parameter ϵ. Another set of parameters are the bar parameters, $\bar{\delta}_\alpha$ and $\bar{\delta}_\beta$. The convection between these two sets are given by (Stapp et al 1957)

$$\sin 2\bar{\epsilon} = (\sin 2\epsilon)\,[\sin(\delta_\alpha-\delta_\beta)],$$

$$\sin(\bar{\delta}_\alpha-\bar{\delta}_\beta) = \frac{\tan 2\bar{\epsilon}}{\tan 2\epsilon}, \qquad \text{(II-64)}$$

and

$$\bar{\delta}_\alpha+\bar{\delta}_\beta = \delta_\alpha+\delta_\beta.$$

We note that the two sets of ϵ, δ_α and δ_β give the same set of $\bar{\epsilon}$, $\bar{\delta}_\alpha$ and $\bar{\delta}_\beta$. For the L=J case as well as the exceptional ^3P_o-state, the nuclear bar phases are the same as the corresponding Blatt and Biedenharn phases.

D. Numerical Details and Results: To get the phase parameters, the Schrödinger equation must first be solved numerically. For the singlet states and the uncoupled triplet L=J states, the equation has the structure

$$a(r)u''+b(r)u'+c(r)u=0. \tag{II-65}$$

The singlet case presents no problem. For the triplet L=J case, a(r) will in general vanish at a certain point $r=r_o$, the precise location of which depends on the potential parameters used. This is to be compared with A. M. Green's velocity-dependent potential. In his case a(r)=(1+2w) is always positive. The nature of our singularity may be ascertained as follows. Let $a(r_o)=0$ and $r-r_o=x$. Since the functions a(r), b(r) and c(r) are Yukawas or derivatives of Yukawas, near x=0 equation (II-65) has the form

$$a_1xu''+b_ou'+c_ou=0 \tag{II-66}$$

where a_1, b_o and c_o are constants. Let

$$u=\sum d_n x_n^{\sigma+n} \tag{II-67}$$

Substituting equation (II-67) into (II-66), and equating the coefficients of the lowest term in x, one gets the indicial equation

$$\sigma[a_1(\sigma-1)+b_o] = 0, \tag{II-68}$$

or the two roots are

$$\sigma_1=0 \text{ and } \sigma_2=\left(1-\frac{b_o}{a_1}\right) = \left(1-\frac{b}{a'}\Big|_{x=o}\right).$$

The last step follows since $a=\sum a_n x^n$ and $b=\sum b_n x^n$. The solution

corresponding to σ_1 is a regular solution at x=0. If $\sigma_2>0$, then this second solution is also regular. If $\sigma_2<0$, then this second solution is irregular. If $\sigma_2=0$, we then have two equal roots to the indicial equation (II-68) and so the second solution has a logarithmic singularity at x=0. With our potential

$$\sigma_2 \equiv 0$$

and thus we have a logarithmic singularity. For instance, for the $^3P_1$ state

$$a= -1-2w-4w_T+2w_{LS}$$

$$b= -2w'-4w'_T+2w'_{LS}$$

and therefore

$$\frac{b}{a'} \equiv 1.$$

With the potential parameters we shall use, this point $r=r_o$ occurs at about .19F for the $^3P_1$ and $^3F_1$ states, whereas for the $^3D_2$ and $^3G_4$ states this occurs at r_o=.11F.

For the coupled triplet states we have the structures

$$a(r)u''_{J,J+1}+b(r)u'_{J,J+1}+c(r)u_{J,J+1}+d(r)u''_{J,J-1}$$

$$+\ell(r)u'_{J,J-1}+f(r)u_{J,J-1}=0, \qquad \text{(II-69a)}$$

and

$$p(r)u''_{J,J+1}+q(r)u'_{J,J+1}+s(r)u_{J,J+1}+t(r)u''_{J,J-1}$$

$$+u(r)u'_{J,J-1}+v(r)u_{J,J-1}=0. \qquad \text{(II-69b)}$$

We reduce this set of two second-order differential equations

into a system of four first-order ones. Let

$$u_{J,J+1} \equiv Y(1),$$

$$u'_{J,J+1} = Y'(1) \equiv Y(2) \equiv Z(1),$$

$$u''_{J,J+1} = Y''(1) \equiv Z(2),$$

$$u_{J,J-1} \equiv Y(3),$$

$$u'_{J,J-1} = Y'(3) \equiv Y(4) \equiv Z(3),$$

and $$u''_{J,J-1} = Y''(3) \equiv Z(4). \qquad \text{(II-70)}$$

We then have

$$Z(1) = Y(2), \qquad \text{(II-71a)}$$

$$Z(2) = \frac{1}{(dp-at)} [(ct-ds)Y(1)+(bt-dq)Y(2)$$

$$+(ft-dv)Y(3)+(et-du)Y(4)], \qquad \text{(II-71b)}$$

$$Z(3) = Y(4), \qquad \text{(II-71c)}$$

$$Z(4) = \frac{1}{(at-dp)} [(cp-as)Y(1)+(bp-aq)Y(2)$$

$$+(fp-av)Y(3)+(ep-au)Y(4)]. \qquad \text{(II-71d)}$$

Again, in general (dp-at) will vanish at a certain point $r=r_o$.

We use the Runge-Kutta-Gill method to solve the differential equations. For computational convenience, we put in a mathematical hard core at $r=r_c$, and start our outward integration at this point. We choose the wave function to vanish at $r=r_c$ but allow it to have arbitrary slope, which is immaterial for phse shift analysis. With respect to the singular point $r=r_o$, this solution will in general be a mixture of regular and irregular solution. Fortunately the singularity does not seem to be too serious, since the results are not sensitive to mesh size ∇(variation around $\nabla \simeq .01F$, say).

We cannot, of course, reduce the mesh indefinitely. One reason why the singularity is not serious is perhaps because logarithmic divergence is relatively weak and therefore the singular admixture in our solution remains finite and small unless we get really close to the singular point. Another reason may be seen if we note that the singular point r_o occurs at small distance (never further out than .49F). In this region, the wave function is still very small and certainly much smaller than that in the asymptotic region. Thus a little change in the wave function in this innermost region due to a change in mesh will be covered up in the asymptotic region, thus producing no difference in the phase shifts. For nuclear matter calculation, we are confronted with a bound state problem; the magnitude of the wave function at large distance is not bigger than that in the inside region. Actual computation indicates again, however, that even here the difference in wave function due to change in mesh is tolerable, and the situation is further helped by noting that the interesting quantity, the G-matrix, is

$$G \sim \int (\phi-\psi)\phi dr$$

where ψ is the actual wave function, and ϕ is the unperturbed wave function. For S-state, ϕ is just a sine wave. For most collision momenta, sin(kr) is quite small for small r, and thus essentially $(\phi-\psi)$ is reduced greatly and the integrand in this region where the singularity occurs has only small contribution to the G-matrix.

The singularity is unphysical. It arises because we have expanded the potential, resulting in a particular type of velocity-dependence. All mesons, including the pion, contribute to it. Thus the velocity-dependence is not necessarily weak or extremely short-ranged.

Otherwise the numerical procedure is straight forward. We only wish to make two practical comments. We have already mentioned in connection with equation (II-55) that, in order to determine A, B, A', and B' we need the asymptotic solutions at two different points r_1 and r_2. In principle these can be any two points in the asymptotic region, but in practice it may be advisable to choose r_1 and r_2 to be approximately one quarter of a wavelength apart. We also wish to emphasize that by the asymptotic region we mean the region where $V(r) << k^2$. Therefore for very low-energy scattering, we have to go out very far for the asymptotic region.

We first fit the T=1 states. The calculated phases are compared with those from Yale or Livermore analysis, and the potential parameters are then adjusted accordingly for the next run on the computer. We fix the pion mass to be .147 nucleon mass, g_π^2 to be 14.0 and the vector meson mass to be .8085 nucleon mass as predetermined. The T=0 states are subsequently investigated. Surprisingly, the parameter values given by Bryan, Dismukes and Ramsay (1963) actually almost give a best over-all fit, except that the singlet $^1P_1$-phase is not even qualitatively correct. The vector meson,

supposedly a mixture of ω and ρ meson, acts as an isoscalar particle. To get any reasonable behaviour for the singlet $^1P_1$-phase, it is necessary to use a vector meson coupling constant more than twice as large as that for the other states. This is perhaps a reflection of the fact that we have an oversimplified model, that in reality there are more mesons. Anyway our necessity of using different g_v^2 for the singlet-odd states may be regarded as a phenomenology. Furthermore, with the above potential parameters, although the high energy region for the singlet-even states are fairly well reproduced, low-energy scattering data such as the scattering length and effective range are not too good. Therefore we again adjusted the scalar meson parameters slightly for these states. The final potential parameters are shown in Table II-1. We have set a hard-core at $r=r_c=.074F$. This is a device partly to cut off the singularity at the origin, and partly for convenience in using the reference spectrum method in a nuclear matter calculation. The S-states are sensitive to r_c and in this sense r_c is an extra parameter. All the other states are not sensitive to r_c, in view of its smallness. The various calculated phase parameters are sketched in Fig. II-2 to Fig. II-15.

Certainly one cannot claim this is the best fit or a unique fit. If we have chosen the vector meson mass to be 1.051 nucleon mass, which is approximately equal to C. Wong's value of $\mu_v=5F^{-1}$, then $g_s^2=3.02, \mu_s=.40$, $g_v^2=42.24$ and $r_c=.052F$

will give almost a perfect fit to the ^1S_o- and $^1D_2$-states. The fit to other states are rather poor. It is found that also in this case the singlet odd states need a much stronger g_v^2 than other states.

POTENTIAL PARAMETERS

TABLE II-1

	Singlet Even	Singlet Odd	Triplet Even	Triplet Odd
μ_π	.147	.147	.147	.147
g_π^2	14.0	14.0	14.0	14.0
μ_V	.8085	.8085	.8085	.8085
g_V^2	34.0	74.0	34.0	34.0
μ_S	.598	.588	.588	.588
g_S^2	15.3	15.4	15.4	15.4

All states have $r_c = .074F$.

μ in units of reciprocal nucleon Compton wavelength.

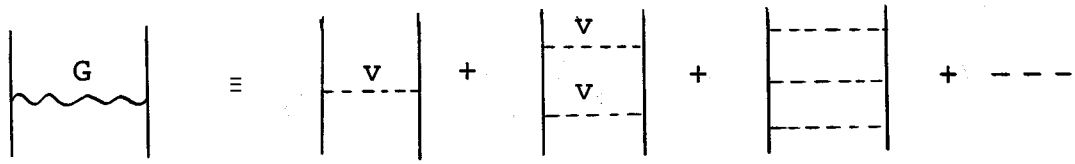
CHAPTER III

NUCLEAR MATTER CALCULATION

A. Brueckner Theory: Nuclear matter is a hypothetical infinite medium of equal number of protons and neutrons. For such a medium, Coulomb force must of course be omitted, and we assume that the two-body interaction v_{ij} to be of a purely nuclear nature. The simplicity of nuclear matter is that it has uniform density, and it has no surface effect. The nearest realization of such a hypothetical medium is perhaps the interior of a heavy nucleus. The study of nuclear matter is the first step towards an understanding of a many-body treatment of actual finite nuclei.

To calculate the interaction energy, one may wish to use the perturbation theory. The nucleon-nucleon interaction is known to be very strong and in fact, is usually taken to consist of an infinite repulsive core in some phenomenological potential models. Thus the matrix elements of the potential become infinite and perturbation method seems meaningless and inapplicable. If the range of the strong interaction is small, however, the interaction energy will also be small. This suggests that a suitably modified perturbation approach may nevertheless still be useful. This is done by the so-called vertex modification, or expressed more

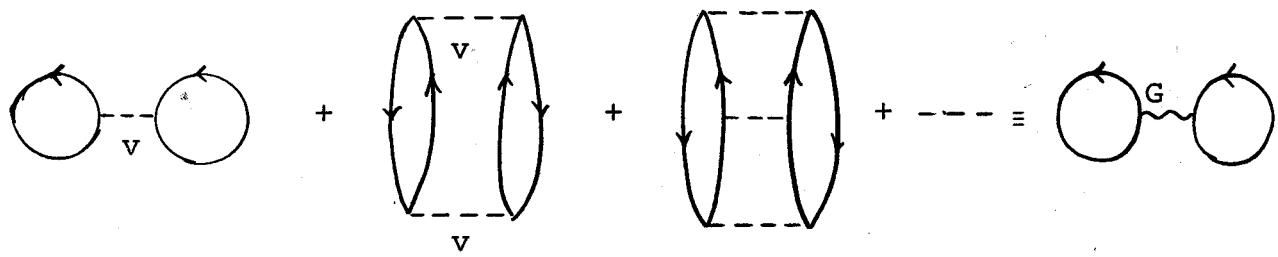
plainly, by defining a new interaction G in terms of the original v:



Each of the terms on the right-hand side is very large or divergent, but their sum may be quite small.

In an infinite medium of fermions, all the levels up to the Fermi momentum k_F are occupied in the ground state. Therefore, when two nucleons scatter against each other, they are excited to states $k>k_F$ only, leaving two holes inside the Fermi sea. We reserve the name particles for those nucleons with $k>k_F$, and holes for those with $k<k_F$. Diagramatically, particles are represented by a solid line with an arrow pointing upwards (particles propagate forward in time ↑), and a hole by a solid line with an arrow pointing downwards (holes propagate backward in time ↓).

Brueckner's theory of nuclear matter consists of selective summation of certain terms in the perturbation series. Precisely, the theory sums all diagrams with only two hole lines and allows the two excited particles to interact any number of times with each other, i.e., it includes diagrams such as



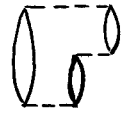
We have not drawn any exchange diagrams explicitly, but they are understood to be included as well. Hole-hole scattering is not included because the phase space for such interactions is more restricted. The total energy of the system is then

$$E \approx \sum_{i \le k_F} (i|T|i) + 1/2 \sum_{i,j \le k_F} (ij|G|ij-ji), \quad \text{(III-1)}$$

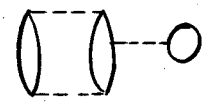
where

$$(ij|G|ij) = (ij|v|ij) - \sum_{m,n>k_F} (ij|v|mn) \frac{1}{T_m + T_n - T_i - T_j} (mn|G|ij). \quad \text{(III-2)}$$

Some of the diagrams omitted are, in the third-order:



(a)



(b)

Diagrams of type (b) are called self-energy insertions and may be taken into account, at least in a sort of average manner, by introducing the Hartree-Fock potential (modification of the propagator, $T_i \to \epsilon_i$). Diagrams of class (a) form the so-called three-body cluster diagrams, and recently Bethe (1965) has shown how to handle such diagrams up to all order. We shall neglect this class of diagrams except that we shall follow Rajaraman's (1963) suggestion of appropriate adjustment of the statistical weight of various states. On the other hand, we shall discuss the self-energy diagrams in some details later.

The original Brueckner approach is to solve directly the integral equation (III-2). This is rather tedious and

painful. Two computationally simpler methods, the separation method and the reference spectrum method, have been developed.

B. The Separation Method: Moszkowski and Scott (1960) have pointed out that, although the nucleon-nucleon interaction is very strong, the repulsive part, which may be infinite, is actually cancelled out to a large extent by part of the attractive part of the potential. Therefore, effectively, only the weak tail part of the attraction is left and so even ordinary perturbation theory might be used.

Consider two nucleons colliding in vacuo (free space). Let ψ_F be the wave function describing this situation of the two nucleons interacting with V, and ϕ be the unperturbed wave function (V=0). Schematically they are shown in Fig. 1. Let us separate the

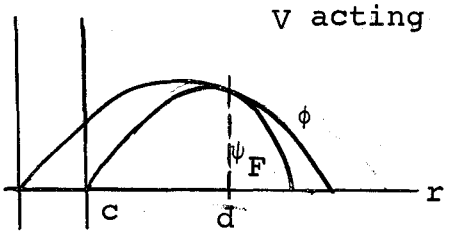


Fig. 1

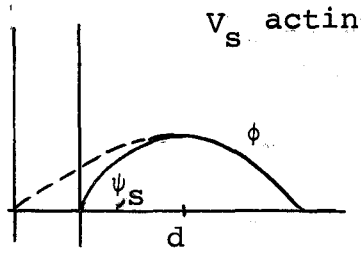


Fig. 2

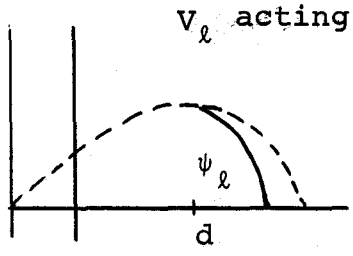


Fig. 3

potential into two parts

$$V = V_s, \quad r \leq d$$
$$= V_\ell, \quad r > d$$

We then see (Fig. 2 and Fig. 3) that, the effect of V_s alone is to distort the wave function at short distance, but causes no phase shift. V_ℓ acting alone, on the other hand, does not distort the wave function at short distance but produces the same phase shift as the complete potential V.

Since V_ℓ is weak and of long range, it causes scattering mainly into inside the Fermi sea, which is forbidden by the Pauli Principle. Thus V_ℓ has little effect on the wave function inside nuclear matter and, therefore, to a very good approximation, it may be assumed

$\psi^N \equiv$ wave function in nuclear matter

$\approx \psi^s$, wave function for two-body scattering in vacuo with interaction V_s.

The G-matrix has been defined as

$$G^N = V - V \frac{Q}{e^N} G^N \qquad \text{(III-3)}$$

where Q is the Pauli operator to ensure all the intermediate states are projected outside the Fermi sea, and the index N refers to nuclear matter. We now define

$$G_s^F \equiv V_s - V_s \frac{1}{e_o} G_s^F , \qquad \text{(III-4)}$$

and

$$G_s^F \equiv V_s \Omega_s^F \qquad \text{(III-5)}$$

where e_o is the propagator in free space, Ω_s^F the wave operator which converts the unperturbed wave function ϕ to the perturbed wave function ψ_s. It can then be shown (Bhaduri 1963)

$$G^N \approx G_s^F + V_\ell - V_\ell \frac{Q}{e^N} V_\ell - (\Omega_s^F - 1)(e_o - e^N)(\Omega_s^F - 1)$$

$$-(\Omega_s^F - 1) e^N (Q-1)(\Omega_s^F - 1) - V_\ell \frac{Q}{e_o} G_s^F - G_s^F \frac{Q}{e_o} V_\ell . \qquad \text{(III-6)}$$

The separation distance d has been so chosen that the local phase shift due to V_s is zero, i.e.

$$\left.\frac{\phi'}{\phi}\right|_d = \left.\frac{\psi_F'}{\psi_F}\right|_d \qquad \text{(III-7)}$$

That the phase shift due to V_s is zero implies $(k|G_s^F|k)=0$, since

$$\frac{\sin\delta_s}{k} = -\frac{m}{\hbar^2}(\phi_k|V_s|\psi_k) = -\frac{m}{\hbar^2}(k|G_s^F|k).$$

It is clear that d will depend on k. For high energy collision, the particles see more repulsion and therefore more of the attractive, outer region is required to cancel the repulsion; hence a larger d is required. For the interesting range of k, however, it turns out that d varies only slowly with k, and it is usually taken to be k-independent.

The terms V_ℓ and $V_\ell \frac{Q}{e} V_\ell$ form the first-order and second-order Born approximation terms. The second-order Born term is small due to the somehow mutually exclusive effects of Q and V_ℓ. The term $(\Omega_s^F-1)(e_o-e^N)(\Omega_s^F-1)$ arises because energy-momentum relation in nuclear matter is different from that in vacuo, and is therefore called the dispersion correction term. The term $(\Omega_s^F-1)e(Q-1)(\Omega_s^F-1)$ is referred to as the Pauli correction term. The last two terms in equation (III-6) are the interference terms between the short-range and the long-range parts of the potential. In vacuo, only the first three terms in (III-6) survive. It is clear now that the advantage of the separation method lies not only in its compu-

tational simplicity, but also in its clarification of the individual role played by the Pauli Principle and by the modified energy spectrum in nuclear matter.

The dispersion term can be written as

$$G_{kk}(D) \equiv \langle k|(\Omega_s^F-1)(e_o-e^N)(\Omega_s^F-1)|k\rangle$$

$$= \sum_{k'} |\langle \psi^s-\phi_k|k'\rangle|^2 \langle k'|e_o-e^N|k'\rangle$$

$$\approx (e_o-e^N)_{ave.} \sum_{k'} |\langle \psi^s-\phi_k|k'\rangle|^2. \qquad \text{(III-8)}$$

We can write

$$G_{kk}(D) = \sum_{\ell} (2\ \ell+1) C_\ell G_{kk}^\ell \qquad \text{(III-9)}$$

where C_ℓ is a statistical factor. For the singlet S state $C_\ell = \frac{6}{16}$. For S-state, we have

$$G_{kk}^o(D) \approx \frac{4\pi}{k^2}(e_o-e^N)_{ave}\int_o^d [u(kr)-\sin(kr)]^2 dr$$

$$\equiv \frac{4\pi}{k^2}(e_o-e^N)_{ave}\int_o^d \chi^2 dr \qquad \text{(III-10)}$$

the notations being $\psi^s(kr) = \frac{u(kr)}{kr}$ and $\phi(kr) = \frac{\sin kr}{kr}$.

To find the dispersion correction to the average potential energy per particle, we multiply $G_{kk}(D)$ by the probability of finding a pair of particles with relative momentum k, and integrate it over the Fermi sphere

$$E_D = \frac{8}{\pi^2}\int_o^{k_F} G_{kk}(D)\left(1-\frac{3}{2}\frac{k}{k_F}+\frac{1}{2}\frac{k^3}{k_F^3}\right)k^2 dk$$

If we assume a potential acting only in the ^1S_o-state, then

$$E_D = \frac{8c_o}{\pi^2}\int_o^{k_F} G^o_{kk}(D)\left[1-\frac{3}{2}\frac{k}{k_F}+\frac{1}{2}\frac{k^3}{k_F^3}\right]k^2dk$$

$$\approx \frac{8c_o}{\pi^2}\frac{k_F^3}{24}\ \overline{G^o_{kk}(D)}$$

$$= \frac{3}{16}(e_o-e^N)_{ave}\frac{4\pi\rho}{k_o^2}\int_o^d \chi^2_{k_o}\,dr \equiv \frac{3}{16}(e_o-e^N)_{ave}D. \qquad \text{(III-11)}$$

where k_o is some average value of k. Moszkowski and Scott take

$$(e_o-e^N)_{ave} = 2\Delta U \qquad \text{(III-12)}$$

where ΔU is the difference in the single-particle potential between an average excited state and an average state inside the Fermi sea.

As already mentioned, the long-range part V_ℓ of the potential acting alone will produce the same phase shift as the whole potential. For two different nucleon-nucleon potentials giving the same phase shifts for two-body scattering in vacuo, it is thus expected that their $(k|V_\ell|k)$ will not differ much. Hence, the difference in binding energy per particle in nuclear matter calculation by using two different potentials producing same phase shifts is mainly due to difference in the dispersion term

$$\Delta E \approx \Delta E_D \approx \frac{3}{16}(e_o-e)_{ave}\Delta D = 26.25\Delta D$$

if ΔU is taken to be 70Mev. C. Wong (1965) obtained several

sets of potentials for the ^1S_o-state and calculated various low-energy scattering data as well as nuclear matter properties. To compare with his three-meson static potential model, we choose for our potential $g_s^2=2.42, \mu_s=.40\lambda\!\!\!^{-}_N^{-1}$ (reciprocal nucleon Compton wavelength), $g_\pi^2=14.0, \mu_\pi=.147\lambda\!\!\!^{-}_N^{-1}$, $g_v^2=42.24$ and $\mu_v=1.0508\lambda\!\!\!^{-}^{-1}$. The vector meson mass is the same as C. Wong's, and we also use $r_c=.0001F.$ as the starting point for numerical integrations. The comparison is shown in Table III-1. The notations here are as follows: a is the scattering length, r_o is the effective range, δ the ^1S_o-state phase shifts, d the separation distance as defined by Moszkowski and Scott, $V_\ell = \frac{4\pi}{k_o^2}\int_d^\infty \sin^2(k_o r)V_\ell(r)dr$ is the first Born term of the long-range part of the potential. We see that relative to C. Wong we have

$$|\Delta E_D| \approx .94\text{Mev}.$$

We must note, however, that comparison in the ^1S_o-state alone may not be very conclusive because of the dependence on r_c and thus the added arbitrariness. Also, all we have seen is that perhaps we shall get more binding, but nothing is said of the saturation problem. Moreover, we have not considered the interference terms.

C. <u>The Reference Spectrum Method</u>: In the Moszkowski-Scott separation method, a G_s^F-matrix is defined

$$G_s^F = V_s - V_s \frac{1}{e^o} G_s^F \qquad \text{(III-4)}$$

to be the G-matrix corresponding to the short-range part V_s of the potential in free space. It is introduced as a first approximation to

$$G_s^N = V_s - V_s \frac{Q}{e} G_s^N$$

the actual G-matrix for the short-range part of interaction in nuclear matter. Here e without any superscript refers to the energy spectrum in nuclear matter. The separation distance d is so chosen that G_s^F vanishes. Then the total G-matrix is made of the first Born approximation for V_ℓ and several small correction terms. Bethe et al (1963) have developed a further method called the reference spectrum method, which improves the accuracy and also is simpler in the sense that it does not require separation into short and long range parts, although this can easily be done if so desired. We shall briefly outline the underlying ideas; detailed derivation of various formulae is given in the original paper by Bethe et al.

Although the contribution of G_s^N to the binding energy is small compared with that of V_ℓ, it is very important for the saturation problem owing to its strong dependence on the density. Therefore one should try to get as good an approximation to it as possible. BBP suggests that, instead of using G_s^F as a first approximation to G_s^N, one uses the reference matrix

$$G_s^R \equiv V_s - V_s \frac{1}{e^R} G_s^R \qquad \text{(III-13)}$$

where e^R is a reference spectrum. If e^R is quadratic in momentum, then G_s^R is quite easy to calculate. Therefore we define the reference energy

$$E_R(k') = A + \frac{k'^2}{2m^*} \tag{III-14}$$

Then

$$e^R = \frac{k'^2}{m^*} + 2A + \frac{P^2}{m^*} - H(k,P) \tag{III-15}$$

where H is the starting energy. Parameters A and m* in the reference spectrum can be so chosen as to fit the energy region of most importance in the nuclear matter calculation. If no separation is intended, then we can still define

$$G^R = V - V \frac{1}{e^R} G^R \tag{III-16}$$

for the whole reaction matrix. Replacing G^R by $V\Omega^R$, where Ω^R is the wave operator, we get

$$V\Omega^R = V - V \frac{1}{e^R} V\Omega^R \tag{III-17}$$

If we apply this to the unperturbed wave function ϕ we have

$$V\psi^R = V\phi - V \frac{1}{e^R} V\psi^R \tag{III-18}$$

Or

$$e^R(\phi - \psi^R) = V\psi^R \tag{III-19}$$

This equation can be easily written in configuration space because of the quadratic momentum-dependence of e^R. The resulting equation is

$$(\gamma^2 - \nabla^2)\, \zeta^R = m^* V\psi^R \tag{III-20}$$

where

$$\gamma^2 \equiv P^2 + m^* [2A - H(k_o, P)] \qquad \text{(III-21)}$$

and

$$\zeta \equiv \phi - \psi \qquad \text{(III-22)}$$

is the wave defect. The reference matrix is given by

$$(k|G^R|k_o,P) = (k|V|\psi^R)$$

$$\equiv \int \phi(k,r) V \psi^R(\underset{\sim}{k}_o, \underset{\sim}{r})\, d\tau$$

$$= \frac{(\gamma^2+k^2)}{m^*} \int \phi(\underset{\sim}{k}, \underset{\sim}{r})\, \zeta^R(k_o,r)\, d\tau . \qquad \text{(III-23)}$$

The actual reaction matrix G^N in nuclear matter is related to G^R by (BBP appendix A)

$$G^N = G^R + G^{R+}\left(\frac{1}{e^R} - \frac{Q}{e}\right) G^N$$

$$\approx G^R + G^{R+}\left(\frac{1}{e^R} - \frac{Q}{e}\right) G^R , \qquad \text{(III-24a)}$$

or equivalently,

$$(k_o|G^N|k_o) \simeq (k_o|G^R|k_o)$$

$$+ \frac{1}{(2\pi)^3} \int d^3k' |(k'|G^R|k_o)|^2 \left(\frac{1}{e^R(k')} - \frac{Q(k')}{e(k')}\right). \qquad \text{(III-24b)}$$

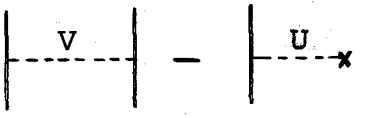
D. <u>On- and Off-energy Shell Propagation</u>: We have mentioned that, in addition to ladder diagrams, the Brueckner theory also attempts to take the third-order self-energy diagrams into account by using adjustable undefined single particle energy. We write the Hamiltonian

$$H=\sum(m|T|n)A_m^+A_n+1/2\sum(ij|V|k\ell-\ell k)A_i^+A_j^+A_\ell A_k$$

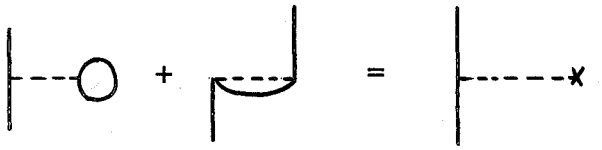
$$=[\sum(m|T+U|n)A_m^+A_n]+[1/2\sum(ij|V|k\ell-\ell k)A_i^+A_j^+A_\ell A_k-\sum(m|U|n)A_m^+A_n]$$

$$\equiv H_o + H_{int}.$$

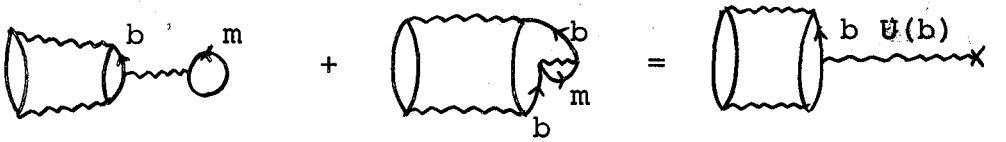
U is some single particle potential yet to be specified. The single-particle energy now is $T+U=\epsilon$, and the interaction H_{int} is



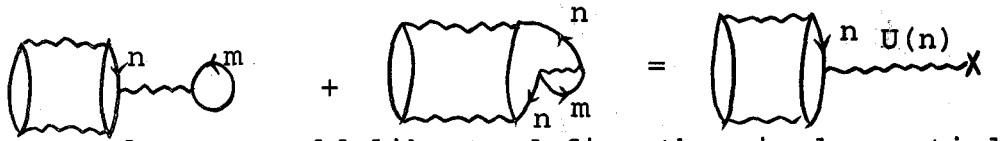
In the Hartree-Fock theory, it is possible to define U in such a way that the particle (also hole) self-energy is exactly cancelled by the corresponding insertion of the single particle potential U, i.e.



in all orders. Our aim here is to cancel self-energy diagrams in the third-order in the G-interaction:



and



In other words, we would like to define the single particle potential U(q) as

$$U(q) = \sum_{m\leq k_F} (qm|G|qm-mq) \qquad \text{(III-25)}$$

It turns out, however, that we are unable to do this exactly.

Let us expand the G-interaction of the inserted bubble in terms of the V-interaction

The contribution to energy of this diagram is

$$\sum (\ell n|G|ab) \frac{1}{\epsilon_a+\epsilon_b-\epsilon_\ell-\epsilon_n} (bm|G|bm) \frac{1}{\epsilon_a+\epsilon_b-\epsilon_\ell-\epsilon_n} (ab|G|\ell n)$$

where

$$(bm|G|bm)=(bm|V|bm) - \sum_{cd>k_F} (bm|V|cd) \frac{1}{\epsilon_a+\epsilon_c+\epsilon_d-\epsilon_\ell-\epsilon_n-\epsilon_m} (cd|G|bm) \qquad \text{(III-26)}$$

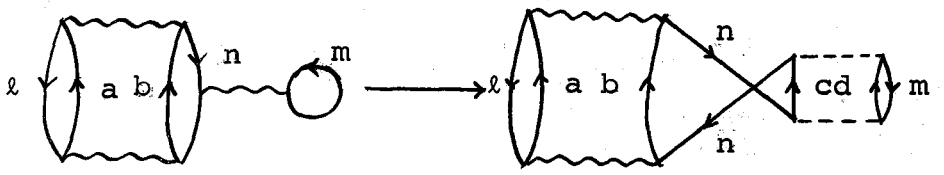
In equation (III-26) we see that the energy denominator is not $\epsilon_c+\epsilon_d-\epsilon_b-\epsilon_m$. Instead it is $\epsilon_c+\epsilon_d-\epsilon_b-\epsilon_m+\delta E$ with $\delta E = \epsilon_a+\epsilon_b-\epsilon_\ell-\epsilon_n$. We say that the particle (in state b) propagates off the energy shell. There is an additional excitation energy δE. Thus, when the G-matrix occurs with self-energy bubble insertion, we should write it as $(bm|G(\delta E)|bm)$ since it depends on other parts of the diagram as well. It is now clear that equation (III-25) cannot be satisfied. The right-hand side depends on a, b, ℓ, n (but only three of these are independent), whereas the left-side is a function of b only. The best we can do is to define

$$U(b) = \underset{\ell,n}{\text{ave}} \sum_{\leq k_F} (bm|G(\delta E)|bm-mb;\ell n) \qquad \text{(III-27)}$$

Therefore the single particle potential U is not really self-consistent. The cancellation of self-energy diagram is only

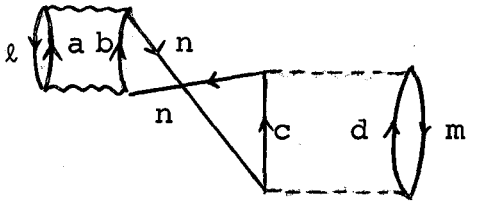
true for some sort of average momentum.

The same thing might also be said of the hole propagation



(III-28)

However, let us consider an additional diagram



(III-29)

Their combined contribution is

$$|(\ell n|V|ab)(nm|V|cd)|^2$$

$$\times\left[\frac{1}{(\epsilon_a+\epsilon_b-\epsilon_\ell-\epsilon_n)}\cdot\frac{1}{(\epsilon_a+\epsilon_b+\epsilon_c+\epsilon_d-\epsilon_\ell-\epsilon_m-2\epsilon_n)}\cdot\frac{1}{(\epsilon_a+\epsilon_b-\epsilon_\ell-\epsilon_n)}\right.$$

$$\left.+\frac{1}{(\epsilon_c+\epsilon_d-\epsilon_m-\epsilon_n)}\cdot\frac{1}{(\epsilon_a+\epsilon_b+\epsilon_c+\epsilon_d-\epsilon_\ell-\epsilon_m-2\epsilon_n)}\cdot\frac{1}{(\epsilon_a+\epsilon_b-\epsilon_\ell-\epsilon_n)}\right]$$

$$=\frac{|(\ell n|V|ab)(nm|V|cd)|^2}{(\epsilon_a+\epsilon_b-\epsilon_\ell-\epsilon_n)(\epsilon_c+\epsilon_d-\epsilon_n-\epsilon_m)(\epsilon_a+\epsilon_b-\epsilon_\ell-\epsilon_n)}$$

Hence bubble insertion to a hole line may be taken to be on the energy shell if we implicitly also include (III-29). This important result was first noted by Brueckner and Goldman (1960), and proved generally by Bethe et al (1963). We may therefore write the single hole potential

$$U(n) = \sum (nm|G(\delta E=0)|nm-mn).$$

E. Calculational Procedure and Numerical Details: Practical techniques for using the reference spectrum method have been lucidly given by Razavy (1963) and by Razavy and Sprung (1964). In particular, we have been greatly benefited from discussions with Dr. Sprung.

I. Static Potential

1. Singlet Case

Let

$$\zeta = \frac{1}{k_o r} \sum (2L+1) i^L X_L(k_o r) P_L(\cos\theta),$$

$$\phi = \frac{1}{k_o r} \sum (2L+1) i^L J_L(k_o r) P_L(\cos\theta),$$

and

$$\psi = \frac{1}{k_o r} \sum (2L+1) i^L U_L(k_o r) P_L(\cos\theta). \qquad \text{(III-30)}$$

Then equation (III-20) becomes

$$\left(\frac{d^2}{dr^2} + g_L\right) X_L = h_L \qquad \text{(III-31)}$$

where

$$g_L \equiv -\left[\frac{L(L+1)}{r^2} + \gamma^2 + m^* V\right] \qquad \text{(III-32a)}$$

and

$$h_L \equiv -m^* V \zeta_L \qquad \text{(III-32b)}$$

In the asymptotic region, $X_L = H_L \equiv \zeta_L(c) \dfrac{H_L^{(-)}(\gamma r)}{H_L^{(-)}(\gamma c)}$ where the $H_L^{(\pm)}$ are related to the usual spherical Hankel functions by

$$H_L^{(\pm)}(x) = i^{(L+1)}(\mp ix)h_L^{(1)}(\pm ix)$$

$$= i^{-(L+1)}(\pm ix)h_L^{(2)}(\pm ix).$$

Therefore we have

$$\frac{dX_L}{dr} - \gamma \frac{H_L^{(-)'}(\gamma r)}{H_L^{(-)}(\gamma r)} X_L = 0, \quad r=\infty \ . \tag{III-33}$$

as one of our boundary conditions. Here the prime denotes differentiation with respect to the argument. Now, although our potential has a soft core, for computational convenience we have put in a very small mathematical hard core at r=c. Thus

$$X_L(c) = J_L(c) \tag{III-34}$$

Equations (III-31), (III-33) and (III-34) constitute a two-point boundary value problem. A convenient way is to use the Ridley method (Ridley 1957), which essentially converts the two-point boundary value problem into an initial value problem. Let

$$\left(\frac{d}{dr} + t_L(r)\right)\left(\frac{d}{dr} + S_L(r)\right) \ X_L = h_L \tag{III-35}$$

Comparison of (III-31) and (III-35) shows

$$S_L(r) + t_L(r) = 0, \tag{III-36}$$

and

$$\frac{dS_L}{dr} + S_L t_L = g_L \tag{III-37}$$

Hence

$$\frac{dS_L}{dr} - S_L^2 = g_L \qquad \text{(III-38)}$$

Let

$$\frac{dX_L}{dr} + S_L X_L = W_L. \qquad \text{(III-39)}$$

Then equation (III-35) can be rewritten as

$$h_L = \frac{dW_L}{dr} + t_L W_L$$

$$= \frac{dW_L}{dr} - S_L W_L. \qquad \text{(III-40)}$$

From equations (III-39) and (III-33) we get

$$S_L \Big|_\infty = -\gamma \left. \frac{H_L'^{(-)}(\gamma r)}{H_L^{(-)}(\gamma r)} \right|_\infty \qquad \text{(III-41a)}$$

and

$$W_L \Big|_\infty = 0. \qquad \text{(III-41b)}$$

With these as the initial conditions, equations (III-38) and (III-40) can be integrated backwards to r=c. At r=c, we use (III-34) as the initial condition and integrate (III-39) forwards, thus obtaining the solution for X_L. Once the wave defect X_L is known, the reaction matrix

$$(k_o | G_L^R | k_o) = \frac{(\gamma^2 + k_o^2)}{m^*} \int \phi_L (k_o r) X_L dr. \qquad \text{(III-42)}$$

can be obtained in a straightforward manner.

2. Triplet Case

The treatment for the uncoupled triplet state is practically identical to that of the singlet case already discussed. We therefore confine ourselves to the coupled states $L=J\pm1$. The basic equation is now

$$(\nabla_J^2-\gamma^2-m^*V^J)X^J= -m^*V^J\mathcal{J}^J \tag{III-43}$$

where

$$\nabla_J^2 = \begin{bmatrix} \frac{d^2}{dr^2} - \frac{J(J-1)}{r^2} & 0 \\ 0 & \frac{d^2}{dr^2} - \frac{(J+1)(J+2)}{r^2} \end{bmatrix}, \tag{III-44}$$

$$\mathcal{J}^J = \begin{bmatrix} \mathcal{J}_{J-1} & 0 \\ 0 & \mathcal{J}_{J+1} \end{bmatrix}, \tag{III-45}$$

$$V^J = \begin{bmatrix} (J,J-1|V_{tot}|J,J-1) & (J,J-1|V_{tot}|J,J+1) \\ (J,J+1|V_{tot}|J,J-1) & (J,J+1|V_{tot}|J,J+1) \end{bmatrix} \tag{III-46}$$

and

$$X^J = \begin{bmatrix} X_{J-1,J-1} & X_{J-1,J+1} \\ X_{J+1,J-1} & X_{J+1,J+1} \end{bmatrix} \tag{III-47}$$

is the solution matrix with

$$\begin{bmatrix} X_{J-1,J-1} \\ \\ X_{J+1,J-1} \end{bmatrix} = \begin{bmatrix} \mathcal{J}_{J-1} - U_{J-1,J-1} \\ \\ - U_{J+1,J-1} \end{bmatrix}$$

and

$$\begin{bmatrix} X_{J-1,J+1} \\ \\ X_{J+1,J+1} \end{bmatrix} = \begin{bmatrix} - U_{J-1,J+1} \\ \\ \mathcal{J}_{J+1} - U_{J+1,J+1} \end{bmatrix}$$

each being a solution: the first one is the (J-1)-wave dominant solution, whereas the second one is the (J+1)-dominant solution. Equation (III-43) in essence contains four equations and can again be solved by Ridley's method with the following generalizations:

$$S_L \longrightarrow S^J = \begin{bmatrix} S_{J-1,J-1} & S_{J-1,J+1} \\ \\ S_{J+1,J-1} & S_{J+1,J+1} \end{bmatrix} \qquad \text{(III-48)}$$

$$W_L \longrightarrow W^J = \begin{bmatrix} W_{J-1,J-1} & W_{J-1,J+1} \\ \\ W_{J+1,J-1} & W_{J+1,J+1} \end{bmatrix} \qquad \text{(III-49)}$$

$$g_L \longrightarrow g^J = - \begin{bmatrix} \frac{(J-1)J}{r^2} + \gamma^2 + m*V_{J-1,J-1} & m*V_{J-1,J+1} \\ \\ m*V_{J+1,J-1} & \frac{(J+1)(J+2)}{r^2} + {}^2 + m*V_{J+1,J+1} \end{bmatrix}$$

(III-50)

and

$$h_L \longrightarrow h^J = -m^*V^J \not{J}^J. \tag{III-51}$$

Equations (III-38), (III-40) and (III-39) are simply replaced, respectively, by

$$\frac{dS^J}{dr} - (S^J)^2 = g^J, \tag{III-52}$$

$$\frac{dW^J}{dr} - S^J W^J = h^J, \tag{III-53}$$

and

$$\frac{dX^J}{dr} + S^J X^J = W^J \tag{III-54}$$

The numerical solution then proceeds as before, and the reaction matrix is given in terms of the diagonal elements of X.

$$(k_o|G_L^R|k_o) = \frac{(\gamma^2+k_o^2)}{m^*}\int \phi_L(k_o r)\, X_{L,L}\, dr \tag{III-55}$$

with $L=J\pm 1$.

II. Velocity-dependent Potentials

1. Singlet Case

For a velocity-dependent potential of the form (II-3) the equation for the wave defect X_L becomes

$$\left(\frac{d^2}{dr^2} + f_L \frac{d}{dr} + g_L\right) X_L = h_L \tag{III-56}$$

where

$$f_L \equiv \frac{2m^*W'}{(1+2m^*W)}, \tag{III-57}$$

$$g_L \equiv \frac{L(L+1)}{r^2} + \frac{(\gamma^2+m^*V-m^*W'')}{(1+2m^*W)}, \tag{III-58}$$

and

$$h_L \equiv \frac{m^*}{(1+2m^*W)} \left\{ 2W \int_L'' + 2W' \int_L' - \left[V - W'' + \frac{2L(L+1)}{r^2} W \right] \int_L \right\}. \tag{III-59}$$

Applying Ridley's method we now get

$$\frac{dS_L}{dr} + (f_L - S_L) S_L = g_L, \tag{III-60}$$

$$\frac{dW_L}{dr} + (f_L - S_L) W_L = h_L, \tag{III-61}$$

and

$$\frac{dX_L}{dr} + S_L X_L = W_L. \tag{III-62}$$

Note that for a static potential $f_L = 0$ and these three equations reduce to the previous set of equations (III-38), (III-39) and (III-40).

2. Triplet Case

For the coupled triplet states we need the matrix elements

$$\begin{aligned}
(J,J-1|V_{tot}|J,J-1) &= V - \frac{2(J-1)}{(2J+1)} V_T + (J-1) V_{LS} \\
&+ \frac{2J(J-1)}{r^2} \left[W - \frac{2(J-1) W_T}{(2J+1)} + (J-1) W_{LS} \right] - W'' \\
&+ \frac{2(J-1)}{2J+1} W_T'' - (J-1) W_{LS}'' \\
&+ \left[-2W' + \frac{4(J-1)}{(2J+1)} W_T' - 2(J-1) W_{LS}' \right] \frac{d}{dr} \\
&+ \left[-2W + \frac{4(J-1)}{(2J+1)} W_T - 2(J-1) W_{LS} \right] \frac{d^2}{dr^2} \\
&\equiv a^J_{J-1,J-1} + b^J_{J-1,J-1} \frac{d}{dr} + c^J_{J-1,J-1} \frac{d^2}{dr^2}
\end{aligned} \tag{III-63}$$

$$
\begin{aligned}
&(J,J+1|V_{tot}|J,J+1) \\
&\quad = V - \frac{2(J+2)}{2J+1} V_T - (J+2)V_{LS} + \frac{2(J+1)(J+2)}{r^2} \\
&\qquad\qquad \left[W - \frac{2(J+2)}{2J+1} W_T - (J+2)W_{LS} \right] \\
&\qquad -W'' + \frac{2(J+2)}{2J+1} W_T'' + (J+2)W_{LS}'' \\
&\qquad + \left(-2W' + \frac{4(J+2)}{2J+1} W_T' + 2(J+2)W_{LS}'\right) \frac{d}{dr} \qquad \text{(III-64)} \\
&\qquad + \left(-2W + \frac{4(J+2)}{2J+1} W_T + 2(J+2)W_{LS}\right) \frac{d^2}{dr^2} \\
&\qquad \equiv a^J_{J+1,J+1} + b^J_{J+1,J+1} \frac{d}{dr} + c^J_{J+1,J+1} \frac{d^2}{dr^2},
\end{aligned}
$$

and

$$
\begin{aligned}
&(J,J+1|V_{tot}|J,J-1) = (J,J-1|V_{tot}|J,J+1) \\
&\quad = \frac{6\sqrt{J(J+1)}}{2J+1} \left[V_T + \frac{2(J^2+J+1)}{r^2} W_T - W_T'' - 2W_T' \frac{d}{dr} - 2W_T \frac{d^2}{dr^2} \right] \\
&\quad \equiv a^J_{J+1,J-1} + b^J_{J+1,J-1} \frac{d}{dr} + c^J_{J+1,J-1} \frac{d^2}{dr^2}. \qquad \text{(III-65)}
\end{aligned}
$$

Equation (III-43) now takes the form

$$
\begin{aligned}
&\left\{ \begin{bmatrix} (1-m^*c^J_{J-1,J-1}), & -m^*c^J_{J-1,J+1} \\ -m^*c^J_{J+1,J-1}, & (1-m^*c^J_{J+1,J+1}) \end{bmatrix} \frac{d^2}{dr^2} + \begin{bmatrix} -m^*b^J_{J-1,J-1}, & -m^*b^J_{J-1,J+1} \\ -m^*b^J_{J+1,J-1}, & -m^*b^J_{J+1,J+1} \end{bmatrix} \frac{d}{dr} \right. \\
&\left. + \begin{bmatrix} -\left(m^*a^J_{J-1,J-1} + \frac{J(J-1)}{r^2} + \gamma^2\right), & -m^*a^J_{J-1,J+1} \\ -m^*a^J_{J+1,J-1}, & -\left(m^*a^J_{J+1,J+1} + \frac{(J+1)(J+2)}{r^2} + \gamma^2\right) \end{bmatrix} \right\} X^J \\
&\qquad = -m^*V^J \mathcal{J}^J. \qquad \text{(III-66)}
\end{aligned}
$$

Or

$$
\left(\frac{d^2}{dr^2} + f^J \frac{d}{dr} + g^J \right) X^J = h^J \qquad \text{(III-67)}
$$

where

$$f^J \equiv M^{-1} \begin{bmatrix} -m^* b^J_{J-1,J-1} & -m^* b^J_{J-1,J+1} \\ -m^* b^J_{J+1,J-1} & -m^* b^J_{J+1,J+1} \end{bmatrix}, \qquad \text{(III-68)}$$

$$g^J \equiv M^{-1} \begin{bmatrix} -\left(m^* a^J_{J-1,J-1} + \frac{J(J-1)}{r^2} + \gamma^2\right) & -m^* a^J_{J-1,J+1} \\ -m^* a^J_{J+1,J-1} & -\left(m^* a^J_{J+1,J+1} + \frac{(J+1)(J+2)}{r^2} + \gamma^2\right) \end{bmatrix} \qquad \text{(III-69)}$$

$$h^J \equiv -M^{-1} m^* V^J \mathcal{J}^J \qquad \text{(III-70)}$$

and

$$M^{-1} \equiv \begin{bmatrix} (1-m^* c^J_{J-1,J-1}) & -m^* c^J_{J-1,J+1} \\ -m^* c^J_{J+1,J-1} & (1-m^* c^J_{J+1,J+1}) \end{bmatrix}^{-1}$$

$$= \frac{1}{(1-m^* c^J_{J-1,J-1})(1-m^* c^J_{J+1,J+1}) - m^* c^J_{J+1,J-1} c^J_{J+1,J+1}} \cdot$$

$$\begin{bmatrix} (1-m^* c^J_{J+1,J+1}) & m^* c^J_{J+1,J+1} \\ m^* c^J_{J+1,J-1} & (1-m^* c^J_{J+1,J+1}) \end{bmatrix} \qquad \text{(III-71)}$$

Equation (III-67) is the coupled triplet counterpart of equation (III-56). We can also immediately write down the

corresponding equations to (III-60), (III-61) and (III-62):

$$\frac{dS^J}{dr} + (f^J - S^J)\, S^J = g^J, \tag{III-72}$$

$$\frac{dW^J}{dr} + (f^J - S^J)\, W^J = h^J, \tag{III-73}$$

and

$$\frac{dX^J}{dr} + S^J X^J = W^J. \tag{III-74}$$

To complete this discussion, we also write down the equation for the triplet L=J case:

$$\left(\frac{d^2}{dr^2} + f_J \frac{d}{dr} + g_J\right) X_J = h_J \tag{III-75}$$

where

$$f_J \equiv \frac{(-2W' - 4W_T' + 2W_{LS}')\, m^*}{-(1+2m^*W) - 4m^*W_T + 2m^*W_{LS}}, \tag{III-76}$$

$$g_J \equiv \frac{1}{-(1+2m^*W) - 4m^*W_T + 2m^*W_{LS}} \left\{ \gamma^2 + \frac{J(J+1)\,(1+2m^*W)}{r^2} + m^* \left[V + 2V_T - V_{LS} - W'' - 2W_T'' + \frac{4J(J+1)\,W_T}{r^2} + W_{LS}'' - \frac{2J(J+1)}{r^2} W_{LS} \right] \right\}, \tag{III-77}$$

and

$$h_J \equiv \frac{m^*}{-(1+2m^*W) - 4m^*W_T + 2m^*W_{LS}} \left\{ (-2W - 4W_T + 2W_{LS}) \oint_J'' + (-2W' - 4W_T' + 2W_{LS}') \oint_J' + \left[\frac{2J(J+1)}{r^2} W + V + 2V_T - V_{LS} - W'' - 2W_T'' + \frac{4J(J+1)W_T}{r^2} + W_{LS}'' - \frac{2J(J+1)}{r^2} W_{LS} \right] \oint_J \right\}. \tag{III-78}$$

The Ridley method described above is a very compact and convenient one. We solve all our singlet states by this method. The equation in the s-function, however, is a non-linear one and may grow quite fast and thus cause an over-flow in the computer. This happens to our triplet case, as we have a weak singularity. We therefore resort to some other conventional methods in treating the two-point boundary-value problem with boundaries at x=a and x=b. One method is the two-point matching method. Consider a single second order, inhomogeneous differential equation. Let y_{ho} be a solution to the corresponding homogeneous differential equation with $y_{ho}(a)=0$ and arbitrary initial slope. We integrate this solution outwards to a point $x=x_2$ (say). Let y_{po} be a particular integral of the inhomogeneous equation with $y_{po}(a)=y(a)$, the given boundary condition at x=a, and also arbitrary initial slope. This is also integrated from x=a to $x=x_2$. Then

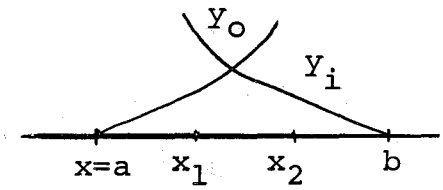
$$y_o=Cy_{ho}+y_{po}$$

is a solution of the inhomogeneous differential equation for $a\leqslant x\leqslant x_2$. C is a yet undetermined coefficient. Now we start at x=b with another set y_{hi} and y_{pi} and integrate inwards each time to $x=x_1$, $x_1<x_2$. Thus,

$$y_i=Dy_{hi}+y_{pi}$$

is another solution of the inhomogeneous differential equation

for $x_1 \leqslant x \leqslant b$. The initial conditions of y_{hi} and y_{pi} are so chosen that y_i will satisfy the required boundary condition at x=b. For example, suppose we want $y(b)=\alpha$. Then we can choose $y_{hi}(b)=0$, $y_{pi}(b)=\alpha$.



The coefficients C and D are determined by matching y_o and y_i at the two points x_1 and x_2. This joins y_o and y_i smoothly, and the boundary conditions are satisfied at x=a and x=b.

Yet another method is as follows. Suppose we now consider a system of two second order inhomogeneous differential equations in the functions u and y. Then

$$y = Ay_{h1} + By_{h2} + y_p$$

$$u = Au_{h1} + Bu_{h2} + u_p$$

is a set of solutions. Here subscript hi denotes the i-th independent set of homogeneous solution, and p the set of particular integrals. Suppose the given boundary conditions are $y(a)=\alpha$, $u(a)=\beta$, $\left.\frac{y'}{y}\right|_b=\sigma$ and $\left.\frac{u'}{u}\right|_b=\tau$. The boundary conditions at x=a can be satisfied if we choose the homogeneous solutions to be zero at x=a and $y_p(a)=\alpha$, $u_p(a)=\beta$. These are all integrated outwards to x=b. The boundary conditions at x=b can be satisfied if we choose A and B to be

$$A = \frac{\begin{vmatrix} y_p'(b)-\sigma y_p(b) & \sigma y_{h2}(b)-y_{h2}'(b) \\ u_p'(b)-\tau u_p(b) & \tau u_{h2}(b)-u_{h2}'(b) \end{vmatrix}}{\det}$$

$$B = \frac{\begin{vmatrix} \sigma y_{h1}(b) - y'_{h1}(b) & y'_p(b) - \sigma y_p(b) \\ Tu_{h1}(b) - u'_{h1}(b) & u'_p(b) - Tu_p(b) \end{vmatrix}}{\det}$$

where

$$\det = \begin{vmatrix} \sigma y_{h1}(b) - y'_{h1}(b) & \sigma y_{h2}(b) - y'_{h2}(b) \\ Tu_{h1}(b) - u'_{h1}(b) & Tu_{h2}(b) - u'_{h2}(b) \end{vmatrix}$$

We use the latter method for the triplet $L=J\pm1$ states, and the former for the $L=J$ states.

Before solving the differential equation for the wave defect X we must first decide on an approximate expression for γ^2. Bethe et al (1963) uses

$$\gamma_m^2 = 2\Delta k_F^2 - k_o^2 \qquad \text{(III-79)}$$

for holes, and

$$\gamma_b^2 = 3k_o^2 + (3\Delta - .6)\, k_F^2 \qquad \text{(III-80)}$$

for particles. Here k_o denotes the difference of the momenta of the two colliding particles. These expressions for γ^2 are meant to take the off-shell propagation effect into account, and are tailored for a hard core potential. In Fig. III-1 we show a plot of X vs. r for our non-static, soft core potential, and in Fig. III-2 we plot the square of the Fourier transform of the wave defect X. We see that our important intermediate states are in comparable momentum region as those for a hard core potential, and therefore we conclude that we may use the same expression as Bethe et al, for γ^2.

As already mentioned, once one has the wave defect the G-matrix has essentially been obtained. To calculate the binding energy of a nucleon in nuclear matter, we follow closely the procedure of Razavy (1963). We differ from him in that we only calculate the hole potential energy for an average collision momentum and that we do not attempt separation in any angular momentum state.

First, we calculate the G-matrix for an average collision momentum $k_o=\sqrt{.3}k_F$. From this we get the mean single-particle potential energy for the hole-state

$$\bar{u}_m = \frac{2k_F^3}{3\pi^2} \quad (k_o=\sqrt{.3}k_F|G^R|k_o) \times 41.5\text{Mev.} \qquad \text{(III-81)}$$

Here G^R denotes the sum of all the partial waves, including their appropriate statistical weights. We then calculate the G-matrix for the particle states. Let

$$W^{\text{particle}}(k_o) = \frac{2k_F^3}{3\pi^2} \times 41.5 \sum_{\text{even } \ell} (k_o|G_\ell^R|k_o) \qquad \text{(III-82)}$$

Following Rajaraman's suggestion (Rajaraman 1963), for particle states we only sum over the even states and with statistical weights equal to one. This serves to take the three-body cluster effect into account to a certain extent. We compute W^{particle} for two k_o's, namely, for $k_o=1.5k_F$ and $2k_F$, and we fit

$$W^{\text{particle}}(k_o) = A' + B' k_o^2 \qquad \text{(III-83)}$$

by a parabola. Then the single particle potential energy for particle states is, as shown in Razavy (1963),

$$u(k_b) = \left(A' + \frac{.6k_F^2 B'}{4} \right) + \frac{B'}{4} k_b^2 \qquad \text{(III-84)}$$

and

$$m_f^* = \frac{1}{1 + \frac{B'M}{2\hbar^2}} \qquad \text{(III-85)}$$

$$\Delta_f = \frac{m_f^*}{41.5} \frac{1}{k_F^2} \left[u(k_b = \sqrt{.6}\, k_F) - \bar{u} \right] \qquad \text{(III-86)}$$

For a given k_F, we choose a quessed input Δ_i and m_i^*, and see if the output Δ_f and m_f^* are consistent with the input. If not, we repeat the cycle. The results are shown in Table III-2.† Table III-3 shows the contribution of various partial waves to $\bar{u}_m$. The average energy per particle is given by the sum of the average kinetic energy and one-half of the average single particle potential energy

$$\bar{E} = \bar{T} + \frac{\bar{u}_m}{2} = \frac{3}{10} \frac{\hbar^2}{M} k_F^2 + \frac{\bar{u}_m}{2}.$$

The above description forms a major cycle in our computation. We execute major cycles for $k_F = 1.25F^{-1}$, $1.36F^{-1}$ and $1.50F^{-1}$. Fig. III-3 shows that we get saturation at $k_F = 1.38F^{-1}$ with an energy minimum of $\bar{E}$ = -13.2Mev per particle.

---

†Due to a mistake in the $^3P_1$ state now corrected, the self-consistent values for Δ will be slightly reduced to .6, but the value for m* would probably remain the same.

COMPARISON OF STATIC AND VELOCITY-DEPENDENT

THREE-MESON POTENTIAL MODELS

TABLE III-1

	C. Wong's Y^2_{np}	Our Model
a	-23.73 F	-23.58 F
r_o	2.67 F	2.78 F
δ (250Mev)	.001 rad.	.019 rad.
d	1.06 F	1.04 F
V_ℓ	-461.3 Mev-F^3	-462.8 Mev-F^3
D	.0537	.0179
Fermi momentum k_F=1.36F^{-1}; Relative momentum k_o = .75 F^{-1}.		

REFERENCE SPECTRUM PARAMETERS

TABLE III-2

$k_F(F^{-1})$	Δ_i	m_i^*	Δ_f	m_f^*
1.25	0.7	0.80	0.68	0.83
1.36	0.7	0.90	0.72	0.80
1.50	0.65	0.85	0.68	0.86

PARTIAL WAVES CONTRIBUTION TO THE G-MATRIX.

$W_\ell(k_o) = \frac{2k_F^3}{3\pi^2} C_\ell \ (k_o = \sqrt{.3}\, k_F | G_\ell | k_o) \times 41.5\text{Mev}$, WHERE

C_ℓ IS THE APPROPRIATE STATISTICAL WEIGHT, $k_F = 1.36\text{F}^{-1}$.

TABLE III-3

States	W_ℓ (Mev)
$^1S_0$	-32.30
$^1P_1$	+ 8.45
$^1D_2$	- 5.29
$^1F_3$	+ 1.79
$^1G_4$	- 0.64
$^3P_0$	- 8.97
$^3P_1$	+ 21.43
$^3D_2$	- 5.67
$^3F_3$	+ 0.84
$^3G_4$	- 0.19
$^3S_1$	-43.41
$^3D_1$	+ 2.98
$^3P_2$	- 9.43
$^3F_2$	- 1.18
$^3D_3$	- 0.08
$^3G_3$	+ 0.28
$^3F_4$	- 0.62
$^3H_4$	- 0.06
Total	-72.02

CHAPTER IV

SUMMARY AND CONCLUSION

We have studied a nucleon-nucleon potential based on the one-boson-exchange model. This potential has a specified type of velocity-dependence, which occurs not only in the central force part but also in the tensor and spin-orbit parts. Moreover, the velocity-dependence is not of short range.

We have fitted the potential parameters to two-body scattering data. In view of the comparatively few parameters we have, our fit is fairly satisfactory, especially for the important S-states.

We have applied this potential to nuclear matter calculation. We come to the following conclusions:

(i) one does not need a hard-core to get saturation at reasonable density. This point has been of considerable interest, as first questioned by Green (1962), and examined in great detail by Bhaduri and Preston (1964) who arrived at the same conclusion as the present investigation. The potential we use nevertheless possesses a very strong soft core, and from the wave defect curve, Fig. III-1, we see that it is almost equivalent to a hard core of 0.3F.

(ii) a velocity-dependent, soft core potential seems to give still more binding than a static, soft core potential. This is borne out by our comparison with C. Wong's work (1965), fixing the same three potential parameters as predetermined and fitting the remaining three to reproduce the same two-body scattering data.

Our calculation gives a binding energy of 13.2Mev per particle in nuclear matter at a saturation density $k_F=1.38F^{-1}$. Preliminary calculations by Patrick Yip indicate that the Pauli and spectral connection terms reduce the binding energy by about 2Mev per particle. Sprung and Bhargava (1966) have pointed out that the contribution from higher partial waves will reduce the binding energy further by about 1.8Mev per particle, whereas an increase of 4 to 5Mev will come about if one treats the three-body clusters properly. Therefore our binding energy would be about 14Mev per particle at $k_F\approx 1.4F^{-1}$. These are comparable to the results obtained by Sprung and Bhargava (1966) for the Hamada-Johnston potential, the Bressel potential, and the Reid potential.

APPENDICES

AND

BIBLIOGRAPHY

APPENDIX A

THE $\mathcal{K}$-FUNCTIONS

Definitions

The $\mathcal{K}$-function is defined by

$$\mathcal{K}_N(x) = C_N \int_1^\infty \frac{dy}{\sqrt{y^2-1}} \quad \frac{e^{-xy}}{y^N} \tag{A-1}$$

where C_N is a normalization constant. For N=0 this function is equivalent to the zeroth order modified Bessel function of the second kind. The normalization constants are chosen such that

$$\int_0^\infty \mathcal{K}_N(x)\,dx \equiv 1/2 \text{ if N even,}$$

and (A-2)

$$\int_0^\infty x\,\mathcal{K}_N(x)\,dx \equiv 1/2 \text{ if N odd.}$$

In particular, this choice gives $C_o = \frac{1}{\pi}$. From equations (A-1) and (A-2), it follows that

$$C_{N+1} = 2C_N\,\mathcal{K}_{N+1}(o) \quad \text{if N even,}$$

and (A-3)

$$C_{N+2} = 2C_N\,\mathcal{K}_{N+2}(o) \quad \text{if N odd.}$$

Determination of the Normalization Constants

We can write alternately

$$\mathcal{K}_N(x) = C_N \int_0^{\pi/2} \cos^{N-1}\theta\; e^{-x\sec\theta}\, d\theta. \tag{A-4}$$

Thus

$$\mathcal{K}_N(o) = C_N \int_o^{\pi/2} \cos^{N-2}\theta \, d(\sin\theta)$$

$$= \frac{C_N}{C_{N-2}} (N-2)\, \mathcal{K}_{N-2}(o) - (N-2)\, \mathcal{K}_N(o), \text{ if } N>2$$

Or

$$\mathcal{K}_N(o) = \frac{(N-2)}{(N-1)} \frac{C_N}{C_{N-2}} \mathcal{K}_{N-2}(o), \quad N>2 \qquad \text{(A-5)}$$

Since

$$\mathcal{K}_1(o) \equiv C_1 \int_1^\infty \frac{dy}{y\sqrt{y^2-1}} = C_1 \frac{\pi}{2},$$

and

$$\int x\, \mathcal{K}_1(x)\, dx \equiv 1/2 = C_1 \int_1^\infty \frac{dy}{y\sqrt{y^2-1}} \int_o^\infty x e^{-xy} dx = C_1 \frac{\pi}{4}$$

we obtain

$$C_1 = \frac{2}{\pi} \quad \text{and} \quad \mathcal{K}_1(o) = 1$$

Similarly we get

$$C_2 = \frac{2}{\pi} \quad \text{and} \quad \mathcal{K}_2(o) = \frac{2}{\pi}$$

Now from equation (A-5) we have

$$\frac{\mathcal{K}_{2n}(o)}{C_{2n}} = \frac{2n-2}{2n-1} \frac{\mathcal{K}_{2n-2}(o)}{C_{2n-2}} = \frac{2n-2}{2n-1} \frac{2n-4}{2n-3} \frac{\mathcal{K}_{2n-4}(o)}{C_{2n-4}}$$

$$= \frac{2n-2}{2n-1} \frac{2n-4}{2n-3} \text{ ---- } \frac{2}{3} \frac{\mathcal{K}_2(o)}{C_2}$$

$$= \frac{(2n-2)!!}{(2n-1)!!}, \quad n>1 \qquad \text{(A-6)}$$

and

$$\frac{\mathcal{K}_{2n+1}(o)}{C_{2n+1}} = \frac{2n-1}{2n} \frac{2n-3}{2n-2} \text{ ---- } \frac{1}{2} \frac{\mathcal{K}_1(o)}{C_1}$$

$$= \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2}, \quad n \geq 1. \qquad \text{(A-7)}$$

Taking equations (A-3) and (A-7) together we get

$$C_{2n} = 1/2 \frac{C_{2n+1}}{\mathcal{K}_{2n+1}(o)} = \frac{(2n)!!}{(2n-1)!!} \frac{1}{\pi}, \quad n>1 \tag{A-8}$$

and

$$C_{2n+1} = \frac{(2n+2)!!}{(2n+1)!!} \frac{1}{\pi}, \quad n \geq 1. \tag{A-9}$$

These together with $C_o = \frac{1}{\pi}$, $C_1 = \frac{2}{\pi}$ and $C_2 = \frac{2}{\pi}$ completely determine all the normalization constants.

Recursion Formulae

From the definition, we get by integration by parts

$$\begin{aligned} \mathcal{K}_N(x) &= C_N \int_o^{\pi/2} \cos^{N-1}\theta \, e^{-x\sec\theta} \, d\theta \\ &= C_N \int_o^{\pi/2} d\theta e^{-x\sec\theta} [-(N-2)\cos^{N-1}\theta + (N-2)\cos^{N-3}\theta \\ &\quad - x\cos^{N-2}\theta + x\cos^{N-4}\theta]. \end{aligned} \tag{A-10}$$

Equation (A-10) holds for any N if $x>0$, and it holds for $N>2$ if $x=0$. It follows then

$$(N-1)\mathcal{K}_N = (N-2)\frac{C_N}{C_{N-2}}\mathcal{K}_{N-2} - x\frac{C_N}{C_{N-1}}\mathcal{K}_{N-1} + x\frac{C_N}{C_{N-3}}\mathcal{K}_{N-3}, \quad \begin{matrix} N>2 \\ x\geq 0 \end{matrix}.$$

Thus we may write

$$\mathcal{K}_{2n}(x) = \frac{(2n-2)(2n)}{(2n-1)^2}\mathcal{K}_{2n-2}(x) - \frac{x}{2n-1}\mathcal{K}_{2n-1}(x) + \frac{2nx}{(2n-1)^2}\mathcal{K}_{2n-3}, \quad n>1$$

and (A-11)

$$\mathcal{K}_{2n+1}(x) = \frac{(2n-1)(n+1)}{n(2n+1)}\mathcal{K}_{2n-1}(x) - \frac{n+1}{n(2n+1)} x \mathcal{K}_{2n}(x) + \frac{2(n+1)x}{(2n+1)(2n-1)}\mathcal{K}_{2n-2}, \quad n\geq 1.$$

These formulae are useful in checking numerical results.

Asymptotic Expansion

$$\mathcal{K}_N(x) \equiv C_N \int_1^\infty \frac{dy}{\sqrt{y^2-1}} \frac{e^{-xy}}{y^N} = C_N e^{-x} \sqrt{\frac{2}{x}} \int_0^\infty \frac{du\, e^{-u^2}}{\left(1+\frac{u^2}{2x}\right)^{\frac{1}{2}} \left(1+\frac{u^2}{x}\right)^N}$$

$$\sim C_N \sqrt{\frac{\pi}{2x}}\, e^{-x} \left(1 - \frac{N+1/4}{2x} + \cdots \right). \qquad \text{(A-12)}$$

Numerical Evaluation

The definition we give for

$$\mathcal{K}_N(x) \equiv C_N \int_1^\infty \frac{dy}{\sqrt{y^2-1}} \frac{e^{-xy}}{y^N}$$

is an improper integral. For numerical purpose it is therefore more convenient to make the change of variable $v=\sqrt{y-1}$. Then

$$\mathcal{K}_N(x) = 2C_N e^{-x} \int_0^\infty dv \frac{e^{-xv^2}}{\sqrt{2+v^2}\,(1+v^2)^N}. \qquad \text{(A-13)}$$

For N=0 and small values of x, the integrand in equation (A-13) may not decay very fast and in such a case it may be more useful to sum the series

$$\mathcal{K}_o(x) = -C_o \sum_0^\infty \frac{\left(\frac{x}{2}\right)^{2m}}{(m!)^2} \left\{ \ell n \frac{x}{2} - \psi(m+1) \right\}, \qquad \text{(A-14)}$$

where

$$\psi(n) = -\gamma + (n-1) \sum_{S=0}^{\infty} \frac{1}{(S+1)(S+n)} \qquad \text{(A-15)}$$

and

$$\gamma = 0.5772157\cdots$$

is Euler's constant. The series converges well for $x \ll 2$.

Some Integrals Containing the $\mathcal{K}$-functions

Here we list some integrals which may be useful:

$$\int_0^\infty x^m \mathcal{K}_N(x)\,dx = C_N m!\,\frac{\mathcal{K}_{N+m+1}(o)}{C_{N+m+1}},$$

$$\int_0^\infty x\,\mathcal{K}_{2n}(x)\,dx = \frac{[(2n)!!]^2}{(2n-1)!!(2n+1)!!}\,\frac{1}{\pi},$$

$$\int_0^\infty x\,\mathcal{K}_{2n+1}(x)\,dx = \frac{1}{2},$$

$$\int_0^\infty x^2\,\mathcal{K}_{2n}(x)\,dx = \frac{2n+1}{2(n+1)}, \qquad \text{(A-16)}$$

$$\int_0^\infty x^2\,\mathcal{K}_{2n+1}(x)\,dx = \frac{[(2n+2)!!]^2}{(2n+1)!!(2n+3)!!}\,\frac{2}{\pi},$$

$$\int_0^\infty x^3\,\mathcal{K}_{2n}(x)\,dx = \frac{(2n)!!(2n+2)!!}{(2n-1)!!(2n+3)!!}\,\frac{6}{\pi},$$

and

$$\int_0^\infty x^3\,\mathcal{K}_{2n+1}(x)\,dx = \frac{(2n+2)!!(2n+3)!!3}{(2n+1)!!(2n+4)!!} = \frac{3(2n+3)}{2n+4}.$$

APPENDIX B

ON CONVERSION OF UNITS

There are many sets of units and all appear to be 'natural' to their own propounders. In principle conversion from one set of units to another is simple and straight forward, but in practice this is always annoying and agonizing. For convenience we therefore give various conversion relations concerning us in our work.

In field theoretic work, one sets $\hbar=c=1$. Then mass, energy, and momentum all have the dimension of inverse length. For energy, one has $1F^{-1}=197.31\text{Mev}$. In our potential we have used $\hbar=c=M=1$, where M is the nucleon mass. This amounts to choosing the nucleon Compton wavelength $\lambda\!\!\!^{-}_N$ as the unit of length:

$$1F=4.7582\lambda\!\!\!^{-}_N$$

and

$$1\lambda\!\!\!^{-}_N{}^{-1}=938.858\text{Mev}.$$

In nuclear matter calculation, it is customary to set $\frac{\hbar}{M}=1$. Energy is then expressed as the square of inverse length. The conversion factor is

$$1F^{-2}=41.47\text{Mev}.$$

Equations in the reference spectrum method are in this system of units. To put our potential (as given by equation (II-39) to (II-42) in these equations we must make the following

conversion:

$V(r) \longrightarrow (4.7582)^2 x\ V(4.7582r)$, r now in F,

$W(r) \longrightarrow W(4.7582r)$,

$W'(r) \longrightarrow (4.7582)\ x\ W'(4.7582r)$,

$W''(r) \longrightarrow (4.7582)^2\ x\ W''(4.7582r)$.

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FIGURE CAPTIONS

Fig. II-1 to II-15 — The $\mathcal{K}$-functions. Phase shift characteristics. Δ are points taken from the analysis of Arndt and MacGregor, Preprint UCRL-14252-T (1965). + are Yale points.

Fig. III-1 — The wave defect for the ^1S_o-state, calculated for $k_F=1.36F^{-1}$, $k_o=.75F^{-1}$, $m^*=.9$ and $\Delta=.75$.

Fig. III-2 — The square of the Fourier Transform of the wave defect.

Fig. III-3 — The binding energy curve.

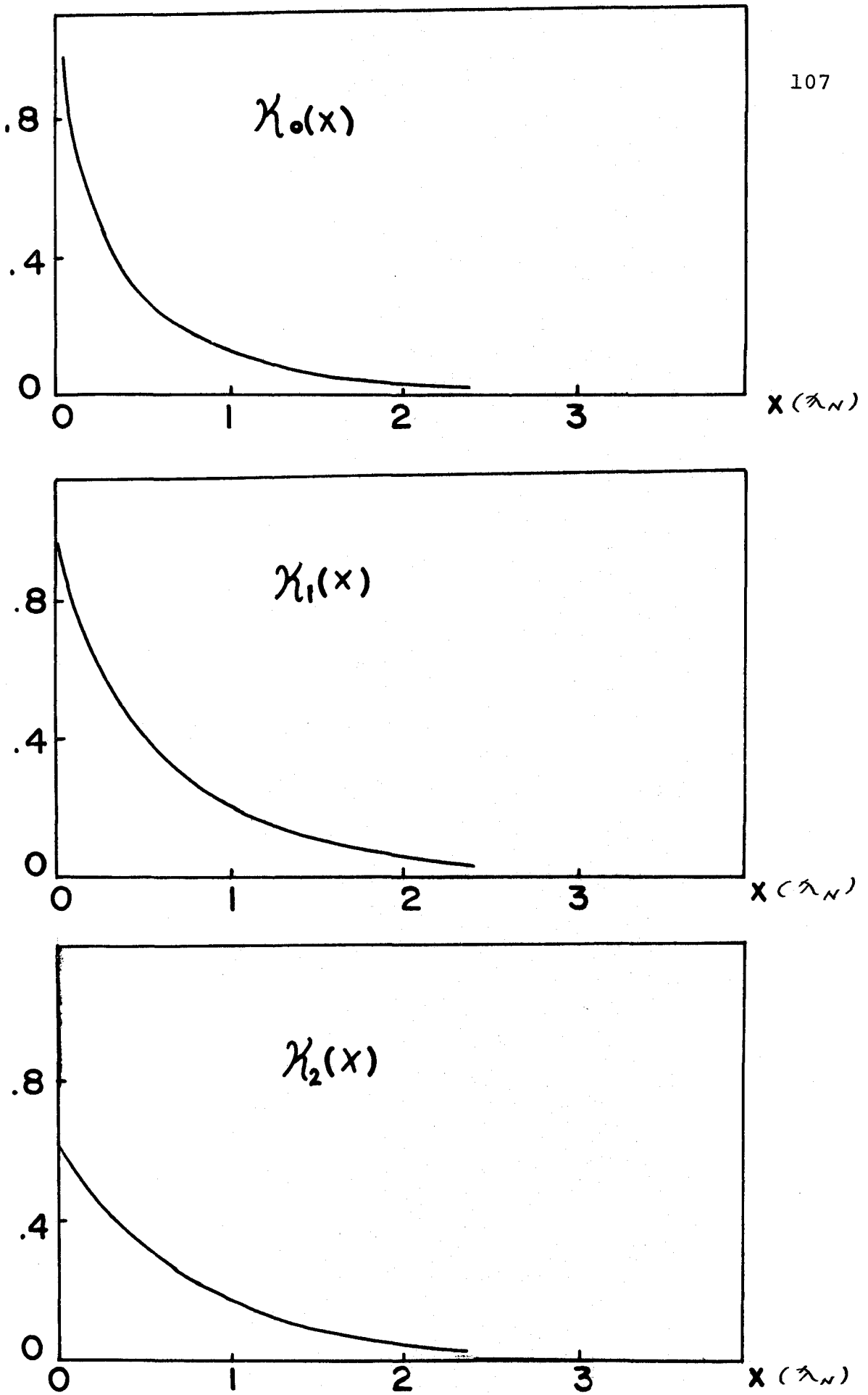
$K_0(x)$
.8
.4
0
0
1
2
3
$x(\lambda_N)$
$K_1(x)$
.8
.4
0
0
1
2
3
$x(\lambda_N)$
$K_2(x)$
.8
.4
0
0
1
2
3
$x(\lambda_N)$

FIG. II-1

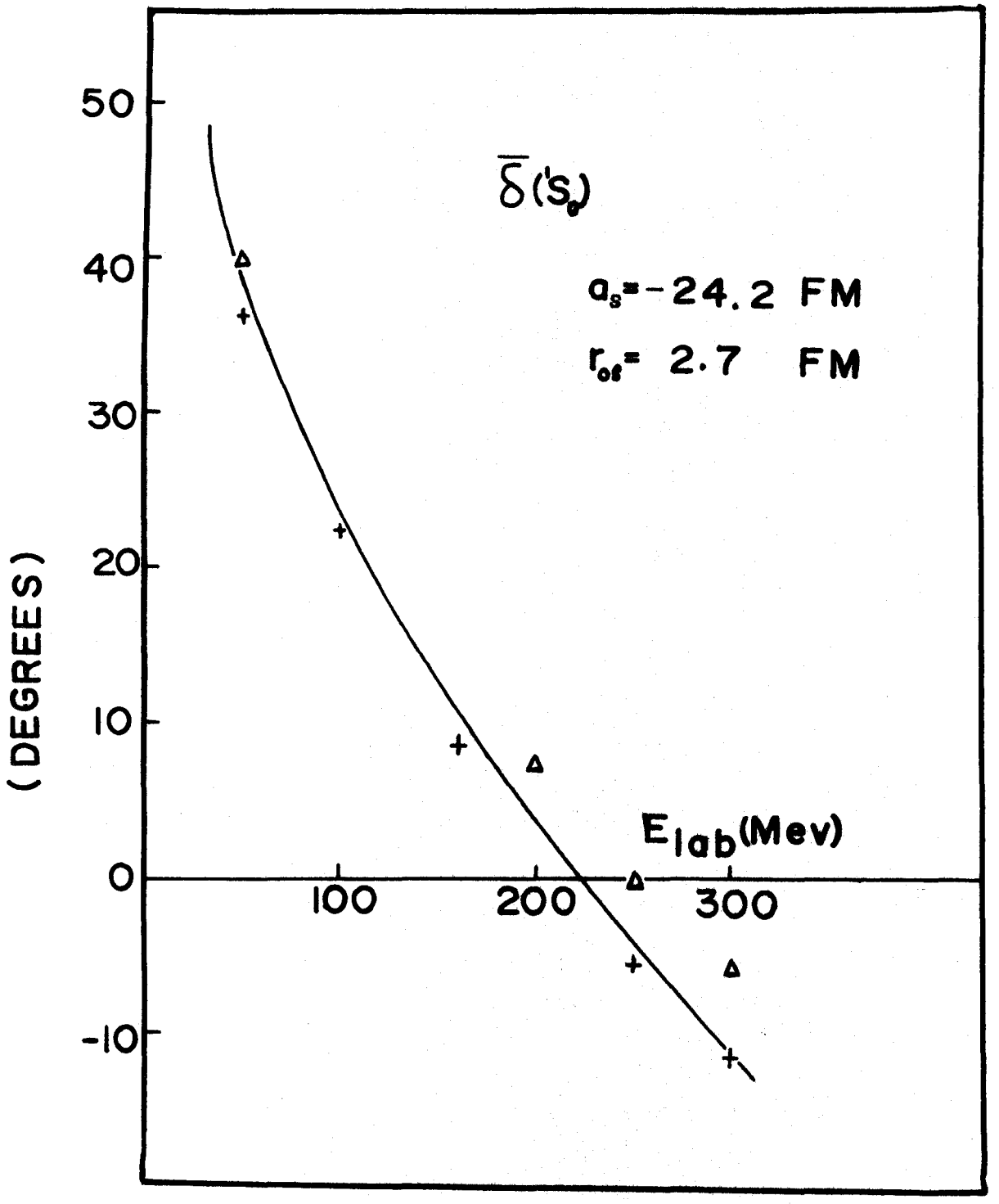
50
40
30
20
10
0
-10
(DEGREES)
$\bar{\delta}(^1S_0)$
$a_s = -24.2$ FM
$r_{os} = 2.7$ FM
E_{lab}(Mev)
100
200
300

FIG. II-2

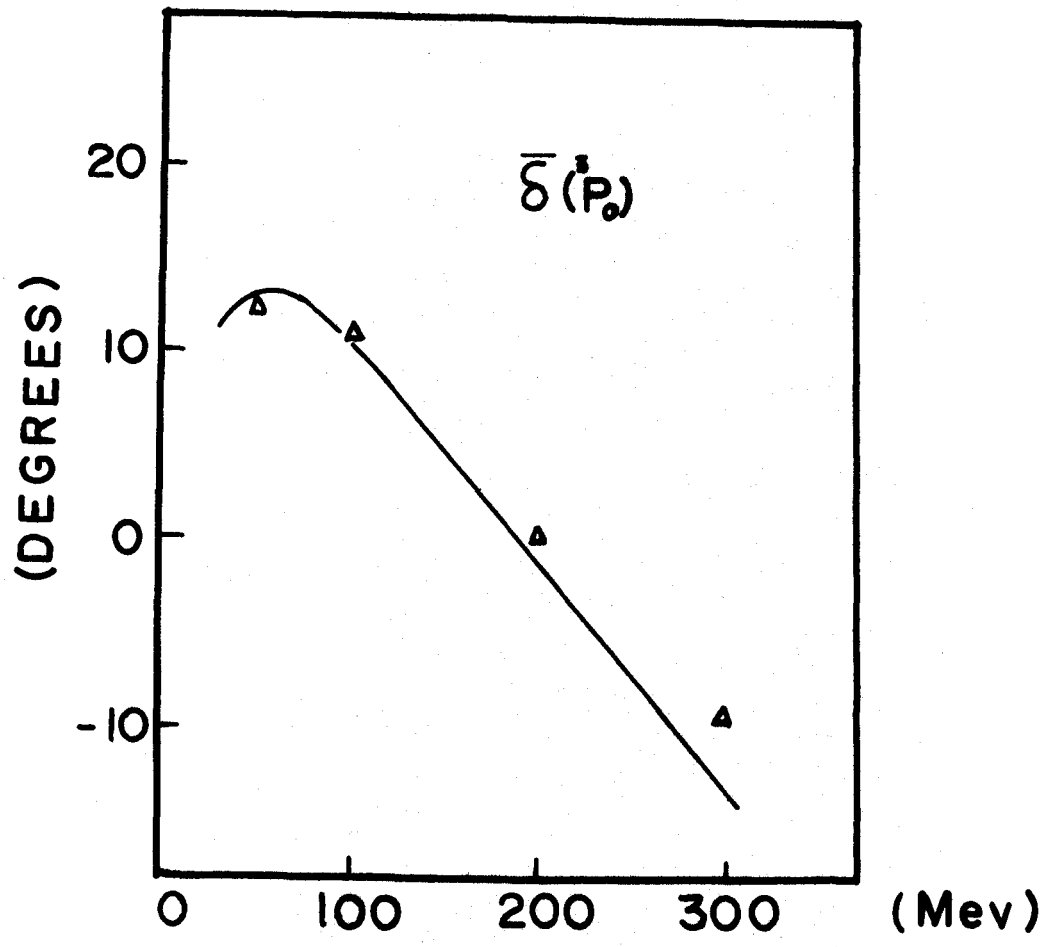

FIG. II-3

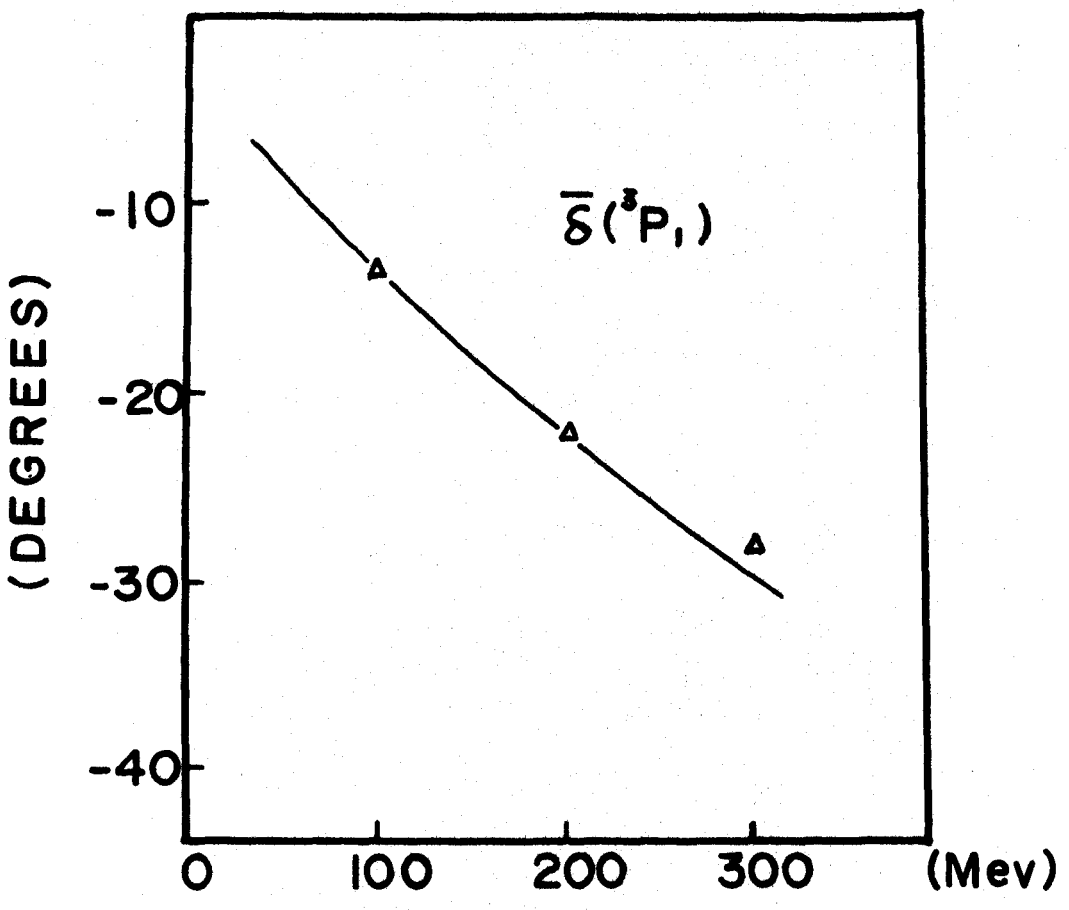

FIG. II-4

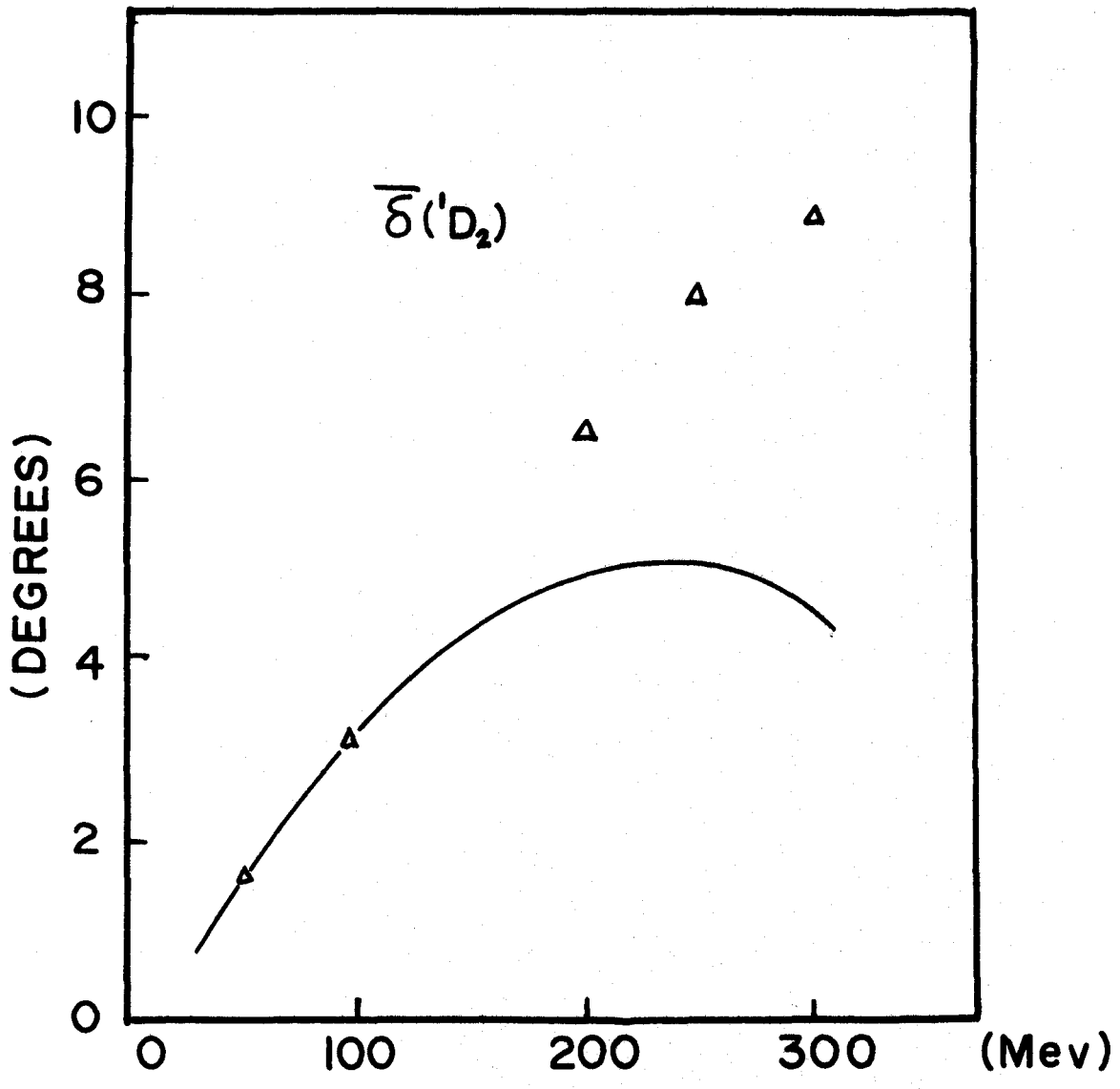

FIG. II-5

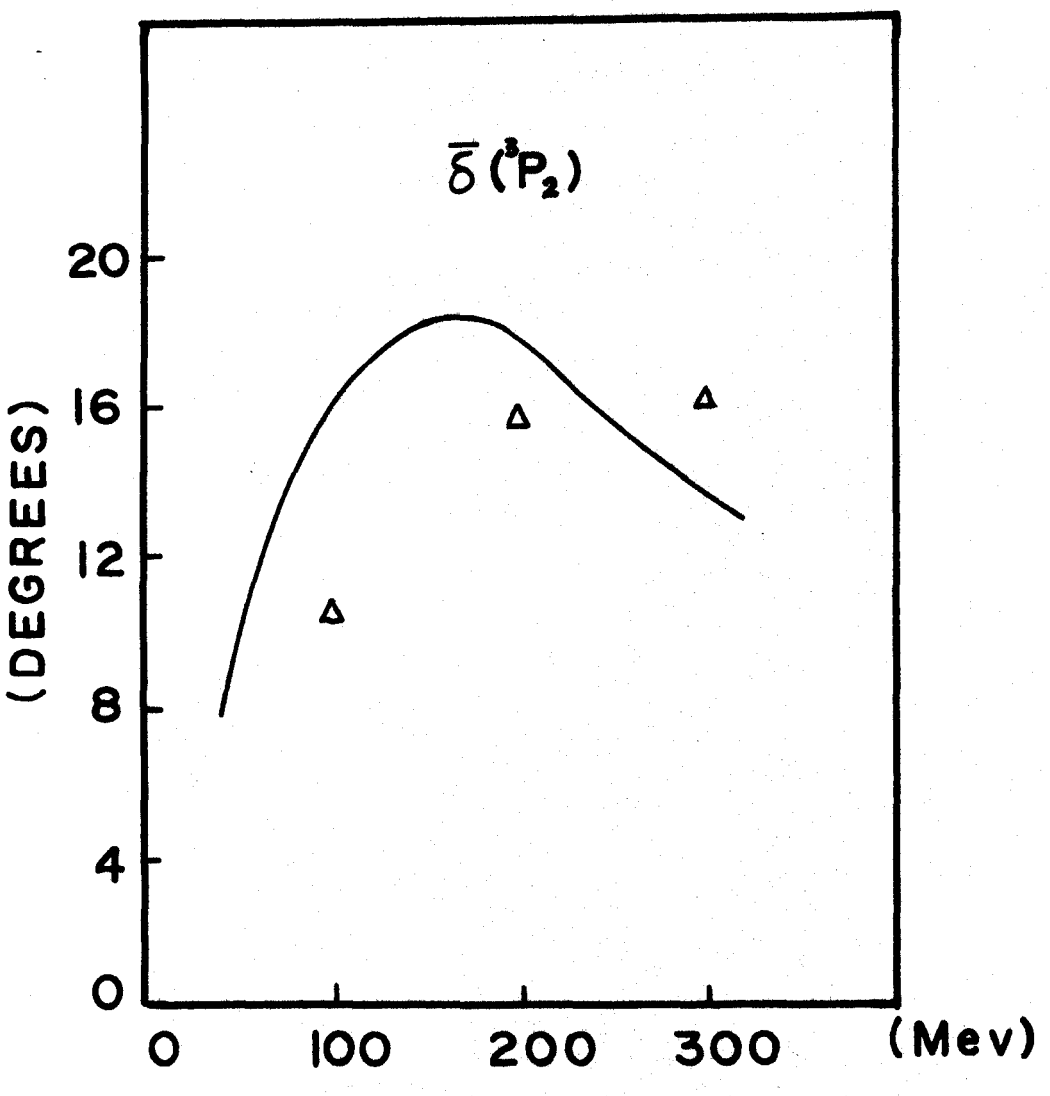

FIG. II-6

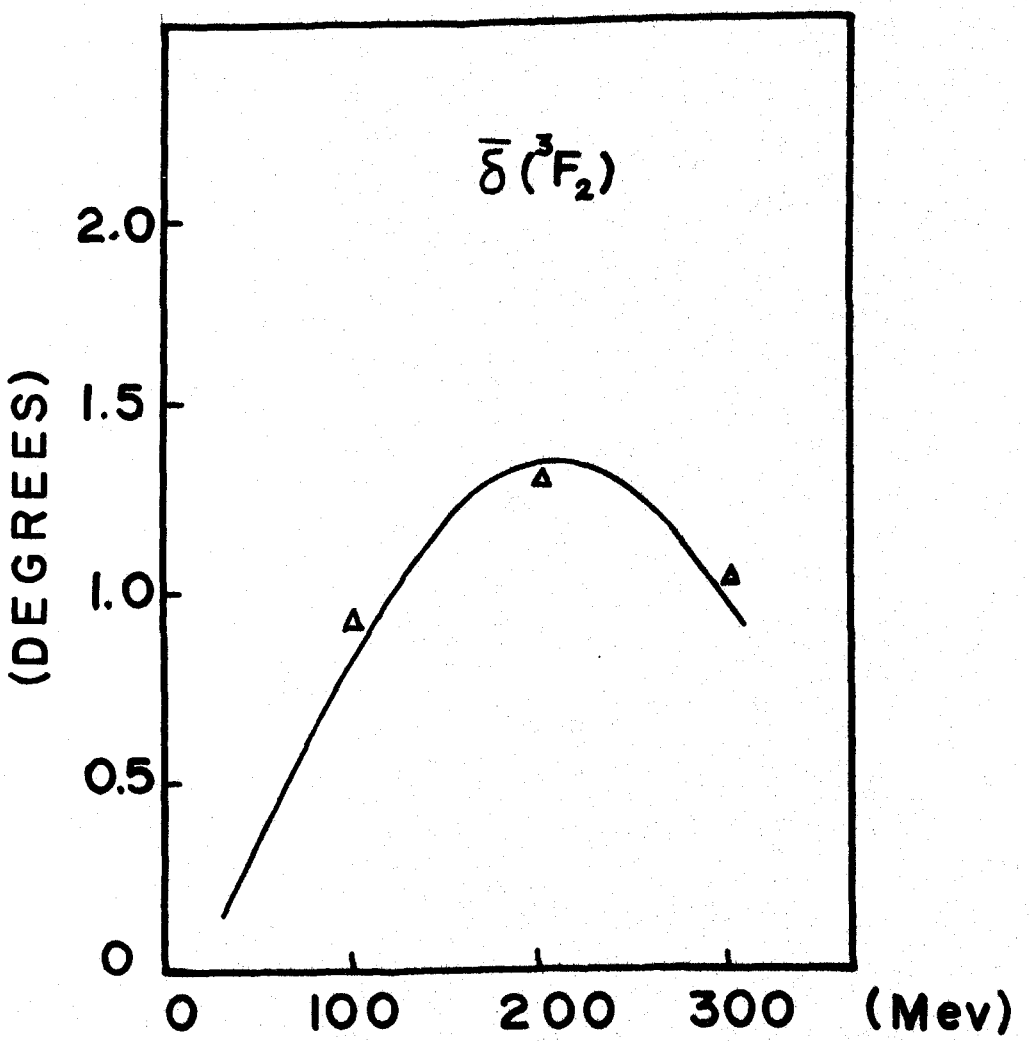

FIG. II-7

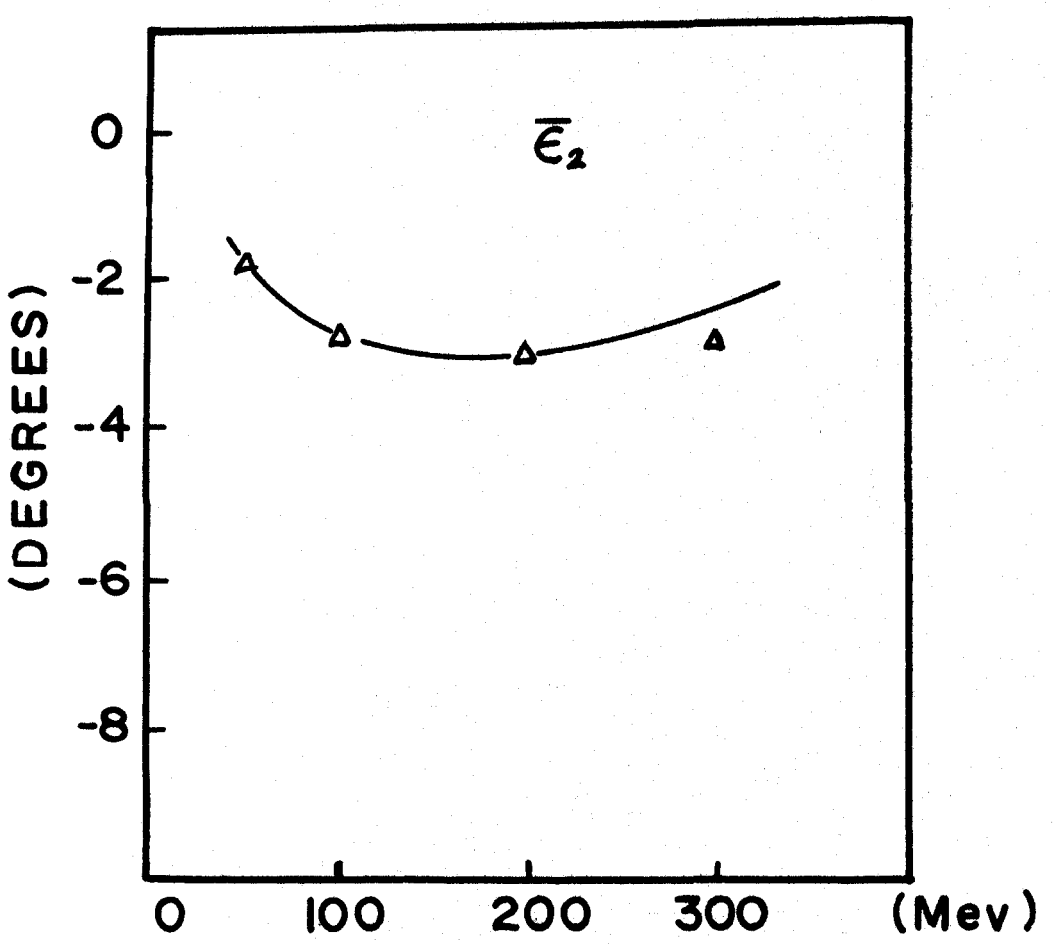

FIG. II-8

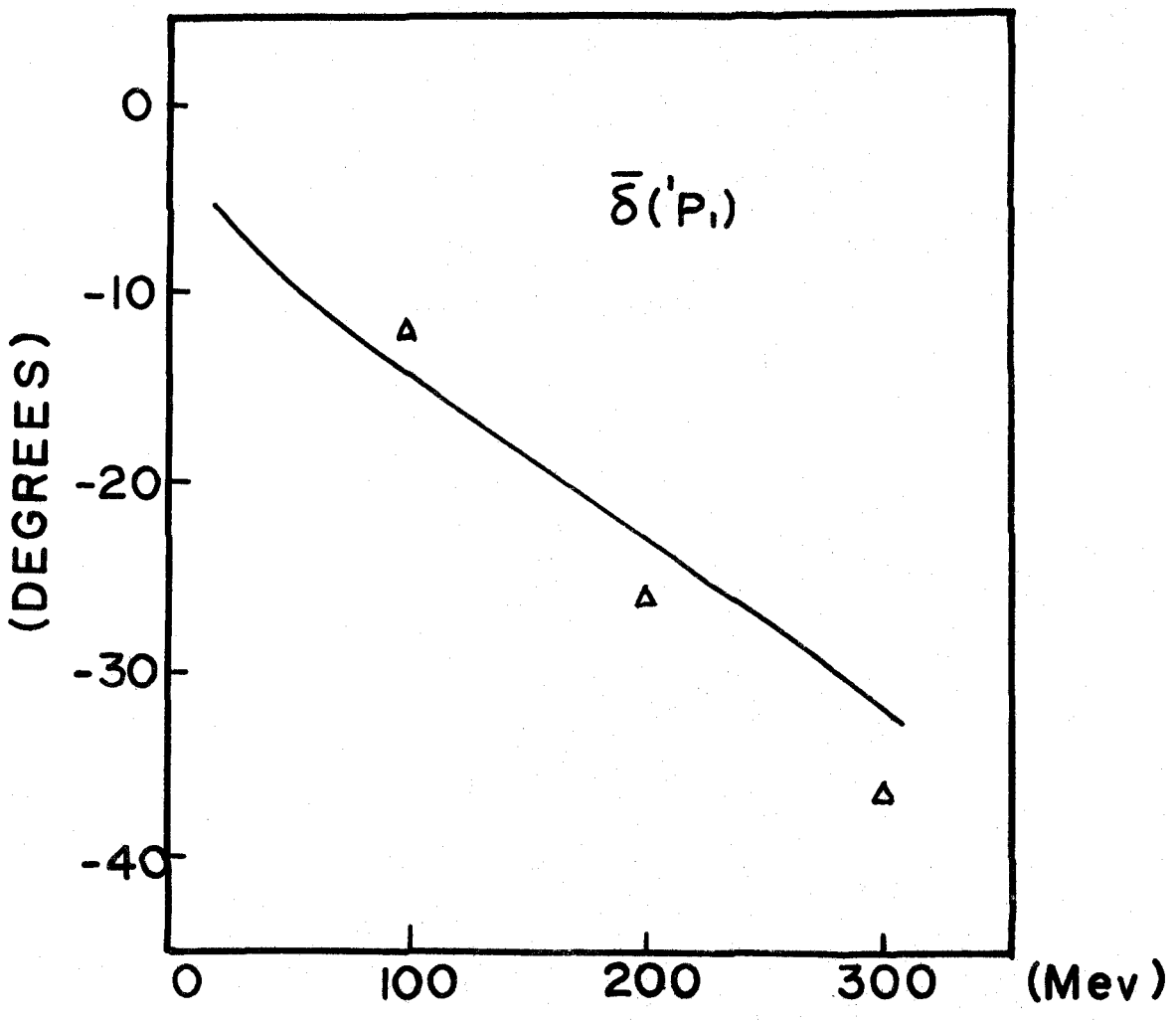

FIG. II-9

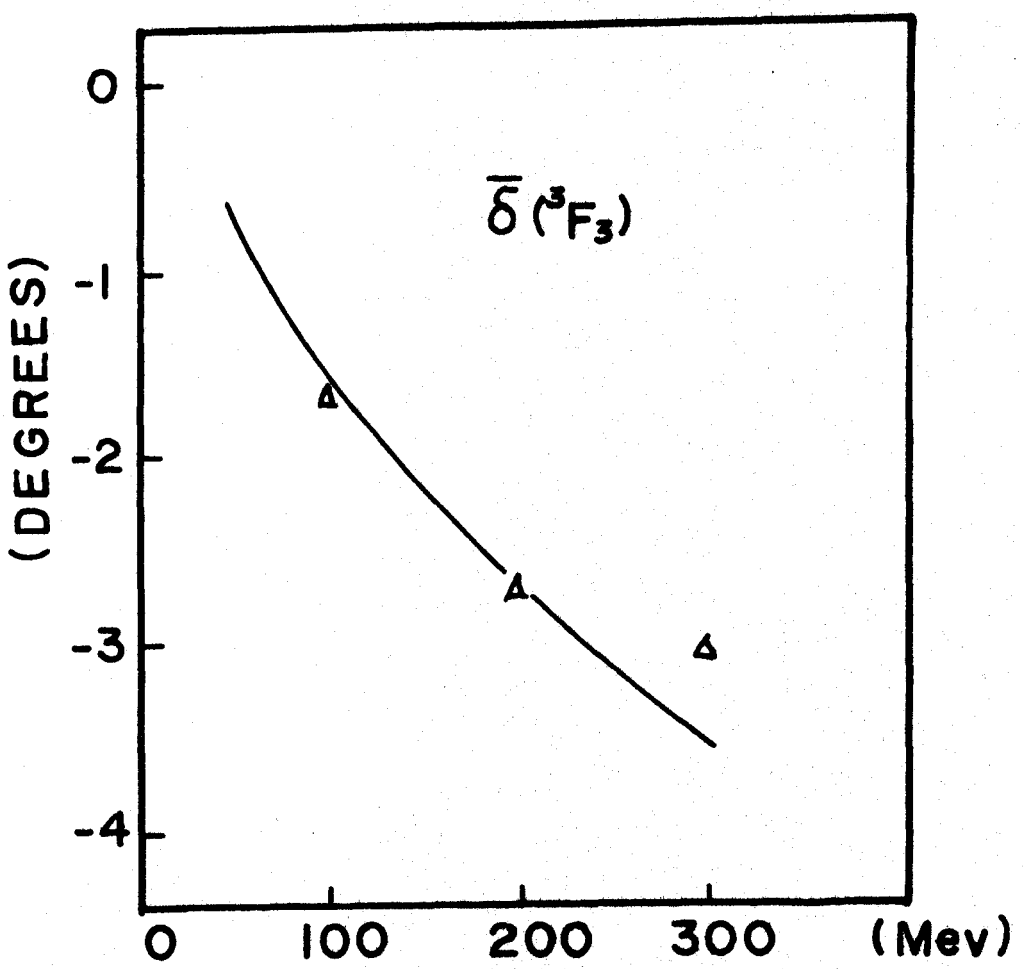

FIG. II-10

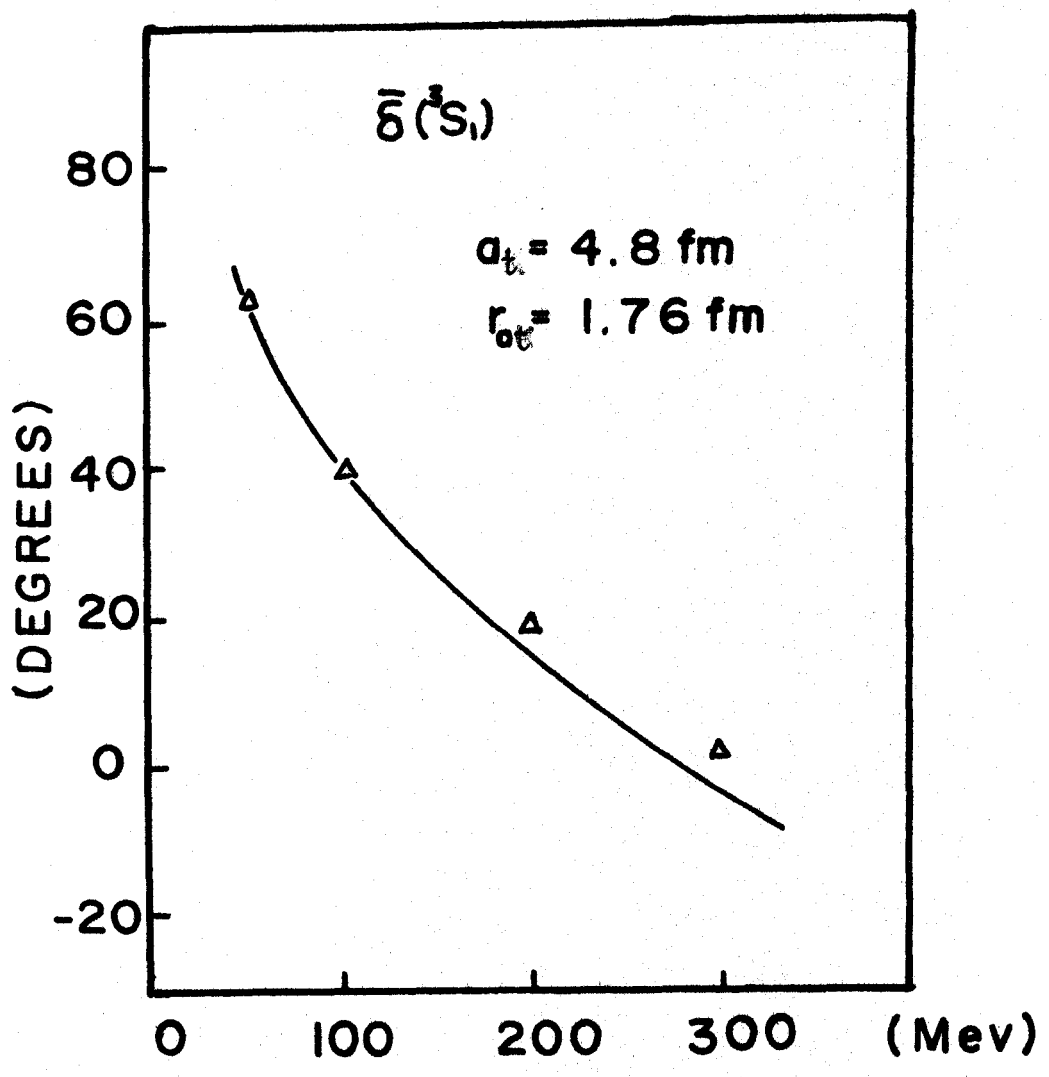
$\bar{\delta}(^3S_1)$
a_t = 4.8 fm
r_{ot} = 1.76 fm
80
60
40
20
0
-20
(DEGREES)
0
100
200
300
(Mev)

FIG. II-11

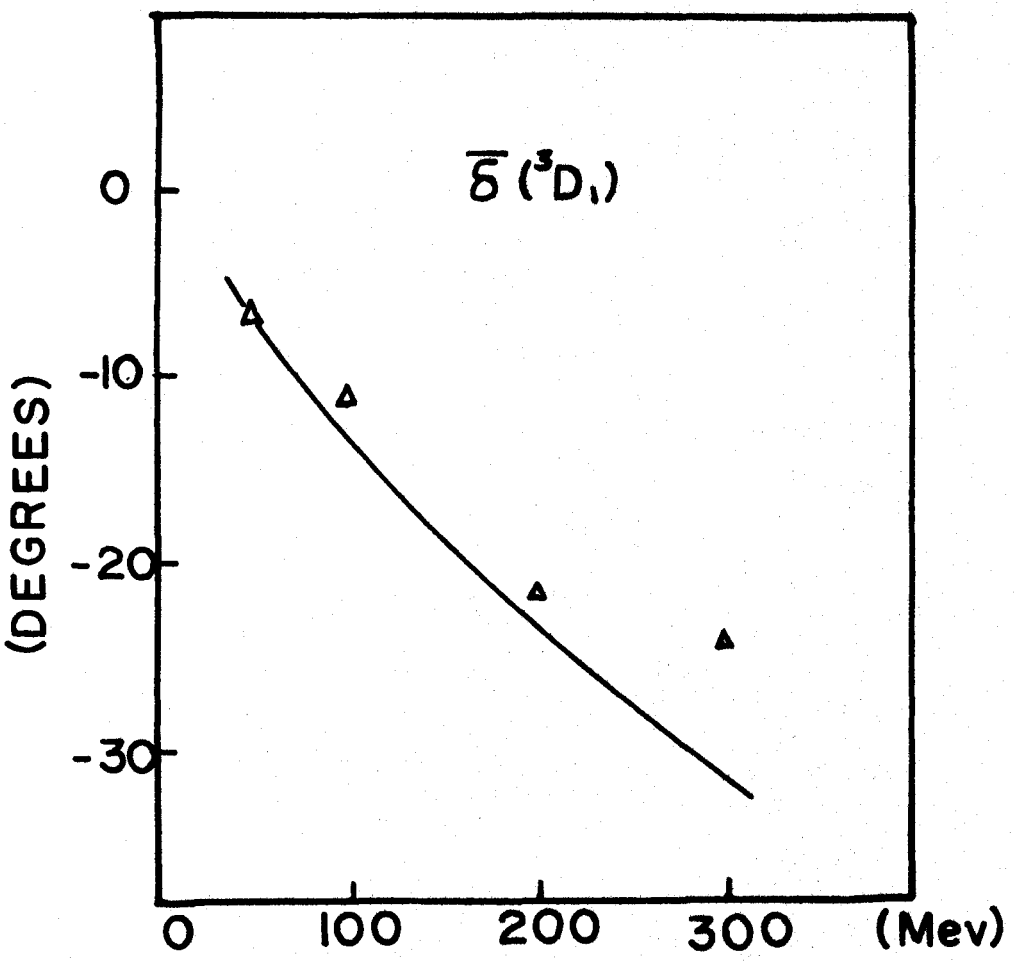

FIG. II-12

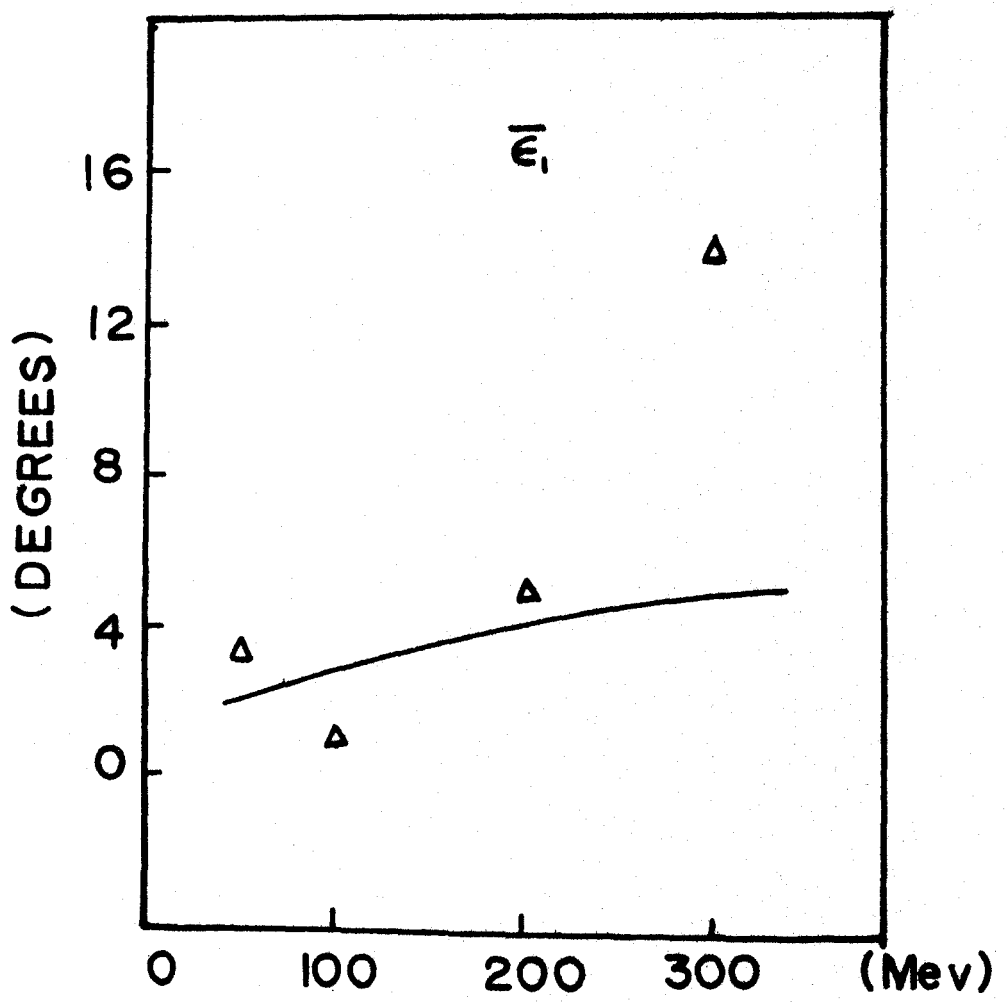

FIG. II-13

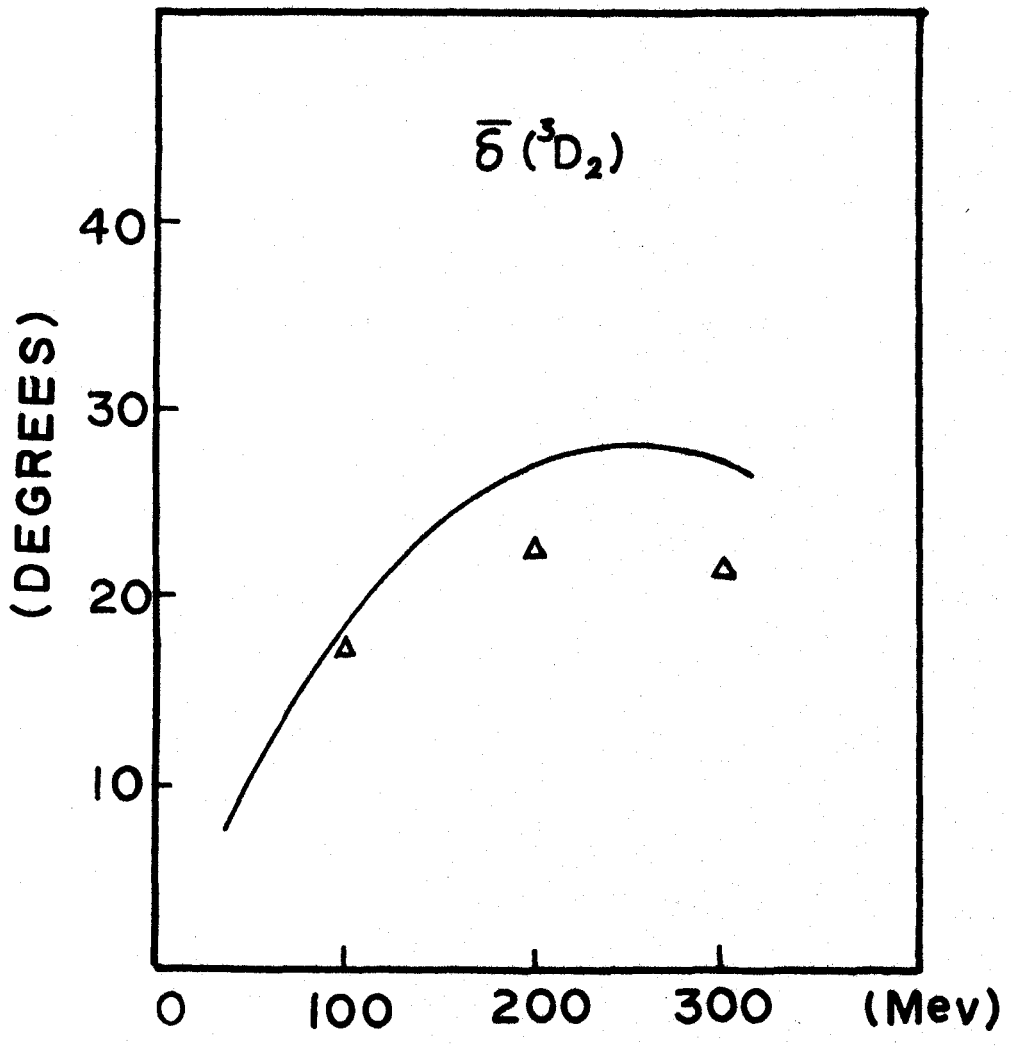

FIG. II-14

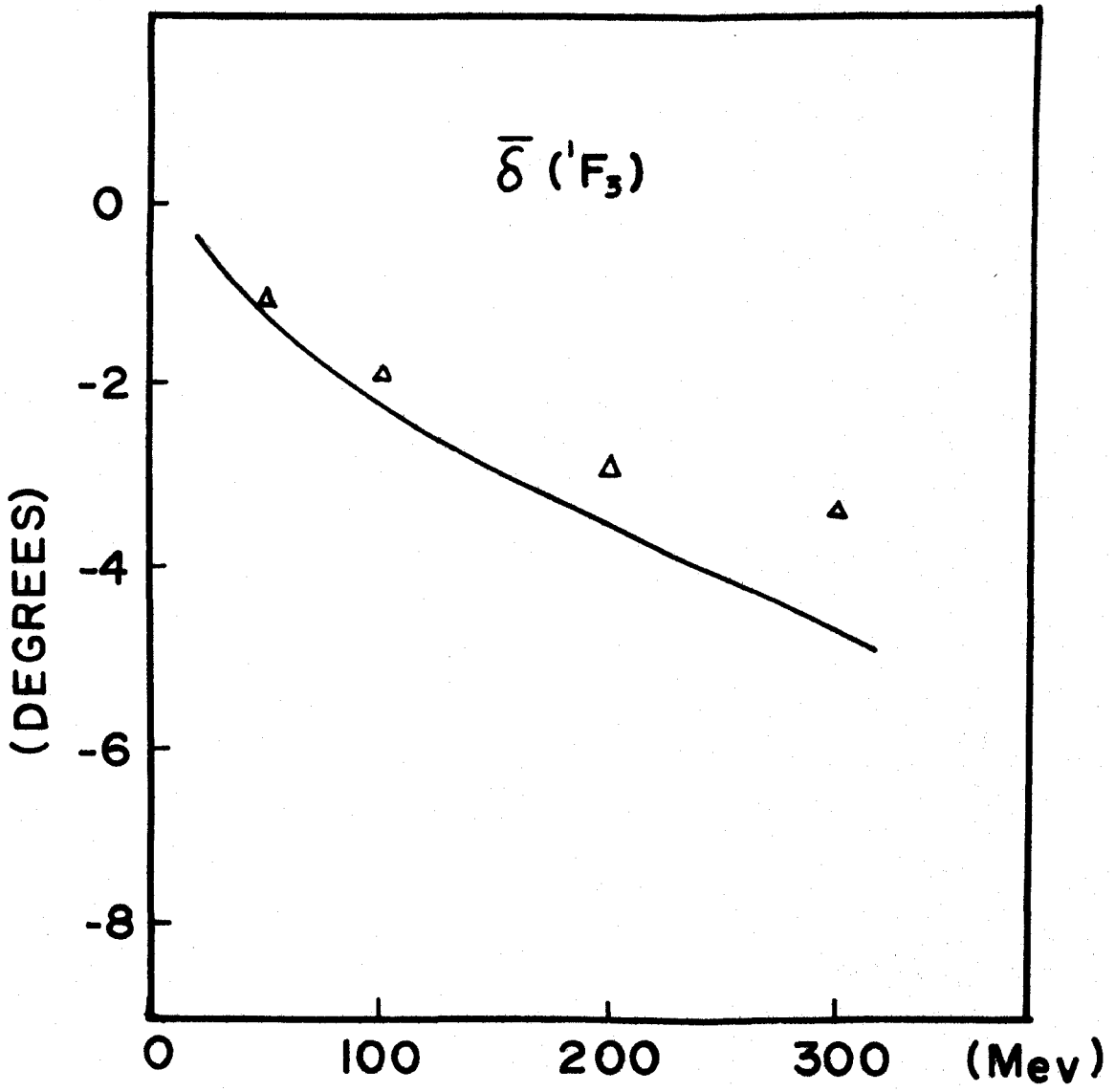

FIG. II-15

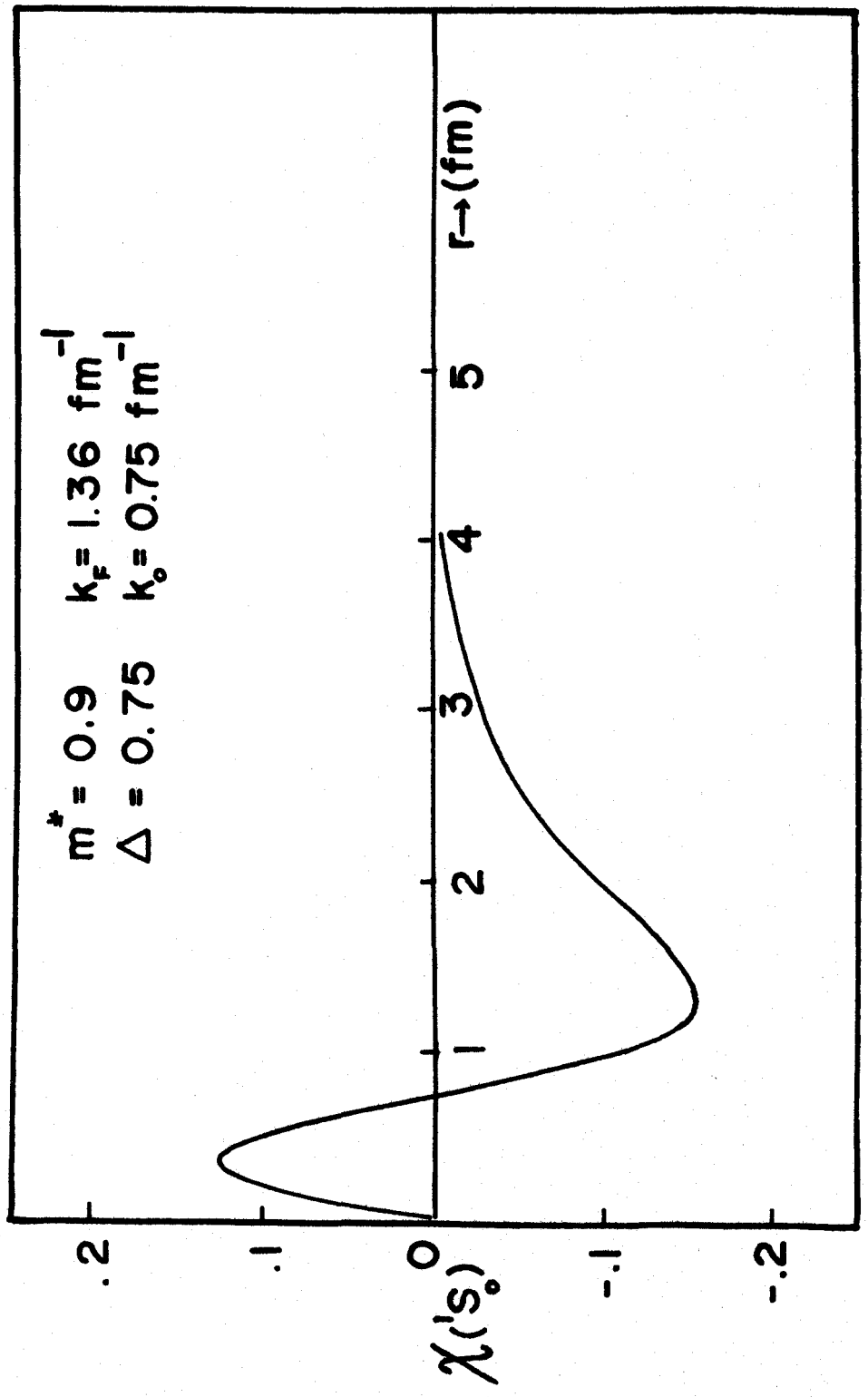

FIG. III-1

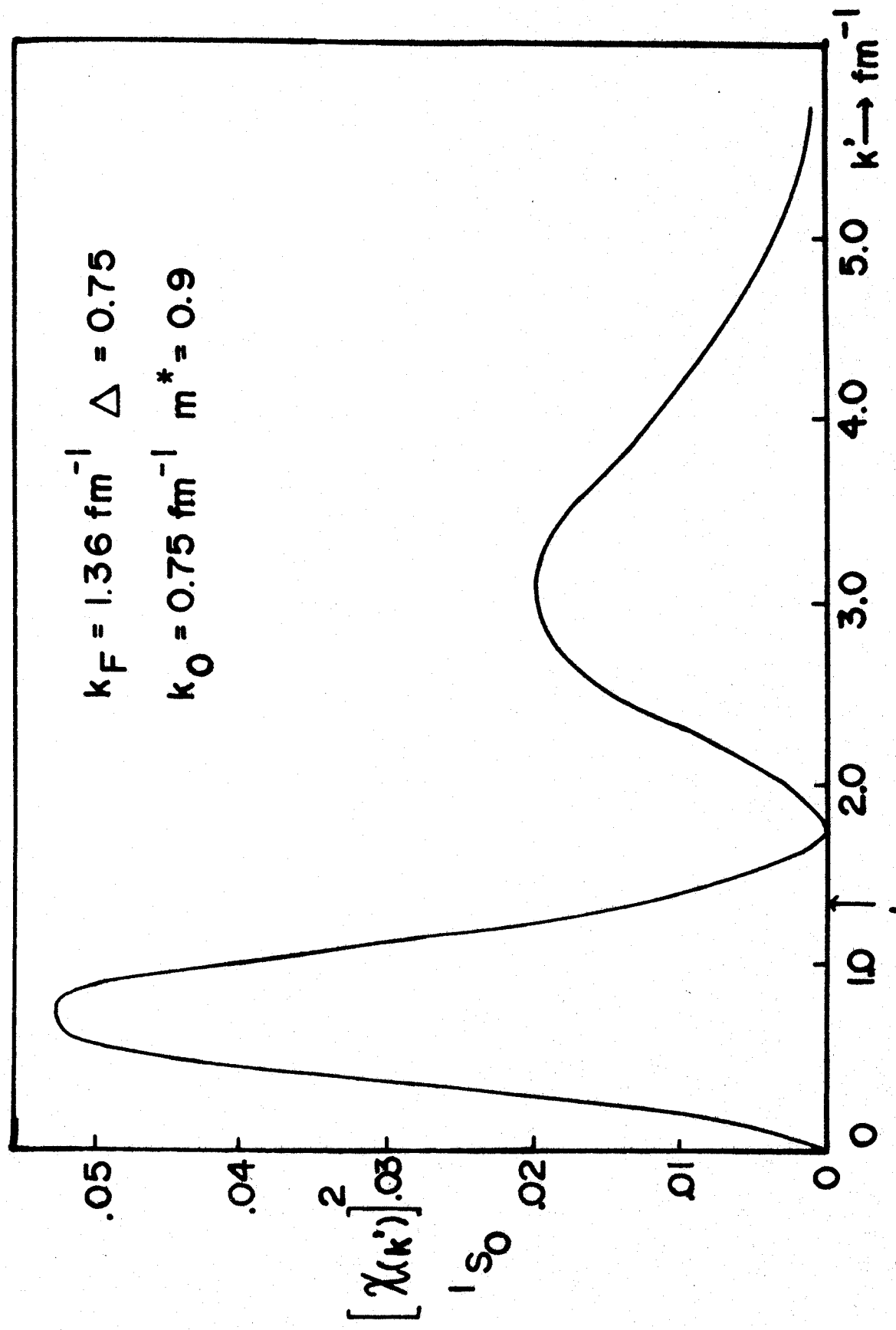
$k_F = 1.36\ fm^{-1}$ $\Delta = 0.75$
$k_0 = 0.75\ fm^{-1}$ $m^* = 0.9$
$[\chi(k')]^2_{^1S_0}$
.05
.04
.03
.02
.01
0
1.0
k_F
2.0
3.0
4.0
5.0
$k' \longrightarrow fm^{-1}$

FIG. III-2

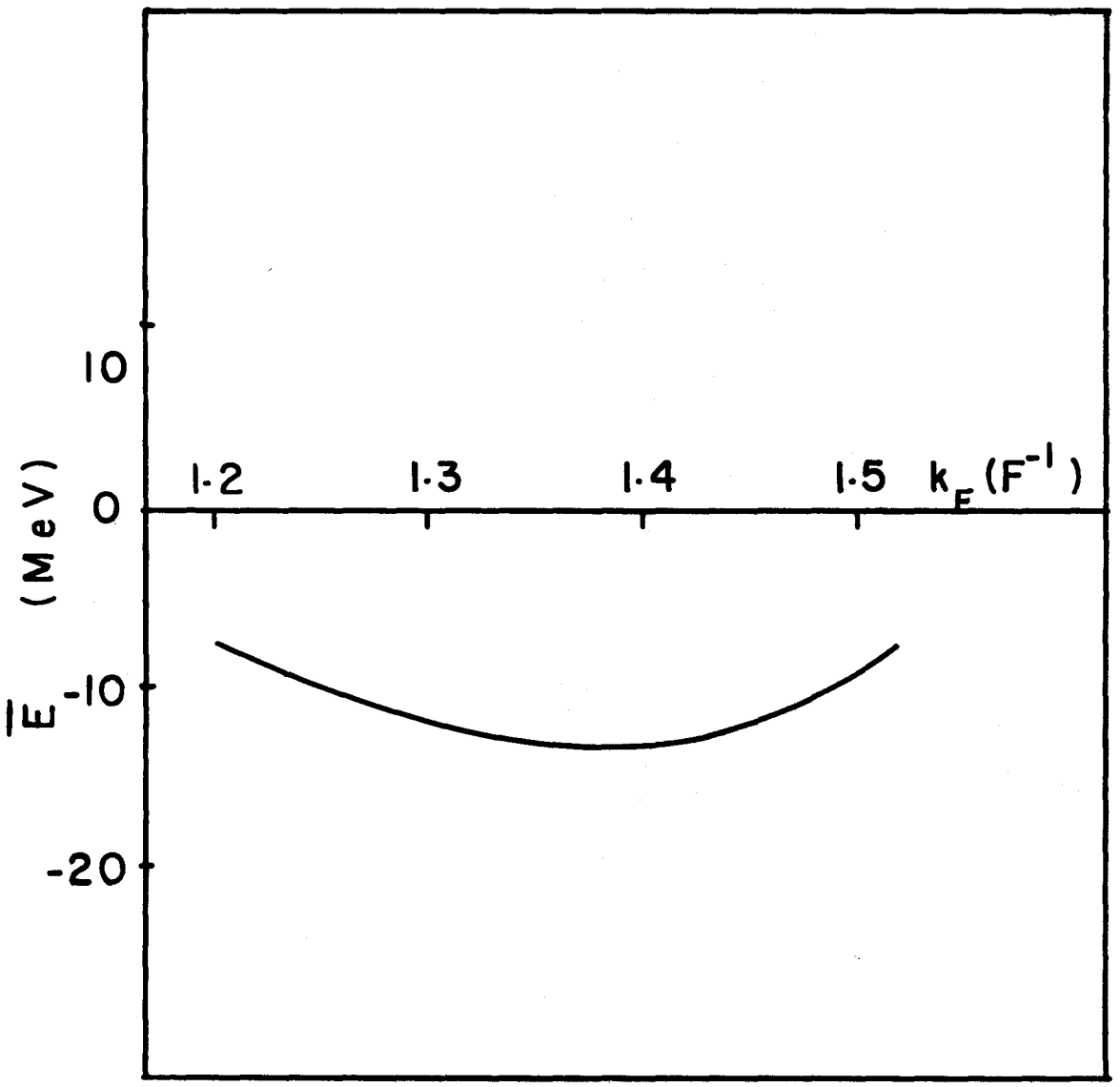

Fig. III - 3