AN EXACT TREATMENT OF THE PAULI EXCLUSION PRINCIPLE AND ITS APPLICATIONS IN NUCLEAR MATTER

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SCOPE AND CONTENTS:

In second order perturbation theory for nuclear matter, an exact treatment of the Pauli exclusion principle is given from a geometrical point of view. All the kinematic effects of the Pauli exclusion principle are then included in a function $K(k,k',q)$, which is related to the Euler's function through a double integration. With this function $K(k,k',q)$, we can treat the Pauli correction in nuclear matter in a more exact way so that a check to the conventional angular average approximation is obtained. For separable core nuclear potential, this function $K(k,k',q)$ serves as a very convenient apparatus for the perturbation calculation of the binding energy in nuclear matter.

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TABLE OF CONTENTS

TABLE OF CONTENTS (cont'd)

LIST OF FIGURES

LIST OF FIGURES (cont'd)

CHAPTER I

INTRODUCTION

The goal of nuclear structure theory is to calculate the properties of real nuclei, given the two-nucleon potential. The two problems are to find what the potential should be and how to carry out nuclear structure calculations. Infinite nuclear matter is one of the simplest systems that can be studied and considerable labour has been devoted to understanding it. The binding energy, density, symmetry energy and compressibility are the quantities which are usually calculated and compared with the semi-empirical mass formula. Assuming that the methods of calculation are valid, a realistic two-nucleon force should give the correct binding energy in nuclear matter, so this study serves either as a testing ground for many body methods or as a check on the validity of two-nucleon forces.

In the calculations of nuclear matter binding energy, perturbation methods have been attempted several times since Euler's[1] first paper in 1937. In his paper, he introduced the so called Euler's function to the first and second order perturbation calculations for

local central potentials. After him, there have been many attempts to apply perturbation theory to well-behaved potentials. An infinite hard core in the two-body potential, however, invalidates perturbation theory in its simplest form. Also, the presence of such a core introduces divergences in the self-consistent potential of the Hartree-Fock method if the usual boundary condition in the zero of the relative coordinates is used. In order to apply the Hartree-Fock method, many other kinds of force have been suggested. For example, Kerman tried to find a weak two-body potential which gave the same scattering data as the hard core in order to avoid the difficulty of estimating two-body correlations due to the strong repulsive core. Following Kerman's proposal, Bressel[2] fitted a static potential from scattering data up to 350 MeV. This static potential has, instead of an infinite hard core, a core with height 648 MeV and width 0.7 fm for the ^1S_o state. Although it is not weak enough, it leads us to the hope that it is possible to remove the infinite hard core and do the standard Hartree-Fock calculations and nuclear matter perturbation calculations.

In 1965, Sprung[3] extended the work of Euler to all types of potential used at that time. In his work, Sprung brought the first and second order perturbation theory into a practical form by introducing a set of "generalized Euler functions". From his work, one can

obtain quickly and accurately contributions to the binding energy of nuclear matter in the first and second order perturbation theory for any local, finite potential.

At the same time when the above work of Sprung was carried out, Tabakin[4] fitted a completely non-local, separable potential from the scattering data. In the binding energy calculation of O^{16}, which has value about 127 MeV, this potential binds too much while Bressel's potential binds too little. So it seems likely that a combination of these two potentials might give the right binding. The theoretical one pion exchange model gives a local potential outside a radius 1.5 fm, while inside the core non-local, many pion effects cannot be ignored. Therefore it seems reasonable to use Tabakin's potential inside the core and Bressel's potential, which agrees with the local part of the OPEP, outside the core. Kerman and Levy[5] thus proposed a potential which, like Bressel's, is the same as Hamada Johnston at large r but has a separable core.

In both standard perturbation theory[6] and Brueckner-Goldstone's theory[7] of a many-body system such as nuclear matter, the Pauli exclusion principle comes into play when we go to the second order term. Because of the Pauli exclusion principle, the intermediate states must be outside the Fermi sea associated with this system. As we will see later in Chapter II, this effect

and the fact that all the initial states must be in the Fermi sea show themselves as the two restrictions in the following integral

$$\int_{\substack{|\underset{\sim}{K}\pm\underset{\sim}{k}|<1 \\ |\underset{\sim}{K}\pm\underset{\sim}{k}'|>1}} d^3\underset{\sim}{k} \tag{1.1}$$

where $\underset{\sim}{K}$ is the center-of-mass momentum of the interacting pair of nucleons while $\underset{\sim}{k}$ and $\underset{\sim}{k}'$ are, respectively, the initial and the intermediate relative momentum of this pair. To interpret this integral, we take a geometrical view by saying that it's value is related to a certain volume between four intersecting unit spheres. The conventional method of treating this problem is either to use the Monte-Carlo method of integration or to perform an angular average over the whole $\underset{\sim}{K}$ space [8] . We will not discuss the Monte-Carlo method in this work but will discuss the angular average method in some detail in Chapter III. As an approximation, the angular average method has been extensively used by people in the calculation of nuclear matter binding energy. We believe that it is a good approximation in certain cases, but this will be doubtful in other cases like the calculation by using a separable core potential which will be carefully discussed in Chapter IV. To prove that what we have said is right, we first perform an analytic evaluation of integral (1.1). The result is

represented by an analytic but fairly complicated function $K(k,k',q)$ where $\underset{\sim}{q} = |\underset{\sim}{k}'-\underset{\sim}{k}|$. Then we calculate the Pauli correction in the "Reference Spectrum Method"* for nuclear matter [9] by using the angular average approximation and also by making use of the function $K(k,k',q)$. We have found that the result obtained from the former doesn't differ much from that obtained from the latter. This means that the angular average approximation is a good approximation in this problem. But when we go to the discussion of the binding energy calculation of nuclear matter for a separable core potential, we are not sure if the angular average approximation is still a good approximation or not as there are certain terms which may badly destroy this approximation. However, we will see in the last Chapter that the function $K(k,k',q)$ will be a very convenient apparatus for treating these terms.

*From now on, it will be abbreviated as RSM.

CHAPTER II

FORMALISM

In this chapter, we are going to discuss explicitly how the Pauli exclusion principle comes into the second order term of the perturbation calculation for nuclear matter, in what form does it appear, and how an exact treatment is possible and performed. As it was said before in the introduction and will be shown later, all kinematic effects can be summed up into a single function of three variables $K(k,k',q)$ with a very complicated form. But it has a simple relation to the well-known Euler's function [1]. The correctness of this relation guarantees that our treatment of Pauli effect is also correct.

In what follows, we will first introduce $K(k,k',q)$ in Sec. 2-1. In Sec. 2-2, we derive this function from a geometrical point of view. Then we discuss the relation between $K(k,k',q)$ and the Euler's function.

2-1. Pauli exclusion principle in nuclear matter.

Nuclear matter is a hypothetical, infinitely extended system of nucleons. Because of the translational invariance of such a system, the wave function of each nucleon in nuclear matter is just a plane wave

$$\phi(x) = \frac{1}{\sqrt{\Omega}} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \qquad (2.1)$$

where $\underset{\sim}{k}$ is the wave vector which runs from 0 to k_F, called the Fermi momentum of nuclear matter. The number density ρ is then related to k_f by

$$\rho = \frac{A}{\Omega} = \frac{2k_f^3}{3\pi^2} \qquad (2.2)$$

In the standard perturbation calculation of the binding energy per nucleon in nuclear matter, the second order direct term is of the form

$$B^{(2)} = -\frac{1}{2\rho\Omega} \sum_{\substack{\underset{\sim}{l},\underset{\sim}{m}<1 \\ \underset{\sim}{a},\underset{\sim}{b}>1 \\ \underset{\sim}{l}+\underset{\sim}{m}=\underset{\sim}{a}+\underset{\sim}{b}}} \frac{\langle \underset{\sim}{l}\underset{\sim}{m}|v|\underset{\sim}{a}\underset{\sim}{b}\rangle \langle \underset{\sim}{a}\underset{\sim}{b}|v|\underset{\sim}{l}\underset{\sim}{m}\rangle}{E_a + E_b - E_l - E_m} \qquad (2.3)$$

The factor 1/2 accounts for the double counting of the interacting pairs. In the second summation over the intermediate states, the restriction that the wave vectors must be greater than Fermi momentum comes from the Pauli exclusion principle and the fact that all the states in the Fermi sea are occupied. It is these restrictions that makes an analytic treatment of $B^{(2)}$ complicated.

Let us first evaluate the matrix element $\langle \underset{\sim}{l}\underset{\sim}{m}|v|\underset{\sim}{a}\underset{\sim}{b}\rangle$ under the condition that $\underset{\sim}{l} + \underset{\sim}{m} = \underset{\sim}{a} + \underset{\sim}{b}$. Using Eq. (2.1), we have

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*Here we take k_f as the unit of momentum.

$$\begin{aligned}&\langle \underset{\sim}{\ell}\underset{\sim}{m}|v|\underset{\sim}{a}\underset{\sim}{b}\rangle \\ &= \frac{1}{\Omega^2}\int d^3\underset{\sim}{r}_\ell \int d^3\underset{\sim}{r}_m \int d^3\underset{\sim}{r}_a \int d^3\underset{\sim}{r}_b \; e^{-i\underset{\sim}{\ell}\cdot\underset{\sim}{r}_\ell} e^{-i\underset{\sim}{m}\cdot\underset{\sim}{r}_m} \times \\ &\times v(\underset{\sim}{r}_\ell,\underset{\sim}{r}_m,\underset{\sim}{r}_a,\underset{\sim}{r}_b) e^{i\underset{\sim}{a}\cdot\underset{\sim}{r}_a} e^{i\underset{\sim}{b}\cdot\underset{\sim}{r}_b}\end{aligned} \tag{2.4}$$

If we perform the transformation

$$\begin{aligned}\underset{\sim}{r} &= \underset{\sim}{r}_\ell - \underset{\sim}{r}_m \quad , \quad \underset{\sim}{R} = (\underset{\sim}{r}_\ell + \underset{\sim}{r}_m)/2 \\ \underset{\sim}{r}' &= \underset{\sim}{r}_a - \underset{\sim}{r}_b \quad , \quad \underset{\sim}{R}' = (\underset{\sim}{r}_a + \underset{\sim}{r}_b)/2\end{aligned} \tag{2.5}$$

to the initial and the intermediate relative and center-of-mass coordinates and also the transformation

$$\begin{aligned}2\underset{\sim}{k} &= \underset{\sim}{\ell} - \underset{\sim}{m} \quad , \quad 2\underset{\sim}{K} = \underset{\sim}{\ell} + \underset{\sim}{m} \\ 2\underset{\sim}{k}' &= \underset{\sim}{a} - \underset{\sim}{b} \quad , \quad 2\underset{\sim}{K}' = \underset{\sim}{a} + \underset{\sim}{b}\end{aligned} \tag{2.6}$$

then Eq. (2.4) becomes

$$\begin{aligned}&\langle \underset{\sim}{\ell}\underset{\sim}{m}|v|\underset{\sim}{a}\underset{\sim}{b}\rangle \\ &= \frac{1}{\Omega}\int d^3\underset{\sim}{r}\int d^3\underset{\sim}{r}' v(\underset{\sim}{r},\underset{\sim}{r}') e^{i(\underset{\sim}{k}'\cdot\underset{\sim}{r}' - \underset{\sim}{k}\cdot\underset{\sim}{r})}\end{aligned} \tag{2.7}$$

We define

$$v(k,k') = \frac{1}{4\pi}\int d^3\underset{\sim}{r}\int d^3\underset{\sim}{r}' \; v(\underset{\sim}{r},\underset{\sim}{r}') e^{i(\underset{\sim}{k}'\cdot\underset{\sim}{r}' - \underset{\sim}{k}\cdot\underset{\sim}{r})} \tag{2.8}$$

then Eq. (2.7) can be written as

$$\langle \underset{\sim}{\ell}\underset{\sim}{m}|v|\underset{\sim}{a}\underset{\sim}{b}\rangle = \frac{4\pi}{\Omega} v(k,k') \tag{2.9}$$

From Eq. (2.6), the energy denominator in Eq. (2.3) is

$$E_a + E_b - E_\ell - E_m = \frac{\hbar^2}{m} \qquad (k'^2 - k^2) \tag{2.10}$$

Using the well-known relation

$$\sum_{\underline{\ell}} \longrightarrow \frac{\Omega}{(2\pi)^3} \int d^3\underline{\ell} \tag{2.11}$$

we then write

$$B^{(2)} = -\frac{3k_f^4}{2^3\pi^5} \frac{m}{\hbar^2} \int_0^1 d^3\underline{\ell} \int_0^1 d^3\underline{m} \int_1^\infty d^3\underline{q} \; \frac{V^2(k,k')}{k'^2 - k^2} \tag{2.12}$$

If we take $\underline{k}, \underline{k}'$ and $\underline{K}$ as the independent variables and put in the transformation Jacobian 8 , then Eq. (2.12) becomes

$$B^{(2)} = -\frac{3k_f^4}{2^4\pi^5} \frac{m}{\hbar^2} \int_0^\infty d^3\underline{k}' \int_0^1 d^3\underline{k} \int_{\substack{|\underline{K}+\underline{k}|<1 \\ |\underline{K}+\underline{k}'|>1}} d^3\underline{K} \; \frac{V^2(k,k')}{k'^2 - k^2} \tag{2.13}$$

The last integral in Eq. (2.13) contains all the kinematic effects due to the Pauli exclusion principle in nuclear matter. The above mentioned integral depends obviously on k,k' and the angle between them $\theta_{k'k}$.

To simplify Eq. (2.13), we take $\underline{k}'$ as the quantization axis and define $\underline{q}$ by

$$\underline{q} = \underline{k}' - \underline{k} \tag{2.14}$$

which has the physical meaning as the momentum transfer between the initial and the intermediate relative momenta. Eq. (2.14) implies that

$$d\cos\theta_{k'k} = \frac{q}{k'k} \, dq \tag{2.15}$$

Using Eq. (2.15), we can replace the integral over $\theta_{k'k}$ by an integral over q and write Eq. (2.13) as

$$B^{(2)} = -\frac{8k_f^4}{2\pi^3}\frac{m}{\hbar^2}\int_0^\infty k'^2 dk' \int_0^1 k\, dk \int_{|k'-k|}^{k'+k} q\, dq\, \frac{v^2(k,k')}{k'^2-k^2} \int_{\substack{|\underline{k}+\underline{k}'|<1 \\ |\underline{k}+\underline{k}'|>1}} d^3\underline{k} \qquad (2.16)$$

So we define

$$K(k,k',q) = \int_{\substack{|\underline{k}+\underline{k}'|<1 \\ |\underline{k}+\underline{k}'|>1}} d^3\underline{k} \qquad (2.17)$$

We see that the Pauli exclusion principle can be treated by the function K(k,k',q) which is a function of the initial relative momentum $\underline{k}$, the intermediate relative momentum $\underline{k}'$, and the momentum transfer between them $\underline{q}$. Once K(k,k',q) is evaluated, $B^{(2)}$ can always be calculated because we didn't specify the potential in our derivation. So our formalism is a very general and exact one suitable for both local and non-local potentials. In later parts of this work, we will apply this formalism to the treatment of separable core potentials as proposed by Kerman and Levy at M.I.T. [5].

2-2. Function K(k,k',q) and its properties.

The function K(k,k',q) which we defined in the last section has a geometrical meaning. For a given set of $\underline{k}$,$\underline{k}'$ and $\underline{q}$, which satisfy the triangle condition $\underline{q} = \underline{k}' - \underline{k}$, the value of K(k,k',q) is the volume inside

the intersection of spheres A and B (the flying saucer), but outside both spheres C and D as shown in Fig. 1 (which means there are bites out of the saucer). All the spheres have a unit radius. Let the centers of these spheres be A,B,C and D respectively, then the distances between them are defined as

$$\begin{aligned} AB &= 2k \\ CD &= 2k' \\ BC &= q \\ AC &= x \end{aligned} \tag{2.18}$$

where

$$\underset{\sim}{x} = \underset{\sim}{k'} + \underset{\sim}{k} \tag{2.19}$$

To evaluate K(k,k',q), we construct all the possible cases in Fig. 2 to Fig. 10. After analyzing these figures, we arrive at Fig. 11 which shows the properties of function K(k,k',q). These properties are summarized as follows:

(1) $0 < k < 1,\ 0 < k < k' < \infty,\ |k'-k| < q < k'+k$ (2.20)

(2) When $k' < k$, $K(k,k',q) = 0$ (2.21)

(3) When $k = 1$, $K(k,k',q) = 0$ (2.22)

(4) When $k' > 1 + \sqrt{1-k^2}$ or $q > 2$

$$K(k,k',q) = \frac{2\pi}{3}(1-k)^2(2+k) \tag{2.23}$$

(5) The values of K(k,k',q) at the different regions are given by Eq.(2.24). The various possible situations are illustrated in Fig.2 to Fig.10 and in Fig.11 we show the boundaries of the regions in kk'q space in which each of the various alternatives is applicable.

Region (1) $K(k,k',q) = 0$ (2.24.1)

Region (2) $K(k,k',q)=VAB-2VABC+2VACD-VCD$ (2.24.2)

Region (3) $K(k,k',q)=VAB-2VABC+VCD$ (2.24.3)

Region (4)R $K(k,k',q)=VAB-2VCB+VCD$ (2.24.4)

Region (4)L $K(k,k',q)=VAB-2VAC+VCD$ (2.24.5)

Region (5)R $K(k,k',q)=VAB-2VBC$ (2.24.6)

Region (5)L $K(k,k',q)=VAB-2VAC$ (2.24.7)

Region (6) $K(k,k',q) = VAB-2VABC$ (2.24.8)

Region (7) $K(k,k',q)=VAB$ (2.24.9)

Region (8) $K(k,k',q)=0$ (2.24.10)

In the above formulas, VAB, VBC, VAC, VABC, and VACD have the simple geometrical meanings as:

VAB: intersection volume between spheres A and B

VBC: intersection volume between spheres B and C

VCD: intersection volume between spheres C and D

VAC: intersection volume between spheres A and C

VABC: intersection volume between spheres A,B, and C

VACD: intersection volume between spheres A,C, and D

Explicitly, they are

$$VAB = \frac{2\pi}{3}(1-k)^2(2+k) \quad (2.25.1)$$

$$VBC = \frac{2\pi}{3}(1-k_2)^2(2+k_2) \quad (2.25.2)$$

$$VCD = \frac{2\pi}{3}(1-k')^2(2+k') \quad (2.25.3)$$

$$VAC = \frac{2\pi}{3}(1-k_3)^2(2+k_3) \quad (2.25.4)$$

$$VABC = \frac{2}{3}M - \frac{2}{3}\sum_{i=1}^{3} k_i (3-k_i^2)\cos^{-1}\left[\frac{k_i(k_j^2+k_2^2-k_i^2)}{M(1-k_i^2)^{1/2}}\right]$$

$$+ \begin{cases} \frac{4}{3}\tan^{-1}\left(\frac{M}{k_1^2+k_2^2+k_3^2-2}\right), & k_1^2+k_2^2+k_3^2-2>0 \\ \frac{4}{3}\left[\tan^{-1}\left(\frac{M}{k_1^2+k_2^2+k_3^2-2}\right)+\pi\right], & k_1^2+k_2^2+k_3^2-2<0 \end{cases} \qquad (2.25.5)$$

where

$$\begin{aligned} k_1 &= k \\ k_2 &= q/2 \\ k_3 &= \tfrac{1}{2}(2k'^2+2k^2-q^2)^{1/2} \end{aligned} \qquad (2.25.6)$$

$$M=(2k_1^2k_2^2+2k_2^2k_3^2+2k_3^2k_1^2-k_1^4-k_2^4-k_3^4)^{1/2}$$

And VACD is similar to VABC and can be obtained from VABC by replacing k_1 by k'. Equations (2.25.1) to (2.25.4) can be easily obtained. VABC, however, is difficult and will be derived in Appendix A.

2-3. Relation between K(k,k',q) and Euler's function P(u).

In section 2-1, K(k,k',q) was introduced into the second order term of the perturbation series for a very general potential. We know, however, that for a local central potential an exact evaluation of the second order term has long been derived in Euler's paper and used in many applications. The result there is*

$$B^{(2)} = -\frac{k_f^4}{10\pi^2}\frac{m}{\hbar^2}\int_0^\infty u\,du\; v^2(u)\,P(u) \qquad (2.26)$$

*Only a direct term is considered.

where u = q/2 and P(u) is the well known Euler's function. It's explicit form is

$$u<1,\quad P(u)=P^{I}(u)=\log(1+u)\cdot\left[4+\frac{15}{2}u-5u^{3}+\frac{3}{2}u^{5}\right]+$$
$$+29u^{2}-3u^{4}+\log(1-u)\cdot\left[4-\frac{15}{2}u+5u^{3}-\frac{3}{2}u^{5}\right]- \quad (2.27.1)$$
$$-40u^{2}\log 2$$

$$u>1,\quad P(u)=P^{II}(u)=\log(u+1)\cdot\left[4-20u^{2}-20u^{3}+4u^{5}\right]+1$$
$$+4u^{3}+22u+\log(u-1)\cdot\left[-4+20u^{2}-20u^{3}+4u^{5}\right]+ \quad (2.27.2)$$
$$+\log u\cdot\left[40u^{3}-8u^{5}\right]$$

From Sec. 2-1, for a local central potential, we have

$$v(k,k')\longrightarrow v(q)=\int_{0}^{\infty}j_{0}(qr)\,v(r)\,r^{2}dr \quad (2.28)$$

Instead of $\underline{k}$, if we take $\underline{q}$ as the quantization axis, then we get an equivalent formula for $B^{(2)}$ as

$$B^{(2)}=-\frac{3kf^{4}}{2\pi^{3}}\frac{m}{\hbar^{2}}\int_{0}^{\infty}q\,dq\int_{0}^{1}k\,dk\int_{\mathrm{Max}[|q-k|,k]}^{q+k}k'dk'\,\frac{v^{2}(q)}{k'^{2}-k^{2}}\,K(k,k';q)$$

$$=-\frac{6kf^{4}}{\pi^{3}}\frac{m}{\hbar^{2}}\int_{0}^{\infty}u\,du\left[\int_{0}^{1}k\,dk\int_{\mathrm{Max}[|q-k|,k]}^{q+k}\frac{k'dk'}{k'^{2}-k^{2}}\,v^{2}(q)\,K(k,k';q)\right]_{q=2u} \quad (2.29)$$

The lower limit of the k' integral being $\mathrm{Max}[|q-k|,k]$ is justified by the fact that K(k,k',q) = 0 for k' < k from Eq. (2.21). If we compare the above two expressions, in Equations (2.26) and (2.29) , we then obtain the following relation between K(k,k',q) and P(u)

$$P(q/2) = \frac{60}{\pi} \int_0^1 dk \int_{\text{Max}[|q-k|,k]}^{q+k} dk' \frac{kk' K(k,k',q)}{k'^2 - k^2} \qquad (2.30)$$

Since we know the analytic forms of both functions, it is possible in principle to derive this relation directly. But K(k,k',q) is a very complicated function unless q > 2; only in that region does it have a simple form. We can thus make an analytic check for q > 2 as is done in Appendix B. For q < 2, we used a numerical method. The results are shown in Table 1 where we can see the good agreement between the two sides of the above equation.

TABLE 1

u	P(u)	Q(u)
0.10000	0.12174	0.12418
0.20000	0.47505	0.47617
0.30000	1.02465	1.02560
0.40000	1.71342	1.71375
0.50000	2.46496	2.46430
0.60000	3.18756	3.18573
0.70000	3.77975	3.77719
0.80000	4.13846	4.13564
0.90000	4.17128	4.16834
1.00000	3.81929	3.81591

$$Q(u) = \frac{60}{\pi} \int_0^1 dk \int_{\mathrm{Max}\left[|q-k|,k\right]}^{q+k} dk' \, \frac{kk'K(k,k',q)}{k'^2 - k^2} \Big/ q = 2u$$

Evaluated by Simpson's rule with dk=dk' = 0.05000

CHAPTER III

APPLICATION TO THE PAULI CORRECTION

As we stated in the introduction, so far all calculations of the Pauli correction were done in an approximate manner. In treating the integration over the center of mass momentum which appears here in the same form as in the previous chapter on the second order term in perturbation theory, people either used numerical Monte-Carlo computation or an angular average approximation[8]. Since we know how to evaluate that integration exactly, we will first calculate the Pauli correction in our formalism, then repeat with the angular average approximation. From these two results, we will know how accurate the angular average approximation is.

Before going to the real calculation, we first derive the Pauli correction by following a method which is similar to that in BBP's paper[9] but differs from it in certain places as shown in the following sections.

3-1. Theory and an exact treatment of the Pauli correction in Reference Spectrum method of nuclear matter.

It is well known that the nuclear G matrix satisfies

$$G = v - v \frac{Q}{e} G \tag{3.1}$$

As defined in BBP's paper, the reference spectrum G^R matrix satisfies

$$G^R = v - v \frac{1}{e^R} G^R \tag{3.2}$$

The difference between these two matrices contains two parts, they are called the "Pauli correction" and the "Spectral correction" respectively. We are only interested in the Pauli correction which is given in BBP's paper as

$$(G-G^R)_P = G^{R+} \frac{1}{e^R} (1-Q) e^R \frac{1}{e^R} G^R \tag{3.3}$$

where Q is called the "Pauli operator". In this work, we consider only the triplet even states $^3S_1$ and $^3D_1$. Let us denote the state vector as $\left| \phi^{M_S}_{S=1,T=0} \right\rangle$ or $\left| \phi^M_{10} \right\rangle$ for simplicity. The Pauli correction is then

$$\langle \phi^M_{10} | (G-G^R)_P | \phi^M_{10} \rangle = \langle \phi^M_{10} | G^{R+} \frac{1}{e^R} (1-Q) e^R \frac{1}{e^R} G^R | \phi^M_{10} \rangle \tag{3.4}$$

We define

$$\frac{1}{e^R} G^R | \phi^M_{10} \rangle = | \chi^M_{10} \rangle \tag{3.5}$$

then

$$\langle \phi^M_{10} | (G-G^R)_P | \phi^M_{10} \rangle = \langle \chi^M_{10} | (1-Q) e^R | \chi^M_{10} \rangle \tag{3.6}$$

Inserting intermediate states $|\underset{\sim}{K},\underset{\sim}{k}',M'\rangle$, we have

$$\langle \phi^M_{10} | (G-G^R)_P | \phi^M_{10} \rangle = \sum_{\underset{\sim}{k}',M'} \langle \underset{\sim}{K},\underset{\sim}{k}',M' | (1-Q) e^R | \underset{\sim}{K},\underset{\sim}{k}',M' \rangle | \langle \underset{\sim}{k}',M' | \chi^M_{10} \rangle |^2 \tag{3.7}$$

The effect of the Pauli operator Q in Eq. (3.7) is that for a given $\underset{\sim}{K},\underset{\sim}{k}'$ must satisfy the condition $|\underset{\sim}{K}\pm\underset{\sim}{k}'|>1$. We have dropped $\underset{\sim}{K}$ in the expression $|\langle \underset{\sim}{k}',M' | \chi^M_{10} \rangle|^2$ because it appears, both in the bra and the ket, as plane waves which cancel each other when we take the scalar product. Using [†]

$$\langle \underset{\sim}{r} | \chi^M_{10} \rangle = (8\pi)^{1/2} (kr)^{-1} \sum_{\substack{\text{even} \\ L,L'}} \sum_J f^M_{L,J} \chi^{(L)}_{L'J} \mathcal{Y}^M_{L',J,\underset{\sim}{k}} \lambda^0_0 \tag{3.8}$$

$$\langle \underset{\sim}{r} | \underset{\sim}{k}',M' \rangle = (4\pi)^{1/2} (k'r)^{-1} \sum_{L''} \sum_{J''} f^{M'}_{L'',J''} j_{L''}(k'r) \mathcal{Y}^{M'}_{L'',J'',\underset{\sim}{k}'} \lambda^0_0 \tag{3.9}$$

$$\langle \mathcal{Y}^{M'}_{L'',J'',\underset{\sim}{k}'} | \mathcal{Y}^M_{L,J,\underset{\sim}{k}} \rangle = \mathcal{D}^J_{M'M} (\hat{\underset{\sim}{k}} \to \hat{\underset{\sim}{k}}') \delta_{L''L'} \delta_{J''J} \tag{3.10}$$

where $\lambda^T_{T_3}$ are isospin states,

$$f^M_{L,J} = i^L (2L+1)^{1/2} \langle L\,1\,0\,M | J\,M \rangle \tag{3.11}$$

and defining the Bessel's transformation of $\chi^{(L)}_{L'J}$ by

$$F^J_{L'L}(k',k) = \int j_{L'}(k'r) \chi^{(L)}_{L'J}(kr)\, dr \tag{3.12}$$

then Eq. (3.7) becomes

$$\langle \phi_{10}^{M} | (G - G_{\rho}^{R}) | \phi_{10}^{M} \rangle$$

$$= \sum_{M'} \sum_{\underline{k}'} (32\pi^2)(kk')^{-2} \sum_{\substack{\text{even} \\ L,L' \\ J,J'}} \sum_{J,\lambda} f_{L'J}^{M'*} f_{LJ}^{M} f_{J\lambda}^{M*} f_{J'\lambda}^{M'} \times$$

$$\times F_{L'L}^{J}(k',k) F_{J'J}^{\lambda}(k',k) \times \mathcal{D}_{M'M}^{J}(\hat{k} \to \hat{k}') \mathcal{D}_{M'M}^{\lambda}(\hat{k} \to \hat{k}') \times$$

$$\times \langle \underline{K}, \underline{k}', M' | (1-Q) e^{R} | \underline{K}, \underline{k}', M' \rangle \qquad (3.13)$$

To calculate the total Pauli correction, we must consider all possible initial states. We thus sum over the spin projection M and the initial momenta $\underline{\ell}$ and $\underline{m}$. Using the transformation Eqs. (2.5) and (2.6) defined in Chapter II, we are allowed to sum over the initial relative and total momenta $\underline{k}$,$\underline{K}$ instead of $\underline{\ell}$,$\underline{m}$. Since the Jacobian of this transformation is 8, we must multiply our new expression by this factor. In addition to this, we must also divide the result by ½ to eliminate the double counting of the interacting pairs. Since the normalization volume here is taken to be 1, we divide the result by ρ in order to obtain the Pauli correction per nucleon. We thus have

$$P.C. = \frac{3k_f^5}{8\pi^5}\frac{\hbar^2}{m}\int\frac{1}{k'^2}dk'\int\frac{1}{k^2}dk\int d^3K\Bigg\{\sum_{M,M'}\sum_{\substack{\text{even}\\L,L'\\J,J'}}$$

$$\sum_{J,\lambda} f_{L'J}^{M'^*} f_{J\lambda}^{M^*} f_{J'\lambda}^{M'} f_{LJ}^{M} \langle \underset{\sim}{K}, \underset{\sim}{k}', M' | (1-Q)e^R |$$

$$\underset{\sim}{K}, \underset{\sim}{k}', M' \rangle \times F_{L'L}^{J}(k',k) F_{J'J}^{\lambda}(k',k) \times$$

$$\times \mathcal{D}_{M'M}^{\lambda\,*}(\hat{\underset{\sim}{k}} \to \hat{\underset{\sim}{k}}') \mathcal{D}_{M'M}^{J}(\hat{\underset{\sim}{k}} \to \hat{\underset{\sim}{k}}')\Bigg\} \qquad (3.14)$$

But the factor $\langle \underset{\sim}{K}, \underset{\sim}{k}', M' | (1-Q)e^R | \underset{\sim}{K}, \underset{\sim}{k}', M' \rangle$ is independent of M' and the direction of $\underset{\sim}{K}$ and $\underset{\sim}{k}'$. It's a function of k,k' and K, and contains two parts. The first part demands that we evaluate the K integration by

$$\int_{|\underset{\sim}{K}\pm\underset{\sim}{k}|<1} d^3K \qquad (3.15)$$

only because all the initial states must be in the Fermi sea. The second part, however, puts one more restriction i.e.

$$\int_{\substack{|\underset{\sim}{K}\pm\underset{\sim}{k}|<1\\ |\underset{\sim}{K}\pm\underset{\sim}{k}'|>1}} d^3K \qquad (3.16)$$

If we take the average value of $\langle k'K|e^R(k)|k'K\rangle$ over the centre of mass coordinate in our considerations, then the whole integrand is independent of K. (See Eq. (3.21)). The second integration is thus just K(k,k',q). The first integration can be easily evaluated, the result is

$$\int_{|\underset{\sim}{K}+\underset{\sim}{k}|<1} d^3\underset{\sim}{K} = \frac{2\pi}{3}(1-k)^2(2+k) = K'(k) \qquad (3.17)$$

K' is the K-function when the Pauli principle is ignored. Writing

$$\langle k',K|e^R(k)|k',K\rangle_{av} = \mathcal{E}(k,k',K_{av}) \qquad (3.18)$$

and restricting to J = λ = 1 as we are dealing with triplet even states, then Eq. (3.14) becomes

$$P.C. = \frac{3k_f^4}{8\pi^5}\frac{\hbar^2}{m}\int\frac{1}{k'^2}d^3\underset{\sim}{k}'\int\frac{1}{k^2}d^3\underset{\sim}{k}\,(K'-K)\times$$
$$\times\,\mathcal{E}(k,k',K_{av})\times\Bigg\{\sum_{M,M'}\sum_{\substack{\text{even}\\ L,L'\\ J,J'}} f_{L'1}^{M'*}f_{J'1}^{M*}f_{J'1}^{M'}f_{L1}^{M}\times$$
$$\times\,F'_{L'L}(k',k)F'_{J'J}(k',k)\times$$
$$\times\,\mathcal{D}'_{M'M}(\hat{k}\to\hat{k}')\mathcal{D}'_{M'M}(\hat{k}\to\hat{k}')\Bigg\} \qquad (3.19)$$

where L,L', μ and μ' can only have values 0 and 2. The term on the bracket can be evaluated by first doing the M and M' summation which is not difficult if we make use of the symmetries of those f's. After this, we do the L,L',μ, and μ' summations term by term, which is fairly tedious. This will be given in Appendix C. The final expression for this term is

$$\{\cdots\} = 3\left[(F_{22}')^2 + (F_{02}')^2 + (F_{20}')^2 + (F_{00}')^2\right] + 3\left[\frac{1}{2}\left(\frac{k'^2+k^2-q^2}{2k'k}\right)^2 - 1\right] \times \left\{\frac{1}{2}(F_{22}')^2 + \frac{1}{\sqrt{2}}(F_{02}'F_{22}' + F_{20}'F_{22}') + (F_{00}'F_{22}' + F_{02}'F_{20}')\right\} \tag{3.20}$$

The explicit form of $\varepsilon(k,k',K_{av})$ in the calculations of Banerjee[10] is

$$\varepsilon(k,k',K_{av}) = \frac{1}{m_p^*}\left[k'^2 - k^2 + 2\Delta - \left(\frac{1}{m_r^*} - 1\right)(K_{av}^2 + k^2 - 0.6)\right] \tag{3.21}$$

where

$$K_{av}^2 = \frac{3}{5}(1-k)\left(1 + \frac{k^2}{3(2+k)}\right)$$

m_r^* = effective mass in the Fermi sea/m_p^* (3.22)

m_p^* = effective mass outside the Fermi sea

$\frac{\hbar^2}{m}\Delta k_F^2$ = gap between the energy spectra of occupied and unoccupied states, taken at $k = \sqrt{0.6}k_F$, the average momentum of occupied states.

Note that on the average, the term in (Eq. (3.21)) proportional to $(\frac{1}{m^*_r} - 1)$ has zero value; this is why we think it is reasonable to consider $\varepsilon(k,k',K_{av})$ while still treating the Pauli operator as a function of k,k' and K. So the Pauli correction per nucleon in RSM is

$$P.C. = \frac{3\rho_j}{\pi^3}\frac{\hbar^2}{m}\int_0^\infty dk'\int_0^1 d\beta \frac{e}{\int_{k'-k}^{k'+k} dq}(K'-K')\varepsilon(k,k',K_{av})\{\cdots\} \quad (3.23)$$

3-2. Angular average approximation of Pauli correction.

In the last section, we have derived the formula for the Pauli correction per nucleon in nuclear matter by using the function K(k,k',q). In this approach, we have made only one approximation of taking the average value over the $\underline{K}$-space for e^R, but otherwise we treated the $\underline{K}$ integration exactly. Conventionally, people have made another approximation, i.e., taking the angular average value for the $\underline{K}$ integration. In this section, we discuss briefly this approximation.

We know that the Pauli exclusion principle imposes one restriction on the $\underline{K}$ integration, i.e., $|\underline{K}\pm\underline{k}'|>1$. The other restriction $|\underline{K}\pm\underline{k}|<1$ simply counts which states are occupied. We can, however, replace these two conditions by two "functions"

$$\int_{\substack{|\underline{K}\pm\underline{k}|<1\\ |\underline{K}\pm\underline{k}'|>1}} d^3\underline{K} = \int d^3\underline{k}\; Q(k',K,\theta_{k'k})\,R(k,K,\theta_{kk}) \quad (3.24)$$

The first function $Q(k',K,\theta_{k'K})$ is called "Pauli operator" which is zero unless $|\underset{\sim}{K}\pm\underset{\sim}{k}'|>1$. The second function is called "Occupation operator" which is 1 if $|\underset{\sim}{K}\pm\underset{\sim}{k}|<1$. The so called angular average approximation is to replace the two functions $Q(k',K,\theta_{k'K})$ and $R(k,K,\theta_{kK})$ in Eq. (3.24) by their angular average functions $Q_{av}(k',K)$ and $R_{av}(k,K)$ respectively. So we write

$$K_{av}(k,k') = \int d^3\underset{\sim}{K}\; Q_{av}(k',K)\, R_{av}(k,K) \qquad (3.25)$$

for angular average approximation. It should be pointed out here that, by treating Q in an average way, the angle between $\underset{\sim}{k}'$ and $\underset{\sim}{K}$ is left undefined. But then also the angle between $\underset{\sim}{K}$ and $\underset{\sim}{k}$ is undefined so we have to treat R in an average way as well. The explicit form of these two functions can easily be derived. The results are

$$\begin{aligned} R_{av}(k,K) &= 1 && \text{when } K<1-k \\ &= \frac{1-k^2-K^2}{2kK} && \text{when } 1-k<K<\sqrt{1-k^2} \\ &= 0 && \text{when } K>\sqrt{1-k}^2 \end{aligned} \qquad (3.26)$$

and

$$\begin{aligned} Q_{av}(k',K) &= 0 && \text{when } K<\sqrt{1-k'}^2 \\ &= \frac{k'^2+K^2-1}{2k'K} && \text{when } \sqrt{1-k'}^2<K<1+k' \\ &= 1 && \text{when } K>1+k' \end{aligned} \qquad (3.27.1)$$

for $k'<1$.

$$Q'_{av}(k',K) = \begin{cases} 1 & \text{when } K<k'-1 \\ \frac{k'^2+K^2-1}{2k'K} & \text{when } k'-1<K<k'+1 \\ 1 & \text{when } K>k'+1 \end{cases} \qquad (3.27.2)$$

for $k'>1$.

With these expressions, we can calculate $K_{av}(k,k')$. The result, after a straightforward but tedious calculation, is listed below.

(1) If $k'<k<1$ $\quad K_{av}=0$ (3.28.1)

(2) If $k<k'<1$

(a) $u<y<x$ $\quad K_{av}= \frac{2\pi}{15kk'}\,(x^5-5x^3y^2+5x^2y^3-y^5)$ (3.28.2)

(b) $y<u<x$ $\quad K_{av}= \frac{\pi}{2k'}\,(u^2-y^2)^2 -$

$$-\left(\frac{\pi}{15kk'}\right)\left[3(x^5-u^5)-5(x^2+y^2)(x^3-u^3)+15x^2y^2(x-u)\right] \qquad (3.28.3)$$

(3) If $k<1<k'$

(a) $u<x<v$ $\quad K_{av}= \frac{2\pi}{3}(1-k)^2(2+k)$ (3.28.4)

(b) $u<v<x$ $\quad K_{av}= \frac{4\pi}{3}\,u^3+\left(\frac{\pi}{2k}\right)(v^2-u^2)(2x^2-u^2-v^2) -$

$$-\left(\frac{\pi}{15kk'}\right)\left[3(x^5-v^5)+5(y'^2-x^2)(x^3-v^3)-15x^2y'^2(x-v)\right] \qquad (3.28.5)$$

(c) $v<u<x$ $\quad K_{av}= \frac{4\pi}{3}\,v^3+\left(\frac{\pi}{2k}\right)(u^2-v^2)(2y'^2+u^2+v^2) -$

$$-\left(\frac{\pi}{15kk'}\right)\left[3(x^5-u^5)+5(y'^2-x^2)(x^3-u^3)-15x^2y'^2(x-u)\right] \qquad (3.28.6)$$

where

$$x = (1-k^2)^{\frac{1}{2}}$$
$$y = (1-k'^2)^{\frac{1}{2}} \quad , \quad y' = (k'^2-1)^{\frac{1}{2}} \tag{3.28.7}$$
$$u = 1-k \quad , \quad v = k'-1$$

Using this angular average function, the Pauli correction per nucleon in RSM is written as

$$P.C. = \frac{6}{\pi^3} k_f^5 \frac{\hbar^2}{m} \int_0^\infty dk' \int_0^1 dk \qquad (K'-K_{av}) \times \varepsilon(k, k', K_{av}) \times \{\cdots\} \tag{3.29}$$

where K', it will be remembered, is the volume of the flying saucer given by Eq. (3.17).

3-3. Computations and comparisons

From Eqs. (3.23) and (3.29) we can calculate numerically the Pauli correction. Since we are more interested in comparing the results from these two methods, we split the term {...}in Eq. (3.20) into two parts, i.e.,

$$\{1\} = 3\left[(F'_{22})^2+(F'_{02})^2+(F'_{20})^2+(F'_{00})^2\right] \tag{3.30}$$

$$\{2\} = 3\left[\left(\frac{k'^2+k^2-q^2}{2kk'}\right)^2-1\right]\left[\frac{1}{4}(F'_{22})^2+\frac{1}{\sqrt{2}}(F'_{02}F'_{22}+F'_{20}F'_{22}) + (F'_{00}F'_{22} + F'_{02}F'_{20})\right] \tag{3.31}$$

In our more sophisticated method, both terms will contribute to the Pauli correction. But in the angular average approximation, only the first term will contribute, the second term will vanish identically because it depends on angle through $P_2(\cos\theta_{kk'})$ which has a zero angular average value. Hence this term did not occur in the paper by BBP.

To perform the calculation, we borrow the numerical results of those F's for Reid's soft core potential from Mr. Banerjee. Using Simpson's integration rule with $k' = 0(0.05)5$, $k = 0(0.05)1$, and $q = (k'-k)(0.05)(k'+k)$, we obtain the result as in Table 2.

TABLE 2

$\int\{1\}_{exact}$	$\int\{1\}_{angular\ average}$	$\int\{1\}_{Banerjee}$	$\int\{2\}_{exact}$	$\int\{2\}_{angular\ average}$
3.49954	3.50025	3.73	-0.00797	0.0

Pauli correction per nucleon for ^3S-^3D states in RSM by using Reid's soft core potential. The unit is in MeV.

From the above table, we see that the angular average approximation is a very good approximation in this calculation.

CHAPTER IV

APPLICATION TO THE SEPARABLE CORE POTENTIAL

For a nuclear potential with a hard core, a perturbation calculation of the binding energy of a nucleus leads to a divergence. To overcome this difficulty, some people have fitted nuclear potentials with soft cores. So far, all the soft cores are still too strong to give a reliable perturbation calculation. As stated in the Introduction, Kerman and Levy at M.I.T. have recently proposed a non-local but separable core for the nuclear potential. For this type of nuclear potential, there will be certain technical difficulties associated with the method of using the Euler function or the angular average approximation. These difficulties will be discussed in the following sections. However, if we make use of our function $K(k,k',q)$, then we can easily evaluate the first order and second order terms in the perturbation series. Since people hope that this kind of semi-dynamic potential will make the perturbation series converge quickly, we are satisfied with only these two terms.

In this chapter, we will deal with the central potential in the $^1S_0$ state only. In section 1, we introduce Kerman and Levy's potential for this state and derive the basic matrix element in the perturbation theory. In Section 2 and 3, we derive the first and second order term, respectively. The same formulation can be extended to tensor and spin-orbit forces; but we will not derive them here. The numerical calculation is not carried out since we don't know Kerman-Levy's potential exactly.

4-1. Kerman-Levy's potential.

Kerman-Levy's potential is taken to be, in the $^1S_0$ state,

$$\begin{aligned} v(r,r') &= v(r)\, v(r') && r,r'<c \\ &= w(r)\,\delta(r-r') && r,r'>c \\ &= 0 && \text{otherwise} \end{aligned} \qquad (4.1)$$

For this type of potential, Eq. (2.8) becomes

$$\begin{aligned} U(k,k') = \frac{1}{4\pi}\Big[& \int_0^c d^3r \int_0^c d^3r'\, v(r)\, v(r')\, e^{i(k'\cdot r' - k\cdot r)} + \\ & + \int_c^\infty d^3r \int_c^\infty d^3r'\, w(r)\,\delta(r-r')\, e^{i(k'\cdot r' - k\cdot r)} \Big] \\ = & \int_0^c j_0(kr)\, v(r)\, r^2 dr \int_0^c j_0(k'r')\, v(r')\, r'^2 dr' + \\ & + \int_c^\infty j_0(kr)\, w(r)\, r^2 dr \end{aligned} \qquad (4.2)$$

where $q = |k'-k|$. We define

$$v_S(k) = \int_0^c j_0(kr)v(r)r^2dr \tag{4.3}$$

$$v_L(q) = \int_c^\infty j_0(qr)W(r)r^2dr \tag{4.4}$$

then Eq. (2.9) gives us the matrix element

$$\langle \underset{\sim}{\ell m}|v|\underset{\sim}{a b}\rangle = \frac{4\pi}{\Omega}\left[v_S(k)v_S(k') + v_L(q)\right] \tag{4.5}$$

4-2. First order perturbation theory.

In the first order perturbation theory, the binding energy per nucleon in nuclear matter is

$$B^{(1)} = \frac{1}{2A}\sum_{\substack{\underset{\sim}{\ell},\underset{\sim}{m}\\ \sigma,\tau}}(\langle \underset{\sim}{\ell m}|v|\underset{\sim}{\ell m}\rangle - \langle \underset{\sim}{\ell m}|v|\underset{\sim}{m \ell}\rangle) \tag{4.6}$$

where σ and τ denote spin and isospin respectively. The label $\underset{\sim}{\ell}$ for a single particle states means $\underset{\sim}{k}_\ell, \sigma_\ell, \tau_\ell$ and all these enter summation.

From Eq. (4.5), we obtain

$$\sum_{\sigma,\tau}\langle \underset{\sim}{\ell m}|v|\underset{\sim}{\ell m}\rangle = \frac{4\pi}{\Omega}\left\{\left[\sum_{\sigma,\tau} v_S^2(k)\right]_D + \left[\sum_{\sigma,\tau} v_L(D)\right]_D\right\} \tag{4.7}$$

$$\sum_{\sigma,\tau}\langle \underset{\sim}{\ell m}|v|\underset{\sim}{m \ell}\rangle = \frac{4\pi}{\Omega}\left\{\left[\sum_{\sigma,\tau} v_S^2(k)\right]_E + \left[\sum_{\sigma,\tau} v_L(2k)\right]_E\right\} \tag{4.8}$$

where

$$\left[\sum_{\sigma,\tau} v_S^2(k)\right]_D = 9\left[v_S^{33}(k)\right]^2 + 3\left[v_S^{31}(k)\right]^2 + 3\left[v_S^{13}(k)\right]^2 + \left[v_S^{11}(k)\right]^2 \quad (4.9)$$

$$\left[\sum_{\sigma,\tau} v_S^2(k)\right]_E = -9\left[v_S^{33}(k)\right]^2 + 3\left[v_S^{31}(k)\right]^2 + 3\left[v_S^{13}(k)\right]^2 - \left[v_S^{11}(k)\right]^2 \quad (4.10)$$

$$\left[\sum_{\sigma,\tau} v_L(0)\right]_D = 9\left[v_L^{33}(0)\right] + 3\left[v_L^{31}(0)\right] + 3\left[v_L^{13}(0)\right] + \left[v_L^{11}(0)\right] \quad (4.11)$$

$$\left[\sum_{\sigma,\tau} v_L(2k)\right]_E = -9\left[v_L^{33}(2k)\right] + 3\left[v_L^{31}(2k)\right] + 3\left[v_L^{13}(2k)\right] - \left[v_L^{11}(2k)\right] \quad (4.12)$$

The superscripts on v are 2S+1, 2T+1 for the interacting pair of nucleons. If we use Eqs. (4.7), (4.8), (2.2) and (2.11), then Eq. (4.6) becomes

$$B^{(1)} = \frac{1}{2\rho}\frac{4\pi}{(2\pi)^6} k_f^6 \int_0^1 d^3\hat{k} \int_0^1 d^3\hat{r} \times$$

$$\times \left\{ \left[\sum_{\sigma,\tau} v_S^2(k)\right]_D + \left[\sum_{\sigma,\tau} v_L(0)\right]_D - \left[\sum_{\sigma,\tau} v_S^2(k)\right]_E - \left[\sum_{\sigma,\tau} v_L(2k)\right]_E \right\} \quad (4.13)$$

For the second term, we can easily evaluate and get the result as

$$\frac{k_f^3}{12\pi}\left[\sum_{\sigma,\tau} v_L(0)\right]_D \quad (4.14)$$

For the other terms, we use Eqs. (2.6) and (3.17). The result is

$$\frac{k_f^3}{\pi}\int_0^1 d\hat{k}\, \hat{k}^2(1-\hat{k})^2(2+\hat{k}) \times$$

$$\times \left\{ \left[\sum_{\sigma,\tau} v_S^2(k)\right]_D - \left[\sum_{\sigma,\tau} v_S^2(k)\right]_E - \left[\sum_{\sigma,\tau} v_L(2k)\right]_E \right\} \quad (4.15)$$

So we finally have

$$B^{(1)} = \frac{k_f^3}{12\pi}\left[\sum_{\sigma\tau} v_L(0)\right]_D + \frac{k_f^3}{\pi}\int_0^1 dk\, k^2(1-k)^2(2+k)\times$$

$$\times\left\{\left[\sum_{\sigma\tau} v_S^2(k)\right]_D - \left[\sum_{\sigma\tau} v_S^2(k)\right]_E - \left[\sum_{\sigma\tau} v_L(2k)\right]_E\right\} \tag{4.16}$$

4-3. Second order perturbation theory.

The second order contribution to the binding energy per particle is

$$B^{(2)} = -\frac{1}{2A}\sum_{\substack{a,b>1\\ \ell,m<1\\ \sigma,\tau}} \frac{\langle \ell m|v|ab\rangle\langle ab|v|\ell m\rangle - \langle \ell m|v|ab\rangle\langle ab|v|m\ell\rangle}{E_a + E_b - E_\ell - E_m} \tag{4.17}$$

We take k_f as the unit of momentum as before and $\underset{\sim}{a}+\underset{\sim}{b}=\underset{\sim}{\ell}+\underset{\sim}{m}$ is implied.

Making a similar transformation as we did in (2.6)

$$\begin{aligned} \underset{\sim}{\ell}-\underset{\sim}{m} &= 2\underset{\sim}{k} \quad , \quad \underset{\sim}{\ell}+\underset{\sim}{m} = 2\underset{\sim}{K} \\ \underset{\sim}{a}-\underset{\sim}{b} &= 2\underset{\sim}{k}' \quad , \quad \underset{\sim}{a}+\underset{\sim}{b} = 2\underset{\sim}{K}' \end{aligned} \tag{4.18}$$

and also defining

$$\underset{\sim}{k}'-\underset{\sim}{k} = \underset{\sim}{q} \quad , \quad \underset{\sim}{k}'+\underset{\sim}{k} = \underset{\sim}{x} \tag{4.19}$$

then we can follow the same method as used in sec. 2-1 to evaluate $B^{(2)}$. The results are

$$B^{(2)}(\text{Direct}) = -\frac{3k_f^4}{2^7\pi^5}\frac{m}{\hbar^2}\int_0^1 d^3\underset{\sim}{\ell}\int_0^1 d^3\underset{\sim}{m}\int_1^\infty d^3\underset{\sim}{q}\,\frac{1}{\underset{\sim}{q}\cdot\underset{\sim}{x}}\times$$

$$\times\left\{\left[\sum_{\sigma\tau} v_S^2(k)v_S^2(k')\right]_D + 2\left[\sum_{\sigma\tau} v_S(k)v_S(k')v_L(q)\right]_D + \left[\sum_{\sigma\tau} v_L(q)\right]_D\right\} \tag{4.20}$$

and

$$B^{(2)}(\text{exchange}) = \frac{3k_f^4}{2^3\pi^5}\frac{m}{\hbar^2}\int_0^1 d^3\underline{s}\int_0^1 d^3\underline{m}\int_1^\infty d^3\underline{q}\;\frac{1}{\underline{s}\cdot\underline{x}}\times$$

$$\times\left\{\left[\sum_{\sigma\tau} v_S^2(k)v_S^2(k')\right]_E + \left[\sum_{\sigma\tau} v_S(k)v_S(k')v_L(x)\right]_E + \left[\sum_{\sigma\tau} v_S(k)v_S(k')v_L(s)\right]_E + \right. \tag{4.21}$$

$$\left. + \left[\sum_{\sigma\tau} v_L(s)v_L(x)\right]_E\right\}$$

The total contribution is therefore given by their sum, i.e.

$$B^{(2)} = -\frac{3k_f^4}{2^3\pi^5}\frac{m}{\hbar^2}\int_0^1 d^3\underline{s}\int_0^1 d^3\underline{m}\int_1^\infty d^3\underline{q}\;\frac{1}{\underline{s}\cdot\underline{x}}\times$$

$$\times\left\{\left[\sum_{\sigma\tau} v_L^2(s)\right]_D + \left[\sum_{\sigma\tau} v_S^2(k)\,v_S^2(k')\right]_D - \right.$$

$$-\left[\sum_{\sigma\tau} v_S^2(k)\,v_S^2(k')\right]_E + 2\left[\sum_{\sigma\tau} v_S(k)v_S(k')v_L(s)\right]_D -$$

$$-\left[\sum_{\sigma\tau} v_S(k)v_S(k')v_L(s)\right]_E - \left[\sum_{\sigma\tau} v_S(k)v_S(k')v_L(x)\right]_E -$$

$$\left. -\left[\sum_{\sigma\tau} v_L(s)v_L(x)\right]_E\right\} \tag{4.22}$$

where the sum over spin and isospin is to be understood as in the previous section and momentum conservation still holds.

Looking at the above expression, there are many terms in the integrand. For the first term we can use Euler's function to evaluate the integrations. For the second and third terms, the angular average approximation may or may not give a good approximation as they do not depend on q. For all the other terms, it's quite possible

that the angular average method will be a poor approximation as they all depend on q. But all these terms can be treated exactly by making use of our function K(k,k',q). The method is strightforward as that used in Sec. 2-1. First we notice that the relation $\underset{\sim}{x}=\underset{\sim}{k}'+\underset{\sim}{k}$ gives

$$x = \left[2(k'^2+k^2)-q^2\right]^{\frac{1}{2}} \tag{4.23}$$

Making use of the fact

$$\int_0^1 d^3\underset{\sim}{\ell} \int_0^1 d^3\underset{\sim}{m} \int_0^\infty d^3\underset{\sim}{a} = 8 \int_0^\infty d^3\underset{\sim}{k}' \int_0^1 d^3\underset{\sim}{k} \int_{\substack{|\underset{\sim}{K}\pm\underset{\sim}{k}|<1 \\ |\underset{\sim}{K}\pm\underset{\sim}{k}'|>1}} d^3\underset{\sim}{K} \tag{4.24}$$

then Eq. (4.22) becomes

$$B^{(2)} = -\frac{3k_f^4}{2\pi^3}\frac{m}{\hbar^2}\int_0^\infty k'dk' \int_0^1 kdk \int_{k'-k}^{k'+k} qdq \; \frac{K(k,k',q)}{k'^2-k^2} \times$$

$$\times \Bigg\{ \left[\sum_{\sigma,2} v_2^2(q)\right]_D + \left[\sum_{\sigma,2} v_3^2(k)\, v_0^2(k')\right]_D -$$

$$- \left[\sum_{\sigma,2} v_3^2(k)\, v_3^2(k')\right]_E + 2\left[\sum_{\sigma,2} v_3(k)\, v_3(k')\, v_2(q)\right]_D -$$

$$- \left[\sum_{\sigma,2} v_3(k)\, v_3(k')\, v_2(q)\right]_E - \left[\sum_{\sigma,2} v_3(k)\, v_3(k')\, v_2(x)\right]_E -$$

$$- \left[\sum_{\sigma,2} v_2(q)\, v_2(x)\right]_E \Bigg\}$$

(4.25)

where x must be replaced by the right hand side of Eq. (4.23) before the integrations are carried out.

This is the final expression for a potential of the Kerman-Levy type with a separable core in the ^1S_o state.

With the above formulas, we can calculate the binding energy per nucleon in nuclear matter for the Kerman-Levy potential by a similar numerical method as being used for calculating the Pauli correction in RSM. We thus conclude that our function K(k,k',q) serves as a very convenient and accurate apparatus in treating this new type of potential.

APPENDIX A

DERIVATION OF THE INTERSECTION VOLUME BETWEEN THREE IDENTICAL SPHERES IN THE PAULI EXCLUSION PRINCIPLE

The three identical spheres all have a unit radius. Let their centers be A,B and C. The distances between them are 2k, q and x for AB, BC and CA respectively. We remember that $\underset{\sim}{q} = \underset{\sim}{k}'-\underset{\sim}{k}$ and $\underset{\sim}{x} = \underset{\sim}{k}'+\underset{\sim}{k}$. These are shown explicitly in Fig. 12. We see there that the intersection volume is divided into six parts. They can all be treated in the same way. Let us consider volume 1. We draw this volume in Fig. 13. The cross-hatched volume element in Fig. 13 can be further considered as the superposition of the shaded volume element in Fig. 14. From Fig. 14 we calculate the cross-hatched volume element by

$$v = \int_a^R A\,dx \tag{A.1}$$

But

$$\phi = \sin^{-1}\sqrt{\left(\frac{R^2-x^2}{R^2}\right)} \tag{A.2}$$

$$A = \pi R^2 \frac{2\phi}{2\pi} - \tfrac{1}{2}\, x \cdot 2R\sin\phi \qquad (A.3)$$
$$= R^2\phi - x\sqrt{R^2-x^2}$$

$$v = \int_a^R \{R^2 \sin^{-1}\sqrt{\left(\frac{R^2-x^2}{R^2}\right)} - x\sqrt{R^2-x^2}\}dx \qquad (A.4)$$

The first integral can be evaluated by integration by parts. The result is

$$v = R^2(R^2-a^2)^{\frac{1}{2}} - \frac{1}{3}(R^2-a^2)^{3/2} - aR^2\sin^{-1}\sqrt{\left(\frac{R^2-a^2}{R^2}\right)} \qquad (A.5)$$

But
$$R^2 = 1-(D\sin\theta)^2 \qquad (A.6)$$
$$a = D\cos\theta \qquad (A.7)$$

Using Eqs. (A.5) to (A.7) and Fig. 2, we have the volume element 1

$$V' = \int_0^\theta v\, d\theta$$
$$= \int_0^\theta \left\{ (1-D^2\sin^2\theta)(1-D^2)^{1/2} - \frac{1}{3}(1-D^2)^{3/2} - \right.$$
$$\left. - D\cos\theta\,(1-D^2\sin^2\theta)\cdot \sin^{-1}\sqrt{\frac{1-D^2}{1-D^2\sin^2\theta}} \right\} d\theta$$
$$= \theta(1-D^2)^{1/2} - \frac{1}{2}D^2(1-D^2)^{1/2}(\theta - \sin\theta\cos\theta) - \frac{1}{3}(1-D^2)^{3/2}\theta -$$
$$- D(\sin\theta - \frac{D^2}{3}\sin^3\theta)\sin^{-1}\sqrt{\frac{1-D^2}{1-D^2\sin^2\theta}} +$$
$$+ \int D(\sin\theta - \frac{D^2}{3}\sin^3\theta)\, d\sin^{-1}\sqrt{\frac{1-D^2}{1-D^2\sin^2\theta}} \qquad (A.8)$$

The last integral in Eq. (A.8) can be simplified as

$$I = D^2(1-D^2)^{\frac{1}{2}} \int \frac{\sin^2\theta - \frac{D^2}{3}\sin^4\theta}{1-D^2\sin^2\theta} d\theta \qquad (A.9)$$

The integral part can be further integrated into

$$II = \frac{1}{6}(\theta - \sin\theta\cos\theta) - \frac{2}{3D^2}\theta + \frac{2}{3D^2}\int \frac{d\theta}{1-D^2\sin\theta} \qquad (A.10)$$

The integral in Eq. (A.10) is

$$III = \frac{1}{\sqrt{1-D^2}} \tan^{-1}\left(\sqrt{1-D^2}\ \tan\theta\right) \qquad (A.11)$$

Combining Eqs. (A.8) to (A.11), we finally obtain the volume as

$$V' = \frac{D^2(1-D^2)^{1/2}}{3}\sin\theta\cos\theta - D\sin\theta\left(1-\frac{D^2}{3}\sin^2\theta\right) \cdot \times \sin^{-1}\sqrt{\frac{1-D^2}{1-D^2\sin^2\theta}} + \frac{2}{3}\tan^{-1}\left(\sqrt{1-D^2}\ \tan\theta\right) \qquad (A.12)$$

Applying Eq. (A.12) to all the six partial volumes, we get the total intersection volume

$$V = 2\sum_{i=1}^{3}\left\{\frac{D^2(1-D^2)^{1/2}}{3}\sin\theta_i\cos\theta_i - D\sin\theta_i\left(1-\frac{D^2}{3}\sin^2\theta_i\right) \cdot \times \sin^{-1}\sqrt{\frac{1-D^2}{1-D^2\sin^2\theta_i}} + \frac{2}{3}\tan^{-1}\left(\sqrt{1-D^2}\ \tan\theta_i\right)\right\} \qquad (A.13)$$

where D and θ_i are shown in Fig. 15 and can be evaluated in terms of k,k', and q through the natural quantities k_1,k_2, and k_3. The result is

$$D = \frac{2k_1 k_2 k_3}{(2k_1^2k_2^2+2k_2^2k_3^2+2k_3^2k_1^2-k_1^4-k_2^4-k_3^4)^{\frac{1}{2}}} \tag{A.14}$$

$$\theta_i = \sin^{-1}\frac{k_i}{D} \tag{A.15}$$

where

$$\begin{aligned} k_1 &= k \\ k_2 &= q/2 \\ k_3 &= \tfrac{1}{2}\,(2k'^2+2k^2-q^2)^{\frac{1}{2}} \end{aligned} \tag{A.16}$$

Making use of Eqs. (A.14) to (A.16), we can rewrite Eq. (A.13) in the following way

$$V = \frac{2}{3}M - \frac{2}{3}\sum_{i=1}^{3} k_i\,(3-k_i^2)\cos^{-1}\left[\frac{k_i\,(k_i^2+k_k^2-k_i^2)}{M\,(1-k_i^2)^{1/2}}\right]$$

$$+\begin{cases} \dfrac{4\pi}{3}\tan^{-1}\left(\dfrac{M}{k_1^2+k_2^2+k_3^2-2}\right), & k_1^2+k_2^2+k_3^2-2>0 \\[2ex] \dfrac{4\pi}{3}\left[\pi+\tan^{-1}\left(\dfrac{M}{k_1^2+k_2^2+k_3^2-2}\right)\right], & k_1^2+k_2^2+k_3^2-2<0 \end{cases} \tag{A.17}$$

with

$$M = (2k_1^2k_2^2+2k_3^2k_3^2+2k_3^2k_1^2-k_1^4-k_2^4-k_3^4)^{\frac{1}{2}} \tag{A.18}$$

This is the formula we have made use of in Chapter II.

APPENDIX B

AN ANALYTIC CHECK OF THE RELATION BETWEEN FUNCTION K(k,k',q) AND EULER'S FUNCTION P(u) FOR q>2

We know from Eq. (2.23) that for $q > 2$

$$K(k,k',q) = \frac{2\pi}{3}(1-k)^2(2+k) \qquad (B.1)$$

Substituting Eq. (B.1) into the right hand side of Eq. (2.30) then

$$RHS = 40\int_0^1 dk\, k(1-k)^2(2+k)\int_{q-k}^{q+k}\frac{k'dk'}{k'^2-k^2}$$

$$= 20\int_0^1 dk\, k(1-k^2)(2+k)\log\left(\frac{u+k}{u-k}\right) \qquad (u = q/2)$$

$$= 20\left\{\frac{u^3}{5}+\frac{11u}{10}+\left(\frac{u^5}{5}-u^3-u^2+\frac{1}{5}\right)\log\left(1+\frac{1}{u}\right)+\right.$$

$$\left.+\left(\frac{u^5}{5}-u^3+u^2-\frac{1}{5}\right)\log\left(1-\frac{1}{u}\right)\right\}$$

$$= \log(u+1)\left[4-20u^2-20u^3+4u^5\right]+4u^3+22u+$$

$$+\log(u-1)\left[-4+20u^2-20u^3+4u^5\right]+\log u\cdot\left[40u^3-8u^5\right]$$

$$= P^{II}(u)$$

The last step can be checked by Eq. (2.27.2). So we have proved the relation in Eq. (2.30) for the case $q>2$.

APPENDIX C

EVALUATION OF SPIN SUM FOR THE PAULI CORRECTION IN COUPLED $^3S_1$-$^3D_1$ STATES

The bracket {....} in Eq. (3.20) represents

$$\sum_{M,M'} \sum_{\substack{\text{even} \\ L,L' \\ \mu,\mu'}} f^{M'*}_{L',1} f^{M*}_{\mu',1} f^{M'}_{\mu',1} f^{M}_{L,1} F'_{L'L}(k',k) F'_{\mu'\mu}(k',k) \times \mathcal{D}'^{*}_{M'M}(\hat{k} \to \hat{k}') \mathcal{D}'_{MM}(\hat{k} \to \hat{k}') \tag{C.1}$$

where

$$f^{M}_{L,l} = i^{L}(2L+1)^{\frac{1}{2}} \langle L10M | 1M \rangle \tag{C.2}$$

$$L,L',\mu, \text{ and } \mu' = 0,2 \tag{C.3}$$

$$M,M' = -1,0,1 \tag{C.4}$$

From Eqs. (C.2) and (C.3), we see that all f's are real. To evaluate Eq. (C.1), we choose $\underset{\sim}{k}$ as the quantization axis, then $\underset{\sim}{k}'$ can be obtained from $\underset{\sim}{k}$ by the rotation $R(\phi, \theta, 0)$ in terms of the Euler's angles (α,β,γ). Then we can write the D-matrix in the following form

$$D^{(1)} = \begin{bmatrix} e^{-i\phi}\frac{1+\cos\theta}{2} & -e^{-i\phi}\frac{\sin\theta}{\sqrt{2}} & e^{-i\phi}\frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ e^{i\phi}\frac{1-\cos\theta}{2} & e^{i\phi}\frac{\sin\theta}{2} & e^{i\phi}\frac{1+\cos\theta}{2} \end{bmatrix} \qquad \text{(C.5)}$$

If we consider the summation over M and M', then we obtain, by using Eqs. (C.4) and (C.5),

$$\sum_{M,M'} f_{L'1}^{M'} f_{J'1}^{M'} f_{L1}^{M} f_{J1}^{M} |\mathcal{D}'_{MM'}|^2$$

$$= \frac{(1+\cos\theta)^2}{4}\left[f_{L'1}^{-1} f_{J'1}^{-1} f_{L1}^{-1} f_{J1}^{-1} + f_{L'1}^{1} f_{J'1}^{1} f_{L1}^{1} f_{J1}^{1} \right]$$

$$+ \frac{\sin^2\theta}{2}\left[f_{L'1}^{-1} f_{J'1}^{-1} f_{L1}^{0} f_{J1}^{0} + f_{L'1}^{0} f_{J'1}^{0} f_{L1}^{-1} f_{J1}^{-1} + \right.$$

$$\left. + f_{L'1}^{0} f_{J'1}^{0} f_{L1}^{1} f_{J1}^{1} + f_{L'1}^{1} f_{J'1}^{1} f_{L1}^{0} f_{J1}^{0} \right]$$

$$+ \frac{(1-\cos\theta)^2}{4}\left[f_{L'1}^{-1} f_{J'1}^{-1} f_{L1}^{1} f_{J1}^{1} + f_{L'1}^{1} f_{J'1}^{1} f_{L1}^{-1} f_{J1}^{-1} \right]$$

$$+ \cos^2\theta\ f_{L'1}^{0} f_{J'1}^{0} f_{L1}^{0} f_{J1}^{0}$$

(C.6)

Since

$$\langle 010-1|1-1\rangle=\langle 0100|10\rangle=\langle 0101|11\rangle=1$$
$$\langle 210-1|1-1\rangle=\langle 2101|11\rangle=\frac{1}{\sqrt{10}} \qquad (C.7)$$
$$\langle 2100|10\rangle=-\frac{2}{\sqrt{10}}$$

we have

$$f_{01}^{-1}=f_{01}^{0}=f_{01}^{1}=1$$
$$f_{21}^{-1}=f_{21}^{1}=-\frac{1}{\sqrt{2}} \qquad (C.8)$$
$$f_{21}^{0}=\sqrt{2}$$

But Eq. (C.8) implies

$$f_{al}^{-1}=f_{al}^{+1} \qquad (C.9)$$

where a can be L,L',μ, and μ'. Using Eq. (C.9) we can reduce Eq. (C.6) into

$$\sum_{M,M'} f_{L'1}^{M'} f_{J'1}^{M'} f_{L1}^{M} f_{J1}^{M} \left|\mathcal{D}'_{M'M}\right|^2$$
$$= (1+\cos^2\theta)\, f_{L'1}^{1} f_{J'1}^{1} f_{L1}^{1} f_{J1}^{1}$$
$$+ \sin^2\theta \left(f_{L'1}^{1} f_{J'1}^{1} f_{L1}^{0} f_{J1}^{0} + f_{L'1}^{0} f_{J'1}^{0} f_{L1}^{1} f_{J1}^{1} \right)$$
$$+ \cos^2\theta\, f_{L'1}^{0} f_{J'1}^{0} f_{L1}^{0} f_{J1}^{0} \qquad (C.10)$$

So we write Eq. (C.1) as

$$\sum_{\substack{\text{even}\\ L,L'\\ \mu,\mu'}} (1+\cos^2\theta)\, f^{1}_{L'1}\, f^{1}_{L1}\, f^{1}_{\mu'1}\, f^{1}_{\mu 1}\, F^{1}_{L'L}\, F^{1}_{\mu'\mu} +$$

$$+ \sin^2\theta \left(f^{1}_{L'1}\, f^{0}_{L1}\, f^{1}_{\mu'1}\, f^{0}_{\mu 1}\, F^{1}_{L'L}\, F^{1}_{\mu'\mu} + \right.$$

$$\left. + f^{0}_{L'1}\, f^{1}_{L1}\, f^{0}_{\mu'1}\, f^{1}_{\mu 1}\, F^{1}_{L'L}\, F^{1}_{\mu'\mu} \right)$$

$$+ \cos^2\theta\; f^{0}_{L'1}\, f^{0}_{L1}\, f^{0}_{\mu'1}\, f^{0}_{\mu 1}\, F^{1}_{L'L}\, F^{1}_{\mu'\mu} \qquad \text{(C.11)}$$

Making use of Eq. (C.8) and summing over L,L',μ, and μ' for each term in Eq. (C.11), we obtain respectively

$$\sum_{\substack{\text{even}\\ L,L'\\ \mu,\mu'}} f^{1}_{L'1}\, f^{1}_{L1}\, f^{1}_{\mu'1}\, f^{1}_{\mu 1}\, F^{1}_{L'L}\, F^{1}_{\mu'\mu}$$

$$= \frac{1}{4}(F^{1}_{22})^2 + \frac{1}{2}(F^{1}_{02})^2 + \frac{1}{2}(F^{1}_{20})^2 + (F^{1}_{00})^2 +$$

$$- \frac{1}{\sqrt{2}} F^{1}_{02} F^{1}_{22} - \frac{1}{\sqrt{2}} F^{1}_{20} F^{1}_{22} + F^{1}_{00} F^{1}_{22} + F^{1}_{02} F^{1}_{20} +$$

$$- \sqrt{2}\, F^{1}_{20} F^{1}_{02} - \sqrt{2}\, F^{1}_{00} F^{1}_{20} \qquad \text{(C.12)}$$

$$\sum_{\substack{\text{even}\\ L,L'\\ \mu,\mu'}} f^{1}_{L'1}\, f^{0}_{L1}\, f^{1}_{\mu'1}\, f^{0}_{\mu 1}\, F^{1}_{L'L}\, F^{1}_{\mu'\mu}$$

$$= (F^{1}_{22})^2 + 2(F^{1}_{02})^2 + \frac{1}{2}(F^{1}_{20})^2 + (F^{1}_{00})^2 +$$

$$- 2\sqrt{2}\, F^{1}_{02} F^{1}_{22} + \sqrt{2}\, F^{1}_{20} F^{1}_{22} - 2 F^{1}_{00} F^{1}_{22} - 2 F^{1}_{02} F^{1}_{20}$$

$$+ 2\sqrt{2}\, F^{1}_{00} F^{1}_{02} - \sqrt{2}\, F^{1}_{00} F^{1}_{20}$$

$$\text{(C.13)}$$

$$\sum_{\substack{\text{even} \\ L,L' \\ \lambda,\lambda'}} f^{0}_{L'1}\, f^{1}_{L1}\, f^{0}_{\lambda'1}\, f^{1}_{\lambda 1}\, F^{1}_{L'L}\, F^{1}_{\lambda'\lambda}$$

$$= (F^{1}_{22})^2 + \tfrac{1}{2}(F^{1}_{02})^2 + \tfrac{1}{2}(F^{1}_{30})^2 + (F^{1}_{00})^2$$

$$+ \sqrt{\tfrac{1}{2}}\, F^{1}_{02} F^{1}_{22} - 2\sqrt{\tfrac{1}{5}}\, F^{1}_{20} F^{1}_{32} - 2 F^{1}_{00} F^{1}_{22} - 2 F^{1}_{02} F^{1}_{30}$$

$$- \sqrt{\tfrac{1}{2}}\, F^{1}_{00} F^{1}_{02} + 2\sqrt{\tfrac{1}{2}}\, F^{1}_{00} F^{1}_{20} \qquad \text{(C.14)}$$

and

$$\sum_{\substack{\text{even} \\ L,L' \\ \lambda,\lambda'}} f^{0}_{L'1}\, f^{0}_{L1}\, f^{0}_{\lambda'1}\, f^{0}_{\lambda 1}\, F^{1}_{L'L}\, F^{1}_{\lambda'\lambda}$$

$$= 4(F^{1}_{32})^2 + 2(F^{1}_{02})^2 + 2(F^{1}_{20})^2 + (F^{1}_{00})^2$$

$$+ 4\sqrt{\tfrac{1}{2}}\, F^{1}_{02} F^{1}_{32} + 4\sqrt{\tfrac{1}{5}}\, F^{1}_{20} F^{1}_{32} + 4 F^{1}_{00} F^{1}_{32} + 4 F^{1}_{02} F^{1}_{30}$$

$$+ 2\sqrt{\tfrac{1}{2}}\, F^{1}_{30} F^{1}_{02} + 2\sqrt{\tfrac{1}{2}}\, F^{1}_{00} F^{1}_{20} \qquad \text{(C.15)}$$

Substituting Eqs. (C.12) to (C.15) into Eq. (C.11), we obtain the final result for the bracket {....}, i.e.,

$$\{\cdots\} = 3\left[(F^{1}_{32})^2 + (F^{1}_{02})^2 + (F^{1}_{30})^2 + (F^{1}_{00})^2\right]$$

$$+ 3(3\cos^2\theta - 1) \times$$

$$\times \left\{\tfrac{1}{4}(F^{1}_{32})^2 + \tfrac{1}{\sqrt{2}}(F^{1}_{02}F^{1}_{32} + F^{1}_{20}F^{1}_{32}) + (F^{1}_{00}F^{1}_{32} + F^{1}_{02}F^{1}_{20})\right\} \qquad \text{(C.16)}$$

The $\cos\theta$ can be written in terms of k,k', and q by using

$$\underset{\sim}{q} = \underset{\sim}{k}' - \underset{\sim}{k} \tag{C.17}$$

i.e.,

$$\cos\theta = \frac{k^2 + k'^2 - q^2}{2kk'} \tag{C.18}$$

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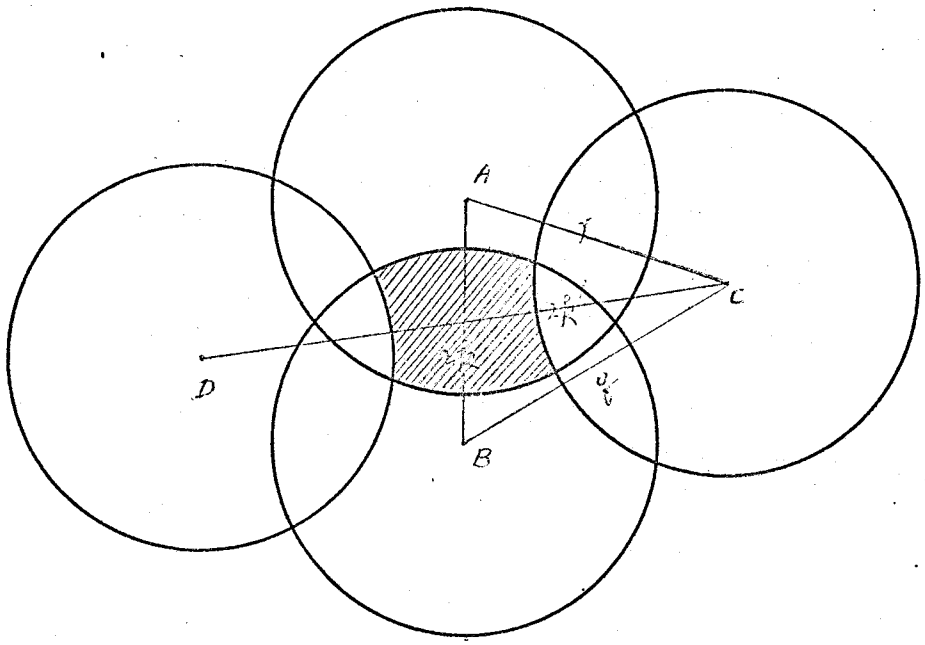

FIGURE 1
Four Intersecting Identical Spheres

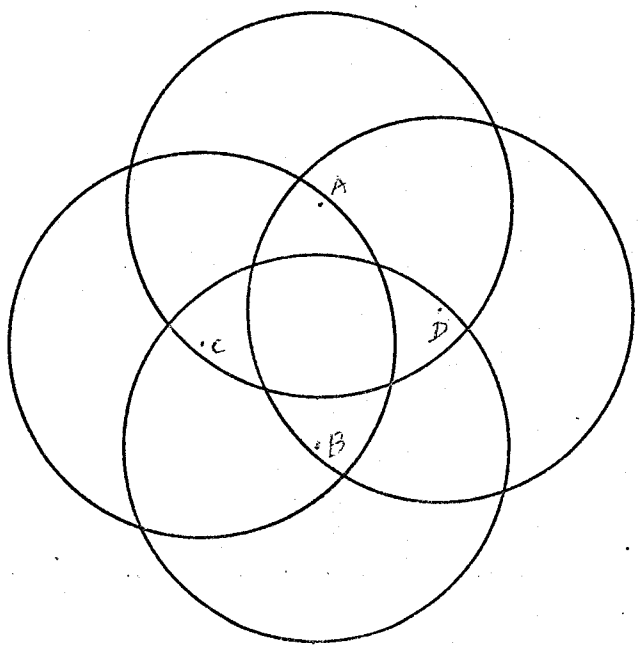

FIGURE 2
Intersection Volume in Region (1) V=0

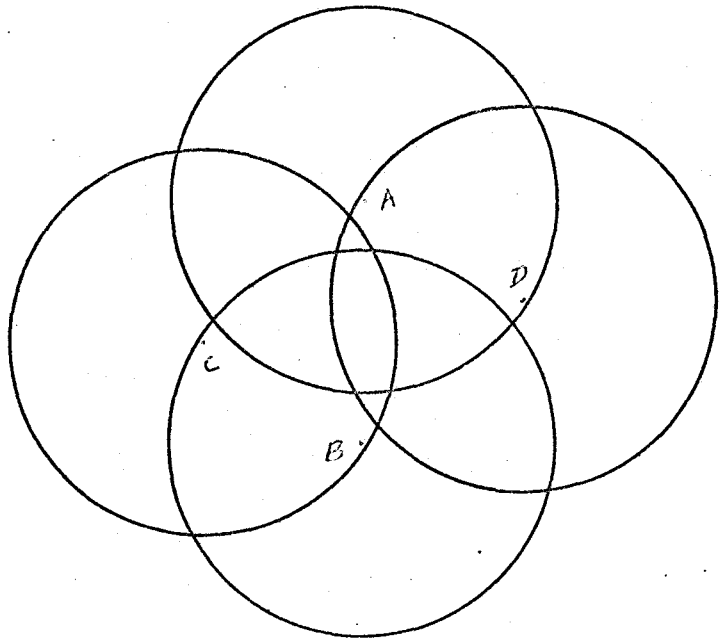

FIGURE 3

Intersection Volume in Region (2)
V=VAB-2VABC+2VACD-VCD

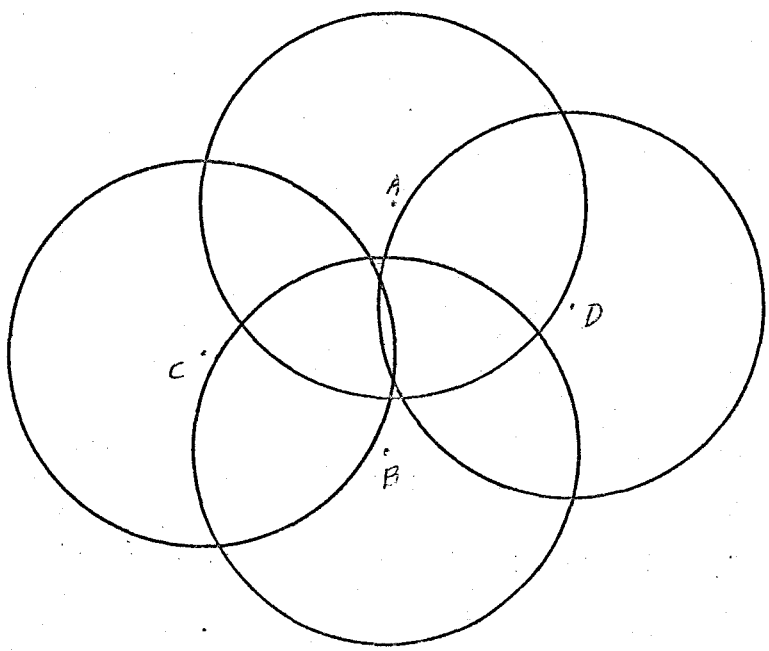

FIGURE 4

Intersection Volume in Region (3)
V=VAB-2VABC+VCD

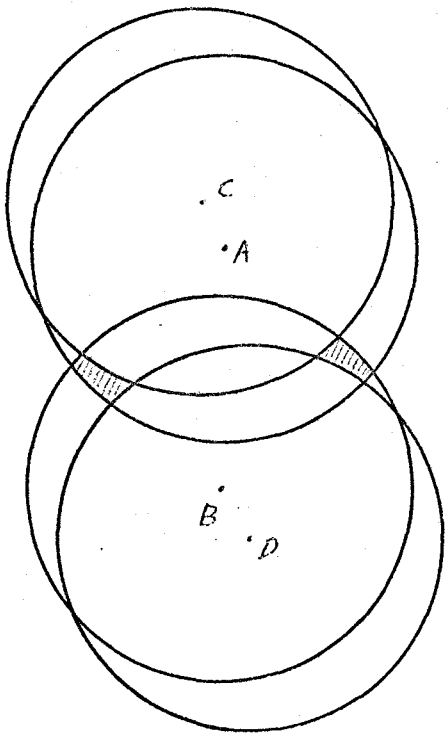

FIGURE 5
Intersection Volume in Region (4)R
V=VAB-2VBC+VCD

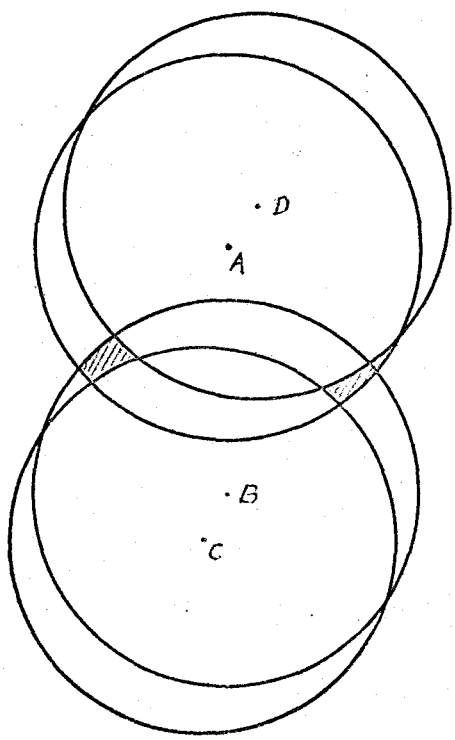

FIGURE 6
Intersection Volume in Region (4)L
V=VAB-2VAC+VCD

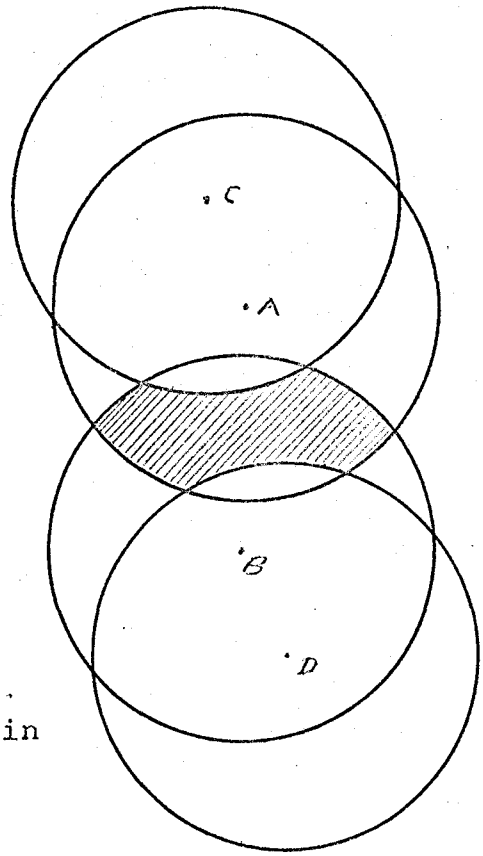

FIGURE 7

Intersection Volume in Region (5)R
V=VAB-2VBC

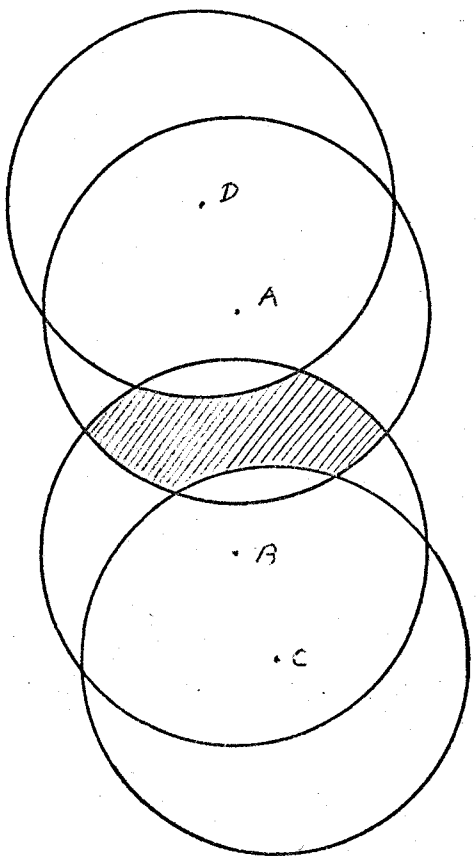

FIGURE 8

Intersection Volume in Region (5)L
V=VAB-2VAC

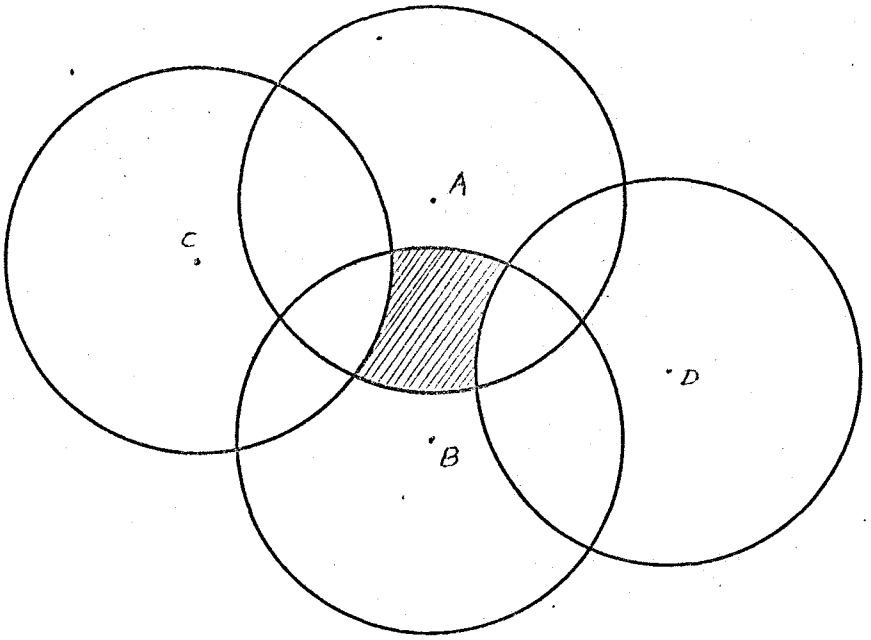

FIGURE 9
Intersection Volume in Region (6)
V=VAB-2VABC

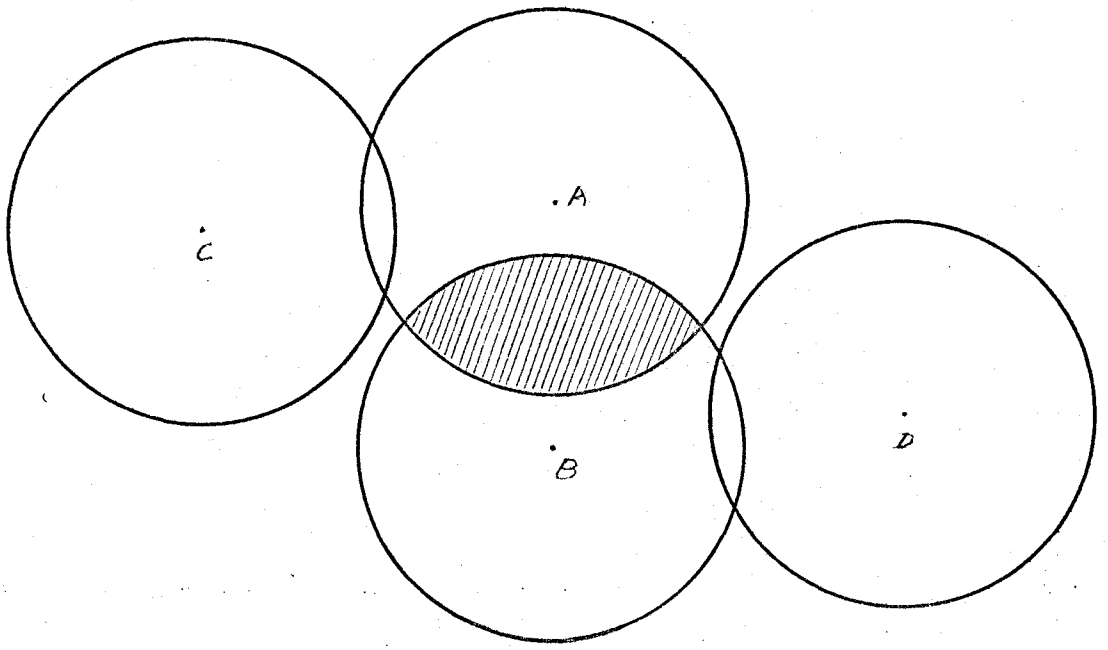

FIGURE 10
Intersection Volume in Region (7)
V=VAB

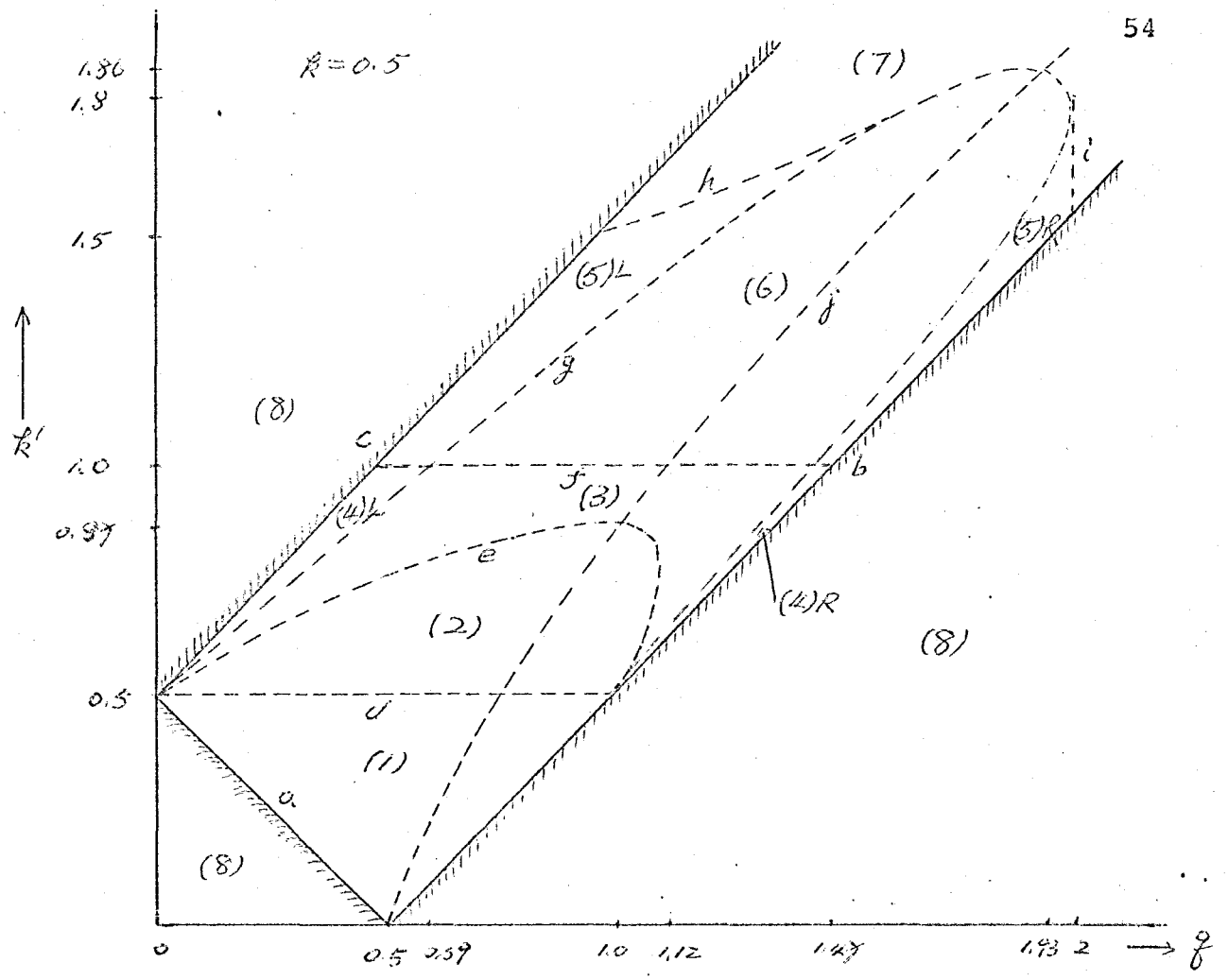

FIGURE 11 Relation Between (k,k',q) and Different Regions

a $\Rightarrow$ $k'=k-q$
b $\Rightarrow$ $k'=q-k$
c $\Rightarrow$ $k'=k+q$
d $\Rightarrow$ $k'=k$
e $\Rightarrow$ $k'^4(1+2q^2)-k'^2(2k^2+2q^2-2k^2q^2+q^4)+(k^2-q^2)^2=0$
f $\Rightarrow$ $k'=1$
g $\Rightarrow$ $k'^2=k^2+q^2(1-k^2)\pm qk\sqrt{1-k^2}\sqrt{4-q^2}$
h $\Rightarrow$ $k'^2=q^2/2-k^2+2$
i $\Rightarrow$ $q=2$
j $\Rightarrow$ $k'^2=q^2-k^2$

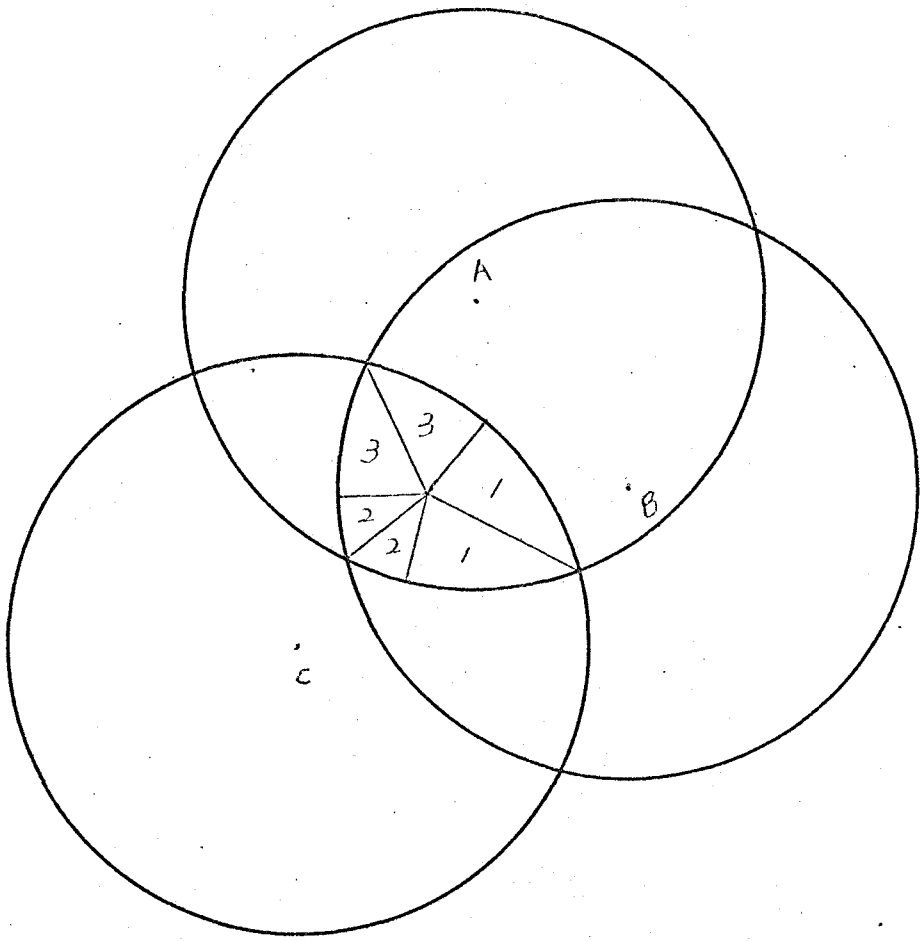

FIGURE 12
Three Intersecting Identical Spheres

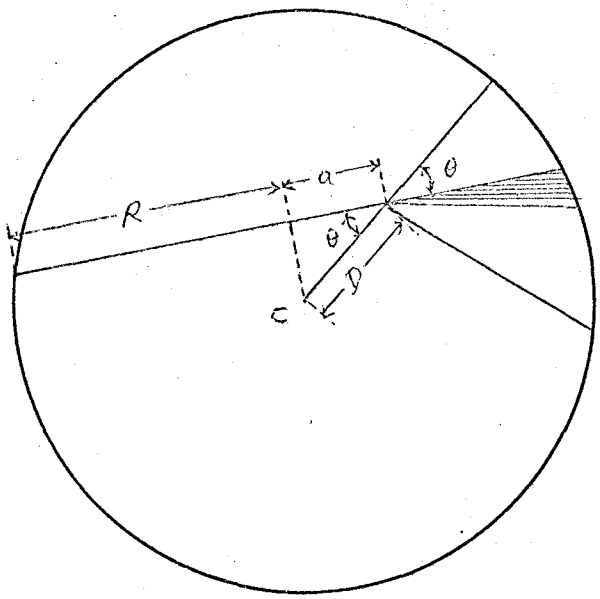

FIGURE 13

A Part of the Intersection Volume
Between Three Identical Spheres

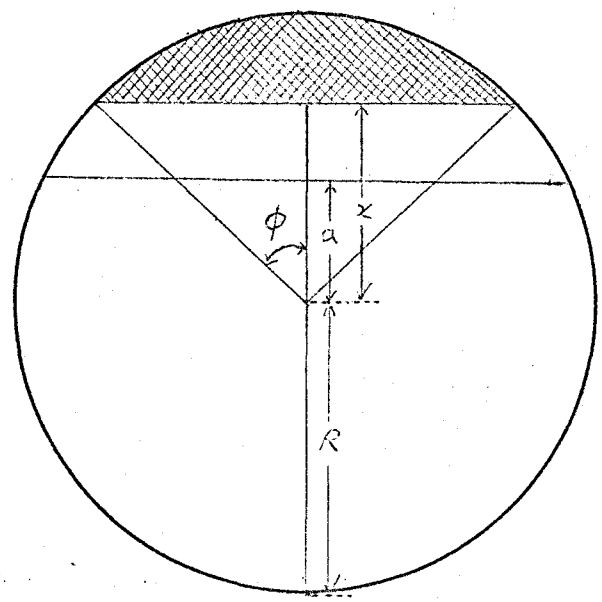

FIGURE 14

A Section of the Part of Fig. 13

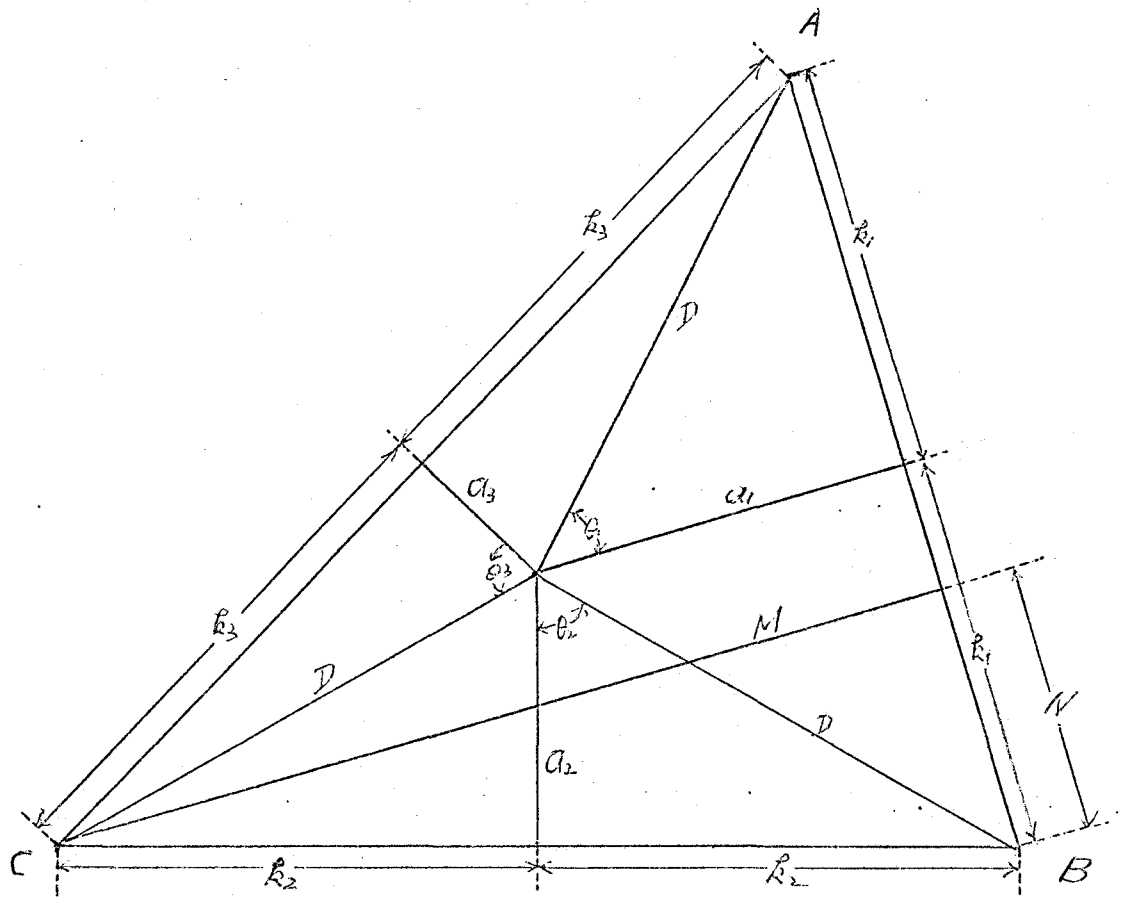

FIGURE 15

Various Quantities in the Three
Intersecting Spheres