MODELING CONCURRENCY WITH INTERVAL TRACES

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By

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I dedicate this thesis to my mother Yirong Guo, my father Xuelong Yin and my wife Jing Sun.

Abstract

When system runs are modeled with interval orders, *interval order structures* are useful tools to model abstract concurrent histories, i.e. sets of equivalent system runs. For the general cases, *Mazurkiewicz traces* allow a representation of the entire partial order by a single sequence with independency relations, and *Comtraces* allow a representation of stratified order structures by single step sequences with appropriate simultaneity and serializability relations. Unfortunately, both of them are unable to clearly describe the abstract interval order semantics of inhibitor nets.

The goal of the thesis is to provide a monoid based model called *Interval Traces* that would allow a single sequence of beginnings and endings to represent the entire stratified order structures as well as all equivalent interval order observations. And the thesis will also show how interval order structures can be modelled by interval traces and how interval traces can be used to describe interval order semantics.

Key words: Mazurkiewicz traces, Comtraces, interval traces, stratified order structures, interval order structures, inhibitor nets.

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Chapter 1

Introduction and Motivation

1.1 Problem Statement

In concurrent systems, most observational semantics are defined either in terms of sequences (i.e. total orders) or step-sequences (i.e. stratified orders). When considering concurrent histories by *causality relations*, i.e. *partial orders*, Mazurkiewicz traces [Diekert and Rozenberg [1995]; Mazurkiewicz [1977, 1995]] allow a representation of the entire partial order by a single sequence (plus *independency* relation), which provides a simple and elegant connection between observational and process semantics of concurrent systems. And *Comtraces* [Janicki and Koutny [1995]] allow a representation of stratified order structures by single step-sequences (with appropriate *simultaneity* and *serializability* relations). Other relevant observations can be derived as just stratified or interval extensions of appropriate partial orders.

However when modelling both causality and "not later than" relationship, we have to use *stratified order structures* [Diekert and Rozenberg [1995]; Mazurkiewicz [1977]], when *all* observations are step-sequences, or *interval order structures* [Lamport [1986]; Janicki and Koutny [1997]], when *all* observations are interval orders.

It was argued by [Wiener [1914]] (and later more formally in [Janicki and Koutny [1993]]) that any execution that can be observed by a single observer must be an interval order, which implies that the most precise observational semantics is defined in terms of interval orders. However, generating interval orders directly is problematic for most models of concurrency. Unfortunately, the only feasible sequence representation of interval order is by using sequences of *beginnings* and *endings* of events involved [Fishburn [1970]; Janicki and Koutny [1993]].

The goal of this thesis is to provide a monoid based model that would allow a single sequence of beginnings and endings (enriched with appropriate *simultaneity* and *serializability* relations) to represent the entire *stratified order structures* as well as all equivalent interval order observations. This will be done by introducing the concept of *interval traces*, a mixture of ideas from both Mazurkiewicz traces [Diekert and Rozenberg [1995]] and comtraces [Janicki and Koutny [1995]], and proving that each interval trace uniquely determines an interval order structure. We also will show how interval traces can define interval order semantics of inhibitor nets. For details regarding order structures models of concurrency and more adequate references the reader is referenced to [Janicki [2008]; Janicki et al. [2010]; Kleijn and Koutny [2008]].

1.2 Contributions

The main contributions of our research are in the following aspects:

1. We introduce the concept of *interval traces*, an extension of both Mazurkiewicz Traces and Comtraces.

2. We discuss properties of *interval traces* and analyze its benefits compared with Mazurkiewicz Traces and Comtraces.

3. We use *interval traces* as a tool for expressing a process semantics when observations are specified using interval orders.

4. We show each *interval trace* uniquely determins an interval order structure.

5. We also show how *interval traces* can define interval order semantics of inhibitor nets.

You could also refer to our published paper [Janicki and Yin [2012]] and submitted journal paper [Janicki et al. [2014]] for details.

1.3 Organization of the Thesis

The remainder of this thesis is organized as follows:

Chapters 2 to 5 deal with the related concepts and previous work. Chapter 2 gives some mathematical preliminaries on partial orders, sequences. In Chapter 3, we talk about stratified order structures and interval order structures. Then, in Chapter 4, Mazurkiewicz traces and comtraces are briefly discussed. Finally, we introduce the background of Petri nets and inhibitor Petri nets in Chapter 5.

Chapter 6 provides a formal introduction to interval traces.

Chapters 7 and 8 constitute the heart of the thesis. In Chapter 7, the properties of interval traces are introduced by analyzing their relationships with interval orders, interval order structures, as well as Comtraces; in Chapter 8, the applications of interval traces are discussed. And we will give a reasonable solution about how to use interval traces to effectively represent the abstract interval order semantics of inhibitor nets.

Chapter 9 makes a summary of our research and concludes our thesis with some final comments.

Chapter 2

Mathmatical Foundations

In this chapter, we would like to introduce some well-known mathematical concepts, notations and results that will be used frequently in the rest of thesis [Burris and Sankappanavar [1981]; Fishburn [1985]].

2.1 Partial Orders

Partial orders are one of the basic tools used in this thesis, which will be used as a full representation of systems runs (or observations) and as a partial representation of concurrent histories.

Definition 1 *A relation $<\subseteq X \times X$ is a (*strict*)* partial order *if it is irreflexive and transitive, i.e. for all $a,c,b \in X$, $a \not< a$ and $a<b<c \implies a<c$. We also define:*

$$a \frown_{<} b \overset{df}{\iff} \neg(a<b) \wedge \neg(b<a) \wedge a \neq b,$$

$$a <^{\frown} b \overset{df}{\iff} a<b \vee a \frown_{<} b.$$

Note that $a \frown_{<} b$ means a and b are incomparable *(w.r.t. $<$) elements of X.* $\square$

Let $<$ be a partial order on a set X. Then:

1. $<$ is *total* if $\frown_< = \emptyset$. In other words, for all $a,b \in X$, $a < b \ \vee \ b < a \ \vee \ a = b$. For clarity, we will reserve the symbol $\lhd$ to denote total orders;

2. $<$ is *stratified* if $a \frown_< b \frown_< c \implies a \frown_< c \vee a = c$, i.e., the relation $\frown_< \cup id_X$ is an equivalence relation on X;

3. $<$ is *interval* if for all $a,b,c,d \in X$, $a < c \ \wedge \ b < d \implies a < d \vee b < c$.

It is clear from these definitions that every total order is stratified and every stratified order is interval. The following simple concept will often be used in this thesis.

Definition 2 *For a relation $R \subseteq X \times X$, any relation $Q \subseteq X \times X$ is an* extension *of R if $R \subseteq Q$.* □

For convenience, we define $\mathsf{Total}(<) \stackrel{df}{=} \{\lhd \subseteq X \times X \mid \lhd \text{ is a total order and } < \subseteq \lhd\}$. In other words, the set $\mathsf{Total}(<)$ consists of all the *total order extensions* of $<$.

By Szpilrajn's Theorem [Szpilrajn [1930]], we know that every partial order $<$ is uniquely represented by the the set $\mathsf{Total}(<)$. Szpilrajn's Theorem can be stated as follows:

Theorem 1 (Szpilrajn [1930]) *For every partial order $<$,*

$$< \; = \bigcap_{\lhd \in \mathsf{Total}(<)} \lhd,$$

i.e. each partial order is the intersection of its all total extensions. □

Stratified orders are often defined in an alternative way, namely, a partial order $<$ on X is stratified if and only if there exists a total order $\lhd$ on some Y and a mapping $\phi : X \to Y$ such that $\forall x,y \in X.\ x < y \iff \phi(x) \lhd \phi(y)$. This definition is illustrated in Figure 2.1, where $\phi(a) = \{a\}$, $\phi(b) = \phi(c) = \{b,c\}$, $\phi(d) = \{d\}$. Note that for all $x,y \in \{a,b,c,d\}$ we have $x <_2 y \iff \phi(x) \lhd_2 \phi(y)$, where the total order $\lhd_2$ can be concisely represented

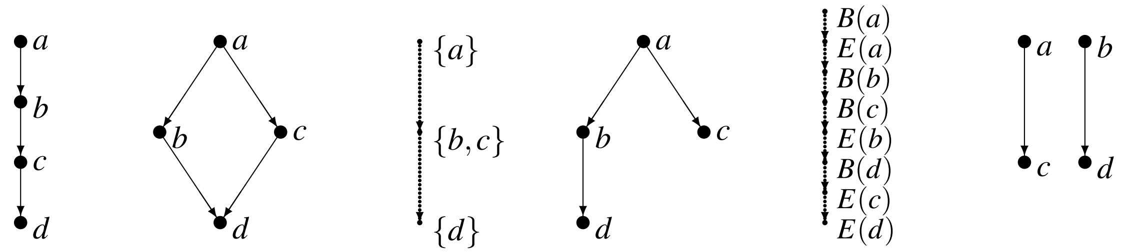


Figure 2.1: Various types of partial orders (represented as Hasse diagrams). The partial order $<_1$ is an extension of $<_2$, $<_2$ is an extension of $<_3$, and $<_3$ is an extension of $<_4$. Note that order $<_1$, being total, is uniquely represented by a sequence *abcd*, the stratified order $<_2$ is uniquely represented by a step sequence $\{a\}\{b,c\}\{d\}$, and the interval order $<_3$ is (*not* uniquely) represented by a sequence that represents $\lhd_3$, i.e. $B(a)E(a)B(b)B(c)E(b)B(d)E(c)E(d)$.

by a *step sequence* $\{a\}\{b,c\}\{d\}$. As a consequence, stratified orders and step sequences can uniquely represent each other (cf. [Janicki and Koutny [1995]; Janicki and Lê [2011]; Lê [2011]]).

For the interval orders, the name and intuition follow from Fishburn's Theorem:

Theorem 2 (Fishburn [1970]) *A partial order $<$ on X is* interval *iff there exists a total order $\lhd$ on some T and two mappings $B,E : X \to T$ such that for all $x,y \in X$,*

1. $B(x) \lhd E(x)$,

2. $x < y \iff E(x) \lhd B(y)$. □

Usually $B(x)$ is interpreted as the beginning and $E(x)$ as the end of an *interval* x.The intuition of Fishburn's theorem is illustrated in Figure 2.1 with $<_3$ and $\lhd_3$. For all $x,y \in \{a,b,c,d\}$, we have $B(x) \lhd_3 E(x)$ and $x <_3 y \iff E(x) \lhd_3 B(y)$. For better readability in the future we will skip parentheses in $B(x)$ and $E(x)$.

2.2 Sequences and Their Relationship to Partial Orders

Sequences are the most obvious and popular tool to define an observational semantics of both sequential and concurrent systems, and they can also conveniently represent finite total, stratified and interval orders.

Let Σ be a finite set (of events) and $\mathscr{P}(\Sigma)$ its power set. The elements of Σ^* are called *sequences* while the elements of $(\mathscr{P}(\Sigma) \setminus \emptyset)^*$ are called *step sequences*.

2.2.1 Enumerated Sequences

When interpreting sequences as partial orders and vice versa is a well-known and established idea, a standard notation has not been set up yet. Below we will define the notation that will be used in this thesis.

For each sequence $x \in \Sigma^*$ or each step sequence $x \in (\mathscr{P}(\Sigma) \setminus \emptyset)^*$, and each $a \in \Sigma$, let $\#_a(x)$ denotes the number of a in x. For example $\#_a(abbaa) = 3$, $\#_b(abbaa) = 2$ and $\#_c(abbaa) = 0$; $\#_a(\{a,b\}\{b,c\}\{a,b,c\}) = 2$, $\#_b(\{a,b\}\{b,c\}\{a,b,c\}) = 3$, $\#_c(\{a,b\}\{b,c\}\{a,b,c\}) = 2$ and $\#_d(\{a,b\}\{b,c\}\{a,b,c\}) = 0$.

The formal relationship between sequences and total orders and between step sequences and stratified orders can be defined as follows.

Definition 3

1. *For each set of events Σ, let $\widehat{\Sigma} = \{a^{(i)} \mid a \in \Sigma, i = 1,2,...,\infty\}$. The elements of $\widehat{\Sigma}$ are called enumerated events.*

2. *For each sequence $x \in \Sigma^*$, its enumerated representation $\widehat{x} \in \widehat{\Sigma}^*$, is defined as follows:*

   - $x = \varepsilon \implies \widehat{x} = \varepsilon$, *and* $x = a \implies \widehat{x} = a^{(1)}$,
   - $x = ya \implies \widehat{x} = \widehat{y}a^{(i)}$, *where* $i = \#_a(y) + 1$.

3. *For each step sequence* $x \in (\mathscr{P}(\Sigma) \setminus \emptyset)^*$*, its enumerated representation* $\widehat{x} \in \widehat{\Sigma}^*$*, is defined as follows:*

   - $x = \varepsilon \implies \widehat{x} = \varepsilon$*, and* $x = \{a_1, ..., a_k\} \implies \widehat{x} = \{a_1^{(1)}, ..., a_k^{(1)}\}$,
   - $x = yA \implies \widehat{x} = \widehat{y}\widehat{A}$*, where* $\widehat{A} = \{a^{(i)} \mid a \in A \wedge i = \#_a(y) + 1\}$.

4. *For each sequence* $x \in \Sigma^*$*, or step sequence* $x \in (\mathscr{P}(\Sigma) \setminus \emptyset)^*$*,* Σ_x *denotes the set of all elements of* Σ *that occur in* x*, and* $\widehat{\Sigma}_x$ *denotes the set of all enumerated events of* $\widehat{x}$*.*

5. *For each sequence* $x \in \Sigma^*$*, we define the following total order* $\lhd_x$ *on* $\widehat{\Sigma}_x$*:*

$$a^{(i)} \lhd_x b^{(j)} \iff \widehat{x} = ua^{(i)}vb^{(j)}w,$$

   where $u, v, w \in (\widehat{\Sigma}_x)^*$.

6. *For each step sequence* $x \in \Sigma^*$*, we define the following stratified order* $\lhd_x$ *on* $\widehat{\Sigma}_x$*:*

$$a^{(i)} \lhd_x b^{(j)} \iff \widehat{x} = uAvBw,$$

   where $a^{(i)} \in A \subseteq \widehat{\Sigma}_x$*,* $b^{(j)} \in B \subseteq \widehat{\Sigma}_x$ *and* $u, v, w \in (\widehat{\Sigma}_x)^*$*.* □

For example, if $x = abbaa$ then $\widehat{x} = a^{(1)}b^{(1)}b^{(2)}a^{(2)}a^{(3)}$, $\Sigma_x = \{a, b\}$ and $\widehat{\Sigma}_x = \{a^{(1)}, a^{(2)}, a^{(3)}, b^{(1)}, b^{(2)}\}$. If $x = \{a, b\}\{b, c\}\{a, b, c\}$, then $\widehat{x} = \{a^{(1)}, b^{(1)}\}\{b^{(2)}, c^{(1)}\}\{a^{(3)}, b^{(2)}, c^{(2)}\}$, $\Sigma_x = \{a, b, c\}$ and $\widehat{\Sigma}_x = \{a^{(1)}, a^{(2)}, a^{(3)}, b^{(1)}, b^{(2)}, c^{(1)}, c^{(2)}\}$.

The sequence $x = abbaa$ represents a total order:

$$\lhd_x = a^{(1)} \to b^{(1)} \to b^{(2)} \to a^{(2)} \to a^{(3)},$$

while the step sequence $x = \{a, b\}\{b, c\}\{a, b, c\}$ represents the stratified order (represented as total order of equivalence classes):

$$\lhd_x = \{a^{(1)}, b^{(1)}\} \to \{b^{(2)}, c^{(1)}\} \to \{a^{(3)}, b^{(2)}, c^{(2)}\}.$$

If $\widehat{\Sigma}_x \subseteq \{a^{(1)} \mid a \in \Sigma\}$, then we will identify x with $\widehat{x}$. More details can be found in for example in [Janicki and Koutny [1995]; Janicki et al. [2010]; Janicki and Lê [2011]].

2.2.2 Interval Sequences

We will now formally show how sequences can represent interval orders. Fishburn total order representation (Theorem 2) will be used.

For a given Σ, let $\mathscr{E}_\Sigma = \{Ba \mid a \in \Sigma\} \cup \{Ea \mid a \in \Sigma\}$, or just $\mathscr{E}$, be the set of all beginnings and ends of events in Σ.

Let $\mathscr{D} \subseteq \mathscr{E}$ and let $s \in \mathscr{E}^*$. We define the projection of s onto $\mathscr{D}$ standardly as:

$$\pi_{\mathscr{D}}(\varepsilon) \stackrel{df}{=} \varepsilon, \qquad \pi_{\mathscr{D}}(s\alpha) \stackrel{df}{=} \begin{cases} \pi_{\mathscr{D}}(s)\alpha & \text{if } \alpha \in \mathscr{D}, \\ \pi_{\mathscr{D}}(s) & \text{if } \alpha \notin \mathscr{D}. \end{cases}$$

For example $\pi_{\{Ba,Ea\}}(BbBaEbBaEaEc) = BaBaEa$, $\pi_{\{Ba,Ea,Bc,Ec\}}(BbBaEbBaEaEc) = BaBaEaEc$.

Definition 4

1. *A string $x \in \mathscr{E}^*$ is an interval sequence iff* $\quad \forall a \in \Sigma.\ \pi_{\{Ba,Ea\}}(x) \in (BaEa)^*$.

   We use $\mathsf{InSeq}(\mathscr{E}^*)$ *to denote the set of all interval sequences of* $\mathscr{E}^*$.

2. *For every $x \in \mathscr{E}_\Sigma^*$, we define $\widehat{\mathscr{E}_x^\Sigma} \subseteq \widehat{\Sigma}^*$ as follows*

   $$\widehat{\mathscr{E}_x^\Sigma} = \{a^{(i)} \mid Ba^{(i)} \in \widehat{\mathscr{E}_x}\} \cup \{a^{(i)} \mid Ea^{(i)} \in \widehat{\mathscr{E}_x}\},$$

3. *Let $x \in \mathsf{InSeq}(\mathscr{E}_\Sigma^*)$, and let $\blacktriangleleft_x$ be a relation on $\widehat{\mathscr{E}_x^\Sigma}$, defined by*

   $$a^{(i)} \blacktriangleleft_x b^{(j)} \iff Ea^{(i)} \lhd_x Bb^{(j)}.$$

   By Theorem 2, the relation $\blacktriangleleft_x$ is an interval order, and it will be called the interval order defined by the sequence x of beginnings and ends. $\square$

Note that if $x \in \mathsf{InSeq}(\mathscr{E}^*)$, then $\widehat{\mathscr{E}}_x^{\Sigma} = \{a^{(i)} \mid Ba^{(i)} \in \widehat{\mathscr{E}}_x\} = \{a^{(i)} \mid Ea^{(i)} \in \widehat{\mathscr{E}}_x\}$. For example $BaBbEbEaBcBaBbEcEbEaBaEa$ is in $\mathsf{InSeq}(\mathscr{E}_{\Sigma}^*)$ for $\Sigma = \{a,b,c\}$, but $EaBbEbBa$ or $BbEbBaEc$ are not. For $x = BaBbEbEaBcBaBbEcEbEaBaEa$, we have $\widehat{\mathscr{E}}_x^{\{a,b,c\}} = \{a^{(1)},a^{(2)},a^{(3)},b^{(1)},b^{(2)},c^{(1)}\}$. For $x = BaEaBbBcEbBdEcEd$, the interval order $\blacktriangleleft_x$ is the same as $<_3$ of Figure 2.1 with $a^{(1)},b^{(1)},c^{(1)}$ and $d^{(1)}$ represented by a,b,c and d, and for $y = BaEaBbBaEbBbEaEb$, the interval order $\blacktriangleleft_y$ is also the same as $<_3$ of Figure 2.1 with $a^{(1)}$ represented by a, $b^{(1)}$ represented by b, $a^{(2)}$ by c, and $b^{(2)}$ by d.

Chapter 3

Order Structures

In this chapter, we will give an overview of stratified order structures and interval order structures.

3.1 Stratified Order Structures

From section 2.1, we have known that partial orders can adequately model 'earlier-later' relationship. However, to model 'not later than' relationship we need more sophisticated tools. The stratified order structures and interval order structures are presented in the following two sections.

Definition 5 (Gaifman and Pratt [1987]; Janicki and Koutny [1991]) *A* stratified order structure *is a relational structure $S = (X, \prec, \sqsubset)$, such that for all $a, b, c \in X$:*

S1: $a \not\sqsubset a$

S2: $a \prec b \implies a \sqsubset b$

S3: $a \sqsubset b \sqsubset c \wedge a \neq c \implies a \sqsubset c$

S4: $a \sqsubset b \prec c \vee a \prec b \sqsubset c \implies a \prec c$.

The relation $\prec$ is called causality *while $\sqsubset$ is called* weak causality. $\square$

Stratified order structures were independently introduced in [Gaifman and Pratt [1987]] and [Janicki and Koutny [1991]]. Their comprehensive theory has been presented in [Janicki [2008]; Janicki and Koutny [1997]; Janicki and Lê [2011]; Kleijn and Koutny [2008]].

In this model the *causality* relation $\prec$ represents the "earlier than" relationship, and the *weak causality* relation $\sqsubset$ represents the "not later than". The relation $\prec$ is always a partial order, while the relation $\sqsubset$ may be not. Moreover, if $<$ is a stratified order on X, then $(X, <, <^{\frown})$ is a stratified order structure, i.e. stratified orders can be interpreted as simple instances of stratified order structures.

Definition 6 (Janicki and Koutny [1997])

1. *A stratified order $<$ on X is an* stratified extension *of a stratified order structure $S = (X, \prec, \sqsubset)$ if $\prec \subseteq <$ and $\sqsubset \subseteq <^{\frown}$, i.e. if $<$ is an extension of $\prec$ and $<^{\frown}$ is an extension of $\sqsubset$.*

2. *The set of all interval extensions of S will be denoted by* $\mathsf{Strat}(S)$. □

Theorem 1 states that each partial order is uniquely represented by its set of total extensions, and there is a similar relationship between stratified order structures and stratified orders.

Theorem 3 (Janicki and Koutny [1997]) *For each stratified order structure $S = (X, \prec, \sqsubset)$, we have*

$$S = \Big(X, \bigcap_{<\in \mathsf{Strat}(S)} <, \bigcap_{<\in \mathsf{Strat}(S)} <^{\frown}\Big),$$

i.e. S is entirely defined by the set of its all extensions. □

The above theorem is a generalization of Szpilrajn's Theorem to stratified order structures. *It is interpreted as the proof of the claim that stratified order structures uniquely* *represent sets of equivalent system runs, provided that the system's operational semantics can be fully described in terms of stratified orders* (see [Janicki [2008]; Janicki and Koutny [1997]; Janicki et al. [2010]; Janicki and Lê [2011]; Kleijn and Koutny [2008]] for details).

The relational structure S from Figure 3.2 is a simple example of a stratified order structure (it is also an interval order structure discussed later) with $\mathsf{Strat}(S) = \{<_1, <_2, <_3, <_4\}$.

3.2 Interval Order Structures

Interval order structures provide a more general formalism for analysis of concurrent systems than partial orders and stratified order structures, as discussed in [Janicki [2008]; Janicki and Koutny [1997]]. They can model concurrent behaviours which cannot be modelled by stratified order structures.

Definition 7 (Janicki and Koutny [1991]; Lamport [1986]) *An* interval order structure *is a relational structure $S = (X, \prec, \sqsubset)$, such that for all $a, b, c, d \in X$:*

I1: $a \not\sqsubset a$

I2: $a \prec b \implies a \sqsubset b$

I3: $a \prec b \prec c \implies a \prec c$

I4: $a \prec b \sqsubset c \vee a \sqsubset b \prec c \implies a \sqsubset c$

I5: $a \prec b \sqsubset c \prec d \implies a \prec d$

I6: $a \sqsubset b \prec c \sqsubset d \implies a \sqsubset d \vee a = d.$

The relation $\prec$ is called causality *and the relation $\sqsubset$ is called* weak causality. □

Interval order structures were introduced in [Lamport [1986]][1] and rediscovered independently in [Janicki and Koutny [1991]]. Some of their properties have been presented in

[1] In a slightly different but equivalent form, with a different interpretation of the relation $\sqsubset$, and initially without Axiom I6, which was added later. Evolution of the definition from [Lamport [1986]] is discussed in [Abraham et al. [1990]]. Definition 7 is a little bit modified version of that from [Janicki and Koutny [1991]]. See also [Janicki and Koutny [1997]].

[Janicki and Koutny [1997]], yet their theory is not as well-developed and much less often applied than for instance simpler stratified order structures (c.f. [Janicki [2008]; Janicki and Koutny [1995]; Kleijn and Koutny [2004]; Lê [2011]]), not to mention just plain partial orders.

In this model the *causality* relation $\prec$ represents the "earlier than" relationship, and the *weak causality* relation $\sqsubset$ represents the "not later than" relationship *but under the assumption that the system runs are interval orders*. The relation $\prec$ is always a partial order, while the relation $\sqsubset$ may not. The main interpretational difference between interval order structures and stratified order structures is that for the latter it is assumed that the systems runs are modeled with stratified orders.

From Definition 7 we can get immediately that $\prec$ is a partial order, and if $<$ is an interval order on X, then $(X,<,<^{\frown})$ is an interval order structure, i.e. interval orders can be interpreted as simple instances of interval order structures.

Definition 8 (Janicki and Koutny [1997])

1. *An interval order $<$ on X is an* interval extension *of an interval order structure $S = (X,\prec,\sqsubset)$ if $\prec \subseteq <$ and $\sqsubset \subseteq <^{\frown}$, i.e. if $<$ is an extension of $\prec$ and $<^{\frown}$ is an extension of $\sqsubset$.*

2. *The set of all interval extensions of S will be denoted by* $\mathsf{Interv}(S)$. $\square$

Similarly as for partial orders, every stratified order structure is also an interval order structure.

Proposition 1 (Janicki and Koutny [1997])

1. *Every stratified order structure S is also an interval order structure.*

2. *For every stratified order structure S,* $\mathsf{Strat}(S) \subseteq \mathsf{Interv}(S)$. $\square$

Theorem 1 states that each partial order is uniquely represented by its set of total extensions, Theorem 3 states that each stratified order structure is uniquely represented by its set of stratified extensions, we have the similar relationship between interval order structures and interval orders.

Theorem 4 (Janicki and Koutny [1997]) *For each interval order structure $S = (X, \prec, \sqsubset)$, we have*

$$S = \Big(X, \bigcap_{<\in \mathsf{Interv}(S)} <, \bigcap_{<\in \mathsf{Interv}(S)} <^{\frown}\Big),$$

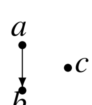
i.e. S is entirely defined by the set of its all extensions. □

The above theorem is a generalization of Szpilrajn's Theorem to interval order structures. It is interpreted as the proof of the claim that interval order structures uniquely represent sets of equivalent system runs, provided that the system's operational semantics can be fully described in terms of interval orders (see [Janicki [2008]; Janicki and Koutny [1997]] for details).

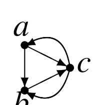
An example of a simple interval order structure which illustrates the main ideas behind this concept is shown in Figure 3.2. The orders $<_1$ and $<_2$ are total, $<_3$ and $<_4$ are stratified and $<_5$ is interval but not stratified.

In the present case $\prec$ equals $<_5$, as there are not so many partial orders over the four elements set, but the interpretations of $<_5$ and $\prec$ are different. The incomparability in $<_5$ is interpreted as *simultaneity* while in $\prec$ as *having no causal relationship*.

The interval order structure from Figure 3.2 is also a stratified order structure. The relational structure $S_0 = (\{a,b,c\}, \prec_0, \sqsubset_0)$, where the relations $\prec_0$ and $\sqsubset_0$ are described below



$\prec_0$



$\sqsubset_0$

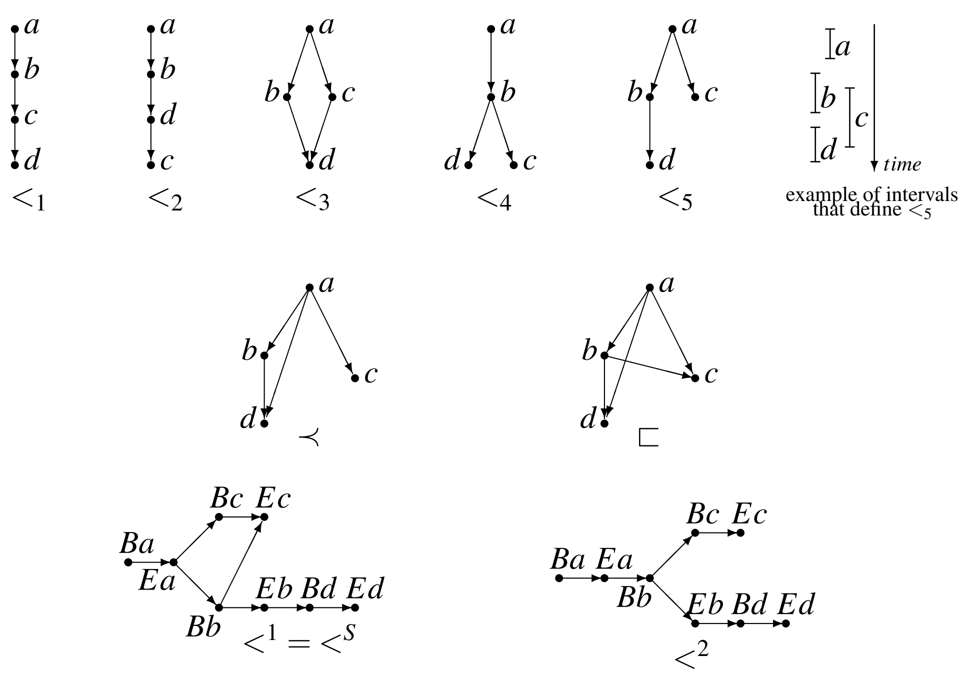


Figure 3.2: An example of a simple interval order structure $S = (X, \prec, \sqsubset)$, with $X = \{a,b,c,d\}$. Its set of all interval extensions $\mathsf{Interv}(S)$ equals to $\{<_1, <_2, <_3, <_4, <_5\}$. Partial orders $<^1$ and $<^2$ (in the form of Hasse Diagrams) represent the interval order structure S via Theorem 5. The partial order $<^1$ is $<^S$, the minimal partial order for S that satisfies Theorem 5. *The relational structure S is also a stratified order structure* with the set of all stratified extensions $\mathsf{Strat}(S) = \{<_1, <_2, <_3, <_4\}$.

is an interval order structure, but not a stratified order structure. We have here $a \prec_0 b \sqsubset_0 c$ but $\neg(a \prec_0 c)$, so the axiom $S4$ of Definition 5 is not satisfied. Note that $\mathsf{Interv}(S_0)$ contains only one element, an interval order which is equal to $\prec_0$, but again the interpretations are different.

It turns out that every interval order structure can be represented by an appropriate partial order of beginnings and ends. We will later use this relationship to construct a monoid model of interval order structures.

Theorem 5 (Abraham, Ben-David, Magidor Abraham et al. [1990]) *A triple $S = (X, \prec, \sqsubset)$ is an interval order structure if and only if there exists a partial order $<$ on some Y and two mappings $B, E : X \to Y$ such that $B(X) \cap E(X) = \emptyset$ and for each $x, y \in X$:*

1. $B(x) < E(x)$,

2. $x \prec y \iff E(x) < B(y)$,

3. $x \sqsubset y \iff B(x) < E(y)$. □

Theorem 5 can be seen as a generalization of Theorem 2 (Fisburn's Theorem) from interval orders to interval order structures.

The partial order in Theorem 5 is not unique (see Figure 3.2), but the least partial order that satisfies Theorem 5 clearly does exist. We will denote it by $<^S$. Moreover one can show that the original construction from [Abraham et al. [1990]] is such least partial order.

Chapter 4

Mazurkiewicz Traces and Comtraces

In Chapter 1, we have mentioned that, *interval traces*, one of the main contributions of our research, stemmed from *Mazurkiewicz traces* and *Comtraces*. Now, it is a good time to give a general introduction to them.

4.1 Trace Theory

Generally, trace theory was first formulated by Antoni Mazurkiewicz in the 1970s, in an attempt to evade some of the problems in the theory of concurrent computation, including the problems of interleaving and non-deterministic choice with regards to refinement in process calculi. Currently, the concept of traces is usually used to describe non-sequential behaviour of concurrent systems via its sequential observations. Traces always represent concurrent processes in the same way as strings represent sequential ones.

Originally, traces theory was motivated by theories of *Petri Nets* [Petri [1962a]] and *formal lanuages with automata*. At the beginning, the most popular approach to deal with concurrency was interleaving. In this approach, concurrency is replaced by non-determinism where concurrent execution of actions is regarded as non-deterministic choice of the order of executions of those actions.

However, later, more and more researchers found that the interleaving approach had some drawbacks concerning: 1) well-visible in the treatment of refinement, see e.g. [Bakker et al. [1989]]; 2) inevitability, see [Mazurkiewicz [1989]]; and 3) serializability of transactions [Fle and Roucairol [1982]]. Therefore, the traditional trace theory had been developed both in breath and in depth. Some of developments have still followed the initial motivation coming from concurrent systems, while other fall within the feilds such as formal language theory, theory of partially commutative monoids, ect.

The rest of this chapter, we would like to introduce two popular traces: Mazurkiewicz Traces and Comtraces.

4.2 Mazurkiewicz Traces

In this section, we define and study the semantic domain of traces which may be employed for describing concurrent systems. Mazurkiewicz traces, usually just called traces, correspond to sequences of atomic actions of concurrent systems and are employed to model executions. Concurrent systems then may be described by sets of executions, called languages in this framework.

4.2.1 Traces Semantics

The behavior of systems may be described by the actions which are executed. The nature of a sequential system is that it can only execute one action after the other. Hence, an execution of a sequential system may be described by a sequence of actions which constitutes a linear order. Given a concurrent system with a fixed notion of dependence, we will no longer expect the different actions of the execution to form a linear order but a partial order, as we pointed out in Chapter 2.

Therefore, we define an execution of a concurrent system to be a partial order.

Definition 9 *Let Σ be an alphabet. A Σ-labeled partially ordered set is a triple $(E, \leq, \lambda)$ such that*

- *$(E, \leq)$ is a partially ordered set (poset), i.e., $\leq \subseteq E \times E$ and $\leq$ is reflexive, transitive, and antisymmetric.*
- *λ is a labeling function from E to Σ which assigns to every element of E a label which is an element of Σ.*

If the alphabet Σ is clear from the context, we may omit it. The labeling function λ can be extended to subsets of E in a straightforward manner viz for C $\subseteq$ E, we define $\lambda(C)$ to denote $\{\lambda(e) \mid e \in C\}$. □

Example 1 *Let us fix the alphabet $\Sigma = \{a, b, c, d\}$ in this example.*

1. *Let $E = \{e_1, \ldots, e_7\}$, and let $\leq$ be the reflexive and transitive closure of $\{(e_i, e_j) \mid i \in \{1, \ldots, 5\}$, where $k = (i+1)$ mod 2, and $j = i+1+k$ or $j = i+2+k\}$, and let λ be defined by $e_1, e_5 \mapsto a, e_2, e_6 \mapsto b, e_3, e_7 \mapsto c$ and $e_4 \mapsto d$. Then $(E, \leq, \lambda)$ is a poset. And its Hasse diagram is shown in 4.3(a). The elements of E are written within circles and their labels are written next to them. To denote that e_i is smaller than e_j with respect to $\leq$, we write $e_i \mapsto e_j$.*

2. *Let us consider a similar example where again $E = \{e_1, \ldots, e_7\}$. Now, let $\leq$ be the reflexive and transitive closure of*
   $(e_1, e_2), (e_2, e_3), (e_3, e_4), (e_3, e_5), (e_4, e_6), (e_4, e_7), (e_5, e_7)$.
   Let λ be defined as before by $e_1, e_5 \mapsto a, e_2, e_6 \mapsto b, e_3, e_7 \mapsto c$ and $e_4 \mapsto d$. Then $(E, \leq, \lambda)$ is the poset depicted in Figure 4.3(b).

3. *Let $E = \{e_i \mid i \in \mathbb{N}\}$ and let $\leq \subseteq E \times E$ be defined by $e_i \leq e_j$ iff*

- *$i, j \in \mathbb{N}\backslash\{0\}$ and j is less or equal to i with respect to the usual order over the naturals or*
- *$i = 0$.*

Let λ send each element simply to a, i.e., for all $i \in \mathbb{N}$, let $\lambda(e_i) = a$. Then $(E, \leq, \lambda)$ is a labeled partially ordered set and its Hasse diagram is indicated in Figure 4.3(c).

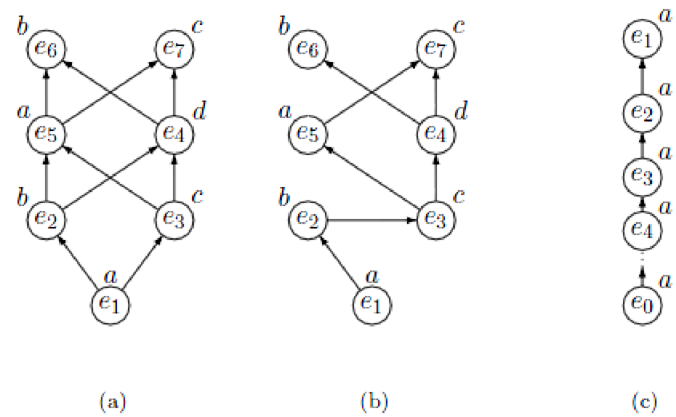


Figure 4.3: Labeled partial orders

For describing executions, arbitrary partial orders are too general. Let us come back to Example 1 . We would like to interpret the elements of the poset as events of the system under consideration. A label of an event is interpreted as the action corresponding to the event. In other words, the actions executed by an underlying system are represented by unique events with corresponding action labels. We call an event e with label a also an a-event.

It is reasonable to assume that our concurrent system has an initial state and, as time proceeds, executes actions. Therefore, we require a poset representing a run to have minimal elements, which denote starting points. Let us look at Figure 4.3(c), the depicted poset has a unique starting point, the event e_0. However, before event e_4 can occur, an infinite number of events $e_5, e_6, \dots$ has to occur, i.e., infinitely many actions have to be executed before. Assuming that every action takes a fixed amount of time, the event e_4 describes a part of the behavior of a system after an infinite amount of time. Since we do not want to deal with these situations, it is natural to require that every event of a run is preceded only by a finite number of events. So Figure 4.3(c) does not represent an execution.

Furthermore, a poset representing an execution of a system should respect its given fixed dependence relation over the actions. Let us assume that we have the independence relation $I = \{(a,d),(d,a),(b,c),(c,b)\}$. We will not consider the poset shown in Figure 4.3(b) to be a run of our system for two reasons: First, the events e_2 and e_3 are ordered although their corresponding actions (their labels) are independent with respect to I. Second, the events e_5 and e_6 are not ordered although their actions are dependent with respect to I. So Figure 4.3(b) does not represent an execution.

We will limit the kind of partial orders we are considering by the items mentioned before and will gain the notion of Mazurkiewicz traces. But let us introduce some definitions before:

Definition 10 *Let $(E, \leq, \lambda)$ be a poset where E is countable.*

- *For $e \in E$, we define $\downarrow e = \{x \in E \mid x \leq e\}$ and $\uparrow e = \{x \in E \mid e \leq x\}$. We call $\downarrow e$ the history of the event e and $\uparrow e$ the future of the event e.*

- *Let $\lessdot$ be the covering relation given by $x \lessdot y$ iff $x \leq y, x \neq y$, and for all $z \in E, x \leq z \leq y$ implies $x = z$ or $z = y$.*

- *Moreover, let the concurrency relation be defined as x co y iff $x \not\leq y$ and $y \not\leq x$.*

□

Now we are ready to give the fundamental definition of Mazurkiewicz trace.

Definition 11 *A Mazurkiewicz trace over the independence alphabet* (Σ, I) *is a* Σ*-labeled poset* $T = (E, \leq, \lambda)$ *satisfying:*

- $\downarrow e$ *is a finite set for each* $e \in E$.
- *For every* $e, e' \in E, e \lessdot e'$ *implies* $\lambda(e) D \lambda(e')$*, where* D *is called dependence relation.*
- *For every* $e, e' \in E$*,* $\lambda(e) D \lambda(e')$ *implies* $e \leq e'$ *or* $e' \leq e$.

$\square$

Example 2 *Let us consider the independence alphabet* (Σ, I) *with a set of actions* $\Sigma = \{a, b, c, d\}$ *and the independence relation* $I = \{(a,d), (d,a), (b,c), (c,b)\}$*. Then, the poset shown in Figure 4.3(a) is a trace while the one shown in Figure 4.3(b) violates (T2) and (T3). The poset shown in Figure 4.3(c) does not satisfy (T1).*

4.2.2 Mazurkiewicz Traces as Monoids

Mazurkiewicz traces rely on the concept of monoids which is a convenient tool for dealing with classes of equivalent sequences.

A triple $(X, *, \mathbb{1})$, where X is a set, $*$ is a total binary operation on X, and $\mathbb{1} \in X$, is called a *monoid* [Burris and Sankappanavar [1981]], if for all $a, b, c \in X$, $(a * b) * c = a * (b * c)$, $a * \mathbb{1} = \mathbb{1} * a = a$, and $a * b \in X$.

A nonempty equivalence relation $\sim \subseteq X \times X$ is a *congruence* in the monoid $(X, *, \mathbb{1})$ if for all $a_1, a_2, b_1, b_2 \in X$, $a_1 \sim b_1 \wedge a_2 \sim b_2 \Rightarrow (a_1 * a_2) \sim (b_1 * b_2)$. Traditionally, $[a]_\sim$ (or just $[a]$) will denote an equivalence class containing a.

The triple $(X/\sim, \circledast, [\mathbb{1}])$, where $[a] \circledast [b] = [a * b]$, is called the *quotient monoid* of $(X, *, \mathbb{1})$ under the congruence $\sim$. The symbols $*$ and $\circledast$ are often omitted if this does not lead to any discrepancy.

Let $M = (X, *, \mathbb{1})$ be a *monoid* and let $EQ = \{\, x_i = y_i \mid x_i, y_i \in X,\ i = 1, \ldots, n \,\}$ be a finite set of *equations*. Define $\equiv_{EQ}$ (or just $\equiv$) to be the *least congruence* on M satisfying, $x_i = y_i \implies x_i \equiv_{EQ} y_i$, for each equation $x_i = y_i \in EQ$. We call the relation $\equiv_{EQ}$ the *congruence defined by EQ*, or *EQ-congruence*.

The *quotient monoid* $M_{\equiv_{EQ}} = (X/{\equiv_{EQ}}, \circledast, [\mathbb{1}])$, where $[x] \circledast [y] = [x * y]$, is called an *equational monoid* (see [Janicki and Lê [2011]; Lê [2011]; Ochmański [1995]] for more details).

Monoids of *Mazurkiewicz traces* (cf. [Diekert and Rozenberg [1995]; Mazurkiewicz [1977]]) are *equational monoids over sequences*. The theory of traces has been utilized to tackle problems from quite diverse areas including combinatorics, graph theory, algebra, logic and, especially concurrency theory [Diekert and Rozenberg [1995]; Mazurkiewicz [1977]]. Applications of traces in concurrency theory are originated from the fact that traces are *sequence representation of partial orders*, which gives traces the ability to model "true concurrency" semantics. We will now recall the definition of a *trace monoid*.

Definition 12 (Diekert and Rozenberg [1995]; Mazurkiewicz [1977]) *Let $M = (\Sigma^*, *, \lambda)$ be the* free monoid *generated by Σ, and let the relation $ind \subseteq \Sigma \times \Sigma$ be an irreflexive and symmetric relation (called* independency*), and $EQ = \{ab = ba \mid (a,b) \in ind\}$. Let $\equiv_{ind}$, called* trace congruence*, be the congruence defined by EQ. Then the equational monoid $M_{\equiv_{ind}} = \big(\Sigma^*/{\equiv_{ind}}, \circledast, [\lambda]\big)$ is a monoid of* traces. *The pair (Σ, ind) is called a* trace alphabet. $\square$

The following folklore result (see for example [Janicki and Lê [2011]] for a proof) allows us to define the congruence $\equiv_{ind}$ explicitly.

Proposition 2 *For every monoid of traces the congruence $\equiv_{ind}$ can be defined explicitly as the reflexive and transitive closure of the relation $\approx$, i.e. $\equiv = (\approx \bigcup \approx^{-1})^*$, where*

$\approx \subseteq \Sigma^* \times \Sigma^*$, *and*

$$x \approx y \iff \exists\, x_1, x_2 \in \Sigma^*.\ \exists\, (u = v) \in EQ.\ \ x = x_1 * u * x_2 \ \wedge\ y = x_1 * v * x_2.$$

Proof Define $\doteq = \approx \bigcup \approx^{-1}$. Clearly, $(\doteq)^*$ is an equivalence relation. Let $x_1 \equiv y_1$ and $x_2 \equiv y_2$. This means $x_1(\doteq)^k y_1$ and $x_2(\doteq)^l y_2$ for some $k, l \geq 0$. Hence, $x_1 * x_2(\doteq)^k y_1 * x_2(\doteq)^l y_1 * y_2$. Thus, $\equiv$ is a congruence. Let $\sim$ be a congruence that satisfies $(u = w) \in EQ \Rightarrow u \sim w$ for each $u = w$ from EQ. Then, clearly, $x \doteq y \Rightarrow x \sim y$. Hence, $x \equiv y \Leftrightarrow x(\doteq)^m y \Rightarrow x \sim^m y \Rightarrow x \sim y$. Thus, $\equiv$ is the least. □

We will omit the subscripts *ind* and $\equiv_{ind}$ from trace congruence if it causes no ambiguity, and often write $[x]_{ind}$, or just $[x]$, instead of $[x]_{\equiv_{ind}}$.

Example 3 *Let* $\Sigma = \{a, b, c\}$*,* $ind = \{(b, c), (c, b)\}$*, i.e.,* $EQ = \{\ bc = cb\ \}$*. Given three sequences* $s = abcbca$*,* $s_1 = abc$ *and* $s_2 = bca$*, we can generate the traces* $[s] = \{abcbca, abccba, acbbca, acbcba, abbcca, accbba\}$*,* $[s_1] = \{abc, acb\}$ *and* $[s_2] = \{bca, cba\}$*. Note that* $[s] = [s_1] \circledast [s_2]$ *since* $[abcbca] = [abc] \circledast [bca] = [abc * bca]$*.* □

Note for each trace $[x]$ its *set of all enumerated events* can be defined as $\widehat{\Sigma}_{[s]} = \widehat{\Sigma}_s$. For the trace $[s]$ from Example 3, we have $\widehat{\Sigma}_{[s]} = \{a^{(1)}, b^{(1)}, c^{(1)}, b^{(2)}, c^{(2)}, a^{(2)}\}$.

Definition 13 *For every trace* $[x]$*, let* $\lessdot_{[x]} \subseteq \widehat{\Sigma}_{[s]} \times \widehat{\Sigma}_{[s]}$ *be a partial order defined as:*

$$\lessdot_{[x]} = \bigcap_{t \in [x]} \lhd_t.$$

The partial order $\lessdot_{[x]}$ *is called generated by the trace* $[x]$*.* □

Theorem 6 (Follows form [Mazurkiewicz [1977, 1995]) *, also Theorem 6.31 in [Janicki et al. [2010]]] For every trace* $[x]$*,* $\mathsf{Total}(\lessdot_{[x]}) = \{\lhd_t \mid t \in [x]\}$*.* □

The partial order defined by the trace $[s]$ from Example 3 is presented in Figure 4.4. By Theorems 1 and 6 each trace $[x]$ *uniquely determines* the partial order $\lessdot_{[s]}$ (that corresponds to *occurrence graph* from [Mazurkiewicz [1995]]), and *vice versa*.

$$\begin{tikzcd}
 & b^{(1)} \arrow[r] & b^{(2)} \arrow[dr] & \\
a^{(1)} \arrow[ur] \arrow[dr] & & & a^{(2)} \\
 & c^{(1)} \arrow[r] & c^{(2)} \arrow[ur] &
\end{tikzcd}$$

Figure 4.4: The partial order $\lessdot_{[s]}$defined by the trace $[s]$ where $s = abcbca$ and $ind = \{(b,c),(c,b)\}$, i.e., $EQ = \{\ bc = cb\ \}$.

4.3 Comtraces

Comtraces are equational monoids that can model 'not later than' relationship in the same way as traces can model causal relationship.

Definition 14 (Janicki and Koutny [1995])

1. *Let Σ be a finite set, $ser \subseteq sim \subset \Sigma \times \Sigma$ be two relations called* serialisability *and* simultaneity *respectively. The triple (Σ, sim, ser) is called* comtrace alphabet. *We assume that sim is irreflexive and symmetric.*

2. *We define $\mathbb{S}$, the set of all (potential)* steps*, as the set of all cliques of the graph (Σ, sim), i.e.* $\mathbb{S} = \{A \mid A \neq \emptyset \wedge (\forall a,b \in A.\ a = b \vee (a,b) \in sim)\}$.

3. *Let EQ be the set of equations defined as ('$*$' denotes concatenation of step sequences and is traditionally represented by juxtaposition):*

$$EQ = \{C = A * B \mid C = A \cup B \in \mathbb{S} \wedge A \cap B = \emptyset \wedge A \times B \subseteq ser\}.$$

Let $\equiv_{(sim,ser)}$ be the EQ-congruence defined by the above set of equations.

4. *The equational monoid $(\mathbb{S}/\equiv_{(sim,ser)}, \circledast, [\lambda])$ is called a monoid of* comtraces. $\square$

We will often write $[x]_{(sim,ser)}$, instead of $[x]_{\equiv_{(sim,ser)}}$.

The comtraces were invented to handle explicitly 'simultaneity' and 'not later than' relationships. The major innovation was to use two relations *sim* and *ser* on a given set of events Σ instead of just one.

If $(a,b) \in sim$ then a and b can be executed simultaneously, while $(a,b) \in ser$ means a and b can either be executed simultaneously, or a precedes b. When operational semantics is expressed in terms of stratified orders or step sequences, $(a,b) \in sim$ means the step $\{a,b\}$ is allowed, and $(a,b) \in ser$ means the both the step $\{a,b\}$ and the sequence $\{a\}\{b\}$ are allowed.

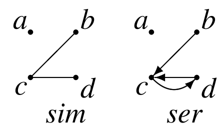
If $sim = ser$ then a comtrace can fully be represented by an appropriate trace with $ind = sim$. It can be shown that each comtrace uniquely defines a stratified order structure (in the same sense as each trace uniquely defines a partial order) that represents the same behaviour(see [Janicki et al. [2010]; Janicki and Koutny [1995]; Janicki and Lê [2011]; Kleijn and Koutny [2008]] for details and applications).

For every comtrace $\mathbf{x} = [x]_{(sim,ser)}$ over (Σ, sim, ser), the set $Strat(\mathbf{x}) = \{\lhd_t \mid t \in \mathbf{x}\}$, is the set of all stratified orders defined by the elements of $\mathbf{x}$, and let $S^{\mathbf{x}} = (\widehat{\Sigma}_{\mathbf{x}}, \prec_{\mathbf{x}}, \sqsubset_{\mathbf{x}})$, be the relational structure given by

$$\prec_{\mathbf{x}} = \bigcap_{< \in Strat(\mathbf{x})} <, \qquad \sqsubset_{\mathbf{x}} = \bigcap_{< \in Strat(\mathbf{x})} <^{\frown}.$$

Proposition 3 (Janicki and Koutny [1995]) *For every comtrace $\mathbf{x} = [x]_{(sim,ser)}$ over (Σ, sim, ser), the relational structure $S^{\mathbf{x}}$ is a* stratified order structure. $\square$

The relational structure $S^{\mathbf{x}}$ is called *stratified order structure generated by the comtrace* $\mathbf{x}$. For example if $\Sigma = \{a,b,c,d\}$, *sim* and *ser* are relations as the ones below:



and $x = \{a\}\{b,c\}\{d\}$ is a step sequence over a comtrace alphabet (Σ, sim, ser), then the set of step sequences

$$[x]_{(sim,ser)} = \{\ \{a\}\{b\}\{c\}\{d\}, \{a\}\{b\}\{d\}\{c\}, \{a\}\{b,c\}\{d\}, \{a\}\{b\}\{c,d\}\ \}$$

is the comtrace generated by the step sequence x. Note that step sequences $[x]_{(sim,ser)}$, when interpreted as stratified orders, i.e. $Strat([x]_{(sim,ser)}$, satisfy

$$Strat([x]_{(sim,ser)} = \{<_1, <_2, <_3, <_4\} = \mathsf{Strat}(S),$$

where S is exactly the stratified order structure from Figure 3.2.

Moreover, $S = S^{[x]_{(sim,ser)}}$ (c.f. [Janicki and Koutny [1995]]).

Chapter 5

Petri Nets and Inhibitor Petri Nets

Petri nets are a widely-used model for concurrency. By modelling the effect of events on local components of state, they reveal how the events of a process interact with each other, and whether they can occur independently of each other by operating on disjoint regions of state.

A Petri net is an abstract, formal model of information flow. The properties, concepts, and techniques of Petri nets are being developed for describing and analyzing the flow of information and control in systems, particularly systems related to concurrent activities. The major use of Petri nets has been the modeling of systems of events in which it is possible for events to occur concurrently with some constraints.

Inhibitor Petri nets, a special type of Petri nets, include one or more inhibitor arcs, and will be widely used in this thesis. Therefore, in this chapter, we want to give a detailed introduction to their fundamentals and properties.

5.1 Petri Nets

5.1.1 Definition of Petri Nets

Petri nets, first formally introduced in 1962 by [Petri [1962b]] is a particular kind of directed graphs comprised of three types of objects: places, transitions, and directed arcs. Directed arcs connect places to transitions or transitions to places.

In its simplest form, a Petri net can be represented by a transition together with an input place and an output place. This elementary net may be used to represent various aspects of the modeled systems. For example, a transition and its input place and output place can be used to represent a data processing event, its input data and output data, respectively, in a data processing system.

In order to study the dynamic behavior of a Petri net modeled system in terms of its states and state changes, each place may potentially hold either none or a positive number of tokens. Tokens are a primitive concept for Petri nets in addition to places and transitions. The presence or absence of a token in a place can indicate whether a condition associated with this place is true or false.

Generally, a Petri net is formally defined as a 5-tuple $N = (P, T, I, O, M_0)$, where

1. $P = \{p_1, p_2, \ldots, p_m\}$ is a finite set of places;
2. $T = \{t_1, t_2, \ldots, t_n\}$ is a finite set of transitions, $P \cup T \neq \emptyset$, and $P \cap T = \emptyset$;
3. $I : P \times T \to N$ is an input function that defines directed arcs from places to transitions, where N is a set of nonnegative integers;
4. $O : T \times P \to N$ is an output function that defines directed arcs from transitions to places; and
5. $M_0 : P \to N$ is the initial marking.

A *marking* in a Petri net is an assignment of tokens to the places. Tokens reside in the places . The number and position of tokens may change during the execution. The tokens are used to define the execution.

Most theoretical work on Petri nets is based on the formal definition of Petri net structures. However, graphical representation of a Petri net structure is much more useful for illustrating the concepts of Petri net theory. Corresponding to the definition of Petri nets, a Petri net graph has two types of nodes: A *circle* represents a place and a *bar* or a *box* represents a transition. Directed arcs (arrows) connect places and transitions, with some arcs directed from places to transitions and other arcs directed from transitions to places. An *arc* directed from a place p_j to a transition t_i defines p_j to be an *input place* of t_i, denoted by $I(t_i, p_j) = 1$. An arc directed from a transition t_i to a place p_j defines p_j to be an *output place* of t_i, denoted by $O(t_i, p_j) = 1$. If $I(t_i, p_j) = k$ or $O(t_i, p_j) = k$, then there exist k directed (parallel) arcs connecting place p_j to transition t_i (or connecting transition t_i to place p_j). Usually, in the graphical representation, parallel arcs connecting a place (transition) to a transition (place) are represented by a single directed arc labeled with its multiplicity, or weight k. A circle containing a dot represents a place containing a token [Peterson [1981]].

Example 4 *The following is a simple example of Petri net, where*

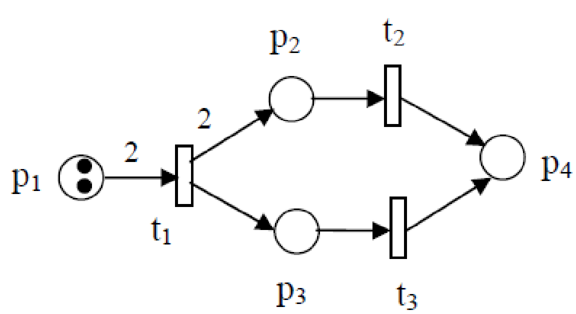


Figure 5.5: A simple Petri net

$P = \{p_1, p_2, p_3, p_4\}$;

$T = \{t_1, t_2, t_3\}$;

$I(t_1, p_1) = 2, I(t_1, p_i) = 0 \, for \; i = 2,3,4$;

$I(t_2, p_2) = 1, I(t_2, p_i) = 0 \, for \; i = 1,3,4$;

$I(t_3, p_3) = 1, I(t_3, p_i) = 0 \, for \; i = 1,2,4$;

$O(t_1, p_2) = 2, O(t_1, p_3) = 1, O(t_1, p_i) = 0 \, for \; i = 1,4$;

$O(t_2, p_4) = 1, O(t_2, p_i) = 0 \, for \; i = 1,2,3$;

$O(t_3, p_4) = 1, O(t_3, p_i) = 0 \, for \; i = 1,2,3$;

$M_0 = (2,0,0,0)$. □

5.1.2 Transition Firing

The execution of a Petri net is controlled by the number and distribution of tokens within the Petri net. A Petri net executes by *firing transitions*. By changing the distribution of tokens in places, Petri net is able to reflect the occurrence of events or execution of operations. We now introduce the enabling rule and firing rule of a transition, which govern the flow of tokens:

1. **Enabling Rule:** A transition t is said to be enabled if each input place p of t contains at least the number of tokens equal to the weight of the directed arc connecting p to t, i.e., $M(p) \geq I(t, p)$ for any $p \in P$.

2. **Firing Rule:** Only enabled transition can fire. The firing of an enabled transition t removes from each input place p the number of tokens equal to the weight of the directed arc connecting p to t. It also deposits in each output place p the number of tokens equal to the weight of the directed arc connecting t to p.

Mathematically, firing t at M yields a new marking: $M'(p) = M(p) - I(t, p) + O(t, p)$ for any p in P.

Notice that since only enabled transitions can fire, the number of tokens in each place always remains non-negative when a transition is fired. Firing transition can never try to remove a token that is not there.

A transition without any input place is called a source transition, and one without any output place is called a *sink transition*. Note that a source transition is unconditionally enabled, and that the firing of a sink transition consumes tokens, but doesn't produce tokens.

A pair of a place p and a transition t is called a *self-loop*, if p is both an input place and an output place of t. A Petri net is said to be pure if it has no self-loops.

Example 5 *The figure below shows a transition firing based on Figure 5.5.*

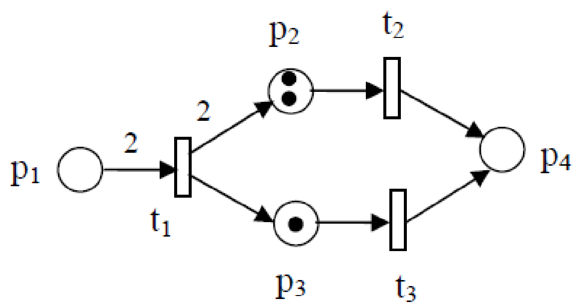


Figure 5.6: Firing of transition t_1

We can find that under the initial marking, $M_0 = (2,0,0,0)$*, only* t_1 *is enabled. Firing of* t_1 *results in a new marking, say* M_1*. It follows from the firing rule that*

$M_1 = (0,2,1,0)$.

The new token distribution of this Petri net is shown in Figure 5.6. Again, in marking M_1*, both transitions of* t_2 *and* t_3 *are enabled. If* t_2 *fires, the new marking, say* M_2*, is:*

$M_2 = (0,1,1,1)$.

If t_3 *fires, the new marking, say* M_3*, is:*

$M_3 = (0,2,0,1)$. □

5.1.3 Modeling Power

The typical characteristics exhibited by the activities in a dynamic event-driven system, such as concurrency, decision making, synchronization and priorities, can be modeled effectively by Petri nets.

1. **Sequential Execution**. In Figure 5.7(a), transition t_2 can fire only after the firing of t_1. This imposes the precedence constraint "t2 later than t1". Such precedence constraints are typical of the execution of the parts in a dynamic system.

2. **Conflict**. Transitions t_1 and t_2 are in conflict in Figure 5.7(b). Both are enabled but the firing of any transition leads to the disabling of the other transition. The resulting conflict may be resolved in a purely non-deterministic way or in a probabilistic way, by assigning appropriate probabilities to the conflicting transitions.

3. **Concurrency**. In Figure 5.7(c), the transitions t_1 and t_2 are concurrent. Concurrency is an important attribute of system interactions. Note that a necessary condition for transitions to be concurrent is the existence of a forking transition that deposits a token in two or more output places.

4. **Synchronization**. It is quite normal in a dynamic system that an event requires multiple resources. The resulting synchronization of resources can be captured by transitions of the type shown in Figure 5.7(d). Here, t_1 is enabled only when each of p_1 and p_2 receives a token. The arrival of a token into each of the two places could be the result a possibly complex sequence of operations elsewhere in the rest of the Petri net model. Essentially, transition t_1 models the joining operation.

5. **Mutually exclusive**. Two processes are mutually exclusive if they cannot be performed at the same time due to constraints on the usage of shared resources. Figure

5.7(e) shows this structure. Two such structures are parallel mutual exclusion and sequential mutual exclusion.

6. **Priorities**. The classical Petri nets discussed so far have no mechanism to represent priorities. Such a modeling power can be achieved by introducing an *inhibitor arc*. The inhibitor arc connects an input place to a transition, and is represented by an arc terminated with a small circle. The presence of an inhibitor arc connecting an input place to a transition changes the transition enabling conditions. In the presence of the inhibitor arc, a transition is regarded as enabled if each input place, connected to the transition by a normal arc (an arc terminated with an arrow), contains at least the number of tokens equal to the weight of the arc, and no tokens are present on each input place connected to the transition by the inhibitor arc. The transition firing rule is the same for normally connected places. The firing, however, does not change the marking in the inhibitor arc connected places. A Petri net with an inhibitor arc is shown in Figure 5.7(f). t_1 is enabled if p_1 contains a token, while t_2 is enabled if p_2 contains a token and p_1 has no token. This gives priority to t_1 over t_2. More details about inhibitor arcs and inhibitor Petri nets will be discussed in the later sections of this Chapter.

5.2 Properties of Petri Nets

As a mathematical tool, essentially, Petri nets have two types of properties: behavioral properties and structural properties. The behavioral properties depend on the initial state or marking of a Petri net. The structural properties, on the other hand, depend only on the net structure of a Petri net. In this section, we provide an overview of some of the most important behavioral properties. They are reachability, safeness, and liveness.

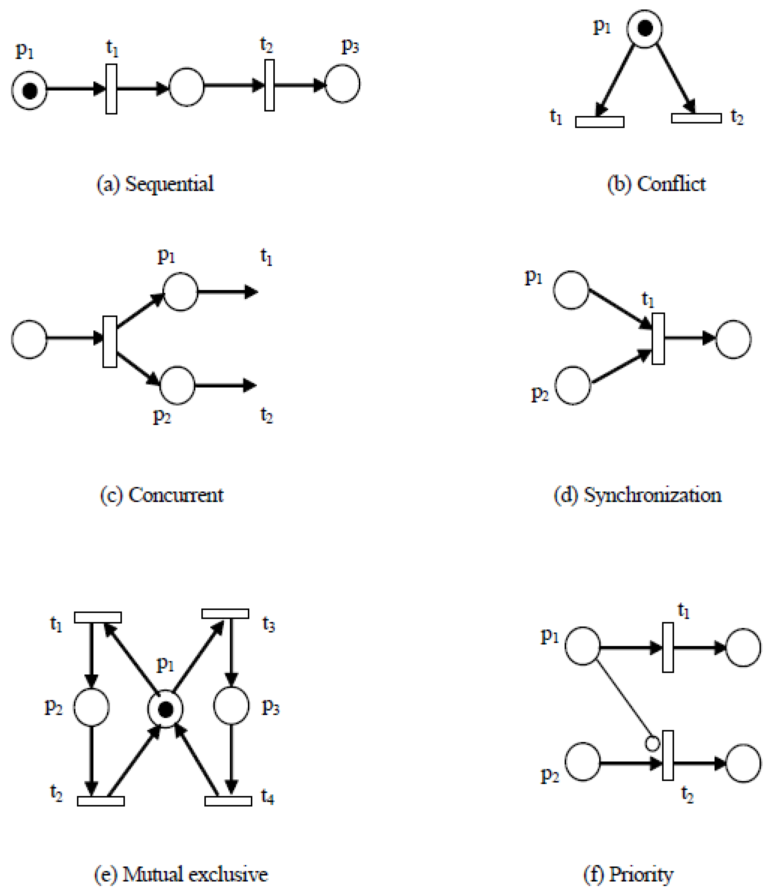


Figure 5.7: Petri net primitives to represent system features

5.2.1 Reachability

An important issue in designing event-driven systems is whether a system can reach a specific state, or exhibit a particular functional behavior. In order to see whether the modeled system can reach a specific state as a result of a required functional behavior, it is necessary to find such a transition firing sequence which would transform a marking M_0 to M_i, where M_i represents the specific state, and the firing sequence represents the required functional behavior.

In general, a marking M_i is said to be *reachable* from a marking M_0 if there exists a sequence of transitions firings which transforms a marking M_0 to M_i. And a marking M_i is said to be *immediately reachable* from M_0 if firing an enabled transition in M_0 results in M_i.

5.2.2 Safeness

In a Petri net, places are often used to represent information storage areas in communication and computer systems, product and tool storage areas in manufacturing systems, etc. It is important to be able to determine whether proposed control strategies prevent from the overflows of these storage areas. The Petri net property which helps to identify the existence of overflows in the modeled system is the concept of boundedness.

A place p is said to be *k-bounded* if the number of tokens in p is always less than or equal to k ($k \geq 0$) for every marking M reachable from the initial marking M_0, i.e., $M \in R(M_0)$.

A Petri net $N = (P, T, I, O, M_0)$ is k-bounded (safe) if each place in P is k-bounded (safe).

5.2.3 Liveness

The concept of liveness is closely related to the deadlock situation, which has been situated extensively in the context of computer operating systems.

A Petri net modeling a deadlock-free system must be live. This implies that for any reachable marking M, it is ultimately possible to fire any transition in the net by progressing through some firing sequence. This requirement, however, might be too strict to represent some real systems or scenarios that exhibit deadlock-free behavior. For instance, the initialization of a system can be modeled by a transition (or a set of transitions) which fire a finite number of times. After initialization, the system may exhibit a deadlock-free behavior, although the Petri net representing this system is no longer live as specified above. For this reason, different levels of liveness for transition t and marking M_0 were defined. Please refer to [Murata [1989]] for details.

5.3 Traces and Nets

In this section, we will discuss the relation between Traces and Nets through parallel factorial scheme [Mazurkiewicz [1977]]. And we will extend its idea to inhibitor nets and interval traces in later chapters.

5.3.1 Action System

Before showing the parallel factorial scheme, we need give some explanation for the action system [Mazurkiewicz [1977]].

Definition 15 *By an action system, fixed for the rest of the section 5.3 , we shall mean a pair* (R,U) *where R is a set (of resource), and* U *is a set (of resource state values). By a state of R, we shall mean any mapping* $s : R \to U$.

□

Then we define the *scope* and *transformation* of a action as follows:

Definition 16 *Let the set of all states of R denoted by* Σ*, and let A denote the action system. Then by an action in A we mean any pair* $\langle X,r\rangle$ *where* $X \subseteq R$ *is a set of resources called the scope of* $\langle X,r\rangle$*, and r is a binary relation over* Σ*,* $r \subseteq \Sigma \times \Sigma$*, which is called the transformation of* $\langle X,r\rangle$*, such that* $(s',s'') \in r \Rightarrow \forall x \in R-X : s'(x) = s''(x)$.

□

The intended meaning of the above condition is that an action does not change any state of resource outside its scope; and hence its transformation can be defined only for its scope.

5.3.2 Parallel Factorial Scheme

The following example will discuss the relation between nets and traces, and we shall extend this idea to inhibitor nets later in Chapter 8.

Example 6 *Let* $A = (R, U)$ *be an action system with*

$R = \{x, y, z, u\}, U = \{\dots, -1, 0, 1, 2, \dots\}$.

Define the following actions:

$x \le 0 \Rightarrow \langle \{x\}, r_1 \rangle, r_1 : s'(x) \le 0, s''(x) = s'(x);$

$x > 0 \Rightarrow \langle \{x\}, r_2 \rangle, r_2 : s'(x) > 0, s''(x) = s'(x);$

$y := x \Rightarrow \langle \{x, y\}, r_3 \rangle, r_3 : s'(x) = s''(x) = s''(y);$

$x := x - 1 \Rightarrow \langle \{x\}, r_4 \rangle, r_4 : s''(x) = s'(x) - 1;$

$z := z * y \Rightarrow \langle \{z, y\}, r_5 \rangle, r_5 : s''(z) = s'(y) * s'(z), s''(y) = s'(y);$

$z := 1 \Rightarrow \langle \{z\}, r_6 \rangle, r_6 : s''(z) = 1.$

And the mapping

$\psi a = x \le 0$

$\psi b = x > 0$

$\psi c = y := x$

$\psi d = x := x - 1$

$\psi e = z := z * y$

$\psi f = z := 1$

is then an interpretation, since it preserves independency relation. System (z, ψ) *can be represented graphically as Figure 5.8 below:*

Since $Res(\{1,5,7\},\{2,5,8\}) = [f(bcde)^* a]$*, by Definition 16, we get*

$$Res_\psi(\{1,5,7\},\{2,5,8\}) = z := 1 \circ (x > 0 \circ y := x \circ x := x - 1 \circ z := z * y)^* \circ x \le 0$$

$$= \langle \{x, y, z\}, r_6 \circ (r_2 \circ r_3 \circ r_4 \circ r_5)^* \circ r_1 \rangle$$

$$= \langle \{x, y, z\}, r_7 \rangle$$

where

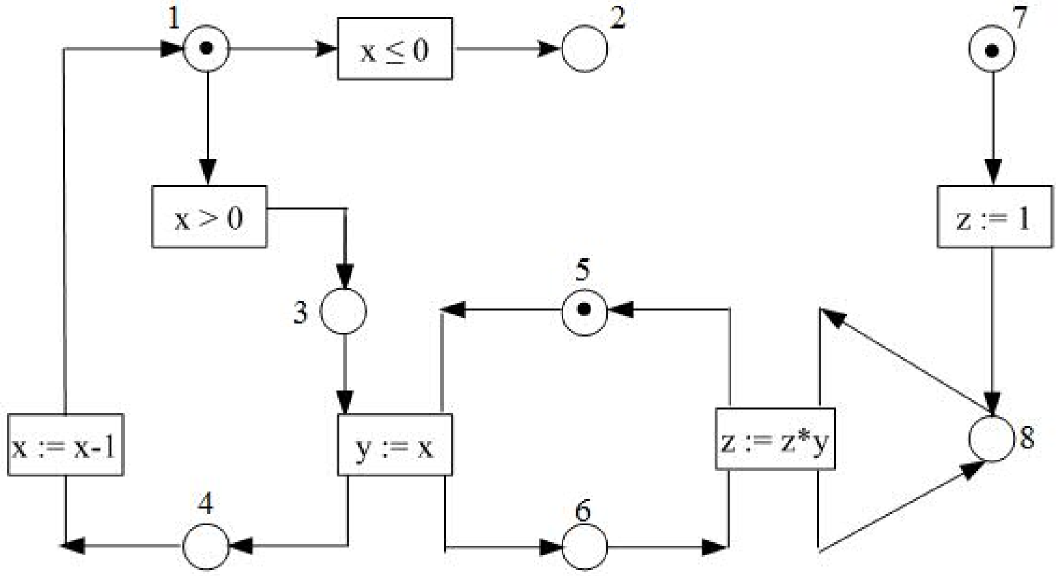


Figure 5.8: The Action System (z, ψ)

$r_7 : s'(x) > 0, s''(x) = s''(y) - 1 = 0, s''(z) = factorial(s'(x))$ *or*
$s'(x) \leq 0, s''(x) = s'(x), s''(y) = s'(y), s''(z) = 1$;

In more familiar form, we can write it as:

$Res_{\psi}(\{1,5,7\},\{2,5,8\}) =$

$(x,y,z) :=$ *if* $x > 0$ *then* $(0,1,factorial(x))$

else $(x,y,1)$.

For more details, you could refer to [Mazurkiewicz [1977]].

5.4 Inhibitor Petri Nets

5.4.1 Inhibitor Arcs

The standard execution rule for inhibitor arcs expresses that an inhibitor arc between a place *s* and an event *e*, satifies that *e* can only be fired if *s* is unmarked. Such a rule is sufficient if one is to define purely sequential (Interleaving) semantics, since a non-interleaving semantics of any kind of nets requires (explictly or implicitly) the definition of a simultaneous step of transitions.

Example 7 *The net from Figure 5.9.*
There is no problem with its interleavings, the net can only generate three valid sequences: e, f, and fe. Moreover, one can observe that the firing of e is completely independent of firing of f, while firing of f depends on the behavior of e since e may disable f by firing first.

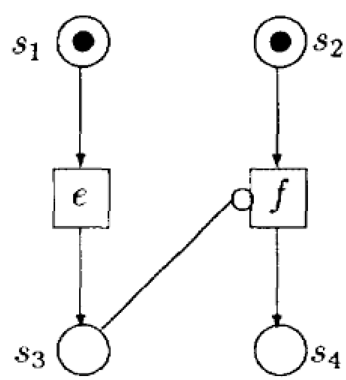


Figure 5.9: Net with Inhibitor Arcs

5.4.2 Inhibitor Nets

Inhibitor Petri nets, introduced in [Agerwala [1974]], are on one hand rather simple, and on the other hand can easily express complex and sophisticated behaviours [Janicki and Koutny [1995]; Kleijn and Koutny [2004, 2008]].

An inhibitor net is a tuple $N = (P, T, F, I, m_0)$, where P is a set of *places*, T is a set of *transitions*, P and T are disjoint, $F \subseteq (P \times T) \cup (T \times P)$ is a *flow relation*, $I \subseteq P \times T$ is a set of *inhibitor arcs* and $m_0 \subset P$ is the *initial marking*. An inhibitor arc $(p, e) \in I$ means that e can be enabled only if p is not marked. In diagrams (p, e) is indicated by an edge with a small circle at the end. Any set of places $m \subseteq P$ is called *marking*.

For every $x \in P \cup T$, the set ${}^\bullet x = \{y \mid (y, x) \in F\}$ denotes *input* nodes of x and the set $x^\bullet = \{y \mid (x, y) \in F\}$ denotes *output* nodes of x. The set $x^\circ = \{y \mid (x, y) \in I \cup I^{-1}\}$ is the set of nodes connected by an inhibitor arc to x. The dot-notation extends to sets in the natural way, e.g. the set $X^\bullet$ comprises all outputs of the nodes in X. We assume that for every $t \in T$, both ${}^\bullet t$ and $t^\bullet$ are non-empty and disjoint. Moreover, both of them must have empty intersection with t°.

Example 8 *The tuple* $N_P = (P, T, F, I, m_0)$, *with* $P = \{s_1, s_2, s_3, s_4, s_5\}$, $T = \{a, b, c\}$, $F = \{(s_1, a), (a, s_3), (s_2, c), (c, s_4), (s_3, b), (b, s_5)\}$, $I = \{(s_3, c)\}$ *and* $m_0 = \{s_1\}$ *is an inhibitor net. This is the net* N_Q *from Figure 6.10. We have here* ${}^\bullet a = \{s_1\}$, $a^\bullet = \{s_3\}$, ${}^\bullet b = \{s_3\}$, $b^\bullet = \{s_5\}$, ${}^\bullet c = \{s_2\}$, $c^\bullet = \{s_4\}$, ${}^\bullet s_1 = \emptyset$, $s_1^\bullet = \{a\}$, ${}^\bullet s_2 = \emptyset$, $s_2^\bullet = \{c\}$, ${}^\bullet s_3 = \{a\}$, $s_3^\bullet = \{c\}$, ${}^\bullet s_4 = \{c\}$, $s_4^\bullet = \emptyset$, ${}^\bullet s_5 = \{b\}$, $s_5^\bullet = \emptyset$, *and* $s_3^\circ = \{c\}$, $c^\circ = \{s_3\}$. $\square$

More information related to inhibitor Petri nets will be shown in section 8.1.

Chapter 6

Interval Traces

So far, we have introduced backgrounds and previous research relative to model observations in the concurrent systems. However, the concept "*Interval Traces*" was still not opened its mysterious veil, and one may have the following questions:

1. Mazurkiewicz Traces and Comtraces are good enough to represent observations in the concurrent systems, why do we still need introduce *Interval Traces*?

2. What is the formal definition of *Interval Traces*?

3. What are the properties of *Interval Traces*?

4. What makes *Interval Traces* "better" than other traces? (i.e., what kinds of problems *Interval Traces* can deal with which cannot be solved by other traces?)

All these questions would be answered in the following three chapters.

6.1 Concurrent Histories

In general, concurrent behaviours can be investigated at the level of individual observations as well as at the level of some structures, such as causal partial orders, stratified order structures, or interval order structures that capture the essential invariant dependencies between events and represent *complete sets of equivalent observations*.

A key link between these two levels comes from the notion of a *concurrent history* [Janicki and Koutny [1993]] which is an *invariant closed* set Δ of observations (system runs). The latter means that Δ can be derived in full from a structure built from simple invariant relationships on events Σ occurring in Δ, such as causality ($a \prec_\Delta b$ if a precedes b in *all* observations in Δ) and weak causality ($a \sqsubset_\Delta b$ if a precedes or is simultaneous with b in *all* observations in Δ).

Formally, for every set of observations (of the same set of event occurrences X) Δ, define: $\prec_\Delta = \bigcap_{<\in\Delta} <$ and $\sqsubset_\Delta = \bigcap_{<^\frown\in\Delta} <^\frown$, and $S_\Delta = (X, \prec_\Delta, \sqsubset_\Delta)$.

It was shown in [Janicki and Koutny [1993, 1997]] that

- if Δ comprises only total orders then Δ is a concurrent history if and only if $\Delta = \mathsf{Total}(\prec_\Delta)$,
- if Δ consists of only stratified orders then Δ is a concurrent history if and only if $\Delta = \mathsf{Strat}(S_\Delta)$, and
- if Δ contains interval orders then Δ is a concurrent history if and only if $\Delta = \mathsf{Interv}(S_\Delta)$.

For example $\Delta_1 = \{<_1, <_2, <_3, <_4\}$ and $\Delta_2 = \{<_1, <_2, <_3, <_4, <_5\}$, where $<_1$, $<_2$, $<_3$, $<_4$ and $<_5$ are from Figure 3.2 are concurrent histories and $S_{\Delta_1} = S_{\Delta_2} = S$ as $\mathsf{Strat}(S_{\Delta_1}) = \Delta_1$ and $\mathsf{Interv}(S_{\Delta_2}) = \Delta_2$. Moreover $\Delta_3 = \{<_1, <_2\}$ is also a concurrent history as $\mathsf{Total}(<_1 \cap <_2) = \Delta_3$. However $\Delta_4 = \{<_2, <_3, <_4\}$ is *not* a concurrent history as $S_{\Delta_4} = S$ and $\mathsf{Strat}(S_{\Delta_4}) = \Delta_1 \neq \Delta_4$.

For more details please refer to [Janicki [2008]; Janicki and Koutny [1993, 1997]; Janicki et al. [2010]].

6.2 Intuition and Motivation of the Model

When a concurrent history (i.e. a set of equivalent systems runs) can fully be represented by a partial order, trace approach allows to represent it by just one sequence. For instance a sequence *abcbca* from Example 3 from Section 4.2 (together with the relation $ind = \{(b,c),(c,b)\}$) defines uniquely the partial order from Figure 4.4. Any particular and legal system run can then be obtained as an extension of the partial order that represent the concurrent history.

If proper modeling of ‘not later than’ relationship is an issue, but possible systems runs are restricted to stratified orders, then concurrent histories can be adequately modeled by *stratified order structures* that can be uniquely represented by equational monoids called *comtraces* [Janicki and Koutny [1995]]. In this case a single step-sequence (together with appropriate simultaneity and serializability relations) uniquely defines the entire concurrent history [Janicki [2008]; Janicki et al. [2010]].

Consider the following simple program written using Dijkstra’s *cobegin*’s and *coend*’s, which is also illustrated in Figure 6.10:

```
Q: cobegin
     a :  begin worka; lock(r) end;
     b :  begin unlock(r); workb end;
     []
     c :  workc
     coend
```

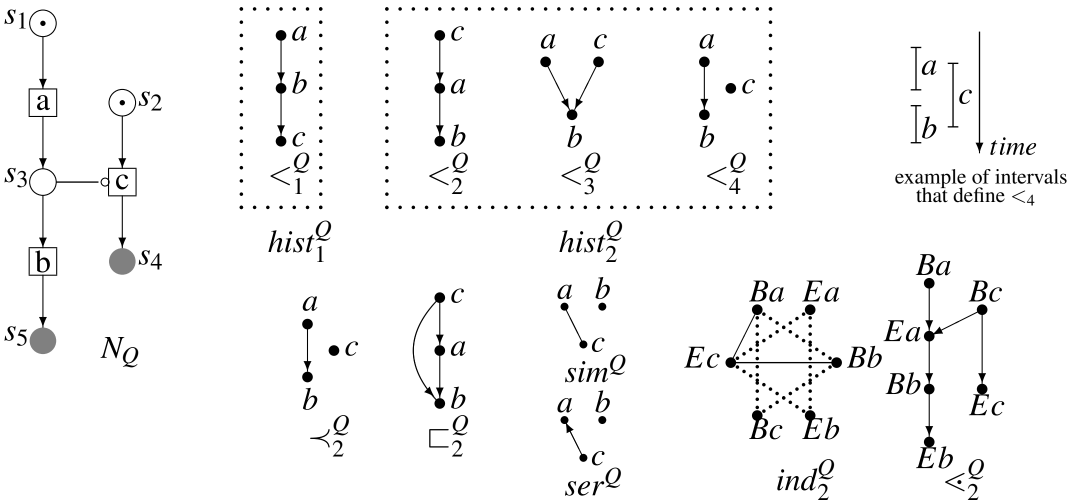


Figure 6.10: Inhibitor net representation of the program Q, two concurrent histories, $hist^Q_1$ and $hist^Q_2$, that both the program Q and the inhibitor net N_Q generate, and the interval order structure $S^Q_2 = (\{a,b,c\}, \prec^Q_2, \sqsubset^Q_2)$ that represents the history $hist^Q_2$. The partial orders in $hist^Q_1$ and $hist^Q_2$ are represented as Hasse diagrams. Also the independency relation ind^Q derived from the program Q and the net N_Q, the partial order $\lessdot^Q_2$ generated by the interval trace $[BaBcEaEbEcEb]_{ind^Q}$, and the relations sim^Q and ser^Q defined by the program Q and the net N_Q.

Assume that the subroutines a, b and c are atomic, *worka*, *workb* and *workc* require the resource r, which can be used simultaneously by any finite number of subroutines. The resource r is initially unlocked and available to use.

The program Q illustrates the difficulties of modeling 'simultaneity' and 'not later than' relationships when no restrictions on the shape of system runs is assumed.

Its inhibitor Petri net representation N_Q is given in Figure 6.10. For both the program Q and the net N_Q, all possible observations (system runs) that involve all three events a, b, c are represented by the set of partial orders $Obs(Q) = \{<^Q_1, <^Q_2, <^Q_3, <^Q_4\}$. The set $Obs(Q)$ is split into two concurrent histories $hist^Q_1$ and $hist^Q_2$, both shown in Figure 6.10. The history $hist^Q_1$ represents system runs (observations) where a occurs before c (or c is later than a), while the history $hist_2^Q$ represents observations where c is not later than a. The history $hist_1^Q$ comprises only one observation, a total order $<_1^Q$, while $hist_2^Q$ contains three observations, a total order $<_2^Q$, a stratified order $<_3^Q$ and an interval (but not stratified) order $<_4^Q$.

However, to derive the observation $<_4^Q$ is in general a non-trivial task, since the event c is executed *simultaneously* with the whole sequence ab. Various types of sequences are a principal tool in defining operational semantics, but $<_4^Q$ is not a stratified order, so it does not have a natural sequences representation. Classical semantics for inhibitor nets generate histories $\{<_1^Q\}$ and $\{<_2^Q, <_3^Q\}$ (c.f. [Janicki and Koutny [1995]]) at most, they are unable to generate $<_4^Q$. The same incompleteness of observations is typical for practically any popular model of concurrency.

The concurrent history $hist_2^Q$ is uniquely represented by the *interval order structure* $S_2^Q = (\{a,b,c\}, \prec_2^Q, \sqsubset_2^Q)$ from Figure 6.10. One can verify by inspection that the set of all interval order extensions of S_2^Q satisfies $\mathsf{Interv}(S_2^Q) = hist_2^Q = \{<_2^Q, <_3^Q, <_4^Q\}$. However, how to derive S_2^Q from either Q or N_Q is not clear (as opposed to both stratified order structures [Janicki et al. [2010]; Kleijn and Koutny [2004]] and partial orders [Mazurkiewicz [1995]; Nielsen et al. [1990]]).

A possible solution is to use Fishburn's Theorem (Theorem 2) to represent interval orders by total orders of beginnings and ends since total orders, i.e. sequences, are easily generated in virtually all formal models of concurrency.

Our goal is to provide a monoid based model that would allow any sequence of beginnings and ends[1] that represent *any* order from $\mathsf{Interv}(S_Q^2)$ to represent the entire $S_2^Q = (\{a,b,c\}, \prec_2^Q, \sqsubset_2^Q)$. For example *BcEcBaEaBbEb*, that represents $<_2$ of Figure 6.10 via Theorem 2, or *BaBcEaEcBbEb*, that represents $<_3$, or *BaBcEaBbEcEb*, that represents $<_4$, should also be able to represent the entire interval order structure S_2^Q (from Figure

[1] A method for generating such sequences of beginning and ends needs to be defined for any specific model of concurrency, we will provide such a method for inhibitor nets in Section 8.1.

6.10).

Since the beginnings and ends of events are *instantaneous* entities, they cannot be executed simultaneously, so we need only express their *independence*, by defining appropriate *independency relation ind*. Of course, for every event a, Ba and Ea cannot be independent, so always $(Ba,Ea) \notin ind$. It can be verified by inspection that ind_2^Q from Figure 6.10 is an appropriate independency relation for the beginnings and ends of the events of the program Q and the net N_Q. We will show later how such independency relations can formally be defined for inhibitor nets.

Having defined independency relation *ind* we can apply Mazurkiewicz trace approach. One can verify by inspection that $[BcEcBaEaBbEb]_{ind_2^Q}$ defines the partial order $\lessdot_2$ from Figure 6.10 and $[BaEaBbEbBcEcBdEd]_{ind^P}$ defines the partial order $<^S$ from Figure 3.2. By Theorem 5, the partial order $\lessdot_2$ defined uniquely the stratified order structure S_2^Q from Figure 6.10, and the partial order $<$ defined uniquely the stratified order structure S_P from Figure 3.2. Hence the trace $[BcEcBaEaBbEb]_{ind_2^Q}$ describes uniquely the concurrent behaviour represented by $hist_2^Q$, and the trace $[BaEaBbEbBcEcBdEd]_{ind^P}$ describes uniquely the concurrent behaviour represented by $hist^P$.

6.3 Interval Traces

By introducing Mazurkiewicz traces, Comtraces, Fishburn's Theorem and Theorem 5 (Abraham, Ben-David and Magidor), we already know that traces utilize Szpirlajn's Theorem (Theorem 1), Fishburn's Theorem allows us to represent interval orders by sequences of beginnings and ends, and Theorem 5 allows us to represent interval order structures by appropriate partial ordering of beginnings and ends.

All of them would be used to generate the formal definition of *Interval Traces* in this section.

6.3.1 Sequence Representations of Interval Orders

Suppose $<$ is a finite interval order. By Theorem 2 we know that that there is its appropriate total order representation $\prec$, which can further be represented as an appropriate sequence. However, neither Theorem 2, nor any of its known proofs (cf. [Fishburn [1970, 1985]; Janicki and Koutny [1993]]), provides an effective method of constructing *all* such total order representations.

In this section we will provide such construction based on the concept of *principal order* [Fishburn [1985]; Janicki and Koutny [1993]]. The construction will be needed to show soundness of our definition of interval traces.

Definition 17 (Fishburn [1985]; Janicki and Koutny [1993]) *Let $<$ be a partial order on X (of any kind, no restrictions).*

1. *A set $A \subseteq X$ is a* **maximal antichain** *of $<$ if and only if*

$$(\forall a,b \in A.\ a \frown_< b \vee a = b) \wedge (\forall a \notin A. \exists b \in A.\ a < b \vee b < a).$$

The set of all maximal antichains of $<$ will be denoted by $\mathscr{A}_<$.

2. *A relation $\ll_< \subseteq \mathscr{A}_< \times \mathscr{A}_<$, defined as*

$$A \ll_< B \iff A \neq B \wedge (\forall a \in A \setminus B. \forall b \in B \setminus A.\ a < b)$$

is called a **principal order** *of $<$* (see [Janicki and Koutny [1993]] for more details). $\square$

It turn out principal orders are always partial orders of maximal antichains and we can always recover the partial order $<$ from its principal order $\ll_<$.

Proposition 4 (Janicki and Koutny [1993]) *Let $<$ be any partial order on X.*

1. *The relation $\ll_<$ is a partial order.*

2. *For all $a,b \in X$:*

$$a < b \iff A \neq B \wedge (\forall A,B \in \mathscr{A}_<.\ a \in A \wedge b \in B \implies A \ll_< B). \qquad \square$$

Maximal antichains and principal orders are also convenient tools for classifying partial orders.

Corollary 1 *A partial order $<$ is stratified if and only if all maximal antichains are equivalence classes of $\frown_<$.* $\square$

Theorem 7 (Fishburn [1985]; Janicki and Koutny [1993]) *A partial order $<$ is an interval order if and only if its corresponding principal order $\ll_<$ is a total order (of maximal antichains).* $\square$

When $\ll_<$ is a total order, it can be represented as an appropriate sequence of antichains of $<$. *We will identify this sequence representation with the total order $\ll_<$ and write $\ll_< = A_1 \ldots A_n$.*

For example, for $<_3$ of Figure 2.1 and $<_4^Q$ of Figure 6.10, we have $\ll_{<_3} = \{a\}\{b,c\}\{c,d\}$ and $\ll_{<_4^Q} = \{a,c\}\{b,c\}$. Both $\ll_{<_3}$ and $\ll_{<_4^Q}$ are total orders of *appropriate maximal antichains*.

Let $<$ be an *interval order* over the set X and let $\ll_< = A_1 \ldots A_n$ be its principal order represented as a sequence of antichains, and let $\mathscr{X} = \{Ba \mid a \in X\} \cup \{Ea \mid a \in X\}$.

For each $a \in X$, we define:

$first_<(a) = A_i$ if $a \in A_i$ and either $i = 1$ or $a \notin A_{i-1}$, and

$last_<(a) = A_i$ if $a \in A_i$ and either $i = n$ or $a \notin A_{i+1}$.

For example for $<_3$ of Figure 2.1 and $<_4^Q$ of Figure 6.10, $first_{<_3}(a) = last_{<_3}(a) = \{a\}$, $first_{<_3}(c) = \{b,c\}$, $last_{<_3}(c) = \{c,d\}$, $first_{<_4^Q}(a) = last_{<_4^Q}(a) = \{a\}$, $first_{<_4^Q}(c) = \{a,c\}$, $last_{<_4^Q}(c) = \{b,c\}$.

For each A_i, we define:

$$B_<(A_i) = \{Ba \mid first_<(a) = A_i\},$$

$$E_<(A_i) = \{Ea \mid last_<(a) = A_i\}.$$

For example, $B_{<_3}(\{b,c\}) = \{Bb, Bc\}$, $E_{<_3}(\{b,c\}) = \{Eb\}$, $B_{<_3}(\{c,d\}) = \{Bd\}$, $E_{<_3}(\{c,d\}) = \{Ec, Ed\}$.

Also, for every set X, let $perm(X)$ denotes the set of all permutations of the elements of X. For example $perm(\{a,b,c\}) = \{abc, acb, bac, bca, cab, cba\}$.

We are now able to provide a constructive definition of all total representations of a given interval order.

Definition 18

1. *A set of sequences $ISR(<) \subseteq \mathscr{X}$ defined as:*

$$ISR(<) = perm(B_<(A_1))perm(E_<(A_1))\dots perm(B_<(A_n))perm(E_<(A_n))$$

is called the set of all **interval sequence representations** *of the interval order $<$.*

2. *A set of total orders $TO(<) \subseteq \mathscr{X} \times \mathscr{X}$ defined as*

$$TO(<) = \{\lhd_x \mid x \in ISR(<)\}$$

is called the set of all **total order representations** *of the interval order $<$.* □

For example for $<_3$ of Figure 2.1 and $<_4^Q$ of Figure 6.10 we have

$$IRS(<_3) = \left\{ \begin{array}{l} BaEaBbBcEbBdEcEd, BaEaBbBcEbBdEdEc, \\ BaEaBcBbEbBdEdEc, BaEaBcBbEbBdEcEd \end{array} \right\}, \text{ and}$$

$$IRS(<_4^Q) = \{BaBcEaBbEbEc, BaBcEaBbEcEb, BcBaEaBbEbEc, BcBaEaBbEcEb\}.$$

The following result justifies Definition 18.

Theorem 8 *Let X be a finite set, $<$ be an interval order over X and $\prec$ be a total order over $\mathscr{X}$. The following two properties are equivalent:*

1. *for each $a \in X$, $Ba \prec Ea$ and for all $a, b \in X$, $a < b \iff Ea \prec Bb$,*
2. *$\prec \in TO(<)$.*

Proof (2)⇒(1) Suppose that $\prec \in TO(<)$ and let $x \in ISR(<)$ be such that $\prec = \lhd_x$. Let $a \in X$. Note that either $first_<(a) = last_<(a)$ or $first_<(a) \lll_< last_<(a)$. Assume $first_<(a) = last_<(a)$=A_i. From Definition 18(1), it follows $x = x_1 y_1\ Ba\ y_2 z_1\ Ea\ z_2 x_2$, where

$x_1 \in perm(B_<(A_1))perm(E_<(A_1)) \ldots perm(B_<(A_{i-1}))perm(E_<(A_{i-1}))$,

$y_1 Ba y_2 \in perm(B_<(A_i))$, $z_1 Ea z_2 \in perm(E_<(A_i))$, and

$x_2 \in perm(B_<(A_{i+1}))perm(E_<(A_{i+1})) \ldots perm(B_<(A_n))perm(E_<(A_n))$,

so $Ba \lhd_x Ea$, i.e. $Ba \prec Ea$.

Assume $first_<(a) \lll_< last_<(a)$, and $first_<(a) = A_i$, $last_<(a) = A_j$. From Definition 18(1), it follows $x = x_1 y_1\ Ba\ y_2 x_2 z_1\ Ea\ z_2 x_3$, where

$x_1 \in perm(B_<(A_1))perm(E_<(A_1)) \ldots perm(B_<(A_{i-1}))perm(E_<(A_{i-1}))$,

$y_1 Ba y_2 \in perm(B_<(A_i))$,

$x_2 \in perm(B_<(A_{i+1}))perm(E_<(A_{i+1})) \ldots perm(B_<(A_{j-1}))perm(E_<(A_{j-1}))$,

$z_1 Eaz_2 \in perm(E_<(A_j))$, and

$x_3 \in perm(B_<(A_{j+1}))perm(E_<(A_{j+1}))\dots perm(B_<(A_n))perm(E_<(A_n))$,

so again $Ba \lhd_x Ea$, i.e. $Ba \prec Ea$.

Now suppose $a < b$. Since $a \in last_<(a)$ and $b \in first_<(b)$, from Proposition 4(2), we have $last_<(a) \lll_< first_<(b)$. But Definition 17(2) implies that $last_<(a) \lll_< first_<(b)$ if and only if $x = \dots Ea \dots Bb \dots$, so $a < b \iff Ea \lhd_x Bb \iff Ea \prec Bb$.

(1)⇒(2) Suppose that each $a \in X$, $Ba \prec Ea$ and for all $a, b \in X$, $a < b \iff Ea \prec Bb$. Let $x \in ISR(<)$ be such that $\prec = \lhd_x$. We just have to show that $x \in ISR(<)$. Suppose $x \notin ISR(<)$. For every $y \in ISR(<)$ we can write $x = v\, x_1, y = v\, y_1$. Let y_0 be such element of $ISR(<)$ that the length of prefix v is maximal.

We have to consider four cases:

Case 1. $x = u\, Ba\, u_x$, $y_0 = u\, Bb\, u_{y_0}$. Hence $x = u\, Ba\, v_1\, Bb\, v_2$ and $y_0 = u\, Bb\, z_1\, Ba\, z_2$. Suppose $z_1 = s\, Ec\, s_1$, i.e. $y_0 = u\, Bb\, s\, Ec\, s_1\, Ba\, z_2$, which means $Ec \lhd_{y_0} Ba$ i.e. $c < a$. Since Ec does not appear in u, we also have $x = u\, Ba\, t\, Ec\, t_1$, which means $Ba \lhd_x Ec$, or $Ba \prec Ec$ i.e. $\neg(c < a)$, a contradiction. This means $z_1 = Ba_1 \dots Ba_m$, so
$y_0 = v\, BbBa_1 \dots Ba_m Ba\, z_2$. But $y_0 \in IRS(<)$, so from Definition 18(2) we have that $y_1 = vBaBbBa_1 \dots Ba_m\, z_2 \in IRS(<)$, so u is not maximal, as uBa is a prefix of both x and y_1. Therefore the Case 1 cannot happen.

Case 2. $x = u\, Ea\, u_x$, $y_0 = u\, Eb\, u_{y_0}$. Hence $x = u\, Ea\, v_1\, Eb\, v_2$ and $y_0 = u\, Eb\, z_1\, Ea\, z_2$. Suppose $z_1 = s\, Bc\, s_1$, i.e. $y_0 = u\, Eb\, s\, Bc\, s_1\, Ea\, z_2$, which means $Bc \lhd_{y_0} Ea$ i.e. $\neg(c < a)$. Since Bc does not appear in u, we also have $x = u\, Ea\, t\, Bc\, t_1$, which means $Ea \lhd_x Bc$, or $Ea \prec Bc$ i.e. $c < a$, a contradiction. This means $z_1 = Ea_1 \dots Ea_m$, so
$y_0 = v\, EbEa_1 \dots Ea_m Ea\, z_2$. But $y_0 \in IRS(<)$, so from Definition 18(2) we have that $y_1 = vEaEbEa_1 \dots Ea_m\, z_2 \in IRS(<)$, so u is not maximal, as uEa is a prefix of both x and y_1. Therefore the Case 2 also cannot happen.

Case 3. $x = u\,Ba\,u_x$, $y_0 = u\,Eb\,u_{y_0}$. Hence $x = u\,Ba\,v_1\,Eb\,v_2$, which means $Ba \lhd_x Eb$, or $Ba \prec Eb$, i.e. $\neg(b < a)$, and $y_0 = u\,Eb\,z_1\,Ba\,z_2$, which means $Eb \lhd_{y_0} Ba$. i.e. $b < a$, a contradiction, so the Case 3 is not valid.

Case 4. $x = u\,Ea\,u_x$, $y_0 = u\,Bb\,u_{y_0}$. Hence $x = u\,Ea\,v_1\,Bb\,v_2$, which means $Ea \lhd_x Bb$, or $Ea \prec Bb$, i.e. $b < a$, and $y_0 = u\,Bb\,z_1\,Ea\,z_2$, which means $Bb \lhd_{y_0} Ea$. i.e. $\neg(b < a)$, a contradiction, so the Case 4 is not valid too. □

The important fact is that the set $TO(<)$ contains *all* (up to name isomorphism) total representations of an interval order $<$, and the set $IRS(<)$ contains *all* sequence representations of $<$.

6.3.2 Constructing Interval Traces

We now have all components needed for a formal definition of Interval Traces.

Let Σ be a set of events, $\mathscr{E} = \{Ba \mid a \in \Sigma\} \cup \{Ea \mid a \in \Sigma\}$, and $\mathsf{InSeq}(\mathscr{E}^*)$ be the set of all sequences over $\mathscr{E}$ that define interval orders (see Definition 4(1)).

Definition 19

Let $ind \subseteq \mathscr{E} \times \mathscr{E}$ be a symmetric and irreflexive relation such that for all $a, b \in \Sigma$, and $a \neq b$,

1. $(Ba, Ea) \notin ind$ *and* $(Ea, Ba) \notin ind$,
2. $(Ba, Bb) \in ind$ *and* $(Ea, Eb) \in ind$.

The relation ind will be called ***interval independency****.* □

The condition (1) above follows from the fact that in any representation of any order, the beginning of an event always precede the end so that cannot commute. The condition (2) follows from the generalization of observation that the interval sequences $BaBbEaEb$,

$BbBaEaEb$, $BaBbEbEa$ and $BbBaEbEa$ represent the same fact, namely that a and b are simultaneous.

Note that $(\mathscr{E}, ind)$ is also a standard trace alphabet, so *we can apply the standard theory of Mazurkiewicz traces*. One of the problems is that not all sequences from $\mathscr{E}^*$ can be interpreted as trace elements, they have to represent interval orders, so only sequences from $\mathsf{InSeq}(\mathscr{E}^*)$ can be used.

Lemma 1 *Let $(\mathscr{E}, ind)$ be an interval trace alphabet.*

1. *For each $x, y \in \mathscr{E}^*$, if $x \in \mathsf{InSeq}(\mathscr{E}^*)$ and $y \in \mathsf{InSeq}(\mathscr{E}^*)$ then $xy \in \mathsf{InSeq}(\mathscr{E}^*)$.*
2. *For each $s \in \mathscr{E}^*$, we have: $s \in \mathsf{InSeq}(\mathscr{E}^*) \iff \forall x \in [s]_{ind}.\ x \in \mathsf{InSeq}(\mathscr{E}^*)$.*
3. *For each $x, y \in \mathscr{E}^*$,*
   if $[x]_{ind} \subseteq \mathsf{InSeq}(\mathscr{E}^*)$ and $[y]_{ind} \subseteq \mathsf{InSeq}(\mathscr{E}^*)$, then $[x]_{ind} \circledast [y]_{ind} = [xy]_{ind} \subseteq \mathsf{InSeq}(\mathscr{E}^*)$.

Proof (1) Since for each $a \in \Sigma$, $(BaEa)^*(BaEa)^* = (BaEa)^*$.
(2) ($\Leftarrow$) Obvious as $s \in [s]_{ind}$.
($\Rightarrow$) By Proposition 2 it suffices to show that if $s \approx x$ then $x \in \mathsf{InSeq}(\mathscr{E}^*)$. Let $s = x_1\alpha\beta x_2$ and $x = x_1\beta\alpha x_2$. This means $(\alpha, \beta) \in ind$. Hence if $\alpha = Ba$ then $\beta \neq Ea$ and $\beta \neq Ba$. Similarly if $\alpha = Ea$ then $\beta \neq Ba$ and $\beta \neq Ea$. Hence, if $\pi_{\{Ba,Ea\}}(s) \in (BaEa)^*$ then also $\pi_{\{Ba,Ea\}}(x) \in (BaEa)^*$.
(3) A consequence of (1) and (2). $\square$

The interval sequence representation of interval orders is not unique, but soundness of the relation $\equiv_{ind}$ requires that all such representations are equivalent, which is given by the following result.

Proposition 5 *Let $(\mathscr{E}, ind)$ be an interval trace alphabet. and $x \in \mathsf{InSeq}(\mathscr{E}^*)$. Then for each $y \in \mathsf{InSeq}(\mathscr{E}^*)$,*

$$\blacktriangleleft_x = \blacktriangleleft_y \implies x \equiv_{ind} y.$$

Proof The operator '$\widehat{\ }$' transforms $\mathscr{E}^*$ into $\widehat{\mathscr{E}}^*$. For example if $x = BaEaBaEa$ then $\widehat{x} = Ba^{(1)}Ea^{(1)}Ba^{(2)}Ea^{(2)}$. Let '$\widetilde{\ }$' be an inverse of '$\widehat{\ }$' that transforms back $\widehat{\mathscr{E}}^*$ into $\mathscr{E}^*$, i.e. if $y = Ba^{(1)}Ea^{(1)}Ba^{(2)}Ea^{(2)}$ then $\widetilde{y} = BaEaBaEa$. It could simply be defined as $x = \varepsilon \implies \widetilde{x} = \varepsilon$ and $x = ya^{(i)} \implies \widetilde{x} = \widetilde{y}a$. Now go back to the proof.

First note that $\blacktriangleleft_x = \blacktriangleleft_{\widehat{x}}$ and $\blacktriangleleft_y = \blacktriangleleft_{\widehat{y}}$. From Theorem 8 it follows that $\widehat{x} \in ISR(\blacktriangleleft_{\widehat{x}})$ and $\widehat{y} \in ISR(\blacktriangleleft_{\widehat{y}})$. Since $\blacktriangleleft_{\widehat{x}} = \blacktriangleleft_{\widehat{y}} = \blacktriangleleft_x = \blacktriangleleft_y$, we have $\widehat{x}, \widehat{y} \in ISR(\blacktriangleleft_x)$. All elements of $ISR(\blacktriangleleft_x)$ satisfy a pattern given by Definition 18(2). Assume $\ll_{\blacktriangleleft_x} = A_1 \dots A_m$. Hence $\widehat{x} = u_1 v_1 \dots u_n v_n$ and $\widehat{y} = s_1 t_1 \dots s_n t_n$, where for all $i = 1, \dots, n$, $u_i, s_i \in perm(B_{\blacktriangleleft_x}(A_i))$ and $v_i, t_i \in perm(E_{\blacktriangleleft_x}(A_i))$. Since $(Ba, Bb) \in ind$ for all $Ba, Bb \in \mathscr{E}$, we have $x = \widetilde{u_1}\widetilde{v_1} \dots \widetilde{u_n}\widetilde{v_n} \equiv_{ind} \widetilde{s_1}\widetilde{v_1} \dots \widetilde{u_n}\widetilde{v_n}$, and, since $(Ea, Eb) \in ind$ for all $Ea, Eb \in \mathscr{E}$, $\widetilde{s_1}\widetilde{v_1} \dots \widetilde{u_n}\widetilde{v_n} \equiv_{ind} \widetilde{s_1}\widetilde{t_1}\widetilde{u_2}\widetilde{v_2} \dots \widetilde{u_n}\widetilde{v_n}$. But this means $x \equiv_{ind} \widetilde{s_1}\widetilde{t_1}\widetilde{u_2}\widetilde{v_2} \dots \widetilde{u_n}\widetilde{v_n}$. Repeating this process $n-1$ times we obtain $x \equiv_{ind} \widetilde{s_1}\widetilde{t_1}\widetilde{u_2}\widetilde{v_2} \dots \widetilde{s_n}\widetilde{t_n} = y$. Hence $x \equiv_{ind} y$. □

We can now define *interval trace* as follows:

Definition 20

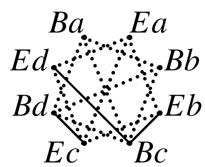
*A trace $[x]_{ind}$ over the interval trace alphabet $(\mathscr{E}, ind)$ is called an **interval trace** if $[x]_{ind} \subseteq$ $\mathsf{InSeq}(\mathscr{E}^*)$.* □

Example 9 *Let $\Sigma = \{a, b, c\}$ and ind is equal to ind_2^Q of Figure 6.10. Then*

$$\mathbf{x} = \left\{ \begin{array}{l} BcEcBaEaBbEb, BaBcEcEaBbEb, BaBcEaEcBbEb, BcBaEcEaBbEb, \\ BcBaEaEcBbEb, BaBcEaBbEbEc, BaBcEaBbEcEb, BcBaEaBbEbEc, \\ BcBaEaBbEcEb \end{array} \right\}.$$

is an interval trace, $\mathbf{x} = [x]_{ind}$ for any $x \in \mathbf{x}$, for example $[x]_{ind} = [BaBcEaBbEbEc]_{ind}$. Moreover one can show by inspection that $\{\blacktriangleleft_x \mid x \in \mathbf{x}\} = hist_2^Q$, so it models one concurrent history generated by the program Q and the net N_Q. □

Example 10 *We can also easily check that if* $\Sigma = \{a, b, c, d\}$, *ind is the relation described below (the default part of the relation ind given by Definition 19(2) is represented by dotted lines):*



and $\mathbf{y}$ *is the following set of sequences:*

$$\mathbf{y} = \left\{ \begin{array}{l} BaEaBbEbBcEcBdEd, BaEaBbEbBdEdBcEc, BaEaBbBcEbEcBdEd, \\ BaEaBcBbEbEcBdEd, BaEaBcBbEcEbBdEd, BaEaBbBcEcEbBdEd, \\ BaEaBbEbBcBdEcEd, BaEaBbEbBdBcEcEd, BaEaBbEbBdBcEdEc, \\ BaEaBbEbBcBdEdEc, BaEaBbBcEbBdEcEd, BaEaBbBcEbBdEdEc, \\ BaEaBcBbEbBdEdEc, BaEaBcBbEbBdEcEd \end{array} \right\},$$

then $\mathbf{y} = [x]_{ind}$ *for any* $x \in \mathbf{y}$, *for example* $[x]_{ind} = [BaEaBbEbBcEcBdEd]_{ind}$. □

Chapter 7

Properties of Interval Traces

In this chapter, we will foucs on exploring the properties of *Interval Traces*. And I will extend Interval Traces by discussing their relationships with *Interval Orders*, *Interval Order Structures* and *Comtraces*.

7.1 Interval Traces and Interval Orders

Since each element of every interval trace is an interval sequence, by Theorem 2, every element of the trace defines a unique interval order. However interval orders that are not total are represented by more than one sequence from the trace.

Definition 21 *For every interval trace* $\mathbf{x} = [x]_{ind}$*, let* $Interv(\mathbf{x}) = \{\blacktriangleleft_t \mid t \in \mathbf{x}\}$ *denote the set of all interval orders defined by the elements of* $\mathbf{x}$ *(see Definition 4(3) for* $\blacktriangleleft_t$*).* $\square$

For the interval trace $\mathbf{x}$ from Example 9, $Interv(\mathbf{x}) = hist_2^Q = \{<_2^Q, <_3^Q, <_4^Q\}$ from Figure 6.10, with $a^{(1)}, b^{(1)}$, and $c^{(1)}$ represented by a, b and c. In this case we have:

- $BcEcBaEaBbEb$ represents the total order $<_2^Q$,

- each of the sequences $BaBcEcEaBbEb, BaBcEaEcBbEb, BcBaEcEaBbEb$ and $BcBaEaEcBbEb$ represents the stratified order $<_3^Q$, and

- each of the sequences $BaBcEaBbEbEc, BaBcEaBbEcEb, BcBaEaBbEbEc$ and $BcBaEaBbEcEb$ represents the stratified order $<_4^Q$.

For the interval trace $\mathbf{y}$ from Example 10, $\mathit{Interv}(\mathbf{y}) = \{<_1, <_2, <_3, <_4, <_5\}$, where $<_1, <_2, <_3, <_4$ and $<_5$ are partial orders from from Figure 3.2, with $a^{(1)}, b^{(1)}, c^{(1)}$, $d^{(1)}$ represented just by a, b, c, d. In this case

- $BaEaBbEbBcEcBdEd$ represents a total order $<_1$,

- $BaEaBbEbBdEdBcEc$ represents a total order $<_2$,

- each of the sequences $BaEaBbBcEbEcBdEd, BaEaBcBbEbEcBdEd$, $BaEaBcBbEcEbBdEd$ and $BaEaBbBcEcEbBdEd$, represents a stratified order $<_3$,

- each of the sequences $BaEaBbEbBcBdEcEd, BaEaBbEbBdBcEcEd$, $BaEaBbEbBdBcEdEc$ and $BaEaBbEbBcBdEdEc$ represents a stratified order $<_4$,

- and each of the sequences $BaEaBbBcEbBdEcEd, BaEaBbBcEbBdEdEc$, $BaEaBcBbEbBdEdEc$ and $BaEaBcBbEbBdEcEd$ represents the interval order $<_5$.

7.2 Interval Order Structures and Interval Traces

We will now show the exact relationship between interval traces and interval order structures. We expect this relationship to be similar to the relationships between Mazurkiewicz traces and partial orders and between comtraces and stratified order structures.

First we recall how one can construct a partial order of beginnings and ends from an interval trace. Assume that a set of events Σ and an interval trace alphabet $(\mathscr{E}, ind)$ are given. Recall that for each sequence $x \in \mathscr{E}^*$, $\widehat{\mathscr{E}_x}$ is the set of all elements of $\hat{x}$, the enumerated version of x, $\lhd_x \subseteq \widehat{\mathscr{E}_x} \times \widehat{\mathscr{E}_x}$ is the *total* order that is equivalent to the sequence x (see Definition 3(5)), and $\lessdot_{[x]} \subseteq \widehat{\mathscr{E}_x} \times \widehat{\mathscr{E}_x}$ is the partial order that is equivalent to the trace $[x]$ (see Definition 13).

We are now ready to define an interval order structure induced by a single sequence $x \in \mathscr{E}^*$.

Definition 22 *For each $x \in \mathscr{E}^*$, let $S^x = (\widehat{\Sigma}^{\mathscr{E}}_x, \prec_x, \sqsubset_x)$, where $\widehat{\Sigma}^{\mathscr{E}}_x = \{a^{(i)} \mid Ba^{(i)} \in \widehat{\mathscr{E}_x}\} \cup \{a^{(i)} \mid Ea^{(i)} \in \widehat{\mathscr{E}_x}\}$, and $\prec_x$ and $\sqsubset_x$ are relations on $\widehat{\Sigma}^{\mathscr{E}}_x$ defined as follows, for all $a, b \in \Sigma$:*

1. $a^{(i)} \prec_x b^{(j)} \overset{df}{\iff} Ea^{(i)} \lessdot_{[x]} Bb^{(j)}$.
2. $a^{(i)} \sqsubset_x b^{(j)} \overset{df}{\iff} Ba^{(i)} \lessdot_{[x]} Eb^{(j)}$. □

The resemblance of Definition 22 to the points (2) and (3) of Theorem 5 is not a coincidence, the triple $S^x = (\widehat{\Sigma}^{\mathscr{E}}_x, \prec_x, \sqsubset_x)$, is indeed an interval order structure.

Proposition 6 *If $x \in \mathsf{InSeq}(\mathscr{E}^*)$ then $S^x = (\widehat{\Sigma}^{\mathscr{E}}_x, \prec_x, \sqsubset_x)$ is an interval order structure.*

Proof Since $x \in \mathsf{InSeq}(\mathscr{E}^*)$, the property (1) of Theorem 5 is satisfied. Definition 22 implies satisfying (2) and (3) of Theorem 5. Hence, by Theorem 5, S^x is an interval order structure. □

We will call $S^x = (\widehat{\Sigma}^{\mathscr{E}}_x, \prec_x, \sqsubset_x)$ the *interval order structure S^x induced by an interval sequence x*. We will show that S^x plays the same role in our model as a partial order derived from a single sequence plays in standard trace theory [Mazurkiewicz [1995]], or a stratified order structure derived from a single step-sequence place the theory of comtraces [Janicki and Koutny [1995]]. To do this we need to show that $x \equiv y \iff S^x = S^y$, and that the set of interval orders $\mathsf{Interv}(S^x)$ is uniquely defined by the elements of $[x]$.

We need the following two lemmas to prove one of our main results. First lemma is quite technical one, it characterizes the relationships $Ba^{(i)} \lessdot_{[x]} Bb^{(j)}$ and $Ea^{(i)} \lessdot_{[x]} Eb^{(j)}$. Because (Ba,Bb) and (Ea,Eb) are in *ind* (Definition 19(2)), these relationships are not arbitrary.

Lemma 2 *For any interval trace alphabet $(\mathscr{E},ind)$, every $x \in \mathsf{InSeq}(\mathscr{E}^*)$, and for all $a^{(i)}, b^{(j)} \in \widehat{\Sigma}_x^{\mathscr{E}}$, we have:*

1. $Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \iff (Ea^{(i)} \lessdot_{[x]} Bb^{(j)}) \vee (\exists c^{(k)} \in \widehat{\Sigma}_x^{\mathscr{E}}.\ Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)})$,

2. $Ea^{(i)} \lessdot_{[x]} Eb^{(j)} \iff (Ea^{(i)} \lessdot_{[x]} Bb^{(j)}) \vee (\exists c^{(k)} \in \widehat{\Sigma}_x^{\mathscr{E}}.\ Ea^{(i)} \lessdot_{[x]} Bc^{(k)} \lessdot_{[x]} Eb^{(j)})$.

Proof 1. ($\Leftarrow$) Since if $\exists c^{(k)} \in \widehat{\Sigma}_x^{\mathscr{E}}.\ Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)}$, obviously, we have $Ba^{(i)} \lessdot_{[x]} Bb^{(j)}$; and if $Ea^{(i)} \lessdot_{[x]} Bb^{(j)}$, by Fishburn Theorem 2.1, we also have $Ba^{(i)} \lessdot_{[x]} Bb^{(j)}$.
($\Rightarrow$) Since $x \in \mathsf{InSeq}(\mathscr{E}^*)$, we have that if $(Ba,Bb) \notin ind$ then a and b never overlap, so $Ea^{(i)} \lessdot_{[x]} Bb^{(j)}$. Suppose that $(Ba,Bb) \in ind$. This means that if there is $x_1 \in [x]$ such that $x_1 = uBaBbw$ and $\hat{x_1} = \hat{u}Ba^{(i)}Bb^{(j)}\tilde{w}$ ($\tilde{w} \neq \hat{w}$ as in $\tilde{w}$ enumeration does not start from one of the symbols that are also in $uBaBb$), then $x_2 = uBbBaw$ is also in $[x]$, so $Ba^{(i)} \lhd_{x_1} Bb^{(j)}$ and $Bb^{(j)} \lhd_{x_2} Ba^{(i)}$. Hence $\neg(Ba^{(i)} \lessdot_{[x]} Bb^{(j)})$. If $Ba^{(i)} \lessdot_{[x]} Bb^{(j)}$ then the situation described above does not happen. This means there is $\gamma \in \widehat{\mathscr{E}}_x$ such that $Ba^{(i)} \lessdot_{[x]} \gamma \lessdot_{[x]} Bb^{(j)}$. If all γ between $Ba^{(i)}$ and $Bb^{(j)}$ are of type $Bc^{(k)}$, by the same reasoning as above we conclude that $\neg(Ba^{(i)} \lessdot_{[x]} Bc^{(k)})$ and $\neg(Bc^{(k)} \lessdot_{[x]} Bb^{(j)})$. Hence at least one γ between $Ba^{(i)}$ and $Bb^{(j)}$ must be equal to $Ec^{(k)}$. If $c^{(k)} = a^{(i)}$, then we have the case $Ea^{(i)} \lessdot_{[x]} Bb^{(j)}$ again.
2. Dually, by exchanging B with E. □

The second lemma shows that the relationship between $\lessdot_{[x]}$ and S^x is a one-to-one correspondence.

Lemma 3 *For all $x,y \in \mathsf{InSeq}(\mathscr{E}^*)$, $\lessdot_{[x]} = \lessdot_{[y]}$ if and only if $S^x = S^y$.* □

The proof is quite technical, and we need use following corollary.

Definition 22 allows us to formulate Lemma 2 in an alternative way.

Corollary 2 *For any interval trace alphabet* $(\mathscr{E}, ind)$*, every* $x \in \mathsf{InSeq}(\mathscr{E}^*)$*, and for all* $a^{(i)}, b^{(j)} \in \widehat{\Sigma}_x^{\mathscr{E}}$*, we have:*

1. $Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \iff (a^{(i)} \prec_x b^{(j)}) \vee (\exists c^{(k)} \in \widehat{\Sigma}_x^{\mathscr{E}}.\ a^{(i)} \sqsubset_x c^{(k)} \wedge c^{(k)} \prec_x b^{(j)})$,
2. $Ea^{(i)} \lessdot_{[x]} Eb^{(j)} \iff (a^{(i)} \prec_x b^{(j)}) \vee (\exists c^{(k)} \in \widehat{\Sigma}_x^{\mathscr{E}}.\ a^{(i)} \prec_x c^{(k)} \wedge c^{(k)} \sqsubset_x b^{(j)})$. □

Now we are able to prove Lemma 3.

Proof ($\Rightarrow$) From Definition 22, we clearly have $S^x = S^y$.

($\Leftarrow$) To prove that $\lessdot_{[x]} = \lessdot_{[y]}$ we need to show that $\alpha \lessdot_{[x]} \beta \iff \alpha \lessdot_{[y]} \beta$ where $\alpha, \beta \in \{Ba^{(i)}, Ea^{(i)}, Bb^{(j)}, Eb^{(j)}\}$ and $a^{(i)}, b^{(j)} \in \widehat{\Sigma}_x^{\mathscr{E}}$. From Theorem 5(1) we have $Ba^{(i)} \lessdot_{[x]} Ea^{(i)}$, $Bb^{(j)} \lessdot_{[x]} Eb^{(j)}$ and $Ba^{(i)} \lessdot_{[y]} Ea^{(i)}$, $Bb^{(j)} \lessdot_{[y]} Eb^{(j)}$.

We have to consider five cases:

(*Case 1*). We have $a^{(i)} \prec_u b^{(j)}$, where $u \in \{x, y\}$. This means, by Definition 22, that $Ea^{(i)} \lessdot_{[x]} Bb^{(j)}$ and $Ea^{(i)} \lessdot_{[y]} Bb^{(j)}$. Hence, $Ba^{(i)} \lessdot_{[x]} Ea^{(i)} \lessdot_{[x]} Bb^{(j)} \lessdot_{[x]} Eb^{(j)}$ and $Ba^{(i)} \lessdot_{[y]} Ea^{(i)} \lessdot_{[y]} Bb^{(j)} \lessdot_{[y]} Eb^{(j)}$, so we indeed have $\alpha \lessdot_{[x]} \beta \iff \alpha \lessdot_{[y]} \beta$ where $\alpha, \beta \in \{Ba^{(i)}, Ea^{(i)},$
$Bb^{(j)}, Eb^{(j)}\}$.

(*Case 2*). We have $a^{(i)} \sqsubset_u b^{(j)}$ and $b^{(j)} \sqsubset_u a^{(i)}$ where $u \in \{x, y\}$. From Definition 22 and Theorem 5(1) we conclude the following: $Ba^{(i)} \lessdot_{[u]} Ea^{(i)}$, $Bb^{(j)} \lessdot_{[u]} Eb^{(j)}$, $Ba^{(i)} \lessdot_{[u]} Eb^{(j)}$, and $Bb^{(j)} \lessdot_{[u]} Ea^{(i)}$. Hence only the relationships between $Ba^{(i)}$ and $Bb^{(j)}$, and between $Ea^{(i)}$ and $Eb^{(j)}$, are not described yet. Suppose $Ba^{(i)} \lessdot_{[x]} Bb^{(j)}$ and $\neg(Ba^{(i)} \lessdot_{[y]} Bb^{(j)})$, i.e. $(Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \wedge Bb^{(j)} \lessdot_{[y]} Ba^{(i)})$ or $(Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \wedge Bb^{(j)} \frown_{\lessdot_{[y]}} Ba^{(i)})$. By Corollary 2, $Ba^{(i)} \lessdot_{[x]} Bb^{(j)}$ implies $(a^{(i)} \prec_x b^{(j)})$ or $(\exists c^{(k)} \in \widehat{\Sigma}_x^{\mathscr{E}}.\ a^{(i)} \sqsubset_x c^{(k)} \wedge c^{(k)} \prec_x b^{(j)})$. Since $\prec_x = \prec_y$ and $\sqsubset_x = \sqsubset_y$, it also implies $(a^{(i)} \prec_y b^{(j)})$ or $(\exists c^{(k)} \in \widehat{\Sigma}_x^{\mathscr{E}}.\ a^{(i)} \sqsubset_y c^{(k)} \wedge c^{(k)} \prec_y b^{(j)})$. Since

$a^{(i)} \prec_y b^{(j)}$ implies $Ea^{(i)} \lessdot_{[y]} Bb^{(j)}$, it clearly contradicts both $Bb^{(j)} \lessdot_{[y]} Ba^{(i)}$ and $Bb^{(j)} \frown_{\lessdot_{[y]}} Ba^{(i)}$. Consider now the case $\exists c^{(k)} \in \widehat{\Sigma}^{\mathscr{E}}_x.\ a^{(i)} \sqsubset_y c^{(k)} \wedge c^{(k)} \prec_y b^{(j)}$. From Proposition 6 and axioms I1, I4 of Definition 7 it follows that $a^{(i)} \sqsubset_y c^{(k)}$ implies $\neg(c^{(k)} \prec_y a^{(i)})$. But $c^{(k)} \prec_y b^{(j)}$ and $Bb^{(j)} \lessdot_{[y]} Ba^{(i)}$ implies $c^{(k)} \prec_y a^{(i)}$, a contradiction; while $c^{(k)} \prec_y b^{(j)}$ and $Bb^{(j)} \frown_{\lessdot_{[y]}} Ba^{(i)})$ implies $\neg(Ba^{(i)} \lessdot_{[y]} Ec^{(k)})$, i.e. $\neg(a^{(i)} \sqsubset_x c^{(k)})$, a contradiction again. Hence $Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \iff Ba^{(i)} \lessdot_{[y]} Bb^{(j)}$. Almost identically we can show that $Ba^{(i)} \frown_{\lessdot_{[x]}} Bb^{(j)} \iff Ba^{(i)} \frown_{\lessdot_{[y]}} Bb^{(j)}$. The proof that $Ea^{(i)} \lessdot_{[x]} Eb^{(j)} \iff Ea^{(i)} \lessdot_{[y]} Eb^{(j)}$ and $Ea^{(i)} \frown_{\lessdot_{[x]}} Eb^{(j)} \iff Ea^{(i)} \frown_{\lessdot_{[y]}} Eb^{(j)}$ is very similar. Hence $\alpha \lessdot_{[x]} \beta \iff \alpha \lessdot_{[y]} \beta$ where $\alpha, \beta \in \{Ba^{(i)}, Ea^{(i)}, Bb^{(j)}, Eb^{(j)}\}$.

(*Case 3*). We have $a^{(i)} \sqsubset_u b^{(j)}$ and $\neg(a^{(i)} \prec_u b^{(j)})$, where $u \in \{x,y\}$. From Definition 22 and Theorem 5(1) we conclude the following: $Ba^{(i)} \lessdot_{[u]} Ea^{(i)}$, $Bb^{(j)} \lessdot_{[u]} Eb^{(j)}$, $Ba^{(i)} \lessdot_{[u]} Eb^{(j)}$, and either $Bb^{(j)} \lessdot_{[u]} Ea^{(i)}$ or $Bb^{(j)} \frown_{\lessdot_{[u]}} Ea^{(i)}$. Suppose $Bb^{(j)} \lessdot_{[x]} Ea^{(i)}$ and $Bb^{(j)} \frown_{\lessdot_{[y]}} Ea^{(i)}$. But the former means $b^{(j)} \sqsubset_x a^{(i)}$, while the latter implies $\neg(b^{(j)} \sqsubset_y a^{(i)})$, a contradiction as $\sqsubset_x = \sqsubset_y$. If $b^{(j)} \sqsubset_x a^{(i)}$ then this case is reduced to Case 2. Suppose $Bb^{(j)} \frown_{\lessdot_{[u]}} Ea^{(i)}$. But now only the relationships between $Ba^{(i)}$ and $Bb^{(j)}$, and between $Ea^{(i)}$ and $Eb^{(j)}$, are not described yet. We can repeat the last part of Case 2 to show that $Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \iff Ba^{(i)} \lessdot_{[y]} Bb^{(j)}$ and $Ea^{(i)} \lessdot_{[x]} Eb^{(j)} \iff Ea^{(i)} \lessdot_{[y]} Eb^{(j)}$.

(*Case 4*). We have $a^{(i)} \sqsubset_u b^{(j)}$ and $\neg(b^{(j)} \sqsubset_u a^{(i)})$, where $u \in \{x,y\}$. Note that $\neg(b^{(j)} \sqsubset_u a^{(i)})$ implies $\neg(Bb^{(j)} \lessdot_{[u]} Ea^{(i)})$, i.e. either $Ea^{(i)} \lessdot_{[u]} Bb^{(j)}$ or $Ea^{(i)} \frown_{\lessdot_{[u]}} Bb^{(j)}$. The former implies $a^{(i)} \prec_u b^{(j)}$, so the case is reduced to Case 1. The latter, by Definition 22 and Theorem 5(1) implies $Ba^{(i)} \lessdot_{[u]} Ea^{(i)}$, $Bb^{(j)} \lessdot_{[u]} Eb^{(j)}$, $Ba^{(i)} \lessdot_{[u]} Eb^{(j)}$, and $Bb^{(j)} \frown_{\lessdot_{[u]}} Ea^{(i)}$, so the case is reduced to Case 2.

(*Case 5*) We have $\neg(a^{(i)} \sqsubset_u b^{(j)})$ and $\neg(b^{(j)} \sqsubset_u a^{(i)})$, where $u \in \{x,y\}$. Hence we have four cases: $(Eb^{(j)} \lessdot_{[u]} Ba^{(i)} \wedge Ea^{(i)} \lessdot_{[u]} Bb^{(j)})$, or $(Eb^{(j)} \lessdot_{[u]} Ba^{(i)} \wedge Ea^{(i)} \frown_{\lessdot_{[u]}} Bb^{(j)})$, or $(Eb^{(j)} \frown_{\lessdot_{[u]}} Ba^{(i)} \wedge Ea^{(i)} \lessdot_{[u]} Bb^{(j)})$, or $(Eb^{(j)} \frown_{\lessdot_{[u]}} Ba^{(i)} \wedge Ea^{(i)} \frown_{\lessdot_{[u]}} Bb^{(j)})$. By Theorem 5(1), only the last case is not a contradiction. By Theorem 5(1) again, we have $Ba^{(i)} \lessdot_{[u]}$

$Ea^{(i)}, Bb^{(j)} \lessdot_{[u]} Eb^{(j)}, Eb^{(j)} \frown_{\lessdot_{[u]}} Ba^{(i)}$, and $Ea^{(i)} \frown_{\lessdot_{[u]}} Bb^{(j)}$. Hence only the relationships between $Ba^{(i)}$ and $Bb^{(j)}$, and between $Ea^{(i)}$ and $Eb^{(j)}$, are not described yet. Again, we can repeat the last part of Case 2 to show that $Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \iff Ba^{(i)} \lessdot_{[y]} Bb^{(j)}$ and $Ea^{(i)} \lessdot_{[x]} Eb^{(j)} \iff Ea^{(i)} \lessdot_{[y]} Eb^{(j)}$. $\square$

We are now able to prove one of our main results, namely that every interval trace uniquely determines an interval order structure.

Theorem 9 *For all $x, y \in \mathsf{InSeq}(\mathscr{E}^*)$, $x \equiv y$ if and only if $S^x = S^y$.*

Proof ($\Rightarrow$) If $x \equiv y$ then $[x] = [y]$, so $\lessdot_{[x]} = \lessdot_{[y]}$. Then by Lemma 3, $S^x = S^y$.
($\Leftarrow$) If $S^x = S^y$ then, by Lemma 3, we have $\lessdot_{[x]} = \lessdot_{[y]}$, and now by Theorem 6, $\{\lhd_t \mid t \in [x]\} = \{\lhd_t \mid t \in [y]\}$. From Definition 3(5) it follows that $t = u \iff \lhd_t = \lhd_u$, so $[x] = [y]$, i.e. $x \equiv y$. $\square$

The above theorem makes possible the following definition.

Definition 23 *For each interval trace $[x]$, the interval order structure $S^{[x]}$ induced by $[x]$ is defined as $S^{[x]} = (\widehat{\Sigma}_x^{\mathscr{E}}, \prec_{[x]}, \sqsubset_{[x]}) = S^t = (\widehat{\Sigma}_x^{\mathscr{E}}, \prec_t, \sqsubset_t)$, where $t \in [x]$.* $\square$

Theorem 9 alone is not enough to claim that interval traces can represent all the properties of interval order structures. We also have to show that for any $x \in \mathsf{InSeq}(\mathscr{E}^*)$, $\mathsf{Interv}(S^x)$, the set of all interval order extensions of S^x (see Definition 8) is equal to the set of all interval orders generated via Fishburn's Theorem (Theorem 2) from all $\hat{t}$ (enumerated version of t) such that $t \in [x]$. Interval orders generated by appropriate sequences from $\mathscr{E}^*$, and denoted by $\blacktriangleleft_x$ for $x \in \mathscr{E}^*$, are described by Definition 4(3).

Our second main result is the following.

Theorem 10 *For every $x \in \mathsf{InSeq}(\mathscr{E}^*)$,*

$$\mathsf{Interv}(S^x) = \mathit{Interv}([x]) = \{\blacktriangleleft_t \mid t \in [x]\}.$$

Proof ($\Leftarrow$) Let $t \in [x]$ and $a^{(i)}, b^{(j)} \in \widehat{\Sigma}_x^{\mathscr{E}}$. Let us consider the relation $\prec_x$ first. We have $a^{(i)} \prec_x b^{(j)} \overset{\text{Def.22}}{\iff} Ea^{(i)} \lessdot_{[x]} Bb^{(j)} \overset{\text{Def.13}}{\implies} Ea^{(i)} \lhd_t Bb^{(j)} \overset{\text{Def.4(3)}}{\iff} a^{(i)} \blacktriangleleft_x b^{(j)}$. Hence, by Definition 2, the relation $\blacktriangleleft_x$ is an extension of $\prec_x$. Let us now consider the relation $\sqsubset_x$. Here we have $a^{(i)} \sqsubset_x b^{(j)} \overset{\text{Def.22}}{\iff} Ba^{(i)} \lessdot_{[x]} Eb^{(j)} \overset{\text{Def.13}}{\implies} Ba^{(i)} \lhd_t Eb^{(j)}$. Because $\lhd_t$ is a total order, $Ba^{(i)} \lhd_t Eb^{(j)} \iff \neg(Eb^{(j)} \lhd_t Ba^{(i)})$. But $\neg(Eb^{(j)} \lhd_t Ba^{(i)}) \overset{\text{Def.4(3)}}{\iff} \neg(b^{(j)} \blacktriangleleft_x a^{(i)}) \iff a^{(i)} \blacktriangleleft_x^{\frown} b^{(j)}$. Hence $a^{(i)} \sqsubset_x b^{(j)} \implies a^{(i)} \blacktriangleleft_x^{\frown} b^{(j)}$, so, by Definition 2, $\blacktriangleleft_x$ is an extension of $\sqsubset_x$ as well, which means, now by Definition 8, $\blacktriangleleft_x \in \mathsf{Interv}(S^x)$.

($\Rightarrow$) Let $< \in \mathsf{Interv}(S^x)$ and let $\lhd_< \subseteq \widehat{\Sigma}_x^{\mathscr{E}} \times \widehat{\Sigma}_x^{\mathscr{E}}$ be a total order representation of $<$ via Fishburn Theorem (Theorem 2), i.e. $a^{(i)} < b^{(j)} \iff Ea^{(i)} \lhd_< Bb^{(j)}$. Furthermore let $t_< \in \mathscr{E}^*$ be the sequence representation of the total order $\lhd_<$, i.e. $\lhd_< = \lhd_{t_<}$, where $\lhd_{t_<}$ is the total order generated by $t_<$ as in Definition 3(5). Note that, by Definition 4(3), the interval order $<$ equals the interval order $\blacktriangleleft_{t_<}$. To show that $< \in \{\blacktriangleleft_t \mid t \in [x]\}$, we have to prove that $t_< \in [x]$.

Since $< \in \mathsf{Interv}(S^x)$ then $<$ is an extension of $\prec_x$ and $\sqsubset_x$, i.e., by Definition 8, $\prec_x \subseteq <$ and $\sqsubset_x \subseteq <^{\frown}$. We will show that $\lhd_<$ is a total extension of $\lessdot_{[x]}$, i.e. $\lhd_< \in \mathsf{Total}(\lessdot_{[x]})$. To prove this we will just show that for all $\alpha, \beta \in \{Ba^{(i)}, Ea^{(i)}, Bb^{(j)}, Eb^{(j)}\}$ we have $\alpha \lessdot_{[x]} \beta \implies \alpha \lhd_< \beta$.

First note that from Theorem 5(1) and Theorem 2(1) we have $Ba^{(i)} \lessdot_{[x]} Ea^{(i)}$, $Bb^{(j)} \lessdot_{[x]} Eb^{(j)}$, and $Ba^{(i)} \lhd_< Ea^{(i)}$, $Bb^{(j)} \lhd_< Eb^{(j)}$. Now we have to consider the remaining four cases.

(*Case 1*). Consider $Ea^{(i)}$ and $Bb^{(j)}$. By Definitions 22, 8 and Theorem 2(2), we have: $Ea^{(i)} \lessdot_{[x]} Bb^{(j)} \overset{\text{Def.22}}{\iff} a^{(i)} \prec_x b^{(j)} \overset{\text{Def.8}}{\implies} a^{(i)} < b^{(j)} \overset{\text{Th.2(2)}}{\iff} Ea^{(i)} \lhd_< Bb^{(j)}$.

(*Case 2*). Consider $Ba^{(i)}$ and $Eb^{(j)}$. Again by Definitions 22, 8 and Theorem 2(2), we have: $Ba^{(i)} \lessdot_{[x]} Eb^{(j)} \overset{\text{Def.22}}{\iff} a^{(i)} \sqsubset_x b^{(j)} \overset{\text{Def.8}}{\implies} a^{(i)} <^{\frown} b^{(j)} \iff \neg(b^{(j)} < a^{(i)}) \overset{\text{Th.2(2)}}{\iff}$

$\neg(Eb^{(j)} \lhd_{<} Ba^{(i)}) \iff Ba^{(i)} \lhd_{<} Eb^{(j)}$.

(*Case 3*). Consider $Ba^{(i)}$ and $Bb^{(j)}$. From Lemma 2(1) it follows: $Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \iff (Ea^{(i)} \lessdot_{[x]} Bb^{(j)}) \vee (\exists c^{(k)} \in \widehat{\Sigma}_x^{\mathscr{E}}.\, Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)})$. From Case 1 we obtain $Ea^{(i)} \lessdot_{[x]} Bb^{(j)} \implies Ea^{(i)} \lhd_{<} Bb^{(j)}$, i.e., by Theorem 2(2), $Ba^{(i)} \lhd_{<} Ea^{(i)} \lhd_{<} Bb^{(j)}$, so $Ba^{(i)} \lhd_{<} Bb^{(j)}$. Similarly from Case 2 and Case 1 we obtain $Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)} \implies Ba^{(i)} \lhd_{<} Ec^{(k)} \lhd_{<} Bb^{(j)} \implies Ba^{(i)} \lhd_{<} Bb^{(j)}$.

(*Case 4*). Consider $Ea^{(i)}$ and $Eb^{(j)}$. Similar to Case 3 but using Lemma 2(2) instead.

This means that indeed $\lhd_{<} \in \mathsf{Total}(\lessdot_{[x]})$. By Theorem 6, $\lhd_{<} \in \{\lhd_t \mid t \in [x]\}$. But $\lhd_{<} = \lhd_{t_x}$, so $t_x \in [x]$, which end the proof of ($\Rightarrow$). □

Theorems 9 and 10 show that interval traces, i.e. sets of legal sequences of beginnings and ends, correspond to interval order structures in the same way as Mazurkiewicz traces correspond to partial orders (dependency graphs of [Mazurkiewicz [1995]]) and comtraces correspond to stratified order structures.

We will now show that the partial order $\lessdot_{[x]}$ equals $<^{S^x}$, i.e. it is the least partial order that satisfies Theorem 5 for S^x.

Proposition 7 *For every $x \in \mathsf{InSeq}(\mathscr{E}^*)$, $\lessdot_{[x]} = <^{S^x}$.*

Proof We will show that for each $<$ that satisfies Theorem 5, and every $\alpha, \beta \in \widehat{\mathscr{E}}_x$, we have $\alpha \lessdot_{[x]} \beta \implies \alpha < \beta$. Since α and β are of the form $Ba^{(i)}$ or $Ea^{(i)}$ where $a \in \Sigma$, we have to consider four cases.

(*Case 1*). $\alpha = Ba^{(i)}, \beta = Eb^{(j)}$. In this case we have

$$Ba^{(i)} \lessdot_{[x]} Eb^{(j)} \overset{\text{Def.22}}{\iff} a^{(i)} \sqsubset_x b^{(j)} \overset{\text{Th.5}}{\iff} Ba^{(i)} < Eb^{(j)}.$$

(*Case 2*). $\alpha = Ea^{(i)}, \beta = Bb^{(j)}$. Now we have

$$Ea^{(i)} \lessdot_{[x]} Bb^{(j)} \overset{\text{Def.22}}{\iff} a^{(i)} \prec_x b^{(j)} \overset{\text{Th.5}}{\iff} Ea^{(i)} < Bb^{(j)}.$$

(*Case 3*). $\alpha = Ba^{(i)}, \beta = Bb^{(j)}$. By Lemma 2 we have

$Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \iff (Ea^{(i)} \lessdot_{[x]} Bb^{(j)}) \vee (\exists c^{(k)} \in \widehat{\Sigma}_x^{\mathscr{E}}.\, Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)})$. If $Ea^{(i)} \lessdot_{[x]}$

$Bb^{(j)}$ the case is reduced to Case 2, so assume $(\exists c^{(k)} \in \widehat{\Sigma}_x^{\mathscr{E}}.\, Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)})$. Thus $Ba^{(i)} \lessdot_{[x]} Ec^{(k)} \lessdot_{[x]} Bb^{(j)} \overset{\text{Def.22}}{\Longleftrightarrow} a^{(i)} \sqsubset_x c^{(k)} \prec_x b^{(j)} \overset{\text{Th.5}}{\Longleftrightarrow} Ba^{(i)} < Ec^{(k)} < Bb^{(j)} \implies Ba^{(i)} < Bb^{(j)}$, so $Ba^{(i)} \lessdot_{[x]} Bb^{(j)} \implies Ba^{(i)} < Bb^{(j)}$.

(*Case 4*). $\alpha = Ba^{(i)}, \beta = Bb^{(j)}$. Dually to Case 3, by exchanging B with E. □

Example 11 *Let $\Sigma = \{a,b,c\}$, so $\mathscr{E} = \{Ba,Ea,Bb,Eb,Bc,Ec\}$. Let ind $\subseteq \mathscr{E} \times \mathscr{E}$ be ind^Q from Figure 6.10, and let $x = BcBaEaBbEcEb \in \mathsf{InSeq}(\mathscr{E}^*)$. Note that $[x] = \mathbf{x}$, where $\mathbf{x}$ is from Example 9 (it contains nine sequences).*

The interval order structure $S^{[x]} = S^x = (\widehat{\Sigma}_x^{\mathscr{E}}, \prec_2^Q, \sqsubset_2^Q)$, where $\widehat{\mathscr{E}}_x^{\Sigma} = \{a^{(1)}, b^{(1)}, c^{(1)}\}$, and the relations $\prec_2^Q$ and $\sqsubset_2^Q$ are these from Figure 6.10, after replacing a with $a^{(1)}$, b with $b^{(1)}$, and c with $c^{(1)}$. The set $\widehat{\mathscr{E}}_x$ is $\{Ba^{(1)}, Ea^{(1)}, Bb^{(1)}, Eb^{(1)}, Bc^{(1)}, Ec^{(1)}\}$ and the relation $\lessdot_{[x]} \subseteq \widehat{\mathscr{E}}_x \times \widehat{\mathscr{E}}_x$ equals $<_2^Q$ also from Figure 6.10, after replacing Ba with $Ba^{(1)}$, Ea with $Ea^{(1)}$, etc.

The set $\mathsf{Interv}(S^{[x]}) = hist_2^Q = \{<_2^Q, <_3^Q, <_4^Q\}$, where $<_2^Q$, $<_3^Q$, and $<_4^Q$ are interval orders from Figure 6.10, again after replacing a with $a^{(1)}$, b with $b^{(1)}$, and c with $c^{(1)}$.

Moreover $<_2^Q = \blacktriangleleft_{BcEcBaEaBbEb}$,

$<_3^Q = \blacktriangleleft_{BaBcEcEaBbEb} = \blacktriangleleft_{BaBcEaEcBbEb} = \blacktriangleleft_{BcBaEcEaBbE} = \blacktriangleleft_{BcBaEaEcBbEb}$,

$<_4^Q = \blacktriangleleft_{BaBcEaBbEbEc} = \blacktriangleleft_{BaBcEaBbEcEb} = \blacktriangleleft_{BcBaEaBbEbEc} = \blacktriangleleft_{BcBaEaBbEcEb}$.

Finally note that the results would be the same if x would be replaced by any $t \in [x]$. □

Example 12 *Let $\Sigma = \{a,b,c,d\}$. Then we have $\mathscr{E} = \{Ba,Ea,Bb,Eb,Bc,Ec,Bd,Ed\}$. Let ind $\subseteq \mathscr{E} \times \mathscr{E}$ be the interval independency from Example 10.*

Take $x = BaEaBbEbBcEcBdEd \in \mathscr{E}^*$. Since $x \in \mathsf{InSeq}(\mathscr{E}^*)$, the interval trace $[x]$ is defined, and $[x] = \mathbf{y}$, where $\mathbf{y}$ is that from Example 10 (it contains fourteen sequences).

The interval order structure $S^{[x]} = S^x = (\widehat{\Sigma}_x^{\mathscr{E}}, \prec, \sqsubset)$, where $\widehat{\mathscr{E}}_x^{\Sigma} = \{a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)}\}$, and the relations $\prec$ and $\sqsubset$ are these from Figure 3.2, after replacing a with $a^{(1)}$, b with $b^{(1)}$, etc. The set $\widehat{\mathscr{E}}_x = \{Ba^{(1)}, Ea^{(1)}, Bb^{(1)}, Eb^{(1)}, Bc^{(1)}, Ec^{(1)}, Bd^{(1)}, Ed^{(1)}\}$ and the relation

$\lessdot_{[x]} \subseteq \widehat{\mathscr{E}}_x \times \widehat{\mathscr{E}}_x$ *equals* $<^1$ *also from Figure 3.2, after replacing Ba with* $Ba^{(1)}$*, Ea with* $Ea^{(1)}$*, etc.*

The set $\mathsf{Interv}(S^{[x]}) = \{<_1, <_2, <_3, <_4, <_5\}$*, where* $<_1$*,* $<_2$*,* $<_3$*,* $<_4$ *and* $<_5$ *are interval orders from Figure 3.2, again after replacing a with* $a^{(1)}$*, b with* $b^{(1)}$*, etc.*

Moreover $<_1 = \blacktriangleleft_{BaEaBbEbBcEcBdEd}$, $<_2 = \blacktriangleleft_{BaEaBbEbBdEdBcEc}$,

$<_3 = \blacktriangleleft_{BaEaBbBcEbEcBdEd} = \blacktriangleleft_{BaEaBcBbEbEcEdEd} = \blacktriangleleft_{BaEaBcBbEcEbBdEd} = \blacktriangleleft_{BaEaBbBcEcEbBdEd}$,

$<_4 = \blacktriangleleft_{BaEaBbEbBcBdEcEd} = \blacktriangleleft_{BaEaBbEbBdBcEcEd} = \blacktriangleleft_{BaEaBbEbBdBcEdEc} = \blacktriangleleft_{BaEaBbEbBcBdEdEc}$,

$<_5 = \blacktriangleleft_{BaEaBbBcEbBdEcEd} = \blacktriangleleft_{BaEaBbBcEbBdEdEc} = \blacktriangleleft_{BaEaBcBbEbBdEdEc} = \blacktriangleleft_{BaEaBcBbEbBdEcEd}$.

Finally note that the results would be the same if x were replaced by any $t \in [x]$*.* □

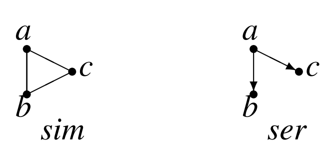
7.3 Comtraces vs Interval Traces

While every stratified order is an interval order, every stratified order structure is an interval order structure and every Mazurkiewicz trace can be interpreted as a simplified comtrace, the similar relationship is much more complex between comtraces and interval traces.

Let (Σ, sim, com) be a comtrace alphabet, x a step sequence and $\mathbf{x} = [x]_{(sim,ser)}$ be a comtrace defined by x.

*It is usually **false** that there is an interval trace alphabet* $(\mathscr{E}, ind)$ *and an interval sequence y such that the interval trace* $\mathbf{y} = [y]_{ind}$ *satisfies* $Strat(\mathbf{x}) = Interv(\mathbf{y})$.

Consider $\Sigma = \{a, b, c\}$, sim and ser as below

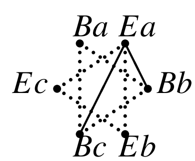


and $\mathbf{x} = [\{a,b,c\}]_{(sim,ser)} = \{\{a,b,c\}, \{a\}\{b,c\}\}$.

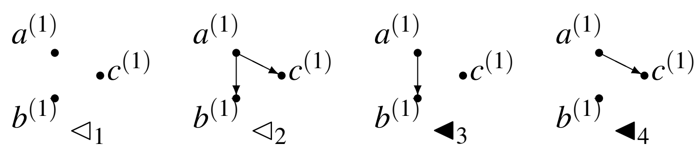
Suppose there is a relation ind on $\mathscr{E} = \{Ba, Bb, Bc, Ea, Eb, Ec\}$ and an interval sequence

$y \in \mathscr{E}^*$ such that the interval trace $\mathbf{y} = [y]_{ind}$ satisfies $Strat(\mathbf{x}) = Interv(\mathbf{y})$. The stratified order $\lhd_{\{a,b,c\}}$ can be represented by the interval sequence $y_1 = BaBbBcEaEbEc$, so $\mathbf{y} = [y_1]_{ind}$, and the stratified order $\lhd_{\{a\}\{b,c\}}$ can be represented by the interval sequence $y_2 = BaEaBbBcEbEc$, so $y_1, y_2 \in \mathbf{y}$, i.e. we must have $y_1 \equiv_{ind} y_2$. To obtain y_1 from y_2 we must move Ea from after Bc to before Bb, hence $(Ea, Bb) \in ind$ and $(Ea, Bc) \in ind$. Then $y_3 = BaBbEaBcEbEc \in \mathbf{y}$ is a contradiction since the order $\blacktriangleleft_{y_3}$ is *not* stratified!

It can be shown by inspection that if $\mathbf{y}$ must contain all interval sequence representations of $\lhd_{\{a,b,c\}}$ and $\lhd_{\{a\}\{b,c\}}$, and the only stratified orders included in $Interv(\mathbf{y})$ are $\lhd_{\{a,b,c\}}$ and $\lhd_{\{a\}\{b,c\}}$, then the relation ind must be as the one below:



The interval trace $\mathbf{y} = [y_1]_{ind}$ generates the set of interval orders $Interv(\mathbf{y}) = \{\lhd_1, \lhd_2, \blacktriangleleft_3, \blacktriangleleft_4\}$, where the orders $\lhd_1 = \lhd_{\{a,b,c\}}$, $\lhd_2 = \lhd_{\{a\}\{b,c\}}$, and $\blacktriangleleft_3$, $\blacktriangleleft_4$ are given below:



The orders $\lhd_1$ and $\lhd_2$ are stratified while $\blacktriangleleft_3$ and $\blacktriangleleft_4$ are not.

However, we have (see Definition 1 for the meaning of $<^\frown$)

$\prec_{\mathbf{x}} = \lhd_1 \cap \lhd_2 = \lhd_1 \cap \lhd_2 \cap \blacktriangleleft_3 \cap \blacktriangleleft_4 = \prec_{\mathbf{y}}$, and

$\sqsubset_{\mathbf{x}} = \lhd_1^\frown \cap \lhd_2^\frown = \lhd_1^\frown \cap \lhd_2^\frown \cap \blacktriangleleft_3^\frown \cap \blacktriangleleft_4^\frown = \sqsubset_{\mathbf{y}}$,

which implies that the stratified order structure $S^{\mathbf{x}} = S^{\mathbf{y}}$ as

$S^{\mathbf{x}} = (\{a^{(1)}, b^{(1)}, c^{(1)}\}, \prec_{\mathbf{x}}, \sqsubset_{\mathbf{x}})$ and $S^{\mathbf{y}} = (\{a^{(1)}, b^{(1)}, c^{(1)}\}, \prec_{\mathbf{y}}, \sqsubset_{\mathbf{y}})$

We show that this pattern holds in general case.

For every comtrace alphabet (Σ, sim, ser), let $(\mathscr{E}, ind_{(sim,ser)})$ be an interval trace alphabet such that the relation $ind_{(sim,ser)}$ satisfies:

$$(Bb, Ea) \in ind_{(sim,ser)} \iff (a,b) \in ser.$$

Theorem 11 *Let (Σ, sim, ser) be a comtrace alphabet, x be a step sequence, y be any interval sequence such that $\lhd_x = \blacktriangleleft_y$, $\mathbf{x} = [x]_{(sim,ser)}$, $\mathbf{y} = [y]_{ind}$. Furthermore let $S^{\mathbf{x}} = (\widehat{\Sigma}_x, \prec_{\mathbf{x}}, \sqsubset_{\mathbf{x}})$ be the stratified order structure generated by the comtrace $\mathbf{x}$, and $S^{\mathbf{y}} = (\widehat{\Sigma}^{\mathscr{E}}_y, \prec_{\mathbf{y}}, \sqsubset_{\mathbf{y}})$ be the interval order structure generated by the interval order $\mathbf{y}$.*

Then we have $S^{\mathbf{x}} = S^{\mathbf{y}}$.

Proof Clearly $\widehat{\Sigma}_x = \widehat{\Sigma}^{\mathscr{E}}_y$. We will show that $\prec_{\mathbf{x}} = \prec_{\mathbf{y}}$ and $\sqsubset_{\mathbf{x}} = \sqsubset_{\mathbf{y}}$.
Let $w = u_1Au_2 \in \mathbf{x}$, $v = u_1BCu_2 \in \mathbf{x}$, $A = B \cup C$, $B \cap C = \emptyset$ and $B \times C \subseteq ser$, i.e. $w \approx_{(sim,ser)} v$. Note that $\lhd_w \subseteq \lhd_v$ and $\lhd_{v}^{\frown} \subseteq \lhd_{w}^{\frown}$. Assume $B = \{b_1, \ldots, b_k\}$, $C = \{c_1, \ldots, c_m\}$, so $A = \{b_1, \ldots, b_k, c_1, \ldots, c_m\}$. For every set $X \subseteq \Sigma$, let $\mathsf{B}(X) = \{Ba \mid a \in X\}$ and $\mathsf{E}(X) = \{Ea \mid a \in X\}$. Let w^{itv}, v^{itv}, u_1^{itv} and u_2^{itv} be *some* interval sequence representations of stratified orders $\lhd_w$, $\lhd_u$, $\lhd_{u_1}$ and $\lhd_{u_2}$ respectively, i.e. $\blacktriangleleft_{w^{itv}} = \lhd_w$, $\blacktriangleleft_{u^{itv}} = \lhd_u$, $\blacktriangleleft_{u_1^{itv}} = \lhd_{u_1}$ and $\blacktriangleleft_{u_2^{itv}} = \lhd_{u_2}$. We may assume that $w^{itv} = u_1^{itv} z_A^{\mathsf{B}} z_A^{\mathsf{E}} u_2^{itv}$, where $z_A^{\mathsf{B}} \in perm(\mathsf{B}(A))$ and $z_A^{\mathsf{E}} \in perm(\mathsf{E}(A))$, and $u^{itv} = u_1^{itv} z_B^{\mathsf{B}} z_B^{\mathsf{E}} z_C^{\mathsf{B}} z_C^{\mathsf{E}} u_2^{itv}$, where $z_B^{\mathsf{B}} \in perm(\mathsf{B}(B))$, $z_B^{\mathsf{E}} \in perm(\mathsf{E}(B))$, $z_C^{\mathsf{B}} \in perm(\mathsf{B}(C))$ and $z_C^{\mathsf{E}} \in perm(\mathsf{E}(C))$.
Because $(Bb, Ea) \in ind_{(sim,ser)} \Leftrightarrow (a,b) \in ser$, and $ind_{(sim,ser)}$ satisfies property (2) of Definition 19, we have $w^{itv} \equiv_{ind_{(sim,ser)}} v^{itv}$. Assume $w^{itv} \approx_{ind_{(sim,ser)}} s_1 \approx_{ind_{(sim,ser)}} \ldots \approx_{ind_{(sim,ser)}} s_n \approx_{ind_{(sim,ser)}} u^{itv}$.

Consider $r = r_1 BaEb r_2$ and $t = r_1 EbBa r_2$ where $(Ba, Eb) \in ind$. Note that $\blacktriangleleft_r \subseteq \blacktriangleleft_t$ and $\blacktriangleleft_{t}^{\frown} \subseteq \blacktriangleleft_{r}^{\frown}$.

But this means the $\blacktriangleleft_{w^{itv}} \subseteq \blacktriangleleft_{s_i} \subseteq \blacktriangleleft_{u^{itv}}$, and $\blacktriangleleft_{u^{itv}}^{\frown} \subseteq \blacktriangleleft_{s_i}^{\frown} \subseteq \blacktriangleleft_{w^{itv}}$, for all $i = 1, \ldots, n$. Hence $\lhd_w \cap \lhd_u = \blacktriangleleft_{w^{itv}} \cap \blacktriangleleft_{s_1} \cap \ldots \cap \blacktriangleleft_{s_n} \cap \blacktriangleleft_{u^{itv}}$ and $\lhd_{w}^{\frown} \cap \lhd_{u}^{\frown} = \blacktriangleleft_{w^{itv}}^{\frown} \cap \blacktriangleleft_{s_1}^{\frown} \cap \ldots \cap \blacktriangleleft_{s_n}^{\frown} \cap \blacktriangleleft_{u^{itv}}^{\frown}$.

Let $x, x', x_1, \ldots, x_l$ be step sequences such that $x \equiv_{(sim,ser)} x'$ and $x \approx_{(sim,ser)} x_1 \approx_{(sim,ser)}$

$\ldots \approx_{(sim,ser)} x_l \approx_{(sim,ser)} x'$, and let y, y' be interval sequences such that $\lhd_x = \blacktriangleleft_y$ and $\lhd_{x'} = \blacktriangleleft_{y'}$. By the property of $ind_{(sim,ser)}$ we have $y \equiv_{ind_{(sim,ser)}} y'$, so let $y_1, \ldots, y_j$ be interval sequences such that $y \approx_{ind_{(sim,ser)}} y_1 \approx_{ind_{(sim,ser)}} \ldots \approx_{ind_{(sim,ser)}} y_j \approx_{ind_{(sim,ser)}} y'$. From what we have proved above, we may conclude that

$\lhd_x \cap \bigcap_{i=1}^{l} \lhd_{x_i} \cap \lhd_{x'} = \blacktriangleleft_y \cap \bigcap_{i=1}^{j} \blacktriangleleft_{y_i} \cap \blacktriangleleft_{y'}$, and

$\lhd_{x}^{\frown} \cap \bigcap_{i=1}^{l} \lhd_{x_i}^{\frown} \cap \lhd_{x'}^{\frown} = \blacktriangleleft_{y}^{\frown} \cap \bigcap_{i=1}^{j} \blacktriangleleft_{y_i}^{\frown} \cap \blacktriangleleft_{y'}^{\frown}$.

Define $\prec_{xx'} = \lhd_x \cap \bigcap_{i=1}^{l} \lhd_{x_i} \cap \lhd_{x'}$, $\sqsubset_{xx''} = \lhd_{x}^{\frown} \cap \bigcap_{i=1}^{l} \lhd_{x_i}^{\frown} \cap \lhd_{x'}^{\frown}$, and $\prec_{yy'} = \blacktriangleleft_y \cap \bigcap_{i=1}^{j} \blacktriangleleft_{y_i} \cap \blacktriangleleft_{y'}$, $\sqsubset_{yy'} = \blacktriangleleft_{y}^{\frown} \cap \bigcap_{i=1}^{j} \blacktriangleleft_{y_i}^{\frown} \cap \blacktriangleleft_{y'}^{\frown}$.

Note that $\prec_{\mathbf{x}} = \bigcap_{t \in [x]_{(sim,ser)}} \lhd_t = \bigcap_{x' \in [x]_{(sim,ser)}} \prec_{xx'}$,

$\prec_{\mathbf{y}} = \bigcap_{r \in [y]_{ind_{(sim,ser)}}} \blacktriangleleft_r = \bigcap_{y' \in [x]_{ind_{(sim,ser)}}} \prec_{yy'}$, so $\prec_{\mathbf{x}} = \prec_{\mathbf{y}}$.

Similarly, $\sqsubset_{\mathbf{x}} = \bigcap_{t \in [x]_{(sim,ser)}} \lhd_{t}^{\frown} = \bigcap_{x' \in [x]_{(sim,ser)}} \sqsubset_{xx'}$,

$\sqsubset_{\mathbf{y}} = \bigcap_{r \in [y]_{ind_{(sim,ser)}}} \blacktriangleleft_{r}^{\frown} = \bigcap_{y' \in [x]_{ind_{(sim,ser)}}} \sqsubset_{yy'}$, so $\sqsubset_{\mathbf{x}} = \sqsubset_{\mathbf{y}}$. Hence $S^{\mathbf{x}} = S^{\mathbf{y}}$. $\square$

Theorem 11 states that while comtraces cannot literally be simulated by interval traces, the stratified order structures they represent, can.

Chapter 8

The Applications of Interval Traces

We focus on discussing the applications of interval traces in this chapter. And let's first start with the analysis of how the full semantics of inhibitor Petri nets can be expressed with interval traces.

8.1 Operational Semantics of Inhibitor Petri Nets

The *operational semantics* of a given net N is defined as the set of appropriate systems runs (observations) the net N generates. In principle we can distinguish three categories of operational semantics: *total orders* or *firing sequences* semantics, *stratified orders* or *firing step sequences* semantics, and *interval orders* semantics. For the net N_Q from Figure 6.10 and the final marking $m_f = \{s_4, s_5\}$, the set of all firing sequences from the initial marking m_0 to the marking m_f is $\mathsf{FS}_{N_Q}(m_0 \to m_f) = \{abc, cab\}$, and the set of appropriate total orders is $\mathsf{TO}_{N_Q}(m_0 \to m_f) = \{<^Q_1, <^Q_2\}$; the set of all firing step sequences from the initial marking m_0 to the marking m_f is $\mathsf{FSS}_{N_Q}(m_0 \to m_f) = \{\{a\}\{b\}\{c\}, \{c\}\{a\}\{b\}, \{a,c\}\{b\}\}$, and the set of appropriate stratified orders is $\mathsf{SO}_{N_Q}(m_0 \to m_f) = \{<^Q_1, <^Q_2, <^Q_3\}$. The set of all interval orders equals $\mathsf{IO}_{N_Q}(m_0 \to m_f) = \{<^Q_1, <^Q_2, <^Q_3, <^Q_4\}$.

While both the firing sequences (total orders) semantics and the firing step sequences (stratified orders) semantics are well established (c.f. [Janicki and Koutny [1995]; Kleijn and Koutny [2004]]), formal description of interval orders semantics is one of the goals of this thesis.

We will now briefly recall the firing sequences and the firing step sequences semantics.

8.1.1 Firing Sequence Semantics

The *firing sequences semantics*, the simplest operational semantics, is almost defined in the same way as for any other kind of Petri nets. The only difference is that for the inhibitor nets a transition can be enabled only if no place with which it is joined by an inhibitor arc is marked.

Formally, a transition t is *enabled* at marking m if ${}^{\bullet}t \subseteq m$ and $(t^{\bullet} \cup t^{\circ}) \cap m = \emptyset$. For each marking m, the set of all enabled transitions at m is denoted by $enabled_N(m)$.

An enabled t can *fire* leading to a new marking $m' = (m \setminus {}^{\bullet} t) \cup t^{\bullet}$, which is denoted by $m[t\rangle m'$.

A *firing sequence* from a marking m_1 to m_{k+1} is any sequence of transitions $t_1...t_k$ for which there are markings $m_2,...,m_k$ satisfying: $m_1[t_1\rangle m_1[t_2\rangle m_2...m_k[t_k\rangle m_{k+1}$. In such case we write: $m_1[t_1...t_k\rangle m_{k+1}$.

The set of all firing sequence from the marking m to the marking m' is defined as

$$\mathsf{FS}_N(m \to m') = \{x \in T^* \mid m[x\rangle m'\},$$

while the appropriate set of total orders is simply given by:

$$\mathsf{TO}_N(m \to m') = \{\lhd_x \mid x \in \mathsf{FS}_N(m \to m')\}.$$

For example for the net N_Q from Figures 6.10 and 8.11 the set of all firing sequences from the initial marking $\{s_1,s_2\}$ to the final marking $\{s_4,s_5\}$, $\mathsf{FS}_{N_Q}(\{s_1,s_2\}\rightarrow\{s_4,s_5\}) = \{abc,cab\}$, and $\mathsf{TO}_{N_Q}(\{s_1,s_2\}\rightarrow\{s_4,s_5\}) = \{<_1^Q,<_2^Q\}$, where $<_1^Q$ and $<_2^Q$ are those from Figure 6.10.

8.1.2 Firing Step Sequence Semantics

The *firing step sequence semantics* is defined in a similar fashion. The only difference is that sets of mutually independent transitions can be fired simultaneously.

Let $A \subseteq T$ be a non-empty set such that for all distinct $t,r \in A$, we have $(t^\bullet \cup {}^\bullet t) \cap (r^\bullet \cup {}^\bullet r) = \emptyset$. Then A is *step enabled* at marking m if ${}^\bullet A \subseteq m$ and $(A^\bullet \cup A^\circ) \cap m = \emptyset$. For each marking m, the set of all step enabled sets of transitions at m is denoted by $senabled_N(m)$.

We also denote $m[A\rangle m'$, where $m' = (m \setminus {}^\bullet A) \cup A^\bullet$.

A *firing step sequence* from the marking m_1 to m_{k+1} is any sequence of non-empty sets of transitions $A_1...A_k$ for which there are markings $m_2,...,m_k$ satisfying:

$m_1[A_1\rangle m_1[A_2\rangle m_2...m_k[A_k\rangle m_{k+1}$.

In such case we may write: $m_1[A_1...A_k\rangle m_{k+1}$.

The set of all firing step sequences from the marking m to the marking m' is defined as follows:

$$\mathsf{FSS}_N(m\rightarrow m') = \{x \in (\mathscr{P}(T) \setminus \emptyset)^* \mid m[x\rangle m'\},$$

while the appropriate set of stratified orders is:

$$\mathsf{SO}_N(m\rightarrow m') = \{\lhd_x \mid x \in \mathsf{FSS}_N(m\rightarrow m')\}.$$

For example, for the net N_Q from Figure 6.10, we have

$\mathsf{FSS}_{N_Q}(\{s_1,s_2\}\rightarrow\{s_4,s_5\}) = \{\ \{a\}\{b\}\{c\}, \{c\}\{a\}\{b\}, \{a,c\}\{b\}\ \}$, and
$\mathsf{SO}_{N_Q}(\{s_1,s_2\}\rightarrow\{s_4,s_6\}) = \{<_1,<_2,<_3\}$, where $<_1,<_2$ and $<_3$ are these from Figure 6.10.

8.2 Firing Interval Sequences Semantics

Since every interval order of events can be represented by some total order (i.e. an appropriate sequence) of event beginnings and ends (Theorem 2), if we figure out how a given inhibitor net can generate appropriate sequences of event beginnings and ends, we might be able to describe all interval orders the net generates.

Let $N = (P,T,F,I,m_0)$ be a given inhibitor net. For each $t \in T$ we define Bt - the beginning of t and Et - the end of t, and let $\mathscr{T} = \{Bt \mid t \in T\} \cup \{Et \mid t \in T\}$. Hence $\mathsf{InSeq}(\mathscr{T}^*)$ is the set of all interval sequences defined by the set T.

We want a formal way to define the set of all *firing interval sequences* from the marking m to the marking m', $\mathsf{FIS}_N(m\rightarrow m')$ such that the set $\mathsf{IO}_N(m\rightarrow m') = \{\blacktriangleleft_x \mid x \in \mathsf{FIS}_N(m\rightarrow m')\}$ is the appropriate set of interval orders. Since total order representations of interval orders via Theorem 2 is not unique, we also want the following relationship between FIS_N and IO_N,

$$\mathsf{FIS}_N(m\rightarrow m') = \{x \mid \blacktriangleleft_x \in \mathsf{IO}_N(m\rightarrow m')\}.$$

In particular, for the net N_Q of Figure 6.10 we want

$\mathsf{FIS}_{N_Q}(\{s_1,s_2\}\rightarrow\{s_4,s_5\}) = \{BaEaBbEbBcEc, BcEcBaEaBbEb,$
$BaBcEcEaBbEb, BaBcEaEcBbEb, BcBaEcEaBbEb, BcBaEaEcBbEb,$
$BaBcEaBbEbEc, BaBcEaBbEcEb, BcBaEaBbEbEc, BcBaEaBbEcEb\}$,

as in such case $\mathsf{IO}_{N_Q}(\{s_1,s_2\}\rightarrow\{s_4,s_5\}) = \{<^Q_1,<^Q_2,<^Q_3,<^Q_4\}$, as expected.

Note that, if $x = BaEaBbEbBcEc$ then $\blacktriangleleft_x = <^Q_1$, if $x = BcEcBaEaBbEb$ then $\blacktriangleleft_x = <^Q_2$, for $x \in \{BaBcEcEaBbEb, BaBcEaEcBbEb, BcBaEcEaBbEb, BcBaEaEcBbEb\}$, $\blacktriangleleft_x = <^Q_3$,

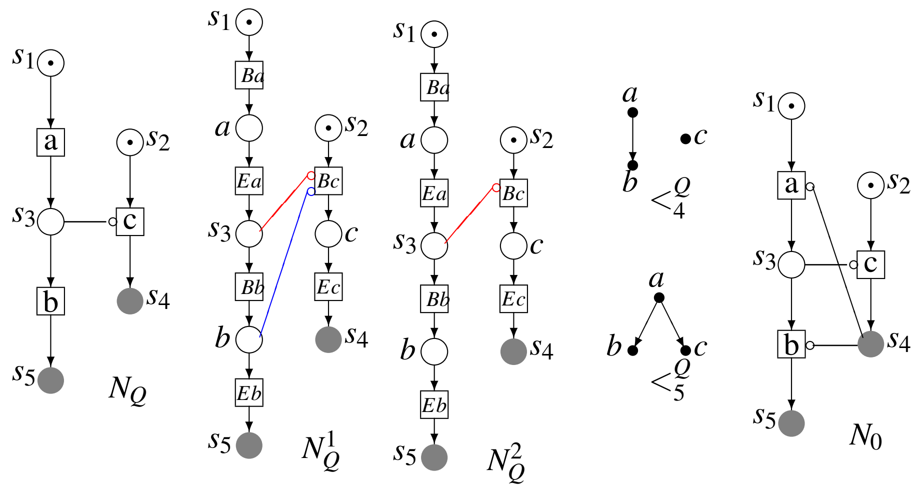


Figure 8.11: Two inhibitor nets, N_Q^1 and N_Q^2, derived from N_Q by straightforward replacing each transition t by Bt and Et (c.f. [Zuberek [1980]]). Note that $\mathsf{FS}_{N_Q^1}(\{s_1,s_2\}\rightarrow\{s_4,s_5\})$ is exactly the same as $\mathsf{FIS}_{N_Q}(\{s_1,s_2\}\rightarrow\{s_4,s_5\})$ - the intended set of firing interval sequences of N_Q, while $\mathsf{FS}_{N_Q^2}(\{s_1,s_2\}\rightarrow\{s_4,s_5\}) = \mathsf{FS}_{N_Q^1}(\{s_1,s_2\}\rightarrow\{s_4,s_5\}) \cup \{BaEaBbBcEcEb, BaEaBbBcEbEc\}$, where for each $x \in \{BaEaBbBcEcEb, BaEaBbBcEbEc\}$, $\blacktriangleleft_x = <_5^Q \notin \mathsf{IO}_{N_Q}(\{s_1,s_2\}\rightarrow\{s_4,s_5\})$. The net N_0 generates only interval order observations, as $\mathsf{IO}_{N_0}(\{s_1,s_2\}\rightarrow\{s_4,s_5\}) = \{<_4^Q\}$ while $\mathsf{TO}_{N_0}(\{s_1,s_2\}\rightarrow\{s_4,s_5\}) = \mathsf{SO}_{N_0}(\{s_1,s_2\}\rightarrow\{s_4,s_5\}) = \emptyset$.

and $x \in \{BaBcEaBbEbEc, BaBcEaBbEcEb, BcBaEaBbEbEc, BcBaEaBbEcEb\}$, $\blacktriangleleft_x = <_4^Q$.

Moreover we also have $\mathsf{FIS}_{N_Q}(\{s_1,s_2\}\rightarrow\{s_4,s_5\}) = \{x \mid \blacktriangleleft_x \in \mathsf{IO}_{N_Q}(\{s_1\}\rightarrow\{s_4,s_5\})\}$.

The basic idea of defining the set of firing interval sequences for a given inhibitor net N is briefly presented in Figure 8.11. If inhibitor arcs are not involved, to represent transitions by their beginnings and ends we might just replace each transition $\boxed{t}$ by the net $\boxed{Bt} \rightarrow (t) \rightarrow \boxed{Et}$ as proposed for example by Zuberek in [Zuberek [1980]] for Timed Petri nets. However, the inhibitor arcs cause some problems. The more natural transformation of the net N_Q into the net N_Q^2 (in N_Q a token in s_3 prevents c from being enabled, so a token in s_3 of N_Q^2

prevents *starting* c, i.e. Bc is not enabled), does not work. For N_Q^2, the interval sequences $BaEaBbBcEcEb$ and $BaEaBbBcEbEc$ are both firing sequences from the initial marking $\{s_1,s_2\}$ to the final marking $\{s_4,s_5\}$ and they both define the interval order $<_5^Q$, which is a stratified order that corresponds to the step sequence $\{a\}\{b,c\}$. However the step sequence $\{a\}\{b,c\}$ *cannot* be generated by the net N_Q. Moreover, the interval sequences $BcBbEcEb$ and $BcBbEbEc$ that also generate the order $<_5^Q$ are *not* firing sequences of N_Q^2.

On the other hand the more complex net N_Q^1 appears to have all the desired properties as

$$\begin{aligned}\mathsf{FS}_{N_Q^1}(\{s_1,s_2\}\to\{s_4,s_5\}) = \{&BaEaBbEbBcEc, BcEcBaEaBbEb,\\ &BaBcEcEaBbEb, BaBcEaEcBbEb, BcBaEcEaBbEb, BcBaEaEcBbEb,\\ &BaBcEaBbEbEc, BaBcEaBbEcEb, BcBaEaBbEbEc, BcBaEaBbEcEb\},\end{aligned}$$

which is exactly the same set as we claim $\mathsf{FIS}_{N_Q}(\{s_1,s_2\}\to\{s_4,s_5\})$ should be.

In this case the inhibitor arc (b,Bc) prevents executing Bc before Eb (providing Ea has been executed and Bc has not), i.e. after execution of a, if c has not started yet, c cannot start until b ends, which is exactly what we want.

While defining the meaning of one entity by transforming it into another is good for providing intuition and motivation, in is not necessarily a good way to do it in a general case. Hence we will formally define $\mathsf{FIS}_N(m\to m')$ in terms of the net N alone, without explicitly using the transformation illustrated in Figure 8.11 (from N_Q into N_Q^1). The key idea is to allow tokens not only in places but in transitions as well. A token in a transition t could be interpreted as 't is active', and removing all tokens from ${}^\bullet t$ and placing one token in t can be interpreted as an execution of Bt, while removing the token from t and placing tokens in $t^\bullet$ can be interpreted as executing Et. The whole definition is given below.

Definition 24 *Let $N = (P,T,F,I,m_0)$ be a given inhibitor net.*

1. For each $t \in T$ we define Bt - the beginning of t and Et - the end of t, and the set

$\mathscr{T} = \{Bt \mid t \in T\} \cup \{Et \mid t \in T\}$. *The elements of* $\mathscr{T}$ *are called* ***BE-transitions***.

2. *For each* $t \in T$ *we define:*
   (a) ${}^{\bullet}Bt = {}^{\bullet}t$,
   (b) $Bt^{\bullet} = \{t\}$,
   (c) ${}^{\bullet}Et = \{t\}$,
   (d) $Et^{\bullet} = t^{\bullet}$,
   (e) $Bt^{\circ} = t^{\circ} \cup (t^{\circ})^{\bullet}$, *and*
   (f) $Et^{\circ} = \emptyset$.
3. *We say that a set* $m \subseteq P \cup T$ *is an* ***extended marking*** *if* $m \cap ({}^{\bullet}m \cup m^{\bullet}) = \emptyset$.
4. *A BE-transition end* $\tau \in \mathscr{T}$ *is* ***enabled*** *at* ***extended marking*** $m \subseteq P \cup T$ *if* ${}^{\bullet}\tau \subseteq m$ *and* $(\tau^{\bullet} \cup \tau^{\circ}) \cap m = \emptyset$. *For each extended marking m, the set of all enabled elements of* $\mathscr{T}$ *at m is denoted by* $enabled_N^{ext}(m)$.
5. *An enabled BE-transition* τ *can* ***occur*** *leading to a* ***new extended marking*** $m' = (m \setminus {}^{\bullet}\tau) \cup \tau^{\bullet}$, *which is denoted by:* $m[\![\tau\rangle\!\rangle m'$.
6. *An* ***extended firing sequence*** *from the extended marking* m_1 *to the extended marking* m_{k+1} *is any sequence of BE-transitions* $\tau_1 \ldots \tau_k$ *for which there are extended markings* $m_2, \ldots, m_k$ *satisfying:* $m_1[\![\tau_1\rangle\!\rangle m_2 \ldots m_k[\![\tau_k\rangle\!\rangle m_{k+1}$.

   In such case we write: $m_1[\![\tau_1 \ldots \tau_k\rangle\!\rangle m_{k+1}$. □

The above definition is pretty much self explanatory as it mimics the standard fining sequence semantics approach, with the exception of condition 2(e). In the standard model, a token in a place $p \in t^{\circ}$ means t cannot be fired until this token is removed. In the new model it means that Bt cannot be fired. But if this token is removed for instance by firing Bt_1 where $t_1 \in p^\bullet$, then Bt could be enabled, and potentially fired *before* firing Et_1, which can be interpreted as simultaneous execution of t and t_1, contrary to the fact that $p \in t^\circ \cap {}^\bullet t_1$ was supposed to prevent it. This is the case for Bc, Bb and Eb in the net N_Q^2 in Figure 8.11. To prevent this we need to extend Bt° (see the rule $(\tau^\bullet \cup \tau^\circ) \cap m = \emptyset$ in condition 4 of Definition 24) by $(t^\circ)^\bullet$, which lead to $Bt^\circ = t^\circ \cup (t^\circ)^\bullet$.

In particular each marking is an extended marking. For the net N_Q from Figure 8.11, for example $\{s_2,s_3\},\{s_1,c\},\{a,c\}$ are extended markings, but $\{s_3,b\}$ is not as $s_3^\bullet = \{b\}$ and ${}^\bullet b = \{s_3\}$.

Corollary 3 *If an extended marking $m \subseteq P$, then for each $a \in T$,*

$$a \in enabled_N(m) \iff Ba \in enabled_N^{ext}(m). \qquad \square$$

We can now formally define the set of firing interval sequences.

Definition 25 *The set of all **firing interval sequence** from the marking m to the marking m' is defined as*

$$\mathsf{FIS}_N(m \to m') = \{x \in \mathscr{T}^* \mid m[\![x\rangle\!\rangle m'\}.$$

Note that we assume $m, m' \subseteq P$. $\square$

For example, for the net N_Q from Figures 6.10 and 8.11, we have $BaBcEaBbEcEb \in \mathsf{FIS}_N(\{s_1,s_2\} \to \{s_4,s_5\})$ since

$\{s_1,s_2\}[\![Ba\rangle\!\rangle\{a,s_2\}[\![Bc\rangle\!\rangle\{a,c\}[\![Ea\rangle\!\rangle\{s_3,c\}[\![Bb\rangle\!\rangle\{b,c\}[\![Ec\rangle\!\rangle\{b,s_4\}[\![Eb\rangle\!\rangle\{s_4,s_5\}$,

so $\{s_1,s_2\}[\![BaBcEaBbEcEb\rangle\!\rangle\{s_4,s_5\}$.

It is not immediately obvious that Definition 25 is sound. The soundness would require the set $\mathsf{FIS}_N(m \to m')$ to satisfy the following two properties

- every element of $\mathsf{FIS}_N(m \to m')$ must be an interval sequence, and

- since all total order representations of a given interval order are considered equivalent and none is preferred, if $x \in \mathsf{FIS}_N(m \to m')$, then $\blacktriangleleft_x = \blacktriangleleft_y$ should imply $y \in \mathsf{FIS}_N(m \to m')$.

Note that if we replace $Bt^\circ = t^\circ \cup (t^\circ)^\bullet$ with $Bt^\circ = t^\circ$ in Definition 24.2(e) (which for the nets in Figure 8.11, corresponds using the net N_Q^2 to represent the net N_Q), the second property does not hold! For example for the net N_Q from Figure 8.11, we have $x = BaEaBbBcEcEb \in \mathsf{FIS}_N(\{s_1,s_2\} \to \{s_4,s_5\})$ and $\blacktriangleleft_x = <_5^Q$. However, $y = BaEaBcBbEcEb \notin \mathsf{FIS}_N(\{s_1,s_2\} \to \{s_4,s_5\})$ but $\blacktriangleleft_y = <_5^Q = \blacktriangleleft_x$.

The following result guarantees the first property.

Proposition 8 *For all markings $m, m' \subseteq P$, we have* $\mathsf{FIS}_N(m \to m') \subseteq \mathsf{InSeq}(\mathscr{T}^*)$.

Proof It suffices to show that if $m[\![x\rangle\!\rangle m'$, then $x \in \mathsf{InSeq}(\mathscr{T}^*)$, or (see Definition 4(1)) to show that for each $a \in T$, $\pi_{\{Ba,Ea\}}(x) \in (BaEa)^*$.

Let $x = y\,Ba\,z$ and $m[\![y\,Ba\rangle\!\rangle m''$. Since $Ba^\bullet = \{a\}$, $a \in m''$. We also have: for any $m_a \subseteq P \cup T$, if $a \in m_a$, then Ba is not enabled in m_a, and the only way to remove a from m_a is to fire Ea (as ${}^\bullet Ea = \{a\}$). Hence we must have $x = y\,Ba\,w\,Ea\,v$, where $\pi_{\{Ba,Ea\}}(w) = \varepsilon$.
□

The second property requires a proposition like the one below.

Proposition 9

If $x \in \mathsf{FIS}_N(m \to m')$, then for every $y \in \mathscr{T}^*$, if $\blacktriangleleft_x = \blacktriangleleft_y$ then $y \in \mathsf{FIS}_N(m \to m')$.

Proof From Theorem 8, it follows that $\blacktriangleleft_{\widehat{x}} = \blacktriangleleft_{\widehat{y}}$ means $\widehat{x} \in ISR(\blacktriangleleft_x)$ and $\widehat{y} \in ISR(\blacktriangleleft_x)$. Since $\blacktriangleleft_x = \blacktriangleleft_{\widehat{x}}$, $\blacktriangleleft_x = \blacktriangleleft_{\widehat{x}}$ and $\blacktriangleleft_x = \blacktriangleleft_y$, it suffices to show that if $\{\widehat{x}, \widehat{y}\} \subseteq ISR(\blacktriangleleft_x)$ then $x \in \mathsf{FIS}_N(m \to m')$ implies $y \in \mathsf{FIS}_N(m \to m')$.

All elements of $ISR(\blacktriangleleft_x)$ satisfy a pattern given by Definition 18(2). Assume $\ll_{\blacktriangleleft_x} = A_1 \ldots A_m$. Hence $\widehat{x} = u_1 v_1 \ldots u_n v_n$ and $\widehat{y} = s_1 t_1 \ldots s_n t_n$, where for all $i = 1, \ldots, n$, $u_i, s_i \in perm(B_{\blacktriangleleft_x}(A_i))$ and $v_i, t_i \in perm(E_{\blacktriangleleft_x}(A_i))$.

Assume that $m[\![\widetilde{u}_1\rangle\!\rangle m_1^1 [\![\widetilde{v}_1\rangle\!\rangle m_1^2 \ldots m_{n-1}^2 [\![\widetilde{u}_n\rangle\!\rangle m_n^1 [\![\widetilde{v}_n\rangle\!\rangle m'$. We need prove that $m[\![\widetilde{s}_1\rangle\!\rangle m_1^1 [\![\widetilde{t}_1\rangle\!\rangle m_1^2 \ldots m_{n-1}^2 [\![\widetilde{s}_n\rangle\!\rangle m_n^1 [\![\widetilde{t}_n\rangle\!\rangle m'$ also hold.

Since both u_1 and s_1 belong to $perm(B_{\blacktriangleleft_x}(A_1))$, from Definition 24 we have that $m[\![\widetilde{u}_1\rangle\!\rangle m_1^1$ implies $m[\![\widetilde{s}_1\rangle\!\rangle m_1^1$. Similarly both v_1 and t_1 belong to $perm(E_{\blacktriangleleft_x}(A_1))$, so from Definition 24 we have that $m_1^1 [\![\widetilde{v}_1\rangle\!\rangle m_1^2$ implies $m_1^1 [\![\widetilde{t}_1\rangle\!\rangle m_1^2$.
Repeating this reasoning $n-1$ times we obtain $m[\![\widetilde{s}_1\rangle\!\rangle m_1^1 [\![\widetilde{t}_1\rangle\!\rangle m_1^2 \ldots m_{n-1}^2 [\![\widetilde{s}_n\rangle\!\rangle m_n^1 [\![\widetilde{t}_n\rangle\!\rangle m'$, i.e. $m[\![y\rangle\!\rangle m'$, i.e. $y \in \mathsf{FIS}_N(m \rightarrow m')$. $\square$

We can now formally define the set of interval orders that is generated by a given inhibitor net.

Definition 26 *The set of all interval orders that lead from the marking m to m' is defined as:*

$$\mathsf{IO}_N(m \rightarrow m') = \{\blacktriangleleft_x \mid x \in \mathsf{FIS}_N(m \rightarrow m')\}. \qquad \square$$

The next result shows that the interval sequence semantics is consistent with both sequence semantics and step sequence semantics. First we show consistency with standard firing sequences and extended firing sequences.

Lemma 4 *For every two $m, m' \subseteq P$, then for each $t \in T$,*

$$m[t\rangle m' \iff m[\![BtEt\rangle\!\rangle m'.$$

ProofSince ${}^\bullet Bt = {}^\bullet t$ and $Bt^\circ = t^\circ \cup (t^\circ)^\bullet$, t is enabled at m if and only if Bt is enabled at m.
$(\Rightarrow)$ If $m[t\rangle m'$ then $m' = (m \setminus {}^\bullet t) \cup t^\bullet$. Let $m[\![Bt\rangle\!\rangle m_B$, i.e. $m_B = (m \setminus {}^\bullet Bt) \cup Bt^\bullet = (m \setminus {}^\bullet t) \cup \{t\}$. Hence Et is enabled at m_B. Let $m_B[\![Et\rangle\!\rangle m_E$. And $m_E = (m_B \setminus {}^\bullet Et) \cup Et^\bullet =$

$(((m \setminus {}^\bullet t) \cup \{t\}) \setminus \{t\}) \cup t^\bullet = (m \setminus {}^\bullet t) \cup t^\bullet = m'$. Hence $m[\![Bt\rangle\!\rangle m_B[\![Et\rangle\!\rangle m'$, i.e. $m[\![BtEt\rangle\!\rangle m'$.

($\Leftarrow$) If $m[\![BtEt\rangle\!\rangle m'$ then reasoning as in the proof of ($\Rightarrow$) we can show that $m' = (m \setminus {}^\bullet t) \cup t^\bullet$. Hence $m[t\rangle m'$. $\square$

We have a similar relationship between firing step sequences and extended firing sequences. For every $A = \{t_1,...,t_k\} \subseteq T$, let $A^{BE} \subseteq \mathscr{T}^*$ be defined as follows

$$A^{BE} = \{Bt_{i_1}...Bt_{i_k}Et_{j_1}...Et_{j_k} \mid i_1,...,i_k \text{ and } j_1,...,j_k \text{ are permutations of } 1,2,...,k\}.$$

For example $\{a,b\}^{BE} = \{BaBbEaEb, BaBbEbEa, BbBaEaEb, BbBaEbEa\}$.

Lemma 5

For every two markings $m, m' \subseteq P$ and every $A \subseteq T$,

$$m\ [A\rangle\ m' \iff \forall x \in A^{BE}.\ m\ [\![x\rangle\!\rangle\ m'.$$

Proof($\Rightarrow$) Let $A = \{t_1,...,t_k\}$. This means, if $i \neq j$ then $(t_i^\bullet \cup {}^\bullet t_i) \cap (t_j^\bullet \cup {}^\bullet t_j) = \emptyset$, ${}^\bullet A \subseteq m$, $(A^\bullet \cap A^\circ) \cap m = \emptyset$, and $m' = (m \setminus {}^\bullet A) \cup A^\bullet$. Let $y = Bt_{i_1}...Bt_{i_k}$ and $z = Et_{j_1}...Et_{j_k}$, where $i_1,...,i_k$ and $j_1,...,j_k$ are permutations of $1,2,...,k$. Since ${}^\bullet t_i = {}^\bullet Bt_i$ and $t_i^\circ = Bt_i^\circ$, we have $m\ [\![y\rangle\!\rangle\ m_B$, where $m_B = (m \setminus ({}^\bullet Bt_{i_1} \cup ... {}^\bullet Bt_{i_k})) \cup (Bt_{i_1}^\bullet \cup ... Bt_{i_k}^\bullet) = (m \setminus {}^\bullet A) \cup (Bt_{i_1}^\bullet \cup ... Bt_{i_k}^\bullet)$. But $Bt_i^\bullet = \{t_i\}$, so $m_B = (m \setminus {}^\bullet A) \cup A$. However, ${}^\bullet Et_i = \{t_i\}$, so $m_B\ [\![z\rangle\!\rangle\ m_E$, where $m_E = (m_B \setminus ({}^\bullet Et_{j_1} \cup ... \cup {}^\bullet Et_{j_k})) \cup (Et_{j_1}^\bullet \cup ... Et_{j_k}^\bullet)$. Since ${}^\bullet Et_i = \{t_i\}$ and $Et_i^\bullet = t_i^\bullet$, $m_E = (m_B \setminus A) \cup A^\bullet = (((m \setminus {}^\bullet A) \cup A) \setminus A) \cup A^\bullet = (m \setminus {}^\bullet A) \cup A^\bullet = m'$.

Hence $m\ [A\rangle\ m' \implies \forall x \in A^{BE}.\ m\ [\![x\rangle\!\rangle\ m'$.

($\Leftarrow$) Let $A = \{t_1,...,t_k\}$ and $m\ [\![yz\rangle\!\rangle\ m'$.

Hence there are markings $m_B^0, m_B^1, ..., m_B^k, m_E^0, m_E^1, ..., m_E^k$ in N^{BE} such that $m = m_B^0$, $m_B^k = m_E^0$, $m_E^k = m'$, and

$$m_B^0 \; [\![Bt_{i_1}\rangle\!\rangle \; m_B^1 \; [\![Bt_{i_2}\rangle\!\rangle \; m_B^2 ... m_B^{k-1} \; [\![Bt_{i_k}\rangle\!\rangle \; m_B^k \; [\![Et_{j_1}\rangle\!\rangle \; m_E^1 \; [\![Et_{j_2}\rangle\!\rangle \; m_E^2 ... m_E^{k-1} \; [\![Et_{j_k}\rangle\!\rangle \; m_E^k.$$

We have $m_B^{l+1} = (m_B^l \setminus {}^\bullet Bt_{i_l}) \cup Bt_{i_l}^\bullet$, and $m_E^{l+1} = (m_E^l \setminus {}^\bullet Et_{j_l}) \cup Et_{j_l}^\bullet$, for $l = 0, ..., k-1$. Since ${}^\bullet Bt_i = {}^\bullet t_i$ and $Bt_i^\bullet = \{t_i\}$, $m_E^0 = m_B^k = (m_B^0 \setminus {}^\bullet A) \cup A = (m \setminus {}^\bullet A) \cup A$. However, ${}^\bullet Et_i = \{t_i\}$ and $Et_i^\bullet = t_i^\bullet$, so $m_E^k = (m_E^0 \setminus A) \cup A^\bullet$.

Thus, $m' = m_E^k = (m_E^0 \setminus A) \cup A^\bullet = (((m \setminus {}^\bullet A) \cup A) \setminus A) \cup A^\bullet = (m \setminus {}^\bullet A) \cup A^\bullet$. But this means $m \, [A\rangle \, m'$. □

Hence:

Proposition 10 *For every inhibitor net:*

$$\mathsf{TO}_N(m \to m') \subseteq \mathsf{SO}_N(m \to m') \subseteq \mathsf{IO}_N(m \to m').$$

Proof Let $\lhd \in \mathsf{TO}_N(m \to m')$. Hence there is $x \in \mathsf{FS}_N(m \to m')$ such that $\lhd = \lhd_x$. Assume that $x = t_1 t_2 \ldots t_n$ and define $x' = \{t_1\}\{t_2\}\ldots\{t_n\}$. Clearly $m[\{t_1\}\{t_2\}\ldots\{t_n\}\rangle m'$ and $\lhd = \lhd_{x'}$, so $x' \in \mathsf{FSS}_N(m \to m')$ and $\lhd \in \mathsf{SO}_N(m \to m')$. Therefore $\mathsf{TO}_N(m \to m') \subseteq \mathsf{SO}_N(m \to m')$.

Let $\lhd \in \mathsf{SO}_N(m \to m')$. Hence there is $x \in \mathsf{FFS}_N(m \to m')$ such that $\lhd = \lhd_x$. Assume that $x = A_1 A_2 \ldots A_n$, and $m = m_0[A_1\rangle m_1 [A_2\rangle m_2 \ldots m_{n-1}[A_n\rangle m_n = m'$. By Lemma 5 we have $m = m_0[\![x_1\rangle\!\rangle m_1 [\![x_2\rangle\!\rangle m_2 \ldots m_{n-1} [\![x_n\rangle\!\rangle m_n = m'$, for some $x_i \in A_i^{BE}$, $i = 1, \ldots, n$.

We define $x' = x_1 x_2 \ldots x_n$. Since each $x_i \in A_i^{BE}$, and in each element of A_i^{BE} all appropriate Bt occur before all appropriate Et we can have

$$a^{(i)} \lhd_x b^{(j)} \iff Ea^{(i)} \lhd_{x'} Bb^{(j)} \iff a^{(i)} \blacktriangleleft_{x'} b^{(j)},$$

i.e., $\lhd_x = \blacktriangleleft_{x'}$. Clearly $m[\![x'\rangle\!\rangle m'$, so $\blacktriangleleft_{x'} \in \mathsf{IO}_N(m \to m')$. Hence we can get $\mathsf{SO}_N(m \to m') \subseteq \mathsf{IO}_N(m \to m')$. □

For the net N_Q from Figures 6.10 and 8.11, we have

$$\mathsf{FIS}_{N_Q}(\{s_1,s_2\}\to\{s_4,s_5\}) = \left\{ \begin{array}{l} BcEcBaEaBbEb, BaBcEcEaBbEb, BaBcEaEcBbEb, \\ BcBaEcEaBbEb, BcBaEaEcBbEb, BaBcEaBbEbEc, \\ BaBcEaBbEcEb, BcBaEaBbEbEc, BcBaEaBbEcEb, \\ BaEaBbEbBcEc \end{array} \right\},$$

and $\mathsf{IO}_{N_Q}(\{s_1\}\to\{s_5,s_6\}) = \{<^Q_1, <^Q_2, <^Q_3, <^Q_4\}$, where $<^Q_1, <^Q_2, <^Q_3$ and $<^Q_4$ are partial orders from Figure 6.10. The detailed relationships between the partial orders $<^Q_1$, $<^Q_2$, $<^Q_3, <^Q_4$ and the elements of $\mathsf{FIS}_{N_P}(\{s_1,s_2\}\to\{s_4,s_5\})$ are discussed at the end of Section 7.1.

Note that there are inhibitor nets that all their observations are interval orders. The net N_0 from Figure 8.11 is such a net. For $m_0 = \{s_1,s_2\}$ and $m_f = \{s_4,s_5\}$, we have $\mathsf{FIS}_{N_0}(m_0\to m_f) = \{BaBcEaBbEbEc, BaBcEaBbEcEb, BcBaEaBbEbEc, BaBcEaBbEcEb\}$ so $\mathsf{IO}_{N_0}(m_0\to m_f) = \{<^Q_4\}$, while $\mathsf{FS}_{N_0}(m_0\to m_f) = \mathsf{FSS}_{N_0}(m_0\to m_f) = \emptyset$, and $\mathsf{TO}_{N_0}(m_0\to m_f) = \mathsf{SO}_{N_0}(m_0\to m_f) = \emptyset$.

8.3 Trace and Comtrace Semantics

One of the disadvantages of any operational semantics is that it does not recognize equivalent executions, so they cannot identify concurrent histories. For instance for the net N_Q of Figures 6.10 and 8.11 the observations $<^Q_2, <^Q_3$ and $<^Q_4$ are equivalent and we have two concurrent histories $\{<^Q_1\}$ and $\{<^Q_2, <^Q_3, <^Q_4\}$.

8.3.1 Trace Semantics

The *trace semantics* (and partially ordered behaviours it generates) can standardly be derived from the firing sequence semantics (c.f. [Janicki and Koutny [1995]]).

Let $N = (P,T,F,I,m_0)$ be an inhibitor net. We define the (*trace*) *independency* relation $ind_N^{tr} \subseteq T \times T$ as (c.f. [Janicki and Koutny [1995]]):

$$\begin{aligned}(a,b) \in ind_N^{tr} \iff & [(a^\bullet \cup {}^\bullet a) \cap (b^\bullet \cup {}^\bullet b) = \emptyset] \wedge \\ & [(a^\circ \cap {}^\bullet b) \cup (b^\circ \cap {}^\bullet a) = \emptyset] \wedge [(a^\bullet \cap b^\circ) \cup (b^\bullet \cap a^\circ) = \emptyset].\end{aligned}$$

The trace alphabet is (T, ind_N^{tr}).

The following result validates the above definition of ind_N^{tr}.

Lemma 6

$$(a,b) \in ind_N^{tr} \iff \exists m,m' \in P.\ \{a,b\} \subseteq enabled_N(m) \wedge m[ab\rangle m' \wedge m[ba\rangle m'.$$

Proof ($\Rightarrow$) If $(a,b) \in ind_N^{tr}$ then $m = {}^\bullet a \cup {}^\bullet b$ and $m' = a^\bullet \cup b^\bullet$ satisfy $\{a,b\} \subseteq enabled_N(m)$, $m[ab\rangle m'$ and $m[ba\rangle m'$.

($\Leftarrow$) If $\{a,b\} \subseteq enabled_N(m)$ then ${}^\bullet a \cup {}^\bullet b \subseteq m$ and $(a^\bullet \cup a^\circ) \cap m = (b^\bullet \cup b^\circ) \cap m = \emptyset$. Let m_a and m_b be such markings that $m[a\rangle m_a[b\rangle m'$ and $m[b\rangle m_b[a\rangle m'$. Hence $m_a = (m \setminus {}^\bullet a) \cup a^\bullet$ and $m_b = (m \setminus {}^\bullet b) \cup b^\bullet$. Since b is enabled at m_a then ${}^\bullet b \subseteq (m \setminus {}^\bullet a) \cup a^\bullet$ and $b^\bullet \cap ((m \setminus {}^\bullet a) \cup a^\bullet) = \emptyset$, so ${}^\bullet a \cap {}^\bullet b = \emptyset$ and $a^\bullet \cap b^\bullet = \emptyset$. Moreover $b^\circ \cap ((m \setminus {}^\bullet a) \cup a^\bullet) = \emptyset$ so $b^\circ \cap a^\bullet = \emptyset$. Similarly, since a is enabled at m_b then $a^\circ \cap b^\bullet = \emptyset$. Suppose that $t \in {}^\bullet a \cap b^\bullet$. Since $m[a\rangle m_a[b\rangle m'$ and $m[b\rangle m_b[a\rangle m'$, then $m' = (((m \setminus {}^\bullet a) \cup a^\bullet) \setminus {}^\bullet b) \cup b^\bullet = (((m \setminus {}^\bullet b) \cup b^\bullet) \setminus {}^\bullet a) \cup a^\bullet$. But $t \in (((m \setminus {}^\bullet a) \cup a^\bullet) \setminus {}^\bullet b) \cup b^\bullet$ and $t \notin (((m \setminus {}^\bullet b) \cup b^\bullet) \setminus {}^\bullet a) \cup a^\bullet$, a contradiction, so ${}^\bullet a \cap b^\bullet = \emptyset$. Symmetrically ${}^\bullet b \cap a^\bullet = \emptyset$. Hence we obtained $(a^\bullet \cup {}^\bullet a) \cap (b^\bullet \cup {}^\bullet b) = \emptyset$.

Suppose $t \in b^{\circ} \cap^{\bullet} a$. But $^{\bullet}a \subseteq m$ while $b^{\circ} \cap m = \emptyset$, a contradiction, so $b^{\circ} \cap^{\bullet} a = \emptyset$. Similarly we can show that $a^{\circ} \cap^{\bullet} b = \emptyset$. Hence $(a^{\circ} \cap {}^{\bullet}b) \cup (b^{\circ} \cap {}^{\bullet}a) = \emptyset$ and $(a^{\bullet} \cap b^{\circ}) \cup (b^{\bullet} \cap a^{\circ}) = \emptyset$. Thus $(a,b) \in ind_N^{tr}$. □

In the above lemma we *do not* assume that either m or m' are reachable from the initial marking m_0, they both may be unreachable.

When we have the relation ind_N^{tr}, then for each firing sequence x, the trace $[x]_{ind_N^{tr}}$ describes a behaviour of the inhibitor net N.

The set of all traces defining behaviours that start from the marking m and end at the marking m' is defined as

$$\mathsf{Tr}_N(m \rightarrow m') = \{[x]_{ind_N^{tr}} \mid x \in \mathsf{FS}_N(m \rightarrow m')\}.$$

For the net N_Q in Figures 6.10 and 8.11, we have $ind_{N_Q}^{tr} = \emptyset$; however for the net N_Q^1 in Figure 8.11, $ind_{N_Q^1}^{tr}$ is the symmetric closure of $\{(Ba,Bc),(Ba,Ec),(Ba,Bb),(Ba,Eb),(Bb,Ec),(Ea,Ec),(Eb,Ec)\}$, and
$\mathsf{Tr}_{N_P}(\{s_1,s_2\} \rightarrow \{s_4,s_5\}) = \{\mathbf{x_1},\mathbf{x_2}\}$, where
$\mathbf{x_1} = [BaEaBbEbBcEc]_{ind_{N_Q}^{tr}} = \{BaEaBbEbBcEc\}$, and

$$\mathbf{x_2} = [BcBaEaBbEbEc]_{ind_{N_Q}^{tr}} = \left\{ \begin{array}{l} BcEcBaEaBbEb, BaBcEcEaBbEb, BaBcEaEcBbEb, \\ BcBaEcEaBbEb, BcBaEaEcBbEb, BaBcEaBbEbEc, \\ BaBcEaBbEcEb, BcBaEaBbEbEc, BcBaEaBbEcEb \end{array} \right\}.$$

The partially ordered behaviour defined by the trace $\mathbf{x_2}$, i.e $\lessdot_{\mathbf{x_2}}$, is just the partial order $\lessdot_2^Q$ from Figure 6.10.

Note that the properties $(Ba,Bb) \in ind_{N_Q}^{tr}$ or $(Ba,Eb) \in ind_{N_Q}^{tr}$ are *never used*, as there is no firing sequence x starting from $\{s_1,s_2\}$ such that $x = uBaBbw$ or $x = uBaEbw$ In other words, the relation $ind^{tr}_{N_P}$ is unnecessarily big, but this is all we can get by using the static structure on the net N to define the relation ind_N (c.f. [Mazurkiewicz [1977, 1995]]).

Traces cannot express simultaneity directly, for this we need the concept of *comtraces* [Janicki and Koutny [1995]].

8.3.2 Comtrace Semantics

The *comtrace semantics* (and behaviours defined by stratified order structures it generates) can standardly be derived from the firing step sequence semantics [Janicki and Koutny [1995]; Kleijn and Koutny [2004]]. Let $N = (P,T,F,I,m_0)$ be an inhibitor net. In this case we define the following relations $sim_N, ser_N \subseteq T \times T$:

$$(a,b) \in sim_N \iff (a^\bullet \cup {}^\bullet a) \cap (b^\bullet \cup {}^\bullet b) = \emptyset \wedge (a^\circ \cap {}^\bullet b) \cup (b^\circ \cap {}^\bullet a) = \emptyset,$$

$$(a,b) \in ser_N \iff (a,b) \in sim_N \wedge a^\bullet \cap b^\circ = \emptyset.$$

The comtrace alphabet here is (T, sim_N, ser_N).

The interpretation of relations sim_N and ser_N is identical to that of *sim* and *ser* for comtraces (see Section 4.3). The relation sim_N, called *simultaneity*, is symmetric and $(a,b) \in sim_N$ means that a and b can potentially be executed simultaneously. The relation ser_N is called *serializability* and is not necessarily symmetric. We say $(a,b) \in ser_N$ if either a and b can execute simultaneously, or a can execute before b. If $(a,b) \in ser_N$ and $(b,a) \in ser_N$, then a and b can potentially either execute simultaneously, or in *any* order (a before b or b before a). The relations sim_N and ser_N represent the potential concurrency structure of a given net.

The validity of sim_N and ser_N follows from the following simple result, where right hand sides of equivalence express the meanings of *simultaneity* and *serializability* in terms of firing step sequence semantics.

Lemma 7

1. $(a,b) \in sim_N \iff \exists A \subseteq T.\exists m,m' \subseteq P.\ a,b \in A \wedge m[A\rangle m'$,

2. $(a,b) \in ser_N \iff \exists A,B,C \subseteq T.\exists m,m' \subseteq P.\ a \in A \wedge b \in B \wedge A \cap B = \emptyset \wedge$
$$A \cup B = C \wedge m[C\rangle m' \wedge m[AB\rangle m'$$

Proof (1)($\Rightarrow$) If $(a,b) \in sim_N$ then $A = \{a,b\}$, $m = {}^\bullet a \cup {}^\bullet b$ and $m' = a^\bullet \cup b^\bullet$ satisfy $m[A\rangle m'$.

(1)($\Leftarrow$) $m[A\rangle m'$ means A is enabled at m, hence $(t^\bullet \cup^\bullet t) \cap (r^\bullet \cup^\bullet r) = \emptyset$ for all distinct $t,r \in A$, so $(a^\bullet \cup {}^\bullet a) \cap (b^\bullet \cup {}^\bullet b) = \emptyset$. Since A is enabled at m, we also have ${}^\bullet A \subseteq m$ and $(A^\bullet \cup A^\circ) \cap m = \emptyset$, which implies $A^\circ \cap^\bullet A = \emptyset$, so $(a^\circ \cap {}^\bullet b) \cup (b^\circ \cap {}^\bullet a) = \emptyset$.

(2)($\Rightarrow$) Let $(a,b) \in ser_N$. Then $A = \{a\}$, $B = \{b\}$, $C = \{a,b\}$, $m = {}^\bullet a \cup {}^\bullet b$ and $m' = a^\bullet \cup b^\bullet$ satisfy $m[C\rangle m'$ and $m[AB\rangle m'$.

(2)($\Leftarrow$) By (1) of this lemma, $m[C\rangle m'$ means that $(a,b) \in sim_N$. Let $m'' = (m \setminus^\bullet A) \cup A^\bullet$, i.e. $m[A\rangle m''$ and $m''[B\rangle m'$. Clearly $a^\bullet \subseteq A^\bullet \subseteq m''$. Since B is enabled at m'' then $B^\circ \cap m'' = \emptyset$, so $b^\circ \cap m'' = \emptyset$. But this means $a^\bullet \cap b^\circ = \emptyset$. $\square$

Again, as in Lemma 6, in the above lemma we *do not* assume that either m or m' are reachable from the initial marking m_0.

In this case for each firing step sequence x, the comtrace $[x]_{(sim_N,ser_N)}$ describes a behaviour of the inhibitor net N.

The set of all comtraces defining behaviours that start from the marking m and end at the marking m' is defined as

$$\mathsf{ComTr}_N(m \rightarrow m') = \{[x]_{(sim_n, ser_N)} \mid x \in \mathsf{FSS}_N(m \rightarrow m')\}.$$

For the net N_Q from Figure 6.10 we have

$$sim_{N_P} = \{(a,c),(c,a)\},$$
$$ser_{N_P} = \{(c,a)\}.$$

i.e. $(a,c) \notin ser_{N_P}$, $\mathsf{ComTr}_{N_P}(\{s_1,s_2\} \rightarrow \{s_4,s_5\}) = \{\mathbf{x_1},\mathbf{x_2}\}$, where $\mathbf{x_1} = \{\{a\}\{b\}\{c\}\}$, $\mathbf{x_2} = \{\{c\}\{a\}\{b\},\{a,c\}\{b\}\}$. When step sequences are interpreted as partial orders, the comtrace $\mathbf{x_1}$ represents the set $\{<^Q_1\}$ and the comtrace $\mathbf{x_2}$ represents the set $\{<^Q_2,<^Q_3\}$, where $<^Q_1,<^Q_2,<^Q_3$ are the partial orders from from Figure 6.10. *Note that* $<^Q_4$ *is also a possible system run of the net* N_Q*, but this is not expressible in this model.*

The *process semantics* (in the sense of [Nielsen et al. [1990]; Reisig [1998]]) has been proposed in [Janicki and Koutny [1995]] and substantially refined in [Kleijn and Koutny [2004]]. It was proven the process semantics and comtrace semantics are equivalent to some extent. The process semantics will not be discussed in this thesis, the details can be found in [Janicki et al. [2010]; Kleijn and Koutny [2008]].

8.4 Interval Trace Semantics and Interval Order Structure Semantics

Since interval traces are just a special kind of general traces, we will just modify the standard trace semantics of inhibitor nets. The main difference is to define the independency relation on *BE*-transitions instead on transitions.

Let $N = (P,T,F,I,m_0)$ be an inhibitor net, and let $\mathscr{T} = \{Bt \mid t \in T\} \cup \{Et \mid t \in T\}$. We define the (*interval trace*) *independency* relation $ind_N \subseteq \mathscr{T} \times \mathscr{T}$ as follows.

Definition 27 *For all distinct $a,b \in T$:*

1. $(Ba,Bb) \in ind_N \wedge (Ea,Eb) \in ind_N$

2. $(Ba,Eb) \in ind_N \iff [(Ba^\bullet \cup {}^\bullet Ba) \cap (Eb^\bullet \cup {}^\bullet Eb) = \emptyset] \wedge$
$$[(Ba^\circ \cap {}^\bullet Eb) \cup (Eb^\circ \cap {}^\bullet Ba) = \emptyset] \wedge$$
$$[(Ba^\bullet \cap Eb^\circ) \cup (Eb^\bullet \cap Ba^\circ) = \emptyset].$$

The interval trace alphabet is $(\mathscr{T}, ind_N)$. □

Corollary 4 *For each $t \in T$, $(Bt,Et) \notin ind_N$ and $(Et,Bt) \notin ind_N$.*

Proof Since $Bt^\bullet \cap {}^\bullet Et = \{t\}$, for each $t \in T$. □

Corollary 4 shows that Definition 19 is satisfied so $(\mathscr{T}, ind_N)$ is indeed an interval trace alphabet indeed.

The following results validate Definition 27. The first result guarantees that the condition (1) of this definition, it does not introduce undesired non-existent behaviours. This result is a consequence of Definition 24(2e) and (2f) which defined Bt° and Et°.

Proposition 11 *For all distinct $a,b \in T$:*

1. *If* $\neg([(Ba^\bullet \cup {}^\bullet Ba) \cap (Bb^\bullet \cup {}^\bullet Bb) = \emptyset] \wedge [(Ba^\circ \cap {}^\bullet Bb) \cup (Bb^\circ \cap {}^\bullet Ba) = \emptyset] \wedge$
$$[(Ba^\bullet \cap Bb^\circ) \cup (Bb^\bullet \cap Ba^\circ) = \emptyset]),$$
then there are no $m,m' \in P \cup T$ such that $m[\![BaBb\rangle\!\rangle m'$ or $m[\![BbBa\rangle\!\rangle m'$, so the relationship $(Ba,Bb) \in ind_N$ can never be used to commute Ba with Bb.

2. $[(Ea^\bullet \cup {}^\bullet Ea) \cap (Eb^\bullet \cup {}^\bullet Eb) = \emptyset] \wedge [(Ea^\circ \cap {}^\bullet Eb) \cup (Eb^\circ \cap {}^\bullet Ea) = \emptyset] \wedge$
$$[(Ea^\bullet \cap Eb^\circ) \cup (Eb^\bullet \cap Ea^\circ) = \emptyset].$$

Proof (1) *Case 1*: $(Ba^{\bullet} \cup {}^{\bullet}Ba) \cap (Bb^{\bullet} \cup {}^{\bullet}Bb) \neq \emptyset$. Suppose that $m[\![Ba\rangle\!\rangle m_1$. Then $Bb \notin enabled_N^{ext}(m_1)$. Similarly for $m[\![Bb\rangle\!\rangle m_1$.

Case 2: $(Ba^{\circ} \cap {}^{\bullet}Bb) \cup (Bb^{\circ} \cap {}^{\bullet}Ba) \neq \emptyset$. Let $r \in Ba^{\circ} \cap {}^{\bullet}Bb$ and $m[\![Ba\rangle\!\rangle m_1$. However firing Ba does not remove any token from Ba°, so $r \in m_1 \cap Ba^{\circ}$, i.e. $Bb \notin enabled_N^{ext}(m_1)$. If $r \in Bb^{\circ} \cap {}^{\bullet}Ba$ then $Ba \notin enabled_N^{ext}(m)$ for any m such that $r \in m$. Hence $m[\![Ba\rangle\!\rangle m_1$ means $r \notin m$. Since $m_1 = (m \setminus^{\bullet} Ba) \cup \{a\}$, then $r \notin m_1$ so $\neg({}^{\bullet}Bb \subseteq m_1)$, which means $Bb \notin enabled_N^{ext}(m_1)$ again.

Case 3: $(Ba^{\bullet} \cap Bb^{\circ}) \cup (Bb^{\bullet} \cap Ba^{\circ}) \neq \emptyset$. Let $Ba^{\bullet} \cap Bb^{\circ} \neq \emptyset$. Since $Ba^{\bullet} = \{a\}$, this means $a \in Bb^{\circ}$. Suppose that $m[\![Ba\rangle\!\rangle m_1$. Hence $m_1 = (m \setminus^{\bullet} Ba) \cup \{a\}$. But $a \in m_1 \cup Bb^{\circ}$ means $Bb \notin enabled_N^{ext}(m_1)$. Now let $Bb^{\bullet} \cap Ba^{\circ} \neq \emptyset$, i.e. $b \in Ba^{\circ}$. By Definition 24(2e), $Ba^{\circ} = a^{\circ} \cup (a^{\circ})^{\bullet}$. Since $b \in a^{\circ}$ then $b \in (a^{\circ})^{\bullet}$. But this means $a^{\circ} \cup^{\bullet} b \neq \emptyset$, or, as ${}^{\bullet}b =^{\bullet} Bb$, $a^{\circ} \cup^{\bullet} Bb \neq \emptyset$, which implies $Ba^{\circ} \cup^{\bullet} Bb \neq \emptyset$, is a part of Case 2.

(2) Since ${}^{\bullet}t = \{t\}$ and $Et^{\circ} = \emptyset$, for each $t \in T$. $\square$

The next result states that interval traces produced by applying the relation ind_N are consistent with the concept of firing interval sequences.

Lemma 8 *For all extended markings m, m':*

$$x \in \mathsf{FIS}_N(m \to m') \iff [x]_{ind_N} \subseteq \mathsf{FIS}_N(m \to m'),$$

Proof ($\Leftarrow$) Because $x \in [x]_{ind_N}$.

($\Rightarrow$) It suffices to show that if $x_a \alpha \beta x_2 \in \mathsf{FIS}_N(m \to m')$ and $(\alpha, \beta) \in ind_N$, then $x_a \beta \alpha x_2 \in \mathsf{FIS}_N(m \to m')$. Assume $m[\![x_a\rangle\!\rangle m_1[\![\alpha\rangle\!\rangle m_2[\![\beta\rangle\!\rangle m_3[\![x_2\rangle\!\rangle m'$, i.e., ${}^{\bullet}\alpha \subseteq m_1$, $(\alpha^{\bullet} \cup \alpha^{\circ}) \cap m_1 = \emptyset$, $m_2 = (m_1 \setminus^{\bullet} \alpha) \cup \alpha^{\bullet}$, and ${}^{\bullet}\beta \subseteq m_2$, $(\beta^{\bullet} \cup \beta^{\circ}) \cap m_2 = \emptyset$, $m_3 = (m_2 \setminus^{\bullet} \beta) \cup \beta^{\bullet}$.

Since $(\alpha, \beta) \in ind_N$ then ${}^{\bullet}\beta \cap ({}^{\bullet}\alpha \cup \alpha^{\bullet}) = \emptyset$, so ${}^{\bullet}\beta \subseteq m_2 = (m_1 \setminus^{\bullet} \alpha) \cup \alpha^{\bullet}$ implies ${}^{\bullet}\beta \subseteq m_1$. If $(\alpha, \beta) \in ind_N$ then also $(\beta^{\bullet} \cap \beta^{\circ}) \cap ({}^{\bullet}\alpha \cup \alpha^{\bullet}) = \emptyset$, which implies:

$(\beta^{\bullet} \cup \beta^{\circ}) \cap m_2 = \emptyset \iff (\beta^{\bullet} \cup \beta^{\circ}) \cap ((m_1 \setminus^{\bullet} \alpha) \cup \alpha^{\bullet}) = \emptyset \iff (\beta^{\bullet} \cup \beta^{\circ}) \cap m_1 = \emptyset$. Hence β is enabled at m_1. Let $m_2' = (m_1 \setminus^{\bullet} \beta) \cup \beta^{\bullet}$. As $(\alpha, \beta) \in ind_N$ then

${}^{\bullet}\alpha \cap ({}^{\bullet}\beta \cup \beta^{\bullet}) = \emptyset$, so ${}^{\bullet}\alpha \subseteq m_1$ implies ${}^{\bullet}\alpha \subseteq (m_1 \setminus {}^{\bullet}\beta) \cup \beta^{\bullet} = m'_2$.

Again, if $(\alpha,\beta) \in ind_N$ then also $(\alpha^{\bullet} \cap \alpha^{\circ}) \cap ({}^{\bullet}\beta \cup \beta^{\bullet}) = \emptyset$, which implies:

$(\alpha^{\bullet} \cup \alpha^{\circ}) \cap m_1 = \emptyset \iff (\alpha^{\bullet} \cup \alpha^{\circ}) \cap ((m_1 \setminus {}^{\bullet}\beta) \cup \beta^{\bullet}) = \emptyset \iff (\alpha^{\bullet} \cup \alpha^{\circ}) \cap m'_2 = \emptyset$. Hence α is enabled at m'_2. Since $(\alpha^{\bullet} \cup {}^{\bullet}\alpha) \cap (\beta^{\bullet} \cup {}^{\bullet}\beta) = \emptyset$, we have:

$(m'_2 \setminus {}^{\bullet}\alpha) \cup \alpha^{\bullet} = (((m_1 \setminus {}^{\bullet}\beta) \cup \beta^{\bullet}) \setminus \alpha^{\bullet}) \cup \alpha^{\bullet} = (((m_1 \setminus {}^{\bullet}\alpha) \cup \alpha^{\bullet}) \setminus \beta^{\bullet}) \cup \beta^{\bullet} =$

$(m_2 \setminus \beta^{\bullet}) \cup \beta^{\bullet} = m_3$.

But this means that $m[\![x_a\rangle\!\rangle m_1 [\![\beta\rangle\!\rangle m'_2 [\![\alpha\rangle\!\rangle m_3 [\![x_2\rangle\!\rangle m'$, i.e. $x_a \beta \alpha x_2 \in \mathsf{FIS}_N(m \to m')$. □

The last result shows that commutation of BE-transitions induced by the relation ind_N apply to, and only to, sequences that can be interpreted as equivalent executions.

Lemma 9

1. $(\alpha,\beta) \in ind_N \implies (\alpha = Ba \wedge \beta = Bb) \vee$
   $$(\exists m, m' \in P \cup T.\ \{\alpha,\beta\} \subseteq enabled_N^{ext}(m) \wedge m[\![\alpha\beta\rangle\!\rangle m' \wedge m[\![\beta\alpha\rangle\!\rangle m').$$
2. $(\alpha,\beta) \in ind_N \Longleftarrow \exists m, m' \in P \cup T.\ \{\alpha,\beta\} \subseteq enabled_N^{ext}(m) \wedge m[\![\alpha\beta\rangle\!\rangle m' \wedge m[\![\beta\alpha\rangle\!\rangle m'$.

Proof In principle this is almost exactly the same proof as the proof of Lemma 6. □

In Lemma 9(2), we *do not* assume that either m or m' are reachable from the initial marking m_0.

When we have the relation ind_N, then for each firing sequence x, the trace $[x]_{ind_N}$ describes a behaviour (concurrent history) of the inhibitor net N.

The set of all interval traces defining behaviours that start from the marking m and end at the marking m' is defined as

$$\mathsf{IntTr}_N(m \to m') = \{[x]_{ind_N} \mid x \in \mathsf{FIS}_N(m \to m')\}.$$

Since every interval trace uniquely defines an interval order structure, we may define the set of all interval order structures defining behaviours that start from the marking m and end at the marking m' as

$$\mathsf{IOS}_N(m \rightarrow m') = \{S^{[x]_{ind_N}} \mid [x]_{ind_N} \in \mathsf{IntTr}_N(m \rightarrow m')\}.$$

By Theorem 9 we can also write $\mathsf{IOS}_N(m \rightarrow m') = \{S^x \mid x \in \mathsf{FIS}_N(m \rightarrow m')\}$.

For the net N_Q from Figures 6.10 and 8.11 we have

$ind_{N_Q} = ind^Q \cup \{(Ba,Bb),(Ea,Eb),(Bb,Ba),(Eb,Ea),(Ba,Eb),(Eb,Ba)\}$,

$[BaEaBbEbBcEc]_{ind_{N_Q}} = \{BaEaBbEbBcEc\}$,

$[BcEcBaEaBbEb]_{ind_{N_Q}} = \{BcEcBaEaBbEb,$

$BaBcEcEaBbEb, BaBcEaEcBbEb, BcBaEcEaBaEb, BaBcEaEcBbEb,$

$BaBcEaBbEbEc, BaBcEaBbEcEb, BcBaEaBbEbEc, BaBcEaBbEcEb\}$,

and $\mathsf{IntTr}_{N_Q}(\{s_1,s_2\} \rightarrow \{s_4,s_5\}) = \{[BaEaBbEbBcEc]_{ind_{N_Q}}, [BcEcBaEaBbEb]_{ind_{N_Q}}\}$.

Also $\mathsf{IOS}_{N_Q}(\{s_1,s_2\} \rightarrow \{s_4,s_5\}) = \{S_1^Q, S_2^Q\}$, where $S_1^Q = (\{a,b,c\}, <_1^Q, <_1^Q)$ and $S_2^Q = (\{a,b,c\}, \prec_2^Q, \sqsubset_2^Q)$, where $<_1^Q$, $\prec_2^Q$ and $\sqsubset_2^Q$ are these from Figure 6.10.

Note that the properties $(Ba,Bb) \in ind_{N_Q}$, $(Ea,Eb) \in ind_{N_Q}$, and $(Ba,Eb) \in ind_{N_Q}$ are *never used*, as there is no extended firing sequence x starting from $\{s_1,s_2\}$ such that $x = uBaBbw$, $x = uEaEbw$ or $x = uBaEbw$, so the relation ind_{N_P} is bigger than needed. Again, this is the price paid for having ind_N derived only from the static structure of the net N.

Chapter 9

Conclusion

In this thesis, we have first introduced the concept of interval traces, a special kind of Mazurkiewicz traces, that can provide an abstract semantics of concurrent systems when the operational semantics involves interval orders.

Then, we proved that interval traces can model interval order structures in the same manner as classical Mazurkiewicz traces can model partial orders [Mazurkiewicz [1995]] and comtraces can model stratified order structures [Janicki and Koutny [1995]]. We also showed each interval trace uniquely determins an interval order structure.

Finally, we discussed the application of interval traces with inhibitor Petri nets. In particular, we have shown how to use interval traces to define interval order semantics of inhibitor nets.

Now we would like to propose some comments for future research of interval traces.

As far as we know, the concept and theory of interval traces stems from three sources: classical traces, comtraces and the representation theorem of Abraham, Ben-David and Magidor ([Abraham et al. [1990]], Theorem 5 in this thesis). Like comtraces, interval traces are generated by two relations *sim* and *ser* on a given set of events, and the interpretation of these relations is the same as for comtraces. However, comtraces are sets of step sequences of event occurrences, interval traces are just sets of ordinary sequences (like classical traces) but with beginnings and ends of event occurrences. Like in classical traces, the structure of interval traces is generated by a single independency relation $ind_{(sim,ser)}$ which is derived from the relations *sim* and *ser*. Technically, interval traces are just a special case of classical traces that are defined on the set of beginnings and ends of events.

The representation theorem of Abraham, Ben-David and Magidor allows representing interval order structures by appropriate partial orders of beginnings and ends. We have already shown that the partial order generated by a given interval trace uniquely defines an interval order structure via the Abraham, Ben-David and Magidor theorem. While, for both Mazurkiewicz traces and comtraces, an equivalent pure process semantics (in a sense of [Nielsen et al. [1990]]) have been constructed [Diekert and Rozenberg [1995]; Janicki and Koutny [1995]; Kleijn and Koutny [2004]]; for interval traces, this remains an open problem for future research.

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