ON THE CLOSED GRAPH AND OPEN MAPPING THEOREMS

By

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SCOPE AND CONTENTS:

The closed graph and open mapping theorems are two of the deeper results in the theory of locally convex spaces. They are very rich in their applications in functional analysis. This thesis contains some extensions of these theorems in locally convex spaces. We begin with a study of α-spaces and γ-spaces, which leads us naturally to a study of δ-spaces. On these spaces, we prove closed graph and open mapping theorems. Similar theorems are also proved for certain classes of $B_r(\mathcal{G})$-spaces. In particular, a closed graph theorem for $B(\mathcal{M})$-spaces enables us to characterise certain classes of $B(\mathcal{L},\mathcal{J})$-spaces. A consideration of countability conditions in locally convex spaces enables us to prove open mapping theorems in $B_r(\mathcal{F})$-spaces. These theorems are then used to relate boundedness of linear mappings and their graphs.

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TABLE OF CONTENTS

In memory of my brother

Ramamoorthi Krishnasamy

INTRODUCTION

The closed graph and open mapping theorems are two of the deeper results in the theory of locally convex spaces. They are very rich in their applications in functional analysis. This thesis contains some extensions of these theorems in locally convex spaces.

The first chapter of this work is composed of those definitions and results from the theory of locally convex spaces which are needed in future chapters. The proofs of many of the results are omitted for the reason that they are easily available in books such as Husain [15], Robertson and Robertson [35] and Köthe [23].

Our investigation of the closed graph and open mapping theorems begins in Chapter II. We begin with a study of α-spaces and γ-spaces which were introduced by Levin [26]. In [26], Levin merely quotes, without proofs, some results he obtained in those spaces. We offer proofs and relate these spaces with B-complete spaces due to Pták [31] and Collins [2]. We are also able to give some characterizations of these spaces.

The study of α-spaces and γ-spaces leads us naturally to a class of locally convex spaces which we call, δ-spaces. On these, we prove closed graph and open mapping theorems.

In his papers [11], [12] and [14] and in his book [15], Husain studied locally convex spaces which he called $B(\mathcal{G})$ and $B_r(\mathcal{G})$-spaces (see Chapter I, Definition 6.1). These are spaces which satisfy weakened forms of B-completeness and B_r-completeness conditions of Pták [31]. For the particular case when $\mathcal{B} = \mathcal{T}$, the class of separated barrelled spaces, a $B_r(\mathcal{T})$-space is characterized by the fact that every continuous, one-to-one, linear mapping from it onto a separated barrelled space, is open. On these spaces, Husain was able to prove a very general closed graph theorem (Chapter I, Theorem 6.2). By appealing to methods in the duality theory of locally convex spaces, we are able to give a shorter proof (Chapter III, Theorem 2.1). Husain's proof, as well as ours, relies heavily on a condition of almost openness of the linear mapping. Husain in [15] had queried as to whether this condition of almost openness could be relaxed. Sulley in [40] showed that this cannot be done, in general. We are, however, able to replace almost openness by some other conditions. For onto mappings, we show that for some classes of $B_r(\mathcal{M})$-spaces and $B_r(\mathcal{S}.\mathcal{T})$-spaces, where $\mathcal{M}$ is the class of metrisable locally convex spaces and $\mathcal{S}.\mathcal{T}$ is the class of separable barrelled spaces, the almost openness condition could be dropped (Chapter III, Theorems 3.1 and 3.2). We are also able to characterize certain classes of $B(\mathcal{S}.\mathcal{T})$-spaces.

The closed graph theorem, proved for α-spaces in Chapter I (Theorem 1.1), enables us to obtain a characterization of semi-reflexive spaces (Chapter III, Lemma 4.1). This characterization enables us to obtain closed graph and open mapping theorems for $B_r(\mathcal{F})$-spaces, where $\mathcal{F}$ is the class of locally convex Fréchet spaces.

In Chapter IV, countability conditions in locally convex spaces are investigated. We show that there are locally convex spaces, more general than metrisable spaces whose strong duals possess a countable fundamental family of bounded sets. We investigate these spaces and prove, among other results, closed graph and open mapping theorems, relaxing the necessary completeness requirement by filter conditions, introduced in [36]. We also investigate conditions that might relax these filter conditions.

A linear mapping is called bounded if it preserves bounded sets. In Chapter V, closed graph and open mapping theorems, proved earlier are used to relate boundedness of linear mappings and their graphs.

The numbering of theorems, propositions and lemmas is started afresh at the beginning of each chapter; a reference not preceded by a chapter number applies to the chapter in which it occurs.

CHAPTER I

PRELIMINARIES

1. Terminology and notation

The graph of a mapping t of a set E into another set F is the subset of $E \times F$ consisting of all elements of the form $(x, t(x))$, with $x \in E$. If E and F are topological spaces, with F separated, and if t is continuous, then its graph G is closed. For if $(x, y) \notin G$ there are disjoint neighbourhoods U of $t(x)$ and V of y, and $(t^{-1}(U), V)$ is a neighbourhood of (x, y) not meeting G. The converse is not true as can be seen easily from the following example:

Define $t: R \longrightarrow R$ by $t(x) = \frac{1}{x}$ for $x \neq 0$ and $t(0) = 0$. Then G is closed but t is not continuous.

We also note that if t is a one-to-one and open mapping of a separated topological space E onto another topological space F then the graphs of t and t^{-1} are homeomorphic and so, if the graph of t is closed, the graph of t^{-1} is also closed.

Let E and F be two separated topological vector spaces and consider the following statements:

(1) A continuous linear mapping t of E onto F is open.

(2) A linear mapping s of F into E, with closed graph, is continuous.

It is well known that the statements (1) and (2) are not true for every pair of topological vector spaces E and F. If they are true for some topological vector spaces E and F, with or without any additional conditions on the mappings, (1) and (2) are called the open mapping and closed graph theorems, respectively.

It will be the purpose of later chapters to investigate some extensions of the above theorems to certain classes of topological vector spaces. We shall confine ourselves for the most part to locally convex topological vector spaces. We shall write: "convex space", to mean: "locally convex topological vector space over the real or complex field". All spaces considered are assumed to be separated, unless otherwise stated. Mostly we shall use the notations and and definitions of [15]. Some of the definitions and notations used here and in later chapters are as follows.

E_u denotes a convex space endowed with a locally-convex topology u. The space $E' = E'_u$ denotes the dual of E_u i.e. the set of all continuous linear functionals on E; E^* denotes the algebraic dual of E i.e. the set of all linear functionals on E. If E and E' are in duality i.e. each is a vector subspace of the algebraic dual of the other, we shall call (E, E') a dual pair. For any vector space E with algebraic dual E^*, (E, E^*) is a dual pair.

If (E, E') is a dual pair, then we have the following topologies:

$\sigma(E, E')$ denotes the weak topology on E determined by E'.

$\sigma(E', E)$ denotes the weak topology on E' determined by E.

$\tau(E, E')$ denotes the Mackey topology on E determined by E'.

$\tau(E', E)$ denotes the Mackey topology on E' determined by E.

$\beta(E, E')$ denotes the strong topology on E determined by E'.

$\beta(E', E)$ denotes the strong topology on E' determined by E.

In view of the fact that (E, E^*) forms a dual pair, we also have the following topologies:

$\sigma(E^*, E)$ denotes the weak topology on E^* determined by E.

$\tau(E, E^*)$ denotes the Mackey topology on E determined by E^*,

i.e. the finest locally convex topology on E

E_σ denotes E endowed with $\sigma(E, E')$.

E_τ denotes E endowed with $\tau(E, E')$.

E_β denotes E endowed with $\beta(E, E')$.

E'^{σ}_u denotes E'_u endowed with $\sigma(E', E)$.

E'^{τ}_u denotes E'_u endowed with $\tau(E', E)$.

E'^{β}_u denotes E'_u endowed with $\beta(E', E)$.

If there is no confusion, E may denote a convex space and E' its dual.

Let (E, E') be a dual pair. If A is a subset of E, the subsets of E' consisting of those x' for which

$$\sup \{ \, |\langle x, x' \rangle| : x \in A \} \leq 1$$

is called the polar of A (in E') and denoted by A^o.

If E is a convex space, we shall call a subset A' of E' almost closed if $A' \cap U^o$ is $\sigma(E', E)$ closed for every neighbourhood U of the origin in E.

It often happens that, for a linear mapping t of E onto F, although we cannot assert that $t(U)$ is a neighbourhood of the origin, we can easily prove that $\overline{t(U)}$ (i.e. the closure of $t(U)$ in F) is. This is the case when F is barrelled for $\overline{t(U)}$ is a barrel, for each absolutely convex neighbourhood U of the origin in E. It is convenient to make the following definitions.

DEFINITION 1.1: A linear mapping t of a convex space E into another convex space F is almost open if $\overline{t(U)}$ is a neighbourhood of the origin in F, for any neighbourhood U of the origin in E.

Similarly, we have:

DEFINITION 1.2: A linear mapping t of a convex space E into another convex space F is almost continuous if $\overline{t^{-1}(V)}$ is a neighbourhood of the origin in E, for every neighbourhood V of the origin in F.

When t is one-to-one and onto, clearly t is almost continuous if and only if t^{-1} is almost open.

2. The transpose of a linear mapping

Suppose that (E, E') and (F, F') are two dual pairs and that t is a linear mapping of E into F. Then $< t(x), y' >$ is a bilinear functional in the two variables x and y'. Denote by $t'(y')$ the linear functional on E which results from this bilinear functional by fixing $y' \varepsilon F'$, so that t' is defined by the identity:

$$< x, t'(y') > = < t(x), y'>$$

valid for all $x \varepsilon E$ and all $y' \varepsilon F'$. Then for each $y' \varepsilon F'$, $t'(y') \varepsilon E^*$ and t' is a linear mapping of F' into E^*. We call t' the transpose of the linear mapping t.

We shall call the linear mapping weakly continuous if it is continuous under the topologies $\sigma(E, E')$ and $\sigma(F, F')$.

The following result is basic.

PROPOSITION 2.1: Let (E, E') and (F, F') be dual pairs and let t be a linear mapping of E into F with transpose t'. Then $t'(F') \subseteq E'$ if and only if t is weakly continuous.

PROOF: First suppose that t is continuous; then for each fixed $y' \in F'$, $< t(x), y' >$ is a continuous linear functional on E. Hence $t'(y') \in E'$.

Next suppose that $t'(F') \subseteq E'$ and let

$$V = \{y: \sup_{1 \leq i \leq n} | < y, y_i' > | \leq 1\}$$

be a $\sigma(F,F')$ basic neighbourhood of the origin in F. Then if

$$U = \{x: \sup_{1 \leq i \leq n} | < x, t'(y_i') > | \leq 1\},$$

U is a $\sigma(E, E')$ basic neighbourhood of the origin in E with $t(U) \subseteq V$. Hence t is weakly continuous. Q.E.D.

If t is a weakly continuous linear mapping of E into F, then t' maps F' into E' and its transpose t'' maps E'' into F'^*. By Proposition 2.1, t'' maps E'' into F'' if and only if t' is continuous when F' and E' have the topologies $\sigma(F', F'')$ and $\sigma(E', E'')$. The simplest case arises when $E'' = E$ and $F'' = F$; then t'' clearly coincides with t, and so we have by Proposition 2.1,

COROLLARY: If t is weakly continuous, so is its transpose t'.

PROPOSITION 2.2: If t is a continuous linear mapping of a convex space E into a convex space F, then t is also weakly continuous.

PROOF: For each fixed $y' \in F'$, $< t(x), y' >$ is a continuous linear functional on E and so $t'(y') \in E'$. Hence $t'(F') \subseteq E'$ and the result follows from Proposition 2.1. Q.E.D.

The converse of this result is not true in general (take, for example, the identity mapping of E under one topology into E under a strictly finer topology of the same dual pair). However, we shall see (Proposition 2.9) that for a suitable topology on E weak continuity implies continuity in the initial topologies.

PROPOSITION 2.3: Let (E, E') be a dual pair. The polars in E' of subsets A and B of E have the following properties:

(i) A^o is absolutely convex and $\sigma(E', E)$ closed;

(ii) if $A \subseteq B$ then $B^o \subseteq A^o$;

(iii) if $\lambda \neq 0$, then $(\lambda A)^o = \frac{1}{|\lambda|} A^o$;

(iv) $(\bigcup_\alpha A_\alpha)^o = \bigcap_\alpha A_\alpha^o$.

PROOF: All are immediate from the definition of polar sets except the $\sigma(E', E)$-closedness of A^o. Now

$$A^o = \bigcap_{x \in A} \{x' : |< x, x' >| \leq 1\}$$

which is an intersection of inverse images of closed sets by $\sigma(E', E)$-continuous functions; hence A^o is $\sigma(E', E)$-closed. Q.E.D.

Another useful result is the following.

PROPOSITION 2.4: If E is a convex space with dual E' and A is a subset of E, then the bipolar A^{oo} of A in E is the $\sigma(E, E')$-closed absolutely convex envelope of A.

PROOF: See [35] (Chapter II, § 4, Theorem 4, Corollary 1).

PROPOSITION 2.5: Let (E, E') and (F, F') be dual pairs and let t be a weakly continuous linear mapping of E into F, with transpose t'. If $A \subset E$ and $B \subset F'$, then:

(i) $(t(A))^o = t'^{-1}(A^o)$

(ii) $t^{-1}(B^o) = (t'(B))^o$.

PROOF: A proof of (i) may be found in [35] (Chapter II, Lemma 6) and (ii) follows on interchanging roles of t and t^{-1}.

PROPOSITION 2.6: Let E and F be convex spaces and t a weakly continuous linear mapping of E into F. Then t is weakly open from E onto $t(E)$ if and only if $t'(F')$ is weakly closed in E'.

PROOF: A proof may be found in [6] (Chapter 8, § 6, Proposition 3).

COROLLARY: If E and F are convex spaces and t is a weakly continuous linear mapping of E into F, then

(i) t is a weak homeomorphism into if and only if $t'(F') = E'$ (i.e. t' is onto).

(ii) $t(E) = F$ (i.e. t is onto) if and only if t' is a weak homeomorphism into.

PROPOSITION 2.7: Let E be a vector space. Then under the topology $\sigma(E^*, E)$, E^* is complete.

PROOF: A proof may be found in [35] (Chapter III, § 6, Proposition 13).

The following result is of fundamental importance in the theory of convex spaces.

THEOREM 2.1: If E_u is a convex space and U is a neighbourhood of the origin then U^o is $\sigma(E', E)$-compact.

PROOF: Give E^* the topology $\sigma(E^*, E)$. Since U is an absorbent set in E, U^o is bounded in E' ([35], Chapter III, § 1, Lemma 2) and therefore precompact ([35], Chapter III, § 7, Proposition 6). Also E^* is complete (Proposition 2.7) and U^o is closed (Proposition 2.3 (i)) and so U^o is complete. Hence U^o is compact. But $U^o \subseteq E'$ and the topologies $\sigma(E', E)$ and $\sigma(E^*, E)$ coincide on E', and so U^o is $\sigma(E', E)$-compact. Q.E.D.

PROPOSITION 2.8: Let E be a separable convex space. Then every equicontinuous subset of E' is weakly metrisable.

PROOF: See [23] (§ 21, 3 (4)).

In Proposition 2.2, we proved that a continuous linear mapping is also weakly continuous. The restricted converse, promised there is:

PROPOSITION 2.9: If E and F are convex spaces, and if E has topology $\tau(E, E')$, then every weakly continuous linear mapping of E into F is also continuous.

PROOF: Let V be a closed absolutely convex neighbourhood of the origin in F. Then by Theorem 2.1, V^o is $\sigma(F', F)$-compact. Since the transpose t' of t is weakly continuous (Proposition 2.1) $t'(V^o)$ is $\sigma(E', E)$-compact. Since $t'(V^o)$ is also absolutely convex, its polar in E is a neighbourhood of the origin in $\tau(E, E')$.

But by Proposition 2.4 (i), $(t'(V^o))^o = t^{-1}(V^{oo}) = t^{-1}(V)$, since V is closed and absolutely convex (Proposition 2.4). Thus t is continuous. Q.E.D.

We shall call a convex space E_u a Mackey space if $u = \tau(E, E')$.

It is clear from the proof of Proposition 2.9 that:

PROPOSITION 2.10: If E and F are convex spaces and t a weakly continuous linear mapping of E into F then t is continuous with respect to the Mackey topologies on E and F i.e. with respect to $\tau(E, E')$ and $\tau(F, F')$:

It is also worth noting that:

PROPOSITION 2.11: If E and F are convex spaces and t a weakly continuous linear mapping of E into F, then t is continuous with respect to the strong topologies on E and F i.e. with respect to $\beta(E, E')$ and $\beta(F, F')$.

PROOF: To show that t is strongly continuous, it suffices to show that if B is a weakly bounded set in F', there exists a bounded set A in E' such that $t^{-1}(B^o) \supseteq A^o$. Now since $t': F' \longrightarrow E'$ is weakly continuous, $t'(B)$ is weakly bounded in E'. It suffices to take $t'(B) = A$. Then we have: $t^{-1}(B^o) = A^o$ (Proposition 2.5, (ii)). Q.E.D.

3. Completeness, B-completeness and B_r-completeness

We begin this section with a very useful characterization of completeness in convex spaces.

THEOREM 3.1: Let E_w be a convex space and E' its dual. Then the following statements are equivalent:

(a) E_w is complete;

(b) Every almost closed hyperplane of E' is $\sigma(E', E)$-closed;

(c) Let $\mathcal{W}$ be a base of neighbourhoods of the origin for w. Then every linear functional on E' that is $\sigma(E',E)$-continuous on each W^o, $W \in \mathcal{W}$, is $\sigma(E',E)$-continuous on E'.

PROOF: A proof of this theorem may be found in [15] (Chapter 5, § 3, Theorem 2).

From this theorem we deduce at once:

PROPOSITION 3.1: If E_w is a complete convex space, then E is complete under any finer topology of the same dual pair.

More generally, we have the following completeness criterion for a convex space E_u in terms of a coarser topology v on E.

PROPOSITION 3.2: Let u and v be two locally convex topologies on a vector space E such that $u > v$. If E_u has a neighbourhood base of the origin consisting of sets complete in v, then E_u is complete.

PROOF: See [38] (Chapter 1, 1.6).

Theorem 3.1 tells us that a hyperplane in the dual of a complete convex space is weakly closed if it has weakly closed intersections with the polars of neighbourhoods of the origin. A theorem of Banach shows that, for a Banach space, this property holds not only for hyperplanes, but for all vector subspaces of its dual. Convex spaces with this property are called B-complete spaces.

It is also possible to characterize B-complete spaces in terms of mappings and range spaces. We begin with the following definition.

DEFINITION 3.1: (a) A convex space E is said to be B-complete if a linear continuous almost open mapping of E onto any convex space is open.

(b) A convex space E is said to be B_r-complete if a linear continuous almost open and one-to-one mapping of E onto any convex space is open.

It follows immediately that:

PROPOSITION 3.3: Every B-complete space is B_r-complete.

PROPOSITION 3.4: Every Fréchet space (in particular, a Banach space) is B-complete.

PROOF: See [15] (Chapter 3, § 2, Theorem 2).

THEOREM 3.2: Let E be a convex space and E' its dual. The following statements are equivalent:

(a) E is B-complete (B_r-complete).

(b) Every almost closed (and dense) subspace of E'^{σ} is $\sigma(E', E)$-closed.

PROOF: See [15] (Chapter 4, § 1, Theorem 1 and § 3, Theorem 5).

Since in any topological vector space a hyperplane is either closed or dense, Theorem 3.1 gives us:

PROPOSITION 3.5: Every B_r-complete space is complete.

We collect now, some properties of B-complete and B_r-complete spaces which we shall have occasion to use in later chapters. The proofs of these may be found in [15] and [38].

PROPOSITION 3.6: Let E be a B-complete (B_r-complete) space and M, a closed subspace. Then M is B-complete (B_r-complete).

PROOF: See [38] (Chapter IV, 8.2).

PROPOSITION 3.7: Let E and F be convex spaces. Let t be a linear, continuous and almost open mapping of E onto F. If E is B-complete, then F is also B-complete.

PROOF: See [15] (Chapter 4, § 1, Proposition 5).

As a particular case of Proposition 3.7, we have:

COROLLARY: Let E be a B-complete space and M a closed subspace of E. Then E/M is B-complete.

4. Linear mappings with closed graphs

When E and F are convex spaces and t is a linear mapping of E into F, the graph of t is a vector subspace of $E \times F$. We shall use the following condition for the graph to be closed:

PROPOSITION 4.1: Let E and F be convex spaces and let t be a linear mapping of E into F, with transpose t' (mapping F' into E^*). The graph of t is closed if and only if $t'^{-1}(E')$ is dense in F'_σ.

PROOF: The set $t'^{-1}(E')$ is dense in F'_σ if and only if $(t'^{-1}(E'))^\circ = \{0\}$. Now if $\mathcal{U}$ is a base of absolutely convex neighbourhoods of the origin in E, then

$$E' = \bigcup_{U \in \mathcal{U}} U^{\circ}$$

and so

$$t'^{-1}(E') = \bigcup_{U \in \mathcal{U}} t'^{-1}(U^{\circ}) = \bigcup_{U \in \mathcal{U}} (t(U))^{\circ}.$$

Hence

$$(t'^{-1}(E'))^{\circ} = \bigcap_{U \in \mathcal{U}} (t(U))^{\circ\circ} = \bigcap_{U \in \mathcal{U}} \overline{t(U)}.$$

But y is in the last set if and only if, for each $U \in \mathcal{U}$ and each neighbourhood V in F, $y + V$ meets $t(U)$ i.e. $(0, y) \in \overline{G}$, where G is the graph of t. Hence if G is closed, $(t'^{-1}(E'))^{\circ} = \{0\}$. Conversely if $(t'^{-1}(E'))^{\circ} = \{0\}$, and if $(x, y) \in \overline{G}$, then $(0, y - t(x)) \in \overline{G}$, and so $y - t(x) = 0$; thus $(x, y) \in G$ and G is closed. Q.E.D.

<u>COROLLARY</u>: If the graph of t is closed, then $t^{-1}(0)$ is closed.

<u>PROOF</u>: For putting $N' = t'^{-1}(E')$, $N'^{\circ} = \{0\}$ by the proposition, and $t^{-1}(0) = t^{-1}(N'^{\circ}) = (t'(N'))^{\circ}$, which is $\sigma(E, E')$-closed, being the polar of a subset of E'. Thus $t^{-1}(0)$ is closed. Q.E.D.

<u>PROPOSITION 4.2</u>: Let E and F be convex spaces and t a linear mapping of E into F and such that $t^{-1}(0)$ is a closed subspace of E. Let $s: E/t^{-1}(0) \longrightarrow F$, where $s(\overline{x}) = t(x)$ for all $x \in F$. Then:

(a) s is almost open if and only if t is almost open.

(b) s is almost continuous if t is almost continuous.

(c) The graph of s is closed if and only if the graph of t is closed.

PROOF: See [15] (Chapter 3, § 2, Proposition 3).

5. The closed graph and open mapping theorems for B_r-complete spaces

Before we prove our main theorem we need:

LEMMA 5.1: Let E and F be convex spaces. If t is an almost continuous linear mapping of E into F, with transpose t' (mapping F' into E^*), and if M' is an almost closed vector subspace of E', then $t'^{-1}(M')$ is almost closed.

PROOF: Let $N' = t'^{-1}(M')$; we have to show that $N' \cap V^o$ is $\sigma(F', F)$-closed for each neighbourhood V of the origin in F. Let $W = t^{-1}(V)$. Then $\overline{W}$ is a neighbourhood of the origin in E since t is almost continuous; therefore, since M' is almost closed, $M' \cap \overline{W}^o$ is $\sigma(E^*, E)$-compact and so $\sigma(E^*, E)$-closed. Hence $t'^{-1}(M' \cap \overline{W}^o)$ is $\sigma(F', F)$-closed. Now

$$t'^{-1}(M' \cap \overline{W}^o) = t'^{-1}(M' \cap W^o) = N' \cap t'^{-1}(W^o)$$

$$= N' \cap (t(W))^o = N' \cap (V \cap t(E))^o.$$

Hence

$$N' \cap V^o = N' \cap V^o \cap (V \cap t(E))^o = t'^{-1}(M' \cap \overline{W}^o) \cap V^o$$

an intersection of $\sigma(F', F)$-closed sets. Q.E.D.

The following closed graph theorem is due to Pták [31].

THEOREM 5.1: Let E_u be a convex space and F_v a B_r-complete space. Let t be a linear mapping of E_u into F_v with closed graph. If t is almost continuous, then t is continuous.

PROOF: We prove first that t is weakly continuous, by showing that $t'^{-1}(E') = F'$, so that $t'(F') = E'$. By Lemma 5.1, with $M' = E'$, $t'^{-1}(E')$ is almost closed. By Proposition 4.1, $t'^{-1}(E')$ is dense in $F_v'^{\sigma}$ since the graph of t is closed. Since F_v is a B_r-complete space, $t'^{-1}(E') = F'$, and so t is weakly continuous. Now if V is a closed absolutely convex neighbourhood of the origin in F_v, V is also weakly closed; therefore $\overline{t^{-1}(V)}$ is weakly closed and so closed. But $\overline{t^{-1}(V)}$ is a neighbourhood of the origin in E_u since t is almost continuous; thus $t^{-1}(V)$ is a neighbourhood, and t is continuous. Q.E.D.

COROLLARY: Let E be a barrelled space and F a B_r-complete space. Let t be a linear mapping of E into F with closed graph. Then t is continuous.

PROOF: Since E is barrelled, t is almost continuous. Q.E.D

In view of Proposition 4.2 and the fact that factors modulo closed subspaces of B-complete spaces are B-complete (Corollary to Proposition 3.7), we have the following open mapping theorem.

THEOREM 5.2: Let E be a B-complete space and F a convex space. Let t be a linear mapping of E onto F with closed graph. If t is almost open, t is open.

COROLLARY: Let E be a B-complete space and F a barrelled space. Let t be a linear mapping of E onto F with closed graph. Then t is open.

6. Convex spaces with the $B(\mathcal{E})$ and the $B_r(\mathcal{E})$-property

In his papers [11], [12] and [14], and in his book [15], Husain studied convex spaces which he called spaces with the $B(\mathscr{C})$ and the $B_r(\mathscr{C})$-property. These are spaces which satisfy weakened forms of the B-completeness and B_r-completeness conditions of Pták [31].

<u>DEFINITION 6.1</u>: (a) Let $\mathscr{C}$ denote a fixed class of convex spaces. A convex space E is said to have the $B(\mathscr{C})$-property or, in short, to be a $B(\mathscr{C})$-space if, for each convex space $F \in \mathscr{C}$, a linear continuous and almost open mapping of E onto F is open.

(b) A convex space E is said to be a $B_r(\mathscr{C})$-space if, for each convex space $F \in \mathscr{C}$, a linear continuous one-to-one and almost open mapping of E onto F is open.

It is clear that:

<u>PROPOSITION 6.1</u>: Every $B(\mathscr{C})$-space is a $B_r(\mathscr{C})$-space.

<u>PROPOSITION 6.2</u>: Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two classes of convex spaces. If $\mathscr{C}_1 \supset \mathscr{C}_2$, then every $B(\mathscr{C}_1)$-space ($B_r(\mathscr{C}_1)$-space) is a $B(\mathscr{C}_2)$-space ($B_r(\mathscr{C}_2)$-space).

The following permanence property for $B(\mathscr{C})$-spaces may be found in [13] (§ 3, Theorem 1).

<u>PROPOSITION 6.3</u>: Let E be a $B(\mathscr{C})$-space and t a linear continuous and almost open mapping of E onto a convex space F. Then F is also a $B(\mathscr{C})$-space.

<u>COROLLARY</u>: Let E be a $B(\mathscr{C})$-space and M a closed subspace of E. Then E/M is also a $B(\mathscr{C})$-space.

In the particular case when $\mathscr{C} = \mathscr{T}$, the class of all barrelled spaces, we have some internal characterizations of $B(\mathscr{C})$ and $B_r(\mathscr{C})$-spaces. For these, it will be convenient to have the following definition.

DEFINITION 6.2: Let E' be the dual of a convex space E_u. A subspace Q of E'^{σ} is said to be boundedly complete if the following conditions are simultaneously satisfied:

(a) For each u-neighbourhood U of the origin in E_u, $Q \cap U^o$ is closed in E'^{σ}.

(b) Every bounded subset of Q is equicontinuous.

It follows immediately that:

PROPOSITION 6.4: Every boundedly complete subspace Q of E'^{σ} of a convex space E is weakly quasi-complete.

We have the following characterization.

THEOREM 6.1: A necessary and sufficient condition for a convex space E to be a $B(\mathcal{J})$-space ($B_r(\mathcal{J})$-space) is that each (dense) boundedly complete subspace of E'^{σ} is closed.

PROOF: See [15] (Chapter 7, § 3, Theorems 2 and 3).

Proposition 6.4 and the above Theorem now give us:

PROPOSITION 6.5: A sufficient condition for a convex space E to be a $B(\mathcal{J})$-space is that each quasi-complete subspace of E'^{σ} is closed.

For $B_r(\mathcal{J})$-spaces we have the following exciting theorem:

THEOREM 6.2: (Husain [15]) Let E_u be a barrelled space and F_v a $B_r(\mathcal{J})$-space. Let t be a linear mapping of E_u into F_v with closed graph. If t is almost open, then t is continuous.

PROOF: Let $\{V\}$ denote a fundamental system of closed, absolutely convex neighbourhoods of the origin in F_v. For each V

in $\{V\}$, let $\overset{*}{V} = \overline{t(\overline{t^{-1}(V)})}$. Then it is easy to verify that $\{\overset{*}{V}\}$ forms a filterbase for closed, absolutely convex neighbourhoods of the origin in F under a locally convex topology w. We show that F_w is separated. Let $y \varepsilon \overset{*}{V}$, for each $\overset{*}{V}$ in $\{\overset{*}{V}\}$. Then $y \varepsilon \frac{1}{2}\overset{*}{V}$. Hence there exists $x_1 \varepsilon \overline{t^{-1}(\frac{1}{2}V)}$ so that $t(x_1) \varepsilon y + \frac{1}{2}V$. But $\overline{t^{-1}(\frac{1}{2}V)} \subset t^{-1}(\frac{1}{2}V) + U$, where U is an arbitrary neighbourhood of the origin in E_u. Hence $x_1 \varepsilon t^{-1}(\frac{1}{2}V) + U$. That means there exists $x_2 \varepsilon U$ so that $x_1 - x_2 \varepsilon t^{-1}(\frac{1}{2}V)$ and hence $t(x_2) \varepsilon t(x_1) + \frac{1}{2}V$. But this shows that $t(x_2) \varepsilon y + V$. In other words, $G \cap (U \times (y + V)) \neq \emptyset$, where G is the graph of t. Since G is closed by hypothesis, $(0, y) \varepsilon G$ and hence $y = 0$.

Further, for each $\overset{*}{V}$, $\overset{*}{V} \supset V$ so $v \supset w$. We wish to show that the identity mapping $i : F_v \longrightarrow F_w$ is almost open. For this it is sufficient to show that if $y \varepsilon \overset{*}{V}$ then $y \varepsilon \overline{V}$, where $\overline{V}$ is the closure of V with respect to the topology w. For each W in $\{V\}$,

$$(y + \frac{1}{2}W) \cap t\,(\overline{t^{-1}(V)}) \neq \emptyset$$

Hence there exists $x_1 \varepsilon \overline{t^{-1}(V)}$ so that $t(x_1) \varepsilon y + \frac{1}{2}W$. But $\overline{t^{-1}(\frac{1}{2}W)}$ being a barrel in E_u, is a neighbourhood of the origin (because E_u is a barrelled space). Hence $x_1 \varepsilon t^{-1}(V) + \overline{t^{-1}(\frac{1}{2}W)}$. Therefore there exists $x_2 \varepsilon t^{-1}(V)$ such that $x_1 - x_2 \varepsilon \overline{t^{-1}(\frac{1}{2}W)}$ and hence $t(x_1) - t(x_2) \varepsilon \frac{1}{2}\overset{*}{W}$. But then $t(x_2) \varepsilon y + \frac{1}{2}W + \frac{1}{2}\overset{*}{W}$ (because $\frac{1}{2}\overset{*}{W}$ is balanced and $t(x_1) \varepsilon y + \frac{1}{2}W$). Hence $t(x_2) \varepsilon y + \overset{*}{W}$, because $W \subset \overset{*}{W}$ and $\overset{*}{W}$ is convex, and therefore $V \cap (y + \overset{*}{W}) \neq \emptyset$ for each $\overset{*}{W}$. This proves that y is a limit point of V under w, or in other words, $y \in \overline{V}$. Thus $i: F_v \longrightarrow F_w$ is continuous and almost open.

Now consider the mapping $t: E_u \longrightarrow F_w$. For each $\overset{*}{V}$ in $\{\overset{*}{V}\}$, $t^{-1}(\overset{*}{V}) = t^{-1}(\overline{t(\overline{t^{-1}(V)})}) \supset \overline{t^{-1}(V)}$. Since $\overline{t^{-1}(V)}$ is a barrel and therefore a neighbourhood of the origin in E_u, $t: E_u \longrightarrow F_w$ is continuous. Further, since $t: E_u \longrightarrow F_v$ is almost open by hypothesis and $i: E_v \longrightarrow F_w$ is almost open, it follows that $t: E_u \longrightarrow F_w$ is also almost open.

Now $t: E_u \longrightarrow F_w$ being a continuous and almost open mapping of a barrelled space E_u into a convex space F_w implies F_w is a barrelled space. Since F_v is a $B_r(\mathfrak{J})$-space, the identity mapping $i: F_v \longrightarrow F_w$, being continuous and almost open, is open. Hence $v = w$. Since $t: E_u \longrightarrow F_w$ has been proved continuous, it follows that $t: E_u \longrightarrow F_v$ is continuous. Q.E.D.

In the case when F_v is barrelled, the almost openness condition imposed on t may be relaxed, because, it turns out that a barrelled $B_r(\mathfrak{J})$-space is B_r-complete ([15], Chapter 7, § 5, Theorem 5). More generally, we have the following:

Let $\mathcal{C}$ be a fixed class of convex spaces which satisfies the following condition: Let E and F be two convex spaces and let t be a linear continuous and almost open mapping of E onto F. If E is in $\mathcal{C}$, then F is also in $\mathcal{C}$.

For such a class $\mathcal{C}$ of convex spaces, we have:

<u>THEOREM 6.3</u>: Every convex space in $\mathcal{C}$ which is a $B(\mathcal{C})$-space ($B_r(\mathcal{C})$-space) is B-complete (B_r-complete).

COROLLARY: Every barrelled space which is a $B(\mathcal{J})$-space ($B_r(\mathcal{J})$-space) is B-complete (B_r-complete).

7. Countably quasi-barrelled and countably barrelled spaces

Countably quasi-barrelled and countably barrelled spaces are due to Husain [16]. Here we collect some properties of these spaces for later use. We begin with:

DEFINITION 7.1: Let E be a convex space and E' its dual. E is a countably quasi-barrelled space if each $\beta(E', E)$-bounded subset of E' which is a countable union of equicontinuous subsets of E' is itself equicontinuous.

(DF)-spaces and quasi-barrelled spaces are countably quasi-barrelled.

DEFINITION 7.2: Let E be a convex space and E' its dual. E is a countably barrelled space if each $\sigma(E', E)$-bounded subset of E' which is a countable union of equicontinuous subsets of E' is itself equicontinuous.

Barrelled spaces are countably barrelled.

It is clear from the above definitions that:

PROPOSITION 7.1: Every countably barrelled space is countably quasi-barrelled.

The following proposition gives a condition under which the converse of Proposition 7.1 is true.

PROPOSITION 7.2: A sequentially-complete convex space is countably barrelled if and only if it is countably quasi-barrelled.

PROOF: Since E is sequentially complete, weakly bounded subsets of E' are strongly bounded. Our proof now follows that of Proposition 4 ([16], § 4).

A particular case of Proposition 7.2 is the following:

COROLLARY: A complete convex space is countably barrelled if and only if it is countably quasi-barrelled.

PROPOSITION 7.3: Let E be a countably barrelled space and E', its dual. Then every $\sigma(E, E')$-bounded subset of E is $\beta(E, E')$-bounded, which is equivalent to every $\sigma(E', E)$-bounded subset of E' is $\beta(E', E)$-bounded.

PROOF: Let A and B be weakly bounded subsets of E and E' respectively, and suppose $\sup_{x\varepsilon A, y\varepsilon B} |<x,y>| = \infty$. Then there exists a sequence $\{y_n\}$, $y_n \varepsilon B$, such that $\sup_{x\varepsilon A} |<x,y_n>| > n^2$ for each n. Now since $\{y_n\}$ is $\sigma(E', E)$-bounded, it is equicontinuous (because E is countably barrelled) and hence strongly bounded. But $\sup_{x\varepsilon A, n} |<x,y_n>| = \infty$, which is a contradiction. Thus, $\sup_{x\varepsilon A, y\varepsilon B} |<x,y>| < \infty$ and therefore A and B are strongly bounded subsets of E and E', respectively. Q.E.D.

Another useful result is the following.

PROPOSITION 7.4: Let E be a countably barrelled space and E', it dual. Then E'^{σ} is sequentially complete.

PROOF: See [16] (§ 6, Theorem 5).

Countably barrelled spaces (countably quasi-barrelled spaces) are preserved by continuous almost open linear mappings. We have:

PROPOSITION 7.5: Let E be a countably barrelled (or countably quasi-barrelled) space and F any convex space. Let t be a linear, continuous and almost open mapping of E onto F. Then F is also countably barrelled (or countably quasi-barrelled).

PROOF: See [16] (§ 8, Proposition 8).

COROLLARY: Factors modulo closed subspaces of countably barrelled (quasi-barrelled) spaces are countably barrelled (quasi-barrelled).

8. On Semi-reflexive spaces

Let E be a convex space and let $\beta(E', E)$ be the strong topology on E'. This is the topology of uniform convergence on the bounded subsets of E. We shall call the dual of E' under this topology the bidual of E. We shall then call E semi-reflexive if the bidual of E is E itself.

By appealing to the Mackey-Arens Theorem ([15], Chapter 2, § 9, Theorem 6), we have the following characterizations of semi-reflexive spaces.

THEOREM 8.1: For any convex space E, the following assertions are equivalent:

(a) E is semi-reflexive.

(b) Every $\beta(E', E)$-continuous linear form on E' is continuous for $\sigma(E', E)$.

(c) E'^{τ} is barrelled i.e. on E', $\tau(E', E) = \beta(E', E)$.

(d) Every bounded subset of E is relatively $\sigma(E, E')$-compact.

(e) E is quasi-complete under $\sigma(E, E')$.

PROOF: See [38] (Chapter IV, 5.5).

Another characterization and one which we shall often refer to in a later chapter, is the following:

THEOREM 8.2: Let E_u be a convex space. If $E_u'^{\beta}$ is separable, then E_u is semi-relexive if and only if $E\sigma$ is sequentially complete.

PROOF: The necessity is obvious. For sufficiency, let B be a bounded, weakly closed set in E. Then B is an equicontinuous subset of $E_u'^{\beta'}$, hence it is $\sigma(E'', E')$-metrisable (because E'^{β} is separable) and the $\sigma(E'', E')$-closure of B is a compact metrisable space (Proposition 2.8). If $\{x_\alpha\}$ is a Cauchy net in B for $\sigma(E, E')$, then there exists a $z \in E_u'^{\beta'}$ such that z is the limit of the Cauchy net $\{x_\alpha\}$. By metrisability, z is the limit of a Cauchy sequence (in the $\sigma(E'', E')$-topology which is the same as the $\sigma(E, E')$-topology on B) in B and by sequential completeness, $z \in B$. Thus B is weakly compact and hence E is semireflexive (Theorem 8.1 (d)). Q.E.D.

CHAPTER II

α-SPACES, γ-SPACES AND δ-SPACES

1. α-spaces

Sequentially closed sets are not in general closed. However, in the case of first countable spaces, sequentially closed sets are closed. In particular, metrisable spaces have this property. In this section we shall concern ourselves with a class of convex spaces whose weak duals are such that every sequentially closed subspace is closed. On these spaces, we prove closed graph and open mapping theorems.

DEFINITION 1.1: A convex space is called an α-space if in its weak dual, every sequentially closed subspace is closed.

The requirement of being an α-space depends only on the dual system and so, if E_u is an α-space, then E_v is an α-space for any locally convex topology v such that $\sigma(E,E') \subset v \subset \tau(E,E')$.

Examples of α-spaces are in abundance. The dual E'_u of a metrisable convex space E_u is a α-space in all locally convex topologies v such that $\sigma(E', E) \subset v \subset \tau(E', E)$. If E_u is a quasi-M-barrelled space ([25], § 3) and E'^{β}_u possess a countable fundamental family of bounded sets, then E'_u is an α-space for any locally convex topology v such that $\sigma(E', E) \subset v \subset \tau(E', E)$, because E_τ is metrisable. $E_w = R^{(N)}$, the space of all finite sequences, with the finest locally convex topology w, is an α-space. For $E'_w = R^N$, the countable product of the reals and R^N with $\sigma(E'_w, E_w)$ is metrisable ([15], Chapter 6 § 1). If E is a convex space with a countable fundamental family of convex compact sets, then E is an α-space. A semi-reflexive (DF)-space ([23], Chapter 6, § 29) is an α-space.

Suppose that E_u is a reflexive Banach space which is not separable. Let v be the topology of uniform convergence on separable bounded subsets of the strong dual E'^{β}_u. Then E_v is a semi-reflexive (DF)-space ([23], Chapter 6, § 29) and therefore an α-space.

Let E and F be convex spaces and t a linear mapping of E into F. Let t' be the transpose of the linear mapping. $t'^{-1}(E')$ is the vector subspace of F' formed by those y' in F' for which the linear functional: $x \longrightarrow < t(x), y' >$ is continuous on E. Let E'^{σ} be the point set E' endowed with the topology $\sigma(E', E)$.

We have the following property of $t'^{-1}(E')$.

<u>PROPOSITION 1.1</u>: If E is a convex space such that E'^{σ} is sequentially complete, then $t'^{-1}(E')$ is sequentially closed in F'^{σ}.

<u>PROOF</u>: Since t': $F' \longrightarrow E^*$ is continuous with respect to $\sigma(F', F)$ and $\sigma(E^*, E)$, $t'^{-1}(E')$ is sequentially closed. Q.E.D.

<u>COROLLARY 1</u>: If E is a countably barrelled space, then $t'^{-1}(E')$ is sequentially closed.

<u>PROOF</u>: Since E is countably barrelled space, E'^{σ} is sequentially complete (Chapter I, Proposition 7.4) and so sequentially closed in $\sigma(E^*, E)$.

Q.E.D.

COROLLARY 2: If E is a barrelled space, then $t'^{-1}(E')$ is sequentially closed.

PROOF: Since barrelled spaces are countably barrelled (Chapter I, § 7), our proof follows as before. Q.E.D.

THEOREM 1.1: Let E be a Mackey space such that E'^{σ} is sequentially complete. Let F be an α-space. If t is a linear mapping of E into F, whose graph is closed in $E \times F$, then t is continuous.

PROOF: Since the graph of t is closed, $t'^{-1}(E')$ is dense in F'^{σ} (Chapter I, Proposition 4.1). Further, since E'^{σ} is sequentially complete, $t'^{-1}(E')$ is sequentially closed, by Proposition 1.1. F is an α-space and therefore $t'^{-1}(E') = F'$. t is therefore weakly continuous but since E is a Mackey space, it is continuous (Chapter I, Proposition 2.9). Q.E.D.

REMARK 1.1: If in the above theorem, t is almost continous, the requirement that E be a Mackey space is not necessary, for then weak continuity implies continuity.

COROLLARY 1: A linear mapping of a countably barrelled Mackey space E into an α-space F, whose graph is closed in $E \times F$ is continuous.

COROLLARY 2: A linear mapping of a barrelled space E into an α-space F, whose graph is closed in $E \times F$, is continuous.

PROPOSITION 1.2: If there is a continuous open linear mapping t of an α-space E onto a convex space F, then F is also an α-space.

PROOF: Since t is onto, the transpose $t': F'^{\sigma} \longrightarrow E'^{\sigma}$ is one-to-one and therefore a homeomorphism into (Chapter I, Proposition 2.6, Corollary (ii)). Let Q be a sequentially closed subspace in F'^{σ}. Then $t'(Q)$ is sequentially closed in $t'(F')$. Since t is open, $t'(F')$ is closed in E'^{σ} (Chapter I, Proposition 2.6) and so $t'(Q)$ is sequentially closed in E'^{σ}. Since E is an α-space, $t'(Q)$ is closed in E'^{σ} and so Q is closed in F'^{σ}. Thus F is an α-space. Q.E.D.

COROLLARY: The factors modulo closed subspaces of α-spaces are also α-spaces.

We thus have the following open mapping theorem:

THEOREM 1.2: Let E be an α-space. Let F be a Mackey space such that F'^{σ} is sequentially complete. If t is a linear mapping of E onto F, whose graph is closed in $E \times F$, t is open.

PROOF: Since the graph of t is closed, $t^{-1}(0)$ is a closed subspace of F (Chapter I, Corollary to Proposition 4.1) and so $E/t^{-1}(0)$ under the quotient topology is separated. Then we can write $t = s \circ k$, where k is the canonical mapping of E onto $E/t^{-1}(0)$ and s is a one-to-one mapping of $E/t^{-1}(0)$ onto F. Since the graph of s is closed (Chapter I, Proposition 4.2 (c)), the graph of s^{-1} is also closed. s^{-1} maps F onto $E/t^{-1}(0)$, which, by the above corollary, is an α-space. Therefore by theorem 1.1, s^{-1} is continuous. Thus s, and so also t, are open. Q.E.D.

REMARK 1.2: If in the above theorem, t is almost open, the requirement that F be a Mackey space is not necessary, for then, s^{-1} is almost continuous and our result follows by Remark 1.1.

COROLLARY: A linear mapping of an α-space E onto a barrelled space F, whose graph is closed in $E \times F$, is open.

REMARK 1.3: Let E and F be convex spaces and t a linear mapping of E into F. If t is almost continuous, then $t'^{-1}(E')$ is almost closed in F'^{σ} (Chapter I, Lemma 5.1). Also if the graph of t is closed in $E \times F$, $t'^{-1}(E')$ is dense in F'^{σ} (Chapter I, Proposition 4.1). Therefore the corollaries to Theorems 1.1 and 1.2 are also true for spaces more general than α-spaces; namely the class of convex spaces whose weak duals are such that every almost closed, sequentially closed, dense subspace is closed and hence coincides with the weak dual.

PROPOSITION 1.3: A closed subspace M of an α-space E is an α-space.

PROOF: Let Q be a weakly sequentially closed subspace of the dual $M' = E'/M^o$ of M.

If ϕ denotes the canonical mapping of $E' \longrightarrow E'/M^o$ continuous with respect to $\sigma(E', E)$ and $\sigma(E'/M^o, M)$, then $\phi^{-1}(Q)$ is a weakly sequentially closed subspace of the dual E' of E. Since E is an α-space, $\phi^{-1}(Q)$ is $\sigma(E', E)$-closed. Hence Q is $\sigma(E'/M^o, M)$-closed and so M is an α-space. Q.E.D.

Let E'' be the bidual of E. With respect to the dual pair E' and E'', let $\sigma(E', E'')$ and $\tau(E', E'')$ be the weak and Mackey topologies on E', respectively.

EXAMPLE 1.1: If E is a non-reflexive Banach space such that $E_{\sigma(E,E')}$ is sequentially complete (such for example is ℓ_1, with the usual norm topology), then $E'^{\sigma(E', E'')}$ is not an α-space and so

$E'_{\tau(E', E'')}$ is not an α-space. But since $E'_{\tau(E',E'')}$ is a Banach space, E' is B-complete under $\tau(E',E'')$. Thus B-complete spaces are not in general α-spaces.

EXAMPLE 1.2: If $E_u = L_1(N)$, the space of all real sequences x_n $(n \geq 1)$ such that $\sum_{n=1}^{\infty} |x_n| < \infty$, where u is the metric topology induced from R^N (the countable product of reals), E_u is not complete ([15], Chapter 7, §6, Example 3) and so not B-complete. But $E'_u = R^{(N)}$, the space of all finite sequences, where $\tau(E', E)$ is the norm topology defined by the norm: $\|x\| = \sup |x_n|$. Furthermore, it is known that $E'^{\tau'}_u = E$ and $\sigma(E, E') = u = \tau(E, E')$ and so E_u is an α-space. Thus α-spaces are not in general B-complete.

The next two propositions give us the particular cases when a B-complete space is a α-space and, when an α-space is B-complete.

PROPOSITION 1.4: A separable B-complete space E_u is an α-space.

PROOF: Suppose Q is a sequentially closed subspace in E'^{σ}_u. Let U be any neighbourhood of the origin in E_u. Since E_u is separable, U^o is weakly metrisable (Chapter I, Proposition 2.8). Thus $Q \cap U^o$ is weakly closed in U^o. Since U^o is $\sigma(E', E)$-compact, $Q \cap U^o$ is also closed in E'^{σ}_u. But since E_u is B-complete, Q is $\sigma(E', E)$-closed. Q.E.D.

DEFINITION 1.1: A convex space E is said to be sequentially barrelled if every sequence in E' which is $\sigma(E',E)$-convergent to zero is equicontinuous.

Since the translation of an equicontinuous set is equicontinuous, E is sequentially barrelled if and only if every sequence in E' which is $\sigma(E',E)$-convergent is equicontinuous.

Sequentially barrelled spaces are due to J. H. Webb [42]. They are more general than barrelled spaces ([42], § 5). Mackey spaces with sequentially complete weak duals are sequentially barrelled ([42], § 4, Proposition 4.2).

The following proposition is due to Webb [42] (§ 4, Corollary 4.14).

PROPOSITION 1.5: An α-space E_u which is sequentially barrelled is B-complete.

PROOF: Suppose Q is a subspace in $E_u'^{\sigma}$ such that $Q \cap U^o$ is $\sigma(E', E)$-closed for each neighbourhood U of the origin in E_u. Let $\{x_n\}$ be a sequence of points of Q such that $x_n \longrightarrow x$ weakly. Since E_u is sequentially barrelled, there exists a neighbourhood U of the origin in E_u such that $\{x_n\} \subseteq U^o$. By hypothesis, $Q \cap U^o$ is closed in $E_u'^{\sigma}$ and therefore $x \in Q$ i.e. Q is sequentially closed. E_u is an α-space and so Q is $\sigma(E', E)$-closed. Q.E.D.

COROLLARY 1: An α-space which is countably barrelled is B-complete.

PROOF: Since countably barrelled spaces are sequentially barrelled our proof follows as before. Q.E.D.

Let E_u be a convex space. We shall denote by $\beta^*(E, E')$, the topology on E of uniform convergence on $\beta(E', E)$-bounded sets of E'. Whenever E_{β^*} is complete, $E_u'^{\tau}$ is sequentially barrelled ([43], Theorem 3.1). This enables us to establish that sequentially barrelled spaces are more general than countably barrelled spaces.

EXAMPLE 1.3: Let $E_u = c_o$, with the usual norm topology. Since E_u is quasi-barrelled, $u = \beta^*(E, E')$. Further, since E_u

is complete, $E_u'^{\tau}$ is sequentially barrelled ([43], Theorem 3.1). However, since E_σ is not sequentially complete, $E_u'^{\tau}$ is not countably barrelled (Chapter I, Proposition 7.4).

COROLLARY 2: An α-space which is barrelled, is B-complete.

PROOF: Since barrelled spaces are countably barrelled (Chapter I, § 7) our proof follows as before. Q.E.D.

The factors modulo closed subspaces of sequentially barrelled spaces are also sequentially barrelled spaces. This follows from the following proposition.

PROPOSITION 1.6: If there is a continuous, almost open linear mapping t of a sequentially barrelled space E_u onto a convex space F, then F is sequentially barrelled.

PROOF: Since t is continuous, the transpose t' maps F' into E' and is continuous with respect to $\sigma(F', F)$ and $\sigma(E', E)$. Let $\{x_n\}$ be a sequence of points of F' such that $x_n \longrightarrow x$ weakly. Then $\{t'(x_n)\}$ is convergent in $E_u'^{\sigma}$. Since E_u is sequentially barrelled, $\{t'(x_n)\} \subset U^o$ for some neighbourhood U of the origin in E_u. Thus

$$\begin{aligned}\{x_n\} &\subset t'^{-1}(U^o) \\ &= (\overline{t(U)})^o .\end{aligned}$$

Since t is almost open, $\{x_n\}$ is equicontinuous and therefore F is sequentially barrelled. Q.E.D.

DEFINITION 1.2: Let $\mathcal{S}q.\mathcal{J}$ denote the class of all sequentially barrelled spaces. A convex space E is said to be a $B(\mathcal{S}q.\mathcal{J})$-space if, for each convex space F in $\mathcal{S}_q \cdot \mathcal{T}$, a linear, continuous and almost open mapping of E onto F is open.

In view of Proposition 1.6, a sequentially barrelled $B(\mathcal{S}_q \cdot \mathcal{T})$-space is B-complete (Chapter I, Theorem 6.3). Thus it follows from Proposition 1.4 that:

PROPOSITION 1.7: A separable, sequentially barrelled $B(\mathcal{S}_q \cdot \mathcal{T})$-space is an α-space.

To obtain a characterization of α-spaces, we consider now the more restricted class $\mathcal{T}$ of all barrelled spaces. Barrelled $B(\mathcal{T})$-spaces are B-complete (Chapter I, Corollary to Theorem 6.3). Since barrelled spaces are Mackey spaces having sequentially complete weak duals, we have the following:

THEOREM 1.3: Let E be a separable barrelled space. Then E is an α-space if and only if for each Mackey space F with a sequentially complete weak dual, a linear mapping of E onto F, whose graph is closed in $E \times F$, is open.

PROOF: This follows from Theorem 1.2 and the remarks above. Q.E.D.

2. γ-spaces

In this section, we shall concern ourselves with a class of convex spaces whose weak duals are such that every subspace that contains the limit points of its bounded subsets is closed. These were first introduced by V. L. Levin [26]. More recently, A. Persson [30] and A. McIntosh [28] have also dealt with such spaces.

DEFINITION 2.1: A convex space is called a γ-space if in its weak dual every subspace containing the limit points of its bounded subsets is closed or equivalently if in its weak dual every subspace which intersects every closed bounded set in a closed set is closed.

It follows immediately that α-spaces and B-complete spaces are γ-spaces.

γ-spaces are more general than α-spaces. This follows from Example 1.1. However we have:

PROPOSITION 2.1: A γ-space E_u which is $\beta(E, E')$-separable, is an α-space.

PROOF: Let B be an absolutely convex $\sigma(E_u', E)$-bounded subset of E_u'. Since $E_u' \subset E_\beta'$ and E is $\beta(E, E')$-separable, the restriction of $\sigma(E_\beta', E)$ to B is metrisable. Hence $\sigma(E_u', E)$ is metrisable on every absolutely convex $\sigma(E_u', E)$-bounded subset of E'. Thus every sequentially closed subspace of $E_u'^{\sigma}$ intersects every closed bounded set in a closed set; E_u is a γ-space and so every sequentially closed space of $E_u'^{\sigma}$ is closed. Q.E.D.

DEFINITION 2.2: A subspace in the weak dual of a convex space is called boundedly closed if it intersects every closed bounded set in a closed set.

A more satisfactory result relating γ-spaces and α-spaces is:

THEOREM 2.1: A convex space E is an α-space if and only if

(a) E is a γ-space

(b) every sequentially closed subspace of E'^{σ} can be strictly separated from any outside point by means of a boundedly closed hyperplane.

PROOF: The necessity of condition (a) is obvious. Assume now that condition (b) is not satisfied. Then there exists in E'^{σ} a sequentially closed subspace Q and $f \not\in Q$, such that Q cannot be strictly separated from f by means of a boundedly closed hyperplane. Since E is a γ-space, this implies that Q cannot be strictly separated from f by means of a weakly closed hyperplane. By the Hahn-Banach theorem, this implies that Q is not weakly closed. Thus E is not an α-space.

Conversely, assume now that we have (a) and (b) and let Q be an arbitrary sequentially closed subspace of E'^{σ}. Consider an arbitrary $f \in E'$ such that $f \not\in Q$. Then by (b), there exists a boundedly closed hyperplane H in E' which strictly separates Q from f. By (a), H is weakly closed. Thus Q can be strictly separated from any $f \not\in Q$ by means of a weakly closed hyperplane, whence Q itself is weakly closed. Thus E is an α-space. Q.E.D.

γ-spaces are also more general than B-complete spaces. This is indicated in the following examples.

EXAMPLE 2.1: Let E be a metrisable convex space which is not barrelled (such for example is $R^{(N)}$, the space of all finite sequences, (x_n), with the norm topology defined by $\| x \| = \sup | x_n |$). Then E'^{τ} is an α-space and so a γ-space, but since E'^{τ} is not complete, it fails to be B-complete.

EXAMPLE 2.2: Let E be any infinite dimensional metrisable convex space. Then E'^{σ} is an α-space and therefore a γ-space. But since E'^{σ} is never complete ([38] Chapter IV, § 6), it is not B-complete.

Example 1.2 is yet another example of a γ-space which is not B-complete. However, it is clear that:

<u>PROPOSITION 2.2</u>: A γ-space which is barrelled is B-complete.

On γ-spaces, the closed graph and open mapping theorems generalise. We now prove these theorems.

<u>THEOREM 2.2</u>: Let E be a barrelled space and F a γ-space. If t is a linear mapping of E into F, whose graph is closed in $E \times F$, then t is continuous.

<u>PROOF</u>: Let $D[t'] = t'^{-1}(E')$. $D[t'] \cap B$ is a bounded subset in F'^{σ} for every bounded subset B in F'^{σ}. Now, since $t'\colon F' \longrightarrow E^*$ is continuous with respect to $\sigma(F', F)$ and $\sigma(E^*, E)$, $t'(D[t'] \cap \bar{B})$ is a bounded subset in E'^{σ}, for each closed bounded subset $\bar{B}$ in F'^{σ}. E is barrelled and so

$$t'(D[t'] \cap \bar{B}) \subset U^o$$

for some neighbourhood U of the origin in E. Now since U^o is compact in E'^{σ} (Chapter I, Theorem 2.1), U^o is weakly closed in E^* and so $t'^{-1}(U^o)$ is closed in F'^{σ}.

But $\quad D[t'] \cap \bar{B} \subseteq t'^{-1}(U^o)$

and so $\quad \overline{D[t'] \cap \bar{B}} \subseteq t'^{-1}(U^o) \subseteq D[t']$.

Hence $\quad D[t'] \cap \bar{B} \subseteq \overline{D[t'] \cap \bar{B}} \subseteq D[t'] \cap \bar{B}$

and therefore $D[t'] \cap \bar{B}$ is closed in F'^{σ}. Since the graph of t is closed in $E \times F$, $t'^{-1}(E')$ is dense in F'^{σ} (Chapter I, Proposition 4.1). Since F is a γ-space, $t'^{-1}(E')$ is closed in F'^{σ} and therefore

$t'^{-1}(E') = F'$. Thus t is weakly continuous. But E is a barrelled space and therefore a Mackey space and so t is continuous (Chapter I, Propostion 2.9). Q.E.D.

Since factors modulo closed subspaces of γ-spaces are also γ-spaces ([30], Lemma 1), we have the following open mapping theorem.

THEOREM 2.3: Let E be a γ-space and F a barrelled space. If t is a linear mapping of E onto F, whose graph is closed in $E \times F$, then t is open.

PROOF: This is similar to that of Theorem 1.2.

It appears worthwhile, at this stage, to investigate, briefly, closed graph and open mapping theorems for more general linear mappings. We could, for instance consider dense linear mappings.

Let E and F be two convex spaces. Let t be a linear mapping defined on a subspace $D[t]$ of E with range $R[t] = t(D[t])$ in F. We call t dense if $D[t]$ is dense in E. As before, $D[t']$ denotes the set of all $y' \in F'$ such that $t'(y') \in E'$. $R[t'] = t'(D[t'])$.

The set of all pairs $(x, t(x)) \in E \times F$, where $x \in D[t]$, is called the graph $G(t)$ of t. Let $\overline{G(t)}$ be the closure of $G(t)$ in $E \times F$. t is said to be closeable if and only if $(0, y) \in \overline{G(t)}$ implies $y = 0$ or equivalently if $\overline{G(t)}$ is the graph of a linear mapping.

PROPOSITION 2.3: Let E and F be convex spaces. Let t be a dense linear mapping of E into F. Then t is closeable in $E \times F$ if and only if $D[t']$ is dense in F'.

PROOF: A proof may be found in [24] ($\S$ 5, (3)).

Theorem 2.2 and Proposition 2.3 give us:

THEOREM 2.4: Let E be a barrelled space and F a γ-space. Let t be a dense linear mapping of E into F, closeable in $E \times F$. Then t is continuous.

We now note the following definitions and results due to Köthe [24].

Let $D[t] \subset E_u$ have the induced topology. Let $\mathcal{U} = \{U\}$ be the filter of all neighbourhoods of the origin in $D[t]$ for the induced topology. The images $t(U)$ for all U's generates a filter $t(\mathcal{U})$ in F. The set of all adherent points of $t(\mathcal{U})$ in F_v is a closed subspace given by: $S[t] = \bigcap_{U \varepsilon \mathcal{U}} \overline{t(U)}$, where $\overline{t(U)}$ is the closure of $t(U)$ in F_v.

Let $\mathcal{V} = \{V\}$ be the neighbourhood filter of the origin in F_v. The inverse images $t^{-1}(V) \subset D[t]$ for all V's generates a filter $t^{-1}(\mathcal{V})$ in $D[t]$. The set of adherent points of $t^{-1}(\mathcal{V})$ in $D[t]$ is a closed subspace in $D[t]$ given by: $K[t] = \bigcap_{V \varepsilon \mathcal{V}} \overline{t^{-1}(V)}$ where $\overline{t^{-1}(V)}$ is the closure of $t^{-1}(V)$ in $D[t]$.

Let $N[t]$ be the kernel of t. If t is continuous, it is clear that $N[t] = K[t]$. Let $\overline{N}[t]$ be the closure of $N[t]$ in $D[t]$. t is said to be weakly singular if $K[t] = \overline{N}[t]$. Since $K[t] = t^{-1}(S[t])$ ([24], § 2, (5)), all closeable linear mappings are weakly singular. $\check{t}$ is called an extension of t if $D[t] \subset D[\check{t}] \subset E$, $R[\check{t}] \subset F$ and $\check{t} = t$ on $D[t]$. A linear mapping is called a maximal mapping[1)] if every extension $\check{t}$ of t, with $\check{t}' = t'$, coincides with t.

1) Köthe in [24] calls such mappings maximal transformations.

<u>PROPOSITION 2.4</u>: Let E and F be convex spaces. Let t be a dense, maximal mapping of E into F. Then the $\sigma(E', E)$-closure of $R[t']$ is $\sigma(E', D[t])$-closed.

<u>PROOF</u>: See [24] (§ 7, (8)).

We are now in a position to prove an open mapping theorem for dense weakly singular maximal mappings.

<u>THEOREM 2.5</u>: Let E be a γ-space and F a barrelled space. Let t be a dense, weakly singular maximal mapping of E onto F. Then t is open.

<u>PROOF</u>: We must show that $R[t']$ is $\sigma(E', D[t])$-closed ([24], § 9). $R[t'] \cap B$ is a bounded subset in E'^{σ} for every bounded subset B in E'^{σ}. Since t maps E onto F, $t'^{-1}: R[t'] \longrightarrow F'$ is continuous with respect to $\sigma(E', D[t])$ and $\sigma(F', F)$ ([24], § 6, (8)). Thus $t'^{-1}(R[t'] \cap \overline{B})$ is a bounded subset in F'^{σ} for each closed bounded subset $\overline{B}$ in E'^{σ}. Since F is barrelled,

$$t'^{-1}(R[t'] \cap \overline{B}) \subset V^{\circ}$$

for some neighbourhood V of the origin in F.

Thus,

$$\begin{aligned} R[t'] \cap \overline{B} &\subset t'(V^{\circ} \cap D[t']) \\ &= (t^{-1}(V))^{\circ} \quad ([24], \S\, 5, (9)). \end{aligned}$$

Now, $(t^{-1}(V))^{\circ}$ is $\sigma(E', E)$-closed. Therefore

$$\begin{aligned} \overline{R[t'] \cap \overline{B}} &\subset t'(V^{\circ} \cap D[t']) \\ &\subset t'(D[t']) \\ &= R[t']. \end{aligned}$$

Hence $R[t'] \cap \bar{B} \subset \overline{R[t'] \cap \bar{B}} \subset R[t'] \cap \bar{B}$ and therefore $R[t'] \cap \bar{B}$ is $\sigma(E', E)$-closed. Since E is a γ-space, $R[t']$ is $\sigma(E', E)$-closed. Further, since t is maximal, $R[t']$ is also $\sigma(E', D[t])$-closed by Proposition 2.4. Q.E.D.

COROLLARY: Let E be an α-space and F a barrelled space. Let t be as in the theorem. Then t is open.

PROOF: Since α-spaces are γ-spaces, this is an immediate consequence of the theorem. Q.E.D.

3. δ-spaces

A convex space is said to be quasi-complete if every closed bounded set is complete.

DEFINITION 3.1: A convex space is called a δ-space if in its weak dual every quasi-complete subspace is closed.

It follows immediately from the definition that α-spaces, B-complete spaces and γ-spaces are δ-spaces.

Example 1.1 shows that δ-spaces are more general than α-spaces. However, we have:

PROPOSITION 3.1: A δ-space E with sequentially complete weak dual and separable in $\beta(E, E')$, is an α-space.

PROOF: We must show that every sequentially closed subspace Q of E'^{σ} is closed. Since E_{β} is separable, it follows from the proof of Proposition 2.1 that Q intersects every closed bounded set in E'^{σ} in a closed metrisable set. Now since E'^{σ} is sequentially complete, it follows that Q is quasi-complete. E is a δ-space and so Q is closed in E'^{σ}. Q.E.D.

Examples 1.2, 2.1 and 2.2 all show that δ-spaces are not in general complete and so may fail to be B-complete. However, it is clear that δ-spaces that are also barrelled are B-complete.

PROPOSITION 3.2: A δ-space E which is barrelled in its Mackey topology is a γ-space.

PROOF: Let Q be a boundedly closed subspace in E'^{σ} (see Definition 2.2). Since E_{τ} is barrelled, E'^{σ} is quasi-complete and so Q is quasi-complete. Now since E is a δ-space, this implies that Q is weakly closed. Thus E is a γ-space. Q.E.D.

A further result that relates γ-spaces and δ-spaces is provided in:

THEOREM 3.1: A convex space E is a γ-space if and only if (a) E is a δ-space

(b) every boundedly closed subspace of E'^{σ} can be strictly separated from any outside point by means of a quasi-complete hyperplane.

PROOF: This is similar to that of Theorem 2.1.

We have the following closed graph theorem on δ-spaces.

THEOREM 3.2: Let E_u be a barrelled space and F_v a δ-space. Let t be a linear mapping of E into F, whose graph is closed in $E_u \times F_v$. If t is almost open from E_u into F_{τ}, t is continuous.

PROOF: Let $D[t'] = t'^{-1}(E'_u)$. Following the proof of Theorem 2.2 we observe that $D[t'] \cap \overline{B}$ is $\sigma(F', F)$-closed for each closed bounded subset $\overline{B}$ in F'^{σ}_v and

$$D[t'] \cap \overline{B} \subseteq t'^{-1}(U^o)$$

$$= \overline{(t(U))}^o \text{ for some neighbourhood } U \text{ of the origin in } E_u.$$

Since t is almost open from E_u into F_τ, $\overline{(t(U))}^o$ is $\sigma(F', F)$-compact and so $D[t'] \cap \overline{B}$ is compact in F'^{σ}_v. Thus $D[t']$ is quasi-complete in F'^{σ}_v. Since the graph of t is closed in $E_u \times F_v$ and F_v is a δ-space, $t'^{-1}(E'_u) = F'_v$. t is therefore weakly continuous but since E_u is a barrelled space and so a Mackey space, t is continuous. Q.E.D.

REMARK 3.1: The above theorem is also true if F_v is such that in its weak dual, every almost closed, dense quasi-complete subspace is closed.

REMARK 3.2: If, in the above theorem, F is barrelled in its Mackey topology, the almost openness condition imposed on t may be relaxed, for then, F_v is a γ-space (Proposition 3.2) and our result follows from Theorem 2.2.

PROPOSITION 3.3: Let E be a δ-space and M a closed subspace of E. Then E/M is a δ-space.

PROOF: It is well known that $(E/M)' = M^o$, with E/M having the quotient topology. Let Q, endowed with the weak topology $\sigma(M^o, E/M)$, be a quasi-complete subspace of M^o. Since $\sigma(E', E)$ coincides with $\sigma(M^o, E/M)$ on M^o, Q can be regarded as a quasi-complete subspace of E'^{σ}. Since E is a δ-space, Q is closed in E'^{σ} and so $\sigma(M^o, E/M)$-closed in M^o. Thus E/M is a δ-space. Q.E.D.

We therefore have the following open mapping theorem.

Theorem 3.3: Let E_u be a δ-space and F_v a barrelled space. Let t be a linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$. If t is almost continuous from E_τ onto F_v, t is open.

Every δ-space is a $B(\mathfrak{J})$-space, where $\mathfrak{J}$ is the class of all barrelled spaces (Chapter I, Proposition 6.5). Conversely we have:

PROPOSITION 3.4: A $B(\mathfrak{J})$-space E which is also a Mackey space is a δ-space.

PROOF: We must show that every quasi-complete subspace Q of E'^{σ} is closed. Since Q contains the limit points of all its bounded subsets, $Q \cap U^o$ is $\sigma(E', E)$-closed, for each neighbourhood U of the origin in E. Now, every bounded subset of Q is contained in a complete convex bounded set (its closed convex envelope). Since E is a Mackey space, it follows that every bounded subset of Q is equicontinuous. Thus Q is boundedly complete (Chapter I, Definition 6.2). E is a $B(\mathfrak{J})$-space and so Q is closed. (Chapter I, Theorem 6.1). Q.E.D

In [40], Sulley observed that there is a Banach space E_u and a $\sigma(E', E)$-dense subspace M of its dual E'_u, with the property that $E_{\sigma(E, M)}$ is not a γ-space. This implies that $E_{\tau(E, M)}$ is also not a γ-space. However, since $\tau(E, M)$ is coarser than u, $E_{\tau(E, M)}$ is a $B(\mathfrak{J})$-space and therefore by Proposition 3.4, a δ-space. Thus δ-spaces are more general than γ-space.

Proposition 3.4 also enables us to give the following partial converse of Theorem 3.3.

THEOREM 3.4: Let E be a Mackey space. Then E is a δ-space if every linear mapping of E onto a barrelled space, with closed graph, is open.

PROOF: If t is a continuous linear mapping of E onto a barrelled space F, the graph of t is closed in E x F. Thus t is open by assumption. This shows that E is a B($\mathcal{J}$)-space. Since E is a Mackey space, it follows from Proposition 3.4 that E is also a δ-space. Q.E.

Another characterization of δ-spaces is provided by:

THEOREM 3.5: A convex space E is a δ-space if and only if

(a) E is a B($\mathcal{J}$)-space

(b) every quasi-complete subspace of E'^{σ} can be strictly separated from any outside point by means of a boundedly complete hyperplane.

PROOF: This is similar to that of Theorem 2.1 .

CHAPTER III

$B_r(\mathscr{C})$-SPACES

The present chapter is devoted to some closed graph theorems on $B_r(\mathscr{C})$-spaces. For the most part, we shall consider the cases when $\mathscr{C}$ is

- $\mathfrak{M}$ the class of all convex metrisable spaces;
- $\mathfrak{N}$ the class of all normed spaces;
- $\mathcal{J}$ the class of all barrelled spaces;
- $\mathcal{S}.\mathcal{J}$ the class of all separable barrelled spaces

and $\mathcal{F}$ the class of all convex Fréchet spaces.

1. Some general properties

While most of the results in this section are preparatory, some are of interest in themselves. We begin with:

PROPOSITION 1.1: If there exists a continuous, almost open linear mapping of a metrisable convex space onto a convex space F, then F is metrisable.

PROOF: Let t be a continuous, almost open linear mapping of a metrisable convex space E onto F. Let $\{U_n\}$ be a countable base of absolutely convex neighbourhoods of the origin in E and let $\{V\}$ be a base of closed, absolutely convex neighbourhoods of the origin in F. Since t is continuous, for each $V \in \{V\}$, there exists $U_i \in \{U_n\}$ such that $U_i \subseteq t^{-1}(V)$. Further, since t is almost open, for each $U_i \varepsilon \{U_n\}$ there exists a $W_i \varepsilon \{V\}$ such that $W_i \subset \overline{t(U_i)}$. Now, since V is closed we have:

$$W_i \subset \overline{t(U_i)} \subseteq \overline{t(t^{-1}(V))} = V.$$

Thus F has a countable base of neighbourhoods of the origin. Since F is assumed separated, it is metrisable ([35], Chapter I, § 4, Theorem 4). Q.E.

We now have:

PROPOSITION 1.2: Every metrisable convex space which is a $B(\mathfrak{M})$-space ($B_r(\mathfrak{M})$-space) is B-complete (B_r-complete).

PROOF: This follows from Proposition 1.1 above and Theorem 6.3 (Chapter I).

It follows therefore that in the class of convex spaces, metrisable $B(\mathfrak{M})$-spaces ($B_r(\mathfrak{M})$-spaces) are $B(\mathfrak{J})$-spaces ($B_r(\mathfrak{J})$-spaces). Conversely we have:

PROPOSITION 1.3: Every sequentially barrelled $B(\mathfrak{J})$-space ($B_r(\mathfrak{J})$-space) is a $B(\mathfrak{M})$-space ($B_r(\mathfrak{M})$-space).

PROOF: Let E be a sequentially barrelled $B(\mathfrak{J})$-space and F a convex metrisable space. If t is a continuous, almost open linear mapping of E onto F, F is sequentially barrelled (Chapter II, Proposition 1.6). Thus every $\sigma(F', F)$-bounded subset of F' is $\beta(F', F)$-bounded ([42], Proposition 4.1). Since metrisable convex spaces are quasi-barrelled, F is barrelled. But then E being a $B(\mathfrak{J})$-space, it follows that t is open, and therefore E is a $B(\mathfrak{M})$-space.

The statement about $B_r(\mathfrak{M})$-spaces follows in a similar way. Q.E.D.

COROLLARY: Every countably barrelled $B(\mathcal{J})$-space ($B_r(\mathcal{J})$-space) is a $B(\mathcal{M})$-space ($B_r(\mathcal{M})$-space).

PROOF: This follows from the fact that countably barrelled spaces are sequentially barrelled. Q.E.D.

Normed spaces are metrisable and therefore $B(\mathcal{M})$-spaces ($B_r(\mathcal{M})$-spaces) are $B(\mathcal{N})$-spaces ($B_r(\mathcal{N})$-spaces) (Chapter I, Proposition 6.2). A condition under which $B(\mathcal{N})$-spaces ($B_r(\mathcal{N})$-spaces) are $B(\mathcal{M})$-spaces ($B_r(\mathcal{M})$-spaces) relies upon the following lemma.

LEMMA 1.1: Let E be a (DF)-space and F a metrisable convex space. If there exists a continuous, almost open linear mapping t of E into F, then F is normable.

PROOF: Let $\{V_i\}$ be a sequence of basic neighbourhoods of the origin in F. Since t is continuous, for any V_i, there exists a neighbourhood U_i of the origin in E such that $t(U_i) \subseteq V_i$. We therefore have a sequence $\{U_i\}$ of neighbourhoods of the origin in E. Since E is a (DF)-space, there exists a convex neighbourhood U of the origin which is absorbed by all U_i ([9], Page 167) i.e. for each i there is a non-zero $\lambda_i \in \mathbb{C}$ for which $\lambda_i U_i \supseteq U$. Thus, we have $t(U) \subseteq \lambda_i t(U_i) \subseteq \lambda_i V_i$ for all i and so $t(U)$ is bounded. Further, since t is almost open, $\overline{t(U)}$ is a bounded neighbourhood of the origin in F. Boundedness of $\overline{t(U)}$ implies that F is normable ([35], Chapter III, § 1, Theorem 1). Q.E.D.

REMARK 1.1: In the particular case when E is a normed space, F could be any convex space. If U is the unit ball in E, $\overline{t(U)}$ is a bounded neighbourhood of the origin in F and therefore F is normable.

We now have:

PROPOSITION 1.4: A (DF)-space which is also a $B(\mathfrak{N})$-space ($B_r(\mathfrak{N})$-space) is a $B(\mathfrak{M})$-space ($B_r(\mathfrak{M})$-space).

PROOF: Let E be a (DF)-space which is also a $B(\mathfrak{N})$-space, and F, a metrisable convex space. If t is a continuous, almost open linear mapping of E onto F, then by Lemma 1.1, F is normable. But then, E being a $B(\mathfrak{N})$-space, it follows that t is open. Thus E is a $B(\mathfrak{M})$-space.

The statement about $B_r(\mathfrak{M})$-spaces follows in a similar way. Q.E.D.

REMARK 1.2: A similar argument as in the above proposition, can be used to show that the class of $B(\mathcal{B})$-spaces ($B_r(\mathcal{B})$-spaces), where $\mathcal{B}$ is the class of all Banach spaces, which are also (DF)-spaces, coincides with the class of $B(\mathfrak{F})$-spaces ($B_r(\mathfrak{F})$-spaces), where $\mathfrak{F}$ is the class of Fréchet spaces.

2. $B_r(\mathfrak{J})$-spaces and $B_r(\mathfrak{N})$-spaces and the closed graph theorem

A $B_r(\mathfrak{J})$-space is characterized by the fact that every continuous, one-to-one, linear mapping from it onto any separated barrelled space, is open. On these spaces, T. Husain was able to prove a very general closed graph theorem (Chapter I, Theorem 6.2). His proof depends upon a connection between the graph of a mapping being closed and a certain topology being separated. By appealing to methods in the duality theory of convex spaces, we are able to give a shorter proof.

THEOREM 2.1: Let E be a barrelled space and F a $B_r(\mathfrak{J})$-space. Let t be a linear mapping of E into F, whose graph is closed in $E \times F$. If t is almost open, then t is continuous.

PROOF: Let $D[t'] = t'^{-1}(E')$. Since $t': F' \longrightarrow E^*$ is continuous with respect to $\sigma(F', F)$ and $\sigma(E^*, E)$ and since E is barrelled, for every bounded subset B of F'^{σ}, there exists a neighbourhood U of the origin in E such that $t'(D[t'] \cap B) \subset U^{\circ}$. Therefore,

$$D[t'] \cap B \subset t'^{-1}(U^{\circ}) = (t(U))^{\circ} = \overline{(t(U))}^{\circ}.$$

But t is almost open and so every bounded subset of $D[t']$ is equicontinuous. Since t is almost continuous, $D[t'] \cap V^{\circ}$ is closed in F'^{σ} for every neighbourhood V of the origin in F (Chapter I, Lemma 5.1). Further, since the graph of t is closed, $D[t']$ is dense in F'^{σ}. Thus $D[t']$ is a dense, boundedly complete subspace of F'^{σ}. Since F is a $B_r(\mathcal{J})$-space, $D[t'] = F'$ (Chapter I, Theorem 6.4).

Thus t is weakly continuous. But E is a barrelled space and therefore a Mackey space and so t is continuous. Q.E.D.

If t is a linear mapping of E_u onto F_v, $\overline{t(U)}$ is a barrel in F_v, for each absolutely convex neighbourhood U of the origin in E_u. Thus $\overline{t(U)}$ is a $\beta(F, F')$ neighbourhood of the origin in F.

THEOREM 2.2: Let E_u be a barrelled space. Let F_v be a convex space such that in its weak dual, every dense, almost closed subspace whose bounded subsets are equicontinuous with respect to $\beta(F, F')$, is closed. Then a linear mapping t of E onto F, whose graph is closed in $E_u \times F_v$, is continuous.

PROOF: Using the same notation as in the proof of Theorem 2.1, we observe that:

$D[t'] \cap B \subset (\overline{t(U)})^{\circ} \subset (\overline{t(\tilde{U})})^{\circ}$ where $\overline{t(U)}$ is the $\sigma(F, F')$-closure of $t(U)$ in F and $(\overline{t(U)})^{\circ}$ is the polar of $\overline{t(U)}$ in F'_{β}. Thus every bounded subset of $D[t']$ is equicontinuous with respect to $\beta(F, F')$ and therefore $D[t']$ is closed in F'^{σ}_{v}. The theorem now follows as in Theorem 2.1. Q.E.D.

We also have:

<u>THEOREM 2.3</u>: Let E_u be a barrelled space and F_v a convex space which is a $B_r(\mathfrak{J})$-space in its strong topology. Then a linear mapping t of E onto F, whose graph is closed in $E_u \times F_v$, is continuous.

<u>PROOF</u>: Since the graph of t is closed in $E_u \times F_v$ and $\beta(F, F')$ is finer than v, the graph of t remains closed in $E_u \times F_{\beta}$. Further, since t is a linear mapping of E onto F, t is almost open from E_u onto F_{β}. By Theorem 2.1, it follows that $t: E_u \longrightarrow F_{\beta}$ is continuous and therefore $t: E_u \longrightarrow F_v$ is also continuous. Q.E.D.

Since $B(\mathfrak{J})$-spaces are $B_r(\mathfrak{J})$-spaces our theorem is also true when F_{β} is a $B(\mathfrak{J})$-space.

<u>COROLLARY</u>: Let E_u be a barrelled space and F_v a convex space which is a $B(\mathfrak{J})$-space in its strong topology. Then a linear mapping t of E onto F, whose graph is closed in $E_u \times F_v$, is continuous.

The above corollary gives the following characterization of spaces which are $B_r(\mathfrak{J})$-spaces in their strong topologies.

<u>THEOREM 2.4</u>: A convex space E_u is a $B(\mathfrak{J})$-space in its strong topology if and only if for each barrelled space F_v, a linear

mapping t of E_β onto F_v, whose graph is closed in $E_\beta \times F_v$, is open.

PROOF: Assume that E_β is a $B(\mathcal{J})$-space. Since the quotient of a $B(\mathcal{J})$-space by a closed subspace is also a $B(\mathcal{J})$-space (Chapter I, Corollary to Proposition 6.3), t is open, by the above **Corollary.** On the other hand, if t is a continuous mapping of E_β onto a barrelled space F_v, the graph of t is closed in $E_\beta \times F_v$ and therefore t is open, by assumption. This shows that E_β is a $B(\mathcal{J})$-space. Q.E.D.

REMARK 2.1: We note here that whenever E_β is a $B(\mathcal{J})$-space, E_u is also a $B(\mathcal{J})$-space.

Husain in [15] queried as to whether almost openness of t could be dropped in Theorem 2.1. Sulley in [40] showed that this cannot be done in general. In that which follows, we shall give another condition on t such that the closed graph theorem is true.

We begin with:

DEFINITION 2.1: Let E and F be two convex spaces t a linear mapping of E into F. We say that t satisfies condition (*) if for some basic neighbourhood V^* of the origin in F, $t^{-1}(V^*)$ is a bounded subset of E.

For example, if E_u is a normed space, the identity mapping $i: E_\sigma \longrightarrow E_u$ satisfies condition (*), because the unit ball in E_u is σ-bounded.

We shall denote by $\mathcal{U}$, the neighbourhood base of the origin in E_u and by $\mathcal{V}$, the neighbourhood base of the origin in F_v.

THEOREM 2.5: Let E_u be a barrelled space and F_v a $B_r(\mathcal{N})$-space. Let t be a linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$. If t satisfies condition (*), then t is continuous.

PROOF: $\left\{\lambda\ \overline{t^{-1}(V^*)}\right\}_{\lambda>0}$ forms a family of absolutely convex absorbent sets and so is a base of neighbourhoods of the origin in a topology u' on E making $E_{u'}$ a convex space. Since $\overline{t^{-1}(V^*)}$ is bounded and balanced, for each $U \varepsilon \mathcal{U}$, there exists a $\mu > 0$ such that

$$\overline{t^{-1}(V^*)} \subset \mu\ U$$

$$\text{i.e.} \quad \frac{1}{\mu}\ \overline{t^{-1}(V^*)} \subset U$$

$$\text{i.e.} \quad \lambda\ \overline{t^{-1}(V^*)} \subset U \quad \text{for} \quad \lambda = \frac{1}{\mu} .$$

Also since E_u is barrelled, for each $\lambda\ \overline{t^{-1}(V^*)}$, $\lambda > 0$, there exists a $U \varepsilon \mathcal{U}$ such that $U \subset \lambda\ \overline{t^{-1}(V^*)}$. Since $\lambda\ \overline{t^{-1}(V^*)}$ is a bounded subset of $E_{u'}$, E_u is normable ([35], Chapter III, § 1, Theorem 1), where the sets $\lambda\ \overline{t^{-1}(V^*)}$, $\lambda > 0$, form a neighbourhood base at the origin.

Let $\overline{t^{-1}(V)}^{u}$ be the u-closure of $t^{-1}(V)$ in E and $\overline{t(\overline{t^{-1}(V)}^{u})}^{v}$ be the v-closure of $t(\overline{t^{-1}(V)}^{u})$ in F. Consider now the family of sets of the form $\tilde{V} = \overline{t(\overline{t^{-1}(V)}^{u})}^{v}$, where $V \varepsilon \mathcal{V}$. $\{\tilde{V}\}$ forms a base of neighbourhoods in a separated topology w on F such that $t\colon E_u \longrightarrow F_w$ is continuous (see the proof of Theorem 6.2, Chapter I). We show now that the mapping $t\colon E_u \longrightarrow F_w$ is almost open.

For each $U \in \mathcal{U}$ and some $\lambda > 0$, we have $\overline{t(\overline{\lambda t^{-1}(V^*)}^{u})}^{w} \subseteq \overline{t(U)}^{w}$. Since λV^* is a neighbourhood of the origin in F_v, there exists a $V \in \mathcal{V}$ such that $V \subset \lambda V^*$.

Thus $\overline{t(\overline{t^{-1}(V)}^{u})}^{v} \subset \overline{t(\overline{\lambda t^{-1}(V^*)}^{u})}^{v}$. Further since $v \supset w$ we have $\overline{t(\overline{t^{-1}(V)}^{u})}^{v} \subset \overline{t(U)}^{w}$. This shows that $t: E_u \longrightarrow F_w$ is almost open. Since E_u is normed, continuity of t from E_u onto F_w now implies that F_w is also a normed space (Remark 1.1). F_v is a $B_r(\mathcal{N})$-space; also the identity mapping $i: F_v \longrightarrow F_w$ is continuous and almost open (see the proof of Theorem 6.2, Chapter I). By the definition of $B_r(\mathcal{N})$-spaces, $v = w$. Now, since $t: E_u \longrightarrow F_w$ is continuous, it follows that $t: E_u \longrightarrow F_v$ is also continuous.

<u>COROLLARY</u>: Let E_u be a barrelled space and F_v a $B_r(\mathcal{M})$-space. Let t be a linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$. If t satisfies condition (*), then t is continuous.

<u>PROOF</u>: Since $B_r(\mathcal{M})$-spaces are $B_r(\mathcal{N})$-spaces, our result follows as before.

We also have:

<u>THEOREM 2.6</u>: Let E_u be a barrelled space and F_v, a $B_r(\mathcal{J})$-space. Let t be a linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$. If t satisfies condition (*), then t is continuous.

<u>PROOF</u>: Using the same notations as in the proof of Theorem 2.5, we observe that:

$t: E_u \longrightarrow F_w$ is continuous and almost open. Since E_u is barrelled, it follows that F_w is also barrelled. Now since F_v is a $B_r(\mathcal{J})$-space and the identity mapping $i: F_v \longrightarrow F_w$ is continuous and almost open, i is open and so $v = w$. Our theorem now follows as before.

REMARK 2.2: In the above theorem, we require that t be onto in order that the sets $\tilde{V}$ be absorbent.

Closed subspaces of $B_r(\mathcal{J})$-spaces are not in general $B_r(\mathcal{J})$ ([40], § 2, Corollary 1). If we restrict to the class of $B_r(\mathcal{J})$-spaces whose closed subspaces are also $B_r(\mathcal{J})$, the requirement that t be onto could be relaxed, because then, $\overline{t(E)}$ is a $B_r(\mathcal{J})$-space and since the graph of t is closed in $E \times \overline{t(E)}$, it is sufficient to prove the theorem when $t(E)$ is dense in F. In that case,
$V = \overline{V \cap t(E)}^{v} \subseteq \overline{t(t^{-1}(V))}^{v} \subseteq \tilde{V}$ and so each $\tilde{V}$ is absorbent.

It is evidently of interest to know the relations between almost openness of t and condition (*) imposed on t.

The following example shows that in general, almost openness does not imply condition (*).

EXAMPLE 2.1: Let E_u be an infinite dimensional Banach space. Consider E under the finest convex topology $\tau(E, E^*)$. Then $E_{\tau(E, E^*)}$ is barrelled ([35], Chapter VI, Supplement (1)). The identity mapping $i: E_{\tau(E, E^*)} \longrightarrow E_u$ is almost open, because E_u is a Banach space but it fails to satisfy condition (*) because otherwise $E_{\tau(E, E^*)}$ would be normable and this is impossible ([35], Chapter IV, Supplement (2) and (3)).

The next example shows that in general, condition (*) does not imply almost openness. This relies upon:

LEMMA 2.1: Let E and F be convex spaces. If t is an almost open linear mapping of E into F, $t(E)$ is dense in F.

PROOF: Let U be a neighbourhood of the origin in E. Since t is almost open $\overline{t(U)}$ is a neighbourhood of the origin in F. But since $\overline{t(U)} \subset \overline{t(E)}$, $\overline{t(E)}$ is also a neighbourhood of the origin in F. Thus $\overline{t(E)}$ is an absorbent subspace and therefore $\overline{t(E)} = F$.

Lemma 2.1 gives us the following example.

EXAMPLE 2.2: Let E be a normed space and S, a non-dense subspace of E. If now we considered the natural injection $i: S \longrightarrow E$, i satisfies condition (*) but by Lemma 2.1, it is not almost open.

3. $B_r(\mathfrak{M})$-spaces and $B_r(\mathcal{S}.\mathcal{T})$-spaces and the closed graph theorem

The almost openness condition imposed on t, in Theorem 2.1, may be dropped in the case when F is a γ-space (see Chapter II, Theorem 2.2). It will be of interest to know for what other classes of spaces, this could be done. In this section, we consider onto mappings and show that for some classes of $B_r(\mathfrak{M})$-spaces and $B_r(\mathcal{S}.\mathcal{T})$-spaces, the almost openness condition could be dropped.

As usual, all spaces considered are separated.

A subfamily $\mathcal{B}$ of all bounded sets of a convex space is said to be a fundamental family of bounded sets, if for each bounded set B

in the space, there exists a set A in $\mathcal{B}$ such that $B \subset A$.

<u>THEOREM 3.1</u>: Let E_u be a Mackey space with the properties that E_β is a separable space and E'^{σ}_u possesses a countable fundamental family of bounded sets. Let F_v be a Mackey space which is also a countably barrelled $B_r(\mathfrak{M})$-space. If t is an almost continuous linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, then t is continuous.

<u>PROOF</u>: Since t is an almost continuous, linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, it is possible to construct a separated locally convex topology w on F such that:

(i) $t: E_u \longrightarrow F_w$ is continuous;

(ii) $i: F_v \longrightarrow F_w$ is continuous and almost open.

(see the proof of Theorem 6.2, Chapter I). Since t is a continuous mapping of E_u onto F_w, $t': F'^{\sigma}_w \longrightarrow E'^{\sigma}_u$ is a homeomorphism into (Chapter I, Corollary (ii) to Proposition 2.6). Since E'^{σ}_u possesses a countable fundamental family of bounded sets, F'^{σ}_w also possesses a countable fundamental family of bounded sets. Thus F is metrisable in its strong topology $\beta(F, F'_w)$. By (ii) $i: F_v \longrightarrow F_w$ is continuous and almost open. Since F_v is a Mackey space, this further implies that $i: F_v \longrightarrow F_{\tau(F,\ F'_w)}$ is continuous (Chapter I, Proposition 2.10) and since $w \subset \tau(F, F'_w)$, it is also almost open. F_v is countably barrelled and so $F_{\tau(F,\ F'_w)}$ is also countably barrelled (Chapter I, Proposition 7.5).

Now, continuity of $t: E_u \longrightarrow F_w$ implies continuity of $t: E_\beta \longrightarrow F_{\beta(F,\ F'_w)}$ (Chapter I, Proposition 2.11). Thus, t is a continuous mapping of a separable space E_β onto $F_{\beta(F, F'_w)}$ and so $F_{\beta(F, F'_w)}$ is also separable.

We have therefore, that $F_{\tau(F, F'_w)}$ is countably barrelled and $F_{\beta(F, F'_w)}$ is separable. Since the weak duals of countably barrelled spaces are sequentially complete, (Chapter I, Proposition 7.4), this gives us that $\tau(F, F'_w) = \beta(F, F'_w)$ (Chapter I, Theorem 8.2 and Theorem 8.1(c)) and so $F_{\tau(F, F'_w)}$ is metrisable. Since $i: F_v \longrightarrow F_{\tau(F, F'_w)}$ is continuous and almost open and F_v is a $B_r(\mathfrak{M})$-space, i is open and so $v = \tau(F, F'_w)$.

Since E_u is a Mackey space, continuity of $t: E_u \longrightarrow F_w$ also implies continuity of $t: E_u \longrightarrow F_{\tau(F, F'_w)}$ and so $t: E_u \longrightarrow F_v$ is continuous. Q.E.D.

<u>COROLLARY 1</u>: Let E_u be a separable barrelled space with the property that E'^{σ}_u possesses a countable fundamental family of bounded sets. Let F_v be as in the theorem. If t is a linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, then t is continuous.

<u>PROOF</u>: Since E_u is barrelled, $\beta(E, E') = u$ and $t: E_u \longrightarrow F_v$ is almost continuous. Thus our result follows as before. Q.E.D.

<u>COROLLARY 2</u>: Let E_u be a separable Fréchet space. Let F_v and t be as in Corollary 1. Then t is continuous.

<u>PROOF</u>: This follows from the fact that E_u is barrelled and E'^{σ}_u possesses a countable fundamental family of bounded sets. ([38], Chapter IV, 6.4). Q.E.D.

COROLLARY 3: Let E_u be as in Corollary 1. Let F_v be a Mackey space which is also a countably barrelled $B_r(\mathcal{J})$-space. If t is a linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, then t is continuous.

PROOF: Since countably barrelled $B_r(\mathcal{J})$-spaces are $B_r(\mathcal{M})$-spaces (Corollary to Proposition 1.3), t is continuous, as before. Q.E.D.

COROLLARY 4: Let E_u be as in Corollary 1. Let F_v be a quasi-complete Mackey space which is also a (DF)-space and a $B_r(\mathcal{N})$-space. If t is a linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, then t is continuous.

PROOF: Since quasi-complete (DF)-spaces are countably barrelled (Chapter I, Proposition 7.2) and (DF)-spaces which are also $B_r(\mathcal{N})$-spaces are $B_r(\mathcal{M})$-spaces (Proposition 1.4), t is continuous as before. Q.E.D.

THEOREM 3.2: Let E_u be a separable barrelled space and F_v a Mackey space which is also a countably barrelled $B_r(\mathcal{S}.\mathcal{J})$-space. If t is a linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, then t is continuous.

PROOF: Using the same notations, as in the proof of Theorem 3.1, we observe that since $\tau(F, F'_w) = \beta(F, F'_w)$, $F_{\tau(F, F'_w)}$ is separable and barrelled. The identity mapping $i: F_v \longrightarrow F_{\tau(F, F'_w)}$ is continuous and almost open and since F_v is a $B_r(\mathcal{S}.\mathcal{J})$-space, i is open. Our theorem follows as before. Q.E.D.

Since $B(\mathcal{S}.\mathcal{J})$-spaces are $B_r(\mathcal{S}.\mathcal{J})$, we have:

COROLLARY 1: Let E_u be a separable barrelled space and F_v a Mackey space which is also a countably barrelled $B(\mathcal{S}.\mathcal{J})$-space. If t is a linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, then t is continuous.

Since $\mathcal{S}.\mathcal{J}$ is a subclass of $\mathcal{J}$, $B_r(\mathcal{J})$-spaces are $B_r(\mathcal{S}.\mathcal{J})$-spaces and this gives us:

COROLLARY 2: Let E_u be a separable barrelled space and F_v a Mackey space which is also a countably barrelled $B_r(\mathcal{J})$-space. If t is a linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, then t is continuous.

Corollary 1 also gives us a characterization of $B(\mathcal{S}.\mathcal{J})$-spaces.

THEOREM 3.3: Let F_v be a countably barrelled Mackey space. Then F_v is a $B(\mathcal{S}.\mathcal{J})$-space if and only if, for each separable barrelled space E_u, a linear mapping t of F_v onto E_u, whose graph is closed in $E_u \times F_v$, is open.

PROOF: Assume F_v is a $B(\mathcal{S}.\mathcal{J})$-space. Since the quotient of a countably barrelled $B(\mathcal{S}.\mathcal{J})$-space by a closed subspace is also countably barrelled and a $B(\mathcal{S}.\mathcal{J})$-space (Chapter I, Corollary to Proposition 7.5 and Corollary to Proposition 6.3), t is open by Corollary 1 of Theorem 3.2. On the other hand, if t is a continuous linear mapping of F_v onto any separable barrelled space E_u, the graph of t is closed in $F_v \times E_u$ and therefore t is open, by assumption. This shows that F_v is a $B(\mathcal{S}.\mathcal{J})$-space. Q.E.D.

4. $B(\mathcal{F})$-spaces and almost open linear mappings with closed graphs

In sections 2 and 3 of this chapter, we considered the situation in which a linear mapping is continuous, whenever its graph is closed. In this section, we consider the situation for which an almost open linear mapping is both continucus and open, whenever it graph is closed. It turns out that certain classes of $B(\mathcal{F})$-spaces play an important role.

We consider, almost entirely, $B(\mathcal{F})$-spaces. But the arguments are also valid for $B_r(\mathcal{F})$-spaces, provided the mappings considered are one-to-one.

As usual, all spaces considered are separated.

THEOREM 4.1: Let E be a countably barrelled Mackey space which is also a $B(\mathcal{F})$-space. Let F be a metrisable α-space. Then a linear, almost open mapping t of E onto F, whose graph is closed in $E \times F$, is continuous and open.

PROOF: Since E is a countably barrelled Mackey space and F is an α-space, $t: E \longrightarrow F$ is continuous (Chapter II, Theorem 1.1, Corollary 1). Now, since t is also almost open, F is countably barrelled (Chapter I, Proposition 7.5) and this together with the fact that F is an α-space implies F is B-complete (Chapter II, Proposition 1.5, Corollary 1) and so complete. But since F is metrisable, it is a Fréchet space. E is a $B(\mathcal{F})$-space and therefore $t: E \longrightarrow F$ is open. Q.E.D.

REMARK 4.1: It is worth noting that a metrisable α-space is not in general, complete. For if $F_w = R^{(N)}$, the space of all finite sequences, where w is the norm topology defined by: $\|x\| = \sup_n |x_n|$, then F_w is a metrisable α-space which is not complete ([15], Chapter 7, § 6, Example 4).

We denote by $\mathcal{B}_c$, the class of all B-complete spaces. Since $\mathcal{F}$ is a subclass of $\mathcal{B}_c$, B($\mathcal{B}_c$)-spaces are B($\mathcal{F}$)-spaces (Chapter I, Proposition 6.2).

Metrisability in Theorem 4.1 may be relaxed if we restrict E to the class of B($\mathcal{B}_c$)-spaces. We have:

THEOREM 4.2: Let E be a countably barrelled Mackey space which is also a B($\mathcal{B}_c$)-space. Let F be an α-space. Then a linear, almost open mapping t of E onto F, whose graph is closed in E x F, is continuous and open.

PROOF: Since a countably barrelled α-space is B-complete, our result follows as before. Q.E.D.

Our next theorem relies upon the following lemma.

LEMMA 4.1: Let E_u be a convex space. If $E_u'^{\beta}$ is an α-space, then E_u is semi-reflexive if and only if E_σ is sequentially complete.

PROOF: The "only if" part of the lemma is obvious. For the "if" part, consider the identity mapping $i': E_u'^{\tau} \longrightarrow E_u'^{\beta}$. Since $\beta(E', E)$ is finer than $\tau(E', E)$, the graph of i' is closed in $E_u'^{\tau} \times E_u'^{\beta}$. Since $E_u'^{\beta}$ is an α-space and E_σ is sequentially complete, it follows that i' is continuous (Chapter II, Theorem 1.1). Thus we have that $\beta(E', E)$ is coarser than $\tau(E', E)$ and so $\beta(E', E) = \tau(E', E)$. This proves that E_u is semi-reflexive (Chapter I, Theorem 8.1 (c)). Q.E.D.

Since $\mathcal{F}$ is a subclass of $\mathcal{J}$, $B(\mathcal{J})$-spaces are $B(\mathcal{F})$-spaces (Chapter I, Proposition 6.2). For this restricted class of $B(\mathcal{F})$-spaces, we have the following situation.

THEOREM 4.3: Let E_u be a countably barrelled Mackey space which is also a $B(\mathcal{J})$-space. Let F_v be a Mackey space which is an α-space in its $\beta(F, F')$-topology. Then a linear, almost open mapping t of E_u onto F_v, whose graph is closed in $E_u \times F_v$, is continuous and open.

PROOF: As in the proof of Theorem 4.1, t is continuous and F_v is countably barrelled. Countably barrelledness of F_v implies that $F_v'^{\sigma}$ is sequentially complete (Chapter I, Proposition 7.4). Thus on F, $v = \tau(F, F') = \beta(F, F')$ (Chapter I, Theorem 8.1 (c)). Therefore F_v is a barrelled space and since E_u is a $B(\mathcal{J})$-space, $t: E_u \longrightarrow F_v$ is open. Q.E.D.

Let $\mathcal{B}_o$ denote the class of all bornological spaces. Since $\mathcal{F}$ is a subclass of $\mathcal{B}_o$, $B(\mathcal{B}_o)$-spaces are $B(\mathcal{F})$-spaces.

Our next theorem is on $B(\mathcal{B}_o)$-spaces and this stems from the fact that separable strong duals of convex metrisable spaces are bornological ([38], Chapter IV, 6.6, Corollary 2).

THEOREM 4.4: Let E_u be a countably barrelled Mackey space which is also a $B(\mathcal{B}_o)$-space. Let F_v be a Mackey space, separable in its $\beta(F, F')$-topology and with $F_v'^{\tau}$ metrisable. Then a linear, almost open mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, is continuous and open.

PROOF: Since $F_v'^{\tau}$ is metrisable, F_v is an α-space. As in the proof of Theorem 4.1, t is continuous and F_v is countably barrelled. Countably barrelledness of F_v implies that F'^{σ} is sequentially complete. This together with the fact that F_β is separable implies that $F_v'^{\tau}$ is semi-reflexive (Chapter I, Theorem 8.2). Thus on F, $v = \tau(F, F') = \beta(F, F')$ and so F_v is separable. Now since $F_v'^{\tau}$ is metrisable, $F_\beta = F_v$ is bornological ([38], Chapter IV, 6.6, Corollary 2). E_u is a $B(\mathfrak{B}_o)$-space and therefore $t: E_u \longrightarrow F_v$, is open. Q.E.D.

The hypothesis that, "F_β is separable", may be replaced by the hypothesis that, "F_β is an α-space". We have:

THEOREM 4.5: Let E_u be a countably barrelled Mackey space, which is also a $B(\mathfrak{B}_o)$-space. Let F_v be a Mackey space, which is an α-space in its $\beta(F, F')$-topology and with $F_v'^{\tau}$ metrisable. Then a linear, almost open mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, is continuous and open.

5. $B(\mathfrak{J})$-spaces and dense, weakly singular, maximal mappings

In this section, we show that $B(\mathfrak{J})$-spaces can be characterized in terms of dense, weakly singular, maximal mappings. These mappings are more general than dense, closeable mappings (see [24], § 2 (3) and (5), § 5 (3) and § 7 (3) and (4)).

THEOREM 5.1: A convex space E is a $B(\mathfrak{J})$-space if and only if each dense, weakly singular maximal mapping t of E onto a barrelled space, is open, whenever it is almost continuous.

PROOF: Assume that E is a $B(\mathcal{J})$-space. Using the same notations as in the proof of Theorem 2.5 (Chapter II) we observe that for each neighbourhood U of the origin in E, $R[t'] \cap U^{\circ}$ is closed in E'^{σ}, since U° is a $\sigma(E', E)$-bounded set. Therefore $R[t']$ is almost closed in E'^{σ}. Also, for each bounded subset B in E'^{σ},

$$R[t'] \cap B \subset (t^{-1}(V))^{\circ}$$

$$= (\overline{t^{-1}(V)})^{\circ}.$$

Since t is almost continuous, every bounded subset of $R[t']$ is equicontinuous. Thus $R[t']$ is boundedly complete and since E is a $B(\mathcal{J})$-space, $R[t']$ is $\sigma(E', E)$-closed (Chapter I, Theorem 6.4). But since t is maximal, $R[t']$ is also $\sigma(E', D[t])$-closed by Proposition 2.3 (Chapter II), and so t is open ([24], § 9).

On the other hand, if t is a continuous, linear mapping of E onto F, t could be regarded as a dense, weakly singular maximal mapping of E onto F. t is open, by assumption. This shows that E is a $B(\mathcal{J})$-space.

CHAPTER IV

COUNTABILITY CONDITIONS

In this chapter, each section contains its own introductory remarks. Countability conditions on bounded sets form the unifying feature of the subject matter.

1. Countability conditions and the closed graph theorem

In this section, we show that there are convex spaces more general than metrisable spaces which have the property that their strong duals possess a countable fundamental family of bounded sets. We investigate these spaces and prove, among other results, closed graph and open mapping theorems, relaxing the necessary completeness requirement by filter conditions introduced in [36].

We begin with the following simple observation.

PROPOSTION 1.1: If E is a metrisable convex space, then E'^{β} possesses a countable fundamental family of bounded sets.

PROOF: Let $\{U_n\}$ be a countable base of neighbourhoods of the origin in E. Since each U_n^o is equicontinuous, it is strongly bounded ([35], Chapter IV, § 3, Corollary to Lemma 2). Also, since E is quasi-barrelled, every strongly bounded set in E' is equicontinuous. Thus $\{B_n\} = \{U_n^o\}$ constitutes a countable fundamental family of bounded sets in E'^{β}. Q.E.D.

The following example shows that the class of convex spaces whose strong duals possess a countable fundamental family of bounded sets is larger than the class of metrisable convex spaces.

EXAMPLE 1.1: Let E_u be a non-reflexive Banach space (such for example is ℓ_1, with the usual norm topology). Then there exists in E_u an absolutely convex bounded set which is not relatively weakly compact (Chapter I, Theorem 8.1 (d)). Thus $E_u'^{\tau}$ is not quasi-barrelled and therefore not metrisable. But $E_u'^{\tau,\beta} = E_u$ is a Banach space and therefore possesses a countable fundamental family of bounded sets.

Suppose E is a convex space and E' its dual. Then in E', every absolutely convex $\sigma(E', E)$-compact set is $\tau(E, E')$-equicontinuous and therefore strongly bounded. If, conversely, we have that every absolutely convex strongly bounded set is relatively $\sigma(E', E)$-compact, E is quasi-barrelled in its Mackey topology. In [25], V. Krishnamurthy calls such spaces, quasi-M-barrelled spaces.

DEFINITION 1.1: A convex space E is called quasi-M-barrelled if in its dual E', every absolutely convex strongly bounded set is relatively weakly compact or equivalently, if the strong bidual induces the Mackey topology on E or if and only if every barrel in E which absorbs all bounded sets is a neighbourhood of the origin for the Mackey topology.

Bornological spaces are quasi-M-barrelled. In fact, every quasi-barrelled space is quasi-M-barrelled. Convex spaces whose strong duals are semi-reflexive are further examples of quasi-M-barrelled spaces.

The requirement of being quasi-M-barrelled depends only on the dual system and so weakening the topology of a quasi-M-barrelled space without affecting the dual, would still have it quasi-M-barrelled. In particular therefore, the weak topology of an infinite dimensional Banach space is an example of a quasi-M-barrelled space which is not quasi-barrelled.

In Example 1.1, we observed that although the strong dual of $E_u'^{\tau}$ possesses a countable fundamental family of bounded sets, $E_u'^{\tau}$ is not metrisable because it is not quasi-barrelled. However, we have:

PROPOSITION 1.2: Let E_u be a quasi-M-barrelled space. Then $E_u'^{\beta}$ possesses a countable fundamental family of bounded sets if and only if E_τ is metrisable.

PROOF: Since $E_u'^{\beta}$ has a countable fundamental family of bounded sets, $E_u'^{\beta,\beta}$ is metrisable. But $E_u'^{\beta,\beta}$ induces the Mackey topology on E, since E_u is quasi-M-barrelled. Thus E_τ, being a subspace of a metrisable space, is metrisable.

The converse follows from Proposition 1.1.

COROLLARY 1: Let E_u be a quasi-barrelled space. Then $E_u'^{\beta}$ possesses a countable fundamental family of bounded sets if and only if E_u is metrisable.

PROOF: This follows, since quasi-barrelled spaces are quasi-M-barrelled spaces and $u = \tau(E, E')$. Q.E.D.

Suppose that E is the strict inductive limit of the convex spaces E_n and that, for each n, E_n is a closed vector subspace of E_{n+1}. Then E is not metrisable ([35], Chapter VII, § 1, Proposition 5).

The condition that each E_n is closed in E_{n+1} is satisfied when the E_n are complete.

From Proposition 1.2, we can also conclude that:

COROLLARY 2: The strong dual of a strict inductive limit of complete barrelled spaces does not possess a countable fundamental family of bounded sets.

PROOF: Since the strict inductive limit of barrelled spaces is barrelled ([35], Chapter V, § 2, Propositon 6) and so quasi-M-barrelled, the existence of a countable fundamental family of bounded sets in its strong dual would imply metrisability, by Proposition 1.2. This is impossible from the above remarks. Q.E.D.

Another result in the direction of Proposition 1.2 is the following.

PROPOSITION 1.3: Let E_u be a convex space with the property that E_β is separable and E'^{σ}_u is sequentially complete. Then E'^{β}_u possesses a countable fundamental family of bounded sets if and only if E_τ is metrisable.

PROOF: The sufficiency follows from Proposition 1.1. On the other hand, since E'^{σ}_u is sequentially complete and E_β is separable, E'^{τ}_u is semi-reflexive (Chapter I, Theorem 8.2) and so $\beta(E, E') = \tau(E, E')$ (Chapter I, Theorem 8.1 (c)). Also, since E'^{σ}_u is sequentially complete, every $\sigma(E', E)$-bounded set in E'_u is $\beta(E', E)$-bounded. Since E'^{β}_u possesses a countable fundamental family of bounded sets, E'^{σ}_u also possesses a countable fundamental family of bounded sets. Thus $\beta(E, E') = \tau(E, E')$ is metrisable on E.

COROLLARY: Let E_u be a countably barrelled space with the property that E_β is separable. Then $E_u'^{\beta}$ possesses a countable fundamental family of bounded sets if and only if E_τ is metrisable.

PROOF: This follows from the fact that the weak duals of countably barrelled spaces are sequentially complete (Chapter I, Proposition 7.4). Q.E.D.

We denote by $\beta^*(E, E')$, the topology on E of uniform convergence on the $\beta(E', E)$-bounded sets of E'. Then $\tau(E, E') \subseteq \beta^*(E, E') \subseteq \beta(E, E')$.

It is clear that $\beta^*(E, E') = \beta(E, E')$ if and only if weakly and strongly bounded sets of E' are identical.

In Example 1.1, we have a convex space in which $\tau(E, E')$ is strictly coarser than $\beta^*(E, E')$. However, $\tau(E, E') = \beta^*(E, E')$ whenever E is quasi-M-barrelled.

DEFINITION 1.2: (See [36]) Let t be a linear mapping of a convex space E into a convex space F. It is said that the inverse filter condition holds if for a convergent filter base $\mathcal{K}$ on E such that $t(\mathcal{K})$ is Cauchy, it follows that $t(\mathcal{K})$ is convergent to a point in $t(E)$.

With the inverse filter condition, we have the following closed graph theorem.

THEOREM 1.1: Let E_u be a barrelled space and F_v a convex space, metrisable in its $\beta^*(F, F')$-topology. Let t be a linear mapping of E_u into F_v, whose graph is closed in $E_u \times F_v$.

Then t is continuous, provided the inverse filter condition holds.

PROOF: Since the graph G of t is closed in $E_u \times F_v$, it is also closed in $E_u \times F_{\beta*}$, because $\beta^*(F, F')$ is finer than v. Let us now regard t as a linear mapping of E_u into $\hat{F}_{\beta*}$, where $\hat{F}_{\beta*}$ is the completion of $F_{\beta*}$. We show that G is closed in $E_u \times \hat{F}_{\beta*}$. Let $(x, y) \in \overline{G}$, where $\overline{G}$ is the closure of G in $E_u \times \hat{F}_{\beta*}$. Then $(x + U) \times (y + W)$ meets G, for U and W, neighbourhoods of the origin in E_u and $\hat{F}_{\beta*}$, respectively. That means, $A_{U,W} = (x+U) \cap t^{-1}(y+W) \neq \emptyset$. Let $\mathcal{K}$ be the filter generated by the sets $A_{U,W}$, where U and W run over the fundamental systems of neighbourhoods of the origin in E_u and $\hat{F}_{\beta*}$, respectively. Then $\mathcal{K} \longrightarrow x$ in E_u; also $t(\mathcal{K}) \longrightarrow y$ in $\hat{F}_{\beta*}$ and therefore $t(\mathcal{K}) \longrightarrow y$ in $\hat{F}_v$, since $\beta^*(F, F')$ is finer than v. Therefore, $t(\mathcal{K})$ is v-Cauchy and so by the inverse filter condition, $t(\mathcal{K})$ converges, with respect to v, to a point in $t(E)$. This shows that $y \in t(E) \subset F$ and so $(x, y) \in \overline{G} \cap (E \times F)$. But G is closed in $E_u \times F_{\beta*}$ and so $(x, y) \in G$. Thus G is closed in $E_u \times \hat{F}_{\beta*}$.

Now, $\hat{F}_{\beta*}$ is a Fréchet space and since the graph of t is closed in $E_u \times \hat{F}_{\beta*}$, $t: E_u \longrightarrow \hat{F}_{\beta*}$ is continuous (Chapter I, Proposition 3.4 and Theorem 5.1). It follows that $t: E_u \longrightarrow F_{\beta*}$ is continuous and since $\beta^*(F, F')$ is finer than v, $t: E_u \longrightarrow F_v$ is also continuous.

COROLLARY: Let E_u be a barrelled space and F_v a convex space with the property that F'^{β}_v possesses a countable fundamental family of bounded sets. Let t be a linear mapping of E_u into F_v, whose graph is closed in $E_u \times F_v$. Then t is continuous, provided the inverse filter condition holds.

PROOF: Since F'^{β}_{v} possesses a countable fundamental family of bounded sets, F_{β^*} is metrisable. Thus, our result follows as before.

In the next theorem below, we give the corresponding open mapping theorem. For this we need:

DEFINITION 1.3: (W. Robertson, [36]) Let E and F be two convex spaces and t, a linear mapping of E into F. It is said that the filter condition holds with respect to t if for each Cauchy filter $\mathcal{K}$ in E, $t(\mathcal{K}) \longrightarrow t(x_o)$ implies $\mathcal{K} \longrightarrow x_o$.

LEMMA 1.1: Let E_u and F_v be two convex spaces and t a linear continuous mapping of E_u into F_v. Let $\hat{t}$ denote the continuous extension of t which maps $\hat{E}_{\beta*}$ into $\hat{F}_v$. Then $t^{-1}(0) = \hat{t}^{-1}(0)$ if the filter condition holds, with respect to E_u and F_v.

PROOF: Clearly, $t^{-1}(0) \subset \hat{t}^{-1}(0)$ because $t(x) = \hat{t}(x)$ for $x \in E$. To show that $t^{-1}(0) \supset \hat{t}^{-1}(0)$, let $\hat{x} \in \hat{E}_{\beta*}$ such that $\hat{t}(\hat{x}) = 0$. Let $\mathcal{K}$ be a Cauchy filter on $E_{\beta*}$ which is a base of a convergent filter $\mathcal{K}'$ which converges to $\hat{x}$. By the continuity of $\hat{t}$, $\hat{t}(\mathcal{K}') \longrightarrow \hat{t}(\hat{x}) = 0$. Consider the sets $t(\mathcal{K})$ and $\hat{t}(\mathcal{K}')$ on $t(E)$. For each $K' \in \mathcal{K}'$, there exists a subset K of K' such that K is in $\mathcal{K}$ and $t(K) = \hat{t}(K) \subset \hat{t}(K') \cap t(E)$. Hence $t(\mathcal{K})$ is finer than the trace of $t(\mathcal{K}')$ on $t(E)$. Hence $t(\mathcal{K}) \longrightarrow 0$.

Now since $\mathcal{K}$ is a β^*-Cauchy filter, it is also a u-Cauchy filter since $\beta^*(E, E')$ is finer than u. Since $t(\mathcal{K}) \longrightarrow 0$ in F_v, by the filter condition, it follows that $\mathcal{K}$ converges, with respect to u, to a point in E. Hence $\hat{x} \in E$ and this implies $t(\hat{x}) = 0$. In other words $\hat{x} \in t^{-1}(0)$. Q.E.D

We are now ready to prove:

THEOREM 1.2: Let E_u be a convex space, metrisable in its $\beta^*(E, E')$-topology, and F_v a barrelled space. Let t be a continuous linear mapping of E_u onto F_v. Then t is open if the filter condition holds.

PROOF: Replacing $E_{\beta*}$ and F_v by their completions $\hat{E}_{\beta*}$ and $\hat{F}_v$ respectively, we observe that the continuous extension $\hat{t}$ maps a Frechet space $\hat{E}_{\beta*}$ onto a barrelled space $\hat{t}(\hat{E}_{\beta*})$ and is therefore open (Chapter I, Proposition 3.4 and Corollary to Theorem 5.2). Let W be a closed neighbourhood of the origin in $E_{\beta*}$. Then $\hat{t}(\hat{W})$ is a neighbourhood of the origin in $\hat{t}(\hat{E})$ and therefore $\hat{t}(\hat{W}) \cap F$ is a neighbourhood of the origin in F_v. We show that $\hat{t}(\hat{W}) \cap F \subset t(W)$. Let $y \in \hat{t}(\hat{W}) \cap F$. Then there exists $\hat{x} \in \hat{W}$ such that $\hat{t}(\hat{x}) = y$, and $x \in E$ such that $t(x) = y$, because t is onto. Therefore $\hat{t}(\hat{x}) - t(x) = \hat{t}(\hat{x} - x) = 0$. By Lemma 1.1, it follows that $\hat{x} = x \in E$. Thus $\hat{x} \in \hat{W} \cap E = W$ (because W is closed in $E_{\beta*}$) shows that

$$y = \hat{t}(\hat{x}) \in \hat{t}(W) = t(W).$$

Q.E.D

COROLLARY: Let E_u be a convex space with the property that E'^{β}_u possesses a countable fundamental family of bounded sets, and F_v a barrelled space. Let t be a continuous linear mapping of E_u onto F_v. Then t is open if the filter condition holds.

It is of interest to know under what conditions $E_{\beta*}$ might be complete. A sufficient condition is given in:

<u>PROPOSITION 1.4</u>: Let E_u be a convex space. Then $E_{\beta*}$ is complete if the polars in E of the $\beta(E', E)$-bounded sets in E' are $\tau(E, E')$-complete.

<u>PROOF</u>: Since $\beta^*(E, E')$ has a neighbourhood base of the origin consisting of sets complete in $\tau(E, E')$ and $\beta^*(E, E')$ is finer than $\tau(E, E')$, $E_{\beta*}$ is complete (Chapter I, Proposition 3.2).

<u>COROLLARY</u>: Let E_u be a convex space. Then $E_{\beta*}$ is complete if E_τ is complete.

In fact, a result due to J. H. Webb ([43], Corollary 4.2) gives us:

<u>PROPOSITION 1.5</u>: Let E_u be a convex space with $E_{\beta*}$ separable and E'^{β}_u complete. Then $E_{\beta*}$ is complete if and only if the polars in E of the $\beta(E', E)$-bounded sets in E' are $\tau(E, E')$-complete.

<u>COROLLARY</u>: Let E_u be as in the proposition. Then $E_{\beta*}$ is complete if and only if E_τ is complete.

In the case when $E_{\beta*}$ is metrisable, we have the following characterization for completeness of $E_{\beta*}$.

<u>PROPOSITION 1.6</u>: Let E_u be a convex space, metrisable in its $\beta^*(E, E')$-topology. Then $E_{\beta*}$ is complete if and only if $E_{\beta*}$ is a $B_r(\mathfrak{M})$-space.

<u>PROOF</u>: Suppose that $E_{\beta*}$ is complete. Since $E_{\beta*}$ is metrisable, by hypothesis, $E_{\beta*}$ is a Fréchet space and therefore a $B_r(\mathfrak{M})$-space.

On the other hand, $B_r(\mathfrak{M})$-spaces which are also metrisable are

B_r-complete (Chapter III, Proposition 1.2) and therefore complete (Chapter I, Proposition 3.5).

COROLLARY 1: Let E_u be a convex space with the property that E'^{β}_u possesses a countable fundamental family of bounded sets. Then $E_{\beta*}$ is complete if and only if $E_{\beta*}$ is a $B_r(\mathcal{M})$-space.

PROOF: This follows since $E_{\beta*}$ is metrisable.

COROLLARY 2: Let E_u be as in the proposition. Then $E_{\beta*}$ is complete if and only if $E_{\beta*}$ is a sequentially barrelled, $B_r(\mathcal{J})$-space.

PROOF: This follows from the fact that sequentially barrelled, $B_r(\mathcal{J})$-spaces are $B_r(\mathcal{M})$-spaces (Chapter III, Proposition 1.3).

REMARK 1.1: We note that in the above proposition, $E_{\beta*}$ is a Frechet space if and only if $E_{\beta*}$ is a $B_r(\mathcal{M})$-space.

Completeness of $E_{\beta*}$ allows us to relax the filter conditions in Theorems 1.1 and 1.2. This follows from the following general theorem.

THEOREM 1.3: Let E_u be a barrelled space and F_v a convex space which is a γ-space in its $\beta^*(F, F')$-topology. If t is a linear mapping of E_u into F_v, whose graph is closed in $E_u \times F_v$, then t is continuous.

PROOF: Since the graph of t is closed in $E_u \times F_v$, it is also closed in $E_u \times F_{\beta*}$, because $\beta^*(F, F')$ is finer than v. By Theorem 2.2 (Chapter II), $t: E_u \longrightarrow F_{\beta*}$ is continuous and therefore $t: E_u \longrightarrow F_v$ is also continuous.

In the corollaries below, let E_u be a barrelled space and F_v, a convex space. Let t be a linear mapping of E_u into F_v, whose graph is closed in $E_u \times F_v$.

COROLLARY 1: If $F_{\beta*}$ is a B-complete space, then $t: E_u \longrightarrow F_v$, is continuous.

PROOF: This follows, since B-complete spaces are γ-spaces (Chapter III, § 2).

COROLLARY 2: If $F_{\beta*}$ is a metrisable B($\mathfrak{M}$)-space, then $t: E_u \longrightarrow F_v$ is continuous.

COROLLARY 3: If $F_{\beta*}$ is a B($\mathfrak{M}$)-space and $F_v'^{\beta}$ possesses a countable fundamental family of bounded sets, then $t: E_u \longrightarrow F_v$ is continuous.

COROLLARY 4: If F_{τ} is sequentially complete and $F_v'^{\beta}$ possesses a countable fundamental family of bounded sets, then $t: E_u \longrightarrow F_v$ is continuous.

THEOREM 1.4: Let E_u be a convex space which is an γ-space in its $\beta^*(E, E')$-topology and F_v a barrelled space. If t is a continuous linear mapping of E_u onto F_v, then t is open.

PROOF: By Theorem 2.3 (Chapter II), $t: E_{\beta*} \longrightarrow F_v$ is open. Since $\beta^*(E, E')$ is finer than u, $t: E_u \longrightarrow F_v$ is also open. Q.E.D

In the corollaries below, let E_u be a convex space and F_v, a barrelled space. Let t be a continuous linear mapping of E_u onto F_v.

COROLLARY 1: If $E_{\beta*}$ is a B-complete space, then $t: E_u \longrightarrow F_v$ is open.

<u>COROLLARY 2</u>: If $E_{\beta*}$ is a metrisable $B(\mathfrak{m})$-space, then $t: E_u \longrightarrow F_v$ is open.

<u>COROLLARY 3</u>: If $E_{\beta*}$ is a $B(\mathfrak{m})$-space and $E_u'^{\beta}$ possesses a countable fundamental family of bounded sets, then $t: E_u \longrightarrow F_v$ is open.

<u>COROLLARY 4</u>: If E_{τ} is sequentially complete and $E_u'^{\beta}$ possesses a countable fundamental family of bounded sets, then $t: E_u \longrightarrow F_v$ is open.

2. Countability conditions and semi-reflexive spaces

In this section, we show that whenever a convex space E_u is semi-reflexive in its $\beta^*(E, E')$-topology, E_u is also semi-reflexive. This enables us to obtain conditions under which E_u might be semi-reflexive. We find that whenever $E_{\beta*}$ is semi-reflexive, $E_{\beta*}$ is metrisable if and only if $E_u'^{\beta}$ possesses a fundamental family of bounded sets.

Since each basic β^*-neighbourhood of the origin in E is the polar of a strongly bounded set in E_u', it absorbs bounded sets in E_u ([38], Chapter IV, § 3, Lemma 2). Thus u-bounded sets are β^*-bounded. Since $\beta^*(E, E')$ is finer than u, this means that u-bounded and β^*-bounded sets are identical. This observation gives us:

<u>PROPOSITION 2.1</u>: If $E_{\beta*}$ is semi-reflexive, E_u is also semi-reflexive.

<u>PROOF</u>: By the above observation, every u-bounded set is β^*-bounded. Now, since $E_{\beta*}$ is semi-reflexive, we further have that every

u-bounded set is relatively $\sigma(E, E'_{\beta*})$-compact (Chapter I, Theorem 8.1 (d)), and therefore relatively $\sigma(E, E'_u)$-compact. Thus E_u is also semi-reflexive. Q.E.D

However, if E_u is semi-reflexive, $E_{\beta*}$ could fail to be semi-reflexive. This follows from:

EXAMPLE 2.1: Let E_u be a non-reflexive Banach space which is weakly sequentially complete (such for example is ℓ_1, with the usual norm topology).

Since E'^{σ}_u is quasi-complete, E'^{τ}_u is semi-reflexive (Chapter I, Theorem 8.1 (e)). Now, if $E'^{\beta*}_u$ $(= E'^{\beta}_u)$ is also semi-reflexive, E'^{β}_u is a reflexive Banach space and therefore an α-space (Chapter II, § 1). Since $E\sigma$ is sequentially complete, it follows from Lemma 4.1 (Chapter III) that E is reflexive!

We now proceed to consider conditions under which E_u might be semi-reflexive. This relies upon:

LEMMA 2.1: If E_u is a B-complete space, then E_u is semi-reflexive if and only if every closed subspace which is an α-space is semi-reflexive.

PROOF: The necessity is obvious, since if E_u is semi-reflexive, every closed subspace is also semi-reflexive ([23], § 23, 3 (5)).

For sufficiency, we may suppose that E_u is endowed with the Mackey topology. Suppose E_u is not semi-reflexive. Then there exists a bounded, closed subset B of E_u which is not weakly compact; in fact due to completeness of E_u, B is not weakly countably compact ([23], § 24, 2 (1)). Let $\{x_n\}$ be a sequence in B which has no weak cluster point in B and let Y be the closed linear span of $\{x_n\}$. Then Y is separable, and since E_u is B-complete, Y is an α-space (Chapter II, Proposition 1.4) and $\{x_n\}$ is a bounded sequence in Y which has no weak cluster point in Y. Hence Y is not semi-reflexive.

<u>THEOREM 2.1</u>: Let E_u be a convex space which is a B-complete space in its $\beta^*(E, E')$-topology. Then E_u is semi-reflexive if every closed α-space in E_{β^*} is semi-reflexive.

<u>PROOF</u>: If every closed α-space in E_{β^*} is semi-reflexive, then by Lemma 2.1, E_{β^*} is semi-reflexive. This implies that E_u is semi-reflexive, by Proposition 2.1. Q.E.D.

In the following corollaries, let E_u be a convex space.

<u>COROLLARY 1</u>: If E_{β^*} is a metrisable $B(\mathcal{M})$-space, then E_u is semi-reflexive if every closed α-space in E_{β^*} is semi-reflexive.

<u>COROLLARY 2</u>: If $E_u'^{\beta}$ possesses a countable fundamental family of bounded sets and E_{β^*} is a $B(\mathcal{M})$-space, then E_u is semi-reflexive if every closed α-space in E_{β^*} is semi-reflexive.

<u>COROLLARY 3</u>: Let $E_u'^{\beta}$ be as in Corollary 2. If E_{β^*} is complete, then E_u is semi-reflexive if every closed α-space in E_{β^*} is semi-reflexive.

<u>COROLLARY 4</u>: Let $E_u'^{\beta}$ be as in Corollary 2. If E_{τ} is sequentially complete, then E_u is semi-reflexive if every closed α-space in E_{β^*} is semi-reflexive.

In the class of convex spaces which are β^*-semi-reflexive, we have the following characterization of metrisability.

<u>THEOREM 2.2</u>: If E_{β^*} is semi-reflexive then E_{β^*} is metrisable if and only if $E_u'^{\beta}$ possesses a countable fundamental family of bounded sets.

<u>PROOF</u>: If $E_u'^{\beta}$ possesses a countable fundamental family of bounded sets, E_{β^*} is clearly metrisable.

Conversely, if E_{β^*} is metrisable, $E'^{\beta}_{\beta^*}$ possesses a countable fundamental family of bounded sets. Since E_{β^*} is semi-reflexive, on E'_{β^*}, $\tau(E'_{\beta^*}, E) = \beta(E'_{\beta^*}, E)$ (Chapter I, Theorem 8.1 (c)) and therefore $E'^{\sigma}_{\beta^*}$ possesses a countable fundamental family of bounded sets. Now, since $E'^{\sigma}_{u} \subseteq E'^{\sigma}_{\beta^*}$, E'^{σ}_{u} possesses a countable fundamental family of bounded sets. But E_u is semi-reflexive (because E_{β^*} is semi-reflexive) and therefore E'^{β}_{u} possesses a countable fundamental family of bounded sets.

3. Countability conditions and the open mapping theorem

The present section is concerned with open mapping theorems on some classes of $B(\mathfrak{F})$-spaces. We show that convex spaces which are metrisable in their β^*-topology play an important role.

As before, we deal only with $B(\mathfrak{F})$-spaces, but the arguments are also valid for $B_r(\mathfrak{F})$-spaces, provided the mappings considered are one-to-one.

All spaces considered are separated.

THEOREM 3.1: Let E_u be a countably quasi-barrelled $B(\mathfrak{F})$-space. Let F_v be a sequentially complete Mackey space, metrisable and separable in its $\beta^*(F, F')$-topology. Then a continuous, almost open linear mapping t of E_u onto F_v is open.

PROOF: Since E_u is countably quasi-barrelled and t is a continuous, almost open mapping, F_v is countably quasi-barrelled (Chapter I, Proposition 7.5); F_v being sequentially complete, it follows that F_v is countably barrelled (Chapter I, Proposition 7.2).

Since $F_v'^{\sigma}$ is now sequentially complete (Chapter I, Proposition 7.4) and F_β is separable (because on a sequentially complete space $\beta^*(F, F') = \beta(F, F')$), $F_v'^{\tau}$ is semi-reflexive (Chapter I, Theorem 8.2). Thus on F, $v = \tau(F, F') = \beta(F, F')$ and so F_v is a Fréchet space. Our result now follows from the definition of $B(\mathcal{F})$-spaces.

REMARK 3.1: A sequentially complete Mackey space, metrisable and separable in its β^*-topology is not in general metrisable. This follows from:

EXAMPLE 3.1: Let $E_w = c_o$, the space of all sequences convergent to zero, with the usual norm topology. Then $E_w'^{\tau}$ is complete ([22], Chapter 5, Problem 19 c) and since $E_w'^{\tau\,'\beta}$ $(= E_w)$ possesses a countable fundamental family of bounded sets, $E_w'^{\beta^*}$ is metrisable. Also, since $E_w'^{\beta^*}$ $(= E_w'^{\beta}) = \ell_1$, with the usual norm topology, $E_w'^{\beta^*}$ is separable. However, since E_w is not reflexive, $E_w'^{\tau}$ is not metrisable.

COROLLARY 1: Let E_u be as in theorem. Let F_v be a sequentially complete Mackey space with the property that $F_v'^{\beta}$ possesses a countable fundamental family of bounded sets and F_{β^*} is separable. Then a continuous, almost open linear mapping of E_u onto F_v, is open.

COROLLARY 2: Let E_u be a countably barrelled $B(\mathcal{F})$-space. Let F_v be as in the theorem. Then a continuous, almost open linear mapping of E_u onto F_v, is open.

PROOF: Since a countably barrelled space is countably quasi-barrelled (Chapter I, Proposition 7.1), the corollary follows.

REMARK 3.2: Countably quasi-barrelled spaces are more general than countably barrelled space ([18], Example (ii)).

THEOREM 3.2: Let E_u be a countably barrelled Mackey space which is also a $B(\mathcal{F})$-space. Let F_v be a Mackey space which is also a metrisable, α-space in its $\beta^*(F, F')$-topology. Then a linear, almost open mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, is continuous and open.

PROOF: Since the graph of t is closed in $E_u \times F_v$, it is also closed in $E_u \times F_{\beta^*}$, because $\beta^*(F, F')$ is finer than v. Since E_u is countably barrelled and F_{β^*} is an α-space, $t: E_u \longrightarrow F_{\beta^*}$ is continuous (Chapter II, Theorem 1.1, Corollary 1). Thus $t: E_u \longrightarrow F_v$ is continuous. Further, since $t: E_u \longrightarrow F_v$ is almost open, F_v is countably barrelled (Chapter I, Proposition 7.5). Since F'^{σ}_v is now sequentially complete and F_β is an α-space (because on a countably barrelled space $\beta^*(F, F') = \beta(F, F')$ - see chapter I, Proposition 7.3). F'^{τ}_v is semi-reflexive (Chapter III, Lemma 4.1). Thus, on F, $v = \tau(F, F') = \beta(F, F')$ and so F_v is a metrisable, barrelled, α-space. Since a barrelled, α-space is B-complete (Chapter II, Proposition 1.5, Corollary 3), F_v is a Fréchet space. E_u is a $B(\mathcal{F})$-space and so t is open.

REMARK 3.3: Since a separable Banach space is an α-space (Chapter II, Proposition 1.4), ℓ_1, with the usual norm topology is an α-space. Example 3.1, therefore, also shows that a Mackey space F_v which is also a metrisable, α-space in its $\beta^*(F, F')$-topology, is not in general, metrisable.

COROLLARY 1: Let E_u be as in the theorem. Let F_v be a Mackey space with the property that F'^{β}_v possesses a countable fundamental family of bounded sets and F_{β^*} is an α-space. Then a linear, almost open mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, is continuous and open.

Quasi-barrelled spaces are countably quasi-barrelled (Chapter I, § 7). For quasi-barrelled $B(\mathcal{F})$-spaces, the requirement, in Theorem 3.1, that F_v be separable in its $\beta^*(F, F')$-topology, may be relaxed. We have:

<u>THEOREM 3.3</u>: Let E_u be a quasi-barrelled $B(\mathcal{F})$-space. Let F_v be a convex space which is a Fréchet space in its $\beta^*(F, F')$-topology. Then a continuous, almost open linear mapping t of E_u onto F_v, is open.

<u>PROOF</u>: Since E_u is quasi-barrelled and t is a continuous, almost open mapping, F_v is quasi-barrelled. Thus on F, $v = \beta^*(F, F')$ and therefore F_v is a Fréchet space. E_u is a $B(\mathcal{F})$-space and so t: $E_u \longrightarrow F_v$ is open. Q.E.D.

For $B(\mathcal{M})$-spaces, completeness of F_{β^*} may be relaxed. We have:

<u>THEOREM 3.4</u>: Let E_u be a quasi-barrelled $B(\mathcal{M})$-space. Let F_v be a convex space, metrisable in its $\beta^*(F, F')$-topology. Then a continuous, almost open linear mapping of E_u into F_v, is open.

<u>PROOF</u>: As in the proof of Theorem 3.3, F_v is metrisable. Therefore, $t(E)$, the range of t, with the induced topology, is also metrisable. Since E_u is a $B(\mathcal{M})$-space and t: $E_u \longrightarrow t(E)$ is also continuous and almost open, t is open.

CHAPTER V

BOUNDED LINEAR MAPPINGS

A linear mapping is called bounded if it preserves bounded sets. In section 1 of this chapter, we investigate the situation in which one could derive that a linear mapping is bounded from the fact that the graph of the mapping is closed. Here, we rely heavily on closed graph theorems proved in earlier chapters. In section 2, we investigate the situation in which one could derive that the graph of a linear mapping is closed from the fact that the mapping is bounded.

Bounded linear mappings with closed graphs are not necessarily continuous. In section 3, we investigate briefly some cases in which bounded linear mappings with closed graphs are continuous.

1. Bounded linear mappings and the closed graph theorem

The following example shows that linear mappings with closed graphs are not necessarily bounded.

EXAMPLE 1.1: Let E_u be a quasi-barrelled space which is not barrelled (such for example is $R^{(N)}$, the space of all finite sequences, (x_n), with the norm topology defined by: $\| x \| = \sup_n |x_n|$; $R^{(N)}$ with this topology is bornological but not barrelled). Since $\tau(E', E)$ is coarser than $\beta(E', E)$, the graph of the identity mapping $i: E'^{\tau}_u \longrightarrow E'^{\beta}_u$ is closed in $E'^{\tau}_u \times E'^{\beta}_u$. But i is not bounded, for otherwise, E_u would be barrelled.

DEFINITION 1.1: We say that a convex space E_u satisfies the Banach-Steinhaus condition, if every $\sigma(E, E')$-bounded subset of E is $\beta(E, E')$-bounded.

In Example 1.1, E'^{τ}_u does not satisfy the Banach-Steinhaus condition. We note also that E'^{β}_u is B_r-complete.

We have the following case which ensures that t is bounded.

THEOREM 1.1: Let E_u be a convex space and F_v a B_r-complete space. Let t be a linear mapping of E_u into F_v, whose graph is closed in $E_u \times F_v$. If E_u satisfies the Banach-Steinhaus condition, then t is bounded.

PROOF: Let V be a neighbourhood of the origin in F_v. Then $\overline{t^{-1}(V)}$ is a barrel in E_u and therefore a neighbourhood of the origin in E_β. Thus $t: E_\beta \longrightarrow F_v$ is almost continuous. Further, since the graph of t is closed in $E_u \times F_v$, it is also closed in $E_\beta \times F_v$ and therefore $t: E_\beta \longrightarrow F_v$ is continuous (Chapter I, Theorem 5.1). Since E_u satisfies the Banach-Steinhaus condition, every u-bounded set is β-bounded and therefore $t: E_u \longrightarrow F_v$ is bounded. Q.E.D.

REMARK 1.1: If in the above theorem, E_u is barrelled, $t: E_u \longrightarrow F_v$ will be continuous by a closed graph theorem (Chapter I, Theorem 5.1, Corollary). However, t as in the theorem, may fail to be continuous. This follows from:

EXAMPLE 1.2: Suppose E_w is a non-reflexive Banach space. Then E'^{τ}_u satisfies the Banach-Steinhaus condition. Consider now, the identity mapping $i: E'^{\tau}_u \longrightarrow E'^{\beta}_u$. i is a mapping of E'^{τ}_u onto a B-complete space, E'^{β}_u. The graph of i is closed in $E'^{\tau}_u \times E'^{\beta}_u$, but

i is not continuous, for otherwise, E_w would be reflexive.

REMARK 1.3: In the above example, E'^{τ}_w is sequentially barrelled ([43], Theorem 3.1). Such spaces satisfy the Banach-Steinhaus condition ([43], § 1).

In the following corollaries, let E_u be a convex space and F_v a B_r-complete space. Let t be a linear mapping of E_u into F_v, whose graph is closed in $E_u \times F_v$.

COROLLARY 1: If E_u is sequentially barrelled, then t is bounded.

COROLLARY 2: If E_u is countably barrelled, then t is bounded.

COROLLARY 3: If $E_{\beta *}$ is complete, then t is bounded.

PROOF: This follows from the fact that if $E_{\beta *}$ is complete E_u satisfies the Banach-Steinhaus condition([43], Corollary 3.2).

COROLLARY 4: If E_{τ} is sequentially complete, then t is bounded.

PROOF: If E_{τ} is sequentially complete, E_u satisfies the Banach-Steinhaus condition and our result follows as before. Q.E.D.

A fundamental sequence of weakly compact sets in a convex space is also a fundamental sequence of bounded sets ([8], § 3, Theorem 2). Therefore a convex space with a fundamental sequence of weakly compact sets is (weakly) sequentially complete, because each Cauchy sequence is bounded and therefore contained in a complete set. This gives us:

COROLLARY 5: If E_u possesses a fundamental sequence of weakly compact sets, then t is bounded.

DEFINITION 1.2: A convex space is said to have the convex compactness property if, whenever A is compact, the closed, absolutely convex envelope of A is also compact.

Any quasi-complete space has this property.

It turns out that a convex space which has the convex compactness property satisfies the Banach-Steinhaus condition ([37], Lemma 4). This gives us:

COROLLARY 6: If E_u has the convex compactness property, then t is bounded.

Let F_v be a convex space. Since v-bounded and $\beta^*(F, F')$-bounded sets are identical, we have the following variation of Theorem 1.1.

THEOREM 1.2: Let E_u be a convex space and F_v another convex space which is a B_r-complete space in its $\beta^*(F, F')$-topology. Let t be a linear mapping of E_u into F_v, whose graph is closed in $E_u \times F_v$. If E_u satisfies the Banach-Steinhaus condition, then t is bounded.

A very useful variation of corollary 4 to Theorem 1.1 is the following:

THEOREM 1.3: Let E_u be a convex space which is sequentially complete in its Mackey topology and F_v a γ-space. If t is a linear mapping of E_u into F_v, whose graph is closed in $E_u \times F_v$, then t is bounded.

PROOF: Since E_τ is sequentially complete, E_{β^*} is also sequentially complete ([23], § 18, 4, (4) b)). If B is a u-closed absolutely convex u-bounded set, then B is also β^*-closed (since $\beta^*(E,E')$ is finer than u) and β^*-bounded. Now, let (E_B, U_B) be the Banach space associated with B, where u_B is the norm topology on E_B. As u_B is finer than the $\beta^*(E, E')$-induced topology on E_B, the graph of the restriction t_B of t to E_B is closed in $(E_B, u_B) \times F_v$. Therefore t_B is continuous by a closed graph theorem (Chapter II, Theorem 2.2). Thus $t(B)$ is bounded in F_v. Q.E.D.

Since α-spaces are γ-spaces, we have:

<u>COROLLARY 1</u>: Let F_v be an α-space. Let E_u and t be as in the theorem. Then t is bounded.

<u>REMARK 1.4</u>: If in Theorem 1.3, E_u is barrelled, then $t: E_u \longrightarrow F_v$ will be continuous (Chapter II, Theorem 2.2). However, the mapping $t: E_u \longrightarrow F_v$, as in the theorem is not necessarily continuous. This follows from:

<u>EXAMPLE 1.3</u>: Let $E_w = c_o$, the space of all sequences convergent to zero, with the usual norm topology. As in Example 3.1 (Chapter IV), E'^{τ}_w is complete and E'^{β}_w is an α-space and therefore a γ-space. The graph of the identity mapping $i: E'^{\tau}_w \longrightarrow E'^{\beta}_w$ is closed in $E'^{\tau}_w \times E'^{\beta}_w$, but i is not continuous because otherwise, E_w would be reflexive.

A convex space E_u which possesses a fundamental sequence of weakly compact sets is semi-reflexive because the sequence of sets is also a fundamental sequence of bounded sets ([8], § 3, Theorem 2). Thus on E'_u, $\tau(E', E) = \beta(E', E)$ (Chapter I, Theorem 8.1 (c)). Since E'^{β}_u is metrisable, E'^{τ}_u is also metrisable and therefore E_u is an α-space. This enables us to state:

COROLLARY 2: Let E_u and F_v be convex spaces, each of which possesses a fundamental sequence of weakly compact sets. If t is a linear mapping of E_u into F_v, whose graph is closed in $E_u \times F_v$, then t is bounded.

PROOF: Since E_u is sequentially complete and F_v is an α-space, our result follows from **Corollary 1.**

Let E_u be a convex space and F_v a B_r-complete space. Let t be a linear mapping of E_u into F_v, whose graph is closed in $E_u \times F_v$. We now show, by giving an example, that boundedness of t does not in general imply that E_u satisfies the Banach-Steinhaus condition.

EXAMPLE 1.4: Let Q be a subspace of a Fréchet space F, which is not barrelled. Then the identity mapping $i: Q \longrightarrow F$ is continuous and therefore bounded but Q does not satisfy the Banach-Steinhaus condition, for otherwise Q would be barrelled.

In the next theorem below, we cite a case when boundedness of t implies that E_u satisfies the Banach-Steinhaus condition. First we need a lemma.

LEMMA 1.1: Let t be a linear mapping of a quasi-barrelled space E into a convex space F. If t is bounded, then t is almost continuous.

PROOF: Let V be an absolutely convex neighbourhood of the origin in F. Then $\overline{t^{-1}(V)}$ is a barrel in E. We show that $\overline{t^{-1}(V)}$ absorbs bounded sets of E. Let B be a bounded set in E. Then $t(B)$ is bounded, by hypothesis. Therefore there exists an $\alpha > 0$ such that

$t(B) \subset \lambda V$ for $|\lambda| \geq \alpha$. This implies that $B \subset \lambda\ t^{-1}(V) \subset \lambda\ \overline{t^{-1}(V)}$. Since E is quasi-barrelled, $\overline{t^{-1}(V)}$ is a neighbourhood of the origin in E. In other words, t is almost continuous.

THEOREM 1.4: Let E_u be a convex space which is a quasi-barrelled, $B_r(\mathfrak{J})$-space in its $\beta^*(E, E')$-topology and F_v a barrelled B_r-complete space. Let t be a one-to-one, linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$. Then t is bounded if and only if E_u satisfies the Banach-Steinhaus condition.

PROOF: If E_u satisfies the Banach-Steinhaus condition, our result follows from Theorem 1.1.

On the other hand, if $t: E_u \longrightarrow F_v$ is bounded, $t: E_{\beta^*} \longrightarrow F_v$ is also bounded. Now, since E_{β^*} is quasi-barrelled, it follows that $t: E_{\beta^*} \longrightarrow F_v$ is almost continuous (Lemma 1.1). Further, since the graph of t is closed in $E_u \times F_v$, it is also closed in $E_{\beta^*} \times F_v$. Since E_{β^*} is a $B_r(\mathfrak{J})$-space, $t: E_{\beta^*} \longrightarrow F_v$ is open ([15], Chapter 7, § 5, Theorem 7). Now, F_v is B_r-complete and since $t^{-1}: F_v \longrightarrow E_{\beta^*}$ is continuous and almost open, E_{β^*} is also B_r-complete ([35], Chapter VI, § 2, Proposition 9) and therefore complete. Completeness of E_{β^*} implies that E_u satisfies the Banach-Steinhaus condition ([43], Corollary 3.2). Q.E.D

REMARK 1.5: If E_u is quasi-barrelled, then $\tau(E, E') = \beta^*(E, E')$. However, if E_{β^*} is quasi-barrelled, E_τ may fail to be quasi-barrelled (see Example 1.1, (Chapter IV)).

It is also worth noting, in connection with Theorem 1.4, that, whenever E_{β^*} is a $B_r(\mathfrak{J})$-space, E_τ is also a $B_r(\mathfrak{J})$-space. This follows directly from the definition of $B_r(\mathfrak{J})$-spaces.

2. Boundedness and the graph of linear mappings

The following example shows that bounded linear mappings do not necessarily have closed graphs.

EXAMPLE 2.1: Let $E_u = \ell_1$, with the usual norm topology. Since E_u is not reflexive, $\tau(E', E)$ is strictly coarser than $\beta(E', E)$ and therefore $E_u'^{\beta'} \supset E_u'^{\tau'} = E$. Let $t \in E_u'^{\beta'} \sim E$; let R_v be the reals, with the usual topology, v. Then: $t: E_u'^{\beta} \longrightarrow R_v$ is continuous and therefore bounded. Since E_u is a Banach space, $\tau(E', E)$-bounded sets are also $\beta(E', E)$-bounded and so: $t: E_u'^{\tau} \longrightarrow R_v$ is also bounded. However, since $E_u'^{\tau'\sigma}$ ($= E_\sigma$) is sequentially complete and R_v is an α-space, the graph of t cannot be closed in $E_u'^{\tau} \times R_v$, because otherwise t would be continuous (Chapter II, Theorem 1.1).

REMARK 2.1: In the above example, $E_u'^{\beta^*}$ ($= E_u'^{\beta}$) is a bornological space; however, $E_u'^{\tau}$ is not, for if so, $t: E_u'^{\tau} \longrightarrow R_v$ would be continuous.

We now investigate the situation in which one could derive that the graph of a linear mapping is closed from the fact that the mapping is bounded.

THEOREM 2.1: Let E_u be a convex space which is a bornological, $B_r(\mathfrak{J})$-space in its $\beta^*(E, E')$-topology and F_v a barrelled space. If t is a one-to-one, bounded linear mapping of E_u onto F_v, then the graph of t is closed in $E_u \times F_v$.

PROOF: Since u-bounded sets are β^*-bounded, $t: E_{\beta^*} \longrightarrow F_v$ is also bounded. E_{β^*} is bornological and therefore $t: E_{\beta^*} \longrightarrow F_v$

is continuous. Now, since $E_{\beta*}$ is a $B_r(\mathfrak{J})$-space and F_v a barrelled space, $t: E_{\beta*} \longrightarrow F_v$ is open. $\beta*(E, E')$ is finer than u and therefore $t: E_u \longrightarrow F_v$ is also open. Thus the graph of t is closed in $E_u \times F_v$. Q.E.D.

In the case when $E_{\beta*}$ is a bornological α-space, the hypothesis that F_v be barrelled, may be weakened. We have:

<u>THEOREM 2.2</u>: Let E_u be a convex space which is a bornological α-space in its $\beta*(E, E')$-topology and F_v, a convex space with sequentially complete weak dual. If t is a one-to-one, bounded linear mapping of E_u onto F_v, then the graph of t is closed in $E_u \times F_v$.

<u>PROOF</u>: Since $t: E_{\beta*} \longrightarrow F_\tau$ is open (Chapter II, Theorem 1.2), $t: E_u \longrightarrow F_\tau$ is also open. Therefore, the graph of t is closed in $E_u \times F_\tau$. Since the closed convex sets in $\tau(E,E')$ and v are the same, the graph of t is also closed in $E_u \times F_v$. Q.E.D.

Our investigations in § 1 and § 2 give us the following characterization.

<u>THEOREM 2.3</u>: Let E_u be a convex space which is a complete, bornological $B_r(\mathfrak{J})$-space in its $\beta*(E, E')$-topology. Let F_v be a barrelled, B_r-complete space. Let t be a one-to-one, linear mapping of E_u onto F_v. Then t is bounded if and only if the graph of t is closed in $E_u \times F_v$.

<u>PROOF</u>: Suppose the graph of t is closed in $E_u \times F_v$. Since $E_{\beta*}$ is complete, E_u satisfies the Banach-Steinhaus condition ([43], Corollary 3.2) and our result follows from Theorem 1.1.

On the other hand, if $t: E_u \longrightarrow F_v$ is bounded, the graph of t is closed in $E_u \times F_v$ by Theorem 2.1.

3. Bounded linear mappings with closed graphs

Several examples can easily be constructed to show that bounded linear mappings with closed graphs are not in general, continuous. Example 1.3 in § 1 is one such example. In that example, $i': E_w'^{\tau} \longrightarrow E_w'^{\beta}$ is bounded (because E_w is a Banach space) and its graph is closed in $E_w'^{\tau} \times E_w'^{\beta}$. But i' fails to be continuous.

However, we have:

THEOREM 3.1: Let E_u be a convex space with sequentially complete weak dual and metrisable in its $\beta^*(E, E')$-topology. Let F_v be a $B(\mathfrak{M})$-space which is separable and barrelled. If t is a bounded, linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, then $t: E_u \longrightarrow F_v$, is continuous.

PROOF: Since $t: E_{\beta^*} \longrightarrow F_v$ is a continuous linear mapping of a metrisable space onto a barrelled space, F_v is metrisable (Chapter III, Proposition 1.1). But F_v is a separable $B(\mathfrak{M})$-space. Therefore F_v is a separable B-complete space (Chapter III, Proposition 1.2) and hence an α-space (Chapter II, Proposition 1.4). Since $E_u'^{\sigma}$ is sequentially complete and the graph of t is closed in $E_u \times F_v$, $t: E_u \longrightarrow F_v$ is now continuous (Chapter II, Theorem 1.1). Q.E.D.

REMARK 3.1: In the above theorem, we can also take a sequentially complete, separable barrelled space for F_v. In that case, our theorem follows from the fact that separable Fréchet spaces are α-spaces.

We also have:

THEOREM 3.2: Let E_u be a convex space which is a separable, bornological space in its $\beta^*(E, E')$-topology. Let F_v be a convex space with sequentially complete weak dual and which is a $B_r(\mathcal{J})$-space in its Mackey topology. If t is a bounded, almost continuous linear mapping of E_u onto F_v, whose graph is closed in $E_u \times F_v$, then $t: E_u \longrightarrow F_v$ is continuous.

PROOF: Since t is bounded and E_{β^*} is bornological, $t: E_{\beta^*} \longrightarrow F_{\beta^*}$ is continuous. Further, since E_{β^*} is separable, F_{β^*} (= F_β, since $F_v'^{\sigma}$ is sequentially complete) is also separable. Thus $F_v'^{\tau}$ is semi-reflexive (Chapter I, Theorem 8.2) and therefore F_τ is barrelled (Chapter I, Theorem 8.1 (c)). Since F_τ is a $B_r(\mathcal{J})$-space by hypothesis, it follows that F_τ is B_r-complete (Chapter I, Theorem 6.3, Corollary). Now, since $t: E_u \longrightarrow F_v$ is almost continuous and its graph is closed in $E_u \times F_v$, $t: E_u \longrightarrow F_v$ is continuous.

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