A Logical Basis for Reasoning With Default Rules

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for

Reasoning with Default Rules

By

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Abstract

This thesis is an investigation into the foundations of reasoning with default rules as presented by Reiter in his seminal 1980 article: 'A Logic for Default Reasoning'. In being such, it opens up with a critical appraisal of the logical underpinnings of Reiter's presentation of the main elements of reasoning with default rules. More precisely, following Reiter's presentation, it discusses the concept of a default rule in comparison with that of a rule of inference, the concept of an extension in comparison with that of a theory, and the concept of 'being a consequence of' for reasoning with default rules. Contrary to the commonly perceived view, the argument put forth is that such a context does not provide sensible logical foundation for reasoning with default rules. As a result, this thesis argues for an alternative interpretation to what is captured by default rules, what is captured by extensions, and what 'being a consequence of' for reasoning with default rules amounts to. In particular, it proposes to treat default rules as premiss-like objects standing for assertions made tentatively, to treat extensions as interpretation structures of a syntactical kind, and to bring the concept of 'being a consequence of' for reasoning with default rules into the foreground by formulating a suitable notion of an entailment relation and its ensuing logical system. Accounting for the fact that in any logical system it is important to have at hand mechanisms for formulating proofs and for structuring large theories, this thesis presents a tableaux based proof calculus for reasoning with default rules and it explores some mappings notions related to the structuring of default presentations, i.e., presentations in the context of reasoning with default rules.

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Contents

Aquí me pongo a cantar
al compás de la vigüela,
que el hombre que lo desvela
una pena estrordinaria
como la ave solitaria
con el cantar se consuela.

Pido a los santos del cielo
que ayuden mi pensamiento:
les pido en este momento
que voy a cantar mi historia
me refresquen la memoria
y aclaren mi entendimiento.

– José Hernández, *El Gaucho Martín Fierro*

Chapter 1

Introduction

This thesis is a logical investigation into the foundations of reasoning with default rules. In being such, it is conceptually divided in two main parts. The first of these parts is a discussion on the traditional presentation of the main elements of reasoning with default rules. This discussion pinpoints what are perceived to be some logical shortcomings of the foundations of the traditional presentation of the main elements of reasoning with default rules. Intended to overcome these logical shortcomings, the second of part of this thesis proposes an alternative presentation of the main elements of reasoning with default rules. This alternative presentation formulates a fairly general notion of a logical system for reasoning with default rules. In addition, accounting for the fact that in any logical system it is important to have at hand mechanisms for formulating proofs and for structuring presentations, this thesis presents a tableaux based proof calculus for one particular logical system for reasoning with default rules and a suitable notion of a mapping between what are referred to as default presentations defined on this logical system. Lastly, as a proof of concept, this thesis discusses a potential application of the ideas hitherto developed in the context of software requirements engineering.

The rest of this chapter is dedicated to sketch the background of this thesis and to formulate a precise problem definition.

1.1 A Foreword on Nonmonotonicity

It can be more or less readily agreed upon that, as a discipline, logic concerns itself with the study of the concept of 'being a consequence of', its various prerequisites and derivatives.

If seen in this light, *a* logical system, *alias*, *a* logic, is a theory about what 'being a consequence of' amounts to. Typically, this theory involves the analysis of the relation that connects premisses with their consequences. Simply put, the formulation and presentation of a logical system is largely an attempt to give a systematic explanation of the extension of this relation.

In this setting, in analogy with a scientific theory, a logical system may be prompted by problems and predictions, i.e., experiments of sorts, the outcome of which attest to the extent of its well-foundedness.

But all this is well-known lore.

Against this background, the origins of nonmonotonic reasoning may be viewed as being based on dissatisfaction with the way in which the problems posed by the Artificial intelligence (AI) community were handled by traditional, i.e., classical, logical systems.

In brief, such a dissatisfaction has its roots in the ordinary understanding of the principle of monotonicity: consequences cannot be undermined by any augmentation of the set of premisses from which they are drawn.

The AI community viewed monotonicity as a fatal flaw of traditional logical systems due to the fact that augmenting the set of premisses of a consequence may indeed undermine said consequence, potentially forcing its withdrawal (a mode of reasoning that is typical of situations in which the premisses of a consequence are non-definite, i.e., they involve a certain degree of uncertainty).

If the principle of monotonicity is to be set aside, i.e., if non monotonicity is to be accounted for, as argued by the AI community, then, a change in perspective about the fundamental aspects of a logical system is necessary. In this respect, it is noted by Bochman in [Boc07, p. 566] that:

> Nonmonotonic reasoning obtained its impetus in 1980 with the publication of a seminal issue of the Artificial Intelligence Journal, devoted to nonmonotonic reasoning. The issue included papers representing three basic approaches to nonmonotonic reasoning: circumscription [McC80], default logic [Rei80], and modal nonmonotonic logic [MD80]. These theories suggested three different ways of [...] developing formalisms that do not have the monotonicity property.

To a large extent, the 1980 issue of the Artificial Intelligence Journal marked the coming of age of what may be dubbed *the logical approach to nonmonotonic reasoning.*

1.2 The Original Motivation and a Rather Different Turn

Logical frameworks are typically considered to be useful aids for the design, specification, analysis, and the validation and verification of software based systems. This is particularly true in early stages of design, where early inspection of the implications of design choices contributes to ensuring that the system being developed is the one that is intended (it is at this stage also where conceptual errors and potential inconsistencies may be elicited before their correction becomes an impossibility).

In this context, this thesis was originally conceived as being an application of Reiter's presentation of reasoning with default rules for the specification and analysis of concurrent systems.

More precisely, the original motivation was that of extending the work reported in [FM92].

In brief, [FM92] brought together the use of temporal logic in the tradition initiated by Pnueli in [Pnu77] and the use of tools from category theory as a means for structuring specifications in the style of Burstall and Goguen in [BG77]. The main message of [FM92] is that the components of a concurrent system should be captured as presentations of theories in a temporal logic, with the interactions between components being modeled in terms of theory preserving mappings. This approach to the design and specification of a concurrent system is elegantly formulated in terms of some basic elements of category theory, q.v. [ML71], as in components and their interactions form a category. Its elegance is obtained from the fact that such a category is co-complete. This allows for a concurrent system to be obtained as a result of an universal construction, i.e., a co-limit, over its architectural description, i.e., a diagram of components and the interactions between them.

There are several advantages of the approach advocated in [FM92]. Of these, worth noting are modularity in the description and analysis of concurrent systems, much needed features due to the inherent complexity of the latter.

Building on some of the ideas presented in [FM92], this thesis intended to study the phenomenon of lack of preservation of liveness properties by mappings: the situation that while safety properties of a component, i.e., nothing bad happens, are preserved at the level of the system containing the component, in general, the same cannot be said about liveness properties, i.e., something good eventually happens, mostly due to the way in which the components of a concurrent system may interact, e.g., a system may exhibit deadlock phenomena, while individual components may terminate properly or run forever.

It is not difficult to see that, as presented above, phenomena such as lack of preservation of liveness properties possess a nonmonotonic logical element: certain properties may be established in a defeasible manner, i.e., those of a component that are not preserved at the level of the system. An advantage of formulating phenomena such as lack of preservation of liveness properties in a nonmonotonic setting is an internalization of the object of study. This view is a move away from the traditional approach to the specification and analysis of concurrent systems and hence a potentially interesting avenue to explore.

The particularities of Reiter's presentation of reasoning with default rules, and, more precisely, the fact that, in the standard sense derived from the work of Reiter, a default rule is intended to be seen as a rule of inference of a defeasible kind, i.e., a rule of inference that is open to revision or annulment, gained in interest when compared to McCarthy's circumscription and McDermott and Doyle's modal nonmonotonic logic in relation to what is known as rely-guarantee reasoning (q.v. [dRdBH+01, Part 3]).

Essentially, rely-guarantee reasoning takes its place in a context where the specification of a component of a concurrent system accounts for certain assumptions about its environment. This enables the analysis of some of the global features of a concurrent system, e.g., component interaction, while maintaining a reasonable degree of localization for reasoning at the level of components – allowing for the implications of design choices to be explored in a modular fashion.

In more detail, rely-guarantee reasoning is based on the premiss that the interaction between a component and its environment is to be specified as a tuple $\langle R, G\rangle$ where R is a rely clause that defines an assumption which the environment of the component is required to satisfy, whereas G is a guarantee clause that defines a property which the component maintains provided that the assumptions on the environment hold.

To be noted at this point is that the interpretation of a rely-guarantee clause depends on the underlying logical framework. In that respect, there are many different approaches to rely-guarantee reasoning (q.v. [dRdBH^{+}01, Part 3] for a brief summary of the main works in the area).

Particularly interesting, since it is much in the style of [FM92], [Dua98] proposes a rely-guarantee clause $\langle R, G\rangle$ to be interpreted as being satisfied in a component provided that the specification of this component entails G as long as its environment does not invalidate R. Given the latter, though not explored in [Dua98], the connection with Reiter's presentation of reasoning with default rules seems more or less direct: a rely-guarantee clause may be specified as a default rule, where the premisses of the default rule characterize the rely clause and the consequent of the default rule characterize the guarantee clause. The logical burden of proof is now placed on finding sensible composition mechanisms that ensure that guarantee clauses of a component, in particular those that refer to liveness, are not undermined at the level of the system.

Altogether, merging the logical framework proposed in [FM92], the ideas about rely-guarantee reasoning introduced in [Dua98], and Reiter's presentation of reasoning with default rules required, as a first step, formulation of Reiter's seminal ideas in the context of temporal logic, and hence an exploration of the logical foundations of reasoning with default rules. The latter brought to the surface problems of a rather different kind which, due to their relevance and importance, took precedence over the original intent of this thesis.

1.3 What is this Thesis About?

Based on his seminal 1980 article, titled *A Logic for Default Reasoning*, q.v. [Rei80], Reiter's approach to reasoning with default rules is viewed as a logic for nonmonotonic reasoning.

Attesting to such a viewpoint is Reiter's presentation of the main elements of reasoning with default rule, i.e., Reiter's attitude towards a default rule being formally treated as a rule of inference of a defeasible kind, his justification of an extension as a theory-like object, and his consideration that, when taken together, default rules and extensions formulate a logic for non-monotonic reasoning.

This thesis is a logical investigation into the foundations of reasoning with default rules.

A first very general problem definition for this Ph.D. thesis is raised by the following question: Can reasoning with default rules, as presented in Reiter's seminal 1980 article: 'A Logic for Default Reasoning', be understood as a logic for non-monotonic reasoning?

This question emerged from a dissatisfaction with the way in which Reiter introduces the main elements of reasoning with default rules. Briefly, since it is elaborated on in detail in Chapter 3, neither does a default rule seemingly formulate a somewhat sensible notion of a rule of inference of a defeasible kind, nor can an extension be justified on the grounds that it characterizes a theory-like object, nor is it at all clear in which sense default rules and extensions explicate what the concept of 'being a consequence of' for reasoning with default rules amounts to.

As a brief parenthetical remark, in defense of reasoning with default rules, the relationship between Reiter's seminal 1980 article and logic is part of a larger history of relationships between AI and logic as a discipline (a summary of which can be found in [San11] and in [Boc07]). This history of relationships should not be underestimated, as it exhibits an interesting interplay between early solutions to the problems posited by the AI community and a rudimentary application of some logical concepts. Moreover, as is noted in [Boc07, p. 557], the field of AI "has practical purposes, which give rise to problems and solutions of a new kind, apparently different from the questions relevant for philosophers".

Following from this parenthetical remark, in light of the observations presented previously, the question to be asked is: Can some of the seminal ideas discussed in Reiter's presentation of reasoning with default rules be reconciled with some general logical principles?

While there is certainly an extensive literature on the subject of reasoning with default rules, a comprehensive list of which can be found in the reference section of [AW07], this question occupies a rather different niche.

Moreover, given that Reiter's seminal 1980 article on the subject of reasoning with default rules and its derivatives are nowadays an essential part of the logical approach to non-monotonic reasoning, there is great value in obtaining a positive answer to the latter question, such is a basic guiding consideration in this thesis.

The message to be retained is not that the contentions and results presented in this thesis are a final word on the subject of reasoning with default rules. Instead, the hope is that they engender some thinking about the (re)conceptualization of the fundamentals of reasoning with default rules, potentially deepening their understanding in light of some newly presented ideas.

1.4 Structure of the Thesis

This thesis is structured as follows:

Chapter 2 introduces the main definitions and concepts that are needed for the understanding of the subject matter of this thesis. In brief, it discusses the basics of classical propositional logic, theories, presentations, consequence operators, be they monotonic (i.e., Tarskian) or not, and reasoning with default rules.

Chapter 3 is a critical appraisal of the main elements of reasoning with default rules. In particular, this chapter discusses the concept of a default rule as a rule of inference and compares it to the typical explanations which justify the idea that a rule of inference is such. In turn, this chapter discusses the concept of an extension in comparison to that of a theory. Lastly, this chapter discusses the concept of 'being a consequence of' for reasoning with default rules.

Based on the observations made in Chapter 3, Chapter 4 proposes an alternative presentation of the main elements of reasoning with default rules. In particular, this chapter presents default rules as being premiss-like objects. In turn, extensions are proposed as interpretation structures of a syntactical kind. Lastly, this chapter discusses the formulation of a logical system for reasoning with default rules.

Of a more technical nature, Chapter 5, building on a tableaux based proof method for classical propositional logic, presents the development of a tableaux based proof method for reasoning with default rules.

In turn, much in the sense of presentations and their corresponding mappings in classical propositional logic, Chapter 6 presents the notion of a mapping between what are therein called default presentations.

Lastly, Chapter 7 presents a potential application of the concepts hitherto developed in the context of software requirements engineering.

Closing this thesis, Chapter 8 offers some general conclusions and comments on some of the further work to be undertaken, both in terms of generalizations and applications.

Chapter 2

Basic Concepts

This chapter is intended to provide a somewhat comprehensive account of the many concepts and definitions that will be used in this thesis.

2.1 Logical Preliminaries

This section has as its sole purpose to briefly introduce those elements of a logical system that will help to clarify the points made in the following chapters of this thesis.

A word of comment is in order to avoid any subsequent misunderstanding. The intention here is not that of providing a definite answer to the rather difficult and controversial, especially if looked at in light of [Gab01], question of *what is a logical system?* Instead, the focus of attention is placed on introducing <u>*some*</u> "logical" elements that, when taken together, arguably yield a logical system.

Just as a comment in passing, the position assumed here is that there needs be no one single and definite answer to the question of what is a logical system. Quite the contrary, this question may have rather different answers depending on how it is approached, e.g., from the perspective of different schools of thought, "extremes" of which are the purely philosophical or the purely mathematical. When viewed in this light, the logical system for reasoning with default rules proposed in Chapter 4 is another contribution to this logical pluralism.

Formal Languages

Perhaps one of the most basic elements of a logical system is the presentation of the notion of a formal language. This presentation typically involves the introduction of linguistic entities of three markedly different kinds: the linguistic entities of the formal language in question, the linguistic entities that refer to the linguistic entities of the formal language in question, and the linguistic entities that the linguistic entities of the formal language in question stand for. The distinction among them is put in more succinct terms after some preliminary definitions.

Definition 2.1.1. Let $\{\S_i\}_{i \in I}$ be a finite set of symbols for n-ary connectives; a propositional language $\mathscr{L}$ is the term algebra of type $\{\S_i\}_{i \in I}$ over a denumerable set $\{p_n\}_{n \in \mathbb{N}}$ of symbols for variables (q.v. [BS12]). The tuple $\langle \{p_n\}_{n \in \mathbb{N}}, \{\S_i\}_{i \in I} \rangle$ is the signature, or alphabet, of $\mathscr{L}$. The set $\{p_n\}_{n \in \mathbb{N}}$ is the set of propositional, proper, or non-logical, symbols of $\mathscr{L}$. The set $\{\S_i\}_{i \in I}$ is the set of logical connectives of $\mathscr{L}$. A sentence is an element of $\mathscr{L}$.

By an abuse of language, for a formal language $\mathscr{L}$, $\mathscr{L}$ indicates the set of all sentences of $\mathscr{L}$.

Regarding Definition 2.1.1, for binary logical connectives, their infix notation is preferred over their prefix one – with parentheses and standard precedences resolving any of the ambiguities that this may introduce. Two sentences of a propositional language are equal iff the same logical and non-logical symbols appear in the same order in both of them.

Definition 2.1.1 follows closely the work of Wójcicki in [Wój88].

A particular propositional language is obtained from Definition 2.1.1 by making precise the set $\{\S_i\}_{i \in I}$ of logical connectives. For instance, the standard propositional language is the one where the set of logical connectives is comprised of $\top$, $\bot$, $\neg$, $\wedge$, $\vee$, and $\supset$ (q.v. Definition 2.1.2).

Definition 2.1.2. The standard propositional language is the propositional language determined by: the 0-ary logical connectives $\top$ and $\bot$ of 'truth' and 'falsity', respectively; the unary logical connective $\neg$ of 'negation'; and the binary logical connectives $\wedge$, $\vee$, and $\supset$ of 'conjunction', 'disjunction', and 'material implication', respectively.

The standard propositional language is given a special place in Definition 2.1.2 because of its ubiquity in this thesis. However, Definition 2.1.1 also encompasses a number of other well-known formal languages, as well as some lesser known ones. For instance, the standard propositional modal language of 'necessity' and 'possibility' obtains if the unary logical connectives $\Box$ and $\Diamond$ are added to those of the standard propositional language. Prior's propositional temporal language of 'sometime in the future' and 'sometime in the past' obtains if the unary logical connectives F and P are added to those of the standard propositional language. The standard propositional deontic language of 'permission' and 'obligation' obtains if the unary logical connectives P and O are added to those of the standard propositional language.

Languages such as those defined according to Definition 2.1.1 are typically referred to as *formal* as a way of distinguishing them from *informal* or *ordinary* languages, such as English.

The sentences of a formal language are the first kind of linguistic entities involved in the presentation of a formal language.

The second kind of linguistic entities involved in the presentation of a formal language are those needed for referring to the sentences of a formal language – the most common of which are variables for sentences. For instance, in expressing that a sentence σ of the standard propositional language $\mathscr{L}$ is such and such, the assumption is not that 'σ' is a sentence of $\mathscr{L}$ but that it is a linguistic entity introduced to indicate one arbitrary sentence of $\mathscr{L}$. Linguistic entities of this second kind are in the meta-language $\mathscr{M}$ of a formal language $\mathscr{L}$.

The third kind of linguistic entities involved in the presentation of a formal language are what the sentences of a formal language stand for. Linguistic entities of this third kind, henceforth referred to as assertions, are truth-bearing statements of an ordinary language, e.g., English.

Regarding assertions, introductions to standard textbooks on logic usually illustrate what they are with the aid of a basic set of examples (q.v. [End01, vD94]). For instance, in [End01, p. 11], relative to the standard propositional language, Enderton explains that:

> [The English assertion] 'Traces of potassium were observed' can be translated into the formal language [the standard propositional language] as, say, the [propositional] symbol p.
>
> For the closely related [assertion] 'Traces of potassium were not observed', we can use [the sentence] $\neg p$. Here $\neg$ is our negation symbol, read as 'not'.

> One might also think of translating 'Traces of potassium were not observed' by some new symbol, for example, q, but we will prefer instead to break such an assertion down into atomic parts as much as possible.
>
> For an unrelated [assertion], 'The sample contained chlorine', we choose, say, the [propositional] symbol q.
>
> Then the following compound [assertions] can be translated as the [sentences] shown at the right:
>
> | | |
> |---|---|
> | 'If traces of potassium were observed, then the sample did not contain chlorine' | $p \supset \neg q$ |
> | 'The sample contained chlorine, and traces of potassium were observed' | $p \wedge q$ |
>
> The second case uses our conjunction symbol $\wedge$ to translate 'and'. The first one uses $\supset$ to translate 'if ..., then ...'.
>
> In the following example the disjunction symbol $\vee$ is used to translate 'or':
>
> | | |
> |---|---|
> | 'Either no traces of potassium were observed, or the sample did not contain chlorine' | $\neg p \vee \neg q$ |

When looked at from the perspective of Enderton's explanation, the translation of an assertion into a sentence of a formal language, its formalization, indicates what assertion this sentence stands for.

It is seldomly made explicit that the process of formalization supposes, in fact requires, a somewhat sensible criterion of adequacy. This criterion of adequate formalization is only touched upon superficially in Enderton's explanation above. In particular, when it is mentioned that the sentence $\neg p$ is preferred over the sentence q as a formalization of the English assertion 'Traces of potassium were not observed'. To a certain extent, providing a somewhat sensible criterion of adequate formalization is a rather important element of a logical system (the work Baumgartner et al. in [BL08] proves to be an interesting read in that respect). Although a precise definition of what such a criterion of adequate formalization may look like falls outside of the scope of this thesis, that one such criterion exists is taken as an assumption that underpins the formulation of a logical system that is considered here.

A second assumption that underpins the formulation of a logical system that is considered here is that the sentences of a formal language, the linguistic entities of the meta-language of this formal language, and the assertions that the sentences of the formal language stand for are all distinct from each other, i.e., the categories to which each of these linguistic entities belong are assumed to be disjoint. Without such a clear distinction, it is well-known that all sort of different problems loom (q.v. [Tar56f]).

The distinction between a sentence and the assertion that this sentence stands for will prove to be useful in Section 4.1 when explicating the notion of an assertion that is held tentatively and how such a notion is to be formalized.

Logical Consequence

The chief subject matter of a logical system is that of providing a precise account of the concept of 'being a consequence of' it intends to capture. This is traditionally referred to as logical consequence.

Because of its importance in this thesis, this section restricts its attention to presenting what the concept of 'being a consequence of' amounts to for the system of classical propositional logic, or CPL in short. Said presentation is done from a syntactical, or deductive-theoretical, perspective. The semantical, or model-theoretic, aspects of CPL are discussed later on this section as a way of justifying the syntactical perspective. In essence, this section establishes a basis for a more general discussion on the concept of 'being a consequence of'.

Definition 2.1.3, due to Wójcicki (q.v. [Wój88]), is preliminary to what follows.

Definition 2.1.3. Let $\mathscr{L}$ be any propositional language; a substitution is an endomorphism s of $\mathscr{L}$.[†] A sentence σ of $\mathscr{L}$ is a substitution instance of another sentence σ' of $\mathscr{L}$ iff $\sigma = s(\sigma')$ for s a substitution. If σ is a sentence of $\mathscr{L}$, then, $S(\sigma)$ is the set of all substitution instances of σ, i.e., $S(\sigma) \stackrel{\text{def}}{=} \{s(\sigma) \mid s \text{ is a substitution}\}$.

It is well-known that any substitution s is completely determined by a mapping from the set of propositional symbols of a formal language into its set of sentences.

Regarding Definition 2.1.3, for the standard propositional language of Definition 2.1.2, $\neg\varphi$ is the set of all substitutions instances of the sentence $\neg p$. Similarly, for two different propositional symbols p and q, $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \supset \psi$, are the sets of all substitution instances of the sentences $p \wedge q$, $p \vee q$, and $p \supset q$, respectively. By an abuse of language, an arbitrary member of $\neg\varphi$ is also indicated by $\neg\varphi$. Similarly, $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \supset \psi$ also indicate an arbitrary member of $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \supset \psi$, respectively. If $\neg\varphi$ is a substitution instance of $\neg p$, then, φ is the substitution instance corresponding to p. And similarly for the remaining logical connectives. It follows more or less directly that, any sentence σ of $\mathscr{L}$ that is not a propositional symbol, $\top$, or $\bot$, belongs to either $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, or $\varphi \supset \psi$.

[†]An endomorphism is an operations preserving mapping from an algebra onto itself.

Unless it is stated otherwise, the rest of this section assumes that $\mathscr{L}$ is the standard propositional language of Definition 2.1.2.

Without further ado, a syntactical presentation of the concept of 'being a consequence of' for CPL is based on two primitive notions: that of an *axiomatic base*, and that of an *inferential base*. These are made precise in Definition 2.1.4.

Definition 2.1.4. The set of sentences $\mathscr{A}$ comprised of the sentences listed below is an axiomatic base for CPL. The elements of $\mathscr{A}$ are called logical axioms.

(A1) $p \supset (q \supset p)$

(A2) $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$

(A3) $(p \wedge q) \supset p$

(A4) $(p \wedge q) \supset q$

(A5) $(p \supset q) \supset ((p \supset r) \supset (p \supset (q \wedge r))$

(A6) $p \supset (p \vee q)$

(A7) $q \supset (p \vee q)$

(A8) $(p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r))$

(A9) $(p \supset \neg q) \supset (q \supset \neg p)$

(A10) $\neg(p \supset p) \supset q$

(A11) $p \vee \neg p$

(A12) $\top$

(A13) $\top \supset \neg\bot$

(A14) $\bot \supset \neg\top$

(A15) $\bot \supset p$

The set $\mathscr{R}$ comprised of the tuple (MP) $(p, p \supset q, q)$, denoted by $\frac{p,\ p \supset q}{q}$, is an inferential base for CPL. The sole element of $\mathscr{R}$ is the logical rule of inference of *modus ponens*.

In an ordinary sense, an axiomatic base is a set of sentences whose members are accepted to stand for assertions that, modulo adequate formalization, are obvious and self-sustaining. In turn, the rule of *modus ponens* is accepted to capture accurately a pattern of reasoning that stands for a canon of proper argumentation, i.e., an argument pattern that is in and of itself "consequence-conducive".

Relative to the notion of an axiomatic base and an inferential base given in Definition 2.1.4, the concept of 'being a consequence of' for CPL is made precise in Definition 2.1.5.

Definition 2.1.5. Let $\mathscr{A}$ and $\mathscr{R}$ be the axiomatic and the inferential base of Definition 2.1.4; in addition, let Cn be an operator having $\wp(\mathscr{L})$ as both its domain and its co-domain; for any set of sentences Γ, $Cn(\Gamma)$ is the smallest set of sentences such that:

$$\Gamma \cup S(\mathscr{A}) \subseteq Cn(\Gamma) \tag{2.1}$$

$$\text{for any substitution } s, \text{ if } s(p) \in Cn(\Gamma) \text{ and } s(p \supset q) \in Cn(\Gamma), \text{ then, } s(q) \in Cn(\Gamma) \tag{2.2}$$

A sentence χ is a consequence of a premiss set Γ iff $\chi \in Cn(\Gamma)$.

As a comment in passing, regarding Definition 2.1.5, it is common for the notion of a substitution to be dropped, or assumed implicitly, in the definition of Cn and instead consider that the axiomatic base $\mathscr{A}$ is it in itself closed under substitutions, i.e., $\mathscr{A} = S(\mathscr{A})$. In this case, one shall speak not of logical axioms but of instances of said axioms. By an abuse of language, the distinction between a logical axiom and its corresponding instances is not made whenever this may be understood from the context. The case of the inferential base $\mathscr{R}$ is similar. Instead of considering $\frac{p,\ p \supset q}{q}$ to be the rule of *modus ponens*, at times, the rule of *modus ponens* is considered to be the set of all substitution instances of $\frac{p,\ p \supset q}{q}$. As before, in the latter case, one shall speak not of the rule of *modus ponens* but of an instance thereof. But again, by an abuse of language, this distinction is not made whenever it can be understood from the context. It is a well-known result that the operator Cn of Definition 2.1.5 is well defined, in fact, for any set of sentences Γ, $Cn(\Gamma)$ corresponds to the intersection of all sets that satisfy Equations 2.1 and 2.2.

In line with the work of Tarski in [Tar56e], the consideration here is that underpinning Definition 2.1.5 is the assumption that the concept of 'being a consequence of' it involves is not an arbitrary one, but that it bears a certain correspondence with its ordinary understanding.

There is then an obvious question to be answered: What ordinary concept of 'being a consequence of' is captured by Definition 2.1.5 and to what extent is this ordinary concept captured accurately? As noted below, the answer to this question may be rather elusive.

In a sense, a case can be made that the ordinary concept of 'being a consequence of' captured in Definition 2.1.5 is that present in a simple form of mathematical reasoning, one that is particularly suitable for metamathematical investigations (q.v. [Tar56b, p. 59]).

"In a sense", for a case can also be made that this metamathematical reasoning is better realized by the intuitionistic school of thought (q.v. [van86]).

If the ordinary understanding of the concept of 'being a consequence of' is taken to refer to everyday, i.e., not necessarily meta-mathematical, assertions, then, the situation is more difficult.

For instance, a case can be made that 'John hit a pedestrian and drove on' is only dubiously an ordinary consequence of 'John drove on and hit a pedestrian' (e.g., if 'and' is conceived as having a certain ordering in time). How to formally treat the implication connective is even more complicated ([Coo68] and [Rea12] prove to be some interesting readings in that regard). In the presence of challenges such as those just presented, it is possible to argue that Definition 2.1.5 does not even come close to capturing the ordinary understanding of the concept of 'being a consequence of' in a sensible manner.

In any case, the latter question is the subject of much philosophical debate ([McK10], [Etc90], [Pat12], [Sim09], [Pri95], are some interesting reads in light of the work of Tarski in [Tar56e]).

To a certain extent, this question is perhaps best answered by Tarski himself:

> With respect to the clarity of its content, the common concept of consequence is in no way superior to other concepts of everyday language. Its extension is not sharply bounded and its usage fluctuates. Any attempt to bring into harmony all possible vague, sometimes contradictory, tendencies which are connected with the use of this concept, is certainly doomed to failure. We must reconcile ourselves from the start to the fact that every precise definition of this concept will show arbitrary features to a greater or less degree. (q.v. [Tar56e].)

As put in quite succinct terms by Tarski, the consideration here is that, while not formulated arbitrarily, it needs not be the case that every feature of a precise definition of the concept of 'being a consequence of' may be a reflection of a feature of the ordinary understanding of this concept, nor that all the features of this ordinary understanding may be reflected in its precise definition. The assumption here is that, while the ordinary understanding of the concept of 'being a consequence of' becomes the cornerstone for its precise definition, it is the role of its precise definition to explicate what this ordinary understanding amounts to. For the case of classical propositional logic this explication corresponds to Definition 2.1.5.

At this point, the discussion is turned into the semantical, or model-theoretic, aspects of CPL.

It is often proposed that syntactic approaches to the formulation of the concept of 'being a consequence of' need of a justification other than that provided by axioms and rules of inference. While debatable, those who sustain the latter position consider that such a justification is readily provided in semantical terms, usually in the form of a model theory.

The semantical aspect of CPL is reduced to what is regarded as the more basic notion of being 'true in a model'. This notion is made precise in Definition 2.1.6.

Definition 2.1.6. Let 1 and 0 be interpreted as 'truth' and 'falsity', respectively; if $\mathscr{L}$ is the standard propositional language, then, a mapping m from $\mathscr{L}$ to $\{1, 0\}$ is a model, or an interpretation structure of a semantical kind, iff:

$$\begin{array}{lll} m(\top) = 1 & m(\neg\varphi) = 1 - m(\varphi) & m(\varphi \wedge \psi) = \mathsf{min}(m(\varphi), m(\psi)) \\ m(\bot) = 0 & m(\varphi \vee \psi) = \mathsf{max}(m(\varphi), m(\psi)) & m(\varphi \supset \psi) = \mathsf{max}(1 - m(\varphi), m(\psi)) \end{array}$$

A sentence σ is true in a model m if $m(\sigma) = 1$, otherwise σ is false in m. If σ is true in m, then, m is a model for σ. In addition, a set of sentences Γ is true in a model m if $m(\sigma) = 1$ for all σ in Γ. If Γ is true in m, then, m is a model for Γ.

Regarding Definition 2.1.6, it is a well-known result that any model is completely determined by an initial mapping of the propositional symbols of $\mathscr{L}$ into $\{1, 0\}$.

Definition 2.1.6 gains in interest as a way of justifying Cn whenever the notion of 'true in a model' is accepted to be in a certain correspondence with the ordinary understanding of an assertion being the case. More precisely, a sentence that is true in a model m is arguably an indication that the assertion that this sentence stands for is the case in a particular state of affairs; whereas a sentence that is false in m is arguably an indication that the assertion that such sentence stands for is not the case in the same particular state of affairs.

In taking such a position, it is worth noting that, a commitment must be made as to an assertion being either the case, or not. There are no grey areas. It is not possible for an assertion to be somewhat the case, to be both the case and not the case, or to be neither the case nor not the case. For such kind of analyses, Definition 2.1.6 is inadequate and a rather different notion of 'true in a model' is needed (q.v. [Mal07] and [FO03], for a discussion on this subject). This said, in what follows assertions are always assumed as either the case or not. Regarding this position, it is worth noting that, it is not the actual determination of whether an assertion is the case in concrete cases what is required, a typical example of which may be taken to be Goldbach's conjecture. Simply that, in principle, an assertion can be taken as either the case, or not – something that may be done by position, hypothesis, fiat, etc. Considerations of this sort correspond typically correspond to an initial mapping of propositional symbols into $\{1, 0\}$.

The way in which the notion of 'truth in a model' formulates the conceptual apparatus for justifying the concept of 'being a consequence of' captured by Cn is made precise in Theorem 2.1.1.

Theorem 2.1.1. For any sentence χ and any set of sentences Γ, it follows that $\chi \in Cn(\Gamma)$ iff for every model m, if m is a model of Γ, then, m is a model of χ.

Following from Theorem 2.1.1, the concepts of soundness and completeness reveal themselves as a cornerstone piece of logical inquiry.

The idea of the concept of 'being a consequence of' being in need of a justification in terms of interpretation structures becomes a basis for the ideas developed in Chapter 4, and more precisely in Section 4.3.

Theories

Based on the presentation of the concept of 'being a consequence of' for CPL of Definition 2.1.5, this section presents some basic concepts regarding theories and presentations of theories. Such concepts will become useful in Chapter 6 in relation to the definition of a default presentation.

Definition 2.1.7, due to Wójcicki (q.v. [Wój88]), is preliminary to what follows.

Definition 2.1.7. Let $\mathscr{L}$ be any propositional language; a propositional symbol p appears in a sentence σ iff $s(\sigma) \neq \sigma$ for all substitutions s such that $s(q) = q$ for all propositional symbols $q \neq p$ and $s(p) \neq p$. If σ is a sentence, then, $\mathcal{L}(\sigma)$ is the set of all propositional symbols that appear in σ. In addition, if Σ is a subset of the set of propositional symbols of $\mathscr{L}$, then, $\mathscr{L}|_\Sigma = \{\sigma \in \mathscr{L} \mid \mathcal{L}(\sigma) \subseteq \Sigma\}$.

By way of introduction, the concept of 'being a consequence of' introduced in Definition 2.1.5 essentially formulates the idea of a sentence following from some other sentences, i.e., a consequence and its corresponding set of premisses, respectively. For certain purposes, however, the interest lies not on premisses and consequences *per se*, but on what are referred to as theories. In ordinary terms, a theory is meant to logically capture a state of affairs in question as a means to rationalize and to explain it.

As an object of formal investigation, the precise definition of a theory is given in Definition 2.1.8.

Definition 2.1.8. Let $\mathscr{L}$ be the standard propositional language of Definition 2.1.2 and Cn be the consequence operator of Definition 2.1.5; the class $\mathfrak{T}$ of all theories of CPL consists of all tuples $\langle \Sigma, \Theta \rangle$ where: (i) Σ, the signature, or language, of $\langle \Sigma, \Theta \rangle$, is a subset of the set of propositional symbols of $\mathscr{L}$, (ii) $\mathcal{L}(\Theta) \subseteq \Sigma$, and (iii) $\Theta = Cn(\Theta)$. A theory is an element of $\mathfrak{T}$.

In certain situations, a point can be made that if the purpose of a theory is that of rationalizing and explaining a state of affairs in question, then, its formulation as in Definition 2.1.8 may prove to be too coarse. In particular, in rationalizing and explaining a state of affairs in question, there may be a need to distinguish between assertions that are particular to such a state of affairs, these correspond to the axioms of the theory, assertions of a more general kind, these correspond to logical axioms, and assertions that follow from them, these corresponds to the theorems of a theory. This finer grained view of a theory is made precise in Definition 2.1.9.

Definition 2.1.9. Let $\mathscr{L}$ be the standard propositional language of Definition 2.1.2 and Cn be the consequence operator of Definition 2.1.5; the class $\mathfrak{P}$ of all theory presentations of CPL, or presentations for short, consists of all tuples $\langle \Sigma, \Gamma \rangle$ where: (i) Σ, the signature, or language, of $\langle \Sigma, \Gamma \rangle$, is a subset of the set of propositional symbols of $\mathscr{L}$, (ii) $\mathcal{L}(\Gamma) \subseteq \Sigma$, and (iii) Γ is a set of sentences. A presentation is an element of $\mathfrak{P}$. A presentation $\langle \Sigma, \Gamma \rangle$ is finite, sometimes called an axiomatization, iff Γ is finite. If $\langle \Sigma, \Gamma \rangle$ is a presentation, then, a (non-logical) axiom of $\langle \Sigma, \Gamma \rangle$ is a sentence α such that $\alpha \in \Gamma$. A theorem of $\langle \Sigma, \Gamma \rangle$ is a sentence τ such that $\tau \in Cn(\Gamma)$.

To be noted at this point is that every theory is a presentation, but not every presentation is a theory. However, accounting for the fact that the purpose of a presentation $\langle \Sigma, \Gamma \rangle$ is that of being a finer grained view of a theory, i.e., the theory generated by Γ with signature Σ, i.e., $\langle \Sigma, Cn(\Gamma) \rangle$, by an abuse of language, presentations are also typically regarded as theories.

Building on Definition 2.1.9, Definition 2.1.10 introduces the notion of a mapping between presentations (q.v. [End01], [Sho67], and [Mes89]).

Definition 2.1.10. Let $\langle \Sigma, \Gamma \rangle$ and $\langle \Sigma', \Gamma' \rangle$ be two presentations in $\mathfrak{P}$ (q.v. Definition 2.1.9); a function m with domain Σ and co-domain Σ' is a mapping between $\langle \Sigma, \Gamma \rangle$ and $\langle \Sigma', \Gamma' \rangle$, i.e., a mapping of presentations, iff $m(Cn(\Gamma)) \subseteq Cn(\Gamma')$. A mapping m between presentations $\langle \Sigma, \Gamma \rangle$ and $\langle \Sigma', \Gamma' \rangle$ is axiom preserving if $m(\Gamma) \subseteq \Gamma'$. A mapping m between presentations $\langle \Sigma, \Gamma \rangle$ and $\langle \Sigma', \Gamma' \rangle$ is a faithful interpretation, a conservative extension if $\Sigma = \Sigma'$, iff $m(Cn(\Gamma)) = Cn(\Gamma') \cap m(\mathscr{L}|_{\Sigma})$.

Definition 2.1.10 provides a basic mechanism on which presentations may be seen as being structured. Underpinning such a standpoint is the idea that presentations are not given whole but instead they are developed from smaller and easier to understand presentations, such is the view advocated in [TM87] and [BG77].

The Abstract Turn

It was Tarski who started a systematic study of the properties of the concept of 'being a consequence of' from a foundational perspective by abstracting some of the "unnecessary" details. In a paper titled *Fundamental Concepts of the Methodology of the Deductive Sciences*, q.v. [Tar56a], Tarski writes:

> The present studies, however, are of a more general character: their aim is *to make precise the meaning of a series of important metamathematical concepts* which are common to the special metadisciplines, *and to establish the fundamental properties of these concepts.* One result of this approach is that some concepts which can be defined on the basis of special metadisciplines will be here regarded as primitive concepts and characterized by a series of axioms. (emphasis in original)

In [Tar56a], Tarski regards two concepts as primitive: (i) that of a sentence, i.e., an object of a formal language $\mathscr{L}$ (cf. Definition 2.1.1); and (ii) that of 'being a consequence of', captured by a consequence operator Cn defined on $\wp(\mathscr{L})$. Regarding (ii), Tarski comments that, for a set of sentences Γ, $Cn(\Gamma)$ indicates the set of consequences of Γ, i.e., those sentences which follow from Γ by virtue of the elements of a well-defined set of rules of inference $\mathscr{R}$.

A brief parenthetical remark is in order here in order to avoid any subsequent misunderstanding. In [Tar56a], Tarski regards the set $Cn(\Gamma)$ of all consequences of Γ to be the intersection of all sets which contain Γ and that are closed under the rules of inference in an inferential base $\mathscr{R}$. In the standard sense deriving from the work of Tarski, a rule of inference may be viewed as a tuple (Π, χ), denoted as $\frac{\Pi}{\chi}$, where $\Pi \cup \{\chi\}$ is a finite subset of $\mathscr{L}$. If $\frac{\Pi}{\chi}$ is one such rule of inference, then, Π is its sets of premisses and χ its consequent. A set of sentences Γ is closed under the rules of inference in $\mathscr{R}$ iff for every $\frac{\Pi}{\chi} \in \mathscr{R}$, if $\Pi \in \Gamma$, then, $\chi \in \Gamma$. In [Tar56a], Tarski considers that it is the task of the particular metadiscipline to establish what the rules of inference in $\mathscr{R}$ are. In this light, a rule of inference such as $\frac{\Pi}{\chi}$ is an indication of the structure of a rule of inference but not a rule of inference in and of itself. A rule of inference in the latter sense would be, e.g., the rule of *modus ponens* (q.v. Definition 2.1.4). Nonetheless, due to the level of abstraction proposed by Tarski there is no alternative other that treating the rules of inference in $\mathscr{R}$ as primitive, with at most their structure being made explicit. To be noted also is that, in [Tar56a], Tarski does not consider Cn to be defined on a axiomatic base $\mathscr{A}$. There is no loss of generality in doing this, formally, logical axioms may be seen as rules of inference whose corresponding sets of premisses are empty and whose consequents are the logical axioms in question.

Following from this parenthetical remark, based on the two primitive concepts of a sentence and that of 'being a consequence of', in [Tar56a, p. 63], Tarski lists and justifies four axioms which he claims express some basic properties of the primitive concepts, as satisfied in all known formalized disciplines. These are made precise in Definition 2.1.11.

Definition 2.1.11. Let $\mathscr{L}$ be a formal language; an operator Cn is a (Tarskian) consequence operator iff it satisfies the following conditions:

$$\mathscr{L} \text{ is denumerable} \tag{2.3}$$

$$\text{if } \Gamma \subseteq \mathscr{L}, \text{ then, } \Gamma \subseteq Cn(\Gamma) \subseteq \mathscr{L} \tag{2.4}$$

$$\text{if } \Gamma \subseteq \mathscr{L}, \text{ then, } Cn(Cn(\Gamma)) = Cn(\Gamma) \tag{2.5}$$

$$\text{if } \Gamma \subseteq \mathscr{L}, \text{ then, } Cn(\Gamma) = \textstyle\bigcup_{\Gamma' \in \wp_{\mathrm{fin}}(\Gamma)} Cn(\Gamma') \tag{2.6}$$

A sentence $\chi \in \mathscr{L}$ is a consequence of a premiss set $\Gamma \subseteq \mathscr{L}$ iff $\chi \in Cn(\Gamma)$.

Regarding Definition 2.1.11, Axioms 2.4 and 2.6 are dubbed the principles of *inclusion* and that of *compactness*, respectively. Axiom 2.5 is a closure property, the consequences of the consequences of a premiss set are the consequences of the premiss set. Axiom 2.5 indicates that $Cn(\Gamma)$ is a fixed-point of Cn.

The first property that Tarski derives from Axioms 2.4 to 2.6 corresponds to what is commonly referred to as the principle of *monotonicity*. This principle is made precise in Equation 2.7.

$$\text{if } \Gamma \subseteq \Gamma' \subseteq \mathscr{L}, \text{ then, } Cn(\Gamma) \subseteq Cn(\Gamma') \tag{2.7}$$

A second property that can be derived immediately from Axioms 2.4 to 2.6 corresponds to what is commonly referred to as the principle of *cut*. This principle is made precise in Equation 2.8.

$$\text{if } X \cup \Gamma \subseteq \mathscr{L}, \text{ then, } Cn(\Gamma) \supseteq Cn(\Gamma \cup X) \text{ whenever } X \subseteq Cn(\Gamma) \tag{2.8}$$

The bulk of the work of Tarski in [Tar56a] consists of a thorough study of some seminal definitions and proofs of properties derived from the formulation of Cn as in Definition 2.1.11.

To be noted at this point, a (Tarskian) consequence operator is not <u>*a*</u> consequence operator of a logical system but an abstraction which encompasses a class of such consequence operators and their corresponding logical systems. A consequence operator of a logical system in the sense of Tarski is a particular instance of Cn, e.g., that presented in Definition 2.1.5. The purpose of the abstract study of Cn is grounded on the fundamental commonalities of all instances of Cn or of particular subclasses thereof that are of interest. The latter being studied by Tarski himself in [Tar56b] and in [Tar56d] (q.v. [Sim09] for a summary of the main contributions of such works).

Instead of a consequence operator, the concept of 'being a consequence of' can be studied in terms of what is typically called an entailment relation (q.v. [Mes89]). This standpoint is made precise in Definition 2.1.12.

Definition 2.1.12. Let $\mathscr{L}$ be a formal language; a relation $\vdash \subseteq \wp(\mathscr{L}) \times \mathscr{L}$ is a (Tarskian) entailment relation iff it satisfies the following conditions:

(2.9) $\mathscr{L}$ is denumerable

(2.10) $\Gamma \vdash \sigma$ for all $\sigma \in \Gamma$

(2.11) $\Gamma \vdash \sigma$ iff $\Gamma' \vdash \sigma$ for some $\Gamma' \in \wp_{\text{fin}}(\Gamma)$

(2.12) $\Gamma \vdash \sigma$ if $\Gamma \cup \Gamma' \vdash \sigma$ and $\Gamma \vdash \sigma'$ for all $\sigma' \in \Gamma'$

A sentence $\chi \in \mathscr{L}$ is a consequence of a premiss set $\Gamma \subseteq \mathscr{L}$ iff $\Gamma \vdash \chi$.

Similarly to what is the case in Definition 2.1.11, Axioms 2.10, 2.11, and 2.12 in Definition 2.1.12 are dubbed the principles of *inclusion*, *compactness*, and *cut*, respectively.

As before, but now for $\vdash$, the principle of *monotonicity* can be derived immediately from Axioms 2.10, 2.11, and 2.12. This principle is made precise in Equation 2.13.

$$\text{if } \Gamma \vdash \chi,\ \text{then,}\ \Gamma' \vdash \chi \text{ for all } \Gamma' \supseteq \Gamma \tag{2.13}$$

Whether the (Tarskian) concept of 'being a consequence of' is presented as a consequence operator Cn or as an entailment relation $\vdash$, i.e., as in Definition 2.1.11 or as in Definition 2.1.12, respectively, is more a matter of style of presentation than of technical difference. It is possible to prove that the entailment relation $\vdash$ defined by $\Gamma \vdash \chi$ iff $\chi \in Cn(\Gamma)$ satisfies Axioms 2.10, 2.11, and 2.12 if Cn satisfies Axioms 2.4, 2.5, and 2.6. As well, it is possible to prove that the consequence operator Cn defined by $Cn(\Gamma) \stackrel{\text{def}}{=} \{\chi \mid \Gamma \vdash \chi\}$ satisfies Axioms 2.4, 2.5, and 2.6 if $\vdash$ satisfies Axioms 2.10, 2.11, and 2.12. It is noted in [Mes89, p. 4] that the formulations of the concept of 'being a consequence of' in terms of a consequence operator are rooted in the work of Tarski, whereas formulations of the concept of 'being a consequence of' in terms of an entailment relation are rooted in the work of Hertz and Gentzen.

The ideas discussed in this section form a basis for discussing the subject matter of Chapters 3 and 4. This corresponds to an exploration of the logical elements of reasoning with default rules.

Nonmonotonicity

This section introduces the basic concepts and definitions that will serve as a technical background for Section 3.3 and Section 4.3.

As reported by Makinson in [Mak94], given the plethora of systems for nonmonotonic reasoning proposed by the Artificial Intelligence (AI) community, a naturally arising question is whether it is possible to find some commonalities among them, i.e., whether there are some fundamental concepts, other that the dislike of the principle of monotonicity, that may help to understand what is going on. It was Gabbay in [Gab85] who first proposed to bring some harmony into the field of nonmonotonic reasoning by focusing, much in the style of [Tar56a], on what may be viewed as some fundamental properties of the nonmonotonic view of the concept of 'being a consequence of'. This approach was further developed by Makinson in [Mak89], [Mak94], [Mak05b], and [Mak05a].

More precisely, in [Gab85], Gabbay noticed that the principle of cut (q.v. Equations 2.8 and 2.12) has an equally important converse, dubbed the principle of *cumulativity*, whose relevance is lessened by the fact that it is a restricted form of the principle of monotonicity. The principle of cumulativity is made precise in Axiom 2.17.

Definition 2.1.13. Let $\mathscr{L}$ be a formal language; a relation $\mathrel{|\!\sim} \subseteq \wp(\mathscr{L}) \times \mathscr{L}$ is a nonmonotonic entailment relation iff it satisfies the following conditions:

$$\mathscr{L} \text{ is denumerable} \tag{2.14}$$

$$\Gamma \mathrel{|\!\sim} \sigma \text{ for all } \sigma \in \Gamma \tag{2.15}$$

$$\Gamma \mathrel{|\!\sim} \sigma \text{ if } \Gamma \cup \Gamma' \mathrel{|\!\sim} \sigma \text{ and } \Gamma \mathrel{|\!\sim} \sigma' \text{ for all } \sigma' \in \Gamma' \tag{2.16}$$

$$\Gamma \cup \Gamma' \mathrel{|\!\sim} \sigma \text{ if } \Gamma \mathrel{|\!\sim} \sigma \text{ and } \Gamma \mathrel{|\!\sim} \sigma' \text{ for all } \sigma' \in \Gamma' \tag{2.17}$$

A sentence $\chi \in \mathscr{L}$ is a nonmonotonic consequence of a premiss set $\Gamma \subseteq \mathscr{L}$ iff $\Gamma \mathrel{|\!\sim} \chi$.

In an ordinary sense, Axioms 2.15, 2.16, and 2.17 in Definition 2.1.13 formulate the basic stock of fundamental principles of what may be taken to be a somewhat sensible formalization of the nonmonotonic view of the concept of 'being a consequence of'. In this respect, it is worth noting that, since the entailment relation $\mathrel{|\!\sim}$ is abstract, i.e., it is independent of any particular set of rules used for its generation, Axioms 2.15, 2.16, and 2.17 are to be viewed as abstract properties of $\mathrel{|\!\sim}$ and therefore should not be confused with the particular set of rules used for generating $\mathrel{|\!\sim}$.

As commented by Makinson in [Mak89], Axioms 2.15, 2.16, and 2.17 gain in importance and interest for they correspond to very natural ways in which nonmonotonic reasoning may be organized. In particular, the principle of cumulativity indicates that nonmonotonic consequences cannot be the origin of nonmonotonicity in reasoning, i.e., if σ' is a nonmonotonic consequence of Γ, i.e., if $\Gamma \mathrel{|\!\sim} \sigma'$, then, if $\Gamma \mathrel{|\!\sim} \sigma$, then, $\Gamma \cup \{\sigma'\} \mathrel{|\!\sim} \sigma$, i.e., nonmonotonic consequences can be accumulated into their sets of premisses without loss of inferential power. Simply put, nonmonotonicity may only originate as a result of augmenting the set of premisses of a nonmonotonic consequence with "new" premisses.

2.2 Reasoning with Default Rules

This section relies heavily on extracts from [Rei87]. Its purpose is to make the reading of this thesis somewhat self-contained by briefly listing some of the main concepts and definitions underpinning reasoning with default rules as presented by Reiter. A lengthier discussion of these matters is presented in Chapter 3.

The next remark is preliminary to what follows. While Reiter formalizes his ideas on reasoning with default rules in the context of first-order logic (FOL), these ideas are here presented in the context of CPL. This avoids certain technical problems which are irrelevant to the points to be discussed. The consideration here is that discussing the works of Reiter in the context of CPL simplifies the points at issue.

In what follows, $\mathscr{L}$ is assumed to indicate the standard propositional language of Definition 2.1.2 and Cn is assumed to indicate the consequence operator of Definition 2.1.5.

Started by Reiter in his seminal 1980 paper titled *A Logic for Default Reasoning*, q.v. [Rei80], reasoning with default rules has nowadays become one of the most prominent approaches to nonmonotonic reasoning.

Essentially, the main idea of reasoning of reasoning with default rules is to begin with a set of first-order sentences. These are taken to denote assertions that are known to be true of the world in a definite sense. Since such a definite knowledge about the world is normally incomplete, i.e., it may contain certain knowledge gaps, i.e., it may be formally described in terms of an incomplete theory,[†] default rules are in place to act as mappings from this incomplete knowledge of the world to a more complete extension of it.

[†]A theory $\langle \Sigma, \Theta \rangle$, q.v. Definition 2.1.8, is incomplete if $\Theta \cap \{\sigma, \neg\sigma\} = \emptyset$ for some sentence σ.

In brief, a default rule is a tuple $\frac{\pi:\rho}{\chi}$, for formatting purposes displayed as $\pi : \rho \mathbin{/} \chi$, where $\{\pi, \rho, \chi\}$ is a subset of $\mathscr{L}$. According to Reiter, default rules are to be formally treated as rules of inference and not as sentences in a theory (q.v. [Rei80, p. 162]). More precisely, in Reiter's sense, a default rule $\pi : \rho \mathbin{/} \chi$ reads as:

> If π holds and ρ can be consistently assumed, then you can infer χ (q.v. [Rei80, p. 162].)

The more complete extension of the incomplete knowledge of the world is then formalized in Definition 2.2.1.

Definition 2.2.1. Let A be a set of sentences and Δ be a set of default rules; in addition, let C be an operator such that, for any set of sentences Γ, $C(\Gamma)$ is the smallest set of sentences satisfying the following properties:

(2.18) $\mathrm{A} \subseteq C(\Gamma)$

(2.19) $Cn(C(\Gamma)) = C(\Gamma)$ where Cn is the consequence operator of FOL

(2.20) for all $\pi : \rho \mathbin{/} \chi \in \Delta,$ if $\pi \in C(\Gamma)$ and $\neg\rho \notin \Gamma,$ then, $\chi \in C(\Gamma)$

A set of sentences E is an extension of Γ and Δ iff $E = C(E)$, i.e., iff E is a fix-point of the operator C.

In [Rei80], Reiter justifies the definition of an extension as being a theory that is further closed under the application of default rules. More precisely, let $\pi : \rho \mathbin{/} \chi$ be a default rule; if an incomplete theory $\langle \Sigma, \Gamma \rangle$ contains π, i.e., if $\pi \in \Gamma$, and if ρ is consistent with this theory, i.e., if $Cn(\Gamma \cup \{\rho\}) \subset \mathscr{L}$ or, alternatively, if $\neg\rho \notin Cn(\Gamma)$, then, this theory can be extended by adding χ to it. This results in the extension $Cn(\Gamma \cup \{\chi\})$. An extension is then a theory-like object, but now in a default sense.

It is worth noting that, when taken as rules of inference, default rules sanction plausible conclusions. These are conclusions that are defeasible in nature, i.e, open to revision or annulment. More precisely, let $\langle \Sigma, \Gamma \rangle$ be as above and $\langle \Sigma, \Gamma' \rangle$ be an incomplete theory such that $\Gamma \subset \Gamma'$ with $Cn(\Gamma' \cup \{\rho\}) = \mathscr{L}$; while $\langle \Sigma, \Gamma \rangle$ can be extended by adding χ to it, $\langle \Sigma, \Gamma' \rangle$ cannot.

Reiter's approach to reasoning with default rules, dubbed *default logic*, has been widely used by the AI community in the field of nonmonotonic reasoning. Examples of this usage range from the early view of 'expert systems', diagnostics, and speech recognition, all mentioned by Reiter in [Rei87]. More modern views of reasoning with default rules are, e.g., discussed by Verheij in [Ver09] in the context of argumentation theory.

This thesis proposes a logical examination of the indubitably logical flavor, in particular if Definitions 2.1.5 and 2.2.1 are taken side by side, of reasoning with default rules.

Chapter 3

An Analysis of the Foundations of Reasoning with Default Rules

Based on his seminal 1980 article: *A Logic for Default Reasoning* (q.v. [Rei80]) Reiter's approach to reasoning with default rules is typically regarded as a logic for nonmonotonic reasoning. This chapter discusses this standpoint from the perspective of those elements that arguably portray Reiter's approach to reasoning with default rules as an example of a logical system. More precisely, Section 3.1 discusses the concept of a default rule in comparison to that of a rule of inference; Section 3.2 discusses the concept of an extension in comparison to that of a theory; and Section 3.3 discusses the concept of 'being a consequence of' for reasoning with default rules.

While not free of omissions, some arguable elements, and perhaps some "biases", the conclusions drawn in Sections 3.1 to 3.3 raise the following question: Does reasoning with default rules, as presented by Reiter in his 1980 article, stand on somewhat sensible logical grounds? Framed somewhat differently: Do the elements presented in

Reiter's 1980 article result in a somewhat sensible notion of a logical system, or, more precisely, a syntactical presentation thereof?

In order to establish a well-defined context for discussion, the series of articles [Rei78], [Rei80], and [Rei87], is taken as a point of departure for analysis. This series corresponds to Reiter's most prominent articles on the subject of reasoning with default rules.

From an historical standpoint, [Rei78] is Reiter's pioneering article on the subject of reasoning with default rules. However, it is not until [Rei80] that Reiter advances the preliminary ideas introduced in [Rei78]. It is in this latter work that Reiter formulates and systematizes what became the basic stock of concepts for reasoning with default rules, with [Rei87] adding some notational simplifications and further remarks to what he himself proposed in [Rei80]. In all these articles, Reiter motivates, presents, and elaborates on the subject of reasoning with default rules resorting to a common, yet incrementally enriched, set of logical concepts. It is such a set of logical concepts which is at issue in this chapter.

More often than not, logical concepts gain in clarity when discussed with respect to their formalization. Therefore, unless it is stated otherwise, the rest of this chapter assumes as given a syntactical presentation of classical propositional logic. More precisely, $\mathscr{L}$ is assumed to indicate the standard propositional language, i.e., the set of all sentences formed resorting to an infinite set of propositional symbols and the standard logical connectives of truth, falsity, negation, conjunction, disjunction, and material implication (q.v. Definition 2.1.2). Arbitrary sets of sentences are assumed to be referred to by capital Greek letters and to be subsets of $\mathscr{L}$. Cn is assumed to indicate a consequence operator for classical propositional logic (q.v. Definition 2.1.5). This consequence operator is assumed to be defined w.r.t. an axiomatic basis $\mathscr{A}$ and an inferential basis $\mathscr{R}$.

By way of disclaimer, although Reiter formalizes his ideas on reasoning with default rules in the context of first-order logic, the consideration here is that discussing the main points of his works in the context of classical propositional logic simplifies what is at issue.

3.1 The Concept of a Default Rule

The concept of a default rule is one of the most prominent elements of Reiter's seminal works on the subject of reasoning with default rules. Regarding this concept, in all his works, Reiter considers that a default rule is to be formally treated as a rule of inference – something that he himself makes precise in his seminal 1980 article (q.v. [Rei80, p. 162]). This section proposes an analysis of such an interpretation of a default rule in comparison with the traditional understanding of a rule of inference. The point to be made is whether it is at all logically tenable to sustain that a default rule is somewhat like a rule of inference.

To begin with, the more or less standard formulation of the concept of a default rule corresponds to the way in which Reiter presents this concept in [Rei87]. This presentation is made precise in Definition 3.1.1.

Definition 3.1.1. Let $\mathscr{L}$ be the language of classical propositional logic; the set $\mathscr{D}$ of all *default rules* defined on $\mathscr{L}$ is the set of all tuples (π, ρ, χ), the latter denoted by $\frac{\pi:\rho}{\chi}$, where π, ρ, and χ are in $\mathscr{L}$. A *default rule* is an element of $\mathscr{D}$.

Regarding Definition 3.1.1, in line with the work of Reiter in [Rei80], for a default rule $\frac{\pi:\rho}{\chi}$, the sentences π and χ are its *prerequisite* and its *consequent*, respectively (q.v. [Rei80, p. 88]). While not mentioned by Reiter himself, as is standard in the literature on reasoning with default rules, the sentence ρ is the *justification* of $\frac{\pi:\rho}{\chi}$. If Δ is a set of default rules, then: $\Pi(\Delta)$ is the set of all prerequisites of the default rules in Δ, i.e., $\Pi(\Delta) \stackrel{\text{def}}{=} \{\pi \mid \frac{\pi:\rho}{\chi} \in \Delta\}$; $P(\Delta)$ is the set of all justifications of the default rules in Δ, i.e., $P(\Delta) \stackrel{\text{def}}{=} \{\rho \mid \frac{\pi:\rho}{\chi} \in \Delta\}$; and $X(\Delta)$ is the set of all consequents of the default rules in Δ, i.e., $X(\Delta) \stackrel{\text{def}}{=} \{\chi \mid \frac{\pi:\rho}{\chi} \in \Delta\}$.

Regarding default rules of a particular form, if π is a tautology of classical propositional logic, i.e., if $\pi \in Cn(\emptyset)$, then, $\frac{\pi:\rho}{\chi}$ is a *prerequisite-free* default rule. A prerequisite-free default rule is abbreviated as $\frac{:\rho}{\chi}$. If $\chi \in Cn(\{\rho\})$, then, $\frac{\pi:\rho}{\chi}$ is *semi-normal*. A default rule $\frac{\pi:\rho}{\chi}$ is *normal* if in addition to being semi-normal, $\rho \in Cn(\{\chi\})$. A normal default rule is abbreviated as $\frac{\pi}{\chi}$.

For inline formatting purposes, a default rule $\frac{\pi:\rho}{\chi}$ is displayed as $\pi : \rho \,/\, \chi$ (with a prerequisite-free default rule being displayed as $: \rho \,/\, \chi$ and a normal default rule $\frac{\pi}{\chi}$ being displayed as $\pi \,/\, \chi$.

At this point, a brief parenthetical remark is in order before continuing. It is worth noting that a default rule as in Definition 3.1.1 differs in certain elements from the manner in which Reiter introduced this notion in some of his earlier works.

More precisely, in [Rei78], for ρ and χ sentences in $\mathscr{L}$, Reiter defines a default rule as a sequent of the following form:

$$\frac{\nvdash \rho}{\chi} \tag{3.1}$$

Regarding Equation 3.1, two observations are to be made. First, Reiter seems to implicitly assume that each such default rule is defined with respect to an underlying set of sentences Γ.† If such a set of sentences is to be accounted for in the formulation of a default rule, then, Equation 3.1 seems to be better stated as:

$$\frac{\Gamma \nvdash \rho}{\Gamma \vdash \chi} \tag{3.2}$$

Second, and a rather problematic point, the role played by $\vdash$ in Equation 3.2 is

†In [Rei78], Reiter calls the underlying set of sentences Γ a *knowledge base*.

unclear. In [Rei78, p. 211], Reiter mentions that "there are some serious difficulties associated with just what exactly is meant by '$\nvdash$' but we shall defer these issues for the moment and rely instead on the reader's intuition". However, nowhere in [Rei78] Reiter provides a precise, i.e., formal, definition of '$\vdash$', leaving this as further work to undertake. This point will be returned to later on in this section.

In turn, in [Rei80], for π, ρ, and χ sentences in $\mathscr{L}$, Reiter defines a default rule as a tuple of the following form:

$$\frac{\pi : M\rho}{\chi} \tag{3.3}$$

Regarding Equation 3.3, Reiter's point is that M is to be read as 'it is consistent to assume' (q.v. [Rei80, p. 82]). The operator M is seemingly in line with the work of McDermott et al. in [MD80]. However, in contrast to the latter work, where M is a logical operator of a modal sort, in the work of Reiter M is more of a syntactical decoration (somewhat stressing the role of the sentence ρ in a default rule). It is perhaps for this reason that M is dropped in Reiter's later works. This point will also be returned to later on in this section.

Resuming from this parethetical remark, to avoid any subsequent misunderstanding, in what follows, default rules are assumed to be as in Definition 3.1.1.

The Formalization of a Default Rule

This subsection presents an analysis of the concept of a default rules in comparison with that of a rule of inference. The argument to be made is that default rules are nothing like rules of inference. Thus, contrary to what is postulated by Reiter in his works, the consideration here is that default rules cannot be treated as rules of inference and that a new basis for interpreting them is needed.

As a basis for analysis, to be made clear first is: What is the notion of a 'rule of inference' at issue?

While a fully comprehensive answer to the question above is outside of the scope of this thesis, the assumption here is that a somewhat traditional understanding of the notion of a rule of inference involves at least two pre-theoretical postulates:† (i) that a rule of inference has to do with deduction, ordinarily understood as the movement from premisses to consequences; and (ii) that a rule of inference captures accurately a pattern of reasoning that stands for a canon of proper argumentation. Implicit in both (i) and (ii) is the assumption that (iii) the formulation of a rule of inference is not done arbitrarily but that its consequent is in some sense involved in its premisses (i.e., there is a certain logical relation between the consequent of a rule of inference and its premisses).‡

For the point to be made in what follows, postulates (i) to (iii) above are simply a first step towards characterizing the notion of a rule of inference that a default rule is meant to capture formally. They are a 'first step' for the consideration here is that such a notion of a rule of inference is not reducible to being formalized as a rule of inference as this concept is classically conceived in logical studies. For one thing, as classically conceived, a rule of inference is (iv) *necessarily truth-preserving*, i.e., it is necessarily the case that if the premisses of the rule of inference are true, then, its consequent is also true. For another thing, such a rule of inference is (v) *definite*, i.e., its consequent cannot be challenged without the rule of inference itself being vitiated. While not made precise in his works, for some obvious reasons neither of (iv) and (v) seem to be what Reiter had in mind when mentioning that default rules are to be formally treated as rules of inference in [Rei87, p. 162].

Based on the previous discussion, the contention here is that the notion of a rule of

†Here 'pretheoretical' is used as in prior to formalization.

‡As discussed by Pelletier in [PH12], the understanding of a rule of inference explained w.r.t. postulates (i) to (iii) is, for instance, present in the work of Gentzen in [Gen64], as well as in many textbooks on natural deduction.

inference that a default rule is meant to capture formally shall subscribe to postulates (i) to (iii) above but, somewhat in opposition to points (iv) and (v), it accommodates for the idea that: (iv') truth needs only be *plausibly* preserved, i.e., granting the plausibility of the premisses the consequent is also plausible; and (v') the consequent can be challenged without the rule of inference itself being vitiated, i.e., the rule of inference is *non-definite*. To avoid any subsequent misunderstanding, a rule of inference is *defeasible* if it is in agreement with (i), (ii), (iii), (iv'), and (v') above.

By way of justification, this rough-and-ready definition of a defeasible rule of inference is based on the consideration that a rule of inference of any sort must, at a minimum, subscribe to points (i) to (iii) in order for it to be called a 'rule of inference' at all. Instead, points (iv') and (v') are meant to capture Reiter's postulate that "inferences sanctioned by default [i.e., by default rules] are best viewed as beliefs which may well be modified or rejected by subsequent observations" (q.v. [Rei80, p. 1]).

The idea that a default rule formalizes a defeasible rule of inference stems from the seminal works of Reiter. More precisely, in [Rei87, p. 162], Reiter comments that a default rule indicates the following inference license:

> (D) If the prerequisite holds and the justification can be consistently assumed, then, infer the consequent.

Extractable from (D), the research question posited in this section becomes: (Q) Is it at all logically tenable to sustain that a default rule formalizes the notion of a defeasible rule of inference and that therefore it may be formally treated as such?

The analysis of (Q) involves at least two main points that are worth elaborating on. The first of these points, labeled (1), hinders the understanding of a default rule as a defeasible rule of inference. The second these points, labeled (2), completely undermines this standpoint.

As to (1), Reiter is somewhat imprecise in his works as to what it means for a 'prerequisite to hold', a 'justification to be consistently assumed', and for a consequent to be 'inferred'.

Namely, a reasonable assumption is that 'to hold', 'to assume consistently', and 'to infer' are concepts defined in relation to an entailment relation, or analogously in terms of a consequence operator.

If the entailment relation, say $\mid\!\sim$, is that for reasoning with default rules, then, for a default rule $\pi : \rho \ / \ \chi$, Reiter's (D) schema above may be formalized as:

$$\text{if } \mid\!\sim \pi \text{ and, for all sentences } \sigma \in \mathscr{L}, \{\rho\} \not\mid\!\sim \sigma, \text{ then, } \mid\!\sim \chi \tag{3.4}$$

Granted, most likely Reiter had in mind a schema of a more general form. It is doubtful that in the context in which Reiter's works were conceived that he would not allow for arbitrary assumptions to be made. If this is the case, then, Equation 3.4 becomes:

$$\text{if } \Gamma \mid\!\sim \pi \text{ and, for all sentences } \sigma \in \mathscr{L}, \Gamma \cup \{\rho\} \not\mid\!\sim \sigma, \text{ then, } \Gamma \mid\!\sim \chi \tag{3.5}$$

where Γ is the set of sentences that accommodates for the sort of assumptions mentioned above.

In any case, if it is further assumed that $\mid\!\sim$ is defined in terms of a default rule, or more precisely, in terms of a set thereof, as is done in any standard syntactical presentation of a logical system, of which Reiter's presentation of reasoning with default rules is arguably an instance, then, both Equations 3.4 and 3.5 are seemingly circular. It is not possible to define $\not\mid\!\sim$ in terms of $\mid\!\sim$, nor viceversa, for it is $\mid\!\sim$ which is defined by a default rule.

If $\mid\!\sim$ is not the entailment relation for reasoning with default rules, then, the notion of a default rule is only dealt with at an intuitive level (cf. Equation 3.2), an indication that a moment shall be taken to make precise what this concept is all about, for otherwise there is a danger of a default rule being a mere shuffle of sentences devoid of any logical interest.

A brief parenthetical remark is in order before addressing (2) above. It may be argued that there are at least three straightforwardly possible workarounds to the circularity issue just mentioned.

The first workaround involves the idea that, instead of its formulation as in Equation 3.5, Reiter's (D) schema above is to be formulated as:

$$\text{if } \Gamma \vdash \pi \text{ and, for all sentences } \sigma \in \mathscr{L},\ \Gamma \cup \{\rho\} \nvdash \sigma, \text{ then, } \Gamma \mathrel{|\!\!\sim} \chi \tag{3.6}$$

or equivalently as either of:

$$\text{if } \Gamma \vdash \pi \text{ and } \Gamma \cup \{\rho\} \nvdash \bot, \text{ then, } \Gamma \mathrel{|\!\!\sim} \chi \tag{3.7}$$

$$\text{if } \Gamma \vdash \pi \text{ and } \Gamma \nvdash \neg\rho, \text{ then, } \Gamma \mathrel{|\!\!\sim} \chi \tag{3.8}$$

where $\vdash$ is an entailment relation defined in terms of Cn, i.e., the consequence operator of classical propositional logic, and $\mathrel{|\!\!\sim}$ is the entailment relation that default rules define. In some sense, Equation 3.6 may be justified on the basis that default rules formulate an inferential basis that is built on top of that of classical propositional logic. However, if this is indeed the case, it is worth noting that this workaround may introduce some new problems; e.g., it is not at all clear how the sequencing of default rules is to be handled.

The second of the workarounds mentioned above, and somewhat addressing the how to handle the sequencing of default rules, involves the idea of there being an ordering relation defined on default rules. While not pursued by Reiter himself, this idea has been elaborated on in the literature of reasoning with default rules (q.v. [AW07, pp. 541–546] for a brief summary of the main works on this approach to reasoning with default rules). If looked at from this perspective, it is worth noting that, unless the ordering defined on default rules is linear and fixed, different paths along the ordering relation may yield different sets of provable consequences in the resulting logical system. With respect to this possibility, Reiter is categorical in [Rei78, pp. 216–217], stating that such a behavior is clearly unacceptable. However, for undisclosed reasons, Reiter relaxes his position on this matter in [Rei80, p. 86] and his subsequent works, regarding it as making perfect sense in the context of reasoning with default rules. To be noted, though interesting in its own right, the idea of an ordering relation defined on default rules places an additional logical burden of proof onto the rationale for such an ordering.

The third of workarounds mentioned above is of a semantical nature. What it means for ‘a prerequisite to hold’, for ‘a justification to be consistently assumed’, and for a consequent to be ‘inferred’ is to be thought of in relation to an appropriately defined model theory. In light of the work of Kramosil in [Kra75], this does not seem to be an straightforward avenue to explore. In approaching the idea of a default rule from a semantical perspective, the work of Kramosil establishes some negative conclusions: either default rules are useless or meaningless. To be noted, the work of Kramosil is mentioned by Reiter in his works. In particular, in [Rei78, pp. 217–218], Reiter comments that: “fortunately, [Kramosil’s] “proof” (sic) relies on an incorrect definition of theoremhood [i.e., on what sort of consequences are sanctioned by default rules]”. A similar claim is made by Reiter in [Rei80, p. 94]. However, in neither [Rei78], nor in [Rei80], is Reiter clear as to whether Kramosil is incorrect at a conceptual level or at a formal one. In any case, as required in a model theoretical approach, it does not seem to be obvious to argue for what sort of relation should exist between an appropriate notion of a model in which the prerequisite of a default rule is taken to hold, one in which the justification is taken to be consistent, and a model of the consequent of this default rule, since, in general, the prerequisite, the justification, and the consequent of a default rule are three arbitrary sentences.

In summary, the remarks made in relation to (1) above raise the question: In which sense are default rules like rules of inference?

As to (2), the work of Toulmin in [Tou03] is briefly introduced here, by way of providing a much needed context for discussion.

Toulmin is perhaps most cited because of his famous notion of an *argument pattern* (q.v. [Tou03, p. 87–134]). In brief, whereas classical deductive logic studies typically appeal to the dichotomy between premisses and consequents in the analysis and formulation of rules of inference, Toulmin pinpoints six elements of what can be taken to be a rule of inference of a somewhat general kind. These elements correspond to what Toulmin refers to as: the *data*; the *claim*; the *warrant*; the *qualifier*; the *rebuttal*; and the *backing*, of an argument pattern.

From a traditional standpoint, Toulmin's notion of a data, a warrant, and a claim may be argued to be already present in the notion of a rule of inference. More precisely, the data stands in correspondence with the premisses of a rule of inference, the claim with the consequent of the rule of inference, and the warrant with the rule of inference itself. Toulmin's notion of a qualifier may be argued to be a sort of modal logical operator acting on claims, akin to the role played by modal logical connectives. In turn, Toulmin mentions that a rebuttal is meant to indicate "circumstances in which the general authority of the warrant would have to be set aside", or more precisely, "exceptional conditions which might be capable of defeating or rebutting the warranted conclusion" (q.v. [Tou03, p. 94]). In what follows, rebuttals are taken as in the latter sense. Lastly, Toulmin's notion of a backing is a sort of soundness condition on the inference license ascribed to the warrant.

Toulmin's notion of an argument pattern gains in interest in the analysis of (Q) for two main reasons. First, because an argument pattern accommodates the analysis of a defeasible rule of inference. Second, because certain elements of an argument pattern are present in the notion of a default rule as formulated in Definition 3.1.1. More precisely, it is possible to make a case that: the prerequisite of a default rule is in some correspondence with the notion of a data; the justification of the default rule with the notion of a rebuttal; the consequent of the default rule with the notion of a claim; and the default rule that these elements belong to with the notion of a warrant itself.

As a comment in passing, in spite of the similarities noted above, as is remarked by Verheij in [Ver09, p. 227], "Reiter does not refer to Toulmin in his highly influential 1980 paper, nor in his other work". In [Ver09, p. 227], Verheij justifies this situation by commenting that "being thoroughly embedded in the fertile logic-based AI community of the time, Reiter does not refer to less formal work".

When the concept of a default rule is looked at from the perspective of Toulmin's notion of an argument patter, to be noted is that Toulmin's notion of an argument pattern is a sort of meta notion that serves as a conceptual tool for the analysis of rules of inference of a somewhat general kind, but in and of itself it is not one such rule of inference. In other words, it is not arbitrary, but only proper, instances of Toulmin's notion of an argument pattern that may be considered to yield a sensible notion of a rule of inference.

The contention here is that the same is true in Reiter's case. It is not arbitrary but only proper instances of Reiter's (D) schema that are in some sensible, or, better yet, fit-for-purpose sense, defeasible rules of inference.

However, formally, there are no conditions imposed on Reiter's (D) schema that would force its instances, as captured by a default rule, to be a defeasible rule of inference. This issue is seemingly overlooked in presentations of the concept of a default rule; not only in the works of Reiter but more generally in the literature on reasoning with default rules. To further complicate matters, nowhere in the literature of reasoning with default rules, other than the occasional mention in passing by, is a defense of the way in which a default rule is a defeasible rule of inference. The position here is that this is a final and irrecoverable blow to the idea that default rules are defeasible rules of inference.

As a result, the contention here is that, in spite of the fact that a particular instance of the general notion of a default rule may be argued to be a defeasible rule of inference, the general treatment of this notion completely undermines the idea of a default rule being a defeasible rule of inference. Simply put, as it stands, the position asserting that default rules are to be treated as rules of inference is logically untenable.

While the discussion above paints a somewhat grim picture of the concept of a default rule, an alternative interpretation of what default rules stand for is offered in Section 4.1. In brief, the standpoint assumed in that section is that, as objects of logical investigation, default rules stand on better logical grounds as the formal counterpart of assertions that are assumed tentatively. This differs from formally treating a default rule as a defeasible rule of inference for the notion of a tentative assertion is a sort of premiss-like object. In that respect, the problem of a default rule being formulated arbitrarily is shifted to a problem with the adequacy of its formalization, something altogether different from the points raised above.

Alternative Standpoints on the Concept of a Default Rule

Reiter himself regarded the general notion of a default rule as particularly problematic. In [Rei78, p. 216], he comments about (I) the easiness with which default rules give rise to inconsistent sets of sentences.

Regarding this possibility, a word of clarification is needed. Implicit in (I) there are two different notions of consistency: consistency as understood in classical propositional logic, and consistency in a default sense. If looked at in light of this clarification, (I) is perhaps better understood as stating that: (I') it is possible for a set of sentences Γ that is consistent in a classical propositional sense to become inconsistent in a default sense when closed under the application of a default rule. But even (I') needs some further clarification. Suppose that $\{p, q\}$ is a consistent set of sentences in a classical propositional sense and that it is contended that $\{p, q\}$ is inconsistent in a default sense if closed under a default rule $p : q \,/\, \neg q$. Underpinning such a contention is the assumption that any sentence σ is provable (in a default sense) from $\{p, q\}$. While this assumption may be easily agreed upon, the situation as described is already grounds for dispute, since it may also be argued that the contrary is the case.

In some sense, situations such as those mentioned above may be avoided if the formulation of default rules is restricted to those that are normal. Regarding normal default rules, in [Rei80], Reiter states that these are the most commonly appearing default rules in practical applications. Even more so, in [Rei80, pp. 94–95] Reiter mentions that he knows “no naturally occurring default which cannot be represented in this form”. However, it is also noted by Reiter et al. in [RC81] that there are cases where non-normal default rules arise naturally. This establishes that, in general, reasoning with default rules is not reducible to reasoning with normal default rules. Hence, in considering the more restricted set of normal default rules one loses expressiveness, perhaps an agreed upon trade off for a formally better behaved notion.

A somewhat alternative standpoint on the notion of a default rule is offered by Łukaszewicz in [Łuk88]. In brief, while Łukaszewicz retains the idea that a default rule is a defeasible rule of inference, in [Łuk88, p. 3], he states that a default rule is to be thought of as indicating the following inference license:

> (L) If the prerequisite of a default is believed (its justification is consistent with what is believed), and adding its consequent to the set of beliefs neither leads to inconsistency nor contradicts the justification of this or any other already applied default, then the consequent of the default is to be believed.

In some sense, contrary to Reiter’s (D) schema, the (L) schema proposed by Łukaszewicz considers the justification of a default rule to be somewhat involved in its consequent. However, such an involvement is at a meta level, i.e., is not made precise in the formulation of a default rule but as conditions that are external to it. Worse yet, when compared to Reiter's (D) schema, Łukaszewicz's (L) schema possesses little to no intuitive clarity as an inference license.

In any case, in spite of the fact that both normal default rules and Łukaszewicz's framework for reasoning with default rules enjoy some desirable properties and are formally well-behaved, neither of them, in and of themselves, addresses the point that an arbitrary formulation cannot make a default rule a defeasible rule of inference.

3.2 The Concept of an Extension

The concept of an extension is another of the most prominent elements of Reiter's work on the subject of reasoning with default rules and perhaps its most controversial one. Regarding this concept, Reiter considers an extension to be naturally defined as a theory, in CPL, that is further closed under the application of some default rules under consideration. Based on such a postulate, this section proposes an analysis of the concept of an extension in comparison with that of a theory as this is usually conceived in logical studies. The argument to be made is that extensions cannot simply be viewed as being grounded on the typical understanding of a theory and that, insofar as logical well-foundedness is concerned, a new basis for interpreting what is captured by an adequate definition of an extension is needed.

The basics of the concept of an extension appear as early as in Reiter's 1978 article (q.v. [Rei78, pp. 216–218]).[†] However, it is not until Reiter's 1980 article that the

[†] As a historical note, in his [Rei78] article, Reiter presents what came to be known as an extension under the name of a 'default theory' – a terminology perhaps originating from the way in which he conceived this concept to be formulated. Later on, i.e., in all of his subsequent works on the subject of reasoning with default rules, Reiter refers to a theory that is closed under the application of a set of default rules as an 'extension', reserving the use of 'default theory' to denote a tuple consisting of a set of sentences and a set of default rules.

concept of an extension is given a precise definition (q.v. [Rei80, pp. 88–94]).† This definition is recalled in Definition 3.2.1.‡

Definition 3.2.1 (Reiter)**.** Let Γ be a set of sentences and Δ be a set of default rules; a set of sentences E is an extension iff

$$E = \bigcup_{i \geq 0} E_i$$

where: (i) $E_0 = \Gamma$; (ii) $E_{i+1} = Cn(E_i) \cup X(\Delta')$;§ and (iii) $\Delta' \stackrel{\text{def}}{=} \{\frac{\pi:\rho}{\chi} \in \Delta \mid \pi \in E_i$ and $\neg\rho \notin E\}$.

It is the concept of an extension as formalized in Definition 3.2.1 and its interpretation as a theory-like object which will be at issue for the rest of this section.

The Formalization of an Extension

Before elaborating on the concept of an extension as formulated in Definition 3.2.1, two preliminary points are worth making as they provide a necessary context for discussion.

In [Rei80, p. 88], Reiter considers that the intuitive idea to be captured formally as an extension E is that of a set Δ of default rules inducing a closure of a set Γ of

†To be precise, Reiter presents in [Rei80] two alternative, yet equivalent, formulations of the concept of an extension. Given that it is more suitable for the points at issue, of these two formulations, only that corresponding to [Rei80, p. 89, Theorem 2.1] is dealt with in this section. The other formulation of an extension, corresponding to [Rei80, p. 89, Definition 1], recalled in Definition 2.2.1, is left to be dealt with in Section 3.3.

‡By way of disclaimer, while not exactly the same, Definition 3.2.1 is in all essential aspects similar to that given by Reiter in [Rei80, pp. 89–90, Theorem 2.1].

§Recall from Section 3.1 that, for a set of default rules Δ, $X(\Delta) = \{\chi \mid \frac{\pi:\rho}{\chi} \in \Delta\}$.

sentences. In that respect, Reiter mentions that there are three properties which can reasonably be expected of such an extension (q.v. [Rei80, p. 89]). These are:

(3.9) Γ should be contained in E

(3.10) E should be a theory in classical propositional logic, i.e., $Cn(E) = E$

(3.11) E should be closed under the application of the default rules in Δ

i.e., for all $\pi : \rho \,/\, \chi \in \Delta$, if $\pi \in E$ and $\neg\rho \notin E$, then, $\chi \in E$

In light of Equations 3.9 to 3.11, there seems to be little doubt that in Reiter's sense an extension is a theory closed under the application of default rules, even more so as Reiter himself took these properties as what motivated the formal definition of an extension (q.v. [Rei80, p. 89]).

In addition, because it influences the analysis of the formalization of the concept of an extension, the consideration here is that, if a default rule is to be taken as a rule of inference of some sort, something that is not challenged here, then, a difference has to be made as to: (a) the notion of the rule of inference captured by a default rule, (b) the formal criterion of applicability of a default rule, (c) the result of applying a default rule, (d) the logical correctness of this result, and (e) the adequacy of this result in relation to a problem at hand. While it may be fairly obvious that points (a) to (e) are all somewhat interrelated, the consideration here is that inquiring about them independently sheds some clarity on what is at issue.

In this context, the research question posited becomes: (Q) Can extensions, as objects of logical investigation, be viewed as being grounded on the typical understanding of a theory?

The analysis of (Q) involves, first, dealing with the following issue: the intuitive flavor that Reiter ascribes to the formalization of an extension as in Definition 3.2.1 is misleading. While outwardly Definition 3.2.1 suggests that an extension E is the result of a stepwise incremental construction from the subsets E_i, after all, such is the intuitive understanding of a generated theory, a more careful reading of Definition 3.2.1 establishes that the definition of an extension E in terms of the sets E_i is ill-founded as E appears in the definition of each E_{i+1}. In that respect, Definition 3.2.1 does not define an extension E as the union of all sets E_i but it expresses an equation which may be satisfied by a set E in relation to the union of said sets E_i.

From a methodological perspective, the latter is troublesome, and actually counter-intuitive, for it indicates that in order to determine whether or not a set of sentences E is an extension, in a sense, what an extension *is* must be "guessed" *a priori*, just to later check whether this guess is indeed correct.

In addition, insofar as the properties expressed in Equations 3.9 to 3.11 are concerned, Definition 3.2.1 is extensionally incorrect as it undergenerates: there are sets of sentences that satisfy these properties but that are not extensions in the sense of Definition 3.2.1.

For instance, given an empty set of sentences and default rules $\frac{p:q}{q}$ and $\frac{q:p}{p}$, it is more or less direct to prove that, while $Cn(\{p, q\})$ satisfies Equations 3.9 to 3.11, $Cn(\{p, q\})$ is not an extension according to Definition 3.2.1.

Arguing that this is to be treated as a circularity issue with the application criterion of the default rules in question, an easy fix if an extension is constructed "from below" from the set of sentences under consideration, still leaves untreated other instances of undergeneration.

For instance, given a set of sentences $\{p\}$ and a set of default rules $\{\frac{p:\neg r}{q}, \frac{p:r}{r}\}$, while $Cn(\{p, q, r\})$ satisfies Equations 3.9 to 3.11, $Cn(\{p, q, r\})$ is not an extension according to Definition 3.2.1.

Arguing that the latter is an issue with the order in which default rules are to be applied, i.e., $\frac{p:r}{r}$ shall take precedence over $\frac{p:\neg r}{q}$, requires an order of application for default rules to be made explicit. But not only this is not present in Equations 3.9 to 3.11, nor for what matters in Definition 3.2.1, orders of application for default rules may give rise to some new issues.

For instance, given a set of sentences $\{p\}$ and a set of default rules $\{\frac{p:\neg r}{q}, \frac{p:r}{r}\}$, a case would have to be made as to why applying $\frac{p:r}{r}$ before $\frac{p:\neg r}{q}$ results in an extension, whereas applying $\frac{p:\neg r}{q}$ before $\frac{p:r}{r}$ does not.

If default rules are to be ordered before being applied, to be noted is that different orders of application of default rules may yield different results.

For instance, given a set of sentences $\{p\}$ and default rules $\frac{p:q}{q}$ and $\frac{p:\neg q}{\neg q}$, applying $\frac{p:q}{q}$ before $\frac{p:\neg q}{\neg q}$ results in $Cn(\{p, q\})$ being an extension, whereas applying $\frac{p:\neg q}{\neg q}$ before $\frac{p:q}{q}$ results in $Cn(\{p, \neg q\})$ being a different extension (i.e., both $Cn(\{p, q\})$ and $Cn(\{p, \neg q\})$ satisfy the properties in Equations 3.9 to 3.11 and are solutions to the equation in Definition 3.2.1).

In [Rei78, p. 217] Reiter is categorical about situations such as that just presented: "[extensions] exhibiting such a behavior are clearly unacceptable. At the very least, we must require of [an extension] to satisfy a certain kind of Church-Rosser property: no matter what the order in which the theorems of the [extension] are derived [resorting to default rules], the resulting set of theorems will be unique". While in subsequent works Reiter abandons the standpoint he himself assumed in [Rei78] allowing for extensions not to be uniquely determined, considering it to be a natural phenomena, Reiter does not disclose the reasons for such a change in standpoint. In a sense, and by way of justification, from a purely formal point of view, if the equation in Definition 3.2.1 is considered in isolation, it is possible to argue that there need be no conditions forcing a solution to such equation to be unique; but this interpretation forces an abandonment of extensions being justified as theory-like objects.

Non-uniqueness of extensions is worth mentioning for it distinguishes the understanding of this notion from the usual understanding of a theory, and more particularly, from the understanding of a generated theory. More precisely, contrary to what is the case with a theory, which is a *least element* of an appropriately defined order, an extension is, in a sense, a *minimal element* of one such order, but not necessarily a least one. As noted by Antoniou et al. in [AW07, p. 522], minimality is not a minor issue.

For instance, given a set of sentences $\{p\}$ and a default rule $\frac{p:q}{q}$, it is possible to make a case that both $Cn(\{p,q\})$ and $Cn(\{p,\neg q\})$ are "minimal" elements which satisfy properties (i) to (iii) above, and hence sensible candidates for an extension. The preference of one over the other rests on a basis other than minimality.

Even if one were able to ignore the points above, the more problematic and puzzling feature of Definition 3.2.1, in particular when compared with the notion of a theory, is that extensions may fail to exist, i.e., for certain sets of sentences and certain sets of default rules, the equation given in Definition 3.2.1 may fail to have a solution.

For instance, it can be proven that given a set of sentences $\{p\}$ and a default rule $\frac{p:q}{\neg q}$, no set of sentences E satisfies the equation in Definition 3.2.1.

What is both problematic and puzzling about extensions not existing is its typical accompanying argument (q.v. [AW07, p. 524] or [Ant05, p. 15]). This argument tends to take the following form:

> If E were an extension of $\{p\}$ and $\{\frac{p:q}{\neg q}\}$, then, either $\frac{p:q}{\neg q}$ is triggered by E or it is not. If $\frac{p:q}{\neg q}$ is triggered by E, then, $\neg q$ belongs to E, and so the justification of $\frac{p:q}{\neg q}$ is inconsistent with E. If $\frac{p:q}{\neg q}$ is not triggered by E, then, $\neg q$ does not belong to E, and so E is not closed under the application of $\frac{p:q}{\neg q}$. Either case is impossible.

What the previous argument fails to consider is that there is a difference between the conditions under which a default rule can be triggered and thereupon applied, and the result of such an application.

In fact, at least intuitively, the first case in the argument above is perfectly tenable. Even more so, E can be constructed from below from $\{p\}$, as in the chain $\{p\} \subseteq Cn(\{p\}) \subseteq Cn(\{p\}) \cup \{q\} \subseteq Cn(Cn(\{p\}) \cup \{q\}) = E$. That while applicable w.r.t. $Cn(\{p\})$, $\frac{p:q}{\neg q}$ is not applicable w.r.t. E is an indication of the defeasible aspect of a default rule. But this is an altogether different discussion.

If the problem is that E is not to be considered an extension, for such resulting theories are unwanted on account of the fact that they are intuitively incorrect, then, it may be argued that, *a priori*, there is no reason why a formal account of an extension must necessarily show all ordinary intuitions about this concept to be correct. In fact, a case can be made that, any precise definition of an extension may show a certain logical arbitrariness to a lesser or greater degree. But again this is an altogether different discussion.

In any case, the non-existence of extensions not only completely undermines the intuitive understanding of an extension as being a theory-like object, but from a methodological standpoint it pinpoints that there is something fundamentally wrong with the definition of this concept in relation to its justification.

Recognizing that providing an extensionally adequate and formally correct definition of the concept an extension is not a trivial matter, this is regarded by Reiter in [Rei78] to be one of the major challenges of reasoning with default rules, the consideration here is that, when taken together, the observations made above make a case against justifying the formal definition of an extension on the understanding of this concept as a theory that is further closed under the application of default rules. In that respect, there are at least two possible alternatives: to consider that an extension is the object that is defined by Definition 3.2.1, rendering its study as “abstract nonsense”; or to justify the definition of an extension on a basis other that this concept being intuitively understood as a theory that is further closed under the application of default rules. The latter alternative is explored in 4.2, where a case is made that, at a conceptual level, extensions, albeit syntactical entities, stand somewhat in analogy with the traditional understanding of models as interpretation structures.

Alternative Standpoints on the Concept of an Extension

Starting from Reiter's 1980 article, the question: What shall count as an adequate and formally correct definition of the concept of an extension? has become one of the most prominent themes in the literature on reasoning with default rules, with several alternative formulations of this concept being proposed.

These alternative formulations of the concept of an extension can be categorized into three, not necessarily disjoint, major classes. These are listed below:

(i) Formulations of the concept of an extension that are intended to provide a better understanding of the objects that are captured by Definition 3.2.1.
(ii) Formulations of the concept of an extension that are intended to cope with what are argued to be some shortcomings of Definition 3.2.1.
(iii) Formulations of the concept of an extension that are intended to obtain a somewhat sensible formalization of the concept of 'being a consequence of' for reasoning with default rules.

All members of these classes regard Definition 3.2.1 as the point of departure for formal investigation. The rest of this subsection is dedicated to commenting on what may be seen as being some of their representative candidates.

Belonging to the first of the classes mentioned above is Antoniou's operational characterization of an extension (q.v. [Ant97] and [AW07]).

Definition 3.2.2 (Antoniou)**.** Let Γ be a set of sentences and Δ be a set of default rules; in addition, for any sequence s of default rules of Δ, let Δ_s be the set of default rules in s; moreover, let $In(s) \stackrel{\text{def}}{=} Cn(\Gamma \cup X(\Delta_s))$ and $Out(s) \stackrel{\text{def}}{=} P(\Delta_s)$; then:

– A default rule $\pi : \rho \ / \ \chi \in \Delta$ is applicable to s iff there is a prefix s' of s such that $\pi : \rho \ / \ \chi \notin \Delta_{s'}$, $\pi \in In(s')$, and $\neg\rho \notin In(s')$.

– A sequence s of default rules of Δ without repetitions is: (a) *a process* iff every default rule in s is applicable to s, (b) *closed* iff every default rule in Δ that is applicable to s already belongs to s, (c) *successful* iff $In(s) \cap Out(s) = \emptyset$.

– A set of sentences E is an *extension* (of Γ and Δ) iff there is a closed and successful process p such that $E = In(p)$.

In Definition 3.2.2, the concept of an extension is defined in terms of a *process* that is both *closed* and *successful*. This rests on the consideration that an extension is to be obtained as the result of applying default rules as if they were arranged in a sequence. The applicability criterion of a default rule is that suggested by Reiter in Section 3.1, i.e., a default rule $\pi : \rho \ / \ \chi$ is applicable to a set of sentences Γ if $\pi \in Cn(\Gamma)$ and $Cn(\Gamma \cup \{\rho\})$ is a consistent theory in classical propositional logic.

In Antoniou's sense, a process is then to be thought of as a possible order of application of the default rules in Δ (with different processes potentially resulting in different extensions). In [AW07, p. 523], Antoniou considers that: "we don't want to apply a default more than once ... because no additional information would be gained by doing so", hence the requirement of a process being a sequence of default rules without repetitions. A process being closed corresponds to the idea of an extension being closed under the application of default rules. Lastly, successfulness indicates a *global* consistency check: if p is a process, then, for every default rule $\pi : \rho \,/\, \chi$ in p, $\neg\rho \notin Cn(In(p))$.

Antoniou's operational characterization of an extension differs from Reiter's equational characterization of this concept in terms of how an extension is formulated but not in terms of the formal objects they both capture. This is proven by Antoniou in [AW07, p. 526, Theorem 2].

In taking Reiter's definition of an extension as basic, Antoniou's operational characterization somewhat departs from justifying an extension as being a theory closed under the application of default rules.

In one sense, Definition 3.2.2 is perhaps better justified in terms of what Sandewall calls the 'systems' approach to nonmonotonic reasoning in [San11]. This approach, while certainly of value in terms of applications, may be seen as regarding extensions as objects of a somewhat different kind, not being necessarily grounded on a logical basis, and even less on the typical understanding of a theory.

In another sense, Antoniou's operational characterization of an extension may be regarded as intended to shed some light on some of the basic concepts of reasoning with default rules, more precisely, on what an extension is. When taken in this sense, in the concluding remarks of [AW07], Antoniou offers the view that the subject of reasoning with default rules, as introduced by Reiter in his 1980 article, is one of the most difficult formalisms to comprehend for those wanting to know what is it about. In that respect, in [AW07, p. 553], Antoniou considers that the contributions oriented towards a better understanding of some of the basic concepts of reasoning with default rules should not be underestimated.

Regarding formulations of the concept of an extension belonging to the second of the classes mentioned above, in general terms, their typical point of departure is either: (D) a dissatisfaction with some of the formal properties that are not satisfied by Reiter's definition of an extension, e.g., with extensions not existing, or (P) what is viewed as some paradoxical behavior exhibited by Reiter's definition of an extension.

A brief parenthetical remark is in order here before continuing. Regarding (D), it is possible to hold the view that the non-existence of extensions is not necessarily a drawback in reasoning with default rules.

There are at least three different lines of argumentation on which such a standpoint may be based. First, argue that the non-existence of extensions is a feature of reasoning with default rules rather that a deficiency (sweep the whole issue under the carpet and ignore the fact that extensions may not exist). Second, argue that in those cases where extensions do not exist, the "default" candidate for an extension is the set of all sentences of the language under consideration. Third, argue for restricting default rules to those that guarantee the existence of extensions. The first of the lines of argumentation above does not need be elaborated at length. There are some obvious reasons why it should be avoided. The second of these lines of argumentation, unless accompanied with a discussion of why the set of all sentences of the language under consideration is a sensible "default" candidate for an extension in those cases in which it may not otherwise exist, appears to be more of a patch to a flawed definition rather than an extensionally adequate and rigorous presentation of a logical concept. The third of these lines of argumentation, definitely more technically sound that the previous two, is, e.g., present in Reiter's 1980 article, where the locus of attention is placed on default rules that are normal. To be noted, however, a potential danger with restricting default rules is that their adequacy for a problem at hand may be hindered, but this is something that may be unhappily accepted as a trade off for a formally better behaved definition of an extension.

Following from this parenthetical remark, the work of Łukaszewicz in [Łuk88] is the first of a series of spin-offs that looked at reasoning with default rules from the point of view of (D), i.e., as there being "something wrong" with Reiter's definition of an extension.

In essence, in light of what are presented as being some deficiencies in Reiter's approach to reasoning with default rules, Łukaszewicz argues for the concept of an extension to be reformulated from the outset. Łukaszewicz's (re)formulation of the concept of an extension is given in Definition 3.2.3.

Definition 3.2.3 (Łukaszewicz)**.** Let Γ be a set of sentences and Δ be a set of default rules; a set of sentences E is an m-extension of Γ and Δ with respect to a set of sentences F iff

$$E = \bigcup_{i \geq 0} E_i \qquad \text{and} \qquad F = \bigcup_{i \geq 0} F_i$$

where: (i) $E_0 = \Gamma$; (ii) $F_0 = \emptyset$, (iii) $E_{i+1} = Cn(E_i) \cup X(\Delta')$; (iv) $F_{i+1} = F_i \cup P(\Delta')$, and (v) $\Delta' \stackrel{\text{def}}{=} \{\pi : \rho \mathbin{/} \chi \in \Delta \mid \pi \in E_i \text{ and, for all } \rho' \in F \cup \{\rho\},\ \neg\rho' \notin Cn(E \cup \{\chi\})\}$.

While Definition 3.2.3 has some standing as an object of formal study, since it can be proven that m-extensions always exists, it is not difficult to see that the system of equations involved in Definition 3.2.3 reveals itself as rather complicated and as departing from the justification of the concept of an extension as a theory that is closed under the application of default rules.

The work of Delgrande et al. in [DSJ94] may also be seen as belonging to the second class of those mentioned at the beginning of this section. In essence, the point of departure of the work of Delgrande et al. in [DSJ94] is grounded on Poole's *broken arms* example (q.v. [Poo89]). Because it touches on some important points that are worthy of discussion, Poole's broken arms example, as presented in [Poo89, p. 334], is reproduced below.

In Poole's terms:

> Suppose by default people's left arms are usable, but a person with a broken left arm is an exception, and similarly people's right arms are, by default, usable, but broken right arms are an exception. In Reiter's notation ... this is:
>
> $$\frac{: u_l \wedge \neg b_l}{u_l} \qquad \frac{: u_r \wedge \neg b_r}{u_r} \tag{3.12}$$
>
> If we know nothing about Matt's left arm, we conclude (correctly as to what we assumed a default was) his left arm is usable. If we know his left arm is broken, we (correctly again) do not conclude his left arm is usable.
>
> Suppose we remember seeing him with a broken left arm or a broken right arm (we cannot remember which). We add
>
> $$b_l \vee b_r \tag{3.13}$$
>
> In this case we cannot conclude he has a broken left arm and so conclude his left arm is usable. We also cannot conclude he has a broken right arm so we conclude his right arm is usable. We thus conclude both his left arm and his right arm are usable.
>
> I would argue that this is definitely a bug, being able to conclude both arms are usable given we know one of his arms is broken. The problem is we have implicitly made an assumption, but have been prevented from considering what other assumptions we made as a side effect of this assumption. Somehow we needed to commit to the implicit assumption that his left arm was not broken when we used the first default. This problem of disjunctive exceptions is endemic to the use of non-normal defaults.

Somewhat in contrast to the work of Łukaszewicz, the work of Delgrande et al. in [DSJ94] addresses what is presented as being a paradoxical situation emerging from Reiter's formulation of an extension. Delgrande et al. argue in [DSJ94] that the broken arms example suggests that the set of justifications used in the formulation of an extension should be consistent, rather than each justification being individually consistent. As a result of which, the following formulation of an extension is introduced.

Definition 3.2.4 (Delgrande)**.** Let Γ be a set of sentences and Δ be a set of default rules; a set of sentences E is a constrained extension of Γ and Δ with respect to a set of sentences C iff

$$E = \bigcup_{i \geq 0} E_i \qquad \text{and} \qquad C = \bigcup_{i \geq 0} C_i$$

where: (i) $E_0 = C_0 = \Gamma$; (ii) $E_{i+1} = Cn(E_i) \cup X(\Delta')$; (iii) $C_{i+1} = Cn(C_i) \cup P(\Delta') \cup X(\Delta')$; and (iv) $\Delta' \stackrel{\text{def}}{=} \{\pi : \rho \;/\; \chi \in \Delta \mid \pi \in E_i \text{ and, } \perp \notin Cn(E \cup \{\rho\} \cup \{\chi\})\}$.

At the same time, it is possible to hold the view that what is paradoxical in Poole's broken arms example is Poole's proposed formalization of the scenario it involves and not Reiter's definition of an extension.

The broken arms paradox is avoided if it is further assumed that an arm is not usable if it is broken, i.e., if in addition to Equation 3.13 it is also added that

$$(b_l \supset \neg u_l) \wedge (b_r \supset \neg u_r) \tag{3.14}$$

something that is seemingly implicit in Poole's presentation of the broken arms example.

Equation 3.14 is also interesting for it is mentioned by Brewka in [Bre91]. In brief, in that work, Brewka argues that underlying the broken arms example is the idea that such an assumption does not hold. In [Bre91, p. 185], Brewka advances that:

"we want to make [one of the default rules in Equation 3.12] inapplicable if we know that a given [arm] is broken, but without stating that [this arm] is not usable in this case". But this is troublesome. If Equation 3.14 is taken not to hold, i.e., if it is possible for an arm to be both broken and usable, then, not only does the paradox seemingly disappear, but the fact that both arms are usable is in fact a desirable consequence.

Alternatively, it is possible to hold the view that what is paradoxical in Poole's broken arms example is its ordinary understanding.

For instance:[†]

> Suppose by default that when Matt leaves his house he takes an umbrella with him unless the weather is good, and that he takes a pair of sunglasses with him unless the weather is bad. As Poole's argues, in Reiter's notion this may be formalized as
>
> $$\frac{: u \wedge \neg g}{u} \qquad \frac{: s \wedge \neg b}{s} \tag{3.15}$$
>
> where: (i) g stands for 'the weather is good'; (ii) b stands for 'the weather is bad'; (iii) u stands for 'Matt takes an umbrella with him'; and (iv) s stands for 'Matt takes a pair of sunglasses with him'.
>
> If nothing is known about the weather being good, it is reasonable to conclude (correctly in relation to what has been assumed by default) that Matt is taking an umbrella with him. If it is known that the weather is good, it is reasonable (correctly again) not to conclude that Matt is taking an umbrella with him.

[†]Adapted from [AW07, p. 532].

> Suppose further that either the weather is good or that it is bad, i.e., that
>
> $$g \vee b \tag{3.16}$$
>
> In this case, since it is not possible to conclude that the weather is good, it may be concluded that Matt is taking an umbrella with him. Moreover, since it is also not possible to conclude the weather is bad, it may be concluded that Matt is taking a pair of sunglasses with him. Thus, it may both be concluded that Matt is taking an umbrella and a pair of sunglasses with him.

In the example above, what is paradoxical of Poole's broken arms example may indeed be argued to be a reasonable consequence, e.g., justified based on Matt being a cautious person.

In any case, it may just as well be accepted that paradoxes are perhaps an intrinsic feature of any formalism to a greater or lesser degree. If looked at from this perspective, whether constrained extensions as defined by Delgrande et al. are to be preferred over extensions as defined by Reiter, or *vice-versa*, may depend on the particularities of the situation at hand.

Several other alternative formulations of the concept of an extension which may also be seen as belonging to the second of the classes mentioned at the introduction of this subsection are discussed by Antoniou et al. in [AW07] and Delgrande et al. in [DSJ94]. A comparison of these alternative formulations of the concept of an extension, in terms of how they correspond to each other, is studied by Delgrande et al. in [DS03].

Acknowledging that there needs to be one formally correct definition of such a concept, in that different formulations may be accepted as being grounded on different intuitions, what becomes clear after analyzing the different formulations of the concept of an extension that belong to this second class is that there is a departure from the idea of an extension being justified on the grounds of it being a theory that is further closed under the application of default rules.

Lastly, the work of Brewka in [Bre91] and that of Antonelli in [Ant05] may be seen as belonging to those formulations of the concept of an extension that are intended to obtain a somewhat sensible formalization of the concept of 'being a consequence of' for reasoning with default rules, i.e., to the third of the classes mentioned at the beginning of this subsection.

In brief, Brewka justifies his formulation of the notion of an extension on the observation that reasoning with default rule is more a logic of reasoned belief than just plain belief. According to Brewka, it is in making this observation explicit that one obtains a somewhat sensible formulation of the concept of 'being a consequence of' for the case of reasoning with default rules (q.v. [Bre91, p. 187]).

To obtain such a *desideratum*, Brewka formalizes a reasoned belief in terms of what he calls an *assertion*, i.e., a tuple $\langle \alpha, \mathrm{A} \rangle$ where: (i) α is a sentence indicating what is to be believed, and (ii) A is a set of sentences indicating the reasons why what is to be believed is to be believed. Based on this notion of an assertion, Brewka formulates the concept of an extension as a set of assertions defined w.r.t. a set of what he calls well-based assertions and a set of default rules (q.v. [Bre91, Section 2]).

Regarding Brewka's formulation of an extension, it is mentioned by Antoniou et al. in [AW07] that it is doubtful whether this is the right way to go, since such a formulation has an additional conceptual and computational load; largely due to the use of assertions rather than plain sentences (q.v. [AW07, p. 533]). In addition, and perhaps more problematic, it is mentioned in [DSJ94] that, since in Brewka's sense extensions are no longer sets of sentences but are sets of assertions, it is unclear (at best) how they can be joined to yield conclusions common to every extension (q.v. [DSJ94, p. 178]). As will be seen in Section 3.3, whether extensions can be joined or not has a direct influence on the formalization of the concept of 'being a consequence of' for reasoning with default rules. In that respect, being unable to join extensions may hinder the consideration that Brewka's definition of an extension in fact yields a somewhat sensible formalization of the concept of 'being a consequence of' for reasoning with default rules.

In turn, the work of Antonelli in [Ant05] originates from the following analysis (q.v. [Ant05, p. 15]).

> Let W contain the sentence η, and let Δ comprise the single default
>
> $$\eta : \theta \,/\, \neg\theta$$
>
> If E were an extension, then the preceding default would have to be either triggered or not triggered by it, and either case is impossible. The default cannot be triggered, for then its justification would be inconsistent with the extension; but then the justification is consistent with E, so that the default is triggered after all.

According to Antonelli, the preceding default rule bears a resemblance to the so-called *Liar's Paradox* (q.v. [Tar56f]).

Based on such a consideration, and in light of the work of Kripke in [Kri75], Antonelli proposes to abandon *bivalence* and to formulate the concept of an extension in a *three-valued* setting (q.v. [Ant05, pp. 15–16, Section 3]). Antonelli's formulation of the concept of an extension is given in Definition 3.2.5.

Antonelli's formulation of the concept of an extension can be understood by analyzing the roles played by the default rules in the sets Δ^e, Δ^p, and Δ^t in Definition 3.2.5. In brief, Δ^e is the set of default rules which generates the extension, Δ^p is the set of those default rules which are preempted in this extension, and Δ^t is the set of rules that may be triggered in this extension.

Definition 3.2.5 (Antonelli)**.** Let Γ be a set of sentences and Δ be a set of default rules; a set of sentences E is a general extension of Γ and Δ iff

$$E = Cn(\Gamma \cup X(\Delta^e))$$

where: (i) $\Delta^e = \bigcup_{i \geq 0} \Delta^e_i$; (ii) $\Delta^p = \bigcup_{i \geq 0} \Delta^p_i$; (iii) $\Delta^t = \bigcup_{i \geq 0} \Delta^t_i$; and:

$$\Delta^e_0 = \emptyset \tag{3.17}$$

$$\Delta^e_{i+1} = \left\{ \frac{\pi : \rho}{\chi} \in \Delta \mid \pi \in Cn(\Gamma \cup X(\Delta^e_i)) \text{ and } \neg\rho \notin Cn(\Gamma \cup X(\Delta^t_i \setminus \Delta^p_i)) \right\} \tag{3.18}$$

$$\Delta^p_0 = \emptyset \tag{3.19}$$

$$\Delta^p_{i+1} = \left\{ \frac{\pi : \rho}{\chi} \in \Delta \mid \neg\rho \in Cn(\Gamma \cup X(\Delta^e_{i+1})) \text{ and } \neg\chi \in Cn(\Gamma \cup X(\Delta^e_{i+1})) \right\} \tag{3.20}$$

$$\Delta^t_0 = \left\{ \frac{\pi : \rho}{\chi} \in \Delta \mid \neg\pi \notin Cn(\Gamma) \right\} \tag{3.21}$$

$$\Delta^t_{i+1} = \left\{ \frac{\pi : \rho}{\chi} \in \Delta \mid \neg\pi \notin Cn(\Gamma \cup X(\Delta^e_{i+1})) \right\} \tag{3.22}$$

It is resorting to Definition 3.2.5 that Antonelli proves that it is possible to obtain a somewhat sensible formalization of the concept of 'being a consequence of' for reasoning with default rules (q.v. [Ant05, Chapter 4]).

A rather obscure feature of Antonelli's formulation of the concept of an extension is what he views as being the "three-valued nature" of Definition 3.2.5. This three-valued nature of Definition 3.2.5 differs from other three-valued approaches in that the underlying logic is bivalued. In that respect, in [Ant05, p. 99], Antonelli comments that, three-valued-ness is cashed out not by adopting a three-valued underlying logical framework, but due to the non-complementary aspect of the sets Δ^e and Δ^p. This implies that the preemption of a default rule does not depend solely on the default rules generating an extension, but also on those default rules which may be neither generating nor excluded from this extension. Antonelli offers the view that it is this the latter that allows for a more concrete representation of extensions, whose properties can then be more easily investigated (q.v. [Ant05, p. 99]).

At the same time, the non-complementary aspect of the sets Δ^e and Δ^p is troublesome. For instance, if it is taken that one of the main purposes of reasoning with default rules is to reason in the presence of contradictory information, and if this contradictory information is to be formalized in terms of default rules, then, if (i) Δ is a set of default rules such that $\perp \in Cn(\Gamma \cup X(\Delta))$ and (ii) Γ is a set of sentences such that for no $\frac{\pi:\rho}{\chi} \in \Delta$, $\neg\pi \notin Cn(\Gamma)$, then, according to Definition 3.2.5, $Cn(\Gamma)$ is always an extension of Γ and Δ. As will be seen in Section 3.3, conditions (i) and (ii) reduce the formulation of the concept of 'being a consequence of' for the case of reasoning with default rules to that captured by Cn. The ubiquity of these conditions may then hinder the consideration that Antonelli's definition of an extension in fact yields a somewhat sensible notion of 'being a default consequence of' for the case of reasoning with default rules.

Even if the remarks made in relation to the works of Brewka and Antonelli are disregarded or bypassed, the point at issue here is that the formulations of the concept of an extension proposed in these works is a clear departure from the idea of an extension being justified on the grounds of it being a theory that is further closed under the application of default rules.

To be made clear at this point is the following observation. If the concept of an extension is simply reduced to what is captured by a particular definition, then, it loses much of its logical interest, i.e., a extension is what a particular definition says that an extension is.

3.3 The Concept of 'Being a Consequence of' for Reasoning with Default Rules

If (non)monotonicity is taken as a property that is possessed (or not) by a consequence operator that is intended to capture a certain notion of the concept of 'being a consequence of', then, extractable from the works of Reiter is the idea that the formulation of the consequence operator capturing the concept of 'being a consequence of' for the case of reasoning with default rules shall be intrinsically nonmonotonic. As is discussed in this section, while built on the maxim that default rules are defeasible rules of inference, what Reiter took as a somewhat sensible formulation of such a consequence operator is a question with a rather elusive answer, for nowhere in his works does Reiter make it explicit.

Perhaps the closest answer to such a question is that, for a sentence δ and a set of sentences Γ,

$$\delta \text{ is a default consequence of } \Gamma \text{ iff } \delta \in C(\Gamma) \tag{3.23}$$

where C is the operator defined in [Rei80, p. 89, Definition 1] (q.v. Definition 2.2.1).

Though exhibiting a certain degree of liberty in its formulation, Equation 3.23 is based on the following two observations.

First, alternatively to being a solution to the equation in Definition 3.2.1, Reiter defines an extension as a fixed-point of C (i.e., E is an extension iff $E = C(E)$). Given that Reiter regards an extension as a theory-like object, an assumption that is not currently being challenged, it is seemingly reasonable to assume that a sentence δ is a default consequence of Γ if $\delta \in C(\Gamma)$, since such is the case in the classical account of theories.

Second, in Definition 2.2.1, C is formulated on an underlying set of sentences A and an underlying set of default rules Δ. Since these sets are fixed in Definition 2.2.1, in analogy with what is the case in Tarskian consequence operators, cf. Definition 2.1.5, it is seemingly reasonable to assume that A and Δ are meant to stand somewhat in correspondence with the roles played by an axiomatic base and an inferential base (an assumption that may be justified on account of Reiter's maxim of a default rule being a defeasible rule of inference, something currently not being challenged). If such is the case, then, C indeed indicates what is sanctioned resorting to the default rules in Δ, and hence that a sentence δ is a default consequence of Γ iff $\delta \in C(\Gamma)$.

It is the concept of 'being a consequence of' for the case of reasoning with default rules as formulated in Equation 3.23 which will be at issue for the rest of this section. In order to avoid being lost in generalities, the locus of attention is placed on the nonmonotonic aspect of reasoning with default rules and what arguably are some desirable properties of any nonmonotonic logical system. The argument to be made is that it is dubious whether Equation 3.23 indeed constitutes a somewhat sensible formulation of the concept of 'being a consequence of' for reasoning with default rules. The question to be asked is: (Q) Does Reiter's approach to reasoning with default rules stands on somewhat sensible logical grounds?

The Formalization of the Concept of 'Being a Consequence of' for Reasoning with Default Rules

A first thing worth noting is that, as formulated by Reiter in Definition 2.2.1, the operator C is abstract in nature. More precisely, as presented in Definition 2.2.1, C is defined w.r.t. an underlying set of sentences A and an underlying set of default rules Δ. While fixed, A and Δ are arbitrary sets. This implies that C is not *a* consequence operator of a logical system for reasoning with default rules, but that *one* such consequence operator is obtained upon instantiating the sets A and Δ; different instances of these sets give rise to different logical systems.

This is worth noting since, from an abstract standpoint, principles such as that of monotonicity, inclusion, cut, cumulativity, etc., are taken to be intrinsic properties of the consequence operator in point. But this is not the case for Reiter's formulation of the operator C in Definition 2.2.1, nowhere in his works does Reiter discuss whether C is to subscribe to any of these principles, nor defends such a position, e.g., by making these principles a requirement of any sensible formulation of C. Worsening this situation, none of these principles follow from Definition 2.2.1.

Given its self-evident importance, to be elaborated on is Reiter's position on the nonmonotonic character of reasoning with default rules.

Let C be as in Definition 2.2.1; if C is considered to present a precise account of what it means 'to be a consequence of' for the case of reasoning with default rules, then, whether C is monotonic or not is equivalent to whether Equation 3.24 is satisfied or not

$$\text{if } \delta \in C(\Gamma) \text{ , then } \delta \in C(\Gamma') \text{ whenever } \Gamma \subseteq \Gamma' \tag{3.24}$$

In this context, as far as the conditions imposed in the formulation of C as in Definition 2.2.1, whether Equation 3.24 is satisfied or not requires some further conditions being imposed on the sets Γ and Γ'. In particular, it is necessary for Γ' not to undermine the applicability of any default rule that is applicable w.r.t. Γ. But this condition is external to Definition 2.2.1. The latter establishes that the principle of monotonicity does not follow from the conditions imposed on C, but not that C is necessarily nonmonotonic, i.e., that this principle is necessarily violated. In any case, if formulated as in Definition 2.2.1, then, C is nonmonotonic.

As a comment in passing, the nonmonotonic aspect of C may be argued to reflect the fact that, although a default rule $\pi : \rho \;/\; \chi$ may be applicable w.r.t. a set of sentences Γ, it is not necessarily the case that this default rule remains applicable w.r.t. all supersets Γ' of Γ, for some of these may contain $\neg\rho$ (a similar argument is used by Reiter in [Rei80, pp. 85–86]).

However, it is dubious whether the considerations above represent what Reiter considered to be the nonmonotonic aspect of reasoning with default rules. More precisely, in [Rei80, p. 94], in relation to the work of McDermott et al. in [MD80], Reiter mentions that:

> Instead our position is that the purpose of default reasoning [i.e., of reasoning with default rules] is to determine *one* consistent set of beliefs about a world, i.e., *one* extension, and to reason within this extension until such time as the evidence at hand forces a revision of those beliefs, in which case a switch to a new extension may be called for. [emphasis in the original]

What is problematic about such a conception of the nonmonotonic aspect of reasoning with default rules is its vagueness. The same can be said in a logic independent fashion of any formalized theory. Moreover, in and of itself, Reiter's position speaks less about the nonmonotonic aspect of the logic in which a formalized theory is formulated and more to the adequacy of a theory as a scientific explanation of a given problem at hand. But this is altogether a radically different point (q.v. [Pop72]).

Regarding the nonmonotonic character of reasoning with default rules, at another point in his 1980 seminal article, q.v. [Rei80, p. 91], Reiter mentions that:

> It is easy to see that in general default theories [i.e., tuples $\langle \mathrm{A}, \Delta \rangle$ where A is a set of sentences and Δ is a set of default rules] are nonmonotonic. By this we mean that if $\langle \mathrm{A}, \Delta \rangle$ is a default theory with an extension E, A' is a set of sentences and Δ' is a set of default rules, then, $\langle \mathrm{A} \cup \mathrm{A}', \Delta \cup \Delta' \rangle$ may have no extension E' such that $E \subseteq E'$.

What is problematic in this latter case is that if C is to be taken as a consequence operator, then, according to Definition 2.2.1, the tuples $\langle \mathrm{A}, \Delta \rangle$ and $\langle \mathrm{A} \cup \mathrm{A}', \Delta \cup \Delta' \rangle$ define two different instances of C, one w.r.t. $\langle \mathrm{A}, \Delta \rangle$ and the other w.r.t. $\langle \mathrm{A} \cup \mathrm{A}', \Delta \cup \Delta' \rangle$, and hence two different logical systems. While it may be readily seen that it is not necessarily the case that all default consequences of one instance of C are default consequences of the other instance of C, and *vice-versa*, this speaks less to the nonmonotonic aspect of C and more to a comparison of what is (or is not) the case in different logical systems. But again this is altogether a radically different point.

Following from Reiter's latter position on the nonmonotonic character of reasoning with default rules, it is possible to hold the view that C is not intended to capture a somewhat sensible formulation of the concept of 'being a consequence of' in point; since whatever such a concept is shall be formulated accounting for the fact that the premisses of a default consequence are comprised of a set of sentences and a set of default rules. This point is set aside momentarily and returned to later on.

Given the observations made above, it is dubious whether C can be taken as providing a somewhat sensible formulation of the concept of 'begin a consequence of' for the case of reasoning with default rules. In response, elaborated in detail in Section Section 4.3, the proposal here is to revise the structure, the nonmonotonic aspect, and the set of properties that ought to be satisfied by a consequence relation for reasoning with default rules.

Alternative Standpoints on the Concept of 'Being a Consequence of' for Reasoning with Default Rules

The first systematic study of the concept of 'being a consequence of' for the case of reasoning with default rules corresponds to the following series of works: [Mak89], [Mak94], [Mak05b], [Mak05a] by Makinson. In brief, Makinson approaches this study from the point of view of a syntactical presentation of an abstract consequence operator. In doing so, he provides two different accounts of the concept of 'being a consequence of' for the case of reasoning with default rules, the first of which corresponds to Definition 3.3.1.

As a minor comment in passing, an interesting feature of Definition 3.3.1 is Equation 3.28. In [GG05, pp. 243–244], Makinson explains that:

> [Equation 3.27] tells us that if everything is in order when we compare the part of the extension so far constructed with the justifications of the current [default] rule, then we are forced to try to apply the [default] rule. We cannot back out. But, having got so far, if [Equation 3.28] is not satisfied, then we are condemned – we are shunted into [Equation 3.30] and the construction aborts.

Therefore, in contrast with the standard definition of a consequence operator, the definition of C is partial, there are inputs for which it is not defined.

It is not difficult to see that if formulated as in Definition 3.3.1, then, C needs not satisfy the principle of monotoncity. In addition, on the positive side, on those inputs for which it is defined, it can be proven both that C is a closure operator and that it satisfies the principle of inclusion. On the negative side, however, C fails to satisfy both the principle of cut and that of cumulativity.

Definition 3.3.1 (Makinson)**.** Let Δ be a set of default rules and ω be a well-ordering of the elements of Δ indexed by the natural numbers. In addition, let C be an operator such that for any set of sentences Γ:

$$C(\Gamma) \stackrel{\text{def}}{=} \bigcup_{i \geq 0} \Gamma_i$$

where:

$$\Gamma_0 \text{ is defined as } Cn(\Gamma) \text{ and} \tag{3.25}$$

$$\Gamma_{i+1} \text{ is defined as follows:} \tag{3.26}$$

$$\text{If there is } \frac{\pi : \rho}{\chi} \in \Delta \text{ s.t. } \frac{\pi : \rho}{\chi} \notin \Delta_i^e,\ \pi \in \Gamma_i,\ \text{and } \neg\rho \notin \Gamma_i \tag{3.27}$$

then:

Let $\frac{\pi : \rho}{\chi}$ be the first such default rule according to ω

$$\text{If for every } \rho' \in (P(\Delta_i^e) \cup \{\rho\}),\ \neg\rho' \notin Cn(\Gamma_i \cup \{\chi\}) \tag{3.28}$$

then:

$$\Gamma_{i+1} \stackrel{\text{def}}{=} Cn(\Gamma_i \cup \{\chi\}) \tag{3.29}$$

else:

$$\Gamma_{i+1} \text{ is undefined (and so is } C(\Gamma)) \tag{3.30}$$

else:

$$\Gamma_{i+1} \stackrel{\text{def}}{=} \Gamma_i \tag{3.31}$$

$$\Delta_0^e \text{ is defined as } \emptyset \text{ and} \tag{3.32}$$

Δ_{i+1}^e is defined as:

$$\Delta_i \text{ if Equation 3.27 is not satisfied} \tag{3.33}$$

$$\Delta_i \cup \left\{\frac{\pi : \rho}{\chi}\right\} \text{ if Equations 3.27 and 3.28 are both satisfied} \tag{3.34}$$

$$\text{and is undefined otherwise} \tag{3.35}$$

A sentence δ is a default consequence of a set of sentences Γ iff $\delta \in C(\Gamma)$.

It is commented by Makinson in [Mak05b] that, given a preferred well-ordering of the default rules in Δ, there may be no need to go any further. If such is the case, then, for the preferred well-ordering of Δ, Definition 3.3.1 arguably presents a somewhat sensible notion of the concept of 'being a default consequence of' for the case of reasoning with default rules. Evidently, this standpoint places the logical burden of proof on the preferred well-ordering of the default rules in Δ.

In the absence of such well-ordering, it seems reasonable to assume a *skeptical* standpoint and to define the concept of 'being a consequence of' for the case of reasoning with default rules in an order independent way. This is made precise in Definition 3.3.2.

Definition 3.3.2 (Makinson)**.** Let Δ be a set of default rules and C be an operator such that for any set of sentences Γ:

$$C(\Gamma) \overset{\text{def}}{=} \bigcap_{\omega \in \Omega} C_\omega(\Gamma)$$

where: (i) Ω is the set of all possible well-oderings of Δ indexed by the natural numbers; and (ii) each C_ω is as in Definition 3.3.1 w.r.t. ω. A sentence δ is a default consequence of a set of sentences Γ iff $\delta \in C(\Gamma)$.

In essence, in comparison to Definition 3.3.1, Definition 3.3.2 avoids placing the logical burden of proof on the well-ordering of the underlying set Δ of default rules by defining a default consequence of a set of sentences w.r.t. what is the case for all possible well-orderings of Δ. Similarly to what is the case in Definition 3.3.1, in Definition 3.3.2, the operator C is partial; there are inputs for which it may not be defined (i.e., those inputs for which some C_ω is not defined).

Again, it is not difficult to see that, if formulated as in Definition 3.3.2, C needs not satisfy the principle of monotoncity. In addition, on the positive side, on those inputs for which it is defined, C is a closure operator and it satisfies the principle of inclusion. On the negative side however, C satisfies neither the principle of cut nor that of cumulativity.

Whether formulated as in Definition 3.3.1 or as in Definition 3.3.2, from the standpoint of a syntactical presentation of a consequence operator, a case can be made that C is formally better behaved than if formulated as in Definition 2.2.1. If looked at in this light, it may be considered that any of these definitions present a somewhat sensible account of the concept of 'being a consequence of' for the case of reasoning with default rules.

However, and taking up a previously left loose end, Definitions 3.3.1 and 3.3.2 fall short of capturing the nonmonotonic aspect of reasoning with default rules if, as hinted by Reiter in [Rei80, p. 91], nonmonotonicity is to be understood in light of the fact that, for sets $\Gamma \subseteq \Gamma'$ of sentences and sets $\Delta \subseteq \Delta'$ of default rules, it may be the case for a default consequence of Γ and Δ not to be a default consequence of Γ' and Δ'. The reason being that, w.r.t. Definitions 3.3.1 and 3.3.2, these sets formulate different consequence operators, and hence different logical systems.

The consideration here is that if Reiter's standpoint on the nonmonotonic aspect of reasoning with default rules is to be accounted for, then, a rather different formalization of the concept of 'being a consequence of' for the case of reasoning with default rules is needed. Essentially, instead of a set of sentences, the idea is for the premisses of a default consequence to be a structure comprised of a set of sentences and a set of default rules. More precisely, if δ is a sentence, Γ a set of sentences, and Δ a set of default rules, then,

$$\delta \text{ is a default consequence of } \langle \Gamma, \Delta \rangle \text{ iff } \langle \Gamma, \Delta \rangle \mid\!\sim \delta \tag{3.36}$$

where: $\mid\!\sim$ is the entailment relation that captures the concept of 'being a consequence of' for the case of reasoning with default rules.

To be noted at this point, unless taken as primitive, Equation 3.36 places the logical burden of proof on making precise the entailment relation capturing the concept of 'being a consequence of' for the case of reasoning with default rules. As is typically done in logical studies, said logical burden of proof may be discharged in terms of an axiomatic and inferential basis, a proof-calculus, or alternatively in terms of what is the case with respect to a given formulation of the concept of an extension. The literature on reasoning with default rules seemingly inclines the balance in favor of the latter approach.

A brief parenthetical remark is in order here before continuing. In hindsight, a case could be made that Reiter himself proposed a possible way of making precise the entailment relation in Equation 3.36 in his 1980 seminal paper; in particular, when discussing the notion of a proof theory for normal default rules. However, from the point of view of Hilbert style proof calculi, natural deduction systems, tableau methods, Gentzen style sequent systems, or resolution systems, Reiter's notion of a proof theory is not entirely transparent and only difficultly expressed in precise terms. In that respect, [BO97] and [BO02] prove to be a more tractable reading.

Resuming from this parenthetical remark, if the entailment relation in Equation 3.36 is taken to be formulated in terms of what is the case w.r.t. the extensions of a set of sentences and a set of default rules, then, to be noted is that different definitions of an extension yield different instances of said entailment relation, and hence different logical systems. Notwithstanding, given a notion of an extension, different sets of sentences and of default rules have different extensions within the same logical system.

The way in which the entailment relation in Equation 3.36 may be formulated w.r.t. a given definition of an extension can be roughly classified as being either *selective*, *credulous*, or *skeptical*. Selectively: the entailment relation in Equation 3.36 is formulated in terms of what follows, relative to an underlying consequence operator, such as that of classical propositional logic, from a particular extension of the premiss structure in question. Credulously: the entailment relation in Equation 3.36 is formulated in terms of what follows, again relative to an underlying consequence operator, from some extension of the premiss structure in question. Skeptically: the entailment relation in Equation 3.36 is formulated in terms of what follows, once more relative to an underlying consequence operator, from all extensions of the premiss structure in question.

As a brief comment in passing, it is worth noting that the selective standpoint places the logical burden of proof on making precise an appropriate criterion for selecting one out of the possibly many, or none, extensions of the premiss structure in question. In the absence of such selection criterion, it seems natural to better consider a skeptical formulation of the entailment relation in Equation 3.36.

Against this background, the rest of this section is dedicated to commenting on some of the challenges posed by formalizing the concept of 'being a consequence of' for reasoning with default rules as in Equation 3.36.

To begin with, the principle of monotonicity for the entailment relation in Equation 3.36 can be expressed as:

$$\text{if } \langle \Gamma, \Delta \rangle \mathrel{|\!\sim} \delta \text{ and } \langle \Gamma, \Delta \rangle \subseteq \langle \Gamma', \Delta' \rangle\text{, then, } \langle \Gamma', \Delta' \rangle \mathrel{|\!\sim} \delta \tag{3.37}$$

where: δ is a sentence, Γ and Γ' are sets of sentences, and Δ and Δ' are sets of default rules.

Non-monotonicity as considered by Reiter in [Rei80, p. 91] is equivalent to Equation 3.37 not being satisfied by the entailment relation in Equation 3.36.

It is not difficult to see that, if the principle of monotonicity is taken as in Equation 3.37, then, whether or not it is satisfied by the entailment relation in Equation 3.36 depends on the definition of an extension in question and on the properties of the underlying consequence operator. Fulfilling Reiter's nonmonotonicity *desideratum*, all best-known definitions of an extension, and in particular those dealt with in Section 3.2, yield an entailment relation that does not satisfy the principle of monotonicity as expressed in Equation 3.37, i.e., an entailment relation $\mid\!\sim$ that is nonmonotonic, if $\mid\!\sim$ is taken skeptically.

At the same time, Equation 3.36 poses a rather new challenge on how the principles of *inclusion*, *cut*, and *cumulativity* are to be expressed. Particularly difficult is how a default consequence is to be accumulated in its premiss structure: If δ is a default consequence of $\langle\Gamma,\Delta\rangle$, i.e., if $\langle\Gamma,\Delta\rangle \mid\!\sim \delta$, then, what is the result of adding δ to $\langle\Gamma,\Delta\rangle$?

Two straightforward answers to the question above are: (i) a default consequence is to be accumulated in the set of sentences of its premiss structure, i.e., if $\langle\Gamma,\Delta\rangle \mid\!\sim \delta$, then $\langle\Gamma\cup\{\delta\},\Delta\rangle$ is the result of accumulating δ into $\langle\Gamma,\Delta\rangle$; (ii) a default consequence is to be accumulated in the set of default rules of its premiss structure, i.e., if $\langle\Gamma,\Delta\rangle \mid\!\sim \delta$, then $\langle\Gamma,\Delta\cup\{\delta\}\rangle$ is the result of accumulating δ into $\langle\Gamma,\Delta\rangle$.

Relative to (i), the principle of cumulativity, and similarly that of cut, for the entailment relation in Equation 3.36 can be expressed as:

$$\text{if } \langle\Gamma,\Delta\rangle \mid\!\sim \delta \text{ and } \langle\Gamma,\Delta\rangle \mid\!\sim \delta', \text{ then, } \langle\Gamma\cup\{\delta\},\Delta\rangle \mid\!\sim \delta' \tag{3.38}$$

where: δ and δ' are sentences, Γ is a set of sentences, and Δ is a set of default rules.

Largely due to the works of Makinson, it is a well-known result that, if an extension is taken to be as defined by Reiter, Łukaszewicz, and Delgrande (q.v. Definitions 3.2.1, 3.2.3, and 3.2.4, respectively), the corresponding instances of the skeptical formulation of the entailment relation in Equation 3.36 do not satisfy the principle of cumulativity as expressed in Equation 3.38 (neither do they satisfy the principle of cut.)

The work of Brewka in [Bre91] is the first in addressing more or less successfully the principle of cumulativity as expressed in Equation 3.38.

More precisely, in [Bre91], Brewka makes a proposal for: (a) reasoning with default rules to be formulated based on the use of assertions, i.e., tuples comprised of a sentence and a set of sentences, rather than sentences; and (b) a definition of an extension accounting for the use of assertions rather than sentences. On this basis, following from [Bre91, p. 191, Proposition 2.13], if formulated credulously, it is possible to obtain an instance of the entailment relation in Equation 3.36 that satisfies the principle of cumulativity as expressed in Equation 3.38. However, as is mentioned by Antoniou et al. in [AW07, p. 533] that, due to the additional conceptual load imposed by the use of assertions rather than sentences, it is doubtful whether Brewka's take on cumulativity is the right way to go. Such also seems to be the case from a selective and a skeptical standpoint. Selectively: for, even having a precise definition of the selection criterion, it is possible for an extension other than that from which a default consequence is taken to follow to be selected w.r.t. the premiss structure resulting from accumulating this default consequence in its premiss structure, case in which there is no guarantee that the default consequence in question remains such. Skeptically: for if extensions are taken as defined by Brewka, due to the use of assertions rather than sentences, it is not at all clear what is to be taken to follow from the intersection of the class of extensions associated to a premiss structure.

To a certain extent, the work of Antonelli in [Ant05] may be seen as addressing some of these observations. More precisely, in [Ant05], based on the notion of a general extension (q.v. Definition 3.2.5), Antonelli presents a skeptical formulation of an entailment relation that satisfies the principle of cumulativity as expressed in Equation 3.38. However, regarding the appropriateness of Antonelli's take on cumulativity, to be accounted for is the three-valued nature of a general extension and its effect on the skeptical formulation of the entailment relation in Equation 3.36 (q.v. Section 3.2).

In any case, and more generally, it is possible to argue that Equation 3.38 is altogether misleading and inherently incorrect since a default consequence cannot simply be accumulated in the set of sentences of a premiss structure. By way of justification, in all best-known definitions of an extension, and in particular those dealt with in Section 3.2, the set of sentences of a premiss structure occupies a distinguished place in the definition of an extension, since the former is always included in the latter. This ascribes a certain definite status to the set of sentences of a premiss structure. In that respect, it may be asked: Can such a definite status be ascribed to any default consequence?

In light of the latter question, and taking up (ii) above, it is seemingly better for said default consequence to be accumulated into the set of default rules of its premiss structure (since it may be argued to respect the defeasible aspect of default consequence). In this case, the principle of cumulativity, and similarly that of cut, for the entailment relation in Equation 3.36 can be expressed as:

$$\text{if } \langle \Gamma, \Delta \rangle \mid\!\sim \delta \text{ and } \langle \Gamma, \Delta \rangle \mid\!\sim \delta', \text{ then, } \langle \Gamma, \Delta \cup \{\delta\} \rangle \mid\!\sim \delta' \tag{3.39}$$

where: δ and δ' are sentences, Γ is a set of sentences, and Δ is a set of default rules.

Regarding Equation 3.39, given that δ is a sentence and Δ is a set of default rules, a first obstacle to overcome is what default rule better captures the fact that δ is a default consequence of $\langle \Gamma, \Delta \rangle$. Schaub addressed this issue in [Sch91]. More precisely, from a credulous perspective, in [Sch91, p. 308, Definition 3.3], Schaub suggests that, if δ is a default consequence of $\langle \Gamma, \Delta \rangle$, then, the default rule that best captures this fact is $\frac{:\Psi}{\delta}$, where Ψ is the set of all the justifications and consequents of all default rules that yield the constrained extension (q.v. Definition 3.2.4) from which δ follows. On this basis, it is possible to obtain an instance of the entailment relation in Equation 3.36 that satisfies the principle of cumulativity as expressed in Equation 3.39.

To be noted, Schaub's take on cumulativity is not easily extended to a selective or a skeptical formulation of the entailment relation in Equation 3.36. Selectively: for, even having a precise definition of the selection criterion, it is possible for a constrained extension other than that from which a default consequence is taken to follow to be selected w.r.t. the premiss structure resulting from accumulating this default consequence in its premiss structure, case in which there is no guarantee that the default consequence in question remains such. Skeptically: for it is not at all clear what the set Ψ in $\frac{:\Psi}{\delta}$ looks like when δ is taken to follow from the intersection of the class of constrained extensions associated to its premiss structure.

Perhaps a more problematic aspect of Schaub's take on cumulativity is its reliance on the particular set of default rules that yield the constrained extension from which the default consequence in question follows. More precisely, Schaub's take on cumulativity may be argued to suffer from being specific to an instance of the entailment relation for reasoning with default rules and not a principle inherent to any instance of such an entailment relation.

In any case, both (i) and (ii) above, and correspondingly Equations 3.38 and 3.39, are in their own right interesting formulations of the principle of cumulativity for the entailment relation in Equation 3.36.

To be made clear at this point is the following observation. If a default consequence is to be formulated as in Equation 3.36, then, both the postulate that a default is a rule of inference and that of an extension being a theory-like object are to be abandoned. Since it forms part of the premisses of a default consequence, a default rule cannot simply be a rule of inference, for the latter is a logic defining and not a premiss-like object. In addition, since it determines what a default consequence is, an extension cannot simply be a theory-like object, for it is an extension that defines what a theory is in the first place.

When taken together, the observations made in this section raise the following question: What counts as a somewhat sensible formulation of the concept of 'being a consequence of' for the case of reasoning with default rules? Section 4.3 addresses this question by placing the notion of an entailment relation for reasoning with default rules in the foreground.

3.4 Discussion

This chapter presents three main arguments whose conclusions weaken the position sustaining that the basic concepts on which Reiter bases reasoning with default rules formulate a logical system. As a result, it is dubious whether these concepts yield *a logic for reasoning with default rules* as indicated in the title of Reiter's seminal 1980 article. In that respect, insofar as logical well-foundedness is concerned, a new basis for interpreting the main elements of reasoning with default rules is necessary if reasoning with default rules is to be regarded as a logical approach to nonmonotonic reasoning.

In summary, the first of said conclusions corresponds to that of Section 3.1. Essentially, this section establishes that a default rule is *not* a defeasible rule of inference, the reason being that the former involves none of what may be seen as some of the basic features of the latter. Most prominently, there needs to be no logical relationship whatsoever between the prerequisite, the justification, and the consequent that form a default rule (in general, these may be any three arbitrary sentences). In that respect, it is dubious whether a default rule is anything like defeasible rule of inference, therefore challenging why it must be treated as such.

As a comment in passing, regarding the conclusion of Section 3.1, it is possible to make a case that a default rule is not to be understood as formalizing a rule of inference of a defeasible kind, but simply as an indication of the structure of such a rule of inference. This position may, for instance, be defended from the perspective of the abstract study of a consequence operation (cf. [Wój88]). If seen in this light, the point that only proper formulations of a default rule are in correspondence with a somewhat sensible notion of a defeasible rule of inference goes away – for this may be argued to be presupposed. However, this position is weakened in light of what is discussed in Sections 3.2 and 3.3, the reason being that neither does this abstract view of a default rule yield a sensible notion of a theory, nor does it appear to be logic defining.

In turn, the second of the conclusions mentioned above corresponds to that of Section 3.2. Essentially, contrary to what Reiter postulates is its intuitive understanding, this section establishes that an extension is *not* a set of sentences closed under the application of an underlying set of default rules, i.e., a theory-like object in a default sense, for there are such sets that are not extensions accordingly to Definition 3.2.1.

Lastly, the third of the conclusions mentioned above corresponds to that of Section 3.3. Essentially, this section raises the question: What counts as a somewhat sensible formulation of the concept of 'being a consequence of' for the case of reasoning with default rules?

To be made clear at this point is that, although Sections 3.1 to 3.3 present what are here viewed as some shortcomings of Reiter's approach to reasoning with default rules, it is also the consideration here that this approach in no way loses its value. Quite the contrary, Reiter contributions to the subject of reasoning with default rules are here regarded as clarifying and as presenting some challenges to what evidently is a difficult state of affairs: to provide a precise account of what does it mean to reason with default rules from a logical perspective. This is the point of departure of the ideas developed in Chapter 4.

Chapter 4

A Logical Basis for Reasoning with Default Rules

In light of the main points presented in Chapter 3, this chapter proposes three mainstays catering for a more tenable logical account of reasoning with default rules. These are:

(i) The consideration that not all premissal assertions, nor their formal counterparts, i.e., the sentences of the formal language that they are indicated by, are accorded the same status. The standpoint assumed here is that whereas some premissal assertions are accorded a *definite* status, others are accorded a less than definite or *tentative* status, such is also the case for their formal counterparts. On this basis, the proposal here is for default rules to be treated as the formal counterpart of premissal assertions that are seen as being tentative. By way of justification, the consideration here is that treating default rules as premiss-like objects stand on better logical grounds that treating them as rules of inference. What tentative assertions are, how they differ from assertions that are taken definitely, and their formalization as default rules is elaborated on in Section 4.1.

(ii) The consideration that, when taken as premisses, a set of definite assertions and a set of tentative assertions formulate a structured set. This structured set is formalized by a tuple comprised of a set of sentences, standing for what is held definitely, and a set of default rules, standing for what is held tentatively. On this basis, the proposal here is for extensions to be formally treated as interpretation structures of a syntactical kind. By way of justification, the consideration here is that classes of extensions may be seen as a possible logical way of interpreting the notion of a structured set of premisses. These ideas are the subject matter of Section 4.2.

(iii) The consideration that reasoning with default rules can be formulated in a more logically sensible manner, i.e., more in line with the (somewhat) traditional, or standard, presentation of a logical system. On this basis, the proposal here is to bring into the foreground the notion of a logical system for reasoning with default rules. By way of justification, the consideration here is that such a notion of a logical system for reasoning with default rules better accommodates for a logical study of the concept of 'being a consequence of' for reasoning with default rules. These ideas are the subject matter of Section 4.3.

By elaborating on (i), (ii), and (iii), this chapter presents what are regarded as being more adequate logical grounds on which to understand reasoning with default rules; thus responding to the observations made in Chapter 3.

More often than not, logical concepts gain in clarity when explained in formal terms. In that respect, unless it is stated otherwise, this chapter assumes that: $\mathscr{L}$ and Cn indicate the language and the consequence operator of classical propositional logic (q.v. Definitions 2.1.2 and 2.1.5, respectively). An arbitrary set of sentences is assumed to be a subset of $\mathscr{L}$. In addition, $\mathscr{D}$ is assumed to be the set of all default rules definable on $\mathscr{L}$ (q.v. Definition 3.1.1). Similarly, to what is the case for sentences, an arbitrary set of default rules is assumed to be a subset of $\mathscr{D}$.

4.1 Default Rules as Premiss-like Objects

Section 3.1 presented a number of observations which weakened the position asserting that default rules are defeasible rules of inference. Motivated by these observations, this section proposes an alternative view of what default rules are.

In brief, the standpoint that is assumed and defended here is that default rules stand on better logical grounds as the formal counterpart of assertions that are assumed tentatively.

As a comment in passing, a word of clarification is in order to make more comprehensible the exposition that follows. This section does not pursue a thorough philosophical account of the various statuses that premissal assertions may have. Instead, its purpose is to make a distinction between premissal assertions that are ascribed a *definite*, or categorical, status, and those that are ascribed a *tentative* status. It is just such a demarcation and its distinguishing elements which will be at issue.

Without further ado, there are two widely accepted postulates that underpin most logical analyses of the process of drawing consequences from premisses. The first postulate, labeled (1), is that upon acceptance, either by conviction, position, or fiat, premisses set the context from which consequences can be drawn. While getting premisses to be accepted is a notoriously difficult task, it is not until after they are accepted that a discussion of what can, and cannot, be concluded from them can begin. The second postulate, labeled (2), more properly applicable to classical deductive studies, is that, upon acceptance, premisses cannot be challenged without vitiating the consequences that can be drawn from them from any logical interest. While not as self-evident as the previous one, this second postulate results from the principle of *ex falso quodlibet*, a principle that typically underpins classical deductive disciplines.

These postulates are so well recognized by logicians that they are seldom made explicit. However, as obvious as they may be argued to be, their implicitness has caused no insignificant degree of confusion. This confusion is particularly visible in relation to (2), since its acceptance accords a *definite* status to the assertions that are involved in a set of premisses, an attribution that is sometimes unwelcome. The latter point is elaborated on below.

That in a classical deductive sense the assertions that form a set of premisses are ascribed a definite status does not need of a lengthy discussion, it can be succinctly exemplified by the pedestal on which the axioms of a formalized theory are held. Typical examples of the latter are listed below.

Definite Assertion
'The number zero is the smallest of the natural numbers'.
'The shortest distance between two points is a straight line'.
'The speed of light is constant'.

Assertions such as those above cannot be challenged, e.g., by asserting that 'There is a natural number that is smaller than zero', without vitiating further reasoning from any logical interest.

While there may be no caveats in ascribing a definite status to premissal assertions if these are of an abstract nature, e.g., if they refer to natural numbers, geometrical objects, laws of physics, or the like, if one considers premisses that involve assertions of a more ordinary nature, though possible in some cases, according them a definite status is most likely either an impossibility or a clear appeal to naiveness.

To make this previous point clear, the consideration here is that only dubiously can the following assertions be taken as having a definite status upon accepting them as premisses.

Tentative Assertion
'Oswald shot Kennedy'.
'Expert E is to be trusted'.
'Contract C is best signed'.
'Event E caused a failure'.
'All observed ravens are black'.
'Hypothesis H is the case'.

Several other examples of assertions like those above can be found in [GW11], [Wal02], their cited works, or can be concocted at will.

Assertions such as these later ones are most likely to be challenged, e.g., by asserting that 'There is evidence that Oswald did not shoot Kennedy'. This does not necessarily vitiate further reasoning from logical interest, it simply means that premisses may be undermined, case in which, the consequences that were drawn from them may be open to revision or annulment (at least in principle). In light of such an observation, the contention here is that, if at all accepted as premisses, assertions of this latter kind are accorded a less than definite, i.e., *tentative*, status.

In essence, the idea is that a tentative assertion differs from its definite counterpart in that, given its tentative formulation, it may be challenged, i.e., *rebutted*, in a way that the former does not. To illustrate the notion of a tentative assertion's rebuttal, potential challenges, listed on the column on the right, to the tentative assertions presented above, listed on the column on the left, are given below.

Tentative Assertion	Rebuttal
‘Oswald shot Kennedy’	‘There is evidence to the contrary’.
‘Expert E is to be trusted’	‘E is proven incompetent’.
‘Contract C is best signed’	‘C has hidden motives’.
‘Event E caused a failure’	‘E is only casual to the failure’.
‘All observed ravens are black’	‘The observation is biased’.
‘Hypothesis H is the case’	‘H is disproved’.

The following point introduces the last element of the notion of a tentative assertion in question here. Contrary to what is the case for a definite assertion, typically taken to be obvious or self-sustaining, the consideration here is that standing behind a tentative assertion there are some *grounds* for its formulation. To illustrate the latter element of a tentative assertion, potential grounds, listed on the leftmost column, for the tentative assertions just presented, listed on the column in the center, are given below.

Grounds	Tentative Assertion	Rebuttal
‘There is evidence pointing to Oswald having shot Kennedy’	‘Oswald shot Kennedy’	‘There is evidence to the contrary’.
‘E has presented some credentials attesting to his expertise’	‘Expert E is to be trusted’	‘E is proven incompetent’.
‘Contract C is beneficial’	‘C is best signed’	‘C has hidden motives’.
‘Operations were normal until E occurred’	‘Event E caused a failure’	‘E is only casual to the failure’.
‘The sample of observed ravens is significant w.r.t. their population’	‘All observed ravens are black’	‘The observation is biased’.
‘H was strenuously tested’	‘Hypothesis H is the case’	‘H is disproved’.

As a minor comment in passing, to be noted is that, in principle, and as the previous examples show, the grounds on which a tentative assertion is formulated and this tentative assertion's rebuttal may be different assertions, which, in turn, may be altogether different from what is held tentatively.

Against this background, the general structure of an assertion that is held tentative is that of a *tuple* comprised of: (i) what is held tentatively, i.e., the *tentative assertion*, (ii) the *grounds* on which this tentative assertion is formulated, and (iii) the way in which this tentative assertion may be *rebutted*.† In a sense, a tentative assertion's grounds and its rebuttal indicate different conditions which, if unfulfilled or established, respectively, set aside what is held tentatively.‡

While there may be potentially many other elements that could be considered to be present in the ordinary understanding of an assertion that is held tentatively, those that have been identified above will suffice for the purposes of this thesis.

It is more or less direct how a default rule may be seen as formally capturing the structure of a tentative assertion. Given any such tentative assertion, (i) the consequent of a default rule stands for what is held tentatively, i.e., the tentative assertion itself, (ii) its prerequisite stands for the grounds on which this tentative assertion is formulated, (iii) and its justification stands for this tentative assertion's rebuttal.

Default rules being premiss-like objects are a direct result of the way in which they are formally treated in Sections 4.2 and 4.3.

†A tentative assertion's rebuttal and its grounds are somewhat in correspondence with Toulmin's notion of a rebuttal and a data, respectively (q.v. [Tou03]).

‡This preliminary distinction between a tentative assertion's rebuttal and its footing, as its implications, will be made precise resorting to conditions of provability and of non-provability in Section 4.2.

4.2 Extensions as Interpretation Structures

Regarding the notion of a tentative assertion presented in Section 4.1, an obvious question to be asked is: How are tentative assertions to be interpreted? Namely: What are the conditions under which a tentative assertion can be accepted as being the case?

Definition 4.2.1 is a first step towards a possible answer for this question.

Definition 4.2.1. Let Γ be a set of sentences; a set of default rules Δ is *tentative* (*w.r.t.* Γ) iff for all $\frac{\pi:\rho}{\chi} \in \Delta$, (i) $\pi \in Cn(\Gamma)$ and (ii) $\rho \notin Cn(\Gamma \cup X(\Delta))$.

The most basic goal of Definition 4.2.1 is that of correlating the somewhat ordinary understanding of a tentative assertion and its formalization as a default rule. More precisely, in Section 4.1, a brief point was made as to the grounds on which a tentative assertion is formulated and this tentative assertion's rebuttal being different conditions which, if unfulfilled or established, respectively, force the setting aside of what is held tentatively. If stated in the form of a logical contraposition, a tentative assertion may be be accepted as being the case if the grounds on which it is formulated are established as such and if its rebuttal is not established. On this basis, conditions (i) and (ii) in Definition 4.2.1 make clear what does it mean for the grounds on which a tentative assertion is formulated to be established as such and for a tentative assertion's rebuttal not to be established, i.e., whether a tentative assertion can be accepted as being the case or not corresponds to whether conditions (i) and (ii) hold or not, respectively.

To be noted at this point is that, in an ordinary sense, it is possible that in establishing the grounds on which a set of tentative assertions is formulated there may be a need to appeal to other assertions. In Definition 4.2.1, these other assertions are denoted by the sentences in Γ. There is a potential danger here: What if some of these assertions being appealed to are also tentative assertions? There may be a need to bring forth some new assertions in order to establish the grounds in which the latter tentative assertions are formulated, without which they cannot be accepted as being the case, and neither can the tentative assertion in question. But for any of the latter assertions the same difficulty may now be presented. To break this potentially infinite regression, at some point, assertions of a rather different kind must be accepted as being the case. The consideration here is that assertions that are accorded a definite status fulfill this last condition.

The previous discussion leads in a more or less natural way to Definition 4.2.2.

Definition 4.2.2. Let Φ be a set of sentences; a set Δ of default rules is *acceptable* (*w.r.t.* Φ) iff there is a chain $\mathbf{C}$ of subsets of Δ ordered by inclusion such that:

(4.1) $\emptyset \in \mathbf{C}$

(4.2) Let $\Delta' \in \mathbf{C}$ and $\delta \in \Delta \setminus \Delta'$; if $\Delta' \cup \{\delta\}$ is tentative w.r.t. $\Phi \cup X(\Delta')$,

then, $\Delta' \cup \{\delta\} \in \mathbf{C}$

(4.3) $\Delta = \bigcup_{\Delta' \in \mathbf{C}} \Delta'$

If Δ is acceptable w.r.t. Φ, then, Φ and $X(\Delta)$ are the sets of *definite* and of *tentative* sentences of $\Phi \cup X(\Delta)$, respectively.

By way of justification, Definition 4.2.2 is based on the consideration that tentative assertions are accepted as being the case w.r.t. a set of definite assertions, i.e., definite assertions, denoted by the sentences in Φ, establish a foundation on which tentative assertions can be made. Given this foundation, tentative assertions can then be accepted on the basis of what has already been accepted, indicated by the fact that the default rules in Δ form a chain $\mathbf{C}$ of tentative sets. In Definition 4.2.2, the definite assertions are denoted by the sentences in Φ and the tentative assertions that are accepted as being are denoted by the sentences in $X(\Delta)$.

The rationale underpinning Definition 4.2.2 can be generalized to accommodate arbitrary sets of default rules, not necessarily those that are acceptable as a whole. This is made precise in Definition 4.2.3.

Definition 4.2.3. Let Φ be a set of sentences and Δ be a set of default rules; the class $\mathscr{E}$ of extensions of Φ and Δ is comprised of all sets $\Phi \cup X(\Delta')$, where Δ' is a subset of Δ such that:

(4.4) Δ' is acceptable w.r.t. Φ

(4.5) for any other $\Delta'' \subseteq \Delta$ that is acceptable w.r.t. Φ, if $\Delta' \subseteq \Delta''$, then, $\Delta'' = \Delta'$

A set E of sentences is an *extension* of Φ and Δ iff $E \in \mathscr{E}$.

By way of justification, as formulated in Definition 4.2.3, an extension is based the consideration it indicates a set of assertions, some of which are taken as being definite, i.e., those indicated by the sentences in Φ, and some of which are tentative assertions that can be taken as being the case, i.e., those indicated by the sentences in $X(\Delta')$. The maximality condition of an extension is an indication that what can be accepted as being the case ought to be accepted as being the case.[†]

The following properties attest to the formal behavior of an extension as formulated in Definition 4.2.3.

Proposition 4.2.1. Let Φ be a set of sentences and Δ a set of default rules; the class $\mathcal{E}$ of all extensions of Φ and Δ is not empty. (q.e.d. in Appendix A)

Proposition 4.2.1 needs no lengthy discussion: it guarantees the existence of extensions

[†]This maximality condition on the notion of an extension bears a certain resemblance with Carnap's *Principle of Total Evidence* (q.v. [Car47, pp. 138–139]).

as objects of formal investigation. The consideration here is that, in contrast to other approaches to reasoning with default rules, in particular that proposed by Reiter in [Rei80], existence of extensions is of utmost importance.

Regarding Proposition 4.2.1, it is worth pointing out that extensions need not be unique. In an ordinary sense, the non-uniqueness of extensions may be understood in terms of tentative assertions that can be *jointly* accepted as being the case versus those tentative assertions that can only be *simultaneously but not jointly* accepted as being the case. More precisely, given an extension E of Φ and Δ, those sentences in $E \setminus \Phi$ denote tentative assertions that can be accepted as being the case jointly (since they correspond to the consequents of an acceptable subset of default rules of Δ that is maximal). Instead, given two different extensions E and E' of Φ and Δ, the tentative sentences that are in one but not in the other, and vice versa, denote tentative assertions that can be accepted as being the case simultaneously but not jointly (since if they were accepted jointly, because of maximality, they would belong to the same extension).

As a comment in passing, the distinction above between tentative assertions that can be simultaneously but not jointly accepted as being the case is perhaps best exemplified in terms of *competing hypotheses*. For instance, tentative assertions such as 'Oswald shot Kennedy' and 'Oswald did not shoot Kennedy' are, e.g., during the course of a trial, competing hypotheses, maybe indicating some sense of logical possibility. These are tentative assertions that can be 'simultaneously but not jointly' accepted as being the case, for accepting them jointly establishes whatever rebuttal conditions these tentative assertions may have. Non-uniqueness of extensions is a reflection of this distinction.

This is a good place to comment on another technical particularity of the notion of an extension. Let $\mathcal{E}$ be the class of all extensions of a set of sentences Φ and a set of default rules Δ; while every extension E in $\mathcal{E}$ is such that it includes all the sentences in Φ, it may be the case that for some default rule $\frac{\pi:\rho}{\chi}$ in Δ, there is no extension

$\Phi \cup X(\Delta')$ in $\mathcal{E}$ is such that $\frac{\pi:\rho}{\chi} \in \Delta'$. A default rule of this latter sort, referred to as *inadmissible*, indicates some sense of logical impossibility: the tentative assertion that $\frac{\pi:\rho}{\chi}$ stands for cannot be accepted, hence it is set aside. In contrast, a default rule that is not inadmissible, i.e., that is *acceptable*, indicates some sense of logical possibility: the tentative assertion this default rule stands for can be accepted as being the case.

As a comment in passing, it is worth pointing out that there being no inadmissible default rules amounts to the simultaneous acceptance of all the tentative assertions that are denoted by these default rules.

As is made precise in Proposition 4.2.2,[†] default rules cannot make other default rules inadmissible.

Proposition 4.2.2. Let Φ be a set of sentences, and $\Delta \subseteq \Delta'$ sets of default rules; for every extension E of Φ and Δ, there is an extension E' of Φ and Δ' such that $E \subseteq E'$. (q.e.d. Appendix A.)

It follows immediately from Proposition 4.2.2 that augmenting the set Φ of definite sentences is the only way in which an acceptable default rule may become inadmissible.[‡] Proposition 4.2.3 states under which conditions an enlargement of a set of definite sentences result in no acceptable default rule becoming inadmissible.

Proposition 4.2.3. Let $\Phi \subseteq \Phi'$ be sets of sentences and Δ be a set of default rules; if for all acceptable default rules $\frac{\pi:\rho}{\chi} \in \Delta$ it follows that $\rho \notin Cn(\Phi')$, then, every extension E of Φ and Δ is included in some extension E' of Φ' and Δ.

[†]Proposition 4.2.2 is dubbed the *Principle of Semimonotonicity* (q.v. [Rei80, p. 96, Theorem 3.2]).

[‡]How this can be done is more or less direct, e.g., by extending Φ with the justification of an acceptable default rule of Δ.

Proposition 4.2.4 touches on a consistency related aspect of extensions.

Proposition 4.2.4. Let Φ be a set of sentences and Δ a set of default rules; an extension E of Φ and Δ is consistent iff Φ is consistent. (q.e.d. Appendix A.)

Proposition 4.2.4 reflects the idea that default rules cannot be a source of inconsistencies.

It follows immediately from Proposition 4.2.4 that if Φ is an inconsistent set of sentences, then, no matter what set of default rules Δ is considered, Φ is the only extension of Φ and Δ. This corollary indicates that default rules are only acceptable on the basis of a consistent set of sentences.

Proposition 4.2.5 is a noteworthy characteristic of the role default rules, i.e., they act as a plausible completion of what is accepted definitely.

Proposition 4.2.5. Let Φ be a set of sentences and Δ a set of default rules; if $\langle \mathcal{L}(\Phi), Cn(\Phi) \rangle$ is a *complete* theory, q.v. Definition 2.1.8, and $\mathcal{L}(X(\Delta)) \subseteq \mathcal{L}(\Phi)$, then, for every extension E of Φ and Δ, $Cn(\Phi) = Cn(E)$.[†]

The following observation is to be made clear at this point. Given a set Φ of sentences and a set Δ of default rules, the class $\mathscr{E}$ of extensions of Φ and Δ constitutes a possible logical interpretation of Φ and Δ whenever these sets are viewed as a structured set of premisses. Namely, each extension in $\mathscr{E}$ may be viewed as a possible interpretation structure of how the tentative assertions denoted by the default rules in Δ can be accepted as being the case with respect to the set of definite assertions denoted by the sentences in Φ. If viewed in this light, the class $\mathscr{E}$ indicates all possible such interpretation structures.

[†]Recall from Definition 2.1.7 that for a set of sentences Γ, $\mathcal{L}(\Gamma)$ is its set of propositional symbols.

A brief parenthetical remark is in order here to avoid any subsequent misunderstanding. Two important points are worth noting about this proposed idea of a class of extensions associated to a structured set of premisses being a logical interpretation. First, as opposed to the more traditional logical notion of an interpretation structure, i.e., a model, an extension is a syntactical entity – formally, it may be treated as a *presentation* of the logical system on top of which reasoning with default rules is being discussed, in this case CPL. Second, instead of accounting for truth, an extension accounts for a possible logical standpoint on the way in which a definite and tentative assertions may be accepted as being the case.

The view of extensions as interpretation structures gains in strength and clarity when taken as a way of *justifying* what can be concluded from a structured set of premisses. This point is elaborated on in Section 4.3.

4.3 A Logical System for Reasoning with Default Rules

This section proposes a formulation of a logical system for reasoning with default rules that is more in line with a somewhat traditional, or standard, presentation of a logical system.

By way of introduction, a preliminary discussion on the starting point assumed here is deemed necessary. Traditionally, a logical system is concocted by putting together two primitive concepts: that of an *entailment relation*, aimed at formalizing the notion of provability (this is known as the syntactical, or deductive-theoretical, approach to the study of logical consequence), and that of a *satisfaction relation*, aimed at making precise the notion of a 'true' sentence (this is known as the semantical, or model-theoretical, approach to the study of logical consequence). On this basis, a logical system is obtained upon relating these two notions via *soundness* and *completeness* conditions. This view of a logical system is put in quite succinct terms by Meseguer in [Mes89], and it is discussed variously and at greater length in [Gab01].

Relative to the previous formulation of a logical system, to be made explicit is that, given necessary and sufficient conditions for their acceptability, models are traditionally used to determine the existence of proofs, i.e., of precise axiomatizations of the entailment relation, or the lack thereof.

While interesting in its own right, taking the latter as being the sole purpose of models falls short in providing a more conceptual analysis of the notion of logical consequence under consideration. Fortunately, and maybe a more interesting standpoint regarding their purpose, models may also be viewed as a way of *justifying* the entailment relation (as in, the entailment relation is justified by models insofar as the latter are taken to validate what is explained by the former). Such a standpoint is, for instance, described and studied in more detail in [Asm08]. The contention here is that, for the case of reasoning with default rules, at a conceptual level, extensions stand somewhat in analogy with the view of models as a way of justifying entailment. A point can be made that classes of extensions serve as way of justifying an entailment relation for reasoning with default rules, with different formulations of an extension serving as a different justification for this entailment relation.

The formulation of the notion of a logical system for reasoning with default rules proposed in Definition 4.3.1 rests on these preliminary observations.

The most basic goal of the notion of a logical system $\mathfrak{L}$ for reasoning with default rules as formulated in Definition 4.3.1 is that of bringing the notion of an entailment relation for reasoning with default rules, and consequently that of an extension, into the foreground.

Definition 4.3.1. A logical system $\mathfrak{L}$ for reasoning with default rules is a tuple $\langle \mathscr{L}, \mathscr{D}, \mathscr{P}, \mathrel{|\!\!\sim}, \mathscr{E}, Cn \rangle$ where:

– $\mathscr{L}$ is the standard propositional language (q.v. Definition 2.1.2). A sentence is a member of $\mathscr{L}$. The set $\mathscr{L}$ is $\mathfrak{L}$'s underlying language.

– $\mathscr{D}$ is the set of all default rules defined on $\mathscr{L}$, i.e., the set of all tuples $\frac{\pi:\rho}{\chi}$ s.t. $\{\pi, \rho, \chi\}$ is a set of sentences of $\mathscr{L}$. A default rule is a member of $\mathscr{D}$. The set $\mathscr{D}$ is $\mathfrak{L}$'s underlying set of default rules.

– $\mathscr{P}$ is the set of all tuples $\langle \Phi, \Delta \rangle$ where Φ is a set of sentences and Δ is a set of default rules. The set $\mathscr{P}$ is $\mathfrak{L}$'s underlying set of premiss structures.

– $\mathrel{|\!\!\sim} \subseteq \mathscr{P} \times \mathscr{L}$. A sentence σ is a default consequence of a premiss structure $\langle \Phi, \Delta \rangle$ iff $\langle \Phi, \Delta \rangle \mathrel{|\!\!\sim} \sigma$. The relation $\mathrel{|\!\!\sim}$ is $\mathfrak{L}$'s underlying entailment relation, or, alternatively, $\mathfrak{L}$'s default entailment relation.

– $\mathscr{E} \subseteq \mathscr{P} \times \wp(\wp(\mathscr{L}))$ is a total mapping. $\mathscr{E}$ associates to every $\langle \Phi, \Delta \rangle \in \mathscr{P}$ the class of all extensions of Φ and Δ, denoted by $\mathscr{E}(\langle \Phi, \Delta \rangle)$. An extension of Φ and Δ is a member of $\mathscr{E}(\langle \Phi, \Delta \rangle)$. $\mathscr{E}$ is $\mathfrak{L}$'s underlying formulation of an extension.

– Cn is the consequence operator of CPL (q.v. Definition 2.1.5). Cn is $\mathfrak{L}$'s underlying consequence operation (sometimes referred to as $\mathfrak{L}$'s underlying logic.)

– The following correctness condition is satisfied: for any premiss structure $\langle \Phi, \Delta \rangle$ and sentence σ, it follows that

$$\langle \Phi, \Delta \rangle \mathrel{|\!\!\sim} \sigma \text{ iff } \sigma \in Cn(E) \text{for every } E \in \mathscr{E}(\langle \Phi, \Delta \rangle)$$

Following from Definition 4.3.1, the consideration here is that, in a logical system $\mathfrak{L}$ for reasoning with default rules, the premisses of $\mathfrak{L}$'s underlying entailment relation are structured sets comprised of a set of sentences and a set of default rules.

To be pointed out is that $\mathfrak{L}$'s underlying entailment relation, i.e., $\mid\!\sim$, is taken to be *primitive*, i.e., no particular set of rules for its generation is presupposed, nor there is an *a priori* commitment to a specific notion of a proof calculus. Rather than committing $\mathfrak{L}$'s underlying entailment relation to be generated by a particular proof calculus, the position assumed here is that it is best to keep these two notions separate, since it is precisely the notion of entailment what remains invariant under the possibly many equivalent proof calculi that can be used for its generation.

In being primitive, $\mathfrak{L}$'s underlying entailment relation $\mid\!\sim$ asserts the provability of a default consequence from a structured set of premisses, but abstracts the structure of proofs. In other words, $\langle \Phi, \Delta \rangle \mid\!\sim \delta$ indicates that δ is a default consequence of $\langle \Phi, \Delta \rangle$, but no mention is made as to how δ is proven from $\langle \Phi, \Delta \rangle$.

A second primitive notion is $\mathfrak{L}$'s underlying formulation of an extension $\mathscr{E}$. As it is the case with $\mathfrak{L}$'s underlying default entailment relation, $\mathscr{E}$ is not *a priori* committed to any particular formulation of an extension.

In being primitive, the mapping $\mathscr{E}$ abstracts the particular structure of extensions. In other words, $\mathscr{E}(\langle \Phi, \Delta \rangle)$ simply indicates that a class of sets of sentences, whose members are called extensions, is associated to a premiss structure $\langle \Phi, \Delta \rangle$, but no commitment is made as to how extensions are to be defined.

While it is possible to define the $\mid\!\sim$ in terms of $\mathscr{E}$ resorting to Cn, i.e., $\mathfrak{L}$'s underlying consequence operator,[†] the consideration here is that both $\mid\!\sim$ and $\mathscr{E}$ are to be seen as

[†]A case could be made that if $\mid\!\sim$ is defined in terms of $\mathscr{E}$, then, *a priori*, no sensible notion of a proof calculus should be expected for its generation.

providing a one-sided account of a logical system for reasoning with default rules. It is the extent to which the notions of provability at the level $\mathfrak{L}$'s underlying entailment relation and provability at the level of $\mathfrak{L}$'s underlying notion of an extension (w.r.t. $\mathfrak{L}$'s underlying consequence operator) are in correspondence with each other which is of importance, not the fact that one notion can be defined in terms of the other. This is precisely the idea captured in $\mathfrak{L}$'s correctness condition. When understood in this way, classes of extensions become a natural way in which to formulate a conceptual apparatus for justifying the default entailment relation.

Even at this early stage, an interesting feature of the notion of a logical system for reasoning with default rules proposed in Definition 4.3.1, and the kind of logical analysis it enables, is noteworthy. If $\mathscr{E}$ is taken as a way of justifying the notion of a default entailment relation, then, the existence of extensions is a fundamental property of any well-behaved logical system for reasoning with default rules. This point is made precise in Definition 4.3.2.

Definition 4.3.2. A logical system $\mathfrak{L}$ for reasoning with default rules is *rational* iff for all $\langle \Phi, \Delta \rangle$ in $\mathscr{P}$, $\mathscr{E}(\langle \Phi, \Delta \rangle) \neq \emptyset$.

By way of justification, the rationale for Definition 4.3.2 originates from the seemingly accepted postulate of default rules not being a source of inconsistencies. For the notion of a logical system for reasoning with default rules proposed in Definition 4.3.2 this postulate implies that if Φ is a CPL-consistent set of sentences, then, for no set of default rules Δ can it be the case that $\langle \Phi, \Delta \rangle \mathrel{|\!\sim} \sigma$ for all sentences σ, i.e., that $\langle \Phi, \Delta \rangle$ is $\mathfrak{L}$-inconsistent. Existence of extensions is a necessary condition for this to happen. If the class of extensions of Φ and Δ is empty for some consistent set of sentences Φ and some set of default rules Δ, it follows vacuously from $\mathfrak{L}$' correctness condition that $\langle \Phi, \Delta \rangle \mathrel{|\!\sim} \sigma$ for all sentences σ.

Intrinsic to the formulation of $\mathfrak{L}$ as a logical system for reasoning with default rules is the idea that the default consequences of premiss structures have a defeasible status.

This is made precise in Definition 4.3.3.

Definition 4.3.3. A logical system $\mathfrak{L}$ for reasoning with default rules is *nonmonotonic* iff, for a sentence σ, there are $\langle \Phi, \Delta \rangle \subseteq \langle \Phi', \Delta' \rangle$ in $\mathscr{P}$ with $\langle \Phi, \Delta \rangle \mathrel{|\!\sim} \sigma$ but $\langle \Phi', \Delta' \rangle \not\mathrel{|\!\sim} \sigma$.[†]

Definition 4.3.3 requires no lengthy explanation. The sought after notion in reasoning with default rules is that of an entailment relation which does not necessarily satisfy the principle of monotonicity. Nonetheless, Definition 4.3.3 gains in importance w.r.t. what is taken to be the source of nonmonotonicity. Definition 4.3.4 is introduced to this end.

Definition 4.3.4. A logical system $\mathfrak{L}$ for reasoning with default rules is *classical* iff, for every $\langle \Phi, \Delta \rangle$ in $\mathscr{P}$, for every E in $\mathscr{E}(\langle \Phi, \Delta \rangle)$, $\Phi \subseteq Cn(E)$.

Proposition 4.3.1 follows immediately from Definitions 4.3.4 and 4.3.5.

Definition 4.3.5. A logical system for reasoning with default rules is *1-monotonic* iff for all $\langle \Phi, \emptyset \rangle \subseteq \langle \Phi', \Delta' \rangle$ in $\mathscr{P}$, if $\langle \Phi, \emptyset \rangle \mathrel{|\!\sim} \sigma$, then, $\langle \Phi', \Delta' \rangle \mathrel{|\!\sim} \sigma$.

Proposition 4.3.1. Let $\mathfrak{L}$ be a logical system for reasoning with default rules; if $\mathfrak{L}$ is classical, then, $\mathfrak{L}$ is 1-monotonic.

Proposition 4.3.1 then reflects the different statuses, definite and tentative, of the elements of a premiss structure (q.v. Section 4.2). In the presence of 1-monotonicity, nonmonotonicity can only originate as a result of what is the case with respect to default rules, hence their tentative status.

Regarding Definition 4.3.4, Proposition 4.3.2 is worth also noting.

[†]For $\langle \Phi, \Delta \rangle$ and $\langle \Phi', \Delta' \rangle$ in $\mathscr{P}$, $\langle \Phi, \Delta \rangle \subseteq \langle \Phi', \Delta' \rangle$ iff $\Phi \subseteq \Phi'$ and $\Delta \subseteq \Delta'$.

Proposition 4.3.2. Let $\mathfrak{L}$ be a logical system for reasoning with default rules; if $\mathfrak{L}$ is classical, then, for every $\langle \Phi, \emptyset \rangle$ in $\mathscr{P}$, if $E \subseteq \Phi$ for every $E \in \mathscr{E}(\langle \Phi, \emptyset \rangle)$, then, $\langle \Phi, \emptyset \rangle \mathrel{|\!\sim} \sigma$ iff $\sigma \in Cn(\Phi)$.

In ordinary terms, Proposition 4.3.2 states that, for the notion of logical system $\mathfrak{L}$ for reasoning with default rules formulated as in Definition 4.3.1, provability at the level of $\mathfrak{L}$'s underlying entailment relation and provability at the level of $\mathfrak{L}$'s underlying notion of an extension, w.r.t. Cn, coincide in the absence of default rules. Proposition 4.3.2 may be thought of as capturing a condition that Makinson calls *supraclassicality* in [Mak94, p. 45].

The rest of this section is dedicated to discussing the manner in which the formal properties of a nonmonotonic entailment relation, i.e., the principles of inclusion, cut, and cumulativity, may be stated for a default entailment relation. These principles were originally advanced by Gabbay in [Gab85] and later on further studied by Makinson in [Mak89] and [Mak94].

As a guiding consideration, the standpoint here is that inclusion, cut, and cumulativity ought to be seen as abstract properties of the default entailment relation (i.e., independent of: the particular set of rules that may be used for generating the default entailment relation, and the particularities of the notion of extension at hand). Definition 4.3.6 makes precise how these principles may be expressed for the notion of a logical system for reasoning with default rules proposed in Definition 4.3.1.

Definition 4.3.6. Let $\mathfrak{L}$ be a logical system for reasoning with default rules; the default entailment relation $\mathrel{|\!\sim}$ satisfies the principle of:

$$\text{inclusion iff if } \sigma \in \Phi, \text{ then, } \langle \Phi, \Delta \rangle \mathrel{|\!\sim} \sigma \tag{4.6}$$

$$\text{cut iff if } \langle \Phi, \emptyset \rangle \mathrel{|\!\sim} \sigma \text{ and } \langle \Phi \cup \{\sigma\}, \Delta \rangle \mathrel{|\!\sim} \psi, \text{ then, } \langle \Phi, \Delta \rangle \mathrel{|\!\sim} \psi \tag{4.7}$$

$$\text{cumulativity iff if } \langle \Phi, \emptyset \rangle \mathrel{|\!\sim} \sigma \text{ and } \langle \Phi, \Delta \rangle \mathrel{|\!\sim} \psi, \text{ then, } \langle \Phi \cup \{\sigma\}, \Delta \rangle \mathrel{|\!\sim} \psi \tag{4.8}$$

In order to justify their formulation as presented in Definition 4.3.6, the principles of inclusion, cut, and cumulativity are discussed in analogy with the manner in which they may be understood in an ordinary sense.

First, intrinsic to the principle of inclusion is the consideration that 'what is accepted as being the case must be able to be concluded'. On this basis, the standpoint here is that, for the case of a default entailment relation, such a claim can only be made with respect to the set of sentences of a premiss structure. This consideration is grounded on the different statuses, definite and tentative, of the assertions denoted by the set of sentences and the set of default rules forming a premiss structure (q.v. Section 4.2). In that respect, the formulation of the principle of inclusion for the default entailment relation proposed in Definition 4.3.6 may be seen as formalizing the consideration that 'what is accepted as being the case definitely must, by virtue of its acceptance, be able to be concluded'. Given that what is accepted definitely is indicated by the set of sentences of a premiss structure, the formulation of the principle of inclusion for a default entailment relation proposed in Definition 4.3.6 refers only to what shall be able to be concluded with respect to the set of sentences of a premiss structure.

In turn, the rationale underpining the formulation of the principles of cut and cumulativity for a default entailment relation proposed in Definition 4.3.6 results from the following understanding of the notion of a lemma. In more or less succinct terms, a lemma is a sentence that is advanced as part of some premisses as an aid for establishing that another sentence follows from them. In an ordinary sense, the proposition of a lemma may ordinarily be seen as an intermediate step in the proof of a conclusion. Notwithstanding, it is not solely its proposition that makes it a lemma to be such. In order for a sentence to be a lemma, it must be discharged, i.e., it itself must be provable from the premisses in question.

On this basis, intrinsic to the principle of cut is the consideration that 'lemmas can be used safely', i.e., whatever can be proved with the use of a lemma can be proved without the lemma, i.e., lemmas are logically dispensable. As for the case of the principle of inclusion, the consideration here is that such a claim can only be made partially for a default entailment relation, i.e., with respect to what is the case for a set of sentences of a premiss structure. In that respect, the formulation of the principle of cut proposed in Definition 4.3.6 may be understood as formalizing the consideration that 'lemmas can be used safely, with the proviso that they are proven solely from what is accepted definitely'.

Lastly, in the context of nonmonotonic reasoning, the principle of cumulativity originates from considering that 'what is defeasibly concluded cannot be a source of nonmonotonicity'. Once again, the consideration here is that such a claim can only be made partially for a default entailment relation, i.e., with respect to what is the case for a set of sentences of a premiss structure. In that respect, the formulation of the principle of cumulativity proposed in Definition 4.3.6 may be understood as formalizing the consideration that 'whatever is concluded from what is accepted definitely cannot be a source of nonmonotonicity'.

Proposition 4.3.3 states some minimal conditions guaranteeing that the principles of inclusion, cut, and cumulativity are satisfied.

Proposition 4.3.3. Let $\mathfrak{L}$ be a logical system for reasoning with default rules; if $\mathfrak{L}$ is classical, then, $\mathrel{|\!\sim}$ satisfies the principles of inclusion, cut, and cumulativity. (q.e.d. Appendix A.)

By way of conclusion, Proposition 4.3.4 summarizes the ideas and results discussed in this chapter by presenting $\underline{a}$ logical system for reasoning with default rules.

Proposition 4.3.4. A logical system $\mathfrak{L}$ for reasoning with default rules is obtained if extensions are defined according to Definition 4.2.3 and if, for $\langle \Phi, \Delta \rangle$ in $\mathscr{P}$, $\mathrel{|\!\sim}$ is defined as $\langle \Phi, \Delta \rangle \mathrel{|\!\sim} \delta$ iff $\delta \in Cn(E)$ for all $E \in \mathscr{E}(\langle \Phi, \Delta \rangle)$.

As a comment in passing, a reasonable objection to the logical system $\mathfrak{L}$ for reasoning with default rules proposed in Proposition 4.3.4 is that $\mathfrak{L}$'s underlying entailment relation is taken to be defined in terms of $\mathscr{E}$, i.e., no sensible notion of a proof calculus is proposed for its generation. This observation is addressed in Chapter 5, Theorem 5.3.1.

It follows immediately that the logical system for reasoning with default rules resulting from Proposition 4.3.4 is classical and correct. In that respect, as per Proposition 4.3.3, its underlying entailment relation satisfies the principles of inclusion, cut, and cumulativity. For this logical system for reasoning with default rules, nonmonotonicity obtains as a result of arbitrary enlargements of either the set of sentences of a premiss structure, or the set of default rules of a premiss structure.

4.4 Discussion

In response to the observations made in Chapter 3, this chapter proposed to treat default rules not as rules of inference but as premiss-like objects standing as the formal counterpart of assertions that are held tentatively. This was the subject matter of Section 4.1. Given this formal treatment of a default rule, the notion of an extension becomes a more or less natural way in which what is held tentatively may be accepted as being the case, i.e., it construes a sort of interpretation structure of a syntactical kind. This was the subject matter of Section 4.2. The notion of a logical system for reasoning with default rules proposed in Section 4.3 is built on these basis.

It is more or less direct how this new interpretation of the main elements of reasoning with default rules addresses the problems and concerns raised in Chapter 3. First, treating default rules as premiss-like objects sets aside considerations of structurality, i.e., there being no logical relationship between the sentences forming a default rules, as a problem of adequacy of formalization, and considerations of default rules not being logic defining rules of inference as premiss formation matters; both of which are radically different from problems concerning the formulation of rules of inference. In turn, treating extensions as interpretation structures of a syntactical kind allows not only to provide a better logical standing to the kind of objects that are captured by extensions, but it also provides more flexibity when it comes to formulating an "adequate" definition of what is an extension. Lastly, bringing into the foreground the notion of a logical system for reasoning with default rules, and its ensuing entailment relation, enables for an adequate formalization of the principles this logical system ought to satisfy.

By way of conclusion, the basic message to be left is not that the contentions and results presented here are a final word on the subject of reasoning with default rules. Instead, given that one of the possibly many purposes of studying the notion of a logical system for reasoning with default rules is that of providing a somewhat logically informative account of the idea of a consequence being defeasible, the hope is that what has been presented, discussed, and elaborated on above has engendered some thinking about the (re)conceptualization of the fundamentals of reasoning with default rules, potentially deepening their understanding in light of the newly presented ideas.

Chapter 5

Accounting for Proofs

Of a technical nature, built on a tableaux based proof method for classical propositional logic, this chapter presents the development of a tableaux based proof method for reasoning with default rules. The locus of attention is on providing a mechanism for proving that a given sentence is a default consequence of a finite premiss structure.

The rest of this chapter assumes that $\mathscr{L}$ and $\mathscr{D}$ indicate the standard propositional language and the set of all default rules defined on $\mathscr{L}$ (q.v. Definitions 2.1.2 and 3.1.1, respectively). Arbitrary sets of sentences are assumed to be subsets of elements of $\mathscr{L}$, i.e., of sentences. Similarly, arbitrary sets of default rules are assumed to be subsets of elements of $\mathscr{D}$, i.e., of default rules. Lastly, the negation of a sentence σ is assumed to be denoted by $\neg\sigma$.

5.1 A Foreword on Labeled Trees

This section summarizes some of the basic definitions underpinning the notion of a *labeled tree*. Their detailed versions can be found in [HJ99] and in [Smu95].

Definition 5.1.1 introduces the precise formulation of a labeled tree.

Definition 5.1.1 (Labeled Tree)**.** A *labeled tree* is a tuple $\langle N, <, f \rangle$ where:

– N is a set of elements called *nodes*. A labeled tree is *finite* iff its set of nodes is finite, otherwise, it is *infinite*.

– $<$ is a strict partial ordering, called a *tree ordering*, defined on N satisfying the following two conditions: (a) there is a least element in N, and (b) for any $n \in N$, the set $\{n' \,|\, n' < n\}$ is well-ordered. The relation $<$ defines the structure of a labeled tree. If $n < n'$, then, n is a *predecessor* of n' and n' is a *successor* of n.

– f is a function from N to a set L of elements called *labels*. The function f defines a labeling on nodes: a node n is *labeled* by l, or l is the *label* of n, iff f maps n to l.

A forest is a set of labeled tress F such that, for any two labeled tress $\langle N, <, f \rangle$ and $\langle N', <', f' \rangle$ in T, $\mathsf{range}(f) = \mathsf{range}(f')$ – i.e., a forest is a set labeled trees defined on the same set of labels.

Unless it is stated otherwise, when speaking of related labeled trees, this chapter assumes that they belong to the same forest.

As is standard in the literature on labeled trees, the *root* of a labeled tree τ is the least element of the set of nodes of τ. If n is a node of τ, then, provided it exists, an *immediate successor* of n is a minimal element of the set $\{n' \mid n < n'\}$. In general, a node may have zero or more immediate successors. A *leaf* of τ is a node that has no immediate successors. It follows from Definition 5.1.1 that the *leaves* of a labeled tree are the maximal elements of its set of nodes.

Let $\tau = \langle N', <', f' \rangle$ and $\tau' = \langle N', <', f' \rangle$ be two labeled trees; τ is a *subtree* of τ', or alternatively τ' is an *expansion* of τ, iff: (a) $N \subseteq N'$, (b) $< \subseteq <'$, (b') for all $n \in N$ and $n' \in N'$, $n' <' n$ implies $n' \in N$, and (c) $f \subseteq f'$. If τ' is an expansion of τ, the set $N' \setminus N'$ is τ''s set of *new* nodes. An *immediate expansion* of a labeled tree τ is a labeled tree τ' that has exactly one new node. A labeled tree is *finitely generated* iff it is the limit of a chain of immediate expansions of a single node labeled tree. If τ is a finitely generated labeled tree, then, the chain of immediate expansions from which τ originates is τ's *formation sequence.*

A *branch* is a labeled tree whose set of nodes is totally ordered. A *branch* π of a labeled tree τ is a subtree of τ that is also a branch. A branch π of a labeled tree τ is *maximal* if no other branch of τ is an expansion of π. A labeled tree τ is *finitely branching* if its set of maximal branches is finite, otherwise, τ is *infinitely branching.* It is possible to prove that, if τ is a labeled tree that is both infinite and finitely branching, then, τ has at least one maximal branch that is infinite. This property is known as *König's Lemma* (q.v. [HJ99, p. 227]).

Let τ be a labeled tree; mostly for explanatory purposes, τ will be depicted in a tabular form. The first row in this table will contain the label of the root r of τ; the second row will contain the labels of the immediate successors of r; the third row will contain the labels of the immediate successors of the immediate successors of r; and so on. Column separators will indicate branching points in τ. Since the nodes of τ are not required to be uniquely labeled, labels may appear repeated. This will mean that τ has different nodes labeled by the same label.

5.2 Propositional Tableaux

This section introduces the basics of the method of tableaux for CPL. Its main purpose is that of explaining the principles which underpin the formulation of the tableaux based proof method for reasoning with default rules of Section 5.3. Definitions 5.2.1 and 5.2.2 are preliminary to what follows.

Definition 5.2.1. Let σ be any sentence of $\mathscr{L}$; σ is: (i) a *literal* if it is either a propositional variable or a negation thereof; (ii) of *linear* type if it is a substitution instance of $p \wedge q$, $\neg(p \vee q)$, $\neg(p \supset q)$, or $\neg\neg p$; (iii) of *branching* type if it is a substitution instance of $\neg(p \wedge q)$, $p \vee q$, or $p \supset q$. By way of notation, α and β indicate arbitrary sentences of linear and branching type, respectively.

Definition 5.2.2. Let σ be a sentence of $\mathscr{L}$; if σ is a substitution instance of the sentence that is located in the first column of the n-th row of Table 5.1 or of Table 5.2, then, the *first* and the *second component* of σ are the corresponding substitution instances of the sentences located in the second and the third column of the same row of the same table, respectively. The first and the second component of a sentence are indicated by subscripts.

α	α_1	α_2
$p \wedge q$	p	q
$\neg(p \vee q)$	$\neg p$	$\neg q$
$\neg(p \supset q)$	p	$\neg q$
$\neg\neg p$	p	p

Table 5.1: Linear

β	β_1	β_2
$\neg(p \wedge q)$	$\neg p$	$\neg q$
$p \vee q$	p	q
$p \supset q$	$\neg p$	q

Table 5.2: Branching

Quoting Smullyan, Definitions 5.2.1 and 5.2.2 "will save us considerable repetition of essentially the same arguments" (q.v. [Smu95], pp. 20-21).

As a brief parenthetical remark, it is possible to think of Tables 5.1 and 5.2 in Definition 5.2.2 as rules for determining the *truth* of a sentence, or more generally as providing a set of rules for syntactically describing a *model* in which a sentence is true.[†] Notwithstanding, this view of Tables 5.1 and 5.2 and their relation to *truth* and *models* is not be elaborated on further here. Instead, the locus of attention is on presenting the method of tableaux from a *proof-theoretical* perspective.

Following from this parenthetical remark, Definition 5.2.3 introduces the main elements of the method of tableaux for CPL.

Definition 5.2.3 (Sentence Labeled Tableau)**.** Let σ be a sentence of $\mathscr{L}$; the set T of *all tableaux for* σ is the smallest set of labeled trees that satisfies the following conditions:

R0 The unique one-node labeled tree with label σ belongs to T.

– Let τ be in T, π be a maximal branch of τ, and τ' a labeled tree:

RA If a node n of π is labeled by a sentence α of linear type, and τ' is obtained from τ by adding a new node n' with label α_1 (or α_2) as an immediate successor of the leaf node of π, then, τ' is in T.

RB If a node n of π is labeled by a sentence β of branching type, and τ' is obtained from τ by adding two new nodes n' and n'' with labels β_1 and β_2, respectively, as an immediate successor of the leaf node of π, then, τ' is in T.

A labeled tree τ is a *tableau for* σ iff it is a member of T.

[†] To be noted is that a sentence of linear type is true iff both of its components are true, and that a sentence of branching type is true iff at least one of its components is true.

Figure 5.1 depicts a tableau for $\neg[(p \supset (q \supset r)) \supset ((p \vee s) \supset ((q \supset r) \vee s))]$.

$$
\begin{array}{l}
\neg[(p \supset (q \supset r)) \supset ((p \vee s) \supset ((q \supset r) \vee s))] \\
p \supset (q \supset r) \\
\neg[(p \vee s) \supset ((q \supset r) \vee s)] \\
\hline
\begin{array}{l|l}
\begin{array}{l}
\neg p \\
p \vee s \\
\neg[(q \supset r) \vee s] \\
\hline
\begin{array}{l|l}
\begin{array}{l} p \\ \neg(q \supset r) \\ \neg s \end{array} & s
\end{array}
\end{array}
&
q \supset r
\end{array}
\end{array}
$$

Figure 5.1: Tableau for $\neg[(p \supset (q \supset r)) \supset ((p \vee s) \supset ((q \supset r) \vee s))]$

The formulation of a tableau for a sentence as presented in Definition 5.2.3 is similar to that presented by Smullyan in [Smu95, p. 24]. While interesting in its own right, however, it is not w.r.t. the formulation of a tableau for a sentence that Smullyan studied and proved several interesting properties of CPL in [Smu95], but with respect to a suitable formulation of a proof based on a tableau for a sentence. This formulation of a proof is presented in Definition 5.2.5.

Definition 5.2.4 (Closed Tableau)**.** Let τ be a tableau for σ; a branch π of τ is *closed* iff one of the following conditions holds:

– There is a node n of π that is labeled by either $\bot$ or $\neg\top$.

– For some sentence σ, there are nodes n and n' of π such that n is labeled by σ and n' is labeled by $\neg\sigma$.

The branch π is *open* iff it is not closed. The tableau τ is *closed* iff all of its maximal branches are closed, otherwise it is *open*.

Definition 5.2.5 (Proof)**.** Let σ be a sentence; a *proof of* σ is a closed tableau for $\neg\sigma$. The sentence σ is *provable* iff there is a proof of σ.

Figure 5.2 depicts a proof of $(p \supset (q \supset r)) \supset ((p \vee s) \supset ((q \supset r) \vee s))$.

$$
\begin{array}{l}
\neg[(p \supset (q \supset r)) \supset ((p \vee s) \supset ((q \supset r) \vee s))] \\
p \supset (q \supset r) \\
\neg[(p \vee s) \supset ((q \supset r) \vee s)] \\
\hline
\begin{array}{l|l}
\neg p & q \supset r \\
p \vee s & p \vee s \\
\neg[(q \supset r) \vee s] & \neg[(q \supset r) \vee s] \\
\hline
\begin{array}{l|l} p & s \\ & \neg(q \supset r) \\ & \neg s \end{array} & \begin{array}{l|l} p & s \\ \neg(q \supset r) & \neg(q \supset r) \\ & \end{array}
\end{array}
\end{array}
$$

Figure 5.2: Proof of $(p \supset (q \supset r)) \supset ((p \vee s) \supset ((q \supset r) \vee s))$

Given Definitions 5.2.3 and 5.2.5, the construction of any tableau for a sentence $\neg\sigma$ may, from a proof-theoretical perspective, be seen as an *attempt* at proving σ, with any closed tableau for $\neg\sigma$ being a *successful* one. When taken in this sense, Definitions 5.2.3 and 5.2.5 can be thought of as a *proof method* for CPL, the *soundness* and *completeness* of which is shown in [Smu95, pp. 25–30].

It should be noted that proofs, as given in Definition 5.2.5, are proofs of sentences. More often than not, however, there is a need to prove that a sentence follows from a set of sentences, to prove *consequences* of sets of *premisses*. In the context of CPL, there is no need to modify the definitions introduced thus far. Following from the *deduction theorem*, proving that a sentence σ is a consequence of a finite set of premisses Γ is reducible to finding a proof of the implication whose antecedent is the conjunction of those sentences in Γ and whose consequent is σ. Despite this facility, from a proof-theoretical perspective, it seems better to modify the definition of a tableau, and that of a proof, so that they can accommodate proofs of consequences in a more direct way. Definitions 5.2.6 and 5.2.7 are introduced to this end.

Definition 5.2.6. Let Γ be a finite set of sentences; the set T of *all tableaux for* Γ is the smallest set of labeled trees that satisfies the following conditions:

R0 If π is a finite branch whose nodes are all labeled by the sentences in Γ, then, π is in T.

R1 is identical to R1 in Definition 5.2.3.

R2 is identical to R2 in Definition 5.2.3.

A labeled tree τ is a *tableau for* Γ iff it is a member of T.

Definition 5.2.7 (Proof)**.** Let σ be a sentence and Γ a finite set of sentences; a *proof of* σ *from* Γ is a closed tableau for $\Gamma \cup \{\neg\sigma\}$. The sentence σ is *provable* from Γ iff there is a proof of σ from Γ. The sentence σ is a *consequence* of, or *follows from*, the set of *premisses* Γ iff it is provable from Γ.

Figure 5.3 depicts a proof of $(q \supset r) \vee s$ from $\{p \supset (q \supset r), p \vee s\}$.

$$
\begin{array}{l}
p \supset (q \supset r) \\
p \vee s \\
\neg[(q \supset r) \vee s] \\
\hline
\begin{array}{l|l}
p & q \supset r \\
\hline
\begin{array}{l|l}
p & \begin{array}{l} s \\ \neg(q \supset r) \\ \neg s \end{array}
\end{array}
&
\begin{array}{l|l}
\begin{array}{l} p \\ \neg(q \supset r) \\ \end{array} & \begin{array}{l} s \\ \neg(q \supset r) \\ \end{array}
\end{array}
\end{array}
\end{array}
$$

Figure 5.3: Proof of $(p \supset (q \supset r)) \supset (p \vee s)$ from $\{p, q \supset r\}$

Definition 5.2.7 provides a way of speaking of a sentence σ following from a set of premisses Γ, or of σ being a consequence of Γ, with all these terms being properly defined and without the need for workarounds. As with Definitions 5.2.3 and 5.2.5, Definitions 5.2.6 and 5.2.7 can also be thought of as a proof method for CPL, the soundness and completeness of which is proven in [Smu95]. In this case, any tableau for $\Gamma \cup \{\neg\sigma\}$ may be seen as an attempt at proving that σ is a consequence of Γ, with any closed tableau for $\Gamma \cup \{\neg\sigma\}$ being a *successful* one.

Given that Definitions 5.2.3 and 5.2.5 are particular cases of Definitions 5.2.6 and 5.2.7, respectively, henceforth, when speaking of the method of tableaux as a proof method for CPL, the formulation of a tableau and that of a proof in mind will be that of a tableau for a set of sentences and that of a proof of a sentence from a finite set of premisses, respectively.

There are two properties worth noting that are satisfied by the method of tableaux as a proof method for CPL: (i) it can be demonstrated that any attempt at proving that σ follows form Γ can be extended to a successful one if a proof were to exist; (ii) and perhaps more interesting, this proof method also makes it possible to discover the nonexistence of proofs by looking at some particular cases of failed attempts. Definition 5.2.8 makes precise the point made in (ii).

Definition 5.2.8 (Completed Tableau)**.** Let Γ be a finite set of sentences and τ be a tableau for Γ; a branch π of τ is *completed* iff the following conditions are met:

– For every sentence α of linear type that is the label of a node n of π, there are nodes n' and n'' of π that are labeled by the components α_1 and α_2 of α, respectively.

– For every sentence β of branching type that is the label of a node n of π, there is a node n' of π that is labeled by one the components β_1 or β_2 of β.

The tableau τ is *completed* iff all of its maximal branches are completed.

Definition 5.2.8 gains in interest when approached from a proof-theoretical perspective for the following reason: there is no need to continue expanding a tableau that is completed. This has two interesting properties. First, it indicates when to stop in the search for a proof. Second, and more importantly, following from a result known as *Hintikka's lemma*, q.v., [Smu95, pp. 26–28]), if a completed tableau has an open branch,then, the sought after proof does not exist. (This result that will be used in the definition of a tableaux method for reasoning with default rules presented in Section 5.3.)

Some Final Remarks

It is possible to consider that though it provides a direct way of proving a consequence from a set of premisses, Definition 5.2.7 does so in a somewhat odd manner. Namely, following Gentzen's maxim, the idea is that the definition of a proof method should resemble, at least as closely as possible, an "actual" process of proving (in the case of tableau constructions, a method of proof by *refutation*). Notwithstanding, it is unlikely that in attempting to prove that a sentence σ is a consequence of a set of premisses Γ one will list the sentences in $\Gamma \cup \{\neg\sigma\}$ in some arbitrary order and then use this listing as a starting point for proceeding with a proof attempt. Instead, a seemingly more reasonable approach would be to begin from $\neg\sigma$ and to proceed from this starting point, resorting to premisses as needed or as found useful.

For instance, if required to prove that $p \wedge q$ is a consequence of $\{p, q\}$, the idea would be to begin with $\neg(p \wedge q)$ (q.v. Figure 5.4); decompose this sentence into $\neg p$ and $\neg q$ (q.v. Figure 5.5); refute that $\neg p$ is the case, appealing to p (q.v. Figure 5.6); refute that $\neg q$ is the case, appealing to q (q.v. Figure 5.7).

However, the formation sequence depicted in Figures 5.4 to 5.7 does not yield a tableau that is well-formed according to Definition 5.2.6. The reason is that, while

$$\neg p \wedge q$$

Figure 5.4: Step 1

$$\begin{array}{c|c} \multicolumn{2}{c}{\neg p \wedge q} \\ \hline \neg p & \neg q \end{array}$$

Figure 5.5: Step 2

$$\begin{array}{c|c} \multicolumn{2}{c}{\neg p \wedge q} \\ \hline \neg p & \neg q \\ p & \end{array}$$

Figure 5.6: Step 3

$$\begin{array}{c|c} \multicolumn{2}{c}{\neg p \wedge q} \\ \hline \neg p & \neg q \\ p & q \end{array}$$

Figure 5.7: Step 4

the formation sequence of any tableau defined according to such a definition will always begin with a listing of the sentences in $\{p, q, \neg(p \wedge q)\}$ in some order, once this initial list is set, it cannot be expanded, i.e., no further sentences can be brought into a proof.

In some cases, starting with the negation of what one is trying to prove and proceeding from there may result in shorter and more readable proofs. Definition 5.2.9 is introduced to this end.

Definition 5.2.9. Let Γ be a finite set of sentences; the set T of *all tableaux* for Γ is the smallest set of labeled trees that satisfies the following conditions:

R0 For any sentence $\sigma \in \Gamma$, if τ is a one-node labeled tree with label σ, then, τ belongs to T.

R1 is identical to R1 in Definition 5.2.3.

R2 is identical to R2 in Definition 5.2.3.

R3 For any sentence $\sigma \in \Gamma$, if τ is in T and τ' is obtained from τ by adding a new node n' with label σ as an immediate successor of a leaf node of τ, then, τ' is in T.

A labeled tree τ is a *tableau for S* iff it is a member of T.

Definition 5.2.9 corresponds to that given by Fitting in [Fit99, p. 67]. It is not difficult to observe how such a definition accommodates for the proof strategy depicted in Figures 5.4 to 5.7. In fact, Figures 5.4 to 5.7 depicts a formation sequence for a tableau for $\{\neg(p \wedge q), p, q\}$ that is formulated according to Definition 5.2.9.[†]

It follows from the above that strategies are a useful and interesting aid when attempting to formally prove whether something is the case, even more so when it comes to systematizing a proof method. In this respect, to be noted is that the tableau expansion rules state *what can be done* not *what must be done*.

To illustrate this latter point, in all of the tableaux presented thus far, expansion rules have been used on sentences as they appear on a branch; starting from the root and moving downwards to the leaves. Nonetheless, in certain cases, shorter proofs can be obtained if they are constructed by: first working on those sentences that are of linear type, until none are left; and only then moving on to work on those sentences that are of branching type; repeating this process if necessary. This strategy for constructing tableaux, labeled LTF, is depicted in Figure 5.8.

$$
\begin{array}{l}
\neg[(p \supset (q \supset r)) \supset ((p \vee s) \supset ((q \supset r) \vee s))]\\
p \supset (q \supset r)\\
\neg[(p \vee s) \supset ((q \supset r) \vee s)]\\
p \vee s\\
\neg[(q \supset r) \vee s]\\
\neg(q \supset r)\\
\neg s\\
q\\
\neg r\\
\hline
\begin{array}{l|l}
\begin{array}{l|l}
\multicolumn{2}{l}{\neg p}\\
\hline
p & s
\end{array} & q \supset r
\end{array}
\end{array}
$$

Figure 5.8: LTF strategy

[†]Regarding Definition 5.2.9, to be noted is that it is not the definition of a proof which changes, but the way in which the tableau involved in that proof may be constructed.

In brief, when compared with the tableau depicted in Figure 5.2, the tableau depicted in Figure 5.8 is quicker to construct – in that, since it does not have repeated sentences in different branches, its formation sequence involves fewer steps.

Though systematicity is important in proof methods, in particular for their machine implementation, it is an issue that will not be dealt with in this chapter. Choosing carefully the steps to be used in the construction of a tableau results in shorter and more readable proofs, an advantage from which some benefit can be derived.

In turn, a point worth commenting on concerns the finitude of sets of premisses. While it is possible to consider that finite sets of premisses are too restrictive, it is also possible to argue otherwise (in fact, one could even go further and say that much of everyday "proving practice" in CPL consists of proving consequences from finite sets of premisses). In addition, for the case of classical propositional logic, there is no loss of "proving power" in restricting proofs of consequences of potentially infinite sets of premisses to proofs of consequences of finite sets of premisses. Following from the *compactness theorem*, if a sentence can be proved from an infinite set of premisses, then, there is a finite subset of these premisses from which it can be proved. Thus, in principle, in proving whether a sentence is a consequence of an infinite set of premisses, proof attempts could proceed by considering finite subsets of these premisses, in an incremental fashion; the termination of such a process would be guaranteed if one such a proof were to exist.

Regarding the finitude of sets of premisses and the finitude of proofs, a quite different situation occurs if a proof cannot be found in a finite number of steps. In this case, it is notoriously difficult, at least *a priori*, to determine whether a proof will be found in subsequent proof attempts, or whether there is no proof at all (a problem partly originating from considering infinite sets of premisses).

In any case, while considering infinite sets of premisses is definitely interesting, in particular in the abstract study of proof methods and consequence relations, in the context of this thesis, attention will be restricted to proving consequences from finite sets of premisses.

As a closing remark, the tableaux formulations thus far presented all have in common that nodes are labeled by sentences. Alternatively, it is possible to consider the nodes of a tableau to be labeled by sets of sentences. While its utility will be made clear later in Section 5.3, for adequacy purposes such a formulation of a tableau is presented below in Definition 5.2.10.

Definition 5.2.10 (Set Labeled Tableau)**.** Let Γ be a finite set of sentences; the set T of *all tableaux* for Γ is the smallest set of labeled trees that satisfies the following conditions:

R0 The unique one-node labeled tree with label Γ belongs to T.

– Let τ be in T, l be a leaf of τ with label Γ', and τ' be a labeled tree:

R1 If a sentence α of linear type belongs to Γ', and τ' is obtained from τ by adding a new node n' with label $\Gamma' \cup \{\alpha_1, \alpha_2\}$ as an immediate successor of l, then, τ' is in T.

R2 If a sentence β of branching type belongs to Γ', and τ' is obtained from τ by adding two new nodes n' and n'' with labels $\Gamma' \cup \{\beta_1\}$ and $\Gamma' \cup \{\beta_2\}$, respectively, as immediate successors of l, then, τ' is in T.

A labeled tree τ is a *tableau for* Γ iff it is a member of T.

Definition 5.2.10 has proven to be quite useful for dealing with modalities (q.v. [DGHP99]). In the context of this thesis, Definition 5.2.10 is worth considering for it localizes the application of a tableau expansion rule to what is the case in the leaf nodes of an already built tableau, avoiding having to refer to path conditions (now accumulated into leaf nodes).

Figure 5.9 depicts a construction of a set labeled tableau for $\{\neg[(p \wedge q) \vee r]\}$.

$$
\begin{array}{ll|ll}
\multicolumn{4}{c}{\begin{array}{ll} \text{(a)} & \neg[(p \wedge q) \vee r] \\ \text{(b)} & \neg[(p \wedge q) \vee r] \\ & \neg(p \wedge q) \\ & \neg r \end{array}} \\
\hline
\text{(c)} & \neg[(p \wedge q) \vee r] & \text{(d)} & \neg[(p \wedge q) \vee r] \\
 & \neg(p \wedge q) & & \neg(p \wedge q) \\
 & \neg r & & \neg r \\
 & \neg p & & \neg q
\end{array}
$$

Figure 5.9: Set labeled tableau

In certain cases, localizing the application of a tableau expansion rule to what is the case in the leaf nodes of an already built tableau helps to simplify some subsequent definitions, an advantage from which some benefits can be derived.

For instance, if a tableau τ is formulated as in Definition 5.2.10, in order to determine whether τ is closed, all that is needed is to check whether all the leaf nodes of τ contain either: the sentence $\perp$, the sentence $\neg\top$, or a substitution instance of the set $\{p, \neg p\}$. Albeit the previous seems to be a triviality in the context of classical propositional logic, such principles of locality and cumulativity will prove to be of great use in the definition of a tableau method for reasoning with default rules, where tableau construction rules depend on conditions of the *non-provability* of what is the case in certain nodes.

5.3 Default Tableaux

Building on the ideas presented in the previous section, this section presents a tableaux based proof method for reasoning with default rules. The locus of attention is on proving that a sentence is a *default consequence* of a *premiss structure* comprised of a *finite* set of sentences and a *finite* set of default rules (q.v. Definition 4.3.1).

The presentation of a tableaux based proof method for reasoning with default rules begins with the introduction of a default tableau in Definition 5.3.1.

Figure 5.10 depicts a default tableau for a sentence $\neg s$ with premisses in the structured set $\langle\{p, p \supset (q \vee r \supset s)\}, \{p : u \ / \ q \wedge t, p : t \ / \ r \wedge u\}\rangle$.

In order to understand the basic ideas underpinning the formulation of a default tableau given in Definition 5.3.1, consider a situation in which it is required to prove that the sentence s is a default consequence of the premiss structure $\langle\Phi, \Delta\rangle = \langle\{p, p \supset (q \vee r \supset s)\}, \{p : u \ / \ q \wedge t, p : t \ / \ r \wedge u\}\rangle$. In attempting such a proof by refutation, there is a need to establish from $\langle\Phi, \Delta\rangle$ that assuming $\neg s$ leads to a contradiction. As a first step, this proof may be attempted by appealing only to the sentences in $\{p, p \supset (q \vee r \supset s)\}$. Given this initial standpoint, the sought after proof begins with a labeled tree with a single node (a) labeled by $L_{(\mathrm{a})} = \{p, p \supset (q \vee r \supset s), \neg s\}$. The proof then proceeds as follows: since $p \supset (q \vee r \supset s)$ belongs to $L_{(\mathrm{a})}$, nodes (b) and (c), labeled by $L_{(\mathrm{b})} = L_{(\mathrm{a})} \cup \{\neg p\}$ and $L_{(\mathrm{c})} = L_{(\mathrm{a})} \cup \{q \vee r \supset s\}$, respectively, are added as immediate successors of (a); next, since $q \vee r \supset s$ belongs to $L_{(\mathrm{c})}$, nodes (d) and (e), labeled by $L_{(\mathrm{d})} = L_{(\mathrm{c})} \cup \{\neg(q \vee r)\}$ and $L_{(\mathrm{e})} = L_{(\mathrm{c})} \cup \{s\}$, respectively, are added as immediate successors of (c). Lastly, since $\neg(q \vee r)$ belongs to $L_{(\mathrm{d})}$, a node (f), labeled by $L_{(\mathrm{f})} = L_{(\mathrm{d})} \cup \{\neg q, \neg r\}$, is added as an immediate successor of (d). The previous default tableau construction steps, corresponding to the labeled tree comprised by nodes (a) to (f) in Figure 5.10, are those of a standard set of sentences labeled tableau.

Definition 5.3.1 (Default Tableau)**.** Let σ be a sentence, and Φ and Δ be finite sets of sentences and default rules, respectively; the set of all default tableaux for σ with premisses in $\langle \Phi, \Delta \rangle$ is the smallest set T of labeled trees that satisfies the following conditions:

R0 The unique one-node labeled tree with label $\langle \Phi \cup \{\sigma\}, \emptyset \rangle$ is in T.

– Let τ be in T, l a leaf node of τ with label $\langle \Phi', \Delta' \rangle$, and τ' a labeled tree:

R1 If a sentence α of linear type belongs to Φ', and τ' is obtained from τ by adding a new node n' with label $\langle \Phi' \cup \{\alpha_1, \alpha_2\}, \Delta' \rangle$ as an immediate successor of l, then, τ' is in T.

R2 If a sentence β of branching type belongs to Φ', and τ' is obtained from τ by adding two new nodes n' and n'' with labels $\langle \Phi' \cup \{\beta_1\}, \Delta' \rangle$ and $\langle \Phi' \cup \{\beta_2\}, \Delta' \rangle$, respectively, as immediate successors of l, then, τ' is in T.

– Let n be a node of τ with label $\langle \Phi', \Delta' \rangle$:

R3 For any default rule $\frac{\pi:\rho}{\chi}$ in Δ, if τ' is obtained from τ by adding a new node n' with label $\langle \Phi' \cup \{\chi\}, \Delta' \cup \{\frac{\pi:\rho}{\chi}\} \rangle$ as an immediate successor of n, then, τ' is in T iff the following side conditions are satisfied:
   (a) there is a closed tableau for $\neg\pi$ with premisses in $\Phi \cup X(\Delta')$, and
   (b) for every $\rho' \in P(\Delta') \cup \{\rho\}$, there is a tableau for $\neg\rho'$ with premisses in $\Phi \cup X(\Delta') \cup \{\chi\}$ that is both complete and open.

A default tableau for σ with premisses in $\langle \Phi, \Delta \rangle$ is a labeled tree τ in T.

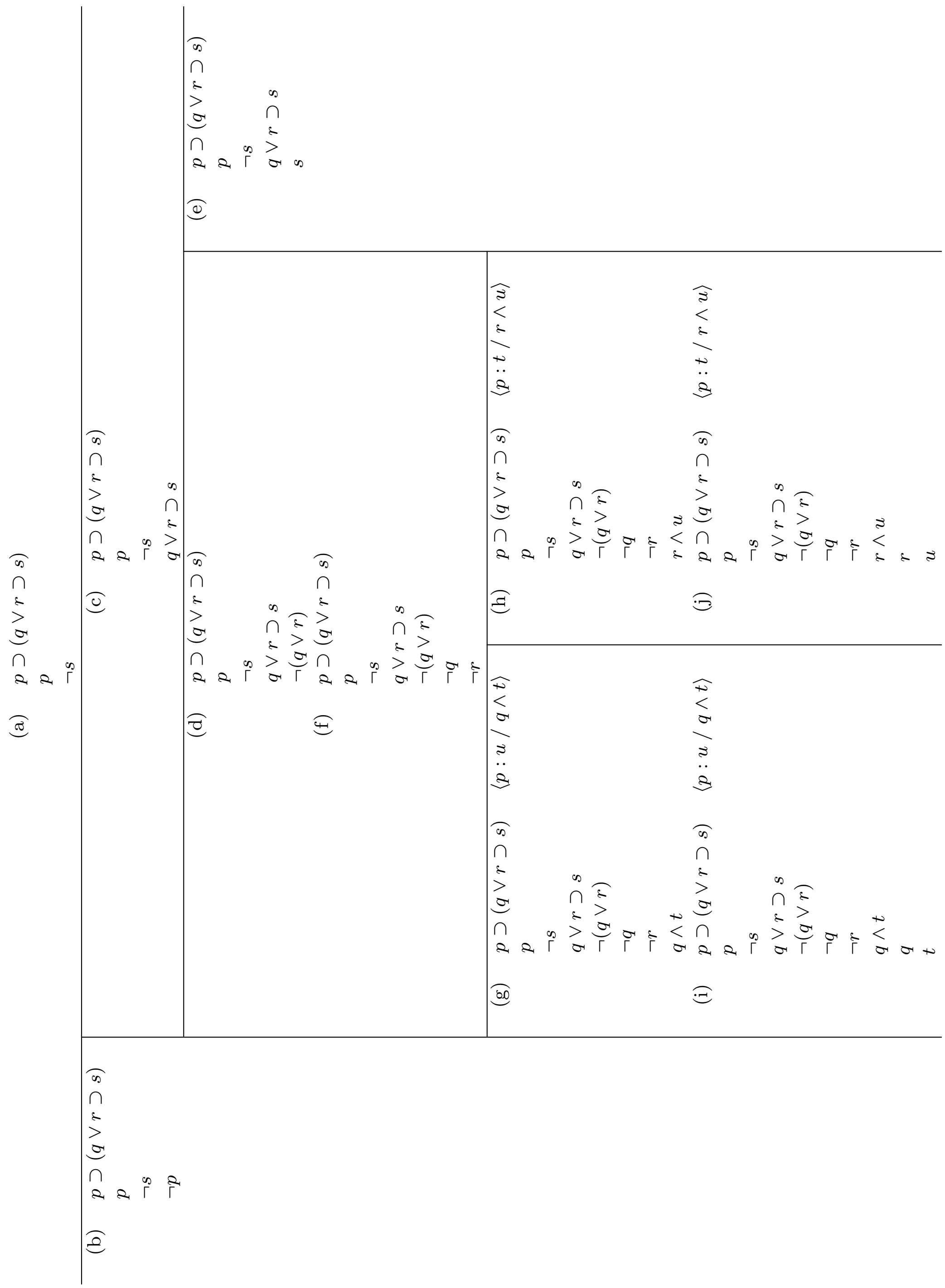

Figure 5.10: Default tableau for $\neg s$ with premisses in $\langle \{p, p \supset (q \vee r \supset s)\}, \{p : u \,/\, q \wedge t, p : t \,/\, r \wedge u\} \rangle$

At this point, it may be observed that (b) and (e) are leaf nodes that are closed, and that (f) is a leaf node that is open and "completed" (q.v. Figure 5.10). However, since the default rules in $\langle \Phi, \Delta \rangle$ remain unused, (f) is only "completed" w.r.t. the tableau construction rules for CPL. In other words, the sought after default proof may still proceed by appealing to the default rules in $\langle \Phi, \Delta \rangle$. Thus, given that, as per *R3* in Definition 5.3.1, the side conditions hold for applying $p : u \ / \ q \wedge t$ hold, a new node (g), labeled by $S_{(\mathrm{g})} = S_{(\mathrm{f})} \cup \{q \wedge t\}$, is added as an immediate successor of (f), simultaneously recording that $p : u \ / \ q \wedge t$ has been used. Next, since $q \wedge t$ belongs to $S_{(\mathrm{g})}$, a new node (i), labeled by $S_{(\mathrm{i})} = S_{(\mathrm{g})} \cup \{q, t\}$, is added as an immediate successor of (g). It is immediate to check that the leaf node (i) is closed and completed in a default tableau sense (since, as per *R3* in Definition 5.3.1, the side conditions for $p : t \ / \ r \wedge u$ do not hold, this branch cannot be extended further).

Even though (i) is closed and completed, the proof that s is a default consequence of $\langle \Phi, \Delta \rangle$ is still unfinished, the reason being that the addition of (g) as an immediate successor of (f) preempted the use of $p : t \ / \ r \wedge u$. Since the main idea underpinning a default tableau is that of systematizing a notion of provability in all extensions, a default proof should not depend on a particular selection of default rules to be applied. This means that it must be checked what would have been the case had $p : t \ / \ r \wedge u$ be chosen instead of $p : u \ / \ q \wedge t$. In that respect, given that, as per *R3* in Definition 5.3.1, the side conditions for applying $p : t \ / \ r \wedge r$ hold, a new node (h), labeled by $S_{(\mathrm{h})} = S_{(\mathrm{f})} \cup \{r \wedge u\}$, is added as an immediate successor of (f), simultaneously recording that $p : t \ / \ r \wedge r$ has now been used. This branch can be completed, in a default tableau sense, by adding a new node (j) with label $S_{(\mathrm{j})} = S_{(\mathrm{h})} \cup \{r, u\}$ as an immediate successor of (h).

The tableau construction steps described in the last two paragraphs complete the default tableau depicted in Figure 5.10.

As made precise in Definition 5.3.4, the default tableau in Figure 5.10 formulates a default proof of s from $\langle \{p, p \supset (q \vee r \supset s)\}, \{p : u \ / \ q \wedge t, p : t \ / \ r \wedge u\} \rangle$.

Definition 5.3.2 (Closedness)**.** Let τ be a default tableau for σ with premisses in $\langle \Phi, \Delta \rangle$; a node n of τ with label $\langle \Phi', \Delta' \rangle$ is closed (otherwise it is open) iff either of the following conditions holds:

– $\{\bot, \neg\top\} \cap \Phi' \neq \emptyset$.

– For some sentence σ, $\{\sigma, \neg\sigma\} \subseteq \Phi'$.

The default tableau τ is closed iff its leaf nodes are closed (otherwise τ is open).

Definition 5.3.3 (d-Saturation)**.** Let τ be a default tableau for σ with premisses in $\langle \Phi, \Delta \rangle$; a node n of τ with label $\langle \Phi', \Delta' \rangle$ is d-branching iff it has an immediate successor a node n' with label $\langle \Phi'', \Delta'' \rangle$ such that $\Delta' \subset \Delta''$. A d-branching node n of τ is d-saturated iff adding a new node n' with label $\langle \Phi' \cup \{\chi\}, \Delta' \cup \{\pi : \rho \,/\, \chi\} \rangle$ as an immediate successor of n, as per *R3* in Definition 5.3.1, results in n having at least two immediate successors labeled with the same label. The default tableau τ is d-saturated iff all of its d-branching nodes are d-saturated.

By way of example, node (f) in Figure 5.10 is both d-branching and d-saturated.

Definition 5.3.4 (Default Proof)**.** Let σ be a sentence, and Φ and Δ be finite sets of sentences and default rules, respectively; a default proof of σ from $\langle \Phi, \Delta \rangle$ is a closed and d-saturated default tableau for $\neg\sigma$ with premisses in $\langle \Phi, \Delta \rangle$. The sentence σ is provable from $\langle \Phi, \Delta \rangle$, i.e., it is a default consequence of $\langle \Phi, \Delta \rangle$, iff there is a default proof of σ from $\langle \Phi, \Delta \rangle$.

The view of default tableau constructions as constituting a proof calculus for default reasoning conforms to the following rationale. For a sentence σ and a finite premiss structure $\langle \Phi, \Delta \rangle$, a default tableau for $\neg\sigma$ with premisses in $\langle \Phi, \Delta \rangle$ may be thought as an attempt at proving that σ is a default consequence of $\langle \Phi, \Delta \rangle$, with any default tableau for $\neg\sigma$ with premisses in $\langle \Phi, \Delta \rangle$ that is d-saturated and closed being a successful proof attempt, i.e., a proof that σ is a default consequence of $\langle \Phi, \Delta \rangle$ (q.v. Figure 5.10).

Perhaps requiring a bit of explanation is the idea of a node of a default tableau being d-branching and d-saturated (q.v. Definition 5.3.3, exemplified by node (f) in Figure 5.10). While there is no similar concept in the construction of a standard set labeled tableaux for classical propositional logic, its underpinning rationale may be understood by drawing the following correspondence. Let β be a sentence of branching type appearing in a tableau τ; if instead of extending τ simultaneously with two nodes, whose labels correspond to the components β_1 and β_2 of β, for whatever reason, τ must be expanded one node at a time, then, there would be an impossibility for proceeding solely at the level of leaves. In such a scenario, there would be a need to take note of which one of the components of β has been used in extending τ, and to consider what would be the case had the other component been used, i.e., construct the alternative branch at the level of some intermediate node of τ. If a tableau is being constructed in this way, then, it would be completed, in a branching sense, once both components of a sentence of branching type have been used. Of course, this explanation is an elaborate way of describing what otherwise is an extremely simple construction which exhausts all possibilities for a sentence of branching type, i.e., "add two different nodes as immediate successors of another one". In this respect, there seems to be no rationale for its preference. However, the situation is rather different for default tableau constructions. In most cases it is necessary to have the flexibility of considering default rules one at a time (recall from the default tableau depicted in Figure 5.10 how using one default rule prohibited the use of another). In such scenarios, d-saturation guarantees that all default rules have been considered (q.v. nodes (g) and (h) in Figure 5.10).

The correctness of default tableau constructions as constituting a proof calculus for default reasoning is stated in Theorem 5.3.1.

Theorem 5.3.1 (Correctness)**.** Let $\mathfrak{L}$ be the logical system for reasoning with default rules of Proposition 4.3.4; for any sentence σ, and for any finite sets Φ and Δ of sentences and default rules, respectively, σ is a default consequence of $\langle\Phi, \Delta\rangle$, i.e., there is a closed and d-saturated default tableau for $\neg\sigma$ with premisses in $\langle\Phi, \Delta\rangle$, iff $\langle\Phi, \Delta\rangle \mathrel{|\!\!\sim} \sigma$, i.e., iff for every extension E of $\langle\Phi, \Delta\rangle$, there is a proof of σ from E, i.e., there is a closed tableau for $\{\neg\sigma\} \cup E$. (q.e.d. in Appendix A.)

As it is typical of tableaux methods, the view of default tableau constructions as constituting a proof calculus further gains in interest for it accommodates for the discovery of default proofs not existing by inspecting some particular cases of proof attempts.

The nonexistence of default proofs is made precise in Proposition 5.3.1 with the aid of Definition 5.3.5.

Definition 5.3.5 (Completed)**.** Let τ be a default tableau for σ with premisses in $\langle\Phi, \Delta\rangle$; a node n of τ with label $\langle\Phi', \Delta'\rangle$ is completed iff:

– For every sentence α of linear type in Φ', the components α_1 and α_2 of α are also in Φ'.

– For every sentence β of branching type in Φ', at least one of the components β_1 or β_2 of β is in Φ'.

– For every default rule $\pi : \rho \;/\; \chi$ in Δ, if $\pi : \rho \;/\; \chi$ meets the side conditions of Definition 5.3.1(Rule c), then, χ is in Φ' and $\pi : \rho \;/\; \chi$ is in Δ'.

The default tableau τ is complete iff all of its leaf nodes are completed.

Proposition 5.3.1. If a default tableau for $\neg\sigma$ with premisses in $\langle \Phi, \Delta \rangle$ has a complete leaf node that is also open, then, σ is not a default consequence of $\langle \Phi, \Delta \rangle$.

The default tableau depicted in Figure 5.11 indicates that t is not a default consequence of $\langle \{p, p \supset (q \vee r \supset s)\}, \{p : u \ / \ q \wedge t, p : t \ / \ r \wedge u\} \rangle$.

(a) $p \supset (q \vee r \supset s)$
p
$\neg t$

(b) $p \supset (q \vee r \supset s)$
p
$\neg t$
$\neg p$

(c) $p \supset (q \vee r \supset s)$
p
$\neg t$
$q \vee r \supset s$

(d) $p \supset (q \vee r \supset s)$
p
$\neg t$
$q \vee r \supset s$
$\neg(q \vee r)$

(e) $p \supset (q \vee r \supset s)$
p
$\neg t$
$q \vee r \supset s$
s

(f) $p \supset (q \vee r \supset s)$ $\langle p : t \ / \ r \wedge u \rangle$
p
$\neg t$
$q \vee r \supset s$
s
$r \wedge u$

(g) $p \supset (q \vee r \supset s)$ $\langle p : t \ / \ r \wedge u \rangle$
p
$\neg t$
$q \vee r \supset s$
s
$r \wedge u$
r
u

Figure 5.11: Default tableau for $\neg t$ with premisses in
$\langle \{p, p \supset (q \vee r \supset s)\}, \{p : u \ / \ q \wedge t, p : t \ / \ r \wedge u\} \rangle$

In essence, a leaf node of a default tableau that is both complete and open constructs an extension from which the alleged default consequence does not follow. For the case of the default tableau depicted in Figure 5.11, i.e., default tableau for $\neg t$ with premisses in $\langle\{p, p \supset (q \vee r \supset s)\}, \{p : u \;/\; q \wedge t, p : t \;/\; r \wedge u\}\rangle$, said extension, the set $E_2 = \{p, p \supset (q \vee r \supset s), r \wedge u\}$, is obtained from the second component of the label of the leaf node (g) in Figure 5.11 together with the set of sentences of the premiss structure in question. That t is not a consequence of this extension is also immediate from the information present in the leaf node (g) in Figure 5.11: the first component of this node corresponds to a leaf node of a tableau for $\neg t$ with premisses in E_2.

5.4 Discussion

One of the most concise descriptions of the rationale underlying tableau methods as proof methods is provided by Fitting in [Fit99]. In Fitting's terms, a tableau method is a formal proof procedure, existing in a variety of forms and for several logics, but always having certain characteristics. First, it is a refutation procedure. In order to prove that something is the case, the initial step is to begin with a syntactical expression intended to assert the contrary. Successive steps then syntactically break down this assertion into cases. Finally, there are impossibility conditions for closing cases. If all cases are closed, then, the initial assertion has been refuted. As a result, it is concluded that what had been taken not to be case is actually the case.

The kind of default tableau constructions presented here operate in the way just described. In order to prove that a sentence σ is a default consequence of a premiss structure $\langle\Phi, \Delta\rangle$, the construction of a default tableau begins with a syntactical expression intended to assert that this is not the case. The set $\Phi \cup \{\neg\sigma\}$ is said syntactical expression. Next, the sentences in this expression are syntactically broken down into their components according to R1 or R2 in Definition 5.3.1 depending on whether they are of linear or of branching type, respectively. R3 in Definition 5.3.1 corresponds to the way in which default rules are to be treated in the construction of a default proof as premiss-like objects. More precisely, the side conditions of R3 in Definition 5.3.1 require the provability and the non-provability of the prerequisite and the justification of a default rule in an underlying proof system (in this case, a tableaux method for CPL). The use of a default rule is either allowed or preempted in a default proof depending on whether these requirements are met or not. Finally, the closedness and d-saturation of a default tableau indicate the impossibility conditions that are needed to establish that what was asserted not to be the case, i.e., that σ is not a default consequence of $\langle \Phi, \Delta \rangle$, is actually the case, altogether establishing whether or not σ is a default consequence of $\langle \Phi, \Delta \rangle$.

In comparison with the tableau constructions presented in Section 5.2, the method of default tableaux may be seen as a combination of Definitions 5.2.6 and 5.2.10 which makes accommodation for the use of default rules. Note that, R1 and R2 in Definition 5.3.1 correspond more or less directly to R1 and R2 in Definition 5.2.10, while R3 in Definition 5.3.1 corresponds, modulo satisfaction of its side conditions, R3 in Definition 5.2.10, i.e., to the manner in which premisses are dealt with in CPL.

The principles underpinning the definition and construction of a default tableau may also be understood in comparison with those intuitions underlying the definition and construction of a tableau for a set of sentences. As briefly mentioned, every branch of a tableau for $\Gamma \cup \{\sigma\}$ may be taken as a partial syntactical description of a model of Γ that is also a model of σ, closed branches indicate that this description is an impossibility, whereas open branches indicate the contrary. In a default tableau for σ with premisses in $\langle \Phi, \Delta \rangle$, the extensions of $\langle \Phi, \Delta \rangle$ play the role of models (q.v. Chapter 4). In this respect, every branch of this default tableau may be taken as a partial description of an extension E of $\langle \Phi, \Delta \rangle$ that has been enlarged by σ; closed branches indicate that this enlargement is an impossibility, whereas open branches indicate the contrary; d-saturation indicates that all extensions have been considered.

In summary, and by way of conclusion, the method of default tableaux developed in Section 5.3 may be seen as a contribution to the proof theory of reasoning with default rules. In a sense, its main features are: (i) its simplicity, in that, as commented on above, it does not deviate from the standard presentation of a method for tableaux;

and (ii) the fact that, in certain cases, default proofs may only involve part of a premiss structure (something which is also true when it comes to showing their nonexistence). The advantages of the latter are immediate.

Related to the ideas presented in Section 5.3 are the following works: [AAGP96], [Ris96], and [Oli99]. However, in contrast to what is the case for the notion of a default tableau presented in this section, the focus of attention of these works is to use tableau constructions for constructing extensions (much in line with Reiter's view of a default proof in the last sections of [Rei80]). In that respect, the ideas presented here extend these previous works by formulating a suitable notion of a proof, therefore establishing a basis for a proof theory for reasoning with default rules. Evidently, much is yet to be done. Issues related to the development of strategies for systematizing default tableau proofs and properties of default tableau proofs are just some preliminary thoughts which have to be developed further.

Chapter 6

Mapping Concepts for Default Presentations

This chapter presents the notion of a default presentation together with an associated notion of a mapping between default presentations. Its main technical results concern the proof of existence of certain kinds of mappings between default presentations. How this concepts may be seen as being applied in practice is the subject matter of Chapter 7.

For the rest of this chapter, the basic definitions concerning theories, presentations, and their corresponding mappings, presented in Chapter 2 are assumed as given. In addition, it is assumed that $\mathfrak{L}$ is the logical system for reasoning with default rules of Proposition 4.3.4. In that respect, $\mathscr{L}$ is $\mathfrak{L}$'s underlying language, i.e., the standard propositional language (q.v. Definition 2.1.2). A sentence is an element of $\mathscr{L}$. Uppercase Greek letters indicate arbitrary set of sentences. In turn, $\mathscr{D}$ is $\mathfrak{L}$'s underlying set of default rules, i.e., the set of all default rules defined on $\mathscr{L}$. A default rule is an element of $\mathscr{D}$. It is further assumed that $\mathscr{P}$ is $\mathfrak{L}$'s underlying set of premiss structures, i.e., the set of all tuples whose first component is a set of sentences and whose second component is a set of default rules. $\mathfrak{L}$'s underlying entailment relation is indicated by $\mid\!\sim$, with $\mathfrak{L}$'s underlying notion of an extension, q.v., Definition 4.2.3, indicated by $\mathscr{E}$. Lastly, Cn indicates $\mathfrak{L}$'s underlying consequence operator, i.e., that of CPL.

6.1 Default Presentations and Mappings for Default Presentations

Following from Definition 2.1.8, given a signature Σ, i.e., a subset of the set of propositional symbols of $\mathscr{L}$, a theory over Σ, or a theory for short, is a set of sentences Θ such that (i) $\mathcal{L}(\Theta) \subseteq \Sigma$ and (ii) $\Theta = Cn(\Theta)$. Theories of this sort are typically denoted by tuples $\langle \Sigma, \Theta \rangle$. For certain purposes, however, the interest lies not in theories *per se* but in what are called (theory) presentations. In brief, a presentation of a theory $\langle \Sigma, \Theta \rangle$ is a set of sentences Γ such that (i) $\mathcal{L}(\Gamma) \subseteq \Sigma$ and (ii) $\Theta = Cn(\Gamma)$. Presentations are also typically denoted by tuples $\langle \Sigma, \Gamma \rangle$.

To be noted at this point is that as objects of logical study theories and presentations are equivalent. Every theory can be seen as a presentation, i.e., a theory is a presentation of itself, and every presentation can itself be seen as a theory, i.e., a presentation is the theory that it generates.

However, as also commented in [Mes89], presentations allow for finer distinctions that are important not only for proof-theoretic or computational purposes, but also for specification purposes.

For instance, let $\langle \Sigma, \Gamma \rangle$ be a presentation of a theory $\langle \Sigma, \Theta \rangle$; resorting to the notion of a presentation it is possible to distinguish between a sentence that is basic, i.e., that it belongs to Γ, and another that is derived, i.e., that it belongs to $\Theta \setminus \Gamma$ (no such a distinction is possible resorting to the notion of a theory in isolation).

The view of presentations as theories forms the basis for Definition 6.1.1.

Definition 6.1.1. The class $\mathfrak{P}_{\mathrm{d}}$ of all default theory presentations, or default presentations for short, of the logical system for default rules $\mathfrak{L}$ of Proposition 4.3.4 consists of all tuples $\langle \Sigma, \langle \Phi, \Delta \rangle \rangle$ where: (i) Σ, the signature of $\langle \Sigma, \langle \Phi, \Delta \rangle \rangle$, is a subset of the set of propositional symbols of $\mathscr{L}$, (ii) $\mathcal{L}(\langle \Phi, \Delta \rangle) \subseteq \Sigma$, and (iii) $\langle \Phi, \Delta \rangle \in \mathscr{P}$. A default presentation is an element of $\mathfrak{P}_{\mathrm{d}}$. A default presentation $\langle \Sigma, \langle \Phi, \Delta \rangle \rangle$ is finite iff both Φ and Δ are finite.

In the standard sense derived from Definition 6.1.1, the idea is that a default presentation $\langle \Sigma, \langle \Phi, \Delta \rangle \rangle$ indicates the set $\{\delta \mid \langle \Phi, \Delta \rangle \mathrel{|\!\sim} \delta\}$ of all default consequences that can be drawn from $\langle \Phi, \Delta \rangle$.

At the same time, in analogy with what is the case for presentations in classical logical systems, it would be tempting to define a notion of a mapping between two default presentations $\langle \Sigma, \langle \Phi, \Delta \rangle \rangle$ and $\langle \Sigma', \langle \Phi', \Delta' \rangle \rangle$ in terms of a function m with domain Σ and co-domain Σ' such that, for all sentences $\delta \in \mathscr{L}|_{\Sigma}$,[†] $\langle \Phi', \Delta' \rangle \mathrel{|\!\sim} m(\delta)$ whenever $\langle \Phi, \Delta \rangle \mathrel{|\!\sim} \delta$, where $m(\delta)$ is the translation of δ along m (q.v. [TM87] and [Mes89]).

Notwithstanding, in the context of reasoning with default rules, a case can be made that the sought after notion of a mapping between default consequences is one that does not require the necessary preservation of default consequences. In other words: What would be the purpose of resorting to a non-monotonic entailment relation if its associated theory presentations are to be structured monotonically?

This last standpoint raises the following question: What sensible notion of structure

[†] $\mathscr{L}|_{\Sigma} = \{\sigma \in \mathscr{L} \mid \mathcal{L}(\sigma) \subseteq \Sigma\}$.

between two default presentations may be preserved by a function on signatures if the *desideratum* of nonmonotonicity is to be accounted for? This question leads in a natural way to focusing attention on the extensions of a default presentation. Definition 6.1.2 is introduced in this light.

Definition 6.1.2. Let $\langle\Sigma,\langle\Phi,\Delta\rangle\rangle$ and $\langle\Sigma',\langle\Phi',\Delta'\rangle\rangle$ be two default presentations; a function m with domain Σ and co-domain Σ' is a mapping between $\langle\Sigma,\langle\Phi,\Delta\rangle\rangle$ and $\langle\Sigma',\langle\Phi',\Delta'\rangle\rangle$, i.e., a mapping of default presentations, iff for every extension $E\in\mathscr{E}(\langle\Phi,\Delta\rangle)$, there is an extension $E'\in\mathscr{E}(\langle\Phi',\Delta'\rangle)$ such that $m(Cn(E))\subseteq Cn(E')$, i.e., if m is a presentation mapping between $\langle\Sigma,E\rangle$ and $\langle\Sigma',E'\rangle$ in CPL.

As a comment in passing, regarding Definition 6.1.2, if m is a mapping between two default presentations $\langle\Sigma,\langle\Phi,\Delta\rangle\rangle$ and $\langle\Sigma',\langle\Phi',\Delta'\rangle\rangle$, then, the aforementioned *desideratum* of nonmonotonicity is realized whenever the cardinality of $\mathscr{E}(\langle\Phi,\Delta\rangle)$ is strictly less than the cardinality of $\mathscr{E}(\langle\Phi',\Delta'\rangle)$. In such situations, it is not necessarily the case that if δ is a default consequence of $\langle\Phi,\Delta\rangle$, then, $m(\delta)$ is a default consequence of $\langle\Phi',\Delta'\rangle$. More or less immediate from this observation, the mapping m is necessarily monotonic whenever $\mathscr{E}(\langle\Phi,\Delta\rangle)$ and $\mathscr{E}(\langle\Phi',\Delta'\rangle)$ have the same cardinality.

At this point, it should be noted that while Definition 6.1.2 defines a somewhat general notion of a mapping between default presentations, in and of itself, it provides no conditions under which such mappings between default presentations may exist. The rest of this section lists some properties on default presentations which guarantee the existence of mappings between them.

Proposition 6.1.1. Let $\langle\Sigma,\Phi\rangle$ and $\langle\Sigma',\Phi'\rangle$ be two presentations in CPL; if m is a presentation mapping between $\langle\Sigma,\Phi\rangle$ and $\langle\Sigma',\Phi'\rangle$ in CPL, then, m is a mapping between the default presentations $\langle\Sigma,\langle\Phi,\emptyset\rangle\rangle$ and $\langle\Sigma',\langle\Phi',\emptyset\rangle\rangle$ in $\mathfrak{P}_{\mathrm{d}}$, i.e., m is a mapping of default presentations in $\mathfrak{L}$. (q.e.d. Appendix A.)

Briefly, Proposition 6.1.1 indicates that, in the absence of default rules, mappings of default presentations are the mappings of their underlying presentations.

Proposition 6.1.2. Let $\langle \Sigma, \Phi \rangle$ and $\langle \Sigma', \Phi' \rangle$ be two presentations in CPL; if m is a presentation mapping between $\langle \Sigma, \Phi \rangle$ and $\langle \Sigma', \Phi' \rangle$ in CPL, then, for any set of default rules Δ' such that $\mathcal{L}(\Delta') \subseteq \Sigma'$, it follows that m is a mapping between the default presentations $\langle \Sigma, \langle \Phi, \emptyset \rangle \rangle$ and $\langle \Sigma', \langle \Phi', \Delta' \rangle \rangle$ in $\mathfrak{P}_{\mathrm{d}}$, i.e., m is a mapping of default presentations in $\mathfrak{L}$. (q.e.d. Appendix A.)

Corollary. If m is an axiom preserving mapping between presentations $\langle \Sigma, \Phi \rangle$ and $\langle \Sigma', \Phi' \rangle$ in CPL, then, for any set of default rules Δ' such that $\mathcal{L}(\Delta') \subseteq \Sigma'$, it follows that m is a mapping between the default presentations $\langle \Sigma, \langle \Phi, \emptyset \rangle \rangle$ and $\langle \Sigma', \langle \Phi', \Delta' \rangle \rangle$ in $\mathfrak{L}$.

In ordinary terms, Proposition 6.1.2 indicates that a basic default presentation, i.e., one whose set of default rules is empty, can be extended by extending both its underlying set of sentences and its underlying set of default rules arbitrarily.

Proposition 6.1.3. Let m be a mapping with domain Σ and co-domain Σ' and let $\langle \Sigma, \Phi \rangle$ and $\langle \Sigma', \Phi' \rangle$ be presentations in CPL; if $\langle \Sigma', \Phi' \rangle$ is a faithful interpretation of $\langle \Sigma, \Phi \rangle$ along m in CPL, then, for any two signatures Σ and Σ' and for any two sets Δ and Δ' of default rules defined on Σ and Σ', respectively, it follows that, if $m(\Delta) \subseteq \Delta'$, then, m is a mapping between $\langle \Sigma, \langle \Phi, \Delta \rangle \rangle$ and $\langle \Sigma', \langle \Phi', \Delta' \rangle \rangle$, i.e., m is a mapping of default presentations in $\mathfrak{L}$. (q.e.d. Appendix A.)

Corollary. If m is an inclusion mapping with domain Σ and co-domain Σ', then, for sets of default rules Δ and Δ' such that $m(\Delta) \subseteq \Delta'$, it follows that m is a mapping between the default presentations $\langle \Sigma, \langle \Phi, \Delta \rangle \rangle$ and $\langle \Sigma', \langle m(\Phi), \Delta' \rangle \rangle$ in $\mathfrak{L}$.

In ordinary terms, Proposition 6.1.3 indicates that a default presentation can be extended by extending its underlying set of sentences faithfully and its underlying set of default rules arbitrarily.

6.2 Discussion

Related to the ideas presented in Section 6.1 is the following series of works: [AMF96], [AM00], and [MA97]. The last of which is particularly relevant here.

Motivated by some of the seminal ideas proposed in [TM87], and with the purpose of applying logical specification techniques in the context of reasoning with default rules, in [MA97], MacNish et. al. present three different notions of a mapping between default presentations. The first two of these different mappings between default presentations, referred to as skeptical and credulous, correspond to whether the mapping accounts for what is the case in all extensions or in some extensions, respectively. The last of these different mappings between default presentations is in correspondence with Definition 6.1.2. Notwithstanding, in [MA97] the focus of attention is placed on proving properties of correspondence between the alternative definitions of the alternative mapping between default presentations. In contrast, as made clear in Propositions 6.1.1 to 6.1.3, the emphasis here is placed on providing some minimal conditions which guarantee the existence of mappings between default presentations. In that respect, these propositions extend the ideas presented in [MA97] and establish a basis on which to structure default specifications. What such a basis may look like is the subject matter of Chapter 7.

Chapter 7

Is There Some Beef?

This chapter presents a potential application of the concepts discussed in the previous chapters of this thesis.

In essence, to be shown is how default presentations and their mappings, as developed in Chapter 6, can be used as modularization units for specification. The latter, while in line with the school of thought started by Burstall et al. in [BG77], and works such as [TM87], [FS87], and [FM92], is a seemingly missing pursuit in the area of reasoning with default rules.

The following excerpt, taken from [Sch08], provides some rationale for the title of this chapter, as well as some context for what is to be developed in what follows.

> At the occasion of the Third International Conference on Principles of Knowledge Representation and Reasoning in 1992, Ray Reiter delivered an invited talk entitled *Twelve Years of Nonmonotonic Reasoning Research:*

> *Where (and What) is the Beef?* reflecting the state and future of the research area of Nonmonotonic Reasoning (NMR). Ray Reiter describes it in as a "flourishing subculture" making many outside researchers "wonder what on earth this stuff is good for".

On challenge put forth by Reiter in [Rei92], Schaub comments in [Sch08] that while Reiter appeared optimistic about the future of NMR, a point was made as its major contributions being on the theoretical side. In Reiter's terms, these contributions have provided some "important insights about, and solutions to, many outstanding problems, not only in Artificial Intelligence but in computer science in general" (q.v. [Sch08, p. 1]).

If the previous remark is to be accounted for, then, Reiter's challenge is perhaps best formulated as: What is reasoning with default rules good for from a somewhat practical perspective, as opposed to a purely theoretical one?

When looked at from that perspective, there are many and quite dissimilar areas of application for reasoning with default rules, ranging from Answer Set Programming to Software Engineering to Law, just to name a few, which may be seen as responding satisfactorily to the last question (q.v. [Sch08, AG99b]). The area of interest here is that of software engineering. More in particular, the locus of attention will be placed on the work of Antoniou et al. in [AG99a].

In brief, in [AG99a], Antoniou et al. present how certain elements of reasoning with default rules can be used as a basis for Formal Requirements Engineering. To this end, they resorted to the well known case of the London Ambulance Service Computer Aided Dispatch System as a basis for discussion (q.v. [Fin93]). This chapter presents how the use of default rules in the context of requirements proposed by Antoniou et al. in [AG99a] may be extended to accommodate for a basic criterion of modularity in reasoning, a much needed feature that is absent in [AG99a]. The London Ambulance Service Computer Aided Dispatch System will also be the case study at issue here, with the approach to Formal Requirements Engineering proposed by Antoniou et al. in [AG99a] as the point for comparison.

7.1 The London Ambulance System

Following closely the work of Antoniou et al. in [AG99a], this section introduces and motivates the selection of the London Ambulance Service Computer Aided Dispatch System as a case study for Formal Requirements Engineering. Its purpose is that of establishing a context for discussion.

First things first, the Computer Aided Dispatch (CAD) system deployed by the London Ambulance Service (LAS) was the case study in point of the 8th International Workshop on Software Specification and Design (IWSSD-8). As commented by Finkelstein et al. in [FD96], underpinning the selection of this case study was the following rationale:

> The International Workshop on Software Specification & Design has established a tradition of using case studies to focus and provide coherence to its intensive working sessions. These case studies, supplied in advance to participants in the various tracks, have proved a fruitful way of working. Evidence of this can be seen most clearly in the proceedings or workshop reports which have followed previous workshops. It was decided for IWSSD-8 that, in order to provide common ground between the tracks, a single shared case study should be used, with each track drawing on it in a manner appropriate to their own interests and concerns. After some discussion we settled on the 'Report of the Inquiry Into the London Ambulance Service [CAD System]' which is interesting in its own right, reflects aspects of requirements architecture, design, concurrency and distribution, and raises significant issues on the relation between these aspects.

Finkelstein et al. mention in [FD96] that the purpose of the LAS CAD system was that of supplanting the then employed manual system in an attempt to improve the existing state of affairs. Essentially, to the in use manual LAS dispatch system – responsible for receiving calls, dispatching ambulances based on an understanding of the nature of the calls and the availability of resources, and monitoring progress of the response to the call – the to be developed CAD system would incorporate an automatic vehicle locating system (AVLS) and mobile data terminals (MDTs) to support automatic communication with ambulances (q.v. [FD96]).

Upon acknowledging that the construction of software based systems typically commences with the collection and elicitation of a set of requirements, Antoniou et al. focus, in [AG99a], on formalizing part of the requirements of the LAS CAD system. Antoniou et al.'s *desideratum* is for the final product of requirements engineering to be a formal specification capturing customer requirements.

Regarding such a *desideratum*, Antoniou et al. recognize in [AG99a] that the process of translating informal, vague, and possibly conflicting requirements into a formal specification is a demanding task in relation to which some central issues arise, particularly noteworthy are: (i) inconsistency management, (ii) evolving requirements and specifications, and (iii) quality of specifications. Elaborating on (i), Antoniou et al. note that while the formalism that has found most application in specification is, in one form or another, classical logic,† said formalism is unsuitable for handling conflicting information. The argument for the latter is that classical logic makes it almost impossible to maintain multiple points of view of the set of requirements being elicited. This is due to the trivialization that occurs in the face of an inconsistency. If different points of view of a set of requirements are inconsistent, either internally or in relation to each other, their specification is a triviality from which anything can be derived. Taking as implicit that a conceptual framework for modeling requirements should provide clear means for representing and reasoning with conflicting information in a coherent fashion, that Antoniou et al. propose in [AG99a] the use of reasoning with default rules for Formal Requirements Engineering.

†In [AG99a], classical logic most likely means classical first order logic.

In this context, taking the requirement statements mentioned in [FD96] as a basis for discussion, Antoniou et al. focus in [AG99a] on their analysis from the points of view of three different stakeholders: an incident room controller, an operations manager, and a logistics manager. These points of view, summarized below, can be found in [AG99a, pp. 9–10].

Incident Room Controller (IRC) 1. A medical emergency is either the result of an illness or an accident. 2. On receipt of a phone call reporting a medical emergency, an ambulance should be dispatched to the scene. 3. On receipt of a phone call, if the incident is judged not to be a medical emergency, then the call should be transferred to another emergency service (such as police or fire brigade).

Operations Manager (OM) 1. On receipt of a phone call reporting an accident, if an ambulance is available then it should be dispatched to the scene. 2. On receipt of a phone call reporting an incident, an ambulance should not be dispatched to the scene if no ambulance is available.

Logistics Manager (LM) 1. If no ambulance operators (drivers, medics) are available, then no ambulance is available. 2. If no ambulances are available, then initiate a search for a free ambulance. 3. If one year has passed since the maintenance work was last done on an ambulance, then perform a safety check on that ambulance.

Turning to their formalization, Antoniou et al. propose in [AG99a] that assertions originating from individual points of view are to be understood as being tentative in nature and, as a result of being such, they ought to be formalized as default rules. In this respect, the following default rule schemata presents a possible formalization of the assertions originating from the points of view of the stakeholders under consideration. In this formalization $\pi \;/\; \chi$ abbreviates $\pi : \chi \;/\; \chi$.

Incident Room Controller (IRC) $a_x \vee i_x \;/\; m_x$; $c_x \wedge m_x \;/\; d_i$; $c_x \wedge \neg m_x \;/\; t_x$.

Operations Manager (OM) $c_x \wedge f_i \;/\; d_i$; $c_x \wedge \neg f_i \;/\; \neg d_i$.

Logistics Manager (LM) $\neg o_i \;/\; \neg f_i$; $\bigwedge_{i \in I} \neg f_i \;/\; f$; $\neg y_i \;/\; s_i$.

In the above default rule schemata: a_x stands for 'x has suffered an accident'; i_x stands for 'x has suffered from an illness'; m_x stands for 'x is having a medical emergency'; c_x stands for 'x has called reporting an incident'; d_i stands for 'ambulance i has been dispatched'; t_x stands for 'x has been transferred to another emergency service'; f_i stands for 'ambulance i is available'; o_i stands for 'there are operators available for ambulance i'; f stands for 'initiate the search for a free ambulance'; y_i stands for 'ambulance i has been serviced within one year'; and lastly, s_i stands for 'safety check ambulance i'.

Furthermore, Antoniou et al. discuss in [AG99a] that in analyzing a requirements specification, there is a need to incorporate information about concrete scenarios in order to determine whether the system under consideration exhibits the desired behavior, or whether there are potential conflicts among requirements. The proposal put forth by Antoniou et al. discuss in [AG99a] is that assertions originating from a concrete scenario are not subject to doubt, as a result of which, they ought to be formalized as sentences that have a definite status (e.g., relative to the case study under consideration, this means that either an accident was reported or not).

Following from the previous considerations, and assuming a scenario where there are two ambulances, one of which is available and one of which has not been serviced for one year, and where a person has called reporting that he has suffered an accident, Antoniou et al. in [AG99a, p. 10], propose a specification that is in all essential aspects similar to that in dr-Spec 1.

spec

$LAS{-}CAD$

signature

$$\begin{array}{llllllll} a_1; & c_1; & d_1; & d_2; & f; & f_1; & f_2; & i_1 \\ m_1; & o_1; & o_2; & s_2; & t_1; & y_1; & y_2. & \end{array}$$

definite

$$a_1; \quad c_1; \quad \neg o_2; \quad f_1; \quad \neg y_2.$$

tentative

$$\begin{array}{rrr} a_1 \vee i_1 \ / \ m_1; & c_1 \wedge m_1 \ / \ d_1; & c_1 \wedge m_1 \ / \ d_2; \\ c_1 \wedge \neg m_1 \ / \ t_1; & c_1 \wedge f_1 \ / \ d_1; & c_1 \wedge f_2 \ / \ d_2; \\ c_1 \wedge \neg f_1 \ / \ \neg d_1; & c_1 \wedge \neg f_2 \ / \ \neg d_2; & \neg o_1 \ / \ \neg f_1; \\ \neg o_2 \ / \ \neg f_2; & \neg f_1 \wedge \neg f_2 \ / \ f; & \neg y_1 \ / \ s_1; \\ \neg y_2 \ / \ s_2. & & \end{array}$$

DR-SPEC 1: LAS-CAD Specification

It is with respect to dr-Spec 1 that Antoniou et al. discuss the use of reasoning with default rules in the context of Formal Requirements Engineering in [AG99a].

Wile there is certainly value in the sort of analysis proposed by Antoniou et al. in [AG99a], something that will be discussed in more detail in Section 7.3, it is possible to make a case that dr-Spec 1 is intrinsically monolithic. As it is argued in [FM92], a monolithic specification of a system is an unstructured, and possibly unmanageably large, collection of sentences, that is difficult to make sense of, hindering the quality of the specification. If specifications are taken to be closely associated with the need of achieving modularity in the system being specified, following Burstall et al. in [BG77], the idea is for the structure of a specification to be made explicit by indicating how said specification, as a logical presentation, is formulated in terms of smaller, more tractable logical presentations, via appropriate mappings. This will be the subject matter of Section 7.2.

7.2 Here is Some Beef

As commented on in Section 7.1, a potential critique of the LAS-CAD specification given in dr-Spec 1 is that, in spite of it being described in a modular fashion, it is intrinsically monolithic.

Following from that remark, it is possible to argue that, while there are obviously major difficulties in producing specifications that adequately reflect users' requirements for complex systems, such difficulties may be mitigated if specifications are made as modular as possible, e.g., if they are built from smaller and, potentially, easier to understand pieces.[†]

It is not difficult to argue that this *desideratum* of modularity in specifications is not reflected in the specification presented in dr-Spec 1. In response, this section shows the way in which the structuring mechanisms presented in Chapter 6 may prove to be a basic guide for fulfilling such a *desideratum*. In brief, the point to be made is that default presentations serve as the modularization units for the formal specification of the set of requirements for a system, as these are understood from the points of view of different stakeholders, with the overall structure of the resulting specification, i.e., the putting together of these points of view, being made explicit as mappings between default presentations.

In order to illustrate this point, the rest of this section is dedicated to show how the set of requirements for the LAS CAD system specified in dr-Spec 1 can be built from the individual specification of the manner in which an incident room controller, an operations manager, and a logistics manager view a particular scenario.

As in Section 7.1, the scenario at hand will be taken to be one where there are two

[†]This is for instance the basic message of [BG77].

ambulances, the first of which is available and the second of which has not been serviced for one year, and where there is a person that has called reporting that he has suffered an accident. This scenario is specified in dr-Spec 2.

spec

$SCEN$

signature

$a_1;\quad c_1;\quad f_1;\quad o_2;\quad y_2.$

definite

$a_1;\quad c_1;\quad f_1;\quad \neg o_2;\quad \neg y_2.$

DR-SPEC 2: Scenario

In turn, the manner in which an incident room controller, an operations manager, and a logistics manager view the particular scenario at hand are specified in dr-Specs 3 to 5, respectively.[†]

spec

IRC

signature

$$\begin{array}{lllll} a_1; & c_1; & d_1; & d_2; & f_1; \\ m_1; & i_1; & o_2; & t_1; & y_2. \end{array}$$

definite

$a_1;\quad c_1;\quad \neg o_2;\quad f_1;\quad \neg y_2.$

tentative

$$\begin{array}{ll} a_1 \vee i_1 \;/\; m_1; & c_1 \wedge m_1 \;/\; d_1; \\ c_1 \wedge m_1 \;/\; d_2; & c_1 \wedge \neg m_1 \;/\; t_1. \end{array}$$

DR-SPEC 3: Incident room controller

[†]Given that in this section extensions are taken to be defined as in Definition 4.2.3, in dr-Specs 3 to 5, $\pi \;/\; \chi$ abbreviate $\pi : \neg\chi \;/\; \chi$.

spec

OM

signature

$$a_1; \quad c_1; \quad d_1; \quad d_2;$$
$$f_1; \quad f_2; \quad o_2; \quad y_2.$$

definite

$$a_1; \quad c_1; \quad \neg o_2; \quad f_1; \quad \neg y_2.$$

tentative

$$c_1 \wedge f_1 \;/\; d_1; \qquad c_1 \wedge f_2 \;/\; d_2;$$
$$c_1 \wedge \neg f_1 \;/\; \neg d_1; \quad c_1 \wedge \neg f_2 \;/\; \neg d_2.$$

DR-SPEC 4: Operations manager

spec

LM

signature

$$a_1; \quad c_1; \quad f; \quad f_1; \quad f_2; \quad o_1;$$
$$o_2; \quad s_2; \quad t_1; \quad y_1; \quad y_2.$$

definite

$$a_1; \quad c_1; \quad \neg o_2; \quad f_1; \quad \neg y_2.$$

tentative

$$\neg o_1 \;/\; \neg f_1; \quad \neg o_2 \;/\; \neg f_2; \quad \neg f_1 \wedge \neg f_2 \;/\; f;$$
$$\neg y_1 \;/\; s_1; \qquad \neg y_2 \;/\; s_2.$$

DR-SPEC 5: Logistics manager

At this point, the relevance of the notion of a mapping defined in Definition 6.1.2 becomes more or less clear: mappings between default presentations are the structuring mechanism catering for the formulation of a complex specification in terms of simpler and easier to understand component specifications. They do so by making precise the sense in which a source default presentation is to be understood as being part of a target default presentation.

More concretely, if dr-Specs 1 to 5 are taken as a collection of modular specifications, then, their view as default presentations, together with the notion of a mapping between two default presentations, define the structure that the formalism developed in Chapter 6 provides for specification.

Moreover, to be noted is that, the notion of a mapping between default presentations is already there in the specifications of the points of view of the incident room controller, the operations manager, and the logistics manager. Given that each of dr-Specs 3 to 5 enlarges dr-Spec 2 with a corresponding instantiation of the default rule schemata of Section 7.1, it is trivial to prove that there are mappings between dr-Spec 2 and each of dr-Specs 3 to 5 when these are seen as default presentations.

Perhaps more importantly, if the interactions between the points of view of the incident room controller, the operations manager, and the logistics manager are taken to be specified as the unions of their corresponding specifications, then, it is also trivial to prove that there are mappings between the specification of each point of view and the specification of the resulting interaction (again, when these are seen as default presentations).

By way of summary, the point at issue in this section, i.e., that default presentations serve as the modularization units for engineering a set of requirements, as these are understood from the point of view that different stakeholders may have of a particular scenario, with the overall structure of the engineered requirements being made explicit in terms of mappings between specifications as default presentations, is, for the case study under consideration, showcased in the form of a diagram in Figure 7.1.

$$
\begin{tikzcd}
 & LAS{-}CAD & \\
IRC \cup OM \arrow[ur] & IRC \cup LM \arrow[u] & OM \cup LM \arrow[ul] \\
IRC \arrow[u] \arrow[ur] & OM \arrow[ul] \arrow[ur] & LM \arrow[ul] \arrow[u] \\
 & SCEN \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

Figure 7.1: LAS-CAD Specification Structure

In Figure 7.1, each node indicates a specification as a default presentation and each arrow indicates a mapping between the corresponding default presentations.

Following from Figure 7.1, the question at hand becomes: What is the added value of making explicit the structure of a specification? This question is the subject matter of Section 7.3.

7.3 Discussion

Even though it consists of single case study, Section 7.2 presented a potential application of the logical concepts that were defined and argued about in the previous chapters of this thesis. The consideration here is that said presentation gains in interest, in particular if seen from a software engineering point of view, if taken not as being solely about the particularities of the case study under consideration, but as introducing some guiding principles for formalizing a system's requirements from a stakeholder point of view. If looked at in this light, the approach proposed in Section 7.2 is more than an exercise in an application of logical concepts. The use of default presentations as modularization units for structuring requirements provides a logical approach to Formal Requirements Engineering that extends in some essential aspects the approach put forth by Antoniou et al. in [AG99a].

In brief, taking as basic that assertions originating from individual points of view are to be formalized as default rules, Antoniou et al. describe how conflicts among requirements can be represented in a particular kind of graph, termed a *blocking graph*. In essence, the idea is that in a blocking graph its nodes are the default rules under consideration and its edges are preemptability conditions, i.e., the source of an edge preempts the supposition of its target. In terms of analyzing requirements, Antoniou et al. argue in [AG99a] that blocking graphs can be a valuable tool in the hands of a requirements engineer. In that respect, this argument makes a case that blocking graphs equip requirement engineers with a concise overview of conflicts among default rules, allowing them to use this information iteratively in discussions with stakeholders.

Though it retains some elements of the work of Antoniou et al. in [AG99a], such as the manner in which assertions originating from individual points of view and assertions originating from particular scenarios are to be formalized, the approach proposed in Section 7.2, differs from the use of blocking graphs in that, contrary to what is the case in the latter, it provides a much needed degree of modularity in (i) description and (ii) analysis. As to (i), modularity is tantamount to parts of a large specification may be given separately and later on brought together. As to (ii), modularity is reflected in that the view of default presentations as modularization units allows for proofs of consequences to be carried on at the level of a part of a large specification, with mappings allowing to establish whether or not these are consequences of the global system. Both modularity in description and analysis are missing elements of a blocking graph, and hence of the approach proposed by Antoniou et al. in [AG99a].

The following analysis is in place to illustrate points (i) and (ii) above.

- From the point of view of the incident room controller, it is possible to infer that, if a call has been received from someone reporting that he has suffered an accident, then, this person is having a medical emergency, allowing for any of the two ambulances to be dispatched.

  The formal counterpart of this analysis corresponds to the following list of results: $IRC \mathrel{|\!\!\sim} m_1$; $IRC \mathrel{|\!\!\sim} d_1$; $IRC \mathrel{|\!\!\sim} d_2$.

- From the point of view of the operations manager, it is possible to infer that, upon receiving a call from someone reporting that he has suffered an accident, if it is known that the first of the two ambulances is free, then, said ambulance could be dispatched. However, without knowing whether the second ambulance is free, it is not possible to infer whether this ambulance is dispatched or not.

  The formal counterpart of this analysis corresponds to the following list of results: $OM \mathrel{|\!\!\sim} d_1$; $OM \not\mathrel{|\!\!\sim} d_2$; $OM \not\mathrel{|\!\!\sim} \neg d_2$.

- Lastly, from the point of view of the logistics manager, it is possible to infer that, if the second of the two ambulances has no operators and if this ambulance has not been serviced for one year, then, this ambulance is not free and it shall be sent to be serviced.

  The formal counterpart of this analysis corresponds to the following list of results: $LM \mathrel{|\!\!\sim} \neg f_2$; $LM \mathrel{|\!\!\sim} s_2$.

In addition, the explicit structure of dr-Spec 1, showcased in Figure 7.1, makes possible the following analysis of the interactions among the points of view that the incident room controller, the operations manager, and the logistics manager have about the particular scenario at hand:

- As mentioned, from the point of view of the operations manager, upon receiving a call from someone reporting that he has suffered an accident, it is not possible to infer whether the second of the two ambulances is dispatched or not without knowing whether this ambulance is free or not. However, if the point of view of the operations manager is taken in conjunction with the point of view of the logistics manager, it is then possible to infer that the second ambulance cannot be dispatched to the scene of the accident. This is an emergent property.

  The formal counterpart of this analysis corresponds to the following list of results: $OM \not\mid\!\sim d_2$; $OM \not\mid\!\sim \neg d_2$; $OM \cup LM \mid\!\sim \neg d_2$;

  In any case, from either the point of view of the operations manager, or the point of view of the operations manager in conjunction with that of the logistics manager, it is possible to infer that the first ambulance is dispatched. This is a preserved property.

  The formal counterpart of this analysis corresponds to the following list of results: $OM \mid\!\sim d_1$; $OM \cup LM \mid\!\sim d_1$;

- In turn, whereas the point of view of the incident room controller allows for any of the two ambulances to be dispatched, from the point of view of the three stakeholders together it is not possible to infer that the second of the two ambulances is dispatched. This is a property that is lost.

  The formal counterpart of this analysis corresponds to the following list of results: $IRC \mid\!\sim d_1$; $IRC \mid\!\sim d_2$; $LAS{-}CAD \not\mid\!\sim d_2$.

– Lastly, that someone has called to report that he has suffered an accident, that this accident has been determined to be a medical emergency, and that the first of the two ambulances is dispatched to the scene of the accident, are inferences that can be drawn locally from the point of view of the incident room controller and that are preserved globally. These are system properties.

The formal counterpart of this analysis corresponds to the following list of results: $IRC \mathrel{|\!\sim} c_1$; $IRC \mathrel{|\!\sim} a_1$; $IRC \mathrel{|\!\sim} m_1$; $IRC \mathrel{|\!\sim} d_1$; $LAS{-}CAD \mathrel{|\!\sim} c_1$; $LAS{-}CAD \mathrel{|\!\sim} a_1$; $LAS{-}CAD \mathrel{|\!\sim} m_1$; $LAS{-}CAD \mathrel{|\!\sim} d_1$;

In terms of analyzing requirements, it is of no doubt here that the previous would be quite useful in the hands of a requirements engineer.

By way of summary, again, the point made is that default presentations serve as the modularization units for the formal specification of the set of requirements as these are understood from the point of view of different stakeholders, with the overall structure of the resulting specification, i.e., the putting together of these points of view, being made explicit as mappings between default presentations. This approach advocates the description of individual points of views as default presentations having a common set of definite sentences, originating from a particular scenario at hand, and the use of mappings between default presentations for expressing their interactions. The consideration here is that making explicit the structure of a requirements specification via appropriate mapping mechanisms provides a much needed degree of modularity in description and analysis. The latter has some immediate and obvious advantages in the context of analyzing requirements.

Notwithstanding, much is yet to be done. Section 7.2 is an elementary showcasing of how some of the ideas and concepts developed in the previous chapters of this thesis may be applied in practice. However, the formalism developed in Chapter 6 and the specifications presented in this chapter are far from being usable for real specifications. It is possible to make a case that the specifications that were presented here contain some unnecessary detail, e.g., signatures, or that they leave some desirable elements out, e.g., clarity at a glance. Perhaps more importantly, in terms of structuring specifications, it possible to make a case that it would be interesting to be able to count with mappings that accommodate for the preservation of certain non-trivial properties of a modularization unit, or to try to categorize what properties of specifications are more generally undermined (both of the things just mentioned will most likely involve an analysis of the structure of a default proof to search for patterns). While these two cases are not developed and discussed further in this thesis, they are certainly being considered as work to undertake.

Chapter 8

Conclusions

Logical studies can be approached from what may be viewed as two rather opposite schools of thought. On the one hand, there is the purely philosophical school of thought, having the conceptual interpretation of the objects of logic as its main subject matter. On the other hand, there is the purely mathematical school of thought, having the formalization of the objects of logic as its main subject matter. Albeit their boundaries are not sharply defined, it can more or less be agreed upon that both of these schools of thoughts are in need of each other and that neither can retain its essence without the other.

Relativised to the subject of reasoning with default rules, this Ph.D. thesis nurtures itself from both of these schools of thought. Its first part is an investigation into foundational matters. In particular, this first part is a critical appraisal of the concept of a default rule being understood as a rule of inference, the concept of an extension being a theory-like object, and the concept of 'being a consequence of' for reasoning with default rules. The second part of this thesis builds on what are perceived as some shortcomings of these foundational matters. More precisely, this second part proposes that the concept of a default rule be formally treated as being a premiss-like object and that of an extension as being an interpretation structure of a syntactical kind. In light of this alternative presentation of default rules and extensions, this second part also proposes a somewhat general view of a logical system for reasoning with default rules – placing its associated concept of 'being a consequence of' in the foreground. Accounting for the fact that in any logical system it is important to have at hand mechanisms for formulating proofs and for structuring large theories, this thesis further presents a tableaux based proof calculus for reasoning with default rules and it explores some mappings notions related to the structuring of default presentations, i.e., presentations in the context of reasoning with default rules. Lastly, this thesis discussed a potential application of its main ideas in the context of software requirements engineering.

Notwithstanding, a Ph.D. thesis is never fully concluded. And this holds particularly true here. A case can be made that each chapter contains many open questions whose subject matter makes them suitable topics for a Ph.D. dissertation in their own right. In this respect, this thesis is the result of a process that exceeds it, and its conclusion, while comprehensive, just an attempt to make this process finite. Hence the despair and the compulsion for (re)reading, correcting, and redoing. As Borges would say,[†] this thesis is finished to put an end to the latter, a sign that it cannot contain what it should contain in a complete form.

8.1 Contributions

Chapter 1 presented, as a first very general problem definition for this Ph.D. thesis, the question of whether reasoning with default rules, as presented in Reiter's seminal 1980 article, can be understood as a logic for nonmonotonic reasoning. While there is certainly an extensive literature on the subject of reasoning with default rules, a

[†] Jorge Luis Borges (Buenos Aires, August 24, 1899 – Geneva, June 14, 1986) Argentine writer.

summary of which can be found in the reference section of [AW07], the main focus of this literature is on discussing the adequacy of (the various variants of) reasoning with default rules for a given problem at hand or on proving theorems of the presented formalizations. In this respect, this thesis occupies a rather different niche. It shifts the focus of attention to some foundational matters.

In the context of this thesis, the perceived need for discussing some foundational matters originated from the way in which the main elements of reasoning with default rules are presented in the literature on the subject, and in particular by Reiter in his seminal 1980 article. In that respect, Section 3.1, made a case against default rules being formally treated as rules of inference of a defeasible kind. This case was grounded on the typical explanations that are provided for a rule of inference to be considered as such, none of which is satisfied by Reiter's presentation of a rule of inference. In turn, Section 3.2, made a case against the concept of an extension being a theory-like object. This case was grounded on the typical presentation of the concept of a theory, of which Reiter's notion of an extension is portrayed as being an instance of. Lastly, Section 3.3, made a case as to whether at all default rules and extensions explicate what the concept of 'being a consequence of' for reasoning with default rules amounts to. This case was grounded on what may be argued to be the main element of a logical system, i.e., an entailment relation (or, alternatively, a consequence operator).

Justified on the consideration that those who intend to use a logic seek logics with sensible theoretical foundations, this thesis proposed an alternative presentation of the main elements for reasoning with default rules. In particular, Section 4.1, advocated for default rules to be formally treated as premiss-like objects. This proposal is grounded on the consideration that default rules may be viewed as indicating those assertion that have a tentative status in a set of premisses. In turn, Section 4.2, advocated for the concept of an extension to be formally treated as an interpretation structure, i.e., a model, of a syntactical kind. This proposal is grounded on the consideration that the class of extensions associated to a set of sentences Φ and a set of default rules Δ may be viewed as all possible realizations of the way in which any maximal subset of the tentative assertions denoted by the default rules in Δ is accepted as being the case relative to the set of definite assertions denoted by the sentences in Φ. Building on these alternative views of a default rule and an extension, Section 4.3 proposed a somewhat general presentation of a logical system for reasoning with default rules, bringing the concept of 'being a consequence of' for reasoning with default rules into the foreground, as well as an instance of what such a logical system for reasoning with default rules may look like (q.v. Proposition 4.3.4).

When taken together, both Chapters 3 and 4 contribute to the foundations of reasoning with default rules, hopefully deepening their understanding in light of the newly presented ideas.

The method of default tableaux developed in Chapter 5 is a contribution to the proof theory of reasoning with default rules.

Aimed at making reasoning with default rules applicable to large specifications, the notion of a mapping between default presentations developed in Chapter 6 introduces a basic mechanism for structuring default presentations. This feature provides the logical system for reasoning with default rules of Proposition 4.3.4 with a support for modularity in specification. An important issue here is the existence of such mappings. The various propositions stated in Chapter 6 contribute to the latter observation.

Lastly, Chapter 7 contributes to the set of potential applications of reasoning with default rules by presenting an example of its use in the context of requirements engineering.

8.2 Future Work

As was duly noted above, each chapter of this thesis contains many open questions whose subject matter makes them suitable topics for a Ph.D. dissertation in their own right.

Particularly interesting is the relationship between Reiter's concept of a default rule and Toulmin's notion of an argument pattern, as commented on in Chapter 3. It would be interesting to study which default rules indeed capture defeasible rules of inference and how these correspond to Toulmin's view of an argument pattern. This may provide both a formal basis for studying the work of Toulmin and a starting point for concocting a logical system which resorts to rules of inference of a defeasible kind.

It would also be interesting to formulate the basic ideas presented in Chapter 4, in particular the definition of a logical system for reasoning with default rules, in categorial terms (much in the style of [Mes89]). Other than the well-known lore of category theory being the appropriate framework for achieving a suitable degree of abstraction and simplicity, an immediate advantage of such an approach would be the possibility of constructing a logical system for reasoning with default rules that is independent of the underlying consequence operator.

The categorial formulation of a logical system for reasoning with default rules may also be useful as a way of extending the default tableaux method of Chapter 5. More precisely, the same ideas used in the development of the default tableaux method presented in Chapter 5 may be used relative to an abstract formulation of a tableaux method, modulo certain properties concerning unprovability are satisfied. This approach may help to elicit whether universal constructions may be used to obtain a default tableau from smaller tableaux, accommodating for a form of compositional reasoning. Moreover, it may yield a somewhat general account of a proof system for reasoning with default rules.

In turn, Chapter 6 investigated some mechanisms for putting together different default presentations. Again, a categorial formulation of such ideas may make these mechanisms independent of the underlying consequence operator of the logical system for reasoning with default rules in question. In this respect, it would be interesting also to explore other composition mechanisms that those present end, e.g., those that preserve non-trivial properties.

Finally, Chapter 7 presented, as a proof of concept, how the ideas developed in the previous chapters of this thesis may be applied in the context of formal requirements engineering. This a topic that is worthy of further exploration.

Appendix A

Proofs of Selected Theorems

Chapter 4

Lemma A.1.1. Let Φ be a set of sentences and Δ be a set of default rules; in addition, w.r.t. Φ, let $\mathbf{P}$ be the set of all acceptable subsets of Δ, q.v. Definition 4.2.2, partially ordered by inclusion; $\mathbf{P}$ has a maximal chain.

Proof of Lemma A.1.1 By definition $\mathbf{P}$ satisfies the following two properties: (i) $\emptyset \in \mathbf{P}$; (ii) let $\Delta_a \in \mathbf{P}$ and $\delta \in \Delta \setminus \Delta_a$, if $\Delta_a \cup \{\delta\}$ is acceptable w.r.t. Φ, then, $\Delta_a \cup \{\delta\} \in \mathbf{P}$; no other element belongs to $\mathbf{P}$.

Let $\mathbf{P}$ be partially ordered by inclusion and $\mathbf{C}$ be the set of all chains of $\mathbf{P}$; $\mathbf{C}$ is not empty.

Let $\mathbf{C}$ be partially ordered by inclusion and let $\mathbf{E}$ be a chain of $\mathbf{C}$; since each element of $\mathbf{E}$ is a chain, the union of all these elements is a chain, and hence an element of $\mathbf{C}$. This shows that $\mathbf{E}$ has an upper bound in $\mathbf{C}$. The result follows from Zorn's Lemma.[‡]

[‡]Cf. the proof of Lemma A.1.1 with the proof that Zorn's Lemma implies Hausdorff's maximal

Lemma A.1.2. Let Φ be a set of sentences and Δ be a set of default rules; in addition, let $\mathbf{P}$ be as in Lemma A.1.1; $\mathbf{P}$ has a maximal element.

Proof of Lemma A.1.2 By definition every chain in $\mathbf{P}$ has an upper bound. The result now follows from Zorn's Lemma.

Definition A.1.1. Let Φ be a set of sentences and Δ a set of default rules; in addition, suppose that Δ is acceptable w.r.t. Φ; a chain $\mathbf{C}$ of subsets of Δ ordered by inclusion is a formation chain for Δ if it satisfies Equations 4.1 to 4.3 of Definition 4.2.2.

As a minor comment in passing, formation chains need not be unique.

Lemma A.1.3. Let Φ be a set of sentences and Δ be a set of default rules; in addition, let $\mathbf{P}$ be as in Lemma A.1.1; every maximal chain of $\mathbf{P}$ is a formation chain of a maximal element of $\mathbf{P}$.

Proof of Lemma A.1.3 Immediate.

Proposition 4.2.1. Let Φ be a set of sentences and Δ a set of default rules; the class $\mathcal{E}$ of all extensions of Φ and Δ is not empty. (q.e.d. in Appendix A)

Proof of Proposition 4.2.1 Immediate from Lemmas A.1.1 to A.1.3.

Proposition 4.2.2. Let Φ be a set of sentences, and $\Delta \subseteq \Delta'$ sets of default rules; for every extension E of Φ and Δ, there is an extension E' of Φ and Δ' such that $E \subseteq E'$. (q.e.d. Appendix A.)

Proof of Proposition 4.2.2 Let $\mathbf{P}$ be set of all acceptable, q.v. Definition 4.2.2, subsets of Δ w.r.t. Φ partially ordered by inclusion; in addition, let $\mathbf{P}'$ be set of all acceptable, q.v. Definition 4.2.2, subsets of Δ' w.r.t. Φ partially ordered by inclusion. Immediately, $\mathbf{P} \subseteq \mathbf{P}'$. The result follows trivially.

chain principle.

Proposition A.1.1. If Φ is a consistent set of sentences, then, for any set of default rules Δ, every extension E of Φ and Δ is consistent.

Proof of Proposition A.1.1 Let Φ be a consistent set of sentences; the proof of Proposition A.1.1 proceeds by contradiction. Suppose that there is an extension $E \stackrel{\text{def}}{=} \Phi \cup C(\Delta_a)$ of Φ and Δ such that E is not consistent. Immediately, Δ_a is an empty set (q.v. Definition 4.2.2). This is a contradiction. Therefore, if Φ is consistent, any extension of Φ and Δ is consistent.

Proposition A.1.2. If Φ is an inconsistent set of sentences, then, for any set of default rules Δ, Φ is the sole extension of Φ and Δ.

Proof of Proposition A.1.2 Immediate.

Proposition 4.2.4. Let Φ be a set of sentences and Δ a set of default rules; an extension E of Φ and Δ is consistent iff Φ is consistent. (q.e.d. Appendix A.)

Proof of Proposition 4.2.4 Direct from Propositions A.1.1 and A.1.2.

Proposition A.1.3. Let Φ be a set of sentences and Δ a set of default rules; if $\Phi \cup X(\Delta_a)$ is a consistent extension of Φ and Δ, then, for all $\pi : \rho \;/\; \chi \in \Delta_a$, it follows that $\{\neg\pi, \rho, \neg\chi\} \nsubseteq Cn(\Phi \cup X(\Delta_a))$.

Proposition 4.2.3. Let $\Phi \subseteq \Phi'$ be sets of sentences and Δ be a set of default rules; if for all acceptable default rules $\frac{\pi:\rho}{\chi} \in \Delta$ it follows that $\rho \notin Cn(\Phi')$, then, every extension E of Φ and Δ is included in some extension E' of Φ' and Δ.

Proof of Proposition 4.2.3 Observe that if $\rho \notin Cn(\Phi')$ for all acceptable default rules $\pi : \rho \;/\; \chi$, then, every subset Δ_a of default rules of Δ that is acceptable w.r.t. Φ is also acceptable w.r.t. Φ'. The proof now follows immediately.

Proposition 4.2.5. Let Φ be a set of sentences and Δ a set of default rules; if $\langle \mathcal{L}(\Phi), Cn(\Phi)\rangle$ is a *complete* theory, q.v. Definition 2.1.8, and $\mathcal{L}(X(\Delta)) \subseteq \mathcal{L}(\Phi)$, then, for every extension E of Φ and Δ, $Cn(\Phi) = Cn(E)$.[†]

Proof of Proposition 4.2.5 Let $Cn(\Phi)$ be a complete theory.

Since for every extension E, $\Phi \subseteq E$, $Cn(\Phi) \subseteq Cn(E)$.

The proof of Proposition 4.2.5 is concluded if for every extension E, $Cn(E) \subseteq Cn(\Phi)$.

Let $\Phi \cup X(\Delta_a)$ be an extension of Φ and Δ; if $X(\Delta_a) \nsubseteq Cn(\Phi)$, then, there is a default rule $\pi : \rho \,/\, \chi \in \Delta_a$ such that $\chi \notin Cn(\Phi)$.

Since $Cn(\Phi)$ is a complete theory, $\neg\chi \in Cn(\Phi)$. This results in a contradiction.

Therefore, if $Cn(\Phi)$ is a complete theory, then, for every extension E of Φ and Δ, $Cn(\Phi) = Cn(E)$.

Proposition 4.3.3. Let $\mathfrak{L}$ be a logical system for reasoning with default rules; if $\mathfrak{L}$ is classical, then, $\mathrel{|\!\sim}$ satisfies the principles of inclusion, cut, and cumulativity. (q.e.d. Appendix A.)

Proof of Proposition 4.3.3 Immediate from the properties of Cn.

Chapter 5

Definition A.2.2 (Projection)**.** Let τ be a default tableau for φ with premises in $\langle \Phi, \Delta\rangle$; the *first projection* of τ, denoted by $(\tau)_1$, is a labeled tree τ' that has the same set of nodes and the same tree ordering of τ, and that it is such that, for every node n, if n is labeled by $\langle \Phi', \Delta'\rangle$ in τ, then, n is labeled by Φ' in τ'. The *second projection* of τ, denoted by $(\tau)_2$, is defined analogously.

[†]Recall from Definition 2.1.7 that for a set of sentences Γ, $\mathcal{L}(\Gamma)$ is its set of propositional symbols.

Definition A.2.3 (Restriction). Let τ be a default tableau for φ with premises in $\langle \Phi, \Delta \rangle$; the *restriction* of τ to a subset Δ' of Δ, denoted by $\tau|_{\Delta'}$, is a labeled tree τ' that is constructed as follows: (Step 1) Define τ' as a single node labeled tree with label $\langle \Phi, \emptyset \rangle$, mark the root of τ as being in *correspondence* with the root of τ', and put every immediate successor of the root of τ in a queue κ of *to be visited nodes of* τ. (Step 2) Provided κ is not empty, dequeue a node of κ. (Step 3) Let n be the dequeued node, $\langle \Phi'', \Delta'' \rangle$ its label, p its unique predecesor, and p' the node of τ' that is in correspondence with p; if $\Delta'' \subseteq \Delta'$, then, add a new node n' with label $\langle \Phi'', \Delta'' \rangle$ as an immediate successor of p', mark n as being in correspondence with n', and put all immediate successors of n in κ. (Step 4) Repeat from (Step 2) until κ is empty.

Proposition A.2.4. The restriction of a default tableau τ for φ with premises in $\langle \Phi, \Delta \rangle$ to a subset Δ' of Δ is a subtree of τ that is maximal in the following sense: for every other subtree τ' of τ, if τ' is a subtree of $\tau|_{\Delta'}$, then, $\tau' = \tau|_{\Delta'}$.

Proof of Proposition A.2.4 Immediate by construction.

Corollary (Proposition A.2.4). If τ is a default tableau for φ with premises in $\langle \Phi, \Delta \rangle$ and $\Delta' \subseteq \Delta$, then, every maximal branch of τ, whose leaf node is such that its second component is included in Δ', is a maximal branch of $\tau|_{\Delta'}$.

Proposition A.2.5. The restriction of a default tableau τ for φ with premises in $\langle \Phi, \Delta \rangle$ to a subset Δ' of Δ is a default tableau for φ with premises in $\langle \Phi, \Delta' \rangle$.

Proof of Proposition A.2.5 Immediate by construction.

Proposition A.2.6. The first projection a default tableau τ for φ with premises in $\langle \Phi, \Delta \rangle$ is a tableau for $\Phi \cup X(\Delta) \cup \{\varphi\}$.

Proof of Proposition A.2.6 Immediate by construction.

Lemma A.2.4. If π is a branch of a default tableau τ for φ with premises in $\langle \Phi, \Delta \rangle$, then, the second projection of π corresponds to a prefix of a formation chain of some extension E of $\langle\Phi,\Delta\rangle$ (q.v. Definition A.1.1).

Proof of Lemma A.2.4 Immediate from Definition 5.3.1.

Corollary (Lemma A.2.4)**.** If τ has a leaf node l with label $\langle\Phi',\Delta'\rangle$, then, the set $\Phi\cup X(\Delta')$ is included in some extension E of $\langle\Phi,\Delta\rangle$.

Corollary (Lemma A.2.4)**.** If τ has a leaf node l with label $\langle\Phi',\Delta'\rangle$ that is completed, then, the set $\Phi\cup X(\Delta')$ is an extension of $\langle\Phi,\Delta\rangle$.

Lemma A.2.5. If τ is a default tableau for φ with premises in $\langle\Phi,\Delta\rangle$, then, any complete and open leaf node of τ is a complete and open leaf node of a tableau for $E\cup\{\varphi\}$, where E is an extension of $\langle\Phi,\Delta\rangle$.

Proof of Lemma A.2.5 Immediate from Proposition A.2.6 and Lemma A.2.4.

Corollary (Lemma A.2.5)**.** If a default tableau for $\neg\varphi$ with premises in $\langle\Phi,\Delta\rangle$ has a complete and open leaf node, there is an extension E of $\langle\Phi,\Delta\rangle$ such that $\varphi\notin Cn(E)$.

Theorem A.2.1. If for every extension E of $\langle\Phi,\Delta\rangle$, $\varphi\in Cn(E)$, then, every completed tableau for $\neg\varphi$ with premises in $\langle\Phi,\Delta\rangle$ must be closed.

Proof of Theorem A.2.1 Suppose that every extension E of $\langle\Phi,\Delta\rangle$ is such that $\varphi\in Cn(E)$ and that τ is a completed default tableau for $\neg\varphi$ with premises in $\langle\Phi,\Delta\rangle$.

τ must be closed.

The proof is by contradiction.

Assume that τ is not closed. From Lemma A.2.5, if τ has a complete and open leaf node, then, there is an extension E of $\langle\Phi,\Delta\rangle$ such that $\varphi\notin Cn(E)$. This is a contradiction. Therefore, τ must be closed.

Lemma A.2.6. If τ is a d-saturated default tableau for φ with premises in $\langle \Phi, \Delta \rangle$, then, for every extension E of $\langle \Phi, \Delta \rangle$, there is a maximal branch π of τ such that its second projection corresponds to a prefix of a formation chain for E (q.v. Definition A.1.1).

Proof of Lemma A.2.6 Suppose that τ is a d-saturated default tableau for φ with premises in $\langle \Phi, \Delta \rangle$ and that E is an extension of $\langle \Phi, \Delta \rangle$.

The proof of Lemma A.2.6 is by construction.

Let E be an extension of $\langle \Phi, \Delta \rangle$. From Definition 4.2.3, $E \stackrel{\text{def}}{=} \Phi \cup X(\Delta')$ for some $\Delta' \subseteq \Delta$. Also from Definition 4.2.3, there is a formation chain $\mathbf{C}$ having Δ' as its last element.

The prefix $\mathbf{P}$ of $\mathbf{C}$ can be constructed as follows:

(Step 1) Mark the root of τ as being *visited*, remove the first element of $\mathbf{C}$ and put it into a *prefix* queue $\mathbf{P}$.

(Step 2) Let n be the last visited node of τ; provided that n is not a leaf node of τ:

If n is a d-branching node, choose an immediate successor of n' whose label is such that its second component is equal to the first element of $\mathbf{C}$, mark n' as visited, remove the first element of $\mathbf{C}$ and put it in $\mathbf{P}$ – since τ is d-saturated, at least one such node n' must exist.

If n is not a d-branching node, choose an immediate successor of n at will and mark it as visited.

(Step 3) Repeat from (Step 2) until reaching a leaf node.

Invariant in (Steps 1 to 3) is that those visited nodes of τ define a path π of τ whose second projection, i.e., the formation chain stored in $\mathbf{P}$, is a prefix of $\mathbf{C}$.

Since the branches of τ are finite, after a finite number of iterations, the process defined above is guaranteed to terminate in a leaf node of τ. Therefore, completing the proof of Lemma A.2.6.

Theorem A.2.2. If τ is a closed and d-saturated default tableau for φ with premises in $\langle \Phi, \Delta \rangle$, then, for every extension E of $\langle \Phi, \Delta \rangle$, $\varphi \in Cn(E)$.

Proof of Theorem A.2.2 Immediate from Lemma A.2.6.

Theorem 5.3.1 (Correctness)**.** Let $\mathfrak{L}$ be the logical system for reasoning with default rules of Proposition 4.3.4; for any sentence σ, and for any finite sets Φ and Δ of sentences and default rules, respectively, σ is a default consequence of $\langle \Phi, \Delta \rangle$, i.e., there is a closed and d-saturated default tableau for $\neg\sigma$ with premisses in $\langle \Phi, \Delta \rangle$, iff $\langle \Phi, \Delta \rangle \mathrel{|\!\!\sim} \sigma$, i.e., iff for every extension E of $\langle \Phi, \Delta \rangle$, there is a proof of σ from E, i.e., there is a closed tableau for $\{\neg\sigma\} \cup E$. (q.e.d. in Appendix A.)

Proof of Theorem 5.3.1 Suppose that φ is a sentence, Φ a set of sentences, and Δ a set of default rules; Theorem 5.3.1 follows from proving that:

($\Rightarrow$) If φ is a default consequence of $\langle \Phi, \Delta \rangle$, i.e., if there is a closed and d-saturated default tableau for $\neg\varphi$ with premises in $\langle \Phi, \Delta \rangle$, then, every extension E of $\langle \Phi, \Delta \rangle$ is such that $\varphi \in Cn(E)$, i.e., there is a closed tableau for $E \cup \{\varphi\}$,.

The proof of ($\Rightarrow$) is immediate from Theorem A.2.2.

($\Leftarrow$) If every extension E of $\langle \Phi, \Delta \rangle$ is such that $\varphi \in Cn(E)$, i.e., there is a closed tableau for $E \cup \{\varphi\}$, then, there is a closed and d-saturated default tableau for $\neg\varphi$ with premises in $\langle \Phi, \Delta \rangle$, i.e., φ is a default consequence of $\langle \Phi, \Delta \rangle$.

The proof of ($\Leftarrow$) is immediate from Theorem A.2.1.

Chapter 6

Proposition 6.1.1. Let $\langle \Sigma, \Phi \rangle$ and $\langle \Sigma', \Phi' \rangle$ be two presentations in CPL; if m is a presentation mapping between $\langle \Sigma, \Phi \rangle$ and $\langle \Sigma', \Phi' \rangle$ in CPL, then, m is a mapping between the default presentations $\langle \Sigma, \langle \Phi, \emptyset \rangle \rangle$ and $\langle \Sigma', \langle \Phi', \emptyset \rangle \rangle$ in $\mathfrak{P}_{\mathrm{d}}$, i.e., m is a mapping of default presentations in $\mathfrak{L}$. (q.e.d. Appendix A.)

Proof of Proposition 6.1.1 Immediate from the definition of a mapping between presentations in CPL (q.v. Definition 2.1.10).

Proposition 6.1.2. Let $\langle\Sigma, \Phi\rangle$ and $\langle\Sigma', \Phi'\rangle$ be two presentations in CPL; if m is a presentation mapping between $\langle\Sigma, \Phi\rangle$ and $\langle\Sigma', \Phi'\rangle$ in CPL, then, for any set of default rules Δ' such that $\mathcal{L}(\Delta') \subseteq \Sigma'$, it follows that m is a mapping between the default presentations $\langle\Sigma, \langle\Phi, \emptyset\rangle\rangle$ and $\langle\Sigma', \langle\Phi', \Delta'\rangle\rangle$ in $\mathfrak{P}_{\mathrm{d}}$, i.e., m is a mapping of default presentations in $\mathfrak{L}$. (q.e.d. Appendix A.)

Proof of Proposition 6.1.2 Immediate from the definition of an extension, q.v., Definition 4.2.3, $\mathscr{E}(\langle\Phi, \emptyset\rangle) = \{\Phi\}$. The conclusion follows trivially from this observation.

Proposition 6.1.3. Let m be a mapping with domain Σ and co-domain Σ' and let $\langle\Sigma, \Phi\rangle$ and $\langle\Sigma', \Phi'\rangle$ be presentations in CPL; if $\langle\Sigma', \Phi'\rangle$ is a faithful interpretation of $\langle\Sigma, \Phi\rangle$ along m in CPL, then, for any two signatures Σ and Σ' and for any two sets Δ and Δ' of default rules defined on Σ and Σ', respectively, it follows that, if $m(\Delta) \subseteq \Delta'$, then, m is a mapping between $\langle\Sigma, \langle\Phi, \Delta\rangle\rangle$ and $\langle\Sigma', \langle\Phi', \Delta'\rangle\rangle$, i.e., m is a mapping of default presentations in $\mathfrak{L}$. (q.e.d. Appendix A.)

Proof of Proposition 6.1.3 Since $\langle\Sigma', \Phi'\rangle$ is a faithful interpretation of $\langle\Sigma, \Phi\rangle$ along m in CPL, $m(Cn(\Phi)) = Cn(\Phi') \cap \mathscr{L}|_{\Sigma}$. This means that no default rule in Δ that is acceptable relative to Φ becomes inadmissible relative to Φ' (q.v. Section 4.2). The conclusion follows trivially from this observation.

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