Γ-CONVERGENCE RESULTS FOR SUPERCONDUCTING THIN FILMS WITH HOLES AND FOR GINZBURG-LANDAU MODELS FOR SUPERCONDUCTORS WITH NORMAL INCLUSIONS.

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To my mother, my husband and my lovely kids.

I couldn't have done this with out you.

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Abstract

We study a Ginzburg–Landau model for an inhomogeneous superconductor in the singular limit as the Ginzburg–Landau parameter $\kappa = 1/\epsilon \to \infty$. The inhomogeneity is represented by a potential term $V(u_\epsilon) = \frac{1}{4}(a(x) - |u_\epsilon|^2)^2$, with a given smooth function $a(x)$ which is assumed to become negative in finitely many smooth subdomains, the "normally included" regions. For $h_{ex} = O(|\ln \epsilon|)$ we study the Gamma-limit of this inhomogeneous Ginzburg-Landau functional. The vanishing of $a(x)$ near the inner boundaries imply that the associated operators are strictly but not uniformly elliptic, leading to many questions to be resolved near the boundaries of the normal regions. The method we use is an extension of many techniques including the product estimate from Sandier-Serfaty, Jacobian estimates from Jerrard-Soner and an appropriate Hodge decomposition adapted to our problem.

To resolve these problems, we first study the Γ-limit in the simpler case when $a(x)$ is varying but bounded below by a positive constant a_0. Second, we consider singular limits of the three-dimensional Ginzburg-Landau functional for a superconductor with thin-film geometry, in a constant external magnetic field, where $d(x)$ is the thickness of the thin film. The superconducting domain is multiply connected and has characteristic thickness on the scale $\epsilon > 0$, and we consider the simultaneous limit as the thickness $\epsilon \to 0$ and the Ginzburg-Landau parameter $\kappa \to \infty$. We assume

that the applied field is strong (on the order of ϵ^{-1} in magnitude) in its components tangential to the film domain, and of order $\log\kappa$ in its dependence on κ. Finally, we study the Γ-limit of the inhomogeneous superconducting Ginzburg-Landau model with $a(x)$ vanishing on the boundary of the normal regions.

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Chapter 1

Introduction

The history of superconductivity began with Dutch physicist Kammerling Onnes's discovery of superconductivity in mercury in 1911. He observed that the electrical resistance of various metals disappeared completely in a small temperature range at a critical temperature T_c, which is a characteristic of the material. Since then, many other superconducting materials have been discovered and the theory of superconductivity has been developed. In 1950 Ginzburg and Landau introduced the Ginzburg-Landau model with magnetic field as a phenomenological model to describe superconductivity. They introduced the complex superconducting order parameter u, which is such that $|u|^2$ represents the density of superconducting charge carriers (the Cooper pairs). The ultimate justification for the Ginzburg–Landau model came in 1957, when Gor′kov and Eliashberg demonstrated that the Ginzburg-Landau equations could be derived as a limit of the microscopic theory of Bardeen, Cooper, and Schrieffer [BCS]. The Ginzburg-Landau model has a great importance in the modelling of superconductivity (with Nobel prizes awarded for it: Ginzburg, Landau, and Abrikosov). Through the influential work of Abrikosov [A], the Ginzburg-Landau model allows one to predict the possibility of a *mixed state* in type II superconductors where triangular vortex lattices appear. These vortices have since been the objects of many observations and experiments.

In addition to its importance in the modelling of superconductivity, the Ginzburg–Landau Model turns out to be the simplest case of gauge invariance, and vortices to be the simplest case of topological solitons ; moreover, it is mathematically extremely close to the Gross-Pitaevskii model for superfluidity, and models for rotating Bose-Einstein condensates, in which quantized vortices are also essential objects, and to which the Ginzburg–Landau techniques have been successfully exported.

1.1 The two-dimensional Ginzburg-Landau model

Let $\mathcal{D}$ be a smooth, bounded, simply connected region in $\mathbf{R}^2$ and $u \in \mathbf{H}^1(\mathcal{D}, \mathbf{C})$ where $\mathbf{H}^1(\mathcal{D}, \mathbf{C})$ is the Sobolev space $\mathbf{W^{1,2}}(\mathcal{D}, \mathbf{C})$ (see [E]). We define the superconducting Ginzburg–Landau energy:

$$E_\epsilon(u_\epsilon, A_\epsilon) := \frac{1}{2}\int_{\mathcal{D}} \Big\{ |\nabla_A u_\epsilon|^2 + \frac{1}{2\epsilon^2}(|u_\epsilon|^2 - 1)^2 + (h_\epsilon - h_{ex})^2 \Big\} dx. \tag{1.1.1}$$

In this expression, $\mathcal{D}$ represents the section of an infinitely long cylinder. The first unknown $u : \mathcal{D} \to \mathbf{C}$ is a complex-valued function, called an "order parameter" and it describes the material phase in the Landau theory of phase transitions: $|u|^2$ is the density of Cooper pairs of superconducting electrons. The material is in the superconducting phase if $|u| \simeq 1$, while it is in the normal phase if $|u| = 0$. The two phases are able to coexist in the sample. The second unknown is A, the electromagnetic vector-potential of the magnetic-field, $A : \mathcal{D} \to \mathbf{R}^2$. The induced magnetic field points in the e_3 direction and is given by $h = \nabla^{\perp} A = \partial_1 A_2 - \partial_2 A_1$, it is a real-valued function in $\mathcal{D}$. The notation ∇_A denotes the covariant gradient $\nabla - iA$; $\nabla_A u$ is a vector with complex components. The parameter ϵ is the inverse of the "Ginzburg-Landau parameter" usually denoted κ, a non dimensional parameter depending only on the material, and related to the ratio of penetration depth (scale of variation of h) and coherence length (scale of variation of u). We will consider the regime of small ϵ, corresponding to large-κ (type-II superconductors). In Chapter 3, we denote the thickness of the thin film by ϵ and the GL-parameter by κ and we take limit when both $\epsilon \to 0$ and $\kappa \to \infty$.

The *superconducting current* is given by

$$j = \langle iu, \nabla_A u \rangle \tag{1.1.2}$$

where $\langle ., . \rangle = \mathrm{Re}\, \overline{a} b$, and the bar denotes the complex conjugation. Note that if we identify $\mathbf{C}$ with $\mathbf{R}^2$ via $a = a_1 + ia_2 \in \mathbf{C}$ and $b = b_1 + ib_2 \in \mathbf{C}$ corresponds to the vectors (a_1, a_2), (b_1, b_2) then $\langle a, b \rangle = (a, b)$ where $(., .)$ is the usual scalar product on $\mathbf{R}^2$. The energy admits a *gauge-invariance*: it is invariant under the action of the unitary group in the form $u \to ue^{if}$, $A \to A + \nabla f$; we will explain this more in Chapter 2. The parameter $h_{ex} > 0$ represents the intensity of the applied field which assumed to be directed in e_3 direction.

In this thesis the energy F_ϵ that we are going to study is slightly different from the classical Ginzburg-Landau energy in the sense that there is a term penalizing the variations of the order parameter u. We denote this function by $a(x) : \mathcal{D} \to \mathbf{R}$ and the energy becomes

$$F_\epsilon(u_\epsilon, A_\epsilon; \mathcal{D}) := \frac{1}{2}\int_{\mathcal{D}} \Big\{ |\nabla_A u_\epsilon|^2 + \frac{1}{2\epsilon^2}(|u_\epsilon|^2 - a(x))^2 + (h_\epsilon - h_{ex})^2 \Big\} dx. \qquad (1.1.3)$$

If the material is homogeneous, the function a in F_ϵ is taken to be a constant, proportional to $T_c - T$. Here T is the body's temperature and T_c is the material's critical temperature. Inhomogeneous superconducting materials can arise naturally due to material defects or the presence of grain boundaries. A consequence of having material inhomogeneities is that they tend to pin or stabilize supercurrent patterns. The classical Ginzburg-Landau theory can be modified to take normal inclusions into account. This is done by having the critical temperature, T_c, depend on position which is equivalent to having $a = a(x)$. It is possible that $a(x)$ may vanish or change sign within the domain (see [CDG96], [CR] and [ABP]).

In our work we define

$$\Omega := \mathcal{D} \setminus \cup_j \omega_j \text{ for } j = 1, ..., m \qquad (1.1.4)$$

where $\omega_j \subset \mathcal{D}$ is smooth, bounded, simply connected, and we allow $a(x)$ to be zero on the inner boundaries $\partial\omega_j,\ \forall\ j = 1, ..., m$ for both superconducting thin film and pinning Ginzburg-Landau. We require throughout the thesis that $a(x)$ satisfies the following:

(H1) $a(x) \in C^2(\mathcal{D})$.

(H2) $\{x \in \Omega, a(x) > 0\}$

(H3) $\nabla a(x) \neq 0$ for all $x \in \partial\omega_i,\ i = 1, ..., m$. More specifically, $\exists\delta > 0$ s.t. there are non-negative constants $m_i, and\ M_i$ such that

$$m_i \leq \frac{a(x)}{\operatorname{dist}(x, \partial\omega_i)} \leq M_i.$$

for $\operatorname{dist}(x, \partial\omega_i) < \delta$.

The Euler-Lagrange equations corresponding to the energy (1.1.3) are,

$$(GL)\begin{cases} -(\nabla_A u)^2 u = \frac{1}{\epsilon^2} u(a(x) - |u|^2) & in\ \Omega \\ -\nabla^{\perp} h = (iu, \nabla_A u) & in\ \Omega \end{cases} \tag{1.1.5}$$

with the boundary conditions,

$$\begin{cases} h = h_{ex} & on\ \partial\mathcal{D} \\ \nabla_A u \cdot \nu = 0 & on\ \partial\mathcal{D} \end{cases} \tag{1.1.6}$$

where $\nabla^{\perp}$ denotes the operator $(-\partial_2, \partial_1)$ and ν is the outward pointing unit normal to $\partial\mathcal{D}$.

1.2 Vortices and critical fields.

The mathematical studies of the superconductors Ginzburg–Landau model started by the pioneering work of Bethuel, Brezis, and Hélein [BBH] on the simpler Ginzburg–Landau model

$$G_\epsilon(u_\epsilon) := \frac{1}{2}\int_{\mathcal{D}} \{|\nabla u_\epsilon|^2 + \frac{1}{2\epsilon^2}(|u_\epsilon|^2 - 1)^2\} dx \tag{1.2.1}$$

over the space $\mathbf{H}^1(\mathcal{D})$, with $u|_{\partial\mathcal{D}} = g$, where $g \in L^2(\mathcal{D})$. This model has been studied by numerous authors (see e.g. [JS], [LM], [DM] and [SS04]), after the work of Bethuel, Brezis, and Hélein [BBH]. In order to pass from (1.2.1) to (1.1.1), it suffices to set the magnetic potential A and the applied field h_{ex} to be zero in E_ϵ. The Euler-Lagrange equation associated with (1.2.1) is

$$\begin{cases} -\Delta u = \frac{u}{\epsilon^2}(1-|u|^2) & \textit{in } \mathcal{D}, \\ \quad u = g & \textit{on } \partial\mathcal{D}. \end{cases} \tag{1.2.2}$$

It is an important model problem in the Calculus of Variations as it contains different length scales such as the vortex core, vortex spacing, and its space of solution has a rich topological structure (see [BBH]). For this problem we define the current of u by

$$ju = (iu, \nabla u), \tag{1.2.3}$$

where $(a, b) = \mathrm{Re}\,\overline{a}b$, and the bar denotes the complex conjugation. Using differential form, we can represents ju as

$$ju = \sum_{k=1}^{n}(iu, \partial_k u)dx_k. \tag{1.2.4}$$

It is related to the Jacobian determinants Ju of u through

$$Ju = \frac{1}{2}d(ju) = \frac{1}{2}d(iu, du), \tag{1.2.5}$$

where

$$Ju = \sum_{j<k} (i\partial_j u, \partial_k u) dx_j \wedge dx_k. \tag{1.2.6}$$

Bethuel, Brezis, and Hélein obtained in their book [BBH] a complete description of the asymptotic behaviour of the minimizers of the functional (1.2.1) with given Dirichlet data g. They proved that asymptotically the minimizers have finitely many singularities called vortices. Each of these vortices carries $\pi|\log\epsilon|$ amount of energy and the number of the vortices is determined by the winding number of the Dirichlet data g. Their results indicate that a natural scaling for this functional is $|\log\epsilon|$. In [JS02] Jerrard and Soner studied the Γ-limit of (1.2.1) divided by the scaling factor $|\log\epsilon|$ and proved that: for a sequence $\{u_\epsilon\}$, $\frac{G_\epsilon(u_\epsilon)}{|\log\epsilon|}$ is uniformly bounded in ϵ. Then, the Jacobian of these functions is precompact in the dual of Hölder continuous functions, and any limit J is an atomic Radon measure with weights equal to an integer multiple of π. The support of J is the asymptotic location of the vortices and the weights of J at these points are related to the limiting degree of u_ϵ. In [SS04] Sandier and Serfaty presented an optimal lower bound of (1.2.1). It is a product-type lower bound on Ginzburg-Landau, a slight improvement of the existing lower bounds by [JS02].

Previous works studied the vortices of the functional (1.2.1) but with a pinning term $a(x)$ instead of 1. This functional with non-constant $a(x)$ was proposed by Rubinstein in [R95] as a model of pinning vortices for Ginzburg-Landau minimizers. André and Shafrir [AS] studied the asymptotics of minimizers for a smooth a. One of the first works to consider a discontinuous pinning term, which models a composite two-phase superconductor, was [LM]. In this work, a single inclusion described by a pinning term independent of the parameter ϵ was considered for a simplified Ginzburg-Landau functional with Dirichlet boundary condition g on $\partial\mathcal{D}$ where $\mathcal{D}$ is a simply connected domain. Namely the pinning term is

$$a(x) = \begin{cases} 1 & if\ x \in \Omega \\ b & if\ x \in \omega \end{cases}$$

with $0 < b < 1$. Here $\Omega = \mathcal{D} \setminus \omega$ where ω is a simply connected open set s.t. $\omega \subset \mathcal{D}$. The main objective of [LM] was to establish that the vortices are attracted (pinned) by the inclusion ω, and their location inside ω can be obtained via minimization of certain finite-dimensional functional of renormalized energy. Dos Santos and Misiats in [DM] proved that for small ϵ, minimizers have d distinct zeros (vortices) which are inside the pinning domains and they have a degree equal to 1. The question of finding the locations of the pinning domains with vortices is reduced to a discrete minimization problem for a finite-dimensional functional of renormalized energy. They found the position of the vortices inside the pinning domains and showed that, asymptotically, this position is determined by local renormalized energy which does not depend on the external boundary conditions.

Given ϵ, the behaviour of minimizers and critical points of the Ginzburg–Landau model for superconductors (1.1.1) is determined by the value of the external field h_{ex}. There are three critical values of h_{ex} or critical fields H_{c_1}, H_{c_2}, and H_{c_3}, for which phase-transitions occur. Below the first critical field, which is of order $O(|\log\epsilon|)$ (as first established by Abrikosov), the superconductor is everywhere in its superconducting phase $|u| \sim 1$ and the magnetic field doesn't penetrate. At H_{c_1}, the first vortices appear. Sandier and Serfaty [SS] showed that there exists a constant H_{c_1}

proportional to $|\log(\epsilon)|$ as $\epsilon \to 0$, such that if $h_{ex} < H_{c_1}$, then minimizers for E_ϵ are purely superconducting, satisfying $|u| > 0$ in Ω where Ω is simply connected domain. By the interesting case where $a(x) = 0$ at finitely many isolated points $\{x_1, ..., x_n\}$ where $\Omega = \mathcal{D} \setminus \{x_1, ..., x_n\}$, André, Bauman, and Phillips [ABP] in this strong pinning case were able to show that the transition threshold for h_{ex}, denoted by H_{c_1}, is of order 1 as $\epsilon \to 0$ instead of $O(|\log \epsilon|)$.

Between H_{c_1} and H_{c_2} the superconducting and normal phases coexist in the sample, and the magnetic field penetrates through the vortices. This is called the *mixed state* and has been studied extensively (see [SS07]). When $H_{c_2} = O(\frac{1}{\epsilon^2})$, the vortices are so densely packed that they overlap each other, and at H_{c_2} a second phase transition occurs, after which $|u| \sim 0$ inside the sample. In the interval $[H_{c_2}, H_{c_3}]$, superconductivity persists near the boundary, this is called *surface superconductivity*, and after $H_{c_3} = O(\frac{1}{\epsilon^2})$, superconductivity is completely destroyed and $u \equiv 0$, so the sample is completely in the normal phase (see [SS07]).

A mathematical study for the Ginzburg-Landau equations corresponding to the energy (1.1.1) with variable $a(x)$ was done by Aftalion, Sandier, and Serfaty in [ASS] where the case $\frac{1}{2} \leq a(x) \leq 1$ was considered. In [JS] Jerrard and Soner proved the compactness of rescaled current and Jacobian, and study the Γ-limit in the critical case when h_{ex} is of order $O(|\log \epsilon|)$. Sandier and Serfaty in [SS00] proved that the induced magnetic fields associated to minimizers of the energy functional converge as $\epsilon \to 0$ to the solution of a free-boundary problem. This free boundary-problem has a nontrivial solution only when the applied magnetic field is of the order of the "first critical field" i.e $O(|\log \epsilon|)$. Alama and Bronsard in [AB06] present two results for different regimes of the applied field h_{ex} in the case where $a(x) \equiv 1$. First, they show that when the applied field is fixed (independent of ϵ) there are no interior vortices in Ω but the holes ω_j act as "Giant Vortices", with non-zero winding of the phase of u for large enough h_{ex}. Next, they show that interior vortices appear at $h_{ex} = O(\log \epsilon)$. Another related problem, arising in the context of Bose-Einstein condensates, is presented by Aftalion, Alama, and Bronsard in [AAB]. In [AAB] the domain $\mathcal{D}$ is a disk, and $a(x)$ is chosen to be radial with a positive in a symmetric circular annulus Ω and negative in the hole. As in [AB06] they present two results one concerning pinning for bounded rotations, and one concerning the breakdown of pinning when the rotation $\omega = O(|\ln \epsilon|)$. In the second result, the vortices again appear far from the hole, and accumulate along a finite number of concentric circles with radii explicitly determined by the function $a(x)$. Alama, and Bronsard combine many aspects of both papers in [AB05], where they study the case of general multiply connected domain where $a(x)$ is positive in Ω and negative in the hole. Full Ginzburg-Landau model (1.1.1) with discontinuous pinning term $a(x)$ was later considered by Kachmar [Kac10] and by Aydi and Kachmar [AK].

In [ABGS10] Alama, Bronsard, and Galvão-Sousa studied thin film limits of the full three-dimensional Ginzburg-Landau model for a superconductor in an applied magnetic field oriented obliquely to the film surface. They obtained Γ-convergence results in several regimes, determined by the asymptotic ratio between the magnitude of the parallel applied magnetic field and the thickness of the film. In their other work [ABGS13] they considered singular limits of the three-dimensional Ginzburg-Landau functional for a superconductor with thin-film geometry, in a constant external magnetic field. They proved that the Ginzburg-Landau energy Γ-converges to an energy associated with a two-obstacle problem, posed on the planar domain which supports the thin film. The same limit is obtained regardless of the relationship between ϵ and κ in the limit. A different type of thin film problem has been studied by Contreras and Sternberg [CS10] and Contreras [Con11]. In their setting, the superconductor is a thin shell, built from depositing an ϵ-thick coating on a fixed two-dimensional surface in $\mathbf{R}^3$. The limiting problem in this case is a Ginzburg-Landau model on an embedded 2-manifold, and they obtain remarkable results connecting the lower critical field and the appearance of vortices to the geometry of the limiting surface.

1.3 The mathematical methods for Ginzburg- Landau models

In this section we give a brief idea of the history of the mathematical methods used to study the Γ-convergence of the Ginzburg–Landau model.

Sandier and Serfaty [SS00] studied the Ginzburg–Landau energy of superconductors submitted to a possibly non-uniform magnetic field (applied fields of intensity $p(x)h_{ex}$), in the limit of a large Ginzburg-Landau parameter κ. They proved that (1.1.1) converges in a sense similar to Γ-convergence to the limiting functional for $h_{ex} = \lambda|\log\epsilon|$ for $\lambda > 0$

$$E(f) = \frac{\lambda}{2}\int_{\mathcal{D}} |-\Delta(f-p)+f| + \frac{1}{2}\int_{\mathcal{D}} |\nabla(f-p)|^2 + |f-p|^2 \tag{1.3.1}$$

defined over

$$V = \{f \in \mathbf{H}^1_p(\Omega)/ -\Delta(f-p)+f \text{ is a Radon measure}\}$$

where Radon measure is the set of all bounded signed measure (see [E]). More precisely, they proved that the induced magnetic fields of minimizers of (1.1.1) converge, after a renormalization, to the minimizer of (1.3.1). For a definition of *Gamma*-convergence see (Definition 2.2.1).

Jerrard and Soner [JS] studied the Γ-convergence of the Ginzburg–Landau energy (1.2.1). Compactness results for the scaled Jacobian of u_ϵ are proved under the assumption that $G_\epsilon(u_\epsilon) \leq C|\log\epsilon|^2$. In addition, the Γ-limit of $\frac{G_\epsilon(u_\epsilon)}{|\log\epsilon|^2}$ is shown to be

$$G(j_*) := \frac{1}{2}\|j_*\|_2^2 + \|\nabla \times j_*\|_{\mathfrak{M}},$$

where j_* is the limit of $\frac{ju_\epsilon}{|\log\epsilon|}$ and and $\|.\|_{\mathfrak{M}}$ is the total variation of a Radon measure. These results are applied to the Ginzburg–Landau functional (1.1.1) with external magnetic field $h_{ex} = \lambda|\log\epsilon|$. The Γ-limit of $\frac{E_\epsilon}{|\log\epsilon|^2}$ is given by

$$E(u, A) := \frac{1}{2}\big[\|j_* - A\|_2^2 + \|\nabla \times j_*\|_{\mathfrak{M}} + \|\nabla \times A - \lambda\|_2^2\big].$$

where j_* as above and A is the limit of $\frac{A_\epsilon}{|\log\epsilon|}$. Proving the Γ-limit can be done in two steps (see Definition 2.2.1).

The main tool to prove the lower bound inequality is the *vortex balls construction*. Each ball will contain amount of energy at least of $\pi|d|\log\frac{r}{\epsilon}$ where $d = \deg(\frac{u}{|u|}, \partial B)$, and r is the radius of B. The vortex balls are not completely intrinsic to (u, A) and not unique, they have a simple relation to the configuration (u, A), namely that the measure $\sum_i 2\pi d_i \delta_{p_i}$ is close in certain norm to the gauge-invariant version of the Jacobian determinant of u, an intrinsic quantity depending on (u, A). We use this relation to find a sharp lower bound of the Jacobian in term of the Ginzburg-Landau energy.

At the same period of time two papers, [Sa] and [Jer99], came with the idea of constructing these balls. Jerrard in [Jer99] showed that the unbounded part of the energy is concentrated on a small number of small sets called "vortex balls". He and Soner later on [JS02] used these balls to find a sharp lower bound of the Jacobian in term of the Ginzburg–Landau energy which they called the "Jacobian estimates". While Sandier in [Sa] showed that given any arbitrary configuration (u, A), one can describe it energetically as a collection of vortices glued together, as long as its Ginzburg–Landau energy is not extremely large, but without assuming that it solves any equation.

It is important to mention the work of Sandier and Serfaty [SS04] where they proved a new inequality for the Jacobian associated to the Ginzburg–Landau energy in any dimension. They proved the lower bound of (1.2.1) in any dimension which is a product-type lower bound called "product estimate".

The proof relies on the same ingredient as the other proof of lower bound , i.e. on the ball construction method of [Jer99] and [Sa], but the main new idea is to use a deformation of the metric, and thus a construction of *growing ellipses* instead of balls. Ellipses allow the freedom necessary to "separate" the directions. This sharp lower bound of the Jacobian in term of the Ginzburg–Landau energy was introduced also by Jerrard and Soner [JS] to study the 2 dimensional case. In our thesis we modify and adapt the "product estimate" method to prove the lower bound.

The main tool used to study the upper bound is the *Hodge decomposition* method.

With this method we decompose the space $L^2(\mathcal{D})$ into three subspaces

$$\begin{aligned}
&\mathcal{U} = \{-\nabla^{\perp}\psi,\ \psi \in \mathbf{H}^1_0(\mathcal{D};\mathbf{R})\},\\
&\mathcal{V} = \{\nabla\zeta,\ \zeta \in \mathbf{H}^1(\mathcal{D};\mathbf{R})\},\\
&\mathcal{W} = \{W \in C^1(\mathcal{D};\mathbf{R}^2),\ \nabla^{\perp}\cdot W = 0,\ \nabla\cdot(W) = 0,\ W\cdot\nu = 0 \text{ on } \partial\mathcal{D}\}.
\end{aligned} \tag{1.3.2}$$

s.t. any $j \in L^2(\mathcal{D})$ can be written as:

$$j = U + V + W$$

where $U \in \mathcal{U},\ V \in \mathcal{V},\ and\ W \in \mathcal{W}$ (see Lemma 3.3.2 in Chapter 3). Then we construct a sequence which converges to the desired limit j.

Jerrard and Soner [JS] came up with the idea of using the Hodge decomposition to construct a sequence of functions to obtain an upper bound that matchs the lower bounds for the Ginzburg–Landau energy. This construction is very similar to the construction given by Sandier and Serfaty [SS00] for the functional with applied magnetic field. [JS] introduced this Hodge-decomposition so that it would be easier to generalize to higher dimensions.

More details and informations on the methods used to study the Ginzburg–Landau model can be found in the book of Sandier and Serfaty [SS07].

1.4 Main results

In our thesis, we concentrate on the Γ-convergence of Ginzburg-Landau energy is related to (1.1.3).

1.4.1 The Ginzburg-Landau model with a pinning term bounded from below.

In chapter 2 we consider the Γ-convergence of $F_\epsilon(u, A; \Omega)$ where Ω is as in (1.1.4) with a pinning term $a(x)$ that is bounded from below by a positive number. The aim of this chapter is to introduce "the methods" we will use to find the Γ-limit in order to be able to adapt them in the case where $a(x)$ is allowed to vanish on the inner boundaries (Chapter 3, 4). The fact that $a(x)$ vanishes on the inner boundaries is technically more difficult and will require several additional steps. We combine many methods in a novel approach for these problems.

The Γ-limit is obtained by finding an appropriate lower bound and then constructing a sequence that gives us the matching upper bound. Most of the previous works have considered the Γ-limit of (1.1.1) when $a(x) \equiv 1$. Our main result in Chapter 2 is Theorem 2.2.2. To prove Theorem 2.2.2, for the lower bound we modified the method of Sandier and Serfaty [SS04] which gives an inequality for the Jacobian associated to the Ginzburg-Landau energy in any dimension. What we had to modify is the vortex balls construction and this is due to the presence of $a(x)$. For the upper bound we use a Hodge decomposition method inspired by [JS] and [ABGS13].

1.4.2 Superconducting thin film

In Chapter 3 we consider the Γ-limit of the $3D$ Ginzburg-Landau functional in a thin film geometry

$$\mathbf{I}_{\epsilon,\kappa}(u, A) := \int_{\mathcal{D}_\epsilon} \Big(|\nabla_A u|^2 + \frac{\kappa^2}{2}(1 - |u|^2)^2 \Big) dx + \frac{1}{2} \int_{\mathbf{R}^3} |h - h_{ex}|^2 dx. \tag{1.4.1}$$

This energy reflects the fact that the magnetic field is present everywhere. The superconducting thin film is given by $\mathcal{D}_\epsilon$ which we assume is multiply connected and the thickness of the film is given by $d(x)$ and is allowed to be zero on the boundaries of the holes. Hence the superconductor domain is multiply-connected and has a characteristic thickness on the scale $\epsilon > 0$. We consider the limiting as the thickness $\epsilon \to 0$ and the Ginzburg-Landau parameter $\kappa \to \infty$. As in [ABGS13] it turns out that the relationship between $\epsilon \to 0$ and $\kappa \to \infty$ doesn't affect the limiting problem.

The superconducting sample is represented by the domain $\mathcal{D}_\epsilon \subset \mathbf{R}^3$,

$$\mathcal{D}_\epsilon = \{(x', x_3) \in \mathbf{R}^3 : x' \in \Omega,\ \epsilon f(x') < x_3 < \epsilon g(x')\},$$

where $\Omega := \omega_0 \backslash \cup_j \omega_j,\ j = 1, ..., m$ and $\omega_j \subset \omega_0 \subset \mathbf{R}^2$ is smooth and simply connected, $f, g : \Omega \to \mathbf{R}$ are smooth functions on Ω with $f(x') < g(x')$ for all $x' \in \Omega$, and $\epsilon > 0$. We denote by

$$d(x') = g(x') - f(x')$$

the thickness of the film for given $x' \in \Omega$. We assume $d(x)$ satisfies (H1)-(H3). We rescale the domain by ϵ in the x_3 direction in order to recognize the correct scaling for h_{ex} in terms of the thickness parameter and of the Ginzburg-Landau parameter κ. The energy transforms as follows:

$$\begin{aligned}\tilde{I}_{\epsilon,\kappa}(u, A) =: \int_{\mathcal{D}} \left(\frac{1}{2} |(\nabla' - iA)u|^2 + \frac{1}{2\epsilon^2} |(\partial_3 - iA_3)u|^2 + \frac{\kappa^2}{4}(1 - |u|^2)^2 \right) dx \\ + \frac{1}{2} \int_{\mathbf{R}^3} \left(|h_3 - h_3^{ex}|^2 + \frac{1}{\epsilon^2} |h' - h'_{ex}|^2 \right) dx,\end{aligned}$$

We choose the strength of the exterior applied field to be related to the thickness parameter ϵ, and to be on the scale of the first critical field in κ via,

$$\mathbf{h}_{ex} = \left(H' \frac{\log \kappa}{\epsilon}, H_3 \log \kappa \right). \tag{1.4.2}$$

where $H = (H', H_3) = (H_1, H_2, H_3) \in \mathbf{R}^3$ is fixed constant vector (independent of ϵ, κ). For applied fields of the form (1.4.2), the energy of minimizers of $\tilde{E}_{\epsilon,\kappa}$ will be on the order of $[\log \kappa]^2$. That leads to introduce the following normalization, and study the family of functionals

$$I_{\epsilon,\kappa}(u, A) := \frac{1}{(\log \kappa)^2} \tilde{I}_{\epsilon,\kappa}(u, A)$$

and configuration (u, A) with bounded values of $E_{\epsilon,\kappa}$.

Define the space,

$$\mathcal{Z} := \{ j \in L^2(\Omega, \mathbf{R}^3) : \ j = (j'(x'), 0), \ J := \frac{1}{2} \nabla \times j \in \mathfrak{M}(\mathcal{D}, \mathbf{R}^3) \},$$

where $\mathfrak{M}(\mathcal{D}, \mathbf{R}^3)$ is the space of vector-valued Radon measures on $\mathcal{D}$. Given $j \in \mathcal{Z}$ and $B' : \mathbf{R}^2 \to \mathbf{R}^2$, we define the limiting functional $I_\infty(j; F)$ where $F = \nabla' \times B'$ as below:

$$I_\infty(j; F) = \begin{cases} \frac{1}{2} ||d(x') \nabla \times j||_{\mathfrak{M}(\Omega)} + \frac{1}{2} \int_\Omega d(x') |j' - B'|^2, & \text{if } j \in \mathcal{Z} \\ \infty & \text{otherwise} \end{cases} \tag{1.4.3}$$

The main result is that the Γ-limit of $I_{\epsilon,\kappa}$ is related to I_∞ (see Theorem 3.1.3). We prove this in the usual two steps: first, bounded sequences are compact and the energy is lower semicontinuous in the energies which yields the Theorem 3.1.3. The proof of Theorem 3.1.3 is a direct application of [SS04] when $n = 3$. To get the lower bound inequality, we integrate out the variable x_3 in the energy to reduce the $3D$ problem to a two-dimensional total variation, weighted by the film thickness function $d(x')$.

The second part of the Γ convergence result is the construction of recovery sequences which is given by Theorem 3.1.4. We had many difficulties proving the upper bound: first, we had to modify the Hodge decomposition Lemma to adapt it to the case where $d(x)$ is zero on the inner boundaries. We did this by introducing the space $\mathbb{H}$ (see Definition 3.3.1) then we define the Hodge decomposition with respect to the weighted inner product,

$$\langle v, w \rangle = \int_\Omega d(x) v \cdot w \; dx'$$

on $L^2(\Omega; \mathbf{R}^2)$.

1.4.3 Pinning in the Ginzburg-Landau Model for Superconductors

In chapter 4 we consider the Γ-convergence of a two-dimensional Ginzburg–Landau model

$$E_\epsilon(\psi, A) = \frac{1}{2} \int_{\mathcal{D}} \left\{ |\nabla_A \psi|^2 + \frac{1}{2\epsilon^2} \left[(|\psi|^2 - a(x))^2 - (a^-)^2 \right] + \left| h - h_{ex} \right|^2 \right\} dx \qquad (1.4.4)$$

for an inhomogeneous superconductor with finitely many "normal regions" in the interior. The inhomogeneity is introduced via a potential term

$$V(\psi) = \frac{1}{4}\big[\left(a(x) - |\psi|^2\right)^2 - (a^-)^2\big],$$

with real-valued function $a(x)$. The presence of the inhomogenity $a(x)$ creates problems near the boundaries of the pinning sites. Indeed, under our hypotheses, $\sqrt{a^+(x)} \notin H^1(\Omega)$, and this results in a singular boundary layer as $\epsilon \to 0$. Following [AB05], a remarkable identity (see Lassoued & Mironescu [LM]) allows us to remove the singular boundary layer part from the rest of the energy. Define a functional,

$$J_\epsilon(\eta) := \int_{\mathcal{D}} \left\{ \frac{1}{2}|\nabla\eta|^2 + \frac{1}{4\epsilon^2}\left[\left(\eta^2 - a(x)\right)^2 - (a^-)^2\right] \right\} dx,$$

and let $\eta_\epsilon \in H^1(\mathcal{D}; \mathbf{R})$ be the (unique) minimizer. With $u = \psi/\eta_\epsilon$ we have:

$$\begin{aligned} E_\epsilon(\psi, A) &= J_\epsilon(\eta_\epsilon) + \int_{\mathcal{D}} \left\{ \frac{\eta_\epsilon^2}{2}|\nabla_A u|^2 + \frac{\eta_\epsilon^4}{4\epsilon^2}(|u|^2 - 1)^2 + \frac{1}{2}(h - h_{ex})^2 \right\} dx \\ &=: J_\epsilon(\eta_\epsilon) + F_\epsilon(u, A), \end{aligned}$$

and the object of interest becomes the reduced energy F_ϵ. We define

$$\Omega_\epsilon = \left\{ x \in \Omega : \text{dist}\,(x, \partial\Omega) > \epsilon^{\frac{1}{3}} \right\}$$

where we can show that

$$\eta_\epsilon^2 \le (1 + \epsilon^{\frac{1}{3}})^2 a(x). \tag{1.4.5}$$

In Theorem 4.2.1 we prove the first part of the Γ-convergence which is the lower bound. We modified the method of the product estimate by Sandier and Serfaty [SS04] to the case when $a(x)$ vanishes. The lower bound of F_ϵ is defined in the whole domain Ω but to be able to see the pinning term $a(x)$ in the limit we have to be away from the normal regions by any $\delta > 0$ for all δ.

For the upper bound which is given in Theorem 4.2.2 , using a result of Montero [M07] we reduced the decomposition of the energy E_ϵ into the multiply connected domain Ω and then we adapt an appropriate Hodge decomposition as in Chapter 3. Having η_ϵ^2 in the energy which depends on ϵ complicate matters. Nevertheless we apply the Hodge decomposition with $a(x)$ in the full domain and adapt the steps of Theorem 3.1.4 in Chapter 3 on those functions.

1.5 Open problems

There are many open problems left. Finding the obstacle problem associated to the Γ-limit in Chapter 3 and 4 is the next step. The solution of the obstacle problem gives us the location of the vortices and hence tells us where they first appear. Previous results (see [ASS], [ABGS13] and [SS00]) found the obstacle problem when $a(x)$ is bounded below by a positive number, their results need to be extended to the case when $a(x)$ vanishes. One of the most interesting cases to study is the case where the pinning term $a(x)$ vanishes at a point in a simply connected domain. The problems studied in Chapter 3 and 4 should be helpful in solving this problem. Indeed we hope to be able to solve this question by shrinking the holes in our results. The problem is that the associated elliptic problems are very degenerate. Finally, we hope to extend the results of [ASS] to the case where $a(x)$ vanishes at isolated points.

Chapter 2

Γ-Convergence of the Ginzburg–Landau Functional with a Pinning Term Bounded by a Positive Number.

This chapter is a warm up for the next 2 chapters. We introduce the methods needed to solve the later problems and we do this by solving a new problem which is less technically difficult but for which we combine the previous results in a new way.

2.1 Introduction

Let $\mathcal{D} \subset \mathbf{R}^2$ be a smooth simply-connected domain, and define $\Omega := \mathcal{D} \setminus \cup_{i=1}^{m} \omega_i$, where i is bounded by n, and $\omega_i \subset \mathcal{D}$ is smooth and simply-connected. The Ginzburg-Landau energy is given by:

$$E_\epsilon(u, A) := \frac{1}{2} \int_\Omega \left[\left| \nabla_A u \right|^2 + \frac{1}{2\epsilon^2} \Big(|u|^2 - a(x) \Big)^2 \right] dx + \frac{1}{2} \int_{\mathcal{D}} \left| h - h_{ex} \right|^2 dx \tag{2.1.1}$$

for $\epsilon := \frac{1}{\kappa} > 0$, where κ is the Ginzburg-Landau parameter, $u \in \mathbf{H}^1(\Omega, \mathbf{C})$, $\nabla_A := \nabla - iA$, and $|u|^2$ represents the density of superconducting election pairs. The function $A \in \mathbf{H}^1(\mathcal{D}, \mathbf{R}^2)$ is the magnetic potential and $h := \nabla \times A$ is the induced magnetic field. The applied magnetic field h_{ex} is a vector field and we take it to be of $O(|\log \epsilon|)$. The energy here is different from the original Ginzburg-Landau Energy where $a(x) \equiv 1$. In our case $a : \Omega \to \mathbf{R}$ is a smooth varying function which has a minimum, the minimum of $a(x)$ represents the pinning sites for the vortices and here are more conditions we consider on $a(x)$:

(H1) $a(x) \in \mathbf{C}^2(\mathcal{D})$.

(H2) There exists a constant $a_0 > 0$ such that $a_0 \leq a(x) \leq 1$.

The Ginzburg-Landau equations associated to the functional (2.1.1) when minimizing for $(u, A) \in \mathcal{H} := \mathbf{H}^1(\Omega, \mathbf{C}) \times \mathbf{H}^1(\mathcal{D}, \mathbf{R}^2)$ are:

$$-\nabla_A^2 u + \tfrac{1}{\epsilon^2}(|u|^2 - a(x))u = 0 \text{ in } \Omega; \tag{2.1.2}$$

$$-\nabla^{\perp} h = j := \langle iu, \nabla_A u\rangle \text{ in } \Omega; \tag{2.1.3}$$

$$h = h_{ex} \text{ on } \partial\mathcal{D}; \tag{2.1.4}$$

$$h = H_j(constant) \text{ in } \omega_j, j = 1, ..., m. \tag{2.1.5}$$

Integrating the second equation around each $\partial\omega_j$ will give us an extra boundary condition,

$$\begin{aligned}\int_{\partial\omega_j} \frac{\partial h}{\partial \nu} ds &= \int_{\partial\omega_j} \nabla^{\perp} h \cdot \tau ds \\ &= -\int_{\partial\omega_j} \operatorname{Im}\{\overline{u}\nabla u\} \cdot \tau ds + \int_{\partial\omega_j} A \cdot \tau ds \\ &= -2\pi \deg(u, \partial\omega_j) + \int_{\omega_j} h dx \\ &= -2\pi \deg(u, \partial\omega_j) + H_j|\omega_j|. \end{aligned} \tag{2.1.6}$$

The functional E_ϵ is gauge-invariant: if $\varphi \in H^2(\mathcal{D}, \mathbf{R})$ is any scalar potential, then $E_\epsilon(u \exp(i\varphi), A + \nabla\varphi) = E(u, A)$. The invariance of the Energy causes a problem for the minimization of GL. Indeed, if $\{u_n, A_n\}_n$ is minimizing sequence, then for any sequence of functions $\{f_n\}_n$, $\{(u_n e^{if_n}, A_n + \nabla f_n)\}_n$ is also a minimizing, for any f_n. Thus no good bounds on $\{u_n, A_n\}_n$ can be deduced from the fact that $GL(u_n, A_n)$ is bounded independently of n. Using a particular gauge transformation, namely Coulomb gauge, solves this problem.

Definition 2.1.1 *Let $\mathcal{D} \subset \mathbf{R}^2$ be a smooth simply-connected domain. We say A :*

$\mathcal{D} \to \mathbf{R}^2$ satisfies the Coulomb gauge condition in $\mathcal{D}$ if

$$\begin{cases} \operatorname{div} A = 0 \quad \text{in } \mathcal{D} \\ A \cdot \nu = 0 \quad \text{on } \partial\mathcal{D} \end{cases} \tag{2.1.7}$$

where ν is the outward pointing unit normal to $\partial\mathcal{D}$.

We will use Proposition 3.3 from [SS07],

Proposition 2.1.2 *Let $\mathcal{D}$ be a smooth, bounded, simply connected domain in $\mathbf{R}^2$. There exists a constant $C > 0$ such that if $A : \mathcal{D} \to \mathbf{R}^2$ satisfies the Coulomb gauge condition, then*

$$\|A\|^2_{\mathbf{H}^1(\mathcal{D},\mathbf{R}^2)} \leq C \|\operatorname{curl} A\|^2_{L^2(\mathcal{D})},$$

and

$$\|A\|^2_{\mathbf{H}^2(\mathcal{D},\mathbf{R}^2)} \leq C \|\operatorname{curl} A\|^2_{\mathbf{H}^1(\mathcal{D})}.$$

Now assume that $|h_{ex}|$ is a function of ϵ and the following limit exists and is finite:

$$\lambda := \lim_{\epsilon \to 0} \frac{|\log \epsilon|}{|h_{ex}|} \tag{2.1.8}$$

and suppose that for a sequence of functions (u_ϵ, A_ϵ),

$$E_\epsilon(u_\epsilon, A_\epsilon) \leq C(\log \epsilon)^2. \tag{2.1.9}$$

Using (2.1.9) and with our choice of gauge we have

$$\Big\| \frac{A_\epsilon}{|\log \epsilon|} \Big\|_{L^\infty(\Omega)} \leq C, \tag{2.1.10}$$

$$\left\| \frac{h_\epsilon}{|\log \epsilon|} \right\|_{L^2(\mathcal{D})} \leq C. \tag{2.1.11}$$

Hence there exist subsequence A_ϵ and h_ϵ which converge to A in $L^\infty(\Omega)$ and h in $L^2(\mathcal{D})$ respectively as $\epsilon \to 0$. We will use these subsequences and their limits later on.

We define the current and Jacobian as follow,

Definition 2.1.3 *For a complex-valued u_ϵ, the current of u_ϵ is defined as the 1-form:*

$$j(u_\epsilon) := (iu_\epsilon, du_\epsilon) \tag{2.1.12}$$

where $(a, b) = \mathrm{Re}\,(\bar{a}b)$. Using differential forms, $j(u_\epsilon)$ can be written as

$$j(u_\epsilon) = \sum_{k=1}^{2} (iu_\epsilon, \partial_k u_\epsilon) dx_k. \tag{2.1.13}$$

It is related to the Jacobian determinant by:

$$Ju_\epsilon = \frac{1}{2} d(ju_\epsilon) = \frac{1}{2} d(iu_\epsilon, du_\epsilon) \tag{2.1.14}$$

where

$$Ju_\epsilon = \sum_{j<k} (i\partial_j u_\epsilon, \partial_k u_\epsilon) dx_j \wedge dx_k.$$

Definition 2.1.4 *Consider $f : \Omega \to \mathbf{R}$.*

1. *If f is bounded and continuous, we write*

$$\|f\|_{C^0(\overline{\Omega})} := \sup_{x \in \Omega} |f(x)|$$

2. *the α^{th}-Hölder seminorm of f is:*

$$[f]_{C^{0,\alpha}(\overline{\Omega})} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \Big(\frac{f(x)-f(y)}{\|x-y\|^{\alpha}}\Big).$$

and the α^{th}-Hölder norm of f is:

$$\|f\|_{C^{0,\alpha}(\overline{\Omega})} = \|f\|_{C^{0}(\overline{\Omega})} + [f]_{C^{0,\alpha}(\overline{\Omega})}.$$

Definition 2.1.5 *For $k \in \mathbf{N}$, the Sobolev spaces $W^{-k,p}(\Omega)$ are defined as dual spaces $(W^{k,q}(\Omega))'$, where q is conjugate to $p : \frac{1}{p}+\frac{1}{q}=1$. Their elements are distributions:*

$$W^{-k,p}(\Omega) := \big\{u \in D'(\Omega),\ u = \sum_{|\alpha|\leq k} \partial^{\alpha} u_{\alpha},\ \textit{for some } u_{\alpha} \in L^{p}(\Omega)\big\}.$$

Naturally $W^{-k,p}(\Omega)$ is a Banach space with the norm:

$$\|u\|_{W^{-k,p}(\Omega)} = \sup_{\substack{v\in W^{k,p}(\Omega) \\ \|v\|_{W^{k,p}(\Omega)}\neq 0}} \frac{\big|\langle u,v\rangle\big|}{\|v\|_{W^{k,p}(\Omega)}} \tag{2.1.15}$$

For any integer k, ∂^{α} is a bounded operator from $W^{k,p}$ to $W^{k-|\alpha|,p}$.

Definition 2.1.6 *A sequence $\{f_n\}$ in $L^p(\Omega)$ is said to converge weakly in $L^p(\Omega)$ to $f \in L^p(\Omega)$ if for any $\phi \in L^q(\Omega)$ where q is the conjugate of p we have*

$$\lim_{n\to\infty} \int_{\Omega} \phi \cdot f_n dx = \int_{\Omega} \phi \cdot f dx$$

We write

$$\{f_n\} \rightharpoonup f \ \textit{in } L^p(\Omega)$$

to mean that f and each f_n belong to $L^p(\Omega)$ and $\{f_n\}$ converges weakly in $L^p(\Omega)$ to f .

Definition 2.1.7 *Let $(X, \mathcal{A}, \mu)$ be a signed measure space. $A \in \mathcal{A}$ is non-negative, (respectively, non-positive) if*

$$\forall E \subseteq A, \textit{for which } E \in \mathcal{A}, \textit{we have } \mu(E) \geq 0, \textit{(respectively, } \mu(E) \leq 0.\textit{)}$$

For each $A \in \mathcal{A}$ we define

$$\mu^+(A) = \sup\{\mu(A),\ 0\},$$

and

$$\mu^-(A) = -\inf\{-\mu(A),\ 0\},$$

μ^+, respectively, μ^-, is the positive, respectively, negative, variation of μ. For any signed measure space $(X, \mathcal{A}, \mu)$, the total variation $|\mu|$ is defined as:

$$\forall A \in \mathcal{A}, \qquad |\mu|(A) = \mu^+(A) + \mu^-(A).$$

Definition 2.1.8 *Let V be an n-dimensional (n finite) vector space with inner product g. The Hodge star operator (denoted by $\star$) is a linear operator mapping p-forms on V to $(n-p)$-forms, i.e.,*

$$\star : \Omega^p \to \Omega^{n-p}. \tag{2.1.16}$$

2.2 Γ-limit and Main results.

The Γ-limit was first introduced by E. de Giorgi and T. Franzoni in 1975 and since then was much developed especially in connection with applications to problems in the Calculus of Variations.

Definition 2.2.1 *(Γ-**Convergence**)*

Let X be a topological space and $F_n : X \to \mathbf{R}^+$ a sequence of positive functionals on X. Then F_n are said to Γ-converge to Γ-limit $F : X \to \mathbf{R}^+$ if the following two conditions hold:

1. *Lower bound inequality: For every sequence $(x_n) \in X$ such that $x_n \to x$ as $n \to +\infty$,*
$$F(x) \leq \liminf_{n\to\infty} F_n(x_n).$$

2. *Upper bound inequality: For every $x \in X$, there is a sequence (x_n) converging to x such that*
$$F(x) \geq \limsup_{n\to\infty} F_n(x_n).$$

Our main result is to find the Γ-limit for the Ginzburg-Landau functional and this is done by finding the lower bound and then the matching upper bound as stated in the following Theorem:

Theorem 2.2.2 *Let (H1)-(H2) be satisfied and assume that (2.1.9) holds. Then there exists $j \in L^2(\Omega)$ and $J \in \mathfrak{M}(\Omega) \cap \mathbf{H}^{-1}(\Omega)$ such that $\frac{j(u_\epsilon)}{h_{ex}} \rightharpoonup j$ in $L^2(\Omega)$, and $\frac{Ju_\epsilon}{h_{ex}} = \nabla \times \frac{j(u_\epsilon)}{h_{ex}} \rightharpoonup J$ in the sense of measure. Moreover,*

$$\liminf_{\epsilon\to 0} \frac{E_\epsilon(u_\epsilon, A_\epsilon)}{(\log\epsilon)^2} \geq \frac{1}{2}\Big[\|aJ\|_{\mathfrak{M}(\Omega)} + \int_\Omega a(j-A)^2 dx + \lambda \int_{\mathcal{D}} |\nabla \times A - 1|^2 dx\Big],$$

where $\mathfrak{M}(\Omega)$ is the space of vector-valued Radon measures on Ω, and A is the limit of $\frac{A_\epsilon}{h_{ex}}$ in $L^\infty(\mathcal{D})$. Finally, If Ω is smooth and bounded, then for any given $j \in L^2$ such that $\nabla \times j/2$ is a Radon measure, there exists a sequence $\{u_\epsilon\}$ in $\mathbf{H}^1(\Omega)$ such that $\frac{j_\epsilon}{h_{ex}}, \frac{J_\epsilon}{h_{ex}}$ defined in (2.1.12) and (2.1.14) converge to j, and J respectively, weakly in L^2 and in $(C^{o,\alpha}(\Omega))'$ for every $0 < \alpha \leq 1$, and for this sequence the above limit is achieved with an equality.

The proof of this Theorem will be done in many steps and we will need first to obtain certain results.

2.3 Jacobian estimate and lower bound

In this section we follow Sandier and Serfaty to find a sharp Jacobian estimate in terms of the magnetic Ginzburg-Landau energy, we do this by modifying Theorem 1 in [SS04].

Theorem 2.3.1 *Let (u_ϵ, A_ϵ) be such that $E_\epsilon(u_\epsilon, A_\epsilon) \leq C|\log \epsilon|^2$ and $h_\epsilon = \operatorname{curl} A_\epsilon$. Then up to extraction the rescaled Jacobians $\frac{J(u_\epsilon, A_\epsilon)}{h_{ex}}$ weakly converge to J, a measure-valued 2-form, in $(C_c^{0,\alpha}(\Omega))'$, where $0 < \alpha \leq 1$, $\frac{(iu_\epsilon, \nabla u_\epsilon)}{h_{ex}} \rightharpoonup aj$ in $L^2(\Omega)$, and*

$$\liminf_{\epsilon \to 0} \frac{1}{|\log \epsilon|^2} E_\epsilon(u_\epsilon, A_\epsilon) \geq \frac{a}{2}|J|(\Omega) + \frac{1}{2}\int_\Omega a|j - A|^2 dx + \frac{\lambda}{2}\int_{\mathcal{D}} |h_* - 1|^2 dx \quad (2.3.1)$$

where A and h_* are the limits of A_ϵ and h_ϵ defined in (2.1.10) and (2.1.11).

Proof:

We prove this Theorem in 4 steps as in [SS04] with a minor modification due to the presence of $a(x)$.

Step 1. (**Modified vortex balls**). Sandier and Serfaty constructed their vortex balls based on the fact that $|u_\epsilon|$ is an S^1 valued in the limit. Since we have $|u_\epsilon|$ is close to $a(x)$ when $\epsilon \to 0$, we used the vortex balls constructed by Aftalion, Sandier, and Serfaty [ASS]. Define the domain

$$\Omega_\epsilon := \{x \in \Omega : \ \mathrm{dist}\,(x, \partial\Omega) > \epsilon\},$$

and recall Proposition 1.1 from [ASS],

Proposition 2.3.2 *Assume $h_{ex} \le \lambda|\log\epsilon|$ for some $\lambda > 0$ and that $(H1)$ to $(H2)$ are satisfied, then there exists a positive constant ϵ_0 such that if $\epsilon < \epsilon_0$ and (u_ϵ, A_ϵ) is a minimizer of E_ϵ, there exists a family of balls of disjoint closures (depending on ϵ) $(B_i)_{i\in I_\epsilon} = (B(p_i, r_i))_{i\in I_\epsilon}$ satisfying.*

$$\{x \in \Omega_\epsilon, |\sqrt{a(x)} - |u_\epsilon(x)|| \ge \tfrac{2}{|\log\epsilon|^2}\} \subset \cup_{i\in I_\epsilon} B(p_i, r_i), \tag{2.3.2}$$

$$\textstyle\sum_{i\in I_\epsilon} r_i \le \frac{1}{|\log\epsilon|^2}, \tag{2.3.3}$$

$$\tfrac{1}{2}\textstyle\int_{B_i} |\nabla u_\epsilon|^2 dx \ge \pi a(p_i)|d_i||\log\epsilon|(1 - o(1)), \tag{2.3.4}$$

where $d_i = \deg(u/|u|, \partial B_i)$ if $\overline{B_i} \subset \Omega$, and 0 otherwise.

Step 2. (**Compactness of the Jacobian**).

Using the points $\{p_i\}$ in Proposition 2.3.2 we define the measure,

$$\mu_\epsilon = \pi \sum_{\{i|p_i\in\Omega_\epsilon\}} d_i\delta_{p_i}. \tag{2.3.5}$$

From Proposition 2.3.3, it follows that

$$\begin{aligned} C|\log\epsilon|^2 \geq E_\epsilon(u_\epsilon, A_\epsilon) &\geq \sum_i \pi a(p_i)|d_i||\log\epsilon|(1-o(1)) \\ &\geq a_0 \sum_i \pi |d_i||\log\epsilon|(1-o(1)) \end{aligned}$$

where a_0 is given by hypothesis (H2) on $a(x)$. Hence

$$\frac{1}{2}\int_\Omega |\mu_\epsilon| dx = \frac{\pi \sum_i |d_i|}{|\log\epsilon|} \leq C,$$

thus (μ_ϵ) is a bounded sequence of measures, and we can assume that μ_ϵ converges to some μ_* in the sense of measures.

$$\mu_\epsilon \rightharpoonup \mu_* \;\text{ in the sense of measure,} \tag{2.3.6}$$

$$|\mu_\epsilon| = \pi \sum_{\{i|p_i\in\Omega\}} |d_i| \rightharpoonup |\mu_*| \;\text{ in the sense of measure.} \tag{2.3.7}$$

We have, following [SS04] and proved in details in Chapter 4 Proposition 4.3.2, that

$$\| \star Ju_\epsilon - \mu_\epsilon\|_{(C_c^{0,\alpha}(\Omega))'} \leq C\frac{E_\epsilon(u_\epsilon, A_\epsilon)}{|\log\epsilon|^2}, \tag{2.3.8}$$

where $\star$ is the Hodge operator with respect to the Euclidean metric (see definition (2.1.8)). The compactness of $\frac{\mu_\epsilon}{|\log\epsilon|}$ in $(C_c^{0,\alpha}(\Omega))'$ for any $0 < \alpha \leq 1$ is true because of its boundedness in $(C^0)'$ and the compact embedding of $(C^{0,\alpha}(\Omega))'$ in $(C^0)'$. It follows that $\frac{J_\epsilon}{|\log\epsilon|}$ subsequentially converges in $(C_c^{0,\alpha}(\Omega))'$ to the same limit as $\frac{\mu_\epsilon}{|\log\epsilon|}$, i.e. to a measure J.

Step 3. (**Jacobian estimate**).

Let X, Y be continuous vector field compactly supported in Ω. It follows from the energy bound (2.1.9) that

$$j_{\epsilon,X} = \frac{|X \cdot \nabla u_\epsilon|}{|\log \epsilon|}, \qquad j_{\epsilon,Y} = \frac{|Y \cdot \nabla u_\epsilon|}{|\log \epsilon|} \tag{2.3.9}$$

are bounded in L^2 and therefore converge weakly subsequentially. Using Proposition 2.3.2, there exist a collection of balls $\{B_i\}$ satisfying (2.3.2), (2.3.3), and (2.3.4). Let X, Y be continuous vector fields compactly supported in Ω, we have

$$\frac{1}{2|\log \epsilon|} \int_{B_i} |X \cdot \nabla u_\epsilon|^2 + |Y \cdot \nabla u_\epsilon|^2 \frac{dx_1 dx_2}{|X \wedge Y|} \geq \pi a(p_i)|d_i|(1 - o(1)) \tag{2.3.10}$$

(as we see $a(x)$ appears in the right hand side because of our vortex balls construction which is different than the one in [SS04]). Using the definition of μ_ϵ (2.3.5), and notting that $a(x)$ is near $a(p_i) - o(1)$, we sum over i and get

$$\frac{1}{2|\log \epsilon|} \int_{\cup_i B_i} |X \cdot \nabla u_\epsilon|^2 + |Y \cdot \nabla u_\epsilon|^2 \frac{dx_1 dx_2}{|X \wedge Y|} \geq \left| \int_\Omega (1 - o(1)) a d\mu_\epsilon \right|, \tag{2.3.11}$$

where $o(1)$ is a quantity that tends to zero when $\epsilon \to 0$. Dividing the above inequality by $|\log \epsilon|$ and using (2.3.8) we find

$$\begin{aligned} \liminf_{\epsilon \to 0} \frac{1}{2|\log \epsilon|^2} \int_{\cup_i B_i} |X \cdot \nabla u_\epsilon|^2 + |Y \cdot \nabla u_\epsilon|^2 &\geq |X \wedge Y| \left| \int_\Omega a J(\partial_{x_1}, \partial_{x_2}) \right| \\ &\geq \left| \int_\Omega a J(X, Y) \right|, \end{aligned} \tag{2.3.12}$$

where J is the limit of $\frac{J u_\epsilon}{|\log \epsilon|}$. Using (2.3.9) we fix a convergent subsequence and let

j_X, j_Y denote the weak limits of the normalized current $\frac{ju_\epsilon}{|u_\epsilon||\log\epsilon|}$. Then

$$|j_{\epsilon,X}|^2 \rightharpoonup |j_X|^2 + \nu_X, \qquad |j_{\epsilon,Y}|^2 \rightharpoonup |j_Y|^2 + \nu_Y, \tag{2.3.13}$$

weakly as measures, where ν_X and ν_Y are positive Radon measures, called the defect measures of the sequences. Following [SS04], (proved in details in Chapter 4 Theorem 4.3.1), we get

$$\frac{1}{2}(\|\nu_X\| + \|\nu_Y\|) \geq \left| \int_\Omega aJ(X,Y) \right|. \tag{2.3.14}$$

Step 4. (**Lower bound**).

We first prove the lower bound of the gradient part $\int_\Omega |\nabla u_\epsilon|^2$ using the Jacobian estimate (2.3.12). We choose e_1, e_2 an orthonormal (moving) frame that may depend on $x \in \Omega$, and $f, g \in C_c^0(\Omega)$ with $|f| \leq 1$ and $|g| \leq 1$. Then, let $X_1 = fe_1, X_2 = ge_2$. The inequality

$$|\nabla u_\epsilon|^2 \geq \left|X_1 \cdot \nabla u_\epsilon\right|^2 + \left|X_2 \cdot \nabla u_\epsilon\right|^2 \tag{2.3.15}$$

holds. Since $\left|X_i \cdot ju_\epsilon\right| \leq \left|X_i \cdot \nabla u_\epsilon\right|\left|u_\epsilon\right|$, we have

$$\left|X_i \cdot ju_\epsilon\right| - \sqrt{a}\left|X_i \cdot \nabla u_\epsilon\right| \leq \left(\left|u_\epsilon\right| - \sqrt{a}\right)\left|X_i \cdot \nabla u_\epsilon\right|$$

Since that $\frac{ju_\epsilon}{|\log\epsilon|}$ is bounded in $L^2(\Omega)$, hence weakly compact, and that

$$\frac{\left(\left|X_i \cdot ju_\epsilon\right| - \sqrt{a}\left|X_i \cdot \nabla u_\epsilon\right|\right)_+}{|\log\epsilon|} \to 0 \tag{2.3.16}$$

as $\epsilon \to 0$ in $L^1(\Omega)$. It follows that denoting by ϕ_{X_i} the weak L^2 limit of

$$\frac{\sqrt{a}\big|X_i \cdot \nabla u_\epsilon\big|}{|\log \epsilon|}, \qquad i = 1, 2$$

and by (2.3.13) we have $\sqrt{a}|X_i \cdot j| \le \phi_{X_i}$ almost everywhere, where j is the weak limit of the normalized currents. Denoting by ν_{X_1} and ν_{X_2} the defect measures of

$$\frac{\big|X_1 \cdot \nabla u_\epsilon\big|}{|\log \epsilon|}, \qquad \frac{\big|X_2 \cdot \nabla u_\epsilon\big|}{|\log \epsilon|}$$

respectively, it follows from (2.3.15) and the definition of defect measure that

$$\liminf_{\epsilon \to 0} \frac{1}{|\log \epsilon|^2} \int_\Omega |\nabla u_\epsilon|^2 \ge \|\nu_{X_1}\| + \|\nu_{X_2}\| + \int_\Omega |\phi_{X_1}|^2 + |\phi_{X_2}|^2$$

using the above result, we are led to

$$\begin{aligned} \liminf_{\epsilon \to 0} \frac{1}{|\log \epsilon|^2} \int_\Omega |\nabla u_\epsilon|^2 &\ge 2\left| \int_\Omega aJ(X_1, X_2) \right| + \int_\Omega a|X_1 \cdot j|^2 + a|X_2 \cdot j|^2 \\ &\ge 2\left| \int_\Omega fgaJ(e_1, e_2) \right| + \int_\Omega a|j|^2 \\ &\qquad + \int_\Omega (|f|^2 - 1)a|j \cdot e_1|^2 + (|g|^2 - 1)a|j \cdot e_2|^2. \end{aligned} \tag{2.3.17}$$

Taking the supremum over all such frames e_1, e_2 and all compactly supported $|f| \le 1,\ |g| \le 1$ proves the lower bound of the gradient part of the energy.

To prove the lower bound (2.3.1) we expand the energy as follow:

$$\begin{aligned}
E_\epsilon(u,A) &= \frac{1}{2}\int_\Omega |\nabla_A u|^2 + \frac{1}{2\epsilon^2}\big(|u|^2 - a\big)^2 dx + \frac{1}{2}\int_{\mathcal{D}} \big|h - h_{ex}|^2 dx \\
&\geq \frac{1}{2}\int_\Omega |\nabla u - iAu|^2 dx + \frac{1}{2}\int_{\mathcal{D}} \big|h - h_{ex}|^2 dx \\
&= \frac{1}{2}\int_\Omega \Big[|\nabla u|^2 + |A|^2|u|^2 - 2A\langle iu, \nabla u\rangle\Big] dx + \frac{1}{2}\int_{\mathcal{D}} \big|h - h_{ex}|^2 dx.
\end{aligned}$$

The lower bound of first integral is given by (2.3.17). For the rest, we use the energy bound (2.1.9) and letting A_ϵ and h_ϵ be the convergent subsequences followed from (2.1.10) and (2.1.11) to A in $L^\infty(\Omega)$ and h_* in $L^2(\Omega)$ respectively and using the fact that $|u|^2 \to a(x)$ *a.e. in* Ω, we have

$$\begin{aligned}
\liminf_{\epsilon\to 0} \frac{1}{2|\log\epsilon|^2}\Big[\int_\Omega \big(|A_\epsilon|^2|u_\epsilon|^2 - 2A_\epsilon j u_\epsilon\big) dx + \int_{\mathcal{D}} |h_\epsilon - h_{ex}|^2 dx\Big] \\
\geq \frac{1}{2}\int_\Omega \Big(a(x)|A|^2 - 2a(x)Aj\Big)\, dx + \frac{1}{2}\int_{\mathcal{D}} \big|h_* - \lambda|^2 dx
\end{aligned} \tag{2.3.18}$$

(2.3.17) and (2.3.18) yield the full conclusion of the Theorem 2.3.1.

$\diamondsuit$

2.4 Upper bound

We first define the space $\mathcal{Z}$ by

$$\mathcal{Z} := \{j \in L^2(\Omega;\mathbf{R}^2) : \ J := \frac{1}{2}\nabla\times j \in \mathfrak{M}(\Omega;\mathbf{R}^2)\}.$$

Proposition 2.4.1 *Let $j \in \mathcal{Z}$ and consider any sequence ϵ_n such that $\epsilon_n \to 0$. Then there exists a sequence $\{u_n, A_n\} \subset \mathbf{H}^1(\Omega; \mathbf{C}) \times \mathbf{H}^1(\mathcal{D}, \mathbf{R}^2)$, satisfying*

$$\begin{aligned}
&\frac{j(u_n)}{|\log \epsilon_n|} \to j \quad in\ L^p(\Omega), for\ all\ p < 2,\\
&\frac{Ju_n}{|\log \epsilon_n|} \to J := \frac{1}{2}\nabla \times j \quad weakly\ in\ \mathfrak{M}(\Omega; \mathbf{R}^2),\ and\ strongly\ in\ (C_0^\alpha(\Omega))',\ 0 < \alpha < 1,\\
&\frac{A_n}{|\log \epsilon|} \to A \quad in\ L^\infty(\mathcal{D}),
\end{aligned}$$

with $j(u_n) := (iu_n, du_n)$ and $Ju_n := \frac{1}{2}dj(u_n)$. Moreover,

$$E_\epsilon(u_n, A_n) \leq \frac{1}{2}\int_\Omega \Big\{a(x)d\mu + a(x)(j - A)^2 \Big\}dx + \frac{1}{2}\int_{\mathcal{D}} |\nabla \times A - \lambda|^2 dx.$$

To prove this proposition we follow [JS] and [ABGS13]. We require the following Hodge decomposition with respect to the weighted inner product ,

$$\langle v, w\rangle = \int_\Omega a(x) v \cdot w \ dx$$

on $L^2(\Omega; \mathbf{R}^2)$. We define the following subspaces:

$$\begin{aligned}
&\mathcal{U} = \{-\frac{1}{a}\nabla^\perp \psi,\ \psi \in \mathbf{H}_0^1(\Omega; \mathbf{R})\},\\
&\mathcal{V} = \{\nabla \zeta,\ \zeta \in \mathbf{H}^1(\Omega; \mathbf{R})\},\\
&\mathcal{W} = \{W \in C^1(\Omega; \mathbf{R}^2),\ \nabla^\perp \cdot W = 0,\ \nabla \cdot (aW) = 0,\ W \cdot \nu = 0\ on\ \partial\Omega\}.
\end{aligned} \tag{2.4.1}$$

Recall Lemma 3.1 and its proof from [ABGS13].

Lemma 2.4.2 *Any $Z \in L^2(\Omega; \mathbf{R}^2)$ admits a unique orthogonal decomposition $Z = U + V + W$ with $U \in \mathcal{U}$, $V \in \mathcal{V}$, and $W \in \mathcal{W}$, with respect to the inner product $\langle ., .\rangle$.*

The space $\mathcal{W}$ is a finite dimensional where $\dim(\mathcal{W}) = m$.

Proof of Lemma 2.4.2. First, we assume $Z \in C^{\infty}(\Omega, \mathbf{R})$ .We define ψ and ζ as the solutions to the boundary-value problems,

$$\begin{cases} -\nabla \cdot \left(\frac{1}{a(x)}\nabla\psi\right) = \operatorname{curl} Z \quad in\ \Omega, \\ \psi = 0 \quad on\ \partial\Omega, \end{cases} \qquad \begin{cases} \nabla \cdot (a(x)\nabla\zeta) = \operatorname{div}[aZ] \quad in\ \Omega, \\ \frac{\partial\zeta}{\partial\nu} = Z\cdot\nu \quad on\ \partial\Omega, \end{cases}$$

the existence of solutions to these boundary-value problems is standard because $a(x)$ is bounded below. Then, it is easy to verify that $W := Z + \frac{1}{a}\nabla^{\perp}\psi - \nabla\zeta$ satisfies $\operatorname{curl} W = 0 = \operatorname{div}[aW]$ in Ω, and $W \cdot \nu = 0$ on $\partial\Omega$. Moreover, by integration by parts we see that $W \perp \frac{1}{a}\nabla^{\perp}\psi \perp \nabla\zeta$ in the inner product $\langle ., . \rangle$.

To identify the space W, we apply Lemma 1.1 of [BBH] and note that any $W \in \mathcal{W}$ may be written as $W = \frac{1}{a}\nabla^{\perp}\xi$ with ξ constant on each component of $\partial\Omega$, and $\nabla \cdot \frac{1}{a}\nabla^{\perp}\xi = 0 \quad in\ \Omega$. Our domain $\Omega = \mathcal{D} \setminus \cup_j \omega_j$ is multiply connected, then we follow the treatment of [ABGS13]. For each fixed $j = 1, ..., m$ we define functions $\xi_i \in \mathbf{H}_0^1(\Omega)$ which solve

$$\left.\begin{aligned} \nabla \cdot \frac{1}{a}\nabla\xi_i = 0, \quad \text{in } \Omega, \\ \xi_i|_{\partial\omega_j} = c_{ij}, \quad j = 1, \ldots, m \\ \xi_i|_{\partial\mathcal{D}} = 0, \\ \frac{1}{2\pi}\oint_{\partial\omega_j} \frac{1}{a}\frac{\partial\xi_i}{\partial\nu}dx = \delta_{i,j} \quad j = 1, \ldots, m, \end{aligned}\right\} \tag{2.4.2}$$

where c_{ij} are constants (determined by the solutions,) and $\delta_{i,j}$ is Kronecker's delta. We got the last equation by integrating around each ω_j as earlier (2.1.5).

We can obtain the existence of such ξ_i by minimizing

$$F_i(\xi) = \frac{1}{2}\int_\Omega \frac{1}{a}|\nabla\xi|^2 dx + 2\pi\xi|_{\omega_i}$$

over $\mathbb{W}$ where

$$\mathbb{W} := \{\xi \in \mathbf{H}_0^1(\mathcal{D}) \ with \ \xi|_{\omega_j} = constant\}. \tag{2.4.3}$$

By the Poincaré inequality and the trace inequalities, F_i is bounded below on $\mathbf{H}_0^1(\Omega)$, and by convexity it attains a unique minimizer ξ_i. A simple computation shows that minimizers give weak solutions to the boundary-value problem (2.4.2). Indeed, the first variation yields,

$$0 = DF_i(\xi_i)u = \int_\Omega [\frac{1}{a}\nabla\xi_i \cdot \nabla u] dx + 2\pi\xi|_{\omega_i}. \tag{2.4.4}$$

for all $u \in \mathbf{H}_0^1(\Omega)$. The equation and boundary conditions then follow from choosing u with values either zero or one in the appropriate domains ω_j.

$$\xi = \sum_{i=1}^m \Phi_i \xi_i(x), \qquad \Phi_i := \Big(\frac{1}{2\pi}\oint_{\partial\omega_j} \frac{1}{a}\frac{\partial\xi}{\partial\nu} dx\Big).$$

Thus, $W = \frac{1}{a}\nabla^\perp\xi \in \mathcal{W}$ is parametrized by the m constants $\Phi_i, i = 1, ..., m$, and $\mathcal{W}$ which is a finite dimensional space of order m. The general result for $Z \in L^2(\Omega; \mathbf{R}^2)$ is obtained by density.

$\diamondsuit$

Proof of Proposition 2.4.1

Let $j \in \mathcal{Z}$ be given, as well as $\epsilon_n \to 0$. From the energy bound (2.1.9), the potential $\frac{A_\epsilon}{|\log\epsilon|}$ is bounded and hence has a limit that we call A. We choose the vector potentials $A_n = |\log \epsilon_n| A$, and construct a sequence of order parameters u_n to satisfy the demands of the theorem.

We apply the Hodge decomposition given in Lemma 2.4.2 to our $j \in \mathcal{Z}$, and write

$$j = U + V + W = -\frac{1}{a}\nabla^{\perp}\psi + \nabla\zeta + W$$

with $\psi \in \mathbf{H}_0^1(\Omega),\ \zeta \in \mathbf{H}^1(\Omega),\ \textit{and}\ W \in \mathcal{W}$, a mutually orthogonal splitting in the inner product $\langle .,. \rangle$. Since $\nabla \times (V + W) \equiv 0$ then $J = \frac{1}{2}\nabla \times j = \frac{1}{2}\nabla \times U$ and $V + W$ doesn't contribute to the weak Jacobian. We need to construct sequences w_ϵ and u_ϵ which converge to $V + W$ and U respectively. As in [JS] we may associate to V, W an S^1-valued map w_ϵ. The singular part of the Jacobian is contained in U; for this part we construct a family u_ϵ with points vortices via an appropriate Green's function. Putting these two parts together, the desired recovery sequence will have the form $v_n = u_{\epsilon_n} w_{\epsilon_n}$.

Constructing the sequences is an adaptation of the proof of Theorem 1.2 in [ABGS10] since they worked with $a(x)$ is bounded below, the main difference is that we work on the full Ginzburg-Landau Energy where $a(x)$ is shown only in the part $\int_\Omega \frac{1}{4\epsilon^2}\left(|u|^2 - a(x)\right)^2$.

Step 1. (Recovering $V + W$)

From the proof of Lemma 2.4.2, we may write $V = \nabla\zeta,\ \zeta \in \mathbf{H}^1(\Omega)$ and $W = \frac{1}{a}\nabla^{\perp}\xi\ \textit{with}\ \xi(x) = \sum_{i=1}^m \Phi_i \xi_i(x),\ \textit{for}\ \xi_i$ as in (2.4.2) with Φ_i real constants. Let $M_{i,n} = [\Phi_i |\log \epsilon_n|],\ i = 1, ..., m$, where brackets denote the integer part, Set

$$\Xi_n := \sum_{i=1}^m M_{i,n}\xi_i, \qquad W_n = -\frac{1}{a}\nabla^{\perp}\Xi_n.$$

We note that

$$\|W_n - W\log\epsilon_n\|_{C^1} \le C, \tag{2.4.5}$$

for constant C depending on W (but independent of n.)

Since

$$\begin{aligned}
\operatorname{curl} W_n &= \sum_{i=1}^m M_{i,n}\nabla^\perp\cdot\frac{1}{a}\nabla^\perp\xi_i = 0,\\
\oint_{\partial\omega_j} W_n\cdot\tau\, ds &= \sum_{i=1}^m M_{i,n}\oint_{\partial\omega_j}\frac{1}{a}\frac{\partial\xi_i}{\partial\nu}ds = 2\pi M_{j,n},
\end{aligned}$$

an integer multiple of 2π for each $j = 1,\dots,m$, it follows that W_n is locally a gradient, $W_n = \nabla\eta_n$ for η_n possibly multiple valued, but for which $e^{i\eta_n}$ is smooth and single-valued in Ω. We may then define the complex order parameter

$$w_n = \exp i(\eta_n + \zeta\log\epsilon_n).$$

By construction,

$$\frac{j(w_n)}{\log\epsilon_n} = \frac{(iw_n,\nabla w_n)}{\log\epsilon_n} \to V + W \tag{2.4.6}$$

in $C^1(\bar\Omega)$. Since $|w_n| = 1$, we may easily calculate the contribution to the energy using the orthogonality:

$$\begin{aligned}
\frac{1}{2}\int_\omega |\nabla w_n|^2\, dx &= \frac{1}{2}\int_\Omega |\nabla\eta_n + \nabla\zeta\log\epsilon_n|^2\, dx\\
&= \frac{1}{2}\int_\Omega |W_n|^2 + \frac{(\log\epsilon_n)^2}{2}\int_\Omega |\nabla\zeta|^2\, dx\\
&\le \frac{(\log\epsilon)^2}{2}\int_\Omega \left\{|W|^2 + |V|^2\right\} dx + O(1),
\end{aligned} \tag{2.4.7}$$

using (2.4.5) in the last line. This completes Step 1.

The treatment of the component $U = -\frac{1}{a}\nabla^{\perp}\psi \in \mathcal{U}$ will require several steps. First, we restrict to $\psi \in C_0^{\infty}(\Omega)$; the result for general $\psi \in H_0^1(\Omega)$ will follow from a diagonal argument.

Step 2: Approximating the measure $\mu := \operatorname{curl} U = -\nabla \times \frac{1}{a}\nabla^{\perp}\psi$ by Dirac masses (representing vortices.) As μ is smooth, and absolutely continuous w.r.t. Lebesgue measure, we will use the notation μ or $\mu(x)dx$ interchangeably.

We will need the result of Lemma 7.5 of [JS] and recall its proof.

Lemma 2.4.3 *There exists families $\{p_i^{\epsilon}\}_{i=1}^{N_\epsilon}$ of points and integers $\sigma_i^{\epsilon} = \pm 1$, satisfying*

$$\mu^{\epsilon} := \frac{\pi}{|\log\epsilon|}\sum \sigma_i^{\epsilon}\delta_{p_i^{\epsilon}} \rightharpoonup \mu(x)dx \quad \text{weakly in } \mathfrak{M}(\Omega) \text{ and strongly in } W^{-1,p}(\Omega) \quad \forall p<2$$

$$\text{and in } \ (C_0^{0,\alpha}(\Omega))' \text{ for } 0<\alpha\leq 1 \tag{2.4.8}$$

$$|\mu^{\epsilon}| := \frac{\pi}{|\log\epsilon|}\sum \delta_{p_i^{\epsilon}} \rightharpoonup |\mu(x)|dx \quad \text{weakly in } \mathfrak{M}(\Omega) \text{ and strongly in } W^{-1,p}(\Omega) \quad \forall p<2$$

$$\text{and in } \ (C_0^{0,\alpha}(\Omega))' \text{ for } 0<\alpha\leq 1 \tag{2.4.9}$$

$$|p_i^{\epsilon} - p_j^{\epsilon}| \geq c_0|\log\epsilon|^{-\frac{1}{2}} \ \forall j\neq i, \quad \operatorname{dist}(p_i^{\epsilon},\partial\Omega) \geq c_0|\log\epsilon|^{-\frac{1}{2}} \quad \forall i \tag{2.4.10}$$

where c_0 is small constant that depend on $\|\mu\|_{\infty}$, and $\mathfrak{M}$ is the space of all bounded Radon measures.

Proof: . Write $\Omega = \cup\Omega_i^{\epsilon}$, where for each i, Ω_i^{ϵ} is a set of the form $\Omega \cap \mathcal{Q}_i^{\epsilon}$, and $\mathcal{Q}_i^{\epsilon}$ is a cube of side length $|\log\epsilon|^{-\frac{1}{4}}$. For each i, let

$$N_i^{\epsilon} = \begin{cases} \lfloor |\log\epsilon| \int_{\Omega_i^{\epsilon}} |\mu| dx \rfloor & \operatorname{dist}(\Omega_i^{\epsilon},\partial\Omega) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Also let $\sigma_i^\epsilon = sgn(\int_{\Omega_i^\epsilon} \mu dx)$. In each Ω_i^ϵ select N_i^ϵ points $\{p_i^\epsilon\}_{i=1}^{N_i^\epsilon}$ that are roughly equally distributed. Note that $N_i^\epsilon = \|\mu\|_\infty |\log\epsilon|^{\frac{1}{2}}$ for all i. This implies that the points can be chosen so that the distances are bounded below as in (2.4.10). Finally, define

$$\mu^\epsilon := \sum_i \sum_{j=1}^{N_i^\epsilon} \sigma_i^\epsilon \delta_{p_{ij}^\epsilon}.$$

Upon relabelling, this collection of points has the same form as in (2.4.8)-(2.4.10).

By construction it is easy to see that this sequence of measures has uniformly bounded mass, so weak convergence in $\mathfrak{M}$ will follow from strong convergence in $W^{-1,p},\ p < 2$. For the latter, since functions in $W^{1,q},\ q > 2$ are Hölder continuous, it suffices to verify that for $0 < \alpha \leq 1$,

$$\sup_{\|\phi\|_{C^{0,\alpha}} \leq 1} \left| \int_\Omega \phi d\mu^\epsilon - \int \phi(x)\mu(x)dx \right| \to 0 \tag{2.4.11}$$

as $\epsilon \to 0$. To verify this, note that if $\mathrm{dist}\,(\Omega_i^\epsilon, \partial\Omega) > 0$ and $\|\phi\|_{C^{0,\alpha}} \leq 1$ also the number of sets Ω_i^ϵ such that $\mathrm{dist}\,(\Omega_i^\epsilon, \partial\Omega) > 0$ is bounded by $C|\log\epsilon|^{\frac{1}{2}}$, then

$$\left| \int_{\Omega_i^\epsilon} \phi d\mu^\epsilon - \int \phi(x)\mu(x)dx \right| \leq C|\log\epsilon|^{-(1/2)-(\alpha/4)}$$

Similarly, one can show the limit at the boundary vanishes.

◇

We go back to our problem of approximating the measure $\mu = \nabla \times U = \nabla \times \frac{1}{a}\nabla^\perp\psi$, let $N_n \in \mathbf{N}$ be any sequence of whole numbers with

$$\frac{N_n}{\log\epsilon_n} \longrightarrow 1$$

where ϵ_n is a subsequence of ϵ which goes to zero when $n \to \infty$.

By using Lemma 2.4.3, there exist families of points $\{p_i^n\}_{i=1,\dots,N_n}$ in the set $K = \operatorname{supp}\psi$ and associated integers $\sigma_i^n \in \{-1,1\}$ with the following properties:

$$|p_i^n - p_j^n| \geq c_0 N_n^{-1/2} \quad \text{for } i \neq j, \text{ for constant } c_0 = c_0(\|\mu\|_\infty) \text{ and } \operatorname{dist}(p_i^n, \partial\Omega) \geq c_0 N_n^{-1/2}; \tag{2.4.12}$$

$$\lim_{\alpha \to 0} R(\alpha) = 0 \quad \text{where} \quad R(\alpha) = \limsup_{n\to\infty} \sum_{\substack{i \neq j: \\ |p_i^n - p_j^n| \leq \alpha}} \frac{\left|\log|p_i^n - p_j^n|\right|}{N_n^2}, \tag{2.4.13}$$

$$\mu_n := \frac{2\pi}{N_n} \sum_{i=1}^{N_n} \sigma_i^n\, \delta_{p_i^n} \rightharpoonup \mu, \tag{2.4.14}$$

$$|\mu_n| = \frac{2\pi}{N_n} \sum_{i=1}^{N_n} \delta_{p_i^n} \rightharpoonup |\mu|, \tag{2.4.15}$$

the convergence in (2.4.14),(2.4.15) is weakly in the sense of measures, and strongly in $(C_0^{0,\alpha})'$ for all $0 < \alpha \leq 1$. By $|\mu|$ we mean the total variation of the measure $\mu = \operatorname{curl} U$ which is smooth and compactly supported in $K \subset \Omega$.

As in [SS00] we modify the measures μ_n by regularizing the Dirac mass. Let $\mu_i^n := \epsilon_n \mathcal{H}^1 \lfloor_{\partial B(p_i^n, \epsilon_n)}$, the element of arclength on $S_i^n := \partial B(p_i^n, \epsilon_n)$, normalized with mass 2π. We define the measures

$$\nu_n = \frac{1}{N_n} \sum_{i=1}^{N_n} \sigma_i^n\, \mu_i^n,$$

with $p_i^n \in K$, $\sigma_i^\epsilon \in \{0,1\}$ as above. Since each $\mu_i^n \longrightarrow \delta_{p_i^n}$ strongly in $(C_0^{0,\alpha}(\Omega))'$ for all $0 < \alpha \leq 1$, and weakly in $\mathfrak{M}(\Omega)$, we may conclude that (2.4.14),(2.4.15) hold as well for ν_n,

$$\nu_n \longrightarrow \mu, \qquad |\nu_n| \longrightarrow |\mu|, \quad \text{strongly in } (C_0^{0,\alpha}(\Omega))' \text{ and weakly in } \mathfrak{M}(\Omega). \quad (2.4.16)$$

By Fubini's theorem we also note that the product measures also converge,

$$\nu_n \otimes \nu_n \longrightarrow \mu \otimes \mu, \quad (2.4.17)$$

strongly in $[C_0^{0,\alpha}(\Omega \times \Omega)]'$ and weakly in $\mathfrak{M}(\Omega \times \Omega)$.

Now for the general $\mu \in \mathbf{H}^{-1}(\Omega) \cap \mathfrak{M}(\Omega)$, we get approximating measures μ^m tending to μ as $m \to \infty$ in $\mathbf{H}^{-1}$ norm and in sense of measure which satisfy properties (2.4.14), and (2.4.15). As in Lemma 2.4.3 we define the sequence

$$\mu_n^m := \frac{2\pi}{N_n} \sum_{i=1}^{N_n} \sigma_i^n \, \delta_{p_i^n}, \quad (2.4.18)$$

and we define

$$\nu_n^m = \frac{1}{N_n} \sum_{i=1}^{N_n} \sigma_i^n \, \mu_i^{nm}. \quad (2.4.19)$$

For fixed n as m tends to ∞, $\mu_n^m \to \mu_n$ and $\nu_n^m \to \nu_n$ from the convergence of $\mu^m \to \mu$. We have for fixed m as n tends to ∞, $\mu_n^m \to \mu^m$ and $\nu_n^m \to \mu^m$ from(2.4.14), (2.4.16) and (2.4.17). We take a diagonal sequences , call it μ_n and ν_n which will tend to μ weakly because of the stronge convergence in $\mathbf{H}^{-1}$.

Step 3: We introduce the Dirichlet Green's function, $G_a(x, y)$ in Ω, which solves

$$\begin{cases} -\nabla_x \cdot \dfrac{1}{a(x)} \nabla_x G_a(x, y) = \delta_y(x), & \text{in } \Omega, \\ G_a(\cdot, y) = 0, & \text{on } \partial\Omega, \end{cases}$$

for each fixed $y \in \Omega$. By standard elliptic theory (see [GT] and recall $a > 0$ is smooth in $\overline{\Omega}$) we may conclude that $G_a(x, y)$ is smooth in $\overline{\Omega} \times \overline{\Omega} \setminus \{y = x\}$, and

$$G_a(x, y) = -\frac{a(x)}{2\pi} \log|x - y| + \gamma(x, y), \tag{2.4.20}$$

where the regular part γ has the property that for every compact set $K \subset\subset \Omega$, there exists $C(K) < \infty$ with

$$\sup_{\substack{y \in K \\ x \in \overline{\Omega}}} |\gamma(x, y)| \leq C(K).$$

Given $U \in \mathcal{U}$, we then obtain the potential function $\psi \in \mathbf{H}_0^1(\Omega)$ from $\operatorname{curl} U = \mu$ by solving

$$\begin{cases} -\nabla \cdot \frac{1}{a(x)} \nabla \psi = \mu & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

and we recover $U = -\frac{1}{a} \nabla^\perp \psi$. Using the Green's function representation, we have

$$\psi(x) = \int_\Omega G_a(x, y) \, d\mu(y).$$

Since $\mu \in H^{-1}(\Omega) \cap \mathfrak{M}(\Omega)$, we may calculate the weighted norm of U in terms of the measure μ as follows:

$$\begin{aligned}\int_\Omega a(x)\,|U|^2\,dx &= \int_\Omega \frac{1}{a}\,|\nabla\psi|^2\,dx\\ &= -\int_\Omega \psi\cdot\nabla^\perp\left(\frac{1}{a}\nabla^\perp\psi\right)\,dx\\ &= \int_\Omega \psi(x)\,d\mu(x)\\ &= \int_\Omega\int_\Omega G_a(x,y)\,d\mu(y)\,d\mu(x). \end{aligned}\tag{2.4.21}$$

Step 4: There exists a sequence $\psi_n \in H^1_0(\Omega)$ for which $-\frac{1}{a}\nabla^\perp\psi_n \longrightarrow U$ strongly in $L^p(\Omega)$ for all $p<2$, and

$$\limsup_{n\to\infty}\int_\Omega \frac{1}{a}|\nabla\psi_n|^2\,dx \le \int_\Omega a(x)\,d|\mu|(x) + \int_\Omega a(x)|U|^2\,dx. \tag{2.4.22}$$

For each n, we define $\psi_n(x) = \int_\Omega G_a(x,y)\,d\nu_n(y)$, and so ψ_n solves

$$\begin{cases} -\nabla\cdot\frac{1}{a(x)}\nabla\psi_n = \nu_n & \text{in } \Omega,\\ \psi_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By (2.4.16) and elliptic regularity, we have $\psi_n \to \psi$ in $W^{1,p}(\Omega)$ for all $p<2$, and thus $-\frac{1}{a}\nabla^\perp\psi_n \to U$ in $L^p(\Omega)$ for all $p<2$ as claimed.

To estimate the energy we use the Green's representation. Since $\nu_n \in H^{-1}(\Omega)$ for fixed n, by (2.4.21) we conclude that

$$\int_\Omega \frac{1}{a}\,|\nabla\psi_n|^2\,dx = \int_\Omega\int_\Omega G_a(x,y)\,d\nu_n(y)\,d\nu_n(x).$$

For any $0<\alpha<1$, let $\Delta_\alpha = \{(x,y)\in\Omega\times\Omega : |x-y|\le\alpha\}$. Fix $\chi_\alpha \in C^\infty(\bar\Omega\times\bar\Omega)$

with $0 \le \chi_\alpha \le 1$, and

$$\chi_\alpha(x,y) = \begin{cases} 1, & \text{if } x \in \Delta_\alpha, \\ 0, & \text{if } x \notin \Delta_{2\alpha}. \end{cases}$$

For any $\alpha \in (0,1)$, $G_a(x,y)(1-\chi_\alpha(x,y))$ is smooth, and hence by the strong $(C_0^{0,\alpha})'$ convergence $\nu_n \to \mu$ we have:

$$\lim_{n\to\infty} \int_\Omega \int_\Omega G_a(x,y)(1-\chi_\alpha(x,y)) d\nu_n(y)\, d\nu_n(x) = \int_\Omega \int_\Omega G_a(x,y)(1-\chi_\alpha(x,y)) d\mu(y)\, d\mu(x). \tag{2.4.23}$$

For the complementary integral, we use (3.3.16) to observe that

$$\begin{aligned} &\int_\Omega \int_\Omega G_a(x,y)\chi_\alpha(x,y) d\nu_n(y)\, d\nu_n(x) \\ &\qquad = \int_K \int_{\Delta_{2\alpha}} \left[\frac{a(x)}{2\pi} \log \frac{1}{|x-y|} + \gamma(x,y) \right] \chi_\alpha \, d\nu_n(y)\, d\nu_n(x) \\ &\qquad \le \int_K \int_{\Delta_{2\alpha}} \frac{a(x)}{2\pi} \log \frac{1}{|x-y|}\, d\nu_n(y)\, d\nu_n(x) + C\alpha \\ &\qquad = \frac{1}{N_n^2} \sum_{i,j=1}^{N_n} \iint_{\Delta_{2\alpha}} \frac{a(x)}{2\pi} \log \frac{1}{|x-y|}\, d\mu_i^n(y)\, d\mu_i^n(x) + C\alpha. \end{aligned} \tag{2.4.24}$$

To evaluate the remaining integral, we consider the contribution due to distinct points $p_i^n \neq p_j^n$ in $\Delta_{2\alpha}$ separately. We adapt an argument in Proposition 7.4 of [SS07]. Define the index set

$$\mathcal{J}_n = \{(i,j) : \ |p_i^n - p_j^n| \le 2\alpha\}.$$

Let $R_n = \frac{1}{4} c_0 N_n^{-1/2}$, where $c_0 = c_0(\psi)$ is the constant in (2.4.12). We also define balls

$\tilde{B}_i^n = B(p_i^n, R_n)$, $i = 1, \ldots, N_n$. By the choice of R_n, they are disjoint, as is the union

$$\bigcup_{(i,j)\in\mathcal{J}_n} \left(\tilde{B}_i \times \tilde{B}_j\right) \subset \Delta_{3\alpha}.$$

We also observe that for any $R \le R_n$ and $(i,j) \in \mathcal{J}_n$, since $R \le \frac{1}{4}|p_i - p_j|$, we have

$$\frac{1}{2} \le \frac{|x-y|}{|p_i^n - p_j^n|} \le \frac{3}{2} \quad \text{for all } x \in B(p_i^n, R),\ y \in B(p_j^n, R). \tag{2.4.25}$$

For $(i,j) \in \mathcal{J}_n$ we then have (recalling that $S_n^i = \partial B(p_i^n, \epsilon_n) = \operatorname{supp} \mu_i^n$,)

$$\begin{aligned}
\iint_{\tilde{B}_i^n \times B_j^n} \log \frac{3}{|x-y|} dx\, dy &\ge \iint_{\tilde{B}_i^n \times B_j^n} \log \frac{2}{|p_i^n - p_j^n|} dx\, dy \\
&= \pi^2 R_n^4 \log \frac{2}{|p_i^n - p_j^n|} \\
&= \frac{R_n^4}{4} \iint_{\tilde{S}_i^n \times S_j^n} \log \frac{2}{|p_i^n - p_j^n|} d\mu_i^n(x)\, d\mu_j^n(y) \\
&\ge \frac{R_n^4}{4} \iint_{\tilde{S}_i^n \times S_j^n} \log \frac{1}{|x-y|} d\mu_i^n(x)\, d\mu_j^n(y),
\end{aligned}$$

using (2.4.22) in the first and last lines. Summing over all pairs $(i,j) \in \mathcal{J}_n$, and using the disjointness of the union of the $\tilde{B}_i^n \times \tilde{B}_j^n$, we obtain:

$$\begin{aligned}
\frac{1}{N_n^2} \sum_{(i,j)\in\mathcal{J}_n} \iint_{S_i^n \times S_j^n} \frac{a(x)}{2\pi} \log \frac{1}{|x-y|} d\mu_i^n(x)\, d\mu_j^n(y) \\
\le \frac{C}{R_n^4 N_n^2} \sum_{(i,j)\in\mathcal{J}_n} \iint_{\tilde{B}_i^n \times B_j^n} \log \frac{3}{|x-y|} dx\, dy \\
\le C \iint_{\Delta_{3\alpha}} \log \frac{3}{|x-y|} dx\, dy =: \mathcal{R}(\alpha).
\end{aligned} \tag{2.4.26}$$

As $|\log|x-y||$ is integrable, the remainder $\mathcal{R}(\alpha) \to 0$ as $\alpha \to 0$, and so this term will not contribute to the limiting energy.

Finally, we consider the contribution from the self-energy of the vortices p_i^n. We parametrize the integrals over $S_i^n = \partial B(p_i^n, \epsilon_n)$ using complex notation, that is we write $x, y \in \partial B(p_i^n, \frac{1}{\epsilon_n})$ as $x = p_i^n + \epsilon_n e^{i\theta}$, $y = p_i^n + \epsilon_n e^{i\tau}$, $0 \le \theta, \tau < 2\pi$. Then we have:

$$\begin{aligned}
&\frac{1}{N_n^2} \iint_\Omega \frac{a(x)}{2\pi} \log \frac{1}{|x-y|}\, d\mu_i^n(y)\, d\mu_i^n(x) \\
&\qquad = \frac{1}{N_n^2} \int_0^{2\pi} \int_0^{2\pi} \frac{a\left(p_i^n + \epsilon_n e^{i\theta}\right)}{2\pi} [\log \epsilon_n + \log |e^{i(\theta-\tau)} - 1|] d\theta\, d\tau \\
&\qquad = \frac{1}{N_n} \int_0^{2\pi} a\left(p_i^n + \epsilon_n e^{i\theta}\right)\, d\theta + O(N_n^{-2}) \\
&\qquad = \frac{1}{N_n} \int_\Omega a(x)\, d|\mu_i^n|(x) + O(N_n^{-2}).
\end{aligned}$$

Summing over all $i = 1, \ldots, N_n$, we arrive at

$$\begin{aligned}
\frac{1}{N_n^2} \sum_{i=1}^{N_n} \iint_\Omega \frac{a(x)}{2\pi} \log \frac{1}{|x-y|}\, d\mu_i^n(y)\, d\mu_i^n(x) &= \frac{1}{N_n} \int_\Omega a(x)\, d|\nu_n|(x) + O(N_n^{-1}) \\
&= \int_\Omega a(x)\, d|\mu|(x) + O(N_n^{-1}). \qquad (2.4.27)
\end{aligned}$$

Passing to the limit $\epsilon_n \to \infty$, we thus obtain from (2.4.23),(2.4.24),(2.4.26), and (2.4.27), that

$$\begin{aligned}
&\limsup_{n\to\infty} \int_\Omega \int_\Omega G_a(x,y) d\nu_n(y)\, d\nu_n(x) \\
&\quad \le \int_\Omega a(x)\, d|\mu|(x) + \int_\Omega \int_\Omega G_a(x,y)(1 - \chi_\alpha(x,y)) d\mu(y)\, d\mu(x) + C\alpha + C\mathcal{R}(\alpha).
\end{aligned}$$

By hypothesis, the measure μ is bounded, and so we may apply dominated convergence to pass to the limit $\alpha \to 0$ and obtain the desired bound (2.4.22), as

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega \frac{1}{a}|\nabla\psi_n|^2\,dx &= \limsup_{n\to\infty} \int_\Omega\int_\Omega G_a(x,y)d\nu_n(y)\,d\nu_n(x) \\ &\le \int_\Omega a(x)\,d|\mu|(x) + \int_\Omega\int_\Omega G_a(x,y)d\mu(y)\,d\mu(x) \\ &= \int_\Omega a(x)\,d|\mu|(x) + \int_\Omega a(x)\,|U|^2\,dx,\end{aligned}$$

by (2.4.21).

Step 5: Let $U_n = -N_n\frac{1}{a}\nabla^\perp\psi_n$. Then, $\nabla^\perp U_n = N_n\,\nabla\cdot\left(\frac{1}{a}\nabla\psi_n\right) = 0$ locally in $\Omega\setminus\cup_i^{N_n}B(p_i^n,\epsilon_n)$. Moreover, if C is a simple closed curve in $\Omega\setminus\cup_i^{N_n}B(p_i^n,\epsilon_n)$, we have

$$\int_C U_n\cdot\tau\,ds \in 2\pi\,\mathbb{Z},$$

by the normalization $|d\mu_i^n| = 2\pi$. Thus, we may write $U_n = \nabla\phi_n$ in $\Omega\setminus\cup_i^{N_n}B(p_i^n,\epsilon_n)$, with ϕ_n which is multiple valued, but for which $\nabla\phi_n$ and $e^{i\phi_n}$ are single-valued in $\Omega\setminus\cup_i^{N_n}B(p_i^n,\epsilon_n)$.

To remove the singularity at each vortex core we define,

$$\rho_i^n(x) := \begin{cases} 0 & \text{if } |x-p_i^n| < \frac{\epsilon_n}{2}, \\ \frac{2\sqrt{a(x)}}{\epsilon_n}\big(|x-p_i^n|-1\big) & \text{if } \frac{\epsilon_n}{2} \le |x-p_i^n| \le \epsilon_n, \\ \sqrt{a(x)} & \text{if } |x-p_i^n| > \epsilon_n, \end{cases}$$

and $\rho_n := \prod_{i=1}^{N_n}\rho_i^n$. The function $\rho_i^n(x)$ here is defined differently than the way Alama, Bronsard, and Galvão-Sousa did in [ABGS13].

A simple computation shows that

$$\int_\Omega \left\{ \frac{1}{2}|\nabla \rho_i^n|^2 + \frac{1}{4\epsilon_n^2}((\rho_i^n)^2 - a(x))^2) \right\} dx \le C_0,$$

with constant C_0 independent of n. Also $\rho_n^2 = \prod_{i=1}^{N_n} (\rho_i^n)^2 \le a(x)$ for all $x \in \Omega$. $(\rho_n^2 - a(x)) \to 0$ in L^q for all $q < \infty$, indeed

$$\begin{aligned} \int_\Omega \left| {\rho_n}^2 - a(x) \right|^q &= \int_\Omega \left| \prod_i (\rho_i^n)^2 - a(x) \right|^q \\ &\le \int_{\{x\in\Omega| \ \frac{\epsilon_n}{2} \le |x-p_i| \le \epsilon_n\}} \left| \prod_i a(x) \big(\frac{2}{\epsilon_n}|x_i - p_i^n| - 1\big)^2 - a(x) \right|^q \\ &\le \int_{\{x\in\Omega| \ \frac{\epsilon_n}{2} \le |x-p_i| \le \epsilon_n\}} \left| \prod_i a(x) \big(\frac{2}{\epsilon_n}\epsilon_n - 1\big)^2 - a(x) \right|^q = 0. \end{aligned}$$

Now define $u_n = \rho_n e^{i\phi_n}$, with ρ_n, ϕ_n as in the preceding paragraphs. We then have:

$$\begin{aligned} \int_\Omega \left\{ \frac{1}{2}|\nabla u_n|^2 + \frac{1}{4\epsilon_n^2}\big(|u_n|^2 - a(x)\big)^2 \right\} dx \\ &= \int_\Omega \left\{ \frac{1}{2}\rho_n^2|\nabla \phi_n|^2 + \frac{1}{2}|\nabla \rho_n|^2 + \frac{1}{4\epsilon_n^2}\big(\rho_n^2 - a(x)\big)^2 \right\} dx \\ &\le \frac{N_n^2}{2} \int_\Omega a(x) \frac{1}{a(x)^2} |\nabla \psi_n|^2 \, dx + C_0 N_n \\ &\le \frac{N_n^2}{2} \int_\Omega \frac{1}{a(x)} |\nabla \psi_n|^2 \, dx + C_0 N_n. \end{aligned}$$

From (2.4.22) we then conclude that

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{(\log \epsilon_n)^2} \int_\Omega \left\{ \frac{1}{2}|\nabla u_n|^2 + \frac{1}{4\epsilon_n^2}\big(|u_n|^2 - a(x)\big)^2 \right\} dx \\ \le \frac{1}{2}\int_\Omega a(x)\, d|\mu|(x) + \frac{1}{2}\int_\Omega a(x)|U|^2\, dx. \quad (2.4.28) \end{aligned}$$

Since $(\rho_n^2 - a(x)) \to 0$ in L^q for all $q < \infty$, we also conclude that

$$\begin{aligned}\frac{j(u_n)}{N_n} &= \frac{1}{N_n}(i\rho_n e^{i\phi_n}, \nabla\rho_n e^{i\phi_n}) \\ &= -\frac{1}{N_n}(\rho_n)^2\nabla\phi_n \\ &= -\frac{1}{N_n}(\rho_n)^2 U_n \\ &= -a(x)\frac{1}{a(x)}\nabla^\perp\psi_n + \frac{(a(x)-\rho_n^2)}{a(x)}\nabla^\perp\psi_n \longrightarrow a(x)U \quad \text{in } L^p(\Omega) \text{ for all } p < 2.\end{aligned} \tag{2.4.29}$$

Step 6: Putting it all together.

This follows as in [JS]; with the modification needed due to the presence of $a(x)$. Write $j \in \mathcal{Z}$ as $j = U + \tilde{W}$ with $U \in \mathcal{U}$ and $\tilde{W} = V + W$, $V \in \mathcal{V}$, $W \in \mathcal{W}$. Let w_n be as defined in Step 1 and u_n as constructed in Step 5, and define $v_n = u_n\, w_n$. Since $|w_n| = 1$, we have

$$j(v_n) = j(u_n) + \rho_n^2 j(w_n) \longrightarrow a(x)U + a(x)\tilde{W} = a(x)j. \tag{2.4.30}$$

in $L^p(\Omega)$ for all $p < 2$.

To estimate the energy, we again use the fact that $|w_n| = 1$ to expand:

$$\frac{1}{N_n^2}\int_\Omega |\nabla v_n|^2\, dx = \frac{1}{N_n^2}\int_\Omega \left\{|\nabla u_n|^2 + \rho_n^2|\nabla w_n|^2 + j(u_n)\cdot j(w_n)\right\} dx.$$

We claim that the last term is negligible. Indeed, from Step 1, $\frac{j(w_n)}{\log\epsilon_n} = \nabla\Phi_n$, with

$\Phi_n = \eta_n + \zeta \log \epsilon_n$ and $\nabla \Phi_n \to \tilde{W}$ in C^1, and therefore,

$$\begin{aligned}\frac{1}{N_n^2}\int_\Omega j(u_n)\cdot j(w_n)\,dx &= -\int_\Omega \frac{1}{a(x)}\nabla^\perp \psi_n \cdot \rho_n^2 \nabla \Phi_n\,dx\\ &= \int_\Omega \nabla^\perp \psi_n \cdot \nabla \Phi_n = 0.\end{aligned}$$

We calculate, using (2.4.28), (2.4.7), and (2.4.30) and that $\rho_n^2 \le a(x)$

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{N_n^2} E_{\epsilon_n}(v_n; A_n)\\ &\quad= \limsup_{n\to\infty} \frac{1}{N_n^2}\int_\Omega \Big\{\frac{1}{2}|\nabla u_n|^2 + \frac{1}{2}\rho_n^2|\nabla w_n|^2 - A_n\cdot j(v_n) + \frac{1}{2}|v_n|^2|A_n|^2\\ &\qquad\qquad + \frac{1}{4\epsilon^2}(|u_n|^2 - a(x))^2\Big\}dx + \frac{1}{2}\int_{\mathcal{D}} |\nabla\times A_n - h_{ex}|^2 dx\\ &\quad= \limsup_{n\to\infty} \frac{1}{N_n^2}\int_\Omega \Big\{\frac{1}{2}|\nabla u_n|^2 + \frac{1}{2}\rho_n^2|\nabla w_n|^2 - A_n\cdot j(v_n) + \frac{1}{2}\rho^2|A_n|^2\\ &\qquad\qquad + \frac{1}{4\epsilon^2}(|u_n|^2 - a(x))^2\Big\}dx + \frac{1}{2}\int_{\mathcal{D}} |\nabla\times A_n - h_{ex}|^2 dx\\ &\quad\le \frac{1}{2}\int_\Omega a(x)\,d|\mu| + a(x)|U|^2 dx + \frac{1}{2}\int_\Omega a(x)\big(|\tilde{W}|^2 - 2A\cdot j + |A|^2\big)dx\\ &\qquad\qquad + \frac{1}{2}\int_{\mathcal{D}} |\nabla\times A - \lambda|^2\,dx\\ &\quad\le \frac{1}{2}\int_\Omega a(x)\,d|\mu| + \frac{1}{2}\int_\Omega a(x)\big(|U|^2 + |\tilde{W}|^2 - 2A\cdot j + |A|^2\big)dx\\ &\qquad\qquad + \frac{1}{2}\int_{\mathcal{D}} |\nabla\times A - \lambda|^2\,dx\\ &\quad= \frac{1}{2}\int_\Omega a(x)\,d|\mu| + \frac{1}{2}\int_\Omega a(x)|j - A|^2 + \frac{1}{2}\int_{\mathcal{D}} |\nabla\times A - \lambda|^2 dx,\end{aligned}$$

This completes the proof of the Γ-convergence result.

$\diamondsuit$

Remark 2.4.4 *The multiply connected domain didn't affect our lower bound since the method of [SS04] used vector fields X and Y which are compactly supported in*

Ω. *On the other hand the boundary conditions will be involved in the equations the magnetic field satisfiy as will be seen in the following section.*

2.5 Generalization to the case when a(x) is rapidly oscillating

In this section we give an idea how the holes affect the problem in the case $a(x)$ oscillates between $\frac{1}{2}$ and 1 in the domain. [ASS] studied this case in a simply connected domain and to generalize it to a multiply connected domain we follow directly the method above and the only different will be the extra conditions of the magnetic field h on the inner boundaries.

Since $a_\epsilon(x)$ is rapidly oscillating function describing impurities, as in [ASS] the frame work for passing to the limit when ϵ is small is that of homogenization theory. When passing to the limit in,

$$-\text{div}\left(\frac{1}{a_\epsilon}\nabla h_\epsilon\right) + h_\epsilon = 2\pi\sum_i d_i\delta_{p_i}.$$

we obtain a different limiting operator, that is

$$-\text{div}\left(\mathcal{A}_0\nabla h_*\right) + h_* = \mu_*,$$

where μ_* is a positive measure which is supported in an inner domain Ω_λ and $\mathcal{A}_0$ is the homogenized limit of the matrix $\mathcal{A}_\epsilon = \frac{1}{a_\epsilon}\mathcal{I}$ in sense of $H-$convergence, (see definition below)

Define the space $\mathbb{V}$

$$\mathbb{V} = \{h \in W_0^{1,q}(\mathcal{D}) \ s.t. \ h|_{\omega_j} = constant\}.$$

Definition 2.5.1 *We say that the family of* 2×2 *matrices* $\mathcal{A}_\epsilon$ $H-$*converges to* $\mathcal{A}_0$ *when* $\epsilon \to 0$, *if and only if, for any* f *in* $\mathbf{H}^{-1}(\Omega)$, *the solution* v_ϵ *in* $\mathbb{V}$

$$-\mathrm{div}\,(\mathcal{A}_\epsilon \nabla v_\epsilon) + v_\epsilon = f \qquad in\ \Omega$$

satifies

$$\begin{aligned} & v_\epsilon \rightharpoonup v_0 \ \ weakly\ in\ \ \mathbb{V} \\ & \mathcal{A}_\epsilon \nabla v_\epsilon \rightharpoonup \mathcal{A}_0 \nabla v_0 \ \ weakly\ in\ (L^2(\Omega))^2, \end{aligned}$$

where v_0 *is the* $\mathbb{V}$ *solution of*

$$-\mathrm{div}\,(\mathcal{A}_0 \nabla v_0) + v_0 = f.$$

As above we assume that h_{ex} is a function of ϵ and that the following limit exists and is finite,

$$\lambda = \lim_{\epsilon \to 0} \frac{|\log \epsilon|}{h_{ex}(x)}.$$

Moreover, we make the following hypotheses on the function $a_\epsilon(x)$:

(H1) There exists a constant $a_0 > 0$ such that $a_0 \le a(x) \le 1$.

(H2) There exists a constant C and a sequence η_ϵ (which may tend to 0 with ϵ) such that $\frac{1}{\eta\epsilon} \ll h_{ex}$ *and* $|\nabla a_\epsilon| \le \frac{C}{\eta_\epsilon}$.

(H3) There exist a continuous function $a(x)$ and a nonnegative functions $\beta_\epsilon(x)$ such that $a_\epsilon(x) = a(x) + \beta_\epsilon(x)$ and for any $\epsilon > 0$ and any $x \in \Omega,\ \min_{B(x,\delta(\epsilon))} \beta_\epsilon = 0$, where

$$\delta(\epsilon) \ll \frac{1}{(\log|\log\epsilon|)^{\frac{1}{2}}}.$$

(H4) The family of matrices $\mathcal{A}_\epsilon$ $H-$converges to $\mathcal{A}_0$.

2.5.1 Deriving the limiting equation

For any (p_i, d_i) satisfying Proposition 2.3.2, we can define

$$\mu_\epsilon = \frac{2\pi}{h_{ex}} \sum_{i \in I_\epsilon} d_i \delta_{p_i}, \tag{2.5.1}$$

a measure of vorticity per unit of applied field. It will remain a bounded family of measures by the following Lemma,

Lemma 2.5.2 *(Lemma 2.1 from [ASS])*

Let (u_ϵ, A_ϵ) be a family of minimizers of E_ϵ with $h_\epsilon = \operatorname{curl} A_\epsilon$, we can extract a sequence $\epsilon_n \to 0$ such that there exists $h_* - 1 \in \mathbb{V}$, and $\mu_* \in \mathfrak{M}(\Omega)$ with

$$\frac{h_{\epsilon_n}}{h_{ex}} - 1 \rightharpoonup h_* - 1 \qquad in \ \ \mathbb{V},$$

$$\mu_{\epsilon_n} \to \mu_* \quad \text{in the sense of measure.}$$

The following Proposition gives the equations that h_* satisfies.

Proposition 2.5.3 *(Proposition 2.1 from [ASS]).*

Let μ_* and h_* be the measures and fields defined in Lemma 2.5.2. then there exists $r_0 < 2$ such that $\mu_* \in W^{-1,r}(\Omega)\ \forall\ r \in (0, r_0)$, and h_* is the unique solution in $W^{1,r}(\Omega)$

of

$$\begin{cases} -\mathrm{div}\,(\mathcal{A}_0 \nabla h_*) + h_* = \mu_* \quad \textit{in } \Omega \\ h_* = 1 \qquad \textit{on } \partial\mathcal{D} \\ h_* = H_j \qquad \textit{on } \partial\omega_j,\ j = 1, ..., m \\ \int_{\omega_j} \mathcal{A}_0 \nabla h_* \cdot \nu = -2\pi\delta_j + H_j|\omega_j|, \quad j = 1, ..., m. \end{cases} \tag{2.5.2}$$

Proof: The proof of this proposition follows exactly [ASS] except for the boundary conditions and we are going to include it for convenience.

Step1. We prove that h_ϵ satisfy

$$\frac{1}{h_{ex}}\left(-\mathrm{div}\left(\frac{\nabla h_\epsilon}{a_\epsilon}\right) + h_\epsilon \right) = f_\epsilon \qquad \textit{in } \Omega, \tag{2.5.3}$$

with $f_\epsilon = \mu_\epsilon + \varphi_\epsilon$, where $\varphi_\epsilon \to 0$ strongly in $(W_0^{1,q})'$ for $q > 2$. We start from the second Ginzburg-Landau equation:

$$-\nabla^\perp h_\epsilon = (iu_\epsilon, \nabla_{A_\epsilon} u_\epsilon),$$

divide by a_ϵ which is positive and take the curl:

$$-\mathrm{div}\left(\frac{\nabla h_\epsilon}{a_\epsilon}\right) = \mathrm{curl}\left(\frac{(iu_\epsilon, \nabla u_\epsilon)}{a_\epsilon} - A_\epsilon \frac{|u_\epsilon|^2}{a_\epsilon}\right),$$

hence

$$-\mathrm{div}\left(\frac{\nabla h_\epsilon}{a_\epsilon}\right) + h_\epsilon = \mathrm{curl}\,\frac{(iu_\epsilon, \nabla u_\epsilon)}{a_\epsilon} + \mathrm{curl}\left(A_\epsilon\left(1 - \frac{|u_\epsilon|^2}{a_\epsilon}\right)\right). \tag{2.5.4}$$

Now consider a test-function $\xi \in W_0^{1,q}(\Omega), q > 2$

$$\Big|\int_\Omega \xi \text{curl}\Big(A_\epsilon\Big(1 - \frac{|u|^2}{a_\epsilon}\Big)\Big)\Big| = \Big|\int_\Omega \nabla^\perp\xi \cdot A_\epsilon\Big(1 - \frac{|u|^2}{a_\epsilon}\Big)\Big)\Big|$$

$$\leq C\|A_\epsilon\|_{L^\infty(\Omega)}\|\nabla\xi\|_{L^2(\Omega)}\|a_\epsilon - |u|^2\|_{L^2(\Omega)}$$

The bound (2.1.2), $\|A_\epsilon\|_{L^\infty(\Omega)} \leq o(h_{ex})$ and the energy bound, $\|a_\epsilon - |u|^2\|_{L^2(\Omega)} \leq C\epsilon h_{ex}$, yield

$$\Big|\int_\Omega \xi \text{curl}\Big(A_\epsilon(1 - \frac{|u|^2}{a_\epsilon})\Big)\Big| \leq o(1)\|\xi\|_{L^2(\Omega)}.$$

consequently, curl $\Big(A_\epsilon(1 - \frac{|u|^2}{a_\epsilon})\Big) \to 0$ strongly in $(W_0^{1,q})'$ for $q > 2$. Combining this with (2.5.4) and lemma 2.5.2 we get the desired result.

Step2. We prove that f_ϵ converges to μ_0, the weak limit of μ_ϵ, in $W^{-1,r}(\Omega)$ for any $r < 2$.

Indeed, from the upper bound of the energy E_ϵ, $\frac{1}{a_\epsilon h_{ex}}\nabla h_\epsilon$ is bounded in $L^2(\Omega)$, hence, using (2.5.3) implies that f_ϵ is bounded in $\mathbf{H}^{-1}$, hence in $W^{-1,p}$ for $p < 2$. But $f_\epsilon = \mu_\epsilon + \varphi_\epsilon$, where φ_ϵ is bounded in $W^{-1,p}$ for $p < 2$ which means that μ_ϵ is also bounded in $W^{-1,p}$ for $p < 2$ and in the sense of measure too, then as in [ASS] we can apply the theorem of Murat which implies that μ_ϵ must be compact in $W^{-1,r}$ for $r < p$. Since it is also the case of φ_ϵ which converges to zero, this implies that f_ϵ is compact in $W^{-1,r}$ for $r < p$. In addition, its limit in the sense of distributions is μ_0, hence it must converge to μ_0 in $W^{-1,r}$.

Step3. As in [ASS] to pass to the limit in (2.5.3) directly we require a right-hand side in H^{-1}. So we are going to pass it in the duality sense as in [ASS] for a fixed right-hand side. Let $g \in W^{-1,q}(\Omega)$ for $q > 2$. Using the hypothesis (H4) on $a(x)$

we can use Meyers [M] and Berestycki and Brezis [BB2] : there exists a $q_0 > 2$, such that if g is in $W^{-1,q}$ with $2 < q \le q_0$, then equation

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla v_\epsilon}{a_\epsilon}\right) + v_\epsilon = g & in\ \Omega \\ v_\epsilon = 0 & on\ \partial\mathcal{D} \\ v_\epsilon = H_j \qquad on\ \partial\omega_j \\ \int_{\omega_j} \frac{1}{a_\epsilon}\frac{\partial v_\epsilon}{\partial\nu} = -2\pi\delta_j + H_j|\omega_j|, \quad j = 1, ..., m. \end{cases} \tag{2.5.5}$$

has a unique solution v_ϵ in $\mathbb{V}$. Thus, we have

$$_{\mathbb{V}'}\left\langle \frac{h_\epsilon}{h_{ex}} - 1, g\right\rangle_{W^{-1,q}(\Omega)} = {}_{W^{-1,q'}(\Omega)}\langle f_\epsilon - 1, v_\epsilon\rangle_{\mathbb{V}}, \tag{2.5.6}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$, and we want to pass to the limit.

More precisely, Meyers' theorem yields that the operator R_ϵ which maps g *to* v_ϵ, is a bounded linear operator from $W^{-1,q}(\Omega)$ *to* $\mathbb{V}$ (for$2 < q \le q_0$), hence up to extraction of a subsequence, v_ϵ has a weak limit v_0 in $\mathbb{V}$. v_0 is the solution of:

$$\begin{cases} -\mathrm{div}\left(\mathcal{A}_0\nabla v_0\right) + v_0 = g & in\ \Omega \\ v_0 = 0 & on\ \partial\Omega \\ v_0 = H_j \qquad on\ \partial\omega_j \\ \int_{\omega_j} \mathcal{A}_0\frac{\partial v_0}{\partial\nu} = -2\pi\delta_j + H_j|\omega_j|, \quad j = 1, ..., m. \end{cases} \tag{2.5.7}$$

Since this possible weak limit v_0 is unique, the whole sequence v_ϵ converges to

v_0 weakly in $\mathbb{V}$. In addition, f_ϵ converges strongly to μ_* in $W^{-1,q'}$, thus we have

$$_{W^{-1,q'}(\Omega)}\big\langle f_\epsilon - 1, v_\epsilon \big\rangle_{\mathbb{V}} \to \langle \mu_* - 1, v_0 \rangle.$$

On the other hand, $\frac{h_\epsilon}{h_{ex}} - 1$ converges weakly to $h_* - 1$ in $\mathbb{V}$. Thus

$$_{\mathbb{V}'}\big\langle \frac{h_\epsilon}{h_{ex}} - 1, g \big\rangle_{W^{-1,q}(\Omega)} \to \langle h_* - 1, g \rangle.$$

Therefore, we pass to the limit in (2.5.3), and we are led to

$$_{\mathbb{V}'}\langle h_* - 1, g \rangle_{W^{-1,q}(\Omega)} =_{W^{-1,q'}(\Omega)} \langle \mu_* - 1, v_0 \rangle_{\mathbb{V}}. \tag{2.5.8}$$

Meyers' aforementioned theorem, also yields that for $q_0' \leq q' < 2$, (2.5.2) has a unique solution in $W^{1,q'}(\Omega)$. Since (2.5.8) holds for any g *in* $W^{-1,q}(\Omega)$, it implies that h_* is this solution.

$\diamondsuit$

2.5.2 Main result

The Γ-limit for the Energy when $a_\epsilon(x)$ is rapidly oscillating is given by the following Theorem,

Theorem 2.5.4 *Let's assume that (H1) to (H4) are satisfied. Let (u_ϵ, A_ϵ) be a family of minimizers of E_ϵ, and $h_\epsilon = \operatorname{curl} A_\epsilon$ the associated magnetic field. Then, as $\epsilon \to 0$,*

$$\frac{h_\epsilon}{h_{ex}} - 1 \rightharpoonup h_* - 1 \quad \textit{weakly in } \mathbb{V},$$

where h_* *is the minimizer of*

$$E(f) = \frac{\lambda}{2}\int_\Omega a(x)|-\operatorname{div}(\mathcal{A}_0\nabla f) + f| + \frac{1}{2}\int_\Omega \nabla f\cdot\mathcal{A}_0\nabla f + \frac{1}{2}\int_{\mathcal{D}} |f-1|^2.$$

Moreover,

$$\lim_{\epsilon\to 0}\frac{E_\epsilon(u_\epsilon, A_\epsilon)}{|h_{ex}|^2} = E(h_*) = \frac{\lambda}{2}\int_\Omega a|\mu_*| + \frac{1}{2}\int_\Omega h_*\cdot\mathcal{A}_0\nabla h_* + \frac{1}{2}\int_{\mathcal{D}}|h_*-1|^2, \quad (2.5.9)$$

$$\frac{1}{a_\epsilon}\frac{|\nabla h_\epsilon|^2}{|h_{ex}|^2} \to \nabla h_*\cdot\mathcal{A}_0\nabla h_* + \lambda a\mu_*, \quad \textit{in the sense of measures.} \quad (2.5.10)$$

The proof follows exactly [ASS].

Chapter 3

Singular Limits for Thin Film Superconductors in Strong Magnetic Fields

In this chapter we study the Γ-limit of the 3D Ginzburg-Landau functional with a constant external magnetic field in a thin film geometry where the thickness of the film which is given by $d(x)$ is allowed to be zero on the boundaries of the holes. The superconducting domain is multiply-connected and has a characteristic thickness on the scale $\epsilon > 0$. We consider the limit as the thickness $\epsilon \to 0$ and the Ginzburg-Landau parameter $\kappa \to \infty$. This model has been studied by Alama, Bronsard, and Galvão-Sousa (see [ABGS13]) when $d(x)$ is strictly positive in a simply-connected domain.

The superconducting sample contains the domain $\mathcal{D}_\epsilon \subset \mathbf{R}^3$,

$$\mathcal{D}_\epsilon = \{(x', x_3) \in \mathbf{R}^3 : x' \in \Omega,\ \epsilon f(x') < x_3 < \epsilon g(x')\},$$

where $\Omega := \omega_0 \setminus \cup_{i=1}^{m} \omega_i \subset \mathbf{R}^2$ is a bounded multiply-connected domain in the plane and $\omega_i \subset \omega_0$ *for all* $i = 1, ..., m$ are simply-connected domains, $f, g : \Omega \to \mathbf{R}$ are smooth functions on Ω with $f(x') < g(x')$ for all $x' \in \Omega$, and $\epsilon > 0$. We denote by

$$d(x') = g(x') - f(x')$$

the thickness of the film for given $x' \in \Omega$. We assume that $d(x)$ satisfies the following conditions:

(H1) $d(x) \in C^2(\mathcal{D})$.

(H2) $d(x) > d_0$ in Ω_δ where

$$\Omega_\delta := \{x \in \Omega \ s.t. \ \text{dist}\,(x, \partial\Omega) > \delta\}$$

for $\delta > 0$ i.e. $d(x)$ vanishes only near $\partial\omega_j \ \forall \ j = 1, ..., n$

(H3) $\nabla d(x) \neq 0$ for all $x \in \partial\omega_i, \ i = 1, ..., m$. More specific, $\exists \delta > 0$ s.t. there are non-negative constants $m_i, and\ M_i$ such that

$$m_i \leq \frac{d(x)}{\text{dist}\,(x, \partial\omega_i)} \leq M_i.$$

for $\text{dist}\,(x, \partial\omega_i) < \delta$.

Note that it follows from the above hypotheses that $d(x)$ is bounded away from zero on the exterior boundary $\partial\mathcal{D}$.

3.1 Introduction

The energy of the configuration (u, A) is given by:

$$\mathbf{I}_{\epsilon,\kappa}(u, A) := \frac{1}{2}\int_{\mathcal{D}_\epsilon}\left(|\nabla_A u|^2 + \frac{\kappa^2}{2}(1-|u|^2)^2\right)dx + \frac{1}{2}\int_{\mathbf{R}^3}|h-h_{ex}|^2dx, \tag{3.1.1}$$

where, $h_{ex} \in \mathbf{R}^3$ is a constant external magnetic field.

3.1.1 Rescaling

Since we have three parameters in our problem, $\epsilon, \kappa, and\ h_{ex}$, we need to identify limiting regimes. We rescale the domain following Alama, Bronsard, and Galvão-Sousa (see [ABGS13]) by ϵ in the x_3 direction in order to recognize the correct scaling for h_{ex} in terms of the thickness parameter.

Let,

$$\begin{aligned}
&x = (x', x_3) = (x_1, x_2, x_3) = \left(\mathbf{x}_1, \mathbf{x}_2, \frac{\mathbf{x}_3}{\epsilon}\right) \in \Omega_1 \\
&A(x) = (\mathbf{A}_1, \mathbf{A}_2, \epsilon\mathbf{A}_3)(\mathbf{x}), \\
&u(x) = \mathbf{u}(\mathbf{x})
\end{aligned}$$

with this scaling we could define u in a fixed $(\epsilon - independent)$ domain

$$\mathcal{D} := \mathcal{D}_1 = \{(x', x_3) : f(x') < x_3 < g(x'),\ x' \in \Omega\}.$$

The magnetic field in the new coordinates is $\mathbf{h}(\mathbf{x}) = \left(\epsilon^{-1}h'(x), h_3(x)\right)$. The energy is

$$\mathbf{I}_{\epsilon,\kappa}(\mathbf{u}, \mathbf{A}) = \epsilon\tilde{I}_{\epsilon,\kappa}(u, A),$$

where

$$\tilde{I}_{\epsilon,\kappa}(u,A) =: \int_{\mathcal{D}} \left(\frac{1}{2}|(\nabla' - iA)u|^2 + \frac{1}{2\epsilon^2}|(\partial_3 - iA_3)u|^2 + \frac{\kappa^2}{4}(1-|u|^2)^2 \right) dx$$
$$+\frac{1}{2}\int_{\mathbf{R}^3} \left(|h_3 - h_3^{ex}|^2 + \frac{1}{\epsilon^2}|h' - h'_{ex}|^2 \right) dx,$$

where $\nabla' = (\partial_1, \partial_2)$ and the rescaled external field takes the form

$$h_{ex} = (h_1^{ex}, h_2^{ex}, h_3^{ex}) = (\epsilon \mathbf{h}_1^{ex}, \epsilon \mathbf{h}_2^{ex}, \mathbf{h}_3^{ex})$$

3.1.2 Introducing the problem

We consider the Γ-limit of the energy as both $\epsilon \to 0$ and $\kappa \to \infty$. We choose an exterior applied field related to the thickness parameter ϵ, and on the scale of the first critical field in κ,

$$\mathbf{h}_{ex} = \left(H' \frac{\log \kappa}{\epsilon}, H_3 \log \kappa \right).$$

In the rescaled functional $\tilde{I}_{\epsilon,\kappa}$ this means

$$h_{ex} = H \log \kappa \tag{3.1.2}$$

where $H = (H', H_3) = (H_1, H_2, H_3) \in \mathbf{R}^3$ is fixed constant vector (independent of ϵ, κ). It is natural for the Ginzburg-Landau Model to choose applied field of order $\log \kappa$ (see [SS07], [AB06], and [BJOS11]) for both the 2D and 3D cases. Since the cost of a vortex p is in an annulus $\mathcal{A} = B(p,R) \setminus B(p, \frac{1}{\kappa})$ is of order $\log \kappa$ in 2D and 3D. For applied fields of the form (3.1.2), the energy of minimizers of $\tilde{\mathbf{I}}_{\epsilon,\kappa}$ will be on the order of $[\log \kappa]^2$. That leads to introduce the following normalization, and study the family of functionals

$$I_{\epsilon,\kappa}(u,A) := \frac{1}{(\log\kappa)^2}\tilde{I}_{\epsilon,\kappa}(u,A)$$

and configuration (u, A) with bounded values of $I_{\epsilon,\kappa}$.

3.1.3 Spaces and gauges

We introduce our spaces for the configuration (u, A).

$u \in \mathbf{H}^1(\mathcal{D}_1, \mathbf{C})$. For A, we first choose a fixed $\hat{A}$ with $\nabla \times \hat{A} = H = (H_1, H_2, H_3)$.A convenient choice is :

$$\hat{A} := \left(H_2x_3 - \frac{1}{2}H_3x_2, \frac{1}{2}H_3x_1 - H_1x_3, 0\right). \tag{3.1.3}$$

which fixes a gauge for $\hat{A}$. Then, we take our A in the following space

$$A \in \mathcal{A} := \{A \in \mathbf{H}^1_{loc}(\mathbf{R}^3, \mathbf{R}^3) : A - \hat{A}\log\kappa \in \breve{\mathbf{H}}^1_{\text{div}}(\mathbf{R}^3, \mathbf{R}^3)\}, \tag{3.1.4}$$

where $\breve{\mathbf{H}}^1_{\text{div}}(\mathbf{R}^3, \mathbf{R}^3)$ is the closure of the space of smooth, compactly supported divergence-free vector fields $F \in \mathbf{C}_0^\infty(\mathbf{R}^3, \mathbf{R}^3)$ in the Dirichlet norm (see [GP99]),

$$||F||_{\breve{\mathbf{H}}^1_{\text{div}}(\mathbf{R}^3,\mathbf{R}^3)} = \left[\int_{\mathbf{R}^3} |DF|^2 dx\right]^2.$$

Next we define the weighted Sobolev space $W^{1,p}_{m,k}$,

Definition 3.1.1 *Consider a real number $1 < p < \infty$, an integer $m \geq 1$, and a real number $k > -1$ satisfying $k + p > m$ (so in particular this condition holds for $p = 2$*

and $k = m$ since m is a positive integer). For such m, k and p, we define the norm

$$\|s\|_{k,m} = \left\{ \int_\Omega \{d^m|\nabla s|^p + d^k|s|^p\}dx \right\}^{\frac{1}{p}}, \tag{3.1.5}$$

for $s \in C^\infty(\overline{\Omega})$, we define $W^{1,p}_{k,m}(\Omega)$ to be the closure of $C^\infty(\overline{\Omega})$ with respect to $\|.\|_{k,m}$. We also define $L^p_k(\Omega)$ to be the closure of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|s\|_k = \left\{ \int_\Omega d^k|s|^p dx \right\}^{\frac{1}{p}}. \tag{3.1.6}$$

The following Theorem gives us facts about $W^{1,p}_{k,m}(\Omega)$. It is taken from [M07] (Theorem A.1).

Theorem 3.1.2 *For $1 < p < \infty$, $k > -1$ and a positive integer $m \geq 1$ so that $k+p > m$, the space $W^{1,p}_{k,m}(\Omega)$ is reflexive, and embeds compactly in $L^p_k(\Omega)$. Furthermore, Poincaré inequality holds for functions $s \in W^{1,p}_{k,m}(\Omega)$ with zero average, that is, there is a constant $C = C(\Omega)$ such that, for any $s \in W^{1,p}_{k,m}(\Omega)$ with*

$$\int_\Omega d^k s \ dx = 0$$

one has

$$\int_\Omega d^k|s|^p \ dx \leq C \int_\Omega d^m|\nabla s|^p dx.$$

Recall the current and the Jacobian. The current (or momentum density) is:

$$j = j(u) = (iu, \nabla u), \quad \textit{where } (a,b) = \operatorname{Re}\overline{a}b,$$

and the weak Jacobian, $J = \frac{1}{2}\nabla \times j$. For $u \in \mathbf{H}^!(\Omega; \mathbf{C})$, $j \in L^2(\Omega; \mathbf{R}^3)$, and so J is defined in the sense of distributions, and we will see that it will also be measure-valued (see [JS02].)

The limiting behaviour of $I_{\epsilon,\kappa}$ will be described in terms of the limit of j and J rather than the order parameter u. The choice of using j and J to study the limiting problem is due to the high increase of the number of vortices in the limit which in fact becomes unbounded.

It will be convenient to represent j and J as differential forms,

$$j = j_1 dx^1 + j_2 dx^2 + j_3 dx^3 \in \Lambda^2(\mathbf{R}^3) \tag{3.1.7}$$

$$J = dj = J_1 dx^2 \wedge dx^3 + J_2 dx^3 \wedge dx^1 + J_3 dx^1 \wedge dx^2 \in \Lambda^2(\mathbf{R}^3) \tag{3.1.8}$$

as the natural mapping between forms and vector fields is an isometry in Euclidean space $\mathbf{R}^3$.

We define the following domain,

$$\mathcal{Z} := \{j \in L^2(\Omega, \mathbf{R}^3) : \ j = (j'(x'), 0), \ J := \frac{1}{2}\nabla \times j \in \mathfrak{M}(\mathcal{D}, \mathbf{R}^3)\}, \tag{3.1.9}$$

where $\mathfrak{M}(\mathcal{D}, \mathbf{R}^3)$ is the space of vector-valued Radon measures on $\mathcal{D}$. For $j \in \mathcal{Z}$, the corresponding Jacobian takes the form $J = (0, 0, J_3(x'))$. we define the functional

$$I_\infty(j; F) = \begin{cases} \frac{1}{2}||d(x')\nabla \times j||_{\mathfrak{M}(\Omega)} + \frac{1}{2}\int_\Omega d(x')|j' - B'|^2, & \text{if } j \in \mathcal{Z} \\ \infty & \text{otherwise} \end{cases} \tag{3.1.10}$$

where

$$F = \nabla' \times B' = h_3^{ex} - (h_1^{ex}, h_2^{ex}) \cdot \nabla' \left(\frac{f(x') + g(x')}{2} \right)$$

with $B' : \mathbf{R}^2 \to \mathbf{R}^2$. This is the vortex density functional which was suggested by previous work of [CRS96]. In this limit, the magnetic field is predetermined by the strength and direction of the original applied field $h_{ex} \in \mathbf{R}^3$ and by the geometry of the domain $\mathcal{D}_\epsilon$, and is part of the variational problem for the thin film limit.

3.1.4 Main results

Our main result is that the Γ-limit of $I_{\epsilon,\kappa}$ is related to I_∞ when we allow $d(x') = 0$ *on* $\partial\omega_i$. We prove this in two steps: first, bounded sequences are compact and the limit is lower semicontinuous in the energies:

Theorem 3.1.3 *For any pair of sequences $\epsilon_n \to 0$ and $\kappa_n \to \infty$, assume $\{(u_n, A_n)\}_{n\in N} \subset$ $\mathbf{H}^1(\mathcal{D}, \mathbf{C}) \times \mathcal{A}$ satisfy the norm bound,*

$$\sup_{n\in\mathbf{N}} I_{\epsilon_n,\kappa_n}(u_n, A_n) < +\infty,$$

and define $j_n = j(u_n) = (u_n, du_n)$ and $J_n = \frac{1}{2} dj_n$ as in (3.1.8). Then there exists a subsequence (which we continue to denote $\{\epsilon_n, \kappa_n\}$) and $j \in \mathcal{Z}$, with $J = \frac{1}{2}\nabla \times j$, such that

1. *along the subsequence,*

$$|u_n|^2 \to 1, \qquad \text{in } L^4(\Omega),, \tag{3.1.11}$$

$$\frac{A_n}{\log\kappa} - \hat{A} \rightharpoonup 0, \qquad \text{in } \breve{\mathbf{H}}^1_{\text{div}}(\mathbf{R}^3, \mathbf{R}^3), \tag{3.1.12}$$

$$\frac{j_n}{|u_n|\log\kappa} \rightharpoonup j, \qquad \text{in } L^2(\mathcal{D}, \mathbf{R}^3), \tag{3.1.13}$$

$$\frac{J_n}{\log\kappa} \rightharpoonup J, \qquad \text{in the weak-}\star \text{ topology on } (C^{0,\gamma}(\mathcal{D}))', \tag{3.1.14}$$

for all $0 < \gamma < 1$.

2. *Furthermore,*

$$\liminf_{n\to\infty} I_{\epsilon_n,\kappa_n}(u_n, A_n) \geq I_\infty(j; F_*) + \frac{1}{2}\int_\Omega \frac{d^3(x')}{12}|H'|^2 dx',$$

and F_* *is defined by*

$$F_*(x') = H_3 - (H_1, H_2)\cdot\nabla'\left(\frac{f(x') + g(x')}{2}\right), \tag{3.1.15}$$

and I_∞ *is defined as in (3.1.10).*

Note that: Following Alama, Bronsard, and Galvão-Sousa [ABGS13] in the case where the applied field is of order $|h_{ex}| = O(\log\kappa)$ the same limit is obtained regardless of the relationships between $\epsilon \to 0$ and $\kappa \to \infty$.

The second part of the Γ-convergence result is the construction of recovery sequences:

Theorem 3.1.4 *Let* $j \in \mathcal{Z}$ *and consider any sequences* $\epsilon_n,\ \kappa_n$ *such that* $\epsilon_n \to 0$ *and*

$\kappa_n \to \infty$. *Then there exists a sequence* $\{(u_n, A_n)\} \subset \mathbf{H}^1(\mathcal{D}; \mathbf{C}) \times \mathcal{A}$, *satisfying*

$$\frac{j_n}{\log \kappa_n} \to j \text{ in } L^p(\mathcal{D}),\ \text{for all } p < 2$$
$$\frac{J_n}{\log \kappa_n} \to J := \frac{1}{2}\nabla \times j \text{ weakly in } \mathfrak{M}(\mathcal{D}; \mathbf{R}^3),\ \text{and strongly in } (C_0^{0,\gamma}(\mathcal{D}))', 0 < \gamma < 1,$$

with $j_n := (iu_n, du_n)$ *and* $J_n := \frac{1}{2} dj_n$. *Moreover*

$$\limsup_{n \to 0} I_{\epsilon_n, \kappa_n}(u_n, A_n) \leq I_\infty(j, F_*) + \frac{1}{2}\int_\Omega \frac{d^3(x')}{12}|H'|^2 dx'$$

with F_* *as in (3.1.15).*

3.2 Compactness and lower bound

In this section we prove the first part of the Γ-convergence. As usual when we deal with the Ginzburg-Landau functional, the gauge invariance is an issue. The choice of $\mathcal{A}$ we made fixes a gauge. We consider the case where the thickness $d(x) = 0$ *on* $\partial\omega_i$. The proof of lower bound is not different than the one in [ABGS13]. They used the method of Sandier and Serfaty [SS04] which works with compactly supported functions and so works for multiply connected domains. We are going to recall the proof for completeness. We recall Lemma 3.1 in [GP99]:

Lemma 3.2.1 *Let* $g \in L^2(\mathbf{R}^3; \mathbf{R}^3)$ *such that* $\operatorname{div} g = 0$ *in* $\mathcal{D}'(\mathbf{R}^3)$. *Then there is a unique* $B \in \mathbf{H}^1(\mathbf{R}^3, \mathbf{R}^3)$ *such that* $\nabla \times B = g$ *and* $\operatorname{div} B = 0$. *As a consequence, it follows that*

$$\|B\|_{\check{\mathbf{H}}^1_{\text{div}}} = \left[\int_{\mathbf{R}^3} |\nabla \times B|^2 dx\right]^{\frac{1}{2}} \tag{3.2.1}$$

is equivalent to the usual (Dirichlet) norm on the space $\breve{\mathbf{H}}^1_{\mathrm{div}}(\mathbf{R}^3, \mathbf{R}^3)$.

Proof of Theorem 3.1.3

Let $K := \sup_{n\in\mathbf{N}} I_{\epsilon_n,\kappa_n}(u_n, A_n) < +\infty$. From the energy bound

$$\int_{\mathcal{D}} (|u_n| - 1)^4 dx \leq \int_{\mathcal{D}} (|u_n|^2 - 1)^2 dx \leq \frac{K(\log \kappa_n)^2}{\kappa_n^2} \to 0 \tag{3.2.2}$$

In particular, $|u_n| \to 1$ *in* $L^4(\mathcal{D})$.

Also from the energy bound,

$$\frac{h'_n}{\log \kappa_n} - H' \to 0 \textit{ in } L^2(\mathbf{R}^3, \mathbf{R}^3). \tag{3.2.3}$$

and

$$\frac{1}{2}\int_{\mathbf{R}^3} \left(|h_3 - h_3^{ex}|^2\right) dx \leq K(\log \kappa)^2$$

thus along a subsequence, we may conclude the weak convergence,

$$\frac{h_{3n}}{\log \kappa_n} - H_3 \rightharpoonup l \textit{ in } L^2(\mathbf{R}^3). \tag{3.2.4}$$

As the vector $[\frac{h_n}{\log \kappa_n} - H] \rightharpoonup \tilde{H} := (0, 0, l)$ *in* $L^2(\mathbf{R}^3, \mathbf{R}^3)$, and each $\operatorname{div} h_n = 0$ in the sense of distributions, we may conclude that $\operatorname{div} \tilde{H} = 0$.

As a consequence of Lemma 3.2.1 and (3.2.4), we conclude that there exists $\tilde{A} \in \breve{\mathbf{H}}^1_{\mathrm{div}}(\mathbf{R}^3, \mathbf{R}^3)$ *with* $\nabla \times \tilde{A} = \tilde{H} = (0, 0, l)$ and

$$\frac{A_n}{\log \kappa_n} - \hat{A} \to \tilde{A}, \tag{3.2.5}$$

weakly in $\breve{H}^1_{\mathrm{div}}(\mathbf{R}^3, \mathbf{R}^3)$, and in the norm topology on $L^p(\mathcal{D})$, $1 \le p < 6$. Since $\tilde{H} := (0,0,l)$ *in* $L^2(\mathbf{R}^3, \mathbf{R}^3)$ with $\operatorname{div} \tilde{H} = 0$, we conclude that $\partial_3 l = 0$ (in the sense of distributions,) and thus $l = 0$, and also $\tilde{H} = 0 = \tilde{A}$ (by Lemma 3.2.1.) In particular, (3.2.5) implies

$$\frac{A_n}{\log \kappa_n} - \hat{A} \to 0 \quad \text{weakly in } \breve{\mathbf{H}}^1_{\mathrm{div}}(\mathbf{R}^3, \mathbf{R}^3), \text{ and in the norm on } L^p(\mathcal{D}),\ 1 \le p < 6. \tag{3.2.6}$$

In particular $h = h_{ex}$ in $\mathcal{D}^c$. To obtain the lower bound we adapt the method of [SS04]. Expanding the quadratic term in the energy bound, we obtain:

$$\begin{aligned}
2K &\ge 2I_{\epsilon_n,\kappa_n}(u_n, A_n) \\
&\ge (\log \kappa_n)^{-2} \int_{\mathcal{D}} (|\nabla' u_n|^2 - 2A'_n \cdot (iu_n, \nabla' u_n) + |u_n|^2 |A'_n|^2) dx \\
&= \int_{\mathcal{D}} \left(\frac{1}{2} \Big| \frac{\nabla' u_n}{\log \kappa_n} \Big|^2 - \Big| \frac{A'_n}{\log \kappa_n} \Big|^2 (|u_n|^2 - 1) - \Big| \frac{A'_n}{\log \kappa_n} \Big|^2 \right) dx
\end{aligned}$$

By (3.2.5), the last term is bounded, and

$$\int_{\mathcal{D}} \Big| \frac{A'_n}{\log \kappa_n} \Big|^2 \big| |u_n|^2 - 1 \big| dx \le \Big\| \frac{A'_n}{\log \kappa_n} \Big\|^2_{L^4} \big\| |u_n|^2 - 1 \big\|_{L^4} \le C \frac{\log \kappa_n}{\kappa_n},$$

by the energy bound and the L^p boundedness of $\frac{A'_n}{\log \kappa_n}$. Thus we have

$$\int_{\mathcal{D}} \Big| \frac{\nabla' u_n}{\log \kappa_n} \Big|^2 dx \le C, \tag{3.2.7}$$

with constant C depending on K. we may also obtain

$$\int_{\mathcal{D}} \frac{1}{\epsilon_n^2}\left|\frac{\partial_3 u_n}{\log \kappa_n}\right|^2 dx \leq C$$

and we have strong convergence in x_3-direction,

$$\frac{\partial_3 u_n}{\log \kappa_n} \to 0 \ \ in \ L^2(\Omega; \mathbf{C}). \tag{3.2.8}$$

We now work on the currents, $j_n := (iu_n, \nabla u_n)$. Following [JS] we normalize the current as follows,

$$\tilde{j} := \frac{j_n}{|u_n| \log \kappa_n} = \frac{(iu_n, \nabla u_n)}{|u_n| \log \kappa_n}.$$

each component of $\tilde{j}_n = (\tilde{j}_{1,n}, \tilde{j}_{2,n}, \tilde{j}_{3,n})$ is (for fixed n) pointwise (a.e.) bounded,

$$|\tilde{j}_{k,n}| \leq \frac{|\partial_k u_n|}{\log \kappa_n}, \quad k = 1, 2, 3, \tag{3.2.9}$$

so $\tilde{j}_n$ is well defined almost everywhere in $\mathcal{D}$. Moreover, from (3.2.7) and (3.2.8) it follows that there exists $j = (j', 0) \in L^2(\mathcal{D}; \mathbf{R}^3)$ such that (along subsequence)

$$\tilde{j}'_n \rightharpoonup j', \quad \tilde{j}'_{3,n} \to 0$$

in $L^2(\mathcal{D})$. Writing

$$\frac{j_n}{\log \kappa_n} = \tilde{j}_n + (|u_n| - 1)\tilde{j}_n,$$

we recall that $(|u_n| - 1) \to 0$ *in* $L^4(\mathcal{D})$ (see (3.2.2)) and thus obtain that

$$\frac{j'_n}{\log \kappa_n} \rightharpoonup j', \quad \frac{j'_{3,n}}{\log \kappa_n} \to 0 \ \ in \ L^{\frac{4}{3}}(\mathcal{D}). \tag{3.2.10}$$

We continue as in the proof of Theorem 2 of [SS04]. Let e_1, e_2, e_3 be the standard basis in $\mathbf{R}^3$, and define vector fields $X_k = h_k e_k, k = 1, 2, 3, with\ h_k \in C^0(\mathcal{D})\ and\ |h_k(x)| \leq 1\ for\ all\ x \in \mathcal{D}, k = 1, 2, 3,$. By (3.2.7), we have

$$\frac{\left|X_k \cdot \nabla' u_n\right|}{\log \kappa_n} = \frac{\left|h_k \partial_k u_n\right|}{\log \kappa_n} \rightharpoonup \phi_{X_k}, \quad k = 1, 2, 3, \tag{3.2.11}$$

weakly in L^2 and pointwise a.e. in $\mathcal{D}$. By (3.2.8), $\phi_{X_3} = 0$. we define the defect measure ν_{X_k} corresponding to the weak convergence in (3.2.11):

$$\left|\frac{X_k \cdot \nabla' u_n}{\log \kappa_n}\right|^2 \rightharpoonup |\phi_{X_k}|^2 + \nu_{X_k} \quad \text{in the sense of measures} \tag{3.2.12}$$

for k=1,2,3. Because of the strong convergence in (3.2.8), it follows that $\nu_{X_3} \equiv 0$.

For the compactness of the Jacobian we recall Theorem 1 from [SS04]

Theorem 3.2.2 *Let* (u_n, A_n) *be a family of* $\mathbf{H}^1(\Omega, \mathbf{C}) \times \mathbf{H}^1(\mathbf{R}^3, \mathbf{R}^3)$ *such that*

$$I_{\epsilon_n, \kappa_n}(u_n, A_n) \leq |\log \kappa_n|^2.$$

Then, up to extraction sequence

$$\frac{ju_n}{|\log \kappa_n|} \rightharpoonup J \quad in\ (C_0^{0,\gamma}(\mathcal{D}))',$$

where J *is a measure-valued 2-form. Moreover, for all continuous vector-fields* X *and* Y *compactly supported in* $\mathcal{D}$,

$$\frac{|X \cdot \nabla u_n|}{|\log \kappa_n|}, \quad \frac{|Y \cdot \nabla u_n|}{|\log \kappa_n|}$$

are bounded in L^2 and if we let ν_X, ν_Y be their defect measures, we have

$$\|\nu_X\|^{\frac{1}{2}}\|\nu_Y\|^{\frac{1}{2}} \geq \Big|\int_{\mathcal{D}} J(X,Y)\Big|. \tag{3.2.13}$$

We apply Theorem 3.2.2: by the energy bound,

$$E_{\kappa_n}(u_n;\mathcal{D}) := \int_{\mathcal{D}} \Big(\frac{1}{2}|\nabla u_n|^2 + \frac{\kappa_n}{4}(|u_n|^2-1)^2\Big)dx \leq C[\log \kappa_n]^2,$$

with constant C independent of n (using the estimates (3.2.7), (3.2.8), and (3.2.2)), we may conclude that

$$\frac{J_n}{\log \kappa_n} \overset{*}{\rightharpoonup} J$$

in the weak* topology on $(C_0^{0,\gamma}(\mathcal{D}))'$, for $0<\gamma\leq 1$. Moreover, the limiting Jacobian is a Radon measure-valued two-form. Furthermore, the same theorem relates the limiting Jacobian to the defect measure ν_{X_k} via a product formula (see Lemma 3.2.3 below.) Recall Lemma 2.2 in [ABGS13]:

Lemma 3.2.3 *The limiting Jacobian $J = \frac{1}{2}dj$ has the form $J = J_3 dx^1 \wedge dx^2$ with $J_3 = J_3(x')$, and the limiting current $j \in \mathcal{Z}$.*

Proof: We make use of the product formula from [SS04] in the case where $N_\kappa = O(\log\kappa)$, which we review here. Let E be a bounded smooth domain in $\mathbf{R}^3$, and $v_\kappa \in H^1(E,\mathbb{C})$ satisfying

$$E_\kappa(v_\kappa;E) := \int_E \left(\frac{1}{2}|\nabla v_\kappa|^2 + \frac{\kappa^2}{4}(|v_\kappa|^2-1)^2\right) \leq C[\log\kappa]^2,$$

for constant C independent of κ. Let X, Y be continuous, compactly supported vector fields in E, and ν_X, ν_Y the defect measures (defined as in 3.2.12) for v_κ as $\kappa\to\infty$.

Then, the normalized Jacobians $\frac{J_\kappa}{\log\kappa} \xrightarrow{\star} J$ in $(C_0^{0,\gamma}(\Omega))'$ for all $0 < \gamma \le 1$, and the defect measures are related to the limiting Jacobian via:

$$|\nu_X|(E)\,|\nu_Y|(E) \ge \left|\int_E J(X,Y)\right|^2. \tag{3.2.14}$$

Here we denote by $|\nu|(E)$ the total variation of the measure ν over the set E.

We note as above that for any $E \subset \Omega$, $E_{\kappa_n}(u_n;E) \le [\log\kappa_n]^2 I_{\epsilon_n,\kappa_n}(u_n,A_n)) \le C[\log\kappa_n]^2$. Let E be any open ball contained in Ω, and $X_k = h_k\,e_k$, with $h_k \in C_0(E)$ and $|h_k| \le 1$, $k = 1,2,3$. Applying the product formula we then obtain,

$$|\nu_{X_1}|^{\frac{1}{2}}(E)|\nu_{X_3}|^{\frac{1}{2}}(E) \ge \left|\int_E J(X_1,X_3)\right| = \left|\int_E h_1\,h_3\,J(e_1,e_3)\right|.$$

Taking the supremum over all such h_1, h_3, and from (3.2.8) we have $\nu_{X_3} \equiv 0$, we conclude that, as a Radon measure, $J(e_1,e_3) = 0$ in the ball E. By an analogous computation with X_2 and X_3 (as above), we also have $J(e_2,e_3) = 0$ in the ball E. This holds for any ball $E \subset \Omega$, and thus these measures vanish identically in Ω, and thus the Jacobian has the form $J = J_3\,dx^1 \wedge dx^2$.

Furthermore, since $J_n = \frac{1}{2}dj_n$ for each n, it follows that $dJ_n = 0$ (in the sense of distributions.) Normalizing by $\log\kappa_n$ and passing to the limit, we retain $dJ = 0$, and hence $\partial_3 J_3 = 0$ in $\mathcal{D}'(\Omega)$, so $J_3 = J_3(x')$. This also implies that (in the sense of $\mathcal{D}'(\Omega)$,)

$$\begin{aligned} 0 &= J_1 = \partial_2 j_3 - \partial_3 j_2 = -\partial_3 j_2, \\ 0 &= J_2 = \partial_3 j_1 - \partial_1 j_3 = \partial_3 j_1. \end{aligned}$$

Thus, the limiting current must have the form $j = \big(j'(x'), 0\big)$ and $J = \frac{1}{2}\nabla \times j \in \mathfrak{M}(\Omega)$, and hence $j \in \mathcal{Z}$.

$\diamondsuit$

It remains to verify the lower bound inequality. First from the definition of the defect measures and the product formula from Theorem 1 of [SS04], we have

$$\begin{aligned}
\liminf_{n\to\infty} \int_{\mathcal{D}} \Big|\frac{\nabla' u_n}{\log \kappa_n}\Big|^2 &\geq \sum_{k=1,2} \liminf_{n\to\infty} \int_{\mathcal{D}} \Big|\frac{X_k \cdot \nabla' u_n}{\log \kappa_n}\Big|^2 \\
&\geq |\nu_{X_1}|(\mathcal{D}) + |\nu_{X_2}|(\mathcal{D}) + \int_{\mathcal{D}} (\phi^2_{X_1} + \phi^2_{X_2}) \\
&\geq 2\Big|\int_{\mathcal{D}} J(X_1, X_2)\Big| + \int_{\mathcal{D}} (X_1 \cdot j)^2 + (X_2 \cdot j)^2 \\
&= 2\Big|\int_{\mathcal{D}} h_1 h_2 J(e_1, e_2)\Big| + \int_{\mathcal{D}} (h_1^2 |j \cdot e_1|^2 + h_2^2 |j \cdot e_2|^2). \qquad (3.2.15)
\end{aligned}$$

The above estimate is valid for any $h_k \in C_0(\Omega)$ with $|h_k(x)| \leq 1,\ k = 1, 2$. We choose these functions to obtain an estimate in terms of the total variation of the measure $J_3 = J(e_1, e_2)$. By the Hahn decomposition, we may write $J_3 = \mu_+ - \mu_-$ for mutually singular, nonnegative finite measures μ_+, μ_-, supported on the disjoint sets $E_+, E_- \in \Omega$, respectively. Take sequence $h_{1,i}, h_{2,i} \in C_0(\Omega)$ with $|h_{k,i}| \leq 1, k = 1, 2$, such that

$$h_{1,i} \to 1 \quad h_{2,i} \to \mathcal{X}_{E_+} - \mathcal{X}_{E_-}$$

pointwise a.e. in Ω. Since that $h_{1,i}h_{2,i}J(e_1, e_2) \to J$ and $h^2_{k,i}|j \cdot e_k|^2 \to |j|^2$ a.e. in $\mathcal{D}$ for $k = 1, 2$ also $h_{1,i}h_{2,i}J \leq J$ and $h^2_{k,i}|j \cdot e_k|^2 \leq |j|^2$ then we may pass to the limit $i \to \infty$ on the right hand side of (3.2.15) (using Lebesque dominated convergence theorem,) and we conclude that

$$\liminf_{n\to\infty} \int_{\mathcal{D}} \Big|\frac{\nabla' u_n}{\log \kappa_n}\Big|^2 \geq 2|J_3|(\mathcal{D}) + \int_{\mathcal{D}} |j|^2. \tag{3.2.16}$$

Finally, we derive the form of the lower bound for the full energy. First,

$$\begin{aligned}\liminf_{n\to\infty} I_{\epsilon_n,\kappa_n}(u_n, A_n) &\geq \liminf_{n\to\infty} \frac{1}{2[\log \kappa_n]^2} \int_{\mathcal{D}} \Big(|\nabla' u_n|^2 - 2A'_n \cdot j'_n + |A'_n|^2 \\ &\qquad + (|u_n|^2 - 1)|A'_n|^2\Big) dx \\ &\geq \|J_3\| + \frac{1}{2}\int_{\mathcal{D}} \big[|j|^2 - 2\hat{A}\cdot j + \hat{A}|^2\big] dx \\ &\qquad + \liminf_{n\to\infty} \frac{1}{2[\log \kappa_n]^2} (|u_n|^2 - 1)|A'_n|^2 dx\end{aligned}$$

the last integral on the right will tend to zero as $n \to \infty$, indeed using Cauchy Schwartz inequality with the energy bound and (3.2.6) we get

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{2[\log \kappa_n]^2} \int_{\mathcal{D}} (|u_n|^2 - 1)|A'_n|^2 dx &\leq \liminf_{n\to\infty} \Big(\int_{\mathcal{D}} (|u|^2 - 1)^2 dx\Big)^{\frac{1}{2}} \Big(\int_{\mathcal{D}} \Big|\frac{A_n}{|\log \kappa_n|}\Big|^4 dx\Big)^{\frac{1}{2}} \\ &\leq \liminf_{n\to\infty} \frac{C}{\kappa_n} |\log \kappa_n| \to 0.\end{aligned}$$

Now to reduce the 3D functional to 2D since both $J_3 = J_3(x')$ and $j' = j'(x')$ we may integrate out the variable x_3, to a two-dimensional total variation, weighted by the film thickness function $d(x')$,

$$\|J_3\|_{\mathfrak{M}(\mathcal{D})} = \|d(x')J_3\|_{\mathfrak{M}(\Omega)},$$

The limiting vector potential $\hat{A}$ (defined in (3.1.3)) is $x_3 - \textit{dependent}$, but this dependence may be averaged out (to produce the desired effective field F_*.) Indeed, we decompose $\hat{A}'$ as follows:

$$\hat{A}' = \Big(-\frac{1}{2}H_3x_2, \frac{1}{2}H_3x_1\Big) + (H_2x_3, -H_1x_3) =: \hat{A}_\perp + (H_2x_3, -H_1x_3).$$

Expanding the energy and integrating out x_3, we have:

$$\begin{aligned}
\int_{\mathcal{D}} |j' - \hat{A}'|^2 &= \int_{\mathcal{D}} |j' - \hat{A}_\perp - (H_2, -H_1)x_3|^2 dx \\
&= \int_\Omega d(x')|j' - \hat{A}_\perp|^2 - 2\int_\Omega (j' - \hat{A}_\perp)\cdot(h_2, -H_1)\int_{f(x')}^{g(x')} x_3 dx_3 dx' \\
&\quad + \int_\Omega |(H_2, -H_1)|^2 \int_{f(x')}^{g(x')} x_3^2 dx_3 dx' \\
&= \int_\Omega d(x')|j' - \hat{A}_\perp|^2 - 2\int_\Omega d(x')\frac{(f+g)}{2}(j' - \hat{A}_\perp)\cdot(H_2, -H_1)dx' \\
&\quad + \int_\Omega d(x')\frac{(f^2 + fg + g^2)}{3}|(H_2, -H_1)|^2 dx' \\
&= \int_\Omega d(x')\big|j' - \hat{A}_\perp - (\frac{f+g}{2})(H_2, -H_1)\big|^2 dx' + \int_\Omega \frac{d^3(x')}{12}\big|H'\big|^2 dx'.
\end{aligned} \tag{3.2.17}$$

We conclude that

$$\liminf_{n\to\infty} I_{\epsilon_n,\kappa_n}(u_n, A_n) \geq ||d(x')J_3||_{\mathfrak{M}}(\Omega) + \frac{1}{2}\int_\Omega d(x')\Big(|j' - B'_*|^2 + \frac{d^2(x')}{12}|H'|^2\Big)dx'$$

with

$$B'_* := \hat{A}_\perp + \Big(\frac{f+g}{2}\Big)(H_2, -H_1). \tag{3.2.18}$$

Since $\nabla' \times B'_* = F_*$ with F_* as given in (3.1.15), this concludes the proof of Theorem 3.1.3.

$\diamondsuit$

3.3 The recovery sequence

In this section we prove the existence of a recovery sequence and the Γ-limsup inequality (Theorem 3.1.4).

We introduce the space $\mathbb{H}$,

Definition 3.3.1 *We define the space* $\mathbb{H}$ *to be the closure in the norm*

$$\|f\|_{\mathbb{H}}^2 := \int_\Omega \frac{1}{d(x)} |\nabla f|^2 dx$$

of the linear subspace of $C_0^\infty(\Omega)$ *consisting of all functions which are constant in a neighbourhood of* ω_i *for each* $i = 1, ..., m$.

Note that: Since $d(x)$ is locally integrable, this norm is well-defined and $\mathbb{H}$ defines a Hilbert space. This follows the work of Trudinger [T] as in Alama-Bronsard [AB05]. As $d(x)$ is bounded above in Ω, the $\mathbb{H}$-norm dominates the usual $\mathbf{H}^1$-norm on Ω, and so the Poincaré, trace, and Sobolev inequalities hold for functions in $\mathbb{H}$.

Now we define the Hodge decomposition with respect to the weighted inner product,

$$\langle v, w \rangle = \int_\Omega d(x) v \cdot w \; dx'$$

on $L^2(\Omega; \mathbf{R}^2)$. We define the following subspaces:

$$\begin{aligned}
&\mathcal{U} = \{-\frac{1}{d}\nabla^\perp \psi, \; \psi \in \mathbb{H}(\Omega; \mathbf{R})\}, \\
&\mathcal{V} = \{\nabla \zeta, \; \zeta \in \mathbf{H}^1(\Omega; \mathbf{R})\}, \qquad (3.3.1) \\
&\mathcal{W} = \{W \in C^1(\Omega; \mathbf{R}^2), \; \nabla^\perp \cdot W = 0, \; \nabla \cdot (dW) = 0, \; W \cdot \nu = 0 \textit{ on } \partial\Omega\}.
\end{aligned}$$

Given what is known regarding Hodge decompositions, the main issue for us is the regularity of ψ, ζ. Assuming that $\psi \in \mathbb{H}$ will guarantee the existence of a unique solution in $\mathbb{H}$. For the subspace $\mathcal{V}$, we use a result by Montero [M07] which proves the existence of a unique solution $\zeta \in \mathbf{H}^1(\Omega, \mathbf{R})$.

Lemma 3.3.2 *Any $Z \in L^2(\Omega; \mathbf{R}^2)$ admits a unique orthogonal decomposition $Z = U + V + W$ with $U \in \mathcal{U}$, $V \in \mathcal{V}$, and $W \in \mathcal{W}$, with respect to the inner product $\langle ., . \rangle$. The space $\mathcal{W}$ is finite dimensional space where* $\dim(\mathcal{W}) = m$.

Proof.

As we saw in Chapter 2, following [JS], the only subspace which is affected by the multiply connected domain is the space $\mathcal{W}$. First, we assume $Z \in C^\infty(\Omega; \mathbf{R}^2)$. we define ψ *and* ζ as the solutions to the boundary-value problems,

$$\begin{cases} -\nabla \cdot \left(\dfrac{1}{d(x)} \nabla \psi \right) = \operatorname{curl} Z \ \ \text{in } \Omega, \\ \psi = 0 \ \ \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \nabla \cdot (d(x) \nabla \zeta) = \operatorname{div} [dZ] \ \ \text{in } \Omega, \\ \dfrac{\partial \zeta}{\partial \nu} = Z \cdot \nu \ \ \text{on } \partial\Omega, \end{cases}$$

Due to our case where $d(x)$ is zero on inner boundaries, the existence and uniqueness of the above systems of equation are not straight forward.

We recall Lemma 2.2 in [M07].

Lemma 3.3.3 *Consider an integer $m \geq 1$, and assume that $d(x)$ satisfies (H1)-(H3). For every $\phi_0 \in C^{0,\alpha}(\Omega)$ such that*

$$\int_\Omega d^m \phi_0 dx = 0$$

there is $\zeta \in W^{1,2}_{m,m}(\Omega)$ (defined as in (3.1.1)), unique up to a constant, that satisfies

$$-\operatorname{div}(d^m \nabla \zeta) = d^m \phi_0 \tag{3.3.2}$$

weakly in Ω, that is,

$$\int_\Omega d^m \nabla s \cdot \nabla \zeta = \int_\Omega d^m \phi_0 s \quad \text{for all } s \in W^{1,2}_{m,m}(\Omega). \tag{3.3.3}$$

Furthermore, $\zeta \in C^{2,\alpha}(\Omega)$ and $\nabla\zeta \cdot \nu = 0$ on Ω.

We conclude that for the second system of equations, there exist a unique solution $\zeta \in C^{2,\alpha}(\Omega)$. Indeed, using the above Lemma, we choose $m = 1$ and let $\phi_0 = \frac{\operatorname{div}[dZ]}{d} \in C^{0,\alpha}(\Omega)$, then

$$\int_\Omega d\phi_0 = \int_\Omega \operatorname{div}[dZ] = \int_{\partial\Omega} dZ \cdot \nu = 0,$$

by the boundary conditions on aZ (i.e. $Z \cdot \nu = 0 \quad on\ \partial\Omega$). We can get the general $\zeta \in \mathbf{H}^1$ by density.

Now for the first system where $\psi \in \mathbb{H}$, because of the definition of our space with the weighted norm the system is well-defined and it has a unique solution by Reisz Representation Theorem.

If $W := Z + \frac{1}{d(x)}\nabla^\perp\psi - \nabla\zeta$, then it is clear that W satisfies $\operatorname{curl} W = 0 = \operatorname{div}[dW]$ *in* Ω, *and* $W.\nu = 0$ *on* $\partial\Omega$. By integration by parts, we could see $W \perp \frac{1}{d}\nabla^\perp\psi \perp \nabla\zeta$ in the inner product $\langle .,. \rangle$.

Finally, applying Lemma 1.1 of [BBH] to identify the space $\mathcal{W}$, any $W \in \mathcal{W}$ can be written as $W = \frac{1}{d}\nabla^\perp\xi$ with ξ constant on each componant of $\partial\Omega$, and $\nabla \cdot \frac{1}{d}\nabla\xi = 0$ in Ω. In our case where $\Omega = \omega_0 \setminus \cup_{i=1}^m \omega_i$ is multiply-connected domain, we follow the treatment of [AB06]. For each fixed $i = 1, ..., m$ we define functions $\xi_i \in \mathbb{H}(\Omega)$ which solve

$$\left.\begin{aligned} \nabla\cdot\frac{1}{d}\nabla\xi_i = 0, \quad \text{in } \Omega, \\ \xi_i|_{\partial\omega_j} = c_{ij}, \quad j=1,\ldots,m \\ \xi_i|_{\partial\omega_0} = 0, \\ \frac{1}{2\pi}\oint_{\partial\omega_j}\frac{1}{d}\frac{\partial\xi_i}{\partial\nu}dx = \delta_{i,j} \quad j=1,\ldots,m, \end{aligned}\right\} \tag{3.3.4}$$

where c_{ij} are constants (determined by the solutions,) and $\delta_{i,j}$ is Kronecker's delta. We got the last equation by integrating around each ω_j.

We can obtain the existence of such ξ_i by minimizing

$$F_i(\xi) = \frac{1}{2}\int_\Omega \frac{1}{d}|\nabla\xi|^2 dx + 2\pi\xi|_{\omega_i}$$

over the class of $\xi \in \mathbb{H}(\omega)$ with $\xi|_{\omega_j}$ constant. By the Poincaré inequality and the trace inequalities, F_i is bounded below on $\mathbb{H}$, and by convexity it attains a unique minimizer ξ_i. A simple computation shows that minimizers give weak solutions to the boundary-value problem (3.3.4). Indeed, the first variation yields,

$$0 = DF_i(\xi_i)u = \int_\Omega [\frac{1}{d}\nabla\xi_i\cdot\nabla u]dx + 2\pi\xi|_{\omega_i}. \tag{3.3.5}$$

for all $u \in \mathbb{H}$. The equation and boundary conditions then follow from choosing u with values either zero or one in the appropriate domains ω_j where each ξ_i is smooth in the interior and on the boundary as in [AB05].

$$\xi = \sum_{i=1}^m \Phi_i\xi_i(x), \qquad \Phi_i := \left(\frac{1}{2\pi}\oint_{\partial\omega_j}\frac{1}{d}\frac{\partial\xi}{\partial\nu}dx\right).$$

Thus, $W = \frac{1}{d}\nabla^\perp\xi \in \mathcal{W}$ is parametrized by the m constants $\Phi_i, i = 1, ..., m$, and $\mathcal{W}$

is finite dimensional of order m. The general result for $Z \in L^2(\Omega; \mathbf{R}^2)$ is obtained by density.

$\diamondsuit$

We are now ready to construct the recovery subsequence,

Proof of Theorem 3.1.4. (Upper bound part of the Γ-convergence)

Let $j \in \mathcal{Z}$ be given, as well as the sequence $\kappa_n \to \infty$. We choose vector potentials $A_n = A_{ex}$ where $h_{ex} = \nabla \times A_{ex}$, and construct sequence of order parameters u_n of the form $u_n(x) = v_n(x')$ which gives us the desired result. As noted in [ABGS10], for configurations of this form, the three-dimensional energy $\tilde{I}_{\epsilon,\kappa}$ reduces to,

$$G_{\kappa_n}(v_n, F) = \int_\Omega d(x')\{\frac{1}{2}|(\nabla - iB')v_n|^2 + \frac{\kappa_n^2}{4}(|v_n|^2-1)^2 + \frac{(d(x'))^2}{24}|h'_{ex}|^2|v_n|^2\}dx', \tag{3.3.6}$$

with B'_* defined as in (3.2.18) and $F_* = \nabla' \times B'_*$. We will drop the prime since we will be working in two dimensions from now on. We apply the Hodge decomposition above to our given $j \in \mathcal{Z}$. and write

$$j = U + V + W = -\frac{1}{d}\nabla^\perp \psi + \nabla\zeta + W.$$

where $\psi \in \mathbb{H}(\Omega) \subset \mathbf{H}^1_0(\Omega),\ \zeta \in \mathbf{H}^1(\Omega)$ *and* $W \in \mathcal{W}$.

We will deal with V *and* W first. Since they are irrotational, they won't affect the weak Jacobian $J = \frac{1}{2}\nabla \times j$, and carry no vorticity. As in [JS02], we may associate to $V,\ W$ an S^1-valued map w^κ. For the singular part of the Jacobian which is contained in U, we first consider smooth U and we construct a family u^κ with points vortices via an appropriate Green's function then use a diagonal argument to get the general case. Putting these two parts together, the desired recovery sequence will have the form $v_n = u^{\kappa_n} w^{\kappa_n}$.

Step 1: The components $V + W \in \mathcal{V} \oplus \mathcal{W}$. This step follows [ABGS13] and we recall the proof for completeness.

From Lemma 3.3.2, we can write $V = \nabla\zeta,\ \zeta \in \mathbf{H}^1(\Omega)$ and $W = \frac{1}{d}\nabla^{\perp}\xi,\ with\ \xi(x) = \sum_{i=1}^{m} \Phi_i \xi_i(x),\ for\ \xi_i\ as\ in$ (3.3.4) *with* Φ_i *real constants*. Let $M_{i,n} = [\Phi_i \log \kappa_n],\ i = 1, ..., m$, where brackets denote the integer part, Set

$$\Xi_n := \sum_{i=1}^{m} M_{i,n}\xi_i, \qquad W_n = -\frac{1}{d}\nabla^{\perp}\Xi_n.$$

We note that

$$\|W_n - W \log \kappa_n\|_{C^1} \leq C, \tag{3.3.7}$$

for constant C depending on W (but independent of n.)

Since

$$\begin{aligned}
\operatorname{curl} W_n &= \sum_{i=1}^{m} M_{i,n} \nabla^{\perp} \cdot \frac{1}{d}\nabla^{\perp}\xi_i = 0, \\
\oint_{\partial\omega_j} W_n \cdot \tau\, ds &= \sum_{i=1}^{m} M_{i,n} \oint_{\partial\omega_j} \frac{1}{d}\frac{\partial \xi_i}{\partial \nu} ds = 2\pi M_{j,n},
\end{aligned}$$

an integer multiple of 2π for each $j = 1, \ldots, m$, it follows that W_n is locally a gradient, $W_n = \nabla\eta_n$ for η_n possibly multiple valued, but for which $e^{i\eta_n}$ is smooth and single-valued in Ω. We may then define the complex order parameter

$$w_n = \exp i(\eta_n + \zeta \log \kappa_n).$$

By construction,

$$\frac{j(w_n)}{\log \kappa_n} = \frac{(iw_n, \nabla w_n)}{\log \kappa_n} \to V + W \tag{3.3.8}$$

in $C^1(\bar{\Omega})$. Since $|w_n| = 1$, we may easily calculate the contribution to the energy using the orthogonality:

$$\begin{aligned}
\frac{1}{2}\int_\omega d(x)|\nabla w_n|^2\,dx &= \frac{1}{2}\int_\Omega d(x)|\nabla \eta_n + \nabla\zeta \log \kappa_n|^2\,dx \\
&= \frac{1}{2}\int_\omega d(x)|W_n|^2 + \frac{(\log \kappa_n)^2}{2}\int_\Omega d(x)|\nabla \zeta|^2\,dx \\
&\le \frac{(\log \kappa)^2}{2}\int_\Omega d(x)\left\{|W|^2 + |V|^2\right\}dx + O(1), \qquad (3.3.9)
\end{aligned}$$

using (3.3.7) in the last line. This completes Step 1.

The treatment of $U = -\frac{1}{d}\nabla^\perp\psi \in \mathcal{U}$ will be done in several steps. First, we start with $\psi \in C_0^\infty(\Omega)$, then use a diagonal argument for $\psi \in \mathbb{H}$.

Step 2: Since $U \in \mathcal{U}$ then there exists a sequence $\{\psi^{t_m}\} \subset C_0^\infty(\Omega)$ *with* $K^{t_m} := \operatorname{supp}\psi^{t_m} \subset\subset \Omega$ *and* $\operatorname{dist}(K^{t_m}, \partial\Omega) \ge t_m$ with $t_m \to 0$.

Define

$$d_{t_m}(x) = \max\{t_m, d(x)\}$$

then $d_{t_m}(x)$ is positive and $d(x) \le d_{t_m}(x)$.

For this t_m we define the measure

$$\mu^{t_m} = -\nabla \times \frac{1}{d_{t_m}}\nabla^\perp\psi^{t_m}$$

since ψ^{t_m} is compactly supported in $K^{t_m} \subset \Omega$ then by (H3) on $d(x)$, we have

$$\mu^{t_m} = -\nabla \times \frac{1}{d} \nabla^{\perp} \psi^{t_m}$$

Let $\Omega_{t_m} = \{x \in \Omega \ s.t. \ \mathrm{dist}\,(x, \partial\Omega) \geq t_m\}$ choose t_m small enough s.t. $\min_{\bar{\Omega}_{t_m}} d > t_m$, and to start with we follow proof of Proposition 2.4.1 in Chapter 2 with minor modifications since we have a positive d_{t_m}.

Let $N_n \in \mathbf{N}$ be any sequences of whole numbers with

$$\frac{N_n}{\log \kappa_n} \longrightarrow 1.$$

We apply Lemma 2.4.3 in Chapter 2 (see [JS]), there exist family of points $\{p_i^n\}_{i=1,\dots,N_n}$ in the set $K^{t_m} = \operatorname{supp} \psi^{t_m}$ and associated integers $\sigma_i^n \in \{-1, 1\}$ with the following properties:

$$|p_i^n - p_j^n| \geq c_0 N_n^{-1/2} \quad \text{for } i \neq j, \text{ for constant } c_0 = c_0(\psi^{t_m}) \ and \ \mathrm{dist}\,(p_i^n, \partial\Omega) > c_0 N_n^{-1/2}; \tag{3.3.10}$$

$$\lim_{\alpha \to 0} R(\alpha) = 0 \quad \text{where} \quad R(\alpha) = \limsup_{n \to \infty} \sum_{\substack{i \neq j: \\ |p_i^n - p_j^n| \leq \alpha}} \frac{\left|\log |p_i^n - p_j^n|\right|}{N_n^2}, \tag{3.3.11}$$

$$\mu_n := \frac{2\pi}{N_n} \sum_{i=1}^{N_n} \sigma_i^n \, \delta_{p_i^n} \rightharpoonup \mu^{t_m}, \tag{3.3.12}$$

$$|\mu_n| = \frac{2\pi}{N_n} \sum_{i=1}^{N_n} \delta_{p_i^n} \rightharpoonup |\mu^{t_m}|, \tag{3.3.13}$$

where the convergence in (3.3.12) and (3.3.13) is weakly in the sense of measures. and strongly in $(C_0^{0,\gamma})'$ for all $0 < \gamma \leq 1$. By $|\mu^{t_m}|$ we mean the total variation of the measure $\mu^{t_m} = \operatorname{curl} U^{t_m}$ (see Definition 2.1.7). Since $\psi^{t_m} \in C_0^\infty(\Omega)$ then μ^{t_m} is smooth and compactly supported in $K^{t_m} \subset \Omega$.

As in [SS00] and Chapter 2 we modify the measures μ_n by regularizing the Dirac mass. Let $\mu_i^n := \kappa_n \mathcal{H}^1 \lfloor_{\partial B(p_i^n, 1/\kappa_n)}$, the element of arclength on $S_i^n := \partial B(p_i^n, 1/\kappa_n)$, normalized with mass 2π. We define the measures

$$\nu_n = \frac{1}{N_n} \sum_{i=1}^{N_n} \sigma_i^n \, \mu_i^n,$$

with $p_i^n \in K$, $\sigma_i^\kappa \in \{0,1\}$ as above. Since each $\mu_i^n \longrightarrow \delta_{p_i^n}$ strongly in $(C_0^{0,\gamma}(\Omega))'$ for all $0 < \gamma \leq 1$, and weakly in $\mathfrak{M}(\Omega)$, we may conclude that (3.3.12),(3.3.13) hold as well for ν_n,

$$\nu_n \longrightarrow \mu^{t_m}, \qquad |\nu_n| \longrightarrow |\mu^{t_m}|, \quad \text{strongly in } [C_0^{0,\gamma}(\Omega)]' \text{ and weakly in } \mathfrak{M}(\Omega). \tag{3.3.14}$$

By Fubini's theorem we also note that the product measures also converge,

$$\nu_n \otimes \nu_n \longrightarrow \mu^{t_m} \otimes \mu^{t_m}, \tag{3.3.15}$$

strongly in $(C_0^{0,\gamma}(\Omega \times \Omega))'$ and weakly in $\mathfrak{M}(\Omega \times \Omega)$.

Step 3: We introduce the Dirichlet Green's function, $G_{d_{t_m}}(x,y)$ in Ω, which solves

$$\begin{cases} -\nabla_x \cdot \dfrac{1}{d_{t_m}(x)} \nabla_x G_{d_{t_m}}(x,y) = \delta_y(x), \quad \text{in } \Omega, \\ G_{d_{t_m}}(\cdot, y) = 0, \quad \text{on } \partial\Omega, \end{cases}$$

for each fixed $y \in \Omega$. By standard elliptic theory, since that the coefficient of the elliptic operator $\frac{1}{d_{t_m}}$ is smooth in $\overline{\Omega}$ and bounded below we conclude that the solution $G_{d_{t_m}}(x, y)$ is smooth in $\overline{\Omega} \times \overline{\Omega} \setminus \{y = x\}$ (see GT), and

$$G_{d_{t_m}}(x, y) = -\frac{d_{t_m}(x)}{2\pi} \log |x - y| + \gamma(x, y), \tag{3.3.16}$$

where the regular part γ has the property that for every compact set $K \subset\subset \Omega$, there exists $C(K) < \infty$ with

$$\sup_{\substack{y \in K \\ x \in \overline{\Omega}}} |\gamma(x, y)| \leq C(K).$$

For $U^{t_m} = \frac{1}{d_{t_m}} \nabla^{\perp} \psi^{t_m}$ with $\operatorname{supp} U^{t_m} \subset K^{t_m}$, we may obtain the potential function ψ^{t_m} from $\operatorname{curl} U^{t_m} = \mu^{t_m}$,where μ^{t_m} is smooth, by solving

$$\begin{cases} -\nabla \cdot \frac{1}{d_{t_m}(x)} \nabla \psi^{t_m} = \mu^{t_m} & \text{in } \Omega, \\ \psi^{t_m} = 0 & \text{on } \partial\Omega, \end{cases}$$

and we recover $U^{t_m} = -\frac{1}{d_{t_m}} \nabla^{\perp} \psi^{t_m}$. Using the Green's function representation, we have

$$\psi^{t_m}(x) = \int_{\Omega} G_{d_{t_m}}(x, y)\, d\mu^{t_m}(y) = \int_{K^{t_m}} G_{d_{t_m}}(x, y)\, d\mu^{t_m}(y).$$

We may calculate the weighted norm of U^{t_m} in terms of the measure μ^{t_m} as follows:

$$\begin{aligned}\int_\Omega d_{t_m}(x)\,|U^{t_m}|^2\,dx &= \int_\Omega \frac{1}{d_{t_m}}\left|\nabla\psi^{t_m}\right|^2\,dx\\ &= -\int_\Omega \psi^{t_m}\cdot\nabla^\perp\left(\frac{1}{d_{t_m}}\nabla^\perp\psi^{t_m}\right)\,dx\\ &= \int_\Omega \psi^{t_m}(x)\,d\mu^{t_m}(x)\\ &= \int_\Omega\int_\Omega G_{d_{t_m}}(x,y)\,d\mu^{t_m}(y)\,d\mu^{t_m}(x). \end{aligned} \tag{3.3.17}$$

Step 4. In this step we prove that there exits a sequence $\psi_n \in \mathbf{H}^1_0(\Omega)$ for which $\frac{1}{d_{t_m}}\nabla^\perp\psi_n \longrightarrow U^{t_m}$ strongly in $L^p(\Omega)$ for all $p<2$, and

$$\limsup_{n\to\infty}\int_\Omega \frac{1}{d_{t_m}}|\nabla\psi_n|^2\,dx \le \int_\Omega d_{t_m}(x)\,d|\mu^{t_m}|(x) + \int_\Omega d_{t_m}(x)|U^{t_m}|^2\,dx. \tag{3.3.18}$$

As you may notice $d_{t_m}(x)$ is positive, then this step follows step 2 of the proof of Proposition 2.4.1 .

For each n, we define $\psi_n(x) = \int_\Omega Gd_{t_m}(x,y)\,d\nu_n(y)$, and so ψ_n solves

$$\begin{cases} -\nabla\cdot\frac{1}{d_{t_m}(x)}\nabla\psi_n = \nu_n & \text{in } \Omega,\\ \psi_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By (3.3.14) and elliptic regularity, we have $\psi_n \to \psi^{t_m}$ in $W^{1,p}(\Omega)$ for $p<2$, and thus $-\frac{1}{d_{t_m}}\nabla^\perp\psi_n \to U^{t_m}$ in $L^p(\Omega)$ for all $p<2$ as claimed.

To estimate the energy we use the Green's representation. Since $\nu_n \in H^{-1}(\Omega)$ for fixed n, following (3.3.17) we conclude that

$$\int_\Omega \frac{1}{d_{t_m}} |\nabla \psi_n|^2 \, dx = \int_\Omega \int_\Omega G_{d_{t_m}}(x,y) \, d\nu_n(y) \, d\nu_n(x).$$

For any $0 < \alpha < 1$, let $\Delta_\alpha = \{(x,y) \in \Omega \times \Omega : |x-y| \leq \alpha\}$. Fix $\chi_\alpha \in C^\infty(\bar{\Omega} \times \bar{\Omega})$ with $0 \leq \chi_\alpha \leq 1$, and

$$\chi_\alpha(x,y) = \begin{cases} 1, & \text{if } x \in \Delta_\alpha, \\ 0, & \text{if } x \notin \Delta_{2\alpha}. \end{cases}$$

For any $\alpha \in (0,1)$, $G_{d_{t_m}}(x,y)(1-\chi_\alpha(x,y))$ is smooth, and hence by the strong $(C_0^{0,\gamma})'$ convergence $\nu_n \to \mu^{t_m}$ we have:

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega \int_\Omega & G_{d_{t_m}}(x,y)(1-\chi_\alpha(x,y)) d\nu_n(y) \, d\nu_n(x) \\ &= \int_\Omega \int_\Omega G_{d_{t_m}}(x,y)(1-\chi_\alpha(x,y)) d\mu^{t_m}(y) \, d\mu^{t_m}(x). \end{aligned} \tag{3.3.19}$$

For the complementary integral, we use (3.3.16) to observe that

$$\begin{aligned} \int_\Omega \int_\Omega & G d_{t_m}(x,y)\chi_\alpha(x,y) d\nu_n(y) \, d\nu_n(x) \\ &= \int_K^{t_m} \int_{\Delta_{2\alpha}} [\frac{d_{t_m}(x)}{2\pi} \log \frac{1}{|x-y|} + \gamma(x,y)] \chi_\alpha \, d\nu_n(y) \, d\nu_n(x) \\ &\leq \int_K^{t_m} \int_{\Delta_{2\alpha}} \frac{d_{t_m}(x)}{2\pi} \log \frac{1}{|x-y|} \, d\nu_n(y) \, d\nu_n(x) + C\alpha \end{aligned} \tag{3.3.20}$$

$$\int_\Omega \int_\Omega G d_{t_m}(x,y) \chi_\alpha(x,y) d\nu_n(y)\, d\nu_n(x)$$
$$= \frac{1}{N_n^2} \sum_{i,j=1}^{N_n} \iint_{\Delta_{2\alpha}} \frac{d_{t_m}(x)}{2\pi} \log \frac{1}{|x-y|}\, d\mu_i^n(y)\, d\mu_i^n(x) + C\alpha. \tag{3.3.21}$$

To evaluate the remaining integral, we consider the contribution due to distinct points $p_i^n \neq p_j^n$ in $\Delta_{2\alpha}$ separately. We adapt an argument in Proposition 7.4 of [SS07] Define the index set

$$\mathcal{J}_n = \{(i,j) : \ |p_i^n - p_j^n| \leq 2\alpha\}.$$

Let $R_n = \frac{1}{4} c_0 N_n^{-1/2}$, where $c_0 = c_0(\psi)$ is the constant in (3.3.10). We also define balls $\tilde{B}_i^n = B(p_i^n, R_n)$, $i = 1, \dots, N_n$. By the choice of R_n, they are disjoint, as is the union

$$\bigcup_{(i,j)\in\mathcal{J}_n} \left(\tilde{B}_i \times \tilde{B}_j\right) \subset \Delta_{3\alpha}.$$

We also observe that for any $R \leq R_n$ and $(i,j) \in \mathcal{J}_n$, since $R \leq \frac{1}{4}|p_i - p_j|$, we have

$$\frac{1}{2} \leq \frac{|x-y|}{|p_i^n - p_j^n|} \leq \frac{3}{2} \quad \text{for all } x \in B(p_i^n, R),\ y \in B(p_j^n, R). \tag{3.3.22}$$

For $(i,j) \in \mathcal{J}_n$ we then have (recalling that $S_n^i = \partial B(p_i^n, \frac{1}{\kappa_n}) = \operatorname{supp} \mu_i^n,$)

$$\begin{aligned}
\iint_{\tilde{B}_i^n \times B_j^n} \log \frac{3}{|x-y|} dx\, dy &\geq \iint_{\tilde{B}_i^n \times B_j^n} \log \frac{2}{|p_i^n - p_j^n|} dx\, dy \\
&= \pi^2 R_n^4 \log \frac{2}{|p_i^n - p_j^n|} \\
&= \frac{R_n^4}{4} \iint_{\tilde{S}_i^n \times S_j^n} \log \frac{2}{|p_i^n - p_j^n|} d\mu_i^n(x)\, d\mu_j^n(y) \\
&\geq \frac{R_n^4}{4} \iint_{\tilde{S}_i^n \times S_j^n} \log \frac{1}{|x-y|} d\mu_i^n(x)\, d\mu_j^n(y),
\end{aligned}$$

using (3.3.22) in the first and last lines. Summing over all pairs $(i,j) \in \mathcal{J}_n$, and using the disjointness of the union of the $\tilde{B}_i^n \times \tilde{B}_j^n$, we obtain:

$$\begin{aligned}
\frac{1}{N_n^2} \sum_{(i,j)\in\mathcal{J}_n} \iint_{S_i^n \times S_j^n} \frac{d_{t_m}(x)}{2\pi} \log \frac{1}{|x-y|} d\mu_i^n(x)\, d\mu_j^n(y) \\
\leq \frac{C}{R_n^4\, N_n^2} \sum_{(i,j)\in\mathcal{J}_n} \iint_{\tilde{B}_i^n \times B_j^n} \log \frac{3}{|x-y|} dx\, dy \\
\leq C \iint_{\Delta_{3\alpha}} \log \frac{3}{|x-y|} dx\, dy =: \mathcal{R}(\alpha). \qquad (3.3.23)
\end{aligned}$$

As $|\log|x-y||$ is integrable, the remainder $\mathcal{R}(\alpha) \to 0$ as $\alpha \to 0$, and so this term will not contribute to the limiting energy.

Finally, we consider the contribution from the self-energy of the vortices p_i^n. We parametrize the integrals over $S_i^n = \partial B(p_i^n, \frac{1}{\kappa_n})$ using complex notation. We write

$x, y \in \partial B(p_i^n, \frac{1}{\kappa_n})$ as $x = p_i^n + \frac{1}{\kappa_n} e^{i\theta}$, $y = p_i^n + \frac{1}{\kappa_n} e^{i\tau}$, $0 \le \theta, \tau < 2\pi$. Then we have:

$$\begin{aligned}
\frac{1}{N_n^2} \iint_\Omega \frac{d_{t_m}(x)}{2\pi} \log \frac{1}{|x-y|}\, d\mu_i^n(y)\, d\mu_i^n(x) \\
&= \frac{1}{N_n^2} \int_0^{2\pi} \int_0^{2\pi} \frac{d_{t_m}(p_i^n + \frac{e^{i\theta}}{\kappa_n})}{2\pi} [\log \kappa_n + \log |e^{i(\theta-\tau)} - 1|] d\theta\, d\tau \\
&= \frac{1}{N_n} \int_0^{2\pi} d_{t_m} \left(p_i^n + \frac{e^{i\theta}}{\kappa_n} \right) d\theta + O(N_n^{-2}) \\
&= \frac{1}{N_n} \int_\Omega d_{t_m}(x)\, d|\mu_i^n|(x) + O(N_n^{-2}).
\end{aligned}$$

Summing over all $i = 1, \ldots, N_n$, we arrive at

$$\begin{aligned}
\frac{1}{N_n^2} \sum_{i=1}^{N_n} \iint_\Omega \frac{d_{t_m}(x)}{2\pi} \log \frac{1}{|x-y|}\, d\mu_i^n(y)\, d\mu_i^n(x) &= \frac{1}{N_n} \int_\Omega d_{t_m}(x)\, d|\nu_n|(x) + O(N_n^{-1}) \\
&= \int_\Omega d_{t_m}(x)\, d|\mu^{t_m}|(x) + O(N_n^{-1}).
\end{aligned} \tag{3.3.24}$$

Passing to the limit $\kappa_n \to \infty$, we thus obtain from (3.3.19),(3.3.20),(3.3.23), and (3.3.24), that

$$\begin{aligned}
\limsup_{n\to\infty} \int_\Omega \int_\Omega G_{d_{t_m}}(x,y) d\nu_n(y)\, d\nu_n(x) \\
&\le \int_\Omega d_{t_m}(x)\, d|\mu^{t_m}|(x) + \int_\Omega \int_\Omega G_{d_{t_m}}(x,y)(1 - \chi_\alpha(x,y)) d\mu^{t_m}(y)\, d\mu^{t_m}(x) \\
&\quad + C\alpha + C\mathcal{R}(\alpha).
\end{aligned}$$

By hypothesis, the measure μ^{t_m} is bounded, and so we may apply dominated convergence to pass to the limit $\alpha \to 0$ and obtain the desired bound (3.3.18), as

$$\begin{aligned}\limsup_{n\to\infty} \int_\Omega \frac{1}{d_{t_m}} |\nabla\psi_n|^2\, dx &= \limsup_{n\to\infty} \int_\Omega\int_\Omega G_{d_{t_m}}(x,y) d\nu_n(y)\, d\nu_n(x) \\ &\le \int_\Omega d_{t_m}(x)\, d|\mu^{t_m}|(x) + \int_\Omega\int_\Omega G(x,y) d\mu^{t_m}(y)\, d\mu^{t_m}(x) \\ &= \int_\Omega d_{t_m}(x)\, d|\mu^{t_m}|(x) + \int_\Omega d_{t_m}(x)\, |U^{t_m}|^2\, dx,\end{aligned}$$

by (3.3.17).

Step 5. Constructing a sequence $u_n \in \mathbf{H}_0^1(\Omega)$.

Let $U_n = -N_n \frac{1}{d_{t_m}} \nabla^\perp \psi_n$. Then, $\nabla^\perp U_n = N_n \nabla \cdot \left(\frac{1}{d_{t_m}} \nabla\psi_n\right) = 0$ locally in $\Omega \setminus \cup_i^{N_n} B(p_i^n, \frac{1}{\kappa_n})$. Moreover, if C is a simple closed curve in $\Omega \setminus \cup_i^{N_n} B(p_i^n, \frac{1}{\kappa_n})$, we have

$$\int_C U_n \cdot \tau\, ds \in 2\pi\, \mathbb{Z},$$

by the normalization $|d\mu_i^n| = 2\pi$. Thus, we may write $U_n = \nabla\phi_n$ in $\Omega \setminus \cup_i^{N_n} B(p_i^n, \frac{1}{\kappa_n})$, with ϕ_n which is multiple valued, but for which $\nabla\phi_n$ and $e^{i\phi_n}$ are single-valued in $\Omega \setminus \cup_i^{N_n} B(p_i^n, \frac{1}{\kappa_n})$.

We now define an auxiliary function ρ_n as in [JS] and [ABGS13] to remove the singularity at each vortex core,

$$\rho_i^n(x) := \begin{cases} 0 & \text{if } |x - p_i^n| < \frac{1}{2\kappa_n}, \\ 2\kappa_n |x - p_i^n| - 1 & \text{if } \frac{1}{2\kappa_n} \le |x - p_i^n| \le \frac{1}{\kappa_n}, \\ 1 & \text{if } |x - p_i^n| > \frac{1}{\kappa_n}, \end{cases} \tag{3.3.25}$$

and $\rho_n := \prod_{i=1}^{N_n} \rho_i^n$. A simple computation shows that

$$\int_\Omega d(x) \left\{ \frac{1}{2}|\nabla \rho_i^n|^2 + \frac{\kappa_n^2}{4}((\rho_i^n)^2 - 1)^2) \right\} dx \leq C_0,$$

with constant C_0 independent of n. Also

$$(\rho_n^2 - 1) \to 0 \;\; in \;\; L^q \;\; for\ all \;\; q < \infty, \tag{3.3.26}$$

indeed from (3.3.25)

$$\begin{aligned}
\int_\Omega \left|{\rho_n}^2 - 1\right|^q &= \int_\Omega \left|\prod_i (\rho_i^n)^2 - 1\right|^q \\
&\leq \int_{\{x\in\Omega|\ \frac{\epsilon_n}{2} \leq |x-p_i| \leq \epsilon_n\}} \Big|\prod_i \big(\frac{2}{\epsilon_n}|x_i - p_i^n| - 1\big)^2 - 1\Big|^q + \int_{\{x\in\Omega|\ |x-p_i|>\epsilon_n\}} |1-1| \\
&\leq \int_{\{x\in\Omega|\ \frac{\epsilon_n}{2} \leq |x-p_i| \leq \epsilon_n\}} \Big|\prod_i \big(\frac{2}{\epsilon_n}\epsilon_n - 1\big)^2 - 1\Big|^q = 0
\end{aligned}$$

Now define $u_n = \rho_n e^{i\phi_n}$, with ρ_n, ϕ_n as in the preceding paragraphs. We then have:

$$\begin{aligned}
&\int_\Omega d_{t_m}(x) \left\{ \frac{1}{2}|\nabla u_n|^2 + \frac{\kappa_n^2}{4}\big(|u_n|^2 - 1\big)^2 \right\} dx \\
&\qquad = \int_\Omega d_{t_m}(x) \left\{ \frac{1}{2}\rho_n^2|\nabla \phi_n|^2 + \frac{1}{2}|\nabla \rho_n|^2 + \frac{\kappa^2}{4}\big(\rho_n^2 - 1\big)^2 \right\} dx \\
&\qquad \leq \frac{N_n^2}{2} \int_\Omega \frac{1}{d_{t_m}(x)} |\nabla \psi^n|^2 \, dx + C_0 N_n.
\end{aligned}$$

From (3.3.18) we then conclude that

$$\limsup_{n\to\infty} \frac{1}{(\log \kappa_n)^2} \int_\Omega d_{t_m}(x)\{\frac{1}{2}|\nabla u_n|^2+\frac{\kappa_n^2}{4}\big(|u_n|^2-1\big)^2\}dx$$

$$\leq \frac{1}{2}\int_\Omega d_{t_m}(x)\, d|\mu^{t_m}|(x) + \frac{1}{2}\int_\Omega d_{t_m}(x)|U^{t_m}|^2\, dx. \qquad (3.3.27)$$

Using (3.3.26) we also conclude that

$$\begin{aligned}
\frac{j(u_n)}{N_n} &= \frac{1}{N_n}(iu_n, \nabla u_n) \\
&= \frac{1}{N_n}(i\rho_n e^{i\phi_n}, \nabla(\rho_n e^{i\phi_n})) \\
&= \frac{1}{N_n}(i\rho_n e^{i\phi_n}, \nabla\rho_n e^{i\phi_n} + \rho_n i \nabla\phi_n e^{i\phi_n}) \\
&= \frac{1}{N_n}(i\rho_n, \nabla\rho_n + \rho_n i\nabla\phi_n) \\
&= \frac{1}{N_n}\big((i\rho_n, \nabla\rho_n) + (i\rho_n, i\rho_n\nabla\phi_n)\big)
\end{aligned}$$

the first part is zero since ρ_n is real-valued and using the fact that $U_n = \nabla\phi_n$, we have

$$\begin{aligned}
\frac{j(u_n)}{N_n} &= \frac{1}{N_n} - N_n\rho_n^2 U_n \\
&= -\frac{\rho_n^2}{d_{t_m}}\nabla^\perp\psi_n \\
&= -\frac{1}{d_{t_m}}\nabla^\perp\psi_n + \frac{(1-\rho_n^2)}{d_{t_m}}\nabla^\perp\psi_n \longrightarrow U^{t_m} \quad \text{in } L^p(\Omega) \text{ for all } p<2. \qquad (3.3.28)
\end{aligned}$$

Step 6. Putting everything together.

Write $j \in \mathcal{Z}$ as $j = U + V + W$ with $U \in \mathcal{U}$, $V \in \mathcal{V}$, and $W \in \mathcal{W}$. Let w_n be defined as in Step 1 and u_n as constructed in Step 5, and define $v_n = u_n w_n$. Since

$|w_n| = 1$, we have

$$\begin{aligned}
j(v_n) &= (iv_n, \nabla v_n) \\
&= (iu_n w_n, w_n \nabla u_n + u_n \nabla w_n) \\
&= (iu_n w_n, w_n \nabla u_n) + (iu_n w_n, u_n \nabla w_n) \\
&= |w_n|^2 (iu_n, \nabla u_n) + |u_n|^2 (iw_n, \nabla w_n) \\
&= j(u_n) + \rho_n^2 j(w_n) \longrightarrow U^{t_m} + V + W = j^{t_m} \qquad (3.3.29)
\end{aligned}$$

in $L^p(\Omega)$ for all $p < 2$.

To estimate the energy, we again use the fact that $|w_n| = 1$ to expand:

$$\frac{1}{N_n^2} \int_\Omega d(x) |\nabla v_n|^2 \, dx = \frac{1}{N_n^2} \int_\Omega d(x) \left\{ |\nabla u_n|^2 + \rho_n^2 |\nabla w_n|^2 + j(u_n) \cdot j(w_n) \right\} dx.$$

We claim that the last term is o(1). Indeed, from Step 1, $\frac{j(w_n)}{\log \kappa_n} = \nabla \Phi_n$, with $\Phi_n = \eta_n + \zeta \log \kappa_n$ and $\nabla \Phi_n \to V + W$ in C^1, and therefore,

$$\begin{aligned}
\frac{1}{N_n^2} \int_\Omega d(x) j(u_n) \cdot j(w_n) \, dx &= - \int_\Omega \nabla^\perp \psi_n \cdot \rho_n^2 \frac{d(x)}{d_{t_m}(x)} \nabla \Phi_n \, dx \\
&= - \int_\Omega \left[\nabla^\perp \psi_n \cdot \nabla \Phi_n - (1 - \frac{d(x)}{d_{t_m}(x)} \rho_n^2) \nabla^\perp \psi_n \cdot \nabla \Phi_n \right] dx \\
&= * + **
\end{aligned}$$

$*$ is zero by integration by parts and the definition of Φ_n. Now to calculate $**$ we use definition of ρ_n and split the integral,

$$\int_\Omega I := \int_\Omega \big(1 - \frac{d(x)}{d_{t_m}(x)} \rho_n^2\big) \nabla^\perp \psi_n \cdot \nabla \Phi_n dx = \int_{\Omega \setminus \cup_i B} I + \int_{\{\frac{1}{2\kappa_n} \le |x - p_i^n| \le \frac{1}{\kappa_n}\}} I$$

The second integral, let $B := \{\frac{1}{2\kappa_n} \le |x - p_i^n| \le \frac{1}{\kappa_n}\}$

$$
\begin{aligned}
\int_B (1 - \frac{d(x)}{d_{t_m}(x)}\rho_n^2)\nabla^\perp \psi_n \cdot \nabla \Phi_n dx &\le \|\nabla \phi_n\|_\infty \int_B \nabla^\perp \psi_n \\
&\le \|\nabla \phi_n\|_\infty \Big(\int_B |\nabla^\perp \psi_n|^2 \Big)^{\frac{1}{2}} |VolB| \\
&\le o(1)
\end{aligned}
$$

The first integral we use the uniform bound of $\nabla \Phi_n$ and we know that $a = d_{t_m}$ except for a small area around the inner boundaries then we use Hölder's inequality

$$
\begin{aligned}
&\int_{\Omega \setminus \cup_i B_i} (1 - \frac{d(x)}{d_{t_m}(x)})\nabla^\perp \psi_n \cdot \nabla \Phi_n dx \\
&\qquad \le \|\nabla \Phi_n\|_\infty \left(\int_{\Omega \setminus \cup_i B_i} |\nabla \psi_n|^2 dx \right)^{\frac{1}{2}} \Big(\int_{\Omega \setminus \cup_i B_i} (1 - \frac{d(x)}{d_{t_m}(x)})^2 dx \Big)^{\frac{1}{2}} \\
&\qquad \le \|\nabla \Phi_n\|_\infty \|\psi_n\|_{\mathbf{H}_0^1} \left(\int_{\{x \in \Omega; d(x) \le t_m\} \setminus \cup_i B_i} \Big(1 - \frac{d(x)}{d_{t_m}(x)}\Big)^2 dx \right)^{\frac{1}{2}} \\
&\qquad \le O(t_m)
\end{aligned}
$$

Next we calculate,

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{1}{N_n^2} G_\kappa(v_n; F_*) &= \limsup_{n\to\infty} \frac{1}{N_n^2} \int_\Omega d(x) \Big\{ \frac{1}{2}|\nabla u_n|^2 + \frac{1}{2}|\nabla w_n|^2 + j(u_n) \cdot j(w_n) \\
&\qquad - B_* \cdot j(v_n) + |B_*|^2 |v_n|^2 + \frac{\kappa_n^2}{4}(|u_n|^2 - 1)^2 + \frac{d(x)^2 |H'|^2}{24} \rho_n^2 \Big\} dx \\
&= \limsup_{n\to\infty} \frac{1}{N_n^2} \int_\Omega d(x) \Big\{ \frac{1}{2}|\nabla u_n|^2 + \frac{1}{2}|\nabla w_n|^2 + O(t_m) - B_* \cdot j(v_n) \\
&\qquad + |B_*|^2 |v_n|^2 + \frac{\kappa_n^2}{4}(|u_n|^2 - 1)^2 + \frac{d(x)^2 |H'|^2}{24} \rho_n^2 \Big\} dx
\end{aligned}
$$

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{N_n^2} G_\kappa(v_n; F_*) \le \limsup_{n\to\infty} \frac{1}{N_n^2} \int_\Omega & d_{t_m}(x) \Big\{ \frac{1}{2} |\nabla u_n|^2 + \frac{\kappa_n^2}{4} (|u_n|^2 - 1)^2 - B_* \cdot j(u_n) \\
& + |B_*|^2 |u_n|^2 + O(t_m) \} + d(x) \{ \frac{1}{2} |\nabla w_n|^2 - B_* \cdot j(w_n) \\
& + |B_*|^2 |w_n|^2 + \frac{d(x)^2 |H'|^2}{12} \rho_n^2 \Big\} dx \\
\le \frac{1}{2} \int_\Omega d_{t_m}(x) d|\mu^{t_m}| + & \frac{1}{2} \int_\Omega d_{t_m}(x) |U^{t_m} - B_*|^2 \, dx + O(t_m) \\
& + d(x) \frac{1}{2} \int_\Omega \Big\{ \big| (V+W) - B_* \big|^2 + \frac{d^2(x)|H'|^2}{12} \Big\} dx.
\end{aligned}$$

where B_* is defined as in (3.2.18), and $d(x) \le d_{t_m}(x)$.

Finally take the limit when $t_m \to 0$. The first term will converge to $\frac{1}{2}\int_\Omega d(x) d|\mu|$ by the uniform convergence of $d_{t_m} \to a$ in Ω and by the convergence of μ^{t_m} in sense of measure and in $H^{-1}(\Omega)$.

To deal with the second integral, by definition of U^{t_m}

$$U^{t_m} = -\frac{1}{d(x)} \nabla^\perp \psi^{t_m} = -\frac{1}{d_{t_m}(x)} \nabla^\perp \psi^{t_m}$$

where $d(x) = d_{t_m}(x) \;\; in \; K^{t_m}$, hence

$$\begin{aligned}
\lim_{t_m\to 0} \int_\Omega d(x) |U^{t_m}|^2 &= \lim_{t_m\to 0} \int_\Omega d(x) |\frac{1}{a} \nabla^\perp \psi^{t_m}|^2 \\
&= \lim_{t_m\to 0} \int_\Omega \frac{1}{d} |\nabla^\perp \psi^{t_m}|^2 \\
&= \int_\Omega \frac{1}{d} |\nabla^\perp \psi|^2 = \int_\Omega d(x) |U|^2
\end{aligned}$$

using the fact that ψ^{t_m} converges to ψ in $\mathbb{H}$.

$$\implies \frac{1}{2} \lim_{m\to\infty} \int_\Omega d_{t_m}(x) |U^{t_m} - B_*|^2 = \frac{1}{2} \lim_{m\to\infty} \int_\Omega d(x) |U^{t_m} - B_*|^2 = \frac{1}{2} \int_\Omega d(x) |U - B_*|^2$$

Back to the upper bound,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{N_n^2} G_\kappa(v_n; F_*) &\leq \frac{1}{2}\int_\Omega d(x) d|\mu| + \frac{1}{2}\int_\Omega d(x)\{|j|^2 - B_* \cdot j + |B_*|^2 + \frac{d^2(x)|H'|^2}{12}\}dx \\
&\leq \frac{1}{2}\int_\Omega d(x) d|\mu| + \frac{1}{2}\int_\Omega d(x)\{|j - B_*|^2 + \frac{d^2(x)|H'|^2}{12}\}dx \\
&= I_\infty(j; F_*) + \int_\Omega \frac{d^2(x)|H'|^2}{24}dx. \qquad (3.3.30)
\end{aligned}$$

with $F_* = \operatorname{curl} B_*$. As in [SS07] to get the upper bound in terms of the general $\mu \in H^{-1}(\Omega) \cap \mathfrak{M}(\Omega)$. A diagonal argument together with (3.3.30), yields a sequence $n_k \to \infty$, that we write in shorthand $\{n\}$, such that, writing $\{u_n, A_n\}$ instead of $\{u_{n_k}, A_{n_k}\}$, both (3.3.30) and (3.3.12) hold. This completes the proof of the Γ-convergence result. $\diamondsuit$

Chapter 4

Γ-Limit For a Ginzburg-Landau Functional with Normal Inclusions.

In this chapter we study the Γ-Limit for the Ginzburg Landau Functional for superconductors with normal inclusions. We will study the full energy with holes, by first decoupling the nergy using the decomposition of Lassoued and Mironescu which will put $a(x)$ in front of the gradient part of the energy. Then, we modify the method of Sandier and Serfaty [SS04] and use it to find the lower bound. Finally, we modify the Hodge decomposition we presented in Chapter 2 for the upper bound.

Let $\mathcal{D} \subset \mathbf{R}^2$ be a smooth simply-connected domain, $\psi \in \mathbf{H}^1(\mathcal{D}, \mathbf{C})$ the complex-valued order parameter, $A \in H^1(\mathcal{D}, \mathbf{R})$ the vector potential, $h = \operatorname{curl} A = \partial_x A_y - \partial_y A_x$, and h_{ex} is a constant applied field. We will study the energy

$$E_\epsilon(\psi, A) := \int_{\mathcal{D}} \left\{ \frac{1}{2}|(\nabla - iA)\psi|^2 + \frac{1}{4\epsilon^2}\left[\left(|\psi|^2 - a(x)\right)^2 - (a^-)^2 \right] + \frac{1}{2}(h - h_{ex})^2 \right\} dx, \tag{4.0.1}$$

Note that subtracting $(a^-)^2$ alters the usual inhomogeneous Ginzburg–Landau energy by a constant, which would give the highest order term $O(\epsilon^{-2})$ as $\epsilon \to 0$. Note that the energy density is still non-negative everywhere in the sample $\mathcal{D}$. We assume the following conditions on $a(x)$;

(H1) $a(x) \in C^2(\mathcal{D})$.

(H2) $\{x \in \overline{\mathcal{D}} : \; a(x) \leq 0\} = \overline{\cup_{j=1}^{n} \omega_j}$, with finitely many smooth, simply connected domains $\omega_j \subset\subset \mathcal{D}$.

(H3) $\nabla a(x) \neq 0$ for all $x \in \partial\omega_j$, $j = 1, \ldots, n$. More specific, $\exists \delta > 0$ s.t. there are non-negative constants $m_i, and\ M_i$ such that

$$m_i \leq \frac{a(x)}{\operatorname{dist}(x, \partial\omega_j)} \leq M_i.$$

for $\operatorname{dist}(x, \partial\omega_j) < \delta$.

(H4) we do not allow any isolated pinning points $a(x_0) = 0$, we admit only normal inclusions with nonempty interior.

We define

$$\Omega = \mathcal{D} \setminus \{x : \; a(x) \leq 0\} = \mathcal{D} \setminus \left(\bigcup_{j=1}^{n} \overline{\omega_j} \right).$$

Note that it follows from the above hypotheses that $a(x)$ is bounded away from zero on the exterior boundary $\partial\mathcal{D}$.

4.1 Preliminaries

4.1.1 Spaces, gauges, and equations

We define here the appropriate spaces and the Euler-Lagrange equations for the minimizers.

Definition 4.1.1 *We define the space $\breve{\mathbf{H}}^1_{div}(\mathbf{R}^2, \mathbf{R}^2)$ to be the closure of the space of the smooth, compactly supported divergence-free vector fields $F \in \mathbf{C}_0^\infty(\mathbf{R}^2, \mathbf{R}^2)$.*

Definition 4.1.2 *We say that $(\psi, A) \in \mathcal{H}$ if $\psi \in \mathbf{H}^1(\Omega, \mathbf{C})$ and $A \in \mathbf{H}^1(\mathcal{D}, \mathbf{R}^2)$ such that*

$$\operatorname{div} A = 0 \ \textit{in}\ \mathcal{D}, \quad A \cdot \nu = 0 \ \textit{on}\ \partial\mathcal{D}. \tag{4.1.2}$$

The functional E_ϵ is gauge-invariant: if $\varphi \in H^2(\mathcal{D}, \mathbf{R})$ any scalar potential, then $E_\epsilon(u \exp(i\varphi), A + \nabla\varphi) = E(u, A)$. The use of particular gauge, namely Coloumb gauge as in (4.1.2) eliminates the degenercy.

Minimizers of E_ϵ satisfy the Ginzburg-Landau Equations in Ω,

$$-\nabla_A^2\psi + \tfrac{1}{\epsilon^2}(|\psi|^2 - a(x))\psi = 0 \text{ in } \mathcal{D}; \tag{4.1.3}$$

$$-\nabla^\perp h = j := \langle i\psi, \nabla_A\psi\rangle \text{ in } \mathcal{D}; \tag{4.1.4}$$

$$h = h_{ex} \text{ on } \partial\mathcal{D}; \tag{4.1.5}$$

$$h = H_i(constant) \text{ in } \omega_j,\ j = 1, ..., m. \tag{4.1.6}$$

4.1.2 Decoupling the density profile

As $\epsilon \to 0$ we expect that the potential term in the energy E_ϵ will force $|\psi|^2 \to a^+(x)$. The hypotheses on $a(x)$ do not allow $\sqrt{a^+} \in H^1(\mathcal{D})$, and so this creates a singular boundary layer near the zero set of a. Here we study this boundary layer, so that it can be effectively removed from our energy calculations in the following sections.

Define a functional,

$$J_\epsilon(\eta) := \int_{\mathcal{D}} \left\{ \frac{1}{2}|\nabla \eta|^2 + \frac{1}{4\epsilon^2}\left[\left(\eta^2 - a(x)\right)^2 - (a^-)^2 \right] \right\} dx, \tag{4.1.7}$$

for real-valued functions $\eta \in H^1(\mathcal{D}; \mathbf{R})$. Critical points of J_ϵ solve the boundary-value problem

$$-\Delta\eta + \frac{1}{\epsilon^2}(\eta^2 - a(x))\eta = 0, \text{ in } \mathcal{D}; \quad \frac{\partial \eta}{\partial \nu} = 0, \text{ on } \partial\mathcal{D}. \tag{4.1.8}$$

Recall Proposition 2.1 in [AB05].

Proposition 4.1.3 *(Proposition 2.1 in [AB05]).*

Problem (4.1.8) admits a unique positive solution η_ϵ, which is the unique minimizer of J_ϵ in $H^1(\mathcal{D})$ up to a complex multiplier of modulus one. In addition,

(i) $0 < \eta_\epsilon(x) \le \max_{\mathcal{D}} a$, *and* $|\nabla \eta_\epsilon| \le C/\epsilon$;

(ii)

$$J_\epsilon(\eta_\epsilon) \le C|\log\epsilon| \quad \textit{and} \quad \eta_\epsilon \quad \textit{is bounded in} \quad L^\infty(\Omega). \tag{4.1.9}$$

(iii) There exists a constant C independent of ϵ so that

$$|\eta_\epsilon(x) - \sqrt{a^+(x)}| \le C\epsilon^{1/3}\sqrt{a^+(x)} \textit{ for every } x \in \Omega \textit{ with } \operatorname{dist}(x, \partial\Omega) \ge \epsilon^{1/3}; \tag{4.1.10}$$

(iv) For every $j = 1, \ldots, n$ and $x \in \omega_j$ with $\operatorname{dist}(x, \partial\omega_j) \ge \epsilon^{1/3}$,

$$0 < \eta_\epsilon(x) \le C\epsilon^{1/6} \exp\left[-\operatorname{dist}(x, \partial\omega_j)/\epsilon^{2/3}\right], \tag{4.1.11}$$

where $C > 0$ is a constant independent of ϵ.

In particular, (iv) implies that $\eta_\epsilon \to 0$ locally uniformly in the holes ω_j. The assertion (iv) implies that $|\eta_\epsilon^2(x) - a^+(x)|$ is small with respect to $a^+(x)$ itself provided we remain at a small distance ($\epsilon^{1/3}$) from the boundary of each $\partial\omega_j$.

In addition we recall the result of Proposition 2.3 of [AAB] which implies that the negativity of $a(x)$ in the normal regions ω_j acts more or less like an imposed Dirichlet condition:

Proposition 4.1.4 *Assume $|h_{ex}| \le C_0 |\ln \epsilon|$ for some constant $C_0 > 0$. Then, for any minimizer (ψ, A) of E_ϵ in $\mathbf{H}$,*

$$\int_\Omega (|\psi|^2 - a)^2 dx + \int_{\cup\omega_j} |\psi|^4 dx \le C_1 \epsilon^2 |\ln \epsilon|^2, \tag{4.1.12}$$

with constant C_1 depending on C_0. Moreover, $|\psi(x)| \to 0$ locally uniformly in $\cup_j \omega_j$ and

$$|\psi(x)| \le C\epsilon^{1/6} \exp\left[-\mathrm{dist}\,(x, \partial\omega_j)/\epsilon^{2/3}\right] \tag{4.1.13}$$

for all $x \in \omega_j$ with $\mathrm{dist}\,(x, \partial\omega_j) \ge \epsilon^{1/3}$ and $j = 1, \ldots, n$.

We define spaces we need for the energy decomposition as follow:

Definition 4.1.5 *We define the space $\mathbf{H}^1_{\eta_\epsilon^2}$ to be*

$$\mathbf{H}^1_{\eta_\epsilon^2} := \{u \in W^1(\Omega; \mathbf{C}) : \ \|u\|^2_{\mathbf{H}^1_{\eta_\epsilon^2}} := \int_\Omega \eta_\epsilon^2 \ |\nabla u|^2 dx < \infty\}$$

and the space $\mathbf{H}^1_a$ to be

$$\mathbf{H}^1_a := \{u \in W^1(\Omega; \mathbf{C}) : \ \|u\|^2_{\mathbf{H}^1_a} := \int_\Omega a |\nabla u|^2 dx < \infty\}.$$

We now apply the remarkable observation (see [LM]) that the energy of the profile η_ϵ and the remaining complex order parameter $u = \psi/\eta_\epsilon$ decouple exactly into two independent pieces. Recall lemma 2.3 in [AB05]

Lemma 4.1.6 *Let $(u, A) \in \mathcal{H}$. Then, $u = \psi/\eta_\epsilon$ is well defined, belongs to $H^1_{\eta_\epsilon^2}$, and*

$$E_\epsilon(\psi, A) = J_\epsilon(\eta_\epsilon) + F_\epsilon(u, A) \tag{4.1.14}$$

where

$$F_\epsilon(u, A) = \int_{\mathcal{D}} \left\{ \frac{\eta_\epsilon^2}{2} |\nabla_A u|^2 + \frac{\eta_\epsilon^4}{4\epsilon^2}(|u|^2 - 1)^2 + \frac{1}{2}(h - h_{ex})^2 \right\} \, dx. \tag{4.1.15}$$

Proof: Note that u is well defined in $\mathcal{D}$, since $\eta_\epsilon > 0$. The decomposition and the fact that $u \in H^1_{\eta_\epsilon^2}(\Omega)$ follow exactly as in Serfaty [S].

In Lemma 4.1.6, the energy is decomposed in the full domain $\mathcal{D}$ we will require the following Lemma to be able to reduce it to the multiply connected domain Ω. We recall Lemma B.1 from [M07].

Lemma 4.1.7 *There is $f \in W_0^{1,2}(\Omega)$, global minimizers of (4.1.7) in $W_0^{1,2}(\Omega)$. Furthermore, there is $\epsilon_1 > 0$ such that f is not constant when $\epsilon \in (0, \epsilon_1)$. In this case, one can choose them to satisfy the following properties:*

(1) $0 < f \leq \eta_\epsilon$ *in* Ω.

(2) For some $C > 0$, $\eta_\epsilon \leq \sqrt{a(x)} + C\epsilon^{\frac{1}{3}}$ *for all* $x \in \mathcal{D}$.

(3) For any $\alpha \in (0, \epsilon^{\frac{1}{3}})$, *there are constants* $C > 0$ *and* $0 < \epsilon_2 \leq \epsilon_1$ *such that, for*

any $0 < \epsilon \leq \epsilon_2$*, and all* $x \in \Omega$*, one has*

$$\sqrt{a(x)} - C\epsilon^{\alpha} \leq f(x).$$

(4) For any $\beta > 0$ *there are numbers* $\epsilon_0, \lambda_0 > 0$ *such that*

$$f \geq \lambda_0 a^{1+\beta}(x)$$

for all $x \in \Omega$ *and* $\epsilon \in (o, \epsilon_0)$*.*

(5) There is a constant $C > 0$ *independent of* $\epsilon > 0$ *such that, for* $0 < \epsilon \leq \epsilon_2$*, one has*

$$0 \leq J(f;\Omega) - J(\eta_\epsilon; \mathcal{D}) \leq C.$$

Proof: the proof follows exactly [M07].

Using Lemma 4.1.7 we reduce the decomposition to the multiply connected domain to use it in the proof of the upper bound. Indeed, let $f \in W_0^{1,2}(\Omega)$ be a real valued function defined as in Lemma 4.1.7. Define $\psi = fu$ in Ω, and $u = 0$ in $\mathcal{D} \setminus \Omega$. Then we conclude that

$$\begin{aligned} E_\epsilon(\psi, A) = &\int_\Omega \left\{ \frac{f^2}{2}|\nabla_A u|^2 + \frac{f^4}{4\epsilon^2}(|u|^2 - 1)^2 + \frac{1}{2}(h - h_{ex})^2 \right\} dx \\ &+ \int_\Omega \left\{ \frac{1}{2}|\nabla f|^2 + \frac{1}{4\epsilon^2}\left(f^2 - a(x)\right)^2 \right\} dx \\ = &F_\epsilon(u, A; \Omega) + J(f). \end{aligned} \tag{4.1.16}$$

We use the decomposition given in Lemma 4.1.6 to decouple the energy in order to be able to adapt our method we used in Chapter 2. Note that $J(\eta_\epsilon) \leq C|\log \epsilon|$ so when we divide by $|\log \epsilon|^2$ this term will tend to zero when $\epsilon \to 0$ and we left with the functional F_ϵ only.

Since $E_\epsilon(u_\epsilon, A_\epsilon) \leq C(\log \epsilon)^2$ and $F_\epsilon \leq E_\epsilon$, then

$$F_\epsilon(u_\epsilon, A_\epsilon : \mathcal{D}) \leq C(\log \epsilon)^2 \tag{4.1.17}$$

Define $\Omega_\epsilon = \{x \in \Omega | \text{dist}\,(x, \partial\Omega) > \epsilon^{\frac{1}{3}}\}$, then using (iii) from Proposition 4.1.3 we can approximate η_ϵ by a in $F_\epsilon(u_\epsilon, A_\epsilon)$ and get

$$\begin{aligned} F_\epsilon(u_\epsilon, A_\epsilon) \geq F^a_\epsilon := F_\epsilon(u_\epsilon, A_\epsilon)\big|_{\Omega_\epsilon} & \\ \geq \frac{1}{2}\int_{\Omega_\epsilon} (1-\epsilon^{\frac{1}{3}})^2 a(x)|\nabla_A u_\epsilon|^2 + \frac{(1-\epsilon^{\frac{1}{3}})^4 a^2(x)}{2\epsilon^2}(|u_\epsilon|^2-1)^2 dx + \frac{1}{2}\int_{\mathcal{D}} (h-h_{ex})^2 dx & \\ = \frac{(1-\epsilon^{\frac{1}{3}})^2}{2}\int_{\Omega_\epsilon} a(x)|\nabla_A u_\epsilon|^2 + \frac{(1-\epsilon^{\frac{1}{3}})^2 a^2(x)}{2\epsilon^2}(|u_\epsilon|^2-1)^2 dx + \frac{1}{2}\int_{\mathcal{D}} (h-h_{ex})^2 dx & \end{aligned} \tag{4.1.18}$$

where F^a_ϵ satisfies the bound (4.1.17).

For the compactness and the lower bound, we will use F^a_ϵ as in (4.1.18) instead of F_ϵ and by taking the limit $\epsilon \to 0$ we will get the desired result in the whole domain. For the upper bound and to match our lower bound we use the decomposition of E_ϵ in the multiply connected domain Ω which is given by (4.1.16).

Let,

$$\lim_{\epsilon \to 0} \frac{h_{ex}}{|\log \epsilon|} = \lambda \geq 0, \tag{4.1.19}$$

then

$$F_\epsilon(u_\epsilon, A_\epsilon) \leq E_\epsilon(u_\epsilon, A_\epsilon) \leq C|\log \epsilon|^2$$

With our choice of gauge (see Definition 2.1.1),

$$\left\|\frac{A_\epsilon}{|\log\epsilon|}\right\|_{L^\infty(\mathcal{D})} \le C. \tag{4.1.20}$$

From the energy bound (4.1.17) we may conclude that,

$$\left\|\frac{h_\epsilon}{|\log\epsilon|^2} - \lambda\right\|_{L^2(\mathcal{D})} \le C. \tag{4.1.21}$$

Hence there exist subsequences A_ϵ and h_ϵ which converge to A and h_* in $L^\infty(\Omega)$ and $L^2(\mathcal{D})$ respectively. We will use these subsequences later on.

4.2 Main results

For any regular complex-valued u, the current of u is defined as

$$ju = (iu, du) = \sum_{k=1}^{2}(iu, \partial_k u)dx_k, \tag{4.2.22}$$

where $(.,.)$ denotes the scalar product in $\mathbf{C}$ identified with $\mathbf{R}^2$ i.e. $(a,b) = \mathrm{Re}\,\bar{a}b$. It is related to the Jacobian determinant Ju of u through

$$Ju = \frac{1}{2}d(ju) = \frac{1}{2}d(iu, du), \tag{4.2.23}$$

where

$$Ju = \sum_{j<k}(i\partial_j u, \partial_k u)dx_j \wedge dx_k.$$

Define the space

$$\mathcal{Z} := \{j \in L^2(\Omega)\ s.t.\ J := \nabla \times j \in \mathfrak{M}(\Omega)\}, \tag{4.2.24}$$

where $\mathfrak{M}(\Omega, \mathbf{R}^3)$ is the space of vector-valued Radon measures on Ω.

We define the functional

$$F_\infty(j; A) = \begin{cases} \frac{1}{2}\|a(x)J\|_{\mathfrak{M}(\Omega)} + \frac{1}{2}\int_\Omega a(x)|j - A|^2 + \frac{1}{2}\int_{\mathcal{D}} |\nabla \times A - \lambda|^2 dx, & \text{if } j \in \mathcal{Z} \\ \infty & \text{otherwise} \end{cases} \tag{4.2.25}$$

Our main result is proving that F_ϵ Γ-converges to F_∞. We prove this in two steps: first, bounded sequences are compact and the limit is lower semicontinuous in the energies:

Theorem 4.2.1 *Let $\{u_\epsilon\}$ be a family such that $F_\epsilon \leq |\log \epsilon|^2$, then for fixed $\delta > 0$ up to extraction,*

$$\frac{Ju_\epsilon}{N_\epsilon} \rightharpoonup J_* \text{ measure-valued 2-form in } (C_c^{0,\gamma}(\Omega))', \gamma > 0$$

$$\left|\frac{ju_\epsilon}{\sqrt{N_\epsilon |\log \epsilon}}\right| \rightharpoonup j_*,\ in\ L^2(\Omega),$$

$$\liminf_{\epsilon \to 0} \frac{1}{N_\epsilon |\log \epsilon|} F_\epsilon(u_\epsilon, A_\epsilon) \geq 2|J_*|(\Omega) + \int_\Omega |j_* - A|^2 + \frac{1}{2}\int_{\mathcal{D}} |\nabla \times A - \lambda|^2 dx, \tag{4.2.26}$$

where ju_ϵ is defined in (4.2.22). Moreover, for a fixed $\delta > 0$ and Ω_δ defined as in (4.3.2) we have

$$\liminf_{\epsilon \to 0} \frac{1}{N_\epsilon |\log \epsilon|} F_\epsilon(u_\epsilon, A_\epsilon) \geq 2a|J_\delta|(\Omega_\delta) + \int_{\Omega_\delta} a|j_\delta - A|^2 + \frac{1}{2}\int_{\mathcal{D}} |\nabla \times A - \lambda|^2 dx, \tag{4.2.27}$$

where j_δ is the limit of ju_ϵ in Ω_δ.

The second part of the Γ-convergence result is the construction of recovery sequences:

Theorem 4.2.2 *Let $j \in \mathcal{Z}$ and consider any sequences ϵ_n such that $\epsilon_n \to 0$. Then there exists a sequence $\{(u_n, A_n)\} \subset \mathbf{H}^1(\Omega;\mathbf{C}) \times \mathbf{H}^1(\mathcal{D};\mathbf{R}^2)$, satisfying*

$$\frac{j_n}{|\log \epsilon_n|} \to j \textit{ in } L^p(\Omega), \textit{ for all } p < 2$$
$$\frac{J_n}{|\log \epsilon_n|} \to J := \frac{1}{2}\nabla \times j \textit{ weakly in } \mathfrak{M}(\Omega;\mathbf{R}^2), \textit{ and strongly in } (C_0^\gamma(\Omega))', 0 < \gamma < 1,$$

with $j_n := (iu_n, du_n)$ and $J_n := \frac{1}{2}dj_n$. Moreover

$$\limsup_{n\to\infty} \frac{1}{N_n^2} F_\epsilon \leq F_\infty(j, A). \tag{4.2.28}$$

4.3 Jacobian Estimate

In this section we modify the method of Sandier and Serfaty [SS04] to find a sharp Jacobian estimate in terms of the Ginzburg-Landau energy. In this section we will be working in the domain Ω_ϵ where we could use F_ϵ^a given by (4.1.18).

Let $M(\epsilon)$ be any function of ϵ satisfying

$$\forall\ \alpha > 0, \quad \lim_{\epsilon\to 0} \epsilon^\alpha M(\epsilon) = 0, \quad \lim_{\epsilon\to 0} \frac{|\log \epsilon|}{M(\epsilon)^\alpha} = 0, \quad \textit{and} \quad \log M(\epsilon) = o(|\log \epsilon|) \textit{ as } \epsilon \to 0. \tag{4.3.1}$$

For example $M(\epsilon) = \exp\sqrt{|\log \epsilon|}$ satisfies this. For any $\delta > 0$, we define the space Ω_δ by,

$$\Omega_\delta := \{x \in \Omega : \text{dist}\,(x, \partial\Omega) > \delta\}, \tag{4.3.2}$$

and Ω_ϵ by

$$\Omega_\epsilon := \{x \in \Omega : \mathrm{dist}\,(x, \partial\Omega) > \epsilon^{\frac{1}{3}}\}. \tag{4.3.3}$$

Theorem 4.3.1 *Let u_ϵ be a family of $\mathbf{H}^1(\Omega, \mathbf{C})$ such that*

$$F_\epsilon(u_\epsilon) \leq N_\epsilon |\log \epsilon| \ll M(\epsilon), \tag{4.3.4}$$

with $M(\epsilon)$ as in (4.3.1). Then, for any given $\delta > 0$ up to extraction of a subsequence

$$\frac{Ju_\epsilon}{N_\epsilon} \rightharpoonup J_* \quad in\ (C_c^{0,\gamma}(\Omega))',\ \forall 0 < \gamma \leq 1,$$

where J_* is a measure-valued 2-form. Moreover, for all continuous vector-fields X and Y compactly supported in Ω,

$$\frac{\sqrt{a}\big|X \cdot \nabla u_\epsilon\big|}{\sqrt{N_\epsilon |\log \epsilon|}}, \quad \frac{\sqrt{a}\big|Y \cdot \nabla u_\epsilon\big|}{\sqrt{N_\epsilon |\log \epsilon|}}$$

are bounded in L^2 and if we let ν_X, ν_Y be their defect measures, we have

$$\|\nu_X\|^{\frac{1}{2}} \|\nu_Y\|^{\frac{1}{2}} \geq \left| \int_\Omega J_*(X, Y) \right|. \tag{4.3.5}$$

Note that: in our thesis $N_\epsilon = |\log \epsilon|$ most of the time. We will recall some results needed to prove our Theorem and the proof will follow after.

4.3.1 Modified vortex-balls

We recall Proposition 5.2 from [AB05],

Proposition 4.3.2 *Assume (4.1.17). For any $C > 0$ there exist positive constants ϵ_0, C_0 so that for any (u, A) satisfying (4.1.17) there exists a finite collection $\{B_i = B(p_i, r_i)\} i = 1, ..., m$ of disjoint balls such that:*

$$[1.]\ \{x \in \Omega_\epsilon : |u| < 1 - (2/M(\epsilon))\} \subset \cup_{i=1}^m B_i; \tag{4.3.6}$$

$$[2.]\ \sum_{i=1}^m r_i \leq 1/M(\epsilon); \tag{4.3.7}$$

$[3.]$ *If* $B_i \subset \Omega_\epsilon$,

$$\int_{B_i} \frac{a(x)}{2} |\nabla_A u|^2 dx \geq \pi a(p_i)|d_i||\log\epsilon|\big(1 - o(1)\big) \quad \textit{for all } i. \tag{4.3.8}$$

where $d_i = \deg(u_\epsilon, \partial B_i)$. The o(1) appearing in the lower bound is a function that goes to zero with ϵ and which depends only on K. Moreover, letting

$$\mu_\epsilon = \pi \sum_{\{i|p_i \in \Omega_\epsilon\}} d_i \delta_{p_i}. \tag{4.3.9}$$

$$\| \star aJu_\epsilon - a\mu_\epsilon\|_{(C_0^{0,1}(\Omega_\epsilon))'} \leq C \frac{F_\epsilon^a(u_\epsilon)}{M(\epsilon)}. \tag{4.3.10}$$

where $\star$ is the Hodge star operator defined in (2.1.8).

Proof: The proof of the existence of these balls is exactly the same as [AB05] but we are going to state it for convenience.

Let $U_{\delta,t} := \{x \in \Omega_\delta : \ f(x) < 1 - t\}$, and $\gamma_t = \partial U_{\delta,t}$. Using the co-area formula as in [SS00], there exists $t_0 \in (0, |\log\epsilon|^{-4})$ and a finite set of balls $B_1, \ldots, B_k$ with radii $s_1, \ldots, s_k$ which cover γ_{t_0}, satisfying $\sum_i s_i \leq C\epsilon|\log\epsilon|^8$. In $\Omega_\delta \setminus U_{\delta,t_0}$ we have $f = |u| \geq 1 - t$, and we may write $u = f\, e^{i\phi}$ for a (possibly multi-valued) H^1_{loc} function $\phi(x)$.

We then let the balls grow continuously, using the process described in, [SS00], to obtain a lower bound in the expanding balls,

$$\int_{B_i \setminus U_{\delta,t_0}} \left[\frac{a}{2}|\nabla\phi|^2\right] \geq \pi \left(\min_{B_i} a\right) |d_i| \left(|\log\epsilon| - \overline{C}_0 o(|\log\epsilon|)\right),$$

with constant $\overline{C}_0$ independent of ϵ. Note that the minimum of $a(x)$ over B_i is non-increasing as the radii increase and as balls are merged (when they touch in the expansion process.) We terminate the process when the sum of the radii of the balls equals $|\log\epsilon|^{-12}$. By continuity of $a(x)$ we may then replace the minimum of a on each ball by the value at its center p_i, making an error which is small compared to $a(p_i)$ itself. This error can then be absorbed into the coefficient of $o(1)$. Finally,

$$\begin{aligned}
&\int_{B_i} \left[\frac{a}{2}|\nabla u|^2\right] \\
&\quad \geq \int_{B_j \setminus U_{\delta,t_0}} \left[\frac{a}{2}(1 + f^2 - 1)|\nabla\phi|^2\right] \\
&\quad \geq (1 - C|\log\epsilon|^{-4}) \int_{B_j \setminus U_{\delta,t_0}} \frac{a}{2}\left[|\nabla\phi|^2\right] \\
&\quad \geq (1 - C|\log\epsilon|^{-4}) \left(\pi a(p_i)|d_i|(|\log\epsilon| - \overline{C}_0 o(|\log\epsilon|))\right) \\
&\quad \geq \pi a(p_i)|d_i|(|\log\epsilon| - C_0 o(|\log\epsilon|)),
\end{aligned}$$

for constant C_0 independent of ϵ, which completes the sketch of the proof of the first part of the Proposition.

To prove (4.3.10), we first consider $\chi : \mathbf{R}_+ \to \mathbf{R}_+$ as follows

$$\begin{cases} \chi(x) = x & if\ |x-1| \geq \frac{1}{2} \\ \chi(x) = 1 & if\ |x-1| \leq 1 - M(\epsilon)^{-1} \\ \chi & \text{is continuous and piecewise affine otherwise.} \end{cases}$$

We then define

$$\tilde{u}_\epsilon = \chi(|u_\epsilon|)\frac{u_\epsilon}{|u_\epsilon|}.$$

It is easy to check that $\|u_\epsilon - \tilde{u}_\epsilon\|_{L^\infty(\Omega)} \leq C/M(\epsilon)$ and defining ju_ϵ *and* $j\tilde{u}_\epsilon$ as in (4.2.22),

$$\|\sqrt{a}(ju_\epsilon - j\tilde{u}_\epsilon)\|^2_{L^2(\Omega_\epsilon)} \leq CM(\epsilon)^{-2}F^a_\epsilon(u_\epsilon),$$

where $|\alpha dx + \beta dy|^2 = \alpha^2 + \beta^2$. It follows that for any smooth compactly supported function ξ

$$\begin{aligned} \left|\int_{\Omega_\epsilon} a\xi(Ju_\epsilon - J\tilde{u}_\epsilon)\right| &= \frac{1}{2}\left|\int_{\Omega_\epsilon}(ju_\epsilon - j\tilde{u}_\epsilon)\wedge d(a\xi)\right| \\ &\leq \frac{1}{2}\left|\int_{\Omega_\epsilon} a(ju_\epsilon - j\tilde{u}_\epsilon)\wedge d(\xi)\right| + \frac{1}{2}\left|\int_{\Omega_\epsilon}\xi(ju_\epsilon - j\tilde{u}_\epsilon)\wedge d(a)\right| \\ &= (1) + (2) \end{aligned}$$

Since $\int_{\Omega_\epsilon} a|\nabla u_\epsilon|^2 \leq \eta^a_\epsilon(u_\epsilon)$, then

$$(1) = \frac{1}{2}\left|\int_{\Omega_\epsilon} a(ju_\epsilon - j\tilde{u}_\epsilon)\wedge d(\xi)\right| \leq CM(\epsilon)^{-1}\sqrt{F^a_\epsilon(u_\epsilon)}\|\xi\|_{C^{0,1}_0(\Omega_\epsilon)}.$$

To estimate (2), let $\rho(x) := \operatorname{dist}(x, \partial\Omega)$ and in Ω_ϵ you can always write $a = \rho(x)b(x)$ where $0 < b_0 < b(x)$ *and* $\nabla\rho = 1$ because of (H3). For some fixed $\delta > 0$, rewrite (2) as

$$\int_{\Omega_\epsilon} \xi(ju_\epsilon - j\tilde{u}_\epsilon) \wedge d(a) = \int_{\epsilon^{\frac{1}{3}} < \rho < \delta} \xi(ju_\epsilon - j\tilde{u}_\epsilon) \wedge d(a) + \int_{\delta \le \rho} \xi(ju_\epsilon - j\tilde{u}_\epsilon) \wedge d(a)$$

$$= (i) + (ii).$$

Since $a(x)$ is bounded below in $\{x \in \Omega |\ \rho \ge \delta\}$, then

$$(ii) = \int_{\delta \le \rho} \xi(ju_\epsilon - j\tilde{u}_\epsilon) \wedge d(a) \le CM(\epsilon)^{-1} \sqrt{F_\epsilon^a(u_\epsilon)} \|\xi\|_{C_0^{0,1}(\Omega_\epsilon)}.$$

For (i), we write

$$\left| \int_{\epsilon^{\frac{1}{3}} < \rho < \delta} \xi(ju_\epsilon - j\tilde{u}_\epsilon) \wedge d(a) \right| = \left| \int_{\epsilon^{\frac{1}{3}} < \rho < \delta} \frac{1}{\sqrt{a}} \xi \sqrt{a} (ju_\epsilon - j\tilde{u}_\epsilon) \wedge d(a) \right|$$

$$\le \Big(\int_{\epsilon^{\frac{1}{3}} < \rho < \delta} (\frac{1}{\sqrt{a}})^2 \Big)^{\frac{1}{2}} \left(\int_{\epsilon^{\frac{1}{3}} < \rho < \delta} \xi^2 a \Big| (ju_\epsilon - j\tilde{u}_\epsilon) \wedge d(a) \Big|^2 \right)^{\frac{1}{2}}$$

$$\le |\log \epsilon| \frac{\sqrt{F_\epsilon^a(u_\epsilon)}}{M(\epsilon)} \|\xi\|_{C_0^{0,1}(\Omega_\epsilon)} \|\nabla a\|_{L^2(\Omega_\epsilon)}.$$

Indeed,

$$\left| \int_{\epsilon^{\frac{1}{3}} < \rho < \delta} (\frac{1}{\sqrt{a}})^2 dx \right| = \left| \int_{\epsilon^{\frac{1}{3}} < \rho < \delta} \frac{1}{b\rho} dx \right| \le C \left| \int_{\epsilon^{\frac{1}{3}} < \rho < \delta} \frac{1}{\rho} dx \right|$$

In the simplest case that $\Omega_\epsilon = B_{r_i}(x_i)$ we use polar coordinates

$$\begin{aligned}\left|\int_{\epsilon^{\frac{1}{3}}<\rho<\delta} \frac{1}{\rho}dx\right| &= \left|2\pi\int_{r_i+\epsilon^{\frac{1}{3}}}^{\delta} \frac{1}{r-r_i} r\,dr\right| \\ &\le 2\pi\delta\log(r-r_i)\Big|_{r_i+\epsilon^{\frac{1}{3}}}^{\delta} \\ &= 2\pi\delta\Big(\ln(\delta-r_i)-\ln\epsilon^{\frac{1}{3}}\Big) \\ &\le C|\log\epsilon|.\end{aligned}$$

In general, near the boundary, one can change coordinate frame to a frame with tangential and normal components to $\partial\Omega$, in this coordinate $\rho(x_1, x_2) = x_2$ which implies that $\int \frac{1}{x_2} = \log x_2$.

So $(i) \le \frac{|\log\epsilon|\sqrt{F_\epsilon^a(u_\epsilon)}}{M(\epsilon)}\|\xi\|$, but by hypotheses $\frac{|\log\epsilon|^2}{M(\epsilon)} \xrightarrow[\epsilon\to 0]{} 0$. Therefore we have,

$$\left|\int_{\Omega_\epsilon} a(Ju_\epsilon - J\tilde{u}_\epsilon)\xi\right| \le CM(\epsilon)^{-1}\sqrt{F_\epsilon^a(u_\epsilon)}\|\xi\|_{C_0^{0,1}(\Omega_\epsilon)}.$$

and therefore

$$\|\star aJu_\epsilon - \star aJ\tilde{u}_\epsilon\|_{(C_0^{0,1})'} \le C\frac{\sqrt{F_\epsilon^a(u_\epsilon)}}{M(\epsilon)}. \tag{4.3.11}$$

We wish to estimate the measure norm of $\star aJ\tilde{u}_\epsilon - a\mu_\epsilon$. Let ξ be a smooth compactly supported function. Since $|\tilde{u}_\epsilon| = 1$ outside of $\Omega_\epsilon \cap (\cup_i B_i)$ we have $J\tilde{u}_\epsilon = 0$ there. Therefore

$$\int_{\Omega_\epsilon} \xi aJ\tilde{u}_\epsilon = \sum_{B_i \not\subset \Omega_\epsilon}\int_{B_i\cap\Omega_\epsilon} \xi aJ\tilde{u}_\epsilon + \sum_{B_i\subset\Omega_\epsilon}\int_{B_i}\xi aJ\tilde{u}_\epsilon = I_1 + I_2 \tag{4.3.12}$$

From the definition of Ω_ϵ and since the Euclidean radius of any ball is less than $M(\epsilon)^{-1}$ it follows that if $B_i \not\subset \Omega_\epsilon$ and $x \in B_i \cap \Omega_\epsilon$ then $|\xi(x)| = \frac{|\xi(x)|}{d(x,\partial\Omega_\epsilon)} d(x, \partial\Omega_\epsilon) \leq \frac{\|\xi\|_{C_0^{0,1}(\Omega)}}{M(\epsilon)}$. It follows that

$$I_1 \leq C \frac{F_\epsilon^a(u_\epsilon)}{M(\epsilon)} \|\xi\|_{C_0^{0,1}}. \tag{4.3.13}$$

To deal with the second integral we define $\bar{\xi}$ to be equal to $\xi(p_i)$ on $B_i = B(p_i, r_i) \subset \Omega_\epsilon$ and $\bar{\xi} = 0$ elsewhere. Then letting U be the union of the B_i's which are included in Ω_ϵ, we have $|\xi - \bar{\xi}| \leq \|\xi\|_{C_0^{0,1}}/M(\epsilon)$ on U while

$$\begin{aligned} \int_U \bar{\xi} a J \tilde{u}_\epsilon &= \sum_{B_i \subset \Omega_\epsilon} \xi(p_i) \int_{B_i} a J \tilde{u}_\epsilon \\ &= \sum_{B_i \subset \Omega_\epsilon} \xi(p_i) \int_{B_i} a J u_\epsilon \\ &= \sum_{B_i \subset \Omega_\epsilon} \pi a(p_i) d_i \xi(p_i) = \int \xi a d\mu_\epsilon, \end{aligned}$$

where we used the fact that $|\tilde{u}_\epsilon| = 1$ *on* ∂B_i. Therefore

$$\left| I_2 - \int \xi a d\mu_\epsilon \right| \leq C \frac{F_\epsilon^a(u_\epsilon)}{M(\epsilon)} \|\xi\|_{C^{0,1}(\Omega_\epsilon)}. \tag{4.3.14}$$

It follows from (4.3.11), (4.3.12), (4.3.13), and (4.3.14) that for any compactly supported smooth ξ with support on Ω_ϵ

$$\left| \int \xi a J u_\epsilon - \int \xi a d\mu_\epsilon \right| \leq C \frac{F_\epsilon^a(u_\epsilon)}{M(\epsilon)} \|\xi\|_{C^{0,1}}.$$

This conclude the proof (4.3.10).

$\diamondsuit$

4.3.2 Proof of Theorem 4.3.1

Let X, Y be continuous vector field compactly supported in Ω. It follows from (4.3.4) that

$$\sqrt{a} j_{\epsilon,X} = \frac{\sqrt{a}\big|X \cdot \nabla u_\epsilon\big|}{\sqrt{N_\epsilon |\log \epsilon|}}, \qquad \sqrt{a} j_{\epsilon,Y} = \frac{\sqrt{a}\big|Y \cdot \nabla u_\epsilon\big|}{\sqrt{N_\epsilon |\log \epsilon|}} \tag{4.3.15}$$

are bounded in L^2 and therefore converge weakly subsequentially. Using Proposition 4.3.2, there exists a collection of balls $\{B_i\}_i$ satisfying the properties (4.3.6), (4.3.7), and (4.3.8). Then it follows from (4.3.8) that

$$\frac{1}{2|\log \epsilon|} \int_{B_i} a|X \cdot \nabla u_\epsilon|^2 + a|Y \cdot \nabla u_\epsilon|^2 \frac{dx\ dy}{|X \wedge Y|} \geq \pi |d_i| a(p_i)\big(1 - o(1)\big). \tag{4.3.16}$$

By the definition of μ_ϵ (4.3.9), $a(x)$ is near $a(p_i) - o(1)$ and summing over i, we have

$$\frac{1}{2|\log \epsilon|} \int_{\cup_i B_i} a|X \cdot \nabla u_\epsilon|^2 + a|Y \cdot \nabla u_\epsilon|^2 \frac{dx\ dy}{|X \wedge Y|} \geq \Big| \int_{\Omega_\epsilon} a(1 - o(1)) d\mu_\epsilon \Big|, \tag{4.3.17}$$

Dividing the above inequality by N_ϵ we find

$$\liminf_{\epsilon \to 0} \frac{1}{2N_\epsilon |\log \epsilon|} \int_{\cup_i B_i} a|X \cdot \nabla u_\epsilon|^2 + a|Y \cdot \nabla u_\epsilon|^2 \geq |X \wedge Y| \Big| \int_\Omega d\mu_* \Big|, \tag{4.3.18}$$

Note that: μ_ϵ converges strongly μ_* in $(C^{0,\gamma}(\Omega))'$ in Ω, it follows that the rescaled Jacobian subsequentailly converges in $(C^{0,\gamma}(\Omega))'$ to the same limit as μ_ϵ i.e to a measure we call it J_*. But to see $a(x)$ in the limit we have to be away from the boundary by $\delta > 0$. (i.e. $\frac{aJu_\epsilon}{N_\epsilon} \rightharpoonup aJ_\delta$ in Ω_δ). This note implies that

$$\liminf_{\epsilon \to 0} \frac{1}{2N_\epsilon |\log \epsilon|} \int_{\cup_i B_i} a|X \cdot \nabla u_\epsilon|^2 + a|Y \cdot \nabla u_\epsilon|^2 \geq |X \wedge Y| \Big| \int_{\Omega_\delta} aJ_\delta(\partial_x, \partial_y) \Big|, \tag{4.3.19}$$

where J_δ is the limit of $N_\epsilon^{-1} J u_\epsilon$ in Ω_δ. Note that

$$J_\delta(X,Y) = |X \wedge Y| J_\delta(\partial_x, \partial_y).$$

Using (4.3.15) we fix a convergent subsequence, and let $j_{\delta X}$, $j_{\delta Y}$ denote the weak L^2 limits of $\frac{|X \cdot \nabla u_\epsilon|}{|\log \epsilon|}$ and $\frac{|Y \cdot \nabla u_\epsilon|}{|\log \epsilon|}$ respectively. Then

$$a|j_{\epsilon,X}|^2 \rightharpoonup a|j_{\delta X}|^2 + \nu_X, \qquad a|j_{\epsilon,Y}|^2 \rightharpoonup a|j_{\delta Y}|^2 + \nu_Y, \tag{4.3.20}$$

weakly as measures, where ν_X *and* ν_Y are positive Radon measures, called the defect measures of the sequence. We claim that

$$\liminf_{\epsilon \to 0} \frac{1}{2N_\epsilon |\log \epsilon|} \int_{\cup_i^m B_i} a|X \cdot \nabla u_\epsilon|^2 + a|Y \cdot \nabla u_\epsilon|^2 \le \frac{1}{2}\big(\|\nu_X\| + \|\nu_Y\|\big) \tag{4.3.21}$$

Using (4.3.19) we find

$$\frac{1}{2}\big(\|\nu_X\| + \|\nu_Y\|\big) \ge \Big| \int_{\Omega_\delta} a J_\delta(X,Y) \Big|. \tag{4.3.22}$$

Also,

$$\frac{1}{2}\big(\|\nu_X\| + \|\nu_Y\|\big) \ge \Big| \int_{\Omega} J_*(X,Y) \Big|. \tag{4.3.23}$$

4.4 Lower bound

In this section we find the lower bound of the energy F_ϵ (Theorem 4.2.1) which is the first part of the Γ-convergent.

Note that: From Theorem 4.2.1 we do have the lower bound (4.2.26) up to the inner boundary (in the whole Ω) but $a(x)$ will not be shown in the inequality. To see $a(x)$ in the lower bound we have to be away from the inner boundaries by δ where $\delta > 0$ as in (4.2.27).

Proof of Theorem 4.2.1: The first part follows from Theorem 4.3.1. Since $\frac{\eta_\epsilon^2}{2}|\nabla_A u|^2$ and $\frac{\eta_\epsilon^4}{4\epsilon^2}(|u|^2-1)^2$ are positive quantities, we have

$$\begin{aligned}
F_\epsilon(u,A) &= \int_{\mathcal{D}} \left\{ \frac{\eta_\epsilon^2}{2}|\nabla_A u|^2 + \frac{\eta_\epsilon^4}{4\epsilon^2}(|u|^2-1)^2 + \frac{1}{2}(h-h_{ex})^2 \right\} \, dx \\
&\geq \left\{ \int_\Omega \frac{\eta_\epsilon^2}{2}|\nabla_A u|^2 + \frac{\eta_\epsilon^4}{4\epsilon^2}(|u|^2-1)^2 + \frac{1}{2}\int_{\mathcal{D}}(h-h_{ex})^2 \right\} \, dx \\
&\geq \frac{(1-\epsilon^{\frac{1}{3}})^2}{2}\int_{\Omega_\epsilon} a(x)|\nabla_A u_\epsilon|^2 + \frac{(1-\epsilon^{\frac{1}{3}})^2 a^2(x)}{2\epsilon^2}(|u_\epsilon|^2-1)^2 dx \\
&\qquad\qquad\qquad + \frac{1}{2}\int_{\mathcal{D}}(h-h_{ex})^2 dx \\
&= F_\epsilon^a(u,A). \qquad (4.4.1)
\end{aligned}$$

Hence it is enough to prove the lower bound of the energy F_ϵ^a. We prove the second part in two steps:

Step 1 We first prove the lower bound of the gradient part of the energy

$$\begin{aligned}
\liminf_{\epsilon\to 0} \frac{1}{2|\log\epsilon|^2}\int_\Omega |\nabla u_\epsilon|^2 &\geq \liminf_{\epsilon\to 0} \frac{1}{2|\log\epsilon|^2}\int_{\Omega_\epsilon} a|\nabla u_\epsilon|^2 \\
&\geq 2a|J_\delta|(\Omega_\delta) + \int_{\Omega_\delta} a|j_\delta|^2. \qquad (4.4.2)
\end{aligned}$$

and we do this by using the compactness result and the Jacobian estimate (4.3.23).

Choose e_1, e_2 an orthonormal (moving) frame that may depend on $x \in \Omega_\epsilon$ and

$f, g \in C_c^0(\Omega)$ with $|f| \leq 1$ and $|g| \leq 1$. Then, let $X_1 = fe_1, X_2 = ge_2$. The inequality

$$|\nabla u_\epsilon|^2 \geq \sum_{i=1}^{n} |X_i \cdot \nabla u_\epsilon|^2 \tag{4.4.3}$$

holds. Since $|X_i \cdot ju_\epsilon| \leq |X_i \cdot \nabla u_\epsilon||u_\epsilon|$, we have

$$\eta_\epsilon^2(x)\Big(|X_i \cdot ju_\epsilon| - |X_i \cdot \nabla u_\epsilon|\Big) \leq \eta_\epsilon^2(|u_\epsilon| - 1)|X_i \cdot \nabla u_\epsilon|.$$

Using the bound on η_ϵ^2 and $|u_\epsilon| \leq C$, we infer directly that $\frac{\eta_\epsilon^2 ju_\epsilon}{\sqrt{N_\epsilon|\log \epsilon}}$ is bounded in $L^2(\Omega)$, hence weakly compact, and that

$$\frac{\eta_\epsilon^2(x)\Big(|X_i \cdot ju_\epsilon| - |X_i \cdot \nabla u_\epsilon|\Big)_+}{\sqrt{N_\epsilon|\log \epsilon|}} \to 0 \tag{4.4.4}$$

as $\epsilon \to 0$ in $L^1(\Omega)$.

Note that in Ω_ϵ we could approximate η^2 by a, it follows that denoting by ϕ_{X_i} the weak L^2 limit of

$$\frac{\sqrt{a}|X_i \cdot \nabla u_\epsilon|}{\sqrt{N_\epsilon|\log \epsilon|}},$$

we have $\sqrt{a}|X_i \cdot j_\delta| \leq \phi_{X_i}$ almost everywhere, where j_δ is the restriction of the weak limit j of the normalized currents in Ω_δ. Denoting by $\nu_{X_1},$ *and* ν_{X_2} the defect measures of

$$\frac{\sqrt{a}|X_1 \cdot \nabla u_\epsilon|}{\sqrt{N_\epsilon|\log \epsilon|}}, \ and \ \frac{\sqrt{a}|X_2 \cdot \nabla u_\epsilon|}{\sqrt{N_\epsilon|\log \epsilon|}}$$

respectively, it follows from (4.4.3) and the very definition of defect measure that

$$\liminf_{\epsilon \to 0} \frac{1}{N_\epsilon|\log \epsilon|} \int_{\Omega_\epsilon} a|\nabla u_\epsilon|^2 \geq \|\nu_{X_1}\| + \|\nu_{X_2}\| + \int_{\Omega_\epsilon} |\phi_{X_1}|^2 + |\phi_{X_2}|,$$

thus using Theorem 4.3.1 and the above, we are led to

$$\begin{aligned}\liminf_{\epsilon\to 0}\frac{1}{N_\epsilon|\log\epsilon|}\int_{\Omega_\epsilon} a|\nabla u_\epsilon|^2 &\geq 2\left|\int_{\Omega_\delta} aJ_\delta(X_1,X_2)\right| + \liminf_{\epsilon\to 0}\int_{\Omega_\delta} a|X_1\cdot ju_\epsilon|^2 + a|X_2\cdot ju_\epsilon|^2\\ &\geq 2\left|\int_{\Omega_\delta} aJ_\delta(X_1,X_2)\right| + \int_{\Omega_\delta} a|j_\delta|^2 + \int_{\Omega_\delta} a(|f|^2-1)|j_\delta\cdot e_1|^2\\ &\quad + \int_{\Omega_\delta} a(|g|^2-1)|j_\delta\cdot e_2|^2.\end{aligned}$$

Taking the supremum over all such frames $e_1,...,e_n$ and all compactly supported $|f|\leq 1,\ |g|\leq 1$ proves the Theorem.

Step 2 Now to complete the proof of the lower bound for the functional F_ϵ (i.e. E_ϵ), we will combine (4.4.2) and the compactness of h_ϵ and A_ϵ given by (4.1.21) and (4.1.20). For any sequence (u_ϵ, A_ϵ) s.t. (4.1.17) holds where (A_ϵ) and (h_ϵ) are the subsequences defined in (4.1.20) and (4.1.21), we have

$$\begin{aligned}\liminf_{\epsilon\to 0}\frac{1}{N_\epsilon|\log\epsilon|}F_\epsilon &\geq \liminf_{\epsilon\to 0}\frac{1}{N_\epsilon|\log\epsilon|}F_\epsilon^a\\ &\geq \liminf_{\epsilon\to 0}\frac{1}{N_\epsilon|\log\epsilon|}\frac{(\epsilon^{\frac{1}{3}}-1)^2}{2}\int_{\Omega_\epsilon} a\Big(|\nabla u_\epsilon|^2 - 2A_\epsilon\cdot ju_\epsilon + |A_\epsilon|^2|u_\epsilon|^2\Big)\\ &\quad+\liminf_{\epsilon\to 0}\frac{1}{N_\epsilon|\log\epsilon|}\frac{1}{2}\int_{\mathcal{D}}(h-h_{ex})^2dx\end{aligned}$$

by (4.3.22), (4.4.4), and (4.1.21), we get

$$\begin{aligned}&\geq 2\left|\int_{\Omega_\delta} aJ_\delta(X_1,X_2)\right| + \int_{\Omega_\delta} a|j_\delta|^2 + \liminf_{\epsilon\to 0}\frac{(\epsilon^{\frac{1}{3}}-1)^2}{2}\int_{\Omega_\epsilon} a\Big(-2A_\epsilon\cdot ju_\epsilon\\ &\quad+|A_\epsilon|^2|u_\epsilon|^2\Big)dx + \frac{1}{2}\int_{\mathcal{D}}(h_*-\lambda)^2dx\end{aligned}$$

By (4.1.20), we get

$$\geq 2\left|\int_{\Omega_\delta} aJ_\delta(X_1, X_2)\right| + \int_{\Omega_\delta} a|j_\delta|^2 + \int_{\Omega_\delta} a\Big(- A\cdot j_\delta + |A|^2\Big)dx$$

$$+\frac{1}{2}\int_{\mathcal{D}}(h_* - \lambda)^2 dx$$

$$\geq 2\left|\int_{\Omega_\delta} aJ_\delta(X_1, X_2)\right| + \int_{\Omega_\delta} a|j_\delta - A|^2 dx + \frac{1}{2}\int_{\mathcal{D}}(h_* - \lambda)^2 dx \tag{4.4.5}$$

We do have the lower bound (4.4.5) up to the inner boundary (in the whole Ω) but $a(x)$ will not be shown in the inequality. i.e.

$$\begin{aligned}\liminf_{\epsilon\to 0}\frac{1}{N_\epsilon|\log\epsilon|}E_\epsilon &= \liminf_{\epsilon\to 0}\frac{1}{N_\epsilon|\log\epsilon|}F_\epsilon \\ &\geq 2\int_\Omega d|\mu_*| + \int_\Omega |j_*|^2 dx + \frac{1}{2}\int_{\mathcal{D}}(h_* - \lambda)^2 dx \end{aligned}\tag{4.4.6}$$

4.5 Upper bound

In this section we find the upper bound for the functional $F_\epsilon(u, A)$.To match it with the lower bound we found earlier we use Lemma 4.1.7 where E_ϵ is decomposed into F_ϵ and $J(f_\epsilon)$ in the multiply connected domain Ω. We use the Hodge decomposition introduced in Chapter 2 and 3. The main difficulty is $a(x)$ near the inner holes. We will apply the Hodge decomposition with $a(x)$ in the full domain Ω. For the subspace $\mathcal{U}$ which contributes the Jacobian, we use one of the advantages of the space $\mathbb{H}$ where any function $\psi \in \mathbb{H}$ can be approximated by compactly supported smooth functions ψ^{t_m}. We apply the same steps of Theorem 3.1.4 in Chapter 3 on those functions. We will get the upper bound of the energy with $U^{t_m} = -\frac{1}{a}\nabla\psi^{t_m}$. Taking $m \to \infty$ with some regularity results we claim the upper bound in the whole domain Ω.

Proof of Theorem 4.2.2. We have used f defined in Lemma 4.1.7 to reduce to the multiply-connected domain Ω and since $f \leq \eta_\epsilon$ in Ω we have

$$\begin{aligned} F_\epsilon(u, A : \Omega) &= \int_\Omega \frac{f_\epsilon^2}{2}|\nabla_A u|^2 + \frac{f_\epsilon^4}{4\epsilon^2}(|u|^2-1)^2 dx + \int_{\mathcal{D}} |h - h_{ex}|^2 dx \\ &\leq \int_\Omega \frac{\eta_\epsilon^2}{2}|\nabla_A u|^2 + \frac{\eta_\epsilon^4}{4\epsilon^2}(|u|^2-1)^2 dx + \int_{\mathcal{D}} |h - h_{ex}|^2 dx. \end{aligned}$$

which allows us to work with η_ϵ instead of f.

For any $j \in \mathcal{Z}$, by using the Hodge decomposition introduced by Lemma 3.3.2 in Chapter 3, j can be written as

$$j = U + V + W = -\frac{1}{a}\nabla^\perp\psi + \nabla\zeta + W. \tag{4.5.1}$$

where

$$\begin{aligned} &U \in \mathcal{U} := \{-\frac{1}{a}\nabla^\perp\psi,\ \psi \in \mathbb{H}(\Omega; \mathbf{R})\} = \overline{\{-\frac{1}{a}\nabla^\perp\psi,\ \psi \in C_0^\infty\}}_{\mathbb{H}}, \\ &V \in \mathcal{V} := \{\nabla\zeta,\ \zeta \in \mathbf{H}^1(\Omega; \mathbf{R})\}, \\ &W \in \mathcal{W} = \{W \in C^1(\Omega; \mathbf{R}^2),\ \nabla^\perp \cdot W = 0,\ \nabla\cdot(aW) = 0,\ W\cdot\nu = 0 \ on\ \partial\Omega\}. \end{aligned} \tag{4.5.2}$$

Since $\nabla \times (V + W) \equiv 0$ then $V + W$ doesn't contribute to the weak Jacobian. We need to construct sequences w_ϵ and u_ϵ which converge to $V + W$ and U consequently. As in [JS] we may associate to V, W an S^1-valued map w_ϵ. The singular part of the Jacobian is contained in U; for this part we construct a family u_ϵ with points vortices via an appropriate Green's function. Putting these two parts together, The desired recovery sequence will have the form $v_n = u_{\epsilon_n} w_{\epsilon_n}$.

Recovering $V+W$. Constructing the sequence w_{ϵ_n} is straight forward and it follows exactly Step 1 of the upper bound in Chapter 3.

From Lemma 3.3.2, we can write $V = \nabla\zeta,\ \zeta \in \mathbf{H}^1(\Omega)$ and $W = \frac{1}{a}\nabla^{\perp}\xi,\ with\ \xi(x) = \sum_{i=1}^{m} \Phi_i \xi_i(x),\ \ for\ \xi_i\ as\ in$ (3.3.4) *with* Φ_i *real constants*. Let $M_{i,n} = [\Phi_i \ln \epsilon_n],\ i = 1, ..., m$, where brackets denote the integer part, Set

$$\Xi_n := \sum_{i=1}^{m} M_{i,n}\xi_i, \qquad W_n = -\frac{1}{a}\nabla^{\perp}\Xi_n.$$

We note that

$$\|W_n - W \ln \epsilon_n\|_{C^1} \leq C, \tag{4.5.3}$$

for constant C depending on W (but independent of n.)

Since

$$\begin{aligned} \operatorname{curl} W_n = \sum_{i=1}^{m} M_{i,n} \nabla^{\perp} \cdot \frac{1}{a}\nabla^{\perp}\xi_i = 0, \\ \oint_{\partial\omega_j} W_n \cdot \tau\, ds = \sum_{i=1}^{m} M_{i,n} \oint_{\partial\omega_j} \frac{1}{a}\frac{\partial \xi_i}{\partial \nu} ds = 2\pi M_{j,n}, \end{aligned}$$

an integer multiple of 2π for each $j = 1, \ldots, m$, it follows that W_n is locally a gradient, $W_n = \nabla\varphi_n$ for φ_n possibly multiple valued, but for which $e^{i\varphi_n}$ is smooth and single-valued in Ω. We may then define the complex order parameter

$$w_n = \exp i(\varphi_n + \zeta \ln \epsilon_n).$$

By construction,

$$\frac{j(w_n)}{\log \epsilon_n} = \frac{(iw_n, \nabla w_n)}{\log \epsilon_n} \to V + W \tag{4.5.4}$$

in $C^1(\bar{\Omega})$. Since $|w_n| = 1$, we may easily calculate the contribution to the energy using the orthogonality:

$$\begin{aligned} \frac{1}{2}\int_\Omega \eta_\epsilon^2(x)|\nabla w_n|^2\,dx &= \frac{1}{2}\int_\Omega \eta_\epsilon^2(x)|\nabla\varphi_n + \nabla\zeta \ln \epsilon_n|^2\,dx \\ &= \frac{1}{2}\int_\Omega \eta_\epsilon^2(x)|W_n|^2 + \frac{(\ln \epsilon_n)^2}{2}\int_\Omega \eta_\epsilon^2(x)|\nabla\zeta|^2\,dx \\ &\le \frac{(\ln \epsilon)^2}{2}\int_\Omega \eta_\epsilon^2\left\{|W|^2 + |V|^2\right\}dx + O(1) \end{aligned} \tag{4.5.5}$$

Recovering U. Constructing the sequence u_n is a little bit delicate since we don't know how η_ϵ is close to $a(x)$ near $\partial\omega_i$ and here we take one of the advantages of the space $\mathbb{H}$ where every function in this space can be approximated by a sequence of compactly supported functions in C_0^∞ i.e. since $U \in \mathcal{U}$ then there exists a sequence $\{\psi^{t_m}\} \subset C_0^\infty(\Omega)$ *with* $K^{t_m} = \operatorname{supp}\psi^{t_m} \subset\subset \Omega$ *and* $\operatorname{dist}(K^{t_m}, \partial\Omega) \ge t_m$ with $t_m \xrightarrow[m\to\infty]{} 0$. s.t.

$$U = \lim_{m\to\infty} \frac{-1}{a}\nabla^\perp \psi^{t_m}$$

Assume that $t_m > \epsilon^{\frac{1}{3}}$ then by (iii) in Proposition 4.1.3 we can write U as

$$U = \lim_{m\to\infty} U^{t_m} = \lim_{m\to\infty}\lim_{\epsilon\to 0} \frac{-1}{\eta_\epsilon^2}\nabla^\perp \psi^{t_m}$$

Indeed,

$$
\begin{aligned}
\left|\Big(\frac{-1}{\eta_\epsilon^2}-\frac{-1}{a}\Big)\Big(\nabla^\perp\psi^{t_m}\Big)\right| &= \left|\Big(\frac{-a+\eta_\epsilon^2}{\eta_\epsilon^2 a}\Big)\Big(\nabla^\perp\psi^{t_m}\Big)\right| \\
&= \left|\Big(\frac{(\eta_\epsilon-\sqrt{a})(\eta_\epsilon+\sqrt{a})}{\eta_\epsilon^2 a}\Big)\Big(\nabla^\perp\psi^{t_m}\Big)\right| \\
&\leq \left|\Big(\frac{(\epsilon^{\frac{1}{3}}\sqrt{a})((2+\epsilon^{\frac{1}{3}})\sqrt{a})}{\eta_\epsilon^2 a}\Big)\Big(\nabla^\perp\psi^{t_m}\Big)\right| \\
&\leq \left|\Big(\frac{(\epsilon^{\frac{1}{3}})(2+\epsilon^{\frac{1}{3}})}{(1-\epsilon^{\frac{1}{3}})\sqrt{a}}\Big)\Big(\nabla^\perp\psi^{t_m}\Big)\right| \\
&\leq \left|\Big(\frac{\epsilon^{\frac{2}{3}}}{t_m^{\frac{1}{2}}}\Big)\Big(\nabla^\perp\psi^{t_m}\Big)\right| \\
&\leq \left|\Big(\epsilon^{\frac{1}{2}}\Big)\Big(\nabla^\perp\psi^{t_m}\Big)\right| \xrightarrow[\epsilon\to 0]{} 0
\end{aligned}
$$

We fix t_m and we construct a sequence u_n as follow,

Step 1. we define the measure

$$\mu^{t_m} = \operatorname{curl} U^{t_m} = -\nabla \times \frac{1}{\eta_\epsilon^2(x)}\nabla^\perp\psi^{t_m}$$

Let $N_n \in \mathbf{N}$ be any sequences of whole numbers with

$$\frac{N_n}{\log \epsilon_n} \longrightarrow 1,$$

and ϵ_n is a subsequence of ϵ which goes to zero when $n \to \infty$.

Applying Lemma 2.4.3 in Chapter 2, there exist family of points $\{p_i^n\}_{i=1,\ldots,N_n}$ in the set $K^{t_m} = \operatorname{supp}\psi^{t_m}$ and associated integers $\sigma_i^n \in \{-1,1\}$ with the following properties:

$$|p_i^n - p_j^n| \geq c_0 N_n^{-1/2} \quad \text{for } i \neq j, \text{ for constant } c_0 = c_0(\psi^{t_m}); \tag{4.5.6}$$

$$\lim_{\alpha \to 0} R(\alpha) = 0 \quad \text{where} \quad R(\alpha) = \limsup_{n \to \infty} \sum_{\substack{i \neq j: \\ |p_i^n - p_j^n| \leq \alpha}} \frac{\big|\log |p_i^n - p_j^n|\big|}{N_n^2}, \tag{4.5.7}$$

$$\mu_n := \frac{2\pi}{N_n} \sum_{i=1}^{N_n} \sigma_i^n \, \delta_{p_i^n} \rightharpoonup \mu^{t_m}, \tag{4.5.8}$$

$$|\mu_n| = \frac{2\pi}{N_n} \sum_{i=1}^{N_n} \delta_{p_i^n} \rightharpoonup |\mu^{t_m}|, \tag{4.5.9}$$

where the convergence in (4.5.6) and (4.5.8) is weakly in the sense of measures and strongly in $(C_0^{0,\gamma})'$ for all $0 < \gamma \leq 1$. By $|\mu^{t_m}|$ we mean the total variation of the measure $\mu^{t_m} = \operatorname{curl} U^{t_m}$. Since that $\psi^{t_m} \in C_0^\infty(\Omega)$ then μ^{t_m} is smooth and compactly supported.

We modify the measures μ_n by regularizing the Dirac mass. Let $\mu_i^n := \epsilon_n \mathcal{H}^1 \lfloor_{\partial B(p_i^n, \epsilon_n)}$, the element of arclength on $S_i^n := \partial B(p_i^n, \epsilon_n)$, normalized with mass 2π. We define the measures

$$\nu_n = \frac{1}{N_n} \sum_{i=1}^{N_n} \sigma_i^n \, \mu_i^n,$$

with $p_i^n \in K$, $\sigma_i^\epsilon \in \{0, 1\}$ as above. Since each $\mu_i^n \longrightarrow \delta_{p_i^n}$ strongly in $[C_0^{0,\gamma}(\Omega)]'$ for all $0 < \gamma \leq 1$, and weakly in $\mathfrak{M}(\Omega)$, we may conclude that (4.5.8),(4.5.9) hold as well for ν_n,

$$\nu_n \longrightarrow \mu^{t_m}, \qquad |\nu_n| \longrightarrow |\mu^{t_m}|, \quad \text{strongly in } [C_0^{0,\gamma}(\Omega)]' \text{ and weakly in } \mathfrak{M}(\Omega). \tag{4.5.10}$$

By Fubini's theorem we also note that the product measures also converge,

$$\nu_n \otimes \nu_n \longrightarrow \mu^{t_m} \otimes \mu^{t_m}, \tag{4.5.11}$$

strongly in $[C_0^{0,\gamma}(\Omega \times \Omega)]'$ and weakly in $\mathfrak{M}(\Omega \times \Omega)$.

Step 2: We introduce the Dirichlet Green's function, $G_{\eta_\epsilon}(x, y)$ in Ω, which solves

$$\begin{cases} -\nabla_x \cdot \dfrac{1}{\eta_\epsilon^2(x)} \nabla_x G_{\eta_\epsilon}(x, y) = \delta_y(x), \quad \text{in } \Omega, \\ \\ G_{\eta_\epsilon}(\cdot, y) = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{4.5.12}$$

for each fixed $y \in \Omega$. By standard elliptic theory (see [GT]) (recall $\eta_\epsilon^2(x)$ is smooth in $\overline{\Omega}$ and positive) we may conclude that $G_{\eta_\epsilon^2(x)}(x, y)$ is smooth in $\overline{\Omega} \times \overline{\Omega} \setminus \{y = x\}$, and

$$G_{\eta_\epsilon}(x, y) = -\frac{\eta_\epsilon^2(x)}{2\pi} \ln|x - y| + \gamma(x, y), \tag{4.5.13}$$

where the regular part γ has the property that for every compact set $K^{t_m} \subset\subset \Omega$, there exists $C(K^{t_m}) < \infty$ with

$$\sup_{\substack{y \in K^{t_m} \\ x \in \overline{\Omega}}} |\gamma(x, y)| \leq C(K^{t_m}).$$

If we are away from the boundary (i.e a is bounded below and positive), then we may define

$$G_a(x, y) = -\frac{a(x)}{2\pi} \ln|x - y| + \gamma(x, y) \tag{4.5.14}$$

to be the solution of the Dirichlet Green's function given by (4.5.12) with a(x) instead of η_ϵ^2.

We then obtain the potential function $\psi^{t_m} \in C_0^\infty(\Omega)$ (by standard elliptic theory and the smoothness of f^2, see [GT]) from $\operatorname{curl} U^{t_m} = \mu^{t_m}$ by solving

$$\begin{cases} -\nabla \cdot \frac{1}{\eta_\epsilon^2(x)} \nabla \psi^{t_m} = \mu^{t_m} & \text{in } \Omega, \\ \psi^{t_m} = 0 & \text{on } \partial\Omega, \end{cases}$$

and we recover $U^{t_m} = -\frac{1}{\eta_\epsilon^2(x)} \nabla^\perp \psi^{t_m}$. From uniqueness of the solution and using the Green's function representation, we have

$$\psi^{t_m}(x) = \int_\Omega G_{\eta_\epsilon}(x,y)\, d\mu^{t_m}(y).$$

We may calculate the weighted norm of U^{t_m} in terms of the measure μ^{t_m} as follows:

$$\begin{aligned} \int_\Omega \eta_\epsilon^2(x)\, |U^{t_m}|^2\, dx &= \int_\Omega \frac{1}{\eta_\epsilon^2(x)} \left|\nabla \psi^{t_m}\right|^2\, dx \\ &= -\int_\Omega \psi^{t_m} \cdot \nabla^\perp \left(\frac{1}{\eta_\epsilon^2(x)} \nabla^\perp \psi^{t_m} \right) dx \\ &= \int_\Omega \psi^{t_m}(x)\, d\mu^{t_m}(x) \\ &= \int_\Omega \int_\Omega G_{\eta_\epsilon}(x,y)\, d\mu^{t_m}(y)\, d\mu^{t_m}(x). \end{aligned} \tag{4.5.15}$$

Step 3. In this step we prove that there exits a sequence $\psi_n \in \mathbf{H}_0^1(\Omega)$ for which $\frac{1}{f_{\epsilon_n}^2(x)} \nabla^\perp \psi_n \longrightarrow U^{t_m}$ strongly in $L^p(\Omega)$ for all $p < 2$, and

$$\limsup_{n\to\infty} \int_\Omega \frac{1}{\eta_\epsilon^2(x)} |\nabla \psi_n|^2\, dx \le \int_\Omega a(x)\, d|\mu^{t_m}|(x) + \int_\Omega a(x) |U^{t_m}|^2\, dx. \tag{4.5.16}$$

For each n, we define $\psi_n(x) = \int_\Omega G_{\eta_\epsilon}(x,y)\,d\nu_n(y)$, and so ψ_n solves

$$\begin{cases} -\nabla \cdot \frac{1}{\eta^2_{\epsilon_n}(x)} \nabla \psi_n = \nu_n & \text{in } \Omega, \\ \psi_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By (4.5.10) and elliptic regularity (see [GT]), we have $\psi_n \to \psi^{t_m}$ in $W^{1,p}(\Omega)$ for all $p < 2$, and thus $-\frac{1}{\eta^2_{\epsilon_n}(x)} \nabla^\perp \psi_n \to U^{t_m}$ in $L^p(\Omega)$ for all $p < 2$ as claimed.

To estimate the energy we use the Green's representation. Since $\nu_n \in H^{-1}(\Omega)$ for fixed n, by (4.5.15) we conclude that

$$\int_\Omega \frac{1}{\eta^2_{\epsilon_n}(x)} |\nabla \psi_n|^2 \, dx = \int_\Omega \int_\Omega G_{\eta_{\epsilon_n}}(x,y)\, d\nu_n(y)\, d\nu_n(x).$$

For any $0 < \alpha < 1$, let $\Delta_\alpha = \{(x,y) \in \Omega \times \Omega : |x-y| \leq \alpha\}$. Fix $\chi_\alpha \in C^\infty(\bar{\Omega} \times \bar{\Omega})$ with $0 \leq \chi_\alpha \leq 1$, and

$$\chi_\alpha(x,y) = \begin{cases} 1, & \text{if } x \in \Delta_\alpha, \\ 0, & \text{if } x \notin \Delta_{2\alpha}. \end{cases}$$

For any $\alpha \in (0,1)$, $G_{\eta_\epsilon}(x,y)(1-\chi_\alpha(x,y))$ is defined away from the singularity so it is smooth, and hence by the strong $[C_0^{0,\gamma}]'$ convergence $\nu_n \to \mu^{t_m}$ we have:

$$\lim_{n\to\infty} \int_\Omega \int_\Omega G_{\eta_\epsilon}(x,y)(1-\chi_\alpha(x,y)) d\nu_n(y)\, d\nu_n(x)$$
$$= \lim_{n\to\infty} \int_\Omega \int_\Omega \{-\frac{\eta^2_\epsilon(x)}{2\pi} \log|x-y| + \gamma(x,y)\}(1-\chi_\alpha(x,y)) d\nu_n(y)\, d\nu_n(x)$$

since we are away from the boundary, then we may use the connection between $a(x)$

and η_ϵ^2 together with (4.5.14) and get

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega\int_\Omega G_{\eta_\epsilon}(x,y)(1-\chi_\alpha(x,y))d\nu_n(y)\,d\nu_n(x)\\
&\quad\le \lim_{n\to\infty}\int_\Omega\int_\Omega\{-\frac{(\epsilon_n^{\frac{1}{3}}+1)a(x)}{2\pi}\log|x-y|+\gamma(x,y)\}(1-\chi_\alpha(x,y))d\nu_n(y)\,d\nu_n(x)\\
&\quad\le \lim_{n\to\infty}\int_\Omega\int_\Omega\Big(G_a(x,y)-\frac{\epsilon_n^{\frac{1}{3}}a(x)}{2\pi}\log|x-y|\Big)(1-\chi_\alpha(x,y))d\nu_n(y)\,d\nu_n(x)\\
&\quad\le \int_\Omega\int_\Omega G_a(x,y)(1-\chi_\alpha(x,y))d\mu^{t_m}(y)\,d\mu^{t_m}(x) \qquad (4.5.17)
\end{aligned}$$

For the complementary integral, we use (4.5.13) to observe that

$$\begin{aligned}
&\int_\Omega\int_\Omega G_{\eta_\epsilon}(x,y)\chi_\alpha(x,y)d\nu_n(y)\,d\nu_n(x)\\
&\qquad= \int_{K^{t_m}}\int_{\Delta_{2\alpha}}\left[\frac{\eta^2_{\epsilon_n}(x)}{2\pi}\log\frac{1}{|x-y|}+\gamma(x,y)\right]\chi_\alpha\,d\nu_n(y)\,d\nu_n(x)\\
&\qquad\le \int_{K^{t_m}}\int_{\Delta_{2\alpha}}\frac{\eta^2_{\epsilon_n}(x)}{2\pi}\log\frac{1}{|x-y|}\,d\nu_n(y)\,d\nu_n(x)+C\alpha\\
&\qquad= \frac{1}{N_n^2}\sum_{i,j=1}^{N_n}\iint_{\Delta_{2\alpha}}\frac{\eta^2_{\epsilon_n}(x)}{2\pi}\log\frac{1}{|x-y|}\,d\mu_i^n(y)\,d\mu_i^n(x)+C\alpha. \qquad (4.5.18)
\end{aligned}$$

To evaluate the remaining integral, we consider the contribution due to distinct points $p_i^n\neq p_j^n$ in $\Delta_{2\alpha}$ separately. Define the index set

$$\mathcal{J}_n=\{(i,j):\ |p_i^n-p_j^n|\le 2\alpha\}.$$

Let $R_n=\frac{1}{4}c_0N_n^{-1/2}$, where $c_0=c_0(\psi)$ is the constant in (4.5.6). We also define balls

$\tilde{B}_i^n = B(p_i^n, R_n)$, $i = 1, \ldots, N_n$. By the choice of R_n, they are disjoint, as is the union

$$\bigcup_{(i,j)\in\mathcal{J}_n} \left(\tilde{B}_i \times \tilde{B}_j\right) \subset \Delta_{3\alpha}.$$

We also observe that for any $R \le R_n$ and $(i,j) \in \mathcal{J}_n$, since $R \le \frac{1}{4}|p_i - p_j|$, we have

$$\frac{1}{2} \le \frac{|x-y|}{|p_i^n - p_j^n|} \le \frac{3}{2} \quad \text{for all } x \in B(p_i^n, R),\ y \in B(p_j^n, R). \tag{4.5.19}$$

For $(i,j) \in \mathcal{J}_n$ we then have (recalling that $S_n^i = \partial B(p_i^n, \epsilon_n) = \operatorname{supp} \mu_i^n$,)

$$\begin{aligned}
\iint_{\tilde{B}_i^n \times B_j^n} \log \frac{3}{|x-y|} dx\, dy &\ge \iint_{\tilde{B}_i^n \times B_j^n} \log \frac{2}{|p_i^n - p_j^n|} dx\, dy \\
&= \pi^2 R_n^4 \log \frac{2}{|p_i^n - p_j^n|} \\
&= \frac{R_n^4}{4} \iint_{\tilde{S}_i^n \times S_j^n} \log \frac{2}{|p_i^n - p_j^n|} d\mu_i^n(x)\, d\mu_j^n(y) \\
&\ge \frac{R_n^4}{4} \iint_{\tilde{S}_i^n \times S_j^n} \log \frac{1}{|x-y|} d\mu_i^n(x)\, d\mu_j^n(y),
\end{aligned}$$

using (4.5.19) in the first and last lines. Summing over all pairs $(i,j) \in \mathcal{J}_n$, and using the disjointness of the union of the $\tilde{B}_i^n \times \tilde{B}_j^n$, we obtain:

$$\begin{aligned}
\frac{1}{N_n^2} \sum_{(i,j)\in\mathcal{J}_n} \iint_{S_i^n \times S_j^n} &\frac{\eta_{\epsilon_n}^2(x)}{2\pi} \log \frac{1}{|x-y|} d\mu_i^n(x)\, d\mu_j^n(y) \\
&\le \frac{C}{R_n^4\, N_n^2} \sum_{(i,j)\in\mathcal{J}_n} \iint_{\tilde{B}_i^n \times B_j^n} \log \frac{3}{|x-y|} dx\, dy \\
&\le C \iint_{\Delta_{3\alpha}} \log \frac{3}{|x-y|} dx\, dy =: \mathcal{R}(\alpha).
\end{aligned} \tag{4.5.20}$$

As $|\log|x-y||$ is integrable in this region, the remainder $\mathcal{R}(\alpha) \to 0$ as $\alpha \to 0$, and so this term will not contribute to the limiting energy.

Finally, we consider the contribution from the self-energy of the vortices p_i^n. We parametrize the integrals over $S_i^n = \partial B(p_i^n, \epsilon_n)$ using complex notation, that is we write $x, y \in \partial B(p_i^n, \epsilon_n)$ as $x = p_i^n + \epsilon_n e^{i\theta}$, $y = p_i^n + \epsilon_n e^{i\tau}$, $0 \le \theta, \tau < 2\pi$. Then we have:

$$\begin{aligned}
&\frac{1}{N_n^2} \iint_\Omega \frac{\eta_{\epsilon_n}^2(x)}{2\pi} \log \frac{1}{|x-y|}\, d\mu_i^n(y)\, d\mu_i^n(x) \\
&\qquad = \frac{1}{N_n^2} \int_0^{2\pi} \int_0^{2\pi} \frac{a\left(p_i^n + e^{i\theta}\epsilon_n\right)}{2\pi} \left[\log \epsilon_n + \log\left|e^{i(\theta-\tau)} - 1\right|\right] d\theta\, d\tau \\
&\qquad = \frac{1}{N_n} \int_0^{2\pi} a\left(p_i^n + e^{i\theta}\epsilon_n\right)\, d\theta + O(N_n^{-2}) \\
&\qquad = \frac{1}{N_n} \int_\Omega \eta_{\epsilon_n}^2(x)\, d|\mu_i^n|(x) + O(N_n^{-2}).
\end{aligned}$$

Note that: we were able to use $a(p_i^n + e^{i\theta}\epsilon_n)$ since that the vortex balls were constructed away from the boundary. Also in the following computation we have μ^{t_m} is away from the boundary, far enough that again we can use $a(x)$.

Summing over all $i = 1, \ldots, N_n$, we arrive at

$$\begin{aligned}
\frac{1}{N_n^2}\sum_{i=1}^{N_n}\iint_\Omega \frac{\eta_{\epsilon_n}^2(x)}{2\pi}\log\frac{1}{|x-y|}\,d\mu_i^n(y)\,d\mu_i^n(x) &= \frac{1}{N_n}\int_\Omega \eta_{\epsilon_n}^2(x)\,d|\nu_n|(x) + O(N_n^{-1}) \\
&= \int_\Omega \eta_{\epsilon_n}^2(x)\,d|\mu^{t_m}|(x) + O(N_n^{-1}) \\
&= \int_\Omega (\eta_{\epsilon_n}^2(x) - a(x))\,d|\mu^{t_m}|(x) \\
&\qquad + \int_\Omega a(x)\,d|\mu^{t_m}|(x) + O(N_n^{-1}). \\
&= \int_\Omega (1+\epsilon_n^{\frac{1}{3}})a(x)\,d|\mu^{t_m}|(x) \\
&\qquad + \int_\Omega a(x)\,d|\mu^{t_m}|(x) + O(N_n^{-1}) \qquad .
\end{aligned} \tag{4.5.21}$$

Passing to the limit $\epsilon_n \to 0$, we thus obtain from (4.5.17),(4.5.18),(4.5.20), and (4.5.21), that

$$\begin{aligned}
&\limsup_{n\to\infty}\int_\Omega\int_\Omega G_{\eta_\epsilon}(x,y)d\nu_n(y)\,d\nu_n(x) \\
&\qquad \le \int_\Omega a(x)\,d|\mu^{t_m}|(x) + \int_\Omega\int_\Omega G_a(x,y)(1-\chi_\alpha(x,y))d\mu^{t_m}(y)\,d\mu^{t_m}(x) + C\alpha + C\mathcal{R}(\alpha).
\end{aligned}$$

By hypothesis, the measure μ^{t_m} is bounded, and so we may apply dominated convergence to pass to the limit $\alpha \to 0$ and obtain the desired bound (4.5.16), as

$$\begin{aligned}
\limsup_{n\to\infty}\int_\Omega \frac{1}{\eta_{\epsilon_n}^2}|\nabla\psi_n|^2\,dx &= \limsup_{n\to\infty}\int_\Omega\int_\Omega G_{\eta_\epsilon}(x,y)d\nu_n(y)\,d\nu_n(x) \\
&\le \int_\Omega a(x)\,d|\mu^{t_m}|(x) + \int_\Omega\int_\Omega G_a(x,y)d\mu^{t_m}(y)\,d\mu^{t_m}(x) \\
&= \int_\Omega a(x)\,d|\mu^{t_m}|(x) + \int_\Omega a(x)\,|U^{t_m}|^2\,dx,
\end{aligned}$$

by (4.5.15).

Step 4. Let $U_n = -N_n \frac{1}{\eta_\epsilon^2(x)} \nabla^\perp \psi_n$. Then, $\nabla^\perp U_n = N_n \nabla \cdot \left(\frac{1}{\eta_\epsilon^2(x)} \nabla \psi_n\right) = 0$ locally in $\Omega \setminus \cup_i^{N_n} B(p_i^n, \epsilon_n)$. Moreover, if C is a simple closed curve in $\Omega \setminus \cup_i^{N_n} B(p_i^n, \frac{1}{\kappa_n})$, we have

$$\int_C U_n \cdot \tau \, ds \in 2\pi \, \mathbb{Z},$$

by the normalization $|d\mu_i^n| = 2\pi$. Thus, we may write $U_n = \nabla \phi_n$ in $\Omega \setminus \cup_i^{N_n} B(p_i^n, \epsilon_n)$, with ϕ_n which is multiple valued, but for which $\nabla \phi_n$ and $e^{i\phi_n}$ are single-valued in $\Omega \setminus \cup_i^{N_n} B(p_i^n, \epsilon_n)$. We now define an auxiliary function ρ_n as in Chapter 2 to remove the singularity at each vortex core,

$$\rho_i^n(x) := \begin{cases} 0 & \text{if } |x - p_i^n| < \frac{1}{2}\epsilon_n, \\ \frac{2}{\epsilon_n}|x - p_i^n| - 1 & \text{if } \frac{1}{2}\epsilon_n \leq |x - p_i^n| \leq \epsilon_n, \\ 1 & \text{if } |x - p_i^n| > \epsilon_n, \end{cases}$$

and $\rho_n := \prod_{i=1}^{N_n} \rho_i^n$. A simple computation shows that

$$\int_\Omega \eta_\epsilon^2(x) \left\{ \frac{1}{2} |\nabla \rho_i^n|^2 + \frac{\eta_\epsilon^2(x)}{4\epsilon_n^2} ((\rho_i^n)^2 - 1)^2) \right\} dx \leq C_0,$$

with constant C_0 independent of n. Also $(\rho_n^2 - 1) \to 0$ in L^q for all $q < \infty$.

Now define $u_n = \rho_n e^{i\phi_n}$, with ρ_n, ϕ_n as in the preceding paragraphs. We then have:

$$\begin{aligned}
&\int_\Omega \eta_\epsilon^2(x)\left\{\frac{1}{2}|\nabla u_n|^2 + \frac{\eta_\epsilon^2(x)}{4\epsilon_n^2}\big(|u_n|^2-1\big)^2\right\}dx \\
&\qquad\qquad = \int_\Omega \eta_\epsilon^2(x)\left\{\frac{1}{2}\rho_n^2|\nabla\phi_n|^2 + \frac{1}{2}|\nabla\rho_n|^2 + \frac{\eta_\epsilon^2}{4\epsilon^2}\big(\rho_n^2-1\big)^2\right\}dx \\
&\qquad\qquad \le \frac{N_n^2}{2}\int_\Omega \frac{1}{\eta_\epsilon^2(x)}|\nabla\psi_n|^2\,dx + C_0N_n.
\end{aligned}$$

From (4.5.16) we then conclude that

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{(\log\epsilon_n)^2}\int_\Omega \eta_\epsilon^2(x)\{\frac{1}{2}|\nabla u_n|^2 + \frac{\eta_\epsilon^2(x)}{4\epsilon^2}\big(|u_n|^2-1\big)^2\}dx \\
&\qquad\qquad \le \frac{1}{2}\int_\Omega a(x)\,d|\mu^{t_m}|(x) + \frac{1}{2}\int_\Omega a(x)|U^{t_m}|^2\,dx.
\end{aligned} \tag{4.5.22}$$

Since $(\rho_n^2-1)\to 0$ in L^q for all $q<\infty$, we also conclude that

$$\frac{j(u_n)}{N_n} = -\frac{1}{\eta_\epsilon^2(x)}\nabla^\perp\psi_n + \frac{(1-\rho_n^2)}{\eta_\epsilon^2(x)}\nabla^\perp\psi_n \longrightarrow U^{t_m} \quad \text{in } L^p(\Omega) \text{ for all } p<2. \tag{4.5.23}$$

Putting everything together. Write $j\in\mathcal{Z}$ as $j := \lim_{m\to\infty} j^{t_m} = \lim_{m\to\infty} U^{t_m} + V + W$ with $\lim_{m\to\infty} U^{t_m} = U\in\mathcal{U}$, $V\in\mathcal{V}$, and $W\in\mathcal{W}$. Let w_n be defined as in Step 1 and u_n as constructed in Step 5, and define $v_n = u_n w_n$. Since $|w_n| = 1$, we have

$$j(v_n)/N_n = \frac{1}{N_n}(j(u_n) + \rho_n^2 j(w_n)) \longrightarrow U^{t_m} + V + W = j^{t_m} \tag{4.5.24}$$

in $L^p(\Omega)$ for all $p<2$.

To estimate the energy, we again use the fact that $|w_n| = 1$ to expand:

$$\frac{1}{N_n^2}\int_\Omega \eta_\epsilon^2(x)|\nabla v_n|^2\,dx = \frac{1}{N_n^2}\int_\Omega \eta_\epsilon^2(x)\left\{|\nabla u_n|^2 + \rho_n^2|\nabla w_n|^2 + j(u_n)\cdot j(w_n)\right\}dx.$$

We claim that the last term is $o(1)$. Indeed, from Step 1, $\frac{j(w_n)}{\log\epsilon_n} = \nabla\Phi_n$, with $\Phi_n = \varphi_n + \zeta\log\kappa_n$ and $\nabla\Phi_n \to V + W$ in C^1, and therefore,

$$\begin{aligned}\frac{1}{N_n^2}\int_\Omega \eta_\epsilon^2 j(u_n)\cdot j(w_n)\,dx &= -\int_\Omega \nabla^\perp\psi_n\cdot\rho_n^2\nabla\Phi_n\,dx\\ &= -\int_\Omega\left[\nabla^\perp\psi_n\cdot\nabla\Phi_n - (1-\rho_n^2)\nabla^\perp\psi_n\cdot\nabla\Phi_n\right]dx \longrightarrow 0\end{aligned}$$

by integration by part and the definition of Φ_n the first integral is tending to zero when $n\to\infty$ and since that $\rho_n^2 \le 1$ the second integral is tending to zero when $n\to\infty$.

The energy bound $\int_{\mathcal{D}}|h-h_{ex}|^2dx \le C|\log\epsilon|^2$ leads to $\|\frac{A^\epsilon}{h_{ex}}\|_{L^\infty} \le C$ which implies that $A^\epsilon/|\log\epsilon|$ has a limit ,say A ,in $L^p,\ p<\infty$. Let $A_n := |\log\epsilon_n|A$ and $h_n = \nabla\times A_n$ then calculate,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{N_n^2}F_\epsilon(v_n;A_n;\Omega) &= \limsup_{n\to\infty}\frac{1}{N_n^2}\Bigg(\int_\Omega f_{\epsilon_n}^2(x)\{\frac{1}{2}|\nabla u_n|^2 + \frac{1}{2}|\nabla w_n|^2 - A_n\cdot j(v_n)\\ &\qquad + |A_n|^2|v_n|^2 + \frac{f_{\epsilon_n}^2(x)}{4\epsilon_n^2}(|u_n|^2-1)^2\} + \frac{1}{2}\int_{\mathcal{D}}(h_n-h_{ex})^2dx\Bigg)\\ &\le \limsup_{n\to\infty}\frac{1}{N_n^2}\Bigg(\int_\Omega \eta_{\epsilon_n}^2(x)\{\frac{1}{2}|\nabla u_n|^2 + \frac{1}{2}|\nabla w_n|^2 - A_n\cdot j(v_n)\\ &\qquad + |A_n|^2|v_n|^2 + \frac{\eta_{\epsilon_n}^2(x)}{4\epsilon_n^2}(|u_n|^2-1)^2\} + \frac{1}{2}\int_{\mathcal{D}}(h_n-h_{ex})^2dx\Bigg)\end{aligned}$$

$$\limsup_{n\to\infty}\frac{1}{N_n^2}F_\epsilon(v_n;A_n;\Omega)=\limsup_{n\to\infty}\frac{1}{N_n^2}\Bigg(\int_\Omega\eta_{\epsilon_n}^2(x)\{\frac{1}{2}|\nabla u_n|^2+\frac{\eta_{\epsilon_n}^2(x)}{4\epsilon_n^2}(|u_n|^2-1)^2+|\nabla w_n|^2$$
$$-A_n\cdot j(v_n)+|A_n|^2\rho_n^2\}+\frac{1}{2}\int_{\mathcal{D}}(h_n-h_{ex})^2dx\Bigg).$$

but

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{N_n^2}\int_\Omega\eta_{\epsilon_n}^2(x)|\nabla w_n|^2&=\limsup_{n\to\infty}\frac{1}{N_n^2}\int_\Omega\eta_{\epsilon_n}^2(x)(|V|^2+|W|^2)\\&=\limsup_{n\to\infty}\int_{\Omega_\epsilon}\eta_{\epsilon_n}^2(x)(|V|^2+|W|^2)\\&\quad+\limsup_{n\to\infty}\int_{\Omega\setminus\Omega_\epsilon}\eta_{\epsilon_n}^2(x)(|V|^2+|W|^2)\\&\le\limsup_{n\to\infty}\int_{\Omega_\epsilon}(1+\epsilon_n^{\frac{1}{3}})^2a(x)(|V|^2+|W|^2)\\&\quad+\limsup_{n\to\infty}\int_{\Omega\setminus\Omega_\epsilon}\eta_{\epsilon_n}^2(x)(|V|^2+|W|^2)\\&\le\int_{\Omega_\epsilon}a(x)(|V|^2+|W|^2)\\&\quad+\limsup_{n\to\infty}\|\max_{\mathcal{D}}a\|\int_{\Omega\setminus\Omega_\epsilon}(|V|^2+|W|^2)\end{aligned}$$

the second integral is vanishingly small since V and W are integrable and $\Omega\setminus\Omega_\epsilon$ is a small set which implies

$$\limsup_{n\to\infty}\frac{1}{N_n^2}\int_\Omega\eta_\epsilon^2(x)(|V|^2+|W|^2)\le\int_\Omega a(x)(|V|^2+|W|^2)+o(1).$$

Hence,

$$\limsup_{n\to\infty} \frac{1}{N_n^2} F_\epsilon(v_n; A_n) \leq \frac{1}{2} \int_\Omega a(x) d|\mu^{t_m}| + a(x)\Big(|U^{t_m}|^2 + |V|^2 + |W|^2 + |A|^2\Big) dx \\ + \frac{1}{2} \int_{\mathcal{D}} (\nabla \times A - \lambda)^2 dx - \limsup_{n\to\infty} \frac{1}{N_n^2} \int_\Omega \eta_{\epsilon_n}^2(x) A_n \cdot j(v_n) dx. \tag{4.5.25}$$

We estimate the last term

$$\limsup_{n\to\infty} \frac{1}{N_n^2} \int_\Omega \eta_{\epsilon_n}^2(x) A_n \cdot j(v_n) dx = \limsup_{n\to\infty} \frac{1}{N_n^2} \int_\Omega \eta_{\epsilon_n}^2(x) A \log \epsilon_n \cdot j(v_n) dx$$

We add and subtract the mix term $\eta_{\epsilon_n}^2(x) j^{t_m}$ in the integral

$$\int_\Omega \eta_{\epsilon_n}^2(x) \cdot \frac{j(v_n)}{N_n} - \eta_{\epsilon_n}^2(x) j^{t_m} + \eta_{\epsilon_n}^2(x) j^{t_m} - a(x) \cdot j^{t_m} dx \\ = \int_\Omega \eta_{\epsilon_n}^2(x) \Big(\frac{j(v_n)}{N_n} - j^{t_m} \Big) + \Big(\eta_{\epsilon_n}^2(x) - a(x) \Big) j^{t_m} dx$$

Since that j^{t_m} is compactly supported in K^{t_m} then we could use (4.1.10) on the second integral

$$\leq \int_\Omega \eta_{\epsilon_n}^2(x) \Big(\frac{j(v_n)}{N_n} - j^{t_m} \Big) + \epsilon^{\frac{1}{3}} (2 + \epsilon^{\frac{1}{3}}) a(x) j^{t_m} dx$$

using (4.5.24), the first part $\int_\Omega \eta_{\epsilon_n}^2(x) \Big(\frac{j(v_n)}{N_n} - j^{t_m} \Big) \xrightarrow[n\to\infty]{} 0$ which implies

$$\limsup_{n\to\infty} \frac{1}{N_n^2} \int_\Omega \eta_{\epsilon_n}^2(x) A_n \cdot j(v_n) dx \leq \int_\Omega a(x) A \cdot j^{t_m} dx$$

Hence

$$\int_\Omega \Big(\eta_{\epsilon_n}^2(x)\cdot \frac{j(v_n)}{N_n} - a(x)\cdot j^{t_m}\Big)dx \xrightarrow{n\to\infty} 0. \tag{4.5.26}$$

We substitute in (4.5.25)

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{N_n^2} F_\epsilon(v_n;A_n) &\le \frac{1}{2}\int_\Omega a(x)d|\mu^{t_m}| + a(x)\Big(|U^{t_m}|^2 + |V|^2 + |W|^2 + |A|^2 + 2A\cdot j^{t_m}\Big)dx \\ &\quad + \frac{1}{2}\int_{\mathcal{D}} (\nabla\times A - \lambda)^2 dx \\ &= \frac{1}{2}\int_\Omega a(x)d|\mu^{t_m}| + a(x)\Big(j^{t_m} - A\Big)^2 dx + \frac{1}{2}\int_{\mathcal{D}} (\nabla\times A-\lambda)^2 dx\end{aligned}$$

Taking the limit when $m\to\infty$. We get the convergence of the first term to $\frac{1}{2}\int_\Omega a(x)d|\mu|$ by the strong convergence of μ^{t_m} in $(C_0^{0,\gamma}))'$.

To deal with the second integral, we know by definition of $U^{t_m} = \lim_{\epsilon\to 0} -\frac{1}{\eta_\epsilon^2(x)}\nabla^\perp\psi^{t_m} = -\frac{1}{a(x)}\nabla^\perp\psi^{t_m}$ where $a(x)$ grows linearly near $\partial\omega_i$ $(i.e.|\nabla a(x)| > \delta > 0)$, we conclude

$$\int_\Omega a(x)|U^{t_m}|^2 \to \int_\Omega a(x)|U|^2. \tag{4.5.27}$$

and

$$\limsup_{n\to\infty}\frac{1}{N_n^2}F_\epsilon(v_n;A_n) \le \frac{1}{2}\int_\Omega a(x)d|\mu| + a(x)\Big(j-A\Big)^2 dx + \frac{1}{2}\int_{\mathcal{D}}(\nabla\times A-\lambda)^2 dx \tag{4.5.28}$$

As in [SS07] to get the upper bound in terms of the general $\mu\in H^{-1}(\Omega)\cap\mathfrak{M}(\Omega)$. A diagonal argument together with (4.5.28), yields a sequence $n_k\to\infty$, that we write in shorthand $\{n\}$, such that, writing $\{u_n, A_n\}$ instead of $\{u_{n_k}, A_{n_k}\}$, both (4.5.28) and

(4.5.8) hold.

Hence,

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{N_n^2}E_\epsilon(v_n;A_n) &= \limsup_{n\to\infty}\frac{1}{N_n^2}F_\epsilon(v_n,A_n:\Omega)\\ &\leq \frac{1}{2}\int_\Omega a(x)d|\mu| + a(x)\Big(j-A\Big)^2dx + \frac{1}{2}\int_{\mathcal{D}}(\nabla\times A-\lambda)^2dx\end{aligned} \tag{4.5.29}$$

This completes the Γ-convergence result.

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