# STUDIES IN

# CATEGORICAL TOPOLOGY

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By

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A Thesis
Submitted to the School of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
Doctor of Philosophy

McMaster University
May 1973

DOCTOR OF PHILOSOPHY (1973)
(Mathematics)

McMaster University
Hamilton, Ontario

TITLE: Studies in Categorical Topology

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NUMBER OF PAGES: xi, 100

SCOPE AND CONTENTS: In this thesis we study extensive subcategories of various categories of Hausdorff spaces and continuous maps, and of Hausdorff uniform spaces and uniformly continuous maps. In particular, we obtain new methods to construct extensive subcategories which can be applied to many categories and give us an inclusive relationship between reflective subcategories of Haus and coreflective subcategories of Top. We consider perfect onto projectivity in those categories. The relationships between n-compact spaces and topologically complete spaces are discussed.

# ACKNOWLEDGEMENTS

I would like to thank my supervisor, Professor B. Banaschewski, for introducing me to the topics discussed in this thesis and encouraging me to study them. I appreciate also his patience in listening to and constructively criticizing every result as it was obtained.

I would like to thank Professor T. H. Choe and Dr. H. Herrlich for their many helpful suggestions.

I thank McMaster University, National Research Council of Canada and Ontario Government for their financial support, which enabled me to devote all my time to this work.

Finally, I wish deeply to express my thanks to my wife, Young Hee, for her limitless patience and encouragement.

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# INTRODUCTION

Our study has its origins in one of the most important aspects of general topology, extensions with the universal mapping property [8].

The main importance of extensions is that, if a space has an extension with nice properties, these properties can to some extent be brought to bear on the study of the original space, or they might lead to replace the original space by its extension. In this connection, it is also very important to know whether a given continuous map in the original space has a continuous extension to the extension of the space.

The well known examples are the Stone-Čech compactifications, realcompactifications, maximal zero-dimensional compactifications, and completions.

In this direction, we have a very essential tool, namely categories which provide a convenient conceptual language, based on the notions of category, reflections.

Using this tool, there have been many efforts made to construct new reflective subcategories in various categories of topological (or uniform) spaces and (uniformly, respectively) continuous maps [15, 19, 23, 24, 30, 38, 44, 45, 48], and find out nice properties in those categories [1, 2, 3, 5, 27, 29, 33, 46, 48]. Comprehensive results and bibliography of papers in this field can be found in [25, 27].

In this work, our basic categories are the category Haus of Hausdorff spaces and continuous maps and the category HUnif of Hausdorff uniform spaces and uniformly continuous maps.

Our main objective is a systematic study of extensive subcategories of various subcategories of Haus or HUnif.

One of the main reasons to take extensive subcategories rather than (epi-) reflective subcategories has been already mentioned. It is known that for every epi-reflective subcategory $\mathcal{L}$ of Haus, there exists an epi-reflective subcategory $R\mathcal{L}$ of Haus such that $\mathcal{L}$ is extensive in $R\mathcal{L}$ and for any X in Haus the $\mathcal{L}$-reflection of X has a factorization through the $R\mathcal{L}$-reflection of X and $\mathcal{L}$-reflection of the $R\mathcal{L}$-reflection space of X. Furthermore, $R\mathcal{L}$-reflections can be easily characterized (see Section 3 in Chap. 0). Hence every epi-reflective subcategory of Haus can be completely determined by a certain extensive subcategory in a subcategory of Haus. This is another essential reason, in view of categorical topology, why our main objective is extensive subcategories.

The contents of our work divide into four parts.

The first, comprising Chapter 0, presents together basic definitions and theorems which will pave the way for the further development of the present thesis.

In particular, we list the definitions of z-ultrafilters, simple and strict extensions, $\mathcal{E}$-compact spaces and reflective subcategories. Also included are discussions of Stone-Čech compactifications, realcompactifications and $\mathcal{E}$-compactifications, and finally some properties of reflective subcategories.

The second part is composed of Chapter I and Chapter II.

In Chapter I, we first consider some properties of extensive subcategories. We observe that every reflective subcategory containing an extensive subcategory in a category is also extensive in the category and that every extensive subcategory is left-fitting with respect to perfect morphisms in the category. Consequently, the left-fitting property is strongly connected to extensiveness.

H. Herrlich has introduced the limit-operators [26] to obtain coreflective subcategories of the category Top of topological spaces and continuous maps. Moreover, he has established a one-one correspondence between idempotent limit-operators and coreflective subcategories of Top. Using limit-operators, we establish a method to construct new extensive subcategories from well known extensive subcategories in various subcategories of Haus or HUnif. The new extensive subcategory is constructed as follows. Let $\ell$ be an idempotent limit-operator on an extensive subcategory $\mathcal{L}$ of a subcategory $\mathcal{A}$ of Haus or HUnif. Let $\mathcal{L}_\ell$ be the subcategory of $\mathcal{A}$ determined by those members of $\mathcal{A}$ which are $\ell$-closed in their $\mathcal{L}$-reflection spaces. It is shown that $\mathcal{L}_\ell$ is also extensive in $\mathcal{A}$ , if $\mathcal{A}$ is hereditary. Hence for any epi-reflective subcategory of Haus, we can associate a reflective subcategory of Haus containing it with each coreflective subcategory of Top, for there is a hereditary subcategory of Haus, in which it is extensive. Consequently, we establish a useful interrelation between coreflective subcategories of Top and epireflective subcategories of Haus. Finally, for any reflective subcategory containing an extensive subcategory in a certain category, we can find a semi-limit-operator on the extensive subcategory which generates the reflective subcategory and an idempotent limit-operator on the extensive subcategory which generates a reflective subcategory containing the reflective subcategory. This gives another method to construct new extensive subcategories from well known reflective subcategories.

In Chapter II, we apply the results of Chapter I to various categories. Using trace filters, we can easily characterize new extensive subcategories and comprehend the interrelations between those categories.

The third part, Chapter III, is devoted to perfect onto projectivity in various categories determined by our setting, which is in some sense complementary to B. Banaschewski's inclusive contribution in this field [5]. It is shown that in every full subcategory of Haus determined by the objects of an extensive subcategory of the category Haus* of Hausdorff spaces and continuous semi-open maps, perfect onto projectivity is properly behaved. Secondly, we generalize the concept of almost realcompactness [16] to almost n-compactness for any infinite cardinal number n. In the category of almost n-compact spaces and continuous maps, perfect onto projectivity is also properly behaved. Finally we consider the category of pseudo-compact spaces and continuous maps which is neither productive nor closed hereditary. It is shown that perfect onto projectivity in this category is again properly behaved using the space of convergent maximal open filters.

Finally, in Chapter IV, we deal with the category of topologically complete spaces and continuous maps. We introduce the concept of n-total boundedness. We show that every complete regular space which admits an admissible n-totally bounded complete uniform structure, is n-compact. It is shown that the categories of topologically n-totally bounded complete Hausdorff spaces and continuous maps coincide with the category of realcompact spaces and continuous maps for the cardinal number n with $\aleph_1 \leqslant n \leqslant m$, where m is the first measurable cardinal number. It is noted that those categories are coincident but the categories of n-totally bounded complete uniform spaces and uniformly continuous maps are different for the different cardinal numbers n. It is also shown that a topologically complete completely regular space is realcompact if and only if every locally finite open covering of the space has a nonmeasurable subcovering.

# CHAPTER 0

## PRELIMINARIES

This chapter is a collection of the basic definitions and results which will be needed in the ensuing chapters.

### Section 1: Completely regular spaces.

1.1 Definition A topological space X is said to be completely regular provided that it is a Hausdorff space such that, whenever F is a closed subset of X and x is a point in its complement, there exists a continuous real-valued map f such that $f(x) = 1$ and $f(F) = \{0\}$.

1.2 Notation For a topological space X, we denote the set of all continuous real-valued maps by C(X) and the set of all bounded continuous real-valued maps by C*(X).

It is obvious that C(X) under the functional operations is a commutative ring with unit 1 and C*(X) is a subring of C(X).

1.3 Theorem For every toplogical space X, there exists a completely regular space Y and a continuous map f of

X onto Y, such that the map $g \leadsto gf$ is an isomorphism of C(Y) onto C(X).

Remark: The space Y in the above theorem is known as the complete regularization of the space X.

1.4 Definition A subset Z of a topological space X is said to be a zero-set in X if $Z = f^{-1}(0)$ for some $f \in C(X)$. In this case, Z is also said to be the zero-set of f and we denote it by Z(f), and the set of all zero-sets in X by Z(X).

A subset C of X is said to be a cozero-set in X if it is the complement of a zero-set Z(f) for some $f \in C(X)$. We denote it by coz(f).

1.5 Theorem For a Hausdorff space X, the following are equivalent:

1) X is completely regular.
2) Z(X) is a base for the closed sets in X.
3) X is uniformizable.
4) X is homeomorphic with a subspace of a product space of the copies of real line.

It is well known that Z(X) is a lattice with respect to the set union and intersection.

1.6 Definition For a space X, a proper filter in the lattice Z(X) is said to be a z-filter on X.

By a z-ultrafilter on X is meant a maximal z-filter, i.e., one not contained in any other z-filter.

A z-filter is said to be <u>fixed</u> if it has a cluster point. Otherwise, it is said to be <u>free</u>.

It is well known that for an ideal I in C(X), $Z(I) = \{Z(f) \mid f \in I\}$ is a z-filter in X and for a z-filter $\mathfrak{F}$ on X, $Z^{-1}(\mathfrak{F}) = \{f \mid Z(f) \in \mathfrak{F}\}$ is an ideal in C(X).

Moreover, if I is maximal, then Z(I) is a z-ultrafilter on X and if $\mathcal{U}$ is a z-ultrafilter, then $Z^{-1}(\mathcal{U})$ is a maximal ideal in C(X) [17].

Hence, one can define that an ideal I in C(X) is fixed (free) according to the z-filter Z(I) being fixed (free, respectively).

1.7 <u>Theorem</u> For a completely regular space X, the following are equivalent:

1) X is compact.
2) Every z-filter on X is fixed, i.e. every ideal in C(X) is fixed.
3) Every z-ultrafilter on X is fixed, i.e. every maximal ideal in C(X) is fixed.

1.8 <u>Definition</u> Let X be a completely regular space. A maximal ideal M in C(X) (C*(X)) is said to be <u>real</u> if the quotient field C(X)/M (C*(X)/M, respectively) is isomorphic with $\mathbb{R}$ . In this case, Z(M) is also said to be real.

Otherwise, it is said to be <u>hyper-real</u>.

1.9 <u>Definition</u> A completely regular space is said to be <u>realcompact</u> if every real maximal ideal is fixed, i.e. every real z-ultrafilter is fixed.

1.10 <u>Definition</u> Let $\mathcal{S}$ be a non-empty family of subsets of a set X, and let n be an infinite cardinal number. $\mathcal{S}$ is said to have the <u>n-intersection property</u> if every fewer than n members of $\mathcal{S}$ has a non-empty intersection.

1.11 <u>Theorem</u> The following are equivalent for any maximal ideal M in C(X).

1) M is real.
2) Z(M) is closed under countable intersections.
3) Z(M) has the $\aleph_1$-intersection property.

## <u>Section 2</u>: <u>Extensions</u>.

2.1 <u>Definition</u> Let X and X' be spaces and f: $X \longrightarrow X'$ a map. The pair (X', f) is said to be an <u>extension space</u> of X if f is a homeomorphism of X with the dense subspace f(X) of X'. In particular, (X', f) is an extension space of X such that $X \subseteq X'$ and f maps X identically, then the reference to f will be omitted and X' will itself be called an extension space of X.

<u>Remark</u>: It is known that without loss of generality, one may always restrict onself to extension spaces of a space

$X$ which contain $X$ as a dense subspace.

Let $(X', f)$ be an extension space of the space $X$. And let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies of $X$ and $X'$ respectively. Then, each point $u \in X'$ determines the proper filter $T(u) = f^{-1}(\mathcal{T}'(u)) = \{f^{-1}(V) \mid V \in \mathcal{T}'(u)\}$ in the lattice $\mathcal{T}$, called the <u>trace filter</u> of $u$ on $X$, where $\mathcal{T}'(u)$ is the filter of open neighborhoods of $u$.

The family $(T(u))_{u \in X'}$ will be called the <u>filter trace</u> of the extension space on $X$. If $X' \supseteq X$ then the filter trace of $X'$ on $X$ extends the family $(\mathcal{T}(x))_{x \in X}$ of neighborhood filters of $X$ to a family of filters in $\mathcal{T}$ with larger indexing set since $T(x) = \mathcal{T}(x)$ for $x \in X$.

Consider any family $(T(u))_{u \in X'}$ of filters in $\mathcal{T}$ which extends the family of neighborhood filters of the space $X$, i.e. $X' \supseteq X$ and $T(x) = \mathcal{T}(x)$ for each $x \in X$.
Then, there exist two natural topologies on the set $X'$ such that the resulting spaces are extensions of $X$ whose filter trace on $X$ is just the given family $(T(u))_{u \in X'}$.
The first of these spaces, called the <u>strict extension</u> of $X$ with filter trace $(T(u))_{u \in X'}$, has its topology $\mathcal{T}_1'$ generated by the sets $V^* = \{u \mid V \in T(u)\}$, $V \in \mathcal{T}$; in the second space, with topology $\mathcal{T}_0'$, here referred to as the <u>simple extension</u> of $X$ with filter trace $(T(u))_{u \in X'}$, each $u \in X'$ has as its basic neighborhoods the sets $V \cup \{u\}$, $V \in T(u)$ [4].

2.2 Definition An extension space Y of a space X is said to be a compactification (realcompactification) of X if Y is compact (realcompact, respectively).

2.3 Definition A subspace S of a space X is said to be C-embedded in X if every map in C(S) can be extended to a map in C(X).

Likewise, we say that S is C*-embedded in X if every map in C*(S) can be extended to a map in C*(X).

2.4 Theorem [9,17,49] Every completely regular space X has a compactification $\beta X$, with the following equivalent properties.

1) Every continuous map f of X into any compact space Y has a continuous extension $\bar{f}$ of $\beta X$ into Y.

2) Every map f in C*(X) has an extension to a map $f^{\beta}$ in C($\beta$ X).

3) Any two disjoint zero-sets in X have disjoint closures in $\beta$ X.

4) For any two zero-sets $Z_1$ and $Z_2$ in X,

$$\Gamma_{\beta X}(Z_1 \cap Z_2) = \Gamma_{\beta X}Z_1 \cap \Gamma_{\beta X}Z_2.$$

5) Distinct z-ultrafilters on X have distinct limits in $\beta$ X.

Furthermore, $\beta$ X is unique, in the following sense: if a compactification T of X satisfies any one of the listed conditions, then there exists a homeomorphism of $\beta$ X onto T that leaves X pointwise fixed.

Remark: The space $\beta X$ is known as the Stone-Čech compactification of X. According to the theorem, it is characterized as that compactification of X in which X is C*-embedded.

2.5 Theorem [17,32] Every completely regular space X has a realcompactification $\upsilon X$, contained in $\beta X$, with the following equivalent properties.

1) Every continuous map f of X into any realcompact space Y has a continuous extension $\bar{f}$ of $\upsilon X$ into Y.

2) Every map f in C(X) has an extension to a map $f^{\upsilon}$ in $C(\upsilon X)$.

3) If a countable family of zero-sets in X has empty intersection, then their closures in $\upsilon X$ have empty intersection.

4) For any countable family of zero-sets $Z_n$ in X,

$$\Gamma_{\upsilon X} \bigcap_n Z_n = \bigcap_n \Gamma_{\upsilon X} Z_n.$$

5) Every point of $\upsilon X$ is the limit of a unique z-ultrafilter on X, and it is a real z-ultrafilter.

Furthermore, the space $\upsilon X$ is unique, in the following sense: if a realcompactification T of X satisfies any one of the listed conditions, then there exists a homeomorphism of $\upsilon X$ onto T that leaves X pointwise fixed.

Remark: The space $\upsilon X$ is called the Hewitt realcompactification of X. By the theorem, it is characterized as that realcompactification in which X is C-embedded.

## Section 3: $\mathcal{E}$-compact spaces.

Every space in this section is assumed to be Hausdorff.

3.1 Definition [23] Let $\mathcal{E}$ be a class of spaces. A space X is said to be $\mathcal{E}$-compact if X is homeomorphic with a closed subspace of a product space of a subfamily of $\mathcal{E}$. We denote the class of all $\mathcal{E}$-compact spaces by $K\mathcal{E}$.

A space X is said to be $\mathcal{E}$-regular if X is homeomorphic with a subspace of a product space of a subfamily of $\mathcal{E}$. We denote the class of all $\mathcal{E}$-regular spaces by $R\mathcal{E}$.

3.2 Definition Two classes $\mathcal{E}$ and $\mathcal{E}'$ are said to be equivalent to each other if $K\mathcal{E} = K\mathcal{E}'$.

A class $\mathcal{E}$ of spaces is said to be simple, if $\mathcal{E}$ is equivalent to a single element class {Y}. In this case, $\mathcal{E}$ is also called Y-simple.

3.3 Theorem Let $\mathcal{E}$ be a class of spaces.

1) Every closed subspace of an $\mathcal{E}$-compact space is again $\mathcal{E}$-compact, i.e. $K\mathcal{E}$ is closed-hereditary.

2) Every subspace of an $\mathcal{E}$-regular space is again $\mathcal{E}$-regular, i.e. $R\mathcal{E}$ is hereditary.

3) Every product space of $\mathcal{E}$-compact ($\mathcal{E}$-regular) spaces is again $\mathcal{E}$-compact ($\mathcal{E}$-regular, respectively), i.e. $K\mathcal{E}$ and $R\mathcal{E}$ are productive. Conversely, if a product space of non-empty spaces is $\mathcal{E}$-compact ($\mathcal{E}$-regular), then each factor space is also $\mathcal{E}$-compact ($\mathcal{E}$-regular, respectively).

3.4 <u>Corollary</u> An arbitrary intersection of $\mathcal{E}$-compact subspaces of a given space is $\mathcal{E}$-compact.

3.5 <u>Corollary</u> Let f be a continuous map of an $\mathcal{E}$-compact space into a space Y. Then the total preimage of each $\mathcal{E}$-compact subset of Y under f is $\mathcal{E}$-compact.

3.6 <u>Corollary</u> Let $(X_\iota)_{\iota \in I}$ be a family of non-empty spaces. Then the sum space $\Sigma X_\iota$ is $\mathcal{E}$-compact if and only if each $X_\iota$ is $\mathcal{E}$-compact and I is $\mathcal{E}$-compact with respect to the discrete topology.

3.7 <u>Theorem</u> [23] For a space X (not necessarily Hausdorff), there exists a pair $(\beta_{\mathcal{E}}X, \beta_{\mathcal{E}})$ such that $\beta_{\mathcal{E}}X$ is $\mathcal{E}$-compact and $\beta_{\mathcal{E}}$ is a continuous map of X onto the dense subset $\beta_{\mathcal{E}}(X)$ of $\beta_{\mathcal{E}}X$ with the following property:
for any $\mathcal{E}$-compact space Y and any continuous map $f: X \rightarrow Y$, there is a continuous map $\bar{f}: \beta_{\mathcal{E}}X \longrightarrow Y$ with $\bar{f}\beta_{\mathcal{E}} = f$.

In particular, if X is $\mathcal{E}$-regular, then $(\beta_{\mathcal{E}}X, \beta_{\mathcal{E}})$ is an $K\mathcal{E}$-compactification, i.e. $(\beta_{\mathcal{E}}X, \beta_{\mathcal{E}})$ is an extension space of X such that $\beta_{\mathcal{E}}X$ is $\mathcal{E}$-compact.

3.8 <u>Theorem</u> [23] For a space X (not necessarily Hausdorff), there exists a pair ($\alpha_{\mathcal{E}}X$, $\alpha_{\mathcal{E}}$) such that $\alpha_{\mathcal{E}}X$ is $\mathcal{E}$-regular and $\alpha_{\mathcal{E}}$ is a continuous map of X onto $\alpha_{\mathcal{E}}X$ with the following property:
for any $\mathcal{E}$-regular space Y and any continuous map $f: X \longrightarrow Y$, there exists a continuous map $\bar{f}: \alpha_{\mathcal{E}}X \longrightarrow Y$ with $\bar{f}\alpha_{\mathcal{E}} = f$.

## Section 4: Categories.

4.1 <u>Definition</u> Let $\mathcal{A}$ be a small category and $\mathcal{L}$ a category. Then a funtor $D: \mathcal{A} \longrightarrow \mathcal{L}$ will be called a <u>diagram</u> in $\mathcal{L}$ over $\mathcal{A}$. A <u>lower bound</u> of the diagram D is a pair $(L, g_A)$, where L is an object of $\mathcal{L}$ and $(g_A: L \longrightarrow DA)_{A \in \mathcal{A}}$ is a family of morphisms in $\mathcal{L}$ such that for any morphism $f: A \longrightarrow A'$ in $\mathcal{A}$, $(Df)g_A = g_{A'}$.

A lower bound $(L, g_A)$ of D is said to be a <u>limit</u> of D if for any lower bound $(L', g'_A)$ of D, there exists a unique morphism $f: L' \longrightarrow L$ in $\mathcal{L}$ such that $g_A f = g'_A$ for each $A \in \mathcal{A}$.

4.2 <u>Definition</u> If every diagram in $\mathcal{L}$ over $\mathcal{A}$ has a limit, then $\mathcal{L}$ is said to be <u>$\mathcal{A}$-complete</u> or to have <u>$\mathcal{A}$-limits</u>. If $\mathcal{L}$ is $\mathcal{A}$-complete for every small category $\mathcal{A}$, then $\mathcal{L}$ is said to be <u>complete</u>.

Dually, one can define an <u>upper bound</u> of a diagram, <u>colimits</u>, and <u>cocompleteness</u>.

4.3 <u>Theorem</u> Let $\mathcal{L}$ be a category. Then the following are equivalent:

1) $\mathcal{L}$ is complete.
2) $\mathcal{L}$ has products and pullbacks.
3) $\mathcal{L}$ has products and equalizers.
4) $\mathcal{L}$ has products and inverse images.

4.4 <u>Definition</u> Let $\mathcal{A}$ and $\mathcal{L}$ be categories. An <u>adjunction</u> from $\mathcal{L}$ to $\mathcal{A}$ is a triple $(F,G,\varphi)$, where F and G are functors

$$\mathcal{L} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{A},$$

while $\varphi$ is a map which assigns to each pair of objects $B \in \mathcal{L}$, $A \in \mathcal{A}$ a bijection

$\varphi = \varphi_{B,A}$: $\mathcal{A}(FB,A) \longrightarrow \mathcal{L}(B,GA)$ which is natural in B and A.

In this case, the functor F is said to be a <u>left adjoint</u> for G, while G is called a <u>right adjoint</u> for F.

4.5 <u>Theorem</u> Let $(F,G,\varphi)$ be an adjunction from $\mathcal{L}$ to $\mathcal{A}$. Then

1) G preserves limits.
2) F preserves colimits.
3) There exist a natural transformation $\eta : 1_{\mathcal{L}} \longrightarrow GF$ and a natural transformation $\varepsilon : FG \longrightarrow 1_{\mathcal{A}}$.

4.6 <u>Definition</u> Let $\mathcal{A}'$ be a subcategory of a category $\mathcal{A}$ and A an object of $\mathcal{A}$ . An $\mathcal{A}'$-reflection of A is an object R(A) of $\mathcal{A}'$ together with a morphism $r_A$: $A \longrightarrow R(A)$ such that for every object A' of $\mathcal{A}'$ and every morphism f: $A \longrightarrow A'$ there exists a unique morphism $\bar{f}$: $R(A) \longrightarrow A'$ in $\mathcal{A}'$ with $\bar{f}r_A = f$.

If every object of $\mathcal{A}$ has a reflection in $\mathcal{A}'$, then $\mathcal{A}'$ is said to be a *reflective subcategory* of $\mathcal{A}$ .

Dually, one can define a *coreflective subcategory* of a category.

*Remark*: In the above definition, R becomes a functor from $\mathcal{A}$ to $\mathcal{A}'$, and R is a left adjoint for the inclusion functor I: $\mathcal{A}' \longrightarrow \mathcal{A}$. In this case, R is called the *reflector* of $\mathcal{A}$ in $\mathcal{A}'$.

On the other hand, if the inclusion functor I: $\mathcal{A}' \longrightarrow \mathcal{A}$ has a left adjoint R, then $\mathcal{A}'$ becomes a reflective subcategory of $\mathcal{A}$ and R becomes a reflector of $\mathcal{A}$ in $\mathcal{A}'$.

4.7 *Definition* Let $\mathcal{A}'$ be a reflective subcategory of $\mathcal{A}$ . If for every object $A \in \mathcal{A}$ , the reflection map $r_A$: $A \longrightarrow R(A)$ is an epimorphism, $\mathcal{A}'$ is said to be an *epireflective subcategory* of $\mathcal{A}$ .

4.8 *Theorem* Let $\mathcal{A}'$ be a full reflective subcategory of $\mathcal{A}$ . If a diagram in $\mathcal{A}'$ has a limit in $\mathcal{A}$ , then it has a limit in $\mathcal{A}'$.

4.9 <u>Theorem</u> Let $\mathcal{A}$ be a full subcategory of <u>Top</u> (<u>Haus</u>), where <u>Top</u> (<u>Haus</u>) is the category of all topological spaces (all Hausdorff spaces, respectively) and continuous maps. Then the following are equivalent.

1) $\mathcal{A}$ is epi-reflective in <u>Top</u> (<u>Haus</u>).

2) $\mathcal{A}$ is productive and hereditary (closed hereditary).

3) $\mathcal{A}$ is productive and for every $X \in \mathcal{A}$, $Y \in$ <u>Top</u> ($Y \in$ <u>Haus</u>), $f \in C(X,Y)$, $A \subseteq Y$ and $A \in \mathcal{A}$ implies $f^{-1}(A) \in \mathcal{A}$.

4) $\mathcal{A}$ is productive and for every $X \in$ <u>Top</u> ($X \in$ <u>Haus</u>), $A_\iota \subseteq X$, $A_\iota \in \mathcal{A}$ implies $\bigcap A_\iota \in \mathcal{A}$.

Proof of the theorem can be found in [25].

In what follows, every (epi-) reflective subcategory of a category is assumed to be full and replete.

# CHAPTER I

## EXTENSIVE SUBCATEGORIES

### Section 1: Extensive subcategories.

The following definition is due to B. Banaschewski [5] for the case of the category Haus.

1.1 Definition Let $\mathcal{A}$ be a subcategory of the category Haus (or HUnif) of Hausdorff (uniform, respectively) spaces and (uniformly, respectively) continuous maps. A subcategory $\mathcal{L}$ of $\mathcal{A}$ is said to be an extensive subcategory of $\mathcal{A}$ if it is a reflective subcategory such that the reflection maps $r_X\colon X \longrightarrow rX$ with respect to $\mathcal{L}$ are dense embeddings for each $X \in \mathcal{A}$ .

Examples: 1) The category of compact spaces and continuous maps is extensive in the category of completely regular spaces and continuous maps via the Stone-Čech compactifications.

2) The category of zero-dimensional compact spaces and continuous maps is extensive in the category of zero-dimensional spaces and continuous maps via the maximal zero-dimensional compactifications [1].

3) The category of complete Hausdorff uniform spaces and uniformly continuous maps is extensive in the category <u>HUnif</u>.

1.2 <u>Theorem</u> Let $\mathcal{L}$ be an extensive subcategory of $\mathcal{A}$. Then every reflective subcategory of $\mathcal{A}$ containing $\mathcal{L}$ is also extensive in $\mathcal{A}$.

<u>Proof</u>: Let $\mathcal{C}$ be a reflective subcategory of $\mathcal{A}$ containing $\mathcal{L}$. For any $X \in \mathcal{A}$, let $r_{\mathcal{L}} : X \longrightarrow r_{\mathcal{L}} X$ and $r_{\mathcal{C}} : X \longrightarrow r_{\mathcal{C}} X$ be reflections of X with respect to $\mathcal{L}$ and $\mathcal{C}$, respectively. Since $\mathcal{L}$ is contained in $\mathcal{C}$, there exists a unique morphism $\bar{r}_{\mathcal{L}} : r_{\mathcal{C}} X \longrightarrow r_{\mathcal{L}} X$ with the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{r_{\mathcal{C}}} & r_{\mathcal{C}} X \\ {\scriptstyle r_{\mathcal{L}}}\downarrow & \swarrow {\scriptstyle \bar{r}_{\mathcal{L}}} & \\ r_{\mathcal{L}} X & & \end{array}$$

Since $r_{\mathcal{L}}$ is an embedding, $r_{\mathcal{C}}$ is also an embedding.

It is easy to show that $\bar{r}_{\mathcal{L}}$ is a $\mathcal{L}$-reflection of $r_{\mathcal{C}} X$. Indeed, for any $Y \in \mathcal{L}$ and for any $f : r_{\mathcal{C}} X \longrightarrow Y$ in $\mathcal{A}$, there is a unique $\bar{f} : r_{\mathcal{L}} X \longrightarrow Y$ such that the outer triangle in the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{r_{\mathcal{C}}} & r_{\mathcal{C}} X & \xrightarrow{\bar{r}_{\mathcal{L}}} & r_{\mathcal{L}} X \\ & {\scriptstyle fr_{\mathcal{C}}}\searrow & \downarrow {\scriptstyle f} & \swarrow {\scriptstyle \bar{f}} & \\ & & Y & & \end{array}$$

Then, $\bar{f}\bar{r}_{\mathcal{L}} = f$ and the uniqueness of $\bar{f}$ follows from that $r_{\mathcal{C}}$ is a reflection. Since $\mathcal{L}$ is extensive in $\mathcal{A}$, $\bar{r}_{\mathcal{L}}$

is also a dense embedding. For any non-empty open set U of $r_{\mathcal{L}}X$, there is an open set V of $r_{\mathcal{L}}X$ with $\bar{r}_{\mathcal{L}}(U) = V \cap \bar{r}_{\mathcal{L}}(r_{\mathcal{L}}X)$. Since $r_{\mathcal{L}}$ is dense, there is an $x \in X$ with $r_{\mathcal{L}}(x) \in V$. It is obvious that $r_{\mathcal{L}}(x) \in U$. Hence $r_{\mathcal{L}}$ is dense.

1.3 <u>Definition</u> Let $\mathcal{A}$ be a subcategory of <u>Haus</u>. A subcategory $\mathcal{L}$ of $\mathcal{A}$ is said to be <u>left-fitting</u> with respect to perfect morphisms if X belongs to $\mathcal{L}$ whenever $f\colon X \longrightarrow Y$ is a perfect morphism in $\mathcal{A}$ and Y belongs to $\mathcal{L}$.

1.4 <u>Lemma</u> Let

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ Z & \xrightarrow{\ g\ } & P \end{array}$$

be a commutative diagram in <u>Haus</u>.

1) Suppose h be dense and k be an embedding. Then $g(Z - h(X)) \subseteq P - k(Y)$ if f is perfect. Furthermore, if h is an embedding, then the diagram is a pullback.

2) Suppose Z be compact and h be an embedding. Then f is perfect if $g(Z - h(X)) \subseteq P - k(Y)$.

<u>Proof</u>: Regarding 1), suppose z be an element of Z with $g(z) = k(y)$ for some $y \in Y$. Since h is dense, there is an ultrafilter $\mathcal{U}$ on X containing $h^{-1}(\mathcal{D}(z))$, where $\mathcal{D}(z)$ is the neighborhood filter of z on Z. Using the commutativity of the diagram and k being an embedding, $f(\mathcal{U})$ converges to y. Since f is perfect, there exists a limit point x of $\mathcal{U}$

such that $f(x) = y$. Hence, $h(\mathcal{U})$ converges to $h(x)$ and $z$ simultaneously, so that $h(x) = z$.

For the second part, suppose $u: U \longrightarrow Z$ and $v: U \longrightarrow Y$ be continuous maps with $gu = kv$. It is easy to show that $u(U) \subseteq h(X)$. Indeed, suppose $u(p) \notin h(X)$ for some $p \in U$. Then, $gu(p) = kv(p) \in k(Y)$ which is a contradiction to $g(Z - h(X)) \subseteq P - k(Y)$. Let $\bar{u}: U \longrightarrow X$ be a map defined by $h\bar{u}(p) = u(p)$ for each $p \in U$. Since $h$ is an embedding, $\bar{u}$ is continuous. Since $kf\bar{u} = gh\bar{u} = gu = kv$ and $k$ is an embedding, $f\bar{u} = v$. The uniqueness of $\bar{u}$ follows from $h$ being an embedding.

Regarding 2), suppose $\mathcal{U}$ be an ultrafilter on $X$ and $y$ be a limit point of $f(\mathcal{U})$. Since $Z$ is compact, there is a limit point of $h(\mathcal{U})$, say $z$ and then $g(z) = g(\lim h(\mathcal{U})) = \lim gh(\mathcal{U}) = \lim kf(\mathcal{U}) = k(y) \in k(Y)$. Since $g^{-1}(k(Y)) \subseteq h(X)$, there is an element $x \in X$ with $h(x) = z$. Since $h$ is an embedding, $x$ is a limit point of $\mathcal{U}$. Hence $f$ is perfect.

1.5 <u>Theorem</u> Every extensive subcategory of a category $\mathcal{A}$ is left-fitting with respect to perfect morphisms.

<u>Proof</u>: Let $\mathcal{L}$ be an extensive subcategory of $\mathcal{A}$. Suppose $f: X \longrightarrow Y$ be a perfect morphism in $\mathcal{A}$ and $Y \in \mathcal{L}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle r_X}\downarrow & & \downarrow{\scriptstyle r_Y} \\ rX & \xrightarrow{\ rf\ } & rY \end{array}$$

, where $r_X$ and $r_Y$ are reflection maps of X and Y respectively. Since Y belongs to $\mathcal{L}$, $r_Y$ is a homeomorphism. By Lemma 1.4, $r_X$ is onto, i.e. a homeomorphism. Hence X belongs to $\mathcal{L}$.

It is well known[7] that a Hausdorff space is compact if and only if the map on the space into the singleton space is perfect. Hence we have the following:

1.6 <u>Theorem</u> Let $\mathcal{A}$ be a subcategory of <u>Haus</u> or <u>HUnif</u> such that it contains a singleton space S and $\mathrm{Hom}_{\mathcal{A}}(X, S) \neq \phi$ for each $X \in \mathcal{A}$. Then any extensive subcategory of $\mathcal{A}$ contains all compact spaces belonging to $\mathcal{A}$, whenever it contains a singleton space.

## <u>Section 2: Limit operators and extensive subcategories.</u>

2.1 <u>Definition</u> Let $\mathcal{A}$ be a subcategory of <u>Top</u> or the category <u>Unif</u> of uniform spaces and uniformly continuous maps. An operator $\ell$ which associates with every pair (X, A), where X is an object of $\mathcal{A}$ and A is a subset of X, a subset $\ell_X A$ of X is said to be a <u>limit-operator</u> on $\mathcal{A}$ if $\ell$ satisfies the following three conditions:

1) If A is a subset of X, then $A \subseteq \ell_X A \subseteq \Gamma_X A$, where $\Gamma_X$ denotes the closure operator on X.

2) If A and B are subsets of X then $\ell_X(A \cup B) = \ell_X A \cup \ell_X B$.

3) If $f: X \longrightarrow Y$ is a morphism in $\mathcal{A}$ and A is a subset of X then $f(\ell_X A) \subseteq \ell_Y f(A)$.

A limit-operator $\ell$ on $\mathcal{A}$ is said to be <u>idempotent</u> if $\ell$ satisfies the following:

4) If A is a subset of X then $\ell_X(\ell_X A) = \ell_X A$.

H. Herrlich has defined (idempotent) limit-operators on <u>Top</u> first [26].

It is obvious that an idempotent limit-operator $\ell$ gives rise to a closure operator $\ell_X$ on every object X of $\mathcal{A}$ and that every morphism $f: X \longrightarrow Y$ in $\mathcal{A}$ is also continuous with respect to the new topologies generated by $\ell_X$ and $\ell_Y$.

2.2 <u>Definition</u> Let $\ell$ be a limit-operator on $\mathcal{A}$. A subset A of an object X of $\mathcal{A}$ is said to be <u>$\ell$-closed</u> if $\ell_X A = A$.

<u>Remark</u>: For any limit-operator $\ell$ on $\mathcal{A}$, the family of all $\ell$-closed subsets of $X \in \mathcal{A}$ forms the family of all closed subsets of X with respect to some topology on X.

For any limit-operator $\ell$ on $\mathcal{A}$, there is an associated idempotent limit-operator $\bar{\ell}$ on $\mathcal{A}$ which is defined as follows:

$$\bar{\ell}_X A = \bigcap \{B \mid A \subseteq B \subseteq X,\ \ell_X B = B\} \text{ for } X \in \mathcal{A}.$$

Then it is obvious that $\bar{\ell}_X$ is exactly the closure operator of the space X with the family of $\ell$-closed subsets of X as the family of closed subsets.

In what follows, every subcategory of a category is assumed to be replete and full. And an extension space of a space is also assumed to be a space of which the space is a dense subspace.

2.3 <u>Definition</u> Let $\mathcal{A}$ be a subcategory of <u>Haus</u>. Then $\mathcal{A}$ is said to be <u>hereditary</u> if for any $X \in \mathcal{A}$ , all natural embeddings of subspaces into X are morphisms in $\mathcal{A}$ .

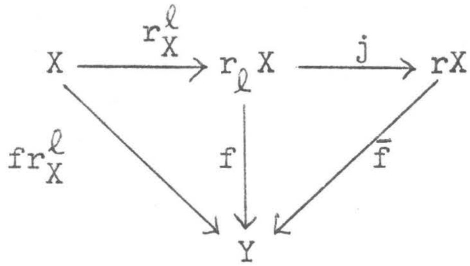
Let $\mathcal{L}$ be an extensive subcategory of a category $\mathcal{A}$ of Hausdorff spaces and continuous maps. For an idempotent limit-operator $\ell$ on $\mathcal{L}$, let $\mathcal{L}_\ell$ be the subcategory determined by those objects of $\mathcal{A}$ which are $\ell$-closed in their $\mathcal{L}$-reflection spaces.

2.4 <u>Theorem</u> If $\mathcal{A}$ is hereditary, then $\mathcal{L}_\ell$ is also an extensive subcategory of $\mathcal{A}$ .

<u>Proof</u>: For every $X \in \mathcal{A}$ , let $r_X\colon X \longrightarrow rX$ be the $\mathcal{L}$-reflection of X such that X is a dense subspace of rX and $r_X$ is the natural embedding. Let $r_\ell X$ be the subspace of rX with $\ell_{rX}X$ as underlying set.

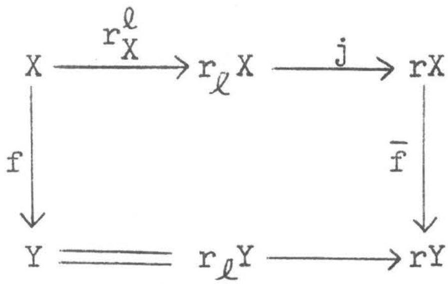
Since $\mathcal{A}$ is hereditary, $r_\ell X$ belongs to $\mathcal{A}$ . It is easy to show that $r_\ell X$ belongs to $\mathcal{L}_\ell$ . Indeed, let $r^\ell_X\colon X \longrightarrow r_\ell X$ and $j\colon r_\ell X \longrightarrow rX$ be the natural embeddings respectively. We wish to show that j is the $\mathcal{L}$-reflection of $r_\ell X$. Because, for any Y in $\mathcal{L}$ , and for any $f\colon r_\ell X \longrightarrow Y$ in $\mathcal{A}$, there is a unique

$\bar{f}$: $rX \longrightarrow Y$ in $\mathcal{L}$ such that the outer triangle in the diagram




commutes. Hence, $\bar{f}j = f$ and the uniqueness of $\bar{f}$ follow from that $r_X^\ell$ and j are dense embeddings. Since $r_\ell X$ is $\ell$-closed in its $\mathcal{L}$-reflection space rX, $r_\ell X$ belongs to $\mathcal{L}_\ell$.

Now, we can conclude that $r_X^\ell$: $X \longrightarrow r_\ell X$ is the $\mathcal{L}_\ell$-reflection. For any Y in $\mathcal{L}_\ell$, and for any morphism f: $X \longrightarrow Y$, there exists a unique $\bar{f}$: $rX \longrightarrow rY$ in $\mathcal{L}$ such that the diagram




commutes. Since $\bar{f}(r_\ell X) = \bar{f}(\ell_{rX}X) \subseteq \ell_{rY}\bar{f}(X) \subseteq \ell_{rY}Y = Y$, $\bar{f}j$: $r_\ell X \longrightarrow Y$ is a well-defined continuous map. Let $f_\ell = \bar{f}j$. Then it is obvious that $f_\ell r_X^\ell = f$. Noting that $r_X^\ell$ is a dense embedding, $f_\ell$ is unique. This completes the proof.

By Propositions and Theorem 1 in [26], every coreflective subcategory $\mathcal{C}$ of the category Top generates an idempotent limit-operator $\ell(\mathcal{C})$ on Top and for every topological space X, $\ell(\mathcal{C})_X$ is precisely the closure operator on the $\mathcal{C}$-coreflection space of X. Hence the following is immediate from Theorem 2.4 for the same categories $\mathcal{A}$ and $\mathcal{L}$ as above.

2.5 Corollary For any coreflective subcategory $\mathcal{C}$ of Top, let $\mathcal{L}_{\mathcal{C}}$ be the subcategory of $\mathcal{A}$ determined by those members of $\mathcal{A}$ which are closed in the $\mathcal{C}$ -coreflection spaces of their $\mathcal{L}$ -reflection spaces. Then $\mathcal{L}_{\mathcal{C}}$ is also an extensive subcategory of $\mathcal{A}$ .

Remark: For any pair of coreflective subcategories $\mathcal{C}$ and $\mathcal{C}'$ of Top, $\mathcal{L}_{\mathcal{C}} \supseteq \mathcal{L}_{\mathcal{C}'}$ if $\mathcal{C}' \supseteq \mathcal{C}$ . In particular, for any coreflective subcategory $\mathcal{C}$ of Top, $\mathcal{L}$ is contained in $\mathcal{L}_{\mathcal{C}}$ , for $\mathcal{L} = \mathcal{L}_{\text{Top}}$.

Proof: For any $X \in \mathcal{A}$ , let $c_{\mathcal{C}} : c_{\mathcal{C}}(r_{\mathcal{L}}X) \longrightarrow r_{\mathcal{L}}X$ and $c_{\mathcal{C}'} : c_{\mathcal{C}'}(r_{\mathcal{L}}X) \longrightarrow r_{\mathcal{L}}X$ be coreflections of the $\mathcal{L}$-reflection space $r_{\mathcal{L}}X$ of X with respect to $\mathcal{C}$ and $\mathcal{C}'$ respectively. Since $\mathcal{C}$ is contained in $\mathcal{C}'$, the map $f: c_{\mathcal{C}}(r_{\mathcal{L}}X) \longrightarrow c_{\mathcal{C}'}(r_{\mathcal{L}}X)$ defined by $x \rightsquigarrow x$ is continuous. Suppose X belongs to $\mathcal{L}_{\mathcal{C}'}$. Then X is closed in $c_{\mathcal{C}'}(r_{\mathcal{L}}X)$. Hence X is also closed in $c_{\mathcal{C}}(r_{\mathcal{L}}X)$, i.e. X belongs to $\mathcal{L}_{\mathcal{C}}$.

The full subcategory of HUnif determined by complete Hausdorff uniform spaces will be denoted by $\mathcal{C}_e$.

For any idempotent limit-operator $\ell$ on $\mathcal{C}_e$ , let $\mathcal{C}_{e\ell}$ be the subcategory determined by the Hausdorff uniform spaces which are $\ell$ -closed in their completions.

2.6 Theorem The subcategory $\mathcal{C}_\ell$ of HUnif is also extensive in HUnif.

Proof: For a Hausdorff uniform space X, let cX be its completion and $c_X\colon X \longrightarrow cX$ the natural embedding. And let $c_\ell X$ be the subspace of cX with $\ell_{cX}X$ as underlying set. By the uniqueness of the completion of a uniform space, cX is isomorphic with $c(c_\ell X)$. Since $c_\ell X$ is $\ell$-closed in cX, $c_\ell X$ belongs to $\mathcal{C}_\ell$. Now, let $c_X^\ell\colon X \longrightarrow c_\ell X$ be the natural embedding of X into $c_\ell X$. For any Y in $\mathcal{C}_\ell$, and any uniformly continuous map $f\colon X \longrightarrow Y$, there is a unique uniformly continuous map $\bar{f}\colon cX \longrightarrow cY$ such that $\bar{f}c_X = c_Y f$, where $c_Y\colon Y \longrightarrow cY$ is the natural embedding of Y into its completion. Since $\bar{f}$ is a morphism in $\mathcal{C}$, $\bar{f}(c_\ell X) = \bar{f}(\ell_{cX}X) \subseteq \ell_{cY}\bar{f}(X) \subseteq \ell_{cY}Y = Y$. Thus $\bar{f}|c_\ell X\colon c_\ell X \longrightarrow Y$ is well defined and uniformly continuous. Moreover, let $f_\ell = \bar{f}|c_\ell X$, then $f_\ell c_X = f$. Finally the uniqueness of $f_\ell$ follows from that $c_X$ is a dense embedding.

2.7 Corollary For every idempotent limit-operator $\ell$ on $\mathcal{C}$, the category $\mathcal{C}_\ell$ is productive and closed hereditary.

2.8 Definition A Hausdorff space X is said to be Hausdorff closed if for every homeomorphism f of X onto a subspace of a Hausdorff space X', f(X) is closed in X'.

2.9 Definition If X and Y are spaces, then a map $f\colon X \longrightarrow Y$ is said to be semi-open if the image under f of each non-empty open set of X has non-empty interior in Y.

Remark: Since the composition of semi-open maps is semi-open and every identity map on a space is semi-open, we can consider the category of Hausdorff spaces and continuous semi-open maps. We will denote it by Haus*.

H. Herrlich and G. E. Strecker have shown [30] that the subcategory $\mathfrak{M}^*$ of Haus* determined by all Hausdorff closed spaces is epi-reflective in Haus* via the Katětov extensions [34].

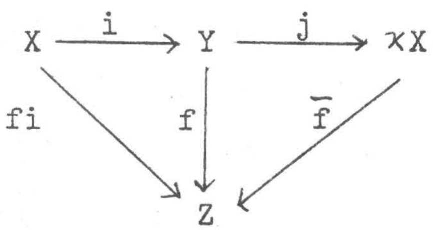
2.10 Lemma [30] Let Z and Y be spaces and $f: Z \longrightarrow Y$ be a map. If X is dense in Z and if $f|X: X \longrightarrow Y$ is semi-open then f is semi-open.

Proof: For any non-empty open set U in Z, $U \cap X$ is again non-empty open in X, for X is dense in Z. Since $f|X$ is semi-open, $f(U \cap X)$ has non-empty interior in Y, so that so does f(U).

The following corollary was proved by B. Banaschewski first [3]. However, we will give here another proof.

2.11 Corollary Let X be a Hausdorff space and let Y be a subspace of the Katětov extension $\kappa X$ of X. If Y contains X, then $\kappa X$ and $\kappa Y$ are homeomorphic.

Proof: It is enough to show that the natural embedding j: $Y \longrightarrow \kappa X$ of Y into $\kappa X$ is a reflection of Y in $\mathcal{A}^*$. By the above lemma, j belongs to Haus*. Let i: $X \longrightarrow Y$ be the natural embedding of X into Y. Since X is open in Y, i also belongs to Haus*. For any Z in $\mathcal{A}^*$ and for any f: $Y \longrightarrow Z$ in Haus*, there is a unique morphism $\bar{f}$: $\kappa X \longrightarrow Z$ in $\mathcal{A}^*$ such that the outer triangle in the diagram




commutes. Since X is dense in Y, $\bar{f}j = f$. The uniqueness of $\bar{f}$ follows from that j is a dense embedding.

Remark: The category Haus* is not hereditary, for the natural embedding of $\mathbb{R}$ into $\mathbb{C}$ is not a morphism in Haus*. Hence we cannot apply Theorem 2.4 for the categories Haus* and $\mathcal{A}^*$.

For any idempotent limit-operator $\ell$ on $\mathcal{A}^*$, let $\mathcal{A}_\ell^*$ be the subcategory of Haus* determined by those members of Haus* which are $\ell$-closed in their Katětov extensions.

2.12 Theorem The subcategory $\mathcal{A}_\ell^*$ of Haus* is also extensive in Haus*.

Proof: For any Hausdorff space X, let $\kappa_X$: $X \longrightarrow \kappa X$ be the reflection of X in $\mathcal{A}^*$, i.e. $\kappa X$ is the Katětov extension of X and $\kappa_X$ is the natural embedding. Let $\kappa_\ell X$ be the subspace of $\kappa X$ with $\ell_{\kappa X} X$ as underlying set and $\kappa_\ell : X \longrightarrow \kappa_\ell X$ the natural embedding of X into $\kappa_\ell X$. By Corollary 2.11, $\kappa X = \kappa \kappa_\ell X$. Being $\ell$-closed in $\kappa X$, hence in $\kappa \kappa_\ell X$, $\kappa_\ell X$ belongs to $\mathfrak{A}_\ell^*$. By Lemma 2.10, $\kappa_\ell$ is a morphism in Haus*. For any Y in $\mathfrak{A}_\ell^*$ and for any $f: X \longrightarrow Y$ in Haus*, there is a morphism $\bar{f}: \kappa X \longrightarrow \kappa Y$ such that $\kappa_Y f = \bar{f} j \kappa_\ell$, where j is the natural embedding of $\kappa_\ell X$ into $\kappa X$. Since $\bar{f}$ is a morphism in $\mathfrak{A}^*$, $\bar{f}(\kappa_\ell X) = \bar{f}(\ell_{\kappa X} X) \subseteq \ell_{\kappa Y} \bar{f}(X) \subseteq \ell_{\kappa Y} Y = Y$. Let $f_\ell = \bar{f} j$. Then $f_\ell$ is a well defined continuous map on $\kappa_\ell X$ into Y and $f_\ell \kappa_\ell = f$. Thus, $f_\ell$ belongs to $\mathfrak{A}_\ell^*$ by Lemma 2.10. Finally, the uniqueness of $f_\ell$ follows from that $\kappa_\ell$ is a dense embedding. This completes the proof.

2.13 Definition A subset of a space is said to be regular-closed if it is the same as the closure of its interior.

2.14 Proposition Every epi-reflective subcategory of Haus* is productive and regular-closed hereditary, i.e. each regular-closed subspace of an object of the subcategory is also its object.

Proof of the proposition can be found in [30].

2.15 Corollary For any idempotent limit-operator $\ell$, the category $\mathfrak{A}_\ell^*$ is productive and regular-closed hereditary.

Remark: D. Harris has considered [19] the category pHaus of all Hausdorff spaces and p-maps, where by a p-map is meant a continuous map such that the inverse image of a p-cover of the codomain is a p-cover of the domain, while a p-cover of a space is an open covering such that the union of some finite subfamily is dense. Then he has shown that the full subcategory of pHaus determined by all Hausdorff closed spaces is also extensive in pHaus via the Katětov extensions and that pHaus is the largest subcategory of Haus in which the full subcategory of Hausdorff closed spaces is extensive via the Katětov extensions. Instead of Haus*, if we consider the category pHaus, Corollary 2.11 and Lemma 2.10 are still true. Hence it is easy to show Theorem 2.12 for the category pHaus.

## Section 3: Limit operators.

In Section 2, we have shown that for any idempotent limit-operator $\ell$ on any extensive subcategory $\mathcal{L}$ of some suitable category $\mathcal{A}$ , $\mathcal{L}_\ell$ is extensive in $\mathcal{A}$ and contains $\mathcal{L}$.

In this section, it will be shown that for any reflective subcategory $\mathcal{E}$ of a certain category $\mathcal{A}$ containing an extensive subcategory $\mathcal{L}$ of $\mathcal{A}$ , there is a limit-operator associated to $\mathcal{E}$ . And it will give another method to construct new extensive subcategories from a well known extensive subcategory.

3.1 <u>Definition</u> Let $\mathcal{A}$ be a subcategory of <u>Top</u> or <u>HUnif</u>. An operator $\ell$ which associates with every pair (X, A), where X is an object of $\mathcal{A}$ and A is a subset of X, a subset $\ell_X A$ of X is said to be a <u>semi-limit-operator</u> on $\mathcal{A}$ if $\ell$ satisfies the conditions 1) and 3) in Definition 2.1, namely

i) if A is a subset of X, then $A \subseteq \ell_X A \subseteq \Gamma_X A$, and

ii) if $f: X \longrightarrow Y$ is a morphism in $\mathcal{A}$ and A is a subset of X then $f(\ell_X A) \subseteq \ell_Y f(A)$.

iii) if $A \subseteq B \subseteq X$, then $\ell_X A \subseteq \ell_X B$.

<u>Remark</u>: For any semi-limit-operator $\ell$ on $\mathcal{A}$, there is an associated idempotent limit-operator $\bar{\ell}$ on $\mathcal{A}$.

<u>Proof</u>: For any $X \in \mathcal{A}$, let $\bar{\ell}_X$ be the closure operator on X with respect to a topology with $\mathcal{S}_\ell(X) = \{A \mid \ell_X A = A\}$ as a subbase for closed sets. Since $\Gamma_X A \subseteq \ell_X(\Gamma_X A) \subseteq \Gamma_X(\Gamma_X A) = \Gamma_X A$, $A \subseteq \ell_X \Gamma_X A = \Gamma_X A \in \mathcal{S}_\ell(X)$. Hence $A \subseteq \bar{\ell}_X A \subseteq \Gamma_X A$. Since $\bar{\ell}_X$ is the closure operator, $\bar{\ell}_X(A \cup B) = \bar{\ell}_X A \cup \bar{\ell}_X B$ and $\bar{\ell}_X(\bar{\ell}_X A) = \bar{\ell}_X A$. For any morphism $f: X \longrightarrow Y$ in $\mathcal{A}$ and for any $F \in \mathcal{S}_\ell(Y)$, $f(\ell_X f^{-1}(F)) \subseteq \ell_Y f(f^{-1}(F)) \subseteq \ell_Y F = F$, so that $\ell_X f^{-1}(F) \subseteq f^{-1}(F)$, i.e. $f^{-1}(F) \in \mathcal{S}_\ell(X)$. Hence f is continuous with respect to the new topologies generated by $\bar{\ell}_X$ and $\bar{\ell}_Y$. Thus for any $A \subseteq X$, $f(\bar{\ell}_X A) \subseteq \bar{\ell}_Y f(A)$.

<u>Remark</u>: For any semi-limit-operator $\ell$ on <u>Top</u>, let $\bar{\ell}$ be the associated idempotent limit-operator with $\ell$. Let $\mathcal{L}(\ell) = \{X \in \underline{\text{Top}} \mid \mathcal{S}_\ell(X) \subseteq \mathfrak{F}(X)\}$ and $\mathcal{L}(\bar{\ell}) = \{X \in \underline{\text{Top}} \mid \{A \mid \bar{\ell}_X A = A\} \subseteq \mathfrak{F}(X)\}$, where $\mathfrak{F}(X)$ is the family of closed

sets of X. Then $\mathcal{L}(\ell) = \mathcal{L}(\bar{\ell})$.

Hence every semi-limit-operator $\ell$ on <u>Top</u> generates a coreflective subcategory of <u>Top</u>.

<u>Proof</u>: It is obvious that $\mathcal{L}(\ell) = \mathcal{L}(\bar{\ell})$. Since $\mathcal{L}(\bar{\ell})$ is a coreflective subcategory of <u>Top</u> [30], $\ell$ generates the coreflective subcategory $\mathcal{L}(\ell) = \mathcal{L}(\bar{\ell})$.

3.2 <u>Theorem</u> Let $\mathcal{A}$ be a subcategory of <u>Haus</u> or <u>HUnif</u> and $\mathcal{L}$ be an extensive subcategory of $\mathcal{A}$.
Suppose $\mathcal{A}$ be hereditary and $\mathcal{E}$ be a reflective subcategory of $\mathcal{A}$ containing $\mathcal{L}$. Then we have the following:

1) There exists a semi-limit-operator $\ell$ on $\mathcal{L}$ such that $\mathcal{E}$ is precisely the subcategory of $\mathcal{A}$ determined by the class $\{X \in \mathcal{A} \mid \ell_{rX} X = X\}$, where $r_X\colon X \longrightarrow rX$ is a $\mathcal{L}$-reflection of X for each $X \in \mathcal{A}$.

2) There exists an idempotent limit-operator $\bar{\ell}$ on $\mathcal{L}$ such that $\mathcal{L}_{\bar{\ell}} \supseteq \mathcal{E}$.

<u>Proof</u>: By Theorem 1.2, $\mathcal{E}$ is also extensive in $\mathcal{A}$. For any $X \in \mathcal{A}$, let $e_X\colon X \longrightarrow eX$ be an $\mathcal{E}$-reflection of X.

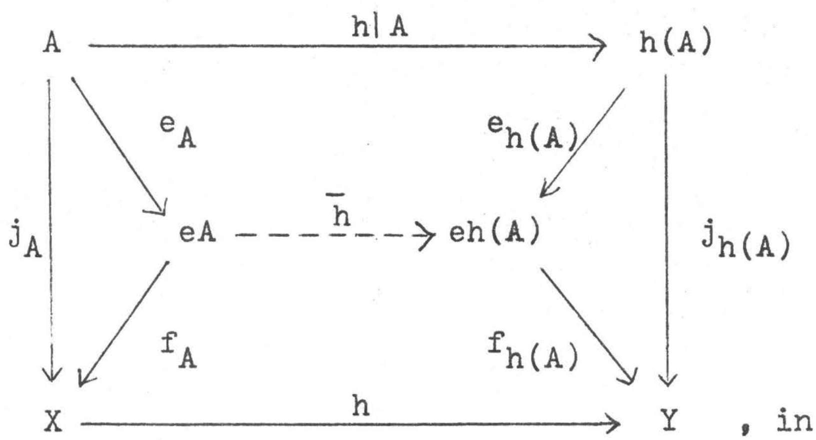
Regarding 1), Let A be a subset of an object X of $\mathcal{L}$. Since $\mathcal{A}$ is hereditary, the subspace A of X belongs to $\mathcal{A}$. For the natural embedding $j_A\colon A \longrightarrow X$, there is a unique morphism $f_A\colon eA \longrightarrow X$ in $\mathcal{E}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{e_A} & eA \\ {\scriptstyle j_A}\downarrow & \swarrow {\scriptstyle f_A} & \\ X & & \end{array}$$

commutes, for $X \in \mathcal{L} \subseteq \mathcal{E}$.
We define $\ell_X A$ by $f_A(eA)$. We wish to show that the operator $\ell$ defined as above is a semi-limit-operator on $\mathcal{L}$.

Firstly, $A = j_A(A) = f_A e_A(A) \subseteq f_A(eA) = \ell_X A$, i.e. $A \subseteq \ell_X A$. And $\ell_X A = f_A(eA) = f_A(\Gamma_{eA}(e_A(A))) \subseteq \Gamma_X f_A(e_A(A)) = \Gamma_X j_A(A) = \Gamma_X A$, i.e. $\ell_X A \subseteq \Gamma_X A$.

Secondly, for any morphism $h\colon X \longrightarrow Y$ in $\mathcal{L}$, we have the following diagram




, in which the outer rectangle and the upper trapezoid commute, where $j_{h(A)}$, $e_{h(A)}$ and $f_{h(A)}$ can be understood such as $j_A$, $e_A$ and $f_A$, and $\bar{h}$ is the unique morphism determined by $e_A$ and $e_{h(A)}(h|A)$.
Since $f_{h(A)}\bar{h}e_A = f_{h(A)}e_{h(A)}(h|A) = j_{h(A)}(h|A) = hj_A = hf_A e_A$, $f_{h(A)}\bar{h} = hf_A$, for $e_A$ is the reflection map.
Hence $h(\ell_X A) = h(f_A(eA)) = f_{h(A)}\bar{h}(eA) \subseteq f_{h(A)}(eh(A)) = \ell_Y h(A)$, i.e. $h(\ell_X A) \subseteq \ell_Y h(A)$. Obviously $\ell$ satisfies iii).

Let $\mathcal{L}_\ell = \{X \in \mathcal{A} \mid \ell_{rX} X = X\}$. Now, we wish to show that $\mathcal{L}_\ell = \mathcal{E}$. For any $X \in \mathcal{E}$, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{e_X = 1_X} & eX \\ {\scriptstyle j_X = r_X}\big\downarrow & \swarrow {\scriptstyle f_X = j_X} & \\ rX & & \end{array}$$

Hence, $\ell_{rX}X = f_X(eX) = j_X(X) = X$, i.e. $X \in \mathcal{L}_\ell$. Conversely, suppose X does not belong to $\mathcal{E}$. Since $\mathcal{E}$ is extensive in $\mathcal{A}$, $e_X$ is not onto. By the proof of Theorem 1.2, the morphism $f_X$: $eX \longrightarrow rX$ defined by $e_X$ and $r_X$ is a $\mathcal{L}$-reflection of eX, so that $f_X$ is a dense embedding. Hence, $\phi \neq f_X(eX - e_X(X)) \subseteq f_X(eX) - f_X(e_X(X)) = \ell_{rX}X - X$. This completes the proof.

Regarding 2), let $\bar{\ell}$ be the associated idempotent limit-operator with $\ell$ in 1). Then it is obvious that $\mathcal{E}$ is contained in $\mathcal{L}_{\bar{\ell}}$.

<u>Remark</u>: 1) For the semi-limit-operator $\ell$ on $\mathcal{L}$ defined in the above theorem, and for any $X \in \mathcal{L}$, $\mathcal{S}_\ell(X)$ is precisely the family of subsets of X which belong to $\mathcal{E}$ as subspaces of X.

2) $\mathcal{L}_{\bar{\ell}}$ may contain $\mathcal{E}$ properly.

<u>Proof</u>: Using the same argument of the proof of Theorem 3.2, one can easily prove 1). We omit the proof of 1).

Regarding 2), let $\mathcal{A}$ be the category of completely regular spaces and continuous maps, $\mathcal{L}$ the subcategory of $\mathcal{A}$ determined by compact spaces and $\mathcal{E}$ the subcategory of $\mathcal{A}$

determined by realcompact spaces.

S. Mrówka has shown [43] that there is a completely regular space M which can be represented as the union of two closed subsets - each of which is realcompact in its relative topology - and which, however, is not realcompact.
Hence, by Remark 1), the space $M \in \mathcal{L}_{\bar{\ell}} - \mathcal{E}$ .

# CHAPTER II

## n-COMPACTLIKE SPACES AND SEQUENTIALLY CLOSED SPACES

In this chapter, we will apply the results of Chap. I to some subcategories of <u>Haus</u> and <u>HUnif</u>.

### Section 1: n-compact spaces.

1.1 <u>Definition</u> Let n be an infinite cardinal number, and let X be a topological space. A subset of X is said to be a <u>$G_n$-set</u> if it is an intersection of fewer than n open subsets of X.

It is clear that the $G_n$-sets of a topological space $(X, \mathfrak{T})$ form a basis for a topology on X. We denote the new topology by $\mathfrak{T}_n$.

Since the inverse image of a $G_n$-set under a continuous map is also a $G_n$-set, the closure operator $\Gamma_n$ on X with respect to $\mathfrak{T}_n$ gives rise to an idempotent limit-operator on <u>Top</u>.

A subset of X is said to be <u>n-closed</u> if it is closed with respect to $\mathfrak{T}_n$. Finally, by the n-closure of a subset A of X is meant $\Gamma_n A$.

The following definition is due to H. Herrlich [24].

1.2 <u>Definition</u> A completely regular space X is said to be <u>n-compact</u> if every z-ultrafilter with the n-intersection property on X is fixed.

<u>Remark</u>: A completely regular space is $\aleph_0$-compact if and only if it is compact. Also, a completely regular space is $\aleph_1$-compact if and only if it is realcompact.

It is known [24] that the category of n-compact spaces is epi-reflective in the category of completely regular spaces and continuous maps. For any completely regular space X, we denote the reflection of X by $\beta_n: X \longrightarrow \beta_n X$.

1.3 <u>Notation</u> For a completely regular space X, the family of all unions of fewer than n cozero sets of $\beta X$, which contain X, will be denoted by $coz_n(X)$.

1.4 <u>Lemma</u> The n-closure of a completely regular space X in $\beta X$ is the intersection of the members of $coz_n(X)$.

<u>Proof</u>: Suppose $p \notin \bigcap coz_n(X)$. Then there is a member S of $coz_n(X)$ such that $p \notin S$. Let $S = \bigcup_{\iota \in I}(\beta X - Z_\iota)$, where each $Z_\iota$ is a zero-set of $\beta X$ and $|I| < n$. Since each $Z_\iota$ is a $G_\delta$-set, $\bigcap_\iota Z_\iota$ is a $G_n$-set containing p and disjoint from X. Hence p does not belong to the n-closure of X in $\beta X$.

Conversely, assume that p does not belong to the n-closure of X in $\beta X$. Then there exists a $G_n$-set G containing p which is disjoint from X. Let $G = \bigcap_{\iota \in I} G_\iota$, where each $G_\iota$

is open in $\beta X$ and $|I| < n$. Since the zero-set neighborhoods form the fundamental system of neighborhoods, there exists a zero-set neighborhood $Z_\iota$ of p in $\beta X$ with $Z_\iota \subseteq G_\iota$ for each $\iota \in I$. Hence $p \notin \bigcup_\iota(\beta X - Z_\iota) \in coz_n(X)$.

Noting that every cozero-set of $\beta X$ is $\sigma$-compact, each member of $coz_n(X)$ is the union of fewer than n compact subsets of $\beta X$. Hence each member of $coz_n(X)$ is n-compact as subspace of $\beta X$. Moreover, the intersection of n-compact subspaces is again n-compact [24]. Thus we have the following:

1.5 <u>Corollary</u> The n-closure of a completely regular space X in $\beta X$ is n-compact.

The following lemma is due to M. Hušek [33].

1.6 <u>Lemma</u> For every infinite cardinal number n, the class of n-compact spaces is $P_n$-simple, where

1) $P_{\aleph_0} = \mathbf{I}$, where $\mathbf{I}$ is the unit interval $[0, 1]$,
2) if n is not a limit cardinal number and $n = t^+$, then $P_n = \mathbf{I}^t - \{p\}$, where p is a point of $\mathbf{I}^t$,
3) if n is a limit cardinal number and $n \neq \aleph_0$, then $P_n = \prod_{t<n} P_{t^+}$.

Proof of the lemma can be found in [33].

1.7 <u>Theorem</u> For a completely regular space X, $\beta_n X$ is precisely the n-closure of X in $\beta X$.

<u>Proof</u>: Since the $\aleph_0$-closure of X in $\beta X$ is $\beta X$ it-

self, we may assume that n is greater than $\aleph_0$. Let Y be the n-closure of X in $\beta X$. By Corollary 1.5 and Theorem 4.3 in [23], it is enough to show that Y is a $P_n$-extendable extension of X. In Lemma 1.6, let p = (1), i.e. all coordinates are 1. Let n be a limit cardinal number. For any continuous map f on X into $P_n$, let $\bar{f}$ be the canonical extension of f to $\beta X$ into $\prod_{t<n} \mathbf{I}^t$. Suppose there be a $q \in Y$ such that $\bar{f}(q) \notin P_n$. Let $\pi_\iota$ and $\pi_t$ be the $\iota$-th projection of $\mathbf{I}^t$ onto $\mathbf{I}$ and the t-th projection of $\prod_{t<n} \mathbf{I}^t$ onto $\mathbf{I}^t$ respectively. Since $\bar{f}(q) \in P_n$, there exists t such that $\pi_t\bar{f}(q) = (1)$. Consider $G_\iota^m = (\pi_\iota \pi_t\bar{f})^{-1}(]1 - 1/m, 1])$ for each natural number m and $\iota < t$. Since $G = \bigcap_{\iota,m} G_\iota^m$ is a $G_n$-set containing q, G meets X, say $x \in G \cap X$. Then $\pi_t f(x) = (1)$, hence $f(x) \in P_n$, which is a contradiction. Hence $\bar{f}|Y$ is the desired extension of f to Y.

By the same argument, one can easily prove that Y is a $P_n$-extendable extension of X for an isolated cardinal number n.

Noting that a completely regular space X is n-compact if and only if $\beta_n X$ is homeomorphic with X and every member of $coz_n(X)$ is a $\sigma_n$-compact subset of $\beta X$, i.e. a subset which is the union of fewer than n compact subsets of $\beta X$, we have the following:

1.8 <u>Corollary</u> For a completely regular space X, the following are equivalent:

1) X is n-compact,

2) X is n-closed in $\beta X$, and

3) X is the intersection of $\sigma_n$-compact subsets of $\beta X$.

1.9 <u>Theorem</u> An n-closed subspace of an n-compact space is again n-compact.

<u>Proof</u>: Let Y be an n-closed subspace of an n-compact space X. Being n-compact, X is n-closed in $\beta X$, and hence Y is n-closed in $\beta X$. Let $\tau$ be the natural embedding of Y into X. We denote the canonical extension of $\tau$ to $\beta Y$ into $\beta X$ by $\bar{\tau}$. Since $\bar{\tau}(\beta Y - Y) \subseteq \beta X - \bar{\tau}(Y) = \beta X - Y$, $\bar{\tau}^{-1}(Y)$ is contained in Y. It is obvious that $Y = \bar{\tau}^{-1}(Y)$ and the inverse image of an n-closed subset under a continuous map is also n-closed. Hence Y is n-closed in $\beta Y$, so that Y is n-compact.

1.10 <u>Proposition</u> Let Y be an extension space of a space X. Then X is n-closed in Y if and only if every point of Y belongs to X, whenever its trace filter has the n-intersection property.

<u>Proof</u>: Suppose that X be n-closed in Y. Take $y \in Y$ whose trace filter has the n-intersection property. For any family $(G_\iota)_{\iota \in I}$ of open neighborhoods of y with $|I| < n$, $G_\iota \cap X \in T(y)$ for each $\iota \in I$, where T(y) denotes the trace filter of y. Hence $\bigcap_\iota G_\iota \cap X \neq \phi$, i.e. $y \in \Gamma_n X = X$.

Conversely, take $y \in \Gamma_n X$. For any subfamily $(G_\iota)_{\iota \in I}$ of $T(y)$ with $|I| < n$, there exists an open neighborhood $V_\iota$ of $y$ for each $\iota \in I$ such that $V_\iota \cap X = G_\iota$. Since $\bigcap_\iota V_\iota$ is a $G_n$-set, $\bigcap_\iota V_\iota \cap X \neq \phi$, i.e. $\bigcap_\iota G \neq \phi$. Hence $T(y)$ has the n-intersection property. Thus $y \in X$.

1.11 <u>Definition</u> A filter $\mathfrak{F}$ on a completely regular space X is said to be <u>completely regular</u>, if there is a base $\mathfrak{B}$ of $\mathfrak{F}$ consisting of open sets such that for any $A \in \mathfrak{B}$, there are a $B \in \mathfrak{B}$ contained in A and a continuous map f on X into $[\![0, 1]\!]$ having the value 0 on B and the value 1 on $\complement A$.

A completely regular filter $\mathfrak{F}$ is said to be <u>maximal</u> if it is not contained in any other completely regular filter.

Since the Stone-Čech compactification $\beta X$ of a completely regular space X is given by the strict extension of X with all maximal completely regular filters on X as the filter trace [4, 7], the following is the immediate consequence of Proposition 1.10.

1.12 <u>Proposition</u> A completely regular space X is n-compact if and only if every maximal completely regular filter with the n-intersection property is convergent.

Section 2: Zero-dimensional spaces.

2.1 Definition A Hausdorff space is said to be zero-dimensional if it has a basis consisting of sets which are both open and closed.

The category of zero-dimensional spaces and continuous maps will be denoted by Zero.

2.2 Definition A filter on a space is said to be open closed if it has a basis consisting of sets which are both open and closed. An open closed filter is said to be maximal if it is not contained in any other open closed filter.

It is known that the subcategory $\mathfrak{Q}$ of Zero determined by all zero-dimensional compact spaces is extensive in Zero [1]. The reflection of a zero-dimensional space X is given by the maximal zero-dimensional compactification $\zeta X$ of X, i.e. the strict extension of X with all maximal open closed filters on X as the filter trace [1, 4].

2.3 Definition Let n be an infinite cardinal number. A zero-dimensional space X is said to be zero-dimensionally n-compact if every maximal open closed filter with the n-intersection property on X is convergent.

Remark: A zero-dimensional space is zero-dimensionally $\aleph_0$($\aleph_1$)-compact if and only if it is compact ($\mathbb{N}$-compact, respectively) [23].

Combining Proposition 1.10 and the fact that the maximal zero-dimensional compactification $\zeta X$ of a zero-dimensional space X is the strict extension of X with all maximal open closed filters on X as the filter trace, we have the following immediately.

2.4 <u>Lemma</u> A zero-dimensional space X is zero-dimensionally n-compact if and only if X is n-closed in $\zeta X$.

Since every n-closed subspace of a compact space is n-compact, we have the following:

2.5 <u>Corollary</u> Every zero-dimensionally n-compact space is also n-compact.

For any infinite cardinal number n, the subcategory of <u>Zero</u> determined by all zero-dimensionally n-compact spaces will be denoted by $\mathcal{Q}_n$. It is noted that $\mathcal{Q}_{\aleph_0}$ is the category of all zero-dimensional compact spaces, while the category $\mathcal{Q}_{\aleph_1}$ is a proper subcategory of all zero-dimensional realcompact (= $\aleph_1$-compact) spaces [46].

Combining Lemma 2.4 and the fact that the category <u>Zero</u> is hereditary, we have the following by Theorem 2.4 in Chap. I.

2.6 <u>Theorem</u> The subcategory $\mathcal{Q}_n$ of <u>Zero</u> is extensive in Zero.

2.7 Corollary The category of all zero-dimensionally n-compact spaces and continuous maps is productive and closed hereditary.

2.8 Corollary Let X be a zero-dimensional space and $f: X \longrightarrow Y$ be a perfect map. Then X is zero-dimensionally n-compact if Y is zero-dimensionally n-compact.

Proof: It is immediate from Theorem 1.5 in Chap. I.

2.9 Corollary Every n-closed subspace of a zero-dimensionally n-compact space is again zero-dimensionally n-compact.

Proof: By the same argument of the proof of Theorem 1.9, one can easily prove the corollary. We omit the proof.

It is well known [7, 23] that the sum space of a family $(X_\iota)_{\iota \in I}$ of Hausdorff spaces is homeomorphic to a closed subspace of $I \times \prod_\iota X_\iota$, where I is endowed with the discrete topology. Hence we have the following by Corollary 2.7.

2.10 Corollary Let $(X_\iota)_{\iota \in I}$ be a family of non-empty zero-dimensionally n-compact spaces. If I is zero-dimensionally n-compact with respect to the discrete topology, then the sum space $\sum X_\iota$ of $(X_\iota)$ is also zero-dimensionally n-compact.

2.11 <u>Definition</u> A space X is said to be <u>strongly zero-dimensional</u> if it is completely regular and $\beta X$ is totally disconnected.

It is obvious [7] that every strongly zero-dimensional space is zero-dimensional. But there is a zero-dimensional space which is not strongly zero-dimensional [17, 46].

2.12 <u>Corollary</u> For a strongly zero-dimensional space X, X is n-compact if and only if it is zero-dimensionally n-compact.

<u>Proof</u>: X is n-compact if and only if X is n-closed in $\beta X = \zeta X$ by the assumption if and only if X is zero-dimensionally n-compact.

2.13 <u>Theorem</u> For every infinite cardinal number n, the class of zero-dimensionally n-compact spaces is $D_n$-simple, where

1) $D_{\aleph_0} = D$, where D is the two point space with the discrete topology,

2) if n is not a limit cardinal number and $n = t^+$, then $D_n = D^t - \{p\}$, where p is a point of $D^t$,

3) if n is a limit cardinal number and $n \neq \aleph_0$, then $D_n = \prod_{t<n} D_{t^+}$.

<u>Proof</u>: Using the same argument of the proof of Lemma 1.6 by replacing the unit interval by D, one can easily prove the theorem. We omit the proof.

## Section 3: n-complete spaces and n-Hausdorff closed spaces.

The following definition is due to M. Hušek [33].

3.1 Definition Let n be an infinite cardinal number. A Hausdorff uniform space X is said to be n-complete if every Cauchy filter with the n-intersection property on X is convergent.

3.2 Definition The minimal elements (by the inclusion relation) of the set of all Cauchy filters on a uniform space X are called minimal Cauchy filters on X.

Recall [7] that for a Hausdorff uniform space, its completion cX is given as follows:

its underlying set is the set of all minimal Cauchy filters on X and its uniform structure is generated by $\{\tilde{V} \mid V:$ symmetric entourage on $X\}$, where $\tilde{V}$ is the set of all pairs $(\xi, \eta)$ of minimal Cauchy filters such that there is a set M in $\xi \cap \eta$ which is a V-small set.

In what follows, we identify each point of X with its neighborhood filter, so that X is a subspace of cX.

Using the fact that each minimal Cauchy filter $\xi$ is generated by $\{V(F) \mid V:$ symmetric entourage on X, $F \in \xi\}$, it is easy to show that the trace filter of $\xi \in cX$ on X generates $\xi$ itself. Moreover, for any Cauchy filter $\eta$, there is a unique minimal Cauchy filter which is coarser than $\eta$ . Hence by Proposition 1.10, we have the following:

3.3 Lemma A Hausdorff uniform space X is n-complete if and only if it is n-closed in cX.

The category of n-complete spaces and uniformly continuous maps will be denoted by $\mathcal{C}_n$.

The following theorem is immediate from Theorem 2.6 in Chap. I and Lemma 3.3.

3.4 Theorem The category $\mathcal{C}_n$ is extensive in HUnif.

3.5 Corollary The category $\mathcal{C}_n$ is productive and closed hereditary.

3.6 Proposition Every n-closed subspace of an n-complete space is again n-complete.

Proof: Since every uniformly continuous map is also continuous, we can easily prove the proposition by the same argument in the proof of Theorem 1.9.

It is well known [7, 17] that for every completely regular space X, $\beta X$ is homeomorphic with the completion cX of X with the uniform structure generated by the set C*(X) of all bounded continuous real-valued maps on X. Hence we have the following:

3.7 <u>Theorem</u> A completely regular space X is n-compact if and only if it is n-complete with respect to the uniform structure on X generated by C*(X).

<u>Proof</u>: X is n-compact if and only if it is n-closed in $\beta X$ if and only if it is n-complete with respect to the uniform structure on X generated by C*(X).

<u>Examples</u>: For any infinite cardinal number n, there is an n-compact space which is not t-compact for $t < n$, namely $P_n$ in Lemma 1.6. Thus there is an n-complete uniform space which is not t-complete, i.e. $\mathcal{C}_t \subset \mathcal{C}_n$ for $t < n$.

It is also well known that for every zero-dimensional space X, $\zeta X$ is homeomorphic with the completion cX of X with the uniform structure generated by C(X, D), where D is the two point space with the discrete topology and C(X, D) is the set of all continuous maps on X into D. Hence we have the following:

3.8 <u>Theorem</u> A zero-dimensional space X is zero-dimensionally n-compact if and only if it is n-complete with respect to the uniform structure on X generated by C(X, D).

3.9 <u>Definition</u> A filter on a space is said to be <u>open</u> if it has a base consisting of open sets. And an open filter is said to be <u>maximal open</u> if it is not contained in any other open filter.

3.10 Definition Let n be an infinite cardinal number. A Hausdorff space X is said to be n-Hausdorff closed if every maximal open filter with the n-intersection property on X is convergent.

It is noted that a Hausdorff space is Hausdorff closed if and only if it is $\aleph_0$-Hausdorff closed.

Since the Katětov extension $\varkappa X$ of a Hausdorff space X is the simple extension space with all non-convergent maximal open filters on X together with all open neighborhood filters on X as the filter trace, we have the following by Proposition 1.10.

3.11 Lemma A Hausdorff space X is n-Hausdorff closed if and only if X is n-closed in $\varkappa X$.

The subcategory of Haus* determined by all n-Hausdorff closed spaces will be denoted by $\mathfrak{A}^*_n$.

3.12 Theorem The category $\mathfrak{A}^*_n$ is extensive in Haus*.

Proof: It is immediate from Theorem 2.12 in Chap. I and Lemma 3.11.

3.13 Corollary The category $\mathfrak{A}^*_n$ is productive and regular closed hereditary.

## Section 4: Sequentially closed spaces.

S. P. Franklin has shown [12, 13, 14] that the category $\mathcal{S}_{\omega_0}$ of sequential spaces and continuous maps is coreflective in Top. Also, H. Herrlich has shown [26] that the category $\mathcal{S}_\alpha$ of $\alpha$-sequential spaces and continuous maps for any regular ordinal $\alpha$ is coreflective in Top.

We will investigate some properties of the associated reflective subcategories in various subcategories of Haus and HUnif with $\mathcal{S}_\alpha$ .

4.1 Definition Let $\alpha$ be a regular ordinal. A net is said to be an $\alpha$-sequence if its domain is the well-ordered index-set $\alpha$ i.e. the set of all ordinals less than $\alpha$ .

It is noted that $\omega_0$-sequences are exactly usual sequences.

4.2 Definition A subset U of a topological space X is $\alpha$-sequentially open if each $\alpha$-sequence in X converging to a point in U is eventually in U. A topological space X is said to be $\alpha$-sequential if each $\alpha$-sequentially open subset of X is open.

It is again noted that $\omega_0$-sequential spaces are exactly sequential spaces.

For a subset A of a topological space X, we define

$l_X^\alpha A$ by $\{x \in X \mid$ there is an $\alpha$-sequence in A converging to $x\}$. Then it is easy to show that the operator $l^\alpha$ defined as above is a limit-operator on Top. Moreover, it is known [26] that $l^\alpha$ is not idempotent and that the category $\mathcal{S}_\alpha$ generates the associated idempotent limit-operator with $l^\alpha$.

4.3 Definition A filter $\mathfrak{F}$ on a set X is said to be an *$\alpha$-filter* for a regular ordinal $\alpha$ if it has a base $(B_\lambda)_{\lambda<\alpha}$ such that $B_\lambda \subseteq B_\mu$ for $\mu \leq \lambda < \alpha$.

It is noted that $\omega_0$-filters on a set are exactly filters with countable bases.

4.4 Definition A filter on a set X is said to be an *$\alpha$-Fréchet filter* for a regular ordinal $\alpha$ if it is generated by the tails of an $\alpha$-sequence on X.

It is obvious that $\omega_0$-Fréchet filters are exactly Fréchet filters.

4.5 Proposition Every $\alpha$-filter on a set X is the intersection of the $\alpha$-Fréchet filters containing it.

Proof: Let $\mathfrak{F}$ be an $\alpha$-filter on X and $(B_\lambda)_{\lambda<\alpha}$ a base of $\mathfrak{F}$ such that $B_\lambda \subseteq B_\mu$ for $\mu \leq \lambda < \alpha$. Let $a_\lambda$ be any element of $B_\lambda$ for each $\lambda < \alpha$; then it is clear that $\mathfrak{F}$ is coarser than the $\alpha$-Fréchet filter generated by the tails of the $\alpha$-sequence $(a_\lambda)_{\lambda<\alpha}$.

Hence the intersection $\mathcal{G}$ of the $\alpha$-Fréchet filters which are finer than $\mathcal{F}$ exists and is finer than $\mathcal{F}$ ; if $\mathcal{G}$ is strictly finer than $\mathcal{F}$ then there exists a set $M \in \mathcal{G}$ such that $B_\lambda \cap \complement M \neq \phi$ for each $\lambda < \alpha$ ; if $b_\lambda \in B_\lambda \cap \complement M$, the $\alpha$-Fréchet filter generated by the tails of the $\alpha$-sequence $(b_\lambda)_{\lambda<\alpha}$ is finer than $\mathcal{F}$ and does not contain M. This contradicts the definition of $\mathcal{G}$ .

4.6 Proposition Let Y be an extension space of a space X. The following are equivalent for the limit-operator $\ell^\alpha$:

1) $\ell^\alpha_Y X = X$.

2) For any $y \in Y$, if there is an $\alpha$-Fréchet filter on X containing its trace filter T(y), then $y \in X$.

3) For any $y \in Y$, if there is an $\alpha$-filter containing its trace filter T(y), then $y \in X$.

Proof: 1) $\Rightarrow$ 2). Let $\mathcal{F}$ be an $\alpha$-Fréchet filter containing T(y) and let $(x_\lambda)_{\lambda<\alpha}$ be an $\alpha$-sequence in X which generates $\mathcal{F}$ . By the definition of the trace filter, it is obvious that the $\alpha$-sequence $(x_\lambda)$ converges to y. Hence y belongs to $\ell^\alpha_Y X = X$.

2)$\Rightarrow$3). It follows immediately from Proposition 4.5.

3)$\Rightarrow$1). For any $y \in \ell^\alpha_Y X$, there is an $\alpha$-sequence $(x_\lambda)_{\lambda<\alpha}$ in X converging to y. It is easy to show that T(y) is contained in the $\alpha$-Fréchet filter generated by the tails of $(x_\lambda)$. Hence $y \in X$.

4.7 <u>Definition</u> A completely regular space X is said to be <u>$\beta$-$\alpha$-sequentially closed</u> if every maximal completely regular filter on X converges, whenever it is contained in an $\alpha$-filter.

Recall that the Stone-Čech compactification $\beta X$ of a completely regular space X is the strict extension space of X with all maximal completely regular filters on X as the filter trace and that $\overline{\ell^\alpha}_{\beta X} X = X$ if and only if X is $\ell^\alpha$-closed in $\beta X$, where $\overline{\ell^\alpha}$ is the associated idempotent limit-operator with $\ell^\alpha$. Hence the following is immediate from Theorem 2.4 in Chap. I and Proposition 4.6.

4.8 <u>Theorem</u> The category <u>Comp</u>$_{\ell^\alpha}$ of $\beta$-$\alpha$-sequentially closed spaces and continuous maps is extensive in the category of completely regular spaces and continuous maps.

4.9 <u>Corollary</u> The category <u>Comp</u>$_{\ell^\alpha}$ is productive and closed-hereditary.

4.10 <u>Definition</u> A zero-dimensional space X is said to be <u>$\zeta$-$\alpha$-sequentially closed</u> if every maximal open closed filter on X converges, whenever it is contained in an $\alpha$-filter.

The category of $\zeta$-$\alpha$-sequentially closed spaces and continuous maps will be denoted by $\mathcal{Q}_{\ell^\alpha}$.

By the same argument of Theorem 4.8, we have the following:

4.11 _Theorem_ The category $\mathcal{D}_{\ell^\alpha}$ is extensive in the category _Zero_.

4.12 _Corollary_ The category $\mathcal{D}_{\ell^\alpha}$ is productive and closed hereditary.

4.13 _Definition_ A Hausdorff uniform space X is said to be _c-$\alpha$-sequentially closed_ if every Cauchy filter on X is convergent, whenever it is contained in an $\alpha$-filter.

We will denote the category of c-$\alpha$-sequentially closed spaces and uniformly continuous maps by $\mathcal{C}_{\ell^\alpha}$.

Using the same argument of Lemma 3.3 and the fact that for any Hausdorff uniform space X, $\bar{\ell}^\alpha_{cX}X = X$ if and only if X is $\ell^\alpha$-closed in cX, we have the following by Theorem 2.6 in Chap. I and Proposition 4.6.

4.14 _Theorem_ The category $\mathcal{C}_{\ell^\alpha}$ is extensive in the category _HUnif_.

4.15 _Corollary_ The category $\mathcal{C}_{\ell^\alpha}$ is productive and closed hereditary.

4.16 _Definition_ A Hausdorff space X is said to be _$\kappa$-$\alpha$-sequentially closed_ if every maximal open filter on X is convergent, whenever it is contained in an $\alpha$-filter.

The category of $\kappa$-$\alpha$-sequentially closed spaces and continuous semi-open maps will be denoted by $\mathcal{O}\mathcal{L}^*_{\ell^\alpha}$.

By the same argument of Lemma 3.11 and the fact that for any Hausdorff space X, $\overline{\ell}^\alpha_{\kappa X}X = X$ if and only if X is $\ell^\alpha$-closed in $\kappa$ X, the following is immediate from Theorem 2.12 in Chap. I and Proposition 4.6.

4.17 Theorem The category $\mathcal{O}\mathcal{L}^*_{\ell^\alpha}$ is extensive in the category Haus*.

4.18 Corollary The category $\mathcal{O}\mathcal{L}^*_{\ell^\alpha}$ is productive and regular-closed hereditary.

The following definition is due to P. Alexandroff and P. Urysohn [0].

4.19 Definition Let $\aleph_\alpha$ be an infinite cardinal number. A Hausdorff space X is said to be $\aleph_\alpha$-$\aleph_0$ compact if every open covering $\mathcal{U}$ of X with $|\mathcal{U}| \leq \aleph_\alpha$, has a finite subcovering.

It is noted that $\aleph_0$-$\aleph_0$ compact spaces are precisely countably compact spaces.

Remark: A Hausdorff space X is $\aleph_\alpha$-$\aleph_0$ compact if and only if every filter with a base whose cardinal number is not greater than $\aleph_\alpha$ has a cluster point, for the condition is the dual statement of the definition.

4.20 <u>Theorem</u> Let $\alpha$ be a regular ordinal and let $\aleph$ be the cardinal number of $\alpha$. Then every $\aleph - \aleph_0$ compact space is $\chi - \alpha$-sequentially closed, and every completely regular (zero-dimensional, Hausdorff uniform) $\aleph - \aleph_0$ compact space is $\beta(\mathfrak{S}$, c, respectively)-$\alpha$-sequentially closed.

<u>Proof</u>: Let X be an $\aleph - \aleph_0$ compact space and $\mathcal{U}$ a maximal open filter on X contained in an $\alpha$-filter $\mathfrak{F}$. Since $\mathfrak{F}$ has a base $(B_\lambda)_{\lambda < \alpha}$, $\mathfrak{F}$ has a cluster point x. Hence the join of $\mathcal{U}$ and $\mathfrak{I}(x)$ exists, where $\mathfrak{I}(x)$ is the neighborhood filter of x. Hence $\mathcal{U} \vee \mathfrak{I}(x) = \mathcal{U}$ by the maximality of $\mathcal{U}$. Thus $\mathcal{U}$ is convergent.

Regarding the second part, let X be an $\aleph - \aleph_0$ compact completely regular (zero-dimensional, Hausdorff uniform) space and $\mathcal{U}$ a maximal completely regular (maximal open closed, Cauchy, respectively) filter on X contained in an $\alpha$-filter $\mathfrak{F}$. Then $\mathfrak{F}$ has a cluster point x. Combining the fact that $\mathcal{U} \vee \mathfrak{I}(x)$ exists and the fact that every neighborhood filter in a completely regular (zero-dimensional, uniform) space is a maximal completely regular (maximal open closed, minimal Cauchy, respectively) filter, we have $\mathcal{U} \supseteq \mathfrak{I}(x)$, for $\mathcal{U} \vee \mathfrak{I}(x) = \mathfrak{I}(x) = \mathcal{U}$ ( $\mathcal{U} \vee \mathfrak{I}(x) = \mathfrak{I}(x) = \mathcal{U}$ , $\mathcal{U} \wedge \mathfrak{I}(x) = \mathfrak{I}(x) \subseteq \mathcal{U}$ , respectively).

4.21 <u>Definition</u> A completely regular space is said to be <u>pseudo-compact</u> if every continuous real-valued map on the space is bounded.

Remark: It is well known [17] that every countably compact space is pseudo-compact. However, there is a pseudo-compact space, namely $\mathbf{I}^{\aleph_1} - \{(1)\}$ which is not $\beta$ - $\omega_0$-sequentially closed.

Since every ultrafilter on a discrete space is convergent, whenever it is a Fréchet filter, we have the following:

4.22 Theorem Every discrete space is $\kappa(\beta, \zeta, c)$-$\omega_0$-sequentially closed.

Example: The smallest ordinal of a cardinal number $\aleph_\lambda$ is denoted by $\omega_\lambda$. Let $\lambda$ be a nonlimit ordinal $> 0$. Let $W(\omega_\lambda)$ be the space of all ordinals less than $\omega_\lambda$ endowed by the interval topology. Then it is well known [17] that no subset of $W(\omega_\lambda)$ of cardinal number $< \aleph_\lambda$ is cofinal, that every bounded subset of $W(\omega_\lambda)$ is relatively compact and that $\beta W(\omega_\lambda) = \zeta W(\omega_\lambda) = W(\omega_\lambda + 1)$. Hence for any regular ordinal $\alpha < \omega_\lambda$, $W(\omega_\lambda) \in$ $\underline{\text{Comp}}_{\ell^\alpha}(\alpha\mathcal{Q}_{\ell^\alpha})$. But for any regular ordinal $\alpha \geqslant \omega_\lambda$, $W(\omega_\lambda) \notin$ $\underline{\text{Comp}}_{\ell^\alpha}(\alpha\mathcal{Q}_{\ell^\alpha})$.

## CHAPTER III

## PROJECTIVE COVERS AND EXTENSIVE SUBCATEGORIES

### Section 1: Extremally disconnected spaces.

1.1 Definition Let $\mathcal{K}$ be a category and $\wp$ a class of morphisms in $\mathcal{K}$ .

An object P of $\mathcal{K}$ is said to be $\wp$-projective if for any morphism $g: P \longrightarrow B$ in $\mathcal{K}$ and for any $f: A \longrightarrow B$ in $\wp$, there exists a morphism $h: P \longrightarrow A$ in $\mathcal{K}$ such that $g = fh$.

A morphism f in $\wp$ is said to be essential if $fg \in \wp$ implies $g \in \wp$ . We denote the class of all essential morphisms in $\mathcal{K}$ by $\wp^*$.

A morphism $f: A \longrightarrow B$ in $\mathcal{K}$ is said to be a $\wp$-projective cover of B if A is $\wp$-projective and $f \in \wp^*$.

The following definition is due to B. Banaschewski [5].

1.2 Definition Let $\mathcal{K}$ be a category and $\wp$ a class of morphisms in $\mathcal{K}$ .

The $\wp$-projectivity is said to behave properly if the following three conditions are fulfilled:

1) The following are equivalent for an object P:

i) P is $\wp$-projective.

ii) Any morphism f: $A \longrightarrow P$ in $\wp$ has a right inverse.

iii) Any morphism f: $A \longrightarrow P$ in $\wp^*$ is an isomorphism.

2) Any object in $\mathcal{X}$ has an essentially unique $\wp$-projective cover.

3) The following are equivalent for a morphism f: $B \longrightarrow A$ in $\wp$:

i) f is a $\wp$-projective cover.

ii) f is an essential morphism and, for any g, if fg is an essential morphism then g is an isomorphism.

iii) B is $\wp$-projective, and if f = hg with morphisms g and h in $\wp$ where h has $\wp$-projective domain then g is an isomorphism.

1.3 Definition A topological space is said to be extremally disconnected if every open set has open closure.

It is well known [5, 40, 50] that the extremally disconnected spaces become $\wp$-projective objects in various categories of topological spaces and some specified classes $\wp$ of morphisms in them.

1.4 Definition Let L be a lattice with 0. A pseudo-complement of an element $a \in L$ is an element $b \in L$ such that for all $x \in L$, $a \wedge x = 0$ is equivalent to that $x \leqslant b$. A lattice L with 0 is said to be pseudo-complemented if every element of L has a pseudo-complement. We denote the pseudo-complement of a by $a^*$ for $a \in L$.

Example: It is known that the lattice of open subsets of a topological space is pseudo-complemented, where for any open set $U$, $\mathbb{I}\mathbb{C}U$ is a pseudo-complement of $U$ in the lattice, while $\mathbb{I}$ denotes the interior operator on the space.

1.5 Definition A distributive pseudo-complemented lattice is said to be a Stone-lattice if $a^* \vee a^{**} = e$ for each element $a$ and the unit $e$ in the lattice.

1.6 Lemma A topological space $(X, \mathfrak{J})$ is extremally disconnected if and only if the lattice $\mathfrak{J}$ is a Stone-lattice.

Proof: Suppose that $X$ be extremally disconnected. For any open set $U \in \mathfrak{J}$, $U^* \vee U^{**} = \mathbb{I}\mathbb{C}U \cup \mathbb{I}\mathbb{C}\mathbb{I}\mathbb{C}U =$ $\mathbb{I}\mathbb{C}U \cup \mathbb{C}\mathbb{I}\mathbb{C}U = X$, for $\Gamma U = \mathbb{C}\mathbb{I}\mathbb{C}U$ and $\mathbb{I}\Gamma U = \Gamma U$. Hence the lattice $\mathfrak{J}$ is a Stone-lattice.

Conversely, for any open set $U$, we have $U^* \vee U^{**} =$ $\mathbb{I}\mathbb{C}U \cup \mathbb{I}\mathbb{C}\mathbb{I}\mathbb{C}U = X$. Hence $\mathbb{C}\mathbb{I}\mathbb{C}U \subseteq \mathbb{I}\mathbb{C}\mathbb{I}\mathbb{C}U$, i.e. $\mathbb{C}\mathbb{I}\mathbb{C}U = \mathbb{I}\Gamma U$. Consequently, $\Gamma U = \mathbb{I}\Gamma U$. This completes the proof.

The following lemma is due to G. Grätzer and E. T. Schmidt.

1.7 Lemma A distributive pseudo-complemented lattice $L$ is a Stone-lattice if and only if every prime filter in $L$ is contained in at most one maximal filter.

Proof of lemma can be found in [18].

Every prime open filter is contained in at least one maximal open filter by Zorn's Lemma. Hence we have the following:

1.8 Theorem A topological space is extremally disconnected if and only if every prime open filter is contained in exactly one maximal open filter.

Proof: The theorem is an immediate consequence of the above two lemmas.

We wish to introduce the maximal open filter space of a topological space which will be used in our subsequent development.

Let X be any Hausdorff space, $\mathfrak{T}$ = its topology, and $\Omega = \Omega(X)$ the set of all maximal open filters on $(X, \mathfrak{T})$. Then for any $V \in \mathfrak{T}$, put $\Omega_V = \{\xi \mid V \in \xi \in \Omega\}$ ; it is obvious that $\Omega_U \cap \Omega_V = \Omega_{U \cap V}$, and the sets $\Omega_V$ form the basis of a topology on $\Omega$ . The space thus given, again denoted by $\Omega$ , is Hausdorff, for $U \cap V = \phi$ implies that $\Omega_U \cap \Omega_V = \phi$ . It is well known that the space $\Omega(X)$ is compact and extremally disconnected.

Now, let $\Lambda = \Lambda(X)$ be the subspace of $\Omega$ given by all convergent $\xi \in \Omega$, i.e. all $\xi \in \Omega$ such that $\xi$ contains a neighborhood filter of a for some $a \in X$. Since $\mathfrak{T}(x)$, $x \in X$, is contained in some $\xi \in \Omega$ by Zorn's Lemma one sees that $\Lambda$ is dense in $\Omega$ ; thus $\Lambda$ is also extremally disconnected.

An obvious map from $\Lambda$ to X is $\xi \rightsquigarrow \lim \xi$, which will be denoted by lim, or $\lim_X$ if reference to the space is required. It follows from what was just said that lim is an onto map.

1.9 Definition Let X and Y be topological spaces, and let $f: X \longrightarrow Y$ be a map. Then f is said to be compact if the inverse images of points are compact.

An onto map $g: X \longrightarrow Y$ is said to be minimal if for any closed subset $A \subseteq X$, $f(A) = Y$ implies $A = X$.

1.10 Theorem The map $\lim: \Lambda(X) \longrightarrow X$ is compact, closed, and minimal onto. And it is continuous if and only if X is regular.

Proofs of the theorem can be found in [5, 50].

## Section 2: Perfect onto projectivity in extensive subcategories.

In what follows, the class of morphisms to play the role of $\mathcal{P}$ in Definition 1.1 and 1.2 will always consist of perfect onto morphisms.

It is known [5] that in a subcategory $\mathcal{K}$ of Haus, perfect onto projectivity is properly behaved if $\mathcal{K}$ is closed hereditary and productive, or $\mathcal{K}$ is a full subcategory of Haus which is left-fitting with respect to essential perfect onto maps, or $\mathcal{K}$ consists of all objects and all perfect maps from a category $\mathcal{L}$ which satisfies one of the above conditions.

Since any extensive subcategory of the category of completely regular spaces (zero-dimensional spaces) and continuous maps is productive and closed hereditary, the perfect onto projectivity in such a category is properly behaved.

However, an extensive subcategory of Haus* need not be closed hereditary.

2.1 Lemma Every minimal closed map is semi-open.

Proof: Let $f: X \longrightarrow Y$ be such a map, and let U be a non-empty open set of X. Since f is minimal closed, $\complement f(\complement U)$ is non-empty open and f(U) contains $\complement f(\complement U)$. Hence f(U) has non-empty interior.

2.2 Corollary Every extensive subcategory of the category Haus* is left-fitting with respect to essential perfect onto maps in Haus.

Proof: It is known [5] that the essential perfect onto maps in Haus are exactly the minimal ones. Hence by Lemma 2.1, every essential perfect onto map in Haus is a morphism in the category Haus*. Thus the corollary is immediate from Theorem 1.5 in Chap. I.

2.3 Theorem Let $\mathcal{L}^*$ be an extensive subcategory of Haus* and let $\mathcal{L}$ be the full subcategory of Haus with the same objects of $\mathcal{L}^*$.

Then perfect onto projectives in $\mathcal{L}$ are precisely the extremally disconnected spaces belonging to $\mathcal{L}$ and the perfect onto projectivity in $\mathcal{L}$ is properly behaved.

And the same holds for the subcategory of $\mathcal{L}$ with the same objects, but only the perfect maps from $\mathcal{L}$.

Proof: It is immediate from Proposition 3, Corollary 3 of Proposition 4 in [5] and Corollary 2.2.

2.4 Corollary In the subcategories of Haus determined by the following classes of spaces together with either all their continuous maps, or all their perfect maps, the perfect onto projectives are precisely the extremally disconnected spaces belonging to them and the perfect onto projectivity is properly behaved:

1) Hausdorff closed spaces.

2) Spaces which are $\ell$-closed in their Katětov extensions for a limit-operator $\ell$ on $\mathfrak{A}^*$.

More specifically, the following are given:

$2_i$) n-Hausdorff closed spaces for each infinite cardinal number n.

$2_{ii}$) $\varkappa$-$\alpha$-sequentially closed spaces for each regular ordinal $\alpha$.

## Section 3: Almost n-compact spaces.

3.1 Definition Let n be an infinite cardinal number. A Hausdorff space X is said to be almost n-compact* if every maximal open filter is convergent, whenever the closures of its members have the n-intersection property.

Remark: Z. Frolik has defined the almost realcompact spaces as the almost $\aleph_1$-compact spaces [16].

A Hausdorff space is almost $\aleph_0$-compact if and only if it is Hausdorff closed.

3.2 Definition Let $\eta$ be a collection of open coverings of a space X. An $\eta$-Cauchy family is a filter subbase $\mathcal{S}$ of open subsets of X such that for every $\mathcal{U}$ in $\eta$, there exist

*Almost n-compactness has independently been defined by R.N. Bhaumik and D.N. Mirsa, Czech. Math. J. 21(96), 625-632(1971).

an A in $\mathcal{U}$ and a B in $\mathcal{S}$ with $B \subseteq A$.

The collection $\eta$ is said to be <u>complete</u> if every $\eta$-Cauchy family has at least one cluster point.

<u>Remark</u>: A uniform space X is complete if and only if the family of all uniform open coverings is complete in the sense of the above definition.

3.3 <u>Theorem</u> Let X be a Hausdorff space and let $\eta_n$ be the family of fewer than n open coverings. Then X is almost n-compact if and only if $\eta_n$ is complete.

Proof of this is essentially contained in the proof of Theorem 1 in [16].

<u>Remark</u>: We note that a maximal open filter $\mathcal{M}$ is $\eta_n$-Cauchy if and only if $\{\Gamma U \mid U \in \mathcal{M}\}$ has the n-intersection property.

3.4 <u>Theorem</u> Let X be a completely regular space and let $\Gamma_n$ be the family of fewer than n cozero set coverings of X. Then X is n-compact if and only if $\Gamma_n$ is complete.

<u>Proof</u>: ($\Longrightarrow$). Let $\mathcal{U}$ be a $\Gamma_n$-Cauchy family in X. Suppose that $\mathcal{U}$ has no cluster point. Since $\mathcal{U}$ is a filter subbase on $\beta X$, $\mathcal{U}$ has a cluster point in $\beta X$, say p. Then it is obvious that $p \notin X$. Since X is n-compact, X is n-closed in $\beta X$. Hence there is a family $\{U_\iota\}_{\iota \in I}$ of open neighborhoods

of p in $\beta X$ such that $\bigcap_\iota U_\iota \cap X = \phi$ and $|I| < n$. Noting that the zero-set neighborhoods of p form a fundamental system of neighborhoods of p, there exists a family $\{Z_\iota\}_{\iota \in I}$ of zero-sets of $\beta X$ such that $p \in \mathbb{I}Z_\iota \subseteq Z_\iota \subseteq U_\iota$ for all $\iota \in I$. Thus, $(\bigcap_\iota Z_\iota) \cap X = \phi$. Since $\bigcup_\iota \mathbb{C}_X(Z_\iota \cap X) = X$ and $\mathbb{C}_X(Z_\iota \cap X)$ is a cozero-set in X for every $\iota \in I$, $\{\mathbb{C}_X(Z_\iota \cap X)\}_{\iota \in I} \in \gamma_n$. There exist a $U \in \mathcal{U}$ and an $\iota \in I$ with $U \subseteq \mathbb{C}_X(Z_\iota \cap X)$, which implies that U does not meet $Z_\iota \cap X$. Hence $U \cap Z_\iota = \phi$, which is a contradiction to that p belongs to the closure of U in $\beta X$.

($\Longleftarrow$). Let $\mathfrak{F}$ be a z-ultrafilter on X with the n-intersection property. Consider $\mathcal{U} = \{U \mid U\text{: open and } U \supseteq Z \text{ for some } Z \in \mathfrak{F}\}$. Show that $\mathcal{U}$ meets every member of $\gamma_n$. Indeed, suppose that there is an $\mathcal{A} \in \gamma_n$ such that each member of $\mathcal{U}$ does not belong to $\mathcal{U}$. Then for every $Z \in \mathfrak{F}$ and every $A \in \mathcal{A}$, $Z \cap \mathbb{C}A \neq \phi$. Noting that $\mathfrak{F}$ is a z-ultrafilter and $\mathbb{C}A$ is a zero-set for each $A \in \mathcal{A}$, $\mathbb{C}A \in \mathfrak{F}$ for all $A \in \mathcal{A}$. But $\bigcap \mathbb{C}A = \phi$ and $|\mathcal{A}| < n$, which is a contradiction to the n-intersection property of $\mathfrak{F}$. Hence $\mathcal{U}$ is $\gamma_n$-Cauchy, so that $\mathcal{U}$ has a cluster point. By the complete regularity of X, we have $\bigcap_{\mathfrak{F}} Z = \bigcap_{U \in \mathcal{U}} \Gamma U \neq \phi$. This completes the proof.

3.5 <u>Corollary</u> Every n-compact space is almost n-compact.

<u>Proof</u>: Since $\eta_n \supseteq \gamma_n$ for any completely regular space, every $\eta_n$-Cauchy family is also $\gamma_n$-Cauchy. Hence

every $\eta_n$-Cauchy family has a cluster point.

3.6 Lemma Let $\omega_\alpha$ be the first ordinal of cardinal number $\aleph_\alpha$. Let $W(\omega_{\alpha+1})$ be the space of ordinals less than $\omega_{\alpha+1}$ endowed with the interval topology. Then every family of fewer than $\aleph_{\alpha+1}$ closed and cofinal subsets of $W(\omega_{\alpha+1})$ has a non-empty intersection.

Proof: Let $(F_\iota)_{\iota \in I}$ be such a family. Choose a relation $<$ which well-orders I. Define an ordered set $\Sigma$ as follows: its underlying set is $I \times \mathbb{N}$ and its order relation $\prec$ is given by: $(\iota, k) \preccurlyeq (\gamma, m)$ iff $k < m$ or $\iota \leqslant \gamma$, if $k = m$. Then by the induction, we can construct a subset $\{\lambda_{\iota,k}\}$ for $(\iota, k) \in I \times \mathbb{N}$, of $W(\omega_{\alpha+1})$ such that $\lambda_{\iota,k} \in F_\iota$ and $(\iota, k) \preccurlyeq (\gamma, m)$ implies $\lambda_{\iota,k} \leqslant \lambda_{\gamma,m}$. Indeed, let $\iota_0$ be the first element of I. Then take any element $\lambda_{\iota_0,0} \in F_{\iota_0}$. Suppose that we have a subset $\{\lambda_{\iota,k} \mid (\iota, k) \prec (\gamma, m)\}$ with the above properties for some $(\gamma, m)$ in $I \times \mathbb{N}$.
Noting that the cardinal number of the set is less than $\aleph_{\alpha+1}$, it is bounded in $W(\omega_{\alpha+1})$. Let $\sigma_{\gamma,m} = \sup\{\lambda_{\iota,k} \mid (\iota, k) \prec (\gamma, m)\}$. Since $F_\gamma$ is cofinal, there exists an element $\lambda_{\gamma,m}$ of $F_\gamma$ such that $\lambda_{\gamma,m} \geqslant \sigma_{\gamma,m}$. Again noting that the cardinal number of the set $\{\lambda_{\iota,k} \mid (\iota, k) \in I \times \mathbb{N}\}$ is less than $\aleph_{\alpha+1}$, it is bounded in $W(\omega_{\alpha+1})$, so that $\sup_{\iota,k} \{\lambda_{\iota,k}\}$ exists. We denote it by $\tau$. Furthermore, we can show that for each $\iota \in I$, $\tau = \sup_m \lambda_{\iota,m}$. It is clear that $\tau \geqslant \sup_m \lambda_{\iota,m}$. For each

$(\gamma, k) \in I \times \mathbb{N}$, $\lambda_{\gamma,k} < \lambda_{\iota,k+1} \leq \sup_m \lambda_{\iota,m}$, so that $\tau \leq \sup_m \lambda_{\iota,m}$. Now we can conclude that $\tau$ belongs to $\bigcap_\iota F_\iota$. Because for any $\sigma < \tau$, there exists an $m \in \mathbb{N}$ such that $\sigma < \lambda_{\iota,m} \leq \tau$, so that $[\![\sigma + 1, \tau]\!] \cap F_\iota \neq \phi$. Therefore, $\tau \in F_\iota$, for it is closed.

3.7 Corollary The space $W(\omega_{\alpha+1})$ is almost $\aleph_{\alpha+2}$-compact but not almost $\aleph_{\alpha+1}$-compact.

Proof: It is known [24] that the space $W(\omega_{\alpha+1})$ is $\aleph_{\alpha+2}$-compact. Hence it is almost $\aleph_{\alpha+2}$-compact by Corollary 3.5. Now let us show that the space is not almost $\aleph_{\alpha+1}$-compact. Let $\mathcal{U}$ be a maximal open filter containing $\{T(\sigma + 1) \mid \sigma < \omega_{\alpha+1}\}$, where $T(\sigma) = \{\sigma \mid \sigma \leq \tau < \omega_{\alpha+1}\}$. Then it is clear that $\mathcal{U}$ is not convergent. By Lemma 3.6, $\{\Gamma U \mid U \in \mathcal{U}\}$ has the $\aleph_{\alpha+1}$-intersection property, for every member of a non-convergent maximal open filter is cofinal.

Z. Frolik has shown [16] that every intersection of almost realcompact subspaces of a REGULAR space is also almost realcompact and every closed subspace of an almost n-compact regular space is again almost realcompact.

With the simple modifications of his proof, we have the following:

3.8 Lemma Every intersection of almost n-compact subspaces of a REGULAR space is also almost n-compact.

And every closed subspace of an almost n-compact regular space is also almost n-compact.

3.9 <u>Theorem</u> Let n be a limit cardinal number. Then there exists an almost n-compact space which is not almost t-compact for every infinite cardinal number $t < n$.

<u>Proof</u>: Let $n = \aleph_\gamma$ and I the set of all isolated infinite cardinal numbers less than n. And let $X = \prod_{\aleph_{\alpha+1} \in I} W(\omega_\alpha + 1)$ with the product topology. It is known [24] that X is n-compact. Hence it is almost n-compact. Suppose that X is almost $\aleph_\alpha$-compact for some $\aleph_\alpha < n$. Since X is regular, the closed subspace of X which is homeomorphic with the space $W(\omega_\alpha + 1)$ is also almost $\aleph_\alpha$-compact, which is a contradiction to Corollary 3.7. This completes the proof.

3.10 <u>Theorem</u> The full subcategory of <u>Haus*</u> determined by all almost n-compact spaces is extensive in <u>Haus*</u>.

<u>Remark</u>: During the preparation of this thesis, it has happened that C-T. Liu and G. E. Strecker have shown that the subcategory of <u>Haus*</u> determined by almost realcompact spaces is extensive in <u>Haus*</u> [38] . Their proof goes almost same as ours. However, for the completeness, we will give here the proof of the theorem.

<u>Proof of Theorem</u>: Let X be a Hausdorff space and $\tilde{X}$ the set of all non-convergent maximal open filters on X.

We define $r_nX = X \cup \{\mathcal{M} \in \tilde{X} \mid \{\Gamma_X U \mid U \in \mathcal{M}\}$ has the n-intersection property$\}$ with the relative topology of $\kappa X$.
It is clear that $r_nX$ is a Hausdorff extension of X and X is open in $r_nX$. Let $r_n: X \longrightarrow r_nX$ be the natural embedding of X into $r_nX$. Then it is obvious that $r_n$ is a morphism in <u>Haus*</u>.
We wish first to show that $r_nX$ is almost n-compact. Let $\bar{\eta}_n$ be the collection of all fewer than n open coverings of $r_nX$. By Theorem 3.3, it is enough to show that $\bar{\eta}_n$ is complete. Suppose that there is a $\bar{\eta}_n$-Cauchy filter subbase $\mathcal{F}$ on $r_nX$ which has no cluster point. Consider $\mathcal{G} = \{F \cap X \mid F \in \mathcal{F}\}$. Clearly, $\mathcal{G}$ is an open filter subbase on X. Moreover, $\mathcal{G}$ is an $\eta_n$-Cauchy filter subbase on X, where $\eta_n$ is the collection of all fewer than n open coverings of X. Indeed, take a member $\mathcal{U} = \{A_\iota\}_{\iota \in I}$ of $\eta_n$. Define $\tilde{A}_\iota = A_\iota \cup \{\mathcal{M} \in r_nX \mid A_\iota \in \mathcal{M}\}$. Noting that $\tilde{A}_\iota$ is the largest open set in $r_nX$ whose intersection with X is $A_\iota$, and every $\mathcal{M} \in r_nX - X$ is $\eta_n$-Cauchy, we can conclude that $\tilde{\mathcal{U}} = \{\tilde{A}_\iota \mid \iota \in I\}$ belongs to $\bar{\eta}_n$. Since $\mathcal{F}$ is $\bar{\eta}_n$-Cauchy, there are an $F \in \mathcal{F}$ and an $\iota \in I$ with $F \subseteq \tilde{A}_\iota$. Hence $F \cap X \subseteq \tilde{A}_\iota \cap X = A_\iota$. Let $\mathcal{M}$ be a maximal open filter containing $\mathcal{G}$. Then $\mathcal{M}$ is also $\eta_n$-Cauchy; therefore, $\{\Gamma_X A \mid A \in \mathcal{M}\}$ has the n-intersection property, and $\mathcal{M}$ does not converge, because $\bigcap_{G \in \mathcal{G}} \Gamma_X G = \phi$. Thus $\mathcal{M}$ is an element of $r_nX$. Moreover $\mathcal{M}$ is a cluster point of $\mathcal{F}$, for $(U \cup \{\mathcal{M}\}) \cap F \supseteq U \cap (F \cap X) \neq \phi$ for all $U \in \mathcal{M}$ and $F \in \mathcal{F}$, which is a contradiction. Hence $r_nX$ is almost n-compact.

For any almost n-compact space Y and for any continuous semi-open map $f: X \longrightarrow Y$, we wish to find an extension of f to $r_nX$. For each $x \in X$, let $\bar{f}(r_n(x)) = f(x)$. If $\mathcal{M} \in r_nX - X$, let $\mathcal{U}_{\mathcal{M}} = \{\mathbb{I}f(U) \mid U \in \mathcal{M}\}$. Since f is semi-open, $\phi \notin \mathcal{U}_{\mathcal{M}}$; for U and V in $\mathcal{M}$, $\mathbb{I}f(U) \cap \mathbb{I}f(V) \supseteq \mathbb{I}f(U \cap V)$, which belongs to $\mathcal{U}_{\mathcal{M}}$. Thus, $\mathcal{U}_{\mathcal{M}}$ is an open filter base on Y. Moreover, $\mathcal{U}_{\mathcal{M}}$ generates a maximal open filter on Y. Let $\mathcal{N}$ be an open filter generated by $\mathcal{U}_{\mathcal{M}}$. Take an open set V in Y such that V meets every member of $\mathcal{N}$. Clearly V meets $\mathbb{I}f(U)$ for all $U \in \mathcal{M}$. Therefore $f^{-1}(V) \cap U \neq \phi$ for all $U \in \mathcal{M}$. By the maximality of $\mathcal{M}$, we have $f^{-1}(V) \in \mathcal{M}$, so that $V \supseteq \mathbb{I}f(f^{-1}(V)) \in \mathcal{U}_{\mathcal{M}}$; hence $V \in \mathcal{N}$. We denote the collection of all fewer than n open coverings of Y by $\eta_n'$. We will show that $\mathcal{N}$ is also $\eta_n'$-Cauchy. Consider a member $\mathcal{V} = (A_\iota)_{\iota \in I}$ of $\eta_n'$. Clearly, $f^{-1}(\mathcal{V}) = \{f^{-1}(A_\iota)\}_{\iota \in I} \in \eta_n$. Since $\mathcal{M}$ is $\eta_n$-Cauchy, there are a $U \in \mathcal{M}$ and an $\iota \in I$ with $U \subseteq f^{-1}(A_\iota)$, so that $\mathbb{I}f(U) \subseteq f(U) \subseteq f(f^{-1}(A_\iota)) \subseteq A_\iota$. Since Y is almost n-compact, $\mathcal{N}$ converges to a unique point $p_{\mathcal{M}} \in Y$ by the maximality of $\mathcal{N}$. We let $\bar{f}(\mathcal{M}) = p_{\mathcal{M}}$; $\bar{f}: r_nX \longrightarrow Y$ is well defined. X being open in $r_nX$, $\bar{f}$ is continuous at each point of X. Take $\mathcal{M} \in r_n - X$, and take an open neighborhood U of $f(\mathcal{M}) = p_{\mathcal{M}}$.
Since $\mathcal{N}$ converges to $p_{\mathcal{M}}$, $U \in \mathcal{N}$. Hence $f^{-1}(U) \in \mathcal{M}$; $f^{-1}(U) \cup \{\mathcal{M}\}$ is an open neighborhood of $\mathcal{M}$, and $\bar{f}(f^{-1}(U) \cup \{\mathcal{M}\}) \subseteq U$; therefore $\bar{f}$ is continuous at $\mathcal{M}$. Since $\bar{f}r_n = f$, $\bar{f}$ is semi-open, i.e. a morphism in Haus*. And the uniqueness of $\bar{f}$

follows from the fact that $r_n$ is dense.

3.11 Corollary The category of almost n-compact spaces and continuous semi-open maps is productive and regular closed hereditary.

3.12 Theorem In the category of almost n-compact spaces and continuous maps or the category of almost n-compact spaces and perfect maps, the perfect onto projectives are precisely extremally disconnected almost n-compact spaces and the perfect onto projectivity is properly behaved.

Proof: It is immediate from Theorem 2.3 and 3.10.

3.13 Lemma Every dense embedding of a Hausdorff space X into an almost n-compact space Y can be continuously extended to $r_n X$.

Proof: Let $j: X \longrightarrow Y$ be the dense embedding. Without loss of generality, we may assume that $j(x) = x$ for each $x \in X$. For each $x \in X$, let $f(x) = x$. If $\mathcal{M} \in r_n X - X$, let $\mathcal{M}' = \{U \mid U\text{: open in } Y \text{ and } U \cap X \in \mathcal{M}\}$. It is obvious that $\mathcal{M}'$ is an open filter on Y. Moreover it is a maximal open filter.
Let $\eta_n$ and $\eta_n'$ be the collections of all fewer than n open coverings of X and Y respectively. Suppose $(A_\iota)_{\iota \in I} \in \eta_n'$. Clearly $(A_\iota \cap X)_{\iota \in I} \in \eta_n$. Since $\mathcal{M}$ is $\eta_n$-Cauchy, there is a $V \in \mathcal{M}$ and there is an $\iota \in I$ such that $V \subseteq A_\iota \cap X$, so that $A_\iota \cap X$ belongs to $\mathcal{M}$. Hence $A_\iota \in \mathcal{M}'$. Thus $\mathcal{M}'$ is $\eta_n'$-Cauchy.

Since Y is almost n-compact, $\mathfrak{M}'$ converges to a unique point $p_{\mathfrak{M}} \in Y$ by the maximality of $\mathfrak{M}'$. We let $f(\mathfrak{M}) = p_{\mathfrak{M}}$; hence $f: r_nX \longrightarrow Y$ is well defined. The continuity of f follows from the exactly same argument in Theorem 3.10.

Remark: We note that a dense embedding need not be semi-open. For example, $f: \mathbb{Q} \longrightarrow \mathbb{R}$ defined by the natural embedding is not semi-open, but a dense embedding.

3.14 Corollary Every continuous map f on a completely regular space X into an n-compact space Y can be continuously extended to $r_nX$.

Proof: It is immediate from the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle r_n}\downarrow & \searrow^{\beta_n} & \uparrow{\scriptstyle \bar{f}} \\ r_nX & \xrightarrow{\ \bar{\beta}_n\ } & \beta_nX \end{array}$$

, where $\bar{\beta}_n$ is determined by $\beta_n$, Lemma 3.13 and Corollary 3.5, and $\bar{f}$ is determined by f and $\beta_n$.

3.15 Theorem For a completely regular space X, $\beta_n \rho r_n X = \beta_n X$, where $\rho Y$ is the complete regularization of a space Y.

Proof: Let $\rho: r_nX \longrightarrow \rho r_nX$ be the reflection map of $r_nX$ in the category of completely regular spaces.
It is easy to show that $\rho r_n$ is the embedding of X into $\rho r_nX$.

Indeed, for any x, y in X and $x \neq y$, there is an $f \in C^*(X)$ such that $f(x) \neq f(y)$. Hence there is a continuous map on $r_nX$ into $\Gamma_{\mathbb{R}} f(X)$, which is an extension of f by Corollary 3.14. Therefore, $\wp(r_n(x)) \neq \wp(r_n(y))$. Clearly, $\wp r_n$ is continuous. Take a zero-set Z in X. Let g be a continuous map on X into the unit interval with $Z(g) = Z$. Then there is a continuous map $g_n$ on $r_nX$ into the unit interval with $g_n|X = g$. Also there is a unique continuous map $\bar{g}$ on $\wp r_nX$ into the unit interval with $\bar{g}\wp = g_n$. It is obvious that $Z(\bar{g}) \cap \wp r_n(X) = \wp r_n(Z)$. Without loss of generality, we may assume that $\wp r_n(x) = x$ for all $x \in X$. Combining the above corollary and the uniqueness of $\beta_nX$, we can conclude that $\beta_n \wp r_nX = \beta_nX$.

Since the almost n-compactness is not closed hereditary, we can not apply the same argument as Corollary 2.10 in Chap II to the case of almost n-compact spaces.

However, we have the following:

3.16 <u>Theorem</u> Let $(X_\iota)_{\iota \in I}$ be a family of non-empty almost n-compact spaces. If I is almost n-compact with respect to the discrete topology, then the sum space $\Sigma X_\iota$ of $(X_\iota)$ is also almost n-compact.

<u>Proof</u>: Let $X = \Sigma X_\iota$ and $\mathcal{M}$ a maximal open filter on X such that $\{\Gamma U \mid U \in \mathcal{M}\}$ has the n-intersection property. We define $I_U = \{\iota \in I \mid U \cap X_\iota \neq \phi\}$ for each $U \in \mathcal{M}$. Then it is obvious that $\mathfrak{F} = \{I_U \mid U \in \mathcal{M}\}$ is an ultrafilter with the

n-intersection property. Hence $\mathfrak{F}$ has a cluster point $\gamma$. Let $\mathfrak{G} = \{U \mid U \in \mathfrak{M}$ and $U \subseteq X_\gamma\}$. Then it is easy to show that $\mathfrak{G}$ is a maximal open filter on $X_\gamma$ such that $\{\Gamma_{X_\gamma} G \mid G \in \mathfrak{G}\}$ has the n-intersection property. Hence $\mathfrak{G}$ has a cluster point in $X_\gamma$; $\mathfrak{M}$ also has a cluster point in X.

For a Hausdorff closed (= almost $\aleph_0$-compact) space X, $\Lambda(X) = \Omega(X)$ is compact, hence almost $\aleph_0$-compact. For any almost n-compact regular space X, lim: $\Lambda(X) \longrightarrow X$ is essential perfect onto, so that $\Lambda(X)$ is also almost n-compact by Corollary 2.2 and Theorem 3.10.

3.17 <u>Theorem</u> If a Hausdorff space X is almost n-compact then $\Lambda(X)$ is also almost n-compact.

<u>Proof</u>: Let $\eta_n$ and $\eta'_n$ be the collections of all fewer than n open coverings of X and $\Lambda(X)$ respectively. Then $\Lambda(X)$ is precisely the set of all $\eta_n$-Cauchy maximal open filters on X, for X is almost n-compact.

Suppose that there is an $\eta'_n$-Cauchy family $\mathfrak{F}$ which has no cluster point. Since lim is minimal closed, $\mathfrak{U} = \{\complement \lim(\complement F) \mid F \in \mathfrak{F}\}$ is an open filter subbase on X. Suppose that $\mathfrak{U}$ has a cluster point a. Then there is a maximal open filter $\mathfrak{M}$ containing $\mathfrak{V}(a)$ and $\mathfrak{U}$.

We wish to show that $\mathfrak{M}$ is a cluster point of $\mathfrak{F}$, which is a contradiction. Indeed, suppose that $\mathfrak{M} \notin \Gamma_{\Lambda(X)} F$ for some $F \in \mathfrak{F}$. Thus there is an open set V in X such that

$\mathfrak{M} \in \Lambda_V = \Omega_V \cap \Lambda(X)$ and $\Lambda_V \cap F = \phi$. Since $V \in \mathfrak{M}$, $V \cap \complement\lim(\complement F) \neq \phi$, say $b \in V \cap \complement\lim(\complement F)$. There exists a maximal open filter $\mathfrak{N}$ on X which converges to b. Hence $\mathfrak{N} \in \Lambda_V$, so that $\mathfrak{N} \notin F$; $\lim \mathfrak{N} = b \in \lim(\complement F)$, which is a contradiction. Thus, $\mathfrak{U}$ has no cluster point.

Let $\mathfrak{M}$ be a maximal open filter on X containing $\mathfrak{U}$. Since X is almost n-compact and $\mathfrak{M}$ has no cluster point, there is a subfamily $(U_\iota)_{\iota \in I}$ of $\mathfrak{M}$ such that $\bigcap_{\iota \in I} \Gamma_X U = \phi$ and $|I| < n$. Since $\bigcup_\iota \complement \Gamma_X U_\iota = X$, we have that $\{\Lambda_{\complement \Gamma_X U_\iota}\}_{\iota \in I}$

is a member of $\eta_n^{\cdot}$. $\mathfrak{F}$ being $\eta_n^{\cdot}$-Cauchy, there are an $F \in \mathfrak{F}$ and an $\iota \in I$ with $F \subseteq \Lambda_{\complement \Gamma_X U_\iota}$.

From this, we can conclude that $\complement\lim(\complement F) \cap U_\iota = \phi$. Indeed, suppose that $\complement\lim(\complement F) \cap U_\iota \neq \phi$. Take an element c of $\complement\lim(\complement F) \cap U_\iota$. Then there exists a maximal open filter $\mathfrak{N}$ on X which converges to c. By the same argument as above, $\mathfrak{N} \in F$; therefore $\mathfrak{N} \in \Lambda_{\complement \Gamma_X U_\iota}$, i.e. $\complement \Gamma_X U_\iota \in \mathfrak{N}$.

But $U_\iota \in \mathfrak{N}$ and $U_\iota \cap \complement \Gamma_X U = \phi$, which is a contradiction.

Since $U_\iota$ and $\complement\lim(\complement F)$ are members of $\mathfrak{M}$, we have a contradiction. This completes the proof.

## Section 4: Category of pseudo-compact spaces.

Every category which we have considered so far in this chapter is either productive or closed hereditary.

In this section, it will be shown that in the category of pseudo-compact spaces and continuous maps which is neither productive nor closed hereditary, the perfect onto projectivity is still properly behaved.

It is known [7] that a completely regular space X is pseudo-compact if and only if every countable open covering of X has a finite subfamily whose union is dense in X. Moreover, it is equivalent to that every countable open filter base on X has a cluster point.

4.1 Definition Let n be an infinite cardinal number. The full subcategory of Haus determined by the spaces with the following property will be denoted by $\mathcal{W}_n$:

Every open filter base on the space whose cardinal number is not greater than n (we will call it simply n-open filter base) has at least one cluster point.

4.2 Lemma Let X be a Hausdorff space. X belongs to $\mathcal{W}_n$ if and only if every open covering $\mathcal{U}$ of X with $|\mathcal{U}| \leq n$ has a finite subfamily whose union is dense in X.

Proof: It is the dual statement of the definition.

4.3 Theorem If a Hausdorff space X belongs to $\mathcal{W}_n$, so does $\Lambda(X)$.

Proof: Let $\mathfrak{F}$ be an n-open filter base on $\Lambda(X)$ and $\mathfrak{G} = \{\complement\lim(\complement U) \mid U \in \mathfrak{F}\}$. Since lim is minimal closed, $\mathfrak{G}$ is also an n-open filter base on X. Since X belongs to $\mathcal{W}_n$, $\mathfrak{G}$ has a cluster point say x. Then there exists a maximal open filter $\mathcal{M}$ which converges to x and contains $\mathfrak{G}$. We wish to show that $\mathcal{M}$ is a cluster point of $\mathfrak{F}$. Suppose that there is an open set V in X such that $\mathcal{M} \in \Lambda_V$ and $\Lambda_V \cap U = \phi$ for some $U \in \mathfrak{F}$. Since $\mathcal{M} \in \Lambda_V$, $V \in \mathcal{M}$; $V \cap \complement\lim(\complement U) \neq \phi$. Let y be an element of $V \cap \complement\lim(\complement U)$, and let $\mathcal{N}$ be a maximal open filter which converges to y. Thus $\mathcal{N} \in \Lambda_V$, so that $\mathcal{N} \notin U$. Hence $y = \lim \mathcal{N} \in \lim(\complement U)$, which is a contradiction.

4.4 Definition An open subset U of a topological space X is said to be regular if U is the interior of its closure; equivalently, U is regular if U is the interior of a closed subset.

A topological space X is said to be semi-regular if the regular open sets of X form a base of the topology on X. In this case, the topology is also said to be semi-regular.

Remark: It is known [7] that the regular open sets of a topological space $(X, \mathcal{J})$ form a base of a topology on X. The topology generated by the regular open sets with respect to $\mathcal{J}$ will be denoted by $\mathcal{J}^*$. Then it is obvious that $\mathcal{J}^*$ is coarser than $\mathcal{J}$ and is semi-regular. The topology $\mathcal{J}^*$ is said to be the semi-regular topology associated with $\mathcal{T}$.

4.5 <u>Lemma</u> Let $(X, \mathcal{T})$ be a Hausdorff space, and let $\mathcal{T}'$ be a topology on X with $\mathcal{T} \supseteq \mathcal{T}' \supseteq \mathcal{T}^*$. Then $(X, \mathcal{T})$ belongs to $\mathcal{W}_n$ if and only if $(X, \mathcal{T}')$ belongs to $\mathcal{W}_n$.

<u>Proof</u>: It is easy to show that $\mathcal{W}_n$ is closed under continuous images. Hence, if $(X, \mathcal{T})$ belongs to $\mathcal{W}_n$, so does $(X, \mathcal{T}')$. Conversely, let $\mathcal{F}$ be an n-open filter base on $(X, \mathcal{T})$. Let $\mathcal{G} = \{\mathbb{I}\Gamma U \mid U \in \mathcal{F}\}$. It is obvious that $\mathcal{G}$ is also an n-open filter base on $(X, \mathcal{T}')$. Since $(X, \mathcal{T}')$ belongs to $\mathcal{W}_n$, $\mathcal{G}$ has a cluster point, say x. It is easy to show that x is also a cluster point of $\mathcal{F}$.

Let X be a Hausdorff space, and $\mathcal{T}(X)$ be its topology. We define $\Lambda'(X)$ as follows:

its underlying set is the same as that of $\Lambda(X)$ and its topology is generated by that of $\Lambda(X)$ together with $\lim_X^{-1}(\mathcal{T}(X))$ [5].

4.6 <u>Lemma</u> For any $X \in \mathcal{W}_n$, the map $\Lambda'(X) \longrightarrow X$ given by $\lim_X$ is a perfect onto projective cover in $\mathcal{W}_n$.

<u>Proof</u>: By Proposition 9 in [5], it is enough to show that $\Lambda'(X)$ belongs to $\mathcal{W}_n$ for any $X \in \mathcal{W}_n$. By Proposition 8 and Lemma 11 in [5], $\Lambda(X)$ and $\Lambda'(X)^*$ are homeomorphic, where $\Lambda'(X)^*$ is the space with the associated semi-regular topology of that of $\Lambda'(X)$. Since X belongs to $\mathcal{W}_n$, so does

$\Lambda(X)$; thus $\Lambda'(X)^*$ also belongs to $\mathcal{W}_n$. By Lemma 4.5, $\Lambda'(X)$ belongs to $\mathcal{W}_n$.

4.7 Theorem For any $X \in \mathcal{W}_n$, X is perfect onto projective if and only if it is extremally disconnected.

In $\mathcal{W}_n$, the perfect onto projectivity is properly behaved.

Proof: The first part is immediate from Proposition 9 in [5] and Lemma 4.6.

The second part follows because perfect onto maps are closed under the composition, for perfect onto maps f and g, gf = f implies g is an identity, and every object of $\mathcal{W}_n$ has a perfect onto projective cover.

Remark: For almost n-compact spaces, it is not difficult to show the corresponding property to Lemma 4.5. Hence one can also prove by Theorem 3.17 and the same argument as the above theorem that the perfect onto projectivity in the category of almost n-compact spaces and continuous maps is properly behaved.

The category of pseudo-compact spaces and continuous maps will be denoted by PComp. Since a completely regular space is pseudo-compact if and only if it belongs to $\mathcal{W}_{\aleph_0}$, and every completely regular space is regular, the following theorem is the immediate consequence of Lemma 4.6 and Theorem

4.7.

4.8 Theorem 1) A completely regular space X is pseudo-compact if and only if $\Lambda(X)$ is pseudo-compact.

2) For any $X \in$ PComp, X is perfect onto projective if and only if it is extremally disconnected.

3) For any $X \in$ PComp, the map $\Lambda(X) \longrightarrow X$ given by $\lim_X$ is a perfect onto projective cover of X in PComp.

4) The perfect onto projectivity in PComp is properly behaved.

# CHAPTER IV

## TOPOLOGICALLY COMPLETE SPACES

### Section 1: n-totally bounded complete spaces.

1.1 Definition Let $(X, \mathcal{U})$ be a uniform space, and let n be an infinite cardinal number. X is said to be n-totally bounded if for each entourage V in $\mathcal{U}$, there exists a subset A of X such that $X = \bigcup_{x \in A} V(x)$ and $|A| < n$.

Remark: A uniform space is $\aleph_0$-totally bounded if and only if it is totally bounded.

1.2 Proposition Let X be a set, let $(Y_\lambda)_{\lambda \in L}$ be a family of uniform spaces, and for each $\lambda \in L$, let $f_\lambda$ be a map on X onto $Y_\lambda$. Let X carry the coarsest uniform structure for which the $f_\lambda$ are uniformly continuous. Then X is n-totally bounded if and only if $Y_\lambda$ is n-totally bounded for each $\lambda \in L$.

Proof: Trivial.

1.3 Corollary A product space of n-totally bounded uniform spaces is again n-totally bounded.

1.4 Proposition Every subspace of an n-totally bounded uniform space is again n-totally bounded.

Proof: Trivial.

The following theorem is the well known Shirota's theorem [17, 47], but we give here the proof of the theorem in the language of entourages.

1.5 Theorem A completely regular space is realcompact if and only if it admits a complete $\aleph_1$-totally bounded uniform structure.

Proof: Since every realcompact space is homeomorphic with a closed subspace of a product space of copies of real line and the real line $\mathbb{R}$ is $\aleph_1$-totally bounded complete with respect to the usual uniform structure of $\mathbb{R}$, every realcompact space admits a complete $\aleph_1$-totally bounded uniform structure.

Conversely, let $\mathcal{U}$ be an admissible complete $\aleph_1$-totally bounded uniform structure on a completely regular space X. Since X is completely regular, X can be considered to be a subspace of the product space $\mathbb{R}^{C(X)}$ under the map $x \rightsquigarrow (f(x))_{f \in C(X)}$. Suppose X is not realcompact. Then there is an element p in the closure of X in $\mathbb{R}^{C(X)}$ but not in X. Let $\mathfrak{F}$ be the trace filter on X of the neighborhood filter of p in $\mathbb{R}^{C(X)}$. Since $\mathfrak{F}$ is not convergent in X, it is not Cauchy. Hence there is an entourage U in $\mathcal{U}$ such that for each $F \in \mathfrak{F}$, $(F \times F) \cap \complement U \neq \phi$. Let d be a uniformly continuous pseudo-metric on X such that $V_{d,\varepsilon} = \{(x,y) \mid d(x,y) < \varepsilon\}$ is contained in U. Since $V_{d,\varepsilon/4}$ is an entourage in $\mathcal{U}$, there exists a sequence $(a_k)$ such that $X = \bigcup_k V_{d,\varepsilon/4}(a_k)$. For each k, $V_{d,\varepsilon/2}(a_k)$ does not belong to $\mathfrak{F}$. Let $f_k$ be a map on X into $\mathbb{R}$ defined by $x \rightsquigarrow ((3 - 6/\varepsilon\, d(x,a_k)) \wedge 1) \vee 0$. Then it is obvious that $f_k$ is continuous and has the value 1 on $V_{d,\varepsilon/3}(a_k)$ and the value 0 on $X - V_{d,\varepsilon/2}(a_k)$. For each $F \in \mathfrak{F}$, $F \cap \complement V_{d,\varepsilon/2}(a_k) \neq \phi$. Hence there is $x \in F$ with $f_k(x) = 0$, and therefore the $f_k$-th coordinate of p must be 0. From this it follows that $\complement V_{d,\varepsilon/3}(a_k) \in \mathfrak{F}$ for each k. As above, we can find a continuous map $h_k$ on X into $\mathbb{R}$ which has the value 1 on $V_{d,\varepsilon/4}(a_k)$ and the value 0 on $\complement V_{d,\varepsilon/3}(a_k)$ for each k. Consider the map $h = \sum_k (h_k \wedge 1/2)$. Then $h \in C(X)$. Since $\{V_{d,\varepsilon/4}(a_k)\}_k$ is a covering of X, there is a k for $x \in X$ such that $h_k(x) = 1$, hence $h(x) > 0$ for each $x \in X$. Since X is C-embedded in $\mathbb{R}^{C(X)}$, h has a continuous extension $\bar{h}$ to $\mathbb{R}^{C(X)}$. Since $h > 0$, $\bar{h}(p) \neq 0$. But no finite union of $V_{d,\varepsilon/3}(a_k)$'s belongs to $\mathfrak{F}$, because $\complement V_{d,\varepsilon/3}(a_k) \in \mathfrak{F}$. Thus h has arbitrarily small values on each member of $\mathfrak{F}$ and so $\bar{h}(p) = 0$, which is a contradiction.

1.6 Lemma For any infinite cardinal number n, let P be the space $P_n$ in Lemma 1.6 in Chap. II. Then every completely regular space X is homeomorphic with a subspace of $P^{C(X,P)}$ such that each continuous map on X into P can be continuously extended to $P^{C(X,P)}$

Proof: It follows immediately from that the class of completely regular spaces is $\mathbf{I}$-regular and that P has a subspace which is homeomorphic with $\mathbf{I}$ .

1.7 Theorem If a completely regular space has an admissible n-totally bounded complete uniform structure, then it is n-compact.

Proof: Let X be a completely regular space and $\mathcal{U}$ an admissible n-totally bounded complete uniform structure on X. We may assume that in Lemma 1.6 in Chap. I and Lemma 1.6, $P = J^t - \{(\infty)\}$ provided $n = t^+$ ($P = \prod_{t<n} (J^t - \{(\infty)\})$ provided n is a limit cardinal number), where $J = [\![0,\infty]\!]$ with the usual topology. Then X can be considered to be a subspace of $P^{C(X,P)}$ such that each continuous map on X into P can be continuously extended to $P^{C(X,P)}$, where P depends on the cardinal number n.

Let q be in the closure of X but not in X and let $\mathfrak{F}$ be the trace filter on X of the neighborhood filter of q in $P^{C(X,P)}$. Since $\mathfrak{F}$ is not convergent in X, it is not Cauchy. Thus there is an entourage U in $\mathcal{U}$ , such that for each F in $\mathfrak{F}$ , $(F \times F) \cap \complement U \neq \phi$. Let d be a uniformly continuous pseudometric on X such that $V_{d,\varepsilon}$ is contained in U. Since $V_{d,\varepsilon/4}$ is an entourage in $\mathcal{U}$ , there exists a subset A of X such that $X = \bigcup_{x \in A} V_{d,\varepsilon/4}(x)$ and $|A| < n$. For each $x \in A$, $V_{d,\varepsilon/2}(x)$

does not belong to $\mathfrak{F}$ , for it is a U-small set. For each x $\in$ A, we can find a continuous map $f_x$ on X into $[\![0,1]\!]$ which has the value 1 on $V_{d,\varepsilon/3}(x)$ and the value 0 on X - $V_{d,\varepsilon/2}(x)$.

Case 1. $n = t^+$. We may assume that $|A| = t$ and $P = J^A - \{(\infty)\}$. Thus there is a continuous map f on X into P such that $\pi_x f = f_x$, where $\pi_x$ is the x-th projection of P, for each $x \in A$. Each F in $\mathfrak{F}$ meets the complement of $V_{d,\varepsilon/2}(x)$ and so there is $y \in F$ with $f_x(y) = 0$. Therefore the f-th coordinate of q must be (0). From this it follows that $X - V_{d,\varepsilon/3}(x) \in \mathfrak{F}$ for each $x \in A$. Similarly, we can find a continuous map $h_x$ on X into $[\![0,1]\!]$ which has the value 1 on $V_{d,\varepsilon/4}(x)$ and the value 0 on $X - V_{d,\varepsilon/3}(x)$ for each $x \in A$. Thus there exists a continuous map h on X into P such that $\pi_x h = h_x$ for each $x \in A$. Let $\bar{h}$ be a continuous extension of h to $P^{C(X,P)}$. Then $\bar{h}$ is the h-th projection of $P^{C(X,P)}$ onto P. Hence $\bar{h}(q) = (0)$, because $X - V_{d,\varepsilon/3}(x) \in \mathfrak{F}$ for each x. Let g be a map on X into P defined by:

$$\text{the x-th coordinate of } g(y) = \begin{cases} \infty & \text{if } h_x(y) = 0, \\ 1/h_x(y) & \text{if otherwise.} \end{cases}$$

Since $\{V_{d,\varepsilon/4}(x) \mid x \in A\}$ is a covering of X, there is for each $y \in X$ an $x \in A$ such that $h_x(y) = 1$. Thus g is well defined on X. And it is easy to show that g is continuous. Hence g has a continuous extension $\bar{g}$ to $P^{C(X,P)}$. For each x $\in$ A, and for each $M > 0$, there is a neighborhood V of q in $P^{C(X.P)}$ such that $\pi_x h(V)$ is contained in $[\![0, 1/M[\![$, for

$\pi_x \bar{h}(q) = 0$. Hence $\pi_x g(V \cap X)$ is contained in $]M, \infty]$.
From this, it follows that $\pi_x \bar{g}(q) = \infty$ for each $x \in A$. Thus $\bar{g}(q)$ does not belong to P, which is a contradiction.

Case 2. Let n be a limit cardinal number. And let $|A| = t < n$. Then there exists a continuous map $f_t$ on X into $P_{t^+} = J^A - \{(\infty)\}$ such that $\pi_x f_t = f_x$ for each $x \in A$, where $\pi_x$ is the x-th projection of $P_{t^+}$.
Define a continuous map f on X into P such that $\pi_m f = (0)$ for $m \neq t$ and $\pi_t f = f_t$, where $\pi_m$ is the m-th projection of P.
By the same argument as Case 1., $\pi_t \pi_f(q) = (0)$, where $\pi_f$ is the f-th projection of $P^{C(X,P)}$. From this, it follows that $X - V_{d,\varepsilon/3}(x) \in \mathfrak{F}$ for each $x \in A$. Again we can find a continuous map $h_x$ on X into $[0,1]$ which has the value 1 on $V_{d,\varepsilon/4}(x)$ and the value 0 on $X - V_{d,\varepsilon/3}(x)$ for each $x \in A$. Thus there exists a continuous map h on X into P such that $\pi_x \pi_t h = h_x$ for each x. Let $\bar{h}$ be a continuous extension of h to $P^{C(X,P)}$.
Then $\bar{h}$ is the h-th projection of $P^{C(X,P)}$ onto P.
Hence $\pi_t \bar{h}(q) = (0)$, for $X - V_{d,\varepsilon/3}(x) \in \mathfrak{F}$ for each $x \in A$.
Let g be a map on X into P defined by:

$$\pi_x \pi_t g(y) = \begin{cases} \infty \text{ if } h_x(y) = 0, \\ 1/h_x(y) \text{ if otherwise, and} \end{cases}$$

$\pi_m g(y) = (0)$ for $m \neq t$.

By the same argument as Case 1., g is well defined and continuous on X. Let $\bar{g}$ be a continuous extension of g to $P^{C(X,P)}$.
By the same argument as Case 1., $\pi_t \bar{g}(q)$ does not belong to

$P_{t^+}$, which is a contradiction.

This completes the proof.

Remark: It is well known [7] that every admissible uniform structure on a pseudo-compact space is totally bounded. Thus, if a pseudo-compact space has an admissible complete uniform structure, then it is compact.

For $n > \aleph_1$, there is an n-compact space which is pseudo-compact but not compact, namely space in Lemma 1.6 in Chap. II. For those spaces, there is no admissible complete uniform structure, so that the converse of Theorem 1.7 need not be true.

## Section 2: Category of n-totally bounded complete spaces.

2.1 Definition A completely regular space is said to be topologically n-totally bounded complete if it has an admissible n-totally bounded complete uniform structure.

In what follows, a topologically n-totally bounded complete space will be called simply n-totally bounded complete.

It is well known that the product space of topologically complete spaces is again topologically complete and a closed subspace of a topologically complete space is also topologically complete. Combining these facts with Proposition 1.2 and 1.4, we have the following:

2.2 Theorem The category of n-totally bounded complete completely regular spaces and continuous maps is complete.

2.3 Lemma Let X be a set and let n be an infinite cardinal number. For an ultrafilter $\mathcal{U}$ on X, the following are equivalent:

1) $\mathcal{U}$ is a Cauchy filter with respect to the uniform structure $\wp_n$ generated by fewer than n partitions of X.

2) $\mathcal{U}$ is closed under the t-intersections for every cardinal number t less than n.

3) $\mathcal{U}$ has the n-intersection property.

Proof: 1)$\Longrightarrow$2). Suppose that $\mathcal{V}$ is a subfamily of $\mathcal{U}$ whose intersecion does not belong to $\mathcal{U}$ and $|\mathcal{V}| < n$. Then there is a partition $\{A_U\}_{U\in\mathcal{V}} \cup \{B\}$ of X such that $A_U \subseteq \complement U$ for all $U \in \mathcal{V}$ and $B \subseteq \bigcap\mathcal{V}$. Since $\mathcal{U}$ is Cauchy with respect to $\wp_n$, there is a $V \in \mathcal{U}$ such that $V \subseteq A_U$ for some $U \in \mathcal{V}$ or $V \subseteq B$. Thus $\complement U$ or $\bigcap\mathcal{V}$ belongs to $\mathcal{U}$, which is a contradiction.

2)$\Longrightarrow$3). Since $\phi \notin \mathcal{U}$, this is trivial.

3)$\Longrightarrow$1). Suppose $\mathcal{U}$ is not Cauchy. Then there is a $V = \bigcup_{\iota\in I} A_\iota \times A_\iota$ in $\wp_n$ such that $\{A_\iota\}$ is a partition of X, $|I| < n$ and $A_\iota \notin \mathcal{U}$ for all $\iota \in I$. Hence $\complement A_\iota \in \mathcal{U}$ and $\bigcap_{\iota\in I} \complement A_\iota = \phi$, which is a contradiction.

Remark: Since every covering of a set has a refinement which is a partition and has the same cardinal number, a discrete space X is n-totally bounded complete if and only if the uniform structure generated by fewer than n partitions of X is n-totally bounded complete.

2.4 Proposition A discrete space is n-totally bounded complete if and only if it is n-compact.

Proof: A discrete space X is n-totally bounded complete if and only if every Cauchy ultrafilter with respect to $\wp_n$ is convergent if and only if every ultrafilter with the n-intersection property is fixed if and only if the space X is n-compact.

2.5 Definition A $\{0,1\}$-valued measure or simply measure on a set X is a countably additive set map defined on the family of all subsets of X into $\{0,1\}$.

A measure $\mu$ on X is said to be n-additive for an infinite cardinal number n if $\mu(\bigcup_{\iota\in I} A_\iota) = 0$ whenever $\{A_\iota\}_{\iota\in I}$ is a family of disjoint subsets of measure zero, with $|I| = n$.

Remark: For an ultrafilter $\mathcal{U}$ on a set X, let $\chi_{\mathcal{U}}$ be its characteristic map defined on the set of all subsets of X. Then the correspondence $\mathcal{U} \mapsto \chi_{\mathcal{U}}$ is one-one from the set of all ultrafilters on X onto the set of all nonzero, finitely additive, $\{0,1\}$-valued set maps defined on X [17].

2.6 Lemma For an ultrafilter $\mathcal{U}$ on a set X, the finitely additive measure $\chi_{\mathcal{U}}$ defined by $\mathcal{U}$ is n-additive if and only if $\mathcal{U}$ is closed under the n-intersection.

Proof: Since in the above definition of n-additive measures, we may drop the requirement that the subsets be disjoint, the map $\chi_{\mathcal{U}}$ is n-additive if and only if $\chi_{\mathcal{U}}(A_\iota) = 0$ ($\iota \in I$ and $|I| = n$) implies $\chi_{\mathcal{U}}(\bigcup_\iota A_\iota) = 0$.
But this is simply the dual of the statement that $\mathcal{U}$ is closed under n-intersection.

Remark: By the above lemma, every measure can be defined by the characteristic map of an ultrafilter with the countable intersection property.

2.7 Definition A cardinal number n is said to be measurable if a set X of cardinal number n admits a measure $\mu$ such that $\mu(X) = 1$, and $\mu(\{x\}) = 0$ for every $x \in X$.

Otherwise, it is said to be nonmeasurable.

2.8 Lemma Each measure is n-additive for every nonmeasurable cardinal number n.

Proof of the lemma can be found in [17].

2.9 Proposition Let X be a discrete space and let m be the first measurable number. Let n be a cardinal number such that $\aleph_1 \leqslant n \leqslant m$. Then the following are equivalent:

1) X is n-totally bounded complete.

2) X is n-compact.

3) X is realcompact.

Proof: It is enough to show that 2) implies 3). Suppose that there is a free ultrafilter on X with the countable intersection property. By Lemma 2.8, the measure defined by the ultrafilter is n-additive for $n < m$. Hence the ultrafilter is closed under the n-intersection. Thus the ultrafilter is fixed, which is a contradiction.

If X is m-compact, then there exists a subfamily $(U_\iota)_{\iota \in I}$ of the ultrafilter with $\bigcap_{\iota \in I} U_\iota = \phi$ and $|I| < m$. Then the measure defined by the ultrafilter is $|I|$-additive, hence the ultrafilter is closed under $|I|$-intersection, which is a contradiction.

The following definition is due to H. Herrlich [24].

2.10 Definition The compactness degree k(X) of a completely regular space X is the smallest cardinal number n such that X is n-compact.

2.11 Corollary If X is a discrete space of cardinal number m, then $k(X) = m^+$.

2.12 Lemma A complete space is realcompact if and only if every closed discrete subspace is realcompact.

Proof of the lemma can be found in [17].

2.13 Theorem Let n be a cardinal number such that $\aleph_1 \leq n \leq m$. A completely regular space is n-totally bounded complete if and only if it is realcompact.

Proof: Let X be an n-totally bounded complete space. Suppose X is not realcompact. Then by Lemma 2.12, there exists a closed discrete subspace F of X which is not realcompact. Since F is a closed subspace, it admits an admissible n-totally bounded complete uniform structure. Hence F is realcompact, which is a contradiction.

Conversely, realcompact space admits an $\aleph_1$-totally bounded complete uniform structure.

The other proof: By Theorem 1.7, every n-totally bounded complete space is n-compact. Hence every closed subspace of X is also n-compact. Thus every closed discrete subspace of X is realcompact by Proposition 2.9. Hence X is realcompact by Lemma 2.12.

## Section 3: A characterization of realcompact spaces.

3.1 Definition A family $\mathcal{S}$ of subsets of a topological space is said to be locally finite (discrete) if each point of the space has a neighborhood which intersects only finitely many members (at most one member, respectively) of $\mathcal{S}$.

3.2 Theorem Let X be a completely regular space and n an infinite cardinal number. If every locally finite open covering of X has a fewer than n subcovering, then every admissible uniform structure on X is n-totally bounded.

Proof: Suppose there is an admissible uniform structure $\mathcal{U}$ on X which is not n-totally bounded.
Let $\alpha$ be the first ordinal whose cardinal number is n. Then there exist an open symmetric entourage V in $\mathcal{U}$ and a net $(x_\lambda)_{\lambda<\alpha}$ on X such that $\{V(x_\lambda)\}_{\lambda<\alpha}$ is a discrete family. Indeed, there exists an entourage U in $\mathcal{U}$ such that for any subset A of X with $|A| < n$, $U(A) \neq X$. Take $x_0 \in X$. Suppose that for $\tau < \alpha$, we have $\{x_\lambda\}_{\lambda<\tau}$ in X such that $x_\mu \notin \bigcup_{\lambda<\mu} U(x_\lambda)$ for any $\mu < \tau$. Since $|\{x_\lambda\}_{\lambda<\tau}| < n$, $\bigcup_{\lambda<\tau} U(x_\lambda) \neq X$. Take an element $x_\tau$ of $\complement \bigcup_{\lambda<\tau} U(x_\lambda)$. Hence by the induction, we have a net $\{x_\lambda\}_{\lambda<\alpha}$ such that for any $\mu < \alpha$, $x_\mu \notin \bigcup_{\lambda<\mu} U(x_\lambda)$. Take a symmetric open entourage V in $\mathcal{U}$ with $V^3 \subseteq U$. Then it is easy to show that $\{V(x_\lambda)\}_{\lambda<\alpha}$ is discrete. Indeed, for any x in X, take the neighborhood

$W(x)$ of $x$, where $W$ is an entourage in $\mathcal{U}$ and $W^2 \subseteq V$. Suppose that $W(x)$ meets $V(x_\lambda)$ and $V(x_\tau)$ for $\lambda < \tau < \alpha$. Since $W^2 \subseteq V$ and $V^3 \subseteq U$, it is clear that $(x_\tau, x_\lambda) \in U$, so that $x_\tau \in U(x_\lambda)$ which is a contradiction.

For each $\lambda < \alpha$, take an open neighborhood $N(x_\lambda)$ of $x_\lambda$ such that $\Gamma N(x_\lambda)$ is contained in $V(x_\lambda)$. Since $\{V(x_\lambda)\}_{\lambda<\alpha}$ is discrete, so is $\{\Gamma N(x_\lambda)\}_{\lambda<\alpha}$, so that $\bigcup_{\lambda<\alpha} \Gamma N(x_\lambda)$ is closed. Hence it is clear that $V(x_\lambda)$ for all $\lambda < \alpha$ together with $\complement \bigcup_{\lambda<\alpha} \Gamma N(x_\lambda)$ form a locally finite open covering of X. Since $\{V(x_\lambda)\}_{\lambda<\alpha}$ is the pairwise disjoint family, the open covering has no proper subcovering. This completes the proof.

3.3 <u>Corollary</u> A completely regular space is pseudo-compact if and only if every locally finite open covering of the space has a finite subcovering.

<u>Proof</u>: It follows immediately from that a completely regular space is pseudo-compact if and only if every locally finite open covering of the space is finite [7] and Theorem 3.2.

Combining the fact that every totally bounded complete space is compact and the fact that every admissible uniform structure on a pseudo-compact space is totally bounded [7], the following is immediate:

3.4 <u>Corollary</u> A topologically complete completely regular space is compact if and only if every locally finite open covering of the space has a finite subcovering.

3.5 Corollary A topologically complete completely regular space is n-compact if every locally finite open covering of the space has a fewer than n subcovering.

Proof: Let $\mathcal{U}$ be an admissible complete uniform structure on such a space. By Theorem 3.2, it is n-totally bounded complete, so that the space is n-compact by Theorem 1.7.

3.6 Theorem A topologically complete completely regular space is realcompact if and only if every locally finite open covering of the space has a non-measurable subcovering.

Proof: ($\Leftarrow$). Let m be the first measurable cardinal number. By Theorem 3.2, the space is m-totally bounded complete. Hence it is realcompact by Theorem 2.13.

($\Rightarrow$). Let X be a realcompact space. Suppose that there is a locally finite open covering $\mathcal{U} = (U_\iota)_{\iota \in I}$ of X which has no non-measurable subcovering. Then we can construct a closed discrete subspace of X whose cardinal number is measurable. Let $\alpha$ be the first ordinal whose cardinal number is m. Let $x_0$ be any element of X. For any $\tau < \alpha$, suppose that we have a net $\{x_\lambda\}_{\lambda<\tau}$ such that $\{x_\lambda\}_{\lambda<\tau}$ is a

discrete family. Since $\mathcal{U}$ is locally finite, there is a neighborhood $V_\lambda$ of $x_\lambda$ such that $V_\lambda$ meets only finitely many members of $\mathcal{U}$. Let $\mathcal{V}_\tau = \{U \mid U \in \mathcal{U}$ and $U \cap V_\lambda \neq \Phi$ for some $\lambda < \tau\}$. Since $|\mathcal{V}_\tau| < m$, $\bigcup \mathcal{V}_\tau \neq X$. Let $x_\tau$ be any element of $\complement \bigcup \mathcal{V}_\tau$. Then $\{x_\lambda\}_{\lambda \leq \tau}$ is also discrete. Indeed, let $U_{\iota_\tau}$ be a member of $\mathcal{U}$ which contains $x_\tau$. Then $U_{\iota_\tau} \cap V_\lambda = \Phi$ for all $\lambda < \tau$, so that every element of $\{x_\lambda\}_{\lambda<\tau}$ has a neighborhood which contains at most one element of $\{x_\lambda\}_{\lambda<\tau}$.
If $x \notin \{x_\lambda\}_{\lambda<\tau}$, then $\complement \bigcup_{\lambda<\tau}\{x_\lambda\}$ is an open neighborhood of x which contains at most one element of $\{x_\lambda\}_{\lambda \leq \tau}$. Hence by the induction, we have a net $\{x_\lambda\}_{\lambda<\alpha}$ which is a discrete family. Thus, $\{x_\lambda\}_{\lambda<\alpha}$ is a closed discrete subspace of X whose cardinal number is measurable, which is a contradiction.

3.7 Corollary For a topologically complete completely regular space, every closed discrete subspace has non-measurable cardinal number if and only if every locally finite open covering of the space has a non-measurable subcovering.

Proof: It follows immediately from that both conditions are equivalent to that the space is realcompact.

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